

Figure 8.7 The impulse force as a function of time for the two colliding particles described in Figure 8.6a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

When two particles of masses m_1 and m_2 collide, the collision forces may vary in time in a complicated way (Fig. 8.7). If $\mathbf{F}_{12}$ is the force on m_1 due to m_2, then the change in momentum of m_1 due to the collision is given by Equation 8.8:

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

Likewise, if $\mathbf{F}_{21}$ is the force exerted on m_2 by m_1, the change in momentum of m_2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Newton's third law states that the force on m_1 by m_2 is equal to and opposite the force exerted on m_2 by m_1; that is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. (This is described graphically in Fig. 8.7.) Hence, we conclude that

$$\Delta \mathbf{p}_1 = \Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect if no external forces are acting on the system (Section 8.2), or if the external forces are assumed to act for too short a time to make a significant change in momentum. However, the result is also valid if we consider the motion just before and just after the collision. Because the forces due to the collision are internal to the system, they do not change the total momentum of the system. Therefore, for any type of collision, the total momentum of a system just before the collision equals the total momentum of a system just after the collision.

CONCEPTUAL PROBLEM 2

You are watching a movie about a superhero and notice that the superhero hovers in the air and throws a piano at some bad guys while remaining stationary in the air. What's wrong with this scenario?

Example 8.4 Carry Collision Insurance

An 1800-kg car stopped at a traffic light is struck from the rear by a 900-kg car and the two become entangled. If the smaller car was moving at 20 m/s before the collision, what is the speed of the entangled mass after the collision?

Reasoning The total momentum of the system (the two cars) before the collision equals the total momentum of the system after the collision because momentum is conserved.

Solution The magnitude of the total momentum of the system before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_i = (900\ \text{kg})(20\ \text{m/s}) = 1.80 \times 10^4\ \text{kg·m/s}$$

After the collision, the mass that moves is the sum of the masses of the cars. The magnitude of the momentum of the combination is

$$p_f = (m_1 + m_2)v_f = (2700\ \text{kg})v_f$$

Equating the momentum before to the momentum after and solving for v_f, the speed of the combined mass, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4\ \text{kg·m/s}}{2700\ \text{kg}} = 6.67\ \text{m/s}$$

8.4 • ELASTIC AND INELASTIC COLLISIONS IN ONE DIMENSION

As we have seen, momentum is conserved in any type of collision. Kinetic energy, however, is generally *not* constant in a collision because some of it is converted to thermal energy, to internal elastic potential energy when the objects are deformed, and to rotational kinetic energy.

• *Inelastic collision*

We define an inelastic collision as one in which total kinetic energy is not constant (even though momentum is constant). The collision of a rubber ball with a hard surface is inelastic, because some of the kinetic energy of the ball is lost when it is deformed while in contact with the surface. When two objects collide and stick together after a collision, some kinetic energy is lost, and the collision is called **perfectly inelastic.** For example, if two vehicles collide and become entangled, as in Example 8.4, they move with some common velocity after the perfectly inelastic collision. If a meteorite collides with the Earth, it becomes buried and the collision is perfectly inelastic.

• *Elastic collision*

An elastic collision is defined as one in which total kinetic energy is constant (as well as momentum). Real collisions in the macroscopic world, such as those between billiard balls, are only approximately elastic because some deformation and loss of kinetic energy takes place. Billiard-ball collisions and the collisions of air molecules with the walls of a container at ordinary temperatures are highly elastic. Truly elastic collisions do occur, however, between atomic and subatomic particles. Elastic and perfectly inelastic collisions are *limiting* cases; most collisions fall in a category between them.

In the remainder of this section we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum is conserved in all cases, but kinetic energy is constant only in elastic collisions.**

Perfectly Inelastic Collisions

Consider two objects of masses m_1 and m_2 moving with initial velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ along a straight line, as in Figure 8.8. If the two objects collide head-on, stick together, and move with some common velocity $\mathbf{v}_f$ after the collision, the collision is perfectly inelastic. Because the total momentum before the collision equals the total momentum of the composite system after the collision, we have

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \qquad [8.13]$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \qquad [8.14]$$

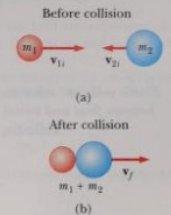


Figure 8.8 A perfectly inelastic head-on collision between two particles.

Elastic Collisions

Now consider two particles that undergo an elastic head-on collision (Fig. 8.9). In this case, both momentum and kinetic energy are constant; therefore, we can write

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} \qquad [8.15]$$

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \qquad [8.16]$$

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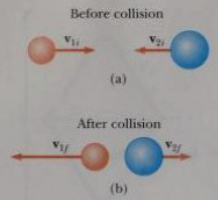


Figure 8.9 An elastic head-on collision between two particles.

where v is positive if a particle moves to the right and negative if it moves to the left.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 8.15 and 8.16 can be solved simultaneously to find them. An alternative approach, employing a little mathematical manipulation of Equation 8.16, often simplifies this process. To see this, let us cancel the factor of $\frac{1}{2}$ in Equation 8.16 and rewrite it as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

Here we have moved the terms containing m_1 to one side of the equation and those containing m_2 to the other. Next, let us factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad [8.17]$$

We now separate the terms containing m_1 and m_2 in the equation for the conservation of momentum (Eq. 8.15) to get

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \qquad [8.18]$$

To obtain our final result, we divide Equation 8.17 by Equation 8.18 and get

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$
$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \qquad [8.19]$$

This equation, in combination with the condition for conservation of momentum, can be used to solve problems dealing with elastic collisions. According to Equation 8.19, the relative speed of the two objects before the collision, $v_{1i} - v_{2i}$, equals the negative of their relative speed after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and the initial velocities of both particles are known. Equations 8.15 and 8.16 can be solved for the final speeds in terms of the initial speeds, because there are two equations and two unknowns.

Elastic collision: relations between final and initial velocities

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \qquad [8.20]$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \qquad [8.21]$$

It is important to remember that the appropriate signs for v_{1i} and v_{2i} must be included in Equations 8.20 and 8.21. For example, if m_2 is moving to the left initially, as in Figure 8.9, then v_{2i} is negative.

Let us consider some special cases. If $m_1 = m_2$, then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange speeds if they have equal masses. This is what one observes in head-on billiard ball collisions.

If m_2 is initially at rest, $v_{2i} = 0$, and Equations 8.20 and 8.21 become

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \qquad [8.22]$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \qquad [8.23]$$

If m_1 is very large compared with m_2, we see from Equations 8.22 and 8.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one initially at rest, the heavy particle continues its motion unaltered after the collision, and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, with a light atom, such as hydrogen.

If m_2 is much larger than m_1, and m_2 is initially at rest, then we find from Equations 8.22 and 8.23 that $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$. That is, when a very light particle collides head-on with a very heavy particle initially at rest, the velocity of the light particle is reversed, and the heavy particle remains approximately at rest. For example, imagine what happens when a marble hits a stationary bowling ball.



The photograph of a car crash test (an inelastic collision) illustrates that much of the car's initial kinetic energy is transformed into the energy it took to damage the vehicle. Why do safety belts and air bags help prevent serious injury in such collisions? *(Courtesy of General Motors)*

CONCEPTUAL PROBLEM 3

A great deal of expense is involved in repairing automobiles after collisions. It seems that much of this money could be saved if cars were built from hard rubber, so that they made approximately elastic collisions and simply bounced off each other without denting. Why are cars not made of hard rubber?

CONCEPTUAL PROBLEM 4

In perfectly inelastic collisions between two objects, under what conditions is all of the original kinetic energy transformed to forms other than kinetic?

Example 8.5 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when a $^{235}_{92}U$ atom splits in a process called fission. These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by being passed through a solid or liquid material called a *moderator*. The slowing-down process involves elastic collisions. Let us show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in "heavy water," D_2O) or carbon (in graphite).

Reasoning Because momentum and energy are constant, Equations 8.22 and 8.23 can be applied to the head-on collision of a neutron with the moderator nucleus.

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that the neutron of mass m_n and initial speed v_{ni} collides head-on with it. The initial kinetic energy of the neutron is

$$K_{ni} = \tfrac{1}{2}m_n v_{ni}^2$$

After the collision, the neutron has a kinetic energy $\frac{1}{2}m_n v_{nf}^2$, where v_{nf} is given by Equation 8.22.

$$K_{nf} = \tfrac{1}{2}m_n v_{nf}^2 = \frac{m_n}{2}\left(\frac{m_n - m_m}{m_n + m_m}\right)^2 v_{ni}^2$$

Therefore, the fraction of the total kinetic energy possessed by the neutron after the collision is

$$(1) \quad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m}\right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_m is close to m_n and is zero when $m_n = m_m$.

We can calculate the kinetic energy of the moderator nucleus after the collision using Equation 8.23:

$$K_{mf} = \tfrac{1}{2}m_m v_{mf}^2 = \frac{2m_n^2 m_m}{(m_n + m_m)^2}v_{ni}^2$$

Hence, the fraction of the total kinetic energy transferred to the moderator nucleus is

$$(2) \quad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4m_n m_m}{(m_n + m_m)^2}$$

Because the total energy is constant, (2) can also be obtained from (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O ($m_m = 2m_n$), $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely to occur.

How do the results differ with graphite as the moderator?

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Example 8.6 A Two-Body Collision with Spring

A block of mass $m_1 = 1.60$ kg, initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track, collides with a spring attached to a second block of mass $m_2 = 2.10$ kg, moving to the left with a speed of 2.50 m/s, as in Figure 8.10a. The spring has a spring constant of 600 N/m. (a) At the instant when m_1 is moving to the right with a speed of 3.00 m/s, as in Figure 8.10b, determine the speed of m_2.

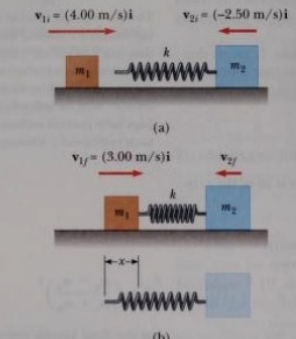


Figure 8.10 (Example 8.6)

Solution First, note that the initial velocity of m_2 is -2.50 m/s because its direction is to the left. Because the total momentum is conserved, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(1.60\ \text{kg})(4.00\ \text{m/s}) + (2.10\ \text{kg})(-2.50\ \text{m/s}) = (1.60\ \text{kg})(3.00\ \text{m/s}) + (2.10\ \text{kg})v_{2f}$$

$$v_{2f} = -1.74\ \text{m/s}$$

The negative value for v_{2f} means that m_2 is still moving to the left at the instant we are considering.

(b) Determine the distance the spring is compressed at that instant.

Solution To determine the compression in the spring, x, shown in Figure 8.10b, we use conservation of energy, because there are no friction or other nonconservative forces acting on the system. Thus, we have

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 + \tfrac{1}{2}kx^2$$

Substituting the given values and the result to part (a) into this expression gives

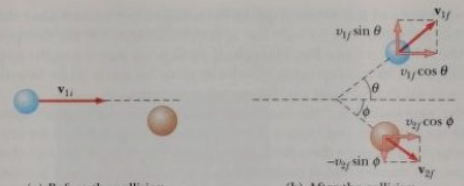
$$x = 0.173\ \text{m}$$

EXERCISE 6 Find the velocity of m_1 and the compression in the spring at the instant that m_2 is at rest.
Answer 0.719 m/s to the right; 0.251 m

EXERCISE 7 A 2.5-kg mass, moving initially with a speed of 10 m/s, makes a perfectly inelastic head-on collision with a 5.0-kg mass that is initially at rest. (a) Find the final velocity of the composite particle. (b) How much kinetic energy is lost in the collision?
Answer (a) 3.33 m/s (b) 83.4 J

8.5 • TWO-DIMENSIONAL COLLISIONS

In Section 8.1 we showed that the total momentum of a system of particles is constant when the system is isolated (that is, when no external forces act on the system). For a general collision of two particles in three-dimensional space, the conservation-of-momentum law implies that the total momentum in each direction is constant. However, an important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for the conservation of momentum:



(a) Before the collision (b) After the collision

Figure 8.11 An elastic glancing collision between two particles.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Consider a two-dimensional problem in which a particle of mass m_1 collides elastically with a particle of mass m_2 that is initially at rest, as in Figure 8.11. After the collision, m_1 moves at an angle θ with respect to the horizontal, and m_2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form, and noting that the total y component of momentum is zero, we get

$$x \text{ component:} \quad m_1 v_{1i} + 0 = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \qquad [8.24]$$

$$y \text{ component:} \quad 0 + 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \qquad [8.25]$$

• *Conservation of momentum*

We now have two independent equations. So long as only two of the preceding quantities are unknown, we can solve the problem completely.

Because the collision is elastic, we can write a third equation for kinetic energy, in the form

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 \qquad [8.26]$$

• *Conservation of energy*

If we know the initial velocity, v_{1i}, and the masses, we are left with four unknowns. Because we have only three equations, one of the four remaining quantities (v_{1f}, v_{2f}, θ, or ϕ) must be given to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* constant, and Equation 8.26 does *not* apply.

PROBLEM-SOLVING STRATEGY • Collisions

The following procedure is recommended when dealing with problems involving collisions between two objects.

1. Set up a coordinate system and define your velocities with respect to that system. It is convenient to have the x axis coincide with one of the initial velocities.
2. In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information. Write expressions for the x and y

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components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors. For example, if an object is moving in the negative x direction, its x component of velocity must be taken to be negative. It is essential that you pay careful attention to signs.

3. Write expressions for the *total* momentum in the x direction *before* and *after* the collision, and equate the two. Repeat this procedure for the total momentum in the y direction. These steps follow from the fact that, because the momentum is conserved in any collision, the total momentum in any direction must be constant. It is important to emphasize that it is the momentum of the *system* (the two colliding objects) that is constant, not the momenta of the individual objects.
4. If the collision is inelastic, kinetic energy is *not* constant, and additional information is probably required. If the collision is perfectly inelastic, the final velocities of the two objects are equal. Proceed to solve the momentum equations for the unknown quantities.
5. If the collision is elastic, kinetic energy is constant, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision. This provides an additional relationship between the velocities.

Example 8.7 Proton–Proton Collision

A proton collides in an elastic fashion with another proton that is initially at rest. The incoming proton has an initial speed of 3.5×10^5 m/s and makes a glancing collision with the second proton, as in Figure 8.11. (At close separations, the protons exert a repulsive electrostatic force on each other.) After the collision, one proton moves off at an angle of 37° to the original direction of motion, and the second deflects at an angle of ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

Solution Both momentum and kinetic energy are constant in this glancing elastic collision. Because $m_1 = m_2$, $\theta = 37°$, and we are given $v_{1i} = 3.5 \times 10^5$ m/s, Equations 8.24, 8.25, and 8.26 become

$$v_{1f} \cos 37° + v_{2f} \cos \phi = 3.5 \times 10^5$$

$$v_{1f} \sin 37° - v_{2f} \sin \phi = 0$$

$$v_{1f}^2 + v_{2f}^2 = (3.5 \times 10^5)^2$$

Simultaneous solution of these three equations with three unknowns gives

$$v_{1f} = 2.80 \times 10^5 \text{ m/s} \qquad v_{2f} = 2.11 \times 10^5 \text{ m/s}$$

$$\phi = 53.0°$$

It is interesting to note that $\theta + \phi = 90°$. This result is *not* accidental. **Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are *always* at right angles to each other.**

Example 8.8 Collision at an Intersection

A 1500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 8.12. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive x direction and north to be along the positive y direction, as in Figure 8.12. Before the collision, the only object having momentum in the x direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the x direction is

$$\sum p_{xi} = (1500 \text{ kg})(25.0 \text{ m/s}) = 3.75 \times 10^4 \text{ kg·m/s}$$

Now let us assume that the wreckage moves at an angle θ and speed v after the collision. The magnitude of the total momentum in the x direction after the collision is

$$\sum p_{xf} = (4000 \text{ kg})\, v \cos \theta$$

Because the total momentum in the x direction is constant, we can equate these two equations to get

$$(1) \quad (3.75 \times 10^4) \text{ kg·m/s} = (4000 \text{ kg})\, v \cos \theta$$

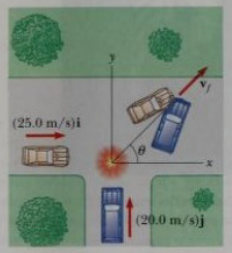


Figure 8.12 (Example 8.8) Top view of an eastbound car colliding with a northbound van.

In a similar way, the total initial momentum of the system in the y direction is that of the van, the magnitude of which is equal to (2500 kg)(20.0 m/s). Applying conservation of momentum to the y direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2500 \text{ kg})(20.0 \text{ m/s}) = (4000 \text{ kg})\, v \sin \theta$$

$$(2) \quad 5.00 \times 10^4 \text{ kg·m/s} = (4000 \text{ kg})\, v \sin \theta$$

If we divide (2) by (1), we get

$$\tan \theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = 53.1°$$

When this angle is substituted into (2), the value of v is

$$v = \frac{5.00 \times 10^4 \text{ kg·m/s}}{(4000 \text{ kg}) \sin 53°} = 15.6 \text{ m/s}$$

EXERCISE 8 A 3.00-kg mass with an initial velocity of 5.00**i** m/s collides with and sticks to a 2.00-kg mass with an initial velocity of −3.00**j** m/s. Find the final velocity of the composite mass. Answer (3.00**i** − 1.20**j**) m/s

EXERCISE 9 A bird and a bee are approaching each other at right angles. The bird has a mass of 0.125 kg and a speed of 0.600 m/s, and the bee has a mass of 5.00×10^{-3} kg and a speed of 15.0 m/s. If the bird catches the bee, what is the new speed of the bird? Answer 0.816 m/s

8.6 • THE CENTER OF MASS

In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a system of particles or any object. We shall see that the center of mass accelerates as if all of the system's mass were concentrated at that point and all external forces acted there. Furthermore, if the resultant external force on the system is $\mathbf{F}$ and the total mass of the system is M, the center of mass moves with an acceleration given by $\mathbf{a} = \mathbf{F}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This result was implicitly assumed in earlier chapters, because nearly all examples referred to the motion of extended objects.

Consider a system consisting of a pair of particles connected by a light, rigid rod (Fig. 8.13). The center of mass is located somewhere on the rod and is closer to the larger mass. If a single force is applied at some point on the rod that is closer to the smaller mass than to the larger one, the system rotates clockwise (Fig. 8.13a). If the force is applied at a point on the rod closer to the larger mass, the system rotates counterclockwise (Fig. 8.13b). If the force is applied at the center of mass,

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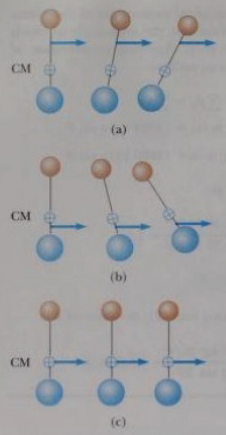


Figure 8.13 Two unequal masses are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the smaller mass and the center of mass. (b) The system rotates counterclockwise when a force is applied between the larger mass and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.

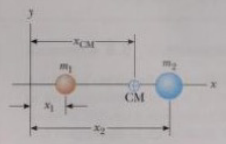


Figure 8.14 The center of mass of two particles on the x axis is located at x_{CM}, a point between the particles, closer to the larger mass.



This multiflash photograph shows that as the diver executes a back flip, his center of mass follows a parabolic path, the same path that a particle would follow. *(© the Harold E. Edgerton 1992 Trust. Courtesy of Palm Press, Inc.)*

the system moves in the direction of **F** without rotating (Fig. 8.13c). Thus, the center of mass can be easily located.

The position of the center of mass of a system can be described as being the *average position* of the system's mass. For example, the center of mass of the pair of particles described in Figure 8.14 is located on the x axis, somewhere between the particles. The x coordinate of the center of mass in this case is

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad [8.27]$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend the center of mass concept to a system of many particles in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum m_i x_i}{\sum m_i} \qquad [8.28]$$

where x_i is the x coordinate of the ith particle and $\sum m_i$ is the *total mass* of the system. For convenience, we shall express the total mass as $M = \sum m_i$, where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{CM} \equiv \frac{\sum m_i y_i}{M} \quad \text{and} \quad z_{CM} \equiv \frac{\sum m_i z_i}{M} \qquad [8.29]$$

The center of mass can also be located by its position vector, $\mathbf{r}_{CM}$. The rectangular coordinates of this vector are x_{CM}, y_{CM}, and z_{CM}, defined in Equations 8.28 and 8.29. Therefore,

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k} = \frac{\sum m_i x_i \mathbf{i} + \sum m_i y_i \mathbf{j} + \sum m_i z_i \mathbf{k}}{M}$$

$$\mathbf{r}_{CM} \equiv \frac{\sum m_i \mathbf{r}_i}{M} \qquad [8.30]$$

where $\mathbf{r}_i$ is the position vector of the ith particle, defined by

$$\mathbf{r}_i \equiv x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

The center of mass of a homogeneous, symmetric body must lie on an axis of symmetry. For example, the center of mass of a homogeneous rod must lie midway between the ends of the rod. The center of mass of a homogeneous sphere or a homogeneous cube must lie at the geometric center of the object. One can experimentally determine the center of mass of an irregularly shaped object, such as a wrench, by suspending the wrench from two different points (Fig. 8.15). The wrench is first hung from point A, and a vertical line, AB, is drawn (which can be established with a plumb bob) when the wrench is in equilibrium. The wrench is then hung from point C, and a second vertical line, CD, is drawn. The center of mass coincides with the intersection of these two lines. In fact, if the wrench is hung freely from any point, the vertical line through that point will pass through the center of mass.

Because a rigid body is a continuous distribution of mass, each portion is acted on by the force of gravity. The net effect of all of these forces is equivalent to the effect of a single force, $M\mathbf{g}$, acting through a special point called the **center of gravity.** If $\mathbf{g}$ is constant over the mass distribution, then the center of gravity coincides with the center of mass. If a rigid body is pivoted at its center of gravity, it will be balanced in any orientation.

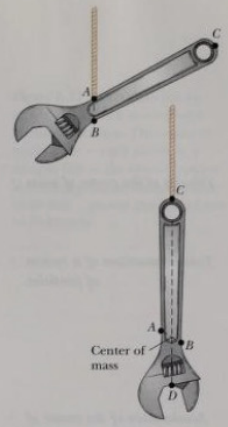


Figure 8.15 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely from two different pivots, A and C. The intersection of the two vertical lines AB and CD locates the center of mass.

Example 8.9 The Center of Mass of Three Particles

A system consists of three particles located at the corners of a right triangle as in Figure 8.16. Find the center of mass of the system.

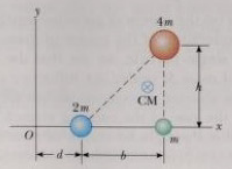


Figure 8.16 (Example 8.9) Locating the center of mass for a system of three particles.

Solution Using the basic defining equations for the coordinates of the center of mass, and noting that $z_{CM} = 0$, we get

$$x_{CM} = \frac{\sum m_i x_i}{M} = \frac{2md + m(d+b) + 4m(d+b)}{7m} = d + \tfrac{5}{7}b$$

$$y_{CM} = \frac{\sum m_i y_i}{M} = \frac{2m(0) + m(0) + 4mh}{7m} = \tfrac{4}{7}h$$

Therefore, we can express the position vector to the center of mass measured from the origin as

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k} = \left(d + \tfrac{5}{7}b\right)\mathbf{i} + \tfrac{4}{7}h\mathbf{j}$$

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8.7 • MOTION OF A SYSTEM OF PARTICLES

We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector of the center of mass $\mathbf{r}_{CM}$, given by Equation 8.30. Assuming that M remains constant—that is, no particles enter or leave the system—we get the following expression for the **velocity of the center of mass:**

Velocity of the center of mass

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M}\sum m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad [8.31]$$

where $\mathbf{v}_i$ is the velocity of the ith particle. Rearranging Equation 8.31 gives

Total momentum of a system of particles

$$M\mathbf{v}_{CM} = \sum m_i \mathbf{v}_i = \sum \mathbf{p}_i = \mathbf{p}_{tot} \qquad [8.32]$$

This result tells us that **the total momentum of the system equals the total mass multiplied by the velocity of the center of mass**—in other words, the total momentum of a single particle of mass M moving with a velocity $\mathbf{v}_{CM}$.

If we now differentiate Equation 8.32 with respect to time, we get the **acceleration of the center of mass:**

Acceleration of the center of mass for a system of particles

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}_{CM}}{dt} = \frac{1}{M}\sum m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum m_i \mathbf{a}_i \qquad [8.33]$$

Rearranging this expression and using Newton's second law, we get

$$M\mathbf{a}_{CM} = \sum m_i \mathbf{a}_i = \sum \mathbf{F}_i \qquad [8.34]$$

where $\mathbf{F}_i$ is the force on particle i.

The forces on any particle in the system may include both external and internal forces. However, by Newton's third law, the force exerted by particle 1 on particle 2, for example, is equal to and opposite the force exerted by particle 2 on particle 1. When we sum over all internal forces in Equation 8.34, they cancel in pairs and the net force on the system is due *only* to external forces. Thus, we can write Equation 8.34 in the form

Newton's second law for a system of particles

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \frac{d\mathbf{p}_{tot}}{dt} \qquad [8.35]$$

That is, the resultant external force on the system of particles equals the total mass of the system multiplied by the acceleration of the center of mass. If we compare this to Newton's second law for a single particle, we see that the center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. In the absence of external forces, the center of mass moves with uniform velocity, as in the case of the rotating wrench in Figure 8.17. If the resultant force acts along a line through the center of mass of an extended body such as the wrench, the body is accelerated without rotation, and its kinetic energy is associated entirely with its linear motion. If the resultant force does not act through the center of mass, the body will be rotationally accelerated, and the body will acquire a rotational kinetic energy in addition to the kinetic energy of its linear motion. The linear acceleration of the center of mass is the same in either case, as given by Equation 8.35.



Figure 8.17 Multiflash photograph of a wrench moving on a horizontal surface. The center of mass of the wrench moves in a straight line as the wrench rotates about this point as marked with a white dot. *(Richard Megna, Fundamental Photographs)*

Finally, we see that if the resultant external force is zero, then from Equation 8.35 it follows that

$$\frac{d\mathbf{p}_{tot}}{dt} = M\mathbf{a}_{CM} = 0$$

so that

$$\mathbf{p}_{tot} = M\mathbf{v}_{CM} = \text{constant} \qquad (\text{when } \sum \mathbf{F}_{ext} = 0) \qquad [8.36]$$

That is, the total linear momentum of a system of particles is constant if there are no external forces acting on the system. It follows that, for an *isolated* system of particles, both the total momentum and the velocity of the center of mass are constant in time. The law of conservation of momentum that was derived in Section 8.1 for a two-particle system is thus generalized to a many-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted on by an external force. For example, consider a system, initially at rest, made up of a swimmer and a raft. When the swimmer dives off the raft, the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the horizontal momentum of the diver is equal in magnitude to the momentum of the raft, but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly decays into two fragments of masses M_1 and M_2, with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Because the total momentum of the system before the decay is zero, the total momentum of the system after the decay must also be zero. Therefore, we see that $M_1\mathbf{v}_1 + M_2\mathbf{v}_2 = 0$. If the velocity of one of the fragments after the decay is known, the recoil velocity of the other fragment can be calculated.

Thinking Physics 3

A boy stands at one end of a floating canoe that is stationary relative to the shore (Fig. 8.18). He then walks to the opposite end of the canoe, away from the shore. Does the canoe move?

Reasoning Yes, the canoe moves toward the shore. Neglecting friction between the canoe and water, there are no horizontal forces acting on the system consisting of the boy and canoe. Therefore, the center of mass of the system remains fixed relative to the shore (or any stationary point). As the boy moves away from the shore, the canoe

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Figure 8.18 (Thinking Physics 3)

must move toward the shore such that the center of mass of the system remains constant. An alternative explanation is that the momentum of the system remains constant if friction is neglected. As the boy acquires a momentum away from the shore, the canoe must acquire an equal momentum toward the shore such that the total momentum of the system is always zero.

CONCEPTUAL PROBLEM 5

Suppose you tranquilize a polar bear on a glacier as part of a research effort. How might you be able to estimate the weight of the polar bear using a measuring tape, a rope, and a knowledge of your own weight?

CONCEPTUAL PROBLEM 6

A projectile is fired into the air and suddenly explodes into several fragments (Fig. 8.19). What can be said about the motion of the center of mass of the fragments after the explosion?

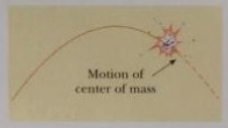


Figure 8.19 (Conceptual Problem 6)

EXERCISE 10 A 5.0-kg particle moves along the x axis with a velocity of 3.0 m/s. A 2.0-kg particle moves along the x axis with a velocity of -2.5 m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system. Answer (a) 1.4 m/s to the right (b) 10 kg·m/s to the right

EXERCISE 11 A 2.0-kg particle has a velocity $(2.0\mathbf{i} - 3.0\mathbf{j})$ m/s, and a 3.0-kg particle has a velocity $(1.0\mathbf{i} + 6.0\mathbf{j})$ m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system. Answer (a) $(1.4\mathbf{i} + 2.4\mathbf{j})$ m/s (b) $(7.0\mathbf{i} + 12\mathbf{j})$ kg·m/s

OPTIONAL

8.8 · ROCKET PROPULSION

When ordinary vehicles, such as automobiles and locomotives, are propelled, the driving force for the motion is one of friction. In the case of the automobile, the driving force is the force exerted by the road on the car. A locomotive "pushes" against the tracks; hence, the driving force is the reaction force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to "push" against. Therefore, the source of the propulsion of a rocket must be different. **The operation of a rocket depends on the law of conservation of momentum as applied to a system of particles, where the system is the rocket plus its ejected exhaust gases.**



Figure 8.20 Lift-off of the space shuttle *Columbia*. Enormous thrust is generated by the shuttle's liquid-fueled engines, aided by the two solid-fuel boosters. Many physical principles from mechanics, thermodynamics, and electricity and magnetism are involved in such a launch. *(Courtesy of NASA)*

The propulsion of a rocket (Fig. 8.20) can be understood by first considering the mechanical system consisting of a machine gun mounted on a cart on wheels. As the machine gun is fired, each bullet receives a momentum $m\mathbf{v}$ in some direction, where $\mathbf{v}$ is the velocity of the bullet relative to the gun. For each bullet that is fired, the gun and cart must receive a compensating momentum in the opposite direction (neglecting the momentum imparted to the escaping gas in the barrel of the gun). That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun. If there are n bullets fired each second, then the average force exerted on the gun is $\mathbf{F}_{av} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space (a vacuum), **its momentum changes when some of its mass is released in the form of ejected gases. Because the ejected gases acquire some momentum, the rocket receives a compensating momentum in the opposite direction.** Therefore, **the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases.** In free space, the center of mass of the system moves uniformly, independent of the propulsion process. It is interesting to note that the rocket and machine gun represent cases of the *inverse* of an inelastic collision—that is, momentum is conserved, but the kinetic energy of the system is *increased* (at the expense of potential energy carried in the fuel).

Suppose that at some time t, the momentum of the rocket plus the fuel is $(M + \Delta m)v$ (Fig. 8.21a). At some short time later, Δt, the rocket ejects fuel of mass Δm and the rocket's speed therefore increases to $v + \Delta v$ (Fig. 8.21b). If the fuel is ejected with a velocity v_e *relative to the rocket*, then the speed of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial momentum of the system with the total final momentum, we get

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

Simplifying this expression gives

$$M\,\Delta v = \Delta m(v_e)$$

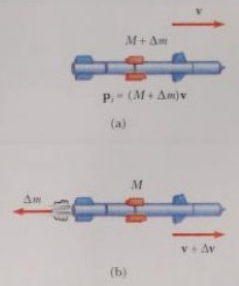


Figure 8.21 Rocket propulsion. (a) The initial mass of the rocket is $M + \Delta m$ at a time t, and its speed is v. (b) At a time $t + \Delta t$, the rocket's mass has reduced to M, and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv.

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We can also arrive at this result by considering the system in the center-of-mass frame of reference—that is, a frame the velocity of which equals the center-of-mass velocity. In this frame, the total momentum is zero; therefore, if the rocket gains a momentum $M\,\Delta v$ by ejecting some fuel, the exhaust gases obtain a momentum $\Delta m(v_e)$ in the *opposite* direction, and so $M\,\Delta v - \Delta m(v_e) = 0$. If we now take the limit as Δt goes to zero, the $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Furthermore, the increase in the exhaust mass, dm, corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is given a negative sign because it represents a decrease in mass. Using this fact, we get

$$M\,dv = -v_e\,dM \qquad [8.37]$$

Integrating this equation and taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f we get

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

Basic expression for rocket propulsion

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \qquad [8.38]$$

This is the basic expression of rocket propulsion. First, it tells us that the increase in speed is proportional to the exhaust speed, v_e. Therefore, the exhaust velocity should be very high. Second, the increase in speed is proportional to the logarithm of the ratio M_i/M_f. Therefore, this ratio should be as large as possible, which means that the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on the rocket by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 8.37:

$$\text{Thrust} = M\frac{dv}{dt} = \left|v_e\frac{dM}{dt}\right| \qquad [8.39]$$

Here we see that the thrust increases as the exhaust speed increases and as the rate of change of mass (burn rate) increases.

Thinking Physics 4

When Robert Goddard proposed the possibility of rocket-propelled vehicles, the *New York Times* agreed that such vehicles would be useful and successful within the Earth's atmosphere ("Topics of the Times," *New York Times*, Jan. 13, 1920, p. 12). But the *Times* balked at the idea of using such a rocket in the vacuum of space, noting that, ". . . its flight would be neither accelerated nor maintained by the explosion of the charges it then might have left. To claim that it would be is to deny a fundamental law of dynamics . . . That Professor Goddard, with his 'chair' in Clark College and the countenancing of the Smithsonian Institution, does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react—to say that would be absurd."

What did the writer of these passages overlook?

Reasoning The writer was making a common mistake in believing that a rocket works by expelling gases that *push* on something, propelling the rocket forward. With this belief, it is impossible to see how a rocket fired in empty space *would* work.

Gases do not need to push on anything—it is the act of expelling the gases itself that pushes the rocket forward. This can be argued from Newton's third law—the rocket pushes the gases backward, resulting in the gases pushing the rocket forward. It can also be argued from conservation of momentum—as the gases gain momentum in one direction, the rocket must gain momentum in the opposite direction to conserve the original momentum of the rocket–gas system.

The *New York Times* did publish a retraction 49 years later ("A Correction," *New York Times*, July 17, 1969, p. 43), at the same time that the Apollo astronauts were on their way to the Moon. It appeared on a page with two other articles, one titled, "Fundamentals of Space Travel," and the other, "Spacecraft, Like Squid, Maneuver by 'Squirts.'" The following passages were part of these articles: ". . . an editorial feature of the *New York Times* dismissed the notion that a rocket could function in a vacuum and commented on the ideas of Robert H. Goddard. . . . Further investigation and experimentation have confirmed the findings of Isaac Newton in the 17th century, and it is now definitely established that a rocket can function in a vacuum as well as in an atmosphere. The *Times* regrets the error."

Example 8.10 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to Earth. Its engines are turned on, and exhaust is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to Earth once its mass is reduced to one half its mass before ignition?

Solution Applying Equation 8.38, we get

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right)$$

$$= 3.0 \times 10^3 + 5.0 \times 10^3 \ln\left(\frac{M_i}{0.5M_i}\right)$$

$$= 6.5 \times 10^3 \text{ m/s}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution

$$\text{Thrust} = \left|v_e\frac{dM}{dt}\right| = \left(5.0 \times 10^3\,\frac{\text{m}}{\text{s}}\right)\left(50\,\frac{\text{kg}}{\text{s}}\right)$$

$$= 2.5 \times 10^5 \text{ N}$$

EXERCISE 12 A rocket engine consumes 80 kg of fuel per second. If the exhaust speed is 2.5×10^3 m/s, calculate the thrust on the rocket. Answer 200 kN

SUMMARY

The **linear momentum** of any object of mass m moving with a velocity $\mathbf{v}$ is

$$\mathbf{p} = m\mathbf{v} \qquad [8.1]$$

Conservation of momentum applied to two interacting objects states that if the two objects form an isolated system, their total momentum is constant regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum:

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \qquad [8.5]$$

The **impulse** of a force $\mathbf{F}$ on any object is equal to the change in the momentum of the object and is given by

$$\mathbf{I} = \int_{t_i}^{t_f} \mathbf{F}\,dt = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \qquad [8.9]$$

This is known as the **impulse-momentum theorem.**

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Impulsive forces are forces that are very strong compared with other forces on the system. They usually act for a short time, as in the case of collisions.

When two bodies collide, the total momentum of the system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which kinetic energy is not constant, but momentum is. A perfectly inelastic collision is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which both momentum and kinetic energy are constant.

In a two- or three-dimensional collision, the components of momentum in each of the directions are conserved independently.

The **vector position of the center of mass of a system of particles** is defined as

$$\mathbf{r}_{CM} \equiv \frac{\sum m_i \mathbf{r}_i}{M} \qquad [8.30]$$

where $M = \sum m_i$ is the total mass of the system and $\mathbf{r}_i$ is the vector position of the ith particle.

