**A–4–9.** Figure 4–36 shows a hydraulic jet-pipe controller. Hydraulic fluid is ejected from the jet pipe. If the jet pipe is shifted to the right from the neutral position, the power piston moves to the left, and vice versa. The jet pipe valve is not used as much as the flapper valve because of large null flow, slower response, and rather unpredictable characteristics. Its main advantage lies in its insensitivity to dirty fluids.

Suppose that the power piston is connected to a light load so that the inertia force of the load element is negligible compared to the hydraulic force developed by the power piston. What type of control action does this controller produce?

**Solution.** Define the displacement of the jet nozzle from the neutral position as $x$ and the displacement of the power piston as $y$. If the jet nozzle is moved to the right by a small displacement $x$, the oil flows to the right side of the power piston, and the oil in the left side of the power piston is returned to the drain. The oil flowing into the power cylinder is at high pressure; the oil flowing out from the power cylinder into the drain is at low pressure. The resulting pressure difference causes the power piston to move to the left.

For a small jet nozzle displacement $x$, the flow rate $q$ to the power cylinder is proportional to $x$; that is,

$$q = K_1 x$$

For the power cylinder,

$$A\rho\, dy = q\, dt$$

where $A$ is the power piston area and $\rho$ is the density of oil. Hence

$$\frac{dy}{dt} = \frac{q}{A\rho} = \frac{K_1}{A\rho} x = Kx$$

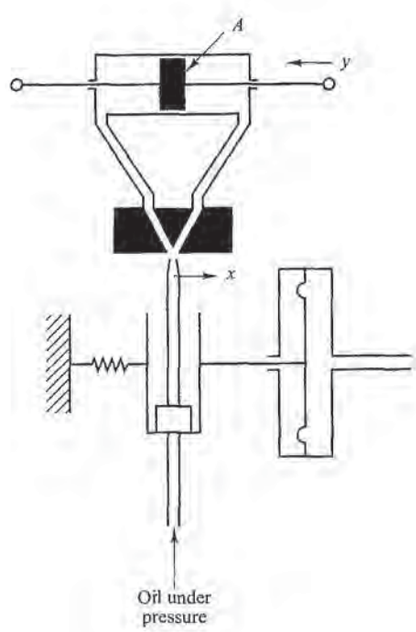


**Figure 4–36**
Hydraulic jet-pipe controller.





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where $K = K_1/(A\rho) =$ constant. The transfer function $Y(s)/X(s)$ is thus

$$\frac{Y(s)}{X(s)} = \frac{K}{s}$$

The controller produces the integral control action.

**A–4–10.** Figure 4–37 shows a hydraulic jet-pipe applied to a flow control system. The jet-pipe controller governs the position of the butterfly valve. Discuss the operation of this system. Plot a possible curve relating the displacement $x$ of the nozzle to the total force $F$ acting on the power piston.

**Solution.** The operation of this system is as follows: The flow rate is measured by the orifice, and the pressure difference produced by this orifice is transmitted to the diaphragm of the pressure measuring device. The diaphragm is connected to the free swinging nozzle, or jet pipe, through a linkage. High-pressure oil ejects from the nozzle all the time. When the nozzle is at a neutral position, no oil flows through either of the pipes to move the power piston. If the nozzle is displaced by the motion of the balance arm to one side, the high-pressure oil flows through the corresponding pipe, and the oil in the power cylinder flows back to the sump through the other pipe.

Assume that the system is initially at rest. If the reference input is changed suddenly to a higher flow rate, then the nozzle is moved in such a direction as to move the power piston and open the butterfly valve. Then the flow rate will increase, the pressure difference across the orifice becomes larger, and the nozzle will move back to the neutral position. The movement of the power piston stops when $x$, the displacement of the nozzle, comes back to and stays at the neutral position. (The jet pipe controller thus possesses an integrating property.)

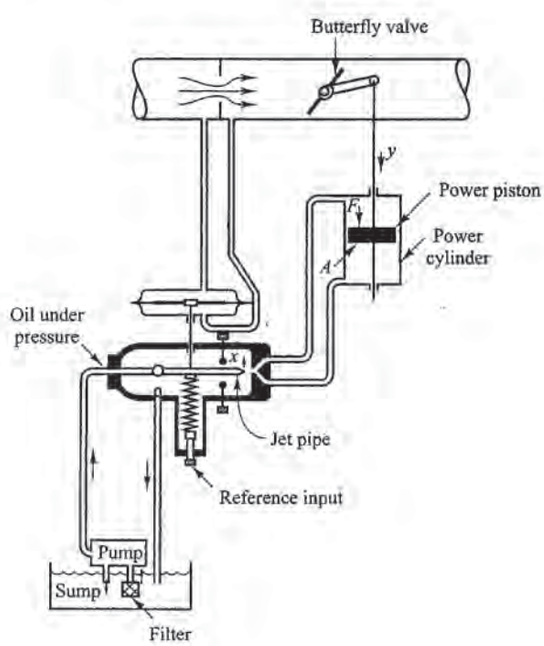


**Figure 4–37** Schematic diagram of a flow control system using a hydraulic jet-pipe controller.





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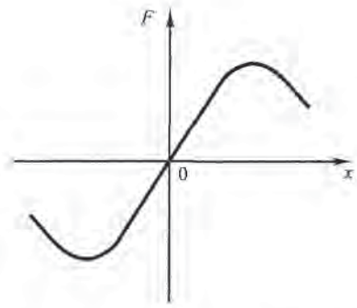


**Figure 4–38**
Force versus diplacement curve.

The relationship between the total force $F$ acting on the power piston and the displacement $x$ of the nozzle is shown in Figure 4–38. The total force is equal to the pressure difference $\Delta P$ across the piston times the area $A$ of the power piston. For a small displacement $x$ of the nozzle, the total force $F$ and displacement $x$ may be considered proportional.

**A–4–11.** Explain the operation of the speed control system shown in Figure 4–39.

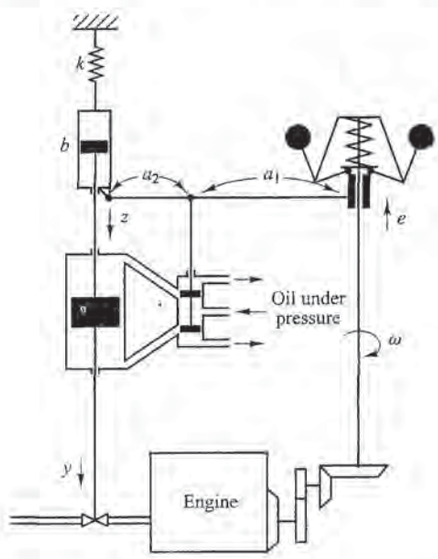


**Figure 4–39**
Speed control system.

**Solution.** If the engine speed increases, the sleeve of the fly-ball governor moves upward. This movement acts as the input to the hydraulic controller. A positive error signal (upward motion of the sleeve) causes the power piston to move downward, reduces the fuel-valve opening, and decreases the engine speed. A block diagram for the system is shown in Figure 4–40.

From the block diagram the transfer function $Y(s)/E(s)$ can be obtained as

$$\frac{Y(s)}{E(s)} = \frac{a_2}{a_1 + a_2} \frac{\dfrac{K}{s}}{1 + \dfrac{a_1}{a_1 + a_2} \dfrac{bs}{bs + k} \dfrac{K}{s}}$$





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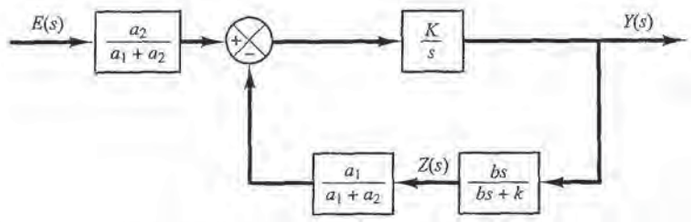


**Figure 4–40**
Block diagram for the speed control system shown in Figure 4–39.

If the following condition applies,

$$\left|\frac{a_1}{a_1 + a_2}\frac{bs}{bs + k}\frac{K}{s}\right| \gg 1$$

the transfer function $Y(s)/E(s)$ becomes

$$\frac{Y(s)}{E(s)} \doteq \frac{a_2}{a_1 + a_2}\frac{a_1 + a_2}{a_1}\frac{bs + k}{bs} = \frac{a_2}{a_1}\left(1 + \frac{k}{bs}\right)$$

The speed controller has proportional-plus-integral control action.

**A–4–12.** Derive the transfer function $Z(s)/Y(s)$ of the hydraulic system shown in Figure 4–41. Assume that the two dashpots in the system are identical ones.

**Solution.** In deriving the equations for the system, we assume that force $F$ is applied at the right end of the shaft causing displacement $y$. (All displacements $y$, $w$, and $z$ are measured from respective equilibrium positions when no force is applied at the right end of the shaft.) When force $F$ is applied, pressure $P_1$ becomes higher than pressure $P_1'$, or $P_1 > P_1'$. Similarly, $P_2 > P_2'$.

For the force balance, we have the following equation:

$$k_2(y - w) = A(P_1 - P_1') + A(P_2 - P_2') \tag{4–48}$$

Since

$$k_1 z = A(P_1 - P_1') \tag{4–49}$$

and

$$q_1 = \frac{P_1 - P_1'}{R}$$

we have

$$k_1 z = ARq_1$$

Also, since

$$q_1\,dt = A(dw - dz)\rho$$

we have

$$q_1 = A(\dot{w} - \dot{z})\rho$$

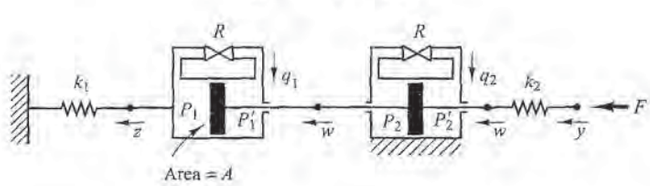


**Figure 4–41**
Hydraulic system.





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or

$$\dot{w} - \dot{z} = \frac{k_1 z}{A^2 R\rho}$$

Define $A^2R\rho = B$. ($B$ is the viscous friction coefficient.) Then

$$\dot{w} - \dot{z} = \frac{k_1}{B} z \tag{4–50}$$

Also, for the right-hand side dashpot we have

$$q_2\, dt = A\, dw\rho$$

Hence

$$\dot{w} = \frac{q_2}{A\rho} = \frac{A(P_2 - P_2')}{A^2 R\rho}$$

or

$$A(P_2 - P_2') = B\dot{w} \tag{4–51}$$

Substituting Equations (4–49) and (4–51) into Equation (4–48) we have

$$k_2 y - k_2 w = k_1 z + B\dot{w}$$

Taking the Laplace transform of this last equation, assuming zero initial condition, we obtain

$$k_2 Y(s) = (k_2 + Bs)W(s) + k_1 Z(s) \tag{4–52}$$

Taking the Laplace transform of Equation (4–50), assuming zero initial condition, we obtain

$$W(s) = \frac{k_1 + Bs}{Bs} Z(s) \tag{4–53}$$

By using Equation (4–53) to eliminate $W(s)$ from Equation (4–52), we obtain

$$k_2 Y(s) = (k_2 + Bs)\frac{k_1 + Bs}{Bs} Z(s) + k_1 Z(s)$$

from which we obtain the transfer function $Z(s)/Y(s)$ to be

$$\frac{Z(s)}{Y(s)} = \frac{k_2 s}{Bs^2 + (2k_1 + k_2)s + \dfrac{k_1 k_2}{B}}$$

Multiplying $B/(k_1 k_2)$ to both the numerator and denominator of this last equation, we get

$$\frac{Z(s)}{Y(s)} = \frac{\dfrac{B}{k_1} s}{\dfrac{B^2}{k_1 k_2} s^2 + \left(\dfrac{2B}{k_2} + \dfrac{B}{k_1}\right)s + 1}$$

Define $B/k_1 = T_1$, $B/k_2 = T_2$. Then the transfer function $Z(s)/Y(s)$ becomes as follows:

$$\frac{Z(s)}{Y(s)} = \frac{T_1 s}{T_1 T_2 s^2 + (T_1 + 2T_2)s + 1}$$





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**A–4–13.** Consider the hydraulic servo system shown in Figure 4–42. Assuming that the load reaction forces are not negligible, derive a mathematical model of the system. Assume also that the mass of the power piston is included in the load mass $m$.

**Solution.** In deriving a mathematical model of the system when the load reactive forces are not negligible, such effects as the pressure drop across the orifice, the leakage of oil around the valve and around the piston, and the compressibility of the oil must be considered.

The pressure drop across the orifice is a function of the supply pressure $p_s$ and the pressure difference $\Delta p = p_1 - p_2$. Thus the flow rate $q$ is a nonlinear function of valve displacement $x$ and pressure difference $\Delta p$ or

$$q = f(x, \Delta p)$$

Linearizing this nonlinear equation about the origin ($x = 0, \Delta p = 0, q = 0$), we obtain, referring to Equation (4–27),

$$q = K_1 x - K_2 \Delta p \tag{4–54}$$

The flow rate $q$ can be considered as consisting of three parts

$$q = q_0 + q_L + q_C \tag{4–55}$$

where $q_0$ = useful flow rate to the power cylinder causing power piston to move, kg/sec

$q_L$ = leakage flow rate, kg/sec

$q_C$ = equivalent compressibility flow rate, kg/sec

Let us obtain specific expressions for $q_0$, $q_L$, and $q_C$. The flow $q_0\,dt$ to the left-hand side of the power piston causes the piston to move to the right by $dy$. So we have

$$A\rho\, dy = q_0\, dt$$

where $A$ (m$^2$) is the power piston area, $\rho$ (kg/m$^3$) the density of oil, and $dy$ (m) the displacement of the power piston. Then

$$q_0 = A\rho \frac{dy}{dt} \tag{4–56}$$

The leakage component $q_L$ can be written

$$q_L = L\,\Delta p \tag{4–57}$$

where $L$ is the leakage coefficient of the system.

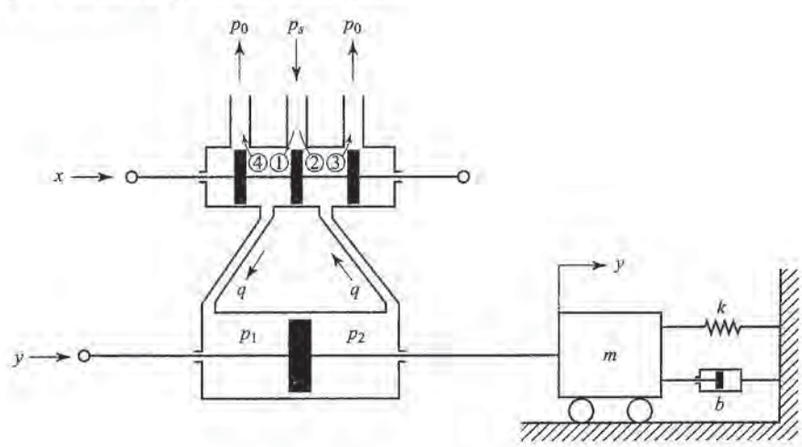


**Figure 4–42**
Hydraulic servo system.





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The equivalent compressibility flow rate $q_C$ can be expressed in terms of the effective bulk modulus $K$ of oil (including the effects of entrapped air, expansion of pipes, etc.), where

$$K = \frac{d\,\Delta p}{-dV/V}$$

(Here $dV$ is negative and so $-dV$ is positive.) Rewriting this last equation gives

$$-dV = \frac{V}{K}\,d\,\Delta p$$

or

$$\rho\frac{-dV}{dt} = \frac{\rho V}{K}\frac{d\,\Delta p}{dt}$$

Noting that $q_C = \rho(-dV)/dt$, we find

$$q_C = \frac{\rho V}{K}\frac{d\,\Delta p}{dt} \tag{4–58}$$

where $V$ is the effective volume of oil under compression (that is, approximately half the total power cylinder volume).

Using Equations (4–54) through (4–58),

$$\begin{aligned} q &= K_1 x - K_2\,\Delta p \\ &= q_0 + q_L + q_C \\ &= A\rho\frac{dy}{dt} + L\,\Delta p + \frac{\rho V}{K}\frac{d\,\Delta p}{dt} \end{aligned}$$

or

$$A\rho\frac{dy}{dt} + \frac{\rho V}{K}\frac{d\,\Delta p}{dt} + (L + K_2)\,\Delta p = K_1 x \tag{4–59}$$

The force developed by the power piston is $A\,\Delta p$, and this force is applied to the load elements. Thus

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = A\,\Delta p \tag{4–60}$$

Eliminating $\Delta p$ from Equations (4–59) and (4–60) results in

$$\begin{aligned} &\frac{\rho V m}{KA}\frac{d^3y}{dt^3} + \left[\frac{\rho V b}{KA} + \frac{(L + K_2)m}{A}\right]\frac{d^2y}{dt^2} \\ &\quad + \left[A\rho + \frac{\rho V k}{KA} + \frac{(L + K_2)b}{A}\right]\frac{dy}{dt} + \frac{(L + K_2)k}{A}\,y = K_1 x \end{aligned}$$

This is a mathematical model of the system relating the valve spool displacement $x$ and the power piston displacement $y$ when the load reactive forces are not negligible.





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**A–4–14.** Considering small deviations from steady-state operation, draw a block diagram of the air heating system shown in Figure 4–43. Assume that the heat loss to the surroundings and the heat capacitance of the metal parts of the heater are negligible.

**Solution.** Let us define

$\bar{\Theta}_i$ = steady-state temperature of inlet air, °C

$\bar{\Theta}_o$ = steady-state temperature of outlet air, °C

$G$ = mass flow rate of air through the heating chamber, kg/sec

$M$ = mass of air contained in the heating chamber, kg

$c$ = specific heat of air, kcal/kg °C

$R$ = thermal resistance, °C sec/kcal

$C$ = thermal capacitance of air contained in the heating chamber = $Mc$, kcal/°C

$\bar{H}$ = steady-state heat input, kcal/sec

Let us assume that the heat input is suddenly changed from $\bar{H}$ to $\bar{H} + h$ and the inlet air temperature is suddenly changed from $\bar{\Theta}_i$ to $\bar{\Theta}_i + \theta_i$. Then the outlet air temperature will be changed from $\bar{\Theta}_o$ to $\bar{\Theta}_o + \theta_o$.

The equation describing the system behavior is

$$C\,d\theta_o = [h + Gc(\theta_i - \theta_o)]\,dt$$

or

$$C\frac{d\theta_o}{dt} = h + Gc(\theta_i - \theta_o)$$

Noting that

$$Gc = \frac{1}{R}$$

we obtain

$$C\frac{d\theta_o}{dt} = h + \frac{1}{R}(\theta_i - \theta_o)$$

or

$$RC\frac{d\theta_o}{dt} + \theta_o = Rh + \theta_i$$

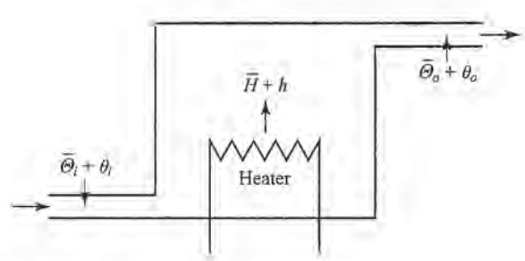


**Figure 4–43**
Air heating system.





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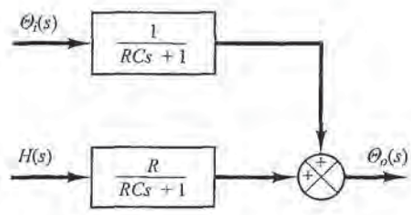


**Figure 4–44**
Block diagram of the air heating system shown in Figure 4–43.

Taking the Laplace transforms of both sides of this last equation and substituting the initial condition that $\theta_0(0) = 0$, we obtain

$$\Theta_o(s) = \frac{R}{RCs + 1} H(s) + \frac{1}{RCs + 1} \Theta_i(s)$$

The block diagram of the system corresponding to this equation is shown in Figure 4–44.

**A–4–15.** Consider the thin, glass-wall, mercury thermometer system shown in Figure 4–45. Assume that the thermometer is at a uniform temperature $\bar{\Theta}$ (ambient temperature) and that at $t = 0$ it is immersed in a bath of temperature $\bar{\Theta} + \theta_b$, where $\theta_b$ is the bath temperature (which may be constant or changing) measured from the ambient temperature $\bar{\Theta}$. Define the instantaneous thermometer temperature by $\bar{\Theta} + \theta$ so that $\theta$ is the change in the thermometer temperature satisfying the condition that $\theta(0) = 0$. Obtain a mathematical model for the system. Also obtain an electrical analog of the thermometer system.

**Solution.** A mathematical model for the system can be derived by considering heat balance as follows: The heat entering the thermometer during $dt$ sec is $q\,dt$, where $q$ is the heat flow rate to the thermometer. This heat is stored in the thermal capacitance $C$ of the thermometer, thereby raising its temperature by $d\theta$. Thus the heat-balance equation is

$$C\,d\theta = q\,dt \tag{4–61}$$

Since thermal resistance $R$ may be written as

$$R = \frac{d(\Delta\theta)}{dq} = \frac{\Delta\theta}{q}$$

heat flow rate $q$ may be given, in terms of thermal resistance $R$, as

$$q = \frac{(\bar{\Theta} + \theta_b) - (\bar{\Theta} + \theta)}{R} = \frac{\theta_b - \theta}{R}$$

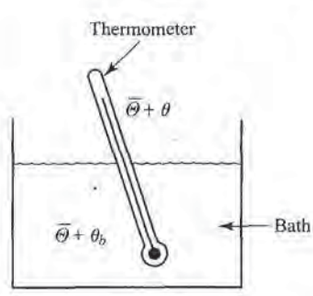


**Figure 4–45**
Thin, glass-wall, mercury thermometer system.





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where $\bar{\Theta} + \theta_b$ is the bath temperature and $\bar{\Theta} + \theta$ is the thermometer temperature. Hence, we can rewrite Equation (4–61) as

$$C\frac{d\theta}{dt} = \frac{\theta_b - \theta}{R}$$

or

$$RC\frac{d\theta}{dt} + \theta = \theta_b \tag{4–62}$$

Equation (4–62) is a mathematical model of the thermometer system.

Referring to Equation (4–62), an electrical analog for the thermometer system can be written as

$$RC\frac{de_o}{dt} + e_o = e_i$$

An electrical circuit represented by this last equation is shown in Figure 4–46.

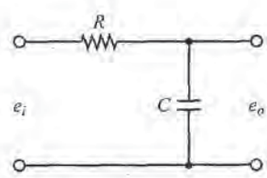


**Figure 4–46**
Electrical analog of the thermometer system shown in Figure 4–45.

## PROBLEMS

**B–4–1.** Consider the liquid-level system shown in Figure 4–47. Assuming that $\bar{H} = 3$ m, $\bar{Q} = 0.02$ m$^3$/sec, and the cross-sectional area of the tank is equal to 5 m$^2$, obtain the time constant of the system at the operating point $(\bar{H}, \bar{Q})$. Assume that the flow through the valve is turbulent.

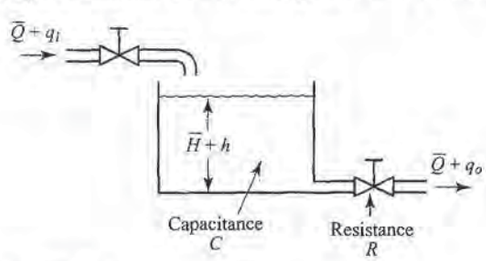


**Figure 4–47**
Liquid-level system.

**B–4–2.** Consider the conical water tank system shown in Figure 4–48; The flow through the valve is turbulent and is related to the head $H$ by

$$Q = 0.005\sqrt{H}$$

where $Q$ is the flow rate measured in m$^3$/sec and $H$ is in meters.

Suppose that the head is 2 m at $t = 0$. What will be the head at $t = 60$ sec?

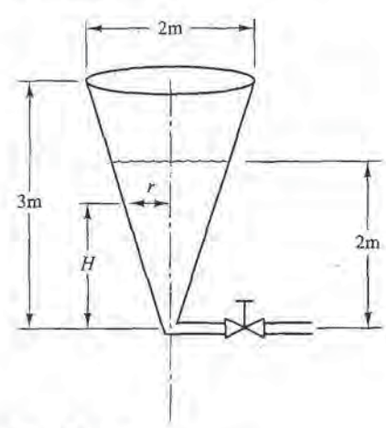


**Figure 4–48**
Conical water tank system.





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**B-4-3.** Consider the liquid-level system shown in Figure 4-49. At steady state the inflow rate is $\bar{Q}$ and the outflow rate is also $\bar{Q}$. Assume that at $t = 0$ the inflow rate is changed from $\bar{Q}$ to $\bar{Q} + q_i$, where $q_i$ is a small quantity. The disturbance input is $q_d$, which is also a small quantity. Draw a block diagram of the system and simplify it to obtain $H_2(s)$ as a function of $Q_i(s)$ and $Q_d(s)$, where $H_2(s) = \mathscr{L}[h_2(t)]$, $Q_i(s) = \mathscr{L}[q_i(t)]$, and $Q_d(s) = \mathscr{L}[q_d(t)]$. The, capacitances of tanks 1 and 2 are $C_1$ and $C_2$, respectively.

**B-4-4.** Consider the liquid-level control system shown in Figure 4-50. The controller is of the proportional type. The set point of the controller is fixed.

Draw a block diagram of the system, assuming that changes in the variables are small. Obtain the transfer function between the level of the second tank and the disturbance input $q_d$. Obtain the steady-state error when the disturbance $q_d$ is a unit-step function.

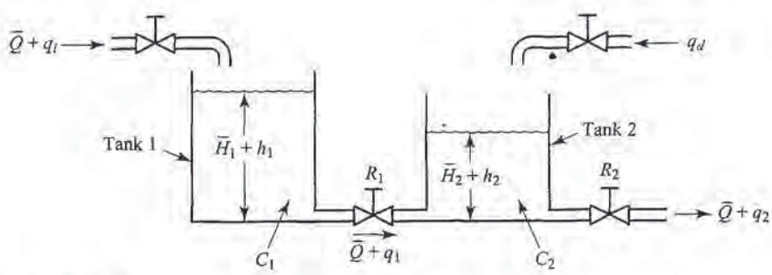


**Figure 4-49**
Liquid-level system.

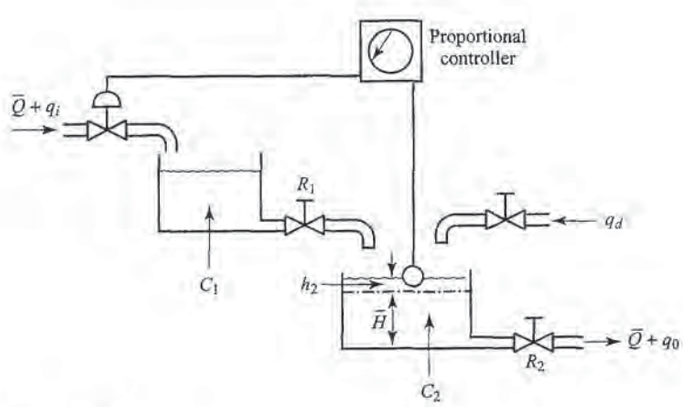


**Figure 4-50**
Liquid-level control system.





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**B–4–5.** For the pneumatic system shown in Figure 4–51, assume that steady-state values of the air pressure and the displacement of the bellows are $\bar{P}$ and $\bar{X}$, respectively. Assume also that the input pressure is changed from $\bar{P}$ to $\bar{P} + p_i$, where $p_i$ is a small change in the input pressure. This change will cause the displacement of the bellows to change a small amount $x$. Assuming that the capacitance of the bellows is $C$ and the resistance of the valve is $R$, obtain the transfer function relating $x$ and $p_i$.

**B–4–6.** Figure 4–52 shows a pneumatic controller. The pneumatic relay has the characteristic that $p_c = Kp_b$, where $K > 0$. What kind of control action does this controller produce? Derive the transfer function $P_c(s)/E(s)$.

**B–4–7.** Consider the pneumatic controller shown in Figure 4–53. Assuming that the pneumatic relay has the characteristics that $p_c = Kp_b$ (where $K > 0$), determine the control action of this controller. The input to the controller is $e$ and the output is $p_c$.

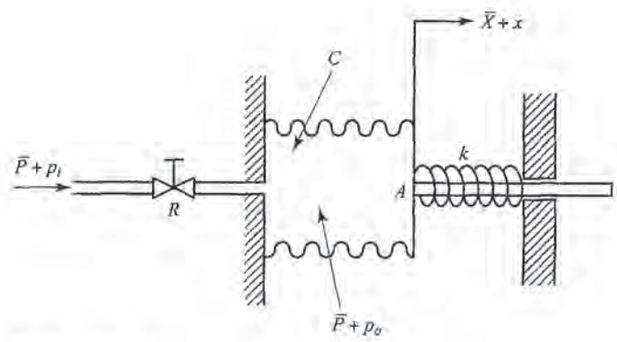


**Figure 4–51**
Pneumatic system.

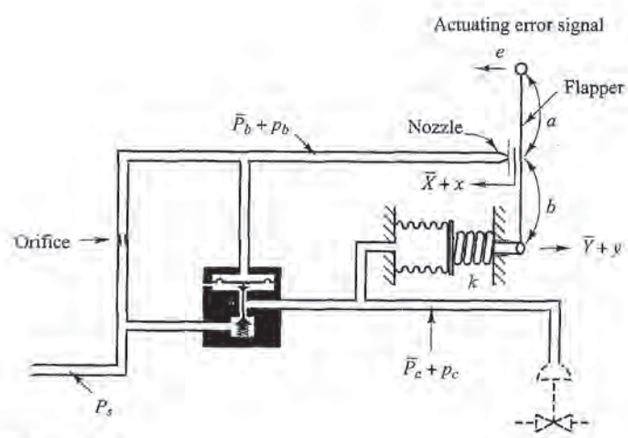


**Figure 4–52**
Pneumatic controller.





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**B–4–8.** Figure 4–54 shows a pneumatic controller. The signal $e$ is the input and the change in the control pressure $p_c$ is the output. Obtain the transfer function $P_c(s)/E(s)$. Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

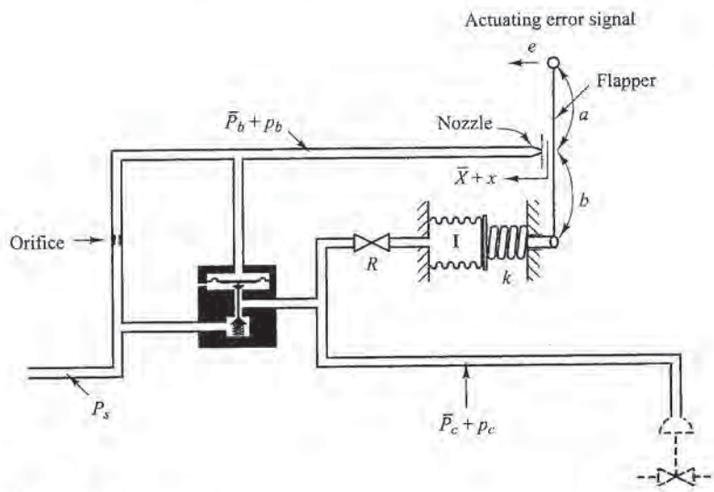


**Figure 4–53**
Pneumatic controller.

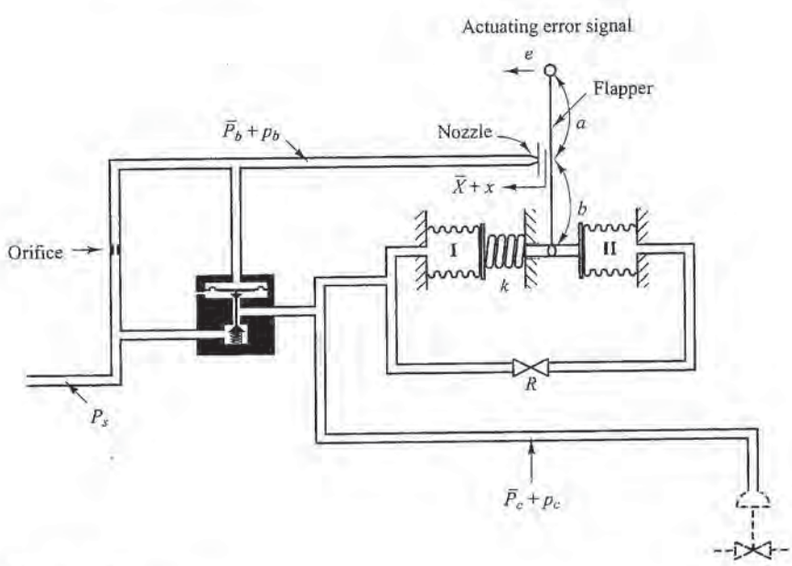


**Figure 4–54**
Pneumatic controller.





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**B–4–9.** Consider the pneumatic controller shown in Figure 4–55. What control action does this controller produce? Assume that the pneumatic relay has the characteristics that $p_c = Kp_b$, where $K > 0$.

**B–4–10.** Figure 4–56 shows a flapper valve. It is placed between two opposing nozzles If the flapper is moved slightly to the right, the pressure unbalance occurs in the nozzles and the power piston moves to the left, and vice versa. Such a device is frequently used in hydraulic servos as the first-stage valve in two-stage servovalves. This usage occurs because considerable force may be needed to stroke larger spool valves that result from the steady-state flow force. To reduce or compensate this force, two-stage valve configuration is often employed; a flapper valve or jet pipe is used as the first-stage valve to provide a necessary force to stroke the second-stage spool valve.

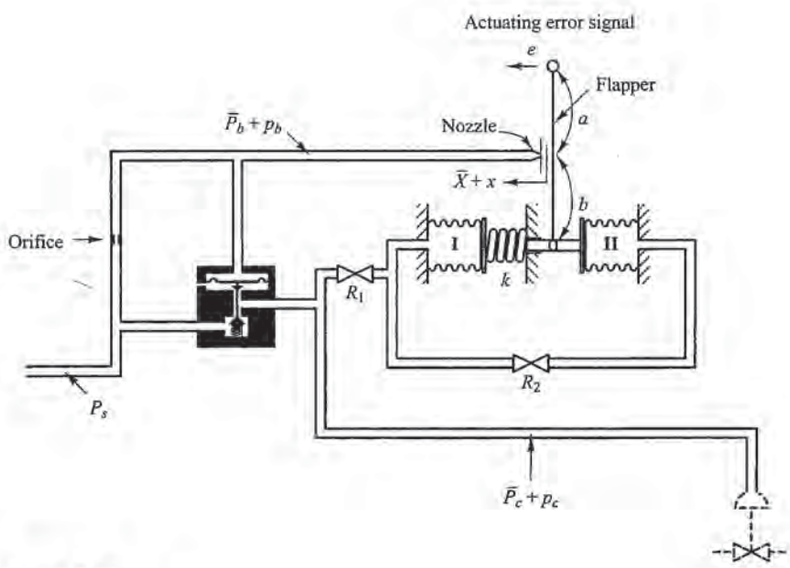


**Figure 4–55**
Pneumatic controller.

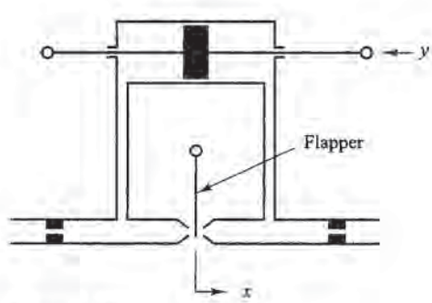


**Figure 4–56**
Flapper valve.





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Figure 4–57 shows a schematic diagram of a hydraulic servomotor in which the error signal is amplified in two stages using a jet pipe and a pilot valve. Draw a block diagram of the system of Figure 4–57 and then find the transfer function between $y$ and $x$, where $x$ is the air pressure and $y$ is the displacement of the power piston.

**B–4–11.** Figure 4–58 is a schematic diagram of an aircraft elevator control system. The input to the system is the deflection angle $\theta$ of the control lever, and the output is the elevator angle $\phi$. Assume that angles $\theta$ and $\phi$ are relatively small. Show that for each angle $\theta$ of the control lever there is a corresponding (steady-state) elevator angle $\phi$.

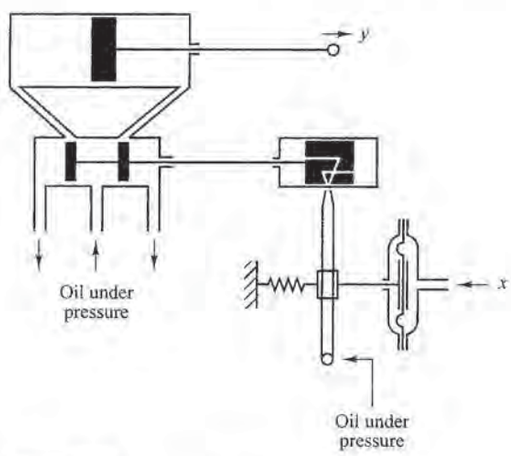


**Figure 4–57**
Schematic diagram of a hydraulic servomotor.

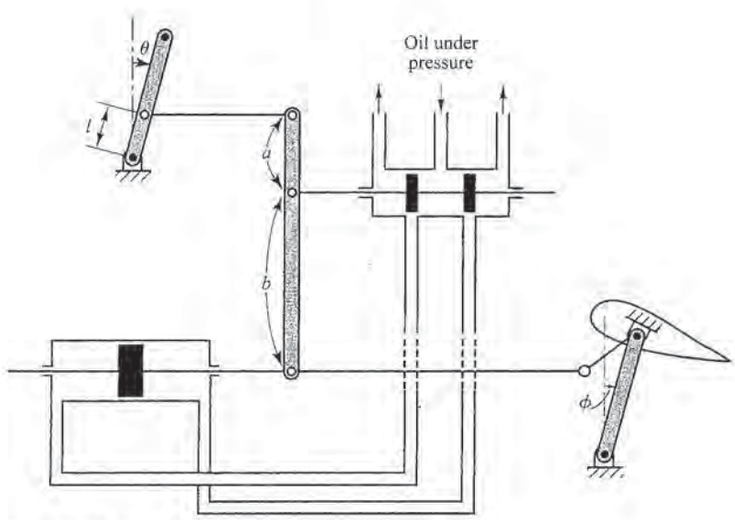


**Figure 4–58**
Aircraft elevator control system.





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**B–4–12.** Consider the liquid-level control system shown in Figure 4–59. The inlet valve is controlled by a hydraulic integral controller. Assume that the steady-state inflow rate is $\bar{Q}$ and steady-state outflow rate is also $\bar{Q}$, the steady-state head is $\bar{H}$, steady-state pilot valve displacement is $\bar{X} = 0$, and steady-state valve position is $\bar{Y}$. We assume that the set point $\bar{R}$ corresponds to the steady-state head $\bar{H}$. The set point is fixed. Assume also that the disturbance inflow rate $q_d$, which is a small quantity, is applied to the water tank at $t = 0$. This disturbance causes the head to change from $\bar{H}$ to $\bar{H} + h$. This change results in a change in the outflow rate by $q_o$. Through the hydraulic controller, the change in head causes a change in the inflow rate from $\bar{Q}$ to $\bar{Q} + q_i$. (The integral controller tends to keep the head constant as much as possible in the presence of disturbances.) We assume that all changes are of small quantities.

We assume that the velocity of the power piston (valve) is proportional to pilot valve displacement $x$, or

$$\frac{dy}{dt} = K_1 x$$

where $K_1$ is a positive constant. We also assume that the change in the inflow rate $q_i$ is negatively proportional to the change in the valve opening $y$, or

$$q_i = -K_v y$$

where $K_v$ is a positive constant.

Assuming the following numerical values for the system,

$C = 2\ \text{m}^2$, $R = 0.5\ \text{sec/m}^2$, $K_v = 1\ \text{m}^2/\text{sec}$

$a = 0.25\ \text{m}$, $b = 0.75\ \text{m}$, $K_1 = 4\ \text{sec}^{-1}$

obtain the transfer function $H(s)/Q_d(s)$.

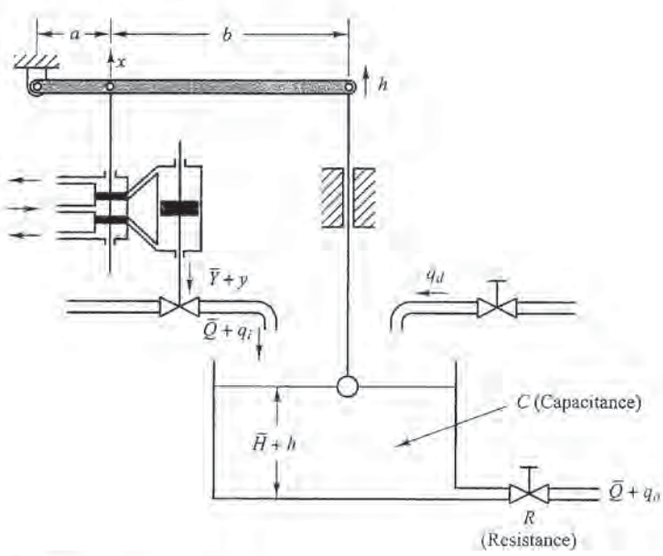


**Figure 4–59**
Liquid-level control system.





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**B–4–13.** Consider the controller shown in Figure 4–60. The input is the air pressure $p_i$ measured from some steady-state reference pressure $\bar{P}$ and the output is the displacement $y$ of the power piston. Obtain the transfer function $Y(s)/P_i(s)$.

**B–4–14.** A thermocouple has a time constant of 2 sec. A thermal well has a time constant of 30 sec. When the thermocouple is inserted into the well, this temperature-measuring device can be considered a two-capacitance system.

Determine the time constants of the combined thermocouple-thermal well system. Assume that the weight of the thermocouple is 8 g and the weight of the thermal well is 40 g. Assume also that the specific heats of the thermocouple and thermal well are the same.

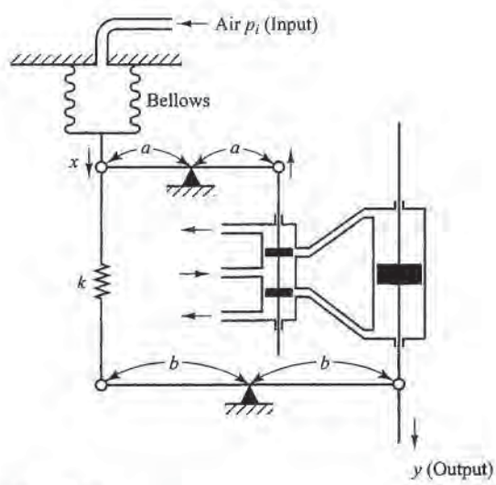


**Figure 4–60**
Controller.





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# 5

# Transient and Steady-State Response Analyses

## 5–1 INTRODUCTION

It was stated in Chapter 3 that the first step in analyzing a control system was to derive a mathematical model of the system. Once such a model is obtained, various methods are available for the analysis of system performance.

In practice, the input signal to a control system is not known ahead of time but is random in nature, and the instantaneous input cannot be expressed analytically. Only in some special cases is the input signal known in advance and expressible analytically or by curves, such as in the case of the automatic control of cutting tools.

In analyzing and designing control systems, we must have a basis of comparison of performance of various control systems. This basis may be set up by specifying particular test input signals and by comparing the responses of various systems to these input signals.

Many design criteria are based on the response to such signals or on the response of systems to changes in initial conditions (without any test signals). The use of test signals can be justified because of a correlation existing between the response characteristics of a system to a typical test input signal and the capability of the system to cope with actual input signals.

**Typical Test Signals.** The commonly used test input signals are those of step functions, ramp functions, acceleration functions, impulse functions, sinusoidal functions, and the like. With these test signals, mathematical and experimental analyses of control systems can be carried out easily since the signals are very simple functions of time.





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Which of these typical input signals to use for analyzing system characteristics may be determined by the form of the input that the system will be subjected to most frequently under normal operation. If the inputs to a control system are gradually changing functions of time, then a ramp function of time may be a good test signal. Similarly, if a system is subjected to sudden disturbances, a step function of time may be a good test signal; and for a system subjected to shock inputs, an impulse function may be best. Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the performance of all systems on the same basis.

**Transient Response and Steady-State Response.** The time response of a control system consists of two parts: the transient response and the steady-state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as $t$ approaches infinity. Thus the system response $c(t)$ may be written as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

where the first term on the right-hand side of the equation is the transient response and the second term is the steady-state response.

**Absolute Stability, Relative Stability, and Steady-State Error.** In designing a control system, we must be able to predict the dynamic behavior of the system from a knowledge of the components. The most important characteristic of the dynamic behavior of a control system is absolute stability, that is, whether the system is stable or unstable. A control system is in equilibrium if, in the absence of any disturbance or input, the output stays in the same state. A linear time-invariant control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition. A linear time-invariant control system is critically stable if oscillations of the output continue forever. It is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition. Actually, the output of a physical system may increase to a certain extent but may be limited by mechanical "stops," or the system may break down or become nonlinear after the output exceeds a certain magnitude so that the linear differential equations no longer apply.

Important system behavior (other than absolute stability) to which we must give careful consideration includes relative stability and steady-state error. Since a physical control system involves energy storage, the output of the system, when subjected to an input, cannot follow the input immediately but exhibits a transient response before a steady state can be reached. The transient response of a practical control system often exhibits damped oscillations before reaching a steady state. If the output of a system at steady state does not exactly agree with the input, the system is said to have steady-state error. This error is indicative of the accuracy of the system. In analyzing a control system, we must examine transient-response behavior and steady-state behavior.

**Outline of the Chapter.** This chapter is concerned with system responses to aperiodic signals (such as step, ramp, acceleration, and impulse functions of time). The outline of the chapter is as follows: Section 5–1 has presented introductory material for the chapter. Section 5–2 treats the response of first-order systems to aperiodic inputs. Section 5–3 deals with the transient response of the second-order systems. Detailed





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analyses of the step response, ramp response, and impulse response of the second-order systems are presented. Section 5–4 discusses the transient response analysis of higher-order systems. Section 5–5 gives an introduction to the MATLAB approach to the solution of transient response problems. Section 5–6 gives an example of a transient-response problem solved with MATLAB. Section 5–7 presents Routh's stability criterion. Section 5–8 discusses effects of integral and derivative control actions on system performance. Finally, Section 5–9 treats steady-state errors in unity-feedback control systems.

## 5–2 FIRST-ORDER SYSTEMS

Consider the first-order system shown in Figure 5–1(a). Physically, this system may represent an $RC$ circuit, thermal system, or the like. A simplified block diagram is shown in Figure 5–1(b). The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \tag{5–1}$$

In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp, and unit-impulse functions. The initial conditions are assumed to be zero.

Note that all systems having the same transfer function will exhibit the same output in response to the same input. For any given physical system, the mathematical response can be given a physical interpretation.

**Unit-Step Response of First-Order Systems.** Since the Laplace transform of the unit-step function is $1/s$, substituting $R(s) = 1/s$ into Equation (5–1), we obtain

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)} \tag{5–2}$$

Taking the inverse Laplace transform of Equation (5–2), we obtain

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–3}$$

Equation (5–3) states that initially the output $c(t)$ is zero and finally it becomes unity. One important characteristic of such an exponential response curve $c(t)$ is that at $t = T$ the value of $c(t)$ is 0.632, or the response $c(t)$ has reached 63.2% of its total change. This may be easily seen by substituting $t = T$ in $c(t)$. That is,

$$c(T) = 1 - e^{-1} = 0.632$$

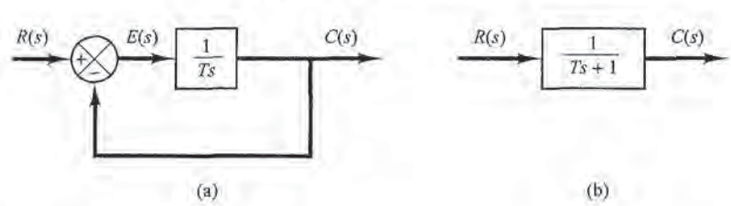


**Figure 5–1**
(a) Block diagram of a first-order system; (b) simplified block diagram.





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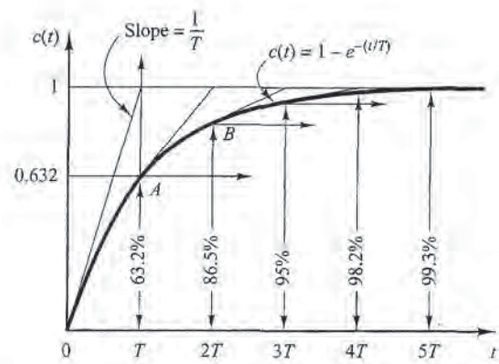


**Figure 5–2**
Exponential response curve.

Note that the smaller the time constant $T$, the faster the system response. Another important characteristic of the exponential response curve is that the slope of the tangent line at $t = 0$ is $1/T$, since

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T}e^{-t/T}\right|_{t=0} = \frac{1}{T} \tag{5–4}$$

The output would reach the final value at $t = T$ if it maintained its initial speed of response. From Equation (5–4) we see that the slope of the response curve $c(t)$ decreases monotonically from $1/T$ at $t = 0$ to zero at $t = \infty$.

The exponential response curve $c(t)$ given by Equation (5–3) is shown in Figure 5–2. In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t = 3T, 4T$, and $5T$, the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value. As seen from Equation (5–3), the steady state is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four time constants.

**Unit-Ramp Response of First-Order Systems.** Since the Laplace transform of the unit-ramp function is $1/s^2$, we obtain the output of the system of Figure 5–1(a) as

$$C(s) = \frac{1}{Ts + 1}\frac{1}{s^2}$$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \tag{5–5}$$

Taking the inverse Laplace transform of Equation (5–5), we obtain

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–6}$$

The error signal $e(t)$ is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T(1 - e^{-t/T}) \end{aligned}$$





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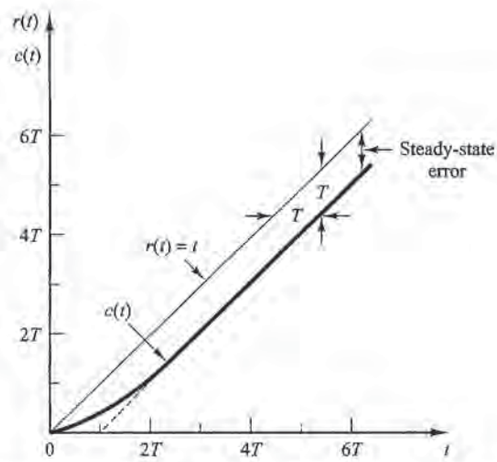


**Figure 5–3**
Unit-ramp response of the system shown in Figure 5–1(a).

As $t$ approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches $T$ or

$$e(\infty) = T$$

The unit-ramp input and the system output are shown in Figure 5–3. The error in following the unit-ramp input is equal to $T$ for sufficiently large $t$. The smaller the time constant $T$, the smaller the steady-state error in following the ramp input.

**Unit-Impulse Response of First-Order Systems.** For the unit-impulse input, $R(s) = 1$ and the output of the system of Figure 5–1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1} \tag{5–7}$$

The inverse Laplace transform of Equation (5–7) gives

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0 \tag{5–8}$$

The response curve given by Equation (5–8) is shown in Figure 5–4.

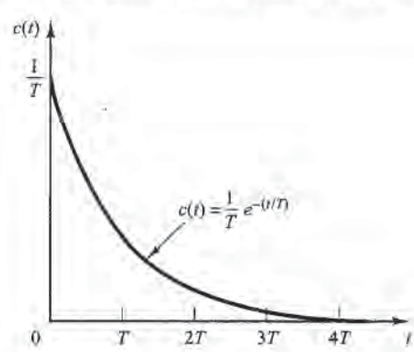


**Figure 5–4**
Unit-impulse response of the system shown in Figure 5–1(a).





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**An Important Property of Linear Time-Invariant Systems.** In the analysis above, it has been shown that for the unit-ramp input the output $c(t)$ is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–6).]}$$

For the unit-step input, which is the derivative of unit-ramp input, the output $c(t)$ is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–3).]}$$

Finally, for the unit-impulse input, which is the derivative of unit-step input, the output $c(t)$ is

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0 \qquad \text{[See Equation (5–8).]}$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero output initial condition. This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess this property.

## 5–3 SECOND-ORDER SYSTEMS

In this section, we shall obtain the response of a typical second-order control system to a step input, ramp input, and impulse input. Here we consider a servo system as an example of a second-order system.

**Servo System.** The servo system shown in Figure 5–5(a) consists of a proportional controller and load elements (inertia and viscous friction elements). Suppose that we wish to control the output position $c$ in accordance with the input position $r$.

The equation for the load elements is

$$J\ddot{c} + B\dot{c} = T$$

where $T$ is the torque produced by the proportional controller whose gain is $K$. By taking Laplace transforms of both sides of this last equation, assuming the zero initial conditions, we obtain

$$Js^2C(s) + BsC(s) = T(s)$$

So the transfer function between $C(s)$ and $T(s)$ is

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js + B)}$$

By using this transfer function. Figure 5–5(a) can be redrawn as in Figure 5–5(b), which can be modified to that shown in Figure 5–5(c). The closed-loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

Such a system where the closed-loop transfer function possesses two poles is called a second-order system. (Some second-order systems may involve one or two zeros.)





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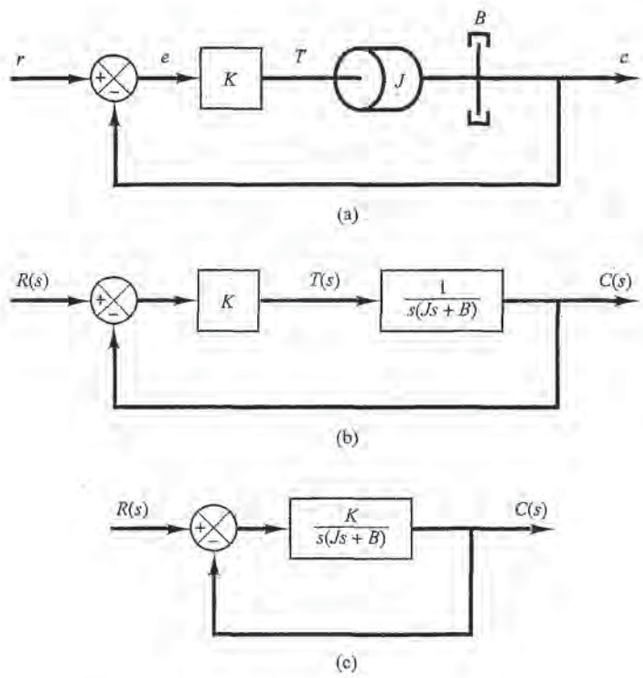


**Figure 5–5**
(a) Servo system;
(b) block diagram;
(c) simplified block diagram.

**Step Response of Second-Order System.** The closed-loop transfer function of the system shown in Figure 5–5(c) is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} \tag{5–9}$$

which can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{J}}{\left[s + \dfrac{B}{2J} + \sqrt{\left(\dfrac{B}{2J}\right)^2 - \dfrac{K}{J}}\right]\left[s + \dfrac{B}{2J} - \sqrt{\left(\dfrac{B}{2J}\right)^2 - \dfrac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2 - 4JK < 0$ and they are real if $B^2 - 4JK \geq 0$. In the transient-response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2, \qquad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where $\sigma$ is called the *attenuation*; $\omega_n$, the *undamped natural frequency*; and $\zeta$, the *damping ratio* of the system. The damping ratio $\zeta$ is the ratio of the actual damping $B$ to the critical damping $B_c = 2\sqrt{JK}$ or

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$





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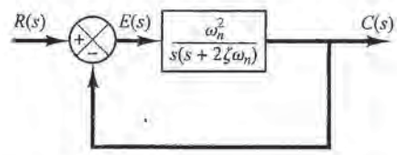


**Figure 5–6**
Second-order system.

In terms of $\zeta$ and $\omega_n$, the system shown in Figure 5–5(c) can be modified to that shown in Figure 5–6, and the closed-loop transfer function $C(s)/R(s)$ given by Equation (5–9) can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5–10}$$

This form is called the *standard form* of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters $\zeta$ and $\omega_n$. If $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half $s$ plane. The system is then called underdamped, and the transient response is oscillatory. If $\zeta = 0$, the transient response does not die out. If $\zeta = 1$, the system is called critically damped. Overdamped systems correspond to $\zeta > 1$.

We shall now solve for the response of the system shown in Figure 5–6 to a unit-step input. We shall consider three different cases: the underdamped $(0 < \zeta < 1)$, critically damped $(\zeta = 1)$, and overdamped $(\zeta > 1)$ cases.

(1) *Underdamped case* $(0 < \zeta < 1)$: In this case, $C(s)/R(s)$ can be written

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The frequency $\omega_d$ is called the *damped natural frequency*. For a unit-step input, $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} \tag{5–11}$$

The inverse Laplace transform of Equation (5–11) can be obtained easily if $C(s)$ is written in the following form:

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

In Chapter 2 it was shown that

$$\mathscr{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathscr{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$





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Hence the inverse Laplace transform of Equation (5–11) is obtained as

$$\begin{aligned}\mathscr{L}^{-1}[C(s)] &= c(t)\\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0 \qquad (5\text{–}12)\end{aligned}$$

This result can be obtained directly by using a table of Laplace transforms. From Equation (5–12), it can be seen that the frequency of transient oscillation is the damped natural frequency $\omega_d$ and thus varies with the damping ratio $\zeta$. The error signal for this system is the difference between the input and output and is

$$\begin{aligned}e(t) &= r(t) - c(t)\\ &= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \qquad \text{for } t \geq 0\end{aligned}$$

This error signal exhibits a damped sinusoidal oscillation. At steady state, or at $t = \infty$, no error exists between the input and output.

If the damping ratio $\zeta$ is equal to zero, the response becomes undamped and oscillations continue indefinitely. The response $c(t)$ for the zero damping case may be obtained by substituting $\zeta = 0$ in Equation (5–12), yielding

$$c(t) = 1 - \cos\omega_n t, \qquad \text{for } t \geq 0 \qquad (5\text{–}13)$$

Thus, from Equation (5–13), we see that $\omega_n$ represents the undamped natural frequency of the system. That is, $\omega_n$ is that frequency at which the system output would oscillate if the damping were decreased to zero. If the linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally. The frequency that may be observed is the damped natural frequency $\omega_d$, which is equal to $\omega_n\sqrt{1-\zeta^2}$. This frequency is always lower than the undamped natural frequency. An increase in $\zeta$ would reduce the damped natural frequency $\omega_d$. If $\zeta$ is increased beyond unity, the response becomes overdamped and will not oscillate.

(2) *Critically damped case* ($\zeta = 1$): If the two poles of $C(s)/R(s)$ are equal, the system is said to be a critically damped one.

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s} \qquad (5\text{–}14)$$

The inverse Laplace transform of Equation (5–14) may be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0 \qquad (5\text{–}15)$$

This result can also be obtained by letting $\zeta$ approach unity in Equation (5–12) and by using the following limit:

$$\lim_{\zeta\to 1}\frac{\sin\omega_d t}{\sqrt{1-\zeta^2}} = \lim_{\zeta\to 1}\frac{\sin\omega_n\sqrt{1-\zeta^2}\,t}{\sqrt{1-\zeta^2}} = \omega_n t$$





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(3) *Overdamped case* ($\zeta > 1$): In this case, the two poles of $C(s)/R(s)$ are negative real and unequal. For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s} \tag{5–16}$$

The inverse Laplace transform of Equation (5–16) is

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0 \end{aligned} \tag{5–17}$$

where $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$. Thus, the response $c(t)$ includes two decaying exponential terms.

When $\zeta$ is appreciably greater than unity, one of the two decaying exponentials decreases much faster than the other, so the faster decaying exponential term (which corresponds to a smaller time constant) may be neglected. That is, if $-s_2$ is located very much closer to the $j\omega$ axis than $-s_1$ (which means $|s_2| \ll |s_1|$), then for an approximate solution we may neglect $-s_1$. This is permissible because the effect of $-s_1$ on the response is much smaller than that of $-s_2$, since the term involving $s_1$ in Equation (5–17) decays much faster than the term involving $s_2$. Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system, and $C(s)/R(s)$ may be approximated by

$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

This approximate form is a direct consequence of the fact that the initial values and final values of both the original $C(s)/R(s)$ and the approximate one agree with each other.

With the approximate transfer function $C(s)/R(s)$, the unit-step response can be obtained as

$$C(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response $c(t)$ is then

$$c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \qquad \text{for } t \geq 0$$

This gives an approximate unit-step response when one of the poles of $C(s)/R(s)$ can be neglected.

