

The Cambridge Companion
to
PASCAL



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3 Pascal's work on probability

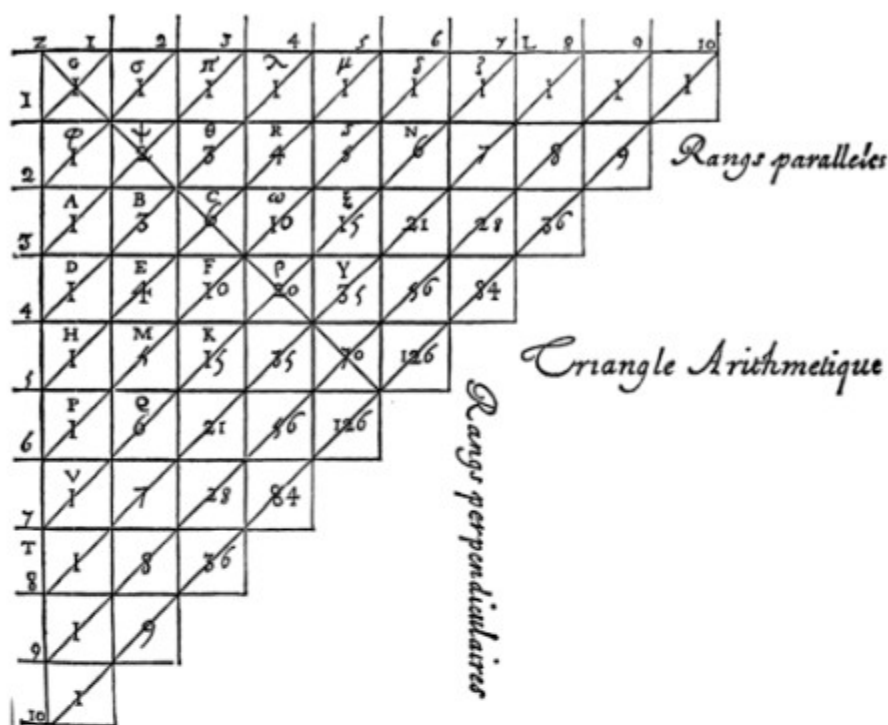
Anceps fortuna aequitate rationis reprimitur

Before the time of Pascal there was no theory of probability, merely an understanding (itself incomplete) of how to compute 'chances' in gaming with dice and cards by counting equally probable outcomes. In addition, problems encountered in the enumeration of dice throws and the counting of arrangements and selections of things had led to an incipient mathematical theory of combinations and permutations, but the rules that appeared in the works of such authors as Tartaglia (1500–57) and Cardano (1501–76) still had the form of recipes rather than as parts of a coherent whole. It fell to Pascal to bring together the separate threads and weave them into a structure that enabled him to progress far beyond his predecessors by introducing entirely new mathematical techniques for the solution of problems that had hitherto resisted solution, techniques which became the foundation of the modern theory of probability.

Pascal's influence was not direct, for none of his writings on probability were published during his lifetime, but instead was transmitted via Huygens to James Bernoulli, where it appeared in the latter's influential *Ars conjectandi* of 1713, and via the *Essay d'analyse sur les jeux de hazard* of Montmort, first published in 1708. These two books, together with De Moivre's *The Doctrine of Chances* (1718), firmly established probability theory as a branch of mathematics. Later scholarship has confirmed the view that Pascal may justly be regarded as the father of the theory of probability.¹

SUMMARY OF PASCAL'S MATHEMATICAL WORK

In 1631 Blaise Pascal's father Etienne (himself an able mathematician who gave his name to the '*limaçon* of Pascal') moved his family

Fig. 1 Pascal's arithmetical triangle from the *Traité*

to Paris in order to secure his son a better education. In 1635 he was one of the founders of Marin Mersenne's 'Academy', the finest exchange of mathematical information in Europe at the time. To this informal academy he introduced his son at the age of 14, and Blaise immediately put his new source of knowledge to good use, producing (at the age of 16) his *Essay pour les coniques*, a single printed sheet enunciating Pascal's Theorem, that the opposite sides of a hexagon inscribed in a conic intersect in three collinear points.