The **velocity of the center of mass for a system of particles** is

$$\mathbf{v}_{CM} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad [8.31]$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass—that is, $\mathbf{p}_{tot} = M\mathbf{v}_{CM}$.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \frac{d\mathbf{p}_{tot}}{dt} \qquad [8.35]$$

where $\mathbf{a}_{CM}$ is the acceleration of the center of mass and the sum is over all external forces. Therefore, the center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. It follows from Equation 8.35 that the total momentum of the system is constant if there are no external forces acting on it.

CONCEPTUAL QUESTIONS

1. Does a large force always produce a larger impulse on a body than a smaller force does? Explain.
2. If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.
3. Explain how linear momentum is conserved when a ball bounces from a floor.
4. Consider a perfectly inelastic collision between a car and a large truck. Which vehicle loses more kinetic energy as a result of the collision?
5. Your physical education teacher, who knows something about physics, throws you a tennis ball at a certain velocity, and you catch it. You are now given the following choice: The teacher can throw you a medicine ball, which is much more massive than the tennis ball, with the same velocity as the tennis ball, the same momentum, or the same kinetic energy. Which choice would you make in order to make the easiest catch, and why?
6. Can the center of mass of a body lie outside the body? If so, give examples.
7. Is the center of mass of a mountain higher or lower than the center of gravity?
8. In golf, novice players are often advised to be sure to "follow through" with their swing. Why does this make the ball travel a longer distance? What about a chip shot? There is very little follow-through here—why?

9. A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why isn't it as dangerous to be hit by the gun as by the bullet?
10. A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Could you calculate the force exerted on him due to the collision? Explain.
11. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
12. An airbag is inflated when a collision occurs, which protects the passenger (the dummy, in this case) from serious injury (Fig. Q8.12). Why does the airbag soften the blow? Discuss the physics involved in this dramatic photograph.
13. A large bedsheet is held vertically by two students. A third student, who happens to be the star pitcher on the baseball team, throws a raw egg at the sheet. Explain why the egg does not break when it hits the sheet, regardless of its initial speed. (If you try this, make sure the pitcher hits the sheet near its center, and do not allow the egg to fall on the floor after being caught in the sheet.)



Figure Q8.12 *(Courtesy of Saab)*

PROBLEMS

Section 8.1 Linear Momentum and Its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\mathbf{i} - 4.00\mathbf{j})$ m/s. Find its x and y components of momentum and the magnitude of its total momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. A 40.0-kg child standing on a frozen pond throws a 0.500-kg stone to the east with a speed of 5.00 m/s. Neglecting friction between child and ice, find the recoil velocity of the child.
4. A pitcher claims she can throw a baseball with as much momentum as a 3.00-g bullet moving with a speed of 1500 m/s. A baseball has a mass of 0.145 kg. What must be its speed if the pitcher's claim is valid?
5. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them, and the blocks are pushed together with the spring between them (Fig. P8.5). A cord holding them together is burned, after which the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the speed of the block of mass M? (b) Find the original elastic potential energy in the spring if $M = 0.350$ kg.

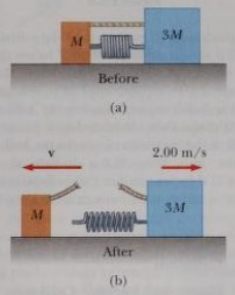


Figure P8.5

Section 8.2 Impulse and Momentum

6. A car is stopped for a traffic signal. When the light turns green, the car accelerates, increasing its speed from 0 to 5.20 m/s in 0.832 s. What linear impulse and average force does a 70.0-kg passenger in the car experience?
7. An estimated force–time curve for a baseball struck by a bat is shown in Figure P8.7. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

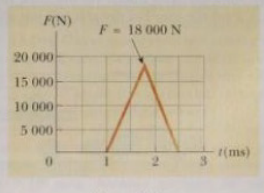


Figure P8.7

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8. A tennis player receives a shot with the ball (0.0600 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racket? (b) What work does the racket do on the ball?

9. A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P8.9). If the ball is in contact with the wall for 0.200 s, what is the average force exerted on the ball by the wall?

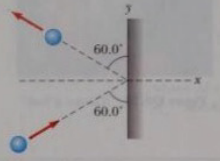


Figure P8.9

10. In a slow-pitch softball game, a 0.200-kg softball crossed the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The ball was hit at 40.0 m/s, 30.0° above the horizontal. (a) Determine the impulse applied to the ball. (b) If the force on the ball increased linearly for 4.00 ms, held constant for 20.0 ms, then decreased to 0 linearly in another 4.00 ms, find the maximum force on the ball.

11. A machine gun fires 35.0 g bullets at a speed of 750.0 m/s. If the gun can fire 200 bullets/min, what is the average force the shooter must exert to keep the gun from moving?

Section 8.3 Collisions

Section 8.4 Elastic and Inelastic Collisions in One Dimension

12. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s. Find the speed of the golf ball just after impact.

13. A 10.0-g bullet is stopped in a block of wood (m = 5.00 kg). The speed of the bullet-plus-wood combination immediately after the collision is 0.600 m/s. What was the original speed of the bullet?

14. A 75.0-kg ice skater, moving at 10.0 m/s, crashes into a stationary skater of equal mass. After the collision, the two skaters move as a unit at 5.00 m/s. The average force a skater can experience without breaking a bone is 4500 N. If the impact time is 0.100 s, does a bone break?

15. As shown in Figure P8.15, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

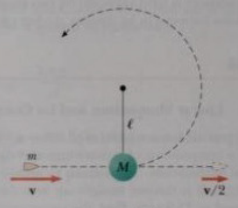


Figure P8.15

16. Gayle runs at a speed of 4.00 m/s and dives on a sled, initially at rest on the top of a frictionless snow-covered hill. After she has descended a vertical distance of 5.00 m, her brother, who is initially at rest, hops on her back and together they continue down the hill. What is their speed at the bottom of the hill if the total vertical drop is 15.0 m? Gayle's mass is 50.0 kg, the sled has a mass of 5.00 kg, and her brother has a mass of 30.0 kg.

17. A 1200-kg car traveling initially with a speed of 25.0 m/s in an easterly direction crashes into the rear end of a 9000-kg truck moving in the same direction at 20.0 m/s (Fig. P8.17). The velocity of the car right after the collision is 18.0 m/s to the east. (a) What is the velocity of the truck right after the collision? (b) How much mechanical energy is lost in the collision? Account for this loss in energy.

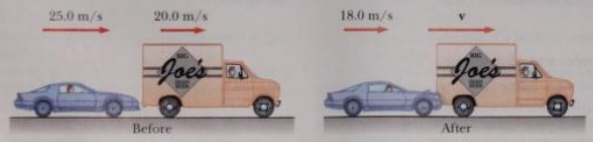


Figure P8.17

18. A railroad car of mass 2.50×10^4 kg, moving with a speed of 4.00 m/s, collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much energy is lost in the collision?

19. A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is about 12.0 times the mass of the neutron.)

20. A 7.00-kg bowling ball collides head-on with a 2.00-kg bowling pin. The pin flies forward with a speed of 3.00 m/s. If the ball continues forward with a speed of 1.80 m/s, what was the initial speed of the ball? Ignore rotation of the ball.

21. A 12.0-g bullet is fired into a 100-g wooden block initially at rest on a horizontal surface. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between block and surface is 0.650, what was the speed of the bullet immediately before impact?

22. A bullet of mass m_1 is fired into a wooden block of mass m_2 initially at rest on a horizontal surface. After impact, the block slides a distance d before coming to rest. If the coefficient of friction between block and surface is μ, what was the speed of the bullet immediately before impact?

23. Consider a frictionless track ABC as shown in Figure P8.23. A block of mass m_1 = 5.00 kg is released from A. It makes a head-on elastic collision with a block of mass m_2 = 10.0 kg at B, initially at rest. Calculate the maximum height to which m_1 rises after the collision.

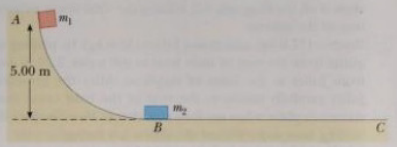


Figure P8.23

24. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, penetrates the block to a depth of 8.00 cm. This block of wood is placed on a frictionless horizontal surface, and a 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

Section 8.5 Two-Dimensional Collisions

25. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the energy lost as a result of the collision. Account for the missing energy.

26. The mass of the blue puck in Figure P8.26 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with equal and opposite momenta, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

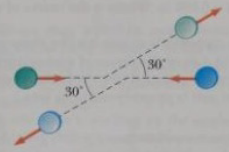


Figure P8.26

27. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves at 4.33 m/s, at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision, find the struck ball's velocity after the collision.

28. A proton, moving with a velocity of $v_0\mathbf{i}$, collides elastically with another proton that is initially at rest. If the two protons have the same speed after the collision, find (a) the speed of each proton after the collision in terms of v_0 and (b) the direction of the velocity vectors after the collision.

29. A 0.300-kg puck, initially at rest on a horizontal, frictionless surface, is struck by a 0.200-kg puck moving initially along the x axis with a speed of 2.00 m/s. After the collision, the 0.200-kg puck has a speed of 1.00 m/s at an angle of θ = 53.0° to the positive x axis (Fig. 8.11). (a) Determine the velocity of the 0.300-kg puck after the collision. (b) Find the fraction of kinetic energy lost in the collision.

30. During the battle of Gettysburg, the gunfire was so intense that several bullets collided in mid-air and fused together. Assume a 5.00-g Union musket ball was moving to the right at a speed of 250 m/s, 20.0° above the horizontal, and that a 3.00-g Confederate ball was moving to the left at a speed of 280 m/s, 15.0° above the horizontal. Immediately after they fuse together, what is their velocity?

31. An unstable nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a velocity of 6.00×10^6 m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of $4.00 \times$

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10^6 m/s. Find (a) the velocity of the third particle and (b) the total energy given off in the process.

32. At an intersection, a 1500-kg car, traveling east with a speed of 20.0 m/s, collides with a 2500-kg van traveling south at a speed of 15.0 m/s. The vehicles undergo a perfectly inelastic collision, and the wreckage slides 6.00 m before coming to rest. Find the magnitude and direction of the constant force that decelerated them.

Section 8.6 The Center of Mass

33. Four objects are situated along the y axis as follows: a 2.00-kg object is at +3.00 m, a 3.00-kg object is at +2.50 m, a 2.50-kg object is at the origin, and a 4.00-kg object is at −0.500 m. Where is the center of mass of these objects?

34. A uniform carpenter's square has the shape of an L, as in Figure P8.34. Locate the center of mass relative to an origin at the lower left corner. (*Hint:* Note that the mass of each rectangular part is proportional to its area.)

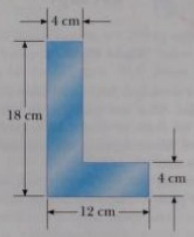


Figure P8.34

35. A uniform piece of sheet steel is shaped as in Figure P8.35. Compute the *x* and *y* coordinates of the center of mass of the piece.

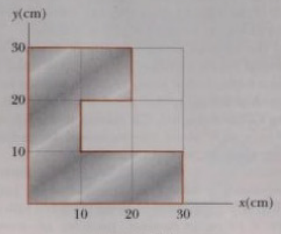


Figure P8.35

36. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^8 m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

37. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P8.37). The angle between the two bonds is 106°. If the bonds are 0.100 nm long, where is the center of mass of the molecule?

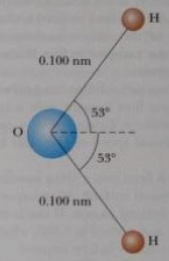


Figure P8.37

Section 8.7 Motion of a System of Particles

38. Consider a system of two particles in the *xy* plane: $m_1 = 2.00$ kg is at $\mathbf{r}_1 = (1.00\mathbf{i} + 2.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} + 0.500\mathbf{j})$ m/s; $m_2 = 3.00$ kg is at $\mathbf{r}_2 = (-4.00\mathbf{i} - 3.00\mathbf{j})$ m and has velocity $(3.00\,\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

39. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat in still water, 2.70 m away from Juliet in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

40. A ball of mass 0.200 kg has velocity $1.50\mathbf{i}$ m/s; a ball of mass 0.300 kg has velocity $-0.400\mathbf{i}$ m/s. Then they meet in a head-on elastic collision. (a) Find their velocities thereafter. (b) Find the velocity of their center of mass before the collision and afterward.

Section 8.8 Rocket Propulsion (Optional)

41. The first stage of a Saturn V space vehicle consumes fuel at the rate of 1.50×10^4 kg/s, with an exhaust speed of 2.60×10^3 m/s. (a) Calculate the thrust produced by these engines. (b) Find the initial acceleration of the vehicle on the launch pad if its initial mass is 3.00×10^6 kg. [*Hint:* You must include the force of gravity to solve part (b).]

42. A large rocket with an exhaust speed of $v_e = 3000$ m/s develops a thrust of 24.0 million newtons. (a) How much mass is being blasted out of the rocket exhaust per second? (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with $v_e = 3.00$ km/s and if 90.0% of its initial mass is fuel?

43. A rocket for use in deep space is to have the capability of accelerating a payload (including the rocket frame and engine) of 3.00 metric tons from rest to a final speed of 10 000 m/s with an engine and fuel designed to produce an exhaust speed of 2000 m/s. (a) How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5000 m/s, what amount of fuel and oxidizer would be required for the same task?

Additional Problems

44. A golf ball ($m = 46.0$ g) is struck a blow that makes an angle of 45.0° with the horizontal. The drive lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

45. An 8.00-g bullet is fired into a 2.50-kg block initially at rest at the edge of a frictionless table of height 1.00 m (Fig. P8.45). The bullet remains in the block, and after impact the block lands 2.00 m from the bottom of the table. Determine the initial speed of the bullet.

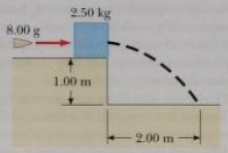


Figure P8.45

46. A spacecraft is stationary in deep space when its rocket engine is ignited for a 100-s "burn." Hot gases are ejected at a constant rate of 150 kg/s, with a speed of 3000 m/s relative to the spacecraft. The initial mass of the spacecraft (plus fuel and oxidizer) is 25 000 kg. Determine the thrust of the rocket engine and the initial acceleration in units of *g*. What is the final speed of the spacecraft?

47. A small block of mass $m_1 = 0.500$ kg is released from rest at the top of a curved-shaped frictionless wedge of mass $m_2 = 3.00$ kg, which sits on a frictionless horizontal surface as in Figure P8.47a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P8.47b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height, *h*, of the wedge?

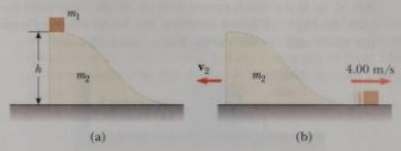


Figure P8.47

48. Tarzan, whose mass is 80.0 kg, swings from a 3.00-m vine that is horizontal when he starts. At the bottom of his arc, he picks up 60.0-kg Jane in a perfectly inelastic collision. What is the height of the highest tree limb they can reach on their upward swing?

49. A 80.0-kg astronaut is working on the engines of her ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds herself 30.0 m behind the ship. Without a thruster, the only way to return to the ship is to throw her 0.500-kg wrench directly away from the ship. If she throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

50. A 40.0-kg child stands at one end of a 70.0-kg boat that is 4.00 m in length (Fig. P8.50). The boat is initially 3.00 m from the pier. The child notices a turtle on a rock at the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction between boat and water, (a) describe the subsequent motion of the system (child plus boat). (b) Where is the child *relative to the pier* when he reaches the far end of the boat? (c) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat.)

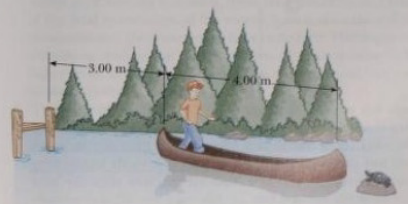


Figure P8.50

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51. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as in Figure P8.51. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the energy lost in the collision.

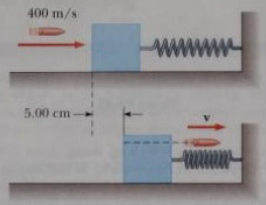


Figure P8.51

52. A student performs a ballistic pendulum experiment using the apparatus shown in Figure P8.52a. A ball of mass 68.8 g is fired into a pendulum of mass 263 g hanging from a low friction pivot. The ball is caught inside the pendulum, which then swings up through a height 8.68 cm. (a) Determine the initial speed v_{1i} of the ball. (b) The second part of her experiment is to obtain v_{1i} by firing the same ball horizontally (with the pendulum removed from the path), by measuring its horizontal displacement, x, and vertical displacement, y (Fig. P8.52b). Show that the initial speed of the ball is related to x and y through the relation

$$v_{1i} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1i} based on her measured values of $x = 257$ cm and $y = 85.3$ cm? What factors might account for the difference in this value compared to that obtained in part (a)?

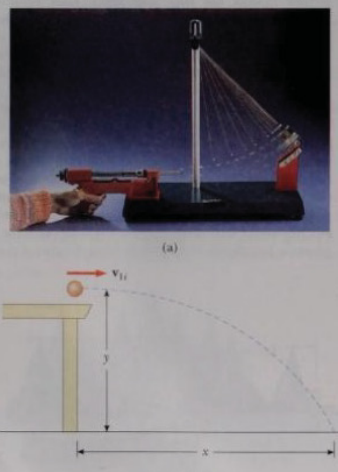


Figure P8.52 (a) and (b) *(Photo courtesy of CENCO)*

53. Two particles of masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_0. Mass m is traveling to the left, and mass $3m$ is traveling to the right. They undergo a head-on elastic collision. Find the final speeds of the particles.

54. Two particles of masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_0. Mass m is traveling to the left, and mass $3m$ is traveling to the right. They undergo an elastic glancing collision such that mass m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two masses. (b) What is the angle θ at which the mass $3m$ is scattered?

55. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, you give the Earth a maximum recoil speed of what order of magnitude? Visualize the Earth as a perfectly rigid object. In your solution state the physical quantities you take as data and the values you measure or estimate for them.

56. There are (one can say) three co-equal theories of motion: Newton's second law, holding that the total force on an object causes its acceleration; the work-kinetic energy theorem, holding that the total work on an object causes its change in kinetic energy; and the impulse-momentum theorem, holding that the total impulse on an object causes its change in momentum. In this problem you compare the predictions of the three theories in one particular case. A 3.00-kg object has velocity $7.00\hat{\mathbf{j}}$ m/s. Then a total force $12.0\hat{\mathbf{i}}$ N acts on the object for 5.00 s. (a) Calculate its final velocity from the impulse-momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_0)/t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma\mathbf{F}/m$. (d) Find the object's vector displacement from $\mathbf{s} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$. (e) Find the work done on the object from $W = \mathbf{F}\cdot\mathbf{s}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f\cdot\mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_0^2 + W$.

Spreadsheet Problems

S1. A rocket has an initial mass of 20 000 kg, of which 20% is the payload. The rocket burns fuel at a rate of 200 kg/s and exhausts gas at a relative speed of 2.00 km/s. Its acceleration dv/dt is determined from the equation of motion

$$M\frac{dv}{dt} = v_e\left|\frac{dM}{dt}\right| + F_{\text{ext}}$$

Assume that there are no external forces and that the rocket's initial velocity is zero. When $F_{\text{ext}} = 0$, the rocket's velocity is $v(t) = v_e \ln (M_i/M)$, where M_i is the rocket's initial mass. Spreadsheet 8.1 calculates the acceleration and velocity as a function of time. (a) Using the given parameters, find the maximum acceleration and velocity. (b) At what time is the velocity equal to half its maximum value? Why isn't this time half of the burn time?

S2. Modify Spreadsheet 8.1 to calculate the distance traveled by the rocket in Problem S1. Add a new column to the spreadsheet to find the new position x_{i+1}. Estimate the new position of the rocket by

$$x_{i+1} = x_i + \tfrac{1}{2}(v_{i+1} + v_i)\Delta t$$

where x_i is the old position, v_i is the old velocity, and v_{i+1} is the new velocity.

S3. If the rocket in the previous problem burns its fuel twice as fast (400 kg/s) while maintaining the same exhaust speed, how far does it travel before it runs out of fuel? The velocity reached by the rocket is the same in both cases; that is

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

Is there an advantage in burning the fuel faster? Is there a disadvantage? (*Hint:* It may be useful to plot the acceleration, velocities, and position for the same time intervals in both cases. In which case does the rocket travel farther? In which case does the rocket have to withstand the largest acceleration? By how much?)

ANSWERS TO CONCEPTUAL PROBLEMS

1. The brain is immersed in a cushioning fluid inside the skull. If the head is struck suddenly by a bare fist, there is a rapid acceleration of the skull. The brain matches this acceleration only because of the large impulsive force of the skull on the brain. This large and sudden force can severely injure the brain. Padded gloves extend the time over which the force is applied to the head. Thus, for a given impulse, the gloves provide a longer time interval than the bare fist, decreasing the average force. Because the average force is decreased, the acceleration of the skull is decreased, reducing (but not eliminating) the chance for brain injury. The same argument can be made for "rolling with the punch." If the head is held steady while being struck, the time interval over which the force is applied is relatively short and the average force is large. If the head is allowed to move in the same direction of the punch, the time interval is lengthened, and the average force reduced.
2. The scenario described violates the principle of conservation of momentum. If the superhero, initially at rest, throws the piano in one direction, he must recoil in the other direction, in order to maintain the zero momentum of the system.
3. Making cars out of hard rubber would save on automobile repair bills, but the results for the drivers and passengers would be disastrous. Imagine two identical cars traveling toward each other at identical speeds, and making a head-on collision. In the approximation that the collision is elastic, the cars bounce off each other and leave the collision with their velocities reversed in direction. Thus, the inhabitants of the cars would suffer a large change in momentum in a short period of time. As a result, the force on the inhabitants would become very large, causing significant bodily injury. For real cars in the above situation, the cars would come to rest during the collision, as the front sections of the cars crumpled. Thus, the change in momentum is only half what it was in the elastic case, and the change occurs over the longer time period during which the cars crumpled. The resulting smaller force on the inhabitants will result in fewer injuries.
4. If all of the kinetic energy is transformed into other forms, then there is no motion of either of the objects after the collision. If nothing is moving, there is no momentum. Thus, if the final momentum of the system is zero, then the initial momentum of the system must have been zero. The required condition, then, is that the two objects are approaching each other so as to make a head-on collision, with equal, but oppositely directed, momenta.
5. Tie one end of the rope around the bear. Lay out the tape measure on the ice between the bear's original position and yours, as you hold the opposite end of the rope. Take off your spiked shoes and pull on the rope. Both you and the bear will slide over the ice until you meet. From the tape observe how far you have slid, x_y, and how far the bear has slid, x_b. The point where you meet the bear is the constant location of the center of mass of the system (bear plus you),

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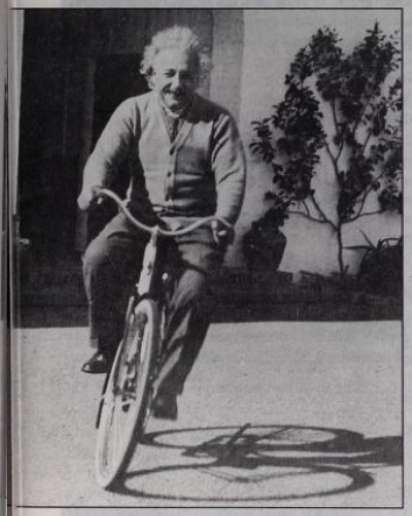
so you can determine the mass of the bear from $m_b x_b = m_y x_y$. (Unfortunately, if the bear now wakes up you cannot get back to your spiked shoes.)

6. Neglecting air resistance, the only external force on the projectile is the force of gravity. Thus, the projectile follows a parabolic path. If the projectile did not explode, it would continue to move along the parabolic path indicated by the broken line in Figure 8.19. Because the forces due to the explosion are internal, they do not affect the motion of the center of mass. Thus, after the explosion, and until the first fragment hits the ground, the center of mass of the fragments follows the same parabolic path the projectile would have followed if there had been no explosion.

9

Relativity

Most of our everyday experiences and observations have to do with objects that move at speeds much less than that of light. Newtonian mechanics and early ideas on space and time were formulated to describe the motion of such objects. This formalism is successful in describing a wide range of phenomena that occur at low speeds. It fails, however, when applied to particles the speeds of which approach that of light. The predictions of Newtonian theory can be tested experimentally at high speeds by accelerating electrons or other charged particles. For example, it is possible to accelerate an electron to a speed of $0.99c$ (where c is the speed of light) by using particle accelerators. According to Newtonian mechanics, if the energy transferred to an electron is increased by a factor of 4, the electron speed should jump to $1.98c$. However, experiments show that the speed of the electron—as well as the speeds of all other particles in the Universe—always remains less than the speed of light, regardless of the particle energy. Because it places no upper limit on speed, New-



◀ **Albert Einstein (1879–1955), one of the greatest physicists of all times, is best known for developing the theory of relativity. He is shown here in a playful mood riding a bicycle. The photograph was taken in 1933 in Santa Barbara, California.** *(From the California Institute of Technology archives)*

"Imagination is more important than knowledge."
Albert Einstein

CHAPTER OUTLINE

9.1 The Principle of Newtonian Relativity
9.2 The Michelson–Morley Experiment
9.3 Einstein's Principle of Relativity
9.4 Consequences of Special Relativity
9.5 The Lorentz Transformation Equations
9.6 Relativistic Momentum and the Relativistic Form of Newton's Laws
9.7 Relativistic Energy
9.8 Mass as a Measure of Energy
9.9 General Relativity (Optional)

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tonian mechanics is contrary to modern experimental results and is clearly a limited theory.

In 1905, at the age of 26, Einstein published his special theory of relativity. Einstein wrote,

> The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions. . . . [1]

Although Einstein made many important contributions to science, special relativity alone represents one of the greatest intellectual achievements of the twentieth century. With special relativity, experimental observations can be correctly predicted over the range of speeds from $v = 0$ to speeds approaching the speed of light. Newtonian mechanics, which was accepted for more than 200 years, is in fact a special case of special relativity. This chapter gives an introduction to special relativity, with emphasis on some of its consequences.

Special relativity covers phenomena such as the slowing down of clocks and the contraction of lengths in moving reference frames as measured by a laboratory observer. We also discuss the relativistic forms of momentum and energy, and some consequences of the famous mass–energy formula, $E = mc^2$.

In addition to its well known and essential role in theoretical physics, special relativity has practical applications, including the design of accelerators and other devices that use high-speed particles. These devices will not work if designed according to nonrelativistic principles. We shall have occasion to use special relativity in some subsequent chapters of this textbook, most often presenting only the outcome of relativistic effects.

9.1 • THE PRINCIPLE OF NEWTONIAN RELATIVITY

To describe a physical event, it is necessary to establish a frame of reference. You should recall from Chapter 5 that Newton's laws are valid in all inertial frames of reference. Because an inertial frame is defined as one in which Newton's first law is valid, it can be said that **an inertial system is one in which a free body experiences no acceleration.** Furthermore, any system moving with constant velocity with respect to an inertial system must also be an inertial system.

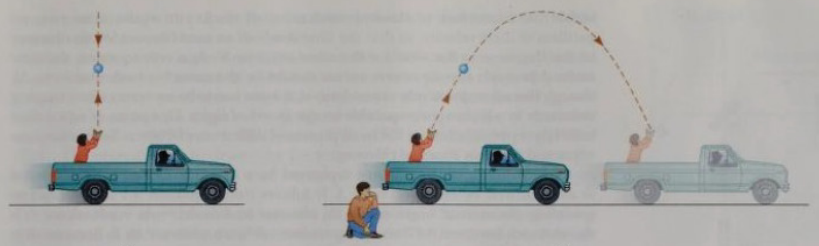
Inertial frame of reference •

There is no preferred frame. This means that the results of an experiment performed in a vehicle moving with uniform velocity will be identical to the results of the same experiment performed in a stationary vehicle. The formal statement of this result is called the principle of **Newtonian relativity:**

> The laws of mechanics must be the same in all inertial frames of reference.

Let us consider an observation that illustrates the equivalence of the laws of mechanics in different inertial frames. Consider a pickup truck moving with a constant velocity, as in Figure 9.1a. If a passenger in the truck throws a ball straight up in the air, the passenger observes that the ball moves in a vertical path. The motion

[1] A. Einstein and L. Infeld, *The Evolution of Physics*, New York, Simon and Schuster, 1961.



(a) (b)

Figure 9.1 (a) The observer in the truck sees the ball move in a vertical path when thrown upward. (b) The Earth observer views the path of the ball to be a parabola.

of the ball appears to be precisely the same as if the ball were thrown by a person at rest on Earth. The law of gravity and the equations of kinematics are obeyed whether the truck is at rest or in uniform motion. Now consider the same experiment viewed by an observer at rest on Earth. This stationary observer sees the path of the ball as a parabola, as in Figure 9.1b. Furthermore, according to this observer, the ball has a horizontal component of velocity equal to the velocity of the truck. Although the two observers see different velocities and rates of energy change, they see the same forces and agree on the validity of Newton's laws as well as on principles like conservation of energy and conservation of momentum. All differences between the two views stem from the relative motion of one frame with respect to the other. That is, the notion of absolute motion through space plays no role in what is seen.

Suppose that some physical phenomenon that we call an *event* occurs in an inertial system. The event's location and time of occurrence can be specified by the coordinates (x, y, z, t). We would like to be able to transform these coordinates from one inertial system to another moving with uniform relative velocity. This is accomplished by using what is called a *Galilean transformation*, which owes its origin to Galileo.

Consider two inertial systems S and S' (Fig. 9.2). The system S' moves with a constant velocity $\mathbf{v}$ along the xx' axes, where $\mathbf{v}$ is measured relative to S. We assume that an event occurs at the point P and that the origins of S and S' coincide at $t = 0$. An observer in S describes the event with space–time coordinates (x, y, z, t), while an observer in S' uses (x', y', z', t') to describe the same event. As we can see from Figure 9.2, these coordinates are related by the equations

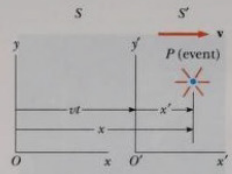


Figure 9.2 An event occurs at a point P. The event is observed by two observers in inertial frames S and S', where S' moves with a velocity $\mathbf{v}$ relative to S.

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \qquad [9.1]$$

• *Galilean transformation of coordinates*

These equations constitute what is known as a **Galilean transformation of coordinates.** Note that time is assumed to be the same in both inertial systems. That is,

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within the framework of classical mechanics, all clocks run at the same rate, regardless of their velocity, so that the time at which an event occurs for an observer in S is the same as the time for the same event in S'. As a consequence, the time interval between two successive events should be the same for both observers. Although this assumption may seem obvious, it turns out to be incorrect when treating situations in which v is comparable to the speed of light. This point of equal time intervals represents one of the most profound differences between Newtonian concepts and Einstein's special relativity.

Now suppose two events are separated by a distance dx and a time interval dt as measured by an observer in S. It follows from Equation 9.1 that the corresponding distance dx' measured by an observer in S' is $dx' = dx - v dt$, where dx is the distance between the two events measured by an observer in S. Because $dt = dt'$, we find that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

or

Galilean addition law for velocities

$$u_x' = u_x - v \qquad [9.2]$$

where u_x and u_x' are the instantaneous velocities of the object relative to S and S', respectively. This result, which is called the **Galilean addition law for velocities** (or Galilean velocity transformation), is used in everyday observations and is consistent with our intuitive notion of time and space. As we will soon see, however, it leads to serious contradictions when applied to electromagnetic waves.



Albert A. Michelson (1852–1931)

A German American physicist, Michelson spent much of his life making accurate measurements of the speed of light. In 1907 he was the first American to be awarded the Nobel prize, which he received for his work in optics. His most famous experiment, conducted with Edward Morley in 1887, implied that it was impossible to measure the absolute velocity of the Earth with respect to the ether. *(AIP Emilio Segrè Visual Archives, Michelson Collection)*

9.2 • THE MICHELSON–MORLEY EXPERIMENT

Many experiments similar to the one described in the preceding section show us that the laws of mechanics are the same in all inertial frames of reference. When similar inquiries are made into the laws of other branches of physics, the results are contradictory. In particular, the laws of electricity and magnetism are found to depend on the frame of reference used. It might be argued that it is these laws that are wrong, but that is difficult to accept because the laws are in total agreement with all known experimental results. The Michelson–Morley experiment was an attempt to explain this dilemma.

The experiment stemmed from a misconception early physicists had concerning the manner in which light propagates. The properties of mechanical waves, such as water and sound waves, were well known, and all of these waves require a medium to support the disturbances. In the nineteenth century, physicists thought that electromagnetic waves also required a medium through which to propagate. They proposed that such a medium existed, and they named it the **luminiferous ether.** This ether was assumed to be present everywhere, even in a vacuum, and to have the unusual property of being a massless but rigid medium (a strange concept indeed!). In addition, it was found that the troublesome laws of electricity and magnetism took on their simplest forms in a frame of reference *at rest* with respect to the luminiferous ether; this frame was called the **absolute frame.** In any reference frame moving with respect to it, the laws of electricity and magnetism would remain valid but would have to be modified.

As a result of the importance attached to this absolute frame, experimental proof of its existence became of considerable interest in physics. However, all attempts to detect the presence of the ether (and hence the absolute frame) proved futile. The most famous experiment designed to show the presence of the ether was performed in 1887 by A. A. Michelson (1852–1931) and E. W. Morley (1838–1923). The objective was to determine the velocity of the Earth with respect to the ether, and the experimental tool used was a device called the interferometer, shown in Figure 9.3.

Suppose one arm of an interferometer (M_0M_2 in Fig. 9.3) were aligned along the direction of the motion of the Earth through space. The Earth moving through the ether would be equivalent to the ether flowing past the Earth in the opposite direction. This "ether wind" blowing in the direction opposite the Earth's motion should cause the speed of light, as measured in the Earth's frame of reference, to be $c - v$ as the light approaches mirror M_2 in Figure 9.3, and $c + v$ after reflection, where c is the speed of light in the ether frame and v is the speed of the Earth through space and hence the speed of the ether wind. The incident and reflected beams of light would recombine, and a pattern would form as the two beams interfere with each other.

During the experiment, the interference pattern was observed while the interferometer was rotated through an angle of 90°. The idea was that this rotation would change the speed of the ether wind along the direction of the arms of the interferometer, and as a consequence the interference pattern would shift slightly but measurably. Measurements failed to show any change in the pattern. The Michelson–Morley experiment was repeated by other researchers under varying conditions and at different locations, but the results were always the same: *no interference-pattern shift of the magnitude required was ever observed.*[2]

The negative result of the Michelson–Morley experiment not only contradicted the ether hypothesis; it also meant that it was impossible to measure the absolute (orbital) velocity of the Earth with respect to the ether frame. From a theoretical viewpoint, this meant that it was impossible to find the absolute frame. However, as we shall see in the next section, Einstein offered a postulate that places a different interpretation on the negative result. In later years, when more was known about the nature of light, the idea of an ether that permeates all space was relegated to the ash heap of worn-out concepts. **Light is now understood to be an electromagnetic wave that requires no medium for its propagation.** As a result, an ether in which light could travel became an unnecessary construct.

Modern versions of the Michelson–Morley experiment have compared the frequencies of resonant laser cavities of identical length, oriented at right angles to each other. Most recently, Doppler shift experiments using gamma rays emitted by a radioactive sample of ^{57}Fe have placed an upper limit of about 5 cm/s on ether wind velocity. These results have shown quite conclusively that the motion of the Earth has no effect on the speed of light!

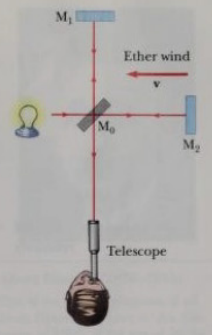


Figure 9.3 According to the ether wind theory, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

[2]From an Earth-observer's point of view, changes in the Earth's speed and direction of motion in the course of a year are viewed as ether wind shifts. Even if the speed of the Earth with respect to the ether were zero at some time, 6 months later the speed of the Earth would be 60 km/s with respect to the ether and a clear fringe shift should be found. None has ever been observed, however.

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9.3 • EINSTEIN'S PRINCIPLE OF RELATIVITY

In the previous section we noted the impossibility of measuring the speed of the ether with respect to the Earth and the failure of the Galilean addition law for velocities in the case of light. Albert Einstein proposed a theory that boldly removed these difficulties and at the same time completely altered our notion of space and time.[3] Einstein based special relativity on two postulates:

The postulates of special relativity •

1. **The Principle of Relativity:** All the laws of physics are the same in all inertial reference frames.
2. **The Constancy of the Speed of Light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

The first postulate asserts that *all* the laws of physics, those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on, are the same in all reference frames moving with constant velocity relative to each other. This postulate is a sweeping generalization of the principle of Newtonian relativity that only refers to the laws of mechanics. From an experimental point of view, Einstein's principle of relativity means that any kind of experiment (measuring the speed of light, for example), performed in a laboratory at rest must give the same result when performed in a laboratory moving at constant velocity past the first one. Hence no preferred inertial reference frame exists and it is impossible to detect absolute motion.