A family of unit-step response curves $c(t)$ with various values of $\zeta$ is shown in Figure 5–7, where the abscissa is the dimensionless variable $\omega_n t$. The curves are functions





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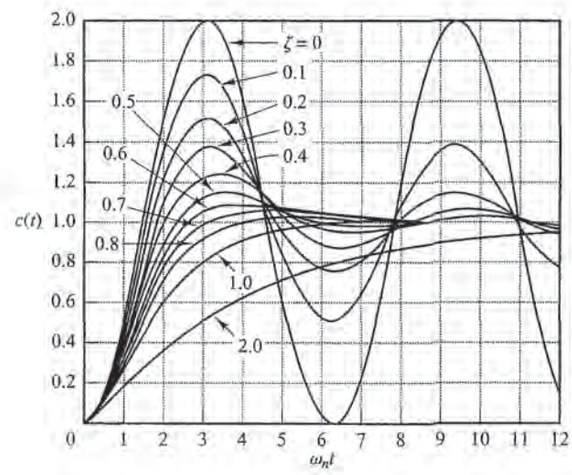


**Figure 5–7**
Unit-step response curves of the system shown in Figure 5–6.

only of $\zeta$. These curves are obtained from Equations (5–12), (5–15), and (5–17). The system described by these equations was initially at rest.

Note that two second-order systems having the same $\zeta$ but different $\omega_n$ will exhibit the same overshoot and the same oscillatory pattern. Such systems are said to have the same relative stability.

It is important to note that, for second-order systems whose closed-loop transfer functions are different from that given by Equation (5–10), the step-response curves may look quite different from those shown in Figure 5–7.

From Figure 5–7, we see that an underdamped system with $\zeta$ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system. Among the systems responding without oscillation, a critically damped system exhibits the fastest response. An overdamped system is always sluggish in responding to any inputs.

**Definitions of Transient-Response Specifications.** In many practical cases, the desired performance characteristics of control systems are specified in terms of time-domain quantities. Systems with energy storage cannot respond instantaneously and will exhibit transient responses whenever they are subjected to inputs or disturbances.

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input since it is easy to generate and is sufficiently drastic. (If the response to a step input is known, it is mathematically possible to compute the response to any input.)

The transient response of a system to a unit-step input depends on the initial conditions. For convenience in comparing transient responses of various systems, it is a common practice to use the standard initial condition that the system is at rest initially with the output and all time derivatives thereof zero. Then the response characteristics of many systems can be easily compared.





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The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, $t_d$
2. Rise time, $t_r$
3. Peak time, $t_p$
4. Maximum overshoot, $M_p$
5. Settling time, $t_s$

These specifications are defined in what follows and are shown graphically in Figure 5–8.

1. Delay time, $t_d$: The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time, $t_r$: The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second-order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
3. Peak time, $t_p$: The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot, $M_p$: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

   The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.
5. Settling time, $t_s$: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

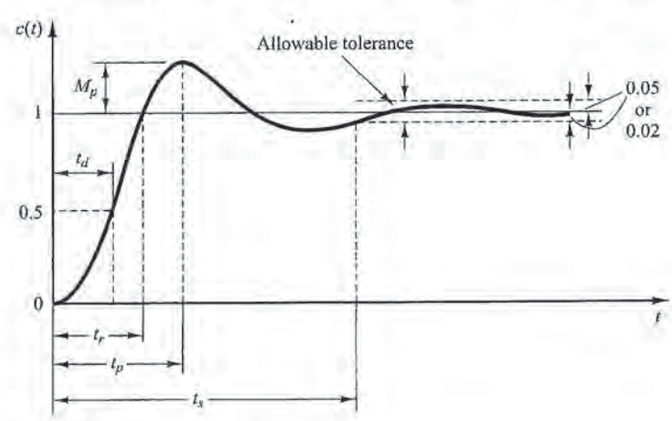


**Figure 5–8**
Unit-step response curve showing $t_d$, $t_r$, $t_p$, $M_p$, and $t_s$.





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The time-domain specifications just given are quite important since most control systems are time-domain systems; that is, they must exhibit acceptable time responses. (This means that, the control system must be modified until the transient response is satisfactory.)

Note that not all these specifications necessarily apply to any given case. For example, for an overdamped system, the terms peak time and maximum overshoot do not apply. (For systems that yield steady-state errors for step inputs, this error must be kept within a specified percentage level. Detailed discussions of steady-state errors are postponed until Section 5–9.)

**A Few Comments on Transient-Response Specifications.** Except for certain applications where oscillations cannot be tolerated, it is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be between 0.4 and 0.8. Small values of $\zeta(\zeta < 0.4)$ yield excessive overshoot in the transient response, and a system with a large value of $\zeta(\zeta > 0.8)$.responds sluggishly.

We shall see later that the maximum overshoot and the rise time conflict with each other. In other words, both the maximum overshoot and the rise time cannot be made smaller simultaneously. If one of them is made smaller, the other necessarily becomes larger.

**Second-Order Systems and Transient-Response Specifications.** In the following, we shall obtain the rise time, peak time, maximum overshoot, and settling time of the second-order system given by Equation (5–10). These values will be obtained in terms of $\zeta$ and $\omega_n$. The system is assumed to be underdamped.

*Rise time $t_r$*: Referring to Equation (5–12), we obtain the rise time $t_r$ by letting $c(t_r) = 1$.

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right) \tag{5–18}$$

Since $e^{-\zeta\omega_n t_r} \neq 0$, we obtain from Equation (5–18) the following equation:

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

or

$$\tan\omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time $t_r$ is

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} \tag{5–19}$$

where $\beta$ is defined in Figure 5–9. Clearly, for a small value of $t_r$, $\omega_d$ must be large.

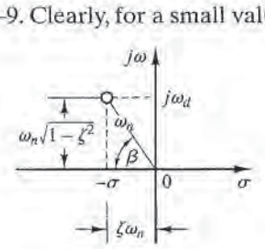


**Figure 5–9**
Definition of the angle $\beta$.





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*Peak time $t_p$:* Referring to Equation (5–12), we may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, $dc/dt$, evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin\omega_d t_p = 0$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot, $\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d} \tag{5–20}$$

The peak time $t_p$ corresponds to one-half cycle of the frequency of damped oscillation.

*Maximum overshoot $M_p$:* The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, $M_p$ is obtained from Equation (5–12) as

$$\begin{aligned} M_p &= c(t_p) - 1 \\ &= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) \\ &= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \end{aligned} \tag{5–21}$$

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

*Settling time $t_s$:* For an underdamped second-order system, the transient response is obtained from Equation (5–12) as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

The curves $1 \pm (e^{-\zeta\omega_n t}/\sqrt{1-\zeta^2})$ are the envelope curves of the transient response to a unit-step input. The response curve $c(t)$ always remains within a pair of the envelope curves, as shown in Figure 5–10. The time constant of these envelope curves is $1/\zeta\omega_n$.





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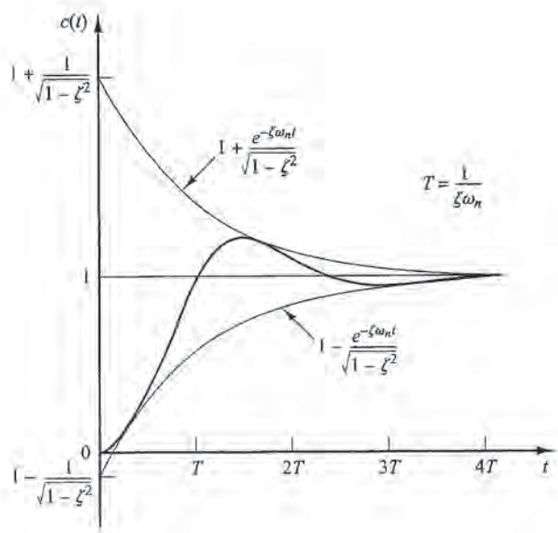


**Figure 5–10**
Pair of envelope curves for the unit-step response curve of the system shown in Figure 5–6.

The speed of decay of the transient response depends on the value of the time constant $1/\zeta\omega_n$. For a given $\omega_n$, the settling time $t_s$ is a function of the damping ratio $\zeta$. From Figure 5–7, we see that for the same $\omega_n$ and for a range of $\zeta$ between 0 and 1 the settling time $t_s$ for a very lightly damped system is larger than that for a properly damped system. For an overdamped system, the settling time $t_s$ becomes large because of the sluggish response.

The settling time corresponding to a $\pm 2\%$ or $\pm 5\%$ tolerance band may be measured in terms of the time constant $T = 1/\zeta\omega_n$ from the curves of Figure 5–7 for different values of $\zeta$. The results are shown in Figure 5–11. For $0 < \zeta < 0.9$, if the 2% criterion is used, $t_s$ is approximately four times the time constant of the system. If the 5% criterion is used, then $t_s$ is approximately three times the time constant. Note that the settling time reaches a minimum value around $\zeta = 0.76$ (for the 2% criterion) or $\zeta = 0.68$ (for the 5% criterion) and then increases almost linearly for large values of $\zeta$. The discontinuities in the curves of Figure 5–11 arise because an infinitesimal change in the value of $\zeta$ can cause a finite change in the settling time.

For convenience in comparing the responses of systems, we commonly define the settling time $t_s$ to be

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)} \tag{5–22}$$

or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)} \tag{5–23}$$

Note that the settling time is inversely proportional to the product of the damping ratio and the undamped natural frequency of the system. Since the value of $\zeta$ is usually determined from the requirement of permissible maximum overshoot, the settling time is determined primarily by the undamped natural frequency $\omega_n$. This means that the





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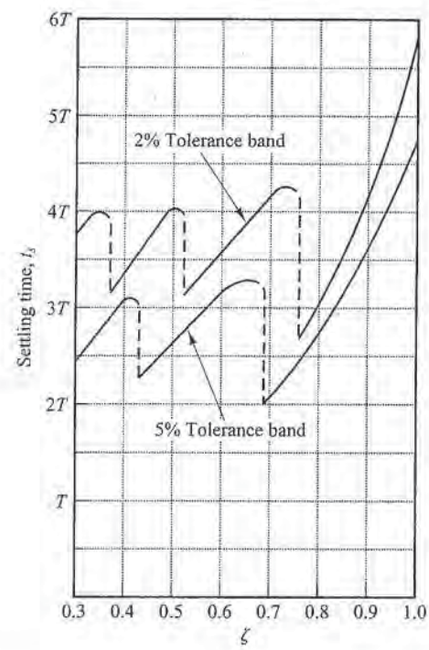


**Figure 5–11**
Settling time $t_s$ versus $\zeta$ curves.

duration of the transient period may be varied, without changing the maximum overshoot, by adjusting the undamped natural frequency $\omega_n$.

From the preceding analysis, it is evident that for rapid response $\omega_n$ must be large. To limit the maximum overshoot $M_p$ and to make the settling time small, the damping ratio $\zeta$ should not be too small. The relationship between the maximum percent overshoot $M_p$ and the damping ratio $\zeta$ is presented in Figure 5–12. Note that if the damping ratio is between 0.4 and 0.7 then the maximum percent overshoot for step response is between 25% and 4%.

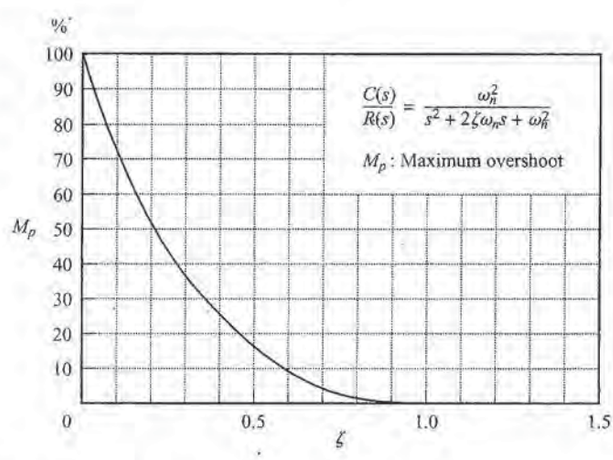


**Figure 5–12**
$M_p$ versus $\zeta$ curve.





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It is important to note that the equations for obtaining the rise time, peak time, maximum overshoot, and settling time are valid only for the standard second-order system defined by Equation (5–10). If the second-order system involves a zero or two zeros, the shape of the unit-step response curve will be quite different from those shown in Figure 5–7.

**EXAMPLE 5–1** Consider the system shown in Figure 5–6, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time $t_r$, peak time $t_p$, maximum overshoot $M_p$, and settling time $t_s$ when the system is subjected to a unit-step input.

From the given values of $\zeta$ and $\omega_n$, we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

*Rise time $t_r$:* The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where $\beta$ is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time $t_r$ is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

*Peak time $t_p$:* The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

*Maximum overshoot $M_p$:* The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

*Settling time $t_s$:* For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

**Servo System with Velocity Feedback.** The derivative of the output signal can be used to improve system performance. In obtaining the derivative of the output position signal, it is desirable to use a tachometer instead of physically differentiating the output signal. (Note that the differentiation amplifies noise effects. In fact, if discontinuous noises are present, differentiation amplifies the discontinuous noises more than the useful signal. For example, the output of a potentiometer is a discontinuous voltage signal because, as the potentiometer brush is moving on the windings, voltages are induced in the switchover turns and thus generate transients. The output of the potentiometer therefore should not be followed by a differentiating element.)





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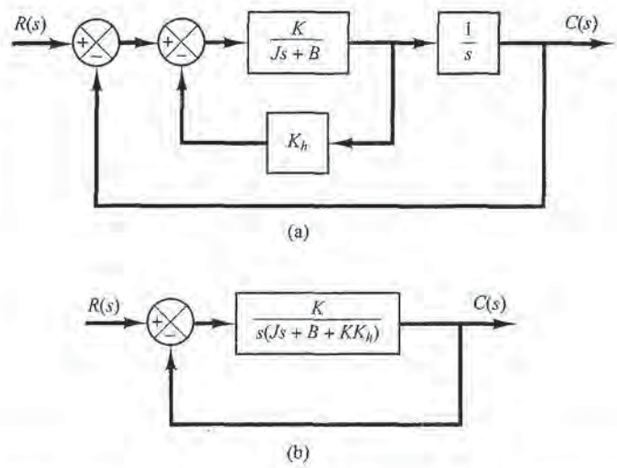


**Figure 5–13**
(a) Block diagram of a servo system;
(b) simplified block diagram.

The tachometer, a special dc generator, is frequently used to measure velocity without differentiation process. The output of a tachometer is proportional to the angular velocity of the motor.

Consider the servo system shown in Figure 5–13(a). In this device, the velocity signal, together with the positional signal, is fed back to the input to produce the actuating error signal. In any servo system, such a velocity signal can be easily generated by a tachometer. The block diagram shown in Figure 5–13(a) can be simplified, as shown in Figure 5–13(b), giving

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K} \tag{5–24}$$

Comparing Equation (5–24) with Equation (5–9), notice that the velocity feedback has the effect of increasing damping. The damping ratio $\zeta$ becomes

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}} \tag{5–25}$$

The undamped natural frequency $\omega_n = \sqrt{K/J}$ is not affected by velocity feedback. Noting that the maximum overshoot for a unit-step input can be controlled by controlling the value of the damping ratio $\zeta$, we can reduce the maximum overshoot by adjusting the velocity feedback constant $K_h$ so that $\zeta$ is between 0.4 and 0.7.

Remember that velocity feedback has the effect of increasing the damping ratio without affecting the undamped natural frequency of the system.

**EXAMPLE 5–2** For the system shown in Figure 5–13(a), determine the values of gain $K$ and velocity feedback constant $K_h$ so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of $K$ and $K_h$, obtain the rise time and settling time. Assume that $J = 1$ kg-m$^2$ and $B = 1$ N-m/rad/sec.

*Determination of the values of $K$ and $K_h$:* The maximum overshoot $M_p$ is given by Equation (5–21) as

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$





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This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

The peak time $t_p$ is specified as 1 sec; therefore, from Equation (5–20),

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since $\zeta$ is 0.456, $\omega_n$ is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency $\omega_n$ is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then, $K_h$ is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

*Rise time $t_r$:* From Equation (5–19), the rise time $t_r$ is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$$

Thus, $t_r$ is

$$t_r = 0.65 \text{ sec}$$

*Settling time $t_s$:* For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$





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**Impulse Response of Second-Order Systems.** For a unit-impulse input $r(t)$, the corresponding Laplace transform is unity, or $R(s) = 1$. The unit-impulse response $C(s)$ of the second-order system shown in Figure 5-6 is

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The inverse Laplace transform of this equation yields the time solution for the response $c(t)$ as follows:

For $0 \le \zeta < 1$,

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \qquad \text{for } t \ge 0 \tag{5–26}$$

For $\zeta = 1$,

$$c(t) = \omega_n^2 t e^{-\omega_n t}, \qquad \text{for } t \ge 0 \tag{5–27}$$

For $\zeta > 1$,

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}, \qquad \text{for } t \ge 0 \tag{5–28}$$

Note that without taking the inverse Laplace transform of $C(s)$ we can also obtain the time response $c(t)$ by differentiating the corresponding unit-step response since the unit-impulse function is the time derivative of the unit-step function. A family of unit-impulse response curves given by Equations (5–26) and (5–27) with various values of $\zeta$ is shown in Figure 5–14. The curves $c(t)/\omega_n$ are plotted against the dimensionless variable $\omega_n t$, and thus they are functions only of $\zeta$. For the critically damped and overdamped cases, the unit-impulse response is always positive or zero; that is, $c(t) \ge 0$. This can be seen from Equations (5–27) and (5–28). For the underdamped case, the unit-impulse response $c(t)$ oscillates about zero and takes both positive and negative values.

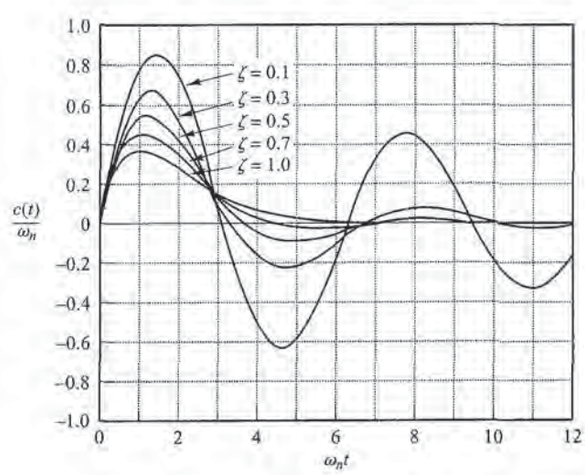


**Figure 5–14**
Unit-impulse response curves of the system shown in Figure 5–6.





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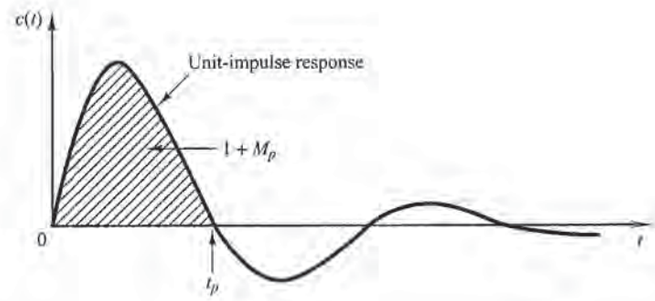


**Figure 5–15**
Unit-impulse response curve of the system shown in Figure 5–6.

From the foregoing analysis, we may conclude that if the impulse response $c(t)$ does not change sign, the system is either critically damped or overdamped, in which case the corresponding step response does not overshoot but increases or decreases monotonically and approaches a constant value.

The maximum overshoot for the unit-impulse response of the underdamped system occurs at

$$t = \frac{\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}, \qquad \text{where } 0 < \zeta < 1 \tag{5–29}$$

[Equation (5–29) can be obtained by equating $dc/dt$ to zero and solving for $t$.] The maximum overshoot is

$$c(t)_{\max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \qquad \text{where } 0 < \zeta < 1 \tag{5–30}$$

[Equation (5–30) can be obtained by substituting Equation (5–29) into Equation (5–26).]

Since the unit-impulse response function is the time derivative of the unit-step response function, the maximum overshoot $M_p$ for the unit-step response can be found from the corresponding unit-impulse response. That is, the area under the unit-impulse response curve from $t = 0$ to the time of the first zero, as shown in Figure 5–15, is $1 + M_p$, where $M_p$ is the maximum overshoot (for the unit-step response) given by Equation (5–21). The peak time $t_p$ (for the unit-step response) given by Equation (5–20) corresponds to the time that the unit-impulse response first crosses the time axis.

## 5–4 HIGHER-ORDER SYSTEMS

In this section we shall present a transient response analysis of higher-order systems in general terms. It will be seen that the response of higher-order systems is the sum of the responses of first-order and second-order systems.





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**Transient Response of Higher-Order Systems.** Consider the system shown in Figure 5–16. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{5–31}$$

In general, $G(s)$ and $H(s)$ are given as ratios of polynomials in $s$, or

$$G(s) = \frac{p(s)}{q(s)} \quad \text{and} \quad H(s) = \frac{n(s)}{d(s)}$$

where $p(s)$, $q(s)$, $n(s)$, and $d(s)$ are polynomials in $s$. The closed-loop transfer function given by Equation (5–31) may then be written

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{p(s)d(s)}{q(s)d(s) + p(s)n(s)} \\ &= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (m \le n)\end{aligned}$$

The transient response of this system to any given input can be obtained by a computer simulation. (See Section 5–5.) If an analytical expression for the transient response is desired, then it is necessary to factor the denominator polynomial. [MATLAB may be used for finding the roots of the denominator polynomial. Use the command roots(den).] Once the numerator and the denominator have been factored, $C(s)/R(s)$ can be written in the form

$$\frac{C(s)}{R(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} \tag{5–32}$$

Let us examine the response behavior of this system to a unit-step input. Consider first the case where the closed-loop poles are all real and distinct. For a unit-step input, Equation (5–32) can be written

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s + p_i} \tag{5–33}$$

where $a_i$ is the residue of the pole at $s = -p_i$. (If the system involves multiple poles, then $C(s)$ will have multiple-pole terms.) [The partial-fraction expansion of $C(s)$, as given by Equation (5–33), can be obtained easily with MATLAB. Use the residue command.]

If all closed-loop poles lie in the left-half $s$ plane, the relative magnitudes of the residues determine the relative importance of the components in the expanded form of

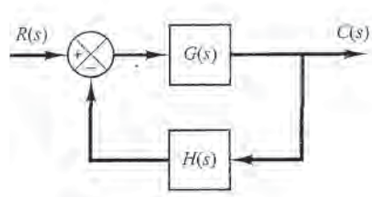


**Figure 5–16**
Control system.





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$C(s)$. If there is a closed-loop zero close to a closed-loop pole, then the residue at this pole is small and the coefficient of the transient-response term corresponding to this pole becomes small. A pair of closely located poles and zeros will effectively cancel each other. If a pole is located very far from the origin, the residue at this pole may be small. The transients corresponding to such a remote pole are small and last a short time. Terms in the expanded form of $C(s)$ having very small residues contribute little to the transient response, and these terms may be neglected. If this is done, the higher-order system may be approximated by a lower-order one. (Such an approximation often enables us to estimate the response characteristics of a higher-order system from those of a simplified one.)

Next, consider the case where the poles of $C(s)$ consist of real poles and pairs of complex-conjugate poles. A pair of complex-conjugate poles yields a second-order term in $s$. Since the factored form of the higher-order characteristic equation consists of first- and second-order terms, Equation (5–33) can be rewritten

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k(s + \zeta_k\omega_k) + c_k\omega_k\sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k\omega_k s + \omega_k^2} \qquad (q + 2r = n)$$

where we assumed all closed-loop poles are distinct. [If the closed-loop poles involve multiple poles, $C(s)$ must have multiple-pole terms.] From this last equation, we see that the response of a higher-order system is composed of a number of terms involving the simple functions found in the responses of first- and second-order systems. The unit-step response $c(t)$, the inverse Laplace transform of $C(s)$, is then

$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t} \cos \omega_k\sqrt{1 - \zeta_k^2}\, t$$

$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t} \sin \omega_k\sqrt{1 - \zeta_k^2}\, t, \qquad \text{for } t \geq 0 \qquad (5\text{–}34)$$

Thus the response curve of a stable higher-order system is the sum of a number of exponential curves and damped sinusoidal curves.

If all closed-loop poles lie in the left-half $s$ plane, then the exponential terms and the damped exponential terms in Equation (5–34) will approach zero as time $t$ increases. The steady-state output is then $c(\infty) = a$.

Let us assume that the system considered is a stable one. Then the closed-loop poles that are located far from the $j\omega$ axis have large negative real parts. The exponential terms that correspond to these poles decay very rapidly to zero. (Note that the horizontal distance from a closed-loop pole to the $j\omega$ axis determines the settling time of transients due to that pole. The smaller the distance is, the longer the settling time.)

Remember that the type of transient response is determined by the closed-loop poles, while the shape of the transient response is primarily determined by the closed-loop zeros. As we have seen earlier, the poles of the input $R(s)$ yield the steady-state response terms in the solution, while the poles of $C(s)/R(s)$ enter into the exponential transient-response terms and/or damped sinusoidal transient-response terms. The zeros of $C(s)/R(s)$ do not affect the exponents in the exponential terms, but they do affect the magnitudes and signs of the residues.





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**Dominant Closed-Loop Poles.** The relative dominance of closed-loop poles is determined by the ratio of the real parts of the closed-loop poles, as well as by the relative magnitudes of the residues evaluated at the closed-loop poles. The magnitudes of the residues depend on both the closed-loop poles and zeros.

If the ratios of the real parts exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient-response behavior because these poles correspond to transient-response terms that decay slowly. Those closed-loop poles that have dominant effects on the transient-response behavior are called *dominant closed-loop* poles. Quite often the dominant closed-loop poles occur in the form of a complex-conjugate pair. The dominant closed-loop poles are most important among all closed-loop poles.

Note that the gain of a higher-order system is often adjusted so that there will exist a pair of dominant complex-conjugate closed-loop poles. The presence of such poles in a stable system reduces the effects of such nonlinearities as dead zone, backlash, and coulomb-friction.

**Stability Analysis in the Complex Plane.** The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the $s$ plane. If any of these poles lie in the right-half $s$ plane, then with increasing time they give rise to the dominant mode, and the transient response increases monotonically or oscillates with increasing amplitude. This represents an unstable system. For such a system, as soon as the power is turned on, the output may increase with time. If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be subjected to damage and fail since the response of a real physical system cannot increase indefinitely. Therefore, closed-loop poles in the right-half $s$ plane are not permissible in the usual linear control system. If all closed-loop poles lie to the left of the $j\omega$ axis, any transient response eventually reaches equilibrium. This represents a stable system.

Whether a linear system is stable or unstable is a property of the system itself and does not depend on the input or driving function of the system. The poles of the input, or driving function, do not affect the property of stability of the system, but they contribute only to steady-state response terms in the solution. Thus, the problem of absolute stability can be solved readily by choosing no closed-loop poles in the right-half $s$ plane, including the $j\omega$ axis. (Mathematically, closed-loop poles on the $j\omega$ axis will yield oscillations, the amplitude of which is neither decaying nor growing with time. In practical cases, where noise is present, however, the amplitude of oscillations may increase at a rate determined by the noise power level. Therefore, a control system should not have closed-loop poles on the $j\omega$ axis.)

Note that the mere fact that all closed-loop poles lie in the left-half $s$ plane does not guarantee satisfactory transient-response characteristics. If dominant complex-conjugate closed-loop poles lie close to the $j\omega$ axis, the transient response may exhibit excessive oscillations or may be very slow. Therefore, to guarantee fast, yet well-damped, transient-response characteristics, it is necessary that the closed-loop poles of the system lie in a particular region in the complex plane, such as the region bounded by the shaded area in Figure 5–17.

Since the relative stability and transient-response performance of a closed-loop control system are directly related to the closed-loop pole-zero configuration in the $s$ plane,





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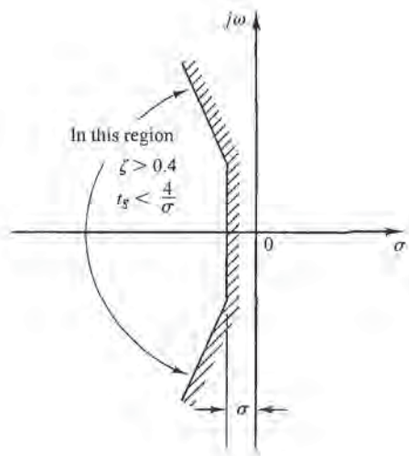


**Figure 5–17**
Region in the complex plane satisfying the conditions $\zeta > 0.4$ and $t_s < 4/\sigma$.

it is frequently necessary to adjust one or more system parameters in order to obtain suitable configurations. The effects of varying system parameters on the closed-loop poles will be discussed in detail in Chapter 6.

## 5–5 TRANSIENT-RESPONSE ANALYSIS WITH MATLAB

**Introduction.** The practical procedure for plotting time response curves of systems higher than second-order is through computer simulation. In this section we present the computational approach to the transient-response analysis with MATLAB. In particular, we discuss step response, impulse response, ramp response, and responses to other simple inputs.

**MATLAB Representation of Linear Systems.** The transfer function of a system is represented by two arrays of numbers. Consider the system

$$\frac{C(s)}{R(s)} = \frac{2s + 25}{s^2 + 4s + 25} \tag{5–35}$$

This system can be represented as two arrays, each containing the coefficients of the polynomials in decreasing powers of $s$ as follows:

```
num = [2  25]
den = [1  4  25]
```

An alternative representation is

```
num = [0  2  25]
den = [1  4  25]
```





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In this expression a zero is padded. Note that if zeros are padded, the dimensions of "num" vector and "den" vector become the same. An advantage of padding zeros is that the "num" vector and "den" vector can be directly added. For example,

$$\begin{aligned} \text{num} + \text{den} &= [0 \;\; 2 \;\; 25] + [1 \;\; 4 \;\; 25] \\ &= [1 \;\; 6 \;\; 50] \end{aligned}$$

If num and den (the numerator and denominator of the closed-loop transfer function) are known, commands such as

```
step(num,den),    step(num,den,t)
```

will generate plots of unit-step responses (t in the step command is the user-specified time.)

For a control system defined in a state-space form, where state matrix **A**, control matrix **B**, output matrix **C**, and direct transmission matrix **D** of state-space equations are known, the command

```
step(A,B,C,D)
```

will generate plots of unit-step responses. The time vector is automatically determined when t is not explicitly included in the step commands.

Note that the command step(sys) may be used to obtain the unit-step response of a system. First, define the system by

```
sys = tf(num,den)
```

or

```
sys = ss(A,B,C,D)
```

Then, to obtain, for example, the unit-step response, enter

```
step(sys)
```

into the computer.

When step commands have left-hand arguments such as

```
[y,x,t] = step(num,den,t)
[y,x,t] = step(A,B,C,D,iu)
[y,x,t] = step(A,B,C,D,iu,t)                    (5–36)
```

no plot is shown on the screen. Hence it is necessary to use a plot command to see the response curves. The matrices y and x contain the output and state response of the system, respectively, evaluated at the computation time points t. (y has as many columns as outputs and one row for each element in t. x has as many columns as states and one row for each element in t.)

Note in Equation (5–36) that the scalar iu is an index into the inputs of the system and specifies which input is to be used for the response, and t is the user-specified time. If the system involves multiple inputs and multiple outputs, the step command, such as given by Equation (5–36), produces a series of step response plots, one for each input and output combination of

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

(For details, see Example 5–3.)





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**EXAMPLE 5–3** Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves.

Although it is not necessary to obtain the transfer matrix expression for the system to obtain the unit-step response curves with MATLAB, we shall derive such an expression for reference. For the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

the transfer matrix $\mathbf{G}(s)$ is a matrix that relates $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ as follows:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

Taking Laplace transforms of the state-space equations, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5–37}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{5–38}$$

In deriving the transfer matrix, we assume that $\mathbf{x}(0) = \mathbf{0}$. Then, from Equation (5–37), we get

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{5–39}$$

Substituting Equation (5–39) into Equation (5–38), we obtain

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

Thus the transfer matrix $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

The transfer matrix $\mathbf{G}(s)$ for the given system becomes

$$\begin{aligned} \mathbf{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s^2 + s + 6.5} \begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2+s+6.5} & \dfrac{s}{s^2+s+6.5} \\ \dfrac{s+7.5}{s^2+s+6.5} & \dfrac{6.5}{s^2+s+6.5} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Since this system involves two inputs and two outputs, four transfer functions may be defined depending on which signals are considered as input and output. Note that, when considering the





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signal $u_1$ as the input, we assume that signal $u_2$ is zero, and vice versa. The four transfer functions are

$$\frac{Y_1(s)}{U_1(s)} = \frac{s-1}{s^2+s+6.5}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s}{s^2+s+6.5}$$

$$\frac{Y_2(s)}{U_1(s)} = \frac{s+7.5}{s^2+s+6.5}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2+s+6.5}$$

The four individual step-response curves can be plotted by use of the command

```
step(A,B,C,D)
```

MATLAB Program 5–1 produces four such step-response curves. The curves are shown in Figure 5–18.

**MATLAB Program 5–1**

```
A = [-1 -1;6.5 0];
B = [1 1;1 0];
C = [1 0;0 1];
D = [0 0;0 0];
step(A,B,C,D)
```

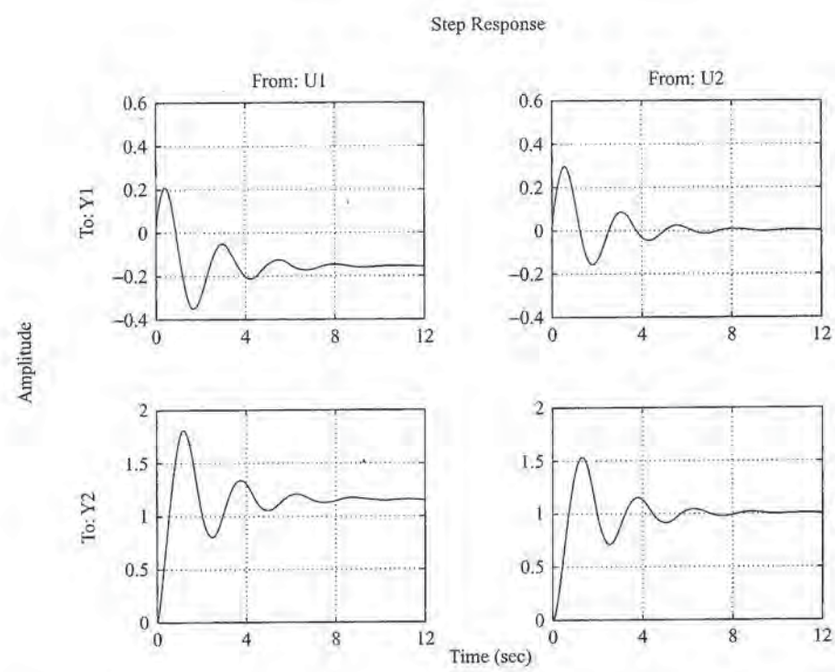


**Figure 5–18**
Unit-step response curves.





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To plot two step-response curves for the input $u_1$ in one diagram and two step-response curves for the input $u_2$ in another diagram, we may use the commands

```
step(A,B,C,D,1)
```

and

```
step(A,B,C,D,2)
```

respectively. MATLAB Program 5–2 is a program to plot two step-response curves for the input $u_1$ in one diagram and two step-response curves for the input $u_2$ in another diagram. Figure 5–19 shows the two diagrams, each consisting of two step-response curves.

**MATLAB Program 5–2**

```
% --------------- Step-response curves for system defined in state
% space ---------------

% ***** In this program we plot step-response curves of a system
% having two inputs (u1 and u2) and two outputs (y1 and y2) *****

% ***** We shall first plot step-response curves when the input is
% u1. Then we shall plot step-response curves when the input is
% u2 *****

% ***** Enter matrices A, B, C, and D *****

A = [-1  -1;6.5  0];
B = [1  1;1  0];
C = [1  0;0  1];
D = [0  0;0  0];

% ***** To plot step-response curves when the input is u1, enter
% the command 'step(A,B,C,D,1)' *****

step(A,B,C,D,1)
grid
title ('Step-Response Plots: Input = u1 (u2 = 0)')
text(3.4, -0.06,'Y1')
text(3.4, 1.4,'Y2')

% ***** Next, we shall plot step-response curves when the input
% is u2. Enter the command 'step(A,B,C,D,2)' *****

step(A,B,C,D,2)
grid
title ('Step-Response Plots: Input = u2 (u1 = 0)')
text(3,0.14,'Y1')
text(2.8,1.1,'Y2')
```





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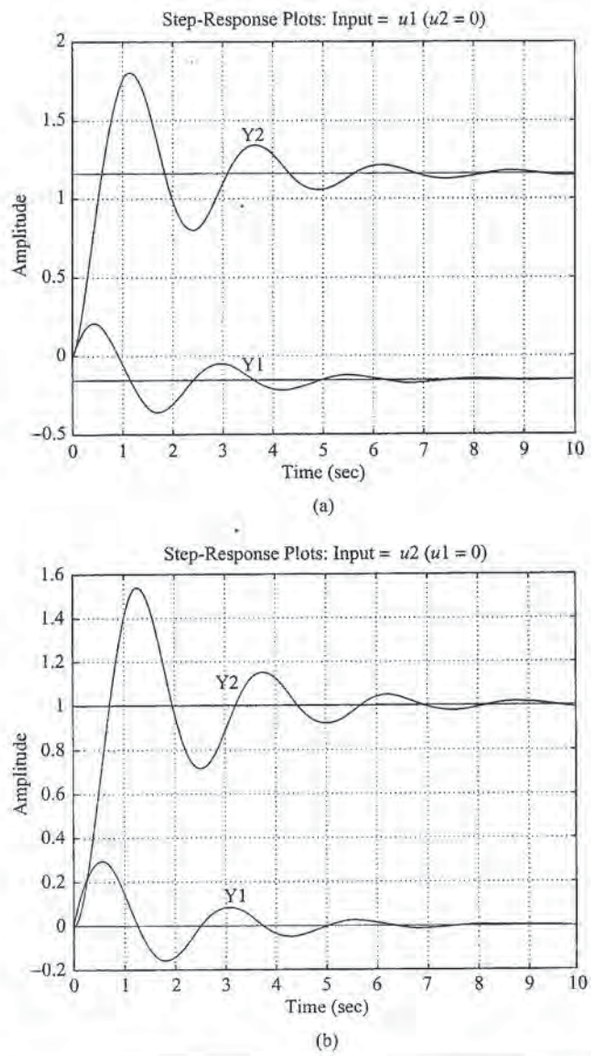


**Figure 5–19**
Unit-step response curves. (a) $u_1$ is the input $(u_2 = 0)$; (b) $u_2$ is the input $(u_1 = 0)$.

**Writing Text on the Graphics Screen.** To write text on the graphics screen, enter, for example, the following statements:

```
text(3.4, -0.06,'Y1')
```

and

```
text(3.4,1.4,'Y2')
```

The first statement tells the computer to write 'Y1' beginning at the coordinates $x = 3.4$, $y = -0.06$. Similarly, the second statement tells the computer to write 'Y2' beginning at the coordinates $x = 3.4$, $y = 1.4$. [See MATLAB Program 5–2 and Figure 5–19(a).]





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Another way to write a text or texts in the plot is to use the gtext command. The syntax is

gtext('text')

When gtext is executed, the computer waits until the cursor is positioned (using a mouse) at the desired position in the screen. When the left mouse button is pressed, the text enclosed in simple quotes is written on the plot at the cursor's position. Any number of gtext commands can be used in a plot. (See, for example, MATLAB Program 5–15.)

**MATLAB Description of Standard Second-Order System.** As noted earlier, the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5–40}$$

is called the standard second-order system. Given $\omega_n$ and $\zeta$, the command

printsys(num,den) or printsys(num,den,s)

prints num/den as a ratio of polynomials in $s$.

Consider, for example, the case where $\omega_n = 5$ rad/sec and $\zeta = 0.4$. MATLAB Program 5–3 generates the standard second-order system where $\omega_n = 5$ rad/sec and $\zeta = 0.4$.

**MATLAB Program 5–3**

```
wn = 5;
damping_ratio = 0.4;
[num0,den] = ord2(wn,damping_ratio);
num = 5^2*num0;
printsys(num,den,'s')
num/den =
                25
          S^2 + 4s + 25
```

**Obtaining the Unit-Step Response of the Transfer-Function System.** Let us consider the unit-step response of the system given by

$$G(s) = \frac{25}{s^2 + 4s + 25}$$

MATLAB Program 5–4 will yield a plot of the unit-step response of this system. A plot of the unit-step response curve is shown in Figure 5–20.

Notice in Figure 5–20 (and many others) that the $x$ axis and $y$ axis labels are automatically determined. If it is desired to label the $x$ axis and $y$ axis differently, we need to modify the step command. For example, if it is desired to label the $x$ axis as 't Sec' and the $y$ axis as 'Input and Output,' then use step-response commands with left-hand arguments, such as

c = step(num,den,t)





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or, more generally,

```
[y,x,t] = step(num,den,t)
```

See, for example, MATLAB Program 5–6.

**MATLAB Program 5–4**

```
% ------------ Unit-step response ------------
% ***** Enter the numerator and denominator of the transfer
% function *****
num = [0  0  25];
 den = [1  4  25];
% ***** Enter the following step-response command *****
step(num,den)
% ***** Enter grid and title of the plot *****
grid
title (' Unit-Step Response of G(s) = 25/(s^2+4s+25)')
```

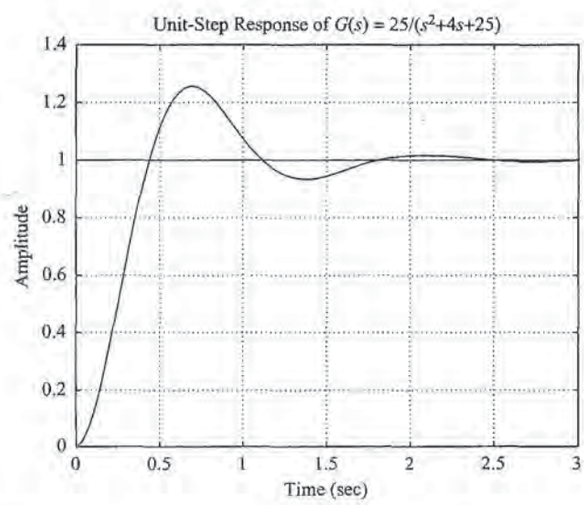


**Figure 5–20**
Unit-step response curve.

**Obtaining Three-Dimensional Plot of Unit-Step Response Curves with MATLAB.** MATLAB enables us to plot three-dimensional plots easily. The command to obtain three-dimensional plots is "mesh."

**EXAMPLE 5–4** Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$





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(The undamped natural frequency $\omega_n$ is normalized to 1.) Plot unit-step response curves $c(t)$ when $\zeta$ assumes the following values:

$$\zeta = 0,\ 0.2,\ 0.4,\ 0.6.\ 0.8,\ 1.0$$

Also plot a three-dimensional plot.

An illustrative MATLAB program for plotting a three-dimensional diagram of unit-step response curves of this second-order system is given in MATLAB 5–5. This program includes a program to obtain a two-dimensional plot (ordinary unit-step response curves) and a program to obtain a three-dimensional plot. The resulting plots are shown in Figures 5–21(a) and (b). [Note that mesh(y) will produce a three-dimensional plot the same as Figure 5–21(b) except that $x$ axis and $y$ axis are interchanged. See Problem A–5–17.]

When a MATLAB program involves repetitive computations, many different MATLAB programs can be written. In this book many different MATLAB programs using loops such as "while loop" and "for loop" are presented for illustration purposes. The reader is advised to study all those programs and try to improve them if possible.

**MATLAB Program 5–5**

```
% ------- Two-dimensional plot and three-dimensional plot of unit-step
% response curves for the standard second-order system with wn = 1
% and zeta = 0, 0.2, 0.4, 0.6, 0.8, and 1. -------

t = 0:0.2:10;
zeta = [0   0.2   0.4   0.6   0.8   1];
    for n = 1:6;
    num = [0   0   1];
    den = [1   2*zeta(n)   1];
    [y(1:51,n),x,t] = step(num,den,t);
    end

% To plot a two-dimensional diagram, enter the command plot(t,y).

plot(t,y)
grid
title('Plot of Unit-Step Response Curves with \omega_n = 1 and \zeta = 0, 0.2, 0.4, 0.6, 0.8, 1')
xlabel('t (sec)')
ylabel('Response')
text(4.1,1.86,'\zeta = 0')
text(3.5,1.5,'0.2')
text(3 .5,1.24,'0.4')
text(3.5,1.08,'0.6')
text(3.5,0.95,'0.8')
text(3.5,0.86,'1.0')

% To plot a three-dimensional diagram, enter the command mesh(t,zeta,y').

mesh(t,zeta,y')
title('Three-Dimensional Plot of Unit-Step Response Curves')
xlabel('t Sec')
ylabel('\zeta')
zlabel('Response')
```





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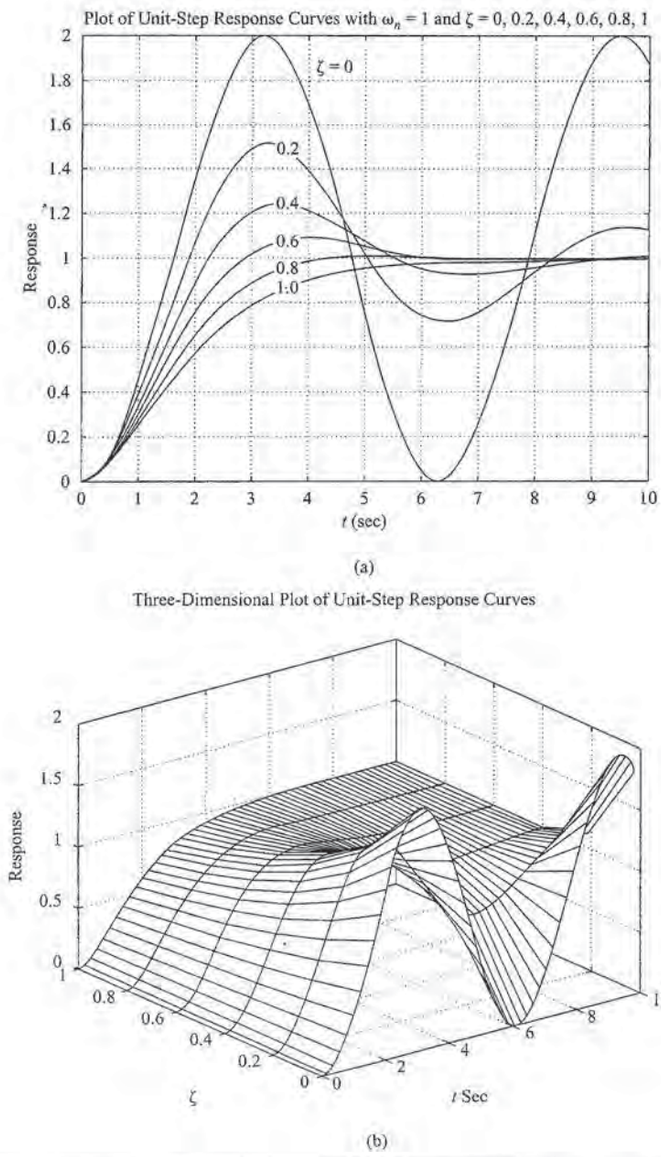


**Figure 5–21**
(a) Two-dimensional plots of unit-step response curves for $\zeta = 0, 0.2, 0.4, 0.6, 0.8$, and 1.0; (b) three-dimensional plots of unit-step response curves.

**Obtaining Rise Time, Peak Time, Maximum Overshoot, and Settling Time with MATLAB.** MATLAB can conveniently be used to obtain the rise time, peak time, maximum overshoot, and settling time. Consider the system defined by

$$\frac{C(s)}{R(s)} = \frac{25}{s^2 + 6s + 25}$$

MATLAB Program 5–6 yields the rise time, peak time, maximum overshoot, and settling time. A unit-step response curve for this system is given in Figure 5–22 to verify the





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results obtained with MATLAB Program 5–6. (Note that this program can also be applied to higher-order systems. See Problem **A–5–12**.)

**MATLAB Program 5–6**

```
% ------- This is a MATLAB program to find the rise time, peak time,
% maximum overshoot, and settling time of the second-order system
% and higher-order system -------
% ------- In this example, we assume zeta = 0.6 and wn = 5 -------
num = [0  0  25];
den = [1  6  25];
t = 0:0.005:5;
[y,x,t] = step(num,den,t);
r = 1; while y(r) < 1.0001; r = r + 1; end;
rise_time = (r - 1)*0.005
rise_time =
   0.5550
[ymax,tp] = max(y);
peak_time = (tp - 1)*0.005
peak_time =
   0.7850
max_overshoot = ymax-1
max_overshoot =
   0.0948
s = 1001; while y(s) > 0.98 & y(s) < 1.02; s = s - 1; end;
settling_time = (s - 1)*0.005
settling_time =
   1.1850
```

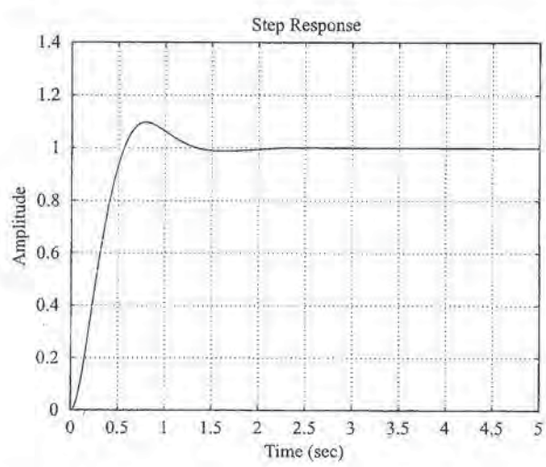


**Figure 5–22**
Unit-step response curve.





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**Impulse Response.** The unit-impulse response of a control system may be obtained by using one of the following MATLAB commands:

```
impulse(num,den)
impulse(A,B,C,D)
[y,x,t] = impulse(num,den)
[y,x,t] = impulse(num,den,t)                    (5–41)
[y,x,t] = impulse(A,B,C,D)
[y,x,t] = impulse(A,B,C,D,iu)                   (5–42)
[y,x,t] = impulse(A,B,C,D,iu,t)                 (5–43)
```

The command impulse(num,den) plots the unit-impulse response on the screen. The command impulse(A,B,C,D) produces a series of unit-impulse-response plots, one for each input and output combination of the system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

with the time vector automatically determined. Note that in Equations (5–42) and (5–43) the scalar iu is an index into the inputs of the system and specifies which input to be used for the impulse response.

Note also that in Equations (5–41) and (5–43) t is the user-supplied time vector. The vector t specifies the times at which the impulse response is to be computed.

If MATLAB is invoked with the left-hand argument [y,x,t], such as in the case of [y,x,t] = impulse(A,B,C,D), the command returns the output and state responses of the system and the time vector t. No plot is drawn on the screen. The matrices y and x contain the output and state responses of the system evaluated at the time points t. (y has as many columns as outputs and one row for each element in t. x has as many columns as state variables and one row for each element in t.)

**EXAMPLE 5–5** Obtain the unit-impulse response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

A possible MATLAB program is shown in MATLAB Program 5–7. The resulting response curve is shown in Figure 5–23.

**MATLAB Program 5–7**

```
A = [0 1;-1 -1];
B = [0;1];
C = [1 0];
D = [0];
impulse(A,B,C,D);
grid;
title('Unit-Impulse Response')
```





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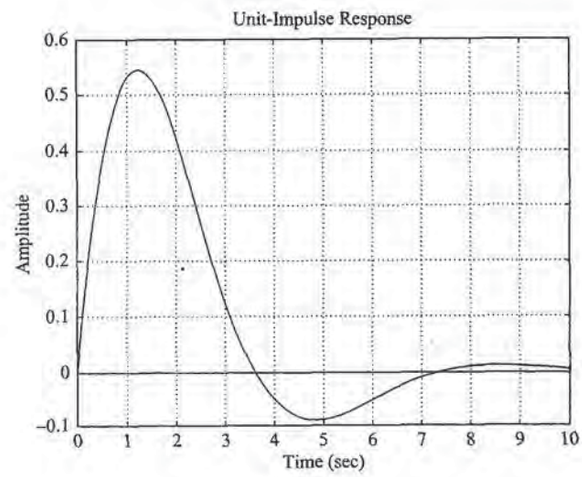


**Figure 5–23**
Unit-impulse response curve.

**EXAMPLE 5–6** Obtain the unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = G(s) = \frac{1}{s^2 + 0.2s + 1}$$

MATLAB Program 5–8 will produce the unit-impulse response. The resulting plot is shown in Figure 5–24.

**MATLAB Program 5–8**

```
num = [0  0  1];
den = [1  0.2  1];
impulse(num,den);
grid
title('Unit-Impulse Response of G(s) = 1/(s^2 + 0.2s + 1)')
```

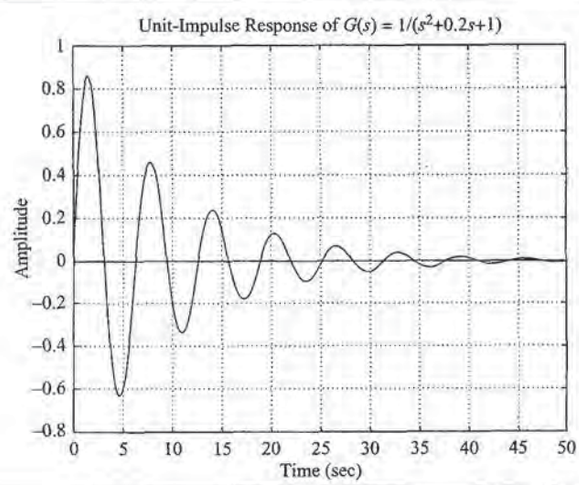


**Figure 5–24**
Unit-impulse response curve.





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**Alternative Approach to Obtain Impulse Response.** Note that when the initial conditions are zero the unit-impulse response of $G(s)$ is the same as the unit-step response of $sG(s)$.

Consider the unit-impulse response of the system considered in Example 5–6. Since $R(s) = 1$ for the unit-impulse input, we have

$$\frac{C(s)}{R(s)} = C(s) = G(s) = \frac{1}{s^2 + 0.2s + 1}$$

$$= \frac{s}{s^2 + 0.2s + 1}\frac{1}{s}$$

We can thus convert the unit-impulse response of $G(s)$ to the unit-step response of $sG(s)$.

If we enter the following num and den into MATLAB,

```
num = [0 1 0]
den = [1 0.2 1]
```

and use the step-response command; as given in MATLAB Program 5–9, we obtain a plot of the unit-impulse response of the system as shown in Figure 5–25.

**MATLAB Program 5–9**

```
num = [0 1 0];
den = [1 0.2 1];
step(num,den);
grid
title('Unit-Step Response of sG(s) = s/(s^2 + 0.2s + 1)')
```

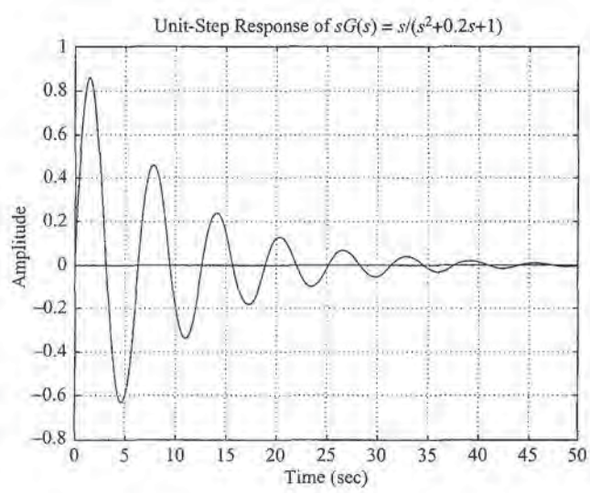


**Figure 5–25** Unit-impulse response curve obtained as the unit-step response of $sG(s) = s/(s^2 + 0.2s + 1)$.





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**Ramp Response.** There is no ramp command in MATLAB. Therefore, we need to use the step command or the lsim command (presented later) to obtain the ramp response. Specifically, to obtain the ramp response of the transfer-function system $G(s)$, divide $G(s)$ by $s$ and use the step-response command. For example, consider the closed-loop system

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

For a unit-ramp input, $R(s) = 1/s^2$. Hence

$$C(s) = \frac{1}{s^2 + s + 1}\frac{1}{s^2} = \frac{1}{(s^2 + s + 1)s}\frac{1}{s}$$

To obtain the unit-ramp response of this system, enter the following numerator and denominator into the MATLAB program,

```
num = [0 0 0 1];
den = [1 1 1 0];
```

and use the step-response command. See MATLAB Program 5–10. The plot obtained by using this program is shown in Figure 5–26.

**MATLAB Program 5–10**

```
% --------------- Unit-ramp response ---------------

% ***** The unit-ramp response is obtained as the unit-step
% response of G(s)/s *****

% ***** Enter the numerator and denominator of G(s)/s *****

num = [0 0 0 1];
den = [1 1 1 0];

% ***** Specify the computing time points (such as t = 0:0.1:7)
% and then enter step-response command: c = step(num,den,t) *****

t = 0:0.1:7;
c = step(num,den,t);

% ***** In plotting the ramp-response curve, add the reference
% input to the plot. The reference input is t. Add to the
% argument of the plot command with the following: t,t,'-'. Thus
% the plot command becomes as follows: plot(t,c,'o',t,t,'-') *****

plot(t,c,'o',t,t,'-')

% ***** Add grid, title, xlabel, and ylabel *****

grid
title('Unit-Ramp Response Curve for System G(s) = 1/(s^2 + s + 1)')
xlabel('t Sec')
ylabel('Input and Output')
```





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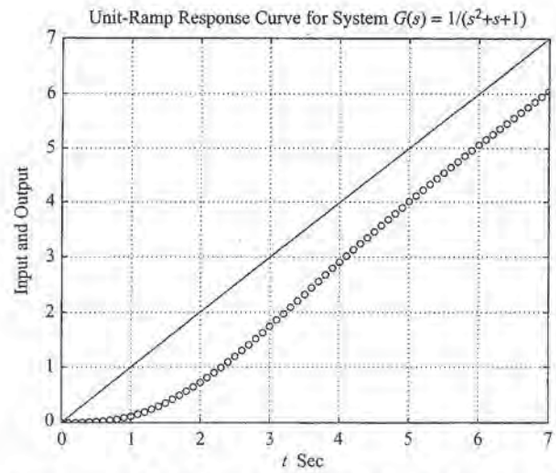


**Figure 5–26**
Unit-ramp response curve.

**Unit-Ramp Response of a System Defined in State Space.** Next, we shall treat the unit-ramp response of the system in state-space form. Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where $u$ is the unit-ramp function. In what follows, we shall consider a simple example to explain the method. Consider the case where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \mathbf{0}$$
$$\mathbf{C} = [1 \quad 0], \qquad D = [0]$$

When the initial conditions are zeros, the unit-ramp response is the integral of the unit-step response. Hence the unit-ramp response can be given by

$$z = \int_0^t y\,dt \tag{5–44}$$

From Equation (5–44), we obtain

$$\dot{z} = y = x_1 \tag{5–45}$$

Let us define

$$z = x_3$$

Then Equation (5–45) becomes

$$\dot{x}_3 = x_1 \tag{5–46}$$

Combining Equation (5–46) with the original state-space equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{5–47}$$

$$z = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5–48}$$





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where $u$ appearing in Equation (5–47) is the unit-step function. These equations can be written as

$$\dot{\mathbf{x}} = \mathbf{AAx} + \mathbf{BB}u$$
$$z = \mathbf{CCx} + DDu$$

where

$$\mathbf{AA} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \mathbf{C} & 0 \end{array}\right]$$

$$\mathbf{BB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \qquad \mathbf{CC} = [0 \quad 0 \quad 1], \qquad DD = [0]$$

Note that $x_3$ is the third element of $\mathbf{x}$. A plot of the unit-ramp response curve $z(t)$ can be obtained by entering MATLAB Program 5–11 into the computer. A plot of the unit-ramp response curve obtained from this MATLAB program is shown in Figure 5–27.

