

The Origin of Probability and The Problem of Points

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The popularity of gambling has been documented throughout history, and even prehistory, over thousands of years. In ancient times, ankle bones of animals, or *astragali*, were used by children as dice are used today. The irregular shapes of the bones provided an element of uncertainty, essential for an interesting game. Here is a description by F.N. David of this primitive toy:

The astragalus has little marrow in it and was possibly not worth cracking for the sake of its contents ... it is knobbly and presents no flat curves for drawing, as does the shoulder blade for example (David 1-2).

The popularity of astragali is seen in the variety of cultures that employed its use. Egyptians played games of "Hounds and Jackals," similar to the present day "Snakes and Ladders", with the help of animal bones. The Greek poet Homer recounted that Patroclus, as a small boy, nearly killed his opponent playing a game of "knucklebones". However, according to Florence N. David's account, based on the literature of the times, the Romans utilized the astragali with "more zeal" than the Greeks (David 5). In fact, this zeal would become so overwhelming, that the Romans eventually banned gaming, except at certain seasons (David 8). Similar attitudes towards gambling would propagate centuries after, to 16th century Europe, and down to the present day.

Thus it may seem natural that mathematicians who lived in the sixteenth century would begin to develop a theory of probability because of their frequent contact with games of chance, and indeed many historians view this development as a result of a few interested mathematicians focusing on particular problems to which their attention had been drawn, relating to games of chance. Indeed, many mathematicians chose problems that arose specifically with relation to gambling as examples to study in regard to the computation of probabilities. However, some historians of mathematics hold the view that other economic and political pressures caused the evolution of probability theory. In fact, according to L.E. Maistrov, "If we assume, as is commonly done, that probability theory owes its origin to gambling, it would be necessary to explain why gambling, which had been in existence for six thousand years, did not stimulate the development of probability theory until the seventeenth century, while in that particular century the theory originated on the basis of the same games of chance." Along the same lines, Ivo Schneider proposes that the economy of the 15th and 16th century was the source of inspiration for the solution of the problem of points. A component of risk, infused with the popularity of games of chance, sparked an interest with the merchants of the time. After Thomas Aquinas' contribution to economic theory, money was seen as "an abstract quantity of reference, and not as a medium of exchange dependent on its value as a metal" (Schneider 223). This idea encouraged the use of a device which Schneider called "an ingenious juridical bridge", called "triple contract." In essence, the European economy relied on a system of insurance. This jurisprudential invention operated as follows:

1. The lender gives a merchant a sum of money as an investment in a commercial enterprise conditional upon the additional payment of a certain proportion of the investment in case of a gain made by the debtor.
2. The lender concedes to the merchant a certain fraction of his assumed profit as a guarantee that the investment will be repaid even in case the capital invested in the enterprise is totally or partially lost.
3. The lender sells the right to the remaining profit for a part of this profit that must be repaid in any case together with the original investment.

(Schneider 224)

At first sight, it would seem that the lender is at a loss in this contract, since if the capital is lost, and no profit is made, the original loan cannot be repaid. What reassurance can the lender receive to compensate for such a misfortune? Domingo de Soto, a member of a small group of influential theologians and jurisconsults, declared in 1559 that: "it is not a simple donation, but a kind of pact 'I give in order that you give'. Thus I expose my

money to danger, just as you in turn expose yours, and the risk of one is judged to be as great as that of the other." In essence, this taking of risk became a large part of the system of loans and insurance. Schneider argues that when gambling was "interpreted in terms of risk taking" in the sixteenth century, problems such as the division of stakes became relevant and of practical importance to the merchants of the time (Schneider 227). In essence, according to this interpretation, whose relationship with the mathematical record is obscure, the problem of points addressed the economic needs of the time, rather than being arbitrarily chosen by a select group of individuals.

Here is a version of the problem of points:

Suppose A and B are playing a series of games where the winner of the overall match is the first player to win a total of k games. However, the contest is interrupted before either competitor has achieved that number. If A needs to take m more games to win the match, while B is n short of the necessary k, how should the stakes be divided? (Larsen 147).

Around 1400, a manuscript written by an unknown author studied by Laura Toti Rigatelli was discovered to deal with the problem of points. Surprisingly enough, this 600-year-old document contains a numerically correct solution of the problem, which some later mathematicians were unable to find.