Note that postulate 2, the principle of the constancy of the speed of light, is required by postulate 1: If the speed of light were not the same in all inertial frames, it would be possible to distinguish between inertial frames and a preferred, absolute frame could be identified in contradiction to postulate 1. Postulate 2 also eliminates the problem of measuring the speed of the ether by denying the existence of the ether and boldly asserting that light always moves with speed c relative to all inertial observers.

Although the Michelson–Morley experiment was performed before Einstein published his work on relativity, it is not clear whether or not Einstein was aware of the details of the experiment. Nonetheless, the null result of the experiment can be readily understood within the framework of Einstein's theory. According to his principle of relativity, the premises of the Michelson–Morley experiment were incorrect. In the process of trying to explain the expected results, we stated that when light traveled against the ether wind its speed was $c - v$, in accordance with the Galilean addition law for velocities. However, if the state of motion of the observer or of the source has no influence on the value found for the speed of light, one will always measure the value to be c. Likewise, the light makes the return trip after reflection from the mirror at speed c, not at speed $c + v$. Thus, the motion of the Earth does not influence the interference pattern observed in the Michelson–Morley experiment, and a null result should be expected.

[3] A. Einstein, "On the Electrodynamics of Moving Bodies," *Ann. Physik* 17:891, 1905. For an English translation of this article and other publications by Einstein, see the book by H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity*, New York, Dover, 1958.

If we accept special relativity, we must conclude that relative motion is unimportant when measuring the speed of light. At the same time, we must alter our common-sense notion of space and time and be prepared for some bizarre consequences.

Thinking Physics 1

Imagine a very powerful lighthouse, with a rotating beacon. Imagine also drawing a horizontal circle around the lighthouse, with the lighthouse at the center. Along the circumference of the circle, the light beam lights up a portion of the circle and the lit portion of the circle moves around the circle at a certain tangential speed. If we now imagine a circle twice as big in radius, the tangential speed of the lit portion is faster, because it must travel a larger circumference in the time of one rotation of the light source. Imagine that we continue to make the circle larger and larger, eventually moving it out into space. The tangential speed of the lit portion will keep increasing. Is it possible that the tangential speed could become larger than the speed of light? Would this violate a principle of special relativity?

Reasoning For a large enough circle, it is possible that the tangential speed of the lit portion of the circle could be larger than the speed of light. This does not violate a principle of special relativity, however, because no matter or information is traveling faster than the speed of light.

CONCEPTUAL PROBLEM 1

You are in a speedboat on a lake. You see ahead of you a wavefront, caused by the previous passage of another boat, moving away from you. You accelerate, catch up with, and pass the wavefront. Is this scenario possible if you are in a rocket and you detect a wavefront of *light* ahead of you?

9.4 • CONSEQUENCES OF SPECIAL RELATIVITY

Before we discuss the consequences of special relativity, we must first understand how an observer located in an inertial reference frame describes an event. As mentioned earlier, an event is an occurrence describable by three space coordinates and one time coordinate. Different observers in different inertial frames usually describe the same event with different space–time coordinates.

The reference frame used to describe an event consists of a coordinate grid and a set of synchronized clocks located at the grid intersections as shown in Figure 9.4 in two dimensions. The clocks can be synchronized in many ways with the help of light signals. For example, suppose the observer is located at the origin with his master clock and sends out a pulse of light at $t = 0$. The light pulse takes a time r/c to reach a second clock located a distance r from the origin. Hence, the second clock is synchronized with the clock at the origin if the second clock reads a time r/c at the instant the pulse reaches it. This procedure of synchronization assumes that the speed of light has the same value in all directions and in all inertial frames. Furthermore, the procedure concerns an event recorded by an observer in a specific inertial reference frame. An observer in some other inertial frame would assign different space–time coordinates to events being observed by using another coordinate grid and another array of clocks.



Albert Einstein (1879–1955)

One of the greatest physicists of all times, Einstein was born in Ulm, Germany. In 1905, at the age of 26, he published four scientific papers that revolutionized physics. Two of these papers were concerned with what is now considered his most important contribution of all, the special theory of relativity. In 1916, Einstein published his work on the general theory of relativity. The most dramatic prediction of this theory is the degree to which light is deflected by a gravitational field. Measurements made by astronomers on bright stars in the vicinity of the eclipsed Sun in 1919 confirmed Einstein's prediction, and Einstein suddenly became a world celebrity. Einstein was deeply disturbed by the development of quantum mechanics in the 1920s despite his own role as a scientific revolutionary. In particular, he could never accept the probabilistic view of events in nature that is a central feature of quantum theory. The last few decades of his life were devoted to an unsuccessful search for a unified theory that would combine gravitation and electromagnetism into one picture.

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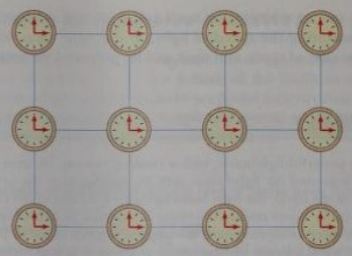


Figure 9.4 In relativity, we use a reference frame consisting of a coordinate grid and a set of synchronized clocks.

Almost everyone who has dabbled even superficially with science is aware of some of the startling predictions that arise because of Einstein's approach to relative motion. As we examine some of the consequences of relativity in the following three sections, we see that they conflict with our basic notions of space and time. We restrict our discussion to the concepts of length, time, and simultaneity, which are quite different in relativistic mechanics than in Newtonian mechanics. For example, **the distance between two points and the time interval between two events depend on the frame of reference in which they are measured.** That is, **there is no such thing as absolute length or absolute time in relativity.** Furthermore, **events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly past the first.**

Simultaneity and the Relativity of Time

A basic premise of Newtonian mechanics is that a universal time scale exists that is the same for all observers. In fact, Newton wrote, "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external." Thus, Newton and his followers simply took simultaneity for granted. In his development of special relativity, Einstein abandoned this assumption. According to Einstein, **a time interval measurement depends on the reference frame in which the measurement is made.**

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike its ends, as in Figure 9.5a, leaving marks on the boxcar and on the ground. The marks on the boxcar are labeled A' and B', and those on the ground are labeled A and B. An observer at O' moving with the boxcar is midway between A' and B', and a ground observer at O is midway between A and B. The events recorded by the observers are the light signals from the lightning bolts.

The two light signals reach observer O at the same time, as indicated in Figure 9.5b. This observer realizes that the light signals have traveled at the same speed over equal distances, and so rightly concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by the observer on the boxcar at O'. The lightning strikes as A' passes A, O' passes O, and B' passes B. By

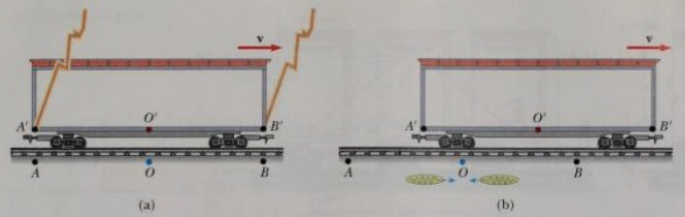


Figure 9.5 Two lightning bolts strike the end of a moving boxcar. (a) The events appear to be simultaneous to the stationary observer at O, who is midway between A and B. (b) The events do not appear to be simultaneous to the observer at O', who claims that the front of the train is struck before the rear.

the time the light has reached observer O, observer O' has moved as indicated in Figure 9.5b. Thus, the light signal from B' has already swept past O', and the light from A' has not yet reached O'. According to Einstein, **observer O' must find that light travels at the same speed as that measured by observer O.** Therefore, the observer O' concludes that the lightning struck the front of the boxcar before it struck the back. This thought experiment clearly demonstrates that the two events, which appear to be simultaneous to observer O, do not appear to be simultaneous to observer O'. In other words,

> two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first. That is, simultaneity is not an absolute concept but one that depends on the state of motion of the observer.

At this point, you might wonder which observer is right concerning the two events. The answer is that *both are correct*, because the principle of relativity states that **there is no preferred inertial frame of reference.** Although the two observers reach different conclusions, both are correct in their own reference frame because the concept of simultaneity is not absolute. This, in fact, is the central point of relativity—any uniformly moving frame of reference can be used to describe events and do physics. However, observers in different inertial frames of reference always measure different time intervals with their clocks and different distances with their meter sticks. Nevertheless, all observers agree on the forms of the laws of physics in their respective frames, because these laws must be the same for all observers in uniform motion. It is the alteration of time and space that allows the laws of physics (including Maxwell's equations) to be the same for all observers in uniform motion.

Time Dilation

The fact that observers in different inertial frames always measure different time intervals between a pair of events can be illustrated by considering a vehicle moving to the right with a speed v, as in Figure 9.6a. A mirror is fixed to the ceiling of the

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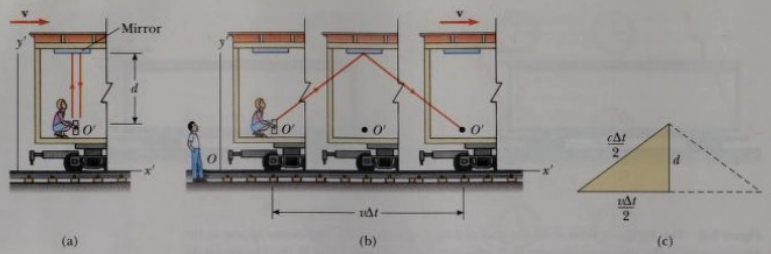


Figure 9.6 (a) A mirror is fixed to a moving vehicle, and a light pulse leaves O' at rest in the vehicle. (b) Relative to a stationary observer on Earth, the mirror and O' move with a speed v. Note that the distance the pulse travels is greater than $2d$ as measured by the stationary observer. (c) The right triangle for calculating the relationship between Δt and $\Delta t'$.

vehicle and observer O' at rest in this system holds a laser a distance d below the mirror. At some instant, the laser emits a pulse of light directed towards the mirror (event 1), and at some later time after reflecting from the mirror, the pulse arrives back at the laser (event 2). Observer O' carries a clock C' that she uses to measure the time interval Δt_p between these two events. Because the light pulse has a speed c, the time it takes the pulse to travel from O' to the mirror and back to O' can be found from the definition of speed:

$$\Delta t_p = \frac{\text{distance traveled}}{\text{speed}} = \frac{2d}{c} \qquad [9.3]$$

This time interval Δt_p measured by O', who is at rest in the moving vehicle, requires only a single clock C' located at the same place in this frame.

Now consider the same pair of events as viewed by observer O in a second frame, as in Figure 9.6b. According to this observer, the mirror and laser are moving to the right with a speed v. The sequence of events appears entirely different as viewed by this observer. By the time the light from the laser reaches the mirror, the mirror has moved a distance $v\,\Delta t/2$, where Δt is the time it takes the light to travel from O' to the mirror and back to O' as measured by observer O. In other words, the second observer concludes that, because of the motion of the vehicle, if the light is to hit the mirror, it must leave the laser at an angle with respect to the vertical direction. Comparing Figures 9.6a and 9.6b, we see that the light must travel farther in the second frame than in the first frame.

According to the second postulate of special relativity, both observers must measure c for the speed of light. Because the light travels farther in the second frame, it follows that the time interval Δt measured by the observer in the second frame is longer than the time interval Δt_p measured by the observer in the first frame. To obtain a relationship between these two time intervals, it is convenient to use the right triangle shown in Figure 9.6c. The Pythagorean theorem gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for Δt gives

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - \frac{v^2}{c^2}}} \qquad [9.4]$$

Because $\Delta t_p = 2d/c$, we can express Equation 9.4 as

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\,\Delta t_p \qquad [9.5]$$

• ***Time dilation***

where $\gamma = (1 - v^2/c^2)^{-1/2}$. This result says that **the time interval Δt measured by an observer moving with respect to the clock is longer than the time interval Δt_p measured by an observer at rest with respect to the clock** because γ is always greater than unity. That is, $\Delta t > \Delta t_p$. This effect is known as **time dilation.**

The time interval Δt_p in Equation 9.5 is called the **proper time.** In general, proper time is defined as **the time interval between two events measured by an observer who sees the events occur at the same point in space.** In our case, observer O' measures the proper time. That is, **proper time is always the time interval measured with a single clock at rest in the frame in which the events take place at the same position.**

• ***Proper time***

Because the time between ticks of a moving clock, $\gamma(2d/c)$, is observed to be longer than the time between ticks of an identical clock at rest, $2d/c$, it is often said, "*A moving clock runs slower than a clock at rest by a factor* γ." This is true for ordinary mechanical clocks as well as for the light clock just described. In fact, we can generalize these results by stating that **all physical processes, including chemical and biological ones, slow down relative to a stationary clock when those processes occur in a moving frame.** For example, the heartbeat of an astronaut moving through space would keep time with a clock inside the spaceship. Both the astronaut's clock and heartbeat are slowed down relative to a stationary clock. The astronaut would not have any sensation of life slowing down in the spaceship. For the astronaut, it is the clock on Earth and the companions at Mission Control that are moving and therefore keep a slow time.

Time dilation is a verifiable phenomenon. For example, muons are unstable elementary particles that have a charge equal to that of the electron and a mass 207 times that of the electron. Muons can be produced by the collision of cosmic radiation with atoms high in the atmosphere. Slow-moving muons have a lifetime of only 2.2 μs in the laboratory. If we take 2.2 μs as the average lifetime of a muon and assume that its speed is close to the speed of light, we find that these particles can travel a distance of only approximately 600 m before they decay (Fig. 9.7a). Hence, they cannot reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons *do* reach the Earth. The phenomenon of time dilation explains this effect. Relative to an observer on Earth, the muons have a lifetime equal to $\gamma\tau$, where $\tau = 2.2$ μs is the lifetime in a frame of reference traveling with the muons. For example, for $v = 0.99c$, $\gamma \approx 7.1$ and $\gamma\tau \approx 16$ μs. Hence, the average distance traveled as measured by an observer on Earth is $\gamma v\tau \approx 4800$ m, as indicated in Figure 9.7b.

In 1976, at the laboratory of the European Council for Nuclear Research in

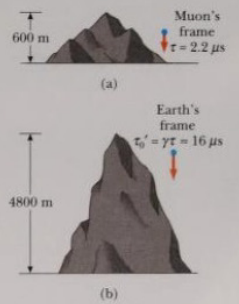


Figure 9.7 (a) Muons traveling with a speed of $0.99c$ travel only approximately 600 m as measured in the muons' reference frame, where their proper lifetime is about 2.2 μs. (b) The muons travel approximately 4800 m as measured by an observer on Earth. Because of time dilation, the muons' lifetime is longer as measured by the Earth observer.

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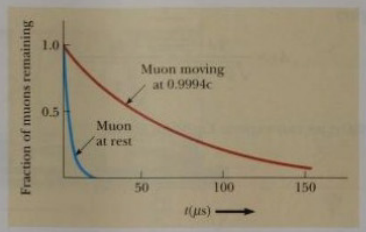


Figure 9.8 Decay curves for muons traveling at a speed of $0.9994c$ and for muons at rest.

Geneva, muons injected into a large storage ring reached speeds of approximately $0.9994c$. Electrons produced by the decaying muons were detected by counters around the ring, enabling scientists to measure the decay rate and hence the muon lifetime. The lifetime of the moving muons was measured to be approximately 30 times as long as that of the stationary muon (Fig. 9.8), in agreement with the prediction of relativity to within 2 parts in 1000.

The results of an experiment reported by Hafele and Keating provided direct evidence of time dilation.[4] The experiment involved the use of very stable cesium-beam atomic clocks. Time intervals measured with four such clocks in jet flight were compared with time intervals measured by reference clocks located at the U.S. Naval Observatory. To compare these results with the theory, many factors had to be considered, including periods of acceleration and deceleration relative to the Earth, variations in direction of travel, and the weaker gravitational field experienced by the flying clocks compared to the Earth-based clock. Their results were in good agreement with the predictions of special relativity and can be explained in terms of the relative motion between the Earth's rotation and the jet aircraft. In their paper, Hafele and Keating report the following: "Relative to the atomic time scale of the U.S. Naval Observatory, the flying clocks lost 59 ± 10 ns during the eastward trip and gained 273 ± 7 ns during the westward trip. . . . These results provide an unambiguous empirical resolution of the famous clock paradox with macroscopic clocks."

Thinking Physics 2

Suppose a student explains time dilation with the following argument: If you start running at $0.99c$ away from a clock at 12:00, you would not see the time change, because the light from the clock representing 12:01 would never reach you. What is the flaw in this argument?

Reasoning The inference in this argument is that the velocity of light relative to the runner is approximately *zero*—"the light . . . would never reach you." This is a Galilean point of view, in which the relative velocity is a simple subtraction of running velocity from the light velocity. From the point of view of special relativity, one of the fundamental postulates is that the speed of light is the same for all observers, *including one running away from the light source at the speed of light.* Thus, the light from 12:01 will move toward the runner at the speed of light.

[4] J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Relativistic Time Gains Observed," *Science*, July 14, 1972, p. 168.

Example 9.1 What Is the Period of the Pendulum?

The period of a pendulum is measured to be 3.0 s in the rest frame of the pendulum's pivot. What is the period when measured by an observer moving at a speed of $0.95c$ relative to the pendulum?

Reasoning Instead of the observer moving at $0.95c$, we can take the equivalent point of view that the observer is at rest and that the pendulum is moving at $0.95c$ past the stationary observer. Hence the pendulum is an example of a moving clock.

Solution The proper time is 3.0 s. Because a moving clock runs slower than a stationary clock by γ, Equation 9.5 gives

$$T = \gamma T_p = \frac{1}{\sqrt{1 - \dfrac{(0.95c)^2}{c^2}}} T_p = (3.2)(3.0\text{ s}) = 9.6\text{ s}$$

That is, a moving pendulum slows down or takes longer to complete a period compared to one at rest.

EXERCISE 1 Muons move in circular orbits at a speed of $0.9994c$ in a storage ring of radius 500 m. If a muon at rest decays into other particles after 2.2 μs (proper time), how many trips around the storage ring do we expect the muons to make before they decay?
Answer 6.1 revolutions

Length Contraction

The measured distance between two points also depends on the frame of reference. The **proper length** of an object is defined as **the length of the object measured by someone who is at rest relative to the object.** The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as **length contraction.**

• *Proper length*

Consider a spaceship traveling with a speed v from one star to another. There are two observers, one on Earth and the other in the spaceship. The observer at rest on Earth (and also assumed to be at rest with respect to the two stars) measures the distance between the stars to be L_p, the proper length. According to this observer, the time it takes the spaceship to complete the voyage is $\Delta t = L_p / v$. What does an observer in the moving spaceship measure for the distance between the stars? Because of time dilation, the space traveler measures a smaller time of travel: $\Delta t_p = \Delta t / \gamma$. The space traveler claims to be at rest and sees the destination star moving toward the spaceship with speed v. Because the space traveler reaches the star in the time Δt_p, she or he concludes that the distance, L, between the stars is shorter than L_p. This distance measured by the space traveler is

$$L = v\,\Delta t_p = v\frac{\Delta t}{\gamma}$$

Because $L_p = v\,\Delta t$, we see that $L = L_p / \gamma$ or

$$L = L_p\left(1 - \frac{v^2}{c^2}\right)^{1/2} \qquad [9.6]$$

• *Length contraction*

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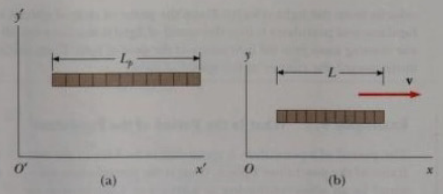


Figure 9.9 (a) A stick as viewed by an observer in a frame attached to the stick (i.e., both have the same velocity). (b) The stick as seen by an observer in a frame in which the stick has a velocity **v** relative to the frame. The length is *shorter* than the proper length, L_p, by a factor $(1 - v^2/c^2)^{1/2}$.

where $(1 - v^2/c^2)^{1/2}$ is a factor less than one. This result may be interpreted as follows:

> A moving observer measures the length L of an object (along the direction of motion) to be shorter than the length L_p measured by an observer at rest with respect to the object (the proper length).

Note that **length contraction takes place only along the direction of motion.** For example, suppose a stick moves past an Earth observer with speed v as in Figure 9.9. The length of the stick as measured by an observer in a frame attached to the stick is the proper length, L_p, as in Figure 9.9a. The length of the stick, L, measured by the Earth observer is shorter than L_p by the factor $(1 - v^2/c^2)^{1/2}$. Furthermore, length contraction is a symmetrical effect: If the stick is at rest on Earth, an observer in the moving frame would also measure its length to be shorter by the same factor $(1 - v^2/c^2)^{1/2}$.

It is important to emphasize that proper length and proper time are measured in different reference frames. As an example of this point, let us return to the decaying muons moving at speeds close to the speed of light. An observer in the muon's reference frame would measure the proper lifetime, whereas an Earth-based observer would measure the proper height of the mountain in Figure 9.7. In the muon's reference frame, there is no time dilation, but the distance of travel is observed to be shorter when measured in this frame. Likewise, in the Earth-based observer's reference frame, there is time dilation, but the distance of travel is measured to be the proper height of the mountain. Thus, when calculations on the muon are performed in both frames, the outcome of the experiment in one frame is the same as the outcome in the other frame!

Example 9.2 A Voyage to Sirius

An astronaut takes a trip to Sirius, located 8 lightyears from Earth. The astronaut measures the time of the one-way journey to be 6 y. If the spaceship moved at a constant speed of $0.8c$, how can the 8-lightyear distance be reconciled with the 6-y duration measured by the astronaut?

Reasoning The 8 lightyears (ly) represents the proper length (distance) from Earth to Sirius measured by an observer seeing both nearly at rest. The astronaut sees Sirius approaching her at $0.8c$ but also sees the distance contracted to

$$\frac{8\text{ ly}}{\gamma} = (8\text{ ly})\sqrt{1 - \frac{v^2}{c^2}} = (8\text{ ly})\sqrt{1 - \frac{(0.8c)^2}{c^2}} = 5\text{ ly}$$

So the travel time measured on the astronaut's clock is

$$\Delta t = \frac{d}{v} = \frac{5\text{ ly}}{0.8c} = 6\text{ y}$$

Example 9.3 The Triangular Spaceship

A spaceship in the form of a triangle flies by an observer with a speed of $0.95c$ along the x-direction. When the ship is at rest relative to the observer (Fig. 9.10a), the distances x and y are measured to be 52 m and 25 m, respectively. What is the shape of the ship as seen by an observer at rest when the ship is in motion along the direction shown in Figure 9.10b?

Solution The observer sees the horizontal length of the ship to be contracted to a length

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} = (52\text{ m})\sqrt{1 - \frac{(0.95c)^2}{c^2}} = 16\text{ m}$$

The 25-m vertical height is unchanged because it is perpendicular to the direction of relative motion between observer and spaceship. Figure 9.10b represents the shape of the spaceship as seen by the observer at rest.

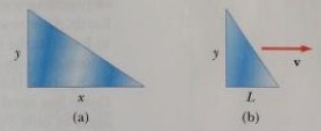


Figure 9.10 (Example 9.3) (a) When the spaceship is at rest, its shape is as shown. (b) The spaceship appears to look like this when it moves to the right with a speed v. Note that only its x dimension is contracted in this case.

EXERCISE 2 A spacecraft moves at $0.9c$. If its length is L_0 when measured from inside the spacecraft, what is its length measured by a ground observer? Answer $0.436L_0$

EXERCISE 3 A spaceship is measured to be 120 m long while at rest relative to an observer. If this spaceship now flies by the observer with a speed $0.99c$, what length does the observer measure? Answer 17 m

The Twins Paradox

OPTIONAL

An intriguing consequence of time dilation is the so-called twins paradox. Consider an experiment involving a set of twins named Speedo and Goslo who are, say, 20 years old. The twins carry with them identical clocks that have been synchronized (Fig. 9.11). Speedo, the more adventuresome of the two, sets out on an epic journey



Figure 9.11 (a) As the twins depart, they are the same age. (b) When Speedo returns from his journey to Planet X, he is younger than his twin Goslo who remained on Earth.

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to Planet X, located 10 lightyears from Earth. Furthermore, his spaceship is capable of reaching a speed of $0.500c$ relative to the inertial frame of his twin brother on Earth. After reaching Planet X, Speedo becomes homesick and immediately returns to Earth at the same high speed he had attained on the outbound journey. On his return, Speedo is shocked to discover that many things have changed in his absence. To Speedo, the most significant change to have occurred is that his twin brother Goslo has aged 40 years and is now 60 years old. Speedo, on the other hand, has aged by only 34.6 years.

At this point, it is fair to raise the following question—which twin is the traveler and which twin is really younger as a result of this experiment? From Goslo's frame of reference, he was at rest while his brother traveled at a high speed. From Speedo's perspective, it is he who is at rest while Goslo is on the high-speed space journey. According to Speedo, it is Goslo and the Earth that have raced away on a 17.3-year journey and then headed back for another 17.3 years. This leads to an apparent contradiction. Which twin has developed signs of excess aging?

To resolve this apparent paradox, recall that relative to any chosen inertial frame (like that of the stay-at-home twin), clocks in a moving object's frame move more slowly. However, the trip situation is not symmetrical. Speedo, the space traveler, must experience a series of accelerations during his journey. As a result, his speed is not always uniform and as a consequence Speedo is not in a single inertial frame. He cannot be regarded as always being at rest and Goslo to be in uniform motion because he changes his state of motion at least once. Therefore there is no paradox.

The conclusion that Speedo is in a noninertial frame is inescapable. The time required to accelerate and decelerate Speedo's spaceship may be made very small using large rockets, so that Speedo can claim that he spends most of his time traveling to Planet X at $0.500c$ in an inertial frame. However, Speedo must slow down, reverse his motion, and return to Earth in an altogether different inertial frame. At the very best, Speedo is in two different inertial frames during his journey. Only Goslo, who is in a single inertial frame, can apply the simple time dilation formula to Speedo's trip. Thus, Goslo finds that instead of aging 40 years, Speedo ages only $(1 - v^2/c^2)^{1/2}(40 \text{ years}) = 34.6$ years. However, Speedo spends 17.3 years traveling to Planet X and 17.3 years returning, for a total travel time of 34.6 years, in agreement with our earlier statement.

CONCEPTUAL PROBLEM 2

You are packing for a trip to another star, to which you will be traveling at $0.99c$. Should you buy smaller sizes of your clothing, since you will be skinnier on your trip? Can you sleep in a smaller cabin than usual, since you will be shorter when you lie down?

CONCEPTUAL PROBLEM 3

Suppose astronauts were paid according to the time spent traveling in space. After a long voyage traveling at a speed near that of light, a crew of astronauts return to Earth and open their pay envelopes. What will their reaction be?

CONCEPTUAL PROBLEM 4

You are observing a rocket moving away from you. You notice that it is measured to be shorter than when it was at rest on the ground next to you, and, through the rocket window, you can see a clock. You observe that the passage of time on the clock is measured to be slower than that of the watch on your wrist. What if the rocket turns around and comes toward you? Will it appear to be *longer* and will the rocketbound clock move *faster*?

9.5 · THE LORENTZ TRANSFORMATION EQUATIONS

We have seen that the Galilean transformation is not valid when v approaches the speed of light. In this section, we state the correct transformation equations that apply for all speeds in the range $0 \leq v < c$.

Suppose an event that occurs at some point P is reported by two observers, one at rest in a frame S and another in a frame S' that is moving to the right with speed v as in Figure 9.12. The observer in S reports the event with space–time coordinates (x, y, z, t), and the observer in S' reports the same event using the coordinates (x', y', z', t'). We would like to find a relationship between these coordinates that is valid for all speeds. In Section 9.1, we found that the Galilean transformation of coordinates, given by Equation 9.1, does not agree with experiment at speeds comparable to the speed of light.

The equations that are valid from $v = 0$ to $v = c$ and enable us to transform coordinates from S to S' are given by the **Lorentz transformation equations:**

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Lorentz transformation for $S \rightarrow S'$ [9.7]

These transformation equations, known as the Lorentz transformation, were developed by Hendrik A. Lorentz (1853–1928) in 1890 in connection with electromagnetism. However, it was Einstein who recognized their physical significance and took the bold step of interpreting them within the framework of special relativity.

We see that the value for t' assigned to an event by an observer standing at O' depends both on the time t and on the coordinate x as measured by an observer at O. This is consistent with the notion that an event is characterized by four space–time coordinates (x, y, z, t). In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other. This is unlike the case of the Galilean transformation in which $t = t'$.

If we wish to transform coordinates in the S' frame to coordinates in the S frame, we simply replace v by $-v$ and interchange the primed and unprimed coordinates in Equation 9.7:

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Inverse Lorentz transformation for $S' \rightarrow S$ [9.8]

When $v \ll c$, the Lorentz transformation should reduce to the Galilean transformation. To check this, note that as $v \rightarrow 0$, $v/c^2 \ll 1$ and $v^2/c^2 \ll 1$, so that

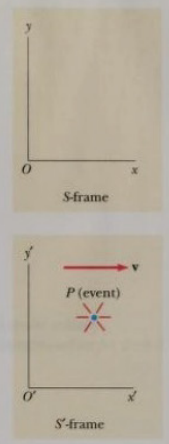


Figure 9.12 Representation of an event that occurs at some point P as observed by an observer at rest in the S frame and another in the S' frame, which is moving to the right with a speed v.

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$\gamma = 1$ and Equation 9.7 reduces in this limit to the Galilean coordinate transformation equations

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

In many situations, we would like to know the difference in coordinates between two events or the time interval between two events as seen by observers at O and O'. This can be accomplished by writing the Lorentz equations in a form suitable for describing pairs of events. From Equations 9.7 and 9.8, we can express the differences between the four variables x, x', t, and t' in the form

$$\Delta x' = \gamma(\Delta x - v\,\Delta t)$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \qquad S \rightarrow S' \qquad [9.9]$$

$$\Delta x = \gamma(\Delta x' + v\,\Delta t')$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \qquad S' \rightarrow S \qquad [9.10]$$

where $\Delta x' = x_2' - x_1'$ and $\Delta t' = t_2' - t_1'$ are the differences measured by the observer at O', and $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ are the differences measured by the observer at O. We have not included the expressions for relating the y and z coordinates because they are unaffected by motion along the x direction.[5]

Example 9.4 Simultaneity and Time Dilation Revisited

Use the Lorentz transformation equations in difference form to show that (a) simultaneity is not an absolute concept and (b) moving clocks run slower than stationary clocks.

Solution (a) Suppose that two events are simultaneous according to a moving observer at O', so that $\Delta t' = 0$. From the expression for Δt given in Equation 9.10, we see that in this case, $\Delta t = \gamma v\,\Delta x'/c^2$. That is, the time interval for the same two events as measured by an observer at O is nonzero, and so they do not appear to be simultaneous in O.

(b) Suppose that an observer at O' finds that two events occur at the same place ($\Delta x' = 0$), but at different times ($\Delta t' \neq 0$). In this situation, the expression for Δt given in Equation 9.10 becomes $\Delta t = \gamma\,\Delta t'$. This is the equation for time dilation found earlier, Equation 9.5, where $\Delta t' = \Delta t$ is the proper time measured by the single clock located at O'.

EXERCISE 4 Use the Lorentz transformation equations in difference form to confirm that $L = L_p/\gamma$.

Lorentz Velocity Transformation

Let us now derive the Lorentz velocity transformation, which is the relativistic counterpart of the Galilean velocity transformation. Once again S is our stationary frame of reference and S' is our frame of reference that moves at a speed v relative to S. Suppose that an object is observed in the S' frame with an instantaneous speed u_x' measured in S' given by

$$u_x' = \frac{dx'}{dt'} \qquad [9.11]$$

[5]Although motion along x does not change y and z coordinates, it does change velocity components along y and z.

Using Equations 9.7, we have

$$dx' = \gamma(dx - v\,dt) \qquad \text{and} \qquad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

Substituting these values into Equation 9.11 gives

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

But dx/dt is just the velocity component u_x of the object measured in S, and so this expression becomes

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad [9.12]$$

• *Lorentz velocity transformation for $S \rightarrow S'$*

In a similar way, if the object has velocity components along y and z, the components in S' are

$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad \text{and} \qquad u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad [9.13]$$

When u_x and v are both much smaller than c (the nonrelativistic case), the denominator of Equation 9.12 approaches unity and so $u_x' \approx u_x - v$. This corresponds to the Galilean velocity transformations. In the other extreme, when $u_x = c$, Equation 9.12 becomes

$$u_x' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c\left(1 - \frac{v}{c}\right)}{1 - \frac{v}{c}} = c$$

From this result, we see that an object whose speed approaches c relative to an observer in S also has a speed approaching c relative to an observer in S'—independent of the relative motion of S and S'. Note that this conclusion is consistent with Einstein's second postulate—namely, that the speed of light must be c relative to all inertial frames of reference. Furthermore, the speed of an object can never exceed c. That is, the speed of light is the ultimate speed. We return to this point later when we consider the energy of a particle.



The speed of light is the speed limit of the Universe.

To obtain u_x in terms of u_x', we replace v by $-v$ in Equation 9.12 and interchange the roles of u_x and u_x':

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} \qquad [9.14]$$

• *Inverse Lorentz velocity transformation for $S' \rightarrow S$*

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Example 9.5 Relative Velocity of Spaceships

Two spaceships A and B are moving directly toward each other, as in Figure 9.13. An observer on Earth measures the speed of A to be $0.750c$ and the speed of B to be $0.850c$. Find the velocity of B with respect to A.

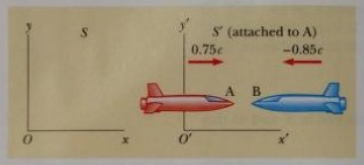


Figure 9.13 (Example 9.5) Two spaceships A and B move in opposite directions. The speed of B relative to A is *less* than c and is obtained by using the relativistic velocity transformation.

Solution This problem can be solved by taking the S' frame as being attached to A, so that $v = 0.750c$ relative to the Earth observer (the S frame). Spaceship B can be considered as an object moving with a velocity $u_x = -0.850c$ relative to the Earth observer. Hence, the velocity of B with respect to A can be obtained using Equation 9.12:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \frac{(-0.850c)(0.750c)}{c^2}} = -0.980c$$

The negative sign indicates that spaceship B is moving in the negative x direction as observed by A. Note that the result is less than c. That is, a body whose speed is less than c in one frame of reference must have a speed less than c in any other frame. (If the Galilean velocity transformation were used in this example, we would find that $u'_x = u_x - v = -0.850c - 0.750c = -1.60c$, which is greater than c. The Galilean transformation does not work in relativistic situations.)

Example 9.6 Relativistic Leaders of the Pack

Two motorcycle pack leaders named David and Emily are racing at relativistic speeds along perpendicular paths, as in Figure 9.14. How fast does Emily recede over David's right shoulder as seen by David?

Solution Figure 9.14 represents the situation as seen by a police officer at rest in frame S, who observes the following:

David: $u_x = 0.75c \qquad u_y = 0$

Emily: $u_x = 0 \qquad u_y = -0.90c$

To get Emily's speed of recession as seen by David, we take S' to move along with David and we calculate u'_x and u'_y for Emily using Equations 9.12 and 9.13:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{0 - 0.75c}{1 - \frac{(0)(0.75c)}{c^2}} = -0.75c$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{\sqrt{1 - \frac{(0.75c)^2}{c^2}}(-0.90c)}{\left(1 - \frac{(0)(0.75c)}{c^2}\right)} = -0.60c$$

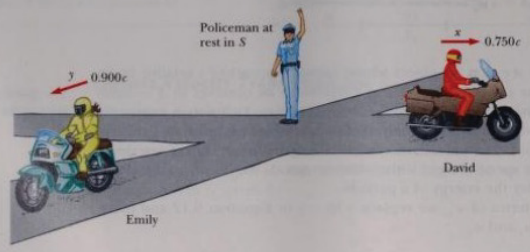


Figure 9.14 (Example 9.6) David moves to the east with a speed $0.750c$ relative to the policeman, while Emily travels south at a speed $0.900c$.

Thus, the speed of Emily as observed by David is

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = \sqrt{(-0.75c)^2 + (-0.60c)^2} = 0.96c$$

Note that this speed is less than c as required by special relativity.

EXERCISE 5 Calculate the classical speed of recession for Emily as observed by David using a Galilean transformation.
Answer $1.2c$

EXERCISE 6 A Klingon spaceship moves away from the Earth at a speed of $0.80c$. The starship *Enterprise* pursues at a speed of $0.90c$ relative to the Earth. Observers on Earth see the *Enterprise* overtaking the Klingon ship at a relative speed of $0.10c$. With what speed is the *Enterprise* overtaking the Klingon ship as seen by the crew of the *Enterprise*?
Answer $0.36c$

9.6 · RELATIVISTIC MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

We have seen that in order to properly describe the motion of particles within the framework of special relativity, the Galilean transformation must be replaced by the Lorentz transformation. Because the laws of physics must remain unchanged under the Lorentz transformation, we must generalize Newton's laws and the definitions of momentum and energy to conform to the Lorentz transform and the principle of relativity. These generalized definitions should reduce to the classical (nonrelativistic) definitions for $v \ll c$.