**MATLAB Program 5–11**

```
% --------------- Unit-ramp response ---------------
% ***** The unit-ramp response is obtained by adding a new
% state variable x3. The dimension of the state equation
% is enlarged by one *****
% ***** Enter matrices A, B, C, and D of the original state
% equation and output equation *****
A = [0  1;-1  -1];
B = [0;  1];
C = [1  0];
D = [0];
% ***** Enter matrices AA, BB, CC, and DD of the new,
% enlarged state equation and output equation *****
AA = [A  zeros(2,1);C  0];
BB = [B;0];
CC = [0  0  1];
DD = [0];
% ***** Enter step-response command: [z,x,t] = step(AA,BB,CC,DD) *****
[z,x,t] = step(AA,BB,CC,DD);
% ***** In plotting x3 add the unit-ramp input t in the plot
% by entering the following command: plot(t,x3,'o',t,t,'-') *****
x3 = [0  0  1]*x'; plot(t,x3,'o',t,t,'-')
grid
title('Unit-Ramp Response')
xlabel('t Sec')
ylabel('Input and Output')
```





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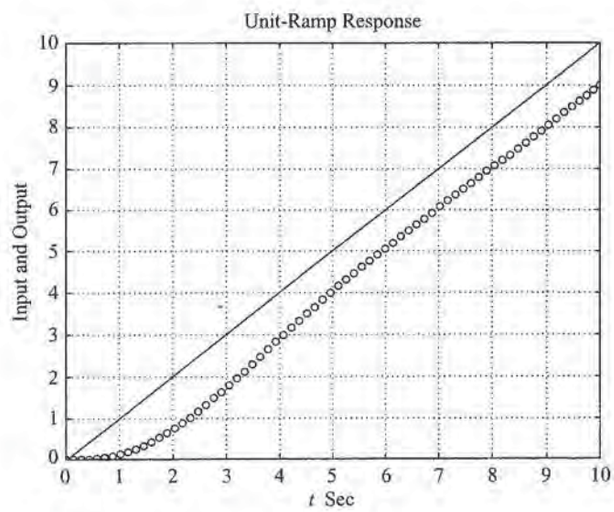


**Figure 5–27**
Unit-ramp response curve.

**Obtaining Response to Arbitrary Input.** To obtain the response to an arbitrary input, the command lsim may be used. The commands like

```
lsim(num,den,r,t)

lsim(A,B,C,D,u,t)

y = lsim(num,den,r,t)

y = lsim(A,B,C,D,u,t)
```

will generate the response to input time function r or u. See the following two examples. (Also, see Problems **A–5–14** through **A–5–16**.)

**EXAMPLE 5–7** Using the lsim command, obtain the unit-ramp response of the following system:

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$





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We may enter MATLAB Program 5–12 into the computer to obtain the unit-ramp response. The resulting plot is shown in Figure 5–28.

**MATLAB Program 5–12**

```
% ------- Ramp Response -------
num = [0  0  1];
 den = [1  1  1];
t = 0:0.1:8;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
title('Unit-Ramp Response Obtained by Use of Command "lsim"')
xlabel('t Sec')
ylabel('Unit-Ramp Input and System Output')
text(2.1,4.65,'Unit-Ramp Input')
text(4.5,2.0,'Output')
```

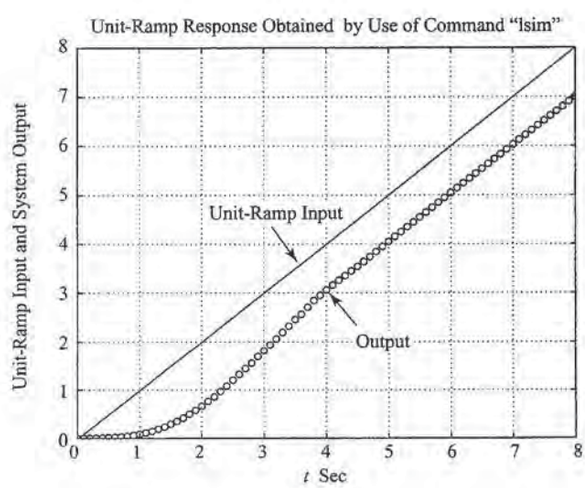


**Figure 5–28**
Unit-ramp response.





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**EXAMPLE 5–8** Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0.5 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using MATLAB, obtain the response curves $y(t)$ when the input $u$ is given by

1. $u$ = unit-step input
2. $u = e^{-t}$

Assume that the initial state is $\mathbf{x}(0) = \mathbf{0}$.

A possible MATLAB program to produce the responses of this system to the unit-step input $[u = 1(t)]$ and the exponential input $[u = e^{-t}]$ is shown in MATLAB Program 5–13. The resulting response curves are shown in Figures 5–29(a) and (b), respectively.

**MATLAB Program 5–13**

```
t = 0:0.1:12;
A = [-1  0.5;-1  0];
B = [0;1];
C = [1  0];
D = [0];

% For the unit-step input u = 1(t), use the command "y = step(A,B,C,D,1,t)".

y = step(A,B,C,D,1,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t Sec')
ylabel('Output')

% For the response to exponential input u = exp(-t), use the command
% z = lsim(A,B,C,D,u,t).

u = exp(-t);
z = lsim(A,B,C,D,u,t);
plot(t,u,'-',t,z,'o')
grid
title('Response to Exponential Input u = exp(-t)')
xlabel('t Sec')
ylabel('Exponential Input and System Output')
text(2.3,0.49,'Exponential input')
text(6.4,0.28,'Output')
```





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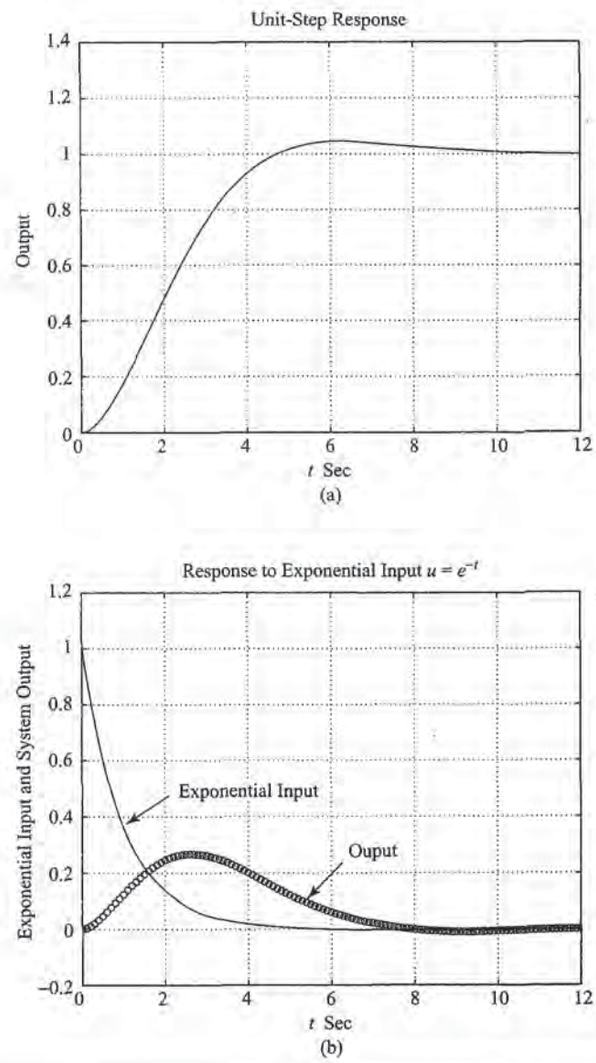


**Figure 5–29**
(a) Unit-step response;
(b) response to input $u = e^{-t}$.

**Response to Initial Condition.** In what follows we shall present a few methods for obtaining the response to an initial condition. Commands that we may use are "step" or "initial". We shall first present a method to obtain the response to initial condition using a simple example. Then we shall discuss the response to initial condition when the system is given in state-space form. Finally, we shall present a command initial to obtain the response of a system given in a state-space form.





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**EXAMPLE 5–9** Consider the mechanical system shown in Figure 5–30, where $m = 1$ kg, $b = 3$ N-sec/m, and $k = 2$ N/m. Assume that at $t = 0$ the mass $m$ is pulled downward such that $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/sec. The displacement $x(t)$ is measured from the equilibrium position before the mass is pulled down. Obtain the motion of the mass subjected to the initial condition. (Assume no external forcing function.)

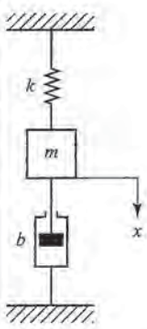


**Figure 5–30**
Mechanical system.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

with the initial conditions $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/sec. ($x$ is measured from the equilibrium position.) The Laplace transform of the system equation gives

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + b[sX(s) - x(0)] + kX(s) = 0$$

or

$$(ms^2 + bs + k)X(s) = mx(0)s + m\dot{x}(0) + bx(0)$$

Solving this last equation for $X(s)$ and substituting the given numerical values, we obtain

$$X(s) = \frac{mx(0)s + m\dot{x}(0) + bx(0)}{ms^2 + bs + k}$$

$$= \frac{0.1s + 0.35}{s^2 + 3s + 2}$$

This equation can be written as

$$X(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}\frac{1}{s}$$

Hence the motion of the mass $m$ may be obtained as the unit-step response of the following system:

$$G(s) = \frac{0.1s^2 + 0.35s}{s^2 + 3s + 2}$$

MATLAB Program 5–14 will give a plot of the motion of the mass. The plot is shown in Figure 5–31.

**MATLAB Program 5–14**

```
% --------------- Response to initial condition ---------------

% ***** System response to initial condition is converted to
% a unit-step response by modifying the numerator polynomial *****

% ***** Enter the numerator and denominator of the transfer
% function G(s) *****

num = [0.1  0.35  0];
den = [1  3  2];

% ***** Enter the following step-response command *****

step(num,den)

% ***** Enter grid and title of the plot *****

grid
title('Response of Spring-Mass-Damper System to Initial Condition')
```





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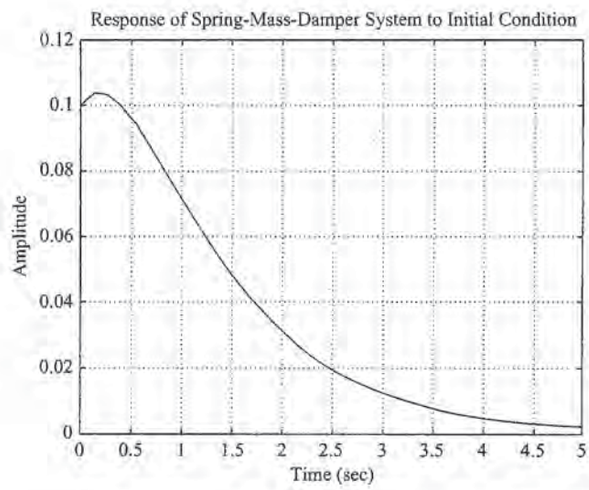


**Figure 5–31**
Response of the mechanical system considered in Example 5–9.

**Response to Initial Condition (State-Space Approach, Case 1).** Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{5–49}$$

Let us obtain the response $\mathbf{x}(t)$ when the initial condition $\mathbf{x}(0)$ is specified. (There is no external input function acting on this system.) Assume that $\mathbf{x}$ is an $n$-vector.

First, take Laplace transforms of both sides of Equation (5–49).

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

This equation can be rewritten as

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{x}(0) \tag{5–50}$$

Taking the inverse Laplace transform of Equation (5–50), we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{x}(0)\,\delta(t) \tag{5–51}$$

(Notice that by taking the Laplace transform of a differential equation and then by taking the inverse Laplace transform of the Laplace-transformed equation we generate a differential equation that involves the initial condition.)

Now define

$$\dot{\mathbf{z}} = \mathbf{x} \tag{5–52}$$

Then Equation (5–51) can be written as

$$\ddot{\mathbf{z}} = \mathbf{A}\dot{\mathbf{z}} + \mathbf{x}(0)\,\delta(t) \tag{5–53}$$

By integrating Equation (5–53) with respect to $t$, we obtain

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–54}$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$





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Referring to Equation (5–52), the state $\mathbf{x}(t)$ is given by $\dot{\mathbf{z}}(t)$. Thus,

$$\mathbf{x} = \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–55}$$

The solution of Equations (5–54) and (5–55) gives the response to the initial condition.

Summarizing, the response of Equation (5–49) to the initial condition $\mathbf{x}(0)$ is obtained by solving the following state-space equations:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

MATLAB commands to obtain the response curves in one diagram are given next.

```
[x,z,t] = step(A,B,A,B);
x1 = [1  0  0 ... 0]*x';
x2 = [0  1  0 ... 0]*x';
.
.
.
xn = [0  0  0 ... 1]*x';
plot(t,x1,t,x2, ... ,t,xn)
```

**Response to Initial Condition (State-Space Approach, Case 2).** Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{5–56}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{5–57}$$

(Assume that $\mathbf{x}$ is an $n$-vector and $\mathbf{y}$ is an $m$-vector.)

Similar to case 1, by defining

$$\dot{\mathbf{z}} = \mathbf{x}$$

we can obtain the following equation:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{5–58}$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Noting that $\mathbf{x} = \dot{\mathbf{z}}$, Equation (5–57) can be written as

$$\mathbf{y} = \mathbf{C}\dot{\mathbf{z}} \tag{5–59}$$

By substituting Equation (5–58) into Equation (5–59), we obtain

$$\mathbf{y} = \mathbf{C}(\mathbf{A}\mathbf{z} + \mathbf{B}u) = \mathbf{C}\mathbf{A}\mathbf{z} + \mathbf{C}\mathbf{B}u \tag{5–60}$$





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The solution of Equations (5–58) and (5–60) gives the response of the system to a given initial condition. MATLAB commands to obtain the response curves (output curves y1 versus t, y2 versus t, ... , ym versus t) are shown next.

```
[y,z,t] = step(A,B,C*A,C*B);
y1 = [1  0  0 ... 0]*y';
y2 = [0  1  0 ... 0]*y';
.
.
.
ym = [0  0  0 ... 1]*y';
plot(t,y1,t,y2, ... ,t,ym)
```

**EXAMPLE 5–10** Obtain the response of the system subjected to the given initial condition.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Obtaining the response of the system to the given initial condition becomes that of solving the unit-step response of the following system:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$
$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Hence a possible MATLAB program for obtaining the response may be given as shown in MATLAB Program 5–15. The resulting response curves are shown in Figure 5–32.

**MATLAB Program 5–15**

```
t = 0:0.01:3;
A = [0  1;-10  -5];
B = [2;1];
[x,z,t] = step(A,B,A,B,1,t);
x1 = [1  0]*x';
x2 = [0  1]*x';
plot(t,x1,'x',t,x2,'-')
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
gtext('x1')
gtext('x2')
```



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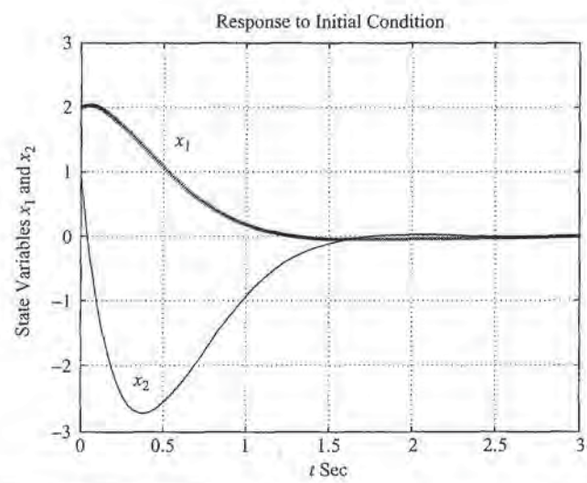


**Figure 5–32**
Response of system in Example 5–10 to initial condition.

**Obtaining Response to Initial Condition by Use of Command Initial.** If the system is given in the state-space form, then the following command

initial(A,B,C,D,[initial condition],t)

will produce the response to the initial condition.

Suppose that we have the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 0], \qquad D = 0$$

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then the command "initial" can be used as shown in MATLAB Program 5–16 to obtain the response to the initial condition. The response curves $x_1(t)$ and $x_2(t)$ are shown in Figure 5–33. They are the same as those shown in Figure 5–32.





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**MATLAB Program 5–16**

```
t = 0:0.05:3;
A = [0  1;-10  -5];
B = [0;0];
C = [0  0];
D = [0];
[y,x] = initial(A,B,C,D,[2;1],t);
x1 = [1  0]*x';
x2 = [0  1]*x';
plot(t,x1,'o',t,x1,t,x2,'x',t,x2)
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
gtext('x1')
gtext('x2')
```

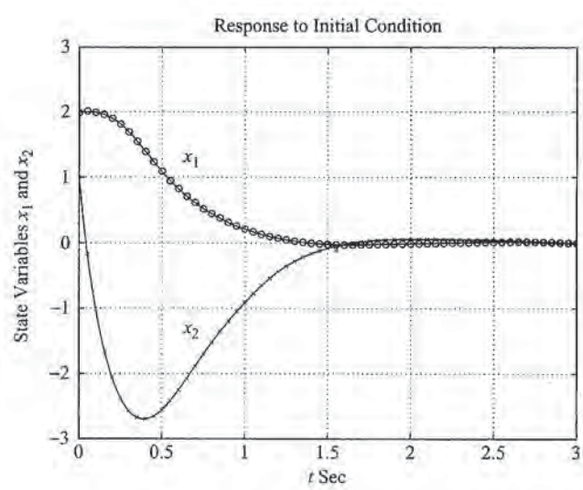


**Figure 5–33**
Response curves to initial condition.

**EXAMPLE 5–11** Consider the following system that is subjected to the initial condition. (No external forcing function is present.)

$$\dddot{y} + 8\ddot{y} + 17\dot{y} + 10y = 0$$

$$y(0) = 2, \qquad \dot{y}(0) = 1, \qquad \ddot{y}(0) = 0.5$$

Obtain the response $y(t)$ to the given initial condition.





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By defining the state variables as

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

we obtain the following state-space representation for the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A possible MATLAB program to obtain the response $y(t)$ is given in MATLAB Program 5–17. The resulting response curve is shown in Figure 5–34.

**MATLAB Program 5–17**

```
t = 0:0.05:10;
A = [0 1 0;0 0 1;-10 -17 -8];
B = [0;0;0];
C = [1 0 0];
D = [0];
y = initial(A,B,C,D,[2;1;0.5],t);
plot(t,y)
grid
title('Response to Initial Condition')
xlabel('t (sec)')
ylabel('Output y')
```

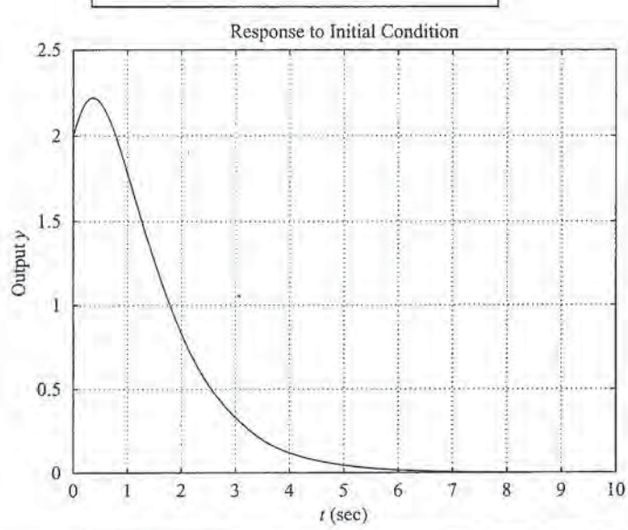


**Figure 5–34**
Response $y(t)$ to initial condition.





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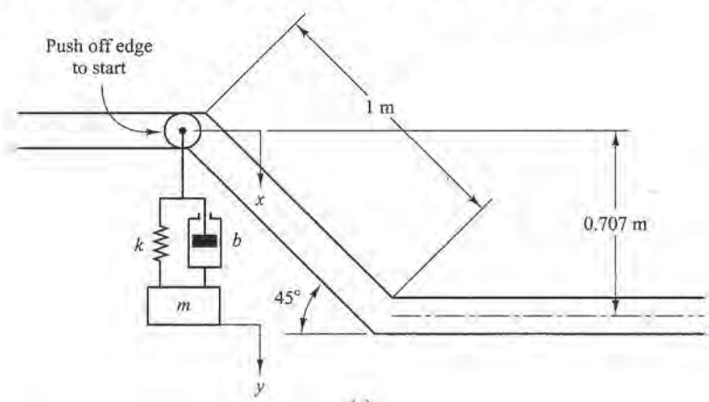
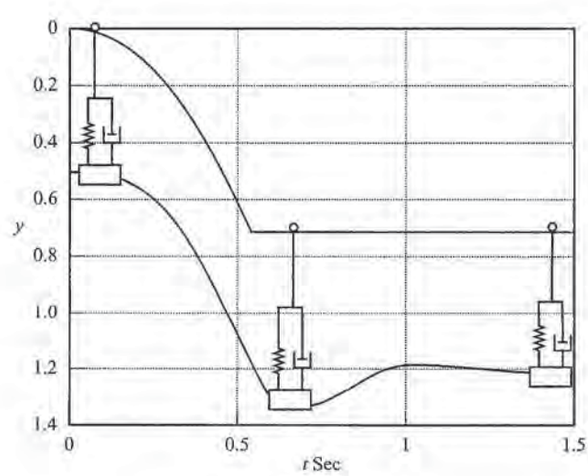
## 5–6 AN EXAMPLE PROBLEM SOLVED WITH MATLAB

The purpose of this section is to present a MATLAB solution to the response of a mechanical vibratory system.

**Mechanical Vibratory System.** Consider the mechanical vibratory system shown in Figure 5–35(a). A wheel has a spring-mass-damper system hanging from it. The wheel is in a track that contains a flat (horizontal) portion, a slanted (downward at 45°) portion, and another flat (horizontal) portion. We start the motion of the system by nudging the wheel over the edge of the ramp. As the wheel drops down the ramp for a total of 0.707 m (vertically measured), the mass $m$ hanging from the spring and damper drops with it, and the mass gains momentum that dissipates gradually. In this problem the wheel is assumed to slide on the slanted portion of the track without friction. On the second flat portion of the track, the wheel slides and rolls. The wheel continues to move on the flat portion of the track until it is stopped by an external means.




(a)




(b)

**Figure 5–35**
(a) Wheel with hanging mass-damper system; (b) dynamic response of the system.





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Assume the following numerical values for $m$, $b$, and $k$:

$$m = 4 \text{ kg}, \qquad b = 40 \text{ N-sec/m}, \qquad k = 400 \text{ N/m}$$

Assume also that the mass $m_p$ of the wheel is negligible compared with the mass $m$. Obtain $x(t)$, the vertical motion of the wheel. Then obtain $Y(s)$, the Laplace transform of $y(t)$, which represents the up and down motion of mass $m$. The coordinate $y$ is attached to the spring-mass-damper system as shown in Figure 5–35(a) and is measured from the equilibrium position of the system. The initial conditions are that $y(0) = 0$ and $\dot{y}(0) = 0$. Note that in this problem we are interested only in the vertical motions of the spring-mass-damper system. Note also that the system is frictionless with the exception of the damper, which relies on viscosity for its operation.

As the spring-mass-damper component travels down the ramp, it will undergo an acceleration produced as a result of the gravity force. When the spring-mass-damper reaches the level region at the bottom of the ramp, a shock will immediately be imposed on the spring-mass-damper component. It will, however, eventually come to a state of equilibrium following the impact due to the settling effects of the damper and spring. The dynamic response of this system is shown in Figure 5–35(b).

***Determination of x(t).*** The system starts with zero initial velocity and follows the track. The input to the system is the vertical position $x$ along the track, and the output is the vertical position $y$ of the mass. Since we assume no sliding friction, referring to Figure 5–36(a) we have in the $z$ direction the following equation:

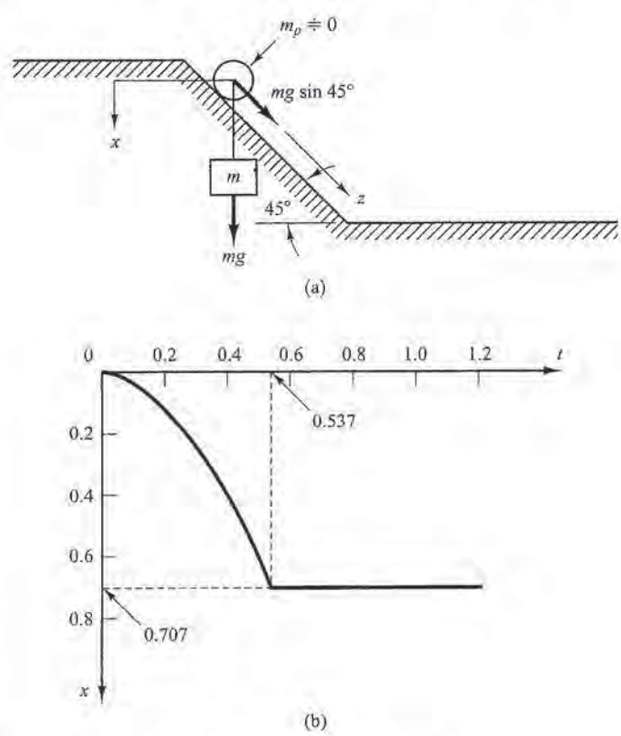


**Figure 5–36**
(a) Wheel with mass $m$ slides on inclined plane; (b) curve $x(t)$ versus $t$.





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$$m\ddot{z} = mg \sin 45^\circ$$

or

$$\ddot{z} = 9.81 \times 0.707 = 6.9357$$

Let us define the time it takes for the wheel to move from $z = 0$ to $z = 1$ m as $t_1$. Then

$$z = 6.9357\frac{t_1^2}{2} = 1$$

which yields

$$t_1 = 0.537 \text{ sec}$$

Thus $x(t)$ can be given as follows:

$$\begin{aligned} x(t) = 0.707z = 0.707 \times 3.4678t^2 &= 2.452t^2, \quad \text{for } 0 \le t \le 0.537 \\ &= 0.707, \quad \text{for } 0.537 < t \end{aligned}$$

It follows that from $t = 0.537$ sec to $t = \infty$ we have an input defined by a constant of 0.707. The position $x$ at the end of the ramp is 0.707 and it takes 0.537 sec to get there. A curve $x(t)$ versus $t$ is shown in Figure 5–36(b). Note that the positive direction of $x(t)$ is vertically downward.

To get a better picture of the events taking place in the system, we need to look at the input, shown in Figure 5–36(b). The effects of gravity do not allow us to model the behavior of the system with an ordinary ramp, but rather a parabolic function, which is followed by the input.

***Determination of Transfer Function Y(s)/X(s).*** Next, we shall obtain the equation of motion for the system and then the transfer function $Y(s)/X(s)$. Since $y$ is measured from the equilibrium position, the system equation becomes

$$m\ddot{y} + b(\dot{y} - \dot{x}) + k(y - x) = 0$$

or

$$m\ddot{y} + b\dot{y} + ky = b\dot{x} + kx$$

where $x$ is the input to the system and $y$ is the output. By substituting the given numerical values for $m$, $b$, and $k$, we obtain

$$4\ddot{y} + 40\dot{y} + 400y = 40\dot{x} + 400x$$

or

$$\ddot{y} + 10\dot{y} + 100y = 10\dot{x} + 100x$$

The transfer function for the system can now be given by

$$\frac{Y(s)}{X(s)} = \frac{10s + 100}{s^2 + 10s + 100}$$

where the input $x(t)$ is given by $2.452t^2$ (for $0 \le t \le 0.537$) and 0.707 (for $0.537 < t$). Our problem here is to plot the response curve with MATLAB.





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**MATLAB Solution.** Since the input consists of two different time functions (one is an acceleration input and the other is a step input,) we shall use "lsim" command in this problem. Since

$$x(t) = 2.452t^2, \qquad \text{for } 0 \le t \le 0.537$$
$$x(t) = 0.707, \qquad \text{for } 0.537 < t$$

we choose the computing time interval to be 0.001 sec. We define $x$ as an array of points in MATLAB. This array initially follows $x(t) = 2.452t^2$ and, after $t = 0.537$ sec, follows $x(t) = 0.707$. We assume that the time region is $0 \le t \le 1.5$.

The acceleration input in the first part can be written as

```
t1 = 0:0.001:0.537
x1 = 2.452*(t1.^2)
```

where t1 represents a time count and x1 is the first part of the complete input function. (There are 538 calculation points from the initial position until the input reaches 0.707 m.) For the second part of the input, we need a step function with magnitude 0.707. After time 0.537 sec,

```
t2 = 0.538:0.001:1.5
x2 = 0.707*ones(size(t2))
```

(There are 963 points from 0.538 sec through 1.5 sec, inclusive.) The next step is to transform both inputs to one complete input:

```
x = [x1  x2]
```

(The two input equations are transformed into a single vector, in order to appear as a single entry in the lsim command argument.)

Now we can use the lsim command

```
y = lsim(num,den,x,t)
```

and plot the response $y(t)$ as well as the input $x(t)$. A possible MATLAB program is shown in MATLAB Program 5–18. Note that the plotted input and output functions are negated. The resulting curves $x(t)$ and $y(t)$ are shown in Figure 5–37.

**MATLAB Program 5–18**

```
num = [0  10  100];
den = [1  10  100];
t1 = 0:0.001:0.537;
x1 = 2.452*(t1.^2);
t2 = 0.538:0.001:1.5;
x2 = 0.707*ones(size(t2));
t = [t1  t2];
x = [x1  x2];
y = lsim(num,den,x,t);
plot(t,-x,'x',t,-y)
grid
title('Response of Hanging Mass-Damper-Spring System')
xlabel('t (sec)')
ylable('Negative of Input x and Negative of Output y')
text(0.07,-0.66,'Negative of Input x')
text(0.55,-0.14,'Negative of Output y')
```





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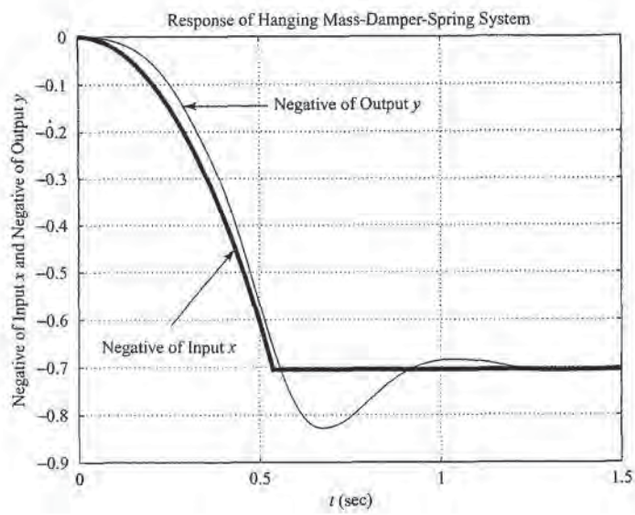


**Figure 5–37**
Curves $-x(t)$ versus $t$ and $-y(t)$ versus $t$.

## 5–7 ROUTH'S STABILITY CRITERION

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system? In Section 5–4 it was stated that a control system is stable if and only if all closed-loop poles lie in the left-half $s$ plane. Most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

where the $a$'s and $b$'s are constants and $m \leq n$. A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half $s$ plane without having to factor the denominator polynomial. (The polynomial may include parameters that MATLAB cannot handle.)

**Routh's Stability Criterion.** Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in $s$ in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{5–61}$$

   where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive





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real parts. Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further. Note that all the coefficients must be positive. This is a necessary condition, as may be seen from the following argument: A polynomial in $s$ having real coefficients can always be factored into linear and quadratic factors, such as $(s + a)$ and $(s^2 + bs + c)$, where $a$, $b$, and $c$ are real. The linear factors yield real roots and the quadratic factors yield complex-conjugate roots of the polynomial. The factor $(s^2 + bs + c)$ yields roots having negative real parts only if $b$ and $c$ are both positive. For all roots to have negative real parts, the constants $a$, $b$, $c$, and so on, in all factors must be positive. The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients. It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. The necessary but not sufficient condition for stability is that the coefficients of Equation (5–61) all be present and all have a positive sign. (If all $a$'s are negative, they can be made positive by multiplying both sides of the equation by $-1$.)

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

The process of forming rows continues until we run out of elements. (The total number of rows is $n + 1$.) The coefficients $b_1$, $b_2$, $b_3$, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$





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The evaluation of the $b$'s is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the $c$'s, $d$'s, $e$'s, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

This process is continued until the $n$th row has been completed. The complete array of coefficients is triangular. Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh's stability criterion states that the number of roots of Equation (5–61) with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed. The necessary and sufficient condition that all roots of Equation (5–61) lie in the left-half $s$ plane is that all the coefficients of Equation (5–61) be positive and all terms in the first column of the array have positive signs.

**EXAMPLE 5–12** Let us apply Routh's stability criterion to the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers. The array of coefficients becomes

$$\begin{array}{ccc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given by

$$a_1 a_2 > a_0 a_3$$





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**EXAMPLE 5–13** Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

$$\begin{array}{cccc||cccc}
s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 \\
s^3 & 2 & 4 & 0 & s^3 & \cancel{2} & \cancel{4} & \cancel{0} \\
 & & & & & 1 & 2 & 0 \\
s^2 & 1 & 5 & & s^2 & 1 & 5 & \\
s^1 & -6 & & & s^1 & -3 & & \\
s^0 & 5 & & & s^0 & 5 & &
\end{array}$$

The second row is divided by 2.

In this example, the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real parts. Note that the result is unchanged when the coefficients of any row are multiplied or divided by a positive number in order to simplify the computation.

**Special Cases.** If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number $\epsilon$ and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0 \tag{5–62}$$

The array of coefficients is

$$\begin{array}{ccc}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
s^1 & 0 \approx \epsilon & \\
s^0 & 2 &
\end{array}$$

If the sign of the coefficient above the zero ($\epsilon$) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (5–62) has two roots at $s = \pm j$.

If, however, the sign of the coefficient above the zero ($\epsilon$) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

the array of coefficients is

$$\begin{array}{lccc}
 & s^3 & 1 & -3 \\
\text{One sign change:} & s^2 & 0 \approx \epsilon & 2 \\
 & s^1 & -3 - \dfrac{2}{\epsilon} & \\
\text{One sign change:} & s^0 & 2 &
\end{array}$$

There are two sign changes of the coefficients in the first column. This agrees with the correct result indicated by the factored form of the polynomial equation.





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If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the $s$ plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots. In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row. Such roots with equal magnitudes and lying radially opposite in the $s$ plane can be found by solving the auxiliary polynomial, which is always even. For a $2n$-degree auxiliary polynomial, there are $n$ pairs of equal and opposite roots. For example, consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

$$\begin{array}{lrrrl} s^5 & 1 & 24 & -25 & \\ s^4 & 2 & 48 & -50 & \leftarrow \text{Auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & & \end{array}$$

The terms in the $s^3$ row are all zero. (Note that such a case occurs only in an odd-numbered row.) The auxiliary polynomial is then formed from the coefficients of the $s^4$ row. The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis). These pairs are obtained by solving the auxiliary polynomial equation $P(s) = 0$. The derivative of $P(s)$ with respect to $s$ is

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

The terms in the $s^3$ row are replaced by the coefficients of the last equation, that is, 8 and 96. The array of coefficients then becomes

$$\begin{array}{lrrrl} s^5 & 1 & 24 & -25 & \\ s^4 & 2 & 48 & -50 & \\ s^3 & 8 & 96 & & \leftarrow \text{Coefficients of } dP(s)/ds \\ s^2 & 24 & -50 & & \\ s^1 & 112.7 & 0 & & \\ s^0 & -50 & & & \end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

we obtain

$$s^2 = 1, \qquad s^2 = -25$$

or

$$s = \pm 1, \qquad s = \pm j5$$





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These two pairs of roots of $P(s)$ are a part of the roots of the original equation. As a matter of fact, the original equation can be written in factored form as follows:

$$(s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0$$

Clearly, the original equation has one root with a positive real part.

**Relative Stability Analysis.** Routh's stability criterion provides the answer to the question of absolute stability. This, in many practical cases, is not sufficient. We usually require information about the relative stability of the system. A useful approach for examining relative stability is to shift the $s$-plane axis and apply Routh's stability criterion. That is, we substitute

$$s = \hat{s} - \sigma \qquad (\sigma = \text{constant})$$

into the characteristic equation of the system, write the polynomial in terms of $\hat{s}$; and apply Routh's stability criterion to the new polynomial in $\hat{s}$. The number of changes of sign in the first column of the array developed for the polynomial in $\hat{s}$ is equal to the number of roots that are located to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots that lie to the right of the vertical line $s = -\sigma$.

**Application of Routh's Stability Criterion to Control System Analysis.** Routh's stability criterion is of limited usefulness in linear control system analysis mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system. It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Consider the system shown in Figure 5–38. Let us determine the range of $K$ for stability. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{cccc} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2 - \frac{9}{7}K & & \\ s^0 & K & & \end{array}$$

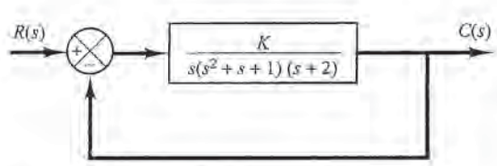


**Figure 5–38**
Control system.





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For stability, $K$ must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

Note that the ranges of design parameters that lead to stability may be determined by use of Routh's stability criterion.

## 5–8 EFFECTS OF INTEGRAL AND DERIVATIVE CONTROL ACTIONS ON SYSTEM PERFORMANCE

In this section, we shall investigate the effects of integral and derivative control actions on the system performance. Here we shall consider only simple systems so that the effects of integral and derivative control actions on system performance can be clearly seen.

**Integral Control Action.** In the proportional control of a plant whose transfer function does not possess an integrator $1/s$, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if the integral control action is included in the controller.

In the integral control of a plant, the control signal, the output signal from the controller, at any instant is the area under the actuating error signal curve up to that instant. The control signal $u(t)$ can have a nonzero value when the actuating error signal $e(t)$ is zero, as shown in Figure 5–39(a). This is impossible in the case of the proportional controller since a nonzero control signal requires a nonzero actuating error signal. (A nonzero actuating error signal at steady state means that there is an offset.) Figure 5–39(b) shows the curve $e(t)$ versus $t$ and the corresponding curve $u(t)$ versus $t$ when the controller is of the proportional type.

Note that integral control action, while removing offset or steady-state error, may lead to oscillatory response of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.

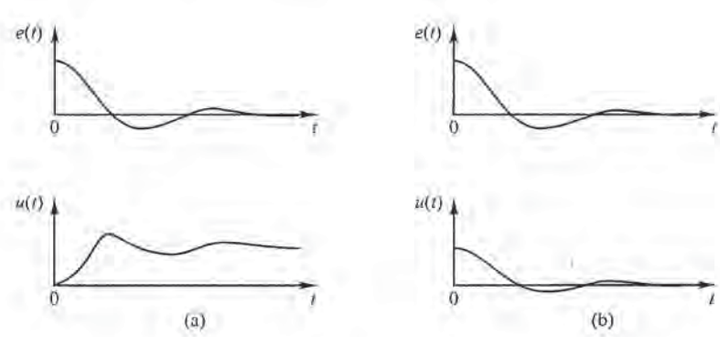


**Figure 5–39**
(a) Plots of $e(t)$ and $u(t)$ curves showing nonzero control signal when the actuating error signal is zero (integral control); (b) plots of $e(t)$ and $u(t)$ curves showing zero control signal when the actuating error signal is zero (proportional control).





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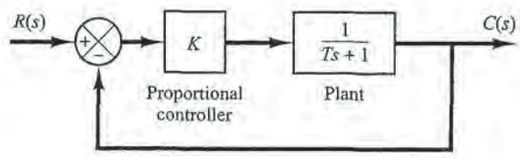


**Figure 5–40**
Proportional control system.

**Proportional Control of Systems.** We shall show that the proportional control of a system without an integrator will result in a steady-state error with a step input. We shall then show that such an error can be eliminated if integral control action is included in the controller.

Consider the system shown in Figure 5–40. Let us obtain the steady-state error in the unit-step response of the system. Define

$$G(s) = \frac{K}{Ts + 1}$$

Since

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

the error $E(s)$ is given by

$$E(s) = \frac{1}{1 + G(s)} R(s) = \frac{1}{1 + \dfrac{K}{Ts + 1}} R(s)$$

For the unit-step input $R(s) = 1/s$, we have

$$E(s) = \frac{Ts + 1}{Ts + 1 + K} \frac{1}{s}$$

The steady-state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{Ts + 1}{Ts + 1 + K} = \frac{1}{K + 1}$$

Such a system without an integrator in the feedforward path always has a steady-state error in the step response. Such a steady-state error is called an offset. Figure 5–41 shows the unit-step response and the offset.

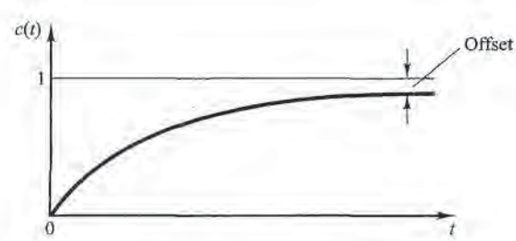


**Figure 5–41**
Unit-step response and offset.





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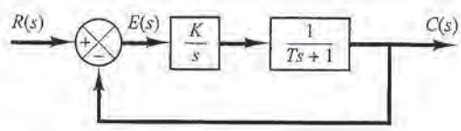


**Figure 5–42**
Integral control system.

**Integral Control of Systems.** Consider the system shown in Figure 5–42. The controller is an integral controller. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts + 1) + K}$$

Hence

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = \frac{s(Ts + 1)}{s(Ts + 1) + K}$$

Since the system is stable, the steady-state error for the unit-step response can be obtained by applying the final-value theorem, as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{s^2(Ts + 1)}{Ts^2 + s + K} \frac{1}{s} \\ &= 0 \end{aligned}$$

Integral control of the system thus eliminates the steady-state error in the response to the step input. This is an important improvement over the proportional control alone, which gives offset.

**Response to Torque Disturbances (Proportional Control).** Let us investigate the effect of a torque disturbance occurring at the load element. Consider the system shown in Figure 5–43. The proportional controller delivers torque $T$ to position the load element, which consists of moment of inertia and viscous friction. Torque disturbance is denoted by $D$.

Assuming that the reference input is zero or $R(s) = 0$, the transfer function between $C(s)$ and $D(s)$ is given by

$$\frac{C(s)}{D(s)} = \frac{1}{Js^2 + bs + K_p}$$

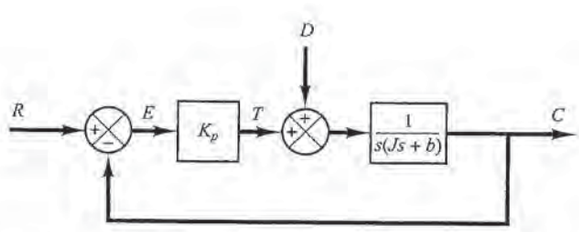


**Figure 5–43**
Control system with a torque disturbance.





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Hence

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{1}{Js^2 + bs + K_p}$$

The steady-state error due to a step disturbance torque of magnitude $T_d$ is given by

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{-s}{Js^2 + bs + K_p} \frac{T_d}{s} \\ &= -\frac{T_d}{K_p} \end{aligned}$$

At steady state, the proportional controller provides the torque $-T_d$, which is equal in magnitude but opposite in sign to the disturbance torque $T_d$. The steady-state output due to the step disturbance torque is

$$c_{ss} = -e_{ss} = \frac{T_d}{K_p}$$

The steady-state error can be reduced by increasing the value of the gain $K_p$. Increasing this value, however, will cause the system response to be more oscillatory.

**Response to Torque Disturbances (Proportional-Plus-Integral Control).** To eliminate offset due to torque disturbance, the proportional controller may be replaced by a proportional-plus-integral controller.

If integral control action is added to the controller, then, as long as there is an error signal, a torque is developed by the controller to reduce this error, provided the control system is a stable one.

Figure 5–44 shows the proportional-plus-integral control of the load element, consisting of moment of inertia and viscous friction.

The closed-loop transfer function between $C(s)$ and $D(s)$ is

$$\frac{C(s)}{D(s)} = \frac{s}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}}$$

In the absence of the reference input, or $r(t) = 0$, the error signal is obtained from

$$E(s) = -\frac{s}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}} D(s)$$

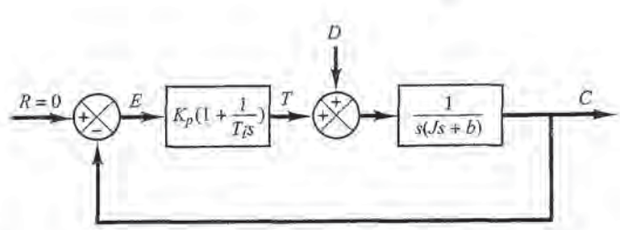


**Figure 5–44** Proportional-plus-integral control of a load element consisting of moment of inertia and viscous friction.





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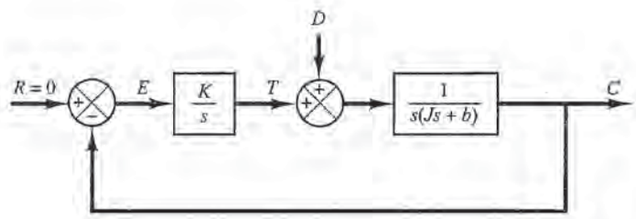


**Figure 5–45**
Integral control of a load element consisting of moment of inertia and viscous friction.

If this control system is stable, that is, if the roots of the characteristic equation

$$Js^3 + bs^2 + K_p s + \frac{K_p}{T_i} = 0$$

have negative real parts, then the steady-state error in the response to a unit-step disturbance torque can be obtained by applying the final-value theorem as follows:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} \frac{-s^2}{Js^3 + bs^2 + K_p s + \dfrac{K_p}{T_i}} \frac{1}{s} \\ &= 0 \end{aligned}$$

Thus steady-state error to the step disturbance torque can be eliminated if the controller is of the proportional-plus-integral type.

Note that the integral control action added to the proportional controller has converted the originally second-order system to a third-order one. Hence the control system may become unstable for a large value of $K_p$ since the roots of the characteristic equation may have positive real parts. (The second-order system is always stable if the coefficients in the system differential equation are all positive.)

It is important to point out that if the controller were an integral controller, as in Figure 5–45, then the system always becomes unstable because the characteristic equation

$$Js^3 + bs^2 + K = 0$$

will have roots with positive real parts. Such an unstable system cannot be used in practice.

Note that in the system of Figure 5–44 the proportional control action tends to stabilize the system, while the integral control action tends to eliminate or reduce steady-state error in response to various inputs.

**Derivative Control Action.** Derivative control action, when added to a proportional controller, provides a means of obtaining a controller with high sensitivity. An advantage of using derivative control action is that it responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large. Derivative control thus anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system.





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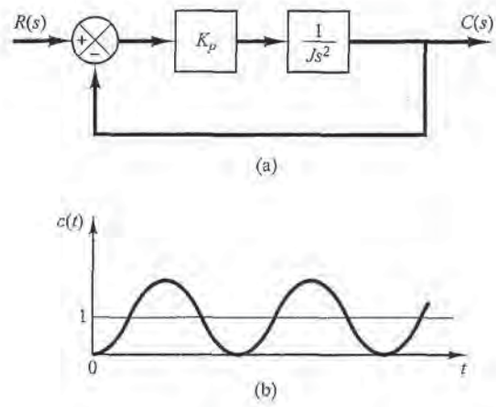


**Figure 5–46**
(a) Proportional control of a system with inertia load; (b) response to a unit-step input.

Although derivative control does not affect the steady-state error directly, it adds damping to the system and thus permits the use of a larger value of the gain $K$, which will result in an improvement in the steady-state accuracy.

Because derivative control operates on the rate of change of the actuating error and not the actuating error itself, this mode is never used alone. It is always used in combination with proportional or proportional-plus-integral control action.

**Proportional Control of Systems with Inertia Load.** Before we discuss the effect of derivative control action on system performance, we shall consider the proportional control of an inertia load.

Consider the system shown in Figure 5–46(a). The closed-loop transfer function is obtained as

$$\frac{C(s)}{R(s)} = \frac{K_p}{Js^2 + K_p}$$

Since the roots of the characteristic equation

$$Js^2 + K_p = 0$$

are imaginary, the response to a unit-step input continues to oscillate indefinitely, as shown in Figure 5–46(b).

Control systems exhibiting such response characteristics are not desirable. We shall see that the addition of derivative control will stabilize the system.

**Proportional-Plus-Derivative Control of a System with Inertia Load.** Let us modify the proportional controller to a proportional-plus-derivative controller whose transfer function is $K_p(1 + T_d s)$. The torque developed by the controller is proportional to $K_p(e + T_d \dot{e})$. Derivative control is essentially anticipatory, measures the instantaneous error velocity, and predicts the large overshoot ahead of time and produces an appropriate counteraction before too large an overshoot occurs.





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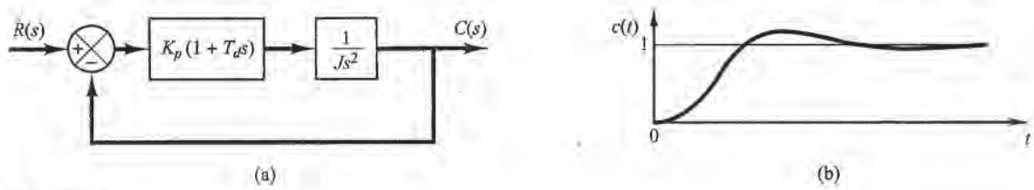


**Figure 5–47**
(a) Proportional-plus-derivative control of a system with inertia load; (b) response to a unit-step input.

Consider the system shown in Figure 5–47(a). The closed-loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T_d s + K_p}$$

The characteristic equation

$$Js^2 + K_p T_d s + K_p = 0$$

now has two roots with negative real parts for positive values of $J$, $K_p$, and $T_d$. Thus derivative control introduces a damping effect. A typical response curve $c(t)$ to a unit-step input is shown in Figure 5–47(b). Clearly, the response curve shows a marked improvement over the original response curve shown in Figure 5–46(b).

**Proportional-Plus-Derivative Control of Second-Order Systems.** A compromise between acceptable transient-response behavior and acceptable steady-state behavior may be achieved by use of proportional-plus-derivative control action.

Consider the system shown in Figure 5–48. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (B + K_d)s + K_p}$$

The steady-state error for a unit-ramp input is

$$e_{ss} = \frac{B}{K_p}$$

The characteristic equation is

$$Js^2 + (B + K_d)s + K_p = 0$$

The effective damping coefficient of this system is thus $B + K_d$ rather than $B$. Since the damping ratio $\zeta$ of this system is

$$\zeta = \frac{B + K_d}{2\sqrt{K_p J}}$$

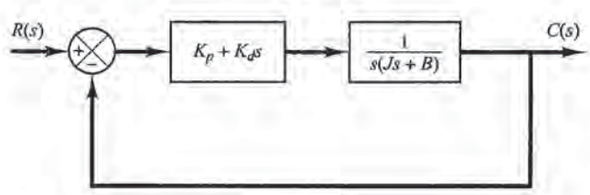


**Figure 5–48**
Control system.





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it is possible to make both the steady-state error $e_{ss}$ for a ramp input and the maximum overshoot for a step input small by making $B$ small, $K_p$ large, and $K_d$ large enough so that $\zeta$ is between 0.4 and 0.7.

## 5–9 STEADY-STATE ERRORS IN UNITY-FEEDBACK CONTROL SYSTEMS

Errors in a control system can be attributed to many factors. Changes in the reference input will cause unavoidable errors during transient periods and may also cause steady-state errors. Imperfections in the system components, such as static friction, backlash, and amplifier drift, as well as aging or deterioration, will cause errors at steady state. In this section, however, we shall not discuss errors due to imperfections in the system components. Rather, we shall investigate a type of steady-state error that is caused by the incapability of a system to follow particular types of inputs.

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. (The only way we may be able to eliminate this error is to modify the system structure.) Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system, to be discussed in what follows.

**Classification of Control Systems.** Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on. This is a reasonable classification scheme because actual inputs may frequently be considered combinations of such inputs. The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Consider the unity-feedback control system with the following open-loop transfer function $G(s)$:

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

It involves the term $s^N$ in the denominator, representing a pole of multiplicity $N$ at the origin. The present classification scheme is based on the number of integrations indicated by the open-loop transfer function. A system is called type 0, type 1, type 2, ..., if $N = 0$, $N = 1$, $N = 2$, ..., respectively. Note that this classification is different from that of the order of a system. As the type number is increased, accuracy is improved; however, increasing the type number aggravates the stability problem. A compromise between steady-state accuracy and relative stability is always necessary.

We shall see later that, if $G(s)$ is written so that each term in the numerator and denominator, except the term $s^N$, approaches unity as $s$ approaches zero, then the open-loop gain $K$ is directly related to the steady-state error.

**Steady-State Errors.** Consider the system shown in Figure 5–49. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$





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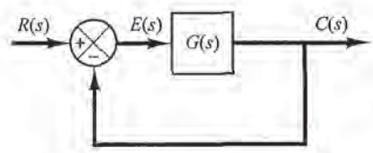


**Figure 5–49**
Control system.

The transfer function between the error signal $e(t)$ and the input signal $r(t)$ is

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)}$$

where the error $e(t)$ is the difference between the input signal and the output signal.

The final-value theorem provides a convenient way to find the steady-state performance of a stable system. Since $E(s)$ is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

the steady-state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

The static error constants defined in the following are figures of merit of control systems. The higher the constants, the smaller the steady-state error. In a given system, the output may be the position, velocity, pressure, temperature, or the like. The physical form of the output, however, is immaterial to the present analysis. Therefore, in what follows, we shall call the output "position," the rate of change of the output "velocity," and so on. This means that in a temperature control system "position" represents the output temperature, "velocity" represents the rate of change of the output temperature, and so on.

**Static Position Error Constant $K_p$.** The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s}$$

$$= \frac{1}{1 + G(0)}$$

The static position error constant $K_p$ is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant $K_p$ is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

For a type 0 system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$





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For a type 1 or higher system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

Hence, for a type 0 system, the static position error constant $K_p$ is finite, while for a type 1 or higher system, $K_p$ is infinite.

For a unit-step input, the steady-state error $e_{ss}$ may be summarized as follows:

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

From the foregoing analysis, it is seen that the response of a feedback control system to a step input involves a steady-state error if there is no integration in the feedforward path. (If small errors for step inputs can be tolerated, then a type 0 system may be permissible, provided that the gain $K$ is sufficiently large. If the gain $K$ is too large, however, it is difficult to obtain reasonable relative stability.) If zero steady-state error for a step input is desired, the type of the system must be one or higher.

**Static Velocity Error Constant $K_v$.** The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{sG(s)}$$

The static velocity error constant $K_v$ is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant $K_v$ is given by

$$e_{ss} = \frac{1}{K_v}$$

The term *velocity error* is used here to express the steady-state error for a ramp input. The dimension of the velocity error is the same as the system error. That is, velocity error is not an error in velocity, but it is an error in position due to a ramp input.

For a type 0 system,

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 1 system,

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$





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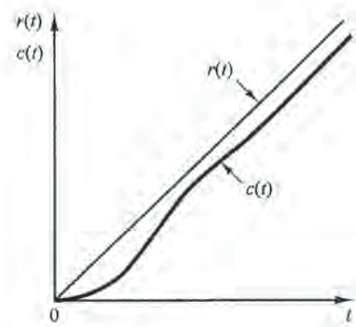


**Figure 5–50**
Response of a type 1 unity-feedback system to a ramp input.

For a type 2 or higher system,

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

The steady-state error $e_{ss}$ for the unit-ramp input can be summarized as follows:

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

The foregoing analysis indicates that a type 0 system is incapable of following a ramp input in the steady state. The type 1 system with unity feedback can follow the ramp input with a finite error. In steady-state operation, the output velocity is exactly the same as the input velocity, but there is a positional error. This error is proportional to the velocity of the input and is inversely proportional to the gain $K$. Figure 5–50 shows an example of the response of a type 1 system with unity feedback to a ramp input. The type 2 or higher system can follow a ramp input with zero error at steady state.

**Static Acceleration Error Constant $K_a$.** The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$r(t) = \frac{t^2}{2}, \qquad \text{for } t \geq 0$$

$$= 0, \qquad \text{for } t < 0$$

is given by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$





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The static acceleration error constant $K_a$ is defined by the equation

$$K_a = \lim_{s \to 0} s^2 G(s)$$

The steady-state error is then

$$e_{ss} = \frac{1}{K_a}$$

Note that the acceleration error, the steady-state error due to a parabolic input, is an error in position.

The values of $K_a$ are obtained as follows:

For a type 0 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 1 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

For a type 2 system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

For a type 3 or higher system,

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Thus, the steady-state error for the unit parabolic input is

$$e_{ss} = \infty, \qquad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \qquad \text{for type 2 systems}$$

$$e_{ss} = 0, \qquad \text{for type 3 or higher systems}$$

Note that both type 0 and type 1 systems are incapable of following a parabolic input in the steady state. The type 2 system with unity feedback can follow a parabolic input with a finite error signal. Figure 5–51 shows an example of the response of a type 2 system with unity feedback to a parabolic input. The type 3 or higher system with unity feedback follows a parabolic input with zero error at steady state.





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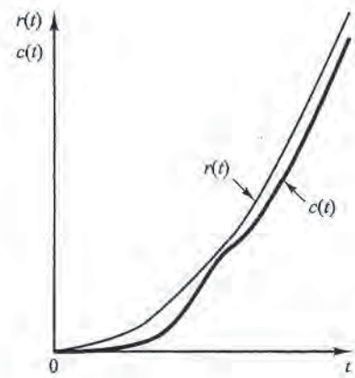


**Figure 5–51**
Response of a type 2 unity-feedback system to a parabolic input.

**Summary.** Table 5–1 summarizes the steady-state errors for type 0, type 1, and type 2 systems when they are subjected to various inputs. The finite values for steady-state errors appear on the diagonal line. Above the diagonal, the steady-state errors are infinity; below the diagonal, they are zero.

**Table 5–1** Steady-State Error in Terms of Gain $K$

| | Step Input $r(t) = 1$ | Ramp Input $r(t) = t$ | Acceleration Input $r(t) = \frac{1}{2}t^2$ |
|---|---|---|---|
| Type 0 system | $\frac{1}{1+K}$ | $\infty$ | $\infty$ |
| Type 1 system | 0 | $\frac{1}{K}$ | $\infty$ |
| Type 2 system | 0 | 0 | $\frac{1}{K}$ |

Remember that the terms *position error, velocity error*, and *acceleration error* mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out the input and output move at the same velocity but have a finite position difference.

The error constants $K_p$, $K_v$, and $K_a$ describe the ability of a unity-feedback system to reduce or eliminate steady-state error. Therefore, they are indicative of the steady-state performance. It is generally desirable to increase the error constants, while maintaining the transient response within an acceptable range. It is noted that to improve the steady-state performance we can increase the type of the system by adding an integrator or integrators to the feedforward path. This, however, introduces an additional stability problem. The design of a satisfactory system with more than two integrators in series in the feedforward path is generally not easy.





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## EXAMPLE PROBLEMS AND SOLUTIONS

**A–5–1.** In the system of Figure 5–52, $x(t)$ is the input displacement and $\theta(t)$ is the output angular displacement. Assume that the masses involved are negligibly small and that all motions are restricted to be small; therefore, the system can be considered linear. The initial conditions for $x$ and $\theta$ are zeros, or $x(0-) = 0$ and $\theta(0-) = 0$. Show that this system is a differentiating element. Then obtain the response $\theta(t)$ when $x(t)$ is a unit-step input.

**Solution.** The equation for the system is

$$b(\dot{x} - L\dot{\theta}) = kL\theta$$

or

$$L\dot{\theta} + \frac{k}{b}L\theta = \dot{x}$$

The Laplace transform of this last equation, using zero initial conditions, gives

$$\left(Ls + \frac{k}{b}L\right)\Theta(s) = sX(s)$$

And so

$$\frac{\Theta(s)}{X(s)} = \frac{1}{L}\frac{s}{s + (k/b)}$$

Thus the system is a differentiating system.

