

Signal Shaping by Non-Uniform QAM for AWGN Channels and Applications Using Turbo Coding

Dirk Sommer and Gerhard P. Fettweis
Dresden University of Technology, Germany
Email: sommer@ifn.et.tu-dresden.de

Abstract: A non-uniform signal set constellation is used in order to provide shaping gain. An approximately Gaussian distribution of the transmitter output signal is achieved by using equally likely signal points with unequal spacing. In this way it is possible to obtain shaping gain without any additional computational effort. The method is also applicable when bit interleaved coded modulation in conjunction with parallel decoding of the individual bit levels is used.

Keywords - Channel capacity, entropy, quadrature amplitude modulation, error correction coding.

I. INTRODUCTION

From information theory it is known that, in addition to coding gain, shaping gain can be obtained if the amplitude of the transmitter output follows a Gaussian distribution.

In most cases, a uniform QAM constellation together with different probabilities for each signal point is used to provide shaping gain. The idea has been investigated in [1]. Other examples are trellis shaping [2], the use of prefix codes [3], or similar, subdivision of the signal constellation into variable-size regions [4]. Recently, shaping also has been used in conjunction with multilevel coding/multistage decoding [5]. While trellis shaping requires an additional Viterbi-algorithm at the transmitter but maintains a fixed data rate, the approaches in [3;4] are conceptually simpler but possibly impractical since the non-constant data rate may involve buffer over- or underrun. Also, resynchronisation has to be carried out frequently in this case to avoid catastrophic error propagation.

Non-uniform (NU) signal point constellations have a long tradition, e.g., in order to create a signal set for hierarchical (rate adaptive) transmission [6; 7], sometimes also called multiresolution modulation. An approach to use an asymmetric coded-modulation scheme in order to optimize trellis coding has been presented in [8].

In this paper we make use of a non-uniform signal point constellation in order to obtain shaping gain. Contrary to other approaches to signal shaping, each signal point is assumed to be chosen with the same

probability. Therefore we arrange the points in a way that a Gaussian distribution of the output signal at the transmitter results, i.e., using more points at the lower power levels and less at higher.

It is shown that for higher order modulation schemes using these constellations, shaping gains in the order of 1 dB are possible. Furthermore, it is demonstrated that the method provides similar gain when bit-interleaved coded modulation (BICM), recently introduced by Caire et. al [9], in conjunction with Gray labeling and simple parallel decoding of the different bit levels is used. Finally, some examples using turbo coding over the AWGN channel are given in order to verify that the principle works when used with common coding schemes.

In the following, all considerations regarding the channel capacity are made with respect to amplitude shift keying (ASK). This can of course simply be extended to QAM by adding a second dimension.

The transmission model is depicted in Fig. 1. The info bits d_k are first coded and bitwise interleaved. Then m interleaved coded bits $x_0, x_1, \dots, x_{m-1}$ are mapped onto a symbol $a_j \in \mathcal{A}$ of the signal space using uniform or non-uniform ASK with Gray labeling. The symbol a_j passes a memoryless white Gaussian noise channel, and y is received.

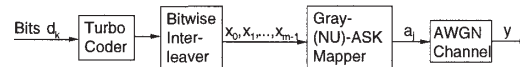


Figure 1. Transmission model

II. SHAPING PRINCIPLE

A. Signal set constellation

In order to find a signal set with the desired properties, an empirical approach can be made as follows. Each point of the signal constellation is assumed to be chosen with the same probability, and we want the output of the transmitter to give an approximately Gaussian distribution. Therefore, using N-ASK, $N = 2^m$, we split the Gaussian cumulative distribution function (CDF) into N sections of equal probability, i.e., the ordinate of the CDF is partitioned into N equal parts. Since each signal point shall represent exactly one of the sections, each signal point (placed on the abscissa)

will be chosen such that the corresponding *probability* is exactly in the middle of a section.

In other words, the points u_j on the abscissa of the Gaussian PDF $f_u(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$ (to be approximated) will be chosen such that

$$\int_{-\infty}^{u_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = p_j \quad (1)$$

for $p_j \in \{\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}\}$. Fig. 2(a) illuminates the procedure. Finally the u_j are normalized by a factor h , $a_j = hu_j$, such that the average power of the transmitted signal is one

$$\frac{1}{N} \sum_{j=0}^{N-1} a_j^2 = 1 \quad (2)$$

Alternatively, it has been tried to find the points u_i by calculating the mean of each probability section, i.e., $u_i = N \int_{u_a}^{u_b} u f_u(u) du$, where u_a and u_b are the limits of the current section, but the results did not change significantly.