First, recall that the law of conservation of momentum states that when two bodies collide, the total momentum remains constant, assuming the bodies are isolated. Suppose the collision is described in a reference frame S in which momentum is conserved. If the velocities in a second reference frame S' are calculated using the Lorentz transformation and the classical definition of momentum, $\mathbf{p} = m\mathbf{u}$, it is found that momentum is *not* conserved in the second reference frame. (We use the symbol $\mathbf{u}$ for particle velocity rather than $\mathbf{v}$, which is used for the relative velocity of two reference frames.) However, because the laws of physics are the same in all inertial frames, the momentum must be conserved in all systems. In view of this condition and assuming the Lorentz transformation is correct, we must modify the definition of momentum to satisfy the following conditions:

- $\mathbf{p}$ must be conserved in all collisions
- $\mathbf{p}$ must approach the classical value $m\mathbf{u}$ as $\mathbf{u} \to 0$

The correct relativistic equation for momentum that satisfies these conditions is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.15]$$

· *Definition of relativistic momentum*

where $\mathbf{u}$ is the velocity of the particle. When u is much less than c, the denominator of Equation 9.15 approaches unity, so that $\mathbf{p}$ approaches $m\mathbf{u}$. Therefore, the rela-

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tivistic equation for $\mathbf{p}$ reduces to the classical expression when u is small compared with c. Because it is simpler, Equation 9.15 is often written as

$$\mathbf{p} = \gamma m\mathbf{u} \qquad [9.16]$$

where $\gamma = (1 - u^2/c^2)^{-1/2}$. Note that γ has the same functional form as the γ in the Lorentz transformation. The transformation is from that of the particle to the frame of the observer moving at speed u relative to the particle.

The relativistic force $\mathbf{F}$ on a particle whose momentum is $\mathbf{p}$ is defined as

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \qquad [9.17]$$

where $\mathbf{p}$ is given by Equation 9.15. This expression is reasonable because it preserves classical mechanics in the limit of low velocities and requires conservation of momentum for an isolated system ($\mathbf{F} = 0$) both relativistically and classically.

It is left as an end-of-chapter problem (Problem 47) to show that the acceleration $\mathbf{a}$ of a particle decreases under the action of a constant force, in which case $a \propto (1 - u^2/c^2)^{3/2}$. From this formula note that as the particle's speed approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed $v \geq c$.

Example 9.7 Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

Solution Using Equation 9.15 with $u = 0.75c$, we have

$$p = \frac{mu}{\sqrt{1 - \dfrac{u^2}{c^2}}}$$

$$p = \frac{(9.11 \times 10^{-31}\ \text{kg})(0.750 \times 3.00 \times 10^{8}\ \text{m/s})}{\sqrt{1 - \dfrac{(0.750c)^2}{c^2}}}$$

$$= 3.10 \times 10^{-22}\ \text{kg·m/s}$$

The incorrect classical expression gives

$$\text{Momentum} = m_e u = 2.05 \times 10^{-22}\ \text{kg·m/s}$$

Hence, the correct relativistic result is 50% greater than the classical result!

9.7 • RELATIVISTIC ENERGY

We have seen that the definition of momentum and the laws of motion require generalization to make them compatible with the principle of relativity. This implies that the definition of kinetic energy must also be modified.

To derive the relativistic form of the work-kinetic energy theorem, let us start with the definition of the work done on a particle by a force F and use the definition of relativistic force, Equation 9.17:

$$W = \int_{x_i}^{x_f} F\,dx = \int_{x_i}^{x_f} \frac{dp}{dt}\,dx \qquad [9.18]$$

for force and motion both along the x axis. In order to perform this integration, and find the work done on a particle, or the relativistic kinetic energy as a function of u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \frac{m(du/dt)}{\left(1 - \dfrac{u^2}{c^2}\right)^{3/2}}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 9.18 gives

$$W = \int_0^t \frac{m(du/dt)\,u\,dt}{\left(1 - \dfrac{u^2}{c^2}\right)^{3/2}} = m\int_0^u \frac{u}{\left(1 - \dfrac{u^2}{c^2}\right)^{3/2}}\,du$$

where we have assumed that the particle is accelerated from rest to some final speed u. Evaluating the integral, we find that

$$W = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} - mc^2 \qquad [9.19]$$

Recall from Chapter 6 that the work done by a force acting on a particle equals the change in kinetic energy of the particle. Because the initial kinetic energy is zero, we conclude that the work W is equivalent to the relativistic kinetic energy K:

$$K = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2 \qquad [9.20]$$

• ***Relativistic kinetic energy***

This equation is routinely confirmed by experiments using high-energy particle accelerators.

At low speeds, where $u/c \ll 1$, Equation 9.20 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \ldots$ for $x \ll 1$, where the higher-order powers of x are neglected in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

Substituting this into Equation 9.20 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 = \frac{1}{2}mu^2$$

which agrees with the classical result. Figure 9.15 shows a comparison of the speed–kinetic energy relationships for a particle using the nonrelativistic expression for K (the blue curve) and the relativistic expression for K (the brown curve). The curves are in good agreement at low speeds but deviate at higher speeds. The nonrelativistic expression indicates a violation of physical law because it suggests that sufficient energy can be added to the particle to accelerate it to a speed larger than c. In the relativistic case, the particle speed never exceeds c, regardless of the kinetic energy. When an object's speed is less than one tenth the speed of light, the classical kinetic energy equation differs by less than 1% from the relativistic

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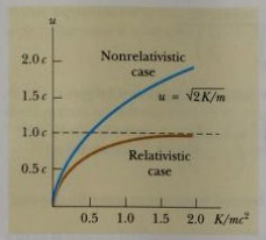


Figure 9.15 A graph comparing relativistic and nonrelativistic kinetic energy. The speeds are plotted versus energy. In the relativistic case, u is always less than c.

equation (which is experimentally verified at all speeds). Thus for practical calculations it is valid to use the classical equation when the object's speed is less than $0.1c$.

The constant term mc^2 in Equation 9.20, which is independent of the speed, is called the **rest energy** of the free particle E_R.

Rest energy

$$E_R = mc^2 \qquad [9.21]$$

The term γmc^2 in Equation 9.20 depends on the particle speed and is the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E—that is, total energy = kinetic energy + rest energy, or

$$E = \gamma mc^2 = K + mc^2 \qquad [9.22]$$

or, when γ is replaced by its equivalent,

Definition of total energy

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.23]$$

This, of course, is Einstein's famous mass–energy equivalence equation. The relation $E = \gamma mc^2 = \gamma E_R$ shows that **mass is one possible manifestation of energy.** Furthermore, this result shows that a small mass corresponds to an enormous amount of energy. This concept is fundamental to much of the field of nuclear physics.

In many situations, the momentum or energy of a particle is measured instead of its speed. It is therefore useful to have an expression relating the total energy E to the relativistic momentum p. This is accomplished by using the expressions $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 35). The result, after some algebra, is

Energy–momentum relationship

$$E^2 = p^2c^2 + (mc^2)^2 \qquad [9.24]$$

When the particle is at rest, $p = 0$, and so $E = E_R = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons

(massless, chargeless particles of light), we set $m = 0$ in Equation 9.24, and we see that

Energy of a photon

$$E = pc \qquad [9.25]$$

This equation is an exact expression relating energy and momentum for photons, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. For this reason, m is often called the *invariant mass*. However, the total energy and momentum of a particle depend on the reference frame in which they are measured, because they both depend on velocity. Because m is a constant, then according to Equation 9.24 the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is invariant (its value remains the same) under a Lorentz transformation. These equations do not yet make provision for potential energy.

When dealing with subatomic particles, it is convenient to express their energy in electron volts (eV), because the particles are usually given this energy in particle accelerators. The conversion factor is

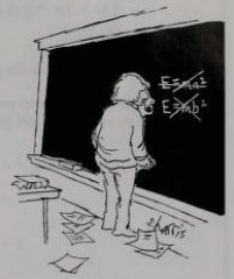
$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

For example, the mass of an electron is 9.11×10^{-31} kg. Hence, the rest energy of the electron is

$$m_e c^2 = (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 8.20 \times 10^{-14} \text{ J}$$

Converting this to eV, we have

$$m_e c^2 = (8.20 \times 10^{-14} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J}) = 0.511 \text{ MeV}$$



Thinking Physics 3

A common principle learned in chemistry is conservation of mass. In practice, if the mass of the reactants is measured before a reaction and the mass of the products is measured afterward, the results will be the same. In light of special relativity, should we stop teaching the principle of conservation of mass in chemistry classes?

Reasoning Consider a reaction that does not require energy input to occur. This type of reaction occurs because the products represent a lower overall rest energy than the reactants, the difference in rest energy carried away as kinetic energy of ejected particles or radiation. Because the rest energy of the reactants is smaller, according to relativity the mass of the reactants should be smaller than that of the products. Thus the law of conservation of mass is violated. The mass changes are so small, however, that, in practice the law of conservation of mass is still useful.

CONCEPTUAL PROBLEM 5

A photon has zero mass. If a photon is reflected from a surface, does it exert a force on the surface?

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Example 9.8 The Energy of a Speedy Proton

The total energy of a proton is three times its rest energy. (a) Find the proton's rest energy in electron volts.

Solution

$$E_R = m_p c^2 = (1.67 \times 10^{-27}\ \text{kg})(3.00 \times 10^8\ \text{m/s})^2$$
$$= (1.50 \times 10^{-10}\ \text{J})(1.00\ \text{eV}/1.60 \times 10^{-19}\ \text{J})$$
$$= 938\ \text{MeV}$$

(b) With what speed is the proton moving?

Solution Because the total energy E is three times the rest energy, $E = \gamma mc^2$ (Eq. 9.23) gives

$$E = 3m_p c^2 = \frac{m_p c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$3 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Solving for u gives

$$\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{9} \quad \text{or} \quad \frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = 2.83 \times 10^8\ \text{m/s}$$

(c) Determine the kinetic energy of the proton in electron volts.

Solution

$$K = E - m_p c^2 = 3m_p c^2 - m_p c^2 = 2m_p c^2$$

Because $m_p c^2 = 938$ MeV, $K = 1876$ MeV

(d) What is the magnitude of the proton's momentum?

Solution We can use Equation 9.24 to calculate the momentum with $E = 3m_p c^2$:

$$E^2 = p^2 c^2 + (m_p c^2)^2 = (3m_p c^2)^2$$

$$p^2 c^2 = 9(m_p c^2)^2 - (m_p c^2)^2 = 8(m_p c^2)^2$$

$$p = \sqrt{8}\,\frac{m_p c^2}{c} = \sqrt{8}\,\frac{(938\ \text{MeV})}{c} = 2650\,\frac{\text{MeV}}{c}$$

The unit of momentum is written MeV/c for convenience.

EXERCISE 7 Find the speed at which the relativistic kinetic energy is two times the nonrelativistic value. Answer 0.786c

EXERCISE 8 Find the speed of a particle whose total energy is twice its rest energy. Answer 0.866c

EXERCISE 9 An electron moves with a speed $u = 0.850c$. Find its (a) total energy and (b) kinetic energy in electron volts. Answer (a) 0.970 MeV (b) 0.459 MeV

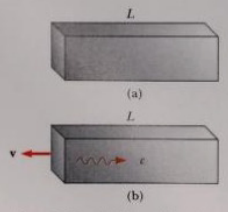


Figure 9.16 (a) A box of length L at rest. (b) When a light pulse is emitted at the left end of the box, the box recoils to the left until the pulse strikes the right end.

9.8 • MASS AS A MEASURE OF ENERGY

To understand the equivalence of mass and energy, consider the following thought experiment proposed by Einstein. Imagine a box of mass M and length L initially at rest, as in Figure 9.16a. Suppose that a pulse of light is emitted from the left side of the box, as in Figure 9.16b. From Equation 9.25, we know that the light of energy E carries momentum $p = E/c$. Hence, the box must recoil to the left with a speed v to conserve momentum. Assuming the box is very massive, the recoil speed is small compared with the speed of light. Conservation of momentum gives $Mv = E/c$, or

$$v = \frac{E}{Mc}$$

The time it takes the light to move the length of the box is approximately $\Delta t = L/c$ (where again, we assume that $v \ll c$). In this time interval, the box moves a small distance Δx to the left, where

$$\Delta x = v\,\Delta t = \frac{EL}{Mc^2}$$

The light then strikes the right end of the box, transfers its momentum to the box, causing the box to stop. With the box in its new position, it appears as if its center of mass has moved to the left. However, its center of mass cannot move because the box is an isolated system. Einstein resolved this perplexing situation by assuming that in addition to energy and momentum, light energy also has kinetic mass. If m_k is the kinetic mass carried by the pulse of light, and the center of mass of the box is to remain fixed, then

$$m_k L = M\,\Delta x$$

Solving for m_k, and using the previous expression for Δx, we get

$$m_k = \frac{M\,\Delta x}{L} = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2}$$

or

$$E = m_k c^2$$

Thus, Einstein reached the profound conclusion, "If a body gives off the energy E in the form of radiation, its mass diminishes by E/c^2 . . . The mass of a body is a measure of its energy content."

Although we derived the relationship $E = mc^2$ for light energy, the equivalence of mass and energy is universal. Equation 9.22, $E = \gamma mc^2$, which represents the total energy of any particle, suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy through its mass. The clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions in which both the conversion of mass into energy and the conversion of energy into mass takes place. Because of this, we can no longer accept the separate classical laws of conservation of mass and conservation of energy; we must instead speak of a unified law of **conservation of mass–energy.** Simply put, this law requires that **the sum of the mass–energy of a system of particles before interaction must equal the sum of the mass–energy of the system after interaction, where the mass–energy of the i^{th} particle is defined as**

• *Conservation of mass–energy*

$$E_i = \frac{m_i c^2}{\sqrt{1 - \frac{u_i^2}{c^2}}} \qquad [9.26]$$

The release of enormous energy, accompanied by the change in masses of particles after they have lost their excess energy as they are brought to rest, is the basis of atomic and hydrogen bombs. In a conventional nuclear reactor, the uranium nucleus undergoes fission, a reaction that results in several lighter fragments having considerable kinetic energy. In the case of ^{235}U (the parent nucleus), which undergoes fission, the fragments are two lighter nuclei and two neutrons. The total mass of the fragments is less than that of the parent nucleus by an amount Δm. The

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corresponding energy Δmc^2 associated with this mass difference is exactly equal to the total kinetic energy of the fragments. This kinetic energy is then used to produce hot water and steam for the generation of electrical power.

Next, consider the basic fusion reaction in which two deuterium atoms combine to form one helium atom. This reaction is of major importance in current research and development of controlled-fusion reactors. The decrease in mass that results from the creation of one helium atom from two deuterium atoms is $\Delta m = 4.25 \times 10^{-29}$ kg. Hence, the corresponding energy that results from one fusion reaction is $\Delta mc^2 = 3.83 \times 10^{-12}$ J = 23.9 MeV. To appreciate the magnitude of this result, if 1 g of deuterium is converted to helium, the energy released is about 10^{12} J! At the 1997 cost of electrical energy, this would be worth about $65 000.

Example 9.9 Binding Energy of the Deuteron

The mass of the deuteron, which is the nucleus of "heavy hydrogen," is not equal to the sum of the masses of its constituents, which are the proton and neutron. Calculate this mass difference and determine its energy equivalence.

Solution Using atomic mass units (u), we have

$$m_p = \text{mass of proton} = 1.007\,276 \text{ u}$$

$$m_n = \text{mass of neutron} = 1.008\,665 \text{ u}$$

$$m_p + m_n = 2.015\,941 \text{ u}$$

Because the mass of the deuteron is 2.013 553 u (Appendix A), we see that the mass difference Δm is 0.002 388 u. By definition, 1 u = 1.66×10^{-27} kg, and therefore

$$\Delta m = 0.002\,388 \text{ u} = 3.96 \times 10^{-30} \text{ kg}$$

Using $E = \Delta mc^2$, we find that

$$E = \Delta mc^2 = (3.96 \times 10^{-30} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2$$

$$= 3.56 \times 10^{-13} \text{ J} = 2.23 \text{ MeV}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

OPTIONAL

9.9 • GENERAL RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that resists acceleration. To designate these two attributes, we use the subscripts g and i and write

$$\text{Gravitational property} \qquad F_g = m_g g$$

$$\text{Inertial property} \qquad F_i = m_i a$$

The value for the gravitational constant G was chosen to make the magnitudes of m_g and m_i numerically equal. Regardless of how G is chosen, however, the strict proportionality of m_g and m_i has been established experimentally to an extremely high degree: a few parts in 10^{12}. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses and the resistance of a single mass to being accelerated, regardless of what kind of force is producing the acceleration. This question, which puzzled Newton and many other physicists over the years, was answered when Einstein published his theory of gravitation, known as *general relativity*, in 1916. Because it is a mathematically complex theory, we merely offer a hint of its elegance and insight.

In Einstein's view, the remarkable coincidence that m_g and m_i seemed to be exactly proportional was evidence for a very intimate and basic connection between the two concepts. He pointed out that no mechanical experiment (such as dropping a mass) could distinguish between the two situations illustrated in Figures 9.17a and 9.17b. In each case, a mass released by the observer undergoes a downward acceleration of g relative to the floor.

Einstein carried this idea further and proposed that *no* experiment, mechanical or otherwise, could distinguish between the two cases. This extension to include all phenomena (not just mechanical ones) has interesting consequences. For example, suppose that a light pulse is sent horizontally across the box, as in Figure 9.17c. The trajectory of the light pulse bends downward as the box accelerates upward to meet it. Therefore, Einstein proposed that a beam of light should also be bent downward by a gravitational field. (No such bending is predicted in Newton's theory of gravitation.)

The two postulates of Einstein's **general relativity** are as follows:

- All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
- In the vicinity of any given point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the *principle of equivalence.*)

The second postulate implies that gravitational mass and inertial mass are completely equivalent, not just proportional. What were thought to be two different types of mass are actually identical.

One interesting effect predicted by general relativity is that time scales are altered by gravity. A clock in the presence of gravity runs more slowly than one in which gravity is negligible. As a consequence, the frequencies of radiation emitted by atoms in the presence of a strong gravitational field are shifted to lower fre-

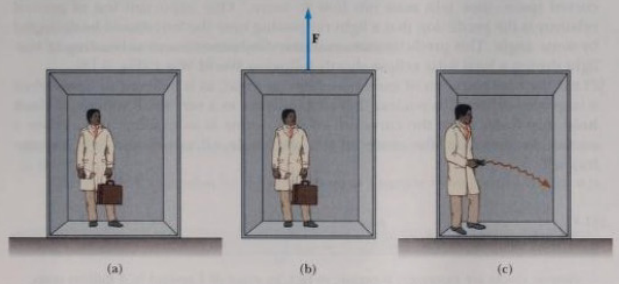


Figure 9.17 (a) The observer is at rest in a uniform gravitational field **g**. (b) The observer is in a region where gravity is negligible, but the frame of reference is accelerated by an external force **F** that produces an acceleration **g**. According to Einstein, the frames of reference in parts (a) and (b) are equivalent in every way. No local experiment could distinguish any difference between the two frames. (c) If parts (a) and (b) are truly equivalent, as Einstein proposed, then a ray of light would bend in a gravitational field.

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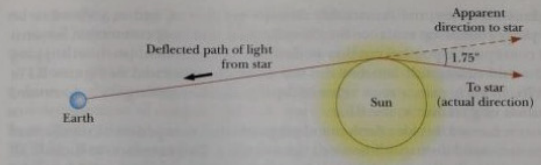


Figure 9.18 Deflection of starlight passing near the Sun. Because of this effect, the Sun and other remote objects can act as a *gravitational lens*. In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75".

quencies when compared with the same emissions in a weak field. This gravitational shift has been detected in spectral lines emitted by atoms in massive stars. It has also been verified on the Earth by comparing the frequencies of gamma rays (a high energy form of electromagnetic radiation) emitted from nuclei separated vertically by about 20 m.

The second postulate suggests that a gravitational field may be "transformed away" at any point if we choose an appropriate accelerated frame of reference—a freely falling one. Einstein developed an ingenious method of describing the acceleration necessary to make the gravitational field "disappear." He specified a certain quantity, the *curvature of space–time*, that describes the gravitational effect at every point. In fact, the curvature of space–time completely replaces Newton's gravitational theory. According to Einstein, there is no such thing as a gravitational force. Rather, the presence of a mass causes a curvature of space–time in the vicinity of the mass, and this curvature dictates the space–time path that all freely moving objects must follow. As one physicist said, "Mass one tells space–time how to curve; curved space–time tells mass two how to move." One important test of general relativity is the prediction that a light ray passing near the Sun should be deflected by some angle. This prediction was confirmed by astronomers as bending of starlight during a total solar eclipse shortly following World War I (Fig. 9.18).

If the concentration of mass becomes very great, as is believed to occur when a large star exhausts its nuclear fuel and collapses to a very small volume, a **black hole** may form. Here the curvature of space–time is so extreme that, within a certain distance from the center of the black hole, all matter and light become trapped.

Thinking Physics 4

Atomic clocks are extremely accurate; in fact, an error of 1 second in 3 million years is typical. This error can be described as about 1 part in 10^{14}. However, the atomic clock in Boulder, Colorado, is often 15 ns faster than the one in Washington after only 1 day. This is an error of about 1 part in 6×10^{12}, which is about 17 times larger than the previously expressed error. If atomic clocks are so accurate, why does a clock in Boulder not remain in synchronism with one in Washington? (*Hint:* Denver, near Boulder, is known as the mile-high city.)

Reasoning According to the general theory of relativity, the rate of passage of time depends on gravity—time runs more slowly in strong gravitational fields. Washington is at an elevation very close to sea level, whereas Boulder is about 1 mile higher in altitude. This will result in a weaker gravitational field at Boulder than at Washington. As a result, time runs more rapidly in Boulder than in Washington.

SUMMARY

The two basic postulates of special relativity are

- All the laws of physics are the same in all inertial reference frames.
- The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Three consequences of special relativity are

- Events that are simultaneous for one observer are not simultaneous for another observer who is in motion relative to the first.
- Clocks in motion relative to an observer appear to be slowed down by a factor γ. This is known as **time dilation.**
- Lengths of objects in motion appear to be contracted in the direction of motion.

To satisfy the postulates of special relativity, the Galilean transformations must be replaced by the **Lorentz transformations:**

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned} \qquad [9.7]$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

The relativistic form of the **velocity transformation** is

$$u_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad [9.13]$$

where u_x is the speed of an object as measured in the S frame and u'_x is its speed measured in the S' frame.

The relativistic expression for the **momentum** of a particle moving with a velocity $\mathbf{u}$ is

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \qquad [9.16]$$

The relativistic expression for the **kinetic energy** of a particle is

$$K = \gamma mc^2 - mc^2 \qquad [9.21]$$

where mc^2 is called the **rest energy** of the particle.

The total energy E of a particle is related to the **mass** through the **energy–mass** equivalence expression:

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$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.23]$$

The relativistic momentum is related to the total energy through the equation

$$E^2 = p^2c^2 + (mc^2)^2 \qquad [9.24]$$

CONCEPTUAL QUESTIONS

1. What two speed measurements do two observers in relative motion *always* agree on?
2. The speed of light in water is 2.3×10^8 m/s. Suppose an electron is moving through water at 2.5×10^8 m/s. Does this violate the principles of relativity?
3. Two identical clocks are synchronized. One is put in orbit directed eastward around the Earth and the other remains on Earth. Which clock runs slower? When the moving clock returns to Earth, are the two still synchronized?
4. A train is approaching you as you stand next to the tracks. Just as an observer on the train passes, you both begin to play the same Beethoven symphony on portable compact disk players. (a) According to you, whose player finishes the symphony first? (b) According to the observer on the train, whose player finishes the symphony first? (c) Whose player *really* finishes the symphony first?
5. A spaceship in the shape of a sphere moves past an observer on Earth with a speed $0.5c$. What shape does the observer see as the spaceship moves past?
6. Explain why it is necessary, when defining length, to specify that the positions of the ends of a rod are to be measured simultaneously.
7. When we say that a moving clock runs slower than a stationary one, does this imply that there is something physically unusual about the moving clock?
8. Give a physical argument that shows that it is impossible to accelerate an object of mass m to the speed of light, even with a continuous force acting on it.
9. List some ways our day-to-day lives would change if the speed of light were only 50 m/s.
10. It is said that Einstein, in his teenage years, asked the question, "What would I see in a mirror if I carried it in my hands and ran at the speed of light?" How would you answer this question?
11. Some of the distant star-like objects, called quasars, are receding from us at half the speed of light (or greater). What is the speed of the light we receive from these quasars?
12. How is it possible that photons of light, which have zero rest mass, have momentum?
13. Relativistic quantities must make a smooth transition to their Newtonian counterparts as the speed of a system becomes small compared to the speed of light. Explain.
14. If the speed of a particle is doubled, what effect does it have on its momentum? (Assume that its initial speed is less than $c/2$.)

PROBLEMS

Section 9.1 The Principle of Newtonian Relativity

1. A 2000-kg car moving at 20.0 m/s collides with and sticks to a 1500-kg car at rest at a stop sign. Show that momentum is conserved in a reference frame moving at 10.0 m/s in the direction of the moving car.
2. A ball is thrown at 20.0 m/s inside a boxcar moving along the tracks at 40.0 m/s. What is the speed of the ball relative to the ground if the ball is thrown (a) forward? (b) backward? (c) out the side door?
3. In a laboratory frame of reference, an observer notes that Newton's second law is valid. Show that it is also valid for an observer moving at a constant speed, small compared to the speed of light, relative to the laboratory frame.
4. Show that Newton's second law is *not* valid in a reference frame moving past the laboratory frame of Problem 3 with a constant acceleration.

Section 9.4 Consequences of Special Relativity

5. How fast must a meter stick be moving if its length is observed to shrink to 0.500 m?
6. At what speed does a clock have to move in order to run at a rate that is one half the rate of a clock at rest?
7. An astronaut is traveling in a space vehicle that has a speed of $0.500c$ relative to the Earth. The astronaut measures his pulse rate at 75.0 per minute. Signals generated by the astronaut's pulse are radioed to Earth when the vehicle is moving perpendicularly to a line that connects the vehicle with an Earth observer. What pulse rate does the Earth observer measure? What would be the pulse rate if the speed of the space vehicle were increased to $0.990c$?
8. The proper length of one spaceship is three times that of another. The two spaceships are traveling in the same direction and, while both are passing overhead, an Earth observer measures the two spaceships to have the same length. If the slower spaceship is moving with a speed of $0.350c$, determine the speed of the faster spaceship.
9. An atomic clock moves at 1000 km/h for 1 hour as measured by an identical clock on Earth. How many nanoseconds slow will the moving clock be at the end of the one-hour interval?
10. If astronauts could travel at $v = 0.950c$, we on Earth would say it takes $(4.20/0.950) = 4.42$ years to reach Alpha Centauri, 4.20 lightyears away. The astronauts disagree. (a) How much time passes on the astronauts' clocks? (b) What distance to Alpha Centauri do the astronauts measure?
11. A spaceship of proper length 300 m takes 0.750 μs to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
12. A spaceship of proper length L_p takes time t to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
13. A muon formed high in the Earth's atmosphere travels at speed $v = 0.990c$ for a distance of 4.60 km before it decays into an electron, a neutrino, and an antineutrino ($\mu^- \rightarrow e^- + \nu + \bar{\nu}$). (a) How long does the muon live, as measured in its reference frame? (b) How far does the muon travel, as measured in its frame?

Section 9.5 The Lorentz Transformation Equations

14. For what value of v does $\gamma = 1.01$? Observe that for speeds lower than this value, time dilation and length contraction will be less-than-one-percent effects.
15. A spaceship travels at $0.750c$ relative to Earth. If the spaceship fires a small rocket in the forward direction, how fast (relative to the ship) must it be fired for it to travel at $0.950c$ relative to Earth?
16. A certain quasar recedes from the Earth at $v = 0.870c$. A jet of material ejected from the quasar toward the Earth moves at $0.550c$ relative to the quasar. Find the speed of the ejected material relative to the Earth.
17. Two jets of material from the center of a radio galaxy fly away in opposite directions. Both jets move at $0.750c$ relative to the galaxy. Determine the speed of one jet relative to the other.
18. A friend travels by you at a high speed in a spaceship. He tells you that his ship is 20.0 m long and that the identically constructed ship you are sitting in is 19.0 m long. According to your observations, (a) how long is your ship, (b) how long is your friend's ship, and (c) what is the speed of your friend's ship?
19. Observer A measures the lengths of two rods, one stationary, the other moving with a speed of $0.955c$. She finds that the rods have the same length. A second observer, B, travels along with the moving rod. What is the ratio of the length of A's rod to the length of B's rod, according to observer B?
20. A moving rod is 2.00 m long, and its length is oriented at an angle of 30.0° with respect to the direction of motion. The rod has a speed of $0.995c$. (a) What is the proper length of the rod? (b) What is the orientation angle in the proper frame?

Section 9.6 Relativistic Momentum and the Relativistic Form of Newton's Laws

21. Calculate the momentum of an electron moving with a speed of (a) $0.0100c$, (b) $0.500c$, (c) $0.900c$.
22. The *nonrelativistic* expression for the momentum of a particle, $p = mv$, can be used if $v \ll c$. For what speed does the use of this formula give an error in the momentum of (a) 1.00% and (b) 10.0%?
23. An electron has a momentum that is 90.0% greater than its classical momentum. (a) Find the speed of the electron. (b) How would your result change if the particle were a proton?
24. A golf ball travels with a speed of 90.0 m/s. By what fraction does its relativistic momentum, p, differ from its classical value, mv? That is, find the ratio $(p - mv)/mv$.
25. An unstable particle at rest breaks into two fragments of *unequal* mass. The rest mass of the lighter fragment is 2.50×10^{-28} kg, and that of the heavier fragment is 1.67×10^{-27} kg. If the lighter fragment has a speed of $0.893c$ after the breakup, what is the speed of the heavier fragment?
26. Show that the speed of an object having momentum p and mass m is

$$v = \frac{c}{\sqrt{1 + (mc/p)^2}}$$

Section 9.7 Relativistic Energy

27. A proton moves at $0.950c$. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy (in eV).
28. Show that, for any object moving at less than one tenth the speed of light, the relativistic kinetic energy agrees with the result of the classical equation $K = \frac{1}{2}mv^2$ to within less than 1%. Thus for most purposes, the classical equation is good enough to describe these objects, whose motion we call *nonrelativistic*.
29. Determine the energy required to accelerate an electron from (a) $0.500c$ to $0.900c$ and (b) $0.900c$ to $0.990c$.
30. A spaceship of mass 1.00×10^6 kg is to be accelerated to $0.600c$. (a) How much energy does this require? (b) How

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many kilograms of matter would it take to provide this much energy?

31. Make an order-of-magnitude estimate of the ratio of mass increase to the rest mass of a flag as you run it up a flagpole. In your solution explain what quantities you take as data and the values you estimate or measure for them.
32. When 1.00 g of hydrogen combines with 8.00 g of oxygen, 9.00 g of water is formed. During this chemical reaction, 2.86×10^5 J of energy is released. How much mass do the constituents of this reaction lose? Is the loss of mass likely to be detectable?
33. A cube of steel has a volume of 1.00 cm^3 and a mass of 8.00 g when at rest on the Earth. If this cube is now given a speed $v = 0.900c$, what is its density as measured by a stationary observer? Note that relativistic density is $m/V = E/c^2V$.
34. An electron has a kinetic energy five times greater than its rest energy. Find (a) its total energy and (b) its speed.
35. Show that the energy–momentum relationship $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.
36. An unstable particle with a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of $0.987c$ and $-0.868c$. Find the rest masses of the fragments. (*Hint:* Conserve both mass–energy and momentum.)

Additional Problems

37. The net nuclear reaction inside the Sun is $4p \rightarrow He^4 + \Delta E$. If the rest mass of each proton is 938.2 MeV and the rest mass of the He^4 nucleus is 3727 MeV, calculate the percentage of the starting mass that is released as energy.
38. An electron has a speed of $0.750c$. Find the speed of a proton that has (a) the same kinetic energy as the electron, (b) the same momentum as the electron.
39. The cosmic rays of highest energy are protons, having kinetic energy on the order of 10^{13} MeV. (a) How long would it take a proton of this energy to travel across the Milky Way galaxy, of diameter 10^5 lightyears, as measured in the proton's frame? (b) From the point of view of the proton, how many kilometers across is the galaxy?
40. A spaceship moves away from the Earth at $0.500c$ and fires a shuttle craft in the forward direction at $0.500c$ relative to the ship. The pilot of the shuttle craft launches a probe at forward speed $0.500c$ relative to the shuttle craft. Determine (a) the speed of the shuttle craft relative to the Earth and (b) the speed of the probe relative to the Earth.
41. The average lifetime of a pi meson in its own frame of reference is 2.60×10^{-8} s. If the meson moves with a speed of $0.950c$, what are (a) its mean lifetime as measured by an observer on Earth and (b) the average distance it travels before decaying, as measured by an observer on Earth?
42. An astronaut wishes to visit the Andromeda galaxy (suppose it is 2.00 million lightyears away), making a one-way trip that will take 30.0 years in the spaceship's frame of reference. Assuming that her speed is constant, how fast must she travel relative to the Earth?
43. A physics professor on Earth gives an exam to her students who are on a rocket ship traveling at speed v relative to Earth. The moment the ship passes the professor, she signals the start of the exam. If she wishes her students to have time T_0 (rocket time) to complete the exam, show that she should wait a time (Earth time) of

$$T = T_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

before sending a light signal telling them to stop. (*Hint:* Remember that it takes some time for the second light signal to travel from the professor to the students.)
44. Spaceship I, which contains students taking a physics exam, approaches Earth with a speed of $0.600c$ (relative to Earth), and spaceship II, which contains professors proctoring the exam, moves at $0.280c$ (relative to Earth) directly toward the students. If the professors stop the exam after 50.0 min have passed on their clock, how long does the exam last as measured by (a) the students? (b) an observer on Earth?
45. A supertrain (rest length = 100 m) travels at a speed of $0.950c$ as it passes through a tunnel (rest length = 50.0 m). As seen by a trackside observer, is the train ever completely within the tunnel? If so, with how much space to spare?
46. Energy reaches the upper atmosphere of the Earth from the Sun at the rate of 1.79×10^{17} W. If all of this energy were absorbed by the Earth and not re-emitted, how much would the mass of the Earth increase in 1 year?
47. A charged particle moves along a straight line in a uniform electric field $\mathbf{E}$ with a speed of v. The electric force on a charge q in an electric field is $q\mathbf{E}$. If the motion and the electric field are both in the x direction, (a) show that the acceleration of the charge q in the x direction is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, how would you proceed to find the speed of the particle and its position after a time t has elapsed?
48. Imagine that the entire Sun collapses to a sphere of radius R_g such that the work required to remove a small mass m from the surface would be equal to its rest energy mc^2. This radius is called the *gravitational radius* for the Sun. Find R_g. (It is believed that the ultimate fate of many stars is to collapse to their gravitational radii or smaller.)

Spreadsheet Problems

S1. Astronomers use the Doppler shift in the Balmer series of the hydrogen spectrum to determine the radial speed of a galaxy. The fractional change in the wavelength of the spectral line is given by

$$Z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1$$

Once this quantity is measured for a particular receding galaxy, the speed of recession can be found by solving for v/c in terms of Z. Spreadsheet 9.1 calculates the speed Z values. (a) What is the speed for galaxies having $Z =$ 0.2, 0.5, 1.0, and 2.0? (b) The largest Z values, $Z \approx 3.8$, have been measured for several quasars (quasi-stellar radio sources). How fast are these quasars moving away from us?

S2. Astronauts in a starship traveling at a speed v relative to a starbase are given instructions from mission control to call back in 1 h as measured by the starship clocks. Spreadsheet 9.2 calculates how long Mission Control has to wait for the call for different starship speeds. How long does Mission Control have to wait if the ship is traveling at $v = 0.1c$, $0.4c$, $0.6c$, $0.8c$, $0.9c$, $0.995c$, $0.9995c$?

S3. Most astronomers believe that the Universe began at some instant with an explosion called the Big Bang and that the observed recession of the galaxies is a direct result of this explosion. If the galaxies recede from each other at a constant rate, then we expect that the galaxies moving fastest are now farthest away from Earth. This result, called Hubble's law after Edwin Hubble, can be written $v = Hr$, where H can be determined from observation, v is the speed of recession of the galaxy, and r is its distance from Earth. Current estimates of H range from 15 to 30 km/s/Mly, where Mly is the distance light travels in one million years, and a conservative estimate is $H = 20$ km/s/Mly. Use Spreadsheet 9.1 to calculate the distance from Earth for each galaxy in Problem S1.

S4. Design a spreadsheet program to calculate and plot the relativistic kinetic energy (Eq. 9.20) and the classic kinetic energy ($\frac{1}{2}mu^2$) of a macroscopic object. Plot the relativistic and classical energies versus speed on the same graph. (a) For an object of mass $m = 3$ kg, at what speed does the classical kinetic energy underestimate the relativistic value by 1%? 5%? 50%? What is the relativistic kinetic energy at these speeds? Repeat part (a) for (b) an electron and (c) a proton.