Mersenne's *Harmonicorum libri XII* of 1636, and the two-volume French version *Harmonie universelle* published in 1636 and 1637, contain the first accounts of the mathematical theory of permutations and combinations in recognisably modern form, applied to musical notes. Included is a table of the number of permutations of r things of one kind and s things of another kind for $r = 0$ to 12 and $s = 0$ to 25. This form of 'arithmetical triangle' was used by the younger Pascal in due course, and became known as 'Pascal's arithmetical triangle' (see figures 1 and 2). It is almost certain that Pascal

1	1	1	1	1	1	1	.	.	.
1	2	3	4	5	6	.	.	.	.
1	3	6	10	15	.	.	.	.	.
1	4	10	20	.	.	.	.	.	.
1	5	15	.	.	.	.	.	.	.
1	6	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Fig. 2 Pascal's arithmetical triangle

learnt of it and its combinatorial uses from the Mersenne books, for in them the author paid tribute to Etienne Pascal's knowledge of music, having previously dedicated his treatise on the organ to him.

At the age of 18 Pascal turned his attention to constructing a calculating machine to help his father in his calculations, and within a few years he had built and sold fifty of them. Some still exist. (The computer programming language PASCAL is named in honour of this achievement.) In 1646 he started work on hydrostatics, determining the weight of air experimentally and writing on the vacuum (leading ultimately to the choice of 'Pascal' as the name for the SI unit of pressure).

In 1654 Pascal returned to mathematics, extending his early work on conics in a manuscript which does not now exist, though it was seen by Leibniz. In the same year he entered into correspondence with Pierre de Fermat of Toulouse about some problems in calculating the odds in games of chance, and this led him to write the *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matière*, probably in August of that year. Not published until 1665, this work, and the correspondence itself which was published in 1679, is the basis of Pascal's reputation in probability theory as the originator of the concept of expectation and its use recursively to solve the 'Problem of Points', as well as the justification for calling the arithmetical triangle 'Pascal's triangle'. His advances, considered to be the foundation of modern probability theory, are described in detail below.

Later in 1654 Pascal underwent a religious experience as a result of which he almost entirely abandoned his scientific work, although in 1656 he posed Fermat a problem in probability which later became well known as the 'Gambler's Ruin' problem. He devoted his remaining years to writing the *Lettres provinciales* and the *Pensées*. The latter includes his famous 'Wager'. In 1658–9 he briefly returned to mathematics, writing on the curve known as the cycloid, but his final input into the development of probability theory arises through his presumed contribution to *La Logique, ou l'art de penser* by Antoine Arnauld and Pierre Nicole, published in 1662 and often referred to in English as the *Port-Royal Logic* through the association of its authors, and Pascal himself, with Port-Royal Abbey. The Gambler's Ruin, the Wager and the *Port-Royal Logic* are considered below.

CORRESPONDENCE WITH FERMAT, 1654

The centrepiece of Pascal's correspondence with Fermat in the summer of 1654 is a gambling problem known in English as the Problem of Points. Also known simply as the 'division problem' (*Problème des partis*), it involves determining how the total stake should be divided in the event of a game of chance being terminated prematurely.

Suppose two players X and Y stake equal money on being the first to win n points in a game in which the winner of each point is decided by the toss of a fair coin. If such a game is interrupted when X still lacks x points and Y lacks y , how should the total stake be divided between them? In the middle of the sixteenth century Tartaglia famously concluded that 'the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation'. A century later the correct solution was derived by three different methods during the correspondence between Pascal and Fermat, after the problem had been brought to Pascal's attention by Antoine Gombaud, chevalier de Méré.