Schneider interprets the manuscript as follows:

The text starts with a description of the following situation: "Two men play schacchi [chess] ... Each of the players stakes one ducat. They play for three games, that is to say the player who first wins three games will receive his opponent's ducat as well as his own. But they have to stop playing at the point when one of the two players has won two games and the other none. The question is, to how much of the opponent's ducat is the leading player entitled? (Schneider 227)

Note the stake now becomes part of 1 unit, rather than a mass of stakes. This may have a psychological impact on the author, since it now becomes more prudent to represent an unknown part of the ducat with a unit of reference, or in keeping with the contemporary "cossic" terminology, a *cosa*. Schneider uses the notation (i,j) to represent a situation in which the first player has won i games, and the second has won j , and writes $DS[a,b]$ to mean a division of the stakes giving a ducats to the first player, and b ducats to the second. Thus at the outset the position is $(0,0)$ and the appropriate division of stakes, if the game is halted at this point, would be $DS[1,1]$. If the first player wins the first game, the score becomes $(1,0)$, and the DS becomes $[1+x; 1-x]$, with x unknown. The author then states, without explanation:

You must understand that by reason he [the winner] must win in the second game as much as he has won in the first game, so that he has won another x and thus he is now entitled to $2x$ after two games, while the second who lost is entitled in total to his ducat minus $2x$.

In symbols:

Score $(2,0)$

$DS[1+2x; 1-2x]$

The author continues to say: It is clear that if the loser of the two games would win two further games from his friend neither one would have won anything from the other. Now let us suppose that the second begins to win a game from the first. I say that he wins in the game one ducat minus $2x$, which the first would have won. The reason is this: if the one who had won two games at first had also won the third game he would win from the second all the rest of the ducat, and this is what conversely the second wins from the first, that is, 1 ducat minus $2x$. Now take 1 ducat minus $2x$ from the amount that the first had won from the second, that is $2x$. There will remain to the first then $4x$ minus 1 ducat. The second, if he continues to play, will have 2 ducats minus $4x$ in the game. (Schneider 228-9)

In symbols, we have:

Score $(2,1)$

$DS[1+2x-(1-2x); 1-2x + (1-2x)]$, or $DS[1+(4x-1); 2-4x]$.

Notice that the unknown author here understands that no matter who wins the next game, the winnings must be equal, or that another x must be won from the other. How do we find x ? The author continues to use this principle: Notice now for the first, the winner of two games, that if the second had won those two games and were to win the third game, he would clearly win the total remaining part of the first's ducat, and if the first were to win this third game he would win 2 ducats minus $4x$. And in this way the second has to proceed against the risk. We assume now that the second wins [his] second game. In this case he is entitled to winnings of 2 ducats minus $4x$ from the first, and he is to collect from the first as much as the first would have won, because now each has won two games. Notice now how much the second wins from the first in the second game; he wins 2 ducats minus $4x$. Now we have to add 1 ducat on either side and we will have on the one side $4x$ and on the other 3 ducats minus $4x$. (Schneider 229)

Based on the obvious fact that when the score is (m,n) with $m=n$ then the DS is $[1, 1]$, the author reaches the "decisive point of the argument," as Schneider puts it. We now have an equation to determine the unknown x . In symbols:

Third game played:

Score(2,1)

$DS[1+(4x-1); 2 - 4x]$

Fourth game played:

Score(2,2)

$DS[1;1]$

The second player ends up winning back $2-4x$, and the first loses $4x-1$. The quantities must then be equal after the depicted fourth game played.

$$4x - 1 = 2 - 4x$$

$$8x = 3$$

$$x = 3/8$$

So going back to the scenario $score(2,0)$, the $DS[1+2x; 1-2x]$ becomes $DS[1+3/4; 1-3/4]$. The additional fraction of the opponent's ducat stands for $7/8$ of the entire stake. The text then continues, and unfortunately falters from the stated principles, and the author breaks off abruptly (Schneider 228-9).

It is unfortunate that this solution went unnoticed for nearly 90 years. The idea that winnings from winning one game to the next would remain equal is such a revolutionary idea that even Blaise Pascal would have trouble reconciling this to himself, 200 years later.