ANSWERS TO CONCEPTUAL PROBLEMS

1. This scenario is not possible with light. Water waves move through a medium—the surface of the water—and therefore have an absolute velocity relative to the reference frame in which this medium is at rest. Thus, the relative velocity of the waves and your boat is determined by Galilean relativity. This makes it possible for you to move faster than the waves. Light waves, which do not require a medium, are described by the principles of special relativity. As you detect the light wave ahead of you and moving away from you (which would be a pretty good trick—think about it!), its velocity relative to you is c. No matter how fast you accelerate in that direction, trying to catch up to it, its velocity relative to you is still c. Thus, you will not be able to catch up to the light wave.
2. The answers to both of these questions is *no*. Both your clothing and your sleeping cabin are at rest in your reference frame, thus, they will have their proper length. There will be no change in measured lengths of objects within your spacecraft. Another observer, on a spacecraft traveling at a high speed relative to yours, will measure you as thinner (if your body is oriented in a direction perpendicular to your velocity vector relative to the other observer) or will claim that you are able to fit into a shorter sleeping cabin (if your body is oriented in a direction parallel to your velocity vector relative to the other observer).
3. Assuming that their on-duty time was kept on Earth, they will be pleasantly surprised with a large paycheck. Less time will have passed for the astronauts in their frame of reference than for their employer back on Earth.
4. The incoming rocket will not appear to have a longer length and a faster clock. Length contraction and time dilation depend only on the magnitude of the relative velocity, not on the direction.
5. A reflected photon does exert a force on a surface. Although a photon has zero mass, a photon does carry momentum. When it reflects from a surface, there is a change in the momentum, just like the change in momentum of a ball bouncing off the floor. According to the momentum interpretation of Newton's second law, a change in momentum results in a force on the surface. This concept is used in theoretical studies of *space sailing*. These studies propose building nonpowered spacecraft with huge reflective sails oriented perpendicularly to the rays from the Sun. The large number of photons from the Sun reflecting from the surface of the sail will exert a force which, although small, will provide a continuous acceleration. This would allow the spacecraft to travel to other planets without fuel.

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10

Rotational Motion

CHAPTER OUTLINE

10.1 Angular Velocity and Angular Acceleration
10.2 Rotational Kinematics
10.3 Relations Between Angular and Linear Quantities
10.4 Rotational Kinetic Energy
10.5 Torque and the Vector Product
10.6 Equilibrium of a Rigid Object
10.7 Relation Between Torque and Angular Acceleration
10.8 Angular Momentum
10.9 Conservation of Angular Momentum
10.10 Quantization of Angular Momentum (Optional)
10.11 Rotation of Rigid Bodies (Optional)

In this chapter we investigate the dynamics of particles and systems of particles moving along a circular path and rigid objects rotating about a fixed axis. We shall encounter such terms as angular displacement, angular velocity, angular acceleration, torque, and angular momentum and show how these quantities are useful in describing rotational motion. Next we shall define a vector product, which is a convenient mathematical tool for expressing such physical quantities as torque and angular momentum.

One of the central points of this chapter is to develop the concept of the angular momentum of a system of particles. By analogy with the conservation of linear momentum (Chapter 8), we shall find that the angular momentum of any isolated system is always constant. The results we derive here will enable us to understand the rotational motions of a diverse range of objects in our environment, from an electron orbiting a nucleus to clusters of galaxies orbiting a common center. Although the focus of this chapter is the circular motion of particles, the extension of this treatment to rigid bodies is reasonably straightforward and is treated in an optional section.



Derek Swinson, professor of physics at the University of New Mexico, demonstrating the "gyro-ski" technique. The skier initiates a turn by lifting the axle of the rotating bicycle wheel. The direction of the turn depends on whether the left or right hand is used to lift the axle from the horizontal. Ignoring friction and gravity, the angular momentum of the system (the skier and the bicycle wheel) remains constant. *(Courtesy of Derek Swinson)*

10.1 • ANGULAR VELOCITY AND ANGULAR ACCELERATION

We began our study of linear motion by defining the terms *displacement*, *velocity*, and *linear acceleration*. We will take the same basic approach now as we turn to the study of rotational motion. Let us begin by considering a circular disk rotating around a fixed axis that is perpendicular to the disk and goes through the point O (Fig. 10.1). A point P on the disk is at a fixed distance r from the origin and rotates about O in a circle of radius r. In fact, **every point on the disk undergoes circular motion about *O*.**

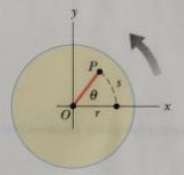


Figure 10.1 Rotation of a disk about a fixed axis through O perpendicular to the plane of the figure. (In other words, the axis of rotation is the z axis.) Note that a particle at P rotates in a circle of radius r centered at O.

It is convenient to represent the position of the point P with its polar coordinates: (r, θ). The origin of the polar coordinates is chosen to coincide with the center of the circle. In this representation, the only coordinate that changes in time is the angle θ; r remains constant. As a point on the disk moves along the circle of radius r from the positive x axis ($\theta = 0$) to the point P, it moves through an arc of length s, which is related to the angular position θ through the relation $s = r\theta$, or

$$\theta = \frac{s}{r} \qquad [10.1]$$

It is important to note the units of θ as expressed by Equation 10.1. The angle θ is the ratio of an arc length and the radius of the circle, and hence is a pure number. However, we commonly refer to the unit of θ as a **radian** (rad). One radian is the angle subtended by an arc length equal to the radius of the arc. Because the circumference of a circle is $2\pi r$, it follows that 360° corresponds to an angle of $2\pi r/r$ rad or 2π rad (one revolution). Hence, 1 rad = $360°/2\pi \approx 57.3°$. To convert an angle in degrees to an angle in radians, we can use the fact that 2π radians = 360°, or π radians = 180°; hence,

$$\theta\,(\text{rad}) = \frac{2\pi}{360°}\,\theta\,(\text{deg}) = \frac{\pi}{180°}\,\theta\,(\text{deg})$$

For example, 60° equals $\pi/3$ rad, and 45° equals $\pi/4$ rad.

In Figure 10.2, as the particle travels from P to Q in a time Δt, the radius vector sweeps out an angle of $\Delta\theta = \theta_2 - \theta_1$, which equals the **angular displacement** during the time interval Δt. We define the **average angular speed,** $\bar{\omega}$ (omega), as the ratio of this angular displacement to the time interval Δt:

$$\bar{\omega} \equiv \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \qquad [10.2]$$

• *Average angular speed*

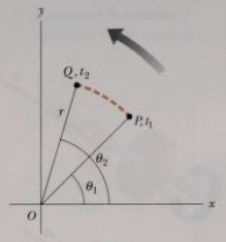


Figure 10.2 A particle on a rotating rigid object moves from P to Q along the arc of a circle. In the time interval $\Delta t = t_2 - t_1$, the radius vector sweeps out an angle $\Delta\theta = \theta_2 - \theta_1$.

By analogy with linear speed, the **instantaneous angular speed,** ω, is defined as the limit of the ratio in Equation 10.2 as Δt approaches zero:

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad [10.3]$$

• *Instantaneous angular speed*

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Angular speed has units of rad/s (or s^{-1} because radians are not dimensional). Let us adopt the convention that the fixed axis of rotation for the disk is the z axis, as in Figure 10.1. We shall take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).

If the instantaneous angular speed of a particle changes from ω_1 to ω_2 in the time interval Δt, the particle has an angular acceleration. The **average angular acceleration,** $\bar{\alpha}$ (alpha), of a rotating particle is defined as the ratio of the change in the angular speed to the time interval, Δt:

Average angular acceleration

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad [10.4]$$

By analogy with linear acceleration, the **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

Instantaneous angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad [10.5]$$

Angular acceleration has units of rad/s^2 or s^{-2}.

Let us now generalize our argument from the circular disk to any rigid body. A rigid body is any object the elements of which remain fixed with respect to one another. **For rotation about a fixed axis, every particle on a rigid body, such as a circular disk, has the same angular velocity and the same angular acceleration.** That is, the vector quantities $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ characterize the rotational motion of the *entire* rigid body. Using these quantities, we can greatly simplify the analysis of rigid-body rotation.

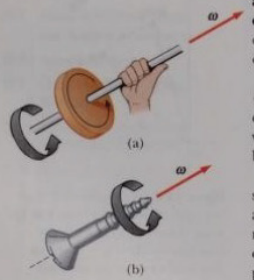


Figure 10.3 (a) The right-hand rule for determining the direction of the angular velocity vector. (b) The direction of $\boldsymbol{\omega}$ is in the direction of advance of a right-handed screw.

The angular displacement ($\boldsymbol{\theta}$), angular velocity ($\boldsymbol{\omega}$), and angular acceleration ($\boldsymbol{\alpha}$) are analogous to linear displacement ($\mathbf{x}$), linear velocity ($\mathbf{v}$), and linear acceleration ($\mathbf{a}$), respectively, for the corresponding motion discussed in Chapter 2. The variables θ, ω, and α differ dimensionally from the linear, variables, x, v, and a, only by a length factor.

We have indicated how the signs for $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are determined, but we have not specified any direction in space associated with these vector quantities.[1] For rotation about a fixed axis, the only direction in space that uniquely specifies the rotational motion is the direction along the axis of rotation. However, we must also decide on the sense of these quantities—that is, whether they point into or out of the plane of Figure 10.1.

The direction of $\boldsymbol{\omega}$ is along the axis of rotation, which is the z axis in Figure 10.1. By convention, we take the direction of $\boldsymbol{\omega}$ to be *out of* the plane of the diagram when the rotation is counterclockwise and *into* the plane of the diagram when the rotation is clockwise. To further illustrate this convention, it is convenient to use the **right-hand rule** illustrated by Figure 10.3a. The four fingers of the right hand are wrapped in the direction of the rotation. The extended right thumb points in the direction of $\boldsymbol{\omega}$. Figure 10.3b illustrates that $\boldsymbol{\omega}$ is also in the direction of advance of a similarly rotating right-handed screw.

The sense of $\boldsymbol{\alpha}$ follows from its definition as $d\boldsymbol{\omega}/dt$. It is the same as $\boldsymbol{\omega}$ if the angular speed (the magnitude of $\boldsymbol{\omega}$) is increasing in time and antiparallel to $\boldsymbol{\omega}$ if the angular speed is decreasing in time.

[1] Although we do not verify it here, the instantaneous angular velocity and instantaneous angular acceleration are vector quantities, but the corresponding average values are not. This is because angular displacement is not a vector quantity for finite rotations.

10.2 • ROTATIONAL KINEMATICS

In our study of linear motion, we found that the simplest accelerated motion to analyze is motion under constant linear acceleration (Chapter 2). Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration. Therefore, we shall next develop kinematic relations for rotational motion under constant angular acceleration.

If we write Equation 10.5 in the form $d\omega = \alpha\, dt$ and let $\omega = \omega_0$ at $t_0 = 0$, we can integrate this expression directly:

$$\omega = \omega_0 + \alpha t \qquad (\alpha = \text{constant}) \qquad [10.6]$$

Likewise, substituting Equation 10.6 into Equation 10.3 and integrating once more (with $\theta = \theta_0$ at $t_0 = 0$), we get

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad [10.7]$$

Rotational kinematic equations (α = constant)

If we eliminate t from Equations 10.6 and 10.7, we get

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad [10.8]$$

If we eliminate α, we obtain

$$\theta = \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \qquad [10.9]$$

Notice that these kinematic expressions for rotational motion under constant angular acceleration are of the *same form* as those for linear motion under constant linear acceleration, with the substitutions $x \to \theta$, $v \to \omega$, and $a \to \alpha$ (Table 10.1). Furthermore, the expressions are valid for both rigid-body rotation about a *fixed* axis and particle motion about a *fixed* axis.

TABLE 10.1 A Comparison of Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis with α = Constant	Linear Motion with a = Constant
Variables: θ and ω	Variables: x and v
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t$	$x = x_0 + \frac{1}{2}(v_0 + v)t$
$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	$v^2 = v_0^2 + 2a(x - x_0)$

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Example 10.1 **Rotating Wheel**

A wheel rotates with a constant angular acceleration of 3.50 rad/s². If the angular speed of the wheel is 2.00 rad/s at $t_0 = 0$, (a) what angle does the wheel rotate through in 2.00 s?

Solution

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 = \left(2.00\ \frac{\text{rad}}{\text{s}}\right)(2.00\ \text{s}) + \tfrac{1}{2}\left(3.50\ \frac{\text{rad}}{\text{s}^2}\right)(2.00\ \text{s})^2$$

$$= 11.0\ \text{rad} = 630^\circ = 1.75\ \text{rev}$$

(b) What is the angular speed at $t = 2.00$ s?

Solution

$$\omega = \omega_0 + \alpha t = 2.00\ \text{rad/s} + \left(3.50\ \frac{\text{rad}}{\text{s}^2}\right)(2.00\ \text{s})$$

$$= 9.00\ \text{rad/s}$$

We could also obtain this result using Equation 10.8 and the results of part (a). Try it!

EXERCISE 1 Find the angle that the wheel rotates through between $t = 2.00$ s and $t = 3.00$ s. Answer 10.8 rad

EXERCISE 2 A wheel starts from rest and rotates with constant angular acceleration to an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.
Answer (a) 4.00 rad/s² (b) 18.0 rad

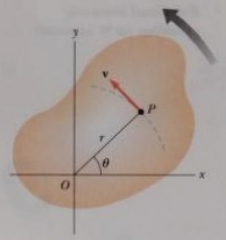


Figure 10.4 As a rigid object rotates about the fixed axis through O, the point P has a linear velocity $\mathbf{v}$ that is always tangent to the circular path of radius r.

10.3 · RELATIONS BETWEEN ANGULAR AND LINEAR QUANTITIES

In this section we shall derive some useful relations between the angular speed and angular acceleration of a rotating particle and its linear speed and linear acceleration. Keep in mind that, when a rigid body rotates about a fixed axis, *every* particle of the body moves in a circle whose center is the axis of rotation.

Consider a particle rotating in a circle of radius r about the z axis, as in Figure 10.4. Because the particle moves along a circular path, its linear velocity vector $\mathbf{v}$ is always tangent to the path; hence, we often call this quantity *tangential velocity*. The magnitude of the tangential velocity of the particle is, by definition, ds/dt, where s is the distance traveled by the particle along the circular path. Recalling from Equation 10.1 that $s = r\theta$, and noting that r is a constant, we get

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Relationship between linear and angular speed

$$v = r\omega \qquad [10.10]$$

That is, the magnitude of the tangential velocity of the particle equals the distance of the particle from the axis of rotation multiplied by the particle's angular speed.

We can relate the angular acceleration of the particle to its tangential acceleration, a_t—which is the component of its acceleration tangent to the path of motion—by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

Relationship between tangential and angular acceleration

$$a_t = r\alpha \qquad [10.11]$$

That is, the tangential component of the linear acceleration of a particle undergoing circular motion equals the distance of the particle from the axis of rotation multiplied by the angular acceleration.

In Chapter 3 we found that a particle rotating in a circular path undergoes a centripetal, or radial, acceleration of magnitude v^2/r directed toward the center of rotation (Fig. 10.5). Because $v = r\omega$, we can express the centripetal acceleration of the particle as

$$a_r = \frac{v^2}{r} = r\omega^2 \qquad [10.12]$$

The *total linear acceleration* of the particle is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$. Therefore, the magnitude of the total linear acceleration of the particle is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \qquad [10.13]$$

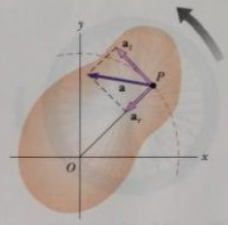


Figure 10.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration, $\mathbf{a}_t$, and a centripetal component of linear acceleration, $\mathbf{a}_r$. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

Thinking Physics 1

A phonograph record (LP—for *long-playing*) is rotated at a constant *angular* speed. A compact disk (CD) is rotated so that the surface sweeps past the laser at a constant *tangential* speed. Consider two circular "grooves" of information on an LP—one near the outer edge and one near the inner edge. Suppose the outer groove "contains" 1.8 seconds of music. Does the inner groove also contain 1.8 seconds of music? How about the CD—do the inner and outer grooves contain the same time interval of music?

Reasoning On the LP, the inner and outer grooves must both rotate once in the same time period. Thus, each groove, regardless of where it is on the record, contains the same time interval of information. Of course, on the inner grooves, this same information must be compressed into a smaller circumference. On a CD, the constant tangential speed requires that there be no such compression—the digital pits representing the information are spaced uniformly everywhere on the surface. Thus, there is more information in an outer groove, due to its larger circumference, and, as a result, a longer time interval of music than the inner groove.

Thinking Physics 2

The launch area for the European Space Agency is not in Europe—it is in French Guiana, near the Equator in South America. Why?

Reasoning Placing a satellite in Earth orbit requires providing a large tangential speed to the satellite. This is the task of the rocket propulsion system. Anything that reduces the requirements on the rocket system is a welcome contribution. The surface of the Earth is already traveling toward the east at a high speed, due to the rotation of the Earth. Thus, if rockets are launched toward the east, the rotation of the Earth provides some initial tangential speed, reducing somewhat the requirements on the rocket. If rockets were launched from Europe, which is at a relatively large angle latitude, the contribution of the Earth's rotation is relatively small, because the distance between Europe and the rotation axis of the Earth is relatively small. The ideal place for launch-

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ing is at the Equator, which is as far as one can be from the rotation axis of the Earth and still be on the surface of the Earth. This results in the largest possible tangential speed due to the Earth's rotation. The European Space Agency exploits this advantage by launching from French Guiana, which is only a few degrees north of the Equator.

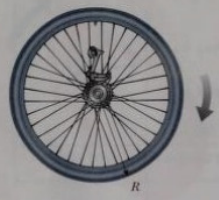


Figure 10.6 (Conceptual Problem 2) A wheel of radius R.

CONCEPTUAL PROBLEM 1

When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground. Why does this happen?

CONCEPTUAL PROBLEM 2

When a wheel of radius R rotates about a fixed axis as in Figure 10.6, do all points on the wheel have the same angular speed? Do they all have the same linear speed? If the angular speed is constant and equal to ω, describe the linear speeds and linear accelerations of the points located at $r = 0$, $r = R/2$, and $r = R$, where the points are measured from the center of the wheel.

EXERCISE 3 A racing car travels on a circular track of radius 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration. Answer (a) 0.180 rad/s (b) 8.10 m/s^2 toward the center of the track

EXERCISE 4 An automobile accelerates from zero to 30 m/s in 6.0 s. The wheels have diameters of 0.40 m. What is the angular acceleration of each wheel?
Answer 25 rad/s^2

10.4 • ROTATIONAL KINETIC ENERGY

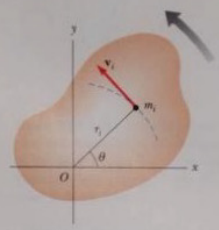


Figure 10.7 A rigid object rotating about the z axis with angular speed ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_iv_i^2$. The total energy of the object is $\frac{1}{2}I\omega^2$.

Let us consider a rigid body as a collection of particles, and let us assume that the body rotates about the fixed z axis with an angular speed of ω (Fig. 10.7). Each particle of the body has some kinetic energy, determined by its mass and speed. If the mass of the ith particle is m_i and its speed is v_i, the kinetic energy of this particle is

$$K_i = \tfrac{1}{2}m_iv_i^2$$

To proceed further, we must recall that, although every particle in the rigid body has the same angular speed ω, the individual linear speeds depend on the distance r_i from the axis of rotation, according to the expression $v_i = r_i\omega$ (Eq. 10.10). The *total* kinetic energy of the rotating rigid body is the sum of the kinetic energies of the individual particles:

$$K_R = \sum K_i = \sum \tfrac{1}{2}m_iv_i^2 = \tfrac{1}{2}\sum m_ir_i^2\omega^2$$

$$K_R = \tfrac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

where we have factored ω^2 from the sum because it is common to every particle. The quantity in parentheses is called the **moment of inertia,** I:

Moment of inertia for a system of particles •

$$I = \sum m_ir_i^2 \qquad [10.14]$$

Therefore, we can express the kinetic energy of the rotating rigid body as

$$K_R = \tfrac{1}{2}I\omega^2 \qquad [10.15]$$

• *Kinetic energy of a rotating rigid body*

From the definition of moment of inertia, we see that it has dimensions of ML^2 (kg·m^2 in SI units). The moment of inertia is a measure of a system's resistance to change in its angular velocity. It plays the role of mass in *all* rotational equations. However, the moment of inertia depends on the location and orientation of the axis, and is not a single property of an object the way its mass is. Although we shall commonly refer to the quantity $\frac{1}{2}I\omega^2$ as the **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy, because it was derived from a sum over individual kinetic energies of the particles contained in the rigid body. However, the form of the kinetic energy given by Equation 10.15 is a convenient one for dealing with rotational motion, provided we know how to calculate I. It is important to recognize the analogy between kinetic energy associated with linear motion, $\frac{1}{2}mv^2$, and rotational kinetic energy, $\frac{1}{2}I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively.

Thinking Physics 3

The temperature of a gas is related to the average *translational* kinetic energy of the molecules of the gas. If we add energy to a gas, the average translational kinetic energy increases and, therefore, the temperature increases. Consider adding the same amount of energy to a monatomic gas, such as helium, and a diatomic gas, such as oxygen (O_2). The temperature of the monatomic gas will rise by more than that of the diatomic gas. Why?

Reasoning In a monatomic gas, the only type of kinetic energy possible is translational kinetic energy. Thus, all of the energy added to the gas appears as translational kinetic energy, with a corresponding large rise in temperature. In the diatomic gas, some of the energy added to the gas can appear as rotational kinetic energy of the molecules. Thus, only part of the energy added appears as translational kinetic energy, and the temperature rises by a lesser amount than for the monatomic gas.

Example 10.2 The Oxygen Molecule

Consider the diatomic molecule oxygen, O_2, which is rotating in the xy plane about the z axis passing through its center, perpendicular to its length. The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature, the average separation between the two oxygen atoms is $d = 1.21 \times 10^{-10}$ m (the atoms are treated as point masses). (a) Calculate the moment of inertia of the molecule about the z axis.

Solution Because the distance of each atom from the z axis is $d/2$, the moment of inertia about the z axis is

$$I = \sum m_ir_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2}$$

$$= \left(\frac{2.66 \times 10^{-26}}{2}\text{ kg}\right)(1.21 \times 10^{-10}\text{ m})^2$$

$$= 1.95 \times 10^{-46}\text{ kg·m}^2$$

(b) If the angular speed of the molecule about the z axis is 4.60×10^{12} rad/s, what is its rotational kinetic energy?

Solution

$$K_R = \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}(1.95 \times 10^{-46}\text{ kg·m}^2)\left(4.60 \times 10^{12}\,\frac{\text{rad}}{\text{s}}\right)^2$$

$$= 2.06 \times 10^{-22}\text{ J}$$

This is approximately equal to the average kinetic energy associated with the linear motion of the molecule at room temperature, which is about 6.2×10^{-21} J.

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Example 10.3 Four Rotating Masses

Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane (Fig. 10.8). (a) If the rotation of the system occurs about the y axis with an angular speed ω, find the moment of inertia about the y axis and the rotational kinetic energy about this axis.

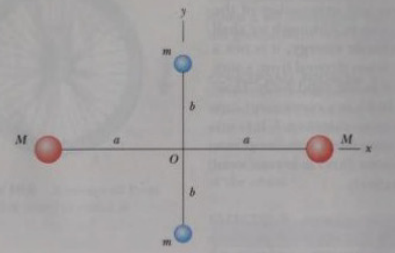


Figure 10.8 (Example 10.3) All four masses are at a fixed separation as shown. The moment of inertia depends on the axis about which it is evaluated.

Solution First, note that the two masses m that lie on the y axis do not contribute to I_y (that is, $r_i = 0$ for these masses about this axis). Applying Equation 10.14, we get

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

Therefore, the rotational kinetic energy about the y axis is

$$K_R = \tfrac{1}{2} I_y \omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2$$

The fact that the masses m do not enter into this result makes sense, because they have no motion about the chosen axis of rotation; hence, they have no kinetic energy.

(b) Suppose the system rotates in the xy plane about an axis through O (the z axis). Calculate the moment of inertia about the z axis and the rotational energy about this axis.

Solution Because r_i in Equation 10.14 is the *perpendicular* distance to the axis of rotation, we get

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2mb^2$$

$$K_R = \tfrac{1}{2} I_z \omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = (Ma^2 + mb^2)\omega^2$$

Comparing the results for (a) and (b), we conclude that the moment of inertia and, therefore, the rotational energy associated with a given angular speed depend on the axis of rotation. In (b), we expect the result to include all masses and distances because all four masses are in motion for rotation in the xy plane. Furthermore, the fact that the rotational energy in (a) is smaller than in (b) indicates that it would take less effort (work) to set the system into rotation about the y axis than about the z axis.

10.5 · TORQUE AND THE VECTOR PRODUCT

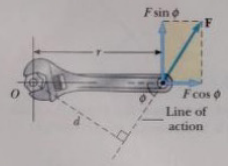


Figure 10.9 The force **F** has a greater rotating tendency about O as F increases and as the moment arm, d, increases. It is the component $F \sin \phi$ that tends to rotate the system about O.

When a force is exerted on a rigid body pivoted about some axis, the body tends to rotate about that axis. The tendency of a force to rotate a body about some axis is measured by a vector quantity called **torque** ($\boldsymbol{\tau}$). Consider the wrench pivoted about the axis through O in Figure 10.9. The applied force **F** generally can act at an angle of ϕ to the horizontal. We define the magnitude of the torque, τ (Greek letter tau), resulting from the force **F** with the expression

$$\tau \equiv rF \sin \phi = Fd \qquad [10.16]$$

It is very important to recognize that **torque is defined only when a reference axis is specified.** The quantity $d = r \sin \phi$, called the **moment arm** (or *lever arm*) of the force **F**, represents the perpendicular distance from the rotation axis to the line of action of **F**. Note that the only component of **F** that tends to cause a rotation is $F \sin \phi$, the component perpendicular to r. The horizontal component, $F \cos \phi$, passes through O and has no tendency to produce a rotation.

If two or more forces are acting on a rigid body, as in Figure 10.10, then each has a tendency to produce a rotation about the pivot at O. For example, $\mathbf{F}_2$ tends to rotate the body clockwise, and $\mathbf{F}_1$ tends to rotate the body counterclockwise. We shall use the convention that the sign of the torque resulting from a force is positive

if its turning tendency is counterclockwise and negative if its turning tendency is clockwise. For example, in Figure 10.10, the torque resulting from $\mathbf{F}_1$, which has a moment arm of d_1, is *positive* and equal to $+F_1 d_1$; the torque from $\mathbf{F}_2$ is *negative* and equal to $-F_2 d_2$. Hence, the *net* torque acting on the rigid body about O is

$$\tau_{net} = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

From the definition of torque, we see that the rotating tendency increases as F increases and as d increases. For example, it is easier to close a door if we push at the doorknob rather than at a point close to the hinge. **Torque should not be confused with force.** Torque has units of force times length—N · m in SI units.

Now consider a force **F** acting on a particle located at the vector position **r** (Fig. 10.11). The origin O is assumed to be in an inertial frame, so Newton's second law is valid. The *magnitude* of the torque due to this force relative to the origin is, by definition, equal to $rF \sin \phi$, where ϕ is the angle between **r** and **F**. The axis about which **F** would tend to produce rotation is perpendicular to the plane formed by **r** and **F**. If the force lies in the xy plane, as in Figure 10.11, then the torque is represented by a vector parallel to the z axis. The force in Figure 10.11 creates a torque that tends to rotate the body counterclockwise looking down the z axis, and so the sense of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is in the positive z direction. If we reversed the direction of **F** in Figure 10.11, $\boldsymbol{\tau}$ would then be in the negative z direction. The torque involves two vectors, **r** and **F**, and is in fact defined to be equal to the **vector product,** or **cross product,** of **r** and **F**:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad [10.17]$$

We now give a formal definition of the vector product. Given any two vectors **A** and **B**, the vector product $\mathbf{A} \times \mathbf{B}$ is defined as a third vector, **C**, the *magnitude* of which is $AB \sin \theta$, where θ is the angle included between **A** and **B**:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad [10.18]$$

$$C \equiv |\mathbf{C}| = |AB \sin \theta| \qquad [10.19]$$

Note that the quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 10.12. The *direction* of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the advance of a right-handed screw when the screw is turned from **A** to **B** through the angle θ. A more convenient rule to use for the direction of $\mathbf{A} \times \mathbf{B}$ is the right-hand rule illustrated in Figure 10.12. The four fingers of the right hand are pointed along **A** and then

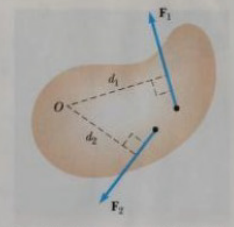


Figure 10.10 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate the object clockwise.

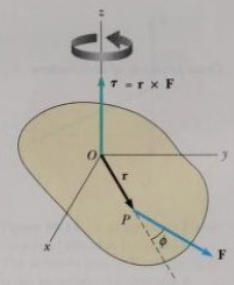


Figure 10.11 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force **F**.

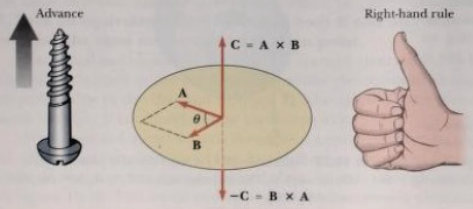


Figure 10.12 The vector product $\mathbf{A} \times \mathbf{B}$ is a third vector **C** having a magnitude $AB \sin \theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the right-hand rule.

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The torque exerted on the nail by the hammer increases as the length of the handle (lever arm) increases. (© Richard Megna, 1991, Fundamental Photographs)

"wrapped" into **B** through the angle θ. The direction of the erect right thumb is the direction of $\mathbf{A} \times \mathbf{B}$. Because of the notation, $\mathbf{A} \times \mathbf{B}$ is often read "**A** cross **B**"; hence the term *cross product.*

Some properties of the vector product follow from its definition:

- Unlike the case of the scalar product, the order in which the two vectors are multiplied in a cross product is important; that is

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A}) \qquad [10.20]$$

  Therefore, if you change the order of the cross product, you must change the sign. One could easily verify this relation with the right-hand rule (Fig. 10.12).
- If **A** is parallel to **B** ($\theta = 0°$ or $180°$), then $\mathbf{A} \times \mathbf{B} = 0$; therefore, it follows that $\mathbf{A} \times \mathbf{A} = 0$.
- If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.

From Equations 10.18 and 10.19 and the definition of unit vectors, one finds that the cross products of the rectangular unit vectors **i**, **j**, and **k** obey the following expressions:

Cross products of unit vectors •

$$\begin{aligned} &\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\ &\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\ &\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\ &\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \end{aligned} \qquad [10.21]$$

Signs are interchangeable. For example, $\mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$.

CONCEPTUAL PROBLEM 3

Both torque and work are products of force and distance. How are they different? Do they have the same units?

Example 10.4 The Net Torque on a Cylinder

A one-piece cylinder is shaped as in Figure 10.13, with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. A rope wrapped around the drum, of radius R_1, exerts a force $\mathbf{F}_1$ to the right on the cylinder. A rope wrapped around the core, of radius R_2, exerts a force $\mathbf{F}_2$ downward on the cylinder. (a) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in Fig. 10.13)?

Solution The torque due to $\mathbf{F}_1$ is $-R_1F_1$ and is negative because it tends to produce a clockwise rotation. The torque due to $\mathbf{F}_2$ is $+R_2F_2$ and is positive because it tends to produce a counterclockwise rotation. Therefore, the net torque about the rotation axis is

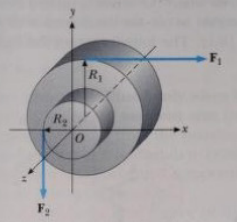


Figure 10.13 (Example 10.4) A solid cylinder pivoted about the z axis through O. The moment arm of $\mathbf{F}_1$ is R_1, and the moment arm of $\mathbf{F}_2$ is R_2.

$$\tau_{net} = \tau_1 + \tau_2 = R_2F_2 - R_1F_1$$

(b) Suppose $F_1 = 5.0$ N, $R_1 = 1.0$ m, $F_2 = 6.0$ N, and $R_2 = 0.50$ m. What is the net torque about the rotation axis and which way does the cylinder rotate?

Solution

$$\tau_{net} = (6.0\text{ N})(0.50\text{ m}) - (5.0\text{ N})(1.0\text{ m}) = -2.0\text{ N}\cdot\text{m}$$

Because the net torque is negative, the cylinder rotates clockwise.

Example 10.5 The Cross Product

Two vectors lying in the xy plane are given by the equations $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{A} \times \mathbf{B}$, and verify explicitly that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 10.21 for the cross product of unit vectors gives

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (2\mathbf{i} + 3\mathbf{j}) \times (-\mathbf{i} + 2\mathbf{j}) \\ &= 2\mathbf{i} \times 2\mathbf{j} + 3\mathbf{j} \times (-\mathbf{i}) = 4\mathbf{k} + 3\mathbf{k} = 7\mathbf{k} \end{aligned}$$

(We have omitted the terms with $\mathbf{i} \times \mathbf{i}$ and $\mathbf{j} \times \mathbf{j}$, because they are zero.)

$$\begin{aligned} \mathbf{B} \times \mathbf{A} &= (-\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j}) \\ &= -\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times 2\mathbf{i} = -3\mathbf{k} - 4\mathbf{k} = -7\mathbf{k} \end{aligned}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

EXERCISE 5 Use the results to this example and Equation 10.19 to find the angle between **A** and **B**. Answer 60.3°

EXERCISE 6 A particle is located at the vector position $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})$ m, and the force acting on it is $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0,6) m? Answer (a) $(-7\mathbf{k})$ N·m (b) $(11\mathbf{k})$ N·m

EXERCISE 7 If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$, what is the angle between **A** and **B**? Answer 45.0°

10.6 • EQUILIBRIUM OF A RIGID OBJECT

Consider a single force **F** acting on a rigid object as in Figure 10.14. The effect of the force depends on its point of application, P. If **r** is the position vector of this point relative to O, the torque associated with the force **F** about O is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

As you can see from Figure 10.14, the tendency of **F** to make the object rotate about an axis through O depends on the moment arm d as well as on the magnitude of **F**. By definition, the magnitude of $\boldsymbol{\tau}$ is Fd.

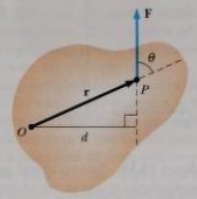


Figure 10.14 A single force **F** acts on a rigid object at the point P. The moment arm of **F** relative to O is the perpendicular distance d from O to the line of action of **F**.

Now suppose either of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, act on a rigid object. The two forces will have the same effect on the object only if they have the same magnitude, the same direction, and the same line of action. In other words, **two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have an equivalent effect on a rigid body if and only if $F_1 = F_2$ and if the two have the same torque about any given point**.

Two equal and opposite forces that are *not* equivalent are shown in Figure 10.15. The force directed to the right tends to rotate the object clockwise about an axis perpendicular to the diagram through O, whereas the force directed to the left tends to rotate it counterclockwise about that axis.

When pivoted about an axis through its center of mass, an object undergoes an angular acceleration about this axis if there is a nonzero torque acting. As an example, suppose an object is pivoted about an axis through its center of mass as in Figure 10.16. Two equal and opposite forces act in the directions shown, such

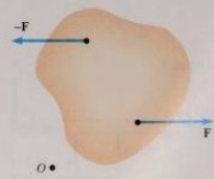


Figure 10.15 The two forces acting on the object are equal in magnitude and opposite in direction, yet the object is not in equilibrium.

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that their lines of action do not pass through the center of mass. A pair of forces acting in this manner form what is called a **couple.** (The two forces shown in Figure 10.15 also form a couple.) Because each force produces the same torque, Fd, the net torque has a magnitude $2Fd$. Clearly, the object rotates clockwise and undergoes an angular acceleration about the axis. This is a nonequilibrium situation as far as the rotational motion is concerned.

In general, an object is in rotational equilibrium only if its angular acceleration $\alpha = 0$. Because torque causes angular acceleration, **a necessary condition of equilibrium is that the net torque about any origin must be zero.** We now have **two necessary conditions for equilibrium of an object,** which can be stated as follows:

Two conditions for equilibrium of an object

The resultant external force must equal zero. $\sum \mathbf{F} = 0$ **[10.22]**

The resultant external torque must be zero about *any* origin. $\sum \tau = 0$ **[10.23]**

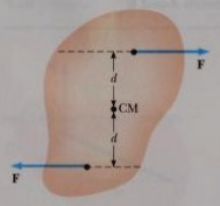


Figure 10.16 Two equal and opposite forces acting on the object form a couple. In this case, the object rotates clockwise. The magnitude of the net torque about the center of mass is $2Fd$.