Fig. 2(b) illustrates how the Gaussian CDF is approximated. The corresponding PDF consists of δ -functions of height $1/N$ at locations u_i . Signal points values for various ASK constellations are listed in the Appendix.

B. Peak-to-average power ratio expansion

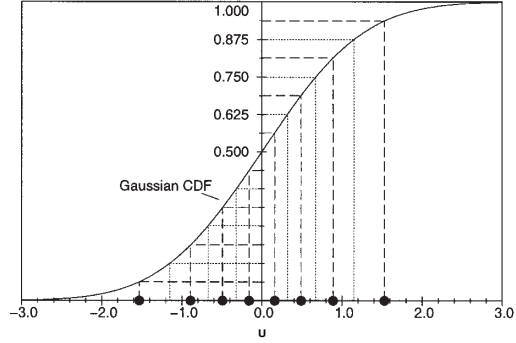
4-ASK	0.1 dB
8-ASK	0.7 dB
16-ASK	1.5 dB
32-ASK	2.3 dB
64-ASK	3.1 dB
128-ASK	3.8 dB

Table 1. PAPR expansion (NU-ASK vs. uniform ASK)

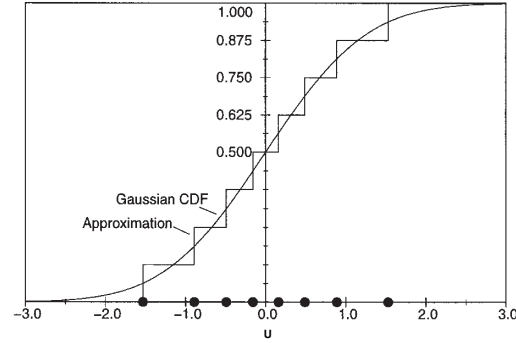
One of the drawbacks of the presented shaping principle is the expansion of the peak-to-average power ratio (PAPR) compared to uniform ASK. If for some reason not the average but the maximum transmission power is limited it might be better to dispense with that sort of signal shaping and exploit the higher signal to noise ratio of a uniformly distributed signal set instead, or to use other methods. Table 1 lists the PAPR expansion for the different modulation schemes. In all cases the PAPR expansion exceeds the shaping gain (see sec. III.C) so the kind of shaping presented here is generally not useful when the power constraint is only on the maximum power.

III. CAPACITY FOR GAUSSIAN NU-ASK

In the following, the capacity of the non-uniform signal constellation is calculated under the assumption of equal probability for each signal point. Two cases are distinguished.



(a) Signal points u , for 8-NU-ASK



(b) Approx. of the Gaussian CDF by 8-NU-ASK

Figure 2. Signal set constellation

A. Signal set capacity

First, the capacity of the signal set that can principally be reached at a certain SNR is calculated, i.e., when all information contained in received signal y is exploited. This could possibly be done using multi-level-coding (MLC) / multi-stage-decoding (MSD) as proposed in [10; 5], or by BICM with additional feedback decoding (AFD) as presented in section IV (see also Fig. 4(b)). This capacity will be referred to as C_{SS} (signal set capacity). Under the assumption of equally likely source signals, C_{SS} can be calculated as follows [11]

$$C_{SS} = I(\mathbf{X}, Y) = H(Y) - H(Y|\mathbf{X}) \quad (3)$$

where the random variable $\mathbf{X} = (X_0, X_1, X_2, \dots, X_{m-1})$ represents the bits of the 2^m -ASK constellation, $I(\dots)$ denotes the mutual information between two random variables, and $H(\dots)$ the entropy.

B. Parallel decoding capacity

Second, each bit-level is considered separately, i.e., the assumption is made that each bit level is decoded without prior knowledge of the decoding results of some other bit levels. The corresponding decoding scenario is depicted in Fig. 4(a). From the received signal y , a metric is calculated for each bit. Subsequent decoding is performed without the attempt to improve that metric using the decoding results. Caire et. al noticed [9] that in this case the capacity strongly depends on the labeling scheme and conjectured that Gray labeling maximizes the capacity. Therefore no other labeling schemes will be considered in the sequel. As remarked in [10] this is equivalent to the capacity when MLC is used with parallel independent decoding of the different bit levels. Adopting the term of [10] the capacity will be referred to as C_{PD} (parallel decoding of individual levels).