For the unit-step input $X(s) = 1/s$, the output $\Theta(s)$ becomes

$$\Theta(s) = \frac{1}{L}\frac{1}{s + (k/b)}$$

The inverse Laplace transform of $\Theta(s)$ gives

$$\theta(t) = \frac{1}{L}e^{-(k/b)t}$$

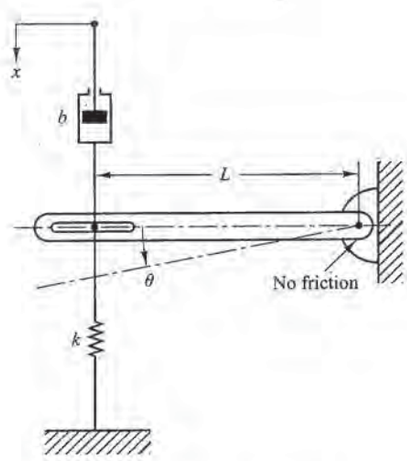


**Figure 5–52**
Mechanical system.





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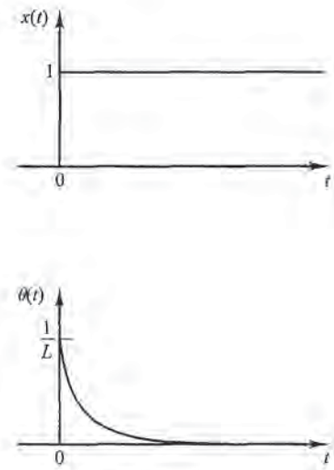


**Figure 5–53**
Unit-step input and the response of the mechanical system shown in Figure 5–52.

Note that if the value of $k/b$ is large the response $\theta(t)$ approaches a pulse signal as shown in Figure 5–53.

**A–5–2.** Consider the mechanical system shown in Figure 5–54. Suppose that the system is at rest initially $[x(0) = 0, \dot{x}(0) = 0]$, and at $t = 0$ it is set into motion by a unit-impulse force. Obtain a mathematical model for the system. Then find the motion of the system.

**Solution.** The system is excited by a unit-impulse input. Hence

$$m\ddot{x} + kx = \delta(t)$$

This is a mathematical model for the system.

Taking the Laplace transform of both sides of this last equation gives

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + kX(s) = 1$$

By substituting the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ into this last equation and solving for $X(s)$, we obtain

$$X(s) = \frac{1}{ms^2 + k}$$

The inverse Laplace transform of $X(s)$ becomes

$$x(t) = \frac{1}{\sqrt{mk}} \sin\sqrt{\frac{k}{m}}\, t$$

The oscillation is a simple harmonic motion. The amplitude of the oscillation is $1/\sqrt{mk}$.

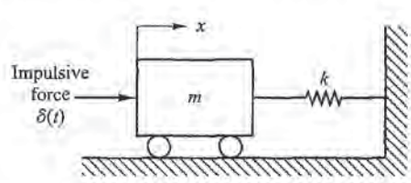


**Figure 5–54**
Mechanical system.





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**A–5–3.** Gear trains are often used in servo systems to reduce speed, to magnify torque, or to obtain the most efficient power transfer by matching the driving member to the given load.

Consider the gear train system shown in Figure 5–55. In this system, a load is driven by a motor through the gear train. Assuming that the stiffness of the shafts of the gear train is infinite (there is neither backlash nor elastic deformation) and that the number of teeth on each gear is proportional to the radius of the gear, obtain the equivalent moment of inertia and equivalent viscous-friction coefficient referred to the motor shaft and referred to the load shaft.

In Figure 5–55 the numbers of teeth on gears 1, 2, 3, and 4 are $N_1, N_2, N_3$, and $N_4$, respectively. The angular displacements of shafts, 1, 2, and 3 are $\theta_1, \theta_2$, and $\theta_3$, respectively. Thus, $\theta_2/\theta_1 = N_1/N_2$ and $\theta_3/\theta_2 = N_3/N_4$. The moment of inertia and viscous-fraction coefficient of each gear train component are denoted by $J_1, b_1; J_2, b_2$; and $J_3, b_3$; respectively. ($J_3$ and $b_3$ include the moment of inertia and friction of the load.)

**Solution.** For this gear train system, we can obtain the following equations: For shaft 1,

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + T_1 = T_m \tag{5–63}$$

where $T_m$ is the torque developed by the motor and $T_1$ is the load torque on gear 1 due to the rest of the gear train. For shaft 2,

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 + T_3 = T_2 \tag{5–64}$$

where $T_2$ is the torque transmitted to gear 2 and $T_3$ is the load torque on gear 3 due to the rest of the gear train. Since the work done by gear 1 is equal to that of gear 2,

$$T_1\theta_1 = T_2\theta_2 \qquad \text{or} \qquad T_2 = T_1\frac{N_2}{N_1}$$

If $N_1/N_2 < 1$, the gear ratio reduces the speed as well as magnifies the torque. For shaft 3,

$$J_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + T_L = T_4 \tag{5–65}$$

where $T_L$ is the load torque and $T_4$ is the torque transmitted to gear 4. $T_3$ and $T_4$ are related by

$$T_4 = T_3\frac{N_4}{N_3}$$

and $\theta_3$ and $\theta_1$ are related by

$$\theta_3 = \theta_2\frac{N_3}{N_4} = \theta_1\frac{N_1}{N_2}\frac{N_3}{N_4}$$

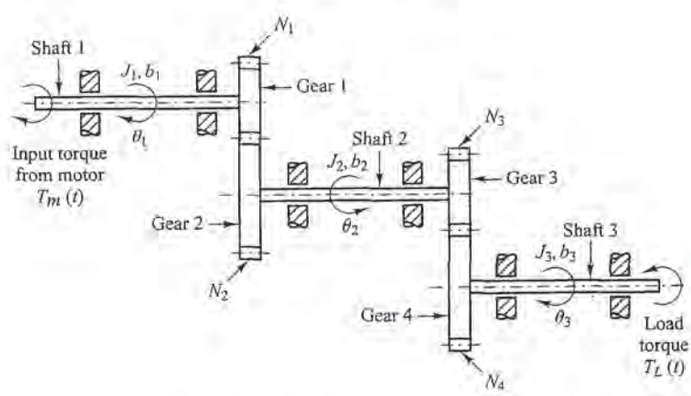


**Figure 5–55**
Gear train system.





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Eliminating $T_1$, $T_2$, $T_3$, and $T_4$ from Equations (5–63), (5–64), and (5–65) yields

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + \frac{N_1}{N_2}(J_2\ddot{\theta}_2 + b_2\dot{\theta}_2) + \frac{N_1N_3}{N_2N_4}(J_3\ddot{\theta}_3 + b_3\dot{\theta}_3 + T_L) = T_m$$

Eliminating $\theta_2$ and $\theta_3$ from this last equation and writing the resulting equation in terms of $\theta_1$ and its time derivatives, we obtain

$$\left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3\right]\ddot{\theta}_1 + \left[b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3\right]\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)\left(\frac{N_3}{N_4}\right)T_L = T_m \tag{5–66}$$

Thus, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to shaft 1 are given, respectively, by

$$J_{1\text{eq}} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_3$$

$$b_{1\text{eq}} = b_1 + \left(\frac{N_1}{N_2}\right)^2 b_2 + \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_3$$

Similarly, the equivalent moment of inertia and viscous-friction coefficient of the gear train referred to the load shaft (shaft 3) are given, respectively, by

$$J_{3\text{eq}} = J_3 + \left(\frac{N_4}{N_3}\right)^2 J_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 J_1$$

$$b_{3\text{eq}} = b_3 + \left(\frac{N_4}{N_3}\right)^2 b_2 + \left(\frac{N_2}{N_1}\right)^2\left(\frac{N_4}{N_3}\right)^2 b_1$$

The relationship between $J_{1\text{eq}}$ and $J_{3\text{eq}}$ is thus

$$J_{1\text{eq}} = \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 J_{3\text{eq}}$$

and that between $b_{1\text{eq}}$ and $b_{3\text{eq}}$ is

$$b_{1\text{eq}} = \left(\frac{N_1}{N_2}\right)^2\left(\frac{N_3}{N_4}\right)^2 b_{3\text{eq}}$$

The effect of $J_2$ and $J_3$ on an equivalent moment of inertia is determined by the gear ratios $N_1/N_2$ and $N_3/N_4$. For speed-reducing gear trains, the ratios, $N_1/N_2$ and $N_3/N_4$ are usually less than unity. If $N_1/N_2 \ll 1$ and $N_3/N_4 \ll 1$, then the effect of $J_2$ and $J_3$ on the equivalent moment of inertia $J_{1\text{eq}}$ is negligible. Similar comments apply to the equivalent viscous-friction coefficient $b_{1\text{eq}}$ of the gear train. In terms of the equivalent moment of inertia $J_{1\text{eq}}$ and equivalent viscous-friction coefficient $b_{1\text{eq}}$, Equation (5–66) can be simplified to give

$$J_{1\text{eq}}\ddot{\theta}_1 + b_{1\text{eq}}\dot{\theta}_1 + nT_L = T_m$$

where

$$n = \frac{N_1N_3}{N_2N_4}$$





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**A–5–4.** When the system shown in Figure 5–56(a) is subjected to a unit-step input, the system output responds as shown in Figure 5–56(b). Determine the values of $K$ and $T$ from the response curve.

**Solution.** The maximum overshoot of 25.4% corresponds to $\zeta = 0.4$. From the response curve we have

$$t_p = 3$$

Consequently,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

It follows that

$$\omega_n = 1.14$$

From the block diagram we have

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

Therefore, the values of $T$ and $K$ are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

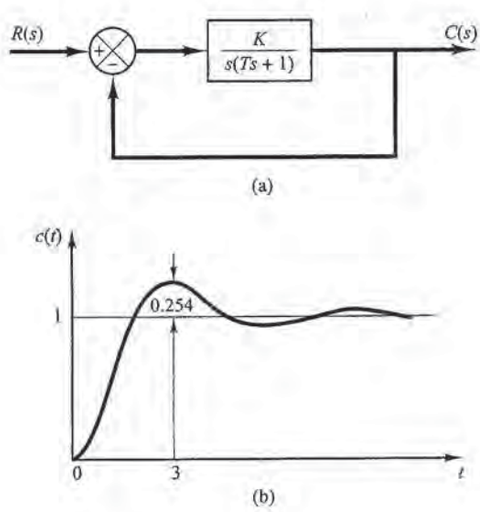


**Figure 5–56**
(a) Closed-loop system; (b) unit-step response curve.





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**A–5–5.** Determine the values, of $K$ and $k$ of the closed-loop system shown in Figure 5–57 so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J = 1$ kg-m$^2$.

**Solution.** The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting $J = 1$ kg-m$^2$ into this last equation, we have

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem

$$\omega_n = \sqrt{K}, \qquad 2\zeta\omega_n = Kk$$

The maximum overshoot $M_p$ is

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

which is specified as 25%. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25$$

from which

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386$$

or

$$\zeta = 0.404$$

The peak time $t_p$ is specified as 2 sec. And so

$$t_p = \frac{\pi}{\omega_d} = 2$$

or

$$\omega_d = 1.57$$

Then the undamped natural frequency $\omega_n$ is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.57}{\sqrt{1-0.404^2}} = 1.72$$

Therefore, we obtain

$$K = \omega_n^2 = 1.72^2 = 2.95 \text{ N-m}$$

$$k = \frac{2\zeta\omega_n}{K} = \frac{2 \times 0.404 \times 1.72}{2.95} = 0.471 \text{ sec}$$

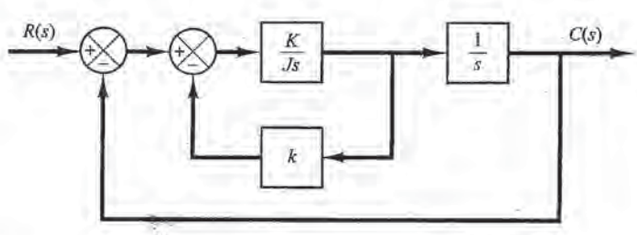


**Figure 5–57**
Closed-loop system.





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**A–5–6.** Figures 5–58(a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure 5–58(b). Determine $m$, $b$, and $k$ of the system from this response curve. The displacement $x$ is measured from the equilibrium position.

**Solution.** The transfer function of this system is

$$\frac{X(s)}{P(s)} = \frac{1}{ms^2 + bs + k}$$

Since

$$P(s) = \frac{2}{s}$$

we obtain

$$X(s) = \frac{2}{s(ms^2 + bs + k)}$$

It follows that the steady-state value of $x$ is

$$x(\infty) = \lim_{s \to 0} sX(s) = \frac{2}{k} = 0.1 \text{ ft}$$

Hence

$$k = 20 \text{ lb}_f/\text{ft}$$

Note that $M_p = 9.5\%$ corresponds to $\zeta = 0.6$. The peak time $t_p$ is given by

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}} = \frac{\pi}{0.8\omega_n}$$

The experimental curve shows that $t_p = 2$ sec. Therefore,

$$\omega_n = \frac{3.14}{2 \times 0.8} = 1.96 \text{ rad/sec}$$

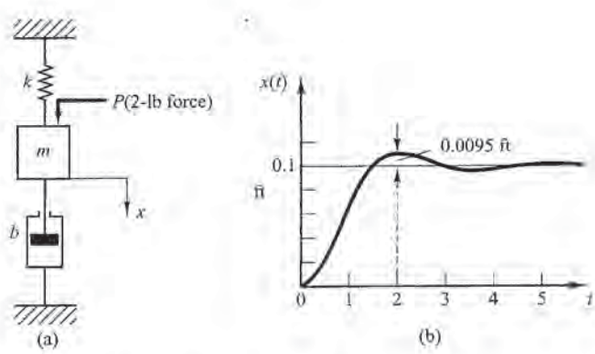


**Figure 5–58**
(a) Mechanical vibratory system; (b) step-response curve.





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Since $\omega^2_n = k/m = 20/m$, we obtain

$$m = \frac{20}{\omega_n^2} = \frac{20}{1.96^2} = 5.2 \text{ slugs} = 166 \text{ lb}$$

(Note that 1 slug = 1 $\text{lb}_f\text{-sec}^2/\text{ft}$.) Then $b$ is determined from

$$2\zeta\omega_n = \frac{b}{m}$$

or

$$b = 2\zeta\omega_n m = 2 \times 0.6 \times 1.96 \times 5.2 = 12.2 \text{ lb}_f/\text{ft/sec}$$

**A–5–7.** Consider the unit-step response of the second-order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The amplitude of the exponentially-damped sinusoid changes as a geometric series. At time $t = t_p = \pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)\pi}$. After one oscillation, or at $t = t_p + 2\pi/\omega_d = 3\pi/\omega_d$, the amplitude is equal to $e^{-(\sigma/\omega_d)3\pi}$; after another cycle of oscillation, the amplitude is $e^{-(\sigma/\omega_d)5\pi}$. The logarithm of the ratio of successive amplitudes is called the *logarithmic decrement*. Determine the logarithmic decrement for this second-order system. Describe a method for experimental determination of the damping ratio from the rate of decay of the oscillation.

**Solution.** Let us define the amplitude of the output oscillation at $t = t_i$ to be $x_i$, where $t_i = t_p + (i - 1)T$ ($T$ = period of oscillation). The amplitude ratio per one period of damped oscillation is

$$\frac{x_1}{x_2} = \frac{e^{-(\sigma/\omega_d)\pi}}{e^{-(\sigma/\omega_d)3\pi}} = e^{2(\sigma/\omega_d)\pi} = e^{2\zeta\pi/\sqrt{1-\zeta^2}}$$

Thus, the logarithmic decrement $\delta$ is

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\zeta\pi}{\sqrt{1-\zeta^2}}$$

It is a function only of the damping ratio $\zeta$. Thus, the damping ratio $\zeta$ can be determined by use of the logarithmic. decrement.

In the experimental determination of the damping ratio $\zeta$ from the rate of decay of the oscillation, we measure the amplitude $x_1$ at $t = t_p$ and amplitude $x_n$ at $t = t_p + (n - 1)T$. Note that it is necessary to choose $n$ large enough so that the ratio o $x_1/x_n$ is not near unity. Then

$$\frac{x_1}{x_n} = e^{(n-1)2\zeta\pi/\sqrt{1-\zeta^2}}$$

or

$$\ln\frac{x_1}{x_n} = (n - 1)\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}$$

Hence

$$\zeta = \frac{\dfrac{1}{n-1}\left(\ln\dfrac{x_1}{x_n}\right)}{\sqrt{4\pi^2 + \left[\dfrac{1}{n-1}\left(\ln\dfrac{x_1}{x_n}\right)\right]^2}}$$





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**A–5–8.** In the system shown in Figure 5–59, the numerical values of $m$, $b$, and $k$ are given as $m = 1$ kg, $b = 2$ N-sec/m, and $k = 100$ N/m. The mass is displaced 0.05 m and released without initial velocity. Find the frequency observed in the vibration. In addition, find the amplitude four cycles later. The displacement $x$ is measured from the equilibrium position.

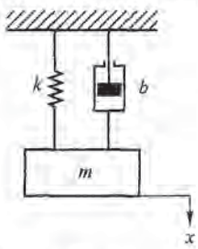


**Figure 5–59**
Spring-mass-damper system.

**Solution.** The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the numerical values for $m$, $b$, and $k$ into this equation gives

$$\ddot{x} + 2\dot{x} + 100x = 0$$

where the initial conditions are $x(0) = 0.05$ and $\dot{x}(0) = 0$. From this last equation the undamped natural frequency $\omega_n$ and the damping ratio $\zeta$ are found to be

$$\omega_n = 10, \qquad \zeta = 0.1$$

The frequency actually observed in the vibration is the damped natural frequency $\omega_d$.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.01} = 9.95 \text{ rad/sec}$$

In the present analysis, $\dot{x}(0)$ is given as zero. Thus, solution $x(t)$ can be written as

$$x(t) = x(0)e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right)$$

It follows that at $t = nT$, where $T = 2\pi/\omega_d$,

$$x(nT) = x(0)e^{-\zeta\omega_n nT}$$

Consequently, the amplitude four cycles later becomes

$$x(4T) = x(0)e^{-\zeta\omega_n 4T} = x(0)e^{-(0.1)(10)(4)(0.6315)}$$
$$= 0.05e^{-2.526} = 0.05 \times 0.07998 = 0.004 \text{ m}$$

**A–5–9.** Obtain both analytically and computationally the unit-step response of the following higher-order system:

$$\frac{C(s)}{R(s)} = \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}$$

[Obtain the partial-fraction expansion of $C(s)$ with MATLAB when $R(s)$ is a unit-step function.]

**Solution.** MATLAB Program 5–19 yields the unit-step response curve shown in Figure 5–60. It also yields the partial-fraction expansion of $C(s)$ as follows:

$$\begin{aligned}
C(s) &= \frac{3s^3 + 25s^2 + 72s + 80}{s^4 + 8s^3 + 40s^2 + 96s + 80}\frac{1}{s} \\
&= \frac{-0.2813 - j0.1719}{s + 2 - j4} + \frac{-0.2813 + j0.1719}{s + 2 + j4} \\
&\quad + \frac{-0.4375}{s + 2} + \frac{-0.375}{(s + 2)^2} + \frac{1}{s} \\
&= \frac{-0.5626(s + 2)}{(s + 2)^2 + 4^2} + \frac{(0.3438) \times 4}{(s + 2)^2 + 4^2} \\
&\quad - \frac{0.4375}{s + 2} - \frac{0.375}{(s + 2)^2} + \frac{1}{s}
\end{aligned}$$





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**MATLAB Program 5–19**

```
% ------- Unit-Step Response of C(s)/R(s) and Partial-Fraction Expansion of C(s) -------

num = [0 3 25 72 80];
den = [1 8 40 96 80];
step(num,den);
v = [0 3 0 1.2]; axis(v), grid

% To obtain the partial-fraction expansion of C(s), enter commands
%     num1 = [0 0 3 25 72 80];
%     den1 = [1 8 40 96 80 0];
%     [r,p,k] = residue(num1,den1)

num1 = [0 0 3 25 72 80];
den1 = [1 8 40 96 80 0];
[r,p,k] = residue(num1,den1)

r =

   -0.2813- 0.1719i
   -0.2813+ 0.1719i
   -0.4375
   -0.3750
    1.0000

p =

   -2.0000+ 4.0000i
   -2.0000- 4.0000i
   -2.0000
   -2.0000
    0

k =

    []
```

Hence, the time response $c(t)$ can be given by

$$c(t) = -0.5626e^{-2t}\cos 4t + 0.3438e^{-2t}\sin 4t$$

$$- 0.4375e^{-2t} - 0.375te^{-2t} + 1$$

The fact that the response curve is an exponential curve superimposed by damped sinusoidal curves can be seen from Figure 5–60.





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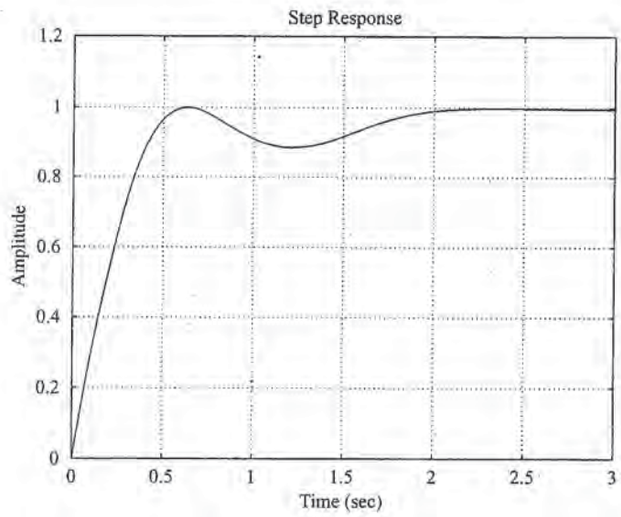


**Figure 5–60**
Unit-step response curve.

**A–5–10.** Obtain both analytical and computational solutions of the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{5(s + 20)}{s(s + 4.59)(s^2 + 3.41s + 16.35)}$$

**Solution.** The closed-loop transfer function is

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{5(s + 20)}{s(s + 4.59)(s^2 + 3.41s + 16.35) + 5(s + 20)} \\ &= \frac{5s + 100}{s^4 + 8s^3 + 32s^2 + 80s + 100} \\ &= \frac{5(s + 20)}{(s^2 + 2s + 10)(s^2 + 6s + 10)}\end{aligned}$$

The unit-step response of this system is then

$$\begin{aligned}C(s) &= \frac{5(s + 20)}{s(s^2 + 2s + 10)(s^2 + 6s + 10)} \\ &= \frac{1}{s} + \frac{\frac{3}{8}(s + 1) - \frac{17}{8}}{(s + 1)^2 + 3^2} + \frac{-\frac{11}{8}(s + 3) - \frac{13}{8}}{(s + 3)^2 + 1^2}\end{aligned}$$

The time response $c(t)$ can be found by taking the inverse Laplace transform of $C(s)$ as follows:

$$c(t) = 1 + \frac{3}{8}e^{-t}\cos 3t - \frac{17}{24}e^{-t}\sin 3t - \frac{11}{8}e^{-3t}\cos t - \frac{13}{8}e^{-3t}\sin t, \qquad \text{for } t \geq 0$$

A MATLAB program to obtain the unit-step response of this system is shown in MATLAB Program 5–20. The resulting unit-step response curve is shown in Figure 5–61.





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**MATLAB Program 5–20**

```
% --------------- Unit-step-response ---------------
num = [0 0 0 5 100];
den = [1 8 32 80 100];
step(num,den)
grid
title('Unit-Step Response of C(s)/R(s) = (5s + 100)/(s^4 + 8s^3 + 32s^2 + 80s + 100)')
```

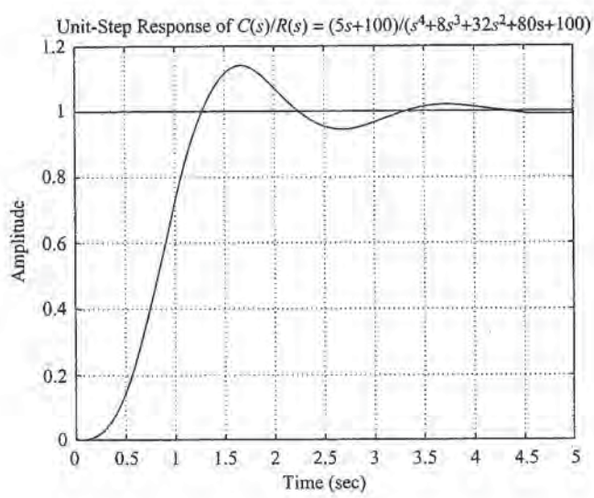


**Figure 5–61**
Unit-step response curve.

**A–5–11.** When the closed-loop system involves a numerator dynamics, the unit-step response curve may exhibit a large overshoot. Obtain the unit-step response of the following system with MATLAB:

$$\frac{C(s)}{R(s)} = \frac{10s + 4}{s^2 + 4s + 4}$$

Obtain also the unit-ramp response with MATLAB.

**Solution.** MATLAB Program 5–21 produces the unit-step response as well as the unit-ramp response of the system. The unit-step response curve and unit-ramp response curve, together with the unit-ramp input, are shown in Figures 5–62(a) and (b), respectively.

Notice that the unit-step response curve exhibits over 215% of overshoot. The unit-ramp response curve leads the input curve. These phenomena occurred because of the presence of a large derivative term in the numerator.





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**MATLAB Program 5–21**

```
num = [0  10  4];
den = [1  4  4];
t = 0:0.02:10;
y = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output')

num1 = [0  0  10  4];
den1 = [1  4  4  0];
y1 = step(num1,den1,t);
plot(t,t,'--',t,y1)
v = [0  10  0  10]; axis(v);
grid
title('Unit-Ramp Response')
xlabel('t (sec)')
ylabel('Unit-Ramp Input and Output')
text(6.1,5.0,'Unit-Ramp Input')
text(3.5,7.1,'Output')
```

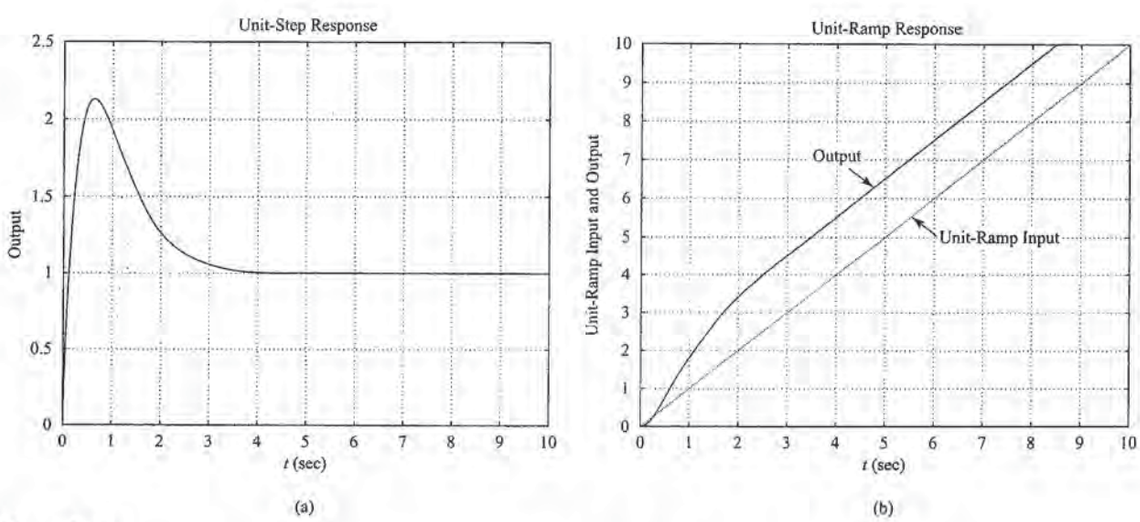


**Figure 5–62**
(a) Unit-step response curve; (b) unit-ramp response curve plotted with unit-ramp input.





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**A–5–12.** Consider a higher-order system defined by

$$\frac{C(s)}{R(s)} = \frac{6.3223s^2 + 18s + 12.811}{s^4 + 6s^3 + 11.3223s^2 + 18s + 12.811}$$

Using MATLAB, plot the unit-step response curve of this system. Using MATLAB, obtain the rise time, peak time, maximum overshoot, and settling time.

**Solution.** MATLAB Program 5–22 plots the unit-step response curve as well as to give the rise time, peak time, maximum overshoot, and settling time. The unit-step response curve is shown in Figure 5–63.

**MATLAB Program 5–22**

```
% ------- This program is to plot the unit-step response curve, as well as to
% find the rise time, peak time, maximum overshoot, and settling time.
% In this program the rise time is calculated as the time required for the
% response to rise from 10% to 90% of its final value. -------

num = [0  0  6.3223  18  12.811];
den = [1  6  11.3223  18  12.811];
t = 0:0.02:20;
[y,x,t] = step(num,den,t);
plot(t,y)
grid
title('Unit-Step Response')
xlabel('t (sec)')
ylabel('Output y(t)')

r1 = 1; while y(r1) < 0.1, r1 = r1+1; end;
r2 = 1; while y(r2) < 0.9, r2 = r2+1; end;
rise_time = (r2-r1)*0. 02

rise_time =

   0.5800

[ymax,tp] = max(y);
peak_time = (tp-1)*0.02

peak_time =

   1.6600

max_overshoot = ymax-1

max_overshoot =

   0.6182

s = 1001; while y(s) > 0.98 & y(s) < 1.02; s = s-1; end;
settling_time = (s-1)*0.02

settling_time =

   10.0200
```





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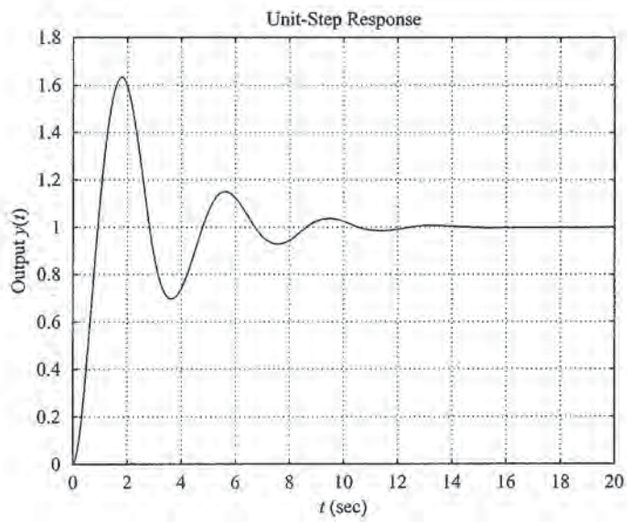


**Figure 5–63**
Unit-step response curve.

**A–5–13.** Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using a "for loop," write a MATLAB program to obtain unit-step response of this system for the following four cases:

Case 1: $\zeta = 0.3, \quad \omega_n = 1$

Case 2: $\zeta = 0.5, \quad \omega_n = 2$

Case 3: $\zeta = 0.7, \quad \omega_n = 4$

Case 4: $\zeta = 0.8, \quad \omega_n = 6$

**Solution.** Define $\omega_n^2 = a$ and $2\zeta\omega_n = b$. Then, $a$ and $b$ each have four elements as follows:

```
a = [1  4  16  36]

b = [0.6  2  5.6  9.6]
```

Using vectors a and b, MATLAB Program 5–23 will produce the unit-step response curves as shown in Figure 5–64.





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**MATLAB Program 5–23**

```
a = [1  4  16  36];
b = [0.6  2  5.6  9.6];
t = 0:0.1:8;
y = zeros(81,4);
    for i = 1:4;
    num = [0  0  a(i)];
    den = [1  b(i)  a(i)];
    y(:,i) = step(num,den,t);
    end
plot(t,y(:,1),'o',t,y(:,2),'x',t,y(:,3),'-',t,y(:,4),'-.')
grid
title('Unit-Step Response Curves for Four Cases')
xlabel('t Sec')
ylabel('Outputs')
gtext('1')
gtext('2')
gtext('3')
gtext('4')
```

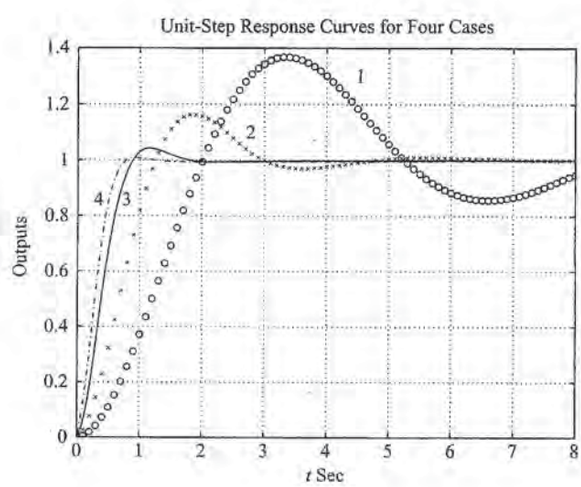


**Figure 5–64**
Unit-step response curves for four cases.





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**A–5–14.** Using MATLAB, obtain the unit-ramp response of the closed-loop control system whose closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{s + 10}{s^3 + 6s^2 + 9s + 10}$$

Also, obtain the response of this system when the input is given by

$$r = e^{-0.5t}$$

**Solution.** MATLAB Program 5–24 produces the unit-ramp response and the response to the exponential input $r = e^{-0.5t}$. The resulting response curves are shown in Figures 5–65(a) and (b), respectively.

**MATLAB Program 5–24**

```
% --------- Unit-Ramp Response ---------

num = [0  0  1  10];
den = [1  6  9  10];
t = 0:0.1:10;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
title('Unit-Ramp Response by Use of Command "lsim"')
xlabel('t Sec')
ylabel('Output')
text(3.2,6.5,'Unit-Ramp Input')
text(6.0,3.1,'Output')

% --------- Response to Input r1 = exp(-0.5t). ---------

num = [0  0  1  10];
den = [1  6  9  10];
t = 0:0.1:12;
r1 = exp(-0.5*t);
y1 = lsim(num,den,r1,t);
plot(t,r1,'-',t,y1,'o')
grid
title('Response to Input r1 = exp(-0.5t)')
xlabel('t Sec')
ylabel('Input and Output')
text(1.4,0.75,'Input r1 = exp(-0.5t)')
text(6.2,0.34,'Output')
```





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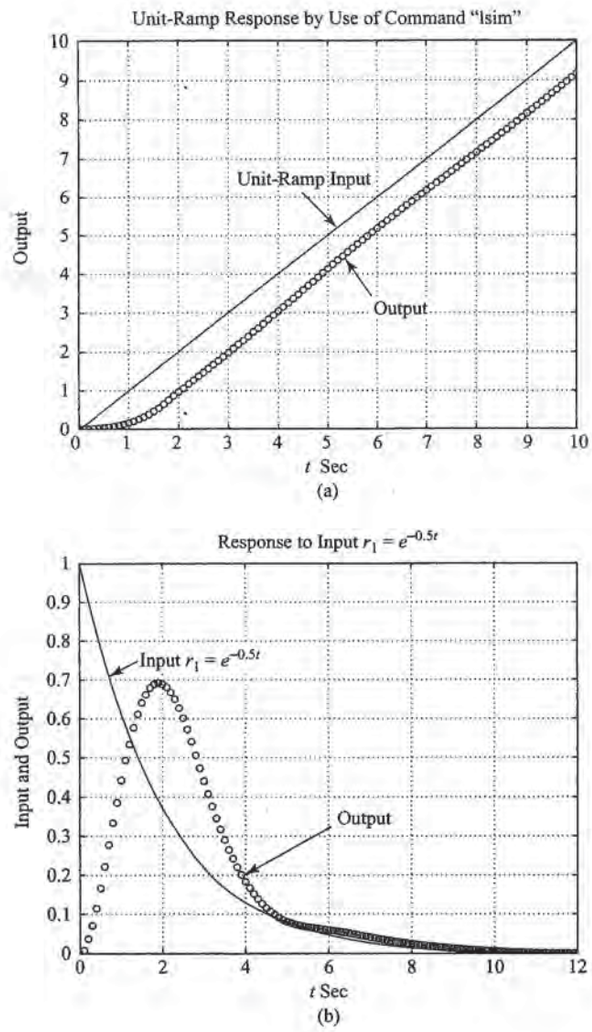


**Figure 5–65**
(a) Unit-ramp response curve; (b) response to exponential input $r = e^{-0.5t}$.

**A–5–15.** Obtain the response of the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{5}{s^2 + s + 5}$$

when the input $r(t)$ is given by

$$r(t) = 2 + t$$

[The input $r(t)$ is a step input of magnitude 2 plus unit-ramp input.]





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**Solution.** A possible MATLAB program is shown in MATLAB Program 5–25. The resulting response curve, together with a plot of the input function, is shown in Figure 5–66.

**MATLAB Program 5–25**

```
num = [0 0 5];
den = [1 1 5];
t = 0:0.05:10;
r = 2+t;
c = lsim(num,den,r,t);
plot(t,r,'-',t,c,'o')
grid
title('Response to Input r(t) = 2 + t')
xlabel('t Sec')
ylabel('Output c(t) and Input r(t) = 2 + t')
```

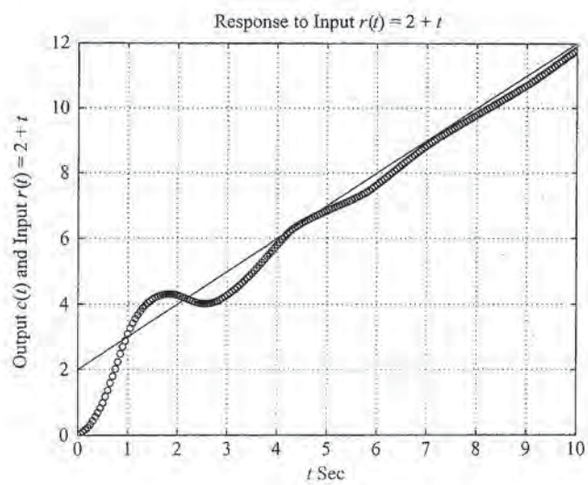


**Figure 5–66**
Response to input $r(t) = 2 + t$.

**A–5–16.** Obtain the response of the system shown in Figure 5–67 when the input $r(t)$ is given by

$$r(t) = \frac{1}{2}t^2$$

[The input $r(t)$ is the unit-acceleration input.]

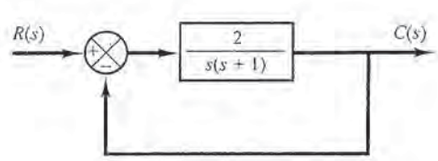


**Figure 5–67**
Control system.





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**Solution.** The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + s + 2}$$

MATLAB Program 5–26 produces the unit-accerelation response. The resulting response, together with the unit-acceleration input, is shown in Figure 5–68.

**MATLAB Program 5–26**

```
num = [0  0  2];
den = [1  1  2];
t = 0:0.2:10;
r = 0.5*t.^2;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o',t,y,'-')
grid
title('Unit-Acceleration Response')
xlabel('t Sec')
ylabel('Input and Output')
text(2.1,27.5,'Unit-Acceleration Input')
text(7.2,7.5,'Output')
```

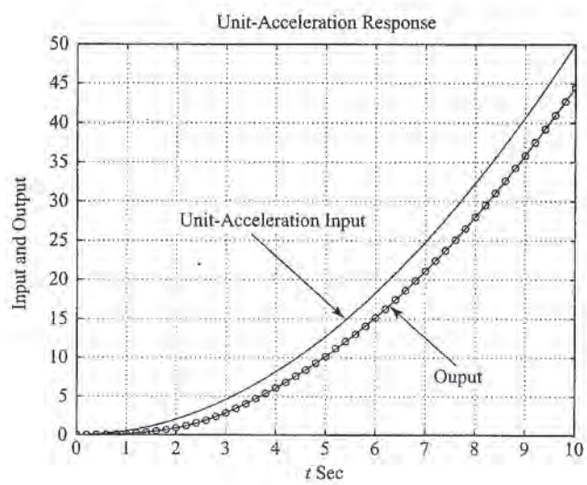


**Figure 5–68**
Response to unit-acceleration input.

**A–5–17.** Consider the system defined by

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta s + 1}$$





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where $\zeta = 0, \quad 0.2, \quad 0.4, \quad 0.6, \quad 0.8$, and 1.0. Write a MATLAB program using a "for loop" to obtain the two-dimensional and three-dimensional plots of the system output. The input is the unit-step function.

**Solution.** MATLAB Program 5–27 is a possible program to obtain two-dimensional and three-dimensional plots. Figure 5–69(a) is the two-dimensional plot of the unit-step response curves for various values of $\zeta$. Figure 5–69(b) is the three-dimensional plot obtained by use of the command "mesh(y)" and Figure 5–69(c) is obtained by use of the command "mesh(y')". (These two three-dimensional plots are basically the same. The only difference is that $x$ axis and $y$ axis are interchanged.)

**MATLAB Program 5–27**

```
t = 0:0.2:12;
    for n = 1:6;
    num = [0  0  1];
    den = [1  2*(n-1)*0.2  1];
    [y(1:61,n),x,t] = step(num,den,t);
    end
plot(t,y)
grid
title('Unit-Step Response Curves')
xlabel('t Sec')
ylabel('Outputs')
gtext('\zeta = 0'),
gtext('0.2')
gtext('0.4')
gtext('0.6')
gtext('0.8')
gtext('1.0')

% To draw a three-dimensional plot, enter the following command: mesh(y) or mesh(y').
% We shall show two three-dimensional plots, one using "mesh(y)" and the other using
% "mesh(y')". These two plots are the same, except that the x axis and y axis are
% interchanged.

mesh(y)
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y)"')
xlabel('n, where n = 1,2,3,4,5,6')
ylabel('Computation Time Points')
zlahel('Outputs')

mesh(y')
title('Three-Dimensional Plot of Unit-Step Response Curves using Command "mesh(y transpose)"')
xlabel('Computation Time Points')
ylabel('n, where n = 1,2,3,4,5,6')
zlabel('Outputs')
```





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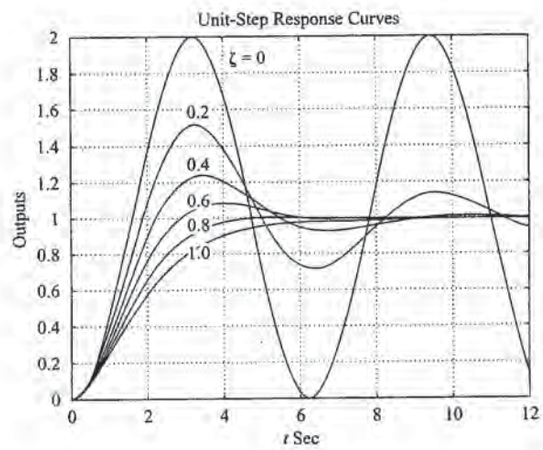
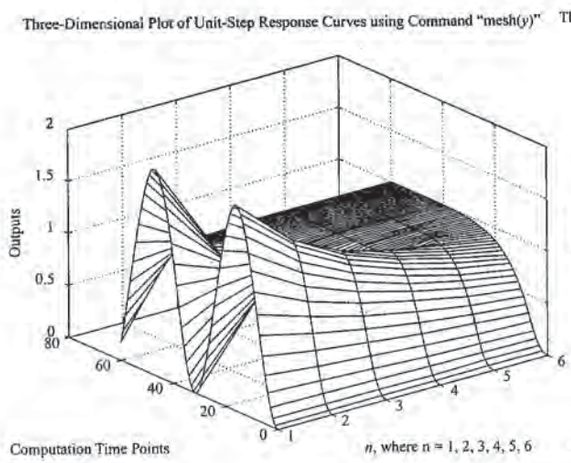
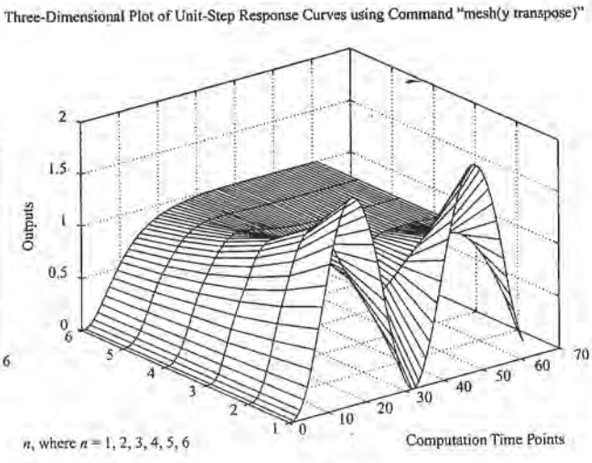
**Figure 5–69**
(a) Two-dimensional plot of unit-step response curves;
(b) three-dimensional plot of unit-step response curves usign command "mesh(y)";
(c) three-dimensional plot of unit-step response curves using command "mesh(y′)".




(a)




(b)




(c)

**A–5–18.** Consider the following characteristic equation:

$$s^4 + Ks^3 + s^2 + s + 1 = 0$$

Determine the range of $K$ for stability.

**Solution.** The Routh array of coefficients is

$$\begin{array}{lll} s^4 & 1 & 1 \quad 1 \\ s^3 & K & 1 \quad 0 \\ s^2 & \dfrac{K-1}{K} & 1 \quad 0 \\ s^1 & 1 - \dfrac{K^2}{K-1} & 0 \\ s^0 & 1 & \end{array}$$





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For stability, we require that

$$K > 0$$
$$\frac{K-1}{K} > 0$$
$$1 - \frac{K^2}{K-1} > 0$$

From the first and second conditions, $K$ must be greater than 1. For $K > 1$, notice that the term $1 - [K^2/(K-1)]$ is always negative, since

$$\frac{K - 1 - K^2}{K-1} = \frac{-1 + K(1-K)}{K-1} < 0$$

Thus, the three conditions cannot be fulfilled simultaneously. Therefore, there is no value of $K$ that allows stability of the system.

**A–5–19.** Consider the characteristic equation given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \tag{5–67}$$

The Hurwitz stability criterion, given next, gives conditions for all the roots to have negative real parts in terms of the coefficients of the polynomial. As stated in the discussions of Routh's stability criterion in Section 5–7, for all the roots to have negative real parts, all the coefficients $a$'s must be positive. This is a necessary condition but not a sufficient condition. If this condition is not satisfied, it indicates that some of the roots have positive real parts or are imaginary or zero. A sufficient condition for all the roots to have negative real parts is given in the following Hurwitz stability criterion: If all the coefficients of the polynomial are positive, arrange these coefficients in the following determinant:

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdots & a_n & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & a_{n-1} & 0 & 0 \\ \cdot & \cdot & \cdot & & a_{n-2} & a_n & 0 \\ \cdot & \cdot & \cdot & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{vmatrix}$$

where we substituted zero for $a_s$ if $s > n$. For all the roots to have negative real parts, it is necessary and sufficient that successive principal minors of $\Delta_n$ be positive. The successive principal minors are the following determinants:

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & \cdots & a_{2i-1} \\ a_0 & a_2 & \cdots & a_{2i-2} \\ 0 & a_1 & \cdots & a_{2i-3} \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & a_i \end{vmatrix} \qquad (i = 1, 2, \ldots, n-1)$$

where $a_s = 0$ if $s > n$. (It is noted that some of the conditions for the lower-order determinants are included in the conditions for the higher-order determinants.) If all these determinants are positive, and $a_0 > 0$ as already assumed, the equilibrium state of the system whose characteristic





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equation is given by Equation (5–67) is asymptotically stable. Note that exact values of determinants are not needed; instead, only signs of these determinants are needed for the stability criterion.

Now consider the following characteristic equation:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

Obtain the conditions for stability using the Hurwitz stability criterion.

**Solution.** The conditions for stability are that all the $a$'s be positive and that

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0$$

$$\begin{aligned} \Delta_3 &= \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \\ &= a_1(a_2 a_3 - a_1 a_4) - a_0 a_3^2 \\ &= a_3(a_1 a_2 - a_0 a_3) - a_1^2 a_4 > 0 \end{aligned}$$

It is clear that, if all the $a$'s are positive and if the condition $\Delta_3 > 0$ is satisfied, the condition $\Delta_2 > 0$ is also satisfied. Therefore, for all the roots of the given characteristic equation to have negative real parts, it is necessary and sufficient that all the coefficients $a$'s are positive and $\Delta_3 > 0$.

**A–5–20.** Show that the first column of the Routh array of

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \ldots, \quad \frac{\Delta_n}{\Delta_{n-1}}$$

where

$$\Delta_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & 0 \\ a_3 & a_2 & a_1 & 1 & \cdot & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ a_{2r-1} & \cdot & \cdot & \cdot & \cdot & a_r \end{vmatrix}, \qquad (n \geq r \geq 1)$$

$$a_k = 0 \qquad \text{if } k > n$$

**Solution.** The Routh array of coefficients has the form

$$\begin{array}{cccccc} 1 & a_2 & a_4 & a_6 & \cdots & a_n \\ a_1 & a_3 & a_5 & \cdots & & \\ b_1 & b_2 & b_3 & \cdots & & \\ c_1 & c_2 & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{array}$$





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The first term in the first column of the Routh array is 1. The next term in the first column is $a_1$, which is equal to $\Delta_1$. The next term is $b_1$, which is equal to

$$\frac{a_1a_2 - a_3}{a_1} = \frac{\Delta_2}{\Delta_1}$$

The next term in the first column is $c_1$, which is equal to

$$\frac{b_1a_3 - a_1b_2}{b_1} = \frac{\left[\dfrac{a_1a_2 - a_3}{a_1}\right]a_3 - a_1\left[\dfrac{a_1a_4 - a_5}{a_1}\right]}{\left[\dfrac{a_1a_2 - a_3}{a_1}\right]}$$

$$= \frac{a_1a_2a_3 - a_3^2 - a_1^2a_4 + a_1a_5}{a_1a_2 - a_3}$$

$$= \frac{\Delta_3}{\Delta_2}.$$

In a similar manner the remaining terms in the first column of the Routh array can be found.

The Routh array has the property that the last nonzero terms of any columns are the same; that is, if the array is given by

$$\begin{array}{cccc} a_0 & a_2 & a_4 & a_6 \\ a_1 & a_3 & a_5 & a_7 \\ b_1 & b_2 & b_3 & \\ c_1 & c_2 & c_3 & \\ d_1 & d_2 & & \\ e_1 & e_2 & & \\ f_1 & & & \\ g_1 & & & \end{array}$$

then

$$a_7 = c_3 = e_2 = g_1$$

and if the array is given by

$$\begin{array}{cccc} a_0 & a_2 & a_4 & a_6 \\ a_1 & a_3 & a_5 & 0 \\ b_1 & b_2 & b_3 & \\ c_1 & c_2 & 0 & \\ d_1 & d_2 & & \\ e_1 & 0 & & \\ f_1 & & & \end{array}$$

then

$$a_6 = b_3 = d_2 = f_1$$

In any case, the last term of the first column is equal to $a_n$, or

$$a_n = \frac{\Delta_{n-1}a_n}{\Delta_{n-1}} = \frac{\Delta_n}{\Delta_{n-1}}$$





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For example, if $n = 4$, then

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix} = \Delta_3 a_4$$

Thus it has been shown that the first column of the Routh array is given by

$$1, \quad \Delta_1, \quad \frac{\Delta_2}{\Delta_1}, \quad \frac{\Delta_3}{\Delta_2}, \quad \ldots, \quad \frac{\Delta_n}{\Delta_{n-1}}$$

**A–5–21.** Show that the Routh's stability criterion and Hurwitz stability criterion are equivalent.

**Solution.** If we write Hurwitz determinants in the triangular form

$$\Delta_i = \begin{vmatrix} a_{11} & & & & * \\ & a_{22} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & a_{ii} \end{vmatrix}, \qquad (i = 1, 2, \ldots, n)$$

where the elements below the diagonal line are all zeros and the elements above the diagonal line any numbers, then the Hurwitz conditions for asymptotic stability become

$$\Delta_i = a_{11} a_{22} \cdots a_{ii} > 0, \qquad (i = 1, 2, \ldots, n)$$

which are equivalent to the conditions

$$a_{11} > 0, \qquad a_{22} > 0, \qquad \ldots, \qquad a_{nn} > 0$$

We shall show that these conditions are equivalent to

$$a_1 > 0, \qquad b_1 > 0, \qquad c_1 > 0, \qquad \ldots$$

where $a_1, b_1, c_1, \ldots,$ are the elements of the first column in the Routh array.

Consider, for example, the following Hurwitz determinant, which corresponds to $i = 4$:

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

The determinant is unchanged if we subtract from the $i$th row $k$ times the $j$th row. By subtracting from the second row $a_0/a_1$ times the first row, we obtain

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$





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where

$$a_{11} = a_1$$

$$a_{22} = a_2 - \frac{a_0}{a_1} a_3$$

$$a_{23} = a_4 - \frac{a_0}{a_1} a_5$$

$$a_{24} = a_6 - \frac{a_0}{a_1} a_7$$

Similarly, subtracting from the fourth row $a_0/a_1$ times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$\hat{a}_{43} = a_2 - \frac{a_0}{a_1} a_3$$

$$\hat{a}_{44} = a_4 - \frac{a_0}{a_1} a_5$$

Next, subtracting from the third row $a_1/a_{22}$ times the second row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & \hat{a}_{43} & \hat{a}_{44} \end{vmatrix}$$

where

$$a_{33} = a_3 - \frac{a_1}{a_{22}} a_{23}$$

$$a_{34} = a_5 - \frac{a_1}{a_{22}} a_{24}$$

Finally, subtracting from the last row $\hat{a}_{43}/a_{33}$ times the third row yields

$$\Delta_4 = \begin{vmatrix} a_{11} & a_3 & a_5 & a_7 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}$$

where

$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}} a_{34}$$





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From this analysis, we see that

$$\Delta_4 = a_{11}a_{22}a_{33}a_{44}$$
$$\Delta_3 = a_{11}a_{22}a_{33}$$
$$\Delta_2 = a_{11}a_{22}$$
$$\Delta_1 = a_{11}$$

The Hurwitz conditions for asymptotic stability

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \Delta_3 > 0, \quad \Delta_4 > 0, \quad \ldots$$

reduce to the conditions

$$a_{11} > 0, \quad a_{22} > 0, \quad a_{33} > 0, \quad a_{44} > 0, \quad \ldots$$

The Routh array for the polynomial

$$a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$$

where $a_0 > 0$ and $n = 4$, is given by

$$\begin{array}{ccc} a_0 & a_2 & a_4 \\ a_1 & a_3 & \\ b_1 & b_2 & \\ c_1 & & \\ d_1 & & \end{array}$$

From this Routh array, we see that

$$a_{11} = a_1$$
$$a_{22} = a_2 - \frac{a_0}{a_1}a_3 = b_1$$
$$a_{33} = a_3 - \frac{a_1}{a_{22}}a_{23} = \frac{a_3b_1 - a_1b_2}{b_1} = c_1$$
$$a_{44} = \hat{a}_{44} - \frac{\hat{a}_{43}}{a_{33}}a_{34} = a_4 = d_1$$

(The last equation is obtained using the fact that $a_{34} = 0$, $\hat{a}_{44} = a_4$, and $a_4 = b_2 = d_1$.) Hence the Hurwitz conditions for asymptotic stability become

$$a_1 > 0, \quad b_1 > 0, \quad c_1 > 0, \quad d_1 > 0$$

Thus we have demonstrated that Hurwitz conditions for asymptotic stability can be reduced to Routh's conditions for asymptotic stability. The same argument can be extended to Hurwitz determinants of any order, and the equivalence of Routh's stability criterion and Hurwitz stability criterion can be established.

**A–5–22.** Consider the characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$

Using the Hurwitz stability criterion, determine the range of $K$ for stability.

**Solution.** Comparing the given characteristic equation

$$s^4 + 2s^3 + (4 + K)s^2 + 9s + 25 = 0$$





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with the following standard fourth-order characteristic equation:

$$a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$$

we find

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4 + K, \quad a_3 = 9, \quad a_4 = 25$$

The Hurwitz stability criterion states that $\Delta_4$ is given by

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

For all the roots to have negative real parts, it is necessary and sufficient that succesive principal minors of $\Delta_4$ be positive. The successive principal minors are

$$\Delta_1 = |a_1| = 2$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 9 \\ 1 & 4 + K \end{vmatrix} = 2K - 1$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 0 \\ 1 & 4 + K & 25 \\ 0 & 2 & 9 \end{vmatrix} = 18K - 109$$

For all principal minors to be positive, we require that $\Delta_i (i = 1, 2, 3)$ be positive. Thus, we require

$$2K - 1 > 0$$

$$18K - 109 > 0$$

from which we obtain the region of $K$ for stability to be

$$K > \frac{109}{18}$$

**A–5–23.** Explain why the proportional control of a plant that does not possess an integrating property (which means that the plant transfer function does not include the factor $1/s$) suffers offset in response to step inputs.

**Solution.** Consider, for example, the system shown in Figure 5–70. At steady state, if $c$ were equal to a nonzero constant $r$, then $e = 0$ and $u = Ke = 0$, resulting in $c = 0$, which contradicts the assumption that $c = r =$ nonzero constant.

A nonzero offset must exist for proper operation of such a control system. In other words, at steady state, if $e$ were equal to $r/(1 + K)$, then $u = Kr/(1 + K)$ and $c = Kr/(1 + K)$, which results in the assumed error signal $e = r/(1 + K)$. Thus the offset of $r/(1 + K)$ must exist in such a system.

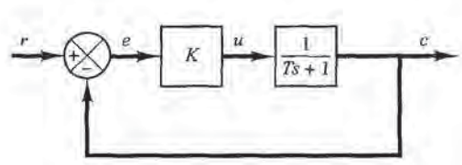


**Figure 5–70**
Control system.





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**A–5–24.** Consider the system shown in Figure 5–71. Show that the steady-state error in following the unit-ramp input is $B/K$. This error can be made smaller by choosing $B$ small and/or $K$ large. However, making $B$ small and/or $K$ large would have the effect of making the damping ratio small, which is normally not desirable. Describe a method or methods to make $B/K$ small and yet make the damping ratio have reasonable value ($0.5 < \zeta < 0.7$).

**Solution.** From Figure 5–71 we obtain

$$E(s) = R(s) - C(s) = \frac{Js^2 + Bs}{Js^2 + Bs + K} R(s)$$

The steady-state error for the unit-ramp response can be obtained as follows: For the unit-ramp input, the steady-state error $e_{ss}$ is

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} s \frac{Js^2 + Bs}{Js^2 + Bs + K} \frac{1}{s^2} \\ &= \frac{B}{K} \\ &= \frac{2\zeta}{\omega_n} \end{aligned}$$

where

$$\zeta = \frac{B}{2\sqrt{KJ}}, \qquad \omega_n = \sqrt{\frac{K}{J}}$$

To assure acceptable transient response and acceptable steady-state error in following a ramp input, $\zeta$ must not be too small and $\omega_n$ must be sufficiently large. It is possible to make the steady-state error $e_{ss}$ small by making the value of the gain $K$ large. (A large value of $K$ has an additional advantage of suppressing undesirable effects caused by dead zone, backlash, coulomb friction, and the like.) A large value of $K$ would, however, make the value of $\zeta$ small and increase the maximum overshoot, which is undesirable.

It is therefore necessary to compromise between the magnitude of the steady-state error to a ramp input and the maximum overshoot to a unit-step input. In the system shown in Figure 5–71, a reasonable compromise may not be reached easily. It is then desirable to consider other types of control action that may improve both the transient-response and steady-state behavior. Two schemes to improve both the transient-response and steady-state behavior are available. One scheme is to use a proportional-plus-derivative controller and the other is to use tachometer feedback.

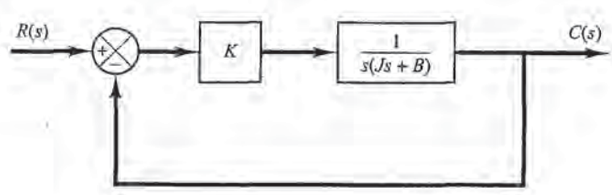


**Figure 5–71**
Control system.





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**A–5–25.** The block diagram of Figure 5–72 shows a speed control system in which the output member of the system is subject to a torque disturbance. In the diagram, $\Omega_r(s)$, $\Omega(s)$, $T(s)$, and $D(s)$ are the Laplace transforms of the reference speed, output speed, driving torque, and disturbance torque, respectively. In the absence of a disturbance torque, the output speed is equal to the reference speed.

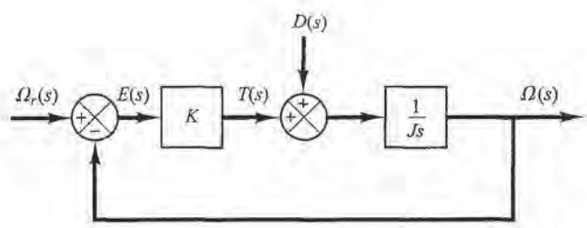


**Figure 5–72**
Block diagram of a speed control system.

Investigate the response of this system to a unit-step disturbance torque. Assume that the reference input is zero, or $\Omega_r(s) = 0$.