The first condition is a statement of translational equilibrium; it tells us that the linear acceleration of the center of mass of the object must be zero when viewed from an inertial reference frame. The second condition is a statement of rotational equilibrium and tells us that the angular acceleration about any axis must be zero. In the special case of **static equilibrium,** the object is at rest so that it has no linear or angular speed (that is, $v_{CM} = 0$ and $\omega = 0$). Note that if the resultant external force is zero, and the resultant external torque is zero about some origin, then the resultant external torque is zero about every other origin.

The two vector expressions given by Equations 10.22 and 10.23 are equivalent, in general, to six scalar equations, three from the first condition of equilibrium and three from the second (corresponding to x, y, and z components). Hence, in a complex system involving several forces acting in various directions, you would be faced with solving a set of equations with many unknowns. Here, we restrict our discussion to situations in which all the forces lie in the xy plane. (When a set of forces lie in the same plane, they are said to be *coplanar.*) With this restriction, we shall have to deal with only three scalar equations. Two of these come from balancing the forces in the x and y directions. The third comes from the torque equation—namely, that the net torque about *any* point in the xy plane must be zero. Hence, the two conditions of equilibrium provide the equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau_z = 0 \qquad \textbf{[10.24]}$$

where the axis of the torque equation is arbitrary, as we show later. In working static equilibrium problems, it is important to recognize external forces acting on the object. Failure to do so will result in an incorrect analysis. The following procedure is recommended when analyzing an object in equilibrium under the action of several external forces:

PROBLEM-SOLVING STRATEGY • Objects in Equilibrium

1. Make a sketch of the object under consideration.
2. Draw a free-body diagram and label all external forces acting on the object. Try to guess the correct direction for each force. If you select a direction that leads to a negative sign in your solution for a force, do not be alarmed; this merely means that the direction of the force is the opposite of what you guessed.
3. Resolve all forces into rectangular components, choosing a convenient coordinate system. Then apply the first condition for equilibrium. Remember to keep track of the signs of the various force components.
4. Choose a convenient axis for calculating the net torque on the object. Remember that the choice of the origin for the torque equation is arbitrary; therefore, choose an origin that will simplify your calculation as much as possible. Usually the most convenient origin for calculating torques is the one at which most of the forces act.
5. The first and second conditions of equilibrium give a set of linear equations with several unknowns. All that is left is to solve the simultaneous equations for the unknowns in terms of the known quantities.

Example 10.6 Standing on a Horizontal Beam

A uniform horizontal beam of length 8.00 m and weight 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the horizontal (Fig. 10.17a). If a 600-N person stands 2.00 m from the wall, find the tension in the cable and the force exerted by the wall on the beam.

Solution First we must identify all the external forces acting on the beam. These are the gravitational force, the force **T** exerted by the cable, the force **R** exerted by the wall at the pivot (the direction of this force is unknown), and the force exerted by the person on the beam. These are all indicated in the free-body diagram for the beam (Fig. 10.17b). If we resolve **T** and **R** into horizontal and vertical components and apply the first condition for equilibrium, we get

(1) $\sum F_x = R\cos\theta - T\cos 53.0° = 0$

(2) $\sum F_y = R\sin\theta + T\sin 53.0° - 600\ \text{N} - 200\ \text{N} = 0$

Because R, T, and θ are all unknown, we cannot obtain a solution from these expressions alone. (The number of simultaneous equations must equal the number of unknowns in order for us to be able to solve for the unknowns.)

Now let us invoke the condition for rotational equilibrium. A convenient axis to choose for our torque equation is the one that passes through the pivot at O. The feature that makes this point so convenient is that the force **R** and the horizontal component of **T** both have a lever arm of zero, and hence zero torque, about this pivot. Recalling our convention for the sign of the torque about an axis and noting that the lever

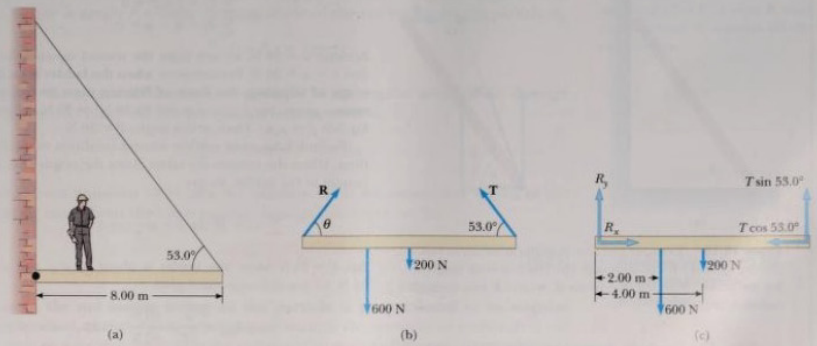


Figure 10.17 (Example 10.6) A uniform beam supported by a cable. (b) and (c) The free-body diagram for the beam.

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arms of the 600-N, 200-N, and T sin 53° forces are 2.00 m, 4.00 m, and 8.00 m, respectively, we get

$$\Sigma\tau_O = (T\sin 53.0°)(8.00\text{ m}) - (600\text{ N})(2.00\text{ m}) - (200\text{ N})(4.00\text{ m}) = 0$$

$$T = 313\text{ N}$$

Thus the torque equation with this axis gives us one of the unknowns directly! This value is substituted into (1) and (2) to give

$$R\cos\theta = 188\text{ N}$$

$$R\sin\theta = 550\text{ N}$$

We divide these two equations and recall the trigonometric identity $\sin\theta/\cos\theta = \tan\theta$ to get

$$\tan\theta = \frac{550\text{ N}}{188\text{ N}} = 2.93$$

$$\theta = 71.1°$$

Finally,

$$R = \frac{188\text{ N}}{\cos\theta} = \frac{188\text{ N}}{\cos 71.1°} = 581\text{ N}$$

If we had selected some other axis for the torque equation, the solution would have been the same. For example, if we had chosen to have the axis pass through the center of gravity of the beam, the torque equation would involve both T and R. However, this equation, coupled with (1) and (2), could still be solved for the unknowns. Try it!

When many forces are involved in a problem of this nature, it is convenient to set up a table of forces, lever arms, and torques. For instance, in the example just given, we would construct the following table. Setting the sum of the terms in the last column equal to zero represents the condition of rotational equilibrium.

Force Component	Lever Arm Relative to O (m)	Torque About O (N·m)
T sin 53.0°	8.00	(8.00) T sin 53°
T cos 53.0°	0	0
200 N	4.00	−(4.00)(200)
600 N	2.00	−(2.00)(600)
R sin θ	0	0
R cos θ	0	0

Example 10.7 The Leaning Ladder

A uniform ladder of length ℓ and weight $w = 50$ N rests against a smooth, vertical wall (Fig. 10.18a). If the coefficient of static friction between ladder and ground is $\mu_s = 0.40$, find the minimum angle θ_{min} such that the ladder does not slip.

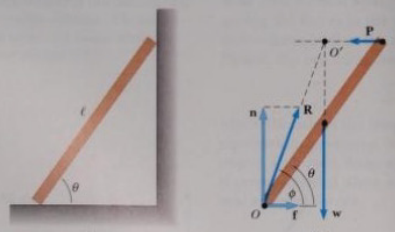


Figure 10.18 (Example 10.7) (a) A uniform ladder at rest, leaning against a frictionless wall. (b) The free-body diagram for the ladder. Note that the forces **R**, **w**, and **P** pass through a common point O'.

Solution The free-body diagram showing all the external forces acting on the ladder is illustrated in Figure 10.18b. The reaction **R** exerted by the ground on the ladder is the vector sum of a normal force, **n**, and the force of friction, **f**. The reaction force **P** exerted by the wall on the ladder is horizontal, because the wall is smooth. From the first condition of equilibrium applied to the ladder, we have

$$\sum F_x = f - P = 0$$

$$\sum F_y = n - w = 0$$

Because $w = 50$ N, we see from the second equation here that $n = w = 50$ N. Furthermore, **when the ladder is on the verge of slipping, the force of friction must be a maximum,** given by $f_{max} = \mu_s n = 0.40(50\text{ N}) = 20$ N. (Recall Eq. 5.9: $f_s \leq \mu_s n$.) Thus, at this angle, $P = 20$ N.

To find θ, we must use the second condition of equilibrium. When the torques are taken about the origin O at the bottom of the ladder, we get

$$\sum\tau_O = P\ell\sin\theta - w\frac{\ell}{2}\cos\theta = 0$$

But $P = 20$ N when the ladder is about to slip and $w = 50$ N, so that this expression gives

$$\tan\theta_{min} = \frac{w}{2P} = \frac{50\text{ N}}{40\text{ N}} = 1.25$$

$$\theta_{min} = 51°$$

It is interesting to note that the result does not depend on ℓ or w. The answer depends only on μ_s.

An alternative approach to analyzing this problem is to consider the intersection O' of the lines of action of forces **w** and **P**. Because the torque about any origin must be zero, the torque about O' must be zero. This requires that the line of action of **R** (the resultant of **n** and **f**) pass through O'. That is, because three forces act on this stationary object, the forces must be concurrent. With this condition, one could then obtain the angle ϕ that **R** makes with the horizontal (where ϕ is greater than θ), assuming the length of the ladder is known.

EXERCISE 8 For the angles labeled in Figure 10.18b, show that $\tan\phi = 2\tan\theta$.

EXERCISE 9 Two children weighing 500 N and 350 N are on a uniform board weighing 40.0 N supported at its center (a see-saw). If the 500-N child is 1.50 m from the center, determine (a) the upward force exerted on the board by the support and (b) where the 350-N child must sit to balance the system. Answer (a) 890 N (b) 2.14 m from the center

10.7 • RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION

In this section we shall show that the angular acceleration of a particle rotating about a fixed axis is proportional to the net torque acting about that axis. The ideas embodied in this situation can easily be extended to the case of a rigid body rotating about a fixed axis.

Consider a particle of mass m rotating in a circle of radius r under the influence of both a tangential force, $\mathbf{F}_t$, and a radial force, $\mathbf{F}_r$, as in Figure 10.19. (The force $\mathbf{F}_r$, whose nature is not yet specified, *must* be present to keep the particle moving in its circular path.) According to Newton's second law, $\mathbf{F}_t = m\mathbf{a}_t$ and $\mathbf{F}_r = m\mathbf{a}_r$, where $\mathbf{a}_t$ is the tangential acceleration of the particle and $\mathbf{a}_r$ is its centripetal acceleration. The torque due to the force $\mathbf{F}_r$ about the origin is zero because $\mathbf{F}_r$ is antiparallel to $\mathbf{r}$; hence, $\mathbf{r} \times \mathbf{F}_r = 0$. The torque about the origin due to the tangential force is $\mathbf{r} \times \mathbf{F}_t$, but because $\mathbf{r}$ is perpendicular to $\mathbf{F}_t$, the magnitude of the torque is simply $F_t r$. Thus, the magnitude of the net torque on the particle is

$$\sum\tau = F_t r = (ma_t)r$$

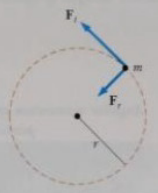


Figure 10.19 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ must also be present to maintain the circular motion.

Because the tangential acceleration is related to the angular acceleration through Equation 10.11, $a_t = r\alpha$, the torque can be expressed as

$$\sum\tau = (mr\alpha)r = (mr^2)\alpha$$

Recall from Equation 10.14 that the quantity mr^2 is the moment of inertia of the rotating mass about the z axis passing through the origin, so that

$$\sum\tau = I\alpha \qquad [10.25]$$

• *Relationship between net torque and angular acceleration*

That is, **the net torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. It is important to note that $\Sigma\tau = I\alpha$ is the rotational analogue of Newton's second law of motion, $\Sigma F = ma$.

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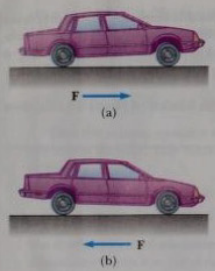


Figure TP10.4
(Thinking Physics 4)

Thinking Physics 4

When an automobile driver steps on the accelerator, the nose of the car moves upward. When the driver brakes, the nose moves downward. Why do these effects occur?

Reasoning When the driver steps on the accelerator, there is an increased force on the tires from the roadway. This force is parallel to the roadway and directed toward the front of the automobile, as suggested in Figure TP10.4a. This force provides a torque about the center of mass that tends to cause the car to rotate in the counterclockwise direction in the diagram. The result of this rotation is a "nosing up" of the car. When the driver steps on the brake, there is an increased force on the tires from the roadway, directed toward the rear of the automobile, as suggested in Figure TP10.4b. This force results in a torque that causes a clockwise rotation and the subsequent "nosing down" of the automobile.

10.8 · ANGULAR MOMENTUM

Again let us consider a particle of mass m, situated at the vector position $\mathbf{r}$ and moving with a momentum $\mathbf{p}$, as shown in Figure 10.20. The **instantaneous angular momentum, L,** of the particle relative to the origin O is defined by the cross product of its instantaneous vector position and the instantaneous linear momentum, $\mathbf{p}$:

Angular momentum of a particle

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [10.26]$$

The SI units of angular momentum are $\text{kg} \cdot \text{m}^2/\text{s}$. It is important to note that both the magnitude and the direction of $\mathbf{L}$ depend on the choice of origin. The direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$, and the sense of $\mathbf{L}$ is governed by the right-hand rule. For example, in Figure 10.20, $\mathbf{r}$ and $\mathbf{p}$ are assumed to be in the xy plane, and $\mathbf{L}$ points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of $\mathbf{L}$ is

$$L = mvr \sin \phi \qquad [10.27]$$

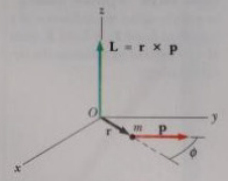


Figure 10.20 The angular momentum $\mathbf{L}$ of a particle of mass m and momentum $\mathbf{p}$ located at the position $\mathbf{r}$ is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of $\mathbf{L}$ depends on the origin and is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$. It follows that $\mathbf{L}$ is zero when $\mathbf{r}$ is parallel to $\mathbf{p}$ ($\phi = 0°$ or $180°$). In other words, when the particle moves along a line that passes through the origin, it has zero angular momentum with respect to the origin. This is equivalent to stating that it has no tendency to rotate about the origin. However, if $\mathbf{r}$ is perpendicular to $\mathbf{p}$ ($\phi = 90°$), L is a maximum and equal to mvr. In this case, the particle has maximum tendency to rotate about the origin. In fact, at that instant the particle moves exactly as though it were on the rim of a wheel rotating about the origin in a plane defined by $\mathbf{r}$ and $\mathbf{p}$. A particle has nonzero angular momentum about some point if the position vector of the particle measured from that point rotates about the point.

For linear motion, we found that the resultant force on a particle equals the time rate of change of the particle's linear momentum (Eq. 8.3). We shall now show that Newton's second law implies that the resultant torque acting on a particle equals the time rate of change of the particle's angular momentum. Let us start by writing the torque on the particle in the form

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad [10.28]$$

where we have used the fact that $\mathbf{F} = d\mathbf{p}/dt$. Now let us differentiate Equation 10.26 with respect to time, using the product rule.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

It is important to adhere to the order of terms, because $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

The last term on the right in the preceding equation is zero because $\mathbf{v} = d\mathbf{r}/dt$ is parallel to $\mathbf{p}$. Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad [10.29]$$

Comparing Equations 10.28 and 10.29, we see that

Torque equals time rate of change of angular momentum

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \qquad [10.30]$$

This result, $\boldsymbol{\tau} = d\mathbf{L}/dt$, is the rotational analog of Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. Equation 10.30 says that the **torque** acting on a particle is equal to the time rate of change of the particle's angular momentum. It is important to note that Equation 10.30 is valid only if the origins of $\boldsymbol{\tau}$ and $\mathbf{L}$ are the *same*. Equation 10.30 is also valid when several forces are acting on the particle, in which case $\boldsymbol{\tau}$ is the *net* torque on the particle. **Furthermore, the expression is valid for any origin fixed in an inertial frame.** Of course, the same origin must be used in calculating all torques as well as the angular momentum.

A System of Particles

The total angular momentum, $\mathbf{L}$, of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum \mathbf{L}_i$$

where the vector sum is over all of the n particles in the system.

Because the individual momenta of the particles may change in time, the total angular momentum may also vary in time. In fact, from Equations 10.28 and 10.29, we find that the time rate of change of the total angular momentum equals the vector sum of *all* torques, including those associated with internal forces between particles and those associated with external forces. However, the net torque associated with internal forces is zero. To understand this, recall that Newton's third law tells us that the internal forces occur in equal and opposite pairs that lie along the line of separation of each pair of particles. Therefore, the torque due to each action–reaction force pair is zero. By summation, we see that *the net internal torque vanishes*. Finally, we conclude that the total angular momentum can vary with time *only* if there is a net *external* torque on the system, so that we have

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt} \sum \mathbf{L}_i = \frac{d\mathbf{L}}{dt} \qquad [10.31]$$



This is an aerial view of Hurricane Fran. The spiral-shaped, nonrigid mass of air undergoes rotation and has angular momentum. *(Courtesy of NASA)*

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That is, the time rate of change of the total angular momentum of the system about some origin in an inertial frame equals the net external torque acting on the system about that origin. Note that Equation 10.31 is the rotational analog of $\mathbf{F}_{ext} = d\mathbf{p}/dt$ (Eq. 8.35) for a system of particles.

Thinking Physics 5

Can a particle moving in a straight line have nonzero angular momentum?

Reasoning A particle has angular momentum about an origin when moving in a straight line as long as its line of motion does not pass through the origin. Its angular momentum is zero only if the line of motion passes through the origin, in which case $\mathbf{r}$ is parallel to $\mathbf{v}$, and $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = 0$.

Example 10.8 Circular Motion

A particle moves in the xy plane in a circular path of radius r, as in Figure 10.21. (a) Find the magnitude and direction of its angular momentum relative to O when its velocity is $\mathbf{v}$.

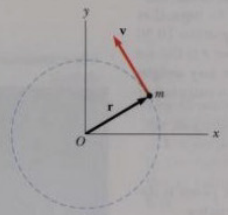


Figure 10.21 (Example 10.8) A particle moving in a circle of radius r has an angular momentum equal in magnitude to mvr relative to the center. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (not shown) points *out* of the diagram in this case.

Solution Because $\mathbf{r}$ is perpendicular to $\mathbf{v}$, $\phi = 90°$ and the magnitude of $\mathbf{L}$ is simply

$$L = mvr \sin 90° = mvr \qquad \text{(for } \mathbf{r} \text{ perpendicular to } \mathbf{v}\text{)}$$

The direction of $\mathbf{L}$ is perpendicular to the plane of the circle, and its sense depends on the direction of $\mathbf{v}$. If the sense of the rotation is counterclockwise, as in Figure 10.21, then by the right-hand rule, the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is *out* of the paper. Hence, we can write the vector expression for the angular momentum as $\mathbf{L} = (mvr)\mathbf{k}$. If the particle were to move clockwise, $\mathbf{L}$ would point into the paper.

(b) Find an alternative expression for L in terms of the angular velocity, ω.

Solution Because $v = r\omega$ for a particle rotating in a circle, we can express L as

$$L = mvr = m(r\omega)r = mr^2\omega = I\omega$$

where I is the moment of inertia of the particle about the z axis through O. In this case the angular momentum is in the *same* direction as the angular velocity vector, $\boldsymbol{\omega}$ (see Section 10.1), and so we can write $\mathbf{L} = I\boldsymbol{\omega} = I\omega\mathbf{k}$.

Example 10.9 Two Connected Masses

Two masses, m_1 and m_2, are connected by a light cord that passes over a pulley of radius R and moment of inertia I about its axle, as in Figure 10.22. The mass m_2 slides on a frictionless, horizontal surface. Determine the acceleration of the two masses using the concepts of angular momentum and torque.

Reasoning and Solution First, calculate the angular momentum of the system, which consists of the two masses plus the pulley. Then calculate the torque about an axis along the axle of the pulley through O. At the instant m_1 and m_2 have a speed v, the angular momentum of m_1 is m_1vR, and that of m_2 is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Therefore, the total angular momentum of the system is

$$(1) \quad L = m_1vR + m_2vR + I\frac{v}{R}$$

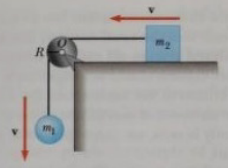


Figure 10.22 (Example 10.9)

Now let us evaluate the total external torque on the system about the axle. Because they have zero moment arms, the forces exerted by the axle on the pulley and by the weight of the pulley do not contribute to the torque. Furthermore, the normal force acting on m_2 is balanced by the force of gravity, $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The external force $m_1\mathbf{g}$ produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about O; that is, $\tau_{ext} = m_1gR$. Using this result, together with (1) and Equation 10.30 gives

$$\tau_{ext} = \frac{dL}{dt}$$

$$m_1gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right]$$

$$(2) \quad m_1gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to get

$$a = \frac{m_1g}{(m_1 + m_2) + I/R^2}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration. Only the external torques contribute to the change in angular momentum.

EXERCISE 10 A car of mass 1500 kg moves on a circular racetrack of radius 50 m with a speed of 40 m/s. What is the magnitude of its angular momentum relative to the center of the track? Answer 3.0×10^5 kg·m^2/s

10.9 · CONSERVATION OF ANGULAR MOMENTUM

In Chapter 8 we found that the total linear momentum of a system of particles remains constant when the net external force acting on the system is zero. In rotational motion, we have an analogous conservation law that states that **the total angular momentum of a system remains constant if the net external torque acting on the system is zero.**

Because the net torque acting on the system equals the time rate of change of the system's angular momentum, we see that if

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = 0 \qquad [10.32]$$

then

$$\mathbf{L} = \text{constant} \qquad [10.33]$$

For a system of particles, we can write this conservation law as $\Sigma\mathbf{L}_i = \text{constant}$, where $\mathbf{L}_i$ is the angular momentum of the ith particle, given by $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$. Hence, we can express the conservation of angular momentum for a system of particles as

$$\sum \mathbf{L}_i = \sum \mathbf{r}_i \times \mathbf{p}_i = \text{constant} \qquad [10.34]$$

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Figure 10.23 Nancy Kerrigan. In this spin, her angular momentum increases when she pulls her arms in close to her body, demonstrating that angular momentum is conserved. *(Reuters/Corbis-Bettmann)*

From Equation 10.34 we derive a third conservation law to add to our list of conserved quantities. We can now state that **the total energy, linear momentum, and angular momentum of an isolated system all remain constant.**

If the system of "particles" under consideration is a rigid body rotating about a fixed axis, then the angular momentum of the body about the axis has a magnitude given by $L = I\omega$, where I is the moment of inertia about the axis. In this case, if the net external torque on the body is zero, we can express the conservation of angular momentum as $I\omega$ = constant.

Although we do not prove it here, an important theorem concerns the angular momentum of a system of particles relative to the center of mass of the system. It can be stated as follows:

> The net torque acting on a system of particles about the center of mass equals the time rate of change of angular momentum, regardless of the motion of the center of mass.

This theorem applies even if the center of mass is accelerating, provided that both $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

Many examples can be used to demonstrate conservation of angular momentum; some of them should be familiar to you. You may have observed a figure skater



Figure 10.24 The Crab Nebula, in the constellation Taurus. This nebula is the remnant of a supernova explosion, which was seen on Earth in the year A.D. 1054. It is located some 6300 lightyears away and is approximately 6 lightyears in diameter, still expanding outward. *(David Malin, Anglo-Australian Observatory)*

spinning (Fig. 10.23). The angular speed of the skater increases as she pulls her hands and feet close to the trunk of her body. Neglecting friction between skater and ice, we see that there are no external torques on the skater. The moment of inertia of her body decreases as her hands and feet are brought in. The resulting change in angular speed is accounted for as follows. Because angular momentum must be conserved, the product $I\omega$ remains constant, and a decrease in I causes a corresponding increase in ω.

An interesting astrophysical example of conservation of angular momentum occurs when, at the end of its lifetime, a massive star uses up all its fuel and collapses under the influence of gravitational forces, causing a gigantic outburst of energy called a supernova explosion. The best-studied example of a remnant of a supernova explosion is the Crab Nebula, a chaotic, expanding mass of gas (Fig. 10.24). Part of the star's mass is released into space, where it eventually condenses into new stars and planets. Most of what is left behind makes up a **neutron star,** an extremely dense sphere of matter with a diameter of about 10 km in comparison with the 10^6-km diameter of the original star. As the rotational inertia of the system decreases, the star's rotational speed increases. More than 300 rapidly rotating neutron stars have been identified, with periods of rotation ranging from 1.6 ms to 4 s. The neutron star is a most dramatic system—an object with a mass greater than the Sun, rotating about its axis a few times each second!

EXERCISE 11 A particle of mass m moves in the xy plane with a velocity $\mathbf{v}$ along a straight line, as in Figure E10.11. What are the magnitude and direction of its angular momentum (a) with respect to the origin O and (b) with respect to the origin O'? Answer (a) mvd into the diagram (b) zero

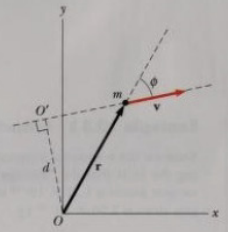


Figure E10.11 (Exercise 11)

Example 10.10 A Revolving Ball on a Horizontal, Frictionless Surface

A ball of mass m on a horizontal, frictionless table is connected to a string that passes through a small hole in the table. The ball is set into circular motion of radius R, at which time its speed is v_0 (Fig. 10.25). If the string is pulled from the bottom so that the radius of the circular path is decreased to r, what is the final speed v of the ball?

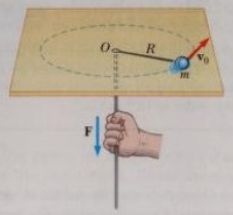


Figure 10.25 (Example 10.10)

Solution Let us take the torque about the center of rotation, O. Note that the force of gravity acting on the ball is balanced by the upward normal force, and so these forces cancel. The force, $\mathbf{F}$, of the string on the ball acts toward the center of rotation, and the vector position $\mathbf{r}$ is directed away from O. Thus, we see that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$. Because $\boldsymbol{\tau} = d\mathbf{L}/dt = 0$, $\mathbf{L}$ is a constant of the motion. That is, $mv_0R = mvr$, or

$$v = \frac{v_0 R}{r}$$

From this result, we see that as r decreases, the speed v increases.

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OPTIONAL

10.10 · QUANTIZATION OF ANGULAR MOMENTUM

We have seen that the concept of angular momentum is very useful for describing the motions of macroscopic systems. The concept is also valid and equally useful on a submicroscopic scale, and it has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it was found that the angular momentum of a system is a *fundamental* quantity. The word "fundamental" in this context implies that angular momentum is an inherent property of atoms, molecules, and their constituents.

In order to explain the results of a variety of experiments on atomic and molecular systems, we must assign discrete values to—in modern terminology, **quantize**—the angular momentum. These discrete values are multiples of a fundamental unit of angular momentum that equals $h/2\pi$, where h is called **Planck's constant.**

$$\text{Fundamental unit of angular momentum} = \frac{h}{2\pi} = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Example 10.11 **Estimating the Rotational Frequency of the Oxygen Molecule**

Estimate the rotational frequency of the oxygen molecule using the facts that the "average" separation between the two oxygen atoms is 1.21×10^{-10} m and that the mass of an oxygen atom is 2.66×10^{-26} kg.

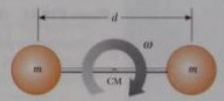


Figure 10.26 (Example 10.11) The rigid-rotor model of the diatomic molecule. The rotation occurs about the center of mass in the plane of the diagram.

Solution Consider the O_2 molecule as a rigid rotor—that is, two oxygen atoms separated by a fixed distance d and rotating about their center of mass (Fig. 10.26). To find the rotational frequency, first we must evaluate the angular momentum of the molecule about an axis through its center of mass, perpendicular to the plane of rotation.

$$L = L_1 + L_2 = mvr + mvr = 2mvr$$

In this example, $r = d/2 = 0.605 \times 10^{-10}$ m. Also, it is useful to use the relation $v = r\omega$ (Eq. 10.10), giving

$$\begin{aligned} L &= 2mvr = 2mr^2\omega \\ &= 2(2.66 \times 10^{-26}\ \text{kg})(0.605 \times 10^{-10}\ \text{m})^2\omega \\ &= (1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2)\omega \end{aligned}$$

We can now estimate the lowest rotational frequency of the molecule by equating the rotational angular momentum with the fundamental unit of angular momentum, $h/2\pi$.

$$\begin{aligned} L &= (1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2)\omega = \frac{h}{2\pi} \\ &= 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s} \end{aligned}$$

Therefore,

$$\omega = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} = 5.41 \times 10^{11}\ \text{rad/s}$$

This result is in good agreement with measured rotational frequencies. The rotational frequencies are much lower than the vibrational frequencies of the molecule, which are typically of the order of 10^{13} rad/s.

Classical concepts and mechanical models can be useful in describing some features of atomic and molecular systems. However, as shown by the preceding example, a wide variety of submicroscopic phenomena can be explained only if one assumes discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962), in his theory of the hydrogen atom, suggested the radical idea of angular momentum quantization. Strictly classical models were unsuccessful in describing many properties of the hydrogen atom, such as the fact that the atom emits radiation at discrete frequencies. Bohr found that the electron could occupy only those orbits about the proton for which the orbital angular momentum was equal to $nh/2\pi$, where n is an integer, or quantum number. From this simple model, one can estimate the rotational frequencies of the electron in the allowed orbits.

Although Bohr's theory provided some insight concerning the behavior of matter at the atomic level, it is basically incorrect. Subsequent developments in quantum mechanics from 1924 to 1930 provided models and interpretations that are still accepted.

Later developments in atomic physics indicated that an electron possesses another kind of angular momentum, called **spin,** which is an inherent property of the electron itself, unrelated to its motion. The spin angular momentum is also restricted to discrete values. Later we shall return to this important property and discuss its great impact on modern physical science.

10.11 · ROTATION OF RIGID BODIES

OPTIONAL

In this section we shall investigate the rotational motion of rigid bodies, using as a basis much of what we have learned concerning the circular motion of particles. First we shall discuss the dynamics of a rigid body rotating about a fixed axis. Next we shall show how to determine the kinetic energy of a rotating rigid body and how its change is related to the work done by external forces. Finally we shall use the principle of conservation of energy to describe an object that rolls on a surface without slipping.

Rotational Dynamics

In Section 10.7 we learned that the magnitude of the net torque acting on a particle is given by Equation 10.25:

$$\tau_{\text{net}} = I\alpha$$

where I is the moment of inertia and α is the angular acceleration. It is important to note that this result also applies to a rigid body of arbitrary shape rotating about a fixed axis, because the body can be regarded as an infinite number of mass elements of infinitesimal size, each rotating in a circle about the rotation axis. The important and strikingly simple result given by $\tau_{\text{net}} = I\alpha$ is in complete agreement with experimental observations. Its simplicity is a result of how the motion is described.

Although the points on a rigid body rotating about a fixed axis may not experience the same force, linear acceleration, or linear velocity, every point on the body has the same angular acceleration and angular velocity at any instant. Therefore, at any instant the rigid body as a whole is characterized by specific values for angular acceleration, net torque, and angular velocity.

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TABLE 10.2 Moments of Inertia of Some Rigid Objects

Hoop or cylindrical shell $I_{CM} = MR^2$	Hollow cylinder $I_{CM} = \frac{1}{2} M(R_1^2 + R_2^2)$
Solid cylinder or disk $I_{CM} = \frac{1}{2} MR^2$	Rectangular plate $I_{CM} = \frac{1}{12} M(a^2 + b^2)$
Long thin rod with rotation axis through center $I_{CM} = \frac{1}{12} ML^2$	Long thin rod with rotation axis through end $I = \frac{1}{3} ML^2$
Solid sphere $I_{CM} = \frac{2}{5} MR^2$	Thin spherical shell $I_{CM} = \frac{2}{3} MR^2$

The specific form of I depends on the axis of rotation and on the size and shape of the body. The moments of inertia of rigid bodies can be evaluated by methods of integral calculus. Table 10.2 gives the moments of inertia for a number of bodies about specific axes. In all cases, note that I is proportional to the mass of the object and to the square of a geometric factor.

Example 10.12 Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as in Figure 10.27. A light cord wrapped around the wheel supports an object of mass m. Calculate the linear acceleration of the object, the angular acceleration of the wheel, and the tension in the cord.

Reasoning and Solution The torque acting on the wheel about its axis of rotation is $\tau = TR$, where T is the force exerted by the cord on the rim of the wheel. (The weight of the wheel and the normal force exerted by the axle on the wheel pass through the axis of rotation and produce no torque.) Because $\tau = I\alpha$, we get

$$\tau = I\alpha = TR$$

$$(1) \quad \alpha = \frac{TR}{I}$$

Now let us apply Newton's second law to the motion of the suspended object, making use of the free-body diagram in Figure 10.27, taking the upward direction to be positive:

$$\sum F_y = T - mg = -ma$$

$$(2) \quad a = \frac{mg - T}{m}$$

The linear acceleration of the suspended object is equal to the tangential acceleration of a point on the rim of the wheel. Therefore, the angular acceleration of the wheel and this linear acceleration are related by $a = R\alpha$. Using this fact together with (1) and (2), gives

$$(3) \quad a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$(4) \quad T = \frac{mg}{1 + mR^2/I}$$

Substituting (4) into (3), and solving for a and α gives

$$a = \frac{g}{1 + I/mR^2}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + I/mR}$$

EXERCISE 12 The wheel in Figure 10.27 is a solid disk of $M = 2.00$ kg, $R = 30.0$ cm, and $I = 0.0900$ kg·m². The suspended object has a mass of $m = 0.500$ kg. Find the tension in the cord and the angular acceleration of the wheel.
Answer 3.27 N; 10.9 rad/s²

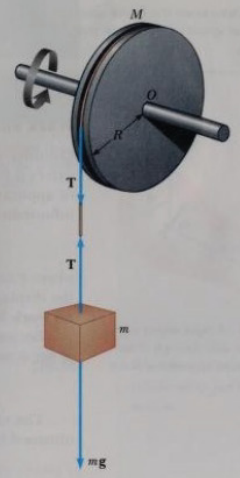


Figure 10.27 (Example 10.12) The cord attached to an object of mass m is wrapped around the wheel, and the tension in the cord produces a torque about the axle through O.

Example 10.13 Rotating Rod Revisited

A uniform rod of length L and mass M is free to rotate on a frictionless pin through one end (Fig. 10.28). The rod is released from rest in the horizontal position. (a) What is the angular speed of the rod at its lowest position?

Reasoning and Solution The question can be answered by considering the mechanical energy of the system. When the rod is horizontal, it has no rotational energy. The potential energy relative to the lowest position of its center of mass (O') is $MgL/2$. When it reaches its lowest position, the energy is entirely rotational energy, $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the pivot. Because $I = \frac{1}{3}ML^2$ (Table 10.2) and because mechanical energy is constant, we have

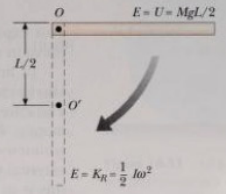


Figure 10.28 (Example 10.13) A uniform rigid rod pivoted at O rotates in a vertical plane under the action of the force of gravity.

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$\frac{1}{2}MgL = \frac{1}{2}I\omega^2 = \frac{1}{2}(\frac{1}{3}ML^2)\ \omega^2$

$\omega = \sqrt{\frac{3g}{L}}$

(b) Determine the linear speed of the center of mass and the linear speed of the lowest point on the rod in the vertical position.

Solution

$v_{CM} = r\omega = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}$

The lowest point on the rod has a linear speed equal to $2v_{CM} = \sqrt{3gL}$.

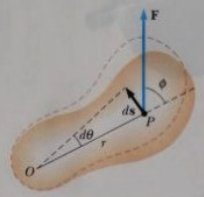


Figure 10.29 A rigid object rotates about an axis through O under the action of an external force **F** applied at P.

Work and Energy in Rotational Motion

Consider a rigid body pivoted at the point O in Figure 10.29. Suppose a single external force, **F**, is applied at the point P and $d\mathbf{s}$ is the displacement of the point of application of the force. The work done by **F** as the body rotates through an infinitesimal distance $ds = r\,d\theta$ in a time dt is

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)\,r\,d\theta$$

where $F\sin\phi$ is the tangential component of **F** or the component of the force along the displacement. Note from Figure 10.29 that **the radial component of F does no work because it is perpendicular to the displacement.**

Because the magnitude of the torque due to **F** about the origin was defined as $rF\sin\phi$, we can write the work done for the infinitesimal rotation in the form

$$dW = \tau d\theta \qquad [10.35]$$

The rate at which work is being done by **F** for rotation about the fixed axis is obtained by formally dividing the left and right sides of Equation 10.35 by dt:

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} \qquad [10.36]$$

But the quantity dW/dt is, by definition, the instantaneous power, P, delivered by the force. Furthermore, because $d\theta/dt = \omega$, Equation 10.36 reduces to

Power delivered to a rigid body •

$$P = \frac{dW}{dt} = \tau\omega \qquad [10.37]$$

This expression is analogous to $P = Fv$ in the case of linear motion, and the expression $dW = \tau d\theta$ is analogous to $dW = F_x dx$.