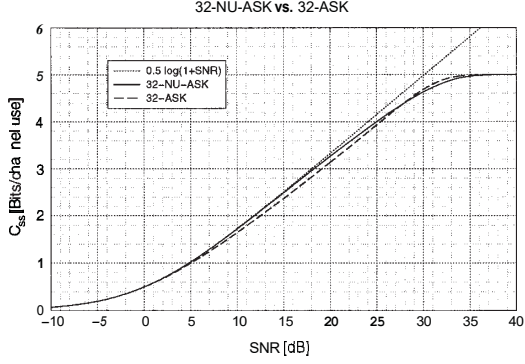
C_{PD} can be calculated as

$$\begin{aligned} C_{PD} &= \sum_{i=0}^{m-1} I(X_i, Y) \\ &= \sum_{i=0}^{m-1} \left[H(Y) - \sum_{k=0}^1 H(Y|x_i = k) \Pr(x_i = k) \right] \end{aligned} \quad (4)$$

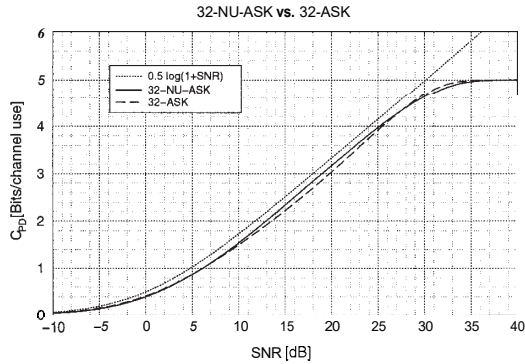
which means that different from (3) one has to sum up the mutual information between Y and the individual bits X_i while averaging the conditional entropy $H(Y|x_i = k)$ over the possibilities $x_i = k$, $k \in \{0, 1\}$. In the sequel, we use the term PD for BICM, too, in order to distinguish from BICM with AFD, although the bit levels are not really decoded in parallel.

C. Shaping gain results

The shaping gain is usually defined as the reduction in average power of the signal set at a fixed data rate [2] compared to a rectangular (hypercube) constellation. With the approach taken here this definition is not applicable since there would be no shaping gain at all. Consequently, we need a different definition and use the term *shaping gain* for the *gain in SNR of a NU-ASK compared to a uniform ASK at a certain capacity*. In order to determine the shaping gain, (3) and (4) have been evaluated for the constellations presented in section II.A. Fig. 3(a) and Fig. 3(b) give an example of the capacity of the 32-NU-ASK constellation for C_{SS} and C_{PD} , respectively. As can be seen from Fig. 3(a), C_{SS} is very close to the optimum if less than ≈ 3 [bits/channel use] are transmitted. In both cases the highest gain occurs at about 2.5 [bits/channel use], i.e., if a code rate of $R \approx 0.5$ is used. Evaluating other NU-ASK constellations confirms that this generally seems to be the case. With turbo codes a powerful class of codes operating at this rate exists that can exploit this gain. However, contrary to common coded modulation or multilevel coding schemes, the number of bits of the



(a) Signal set capacity



(b) Parallel decoding capacity

Figure 3. Capacities for NU-ASK vs. ASK

signal set has to be doubled instead being expanded by only 1 bit, since there is no shaping gain at code rates close to 1. This is a further drawback if the PAPR is of some importance.

Gain (dB)	C_{SS}	C_{PD}
4-ASK	0.07	0.01
8-ASK	0.32	0.15
16-ASK	0.59	0.50
32-ASK	0.82	0.81
128-ASK	1.14	1.18

Table 2. Shaping gain at $R = 0.5$

Table 2 lists the capacity gains of C_{SS} and C_{PD} at $R = 0.5$. Clearly, the use of NU-ASK for shaping offers advantages only for higher order modulation schemes. However, it can provide shaping gains up to ≈ 1 dB independent of whether BICM with PD or a coding/modulation scheme exploiting C_{SS} is used.