**Solution.** Figure 5–73 is a modified block diagram convenient for the present analysis. The closed-loop transfer function is

$$\frac{\Omega_D(s)}{D(s)} = \frac{1}{Js + K}$$

where $\Omega_D(s)$ is the Laplace transform of the output speed due to the disturbance torque. For a unit-step disturbance torque, the steady-state output velocity is

$$\begin{aligned}\omega_D(\infty) &= \lim_{s\to 0} s\Omega_D(s)\\ &= \lim_{s\to 0} \frac{s}{Js + K}\frac{1}{s}\\ &= \frac{1}{K}\end{aligned}$$

From this analysis, we conclude that, if a step disturbance torque is applied to the output member of the system, an error speed will result so that the ensuing motor torque will exactly cancel the disturbance torque. To develop this motor torque, it is necessary that there be an error in speed so that nonzero torque will result.

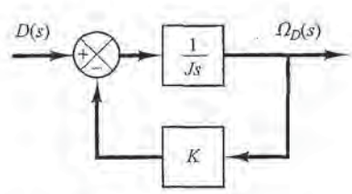


**Figure 5–73**
Block diagram of the speed control system of Figure 5–72 when $\Omega_r(s) = 0$.





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**A–5–26.** In the system considered in Problem **A–5–25**, it is desired to eliminate as much as possible the speed errors due to torque disturbances.

Is it possible to cancel the effect of a disturbance torque at steady state so that a constant disturbance torque applied to the output member will cause no speed change at steady state?

**Solution.** Suppose that we choose a suitable controller whose transfer function is $G_c(s)$, as shown in Figure 5–74. Then in the absence of the reference input the closed-loop transfer function between the output velocity $\Omega_D(s)$ and the disturbance torque $D(s)$ is

$$\frac{\Omega_D(s)}{D(s)} = \frac{\dfrac{1}{Js}}{1 + \dfrac{1}{Js}G_c(s)}$$

$$= \frac{1}{Js + G_c(s)}$$

The steady-state output speed due to a unit-step disturbance torque is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s)$$

$$= \lim_{s \to 0} \frac{s}{Js + G_c(s)} \frac{1}{s}$$

$$= \frac{1}{G_c(0)}$$

To satisfy the requirement that

$$\omega_D(\infty) = 0$$

we must choose $G_c(0) = \infty$. This can be realized if we choose

$$G_c(s) = \frac{K}{s}$$

Integral control action will continue to correct until the error is zero. This controller, however, presents a stability problem because the characteristic equation will have two imaginary roots.

One method of stabilizing such a system is to add a proportional mode to the controller or choose

$$G_c(s) = K_p + \frac{K}{s}$$

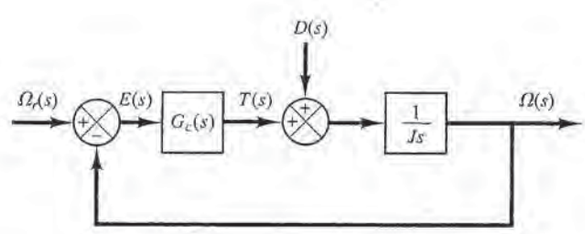


**Figure 5–74**
Block diagram of a speed control system.





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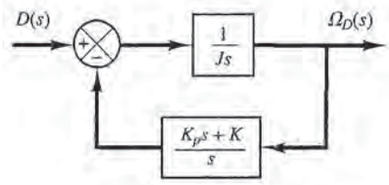



**Figure 5–75**
Block diagram of the speed control system of Figure 5–74 when $G_c(s) = K_p + (K/s)$ and $\Omega_r(s) = 0$.

With this controller, the block diagram of Figure 5–74 in the absence of the reference input can be modified to that of Figure 5–75. The closed-loop transfer function $\Omega_D(s)/D(s)$ becomes

$$\frac{\Omega_D(s)}{D(s)} = \frac{s}{Js^2 + K_p s + K}$$

For a unit-step disturbance torque, the steady-state output speed is

$$\omega_D(\infty) = \lim_{s \to 0} s\Omega_D(s) = \lim_{s \to 0} \frac{s^2}{Js^2 + K_p s + K} \frac{1}{s} = 0$$

Thus, we see that the proportional-plus-integral controller eliminates speed error at steady state.

The use of integral control action has increased the order of the system by 1. (This tends to produce an oscillatory response.)

In the present system, a step disturbance torque will cause a transient error in the output speed, but the error will become zero at steady state. The integrator provides a nonzero output with zero error. (The nonzero output of the integrator produces a motor torque that exactly cancels the disturbance torque.)

Note that the integrator in the transfer function of the plant does not eliminate the steady-state error due to a step disturbance torque. To eliminate this, we must have an integrator before the point where the disturbance torque enters.

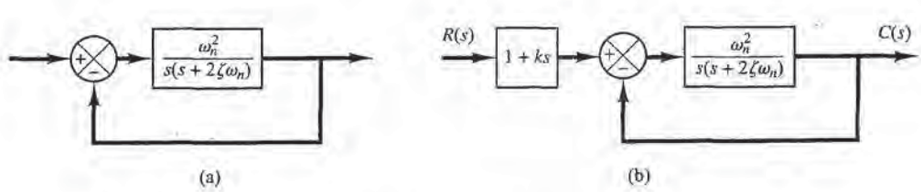
**A–5–27.** Consider the system shown in Figure 5–76(a). The steady-state error to a unit-ramp input is $e_{ss} = 2\zeta/\omega_n$. Show that the steady-state error for following a ramp input may be eliminated if the input is introduced to the system through a proportional-plus-derivative filter, as shown in Figure 5–76(b), and the value of $k$ is properly set. Note that the error $e(t)$ is given by $r(t) - c(t)$.

**Solution.** The closed-loop transfer function of the system shown in Figure 5–76(b) is

$$\frac{C(s)}{R(s)} = \frac{(1 + ks)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Then

$$R(s) - C(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) R(s)$$




(a) (b)

**Figure 5–76**
(a) Control system; (b) control system with input filter.





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If the input is a unit ramp, then the steady-state error is

$$
\begin{aligned}
e(\infty) &= r(\infty) - c(\infty) \\
&= \lim_{s \to 0} s \left( \frac{s^2 + 2\zeta\omega_n s - \omega_n^2 k s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \frac{1}{s^2} \\
&= \frac{2\zeta\omega_n - \omega_n^2 k}{\omega_n^2}
\end{aligned}
$$

Therefore, if $k$ is chosen as

$$k = \frac{2\zeta}{\omega_n}$$

then the steady-state error for following a ramp input can be made equal to zero. Note that, if there are any variations in the values of $\zeta$ and/or $\omega_n$ due to environmental changes or aging, then a nonzero steady-state error for a ramp response may result.

**A–5–28.** Consider the servo system with tachometer feedback shown in Figure 5–77. Obtain the error signal $E(s)$ when both the reference input $R(s)$ and disturbance input $D(s)$ are present. Obtain also the steady-state error when the system is subjected to a reference input (unit-ramp input) and disturbance input (step input of magnitude $d$).

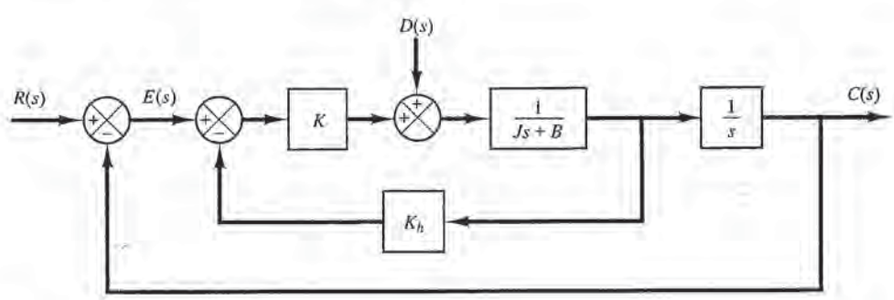


**Figure 5–77**
Servo system with tachometer feedback.

**Solution.** When we consider the reference input $R(s)$ we can assume that the disturbance input $D(s)$ is zero, and vice versa. Then, a block diagram that relates the reference input $R(s)$ and the output $C(s)$ may be drawn as shown in Figure 5–78(a). Similarly, Figure 5–78(b) relates the disturbance input $D(s)$ and the output $C(s)$.

The closed-loop transfer function $C(s)/R(s)$ can be obtained from Figure 5–78(a) as follows:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Similarly, the closed-loop transfer function $C(s)/D(s)$ can be obtained from Figure 5–78(b) as

$$\frac{C(s)}{D(s)} = \frac{1}{Js^2 + (B + KK_h)s + K}$$

If both $R(s)$ and $D(s)$ are present, then

$$C(s) = \frac{1}{Js^2 + (B + KK_h)s + K} [KR(s) + D(s)]$$





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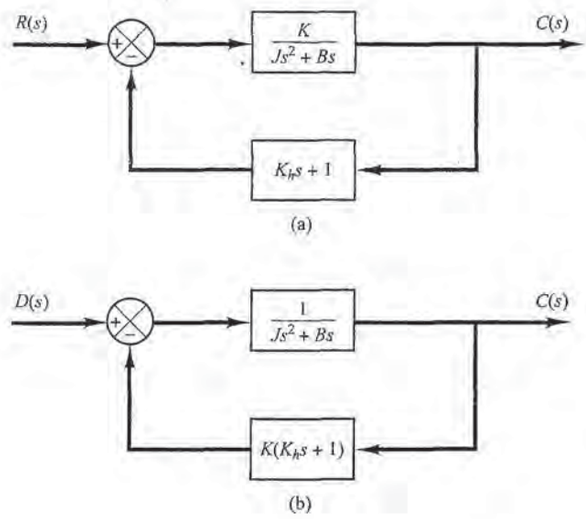


**Figure 5–78**
(a) Block diagram that relates reference input $R(s)$ and output $C(s)$; (b) block diagram that relates disturbance input $D(s)$ and output $C(s)$.

Since

$$E(s) = R(s) - C(s)$$
$$= R(s) - \frac{KR(s) + D(s)}{Js^2 + (B + KK_h)s + K}$$

we obtain

$$E(s) = \frac{1}{Js^2 + (B + KK_h)s + K}\{[Js^2 + (B + KK_h)s]R(s) - D(s)\}$$

Hence the steady-state error can be obtained as follows:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$
$$= \lim_{s\to 0} \frac{s}{Js^2 + (B + KK_h)s + K}[s(Js + B + KK_h)R(s) - D(s)]$$
$$= \lim_{s\to 0}\left[\frac{B + KK_h}{K}s^2R(s) - \frac{1}{K}sD(s)\right]$$

Since $R(s) = 1/s^2$ (unit-ramp input) and $D(s) = d/s$ (step input of magnitude $d$) the steady-state error is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0}\left[\frac{B + KK_h}{K}\frac{s^2}{s^2} - \frac{1}{K}\frac{sd}{s}\right]$$
$$= \frac{B + KK_h}{K} - \frac{d}{K}$$





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**A–5–29.** Consider the stable unity-feedback control system with feedforward transfer function $G(s)$. Suppose that the closed-loop transfer function can be written

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1)} \qquad (m \le n)$$

Show that

$$\int_0^\infty e(t)\,dt = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

where $e(t)$ is the error in the unit-step response. Show also that

$$\frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

**Solution.** Let us define

$$(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1) = P(s)$$

and

$$(T_1 s + 1)(T_2 s + 1)\cdots(T_n s + 1) = Q(s)$$

Then

$$\frac{C(s)}{R(s)} = \frac{P(s)}{Q(s)}$$

and

$$E(s) = \frac{Q(s) - P(s)}{Q(s)} R(s)$$

For a unit-step input, $R(s) = 1/s$ and

$$E(s) = \frac{Q(s) - P(s)}{sQ(s)}$$

Since the system is stable, $\int_0^\infty e(t)\,dt$ converges to a constant value. Referring to Table 2–2 (item 9), we have

$$\int_0^\infty e(t)\,dt = \lim_{s\to 0} s\frac{E(s)}{s} = \lim_{s\to 0} E(s)$$

Hence

$$\begin{aligned}\int_0^\infty e(t)\,dt &= \lim_{s\to 0}\frac{Q(s) - P(s)}{sQ(s)}\\ &= \lim_{s\to 0}\frac{Q'(s) - P'(s)}{Q(s) + sQ'(s)}\\ &= \lim_{s\to 0}[Q'(s) - P'(s)]\end{aligned}$$





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Since

$$\lim_{s\to 0} P'(s) = T_a + T_b + \cdots + T_m$$

$$\lim_{s\to 0} Q'(s) = T_1 + T_2 + \cdots + T_n$$

we have

$$\int_0^\infty e(t)\,dt = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

For a unit-step input $r(t)$, since

$$\int_0^\infty e(t)\,dt = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{1}{1+G(s)} R(s) = \lim_{s\to 0} \frac{1}{1+G(s)} \frac{1}{s} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$$

we have

$$\frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = (T_1 + T_2 + \cdots + T_n) - (T_a + T_b + \cdots + T_m)$$

Note that zeros in the left half-plane (that is, positive $T_a, T_b, \ldots, T_m$) will improve $K_v$. Poles close to the origin cause low velocity-error constants unless there are zeros nearby.

## PROBLEMS

**B–5–1.** A thermometer requires 1 min to indicate 98% of the response to a step input. Assuming the thermometer to be a first-order system, find the time constant.

If the thermometer is placed in a bath, the temperature of which is changing linearly at a rate of 10°/min, how much error does the thermometer show?

**B–5–2.** Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{s(s+1)}$$

Obtain the rise time, peak time, maximum overshoot, and settling time.

**B–5–3.** Consider the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Determine the values of $\zeta$ and $\omega_n$ so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Use the 2% criterion.)

**B–5–4.** Figure 5–79 is a block diagram of a space-vehicle attitude-control system. Assuming the time constant $T$ of the controller to be 3 sec and the ratio $K/J$ to be $\frac{2}{9}$ rad²/sec², find the damping ratio of the system.

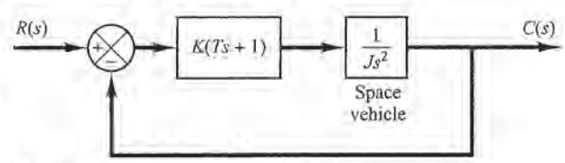


**Figure 5–79**
Space-vehicle attitude-control system.

**B–5–5.** Consider the system shown in Figure 5–80. The system is initially at rest. Suppose that the cart is set into motion by an impulsive force whose strength is unity. Can it be stopped by another such impulsive force?

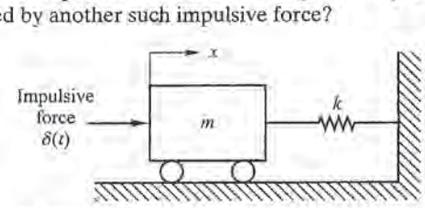


**Figure 5–80**
Mechanical system.





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**B–5–6.** Obtain the unit-impulse response and the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{2s + 1}{s^2}$$

**B–5–7.** Consider the system shown in Figure 5–81. Show that the transfer function $Y(s)/X(s)$ has a zero in the right-half $s$ plane. Then obtain $y(t)$ when $x(t)$ is a unit step. Plot $y(t)$ versus $t$.

**B–5–8.** An oscillatory system is known to have a transfer function of the following form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Assume that a record of a damped oscillation is available as shown in Figure 5–82. Determine the damping ratio $\zeta$ of the system from the graph.

**B–5–9.** Consider the system shown in Figure 5–83(a). The damping ratio of this system is 0.158 and the undamped natural frequency is 3.16 rad/sec. To improve the relative stability, we employ tachometer feedback. Figure 5–83(b) shows such a tachometer-feedback system.

Determine the value of $K_h$ so that the damping ratio of the system is 0.5. Draw unit-step response curves of both the original and tachometer-feedback systems. Also draw the error-versus-time curves for the unit-ramp response of both systems.

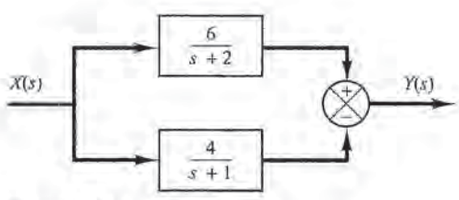


**Figure 5–81**
System with zero in the right-half $s$ plane.

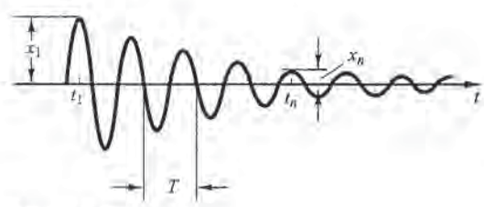


**Figure 5–82**
Decaying oscillation.

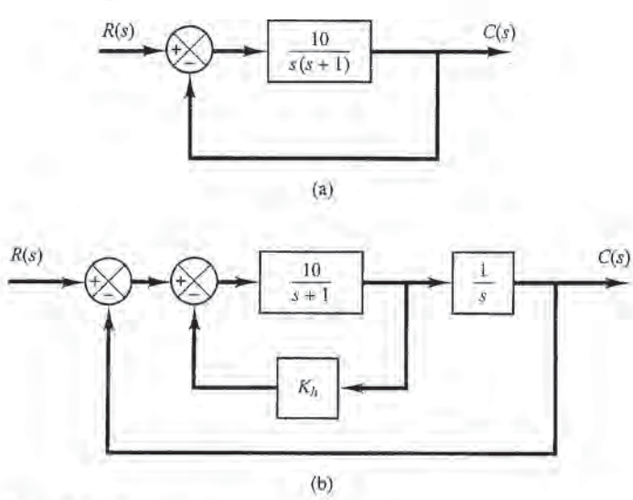


**Figure 5–83**
(a) Control system; (b) control system with tachometer feedback.





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**B–5–10.** Referring to the system shown in Figure 5–84, determine the values of $K$ and $k$ such that the system has a damping ratio $\zeta$ of 0.7 and an undamped natural frequency $\omega_n$ of 4 rad/sec.

**B–5–11.** Consider the system shown in Figure 5–85. Determine the value of $k$ such that the damping ratio $\zeta$ is 0.5. Then obtain the rise time $t_r$, peak time $t_p$, maximum overshoot $M_p$, and settling time $t_s$ in the unit-step response.

**B–5–12.** Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}$$

where $R(s)$ and $C(s)$ are Laplace transforms of the input $r(t)$ and output $c(t)$, respectively.

**B–5–13.** Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $u$ is the input and $y$ is the output.

**B–5–14.** Obtain both analytically and computationally the rise time, peak time, maximum overshoot, and settling time in the unit-step response of a closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{36}{s^2 + 2s + 36}$$

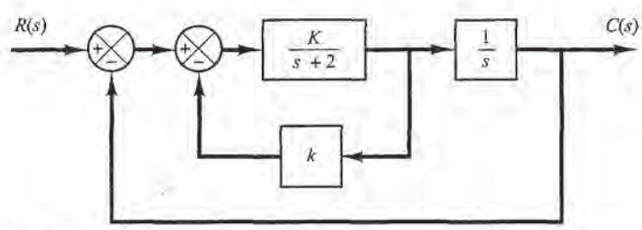


**Figure 5–84**
Closed-loop system.

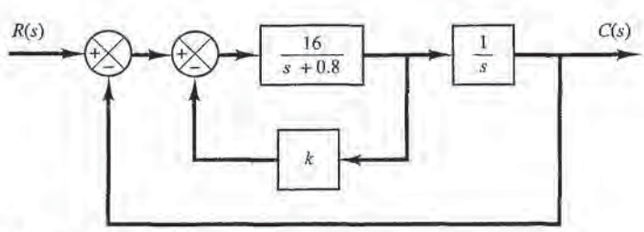


**Figure 5–85**
Block diagram of a system.





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**B–5–15.** Figure 5–86 shows three systems. System I is a positional servo system. System II is a positional servo system with PD control action. System III is a positional servo system with velocity feedback. Compare the unit-step, unit-impulse, and unit-ramp responses of the three systems. Which system is best with respect to the speed of response and maximum overshoot in the step response?

**B–5–16.** Consider the position control system shown in Figure 5–87. Write a MATLAB program to obtain a unit-step response and a unit-ramp response of the system. Plot curves $x_1(t)$ versus $t$, $x_2(t)$ versus $t$, $x_3(t)$ versus $t$, and $e(t)$ versus $t$ [where $e(t) = r(t) - x_1(t)$] for both the unit-step response and the unit-ramp response.

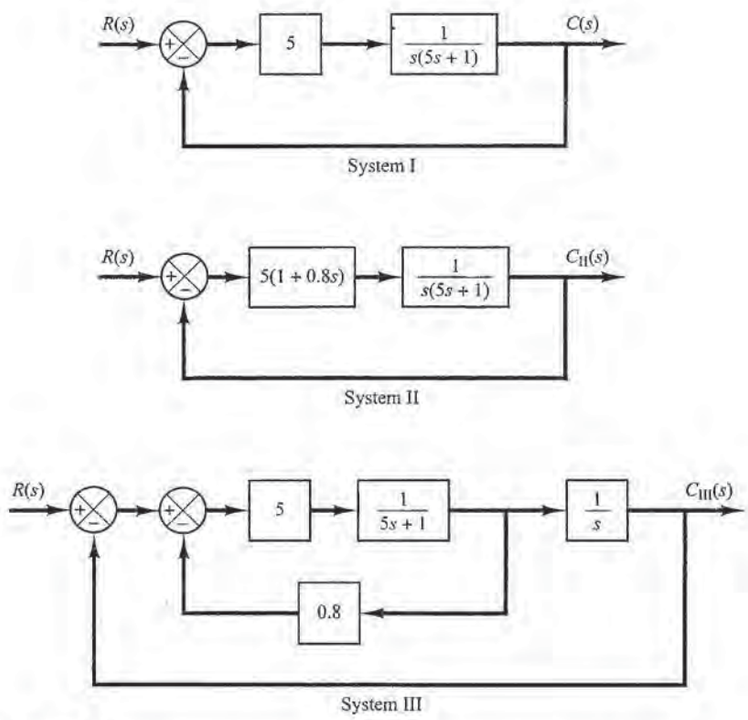


**Figure 5–86**
Positional servo system (System I), positional servo system with PD control action (System II), and positional servo system with velocity feedback (System III).

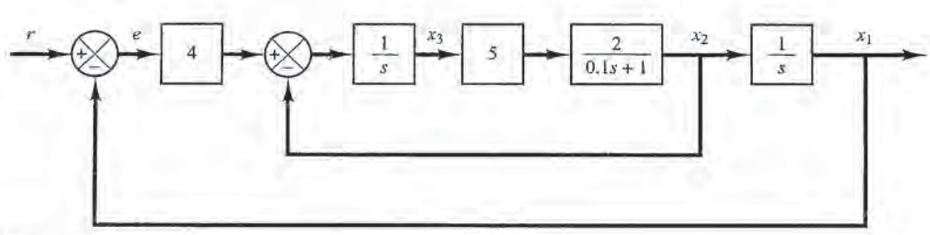


**Figure 5–87**
Position control system.





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**B–5–17.** Using MATLAB, obtain the unit-step response curve for the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10}{s(s+2)(s+4)}$$

Using MATLAB, obtain also the rise time, peak time, maximum overshoot, and settling time in the unit-step response curve.

**B–5–18.** Consider the closed-loop system defined by

$$\frac{C(s)}{R(s)} = \frac{2\zeta s + 1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8$, and 1.0. Using MATLAB, plot a two-dimensional diagram of unit-impulse response curves. Also plot a three-dimensional plot of the response curves.

**B–5–19.** Consider the second-order system defined by

$$\frac{C(s)}{R(s)} = \frac{s+1}{s^2 + 2\zeta s + 1}$$

where $\zeta = 0.2, 0.4, 0.6, 0.8, 1.0$. Plot a three-dimensional diagram of the unit-step response curves.

**B–5–20.** Obtain the unit-ramp response of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $u$ is the unit-ramp input. Use lsim command to obtain the response.

**B–5–21.** Using MATLAB obtain the unit acceleration response curve of the unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{10(s+1)}{s^2(s+4)}$$

The unit acceleration input is defined by

$$r(t) = \frac{1}{2}t^2 \qquad (t \geq 0)$$

**B–5–22.** Consider the differential equation system given by

$$\ddot{y} + 3\dot{y} + 2y = 0, \qquad y(0) = 0.1, \qquad \dot{y}(0) = 0.05$$

Obtain the response $y(t)$, subject to the given initial condition.

**B–5–23.** Determine the range of $K$ for stability of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

**B–5–24.** Consider the unity-feedback control system with the following open-loop transfer function:

$$G(s) = \frac{10}{s(s-1)(2s+3)}$$

Is this system stable?

**B–5–25.** Consider the following characteristic equation:

$$s^4 + 2s^3 + (4+K)s^2 + 9s + 25 = 0$$

Using Routh stability criterion, determine the range of $K$ for stability.

**B–5–26.** Consider the closed-loop system shown in Figure 5–88. Determine the range of $K$ for stability. Assume that $K > 0$.

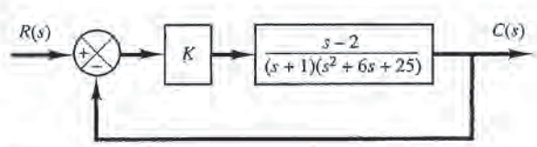


**Figure 5–88**
Closed-loop system.





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**B–5–27.** Consider the satellite attitude control system shown in Figure 5–89(a). The output of this system exhibits continued oscillations and is not desirable. This system can be stabilized by use of tachometer feedback, as shown in Figure 5–89(b). If $K/J = 4$, what value of $K_h$ will yield the damping ratio to be 0.6?

**B–5–28.** Consider the servo system with tachometer feedback shown in Figure 5–90. Determine the ranges of stability for $K$ and $K_h$. (Note that $K_h$ must be positive.)

**B–5–29.** Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & 1 \\ 0 & -b_2 & -b_1 \end{bmatrix}$$

($\mathbf{A}$ is called Schwarz matrix.) Show that the first column of the Routh's array of the characteristic equation $|s\mathbf{I} - \mathbf{A}| = 0$ consists of $1, b_1, b_2$, and $b_1 b_3$.

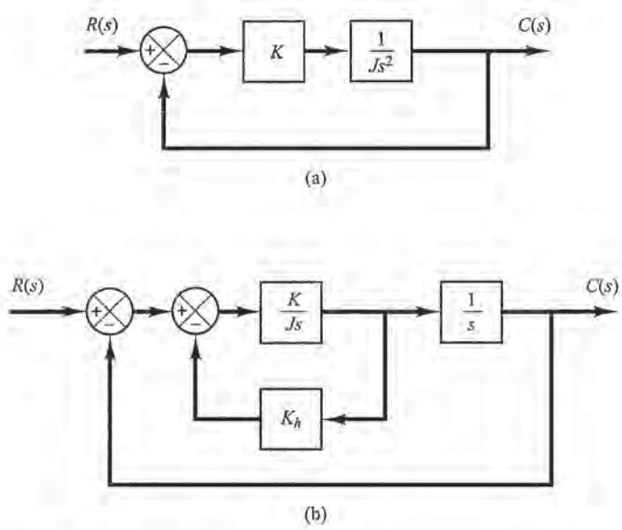


**Figure 5–89**
(a) Unstable satellite attitude control system; (b) stabilized system.

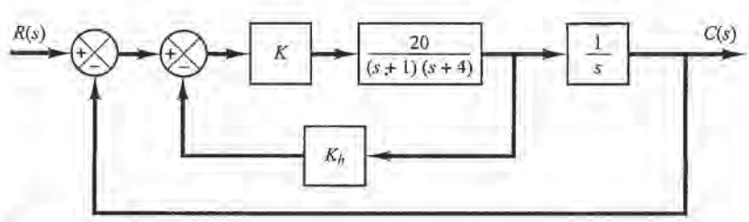


**Figure 5–90**
Servo system with tachometer feedback.





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**B–5–30.** Consider a unity-feedback control system with the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

Determine the open-loop transfer function $G(s)$.

Show that the steady-state error in the unit-ramp response is given by

$$e_{ss} = \frac{1}{K_v} = \frac{a - K}{b}$$

**B–5–31.** Consider a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(Js + B)}$$

Discuss the effects that varying the values of $K$ and $B$ has on the steady-state error in unit-ramp response. Sketch typical unit-ramp response curves for a small value, medium value, and large value of $K$, assuming that $B$ is constant.

**B–5–32.** If the feedforward path of a control system contains at least one integrating element, then the output continues to change as long as an error is present. The output stops when the error is precisely zero. If an external disturbance enters the system, it is desirable to have an integrating element between the error-measuring element and the point where the disturbance enters so that the effect of the external disturbance may be made zero at steady state.

Show that, if the disturbance is a ramp function, then the steady-state error due to this ramp disturbance may be eliminated only if two integrators precede the point where the disturbance enters.





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# 6

# Root-Locus Analysis

## 6–1 INTRODUCTION

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the $s$ plane as the loop gain is varied.

From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations. Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary. (This subject is discussed in detail in Chapter 7.)

The closed-loop poles are the roots of the characteristic equation. Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution. (MATLAB provides a simple solution to this problem.) However, just finding the roots of the characteristic equation may be of limited value, because as the gain of the open-loop transfer function varies the characteristic equation changes and the computations must be repeated.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the *root-locus method,* is one in which the roots of the characteristic equation are





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plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.(See Chapter 7.) Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros. Therefore, it is desired that the designer have a good understanding of the method for generating the root loci of the closed-loop system, both by hand and by use of a computer software like MATLAB.

**Root-Locus Method.** The basic idea behind the root-locus method is that the values of $s$ that make the transfer function around the loop equal $-1$ must satisfy the characteristic equation of the system.

The root locus is the locus of roots of the characteristic equation of the closed-loop system as a specific parameter (usually, gain $K$) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-loop pole or zero to the locations of the closed-loop poles.

In designing a linear control system, we find that the root-locus method proves quite useful since it indicates the manner in which the open-loop poles and zeros should be modified so that the response meets system performance specifications. This method is particularly suited to obtaining approximate results very quickly.

Because generating the root loci by use of MATLAB is very simple, one may think sketching the root loci by hand is a waste of time and effort. However, experience in sketching the root loci by hand is invaluable for interpreting computer-generated root loci, as well as for getting a rough idea of the root loci very quickly.

By using the root-locus method, it is possible to determine the value of the loop gain $K$ that will make the damping ratio of the dominant closed-loop poles as prescribed. If the location of an open-loop pole or zero is a system variable, then the root-locus method suggests the way to choose the location of an open-loop pole or zero.

**Outline of the Chapter.** This chapter introduces the basic concept of the root-locus method and presents useful rules for graphically constructing the root loci, as well as the generation of root loci with MATLAB.

The outline of the chapter is as follows: Section 6–1 has presented an introduction to the root-locus method. Section 6–2 details the concepts underlying the root-locus method and presents the general procedure for sketching root loci using illustrative examples. Section 6–3 summarizes general rules for constructing root loci. (If the designer follows the general rules for constructing the root loci, sketching the root loci for a given system will become a simple matter.) Section 6–4 discusses generating root-locus plots with MATLAB. Section 6–5 treats a special case when the closed-loop system has positive feedback. Section 6–6 treats conditionally stable systems. Finally, Section 6–7 extends the root-locus method to treat closed-loop systems with transport lag.





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**Angle and Magnitude Conditions.** Consider the system shown in Figure 6–1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{6–1}$$

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (6–1) equal to zero. That is,

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1 \tag{6–2}$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in $s$. [Later, in Section 6–7, we extend the analysis to the case when $G(s)H(s)$ involves the transport lag $e^{-Ts}$.] Since $G(s)H(s)$ is a complex quantity, Equation (6–2) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\underline{/G(s)H(s)} = \pm 180°(2k + 1) \qquad (k = 0, 1, 2, \dots) \tag{6–3}$$

Magnitude condition:

$$|G(s)H(s)| = 1 \tag{6–4}$$

The values of $s$ that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition. The details of applying the angle and magnitude conditions to obtain the closed-loop poles are presented later in this section.

In many cases, $G(s)H(s)$ involves a gain parameter $K$, and the characteristic equation may be written as

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0$$

Then the root loci for the system are the loci of the closed-loop poles as the gain $K$ is varied from zero to infinity.

Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of $G(s)H(s)$. Remember that the angles

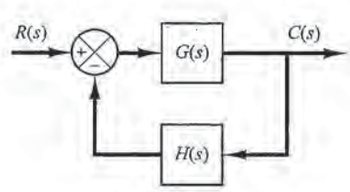


**Figure 6–1**
Control system.





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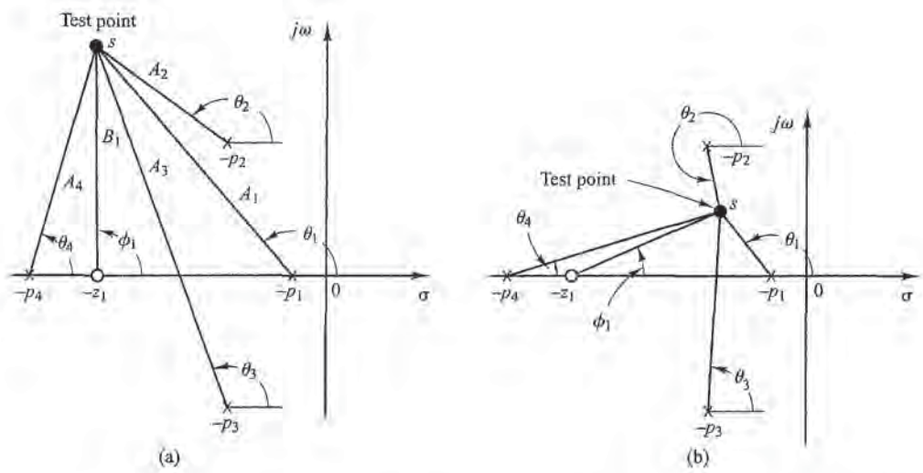


**Figure 6–2**
(a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point $s$.

of the complex quantities originating from the open-loop poles and open-loop zeros to the test point $s$ are measured in the counterclockwise direction. For example, if $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$

where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where $\phi_1$, $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ are measured counterclockwise as shown in Figures 6–2(a) and (b). The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

where $A_1$, $A_2$, $A_3$, $A_4$, and $B_1$ are the magnitudes of the complex quantities $s + p_1$, $s + p_2$, $s + p_3$, $s + p_4$, and $s + z_1$, respectively, as shown in Figure 6–2(a).

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half $s$ plane.

**Illustrative Examples.** In what follows, two illustrative examples for constructing root-locus plots will be presented. Although computer approaches to the construction of the root loci are easily available, here we shall use graphical computation, combined with inspection, to determine the root loci upon which the roots of the characteristic equation of the closed-loop system must lie. Such a graphical approach will enhance understanding of how the closed-loop poles move in the complex plane as the open-loop poles and zeros are moved. Although we employ only simple systems for illustrative purposes, the procedure for finding the root loci is no more complicated for higher-order systems.





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The first step in the procedure for constructing a root-locus plot is to seek out the loci of possible roots using the angle condition. Then, if necessary, the loci can be scaled, or graduated, in gain using the magnitude condition.

Because graphical measurements of angles and magnitudes are involved in the analysis, we find it necessary to use the same divisions on the abscissa as on the ordinate axis when sketching the root locus on graph paper.

**EXAMPLE 6–1** Consider the system shown in Figure 6–3. (We assume that the value of gain $K$ is nonnegative.) For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)}, \qquad H(s) = 1$$

Let us sketch the root-locus plot and then determine the value of $K$ such that the damping ratio $\zeta$ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

For the given system, the angle condition becomes

$$\begin{aligned} \underline{/G(s)} &= \underline{\left/ \frac{K}{s(s+1)(s+2)} \right.} \\ &= -\underline{/s} - \underline{/s+1} - \underline{/s+2} \\ &= \pm 180°(2k+1) \qquad (k = 0, 1, 2, \ldots) \end{aligned}$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$$

A typical procedure for sketching the root-locus plot is as follows:

**1.** *Determine the root loci on the real axis.* The first step in constructing a root-locus plot is to locate the open-loop poles, $s = 0$, $s = -1$, and $s = -2$, in the complex plane. (There are no open-loop zeros in this system.) The locations of the open-loop poles are indicated by crosses. (The locations of the open-loop zeros in this book will be indicated by small circles.) Note that the starting points of the root loci (the points corresponding to $K = 0$) are open-loop poles. The number of individual root loci for this system is three, which is the same as the number of open-loop poles.

To determine the root loci on the real axis, we select a test point, $s$. If the test point is on the positive real axis, then

$$\underline{/s} = \underline{/s+1} = \underline{/s+2} = 0°$$

This shows that the angle condition cannot be satisfied. Hence, there is no root locus on the positive real axis. Next, select a test point on the negative real axis between 0 and $-1$. Then

$$\underline{/s} = 180°, \qquad \underline{/s+1} = \underline{/s+2} = 0°$$

Thus

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -180°$$

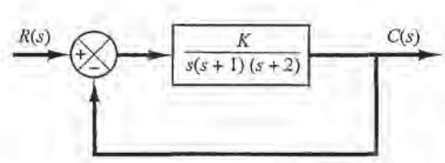


**Figure 6–3**
Control system.





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and the angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and −1 forms a portion of the root locus. If a test point is selected between −1 and −2, then

$$\underline{/s} = \underline{/s+1} = 180°, \qquad \underline{/s+2} = 0°$$

and

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -360°$$

It can be seen that the angle condition is not satisfied. Therefore, the negative real axis from −1 to −2 is not a part of the root locus. Similarly, if a test point is located on the negative real axis from −2 to −∞, the angle condition is satisfied. Thus, root loci exist on the negative real axis between 0 and −1 and between −2 and −∞.

**2.** *Determine the asymptotes of the root loci.* The asymptotes of the root loci as $s$ approaches infinity can be determined as follows: If a test point $s$ is selected very far from the origin, then

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

and the angle condition becomes

$$-3\underline{/s} = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \ldots)$$

or

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} \qquad (k = 0, 1, 2, \ldots)$$

Since the angle repeats itself as $k$ is varied, the distinct angles for the asymptotes are determined as 60°, −60°, and 180°. Thus, there are three asymptotes. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis. Since

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

if a test point is located very far from the origin, then $G(s)$ may be written as

$$G(s) = \frac{K}{s^3 + 3s^2 + \cdots}$$

For large values of $s$, this last equation may be approximated by

$$G(s) \doteqdot \frac{K}{(s+1)^3} \tag{6–5}$$

A root-locus diagram of $G(s)$ given by Equation (6–5) consists of three straight lines. This can be seen as follows: The equation of the root locus is

$$\underline{\left/\frac{K}{(s+1)^3}\right.} = \pm 180°(2k+1)$$

or

$$-3\underline{/s+1} = \pm 180°(2k+1)$$

which can be written as

$$\underline{/s+1} = \pm 60°(2k+1)$$





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By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\underline{/\sigma + j\omega + 1} = \pm 60°(2k + 1)$$

or

$$\tan^{-1}\frac{\omega}{\sigma + 1} = 60°, \qquad -60°, \qquad 0°$$

Taking the tangent of both sides of this last equation,

$$\frac{\omega}{\sigma + 1} = \sqrt{3}, \qquad -\sqrt{3}, \qquad 0$$

which can be written as

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0, \qquad \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0, \qquad \omega = 0$$

These three equations represent three straight lines, as shown in Figure 6–4. The three straight lines shown are the asymptotes. They meet at point $s = -1$. Thus, the abscissa of the intersection of the asymptotes and the real axis is obtained by setting the denominator of the right-hand side of Equation (6–5) equal to zero and solving for $s$. The asymptotes are almost parts of the root loci in regions very far from the origin.

**3.** *Determine the breakaway point.* To plot root loci accurately, we must find the breakaway point, where the root-locus branches originating from the poles at 0 and −1 break away (as $K$ is increased) from the real axis and move into the complex plane. The breakaway point corresponds to a point in the $s$ plane where multiple roots of the characteristic equation occur.

A simple method for finding the breakaway point is available. We shall present this method in the following: Let us write the characteristic equation as

$$f(s) = B(s) + KA(s) = 0 \tag{6–6}$$

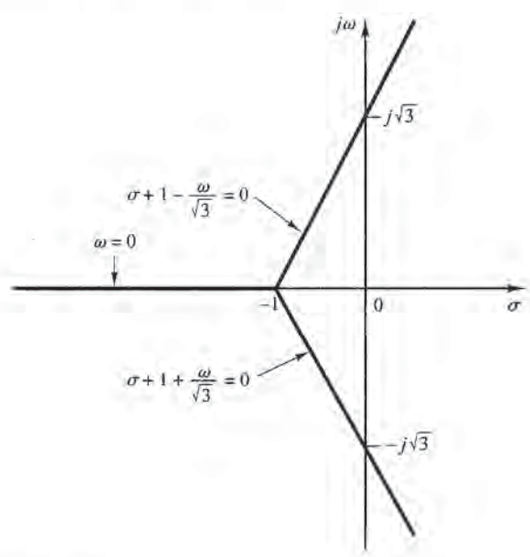


**Figure 6–4**
Three asymptotes.





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where $A(s)$ and $B(s)$ do not contain $K$. Note that $f(s) = 0$ has multiple roots at points where

$$\frac{df(s)}{ds} = 0$$

This can be seen as follows: Suppose that $f(s)$ has multiple roots of order $r$. Then $f(s)$ may be written as

$$f(s) = (s - s_1)^r(s - s_2)\cdots(s - s_n)$$

If we differentiate this equation with respect to $s$ and set $s = s_1$, then we get

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0 \tag{6–7}$$

This means that multiple roots of $f(s)$ will satisfy Equation (6–7). From Equation (6–6), we obtain

$$\frac{df(s)}{ds} = B'(s) + KA'(s) = 0 \tag{6–8}$$

where

$$A'(s) = \frac{dA(s)}{ds}, \qquad B'(s) = \frac{dB(s)}{ds}$$

The particular value of $K$ that will yield multiple roots of the characteristic equation is obtained from Equation (6–8) as

$$K = -\frac{B'(s)}{A'(s)}$$

If we substitute this value of $K$ into Equation (6–6), we get

$$f(s) = B(s) - \frac{B'(s)}{A'(s)}A(s) = 0$$

or

$$B(s)A'(s) - B'(s)A(s) = 0 \tag{6–9}$$

If Equation (6–9) is solved for $s$, the points where multiple roots occur can be obtained. On the other hand, from Equation (6–6) we obtain

$$K = -\frac{B(s)}{A(s)}$$

and

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

If $dK/ds$ is set equal to zero, we get the same equation as Equation (6–9). Therefore, the breakaway points can be simply determined from the roots of

$$\frac{dK}{ds} = 0$$

It should be noted that not all the solutions of Equation (6–9) or of $dK/ds = 0$ correspond to actual breakaway points. If a point at which $dK/ds = 0$ is on a root locus, it is an actual breakaway or break-in point. Stated differently, if at a point at which $dK/ds = 0$ the value of $K$ takes a real positive value then that point is an actual breakaway or break-in point.





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For the present example, the characteristic equation $G(s) + 1 = 0$ is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0$$

or

$$K = -(s^3 + 3s^2 + 2s)$$

By setting $dK/ds = 0$, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

or

$$s = -0.4226, \qquad s = -1.5774$$

Since the breakaway point must lie on a root locus between 0 and $-1$, it is clear that $s = -0.4226$ corresponds to the actual breakaway point. Point $s = -1.5774$ is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of $K$ corresponding to $s = -0.4226$ and $s = -1.5774$ yields

$$K = 0.3849, \qquad \text{for } s = -0.4226$$
$$K = -0.3849, \qquad \text{for } s = -1.5774$$

**4.** *Determine the points where the root loci cross the imaginary axis.* These points can be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

the Routh array becomes

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

The value of $K$ that makes the $s^1$ term in the first column equal zero is $K = 6$. The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the $s^2$ row; that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

which yields

$$s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$. The gain value corresponding to the crossing points is $K = 6$.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for $\omega$ and $K$. For the present system, the characteristic equation, with $s = j\omega$, is

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

or

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

Equating both the real and imaginary parts of this last equation to zero, we obtain

$$K - 3\omega^2 = 0, \qquad 2\omega - \omega^3 = 0$$





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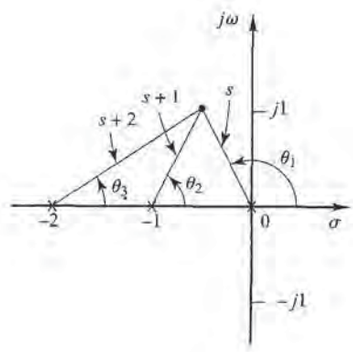


**Figure 6–5**
Construction of root locus.

from which

$$\omega = \pm\sqrt{2}, \qquad K = 6 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Thus, root loci cross the imaginary axis at $\omega = \pm\sqrt{2}$, and the value of $K$ at the crossing points is 6. Also, a root-locus branch on the real axis touches the imaginary axis at $\omega = 0$.

**5.** *Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin,* as shown in Figure 6–5, and apply the angle condition. If a test point is on the root loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must be 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition. (The sum of the angles at the test point will indicate which direction the test point should be moved.) Continue this process and locate a sufficient number of points satisfying the angle condition.

**6.** *Draw the root loci,* based on the information obtained in the foregoing steps, as shown in Figure 6–6.

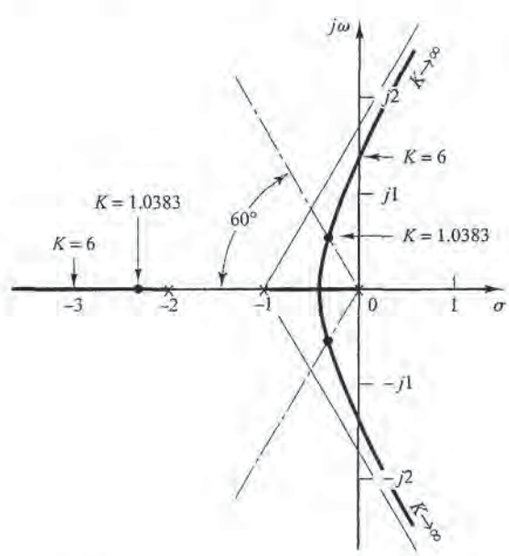


**Figure 6–6**
Root-locus plot.





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**7.** *Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio $\zeta$ is 0.5.* Closed-loop poles with $\zeta = 0.5$ lie on lines passing through the origin and making the angles $\pm\cos^{-1}\zeta = \pm\cos^{-1}0.5 = \pm 60°$ with the negative real axis. From Figure 6–6, such closed-loop poles having $\zeta = 0.5$ are obtained as follows:

$$s_1 = -0.3337 + j0.5780, \qquad s_2 = -0.3337 - j0.5780$$

The value of $K$ that yields such poles is found from the magnitude condition as follows:

$$\begin{aligned} K &= |s(s+1)(s+2)|_{s=-0.3337+j0.5780} \\ &= 1.0383 \end{aligned}$$

Using this value of $K$, the third pole is found at $s = -2.3326$.

Note that, from step 4, it can be seen that for $K = 6$ the dominant closed-loop poles lie on the imaginary axis at $s = \pm j\sqrt{2}$. With this value of $K$, the system will exhibit sustained oscillations. For $K > 6$, the dominant closed-loop poles lie in the right-half $s$ plane, resulting in an unstable system.

Finally, note that, if necessary, the root loci can be easily graduated in terms of $K$ by use of the magnitude condition. We simply pick out a point on a root locus, measure the magnitudes of the three complex quantities $s$, $s + 1$, and $s + 2$, and multiply these magnitudes; the product is equal to the gain value $K$ at that point, or

$$|s| \cdot |s+1| \cdot |s+2| = K$$

Graduation of the root loci can be done easily by use of MATLAB. (See Section 6–4.)

**EXAMPLE 6–2** In this example, we shall sketch the root-locus plot of a system with complex-conjugate open-loop poles. Consider the system shown in Figure 6–7. For this system,

$$G(s) = \frac{K(s+2)}{s^2 + 2s + 3}, \qquad H(s) = 1$$

where $K \geq 0$. It is seen that $G(s)$ has a pair of complex conjugate poles at

$$s = -1 + j\sqrt{2}, \qquad s = -1 - j\sqrt{2}$$

A typical procedure for sketching the root-locus plot is as follows:

**1.** *Determine the root loci on the real axis.* For any test point $s$ on the real axis, the sum of the angular contributions of the complex-conjugate poles is 360°, as shown in Figure 6–8. Thus the net effect of the complex-conjugate poles is zero on the real axis. The location of the root locus on the real axis is determined from the open-loop zero on the negative real axis. A simple test reveals that a section of the negative real axis, that between $-2$ and $-\infty$, is a part of the root locus. It is noted that, since this locus lies between two zeros (at $s = -2$ and $s = -\infty$), it is actually a part of two root loci, each of which starts from one of the two complex-conjugate poles. In other words, two root loci break in the part of the negative real axis between $-2$ and $-\infty$.

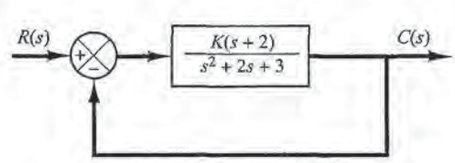


**Figure 6–7**
Control system.





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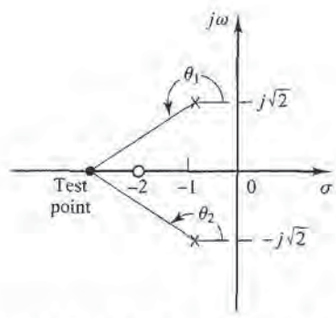


**Figure 6–8**
Determination of the root locus on the real axis.

Since there are two open-loop poles and one zero, there is one asymptote, which coincides with the negative real axis.

**2.** *Determine the angle of departure from the complex-conjugate open-loop poles.* The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles. Knowledge of this angle is important since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Referring to Figure 6–9, if we choose a test point and move it in the very vicinity of the complex open-loop pole at $s = -p_1$, we find that the sum of the angular contributions from the pole at $s = p_2$ and zero at $s = -z_1$ to the test point can be considered remaining the same. If the test point is to be on the root locus, then the sum of $\phi_1'$, $-\theta_1$, and $-\theta_2'$ must be $\pm 180°(2k + 1)$, where $k = 0, 1, 2, \ldots$. Thus, in the example,

$$\phi_1' - (\theta_1 + \theta_2') = \pm 180°(2k + 1)$$

or

$$\theta_1 = 180° - \theta_2' + \phi_1' = 180° - \theta_2 + \phi_1$$

The angle of departure is then

$$\theta_1 = 180° - \theta_2 + \phi_1 = 180° - 90° + 55° = 145°$$

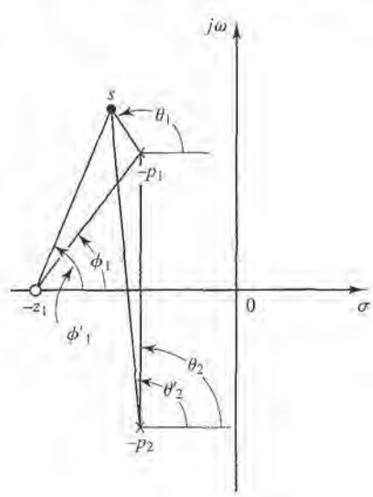


**Figure 6–9**
Determination of the angle of departure.





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Since the root locus is symmetric about the real axis, the angle of departure from the pole at $s = -p_2$ is $-145°$.

**3.** *Determine the break-in point.* A break-in point exists where a pair of root-locus branches coalesces as $K$ is increased. For this problem, the break-in point can be found as follows: Since

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

we have

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

which gives

$$s^2 + 4s + 1 = 0$$

or

$$s = -3.7320 \qquad \text{or} \qquad s = -0.2680$$

Notice that point $s = -3.7320$ is on the root locus. Hence this point is an actual break-in point. (Note that at point $s = -3.7320$ the corresponding gain value is $K = 5.4641$.) Since point $s = -0.2680$ is not on the root locus, it cannot be a break-in point. (For point $s = -0.2680$, the corresponding gain value is $K = -1.4641$.)

**4.** *Sketch a root-locus plot, based on the information obtained in the foregoing steps.* To determine accurate root loci, several points must be found by trial and error between the break-in point and the complex open-loop poles. (To facilitate sketching the root-locus plot, we should find the direction in which the test point should be moved by mentally summing up the changes on the angles of the poles and zeros.) Figure 6–10 shows a complete root-locus plot for the system considered.

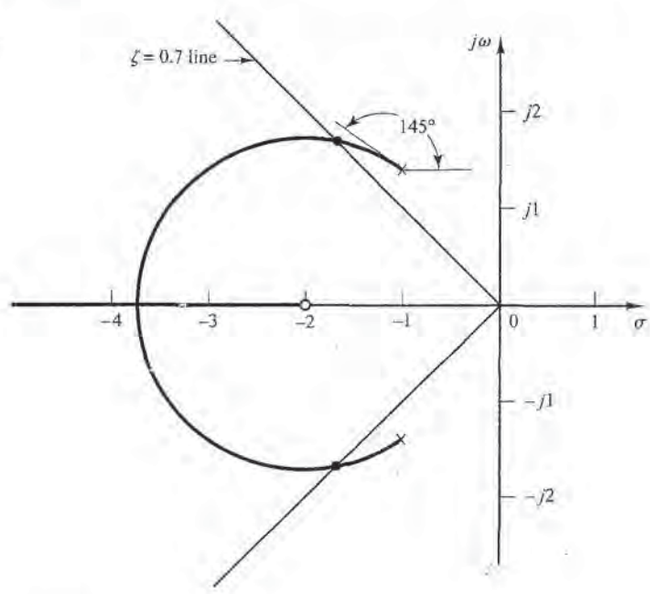


**Figure 6–10**
Root-locus plot.





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The value of the gain $K$ at any point on root locus can be found by applying the magnitude condition or by use of MATLAB (see Section 6–4). For example, the value of $K$ at which the complex-conjugate closed-loop poles have the damping ratio $\zeta = 0.7$ can be found by locating the roots, as shown in Figure 6–10, and computing the value of $K$ as follows:

$$K = \left| \frac{(s + 1 - j\sqrt{2})(s + 1 + j\sqrt{2})}{s + 2} \right|_{s=-1.67+j1.70} = 1.34$$

Or use MATLAB to find the value of $K$. (See Section 6–4.)

It is noted that in this system the root locus in the complex plane is a part of a circle. Such a circular root locus will not occur in most systems. Circular root loci may occur in systems that involve two poles and one zero, two poles and two zeros, or one pole and two zeros. Even in such systems, whether circular root loci occur depends on the locations of poles and zeros involved.

To show the occurrence of a circular root locus in the present system, we need to derive the equation for the root locus. For the present system, the angle condition is

$$\underline{/s + 2} - \underline{/s + 1 - j\sqrt{2}} - \underline{/s + 1 + j\sqrt{2}} = \pm 180°(2k + 1)$$

If $s = \sigma + j\omega$ is substituted into this last equation, we obtain

$$\underline{/\sigma + 2 + j\omega} - \underline{/\sigma + 1 + j\omega - j\sqrt{2}} - \underline{/\sigma + 1 + j\omega + j\sqrt{2}} = \pm 180°(2k + 1)$$

which can be written as

$$\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) - \tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) - \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \pm 180°(2k + 1)$$

or

$$\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180°(2k + 1)$$

Taking tangents of both sides of this last equation using the relationship

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \tag{6–10}$$

we obtain

$$\tan\left[\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right)\right] = \tan\left[\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180°(2k + 1)\right]$$

or

$$\frac{\dfrac{\omega - \sqrt{2}}{\sigma + 1} + \dfrac{\omega + \sqrt{2}}{\sigma + 1}}{1 - \left(\dfrac{\omega - \sqrt{2}}{\sigma + 1}\right)\left(\dfrac{\omega + \sqrt{2}}{\sigma + 1}\right)} = \frac{\dfrac{\omega}{\sigma + 2} \pm 0}{1 \mp \dfrac{\omega}{\sigma + 2} \times 0}$$

which can be simplified to

$$\frac{2\omega(\sigma + 1)}{(\sigma + 1)^2 - (\omega^2 - 2)} = \frac{\omega}{\sigma + 2}$$

or

$$\omega[(\sigma + 2)^2 + \omega^2 - 3] = 0$$

This last equation is equivalent to

$$\omega = 0 \qquad \text{or} \qquad (\sigma + 2)^2 + \omega^2 = (\sqrt{3})^2$$





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These two equations are the equations for the root loci for the present system. Notice that the first equation, $\omega = 0$, is the equation for the real axis. The real axis from $s = -2$ to $s = -\infty$ corresponds to a root locus for $K \geq 0$. The remaining part of the real axis corresponds to a root locus when $K$ is negative. (In the present system, $K$ is nonnegative.) The second equation for the root locus is an equation of a circle with center at $\sigma = -2$, $\omega = 0$ and the radius equal to $\sqrt{3}$. That part of the circle to the left of the complex-conjugate poles corresponds to a root locus for $K \geq 0$. The remaining part of the circle corresponds to a root locus when $K$ is negative.

It is important to note that easily interpretable equations for the root locus can be derived for simple systems only. For complicated systems having many poles and zeros, any attempt to derive equations for the root loci is discouraged. Such derived equations are very complicated and their configuration in the complex plane is difficult to visualize.

## 6–3 SUMMARY OF GENERAL RULES FOR CONSTRUCTING ROOT LOCI

For a complicated system with many open-loop poles and zeros, constructing a root-locus plot may seem complicated, but actually it is not difficult if the rules for constructing the root loci are applied. By locating particular points and asymptotes and by computing angles of departure from complex poles and angles of arrival at complex zeros, we can construct the general form of the root loci without difficulty.

Some of the rules for constructing root loci were given in Section 6–2. The purpose of this section is to summarize the general rules for constructing root loci of the system shown in Figure 6–11. While the root-locus method is essentially based on a trial-and-error technique, the number of trials required can be greatly reduced if we use these rules.

**General Rules for Constructing Root Loci.** We shall now summarize the general rules and procedure for constructing the root loci of the system shown in Figure 6–11.

First, obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0 \tag{6–11}$$

In the present discussions, we assume that the parameter of interest is the gain $K$, where $K > 0$. (If $K < 0$, which corresponds to the positive-feedback case, the angle condition must be modified. See Section 6–5.) Note, however, that the method is still applicable to systems with parameters of interest other than gain. (See Section 7–6.)

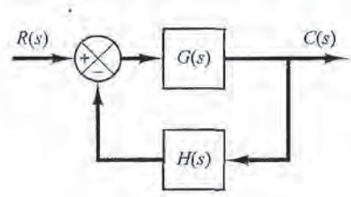


**Figure 6–11**
Control system.





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**1.** *Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity).* From the factored form of the open-loop transfer function, locate the open-loop poles and zeros in the $s$ plane. [Note that the open-loop zeros are the zeros of $G(s)H(s)$, while the closed-loop zeros consist of the zeros of $G(s)$ and the poles of $H(s)$.]

Note that the root loci are symmetrical about the real axis of the $s$ plane, because the complex poles and complex zeros occur only in conjugate pairs.

A root-locus plot will have just as many branches as there are roots of the characteristic equation. Since the number of open-loop poles generally exceeds that of zeros, the number of branches equals that of poles. If the number of closed-loop poles is the same as the number of open-loop poles, then the number of individual root-locus branches terminating at finite open-loop zeros is equal to the number $m$ of the open-loop zeros. The remaining $n - m$ branches terminate at infinity ($n - m$ implicit zeros at infinity) along asymptotes.

If we include poles and zeros at infinity, the number of open-loop poles is equal to that of open-loop zeros. Hence we can always state that the root loci start at the poles of $G(s)H(s)$ and end at the zeros of $G(s)H(s)$, as $K$ increases from zero to infinity, where the poles and zeros include both those in the finite $s$ plane and those at infinity.

**2.** *Determine the root loci on the real axis.* Root loci on the real axis are determined by open-loop poles and zeros lying on it. The complex-conjugate poles and zeros of the open-loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or zeros is 360° on the real axis. Each portion of the root locus on the real axis extends over a range from a pole or zero to another pole or zero. In constructing the root loci on the real axis, choose a test point on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.

**3.** *Determine the asymptotes of root loci.* If the test point $s$ is located far from the origin, then the angle of each complex quantity may be considered the same. One open-loop zero and one open-loop pole then cancel the effects of the other. Therefore, the root loci for very large values of $s$ must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{n-m} \qquad (k = 0, 1, 2, \dots)$$

where $n$ = number of finite poles of $G(s)H(s)$

$m$ = number of finite zeros of $G(s)H(s)$

Here, $k = 0$ corresponds to the asymptotes with the smallest angle with the real axis. Although $k$ assumes an infinite number of values, as $k$ is increased the angle repeats itself, and the number of distinct asymptotes is $n - m$.