In linear motion, we found the energy concept, and in particular the work-kinetic energy theorem, to be extremely useful in describing the motion of a system. The energy concept can be equally useful in simplifying the analysis of rotational motion. From what we learned of linear motion, we expect that for rotation of a symmetric object (such as a symmetric wheel) about a fixed axis, the work done by external forces will equal the change in the rotational kinetic energy. To show that this is in fact the case, let us begin with $\tau = I\alpha$. Using the chain rule from the calculus, we can express the torque as

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

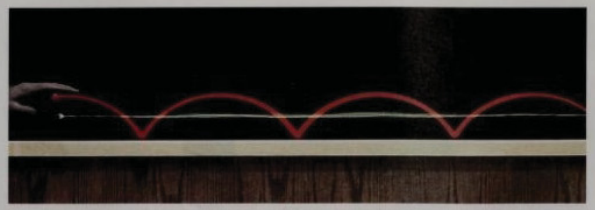


Figure 10.30 Light sources attached at the center and rim of a rolling cylinder illustrate the different paths these points take. The center moves in a straight line, as indicated by the green line, and a point on the rim moves in the path of a cycloid, as indicated by the red curve. *(Henry Leap and Jim Lehman)*

Rearranging this expression and noting that $\tau d\theta = dW$, we get

$$\tau d\theta = dW = I\omega\,d\omega$$

Integrating this expression, we get the total work done:

$$W = \int_{\theta_0}^{\theta}\tau\,d\theta = \int_{\omega_0}^{\omega} I\omega\,d\omega = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 \qquad [10.38]$$

• *Work-kinetic energy relationship for rotational motion*

Rolling Motion of a Rigid Body

Suppose a cylinder is rolling on a straight path as in Figure 10.30. The center of mass moves in a straight line, and a point on the rim moves in a more complex path corresponding to the shape of a cycloid. Let us further assume that the cylinder of radius R is uniform and rolls on a surface with friction. As the cylinder rotates through an angle of θ, its center of mass moves a distance of $s = r\theta$. Therefore, the speed and acceleration of the center of mass for **pure rolling motion** are

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad [10.39]$$

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \qquad [10.40]$$

The linear velocities of various points on the rolling cylinder are illustrated in Figure 10.31. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point. At any instant, the point P is at rest relative to the surface, because sliding does not occur.

A general point on the cylinder, such as Q, has both horizontal and vertical components of velocity. However, the points P and P' and the point at the center of mass are unique and of special interest. Relative to the surface on which the cylinder is moving, the center of mass moves with a speed of $v_{CM} = R\omega$, whereas the contact point P has zero velocity. The point P' has a speed equal to $2v_{CM} = 2R\omega$, because all points on the cylinder have the same angular speed.

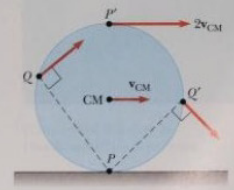


Figure 10.31 All points on a rolling body move in a direction perpendicular to an axis through the contact point P. The center of the body moves with a velocity $\mathbf{v}_{CM}$, and the point P' moves with a velocity $2\mathbf{v}_{CM}$.

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We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2} I_P \omega^2 \qquad [10.41]$$

where I_P is the moment of inertia about the axis through P.

A useful theorem called the **parallel axis theorem** enables us to calculate the moment of inertia, I_P, through any axis parallel to the axis through the center of mass of a body. This theorem states that

$$I_P = I_{CM} + MR^2 \qquad [10.42]$$

where R is the distance from the center-of-mass axis to the parallel axis, and M is the total mass of the body. Substitution of this expression into Equation 10.41 gives

$$K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} MR^2 \omega^2$$

Total kinetic energy of a rolling body

$$K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M v_{CM}^2 \qquad [10.43]$$

where we have used the fact that $v_{CM} = R\omega$.

We can think of Equation 10.43 as follows: The first term on the right, $\frac{1}{2} I_{CM} \omega^2$, represents the rotational kinetic energy about the center of mass, and the term $\frac{1}{2} M v_{CM}^2$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that **the total kinetic energy of an object undergoing rolling motion is the sum of a rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

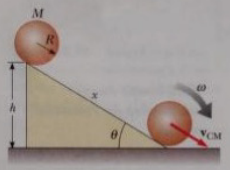


Figure 10.32 A round object rolling down an incline. Mechanical energy remains constant if no slipping occurs and there is no rolling friction.

We can use energy methods to treat a class of problems concerning the rolling motion of a rigid body down a rough incline. We shall assume that the rigid body in Figure 10.32 does not slip and is released from rest at the top of the incline. Note that rolling motion is possible only if a frictional force is present between the object and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy takes place because the contact point is at rest relative to the surface at any instant. However, if the rigid body were to slide, mechanical energy would decrease as motion progressed.

Thinking Physics 6

A common classroom demonstration is to spin on a rotating stool with heavy weights in your outstretched hands and bring the weights close to your body. The result is an increase in the angular velocity, in agreement with the principle of conservation of angular momentum (see Fig. 10.33). Let us imagine we perform such a demonstration, and moving the weights inward results in halving the moment of inertia, and, therefore, doubling the angular velocity. If we consider the rotational kinetic energy, we see that the energy is *doubled* in this situation. Thus, angular momentum is conserved, but kinetic energy is not. Where does this extra energy come from?

Reasoning As you spin with the weights in your hands, there is a tension force in your arms that is causing the weights to move in a circular path. As you bring the weights in, this force from your arms is doing work on the weights, because it is moving them through a radial displacement. As the angular velocity increases in response to the weights moving inward, you must apply more and more force to keep the weights moving in the smaller radius circular path. Thus, the work that you do in bringing the weights in increases as they are brought in. It is this work that you do with your arms that appears as the increase in rotational kinetic energy of the system. You can perform this work due to the energy stored in your body from previous meals, so you are transforming potential energy stored in your body into rotational kinetic energy.

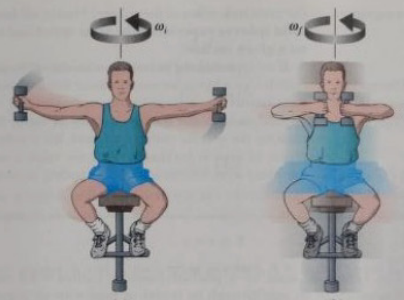


Figure 10.33 (Thinking Physics 6) (a) The student is given an initial angular speed while holding two masses as shown. (b) When the masses are pulled in close to the body, the angular speed of the system increases.

CONCEPTUAL PROBLEM 4

Suppose a pencil is balanced on a perfectly frictionless table. If it falls over, what is the path followed by the center of mass of the pencil?

CONCEPTUAL PROBLEM 5

In some motorcycle races, the riders drive over small hills, and the motorcycle becomes airborne for a short time. If the motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?

CONCEPTUAL PROBLEM 6

Suppose you are designing a car for a coasting race—the cars in this race have no engines, they simply coast down a hill. Do you want large wheels or small wheels? Do you want solid, disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?



(Conceptual Problem 5)
(Tom Raymond/Tony Stone Images)

Example 10.14 Sphere Rolling Down an Incline

If the object in Figure 10.32 is a solid sphere, calculate the speed of its center of mass at the bottom and determine the magnitude of the linear acceleration of the center of mass.

Solution For a uniform solid sphere, $I_{CM} = \frac{2}{5} MR^2$, and therefore, conservation of mechanical energy gives

$$v_{CM} = \left(\frac{2gh}{1 + \frac{\frac{2}{5} MR^2}{MR^2}} \right)^{1/2} = \left(\tfrac{10}{7} gh \right)^{1/2}$$

The vertical displacement is related to the displacement x along the incline through the relationship $h = x \sin \theta$.

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Hence, after squaring both sides, we can express the equation above as

$$v_{CM}^2 = \frac{10}{7} g x \sin\theta$$

Comparing this with the familiar expression from kinematics, $v_{CM}^2 = 2a_{CM}x$, we see that the acceleration of the center of mass is

$$a_{CM} = \frac{5}{7} g \sin\theta$$

These results are quite interesting in that both the speed and the acceleration of the center of mass are *independent* of the mass and radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline.**

If we repeated the calculations for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific body. In all cases, the acceleration of the center of mass is *less* than $g \sin\theta$, the value it would have if the plane were frictionless and no rolling occurred.

EXERCISE 13 The center of mass of a pitched baseball (radius = 3.8 cm) moves at 38 m/s. The ball spins about an axis through its center of mass with an angular speed of 125 rad/s. Calculate the ratio of the rotational energy to the translational kinetic energy. Treat the ball as a uniform sphere. Answer 1/160

EXERCISE 14 An automobile tire, considered as a solid disk, has a radius of 25 cm and a mass of 6.0 kg. Find its rotational kinetic energy when it is rotating about an axis through its center at an angular speed of 2.0 rev/s. Answer 15 J

SUMMARY

The **instantaneous angular speed** of a particle rotating in a circle or of a rigid body rotating about a fixed axis is

$$\omega = \frac{d\theta}{dt} \quad [10.3]$$

where ω is in rad/s or in s^{-1}.

The **instantaneous angular acceleration** of a rotating body is

$$\alpha = \frac{d\omega}{dt} \quad [10.5]$$

and has units of rad/s^2 or s^{-2}.

When a rigid body rotates about a fixed axis, every part of the body has the same angular velocity and the same angular acceleration. However, different parts of the body, in general, have different linear velocities and different linear accelerations.

If a particle (or body) undergoes rotational motion about a fixed axis under constant angular acceleration, α, one can apply equations of kinematics by analogy with kinematic equations for linear motion with constant linear acceleration:

$$\omega = \omega_0 + \alpha t \quad [10.6]$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \quad [10.7]$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad [10.8]$$

$$\theta = \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \quad [10.9]$$

When a particle rotates about a fixed axis, the angular speed and angular acceleration are related to the linear speed and tangential linear acceleration through the relationships

$$v = r\omega \quad [10.10]$$

$$a_t = r\alpha \quad [10.11]$$

The **moment of inertia of a system of particles** is

$$I = \sum m_i r_i^2 \quad [10.14]$$

If a rigid body rotates about a fixed axis with angular speed ω, its **kinetic energy** can be written

$$K = \tfrac{1}{2} I\omega^2 \quad [10.15]$$

where I is the moment of inertia about the axis of rotation.

The **torque,** τ, due to a force **F** about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \quad [10.17]$$

Given two vectors, **A** and **B**, their **cross product, A × B,** is a vector **C** having the magnitude

$$C = |AB \sin\theta| \quad [10.19]$$

where θ is the angle between **A** and **B**. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the right-hand rule. Some properties of the cross product include the facts that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{A} = 0$.

The net torque acting on a particle is proportional to the angular acceleration of the particle, and the proportionality constant is the moment of inertia, I:

$$\sum \tau = I\alpha \quad [10.22]$$

The **angular momentum, L,** of a particle of linear momentum $\mathbf{p} = m\mathbf{v}$ is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad [10.23]$$

where **r** is the vector position of the particle relative to an origin in an inertial frame. If ϕ is the angle between **r** and **p**, the magnitude of **L** is

$$L = mvr \sin\phi = I\omega \quad [10.24]$$

The **net torque** acting on a particle is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \quad [10.27]$$

The law of conservation of angular momentum states that the total angular momentum of a system remains constant if the net external torque acting on the system is zero:

$$\text{When } \sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = 0, \text{ then} \quad [10.29]$$

$$\mathbf{L} = \text{constant} \quad [10.30]$$

Angular momentum is an inherent property of atoms, molecules, and their constituents. Such systems have discrete (quantized) values of angular momenta that are integer multiples of $h/2\pi$, where h is Planck's constant.

The **total kinetic energy** of a rigid body, such as a cylinder, that is rolling on a rough surface without slipping equals the rotational kinetic energy about the body's center of mass $\frac{1}{2} I_{CM}\omega^2$ plus the translational kinetic energy of the center of mass $\frac{1}{2} M v_{CM}^2$:

$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M v_{CM}^2 \quad [10.43]$$

In this expression, v_{CM} is the speed of the center of mass and $v_{CM} = R\omega$ for pure rolling

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CONCEPTUAL QUESTIONS

1. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled—as the tape is pulled off it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular velocity of the reel change with time? If the tape mechanism is suddenly turned on so that the tape is quickly pulled with a large force, is the tape more likely to break when pulled from a nearly full reel or a nearly empty reel?
2. If a car's wheels are replaced with wheels of a larger diameter, will the reading of the speedometer change? Explain.
3. What is the magnitude of the angular velocity, ω, of the second hand of a clock? What is the direction of ω as you view a clock hanging vertically? What is the magnitude of the angular acceleration, α, of the second hand?
4. If you see an object rotating, is there necessarily a net torque acting on it?
5. In order for a helicopter to be stable as it flies, it must have two propellers. Why?
6. The moment of inertia of an object depends on the choice of rotation axis, as suggested by the parallel axis theorem. Argue that an axis passing through the center of mass of an object must be the axis with the smallest moment of inertia.
7. Suppose you remove two eggs from the refrigerator, one hard-boiled and the other uncooked. You wish to determine which is the hard-boiled egg without breaking the eggs. This can be done by spinning the two eggs on the floor and comparing the rotational motions. Which egg spins faster? Which rotates more uniformly? Explain.
8. A ladder rests inclined against a wall. Would you feel safer climbing up the ladder if you were told that the floor is frictionless but the wall is rough, or that the wall is frictionless but the floor is rough? Justify your answer.
9. Often when a high diver wants to flip in midair, she will draw her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?
10. If the net force acting on a system is zero, then is it necessarily true that the net torque on it is also zero?
11. Why do tightrope walkers carry a long pole to help balance themselves?
12. Two solid spheres are rolled down a hill—a large, massive sphere and a small sphere with low mass. Which one reaches the bottom of the hill first? Next, we roll a large, low density sphere, and a small high density sphere, and both spheres have the same mass. Which one wins in this case?
13. Consider an object in the shape of a hoop lying in the xy plane with all of its mass concentrated on its rim. In two separate experiments, the hoop is rotated by an external agent from rest to an angular speed ω. In one experiment, the rotation occurs about the z axis through the center of the hoop. In the other experiment, the rotation occurs about an axis parallel to z passing through a point P on the rim of the hoop. Which rotation requires more work?
14. Vector **A** is in the negative y direction and vector **B** is in the negative x direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ and (b) $\mathbf{B} \times \mathbf{A}$?
15. If global warming occurs over the next century, it is likely that the polar ice caps of the Earth will melt and the water will be distributed closer to the Equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

PROBLEMS

Section 10.2 Rotational Kinematics

1. What is the angular speed in radians per second of (a) the Earth in its orbit about the Sun and (b) the Moon in its orbit about the Earth?
2. An airliner arrives at the terminal and the engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take the rotor to come to rest?
3. An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn during the time found in (a)?
4. A centrifuge in a medical laboratory rotates at an angular speed of 3600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.
5. A dentist's drill starts from rest. After 3.20 s of constant angular acceleration, the drill rotates at a rate of 2.51×10^4 rev/min. (a) Find the drill's angular acceleration. (b) Determine the angle (in radians) through which the drill rotates during this period.
6. The angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$ rad. Determine the angular position, angular speed, and angular acceleration of the door at $t = 0$ and $t = 3.00$ s.
7. The tub of a washer goes into its spin cycle, starting from rest and reaching an angular speed of 5.00 rev/s in 8.00 s. At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The tub slows to rest in 12.0 s. Through how many revolutions does the tub turn? Assume constant angular acceleration while the machine is starting and stopping.
8. A rotating wheel requires 3.00 s to rotate 37.0 revolutions. Its angular velocity at the end of the 3.00 s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?

Section 10.3 Relations Between Angular and Linear Quantities

9. A discus thrower accelerates a discus from rest to a speed of 25.0 m/s by whirling it through 1.25 rev. Assume the discus moves on the arc of a circle 1.00 m in radius. (a) Calculate the final angular speed of the discus. (b) Determine the magnitude of the angular acceleration of the discus, assuming it to be constant. (c) Calculate the acceleration time.
10. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming no slipping. (b) What is the final rotational speed of a tire in revolutions per second?
11. A disk 8.00 cm in radius rotates at a constant rate of 1200 rev/min about its central axis. Determine (a) its angular speed, (b) the linear speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.
12. A 6.00-kg block is released from A on a frictionless track shown in Figure P10.12. Determine the radial and tangential components of acceleration for the block at P.

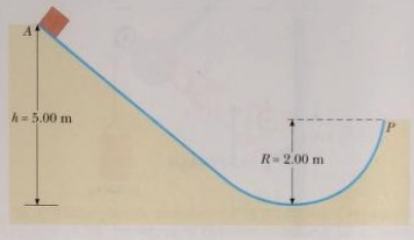


Figure P10.12

13. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s^2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

Section 10.4 Rotational Kinetic Energy

14. Three particles are connected by rigid rods of negligible mass lying along the y axis (Fig. P10.14). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the linear speed of each particle and the total energy evaluated from $\sum \frac{1}{2} m_i v_i^2$.

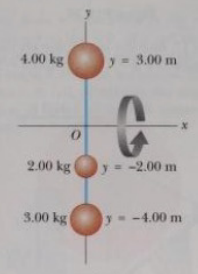


Figure P10.14

15. The four particles in Figure P10.15 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy plane about the z axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the z axis and (b) the rotational energy of the system.

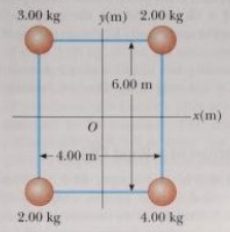


Figure **P10.15**

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Section 10.5 Torque and the Vector Product

16. The fishing pole in Figure P10.16 makes an angle of 20.0° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the fisher's hand?

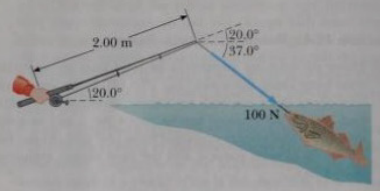


Figure P10.16

17. In Figure P10.17, find the net torque on the wheel about the axle through O if $a = 10.0$ cm and $b = 25.0$ cm.

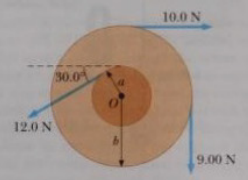


Figure P10.17

18. Given $\mathbf{M} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{N} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.
19. Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between $\mathbf{A}$ and $\mathbf{B}$.
20. A force of $\mathbf{F} = 2.00\mathbf{i} + 3.00\mathbf{j}$ N is applied to an object that is pivoted about a fixed axis aligned along the z coordinate axis. If the force is applied at the point whose vector position is $\mathbf{r} = (4.00\mathbf{i} + 5.00\mathbf{j} + 0\mathbf{k})$ m, find (a) the magnitude of the net torque about the z axis and (b) the direction of the torque vector, $\boldsymbol{\tau}$.
21. A student claims that she has found a vector $\mathbf{A}$ such that $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Do you believe this claim? Explain.

Section 10.6 Equilibrium of a Rigid Object

22. A ladder of weight 400 N and length 10.0 m is placed against a smooth vertical wall. A person weighing 800 N stands on the ladder 2.00 m from the bottom as measured along the ladder. The foot of the ladder is 8.00 m from the bottom of the wall. Draw a free-body diagram of the ladder. Calculate the force exerted by the wall and the normal force exerted by the floor on the ladder.
23. A 1500 kg automobile has a wheel base (the distance between the axles) of 3.00 m. The center of mass of the automobile is on the center line at a point 1.20 m behind the front axle. Find the force exerted by the ground on each wheel.
24. A uniform plank of length 6.00 m and mass 30.0 kg rests horizontally on a scaffold, with 1.50 m of the plank hanging over one end of the scaffold. Draw a free-body diagram of the plank. How far can a painter of mass 70.0 kg walk on the overhanging part of the plank before it tips?
25. A 1200-N uniform boom is supported by a cable as in Figure P10.25. The boom is pivoted at the bottom, and a 2000-N object hangs from its top. Find the tension in the cable and the components of the reaction force on the boom by the floor.

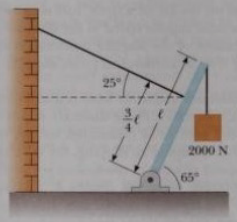


Figure P10.25

26. A crane of mass 3000 kg supports a load of 10 000 kg as in Figure P10.26. The crane is pivoted with a smooth pin at A and rests against a smooth support at B. Find the reaction forces at A and B.

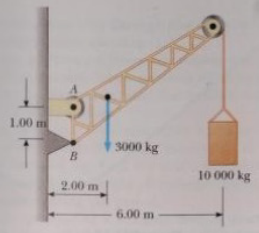


Figure P10.26

Section 10.7 Relation Between Torque and Angular Acceleration

27. A model airplane the mass of which is 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque that the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.
28. The combination of an applied force and a frictional force produces a constant total torque of 36.0 N·m on a wheel rotating about a fixed axis. The applied force acts for 6.00 s, during which time the angular speed of the wheel increases from 0 to 10.0 rad/s. The applied force is then removed, and the wheel comes to rest in 60.0 s. Find (a) the moment of inertia of the wheel, (b) the magnitude of the frictional torque, and (c) the total number of revolutions of the wheel.

Section 10.8 Angular Momentum

29. A light rigid rod 1.00 m in length rotates in the xy plane about a pivot through the rod's center. Two particles of masses of 4.00 kg and 3.00 kg are connected to its ends (Fig. P10.29). Determine the angular momentum of the system about the origin at the instant the speed of each particle is 5.00 m/s.

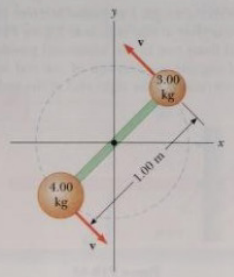


Figure P10.29

30. A 1.50-kg particle moves in the xy plane with a velocity of $\mathbf{v} = (4.20\mathbf{i} - 3.60\mathbf{j})$ m/s. Determine the angular momentum of the particle when its position vector is given by $\mathbf{r} = (1.50\mathbf{i} + 2.20\mathbf{j})$ m.
31. The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\,\mathbf{i} + 5.00\mathbf{j})$ m. Determine the angular momentum of the particle as a function of time.
32. An airplane of mass 12 000 kg flies level to the ground at an altitude of 10.0 km with a constant speed of 175 m/s relative to the Earth. (a) What is the magnitude of the airplane's angular momentum relative to a ground observer directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line?

Section 10.9 Conservation of Angular Momentum

Section 10.10 Quantization of Angular Momentum (Optional)

33. A ball having mass m is fastened at the end of a flagpole that is connected to the side of a tall building at point P shown in Figure P10.33. The length of the flagpole is ℓ and it makes an angle θ with the horizontal. If the ball become loose and starts to fall, determine the angular momentum (as a function of time) of the ball about point P. Neglect air resistance.

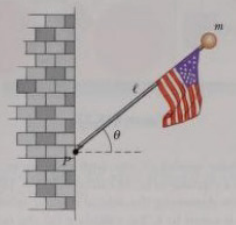


Figure P10.33

34. A student sits on a rotating stool holding two weights, each of mass 3.00 kg. When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is 3.00 kg·m² and is assumed to be constant. The student pulls the weights horizontally to 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the student before and after the weights are pulled in.
35. A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of 500 kg·m² and a radius of

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2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

36. The ball in Figure 10.25 has a mass of 0.120 kg. The distance of the ball from the center of rotation is originally 40.0 cm, and the ball is moving with a speed of 80.0 cm/s. The string is pulled downward 15.0 cm through the hole in the frictionless table. Determine the work done on the ball. (*Hint:* Consider the change of kinetic energy.)

37. A puck of mass 80.0 g and radius 4.00 cm slides along an air table at a speed of 1.50 m/s as shown in Figure P10.37a. It makes a glancing collision with a second puck of radius 6.00 cm and mass 120 g (initially at rest) such that their rims just touch. The pucks stick together and spin after the collision (Fig. P10.37b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular velocity about the center of mass?



Figure P10.37

38. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular frequency of the electron's motion.

Section 10.11 Rotation of Rigid Bodies (Optional)

39. Attention! About face! Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn around an axis through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

40. A horizontal 800-N merry-go-round is a solid disk of radius 1.50 m, started from rest by a constant horizontal force of 50.0 N applied tangentially at its edge. Find the kinetic energy of the disk after 3.00 s.

41. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational energy about its center of mass, and (c) its total energy.

42. Consider two masses connected by a string that passes over a pulley having a moment of inertia of I about its axis of rotation, as in Figure P10.42. The string does not slip on the pulley, and the system is released from rest. Use the principle of conservation of energy to find the linear speeds of the masses after m_2 descends through a distance h and the angular speed of the pulley at this time.

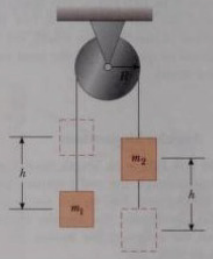


Figure P10.42

43. A uniform rod of length L and mass M is free to rotate about a frictionless pivot at one end, as in Figure P10.43. The rod is released from rest in the horizontal position. What are the *initial* angular acceleration of the rod and the *initial* linear acceleration of the right end of the rod?

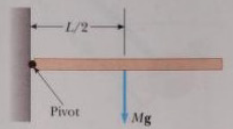


Figure P10.43

44. A bicycle wheel has a diameter of 64.0 cm and a mass of 1.80 kg. The bicycle is placed on a stationary stand on rollers and a resistive force of 120 N is applied to the rim of the tire. (Assume that the wheel is a hoop with all the mass concentrated on the outside radius.) What force must be applied by a chain passing over a 9.00-cm diameter sprocket in order to give the wheel an acceleration of 4.50 rad/s^2? What if you shifted to a 5.60-cm diameter sprocket?

45. (a) A uniform solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.45). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) Repeat part (a) using a uniform hoop.

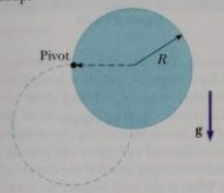


Figure P10.45

46. A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a disk having radius $R = 0.25$ m, mass $M = 10.0$ kg. In addition, they are allowed to move on a fixed block-wedge of angle $\theta = 30.0°$ as in Figure P10.46. The coefficient of kinetic friction is 0.360 for both blocks. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks, and (b) the tensions in the string on both sides of the pulley.

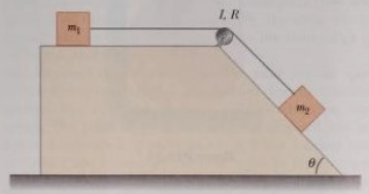


Figure P10.46

47. A can of condensed mushroom soup has mass 215 g, height 10.8 cm, and outer diameter 6.38 cm. It is placed at rest on the top of a 3.00-m long incline, which is at 25.0° to the horizontal. Assuming energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline.

Additional Problems

48. An electric motor can accelerate a Ferris wheel of moment of inertia $I = 20\,000$ kg·m^2 from rest to 10.0 rev/min in 12.0 s. When the motor is turned off, friction causes the wheel to slow down from 10.0 to 8.00 rev/min in 10.0 s. Determine (a) the torque generated by the motor to bring the wheel to 10.0 rev/min and (b) the power needed to maintain this rotational speed.

49. Two blocks, as shown in Figure P10.49, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving up with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

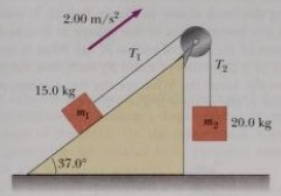


Figure P10.49

50. As a result of friction, the angular speed of a wheel changes with time according to

$$\frac{d\theta}{dt} = \omega_0 e^{-\sigma t}$$

where ω_0 and σ are constants. The angular speed changes from 3.50 rad/s at $t = 0$ to 2.00 rad/s at $t = 9.30$ s. Use this information to determine σ and ω_0. Then determine (a) the magnitude of the angular acceleration at $t = 3.00$ s, (b) the number of revolutions the wheel makes in the first 2.50 s, and (c) the number of revolutions it makes before coming to rest.

51. The pulley shown in Figure P10.51 has radius R and moment of inertia I. One end of the mass m is connected to a spring of force constant k, and the other end is fastened to a cord wrapped around the pulley. The pulley axle and the incline are frictionless. If the pulley is wound counterclock-

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wise in order to stretch the spring a distance d from its unstretched position and then released from rest, find (a) the angular speed of the pulley when the spring is again unstretched and (b) a numerical value for the angular speed at this point if $I = 1.00\ \text{kg} \cdot \text{m}^2$, $R = 0.300$ m, $k = 50.0$ N/m, $m = 0.500$ kg, $d = 0.200$ m, and $\theta = 37.0°$.

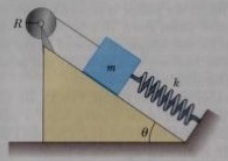


Figure P10.51

52. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P10.52. It starts from rest with the lowest point of the sphere at height h above the bottom of the loop of radius R. (a) What is the minimum value of h (in terms of r and R) such that the sphere completes the loop? (b) What are the force components on the sphere at the point P if $h = 3R$?

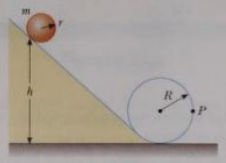


Figure P10.52

53. Two astronauts (Fig. P10.53), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. (a) Calculate the magnitude of the angular momentum of the system by treating the astronauts as particles and (b) the rotational energy of the system. By pulling on the rope, the astronauts shorten the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are their new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronauts in shortening the rope?

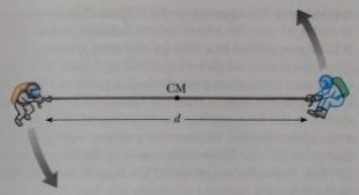


Figure P10.53

54. Two astronauts (Fig. P10.53), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Calculate (a) the magnitude of the angular momentum of the system by treating the astronauts as particles and (b) the rotational energy of the system. By pulling on the rope, the astronauts shorten the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are their new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronauts in shortening the rope?

55. Figure P10.55 shows a vertical force applied tangentially to a uniform cylinder of weight w. The coefficient of static friction between the cylinder and all surfaces is 0.500. Find, in terms of w, the maximum force $\mathbf{F}$ that can be applied without causing the cylinder to rotate. (*Hint:* When the cylinder is on the verge of slipping, both friction forces are at their maximum values. Why?)

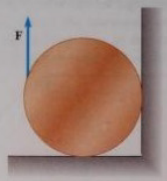


Figure P10.55

56. A common physics demonstration (Fig. P10.56) consists of a ball resting at the end of a board of length ℓ that is elevated at angle θ with the horizontal. A light cup is attached to the board at r_c so that it will catch the ball when the support stick is suddenly removed. (a) Show that the ball will lag behind the falling board when θ is less than 35.3°, and (b) the ball will fall into the cup when the board is supported at this limiting angle and the cup is placed at

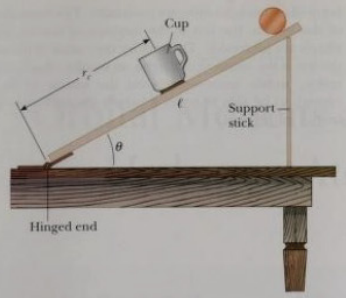


Figure P10.56

$$r_c = \frac{2\ell}{3\cos\theta}$$

(c) If a ball is at the end of a 1.00-m stick at this critical angle, show that the cup must be 18.4 cm from the moving end.

Spreadsheet Problem

S1. A disk having a moment of inertia of 100 kg·m² is free to rotate about a fixed axis through its center as in Figure 10.1. A tangential force whose magnitude ranges from $F = 0$ to $F = 50.0$ N can be applied at any distance ranging from $R = 0$ to $R = 3.00$ m measured from the axis of rotation. Using Spreadsheet 10.1, find values of F and R that will cause the disk to complete two revolutions in 10.0 s. Are there unique values for F and R?

ANSWERS TO CONCEPTUAL PROBLEMS

1. As the smokestack rotates around its base, each higher portion of the smokestack falls with an increasing tangential acceleration. The tangential acceleration of a given point on the smokestack is proportional to the distance of that portion from the base. As the acceleration increases, eventually higher portions of the smokestack will experience an acceleration higher than that which could result from gravity alone. This can only happen if these portions are being pulled downward by a force in addition to the gravitational force. The force that does this is the shear force from lower portions of the smokestack. Eventually, the shear force to provide this acceleration is larger than the smokestack can withstand, and it breaks.
2. Yes, all points on the wheel have the same angular speed. This is why we use angular quantities to describe rotational motion. Not all points on the wheel have the same linear speed. The point at $r = 0$ has zero linear speed and zero linear acceleration; a point at $r = R/2$ has a linear speed $v = R\omega/2$ and a linear acceleration equal to the centripetal acceleration $v^2/(R/2) = R\omega^2/2$ (the tangential acceleration is zero at all points because ω is constant). A point on the rim at $r = R$ has a linear speed $v = R\omega$ and a linear acceleration $R\omega^2$.
3. There are two major differences between torque and work. The primary difference is that the displacement in the expression for work is directed *along* the force, while the important distance in the torque expression is *perpendicular* to the force. The second difference involves whether there is motion or not—in the case of work, there is work done only if the force succeeds in causing a displacement of the point of application of the force. On the other hand, a force applied at a perpendicular distance from a rotation axis results in a torque whether there is motion or not.

   As far as units are concerned, the mathematical expressions for both work and torque result in the product of newtons and meters, but this product is called a joule in the case of work and remains as a newton-meter in the case of torque.
4. On a frictionless table, the only forces on the pencil are the gravitational force, at the center of mass, and the normal force from the table, at the tip. These forces result in a torque on the pencil, causing it to rotate. Both of these forces, however, are vertical—there are no horizontal forces on the pencil. As a result, the center of mass of the pencil must fall straight downward. As the pencil falls, the tip slides to the side, so that the center of mass of the pencil follows a straight line downward to the table.
5. As the motorcycle tire leaves the ground, the friction between the tire and the ground suddenly disappears. If the motorcycle driver keeps the throttle open while leaving the ground, the drive tire will increase its angular velocity, since it no longer experiences the friction force from the ground. The airborne motorcycle is now an isolated system, and its angular momentum must be conserved. The increase in angular momentum of the tire (directed to the left of the motorcycle) must be compensated by an increase in angular momentum of the entire motorcycle (to the right). This ro-

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tation results in the nose of the motorcycle rising and the tail dropping.

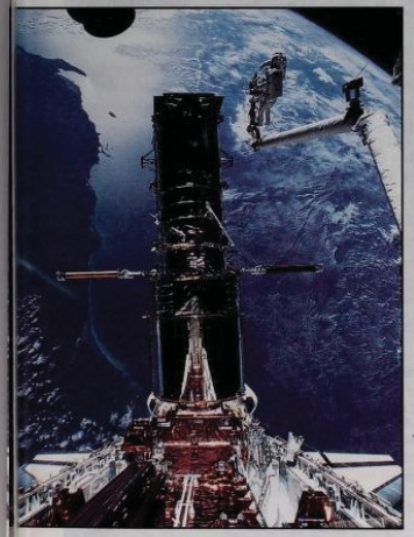
6. In general, you want the rotational kinetic energy of the system to be as small a fraction as possible of the total energy—you want translation, not rotation! You want the wheels to have as little moment of inertia as possible, so that they represent the lowest resistance to changes in rotational motion. Disk-like wheels would have lower moment of inertia than hoop-like wheels, so disks are preferable. The lower the mass of the wheels, the less is the moment of inertia, so light wheels are preferable. The smaller the radius of the wheels, the less is the moment of inertia, so smaller wheels are preferable, within limits—you want the wheels to be large enough to be able to travel relatively smoothly over irregularities in the road.

11

Orbital Motions and the Hydrogen Atom

We began our study of mechanics with translational motion and the forces that cause it. If we know the forces acting on a system and its initial conditions, we can predict the future of the system. However, describing motion in this manner is often tedious and time consuming. Fortunately, we can often follow the simpler approach of using conservation principles. We easily solved many interesting problems involving motion by recognizing that certain fundamental quantities—such as energy and momentum—were conserved.

In this chapter we return to Newton's universal law of gravitation—one of the fundamental force laws in nature—and show how it, together with Newton's laws of motion, enables us to understand a variety of familiar orbital motions, such as the motions of planets and Earth satellites. Newton's theory of gravitation evolved out of his studies of the motions of the



◀ **Astronauts F. Story Musgrave and Jeffrey A. Hoffman complete the final space walks needed to repair the Hubble Space Telescope in December 1993.** *(Courtesy NASA)*

CHAPTER OUTLINE

11.1 Newton's Universal Law of Gravity Revisited
11.2 Kepler's Laws
11.3 The Universal Law of Gravity and the Motions of Planets
11.4 Energy Considerations in Planetary and Satellite Motion
11.5 Atomic Spectra and the Bohr Theory of Hydrogen

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Moon and planets, for which he used the groundwork provided by Copernicus, Brahe, Kepler, and other astronomers. In examining this evolution, we shall once again make use of conservation of energy and angular momentum.

We conclude this chapter with a discussion of Niels Bohr's famous model of the hydrogen atom, which represents an interesting mixture of classical physics (Newton's laws of motion and the electromagnetic force) and quantum physics (quantization of angular momentum). Although Bohr's theory contains ideas that are contrary to classical physics, his model successfully predicts the observed spectral lines of hydrogen.