The first method involves a straightforward enumeration of the possible ways the game could have been completed. At most $(x + y - 1)$ more tosses would have settled the game, and if this number of tosses is imagined to have been made, the resulting $2^{(x+y-1)}$ possible games, each equally probable, may be classified into those which X wins and those which Y wins, the stakes then being divided

in this proportion. Thus the real game, of indeterminate length, is embedded in an imaginary game of fixed length. This method of solution depends upon the peculiar fact that the order of occurrence of the heads and tails is of no significance, only their total numbers, as both Pascal and Fermat had realised. The solution thus involves counting combinations and summing binomial coefficients, which will be explained more fully in the next section. 'But', wrote Pascal, 'because the labour of the combinations is excessive I have found a shortcut or, more exactly, an alternative method which is much quicker and neater' (OC I, 146). This method involves the path-breaking procedure of computing expectations recursively.

Pascal's key advance was to understand that the value of a gamble is equal to its mathematical expectation computed as the average of the values of each of two equally probable outcomes and that this precise definition of value lends itself to recursive computation, because the value of a gamble that one is certain to win is undoubtedly the total stake itself. Thus, if the probabilities of winning a or b units are each one-half, the expectation is $\frac{1}{2}(a + b)$ units, which is then the value of the gamble. In *Ars conjectandi* (1713) James Bernoulli called this 'the fundamental principle of the whole art'. Pascal has invented the concept of 'expected value', that is the probability of a win multiplied by its value, and has understood that it is an exact mathematical concept that can be manipulated.

In his letter to Fermat of 29 July (OC I, 146), Pascal develops the recursive argument applied to expected values in order to find the correct division of the stake money, and thus computes the 'value' of each successive throw. As I shall show, in the *Traité* the same idea is more formally expressed, and in particular Pascal there gives as a principle the value of the expectation when the chances are equal. He remarks at one point that 'the division has to be proportional to the chances' (OC I, 305), but in the solution to the problem when the players have equal chances the question of computing an expectation for unequal chances does not arise; that extension was first formally made by Huygens in his *De ratiociniis in ludo aleae* of 1657. It is known that when Huygens spent July–September 1655 in Paris he had the opportunity to discuss Pascal's work on probability problems with Roberval, and presumably learnt of the concept of mathematical expectation then, though it is often attributed to him.

The easiest way to understand how Pascal used expectation and recursion to solve the problem of points is to visualise the event tree of possible further results. Each bifurcation corresponds to a toss, one branch for X winning it and the other for Y , and successive bifurcations must lead eventually to tips corresponding to the whole game being won by either X or Y . Considering now the expectation of X (say), each tip can be labelled with his expectation, either S (the total stake) or 0 , as the case may be. Applying now Pascal's expectation rule for equal chances, each bifurcation has an expectation associated with it. Working recursively down the tree from the tips to the root, we arrive at the solution to the problem. If $E(x, y)$ be the expectation of player X when he lacks x points and Y lacks y , then the recursion is

$$E(x, y) = \frac{1}{2}E(x-1, y) + \frac{1}{2}E(x, y-1)$$

By these methods, and his knowledge of the arithmetical triangle, Pascal was able to demonstrate how the stake should be divided between the players according to the partial sums of the binomial coefficients, a result which he had already obtained by enumeration, for both he and Fermat had realised that the actual game of uncertain length could be embedded in a game of fixed length to which the binomial coefficients could then be applied.

Pascal formally proved his solution in the third section of part 2 of his *Treatise on the Arithmetical Triangle*, to which we now turn.

TRAITÉ DU TRIANGLE ARITHMÉTIQUE, 1654

The *Traité du triangle arithmétique* itself is 36 pages long (setting aside *quelques autres petits traitez sur la mesme matière*) and consists of two parts. The first carries the title by which the whole is usually known, in English translation *A Treatise on the Arithmetical Triangle*, and is an account of the arithmetical triangle as a piece of pure mathematics. The second part, *Uses of the Arithmetical Triangle*, consists of four sections:

- Use (1) ... in the theory of figurate numbers
 (2) ... in the theory of combinations
 (3) ... in dividing the stakes in games of chance
 (4) ... in finding the powers of binomial expressions