In 1487, the then prominent mathematician Luca Paccioli published *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*. This work was the oldest known printed source for the treatment of the problem of points (Schneider 230) until the discovery related above. Paccioli, frequently referred to in a 16th century citation index under the rubric "error di Fra Luca," was a very careless writer (Schneider 230). In fact, Girolamo Cardano devoted an entire chapter in his *Practica Arithmetica* to Brother Luca's errors (Smith 253). In addition, Paccioli "borrowed freely from various sources, often without giving the slightest credit -" (Smith 252), as textbook writers continue to do down to the present day, with similar pernicious results. Paccioli "was convinced of the existence of a uniquely determined solution to the problem." (Schneider 230) And in exploring such a solution, he proposed the following problem: A team plays ball so that a total of 60 points is required to win and the stakes are 22 ducats. Due to circumstances, they cannot finish the game and one side has 50 points, and the other 30. What share of the prize money belongs to each side? (Maistrov 17-18)

Paccioli's solution is described by L.E. Maistrov as follows: Paccioli carries out the following calculation: $5/11 + 3/11 = 8/11$; since $8/11$ corresponds to 22 ducats, $5/11$ will correspond to 13.75 ducats, and $3/11$ to 8.25 ducats. (Maistrov 18)

Paccioli believed that the stake should be divided proportionally to the scored points. Contemporaries of the time believed that such proposal was invalid, and that solutions from this method were questionable (Schneider 231).

However, Brother Luca was convinced that his method was correct, and that he had found a "uniquely determined solution to the problem." He had also referred to older opinions on how to resolve the problem of points as "preposterous" (Schneider 230). This unwavering faith in his solution can possibly be mirrored by the following quotation from Paccioli, when confronted with an contrary opinion. Do not proceed like some others who refer to the game of morra and say, if one in game with five fingers has 4 and the other 3, 'let us go back by one' so that one has 2 and the other 3. For this is not fair, because one resigns $1/3$ and the other less of his claim, so that they don't resign to the same extent. (Schneider 231)

Schneider speculates: Paccioli demonstrates that he was inextricably bound to his own economic model ... One understands from this quotation that it was not the logical inconsistencies of this 'reduction method' that drove Paccioli to reject it, but only its incompatibility with his own solution, which prescribes sharing the stakes in proportion to the number of games won. Only to those who accept Paccioli's way of sharing the stakes as reasonable does this refutation appear convincing.

Schneider refers to the fact that Paccioli makes the fair distribution of stakes correspond to the number of games already one to each player. And although this point of view does not offer a just division of stakes, Paccioli had contributed significantly to the problem of points in two significant ways. First, by simply offering a solution, Paccioli would have put forward a problem that would later be brought to Cardano's attention. Secondly, and most important of all, Paccioli advocated the view that a definite and unique solution exists for the problem of points. In Cardano's "Practica Arithmetica," published in 1539, he questions Paccioli's solution. He observes that Paccioli does not take into account the number of games yet to be won by the players (Maistrov 24). In one scenario of a game played up to 19 games, interrupted at a score (18:9), Cardano argues: The winner of eighteen games will have gained from his opponent only four gold coins, which is $1/3$ of his stake, and yet he lacks only one game to obtain the total winnings, whereas the second lacks ten. This is totally absurd. (Schneider 231)

Cardano proposes that the stakes should be divided in the ratio of $[1+2+3+ \dots + (S-q)]:[1+2+3+ \dots + (S-p)]$ (Maistrov 24), where S is the total number of rounds needed to win, and q and p are the numbers of games won already by each of the players. Using Paccioli's example, we get the ratio of $[55; 465]$ or $[11:93]$. Subsequent to his infamous dispute with Cardano on August 10, 1548, Niccolo Tartaglia published the work "Trattato generale di numerie misure," in 1556, in which dealt with the same problem Cardano had addressed. In section 20, titled "Error di Fra Luca dal Borgo," Tartaglia made the following remark against Paccioli: *His rule seems neither agreeable nor good, since, if one player has, by chance, ten points and the other no points, then following this rule, the player who has the ten points would take all the stakes which obviously does not make sense.* (Maistrov 24)