11.1 • NEWTON'S UNIVERSAL LAW OF GRAVITY REVISITED

Prior to 1686, many data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces that caused those motions was not yet attainable. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from the first law, that a net force had to be acting on the Moon. If not, the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force between Moon and Earth was an attractive force. He also concluded that there could be nothing special about the Earth–Moon system or the Sun and its planets that would cause gravitational forces to act on them alone. He wrote,

> I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with force of gravity at the surface of the Earth; and found them answer pretty nearly.

As you should recall from Chapter 5, every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. If two particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of the gravitational force between them is

Universal law of gravity •

$$F_g = G\frac{m_1 m_2}{r^2} \qquad [11.1]$$

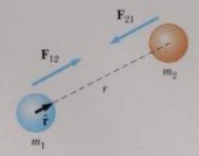


Figure 11.1 The gravitational force between two particles is attractive. The unit vector $\hat{\mathbf{r}}$ is directed from m_1 to m_2. Note that $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

where G is the *gravitational constant* the value of which in SI units is

$$G = 6.672 \times 10^{-11}\,\frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}} \qquad [11.2]$$

The force law given by Equation 11.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles. We can express this attractive force in vector form by defining a unit vector, $\hat{\mathbf{r}}$, directed from m_1 to m_2, as shown in Figure 11.1. The force on m_1 due to m_2 is

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \qquad [11.3]$$

Likewise, by Newton's third law, the force on m_2 due to m_1, designated $\mathbf{F}_{21}$, is equal in magnitude to $\mathbf{F}_{12}$ and in the opposite direction. That is, these forces form an action–reaction pair, and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

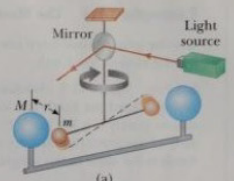
The gravitational force exerted by a finite-sized, spherically symmetric mass distribution on a particle outside the sphere is the same as if the entire mass of the sphere were concentrated at its center. For example, the force on a particle of mass m at the Earth's surface has the magnitude

$$F_g = G\frac{M_E m}{R_E^2}$$

where M_E is the Earth's mass and R_E is the Earth's radius. This force is directed toward the center of the Earth.

Measurement of the Gravitational Constant

The gravitational constant, G, was first measured in an important experiment by Sir Henry Cavendish in 1798. The apparatus he used consisted of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a thin wire, as in Figure 11.2a. Two large spheres, each of mass M, are then placed near the smaller spheres. The attractive force between the smaller and larger spheres causes the rod to rotate and twist the wire. If the system is oriented as shown in Figure 11.2a, the rod rotates clockwise when viewed from the top. The angle through which it rotates is measured by the deflection of a light beam that is reflected from a mirror attached to the wire. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.



(a)



(b)

Figure 11.2 (a) Schematic diagram of the Cavendish apparatus for measuring G. The smaller spheres of mass m are attracted to the large spheres of mass M, and the rod rotates through a small angle. A light beam reflected from a mirror on the rotating apparatus measures the angle of rotation. The dashed line represents the original position of the rod. (b) Photograph of a student Cavendish apparatus. *(Courtesy of PASCO Scientific)*

Thinking Physics 1

A student drops a ball and considers why the ball falls to the ground as opposed to the situation in which the ball stays stationary and the Earth moves up to meet it. The student comes up with the following explanation: "The Earth is much more massive than the ball, so the Earth pulls much harder on the ball than the ball pulls on the Earth. Thus, the ball falls while the Earth remains stationary." What do you think about this explanation?

Reasoning According to Newton's universal law of gravity, the force between the ball and the Earth depends on the product of their masses, so both forces—that of the ball on the Earth and that of the Earth on the ball—are equal in magnitude. This follows also, of course, from Newton's third law. The ball has large motion compared to the Earth because, according to Newton's second law, the force of equal magnitude gives a much greater acceleration to the small mass of the ball.

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Example 11.1 The Mass of the Earth

Use the gravitational force law to find an approximate value for the mass of the Earth.

Solution Figure 11.3, obviously not to scale, shows a baseball falling toward the Earth at a location where the acceleration due to gravity is g. We know from Chapter 5 that the magnitude of the gravitational force exerted on the baseball by the Earth is the same as the weight of the ball, $w = m_b g$. That is,

$$m_b g = G\frac{M_E m_b}{R_E^2}$$

We can divide each side of this equation by m_b and solve for the mass of the Earth, M_E:

$$M_E = \frac{gR_E^2}{G}$$

The falling baseball is close enough to the Earth so that the distance of separation between the center of the ball and the center of the Earth can be taken as the radius of the Earth, 6.38×10^6 m. Thus, the mass of the Earth is

$$M_E = \frac{(9.80\ \text{m/s}^2)(6.38 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2} = 5.98 \times 10^{24}\ \text{kg}$$



Figure 11.3 (Example 11.2) A baseball falling toward the Earth (not drawn to scale).

Example 11.2 Gravity and Altitude

Derive an expression that shows, for any point above the Earth's surface, how the acceleration due to gravity varies with distance from the center of the Earth.

Solution The falling baseball of Example 11.1 can be used here also. Now, however, assume that the ball is situated some arbitrary distance, $r > R_E$, from the Earth's center. The first equation in Example 11.1, with r replacing R_E and m_b removed from both sides, becomes

$$g = G\frac{M_E}{r^2}$$

This indicates that the acceleration due to gravity at a point above the Earth's surface decreases as the inverse square of the distance between the point and the center of the Earth. Our assumption of Chapter 2 that objects fall with a constant acceleration is obviously incorrect in light of this example. For short falls, however, the change in g is so small that neglecting the variation does not introduce a significant effect in our results.

Because the weight of an object is mg, a change in the value of g produces a change in weight. For example, if you weigh 800 N at the surface of the Earth, you will weigh only 200 N at a height above the Earth equal to the radius of the Earth. Also, if the distance of an object from the Earth becomes infinitely large, the weight approaches zero. Values of g at various altitudes are given in Table 11.1.

TABLE 11.1 Free-Fall Acceleration, g, at Various Altitudes Above the Earth's Surface

Altitude h (km)	g(m/s^2)
1000	7.33
2000	5.68
3000	4.53
4000	3.70
5000	3.08
6000	2.60
7000	2.23
8000	1.93
9000	1.69
10 000	1.49
50 000	0.13
∞	0

EXERCISE 1 If an object weighs 270 N at the Earth's surface, what will it weigh at an altitude equal to twice the radius of the Earth? Answer 30 N

EXERCISE 2 Determine the magnitude of the acceleration due to gravity at an altitude of 500 km. By what percentage is the weight of a body reduced at this altitude?
Answer 8.43 m/s^2; 14%

11.2 · KEPLER'S LAWS

The movements of the planets, stars, and other celestial bodies have been observed by people for thousands of years. Early in history, scientists regarded the Earth as the center of the Universe. This so-called geocentric model was elaborated and formalized by the Greek astronomer Claudius Ptolemy in the second century A.D. and was accepted for the next 1400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolve in circular orbits about the Sun (the heliocentric hypothesis).

The Danish astronomer Tycho Brahe (1546–1601) made accurate astronomical measurements over a period of 20 years and provided the basis for the currently accepted model of the Solar System. It is interesting to note that these precise observations, made on the planets and 777 stars, were carried out with nothing more elaborate than a large sextant and compass; the telescope had not yet been invented.

The German astronomer Johannes Kepler, who was Brahe's assistant, acquired Brahe's astronomical data and spent about 16 years trying to deduce a mathematical model for the motions of the planets. After many laborious calculations, he found that Brahe's precise data on the revolution of Mars about the Sun provided the answer. Kepler's analysis first showed that the concept of circular orbits about the Sun had to be abandoned. He eventually discovered that the orbit of Mars could be accurately described by an ellipse with the Sun at one focus. He then generalized this analysis to include the motions of all planets. The complete analysis is summarized in three statements, known as **Kepler's laws:**



Johannes Kepler (1571–1630)
A German astronomer, Kepler is best known for developing the laws of planetary motion based on the careful observations of Tycho Brahe. After spending several years trying to work out a "regular-solid theory" of the planets, he concluded that the Copernican view of circular planetary orbits had to be abandoned for the view that the planetary orbits are ellipses with the Sun always at one of the foci. *(Art Resource)*

1. Every planet moves in an elliptical orbit with the Sun at one of the focal points.
2. The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

• ***Kepler's laws***

Newton demonstrated that these laws were consequences of a simple force that exists between any two masses. Newton's universal law of gravity, together with his laws of motion, provides the basis for a full mathematical solution to the motion of planets and satellites. More important, Newton's universal law of gravity correctly describes the gravitational attractive force between *any* two masses.

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11.3 • THE UNIVERSAL LAW OF GRAVITY AND THE MOTIONS OF PLANETS

In formulating his universal law of gravity, Newton built on an observation that suggests that the gravitational force between two bodies is proportional to the inverse square of the separation. Let us compare the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as an apple (Fig. 11.4). Assume that the two accelerations have the same cause—namely, the gravitational attraction of the Earth. From the inverse-square law, Newton found that the acceleration of the Moon toward the Earth should be proportional to $1/r_M^2$, where r_M is the separation between centers of the Earth and Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $1/R_E^2$, where R_E is the radius of the Earth. When the values $r_M = 3.84 \times 10^8$ m and $R_E = 6.37 \times 10^6$ m are used, the ratio of the Moon's acceleration, a_M, to the apple's acceleration, g, is predicted to be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

Therefore,

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

The centripetal acceleration of the Moon can also be calculated kinematically from a knowledge of the Moon's orbital period, T, where $T = 27.32$ days $= 2.36 \times 10^6$ s and its mean distance from the Earth, r_M. In the time T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$, and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2} = 2.72 \times 10^{-3} \text{ m/s}^2$$

This agreement provides strong evidence that the inverse-square law of force is correct.

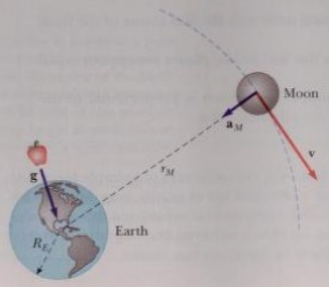


Figure 11.4 As it revolves about the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration $\mathbf{g}$. (Dimensions are not to scale.)

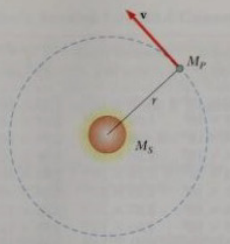


Figure 11.5 A planet of mass M_P moving in a circular orbit about the Sun. The orbits of all planets except Mars, Mercury, and Pluto are nearly circular.

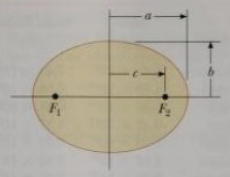


Figure 11.6 Plot of an ellipse. The semimajor axis has a length a, and the semiminor axis has a length b. The focal points are located at a distance c from the center, where $a^2 = b^2 + c^2$, and the eccentricity is defined as $e = c/a$.

Kepler's Third Law

Kepler's third law can be predicted from the inverse-square law for circular orbits.[1] Consider a planet of mass M_P which is assumed to be moving about the Sun (mass M_S) in a circular orbit, as in Figure 11.5. Because the gravitational force on the planet provides the centripetal acceleration as it moves in a circle, we can use Equation 5.3 and write

$$\frac{GM_SM_P}{r^2} = \frac{M_Pv^2}{r}$$

But the orbital speed of the planet is simply $2\pi r/T$, where T is the period; therefore, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3 \qquad [11.4] \quad \bullet \textit{ Kepler's third law}$$

where K_S is a constant given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19} \text{ s}^2/\text{m}^3$$

Equation 11.4 is Kepler's third law. It is also valid for elliptical orbits if we replace r with the length of the semimajor axis, a (Fig. 11.6). Note that the constant of proportionality, K_S, is independent of the mass of the planet. Equation 11.4 is therefore valid for *any* planet. If we were to consider the orbit of a satellite about the Earth, such as the Moon, then the constant would have a different value, with

[1]The orbits of all planets except Mars, Mercury, and Pluto are very close to being circular. For example, the ratio of the semiminor axis to the semimajor axis for the Earth is $b/a = 0.999\,86$. The closeness of the ratio b/a to 1 indicates that the difference between perihelion and aphelion is small as expected.

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TABLE 11.2 Useful Planetary Data

Body	Mass (kg)	Mean Radius (m)	Period (s)	Distance from Sun (m)	$\frac{T^2}{r^3}\left(\frac{s^2}{m^3}\right)$
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

the Sun's mass replaced by the Earth's mass. In this case, the proportionality constant would equal $4\pi^2/GM_E$.

Table 11.2 is a collection of useful planetary data. The last column verifies that the ratio T^2/r^3 is constant.

Example 11.3 An Earth Satellite

A satellite of mass m moves in a circular orbit about the Earth with a constant speed of v, at a height of $h = 1000$ km above the Earth's surface, as in Figure 11.7. (For clarity, this figure is not drawn to scale.) Find the orbital speed of the satellite. The radius of the Earth is 6.37×10^6 m, and its mass is 5.98×10^{24} kg.

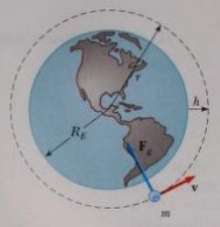


Figure 11.7 (Example 11.3) A satellite of mass m moving around the Earth in a circular orbit of radius r and with constant speed v. The only force acting on the satellite is the gravitational force $\mathbf{F}_g$. (Not drawn to scale.)

Solution The only external force on the satellite is the gravitational attraction exerted by the Earth. This force is directed toward the center of the satellite's circular path and is the force causing the centripetal acceleration of the satellite. Because the magnitude of the force of gravity is GM_Em/r^2, we find that

$$F_g = G\frac{M_E m}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM_E}{r}$$

In this expression, the distance r is the Earth's radius plus the height of the satellite; that is, $r = R_E + h = 7.37 \times 10^6$ m, so that

$$v^2 = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{7.37 \times 10^6\ \text{m}}$$

$$= 5.41 \times 10^7\ \text{m}^2/\text{s}^2$$

Therefore,

$$v = 7.36 \times 10^3\ \text{m/s} \approx 16\,400\ \text{mi/h}$$

Note that *v is independent of the mass of the satellite!*

EXERCISE 3 Calculate the period of revolution, T, of the satellite. Answer 105 min

Kepler's Second Law and Conservation of Angular Momentum

Consider a planet of mass M_P moving about the Sun in an elliptical orbit (Fig. 11.8). The gravitational force acting on the planet is a central force, always along the radius vector, directed toward the Sun. The torque on the planet due to this central force is clearly zero, because $\mathbf{F}$ is parallel to $\mathbf{r}$. That is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F(r)\hat{\mathbf{r}} = 0$$

But recall that the torque equals the time rate of change of angular momentum; that is, $\boldsymbol{\tau} = d\mathbf{L}/dt$. Therefore,

because $\boldsymbol{\tau} = 0$, the angular momentum, $\mathbf{L}$, of the planet is a constant of the motion:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = \text{constant}$$

Because $\mathbf{L}$ is a constant of the motion, the planet's motion at any instant is restricted to the plane formed by $\mathbf{r}$ and $\mathbf{v}$.

We can relate this result to the following geometric consideration. In a time of dt, the radius vector $\mathbf{r}$ in Figure 11.9b sweeps out the area dA, which equals one half the area $|\mathbf{r} \times d\mathbf{r}|$ of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r}$. Because the displacement of the planet in the time dt is given by $d\mathbf{r} = \mathbf{v}\,dt$, we get

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\,dt| = \frac{L}{2m}\,dt$$

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant} \qquad [11.5]$$

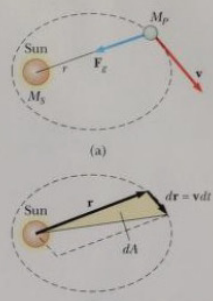
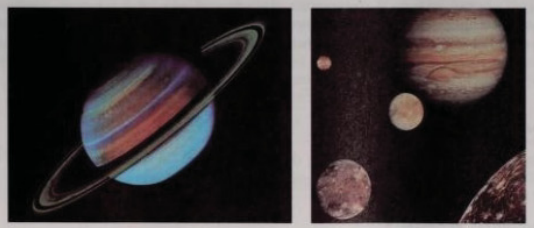


Figure 11.8 (a) The gravitational force acting on a planet acts toward the Sun, along the radius vector. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time dt is equal to one half the area of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r} = \mathbf{v}\,dt$.

• *Kepler's second law*



(Left) This image of Saturn, taken by the *Voyager I* spacecraft on October 18, 1980, was color-enhanced to increase the visibility of large, bright features in Saturn's North Temperate Belt. The distinct color difference between the North Equatorial Belt and other belts may be due to a thicker haze layer covering the northern belt. *(Right)* Jupiter and its four planet-size moons were photographed by *Voyager I* and assembled into this collage. The moons are not to scale but are in their relative positions. Nine other smaller moons circle Jupiter. Not visible is Jupiter's faint ring of particles, seen for the first time by *Voyager I*. *(NASA photos)*

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where L and m are both constants of the motion. Thus, we conclude that

> the radius vector from the Sun to any planet sweeps out equal areas in equal times.

It is important to recognize that this result, which is Kepler's second law, is a consequence of the fact that the force of gravity is a central force, which in turn implies that angular momentum of the planet is constant. Therefore, the law applies to *any* situation that involves a central force, whether inverse square or not.

The inverse-square nature of the force of gravity is not revealed by Kepler's second law. Although we do not prove it here, Kepler's first law (as well as his third law) is a direct consequence of the fact that the gravitational force varies as $1/r^2$. That is, under an inverse-square force law, the orbits of the planets can be shown to be ellipses with the Sun at one focus.

Thinking Physics 2

The Earth is closer to the Sun when it is winter in the Northern Hemisphere than when it is summer. July and January both have 31 days. In which month, if either, does the Earth move through a longer distance in its orbit?

Reasoning The Earth is in a slightly elliptical orbit around the Sun. In order to conserve angular momentum, the Earth must move more rapidly when it is close to the Sun and more slowly when it is farther away. Thus, since it is closer to the Sun in January, it is moving faster, and will cover more distance in its orbit than it will in July.

CONCEPTUAL PROBLEM 1

Kepler's first law indicates that planets travel in elliptical orbits. This is intimately related to the fact that the gravitational force varies as the inverse square of the separation between the Sun and the planet. Suppose the gravitational force varied as the inverse *cube*, rather than the inverse square. This opens up some new types of orbits. What about the second and third laws? Will they change if the force depends on the inverse cube of the separation?

CONCEPTUAL PROBLEM 2

A satellite in orbit is not truly traveling through a vacuum—it is moving through very thin air. Does the resulting air friction cause the satellite to slow down?

CONCEPTUAL PROBLEM 3

How would you explain the fact that Saturn and Jupiter have periods much greater than one year?

EXERCISE 4 At its aphelion the planet Mercury is 6.99×10^{10} m from the Sun, and at its perihelion it is 4.60×10^{10} m from the Sun. If its orbital speed is 3.88×10^4 m/s at the aphelion, what is its orbital speed at the perihelion? Answer 5.90×10^4 m/s

11.4 · ENERGY CONSIDERATIONS IN PLANETARY AND SATELLITE MOTION

Consider a body of mass m moving with a speed of v in the vicinity of a massive body of mass $M \gg m$. The system might be a planet moving around the Sun or a satellite in orbit around the Earth. If we assume that M is at rest in an inertial reference frame,[2] then the total mechanical energy, E, of the two-body system when the bodies are separated by a distance r is the sum of the kinetic energy of the mass m and the potential energy of the system:

$$E = K + U$$

Recall from Chapter 7, Equation 7.22, that the gravitational potential energy, U_g, associated with *any pair* of particles of masses m_1 and m_2 separated by a distance r is given by

$$U_g = -\frac{Gm_1m_2}{r}$$

Therefore, in our case,

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad [11.6]$$

Furthermore, the total mechanical energy is constant if we assume that the system is isolated. Therefore, as the mass m moves from P to Q in Figure 11.9, the total energy remains constant and Equation 11.6 gives

$$E = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GMm}{r_f} \qquad [11.7]$$

Equation 11.6 shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system, such as the Earth and Sun, E is necessarily *less than zero* if we use the arbitrary convention $U \to 0$ as $r \to \infty$. We can easily establish that $E < 0$ for the system consisting of a mass m moving in a circular orbit about a body of mass M. Newton's second law applied to the body of mass m gives

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

Multiplying both sides by r and dividing by 2,

$$\tfrac{1}{2}mv^2 = \frac{GMm}{2r} \qquad [11.8]$$

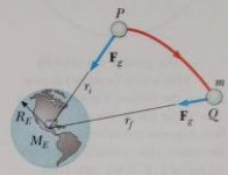


Figure 11.9 As a particle of mass m moves from P to Q above the Earth's surface, the total mechanical energy of the system remains constant.

[2]To see that this is reasonable, consider an object of mass m falling toward the Earth. Because the center of mass of the object–Earth system is stationary, it follows that $mv = M_E v_E$. Thus, the Earth acquires a kinetic energy equal to

$$\tfrac{1}{2}M_E v_E^2 = \tfrac{1}{2}\frac{m^2}{M_E}v^2 = \frac{m}{M_E}K$$

where K is the kinetic energy of the object. Because $M_E \gg m$, the kinetic energy of the Earth is negligible.

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Substituting this into Equation 11.6, we obtain

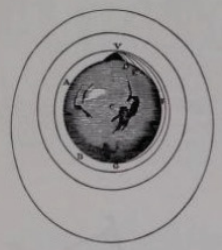
$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total mechanical energy for circular orbits

$$E = -\frac{GMm}{2r} \qquad [11.9]$$

This clearly shows that **the total mechanical energy must be negative in the case of circular orbits.** Furthermore, **the kinetic energy is positive and equal to one half the magnitude of the potential energy** (when the potential energy is chosen to be zero at infinity). The absolute value of E is also equal to the binding energy of the system.

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 11.9, with r replaced by the semimajor axis length, a. The total energy, the total angular momentum, and the total linear momentum of a planet–Sun system are constants of the motion.



"... the greater the velocity ... with which (a stone) is projected, the farther it goes before it falls to the Earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at last, exceeding the limits of the Earth, it should pass into space without touching."—Newton, *System of the World.*

Thinking Physics 3

Icebound, by Dean Koontz (Ballantine Books, New York, 1995), is a story of a group of scientists trapped on a floating iceberg near the North Pole. One of the devices that the scientists have with them on their task is a transmitter with which they can fix their position with "the aid of a *geosynchronous polar satellite.*" Can a satellite in a *polar* orbit be *geosynchronous?*

Reasoning A geosynchronous satellite is one that stays over one location on the Earth's surface at all times. Thus, an antenna receiving signals from the satellite, such as a television dish, can stay pointed in a fixed direction toward the sky. The satellite must be in an orbit with the correct radius such that its orbital period is the same as that of the Earth's rotation. This would result in the satellite appearing to have no east–west motion relative to the observer at the chosen location. Another requirement is that a geosynchronous satellite *must be in orbit over the Equator.* Otherwise it would appear to undergo a north–south oscillation during one orbit. Thus, it would be impossible to have a geosynchronous satellite in a *polar* orbit. Even if such a satellite were at the proper distance from the Earth, it would be moving rapidly in the north–south direction, resulting in the necessity of accurate tracking equipment. What's more, it would be below the horizon for long periods of time, making it useless for determining one's position.

Thinking Physics 4

A satellite is in a circular orbit in space. Is it necessary that there be a massive object at the center of the orbit?

Reasoning This is the usual situation—satellites are normally in orbit around the Earth, a massive object. Imagine, however, tying a rock somewhere along a length of string and twirling the rock in a circle as shown in Figure 11.10. In this case, there is no entity at the center of the circular path of the rock, nor along its radius, that is exerting a force on the rock. The net force holding the rock in the circular orbit is a

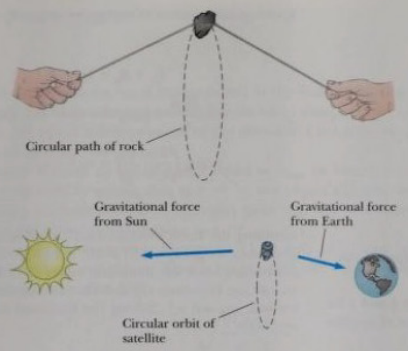


Figure 11.10 (Thinking Physics 4)

combination of the radial components of two forces—one from the string on each side of the rock.

The question posed was not about rocks on strings; it was about satellites. Is there a gravitational counterpart to this situation? There is. Imagine moving from the Earth to the Sun along the line between them. At some point along that line, the gravitational force of attraction to the Earth will balance that toward the Sun. This is similar to the balance along the string of the tension on either side of the rock—the rock is in equilibrium between two balanced forces. Now imagine pulling the satellite perpendicularly off the Earth–Sun line a small amount—as long as the distance is relatively small, the components of the forces from the Earth and Sun along the Sun–Earth centerline will cancel. The components of the two forces perpendicular to the centerline will add, to provide a net force pulling the satellite back toward the centerline. Now, suppose we set the satellite into circular motion with just the right velocity. The satellite will be in orbit around a center point with no mass located there.

The International Sun–Earth Explorer 3 satellite (ISEE-3) was launched in 1978 and placed in just such an orbit in order to study the solar wind. The advantage over a Sun-centered orbit is that the satellite always resides on the side of the Sun toward the Earth, so that it is constantly in communication with Earth. In the absence of a specific mission to Comet Halley, the ISEE-3 satellite was removed from this orbit in 1982 and placed in orbit around the Moon, in anticipation of a flyby of Comet Halley. On the way to this flyby, it passed through the tail of Comet Giacobini-Zinner and successfully gathered important data on both comets.

Example 11.4 A Satellite in an Elliptical Orbit

A satellite moves in an elliptical orbit about the Earth, as in Figure 11.11. The minimum and maximum distances from the surface of the Earth are 400 km and 3000 km. Find the speeds of the satellite at apogee and perigee.

Solution Because the mass of the satellite is negligible compared with the Earth's mass, we take the center of mass of the Earth to be at rest. Gravity is a *central* force, and so the angular momentum of the satellite about the Earth's center of

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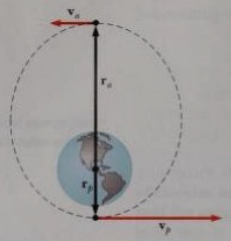


Figure 11.11 (Example 11.4) A satellite in an elliptical orbit about the Earth.

mass remains constant in time. With subscripts a and p for the apogee and perigee positions, conservation of angular momentum gives $L_p = L_a$, or

$$mv_p r_p = mv_a r_a$$
$$v_p r_p = v_a r_a \qquad [1]$$

Applying conservation of energy, we obtain $E_p = E_a$, or

$$U_p + K_p = U_a + K_a$$
$$-G\frac{M_E m}{r_p} + \tfrac{1}{2}mv_p^2 = -G\frac{M_E m}{r_a} + \tfrac{1}{2}mv_a^2$$
$$2GM_E\left(\frac{1}{r_a} - \frac{1}{r_p}\right) = (v_a^2 - v_p^2) \qquad [2]$$

Assuming the Earth's radius is 6.37×10^6 m and using the given data, we get $r_a = 9.37 \times 10^6$ m and $r_p = 6.77 \times 10^6$ m. Because we know the numerical values of G, M_E, r_p, and r_a, we can use Equations (1) and (2) to determine the two unknowns, (v_p and v_a). Solving the equations simultaneously, we obtain

$$v_p = 8.27 \text{ km/s} \qquad v_a = 5.98 \text{ km/s}$$

Escape Speed

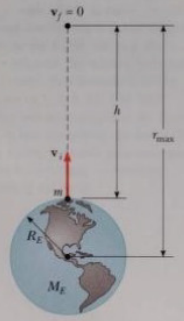


Figure 11.12 An object of mass m projected upward from the Earth's surface with an initial velocity $\mathbf{v}_i$ reaches a maximum altitude h.

Suppose an object of mass m is projected vertically from the Earth's surface with an initial speed v_i, as in Figure 11.12. We can use energy considerations to find the minimum value of the initial speed such that the object will escape the Earth's gravitational field. Equation 11.6 gives the total energy of the object at any point when its speed and distance from the center of the Earth are known. At the surface of the Earth, where $v_i = v$, $r_i = R_E$. When the object reaches its maximum altitude, $v_f = 0$ and $r_f = r_{max}$. Because the total energy of the system is constant, substitution of these conditions into Equation 11.7 gives

$$\tfrac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{max}}$$

Solving for v_i^2 gives

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right) \qquad [11.10]$$

Therefore, if the initial speed is known, this expression can be used to calculate the maximum altitude, h, because we know that $h = r_{max} - R_E$.

We are now in a position to calculate the minimum speed the object must have at the Earth's surface in order to escape from the influence of the Earth's gravitational field. This corresponds to the situation in which the object will continue to move away forever, with the speed asymptotically approaching *zero*. Setting $r_{max} = \infty$ in Equation 11.10 and taking $v_i = v_{esc}$ (the escape speed), we get

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} \qquad [11.11]$$

• *Escape speed*

Note that this expression for v_{esc} is independent of the mass of the object projected from the Earth. For example, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the *direction* of the velocity, provided the object is not thrown into the ground.

If the object is given an initial speed equal to v_{esc}, its *total* energy is equal to zero. This can be seen by noting that as $r \to \infty$, the object's kinetic energy and its potential energy are both zero. If v_i is greater than v_{esc}, the *total* energy is greater than zero and the object is still moving as $r \to \infty$.

Finally, you should note that Equations 11.10 and 11.11 can be applied to objects projected from *any* planet. That is, in general, the escape speed from any planet of mass M and radius R is

$$v_{esc} = \sqrt{\frac{2GM}{R}} \qquad [11.12]$$

A list of escape speeds for the planets, the Moon, and the Sun is given in Table 11.3. Note that the values vary from 1.1 km/s for Pluto to about 618 km/s for the Sun. These results, together with some ideas from the kinetic theory of gases (Chapter 16), explain why some planets have atmospheres and others do not. As we shall see later, a gas molecule has an average kinetic energy that depends on its temperature. Lighter atoms, such as hydrogen and helium, have higher average speeds than do the heavier species. When the average speed of the lighter atoms is not much less than the escape speed, a significant fraction of the molecules have a chance to escape from the planet. This mechanism also explains why the Earth does not retain hydrogen and helium molecules in its atmosphere but does retain much heavier molecules, such as oxygen and nitrogen. By contrast, Jupiter has a very large escape speed (60 km/s), which enables it to retain hydrogen, the primary component of its atmosphere.

TABLE 11.3 Escape Speeds from the Surfaces of the Planets, the Moon, and the Sun

Planet	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Mars	5.0
Jupiter	60.0
Saturn	36.0
Uranus	22.0
Neptune	24.0
Pluto	1.1
Moon	2.3
Sun	618.0

Black Holes

In Chapter 10 we briefly described a rare event called a supernova—the catastrophic explosion of a very massive star. The material that remains in the central core of such an object continues to collapse, but the core's ultimate fate depends on its mass. If the core has a mass less than 1.4 times the mass of our Sun, it gradually cools down and ends its life as a white dwarf star. However, if the core's mass is greater than this, the star may collapse further due to gravitational forces. What remains is a neutron star, a region compressed to a radius of about 10 km. (On Earth, a teaspoon of this material would weigh about 5 billion tons!)

An even more unusual star death may occur when the core has a mass greater than about three solar masses. No known forces in nature are strong enough to prevent the collapse of such a star. The collapse may continue until the star becomes a small object in space, commonly referred to as a **black hole.** In effect, black holes are remains of stars that have collapsed under their own weight. Once an object such as a spaceship comes near a black hole, it experiences an extremely strong gravitational force and is trapped forever (Fig. 11.13).

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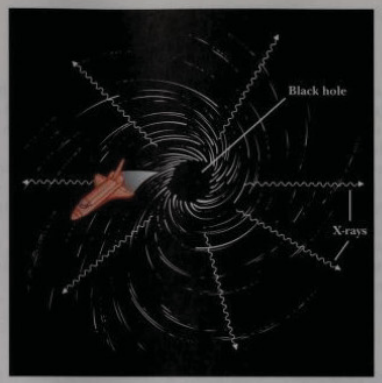


Figure 11.13 The gravitational field in the vicinity of a black hole is so strong that nothing can escape. Any object moving close to a black hole can emit x-rays.

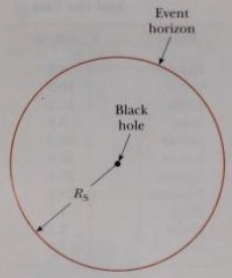


Figure 11.14 A black hole. The distance R_S equals the Schwarzschild radius. Any event occurring within the boundary of radius R_S, called the event horizon, is invisible to an outside observer.

The escape speed from any spherical body can be calculated from Equation 11.12. If the escape speed exceeds the speed of light, c, radiation within the body (such as visible light) cannot escape and the body appears to be black; hence the origin of the terminology "black hole." The critical radius, R_S, for which this occurs is called the **Schwarzschild radius** (Fig. 11.14). Taking $v_{esc} = c$ in Equation 11.12 and solving for R_S, we obtain $R_S = 2GM/c^2$. For example, the value for R_S for a black hole with a mass equal to that of the Sun is calculated to be 3.0 km (about 2 mi); a black hole the mass of which equals that of the Earth has a radius of about 9 mm (about the size of a dime).

Although light from a black hole cannot escape, light from events taking place near the black hole should be visible. A companion star captured by the strong gravitational field of a black hole should emit x-rays (Fig. 11.13). Based on this reasoning, several candidates for black holes have been detected, the most famous being Cygnus X-1, the first x-ray source detected in the constellation Cygnus. There is also evidence that supermassive black holes exist at the centers of galaxies.

EXERCISE 5 The escape speed from the surface of the Earth is 11.2 km/s. Estimate the escape speed for a spacecraft from the surface of the Moon. The Moon has a mass $\frac{1}{81}$ that of Earth and a radius $\frac{1}{4}$ that of Earth. Answer 2.49 km/s

EXERCISE 6 (a) Calculate the minimum energy required to send a 3000-kg spacecraft from the Earth to a distant point in space where Earth's gravity is negligible. (b) If the journey is to take three weeks, what *average* power will the engines have to supply?
Answer (a) 1.88×10^{11} J (b) 103 kW

EXERCISE 7 A satellite moves in an elliptical orbit about the Earth such that, at perigee and apogee positions, the distances from the Earth's center are, respectively, D and $4D$. Find the ratio of the speeds at the two positions, v_p/v_a. Answer 4

11.5 · ATOMIC SPECTRA AND THE BOHR THEORY OF HYDROGEN

As you may have already learned in chemistry, the hydrogen atom is the simplest known atomic system and an especially important one to understand. Much of what is learned about the hydrogen atom (which consists of one proton and one electron) can be extended to other single-electron ions such as He^+ and Li^{2+}. Furthermore, a thorough understanding of the physics underlying the hydrogen atom can then be used to describe more complex atoms and the periodic table of the elements.

Suppose an evacuated glass tube is filled with hydrogen (or some other gas). If a voltage applied between metal electrodes in the tube is great enough to produce an electric current in the gas, the tube emits light the color of which is characteristic of the gas (this is how a neon sign works). When the emitted light is analyzed with a device called a spectroscope, a series of discrete lines is observed, each line corresponding to a different wavelength, or color, of light. Such a series of spectral lines is commonly referred to as an **emission spectrum.** The wavelengths contained in a given line spectrum are characteristic of the element emitting the light (Fig. 11.15). Because no two elements emit the same line spectrum, this phenomenon represents a marvelous and reliable technique for identifying elements in a substance.

As you shall learn in more detail in Chapter 13, a mechanical wave is a disturbance that transports energy as it moves through a system without transporting matter. A common form of periodic wave is the sinusoidal wave, the shape of which

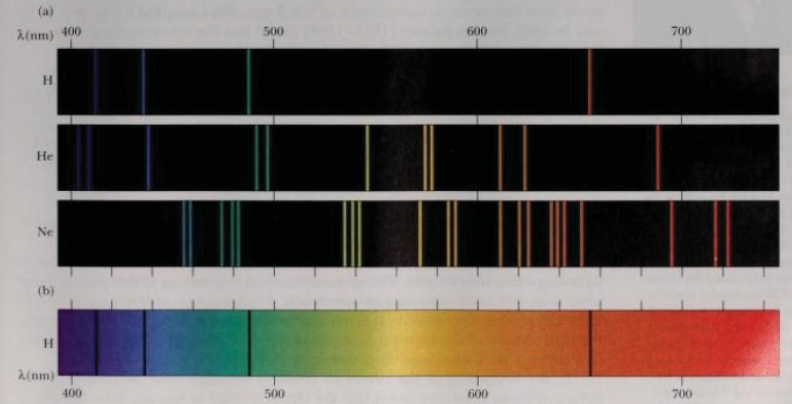


Figure 11.15 Visible spectra. (a) Line spectra produced by emission in the visible range for the elements hydrogen, helium, and neon. (b) The absorption spectrum for hydrogen. The dark absorption lines occur at the same wavelengths as the emission line for hydrogen shown in (a).

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