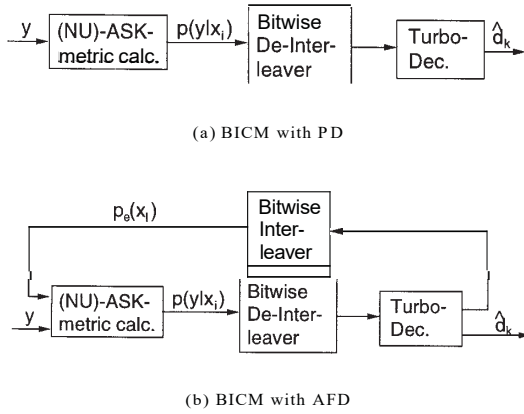


Figure 4. Decoder structure

IV. APPLICATIONS USING TURBO CODES

Turbo codes, first introduced in [12], are known as a class of powerful codes that can approach the channel capacity relatively close. The BCJR-algorithm [13] used for turbo decoding requires $\Pr(y|x_i = k)$ for the branch metrics. When all signal points $a_j \in A$ are assumed to be equally likely, $\Pr(y|x_i = k)$ can be simply determined by summing up the transition probabilities over all signal points a_j with $x_i = k, k \in \{0, 1\}$, for each bit of the ASK symbol,

$$\Pr(y|x_i = k) = c_1 \sum_{a_j: x_i = k} e^{-\frac{(y-a_j)^2}{2\sigma^2}} \quad (5)$$

where σ^2 is the (one dimensional) noise variance, assumed to be known at the receiver. If a-priori information of the bits x_i is provided, then $\Pr(y|x_i = k)$ can be calculated as

$$\Pr(y|x_i = k) = c_2 \sum_{a_j: x_i = k} \left(e^{-\frac{(y-a_j)^2}{2\sigma^2}} \prod_{\substack{l=0 \\ l \neq i}}^{m-1} \Pr(x_l) \right) \quad (6)$$

which means that additionally, the probability of the current signal point a_j is taken into account, not using the input probability of the bit, for which the output soft value has to be determined. The expression in eq. (6) is a variant of the method presented in [14] for GSM speech frame decoding, and has also recently been used in [15] for feedback demodulation of BICM. Constant factors (c_1, c_2) in (5) and (6) can be neglected since they cancel out when likelihood ratios $\Pr(y|x_i = 0)/\Pr(y|x_i = 1)$ are computed for subsequent decoding.

In the following, the theoretical results have been tested using turbo codes over the AWGN channel. Two cases have been considered. First, BICM with simple

subsequent turbo decoding was used, see Fig. 4(a). In this case, the maximum transmission rate is bounded by C_{PD} .

Second, the extrinsic soft values of the turbo decoder for info and parity bits were fed back into the metric calculator in order to serve as a-priori information according to (6), see Fig. 4(b), (AFD, additional feedback decoding). The corresponding capacity is C_{SS} . The soft values were calculated according to the method presented in [16].

The interleaver length was ≈ 8000 bits (slightly adapted to give a multiple of the actual QAM constellation size) for the turbo coding and twice that much (one complete coded burst) between coding and modulation. Both interleavers performed random permutation of the bits/soft values. The turbo code was the one presented in [12], i.e., a 16-state code with polynomials 21 and 37 (octal representation) and every second parity bit punctured.

V. CODING RESULTS

Fig. 5(a) and Fig. 5(b) compare turbo coding results for the AWGN channel using different modulation orders without and with AFD, respectively.

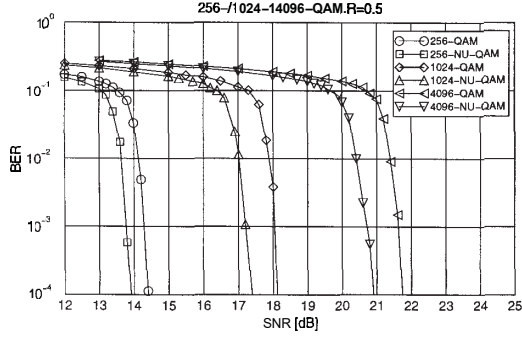
As can be seen, the difference between QAM and NU-&AM matches the shaping gain results of table 2 relatively close at $\text{BER}=10^{-4}$ independent of whether AFD was used or not. Lower BER have not been considered since the typical turbo code error floor starts already in that region. If additional feedback for the metric calculation is provided, the results improve by about 0.5 – 1 dB. The necessary SNR for 16-/32-/64-ASK with $C_{SS}(\text{NU-ASK}), C_{SS}(\text{ASK}), C_{PD}(\text{NU-ASK})$, and $C_{PD}(\text{ASK})$, respectively, are listed in table 3. The

SNR (dB)	$C_{SS}(\text{NU-ASK})$	$C_{SS}(\text{ASK})$
16-ASK (256-&AM)	11.94	12.53
32-ASK (1024-QAM)	15.05	15.87
64-ASK (4096-QAM)	18.10	19.10
	$C_{PD}(\text{NU-ASK})$	$C_{PD}(\text{ASK})$
16-ASK (256-&AM)	12.75	13.25
32-ASK (1024-QAM)	15.96	16.77