All the asymptotes intersect on the real axis. The point at which they do so is obtained as follows: If both the numerator and denominator of the open-loop transfer function are expanded, the result is

$$G(s)H(s) = \frac{K[s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m]}{s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n}$$





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If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)]s^{n-m-1} + \cdots}$$

or

$$G(s)H(s) = \frac{K}{\left[s + \dfrac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m}\right]^{n-m}} \tag{6–12}$$

The abscissa of the intersection of the asymptotes and the real axis is then obtained by setting the denominator of the right-hand side of Equation (6–12) equal to zero and solving for $s$, or

$$s = -\frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \tag{6–13}$$

[Example 6–1 shows why Equation (6–13) gives the intersection.] Once this intersection is determined, the asymptotes can be readily drawn in the complex plane.

It is important to note that the asymptotes show the behavior of the root loci for $|s| \gg 1$. A root locus branch may lie on one side of the corresponding asymptote or may cross the corresponding asymptote from one side to the other side.

**4.** *Find the breakaway and break-in points.* Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in complex-conjugate pairs.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros. If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist no breakaway or break-in points or there may exist both breakaway and break-in points.

Suppose that the characteristic equation is given by

$$B(s) + KA(s) = 0$$

The breakaway points and break-in points correspond to multiple roots of the characteristic equation. Hence, as discussed in Example 6–1, the breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0 \tag{6–14}$$

where the prime indicates differentiation with respect to $s$. It is important to note that the breakaway points and break-in points must be the roots of Equation (6–14), but not all roots of Equation (6–14) are breakaway or break-in points. If a real root of Equation (6–14) lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point. If a real root of Equation (6–14) is not on the root-locus portion of the real axis, then this root corresponds to neither a breakaway point nor a break-in point.





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If two roots $s = s_1$ and $s = -s_1$ of Equation (6–14) are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding $K$ value. If the value of $K$ corresponding to a root $s = s_1$ of $dK/ds = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. (Since $K$ is assumed to be nonnegative, if the value of $K$ thus obtained is negative, or a complex quantity, then point $s = s_1$ is neither a breakaway nor break-in point.)

**5.** *Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero).* To sketch the root loci with reasonable accuracy, we must find the directions of the root loci near the complex poles and zeros. If a test point is chosen and moved in the very vicinity of a complex pole (or complex zero), the sum of the angular contributions from all other poles and zeros can be considered to remain the same. Therefore, the angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180° the sum of all the angles of vectors from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Angle of departure from a complex pole = 180°
− (sum of the angles of vectors to a complex pole in question from other poles)
+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
− (sum of the angles of vectors to a complex zero in question from other zeros)
+ (sum of the angles of vectors to a complex zero in question from poles)

The angle of departure is shown in Figure 6–12.

**6.** *Find the points where the root loci may cross the imaginary axis.* The points where the root loci intersect the $j\omega$ axis can be found easily by (a) use of Routh's stability criterion or (b) letting $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for $\omega$ and $K$. The values of $\omega$ thus found give the frequencies at which root loci cross the imaginary axis. The $K$ value corresponding to each crossing frequency gives the gain at the crossing point.

**7.** *Taking a series of test points in the broad neighborhood of the origin of the s plane, sketch the root loci.* Determine the root loci in the broad neighborhood of the $j\omega$ axis and the origin. The most important part of the root loci is on neither the real axis nor the asymptotes, but the part in the broad neighborhood of the $j\omega$ axis and the origin. The

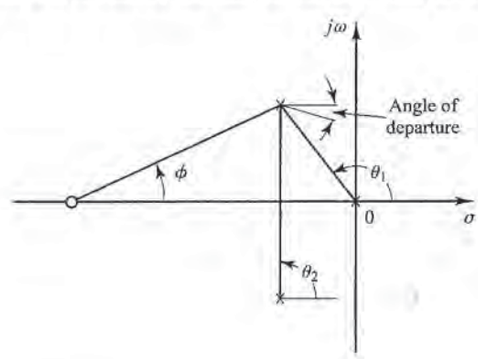


**Figure 6–12**
Construction of the root locus. [Angle of departure $= 180° - [\theta_1 + \theta_2] + \phi$.]





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shape of the root loci in this important region in the $s$ plane must be obtained with reasonable accuracy. (If accurate shape of the root loci is needed, MATLAB may be used rather than hand calculations of the exact shape of the root loci.)

**8.** *Determine closed-loop poles.* A particular point on each root-locus branch will be a closed-loop pole if the value of $K$ at that point satisfies the magnitude condition. Conversely, the magnitude condition enables us to determine the value of the gain $K$ at any specific root location on the locus. (If necessary, the root loci may be graduated in terms of $K$. The root loci are continuous with $K$.)

The value of $K$ corresponding to any point $s$ on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between point } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

This value can be evaluated either graphically or analytically. (MATLAB can be used for graduating the root loci with $K$. See Section 6–4.)

If the gain $K$ of the open-loop transfer function is given in the problem, then by applying the magnitude condition, we can find the correct locations of the closed-loop poles for a given $K$ on each branch of the root loci by a trial-and-error approach or by use of MATLAB, which will be presented in Section 6–4.

**Comments on the Root-Locus Plots.** It is noted that the characteristic equation of the system whose open-loop transfer function is

$$G(s)H(s) = \frac{K(s^m + b_1 s^{m-1} + \cdots + b_m)}{s^n + a_1 s^{n-1} + \cdots + a_n} \qquad (n \geq m)$$

is an $n$th-degree algebraic equation in $s$. If the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more (which means that there are two or more zeros at infinity), then the coefficient $a_1$ is the negative sum of the roots of the equation and is independent of $K$. In such a case, if some of the roots move on the locus toward the left as $K$ is increased, then the other roots must move toward the right as $K$ is increased. This information is helpful in finding the general shape of the root loci.

It is also noted that a slight change in the pole–zero configuration may cause significant changes in the root-locus configurations. Figure 6–13 demonstrates the fact that a slight change in the location of a zero or pole will make the root-locus configuration look quite different.

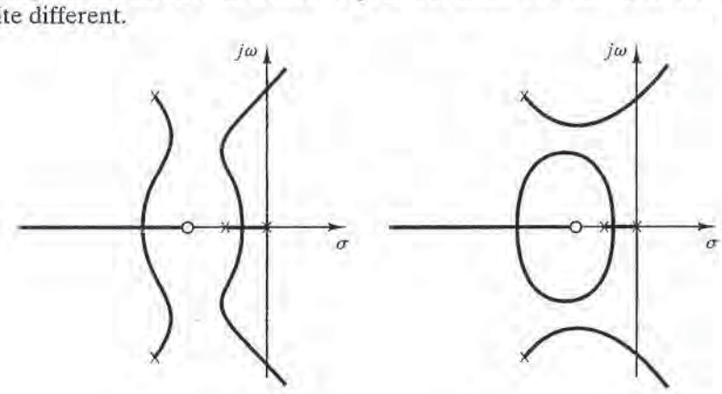


**Figure 6–13**
Root-locus plots.





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**Cancellation of Poles of $G(s)$ with Zeros of $H(s)$.** It is important to note that if the denominator of $G(s)$ and the numerator of $H(s)$ involve common factors then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. For example, consider the system shown in Figure 6–14(a). (This system has velocity feedback.) By modifying the block diagram of Figure 6–14(a) to that shown in Figure 6–14(b), it is clearly seen that $G(s)$ and $H(s)$ have a common factor $s + 1$. The closed-loop transfer function $C(s)/R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2) + K(s+1)}$$

The characteristic equation is

$$[s(s+2) + K](s+1) = 0$$

Because of the cancellation of the terms $(s + 1)$ appearing in $G(s)$ and $H(s)$, however, we have

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+1)(s+2)}$$
$$= \frac{s(s+2) + K}{s(s+2)}$$

The reduced characteristic equation is

$$s(s+2) + K = 0$$

The root-locus plot of $G(s)H(s)$ does not show all the roots of the characteristic equation, only the roots of the reduced equation.

To obtain the complete set of closed-loop poles, we must add the canceled pole of $G(s)H(s)$ to those closed-loop poles obtained from the root-locus plot of $G(s)H(s)$. The important thing to remember is that the canceled pole of $G(s)H(s)$ is a closed-loop pole of the system, as seen from Figure 6–14(c).

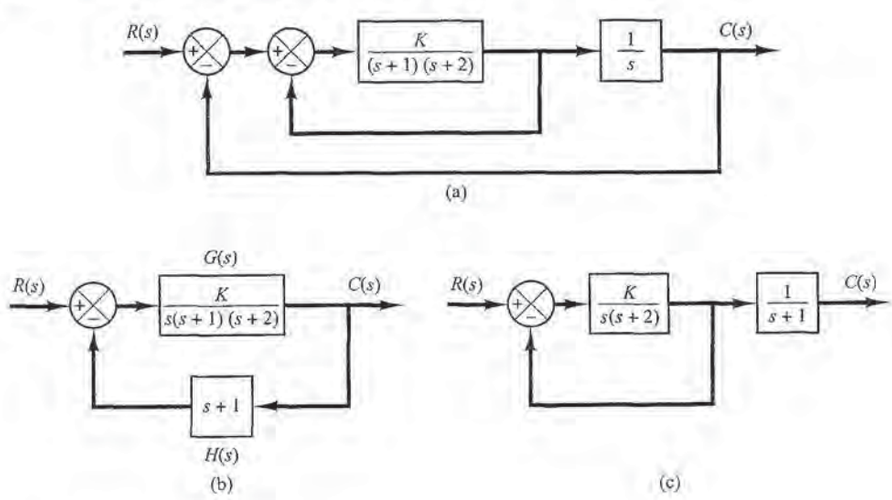


**Figure 6–14**
(a) Control system with velocity feedback; (b) and (c) modified block diagrams.





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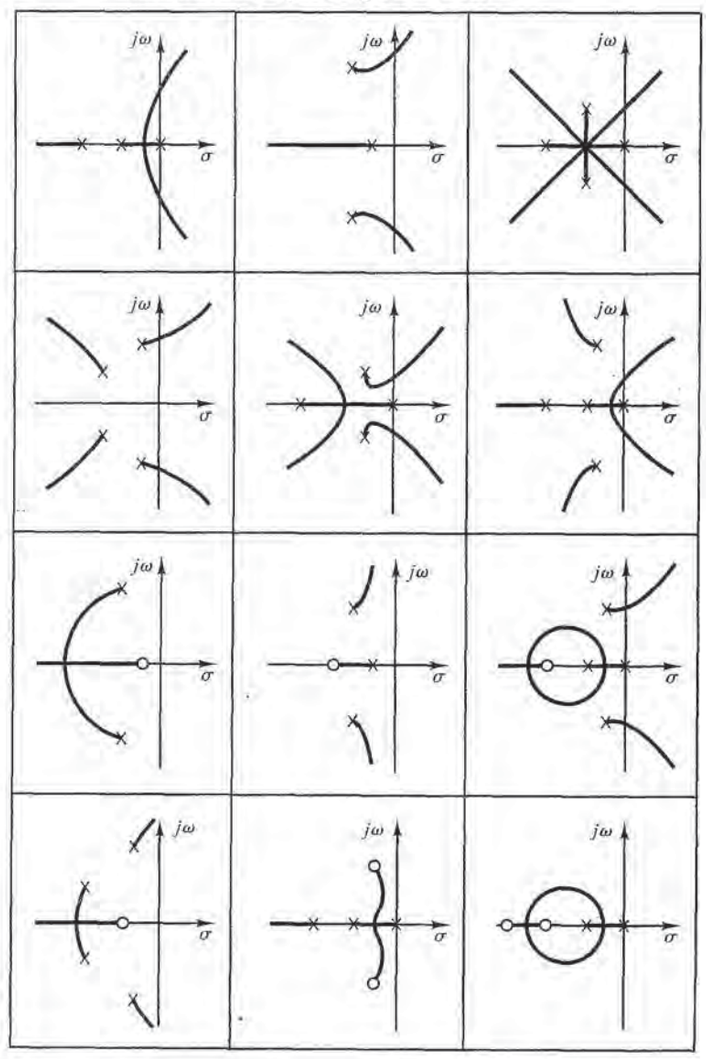
**Typical Pole–Zero Configurations and Corresponding Root Loci.** In summarizing, we show several open-loop pole–zero configurations and their corresponding root loci in Table 6–1. The pattern of the root loci depends only on the relative separation of the open-loop poles and zeros. If the number of open-loop poles exceeds the number of finite zeros by three or more, there is a value of the gain $K$ beyond which root loci enter the right-half $s$ plane, and thus the system can become unstable. A stable system must have all its closed-loop poles in the left-half $s$ plane.

**Table 6–1** Open-Loop Pole–Zero Configurations and the Corresponding Root Loci





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Note that once we have some experience with the method, we can easily evaluate the changes in the root loci due to the changes in the number and location of the open-loop poles and zeros by visualizing the root-locus plots resulting from various pole–zero configurations.

**Summary.** From the preceding discussions, it should be clear that it is possible to sketch a reasonably accurate root-locus diagram for a given system by following simple rules. (The reader should study the various root-locus diagrams shown in the solved problems at the end of the chapter.) At preliminary design stages, we may not need the precise locations of the closed-loop poles. Often their approximate locations are all that is needed to make an estimate of system performance. Thus, it is important that the designer have the capability of quickly sketching the root loci for a given system.

## 6–4 ROOT-LOCUS PLOTS WITH MATLAB

In this section we present the MATLAB approach to the generation of root-locus plots and finding relevant information from the root-locus plots.

**Plotting Root Loci with MATLAB.** In plotting root loci with MATLAB we deal with the system equation given in the form of Equation (6–11), which may be written as

$$1 + K\frac{\text{num}}{\text{den}} = 0$$

where num is the numerator polynomial and den is the denominator polynomial. That is,

$$\begin{aligned}
\text{num} &= (s + z_1)(s + z_2)\cdots(s + z_m) \\
&= s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m \\
\text{den} &= (s + p_1)(s + p_2)\cdots(s + p_n) \\
&= s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n
\end{aligned}$$

Note that both vectors num and den must be written in descending powers of $s$.

A MATLAB command commonly used for plotting root loci is

```
rlocus(num,den)
```

Using this command, the root-locus plot is drawn on the screen. The gain vector K is automatically determined. (The vector K contains all the gain values for which the closed-loop poles are to be computed.)

For the systems defined in state space, rlocus(A,B,C,D) plots the root locus of the system with the gain vector automatically determined.

Note that commands

```
rlocus(num,den,K)    and    rlocus(A,B,C,D,K)
```

use the user-supplied gain vector K.

If invoked with left-hand arguments

```
[r,K] = rlocus(num,den)
[r,K] = rlocus(num,den,K)
```





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```
[r,K] = rlocus(A,B,C,D)
[r,K] = rlocus(A,B,C,D,K)
[r,K] = rlocus(sys)
```

the screen will show the matrix r and gain vector K. (r has length K rows and length den − 1 columns containing the complex root locations. Each row of the matrix corresponds to a gain from vector K.) The plot command

```
plot(r,'-')
```

plots the root loci.

If it is desired to plot the root loci with marks 'o' or 'x', it is necessary to use the following command:

```
r = rlocus(num,den)
plot(r,'o')    or    plot(r,'x')
```

Plotting root loci using marks o or x is instructive, since each calculated closed-loop pole is graphically shown; in some portion of the root loci those marks are densely placed and in another portion of the root loci they are sparsely placed. MATLAB supplies its own set of gain values used to calculate a root-locus plot. It does so by an internal adaptive step-size routine. Also, MATLAB uses the automatic axis-scaling feature of the plot command.

Finally, note that, since the gain vector is automatically determined, root-locus plots of

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

$$G(s)H(s) = \frac{10K(s+1)}{s(s+2)(s+3)}$$

$$G(s)H(s) = \frac{200K(s+1)}{s(s+2)(s+3)}$$

are all the same. The num and den set of the system is the same for all three systems. The num and den are

```
num = [0 0 1 1]
den = [1 5 6 0]
```

**EXAMPLE 6–3** Consider the system shown in Figure 6–15. Plot root loci with a square aspect ratio so that a line with slope 1 is a true 45° line. Choose the region of root-locus plot to be

$$-6 \le x \le 6, \qquad -6 \le y \le 6$$

where $x$ and $y$ are the real-axis coordinate and imaginary-axis coordinate, respectively.

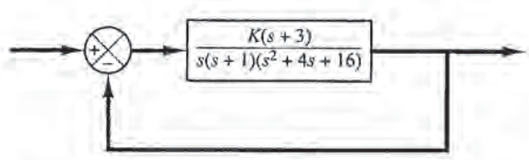


**Figure 6–15**
Control system.





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To set the given plot region on the screen to be square, enter the command

```
v = [-6  6  -6  6]; axis (v); axis('square')
```

With this command, the region of the plot is as specified and a line with slope 1 is at a true 45°, not skewed by the irregular shape of the screen.

For this problem, the denominator is given as a product of first- and second-order terms. So we must multiply these terms to get a polynomial in $s$. The multiplication of these terms can be done easily by use of the convolution command, as shown next.

Define

```
a = s (s + 1):        a = [1  1  0]
b = s^2 + 4s + 16:    b = [1  4  16]
```

Then we use the following command:

```
c = conv(a, b)
```

Note that conv(a, b) gives the product of two polynomials a and b. See the following computer output:

```
a = [1  1  0];
b = [1  4  16];
c = conv (a,b)
c =
     1  5  20  16  0
```

The denominator polynomial is thus found to be

```
den = [1  5  20  16  0]
```

To find the complex-conjugate open-loop poles (the roots of $s^2 + 4s + 16 = 0$), we may enter the roots command as follows:

```
r = roots(b)
r =
  -2.0000 + 3.464li
  -2.0000 - 3.464li
```

Thus, the system has the following open-loop zero and open-loop poles:

Open-loop zero: $s = -3$

Open-loop poles: $s = 0, \quad s = -1, \quad s = -2 \pm j3.4641$

MATLAB Program 6–1 will plot the root-locus diagram for this system. The plot is shown in Figure 6–16.

**MATLAB Program 6–1**

```
% --------- Root-locus plot ---------
num = [0  0  0  1  3];
den = [1  5  20  16  0];
rlocus(num,den)
v = [-6  6  -6  6];
axis(v); axis('square')
grid;
title ('Root-Locus Plot of G(s) = K(s + 3)/[s(s + 1)(s^2 + 4s + 16)]')
```





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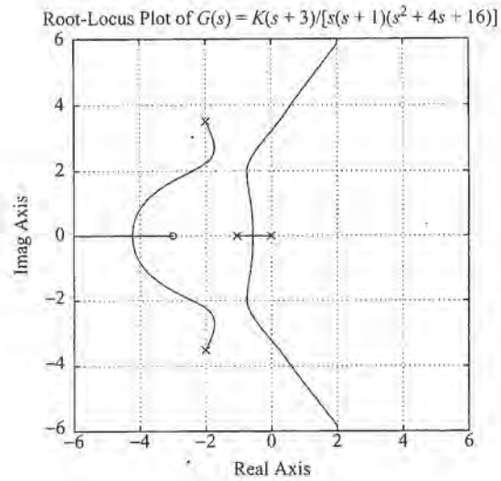


**Figure 6–16**
Root-locus plot.

Note that in MATLAB Program 6–1, instead of

```
den = [1  5  20  16  0]
```

we may enter

```
den = conv ([1  1  0], [1  4  16])
```

The results are the same.

**EXAMPLE 6–4** Consider the system whose open-loop transfer function $G(s)H(s)$ is

$$G(s)H(s) = \frac{K}{s(s + 0.5)(s^2 + 0.6s + 10)}$$
$$= \frac{K}{s^4 + 1.1s^3 + 10.3s^2 + 5s}$$

There are no open-loop zeros. Open-loop poles are located at $s = -0.3 + j3.1480$, $s = -0.3 - j3.1480$, $s = -0.5$, and $s = 0$.

Entering MATLAB Program 6–2 into the computer, we obtain the root-locus plot shown in Figure 6–17.

**MATLAB Program 6–2**

```
% --------- Root-locus plot ---------

num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
r = rlocus(num,den);
plot(r,'o')
v = [-6  6  -6  6]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s+10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```





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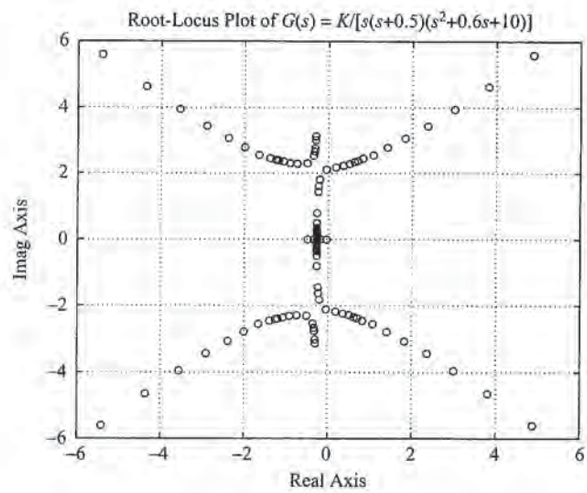


**Figure 6–17**
Root-locus plot.

Notice that in the regions near $x = -0.3$, $y = 2.3$ and $x = -0.3$, $y = -2.3$ two loci approach each other. We may wonder if these two branches should touch or not. To explore this situation, we may plot the root loci using smaller increments of $K$ in the critical region.

By a conventional trial-and-error approach or using the command rlocfind to be presented later in this section, we find the particular region of interest to be $20 \leq K \leq 30$. By entering MATLAB Program 6–3, we obtain the root-locus plot shown in Figure 6–18. From this plot, it is clear that the two branches that approach in the upper half-plane (or in the lower half-plane) do not touch.

**MATLAB Program 6–3**

```
% --------- Root-locus plot ---------

num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
K1 = 0:0.2:20;
K2 = 20:0.1:30;
K3 = 30:5:1000;
K = [K1  K2  K3];
r = rlocus(num,den,K);
plot(r, 'o')
v = [-4  4  -4  4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s + 10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```





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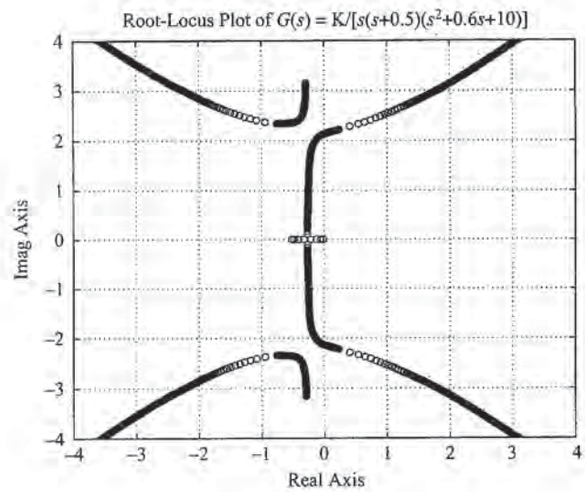


**Figure 6–18**
Root-locus plot.

**EXAMPLE 6–5** Consider the system shown in Figure 6–19. The system equations are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

$$u = r - y$$

In this example problem we shall obtain the root-locus diagram of the system defined in state space. As an example let us consider the case where matrices **A**, **B**, **C**, and $D$ are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} \tag{6–15}$$

$$\mathbf{C} = [1 \quad 0 \quad 0], \qquad D = [0]$$

The root-locus plot for this system can be obtained with MATLAB by use of the following command:

```
rlocus(A,B,C,D)
```

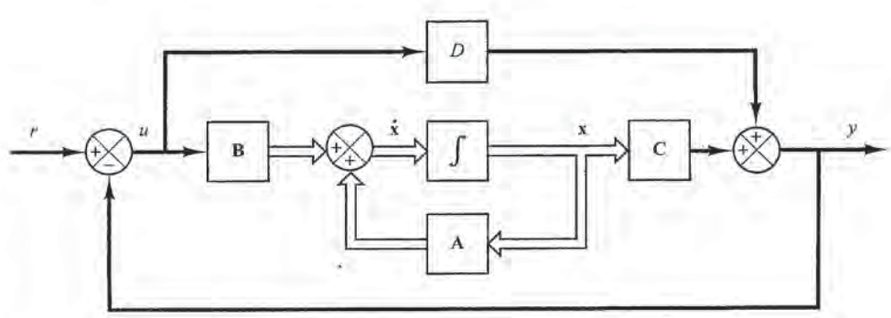


**Figure 6–19**
Closed-loop control system.





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This command will produce the same root-locus plot as can be obtained by use of the rlocus (num,den) command, where num and den are obtained from

```
[num,den] = ss2tf(A,B,C,D)
```

as follows:

```
num = [0 0 1 0]
den = [1 14 56 160]
```

MATLAB Program 6–4 is a program that will generate the root-locus plot as shown in Figure 6–20.

**MATLAB Program 6–4**

```
% --------- Root-locus plot ---------
A = [0 1 0;0 0 1;-160 -56 -14];
B = [0;1;-14];
C = [1 0 0];
D = [0];
K = 0:0.1:400;
rlocus(A,B,C,D,K);
v = [-20 20 -20 20]; axis(v)
grid
title('Root-Locus Plot of System Defined in State Space')
```

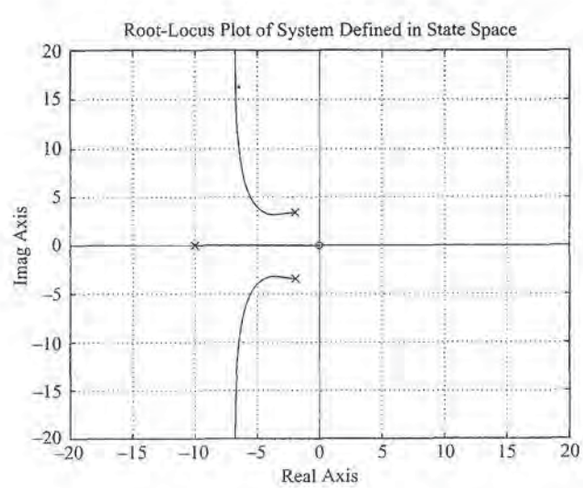


**Figure 6–20**
Root-locus plot of system defined in state space, where **A**, **B**, **C**, and $D$ are as given by Equation (6–15).





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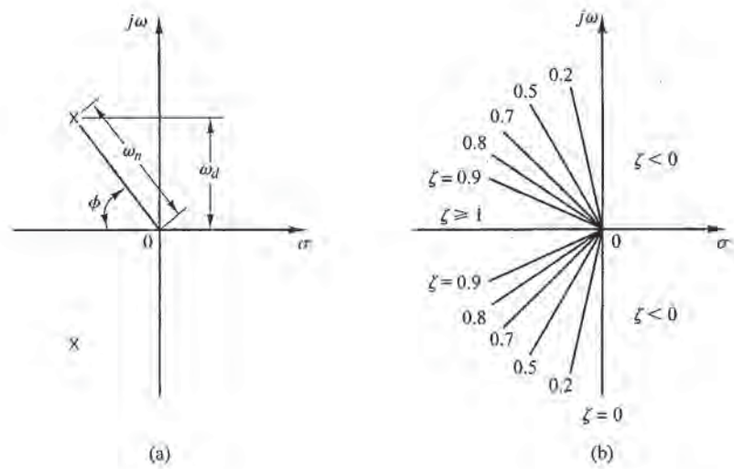


**Figure 6–21**
(a) Complex poles;
(b) lines of constant damping ratio $\zeta$.

**Constant $\zeta$ Loci and Constant $\omega_n$ Loci.** Recall that in the complex plane the damping ratio $\zeta$ of a pair of complex-conjugate poles can be expressed in terms of the angle $\phi$, which is measured from the negative real axis, as shown in Figure 6–21(a) with

$$\zeta = \cos\phi$$

In other words, lines of constant damping ratio $\zeta$ are radial lines passing through the origin as shown in Figure 6–21(b). For example, a damping ratio of 0.5 requires that the complex poles lie on the lines drawn through the origin making angles of $\pm 60°$ with the negative real axis. (If the real part of a pair of complex poles is positive, which means that the system is unstable, the corresponding $\zeta$ is negative.) the damping ratio determines the angular location of the poles, while the distance of the pole from the origin is determined by the undamped natural frequency $\omega_n$. The constant $\omega_n$ loci are circles.

To draw constant $\zeta$ lines and constant $\omega_n$ circles on the root-locus diagram with MATLAB, use the command sgrid.

**Plotting Polar Grids in the Root-Locus Diagam.** The command

```
sgrid
```

overlays lines of constant damping ratio ($\zeta = 0 \sim 1$ with 0.1 increment) and circles of constant $\omega_n$ on the root-locus plot. See MATLAB Program 6–5 and the resulting diagram shown in Figure 6–22.

If only particular constant $\zeta$ lines (such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line) and particular constant $\omega_n$ circles (such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle) are desired, use the following command:

```
sgrid([0.5, 0.707], [0.5, 1, 2])
```





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**MATLAB Program 6–5**

```
sgrid
v = [-2 2 -2 2]; axis(v); axis('square')
title('Constant \zeta Lines and Constant \omega_n Circles')
xlabel(Real Axis')
ylabel('Imag Axis')
gtext('\zeta = 0.9')
gtext('0.8')
gtext('0.7')
gtext('0.6')
gtext('0.5')
gtext('0.4')
gtext('0.3')
gtext('0.2')
gtext('0.1')
gtext('\omega_n = 1')
gtext('\omega_n = 2')
```

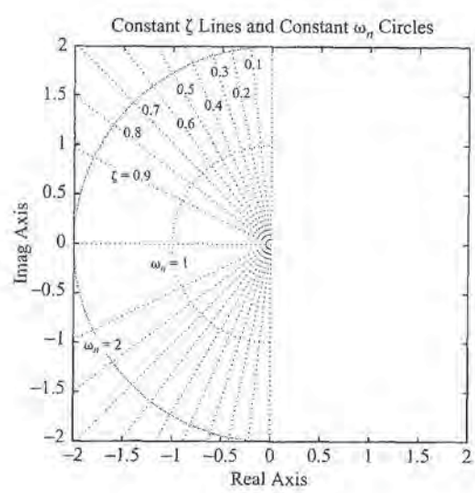


**Figure 6–22**
Constant $\zeta$ lines and constant $\omega_n$ circles.

If we wish to overlay lines of constant $\zeta$ and circles of constant $\omega_n$ as given above to a root-locus plot of a system with

```
num = [0 0 0 1]
den = [1 4 5 0]
```

then enter MATLAB Program 6–6 into the computer. The resulting root-locus plot is shown in Figure 6–23.





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**MATLAB Program 6–6**

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den);
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid([0.5,0.707], [0.5,1,2])
title('Root-Locus Plot with \zeta = 0.5 and 0.707 Lines and \omega_n = 0.5, 1, and 2 Circles')
gtext('\zeta = 0.5')
gtext('\zeta = 0.707')
gtext('\omega_n = 2')
gtext('\omega_n = 1')
gtext('\omega_n = 0.5')
```

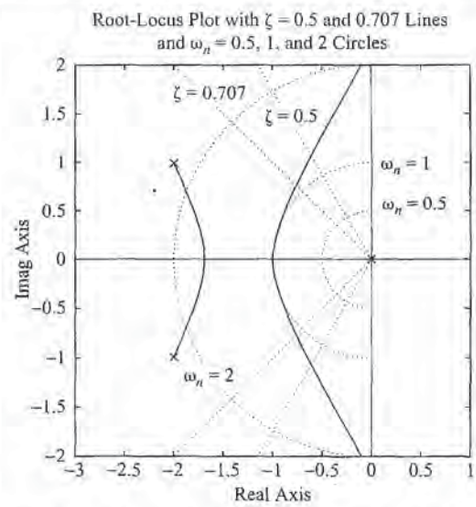


**Figure 6–23**
Constant $\zeta$ lines and constant $\omega_n$ circles superimposed on a root-locus plot.

If we want to omit either the entire constant $\zeta$ lines or entire constant $\omega_n$ circles, we may use empty brackets [ ] in the arguments of the sgrid command. For example, if we want to overlay only the constant damping ratio line corresponding to $\zeta = 0.5$ and no constant $\omega_n$ circles to the root-locus plot shown in Figure 6–23, then we may use the command

```
sgrid(0.5, [])
```

See MATLAB Program 6–7 and the resulting plot shown in Figure 6–24.

**MATLAB Program 6–7**

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den)
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid(0.5, [])
title('Root-Locus Plot and \zeta = 0.5 Line')
gtext('\zeta = 0.5')
```





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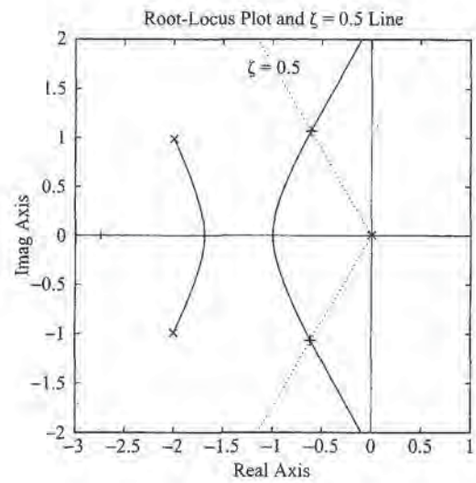


**Figure 6–24**
Root-locus plot with $\zeta = 0.5$ line.

**Orthogonality of Root Loci and Constant-Gain Loci.** Consider the system whose open-loop transfer function is $G(s)H(s)$. In the $G(s)H(s)$ plane, the loci of $|G(s)H(s)| =$ constant are circles centered at the origin, and the loci corresponding to $\underline{/G(s)H(s)} = \pm 180°(2k + 1)$ $(k = 0, 1, 2, \ldots)$ lie on the negative real axis of the $G(s)H(s)$ plane, as shown in Figure 6–25. [Note that the complex plane employed here is not the $s$ plane, but the $G(s)H(s)$ plane.]

The root loci and constant-gain loci in the $s$ plane are conformal mappings of the loci of $\underline{/G(s)H(s)} = \pm 180°(2k + 1)$ and of $|G(s)H(s)| =$ constant in the $G(s)H(s)$ plane.

Since the constant-phase and constant-gain loci in the $G(s)H(s)$ plane are orthogonal, the root loci and constant-gain loci in the $s$ plane are orthogonal. Figure 6–26(a) shows the root loci and constant-gain loci for the following system:

$$G(s) = \frac{K(s + 2)}{s^2 + 2s + 3}, \qquad H(s) = 1$$

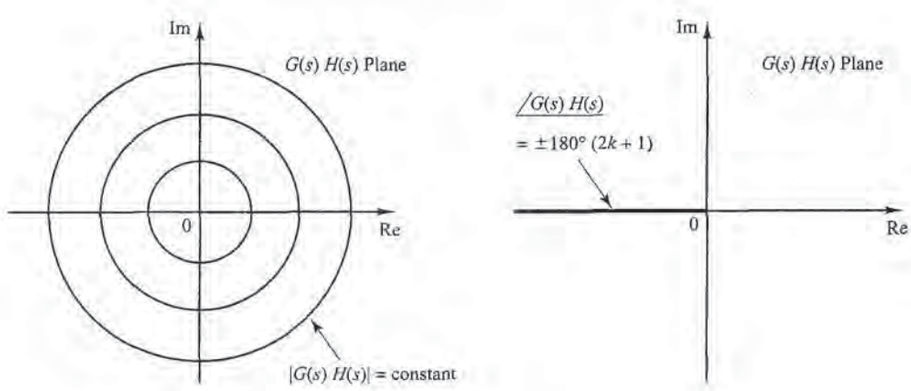


**Figure 6–25**
Plots of constant-gain and constant-phase loci in the $G(s)H(s)$ plane.





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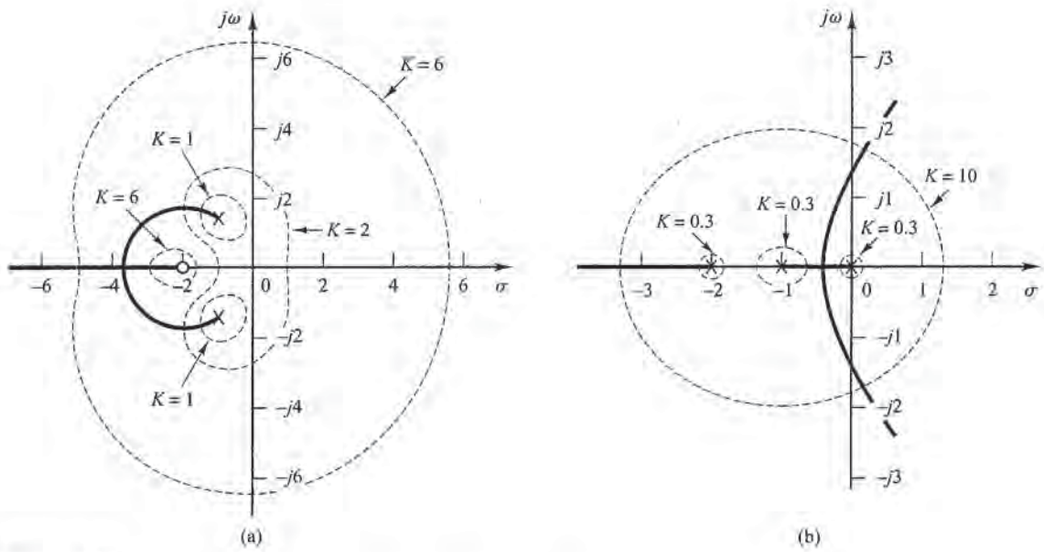


**Figure 6–26**
Plots of root loci and constant-gain loci. (a) System with $G(s) = K(s + 2)/(s^2 + 2s + 3)$, $H(s) = 1$; (b) system with $G(s) = K/[s(s + 1)(s + 2)]$, $H(s) = 1$.

Notice that since the pole–zero configuration is symmetrical about the real axis the constant-gain loci are also symmetrical about the real axis.

Figure 6–26(b) shows the root loci and constant-gain loci for the system:

$$G(s) = \frac{K}{s(s + 1)(s + 2)}, \qquad H(s) = 1$$

Notice that since the configuration of the poles in the $s$ plane is symmetrical about the real axis and the line parallel to the imaginary axis passing through point ($\sigma = -1$, $\omega = 0$), the constant-gain loci are symmetrical about the $\omega = 0$ line (real axis) and the $\sigma = -1$ line.

From Figures 6–26(a) and (b), notice that every point in the $s$ plane has the corresponding $K$ value. If we use a command rlocfind (presented next), MATLAB will give the $K$ value of the specified point as well as the nearest closed-loop poles corresponding to this $K$ value.

**Finding the Gain Value $K$ at an Arbitrary Point on the Root Loci.** In MATLAB analysis of closed-loop systems, it is frequently desired to find the gain value $K$ at an arbitrary point on the root locus. This can be accomplished by using the following rlocfind command:

```
[K, r] = rlocfind(num, den)
```

The rlocfind command, which must follow an rlocus command, overlays movable $x$-$y$ coordinates on the screen. Using the mouse, we position the origin of the $x$-$y$ coordinates over the desired point on the root locus and press the mouse button. Then MATLAB





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displays on the screen the coordinates of that point, the gain value at that point, and the closed-loop poles corresponding to this gain value.

If the selected point is not on the root locus, the rlocfind command gives the coordinates of this selected point, the gain value of this point, and the locations of the closed-loop poles corresponding to this $K$ value. [Note that every point on the $s$ plane has a gain value. See, for example, Figures 6–26 (a) and (b).]

**EXAMPLE 6–6** Consider the unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 4s + 5)}$$

Plot the root loci with MATLAB. Determine closed-loop poles that have the damping ratio of 0.5. Find the gain value $K$ at this point.

We first plot a root-locus diagram as shown in Figure 6–27. Then enter the rlocfind command as shown in MATLAB Program 6–8. Position the origin of the $x$-$y$ coordinates over the intersection of the upper root-locus branch and the $\zeta = 0.5$ line. Then press the button of the mouse. The screen shows the coordinates of this point, the gain value at this point, and the closed-loop poles corresponding to this gain value.

The plot shows the closed-loop poles by a plus sign (+). The three closed-loop poles obtained are

$$s = -2.7474, \qquad s = -0.6263 + j1.0800, \qquad s = -0.6263 - j1.0800$$

Note that the three closed-loop poles are slightly off the exact locations obtained by the analytic method. The reason is that we cannot position the origin of the movable $x$-$y$ coordinates exactly at the intersection of the upper root-locus branch and the $\zeta = 0.5$ line.

**MATLAB Program 6–8**

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den);
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid(0.5, [])
[K,r] = rlocfind(num, den)
Select a point in the graphics window

selected_point =

-0.6246 + 1.0792i

K =

  4.2823

r =

  -2.7474
  -0.6263 + 1.0800i
  -0.6263 - 1.0800i
```





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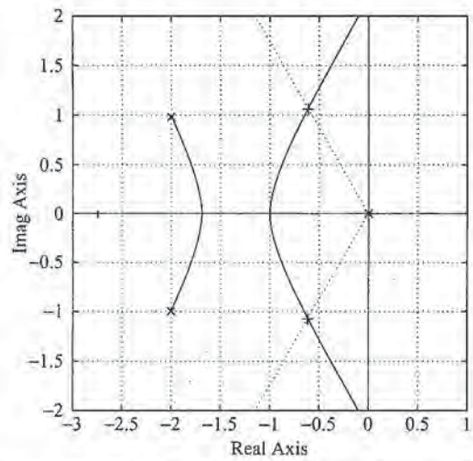


**Figure 6–27**
Root-locus plot with $\zeta = 0.5$ line.

**Nonminimum-Phase Systems.** If all the poles and zeros of a system lie in the left-half $s$ plane, then the system is called *minimum phase.* If a system has at least one pole or zero in the right-half $s$ plane, then the system is called *nonminimum phase.* The term nonminimum phase comes from the phase-shift characteristics of such a system when subjected to sinusoidal inputs.

Consider the system shown in Figure 6–28(a). For this system

$$G(s) = \frac{K(1 - T_a s)}{s(Ts + 1)} \qquad (T_a > 0), \qquad H(s) = 1$$

This is a nonminimum-phase system since there is one zero in the right-half $s$ plane. For this system, the angle condition becomes

$$\begin{aligned} \angle G(s) &= \angle -\frac{K(T_a s - 1)}{s(Ts + 1)} \\ &= \angle \frac{K(T_a s - 1)}{s(Ts + 1)} + 180^\circ \\ &= \pm 180^\circ(2k + 1) \qquad (k = 0, 1, 2, \dots) \end{aligned}$$

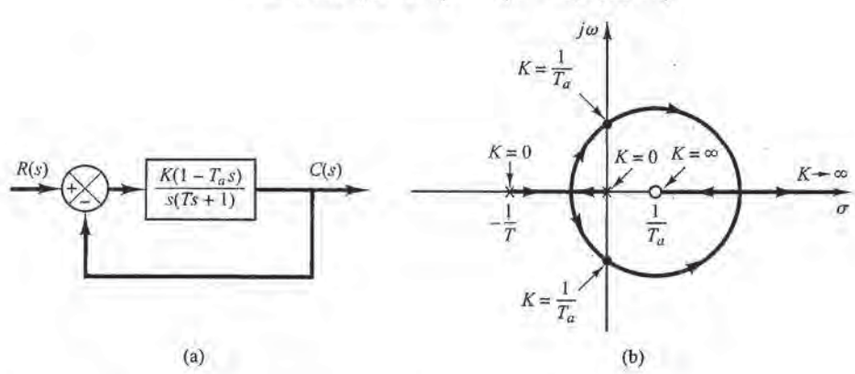


**Figure 6–28**
(a) Nonminimum-phase system;
(b) root-locus plot.





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or

$$\angle \frac{K(T_a s - 1)}{s(Ts + 1)} = 0° \tag{6–16}$$

The root loci can be obtained from Equation (6–16). Figure 6–28(b) shows a root-locus plot for this system. From the diagram, we see that the system is stable if the gain $K$ is less than $1/T_a$.

To obtain a root-locus plot with MATLAB, enter the numerator and denominator as usual. For example, if $T = 1$ sec and $T_a = 0.5$ sec, enter the following num and den in the program:

```
num = [0  -0.5  1]
den = [1  1  0]
```

MATLAB Program 6–9 gives the plot of the root loci shown in Figure 6–29.

**MATLAB Program 6–9**

```
num = [0  -0.5  1];
den = [1  1  0];
k1 = 0:0.01:30;
k2 = 30:1:100;
K3 = 100:5:500;
K = [k1  k2  k3];
rlocus(num,den,K)
v = [-2  6  -4  4]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(1 - 0.5s)/[s(s + 1)]')
```

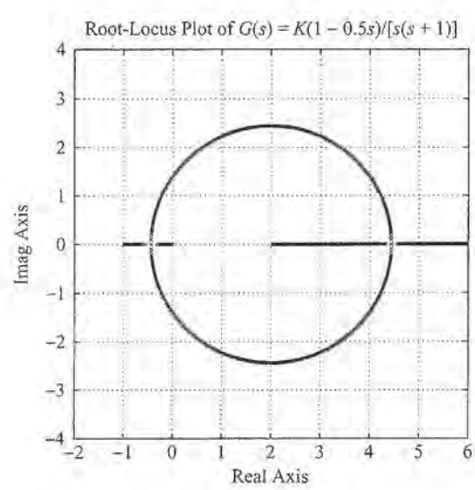


**Figure 6–29**
Root-locus plot of $G(s) = \dfrac{K(1 - 0.5s)}{s(s + 1)}$.





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## 6–5 POSITIVE-FEEDBACK SYSTEMS

**Root Loci for Positive-Feedback Systems.*** In a complex control system, there may be a positive-feedback inner loop as shown in Figure 6–30. Such a loop is usually stabilized by the outer loop. In what follows, we shall be concerned only with the positive-feedback inner loop. The closed-loop transfer function of the inner loop is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

The characteristic equation is

$$1 - G(s)H(s) = 0 \tag{6–17}$$

This equation can be solved in a manner similar to the development of the root-locus method in Section 6–2. The angle condition, however, must be altered.

Equation (6–17) can be rewritten as

$$G(s)H(s) = 1$$

which is equivalent to the following two equations:

$$\underline{/G(s)H(s)} = 0° \pm k360° \qquad (k = 0, 1, 2, \dots)$$

$$|G(s)H(s)| = 1$$

The total sum of all angles from the open-loop poles and zeros must be equal to $0° \pm k360°$. Thus the root locus follows a 0° locus in contrast to the 180° locus considered previously. The magnitude condition remains unaltered.

To illustrate the root-locus plot for the positive-feedback system, we shall use the following transfer functions $G(s)$ and $H(s)$ as an example.

$$G(s) = \frac{K(s + 2)}{(s + 3)(s^2 + 2s + 2)}, \qquad H(s) = 1$$

The gain $K$ is assumed to be positive.

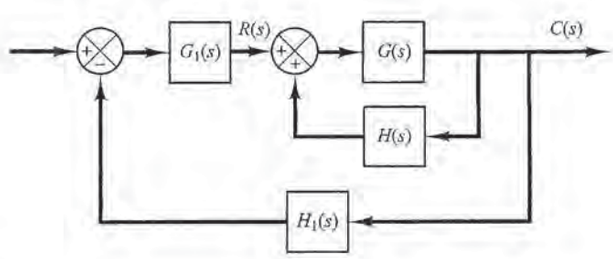


**Figure 6–30**
Control system.

* Reference W-4





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The general rules for constructing root loci given in Section 6–3 must be modified in the following way:

*Rule 2 is Modified as Follows:* If the total number of real poles and real zeros to the right of a test point on the real axis is even, then this test point lies on the root locus.

*Rule 3 is Modified as Follows:*

$$\text{Angles of asymptotes} = \frac{\pm k360^\circ}{n - m} \qquad (k = 0, 1, 2, \ldots)$$

where $n$ = number of finite poles of $G(s)H(s)$
$m$ = number of finite zeros of $G(s)H(s)$

*Rule 5 is Modified as Follows:* When calculating the angle of departure (or angle of arrival) from a complex open-loop pole (or at a complex zero), subtract from 0° the sum of all angles of the vectors from all the other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Other rules for constructing the root-locus plot remain the same. We shall now apply the modified rules to construct the root-locus plot.

1. Plot the open-loop poles ($s = -1 + j, s = -1 - j, s = -3$) and zero ($s = -2$) in the complex plane. As $K$ is increased from 0 to $\infty$, the closed-loop poles start at the open-loop poles and terminate at the open-loop zeros (finite or infinite), just as in the case of negative-feedback systems.
2. Determine the root loci on the real axis. Root loci exist on the real axis between $-2$ and $+\infty$ and between $-3$ and $-\infty$.
3. Determine the asymptotes of the root loci. For the present system,

$$\text{Angles of asymptote} = \frac{\pm k360^\circ}{3 - 1} = \pm 180^\circ$$

This simply means that asymptotes are on the real axis.

4. Determine the breakaway and break-in points. Since the characteristic equation is

$$(s + 3)(s^2 + 2s + 2) - K(s + 2) = 0$$

we obtain

$$K = \frac{(s + 3)(s^2 + 2s + 2)}{s + 2}$$

By differentiating $K$ with respect to $s$, we obtain

$$\frac{dK}{ds} = \frac{2s^3 + 11s^2 + 20s + 10}{(s + 2)^2}$$





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Note that

$$2s^3 + 11s^2 + 20s + 10 = 2(s + 0.8)(s^2 + 4.7s + 6.24)$$
$$= 2(s + 0.8)(s + 2.35 + j0.77)(s + 2.35 - j0.77)$$

Point $s = -0.8$ is on the root locus. Since this point lies between two zeros (a finite zero and an infinite zero), it is an actual break-in point. Points $s = -2.35 \pm j0.77$ do not satisfy the angle condition and, therefore, they are neither breakaway nor break-in points.

5. Find the angle of departure of the root locus from a complex pole. For the complex pole at $s = -1 + j$, the angle of departure $\theta$ is

$$\theta = 0° - 27° - 90° + 45°$$

or

$$\theta = -72°$$

(The angle of departure from the complex pole at $s = -1 - j$ is 72°.)

6. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin and apply the angle condition. Locate a sufficient number of points that satisfy the angle condition.

Figure 6–31 shows the root loci for the given positive-feedback system. The root loci are shown with dashed lines and a curve.

Note that if

$$K > \left.\frac{(s + 3)(s^2 + 2s + 2)}{s + 2}\right|_{s=0} = 3$$

one real root enters the right-half $s$ plane. Hence, for values of $K$ greater than 3, the system becomes unstable. (For $K > 3$, the system must be stabilized with an outer loop.)

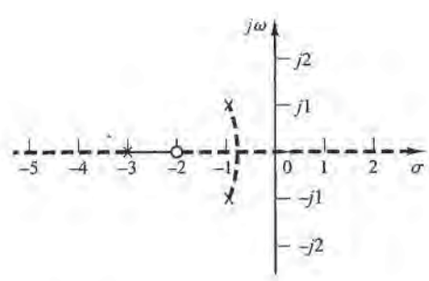


**Figure 6–31**
Root-locus plot for the positive-feedback system with $G(s) = K(s + 2)/[(s + 3)(s^2 + 2s + 2)]$, $H(s) = 1$.





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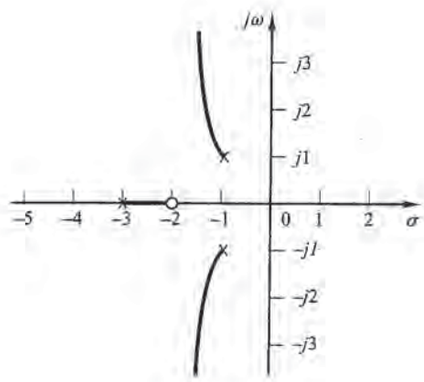


**Figure 6–32**
Root-locus plot for the negative-feedback system with $G(s) = K(s + 2)/[(s + 3)(s^2 + 2s + 2)]$, $H(s) = 1$.

Note that the closed-loop transfer function for the positive-feedback system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

$$= \frac{K(s + 2)}{(s + 3)(s^2 + 2s + 2) - K(s + 2)}$$

To compare this root-locus plot with that of the corresponding negative-feedback system, we show in Figure 6–32 the root loci for the negative-feedback system whose closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K(s + 2)}{(s + 3)(s^2 + 2s + 2) + K(s + 2)}$$

Table 6–2 shows various root-locus plots of negative-feedback and positive-feedback systems. The closed-loop transfer functions are given by

$$\frac{C}{R} = \frac{G}{1 + GH}, \qquad \text{for negative-feedback systems}$$

$$\frac{C}{R} = \frac{G}{1 - GH}, \qquad \text{for positive-feedback systems}$$

where $GH$ is the open-loop transfer function. In Table 6–2, the root loci for negative-feedback systems are drawn with heavy lines and curves, and those for positive-feedback systems are drawn with dashed lines and curves.





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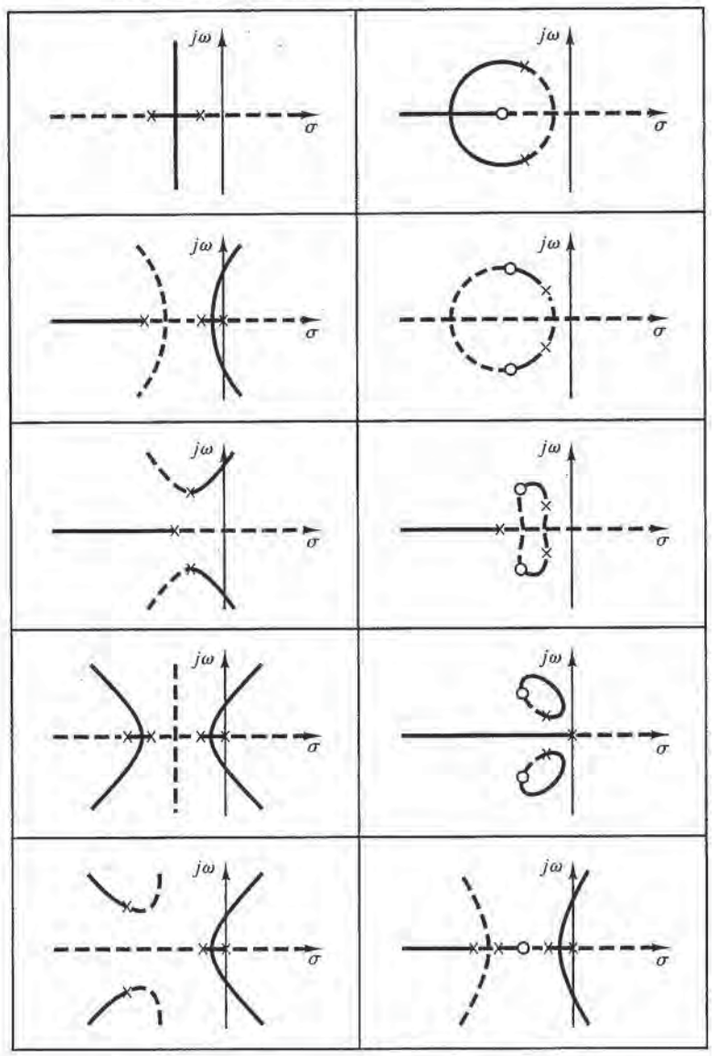
**Table 6–2** Root-Locus Plots of Negative-Feedback and Positive-Feedback Systems




Heavy lines and curves correspond to negative-feedback systems; dashed lines and curves correspond to positive-feedback systems.





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## 6–6 CONDITIONALLY STABLE SYSTEMS

Consider the system shown in Figure 6–33. We can plot the root loci for this system by applying the general rules and procedure for constructing root loci, or use MATLAB to get root-locus plots. MATLAB Program 6–10 will plot the root-locus diagram for the system. The plot is shown in Figure 6–34.

It can be seen from the root-locus plot of Figure 6–34 that this system is stable only for limited ranges of the value of $K$—that is, $0 < K < 12$ and $73 < K < 154$. The system becomes unstable for $12 < K < 73$ and $154 < K$. (If $K$ assumes a value corre-

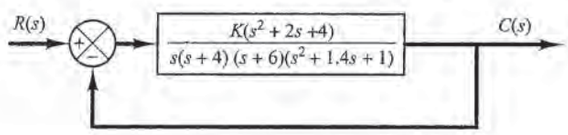


**Figure 6–33**
Control system.

**MATLAB Program 6–10**

```
num = [0 0 0 1 2 4];
den = conv(conv([1 4 0],[1 6]), [1 1.4 1]);
rlocus(num, den)
v = [-7 3 -5 5]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(s^2 + 2s + 4)/[s(s + 4)(s + 6)(s^2 + 1.4s + 1)]')
text(1.0, 0.55,'K = 12')
text(1.0,3.0,'K = 73')
text(1.0,4.15,'K = 154')
```

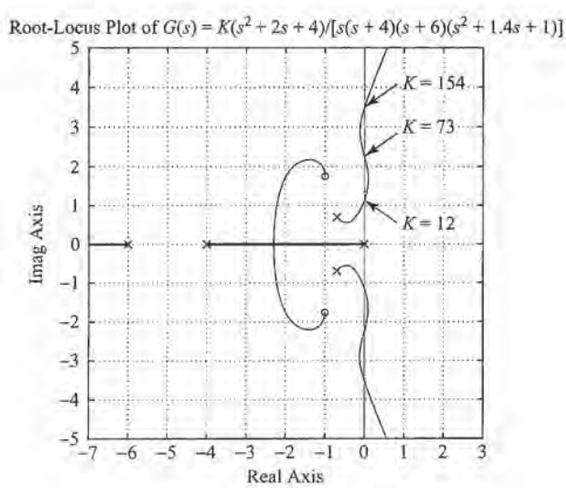


**Figure 6–34**
Root-locus plot of conditionally stable system.





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sponding to unstable operation, the system may break down or may become nonlinear due to a saturation nonlinearity that may exist.) Such a system is called conditionally stable.

In practice, conditionally stable systems are not desirable. Conditional stability is dangerous but does occur in certain systems, in particular, a system that has an unstable feedforward path. Such an unstable feedforward path may occur if the system has a minor loop. It is advisable to avoid such conditional stability since, if the gain drops beyond the critical value for any reason, the system becomes unstable. Note that the addition of a proper compensating network will eliminate conditional stability. [An addition of a zero will cause the root loci to bend to the left.(See Section 7–2.) Hence conditional stability may be eliminated by adding proper compensation.]

## 6–7 ROOT LOCI FOR SYSTEMS WITH TRANSPORT LAG

Figure 6–35 shows a thermal system in which hot air is circulated to keep the temperature of a chamber constant. In this system, the measuring element is placed downstream a distance $L$ ft from the furnace, the air velocity is $v$ ft/sec, and $T = L/v$ sec would elapse before any change in the furnace temperature is sensed by the thermometer. Such a delay in measuring, delay in controller action, or delay in actuator operation, and the like, is called *transport lag* or *dead time*. Dead time is present in most process control systems.

The input $x(t)$ and the output $y(t)$ of a transport-lag or dead-time element are related by

$$y(t) = x(t - T)$$

where $T$ is dead time. The transfer function of transport lag or dead time is given by

$$\text{Transfer function of transport lag or dead time} = \frac{\mathscr{L}[x(t-T)1(t-T)]}{\mathscr{L}[x(t)1(t)]} = \frac{X(s)e^{-Ts}}{X(s)} = e^{-Ts}$$

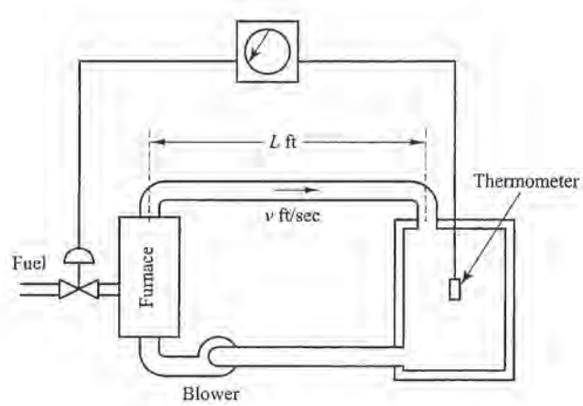


**Figure 6–35**
Thermal system.





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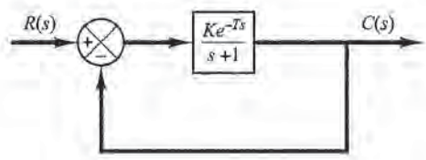


**Figure 6–36**
Block diagram of the system shown in Figure 6–35.

Suppose that the feedforward transfer function of this thermal system can be approximated by

$$G(s) = \frac{Ke^{-Ts}}{s+1}$$

as shown in Figure 6–36. Let us construct a root-locus plot for this system. The characteristic equation for this closed-loop system is

$$1 + \frac{Ke^{-Ts}}{s+1} = 0 \tag{6–18}$$

It is noted that for systems with transport lag the rules of construction presented earlier need to be modified. For example, the number of the root-locus branches is infinite, since the characteristic equation has an infinite number of roots. The number of asymptotes is infinite. They are all parallel to the real axis of the $s$ plane, as will be seen later.