Tartaglia proposes using the difference of the scores already won to solve the problem. Using Paccioli's example to illustrate, Tartaglia performs the following calculations:

$$\begin{aligned} 50-30 &= 20; \\ 20/60 &= 1/3; \\ 22/3 &= 7+(1/3) \end{aligned}$$

So the "winner" would receive $22+7+(1/3) = 29+(1/3)$ ducats. Unfortunately, neither of the solutions posed by Cardano nor Tartaglia is correct, as neither had the idea that a fair division of stakes should be proportional to the *probability of winning* the whole stake by each player (Maistrov 27). Nearly another century would pass before a further controversy over the problem of points would rekindle the discussion. Around 1654, the Chevalier de Méré, a man described as a "con-man", nobleman, knight, fervent gambler, and a philosopher, would attempt to solve the problem of dice, which would lead to the resolution of the problem of points. This story began when de Méré proposed the analysis of a game of dice to Pascal, to which he would later implicitly refer as the *problem of the dice*. This event would eventually be recognized as the spark that enflamed Fermat and Pascal to resolve the problem of points. Unfortunately we can only speculate as to the exact conditions of their problem of points, since the first letter(s) of their correspondence are missing (Smith, 546). However, two interesting points can be seen from the remaining letters:

First, Pascal gave two methods to find the solution to the problem of points. This is astonishing since after his Italian predecessors had given up on the problem of points, they had declared that no uniquely determined

solution can be derived for the division of stakes (Schneider 223). Pascal's first solution gives a general formula for the computation of DS. He asserted that if one player requires n more victories to win the stakes, and the other lacks m victories, then the amount of the stakes given to the former should be proportional to sum of the first m entries in the $(n+m)$ th row of Pascal's triangle, divided by the sum of the entries in that row. Similarly the second player should receive the proportion of the sum of the last n entries of that row, divided by the sum of that same row. Modifying the units in Paccioli's example by a factor of 10, we illustrate the following:

A team plays ball so that a total of 6 points is required to win and the stakes are 22 ducats. The game is interrupted when one side has 5 points, and the other 3, leaving the first player one game short of victory, while the second requires three more wins. What share of the prize money belongs to each side? (Maistrov 17-18)

Here is Pascal's triangle for row $(1+3) = 4$, arranged in the matrix format used by Pascal, so that the "rows" involved are diagonal rows, read from lower left to upper right.

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1 1 1 1
1 2 3
1 3
1
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The first player would receive $7/8$ of the share, while the second would receive $1/8$ th. Note the same results appear for Pascal as it did for the anonymous author of 1400. The results are exactly the same, since as we now understand it does not matter how many points were already won, but rather how many points are left to be won, if the game had continued. And this principle exactly violates what Paccioli warned what should not be done.

A second interesting point comes from Pascal's second solution. The "geometrician" enumerates all possible events once the game had continued, and reflects on the previous idea. In symbols, a is the number of games won by player A, and b is the number of games won by B. A maximum of 3 games are yet to be played, which could progress in the following equally probable ways:

aaa aab aba abb baa bab bba *bbb

From the above, we can see that there is only 1 chance out of 8 (indicated by the asterisk) that the second player would win, so $1/8$ of the stakes should go to him. Here similarities between Pascal and the unknown author emerge again. We see more clearly here that the winnings for each game won should be the same, just as in 1400 the *una cosa*, though unknown, stayed the same. (Math Forum, 1 - 2) After the correspondence between Fermat and Pascal, Christian Huygens, a contemporary of Fermat, would independently solve the problem of points without influence from that correspondence. Huygens would then go on to develop a list of further propositions that would become essential to the foundations of probability, involving an intuitive notion of expectation, in addition to the intuitive notion of probability which was used by Fermat and Pascal. It is interesting to see how after 200 years, a full circle should come around so dramatically, that no one should realize its significance for another 3.5 centuries, after the discovery of the original manuscript. Perhaps since the unknown author has unsuccessfully produced the solution using his original principles, that his version did not become popular. There is no evidence that his solution had a significant impact on 15th century European mathematicians, and Paccioli probably was not aware of his analysis, though this cannot be definitely known. However, Fermat and Pascal's attempt to solve the same problem in terms of the likelihood of a victory for either side would eventually be credited as the critical step toward the creation of the field of probability.

Works Cited

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