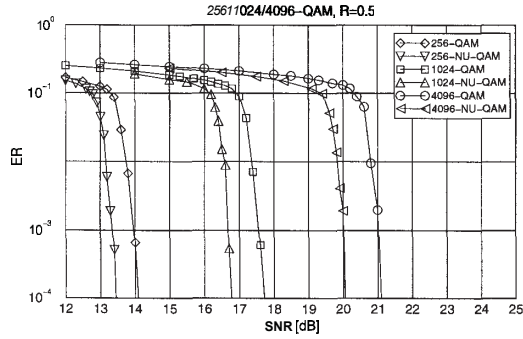
Table 3. Capacity at $R = 0.5$

SNR necessary to reach $\text{BER}=10^{-4}$ are in all cases within 1 – 2 dB from the capacity result.

For all simulation, 12 iterations of inner turbo decoding have been used. With AFD, 5 outer iterations have been made additionally resulting in a total of 60 iterations. This is impractical, however, the number could be dramatically lowered, if some suitable criterion on when to stop an inner (turbo code) iteration was applied. Simulations show that for the first outer iterations only a few inner iterations are necessary.



(a) With PD (without AFD)



(b) With AFD

Figure 5. Comparison between NU-SAM and uniform QAM for BICM

VI. CONCLUSIONS

A simple method to obtain shaping gain for higher order ASK modulation on AWGN channels was presented.

Using non-uniform ASK in conjunction with BICM, shaping gains in the order of the theoretically predicted capacity gains can be achieved with a real coding system, independent of whether parallel decoding (no AFD) or additional feedback decoding is used. Improvements due to AFD are within 0.5 - 1 dB.

Compared to other shaping methods, the achieved gain is lower, and even for higher order modulation scheme it does not seem approach the ultimate gain of 1.53 dB. Also, the fact that shaping gain is exhibited only when the constellations are used with code rates $R = 0.5$ might lead to a higher decoding effort in comparison to higher rate coding schemes. Furthermore, the enlarged signal point constellation, entailing a considerably higher peak-to-average power ratio, can be prohibitive in some applications.

The main Advantage is the simplicity of the method.

Compared to uniform ASK with $R = 0.5$, shaping by NU-ASK does not entail any additional computational effort. Only the lookup tables for signal transmission and metric calculation have to be replaced, so signal shaping with gains up to about 1 dB has been demonstrated without changing the coder/decoder structure.

VII. APPENDIX

Signal point constellations for NU-ASK

4-NU-ASK

f0.377525	41.362892
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8-NU-ASK

f0.170563	±0.529790	f0.961643	f1.662971
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16-NU-ASK

f0.081626	±0.246854	f0.418533	±0.602594
f0.807881	f1.050833	51.371404	±1.938154

f0.039934	50.120107	f0.200994	f0.283207
f0.367358	f0.453957	f0.544127	f0.638580
f0.739052	f0.847173	±0.965699	f1.099117
f1.254465	f1.446126	f1.709493	f2.196958

64-NU-ASK

±0.019745	f0.059337	410.099030	±0.138925
±0.178920	±0.219320	±0.259922	±0.301130
±0.342742	±0.384959	±0.427783	±0.471516
±0.516057	±0.561708	±0.608471	±0.656648
±0.706238	±0.757748	±0.811177	±0.867029
±0.925710	±0.987623	±1.053575	±1.124275
f1.201035	f1.285369	f1.379905	f1.488176
f1.617051	f1.779256	f2.007313	f2.441711

128-NU-ASK

±0.00984409	f0.0295222	f0.0492204	f0.0689388
±0.0886772	±0.108456	±0.128275	±0.148144
±0.168073	±0.188063	±0.208133	±0.228283
±0.248524	±0.268866	±0.289328	±0.3099
±0.330604	±0.351458	±0.372452	±0.393618
±0.414954	±0.436492	±0.45822	±0.480169
±0.50235	±0.524772	±0.547465	±0.57044
±0.593716	±0.617313	±0.641253	±0.665564
±0.690277	±0.715413	±0.741	±0.76708
±0.793683	±0.820858	±0.848647	±0.877099
±0.906274	±0.936234	±0.967047	±0.998796
±1.03158	±1.0655	±1.10067	±1.13725
±1.1754	±1.21533	±1.25726	±1.30152
±1.34844	±1.39851	±1.45232	±1.51064
±1.57454	±1.64553	±1.72583	±1.81897
±1.93098	±2.07387	±2.27758	±2.6734

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