From Equation (6–18), we obtain

$$\frac{Ke^{-Ts}}{s+1} = -1$$

Thus, the angle condition becomes

$$\underline{\left/\frac{Ke^{-Ts}}{s+1}\right.} = \underline{/e^{-Ts}} - \underline{/s+1} = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \ldots) \tag{6–19}$$

To find the angle of $e^{-Ts}$, substitute $s = \sigma + j\omega$. Then we obtain

$$e^{-Ts} = e^{-T\sigma - j\omega T}$$

Since $e^{-T\sigma}$ is a real quantity, the angle of $e^{-T\sigma}$ is zero. Hence

$$\begin{aligned} \underline{/e^{-Ts}} &= \underline{/e^{-j\omega T}} = \underline{/\cos\omega T - j\sin\omega T} \\ &= -\omega T \qquad \text{(radians)} \\ &= -57.3\omega T \qquad \text{(degrees)} \end{aligned}$$

Since $T$ is a given constant, the angle of $e^{-Ts}$ is a function of $\omega$ only. The angle condition, Equation (6–19), then becomes

$$-57.3\omega T - \underline{/s+1} = \pm 180°(2k+1)$$

We shall next determine the angle contribution due to $e^{-Ts}$ as given by Equation (6–19). For $k = 0$, the angle condition may be written

$$\underline{/s+1} = \pm 180° - 57.3°\omega T \tag{6–20}$$





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Since the angle contribution of $e^{-Ts}$ is zero for $\omega = 0$, the real axis from $-1$ to $-\infty$ forms a part of the root loci. Now assume a value $\omega_1$ for $\omega$ and compute $57.3°\omega_1 T$. At point $-1$ on the negative real axis, draw a line that makes an angle of $180° - 57.3°\omega_1 T$ with the real axis. Find the intersection of this line and the horizontal line $\omega = \omega_1$. This intersection, point $P$ in Figure 6–37(a), is a point satisfying Equation (6–20) and hence is on a root locus. Continuing the same process, we obtain the root-locus plot as shown in Figure 6–37(b).

Note that as $s$ approaches minus infinity, the open-loop transfer function

$$\frac{Ke^{-Ts}}{s+1}$$

approaches minus infinity since

$$\begin{aligned}\lim_{s=-\infty} \frac{Ke^{-Ts}}{s+1} &= \left.\frac{\dfrac{d}{ds}(Ke^{-Ts})}{\dfrac{d}{ds}(s+1)}\right|_{s=-\infty} \\ &= -KTe^{-Ts}|_{s=-\infty} \\ &= -\infty\end{aligned}$$

Therefore, $s = -\infty$ is a pole of the open-loop transfer function. Thus, root loci start from $s = -1$ or $s = -\infty$ and terminate at $s = \infty$, as $K$ increases from zero to infinity. Since the right-hand side of the angle condition given by Equation (6–19) has an infinite number of values, there are an infinite number of root loci, as the value of $k$ ($k = 0, 1, 2, \ldots$) goes from zero to infinity. For example, if $k = 1$, the angle condition becomes

$$\begin{aligned}\underline{/s+1} &= \pm 540° - 57.3°\omega T \qquad \text{(degrees)} \\ &= \pm 3\pi - \omega T \qquad \text{(radians)}\end{aligned}$$

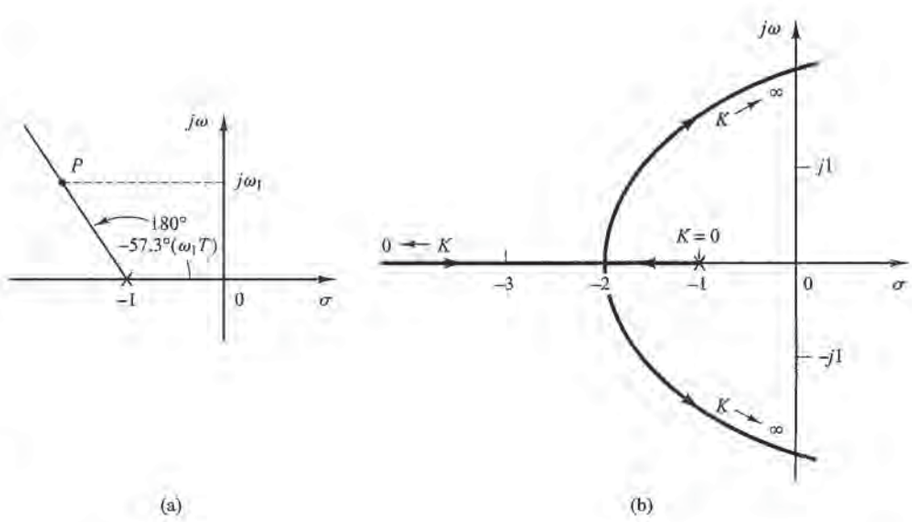


**Figure 6–37**
(a) Construction of the root locus;
(b) root-locus plot.





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The construction of the root loci for $k = 1$ is the same as that for $k = 0$. A plot of root loci for $k = 0, 1$, and 2 when $T = 1$ sec is shown in Figure 6–38.

The magnitude condition states that

$$\left|\frac{Ke^{-Ts}}{s+1}\right| = 1$$

Since the magnitude of $e^{-Ts}$ is equal to that of $e^{-T\sigma}$ or

$$|e^{-Ts}| = |e^{-T\sigma}| \cdot |e^{-j\omega T}| = e^{-T\sigma}$$

the magnitude condition becomes

$$|s + 1| = Ke^{-T\sigma}$$

The root loci shown in Figure 6–38 are graduated in terms of $K$ when $T = 1$ sec.

Although there are an infinite number of root-locus branches, the primary branch that lies between $-j\pi$ and $j\pi$ is most important. Referring to Figure 6–38, the critical value of $K$ at the primary branch is equal to 2, while the critical values of $K$ at other branches are much higher (8, 14, ...). Therefore, the critical value $K = 2$ on the primary branch is most significant from the stability viewpoint. The transient response of the system is determined by the roots located closest to the $j\omega$ axis and lie on the primary branch. In summary, the root-locus branch corresponding to $k = 0$ is the dominant one; other branches corresponding to $k = 1, 2, 3, \ldots$ are not so important and may be neglected.

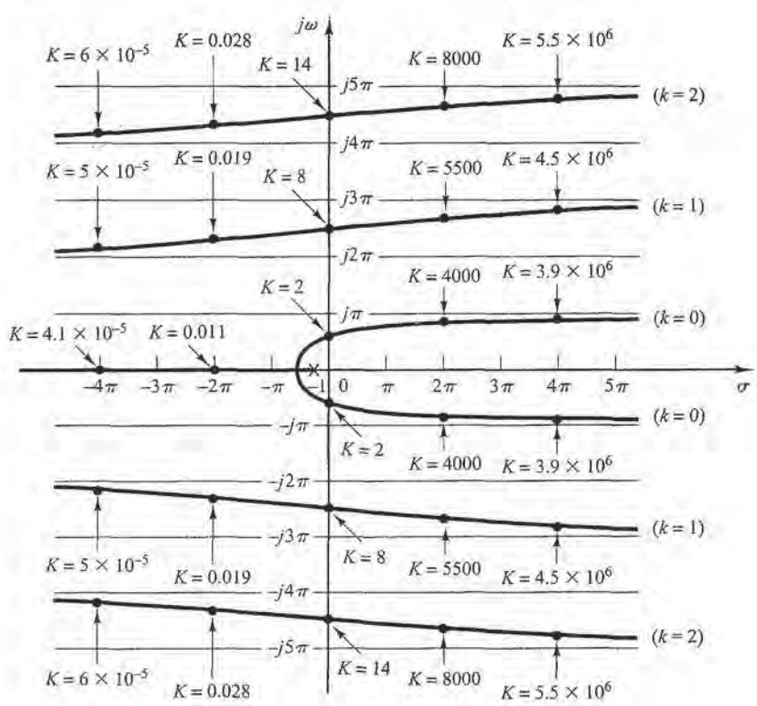


**Figure 6–38**
Root-locus plot for the system shown in Figure 6–36 ($T = 1$ sec).





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This example illustrates the fact that dead time can cause instability even in the first-order system because the root loci enter the right-half $s$ plane for large values of $K$. Therefore, although the gain $K$ of the first-order system can be set at a high value in the absence of dead time, it cannot be set too high if dead time is present. (For the system considered here, the value of gain $K$ must be considerably less than 2 for a satisfactory operation.)

**Approximation of Transport Lag or Dead Time.** If the dead time $T$ is very small, then $e^{-Ts}$ is frequently approximated by

$$e^{-Ts} \doteqdot 1 - Ts$$

or

$$e^{-Ts} \doteqdot \frac{1}{Ts + 1}$$

Such approximations are good if the dead time is very small and, in addition, the input time function $f(t)$ to the dead-time element is smooth and continuous. [This means that the second- and higher-order derivatives of $f(t)$ are small.]

A more elaborate expression to approximate $e^{-Ts}$ is available and is

$$e^{-Ts} = \frac{1 - \dfrac{Ts}{2} + \dfrac{(Ts)^2}{8} - \dfrac{(Ts)^3}{48} + \cdots}{1 + \dfrac{Ts}{2} + \dfrac{(Ts)^2}{8} + \dfrac{(Ts)^3}{48} + \cdots}$$

If only the first two terms in the numerator and denominator are taken, then

$$e^{-Ts} \doteqdot \frac{1 - \dfrac{Ts}{2}}{1 + \dfrac{Ts}{2}} = \frac{2 - Ts}{2 + Ts}$$

This approximation is also used frequently.

**MATLAB Approximation of Dead Time.** To handle dead time $e^{-sT}$, MATLAB uses the pade approximation. For example, if $T = 0.1$ sec, then using the third-order transfer function as an approximation to $e^{-sT}$, enter the following MATLAB program into the computer.

```
[num,den] = pade(0.1, 3);
printsys(num, den, 's')
  num/den =
  -1s^3 + 120s^2 - 6000s + 120000
  ------------------------------------
  s^3 + 120s^2 + 6000s + 120000
```





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Similarly, the program for the fourth-order transfer function approximation with $T = 0.1$ sec is

```
[num,den] = pade(0.1, 4);
printsys(num, den, 's')

num/den =

s^4 - 200s^3 + 18000s^2 - 840000s + 16800000
---------------------------------------------
s^4 + 200s^3 + 18000s^2 + 840000s + 16800000
```

Notice that the pade approximation depends on the dead time $T$ and the desired order for the approximating transfer function.

## EXAMPLE PROBLEMS AND SOLUTIONS

**A–6–1.** Sketch the root loci for the system shown in Figure 6–39(a). (The gain $K$ is assumed to be positive.) Observe that for small or large values of $K$ the system is overdamped and for medium values of $K$ it is underdamped.

**Solution.** The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and −1 and between −2 and −3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the $s$ plane.

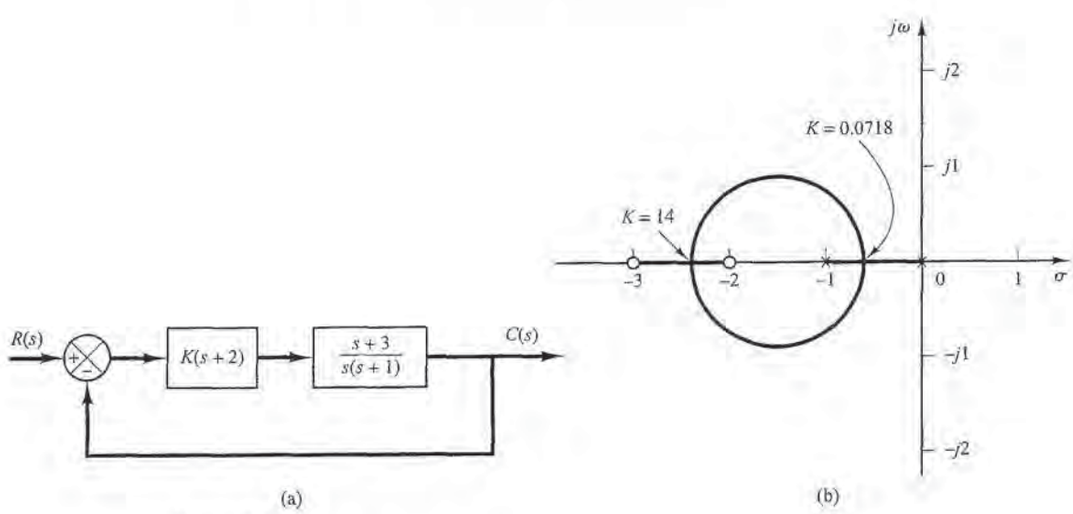


**Figure 6–39**
(a) Control system; (b) root-locus plot.





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3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and break-in points are determined from

$$\begin{aligned}\frac{dK}{ds} &= -\frac{(2s+1)(s+2)(s+3) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2} \\ &= -\frac{4(s+0.634)(s+2.366)}{[(s+2)(s+3)]^2} \\ &= 0\end{aligned}$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of $K$ is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at $-1.5$ that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–39(b).

Note that this system is stable for any positive value of $K$ since all the root loci lie in the left-half $s$ plane.

Small values of $K$ ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of $K$ ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of $K$ ($14 < K$) correspond to an overdamped system. With a large value of $K$, the steady state can be reached in much shorter time than with a small value of $K$.

The value of $K$ should be adjusted so that system performance is optimum according to a given performance index.





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**A–6–2.** Sketch the root loci of the control system shown in Figure 6–40(a).

**Solution.** The open-loop poles are located at $s = 0$, $s = -3 + j4$, and $s = -3 - j4$. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

Referring to Equation (6–13), the intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{0 + 3 + 3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

$$K = -s(s^2 + 6s + 25)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 12s + 25) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$

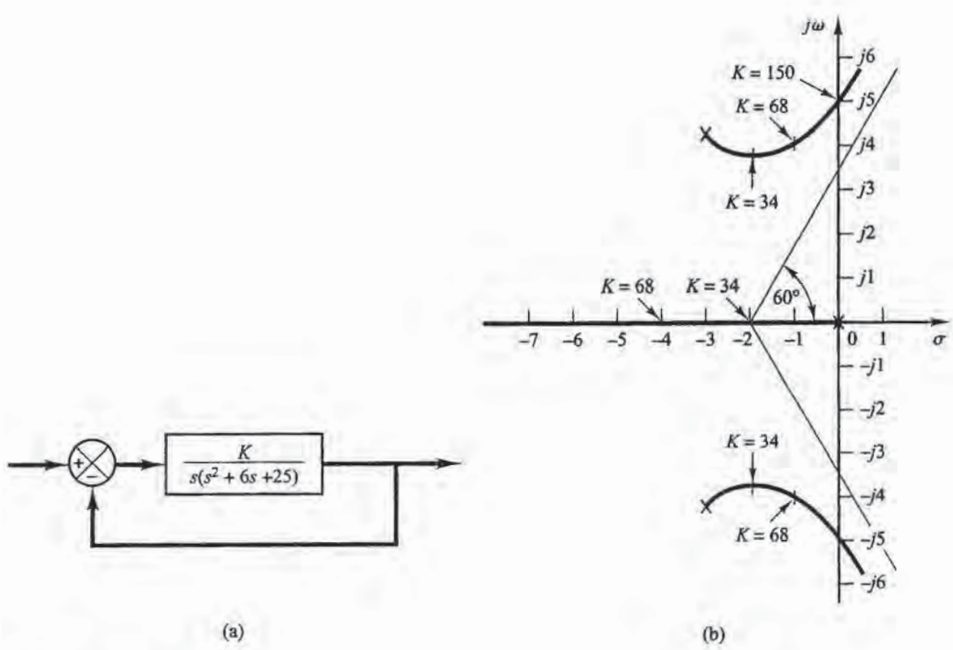


**Figure 6–40**
(a) Control system; (b) root-locus plot.





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Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of $K$, we obtain

$$K = -s(s^2 + 6s + 25)\Big|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of $K$ must be real and positive.)

The angle of departure from the complex pole in the upper half $s$ plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for $\omega$ and $K$ as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \qquad K = 150 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain $K$ at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain $K$ is increased, then other closed-loop poles must move toward the left as gain $K$ is increased. This fact can be seen clearly in this problem. If the gain $K$ is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.

**A–6–3.** Consider the system shown in Figure 6–41(a). Sketch the root loci for the system. Observe that for small or large values of $K$ the system is underdamped and for medium values of $K$ it is overdamped.

**Solution.** A root locus exists on the real axis between the origin and $-\infty$. The angles of asymptotes of the root-locus branches are obtained as

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, -180°$$

The intersection of the asymptotes and the real axis is located on the real axis at

$$s = -\frac{0 + 2 + 2}{3} = -1.3333$$





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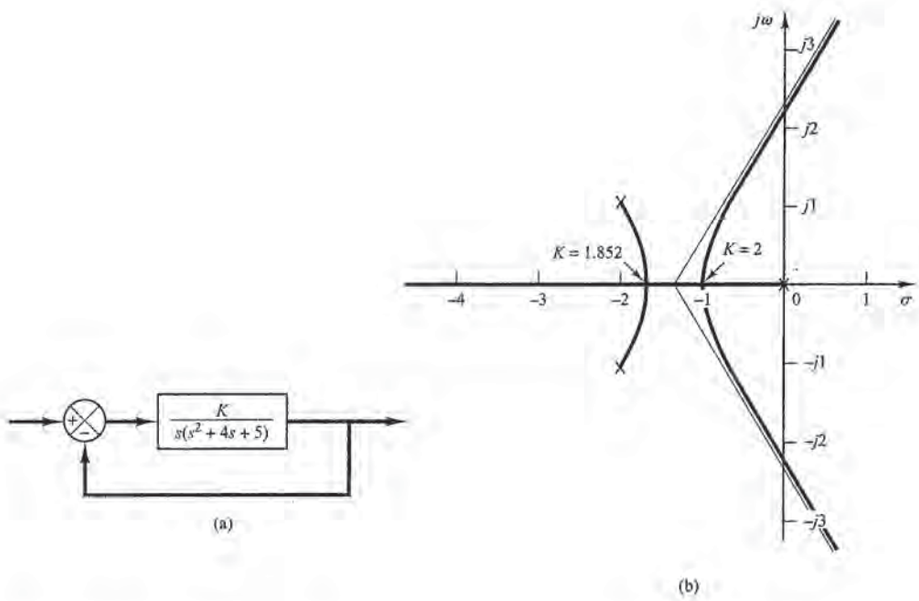


**Figure 6–41**
(a) Control system;
(b) root-locus plot.

The breakaway and break-in points are found from $dK/ds = 0$. Since the characteristic equation is

$$s^3 + 4s^2 + 5s + K = 0$$

we have

$$K = -(s^3 + 4s^2 + 5s)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 8s + 5) = 0$$

which yields

$$s = -1, \qquad s = -1.6667$$

Since these points are on root loci, they are actual breakaway or break-in points. (At point $s = -1$, the value of $K$ is 2, and at point $s = -1.6667$, the value of $K$ is 1.852.)

The angle of departure from a complex pole in the upper half $s$ plane is obtained from

$$\theta = 180° - 153.43° - 90°$$

or

$$\theta = -63.43°$$

The root-locus branch from the complex pole in the upper half $s$ plane breaks into the real axis at $s = -1.6667$.

Next we determine the points where root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + K = 0$$

or

$$(K - 4\omega^2) + j\omega(5 - \omega^2) = 0$$

from which we obtain

$$\omega = \pm\sqrt{5}, \qquad K = 20 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$





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Root-locus branches cross the imaginary axis at $\omega = \sqrt{5}$ and $\omega = -\sqrt{5}$. The root-locus branch on the real axis touches the $j\omega$ axis at $\omega = 0$. A sketch of the root loci for the system is shown in Figure 6–41(b).

Note that since this system is of third order, there are three closed-loop poles. The nature of the system response to a given input depends on the locations of the closed-loop poles.

For $0 < K < 1.852$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. For $1.852 \leq K \leq 2$, there are three real closed-loop poles. For example, the closed-loop poles are located at

$$s = -1.667, \quad s = -1.667, \quad s = -0.667, \quad \text{for } K = 1.852$$

$$s = -1, \quad s = -1, \quad s = -2, \quad \text{for } K = 2$$

For $2 < K$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. Thus, small values of $K$ $(0 < K < 1.852)$ correspond to an underdamped system. (Since the real closed-loop pole dominates, only a small ripple may show up in the transient response.) Medium values of $K$ $(1.852 \leq K \leq 2)$ correspond to an overdamped system. Large values of $K$ $(2 < K)$ correspond to an underdamped system. With a large value of $K$, the system responds much faster than with a smaller value of $K$.

**A–6–4.** Sketch the root loci for the system shown in Figure 6–42(a).

**Solution.** The open-loop poles are located at $s = 0$, $s = -1$, $s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{4} = 45°, -45°, 135°, -135°$$

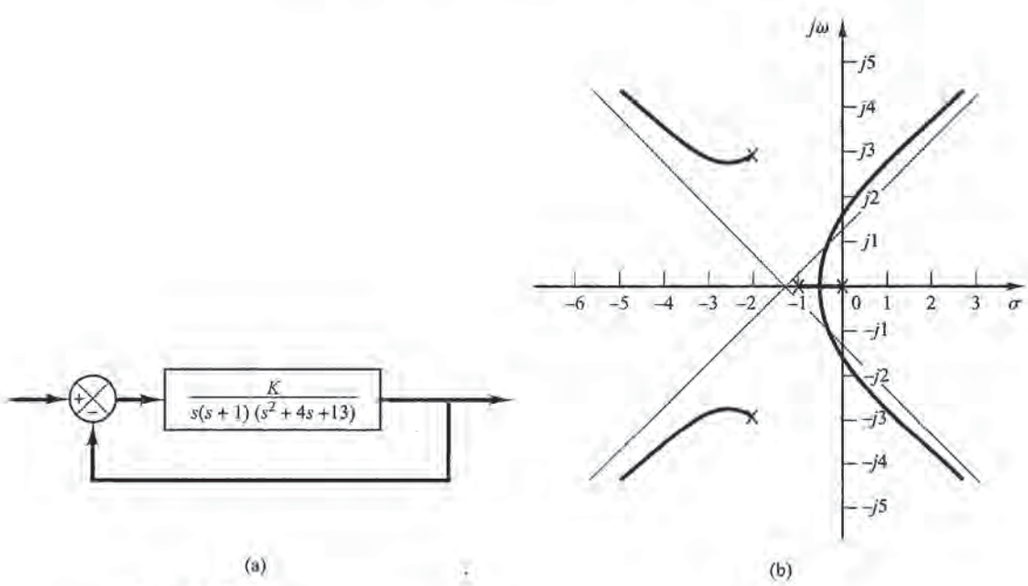


**Figure 6–42**
(a) Control system; (b) root-locus plot.





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The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 1 + 2 + 2}{4} = -1.25$$

The breakaway and break-in points are found from $dK/ds = 0$. Noting that

$$K = -s(s + 1)(s^2 + 4s + 13) = -(s^4 + 5s^3 + 17s^2 + 13s)$$

we have

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 34s + 13) = 0$$

from which we get

$$s = -0.467, \qquad s = -1.642 + j2.067, \qquad s = -1.642 - j2.067$$

Point $s = -0.467$ is on a root locus. Therefore, it is an actual breakaway point. The gain values $K$ corresponding to points $s = -1.642 \pm j2.067$ are complex quantities. Since the gain values are not real positive, these points are neither breakaway nor break-in points.

The angle of departure from the complex pole in the upper half $s$ plane is

$$\theta = 180° - 123.69° - 108.44° - 90°$$

or

$$\theta = -142.13°$$

Next we shall find the points where root loci may cross the $j\omega$ axis. Since the characteristic equation is

$$s^4 + 5s^3 + 17s^2 + 13s + K = 0$$

by substituting $s = j\omega$ into it we obtain

$$(j\omega)^4 + 5(j\omega)^3 + 17(j\omega)^2 + 13(j\omega) + K = 0$$

or

$$(K + \omega^4 - 17\omega^2) + j\omega(13 - 5\omega^2) = 0$$

from which we obtain

$$\omega = \pm 1.6125, \qquad K = 37.44 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

The root-locus branches that extend to the right-half $s$ plane cross the imaginary axis at $\omega = \pm 1.6125$. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–42(b) shows a sketch of the root loci for the system. Notice that each root-locus branch that extends to the right half $s$ plane crosses its own asymptote.





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**A–6–5.** Sketch the root loci for the system shown in Figure 6–43(a).

**Solution.** A root locus exists on the real axis between points $s = -1$ and $s = -3.6$. The asymptotes can be determined as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3 - 1} = 90°, -90°$$

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 0 + 3.6 - 1}{3 - 1} = -1.3$$

Since the characteristic equation is

$$s^3 + 3.6s^2 + K(s + 1) = 0$$

we have

$$K = -\frac{s^3 + 3.6s^2}{s + 1}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 1) - (s^3 + 3.6s^2)}{(s + 1)^2} = 0$$

or

$$s^3 + 3.3s^2 + 3.6s = 0$$

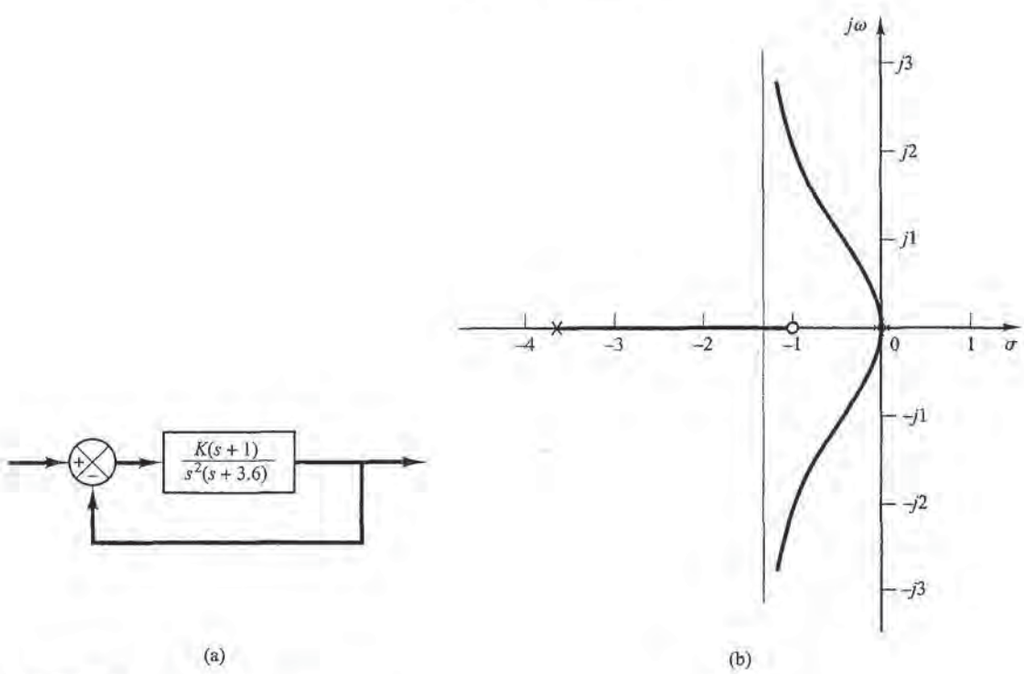


**Figure 6–43**
(a) Control system; (b) root-locus plot.





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from which we get

$$s = 0, \qquad s = -1.65 + j0.9367, \qquad s = -1.65 - j0.9367$$

Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ are neither breakaway nor break-in points, because the corresponding gain values $K$ become complex quantities.

To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

$$(K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

Notice that this equation can be satisfied only if $\omega = 0$, $K = 0$. Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$ axis at $\omega = 0$. The root-locus branches do not cross the $j\omega$ axis. Figure 6–43(b) is a sketch of the root loci for this system.

**A–6–6.** Sketch the root loci for the system shown in Figure 6–44(a).

**Solution.** A root locus exists on the real axis between point $s = -0.4$ and $s = -3.6$. The angles of asymptotes can be found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3 - 1} = 90°, -90°$$

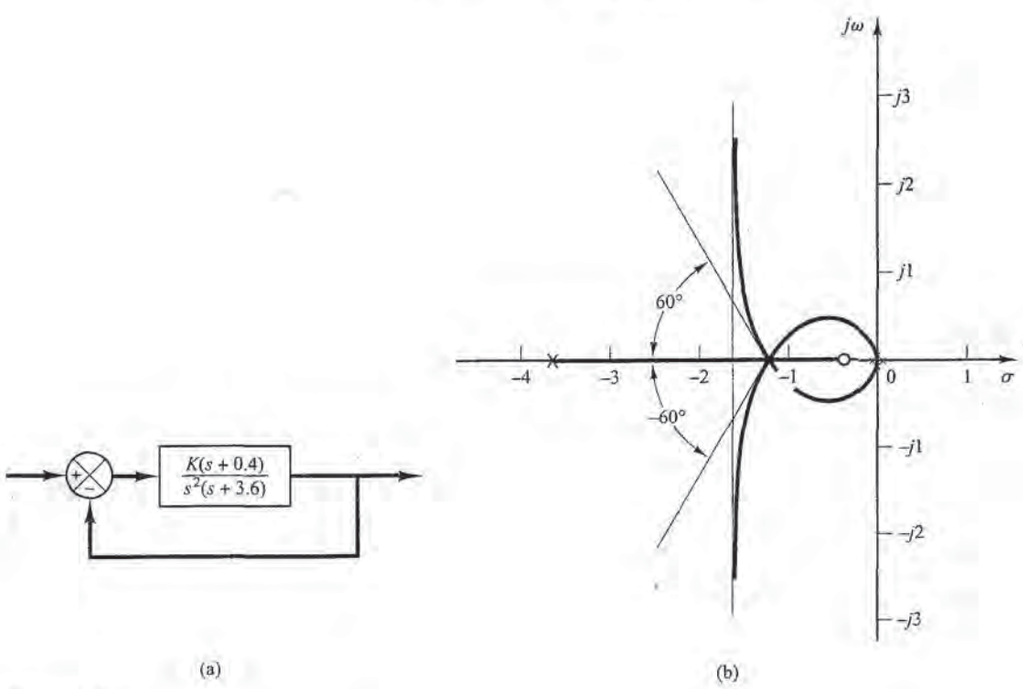


**Figure 6–44**
(a) Control system; (b) root-locus plot.



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The intersection of the asymptotes and the real axis is obtained from

$$s = -\frac{0 + 0 + 3.6 - 0.4}{3 - 1} = -1.6$$

Next we shall find the breakaway points. Since the characteristic equation is

$$s^3 + 3.6s^2 + Ks + 0.4K = 0$$

we have

$$K = -\frac{s^3 + 3.6s^2}{s + 0.4}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 0.4) - (s^3 + 3.6s^2)}{(s + 0.4)^2} = 0$$

from which we get

$$s^3 + 2.4s^2 + 1.44s = 0$$

or

$$s(s + 1.2)^2 = 0$$

Thus, the breakaway or break-in points are at $s = 0$ and $s = -1.2$. Note that $s = -1.2$ is a double root. When a double root occurs in $dK/ds = 0$ at point $s = -1.2$, $d^2K/(ds^2) = 0$ at this point. The value of gain $K$ at point $s = -1.2$ is

$$K = -\frac{s^3 + 3.6s^2}{s + 4}\bigg|_{s=-1.2} = 4.32$$

This means that with $K = 4.32$ the characteristic equation has a triple root at point $s = -1.2$. This can be easily verified as follows:

$$s^3 + 3.6s^2 + 4.32s + 1.728 = (s + 1.2)^3 = 0$$

Hence, three root-locus branches meet at point $s = -1.2$. The angles of departures at point $s = -1.2$ of the root locus branches that approach the asymptotes are $\pm 180°/3$, that is, 60° and −60°. (See Problem **A–6–7**.)

Finally, we shall examine if root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 3.6(j\omega)^2 + K(j\omega) + 0.4K = 0$$

or

$$(0.4K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

This equation can be satisfied only if $\omega = 0$, $K = 0$. At point $\omega = 0$, the root locus is tangent to the $j\omega$ axis because of the presence of a double pole at the origin. There are no points that root-locus branches cross the imaginary axis.

A sketch of the root loci for this system is shown in Figure 6–44(b).





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**A–6–7.** Referring to Problem **A–6–6**, obtain the equations for the root-locus branches for the system shown in Figure 6–44(a). Show that the root-locus branches cross the real axis at the breakaway point at angles $\pm 60°$.

**Solution.** The equations for the root-locus branches can be obtained from the angle condition

$$\underline{/\frac{K(s+0.4)}{s^2(s+3.6)}} = \pm 180°(2k+1)$$

which can be rewritten as

$$\underline{/s+0.4} - 2\underline{/s} - \underline{/s+3.6} = \pm 180°(2k+1)$$

By substituting $s = \sigma + j\omega$, we obtain

$$\underline{/\sigma + j\omega + 0.4} - 2\underline{/\sigma + j\omega} - \underline{/\sigma + j\omega + 3.6} = \pm 180°(2k+1)$$

or

$$\tan^{-1}\left(\frac{\omega}{\sigma+0.4}\right) - 2\tan^{-1}\left(\frac{\omega}{\sigma}\right) - \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) = \pm 180°(2k+1)$$

By rearranging, we have

$$\tan^{-1}\left(\frac{\omega}{\sigma+0.4}\right) - \tan^{-1}\left(\frac{\omega}{\sigma}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) \pm 180°(2k+1)$$

Taking tangents of both sides of this last equation, and noting that

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+3.6}\right) \pm 180°(2k+1)\right] = \frac{\omega}{\sigma+3.6}$$

we obtain

$$\frac{\dfrac{\omega}{\sigma+0.4} - \dfrac{\omega}{\sigma}}{1 + \dfrac{\omega}{\sigma+0.4}\dfrac{\omega}{\sigma}} = \frac{\dfrac{\omega}{\sigma} + \dfrac{\omega}{\sigma+3.6}}{1 - \dfrac{\omega}{\sigma}\dfrac{\omega}{\sigma+3.6}}$$

which can be simplified to

$$\frac{\omega\sigma - \omega(\sigma+0.4)}{(\sigma+0.4)\sigma + \omega^2} = \frac{\omega(\sigma+3.6) + \omega\sigma}{\sigma(\sigma+3.6) - \omega^2}$$

or

$$\omega(\sigma^3 + 2.4\sigma^2 + 1.44\sigma + 1.6\omega^2 + \sigma\omega^2) = 0$$

which can be further simplified to

$$\omega[\sigma(\sigma+1.2)^2 + (\sigma+1.6)\omega^2] = 0$$

For $\sigma \neq -1.6$, we may write this last equation as

$$\omega\left[\omega - (\sigma+1.2)\sqrt{\frac{-\sigma}{\sigma+1.6}}\right]\left[\omega + (\sigma+1.2)\sqrt{\frac{-\sigma}{\sigma+1.6}}\right] = 0$$





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which gives the equations for the root-locus as follows:

$$\omega = 0$$

$$\omega = (\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}}$$

$$\omega = -(\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}}$$

The equation $\omega = 0$ represents the real axis. The root locus for $0 \le K \le \infty$ is between points $s = -0.4$ and $s = -3.6$. (The real axis other than this line segment and the origin $s = 0$ corresponds to the root locus for $-\infty \le K < 0$.)

The equations

$$\omega = \pm(\sigma + 1.2)\sqrt{\frac{-\sigma}{\sigma + 1.6}} \tag{6–21}$$

represent the complex branches for $0 \le K \le \infty$. These two branches lie between $\sigma = -1.6$ and $\sigma = 0$. [See Figure 6–44(b).] The slopes of the complex root-locus branches at the breakaway point ($\sigma = -1.2$) can be found by evaluating $d\omega/d\sigma$ of Equation (6–21) at point $\sigma = -1.2$.

$$\left.\frac{d\omega}{d\sigma}\right|_{\sigma=-1.2} = \pm\left.\sqrt{\frac{-\sigma}{\sigma + 1.6}}\right|_{\sigma=-1.2} = \pm\sqrt{\frac{1.2}{0.4}} = \pm\sqrt{3}$$

Since $\tan^{-1}\sqrt{3} = 60°$, the root-locus branches intersect the real axis with angles $\pm 60°$.

**A–6–8.** Consider the system shown in Figure 6–45(a), which has an unstable feedforward transfer function. Sketch the root-locus plot and locate the closed-loop poles. Show that, although the closed-loop poles lie on the negative real axis and the system is not oscillatory, the unit-step response curve will exhibit overshoot.

**Solution.** The root-locus plot for this system is shown in Figure 6–45(b). The closed-loop poles are located at $s = -2$ and $s = -5$.

The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{10(s + 1)}{s^2 + 7s + 10}$$

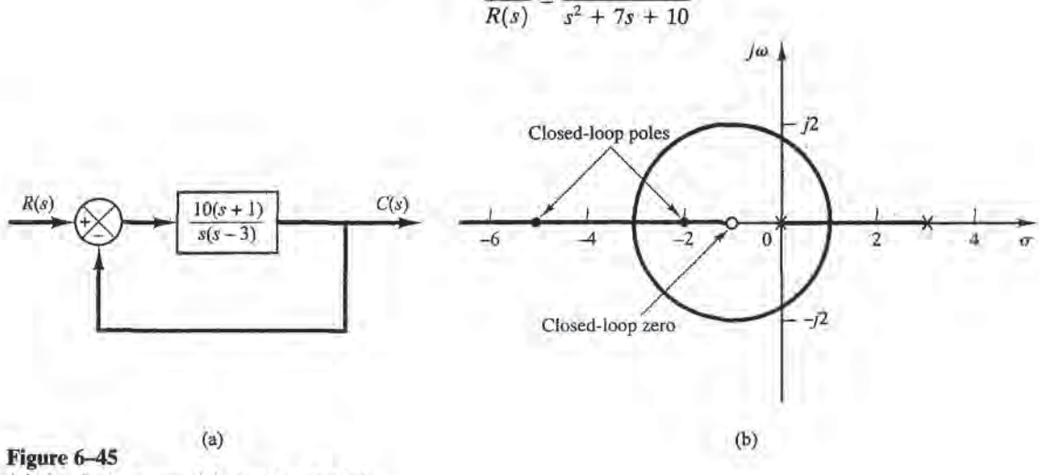


**Figure 6–45**
(a) Control system; (b) root-locus plot.





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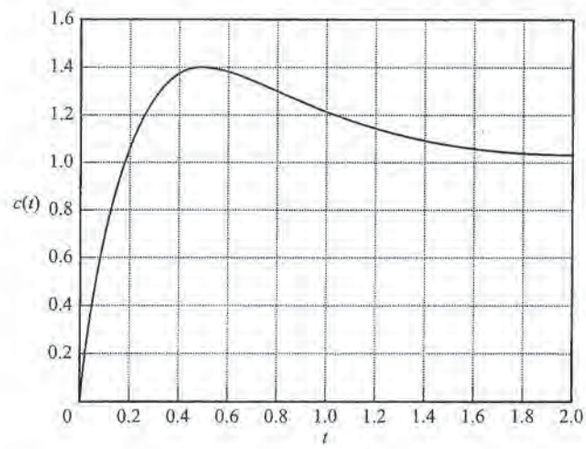


**Figure 6–46**
Unit-step response curve for the system shown in Figure 6–45(a).

The unit-step response of this system is

$$C(s) = \frac{10(s+1)}{s(s+2)(s+5)}$$

The inverse Laplace transform of $C(s)$ gives

$$c(t) = 1 + 1.666e^{-2t} - 2.666e^{-5t}, \qquad \text{for } t \geq 0$$

The unit-step response curve is shown in Figure 6–46. Although the system is not oscillatory, the unit-step response curve exhibits overshoot. (This is due to the presence of a zero at $s = -1$.)

**A–6–9.** Sketch the root loci of the control system shown in Figure 6–47(a). Determine the range of gain $K$ for stability.

**Solution.** Open-loop poles are located at $s = 1$, $s = -2 + j\sqrt{3}$, and $s = -2 - j\sqrt{3}$. A root locus exists on the real axis between points $s = 1$ and $s = -\infty$. The asymptotes of the root-locus branches are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} = 60°, -60°, 180°$$

The intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{-1+2+2}{3} = -1$$

The breakaway and break-in points can be located from $dK/ds = 0$. Since

$$K = -(s-1)(s^2+4s+7) = -(s^3+3s^2+3s-7)$$

we have

$$\frac{dK}{ds} = -(3s^2+6s+3) = 0$$

which yields

$$(s+1)^2 = 0$$





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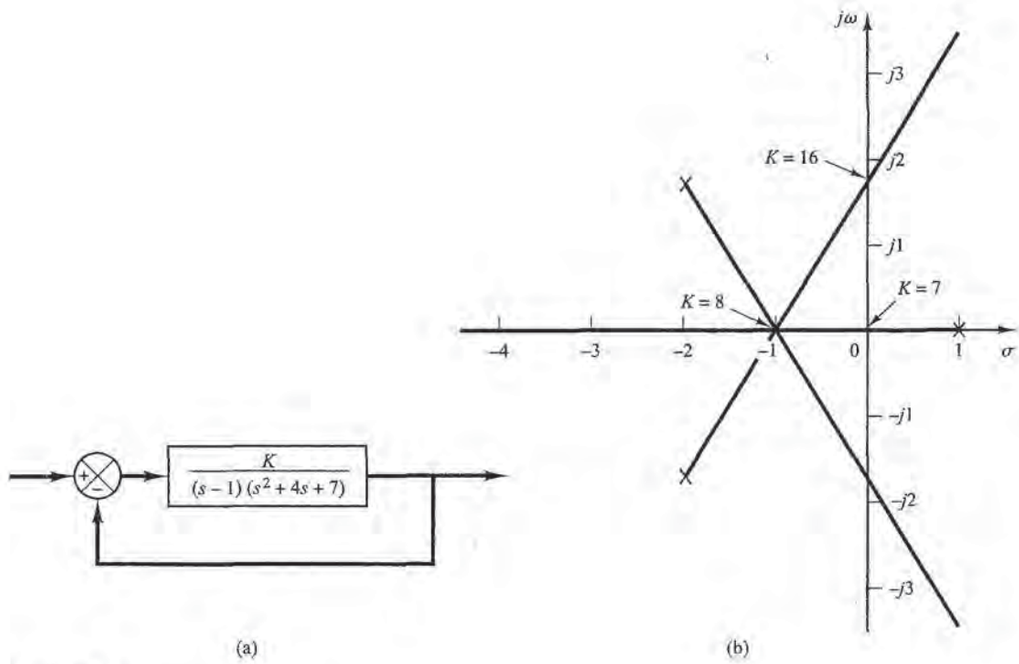


**Figure 6–47**
(a) Control system; (b) root-locus plot.

Thus the equation $dK/ds = 0$ has a double root at $s = -1$. (This means that the characteristic equation has a triple root at $s = -1$.) The breakaway point is located at $s = -1$. Three root-locus branches meet at this breakaway point. The angles of departure of the branches at the breakaway point are $\pm 180°/3$, that is, 60° and −60°.

We shall next determine the points where root-locus branches may cross the imaginary axis. Noting that the characteristic equation is

$$(s-1)(s^2+4s+7)+K=0$$

or

$$s^3+3s^2+3s-7+K=0$$

we substitute $s = j\omega$ into it and obtain

$$(j\omega)^3+3(j\omega)^2+3(j\omega)-7+K=0$$

By rewriting this last equation, we have

$$(K-7-3\omega^2)+j\omega(3-\omega^2)=0$$

This equation is satisfied when

$$\omega=\pm\sqrt{3}, \qquad K=7+3\omega^2=16 \qquad \text{or} \qquad \omega=0, \qquad K=7$$





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The root-locus branches cross the imaginary axis at $\omega = \pm\sqrt{3}$ (where $K = 16$) and $\omega = 0$ (where $K = 7$). Since the value of gain $K$ at the origin is 7, the range of gain value $K$ for stability is

$$7 < K < 16$$

Figure 6–47(b) shows a sketch of the root loci for the system. Notice that all branches consist of parts of straight lines.

The fact that the root-locus branches consist of straight lines can be verified as follows: Since the angle condition is

$$\underline{\left/ \frac{K}{(s-1)(s+2+j\sqrt{3})(s+2-j\sqrt{3})} \right.} = \pm 180°(2k+1)$$

we have

$$-\underline{/s-1} - \underline{/s+2+j\sqrt{3}} - \underline{/s+2-j\sqrt{3}} = \pm 180°(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation,

$$\underline{/\sigma - 1 + j\omega} + \underline{/\sigma + 2 + j\omega + j\sqrt{3}} + \underline{/\sigma + 2 + j\omega - j\sqrt{3}} = \pm 180°(2k+1)$$

or

$$\underline{/\sigma + 2 + j(\omega + \sqrt{3})} + \underline{/\sigma + 2 + j(\omega - \sqrt{3})} = -\underline{/\sigma - 1 + j\omega} \pm 180°(2k+1)$$

which can be rewritten as

$$\tan^{-1}\left(\frac{\omega + \sqrt{3}}{\sigma + 2}\right) + \tan^{-1}\left(\frac{\omega - \sqrt{3}}{\sigma + 2}\right) = -\tan^{-1}\left(\frac{\omega}{\sigma - 1}\right) \pm 180°(2k+1)$$

Taking tangents of both sides of this last equation, we obtain

$$\frac{\dfrac{\omega + \sqrt{3}}{\sigma + 2} + \dfrac{\omega - \sqrt{3}}{\sigma + 2}}{1 - \left(\dfrac{\omega + \sqrt{3}}{\sigma + 2}\right)\left(\dfrac{\omega - \sqrt{3}}{\sigma + 2}\right)} = -\frac{\omega}{\sigma - 1}$$

or

$$\frac{2\omega(\sigma + 2)}{\sigma^2 + 4\sigma + 4 - \omega^2 + 3} = -\frac{\omega}{\sigma - 1}$$

which can be simplified to

$$2\omega(\sigma + 2)(\sigma - 1) = -\omega(\sigma^2 + 4\sigma + 7 - \omega^2)$$

or

$$\omega(3\sigma^2 + 6\sigma + 3 - \omega^2) = 0$$

Further simplification of this last equation yields

$$\omega\left(\sigma + 1 + \frac{1}{\sqrt{3}\,\omega}\right)\left(\sigma + 1 - \frac{1}{\sqrt{3}\,\omega}\right) = 0$$

which defines three lines:

$$\omega = 0, \qquad \sigma + 1 + \frac{1}{\sqrt{3}}\omega = 0, \qquad \sigma + 1 - \frac{1}{\sqrt{3}}\omega = 0$$





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Thus the root-locus branches consist of three lines. Note that the root loci for $K > 0$ consist of portions of the straight lines as shown in Figure 6–47(b). (Note that each straight line starts from an open-loop pole and extends to infinity in the direction of 180°, 60°, or −60° measured from the real axis.) The remaining portion of each straight line corresponds to $K < 0$.

**A–6–10.** Consider the system shown in Figure 6–48(a). Sketch the root loci.

**Solution.** The open-loop zeros of the system are located at $s = \pm j$. The open-loop poles are located at $s = 0$ and $s = -2$. This system involves two poles and two zeros. Hence, there is a possibility that a circular root-locus branch exists. In fact, such a circular root locus exists in this case, as shown in the following. The angle condition is

$$\underline{\left/\frac{K(s+j)(s-j)}{s(s+2)}\right.} = \pm 180°(2k+1)$$

or

$$\underline{/s+j} + \underline{/s-j} - \underline{/s} - \underline{/s+2} = \pm 180°(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\underline{/\sigma + j\omega + j} + \underline{/\sigma + j\omega - j} = \underline{/\sigma + j\omega} + \underline{/\sigma + 2 + j\omega} \pm 180°(2k+1)$$

or

$$\tan^{-1}\left(\frac{\omega+1}{\sigma}\right) + \tan^{-1}\left(\frac{\omega-1}{\sigma}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) + \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \pm 180°(2k+1)$$

Taking tangents of both sides of this equation and noting that

$$\tan\left[\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \pm 180°\right] = \frac{\omega}{\sigma+2}$$

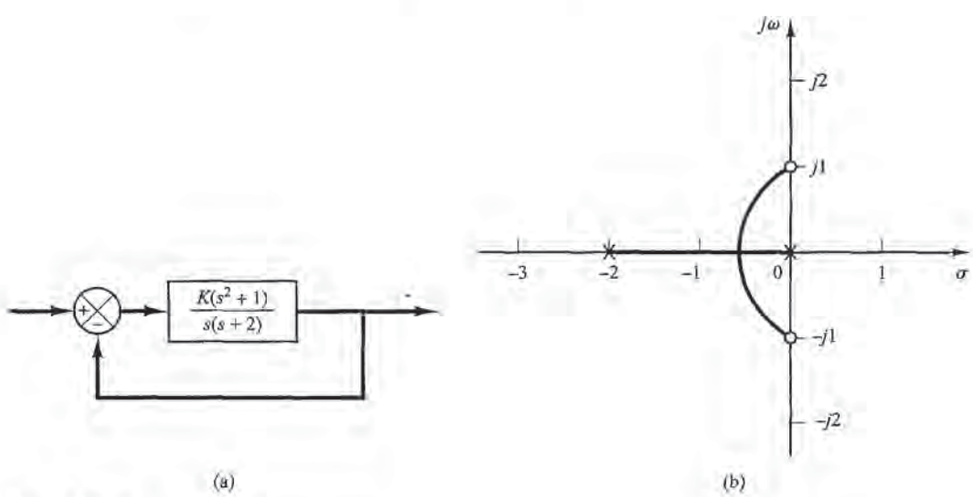


**Figure 6–48**
(a) Control system; (b) root-locus plot.





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we obtain

$$\frac{\dfrac{\omega+1}{\sigma}+\dfrac{\omega-1}{\sigma}}{1-\dfrac{\omega+1}{\sigma}\dfrac{\omega-1}{\sigma}}=\frac{\dfrac{\omega}{\sigma}+\dfrac{\omega}{\sigma+2}}{1-\dfrac{\omega}{\sigma}\dfrac{\omega}{\sigma+2}}$$

or

$$\omega\left[\left(\sigma-\frac{1}{2}\right)^2+\omega^2-\frac{5}{4}\right]=0$$

which is equivalent to

$$\omega=0 \qquad \text{or} \qquad \left(\sigma-\frac{1}{2}\right)^2+\omega^2=\frac{5}{4}$$

These two equations are equations for the root loci. The first equation corresponds to the root locus on the real axis. (The segment between $s = 0$ and $s = -2$ corresponds to the root locus for $0 \le K < \infty$. The remaining parts of the real axis correspond to the root locus for $K < 0$.) The second equation is an equation for a circle. Thus, there exists a circular root locus with center at $\sigma = \frac{1}{2}$, $\omega = 0$ and the radius equal to $\sqrt{5}/2$. The root loci are sketched in Figure 6–48(b). [That part of the circular locus to the left of the imaginary zeros corresponds to $K > 0$. The portion of the circular locus not shown in Figure 6–48(b) corresponds to $K < 0$.]

**A–6–11.** Consider the control system shown in Figure 6–49. Plot the root loci with MATLAB.

**Solution.** MATLAB Program 6–11 generates a root-locus plot as shown in Figure 6–50. The root loci must be symmetric about the real axis. However, Figure 6–50 shows otherwise.

MATLAB supplies its own set of gain values that are used to calculate a root-locus plot. It does so by an internal adaptive step-size routine. However, in certain systems, very small changes in the gain cause drastic changes in root locations within a certain range of gains. Thus, MATLAB takes too big a jump in its gain values when calculating the roots, and root locations change by a relatively large amount. When plotting, MATLAB connects these points and causes a strange-looking graph at the location of sensitive gains. Such erroneous root-locus plots typically occur when the loci approach a double pole (or triple or higher pole), since the locus is very sensitive to small gain changes.

**MATLAB Program 6–11**

```
% --------- Root-locus plot ---------
num = [0 0 1 0.4];
den = [1 3.6 0 0];
rlocus(num,den);
v = [-5 1 -3 3]; axis(v)
grid
title('Root-Locus Plot of G(s) = K(s + 0.4)/[s^2(s + 3.6)]')
```

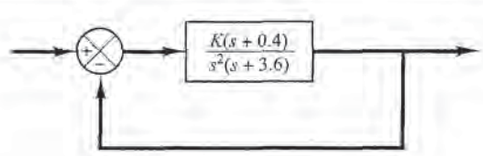


**Figure 6–49**
Control system.





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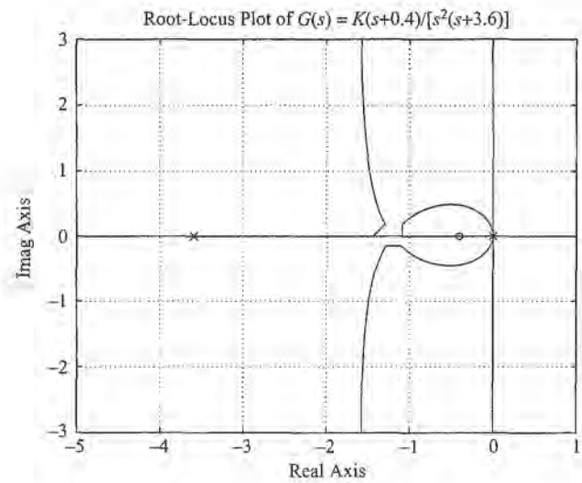


**Figure 6–50**
Root-locus plot.

In the problem considered here, the critical region of gain $K$ is between 4.2 and 4.4. Thus we need to set the step size small enough in this region. We may divide the region for $K$ as follows:

```
K1 = [0:0.2:4.2];
K2 = [4.2:0.002:4.4];
K3 = [4.4:0.2:10];
K4 = [10:5:200];
K = [K1  K2  K3  K4];
```

Entering MATLAB Program 6–12 into the computer, we obtain the plot as shown in Figure 6–51. If we change the plot command plot(r,'o') in MATLAB Program 6–12 to plot(r,'-'), we obtain Figure 6–52. Figures 6–51 and 6–52 respectively, show satisfactory root-locus plots.

**MATLAB Program 6–12**

```
% ---------- Root-locus plot ----------
num = [0  0  1  0.4];
den = [1  3.6  0  0];
K1 = [0:0.2:4.2];
K2 = [4.2:0.002:4.4];
K3 = [4.4:0.2:10];
K4 = [10:5:200];
K = [K1  K2  K3  K4];
r = rlocus(num,den,K);
plot(r,'o')
v = [-5  1  -5  5]; axis(v)
grid
title('Root-Locus Plot of G(s) = K(s + 0.4)/[s^2(s + 3.6)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```





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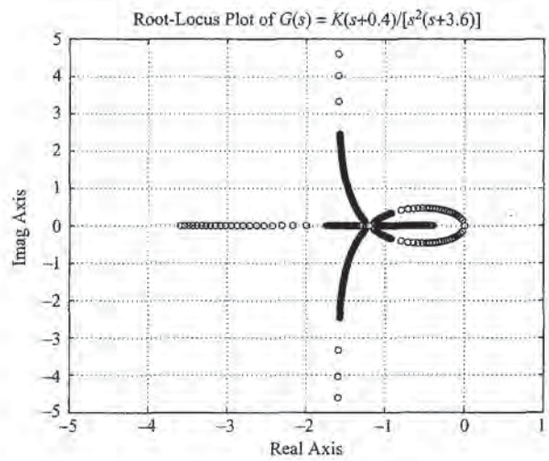


**Figure 6–51**
Root-locus plot.

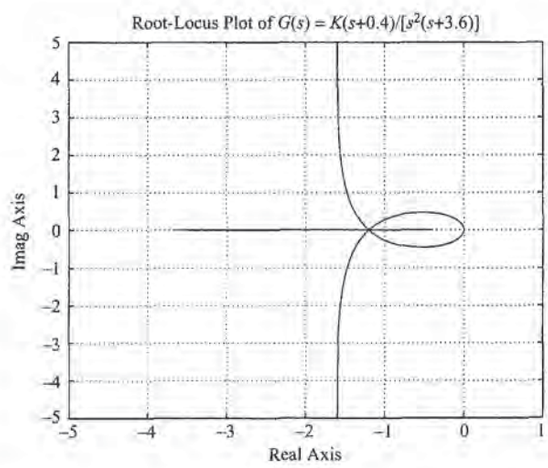


**Figure 6–52**
Root-locus plot.

**A–6–12.** Consider the system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Using MATLAB, plot root loci and their asymptotes.

**Solution.** We shall plot the root loci and asymptotes on one diagram. Since the open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)} = \frac{K}{s^3 + 3s^2 + 2s}$$

the equation for the asymptotes may be obtained as follows: Noting that

$$\lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 2s} \doteq \lim_{s\to\infty} \frac{K}{s^3 + 3s^2 + 3s + 1} = \frac{K}{(s+1)^3}$$





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the equation for the asymptotes may be given by

$$G_a(s)H_a(s) = \frac{K}{(s+1)^3}$$

Hence, for the system we have

```
num = [0 0 0 1]
den = [1 3 2 0]
```

and for the asymptotes,

```
numa = [0 0 0 1]
dena = [1 3 3 1]
```

In using the following root-locus and plot commands

```
r = rlocus(num,den)
a = rlocus(numa,dena)
plot([r a])
```

the number of rows of r and that of a must be the same. To ensure this, we include the gain constant K in the commands. For example,

```
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5:
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1 K2 K3 K4]
r = rlocus(num,den,K)
a = rlocus(numa,dena,K)
y = [r  a]
plot(y, '-')
```

**MATLAB Program 6–13**

```
% ---------- Root-Locus Plots ----------
num = [0 0 0 1];
den = [1 3 2 0];
numa = [0 0 0 1];
dena = [1 3 3 1];
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5;
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1 K2 K3 K4];
r = rlocus(num,den,K);
a = rlocus(numa,dena,K);
y = [r  a];
plot(y,'-')
v = [-4 4 -4 4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 1)(s + 2)] and Asymptotes')
xlabel('Real Axis')
ylabel('Imag Axis')
% ***** Manually draw open-loop poles in the hard copy *****
```





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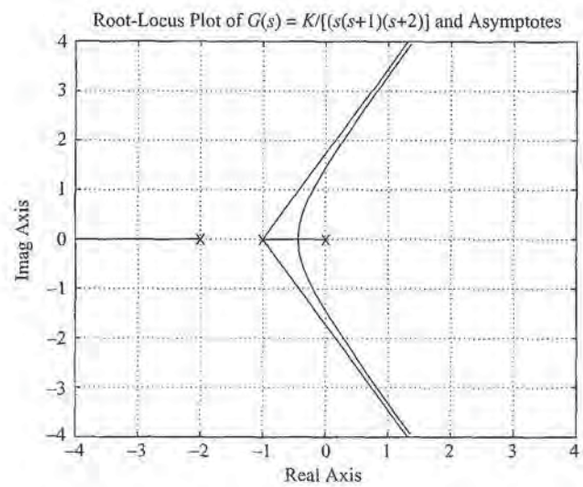


**Figure 6–53**
Root-locus plot.

Including gain K in rlocus command ensures that the r matrix and a matrix have the same number of rows. MATLAB Program 6–13 will generate a plot of root loci and their asymptotes. See Figure 6–53.

Drawing two or more plots in one diagram can also be accomplished by using the hold command. MATLAB Program 6–14 uses the hold command. The resulting root-locus plot is shown in Figure 6–54.

**MATLAB Program 6–14**

```
% ------------ Root-Locus Plots ------------
num = [0  0  0  1];
den = [1  3  2  0];
numa = [0  0  0  1];
dena = [1  3  3  1];
K1 = 0:0.1:0.3;
K2 = 0.3:0.005:0.5;
K3 = 0.5:0.5:10;
K4 = 10:5:100;
K = [K1  K2  K3  K4];
r = rlocus(num,den,K);
a = rlocus(numa,dena,K);
plot(r,'o')
hold
Current plot held
plot(a,'-')
v = [-4  4  -4  4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s+1)(s+2)] and Asymptotes')
xlabel('Real Axis')
ylabel('Imag Axis')
```





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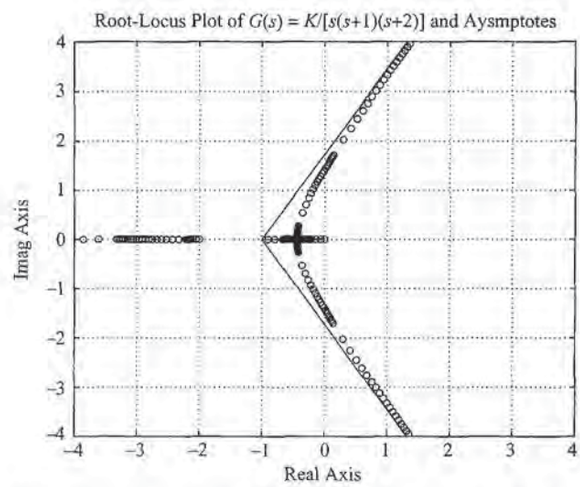


**Figure 6–54**
Root-locus plot.

**A–6–13.** Consider a unity-feedback system with the following feedforward transfer function $G(s)$:

$$G(s) = \frac{K(s+2)^2}{(s^2+4)(s+5)^2}$$

Plot root loci for the system with MATLAB.

**Solution.** A MATLAB program to plot the root loci is given as MATLAB Program 6–15. The resulting root-locus plot is shown in Figure 6–55.

Notice that this is a special case where no root locus exists on the real axis. This means that for any value of $K > 0$ the closed-loop poles of the system are two sets of complex-conjugate poles. (No real closed-loop poles exist.) For example, with $K = 25$, the characteristic equation for the system becomes

$$\begin{aligned} &s^4 + 10s^3 + 54s^2 + 140s + 200 \\ &= (s^2 + 4s + 10)(s^2 + 6s + 20) \\ &= (s + 2 + j2.4495)(s + 2 - j2.4495)(s + 3 + j3.3166)(s + 3 - j3.3166) \end{aligned}$$

**MATLAB Program 6–15**

```
% ------------ Root-Locus Plot ------------
num = [0  0  1  4  4];
den = [1  10  29  40  100];
r = rlocus(num,den);
plot(r,'o')
hold
current plot held
plot(r,'-')
v = [-8  4  -6  6]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = (s + 2)^2/[(s^2 + 4)(s + 5)^2]')
xlabel('Real Axis')
ylabel('Imag Axis')
```





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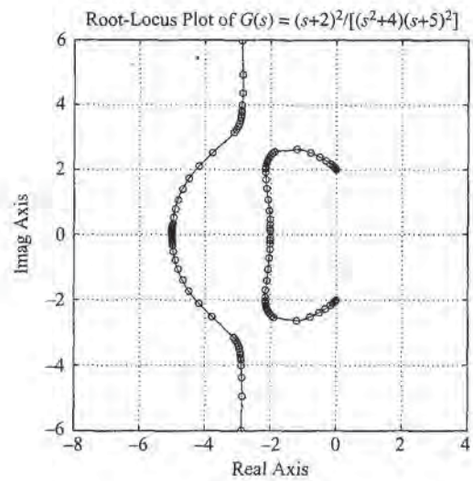


**Figure 6–55**
Root-locus plot.

Since no closed-loop poles exist in the right-half $s$ plane, the system is stable for all values of $K > 0$.

**A–6–14.** Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{s + 2}{s^3 + 9s^2 + 8s}$$

Plot a root-locus diagram with MATLAB. Superimpose on the $s$ plane constant $\zeta$ lines and constant $\omega_n$ circles.

**Solution.** MATLAB Program 6–16 produces the desired plot as shown in Figure 6–56.

**MATLAB Program 6–16**

```
num = [0 0 1 2];
den = [1 9 8 0];
K = 0:0.2:200;
rlocus(num,den,K)
v = [-10 2 -6 6]; axis(v); axis('square')
sgrid
title('Root-Locus Plot with Constant \zeta Lines and Constant \omega_n Circles')
gtext('\zeta = 0.9')
gtext('0.7')
gtext('0.5')
gtext('0.3')
gtext('\omega_n = 10')
gtext('8')
gtext('6')
gtext('4')
gtext('2')
```





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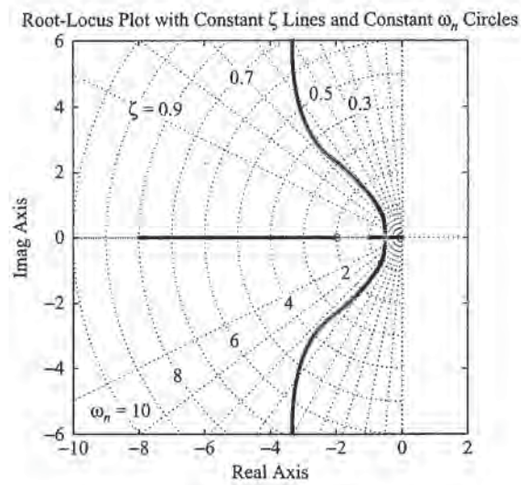


**Figure 6–56**
Root-locus plot with constant $\zeta$ lines and constant $\omega_n$ circles.

**A–6–15.** Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K(s^2 + 25)s}{s^4 + 404s^2 + 1600}$$

Plot root loci for the system with MATLAB. Show that the system is stable for all values of $K > 0$.

**Solution.** MATLAB Program 6–17 gives a plot of root loci as shown in Figure 6–57. Since the root loci are entirely in the left-half $s$ plane, the system is stable for all $K > 0$.

**MATLAB Program 6–17**

```
num = [0 1 0 25 0];
den = [1 0 404 0 1600];
K = 0:0.4:1000;
rlocus(num,den,K)
v = [-30 20 -25 25]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(s^2 + 25)s/(s^4 + 404s^2 + 1600)')
```

**A–6–16.** A simplified form of the open-loop transfer function of an airplane with an autopilot in the longitudinal mode is

$$G(s)H(s) = \frac{K(s + a)}{s(s - b)(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \qquad a > 0, \qquad b > 0$$

Such a system involving an open-loop pole in the right-half $s$ plane may be conditionally stable. Sketch the root loci when $a = b = 1$, $\zeta = 0.5$, and $\omega_n = 4$. Find the range of gain $K$ for stability.





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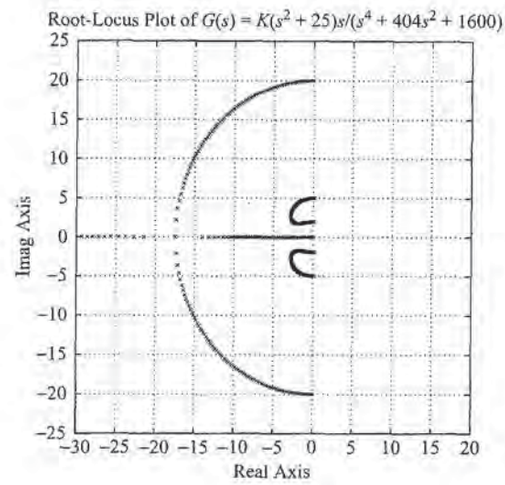


**Figure 6–57**
Root-locus plot.

**Solution.** The open-loop transfer function for the system is

$$G(s)H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

To sketch the root loci, we follow this procedure:

1. Locate the open-loop poles and zero in the complex plane. Root loci exist on the real axis between 1 and 0 and between −1 and −∞.
2. Determine the asymptotes of the root loci. There are three asymptotes whose angles can be determined as

$$\text{Angles of asymptotes} = \frac{180°(2k+1)}{4-1} = 60°, -60°, 180°$$

Referring to Equation (6–13), the abscissa of the intersection of the asymptotes and the real axis is

$$s = -\frac{(0 - 1 + 2 + j2\sqrt{3} + 2 - j2\sqrt{3}) - 1}{4-1} = -\frac{2}{3}$$

3. Determine the breakaway and break-in points. Since the characteristic equation is

$$1 + \frac{K(s+1)}{s(s-1)(s^2+4s+16)} = 0$$

we obtain

$$K = -\frac{s(s-1)(s^2+4s+16)}{s+1}$$

By differentiating $K$ with respect to $s$, we get

$$\frac{dK}{ds} = -\frac{3s^4 + 10s^3 + 21s^2 + 24s - 16}{(s+1)^2}$$





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The numerator can be factored as follows:

$$3s^4 + 10s^3 + 21s^2 + 24s - 16$$
$$= 3(s + 0.76 + j2.16)(s + 0.76 - j2.16)(s + 2.26)(s - 0.45)$$

Points $s = 0.45$ and $s = -2.26$ are on root loci on the real axis. Hence, these points are actual breakaway and break-in points, respectively. Points $s = -0.76 \pm j2.16$ do not satisfy the angle condition. Hence, they are neither breakaway nor break-in points.

**4.** Using Routh's stability criterion, determine the value of $K$ at which the root loci cross the imaginary axis. Since the characteristic equation is

$$s^4 + 3s^3 + 12s^2 + (K - 16)s + K = 0$$

the Routh array becomes

$$\begin{array}{cccc}
s^4 & 1 & 12 & K \\
s^3 & 3 & K - 16 & 0 \\
s^2 & \dfrac{52 - K}{3} & K & 0 \\
s^1 & \dfrac{-K^2 + 59K - 832}{52 - K} & 0 & \\
s^0 & K & &
\end{array}$$

The values of $K$ that make the $s^1$ term in the first column equal zero are $K = 35.7$ and $K = 23.3$.

The crossing points on the imaginary axis can be found by solving the auxiliary equation obtained from the $s^2$ row, that is, by solving the following equation for $s$:

$$\frac{52 - K}{3} s^2 + K = 0$$

The results are

$$s = \pm j2.56, \qquad \text{for } K = 35.7$$

$$s = \pm j1.56, \qquad \text{for } K = 23.3$$

The crossing points on the imaginary axis are thus $s = \pm j2.56$ and $s = \pm j1.56$.

**5.** Find the angles of departure of the root loci from the complex poles. For the open-loop pole at $s = -2 + j2\sqrt{3}$, the angle of departure $\theta$ is

$$\theta = 180° - 120° - 130.5° - 90° + 106°$$

or

$$\theta = -54.5°$$

(The angle of departure from the open-loop pole at $s = -2 - j2\sqrt{3}$ is 54.5°.)

**6.** Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin, and apply the angle condition. If the test point does not satisfy the angle condition, select another test point until it does. Continue the same process and locate a sufficient number of points that satisfy the angle condition.





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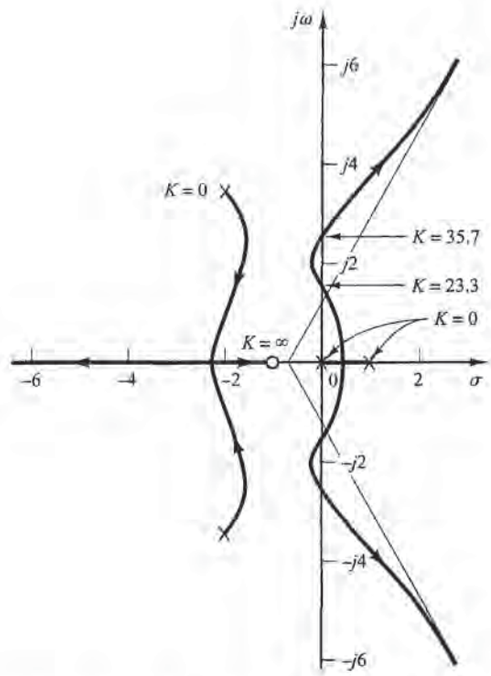


**Figure 6–58**
Root-locus plot.

Figure 6–58 shows the root loci for this system. From step 4, the system is stable for $23.3 < K < 35.7$. Otherwise, it is unstable. Thus, the system is conditionally stable.

**A–6–17.** Consider the system shown in Figure 6–59, where the dead time $T$ is 1 sec. Suppose that we approximate the dead time by the second-order pade approximation. The expression for this approximation can be obtained with MATLAB as follows:

```
[num,den] = pade(1, 2);
printsys(num, den)
num/den =
    s^2 - 6s + 12
    ---------------
    s^2 + 6s + 12
```

Hence

$$e^{-s} = \frac{s^2 - 6s + 12}{s^2 + 6s + 12} \tag{6–22}$$

Using this approximation, determine the critical value of $K$ (where $K > 0$) for stability.

**Solution.** Since the characteristic equation for the system is

$$s + 1 + Ke^{-s} = 0$$





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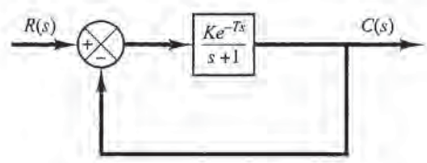


**Figure 6–59**
A control system with dead time.

by substituting Equation (6–22) into this characteristic equation, we obtain

$$s + 1 + K\frac{s^2 - 6s + 12}{s^2 + 6s + 12} = 0$$

or

$$s^3 + (7 + K)s^2 + (18 - 6K)s + 12(1 + K) = 0$$

Applying the Routh stability criterion, we get the Routh table as follows:

$$\begin{array}{ccc} s^3 & 1 & 18 - 6K \\ s^2 & 7 + K & 12(1 + K) \\ s^1 & \dfrac{-6K^2 - 36K + 114}{7 + K} & 0 \\ s^0 & 12(1 + K) & \end{array}$$

Hence, for stability we require

$$-6K^2 - 36K + 114 > 0$$

which can be written as

$$(K + 8.2915)(K - 2.2915) < 0$$

or

$$K < 2.2915$$

Since $K$ must be positive, the range of $K$ for stability is

$$0 < K < 2.2915$$

Notice that according to the present analysis, the upper limit of $K$ for stability is 2.2915. This value is greater than the exact upper limit of $K$. (Earlier, we obtained the exact upper limit of $K$ to be 2, as shown in Figure 6–38.) This is because we approximated $e^{-s}$ by the second-order pade approximation. A higher-order pade approximation will improve the accuracy. However, the computations involved increase considerably.

**A–6–18.** Consider the system shown in Figure 6–60. The plant involves the dead time of $T$ sec. Design a suitable controller $G_c(s)$ for the system.

**Solution.** We shall present the Smith predictor approach to design a controller. The first step to design the controller $G_c(s)$ is to design a suitable controller $\hat{G}_c(s)$ when the system has no dead time. Otto J. M. Smith designed an innovative controller scheme, now called the "Smith predictor,"





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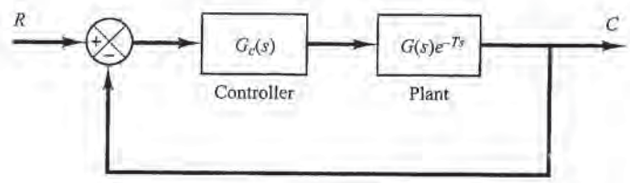


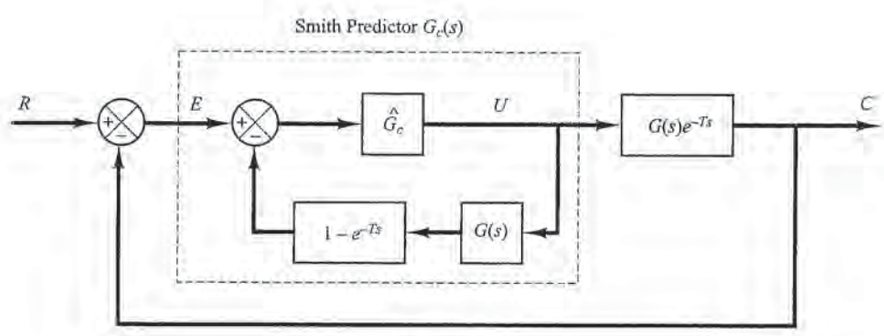

**Figure 6–60**
Control system with plant with dead time.

to control the plant with dead time. The Smith predictor consists of $G_c(s)$, dead time $e^{-Ts}$, and the plant transfer function $G(s)$. It has the form

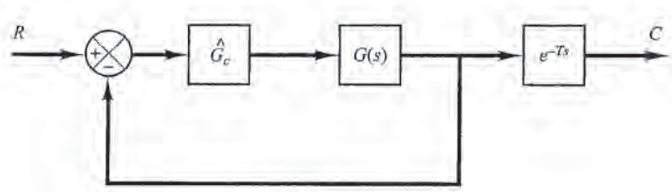
$$G_c(s) = \frac{\hat{G}_c(s)}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s)}$$

Figure 6-61(a) shows the Smith predictor as a minor loop in the block diagram. The transfer function between $U(s)$ and $E(s)$ is

$$\frac{U(s)}{E(s)} = \frac{\hat{G}_c(s)}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s)}$$




(a)




(b)

**Figure 6–61**
(a) Control system with Smith predictor; (b) equivalent block diagram for Smith predictor controlled system shown in (a).





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Then the closed-loop transfer function $C(s)/R(s)$ can be given by

$$\frac{C(s)}{R(s)} = \frac{\hat{G}_c(s)G(s)e^{-Ts}}{1 + (1 - e^{-Ts})\hat{G}_c(s)G(s) + \hat{G}_c(s)G(s)e^{-Ts}}$$

$$= \frac{\hat{G}_c(s)G(s)}{1 + \hat{G}_c(s)G(s)} e^{-Ts}$$

Hence, the block diagram of Figure 6–61(a) can be modified to that of Figure 6–61(b). The closed-loop response of the system with dead time $e^{-Ts}$ is the same as the response of the system without dead time $e^{-Ts}$, except that the response is delayed by $T$ sec.

Typical step-response curves of the system without dead time controlled by the controller $\hat{G}_c(s)$ and of the system with dead time controlled by the Smith predictor type controller are shown in Figure 6–62.

It is noted that implementing the Smith predictor in digital form is not difficult, because dead time can be handled easily in digital control. However, implementing the Smith predictor in an analog form creates some difficulty.

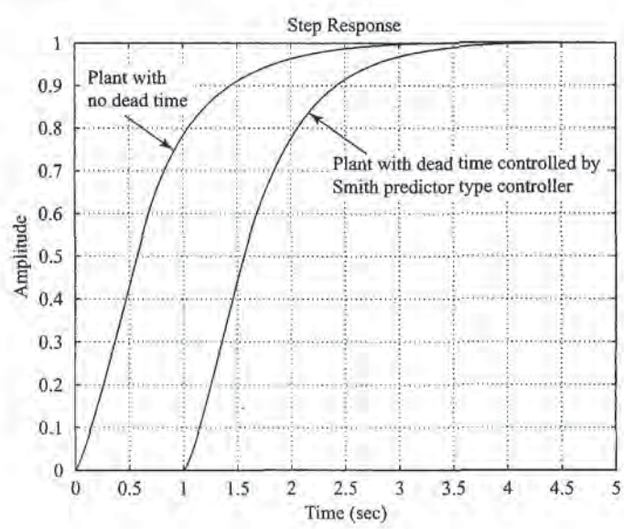


**Figure 6–62**
Step-response curves.

## PROBLEMS

**B–6–1.** Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K(s+1)}{s^2}, \qquad H(s) = 1$$

**B–6–2.** Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K(s+4)}{(s+1)^2}, \qquad H(s) = 1$$

**B–6–3.** Plot the root loci for the closed-loop control system with

$$G(s) = \frac{K}{s(s+1)(s^2+4s+5)}, \qquad H(s) = 1$$

**B–6–4.** Plot the root loci for the system with

$$G(s) = \frac{K}{s(s+0.5)(s^2+0.6s+10)}, \qquad H(s) = 1$$





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**B–6–5.** Plot the root loci for a system with

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s^2 + 2s + 5)}, \qquad H(s) = 1$$

Determine the exact points where the root loci cross the $j\omega$ axis.

**B–6–6.** Show that the root loci for a control system with

$$G(s) = \frac{K(s^2 + 6s + 10)}{s^2 + 2s + 10}, \qquad H(s) = 1$$

are arcs of the circle centered at the origin with radius equal to $\sqrt{10}$.

**B–6–7.** Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 0.2)}{s^2(s + 3.6)}, \qquad H(s) = 1$$

**B–6–8.** Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 0.5)}{s^3 + s^2 + 1}, \qquad H(s) = 1$$

**B–6–9.** Plot the root loci for a closed-loop control system with

$$G(s) = \frac{K(s + 9)}{s(s^2 + 4s + 11)}, \qquad H(s) = 1$$

Locate the closed-loop poles on the root loci such that the dominant closed-loop poles have a damping ratio equal to 0.5. Determine the corresponding value of gain $K$.

**B–6–10.** Plot the root loci for the system shown in Figure 6–63. Determine the range of gain $K$ for stability.

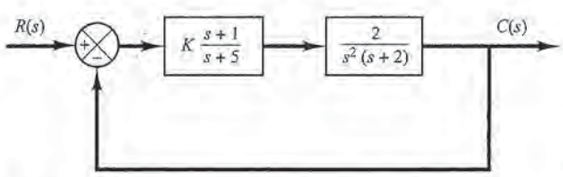


**Figure 6–63**
Control system.

**B–6–11.** Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 4s + 8)}$$

Plot the root loci for the system. If the value of gain $K$ is set equal to 2, where are the closed-loop poles located?

**B–6–12.** Consider the system whose open-loop transfer function $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + 1)}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

Plot a root-locus diagram with MATLAB.

**B–6–13.** Consider the system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K(s - 0.6667)}{s^4 + 3.3401s^3 + 7.0325s^2}$$

Show that the equation for the asymptotes is given by

$$G_a(s)H_a(s) = \frac{K}{s^3 + 4.0068s^2 + 5.3515s + 2.3825}$$

Using MATLAB, plot the root loci and asymptotes for the system.

**B–6–14.** Consider the unity-feedback system whose feedforward transfer function is

$$G(s) = \frac{K}{s(s + 1)}$$

The constant-gain locus for the system for a given value of $K$ is defined by the following equation:

$$\left|\frac{K}{s(s + 1)}\right| = 1$$

Show that the constant-gain loci for $0 \le K \le \infty$ may be given by

$$[\sigma(\sigma + 1) + \omega^2]^2 + \omega^2 = K^2$$

Sketch the constant-gain loci for $K = 1, 2, 5, 10$, and 20 on the $s$ plane.

**B–6–15.** Consider the system shown in Figure 6–64. Plot the root loci with MATLAB. Locate the closed-loop poles when the gain $K$ is set equal to 2.

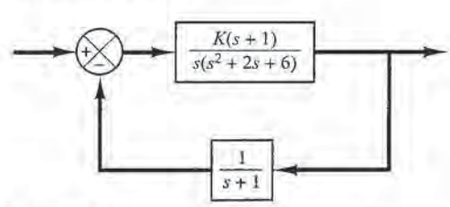


**Figure 6–64**
Control system.





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**B–6–16.** Plot root-locus diagrams for the nonminimum-phase systems shown in Figures 6–65(a) and (b), respectively.

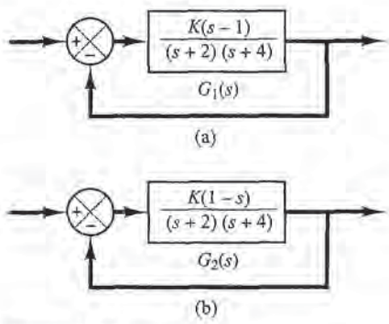


**Figure 6–65**
(a) and (b) Nonminimum-phase systems.

**B–6–17.** Consider the closed-loop system with transport lag shown in Figure 6–66. Determine the stability range for gain $K$.

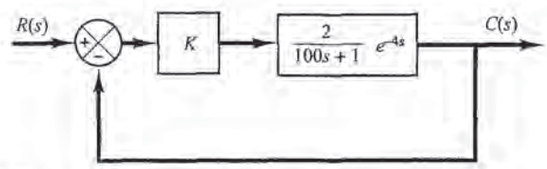


**Figure 6–66**
Control system.





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# 7

# Control Systems Design by the Root-Locus Method

## 7-1 INTRODUCTION

The primary objective of this chapter is to present procedures for the design and compensation of single-input–single-output linear time-invariant control systems. Compensation is the modification of the system dynamics to satisfy the given specifications. The approach to the control system design and compensation used in this chapter is the root-locus approach. (The frequency-response approach and the state-space approach to the control systems design and compensation will be presented in Chapters 9 and 12, respectively.) The PID-based computational approach to control systems design is given in Chapter 10.

**Performance Specifications.** Control systems are designed to perform specific tasks. The requirements imposed on the control system are usually spelled out as performance specifications. The specifications may be given in terms of transient response requirements (such as the maximum overshoot and settling time in step response) and of steady-state requirements (such as steady-state error in following ramp input). The specifications of a control system must be given before the design process begins.

For routine design problems, the performance specifications (which relate to accuracy, relative stability, and speed of response) may be given in terms of precise numerical values. In other cases they may be given partially in terms of precise numerical values and partially in terms of qualitative statements. In the latter case the specifications may have to be modified during the course of design, since the given specifications





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may never be satisfied (because of conflicting requirements) or may lead to a very expensive system.

Generally, the performance specifications should not be more stringent than necessary to perform the given task. If the accuracy at steady-state operation is of prime importance in a given control system, then we should not require unnecessarily rigid performance specifications on the transient response since such specifications will require expensive components. Remember that the most important part of control system design is to state the performance specifications precisely so that they will yield an optimal control system for the given purpose.

**Design by Root-Locus Method.** The design by the root-locus method is based on reshaping the root locus of the system by adding poles and zeros to the system's open-loop transfer function and forcing the root loci to pass through desired closed-loop poles in the $s$ plane. The characteristic of the root-locus design is its being based on the assumption that the closed-loop system has a pair of dominant closed-loop poles. (Zeros and additional poles affect the response characteristics.)

**System Compensation.** Setting the gain is the first step in adjusting the system for satisfactory performance. In many practical cases, however, the adjustment of the gain alone may not provide sufficient alteration of the system behavior to meet the given specifications. As is frequently the case, increasing the gain value will improve the steady-state behavior but will result in poor stability or even instability. It is then necessary to redesign the system (by modifying the structure or by incorporating additional devices or components) to alter the overall behavior so that the system will behave as desired. Such a redesign or addition of a suitable device is called *compensation*. A device inserted into the system for the purpose of satisfying the specifications is called a *compensator*. The compensator compensates for deficit performance of the original system.

Commonly used compensators and controllers are lead, lag, and lag–lead compensators and PID controllers. This chapter discusses the root-locus approach to the design of lead, lag, and lag–lead compensators. PID controllers will be discussed in Chapter 10.

**Series Compensation and Parallel (or Feedback) Compensation.** Figures 7–1(a) and (b) show compensation schemes commonly used for feedback control systems. Figure 7–1(a) shows the configuration where the compensator $G_c(s)$ is placed in series with the plant. This scheme is called *series compensation*.

An alternative to series compensation is to feed back the signal(s) from some element(s) and place a compensator in the resulting inner feedback path, as shown in Figure 7–1(b). Such compensation is called *parallel compensation* or *feedback compensation*.

In compensating control systems, we see that the problem usually boils down to a suitable design of a series or parallel compensator. The choice between series compensation and parallel compensation depends on the nature of the signals in the system, the power levels at various points, available components, the designer's experience, economic considerations, and so on.

In general, series compensation may be simpler than parallel compensation; however, series compensation frequently requires additional amplifiers to increase the gain and/or to provide isolation. (To avoid power dissipation, the series compensator is inserted at the lowest energy point in the feedforward path.) Note that, in general, the number of





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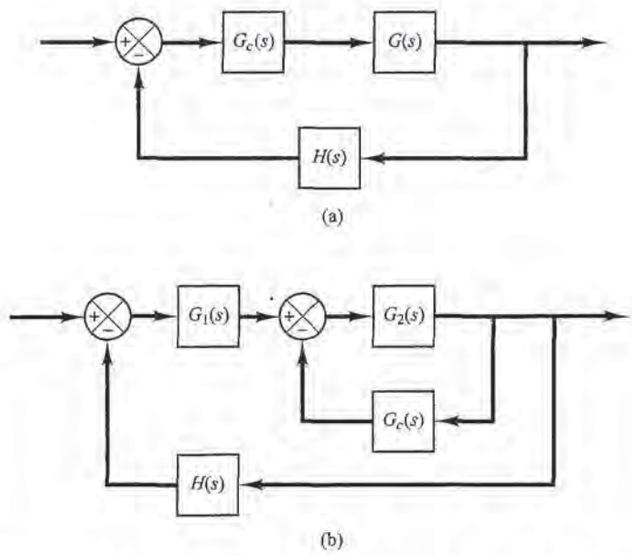


**Figure 7–1**
(a) Series compensation;
(b) parallel or feedback compensation.

components required in parallel compensation will be less than the number of components in series compensation, provided a suitable signal is available, because the energy transfer is from a higher power level to a lower level. (This means that additional amplifiers may not be necessary.)

In this chapter we discuss series compensation in detail. As for parallel compensation, we shall discuss the design of a velocity-feedback control system in Section 7–6.

**Compensators.** If a compensator is needed to meet the performance specifications, the designer must realize a physical device that has the prescribed transfer function of the compensator.

Numerous physical devices have been used for such purposes. In fact, many noble and useful ideas for physically constructing compensators may be found in the literature.

If a sinusoidal input is applied to the input of a network, and the steady-state output (which is also sinusoidal) has a phase lead, then the network is called a lead network. (The amount of phase lead angle is a function of the input frequency.) If the steady-state output has a phase lag, then the network is called a lag network. In a lag–lead network, both phase lag and phase lead occur in the output but in different frequency regions; phase lag occurs in the low-frequency region and phase lead occurs in the high-frequency region. A compensator having a characteristic of a lead network, lag network, or lag–lead network is called a lead compensator, lag compensator, or lag–lead compensator.

Among the many kinds of compensators, widely employed compensators are the lead compensators, lag compensators, lag–lead compensators, and velocity-feedback (tachometer) compensators. In this chapter we shall limit our discussions mostly to these types. Lead, lag, and lag–lead compensators may be electronic devices (such as circuits using operational amplifiers) or *RC* networks (electrical, mechanical, pneumatic, hydraulic, or combinations thereof) and amplifiers.

In the actual design of a control system, whether to use an electronic, pneumatic, or hydraulic compensator is a matter that must be decided partially based on the nature of





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the controlled plant. For example, if the controlled plant involves flammable fluid, then we have to choose pneumatic components (both a compensator and an actuator) to avoid the possibility of sparks. If, however, no fire hazard exists, then electronic compensators are most commonly used. (In fact, we often transform nonelectrical signals into electrical signals because of the simplicity of transmission, increased accuracy, increased reliability, ease of compensation, and the like.)

**Design Procedures.** In the trial-and-error approach to system design, we set up a mathematical model of the control system and adjust the parameters of a compensator. The most time-consuming part of such work is the checking of the system performance by analysis with each adjustment of the parameters. The designer should make use of a computer package such as MATLAB to avoid much of the numerical drudgery necessary for this checking.

Once a satisfactory mathematical model has been obtained, the designer must construct a prototype and test the open-loop system. If absolute stability of the closed loop is assured, the designer closes the loop and tests the performance of the resulting closed-loop system. Because of the neglected loading effects among the components, nonlinearities, distributed parameters, and so on, which were not taken into consideration in the original design work, the actual performance of the prototype system will probably differ from the theoretical predictions. Thus the first design may not satisfy all the requirements on performance. By trial and error, the designer must make changes in the prototype until the system meets the specifications. In doing this, he or she must analyze each trial, and the results of the analysis must be incorporated into the next trial. The designer must see that the final system meets the performance specifications and, at the same time, is reliable and economical.

It is noted that in designing control systems by the root-locus or frequency-response methods the final result is not unique, because the best or optimal solution may not be precisely defined if the time-domain specifications or frequency-domain specifications are given.

**Outline of the Chapter.** Section 7–1 has presented an introduction to the compensation of control systems. Section 7–2 discusses preliminary considerations for the root-locus approach to the control systems design. Section 7–3 treats details of the lead compensation techniques based on the root-locus method. Section 7–4 deals with the lag compensation techniques by the root-locus method. Section 7–5 presents lag–lead compensation techniques. Finally, Section 7–6 discusses the parallel compensation technique using an example.

## 7–2 PRELIMINARY DESIGN CONSIDERATIONS

In building a control system, we know that proper modification of the plant dynamics may be a simple way to meet the performance specifications. This, however, may not be possible in many practical situations because the plant may be fixed and may not be modified. Then we must adjust parameters other than those in the fixed plant. In this book, we assume that the plant is given and unalterable.

The design problems, therefore, become those of improving system performance by insertion of a compensator. Compensation of a control system is reduced to the design





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of a filter whose characteristics tend to compensate for the undesirable and unalterable characteristics of the plant. Our discussions are limited to continuous-time compensators.

In Sections 7–3 through 7–5, we shall specifically consider the design of lead compensators, lag compensators, and lag–lead compensators. In such design problems, we place a compensator in series with the unalterable transfer function $G(s)$ to obtain desirable behavior. The main problem then involves the judicious choice of the pole(s) and zero(s) of the compensator $G_c(s)$ to alter the root loci so that the performance specifications will be met. The parallel compensation technique is treated in Section 7–6.

**Root-Locus Approach to Control System Design.** The root-locus method is a graphical method for determining the locations of all closed-loop poles from knowledge of the locations of the open-loop poles and zeros as some parameter (usually the gain) is varied from zero to infinity. The method yields a clear indication of the effects of parameter adjustment.

In practice, the root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain. In fact, in some cases, the system may not be stable for all values of gain. Then it is necessary to reshape the root loci to meet the performance specifications.

In designing a control system, if other than a gain adjustment is required, we must modify the original root loci by inserting a suitable compensator. Once the effects on the root locus of the addition of poles and/or zeros are fully understood, we can readily determine the locations of the pole(s) and zero(s) of the compensator that will reshape the root locus as desired. In essence, in the design by the root-locus method, the root loci of the system are reshaped through the use of a compensator so that a pair of dominant closed-loop poles can be placed at the desired location. (Often, the damping ratio and undamped natural frequency of a pair of dominant closed-loop poles are specified.)

**Effects of the Addition of Poles.** The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response. (Remember that the addition of integral control adds a pole at the origin, thus making the system less stable.) Figure 7–2 shows examples of root loci illustrating the effects of the addition of a pole to a single-pole system and the addition of two poles to a single-pole system.

**Effects of the Addition of Zeros.** The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system

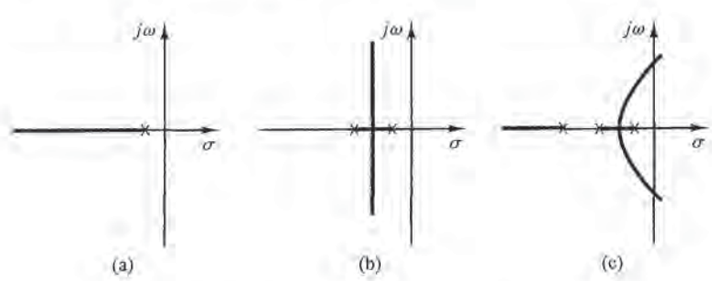


**Figure 7–2**
(a) Root-locus plot of a single-pole system;
(b) root-locus plot of a two-pole system;
(c) root-locus plot of a three-pole system.





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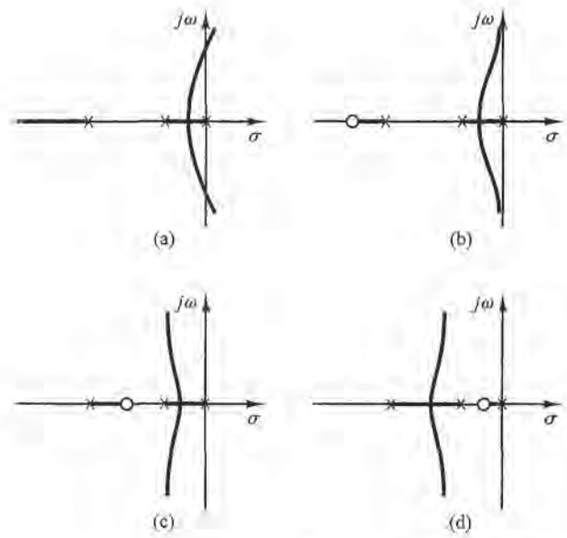


**Figure 7–3**
(a) Root-locus plot of a three-pole system; (b), (c), and (d) root-locus plots showing effects of addition of a zero to the three-pole system.

more stable and to speed up the settling of the response. (Physically, the addition of a zero in the feedforward transfer function means the addition of derivative control to the system. The effect of such control is to introduce a degree of anticipation into the system and speed up the transient response.) Figure 7–3(a) shows the root loci for a system that is stable for small gain but unstable for large gain. Figures 7–3(b), (c), and (d) show root-locus plots for the system when a zero is added to the open-loop transfer function. Notice that when a zero is added to the system of Figure 7–3(a), it becomes stable for all values of gain.

## 7–3 LEAD COMPENSATION

**Lead Compensators.** There are many ways to realize continuous-time (or analog) lead compensators, such as electronic networks using operational amplifiers, electrical *RC* networks, and mechanical spring-dashpot systems.

Figure 7–4 shows an electronic circuit using operational amplifiers. The transfer function for this circuit was obtained in Chapter 3 as follows:

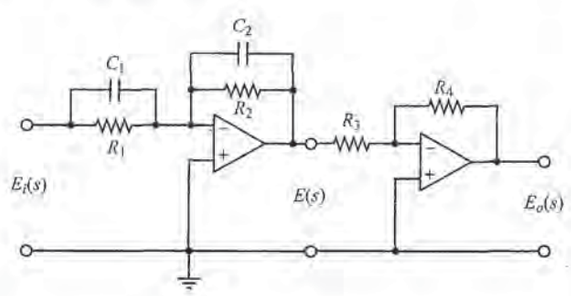


**Figure 7–4**
Electronic circuit that is a lead network if $R_1C_1 > R_2C_2$ and a lag network if $R_1C_1 < R_2C_2$.





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$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\frac{R_1C_1s+1}{R_2C_2s+1} = \frac{R_4C_1}{R_3C_2}\frac{s+\dfrac{1}{R_1C_1}}{s+\dfrac{1}{R_2C_2}}$$

$$= K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\alpha T}} \tag{7-1}$$

where

$$T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \frac{R_4C_1}{R_3C_2}$$

Notice that

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}, \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

This network has a dc gain of $K_c\alpha = R_2R_4/(R_1R_3)$.

From Equation (7–1), we see that this network is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$. The pole-zero configurations of this network when $R_1C_1 > R_2C_2$ and $R_1C_1 < R_2C_2$ are shown in Figure 7–5(a) and (b), respectively.

**Lead Compensation Techniques Based on the Root-Locus Approach.** The root-locus approach to design is very powerful when the specifications are given in terms of time-domain quantities, such as the damping ratio and undamped natural frequency of the desired dominant closed-loop poles, maximum overshoot, rise time, and settling time.

Consider a design problem in which the original system either is unstable for all values of gain or is stable but has undesirable transient-response characteristics. In such a case, the reshaping of the root locus is necessary in the broad neighborhood of the $j\omega$ axis and the origin in order that the dominant closed-loop poles be at desired locations in the complex plane. This problem may be solved by inserting an appropriate lead compensator in cascade with the feedforward transfer function.

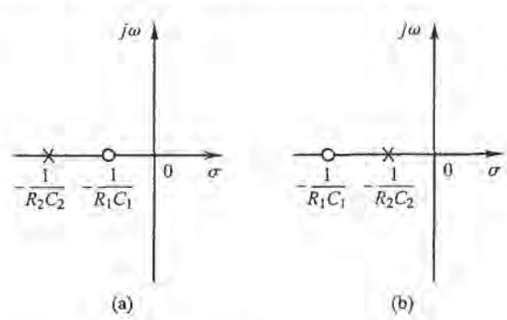


**Figure 7–5** Pole-zero configurations: (a) lead network; (b) lag network.





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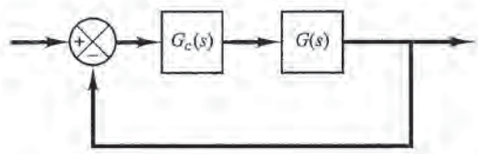


**Figure 7–6**
Control system.

The procedures for designing a lead compensator for the system shown in Figure 7–6 by the root-locus method may be stated as follows:

1. From the performance specifications, determine the desired location for the dominant closed-loop poles.
2. By drawing the root-locus plot of the uncompensated system (original system), ascertain whether or not the gain adjustment alone can yield the desired closed-loop poles. If not, calculate the angle deficiency $\phi$. This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.
3. Assume the lead compensator $G_c(s)$ to be

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\alpha T}}, \qquad (0 < \alpha < 1)$$

   where $\alpha$ and $T$ are determined from the angle deficiency. $K_c$ is determined from the requirement of the open-loop gain.
4. If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle $\phi$. If no other requirements are imposed on the system, try to make the value of $\alpha$ as large as possible. A larger value of $\alpha$ generally results in a larger value of $K_v$, which is desirable. (If a particular static error constant is specified, it is generally simpler to use the frequency-response approach.)
5. Determine the open-loop gain of the compensated system from the magnitude condition.

Once a compensator has been designed, check to see whether all performance specifications have been met. If the compensated system does not meet the performance specifications, then repeat the design procedure by adjusting the compensator pole and zero until all such specifications are met. If a large static error constant is required, cascade a lag network or alter the lead compensator to a lag–lead compensator.

Note that if the selected dominant closed-loop poles are not really dominant, it will be necessary to modify the location of the pair of such selected dominant closed-loop poles. (The closed-loop poles other than dominant ones modify the response obtained from the dominant closed-loop poles alone. The amount of modification depends on the location of these remaining closed-loop poles.) Also, the closed-loop zeros affect the response if they are located near the origin.





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**EXAMPLE 7–1** Consider the system shown in Figure 7–7(a). The feedforward transfer function is

$$G(s) = \frac{4}{s(s+2)}$$

The root-locus plot for this system is shown in Figure 7–7(b). The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

$$= \frac{4}{(s + 1 + j\sqrt{3})(s + 1 - j\sqrt{3})}$$

The closed-loop poles are located at

$$s = -1 \pm j\sqrt{3}$$

The damping ratio of the closed-loop poles is 0.5. The undamped natural frequency of the closed-loop poles is 2 rad/sec. The static velocity error constant is 2 $\sec^{-1}$.

It is desired to modify the closed-loop poles so that an undamped natural frequency $\omega_n = 4$ rad/sec is obtained, without changing the value of the damping ratio, $\zeta = 0.5$.

The damping ratio of 0.5 requires that the complex-conjugate poles lie on the lines drawn through the origin making angles of $\pm 60°$ with the negative real axis.

Since the damping ratio determines the angular location of the complex-conjugate closed-loop poles, while the distance of the pole from the origin is determined by the undamped natural frequency $\omega_n$, the desired locations of the closed-loop poles of this example problem are

$$s = -2 \pm j2\sqrt{3}$$

In some cases, after the root loci of the original system have been obtained, the dominant closed-loop poles may be moved to the desired location by simple gain adjustment. This is, however, not the case for the present system. Therefore, we shall insert a lead compensator in the feedforward path.

A general procedure for determining the lead compensator is as follows: First, find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and determine the necessary angle $\phi$ to be added so that the total sum of the angles is equal to $\pm 180°(2k + 1)$. The lead compensator must contribute this angle $\phi$. (If the angle $\phi$ is quite large, then two or more lead networks may be needed rather than a single one.)

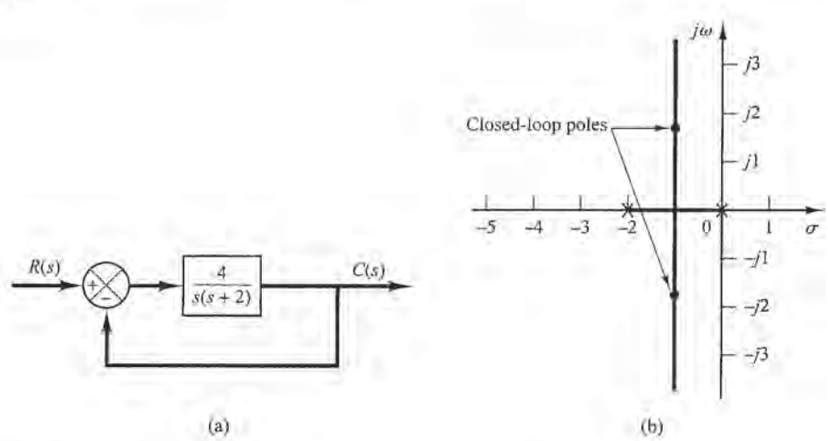


**Figure 7–7**
(a) Control system;
(b) root-locus plot.





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If the original system has the open-loop transfer function $G(s)$, then the compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \left( K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \right) G(s)$$

where

$$G_c(s) = K_c\alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}, \qquad (0 < \alpha < 1)$$

Notice that there are many possible values for $T$ and $\alpha$ that will yield the necessary angle contribution at the desired closed-loop poles.

The next step is to determine the locations of the zero and pole of the lead compensator. There are many possibilities for the choice of such locations. (See the comments at the end of this example problem.) In what follows, we shall introduce a procedure to obtain the largest possible value for $\alpha$. (Note that a larger value of $\alpha$ will produce a larger value of $K_v$. In most cases, the larger the $K_v$ is, the better the system performance.) First, draw a horizontal line passing through point $P$, the desired location for one of the dominant closed-loop poles. This is shown as line $PA$ in Figure 7–8. Draw also a line connecting point $P$ and the origin. Bisect the angle between the lines $PA$ and $PO$, as shown in Figure 7–8. Draw two lines $PC$ and $PD$ that make angles $\pm\phi/2$ with the bisector $PB$. The intersections of $PC$ and $PD$ with the negative real axis give the necessary locations for the pole and zero of the lead network. The compensator thus designed will make point $P$ a point on the root locus of the compensated system. The open-loop gain is determined by use of the magnitude condition.

In the present system, the angle of $G(s)$ at the desired closed-loop pole is

$$\left. \angle \frac{4}{s(s+2)} \right|_{s=-2+j2\sqrt{3}} = -210^\circ$$

Thus, if we need to force the root locus to go through the desired closed-loop pole, the lead compensator must contribute $\phi = 30^\circ$ at this point. By following the foregoing design procedure, we determine the zero and pole of the lead compensator, as shown in Figure 7–9, to be

$$\text{Zero at } s = -2.9, \qquad \text{Pole at } s = -5.4$$

or

$$T = \frac{1}{2.9} = 0.345, \qquad \alpha T = \frac{1}{5.4} = 0.185$$

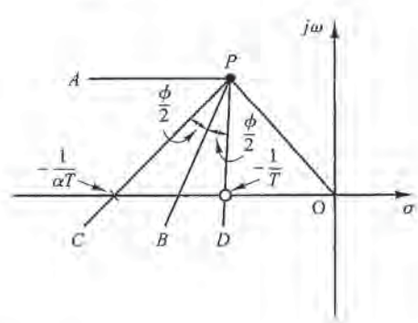


**Figure 7–8**
Determination of the pole and zero of a lead network.





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Thus $\alpha = 0.537$. The open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = K_c \frac{s + 2.9}{s + 5.4} \frac{4}{s(s + 2)} = \frac{K(s + 2.9)}{s(s + 2)(s + 5.4)}$$

where $K = 4K_c$. The root-locus plot for the compensated system is shown in Figure 7–9. The gain $K$ is evaluated from the magnitude condition as follows: Referring to the root-locus plot for the compensated system shown in Figure 7–9, the gain $K$ is evaluated from the magnitude condition as

$$\left| \frac{K(s + 2.9)}{s(s + 2)(s + 5.4)} \right|_{s=-2+j2\sqrt{3}} = 1$$

or

$$K = 18.7$$

It follows that

$$G_c(s)G(s) = \frac{18.7(s + 2.9)}{s(s + 2)(s + 5.4)}$$

The constant $K_c$ of the lead compensator is

$$K_c = \frac{18.7}{4} = 4.68$$

Hence, $K_c\alpha = 2.51$. The lead compensator, therefore, has the transfer function

$$G_s(s) = 2.51 \frac{0.345s + 1}{0.185s + 1} = 4.68 \frac{s + 2.9}{s + 5.4}$$

If the electronic circuit using operational amplifiers as shown in Figure 7–4 is used as the lead compensator just designed, then the parameter values of the lead compensator are determined from

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3} \frac{R_1C_1s + 1}{R_2C_2s + 1} = 2.51 \frac{0.345s + 1}{0.185s + 1}$$

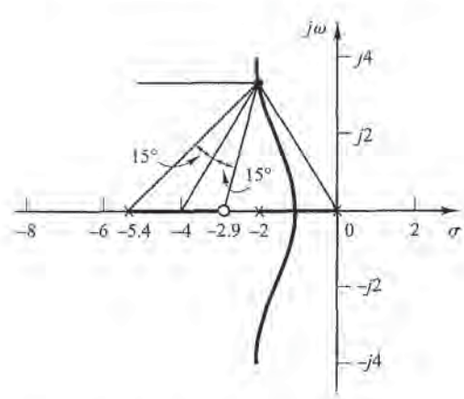


**Figure 7–9**
Root-locus plot of the compensated system.





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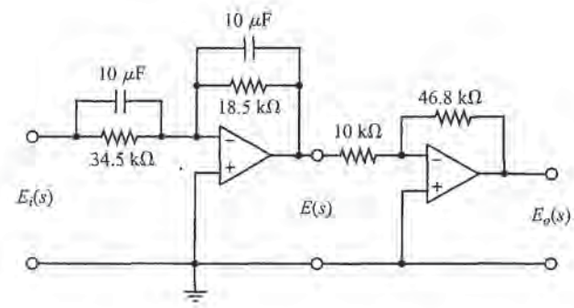


**Figure 7–10**
Lead compensator.

as shown in Figure 7–10, where we have arbitrarily chosen $C_1 = C_2 = 10\ \mu\text{F}$ and $R_3 = 10\ \text{k}\Omega$.

The static velocity error constant $K_v$ is obtained from the expression

$$K_v = \lim_{s \to 0} sG_c(s)G(s)$$

$$= \lim_{s \to 0} \frac{s18.7(s + 2.9)}{s(s + 2)(s + 5.4)}$$

$$= 5.02\ \text{sec}^{-1}$$

Note that the third closed-loop pole of the designed system is found by dividing the characteristic equation by the known factors as follows:

$$s(s + 2)(s + 5.4) + 18.7(s + 2.9) = (s + 2 + j2\sqrt{3})(s + 2 - j2\sqrt{3})(s + 3.4)$$

The foregoing compensation method enables us to place the dominant closed-loop poles at the desired points in the complex plane. The third pole at $s = -3.4$ is close to the added zero at $s = -2.9$. Therefore, the effect of this pole on the transient response is relatively small. Since no restriction has been imposed on the nondominant pole and no specification has been given concerning the value of the static velocity error coefficient, we conclude that the present design is satisfactory.

**Comments.** We may place the zero of the compensator at $s = -2$ and pole at $s = -4$ so that the angle contribution of the lead compensator is 30°. (In this case the zero of the lead compensator will cancel a pole of the plant, resulting in the second-order system, rather than the third-order system as we designed.) It can be seen that the $K_v$ value in this case is 4 $\text{sec}^{-1}$. Other combinations can be selected that will yield 30° phase lead. (For different combinations of a zero and pole of the compensator that contribute 30°, the value of $\alpha$ will be different and the value of $K_v$ will also be different.) Although a certain change in the value of $K_v$ can be made by altering the pole-zero location of the lead compensator, if a large increase in the value of $K_v$ is desired, then we must alter the lead compensator to a lag–lead compensator. (See Section 7–5 for lag–lead compensation.)

**Comparison of step responses of the compensated and uncompensated systems.** In what follows we shall examine the unit-step responses of the compensated and uncompensated systems with MATLAB.





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The closed-loop transfer function of the compensated system is

$$\frac{C(s)}{R(s)} = \frac{18.7(s + 2.9)}{s(s + 2)(s + 5.4) + 18.7(s + 2.9)}$$

$$= \frac{18.7s + 54.23}{s^3 + 7.4s^2 + 29.5s + 54.23}$$

Hence,

numc = [0 0 18.7 54.23]
denc = [1 7.4 29.5 54.23]

For the uncompensated system, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

Hence,

num = [0 0 4]
den = [1 2 4]

MATLAB Program 7–1 produces the unit-step response curves for the two systems. The resulting plot is shown in Figure 7–11. Notice that the compensated system exhibits slightly larger maximum overshoot. The settling time of the compensated system is one-half that of the original system, as expected.

**MATLAB Program 7–1**

```
% ----------Unit-step response ----------
% ***** Unit-step responses of compensated and uncompensated
% systems *****
numc = [0 0 18.7 54.23];
denc = [1 7.4 29.5 54.23];
num = [0 0 4];
den = [1 2 4];
t = 0:0.05:5;
[c1,x1,t] = step(numc,denc,t);
[c2,x2,t] = step(num,den,t);
plot(t,c1,t,c1,'o',t,c2,t,c2,'x')
grid
title('Unit-Step Responses of Compensated and Uncompensated Systems')
xlabel('t Sec')
ylabel('Outputs c1 and c2')
text(0.6,1.32,'Compensated system')
text(1.3,0.68,'Uncompensated system')
```





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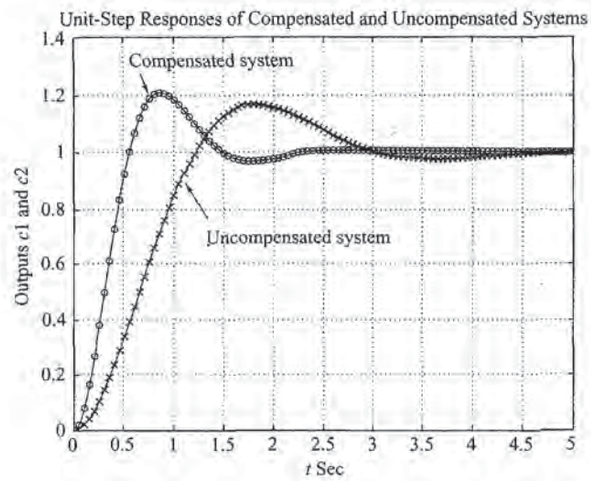


**Figure 7–11**
Unit-step responses of compensated and uncompensated systems.

## 7–4 LAG COMPENSATION

**Electronic Lag Compensator Using Operational Amplifiers.** The configuration of the electronic lag compensator using operational amplifiers is the same as that for the lead compensator shown in Figure 7–4. If we choose $R_2C_2 > R_1C_1$ in the circuit shown in Figure 7–4, it becomes a lag compensator. Referring to Figure 7–4, the transfer function of the lag compensator is given by

$$\frac{E_o(s)}{E_i(s)} = \hat{K}_c\beta\frac{Ts+1}{\beta Ts+1} = \hat{K}_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}}$$

where

$$T = R_1C_1, \qquad \beta T = R_2C_2, \qquad \beta = \frac{R_2C_2}{R_1C_1} > 1, \qquad \hat{K}_c = \frac{R_4C_1}{R_3C_2}$$

Note that we use $\beta$ instead of $\alpha$ in the above expressions. [In the lead compensator we used $\alpha$ to indicate the ratio $R_2C_2/(R_1C_1)$, which was less than 1, or $0 < \alpha < 1$.] In this chapter we always assume that $0 < \alpha < 1$ and $\beta > 1$.

**Lag Compensation Techniques Based on the Root-Locus Approach.** Consider the problem of finding a suitable compensation network for the case where the system exhibits satisfactory transient-response characteristics but unsatisfactory steady-state characteristics. Compensation in this case essentially consists of increasing the open-loop gain without appreciably changing the transient-response characteristics. This means





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that the root locus in the neighborhood of the dominant closed-loop poles should not be changed appreciably, but the open-loop gain should be increased as much as needed. This can be accomplished if a lag compensator is put in cascade with the given feedforward transfer function.

To avoid an appreciable change in the root loci, the angle contribution of the lag network should be limited to a small amount, say 5°. To assure this, we place the pole and zero of the lag network relatively close together and near the origin of the $s$ plane. Then the closed-loop poles of the compensated system will be shifted only slightly from their original locations. Hence, the transient-response characteristics will be changed only slightly.

Consider a lag compensator $G_c(s)$, where

$$G_c(s) = \hat{K}_c\beta\frac{Ts+1}{\beta Ts+1} = \hat{K}_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}} \tag{7-2}$$

If we place the zero and pole of the lag compensator very close to each other, then at $s = s_1$, where $s_1$ is one of the dominant closed-loop poles, the magnitudes $s_1 + (1/T)$ and $s_1 + [1/(\beta T)]$ are almost equal, or

$$|G_c(s_1)| = \left|\hat{K}_c\frac{s_1+\dfrac{1}{T}}{s_1+\dfrac{1}{\beta T}}\right| \doteq \hat{K}_c$$

To make the angle contribution of the lag portion of the compensator to be small, we require

$$-5^\circ < \left\angle\frac{s_1+\dfrac{1}{T}}{s_1+\dfrac{1}{\beta T}}\right. < 0^\circ$$

This implies that if gain $\hat{K}_c$ of the lag compensator is set equal to 1, then the transient-response characteristics will not be altered. (This means that the overall gain of the open-loop transfer function can be increased by a factor of $\beta$ where $\beta > 1$.) If the pole and zero are placed very close to the origin, then the value of $\beta$ can be made large. (A large value of $\beta$ may be used, provided physical realization of the lag compensator is possible.) It is noted that the value of $T$ must be large, but its exact value is not critical. However, it should not be too large in order to avoid difficulties in realizing the phase lag compensator by physical components.

An increase in the gain means an increase in the static error constants. If the open-loop transfer function of the uncompensated system is $G(s)$, then the static velocity error constant $K_v$ of the uncompensated system is

$$K_v = \lim_{s\to 0} sG(s)$$





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If the compensator is chosen as given by Equation (7–2), then for the compensated system with the open-loop transfer function $G_c(s)G(s)$ the static velocity error constant $\hat{K}_v$ becomes

$$\begin{aligned}\hat{K}_v &= \lim_{s\to 0} sG_c(s)G(s)\\ &= \lim_{s\to 0} G_c(s)K_v\\ &= \hat{K}_c\beta K_v\end{aligned}$$

Thus if the compensator is given by Equation (7–2), then the static velocity error constant is increased by a factor of $\hat{K}_c\,\beta$, where $\hat{K}_c$ is approximately unity.

The main negative effect of the lag compensation is that the compensator zero that will be generated near the origin creates a closed-loop pole near the origin. This closed-loop pole and compensator zero will generate a long tail of small amplitude in the step response, thus increasing the settling time.

**Design Procedures for Lag Compensation by the Root-Locus Method.** The procedure for designing lag compensators for the system shown in Figure 7–12 by the root-locus method may be stated as follows (we assume that the uncompensated system meets the transient-response specifications by simple gain adjustment; if this is not the case, refer to Section 7–5):

1. Draw the root-locus plot for the uncompensated system whose open-loop transfer function is $G(s)$. Based on the transient-response specifications, locate the dominant closed-loop poles on the root locus.
2. Assume the transfer function of the lag compensator to be

$$G_c(s) = \hat{K}_c\beta\frac{Ts+1}{\beta Ts+1} = \hat{K}_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}}$$

   Then the open-loop transfer function of the compensated system becomes $G_c(s)G(s)$.
3. Evaluate the particular static error constant specified in the problem.
4. Determine the amount of increase in the static error constant necessary to satisfy the specifications.
5. Determine the pole and zero of the lag compensator that produce the necessary increase in the particular static error constant without appreciably altering the

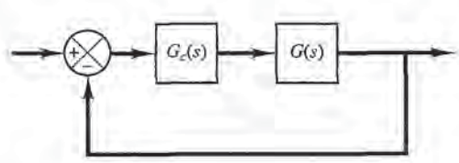


**Figure 7–12**
Control system.





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original root loci. (Note that the ratio of the value of gain required in the specifications and the gain found in the uncompensated system is the required ratio between the distance of the zero from the origin and that of the pole from the origin.)

6. Draw a new root-locus plot for the compensated system. Locate the desired dominant closed-loop poles on the root locus. (If the angle contribution of the lag network is very small, that is, a few degrees, then the original and new root loci are almost identical. Otherwise, there will be a slight discrepancy between them. Then locate, on the new root locus, the desired dominant closed-loop poles based on the transient-response specifications.)
7. Adjust gain $\hat{K}_c$ of the compensator from the magnitude condition so that the dominant closed-loop poles lie at the desired location. ($\hat{K}_c$ will be approximately 1.)

**EXAMPLE 7–2** Consider the system shown in Figure 7–13(a). The feedforward transfer function is

$$G(s) = \frac{1.06}{s(s+1)(s+2)}$$

The root-locus plot for the system is shown in Figure 7–13(b). The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

$$= \frac{1.06}{(s+0.3307-j0.5864)(s+0.3307+j0.5864)(s+2.3386)}$$

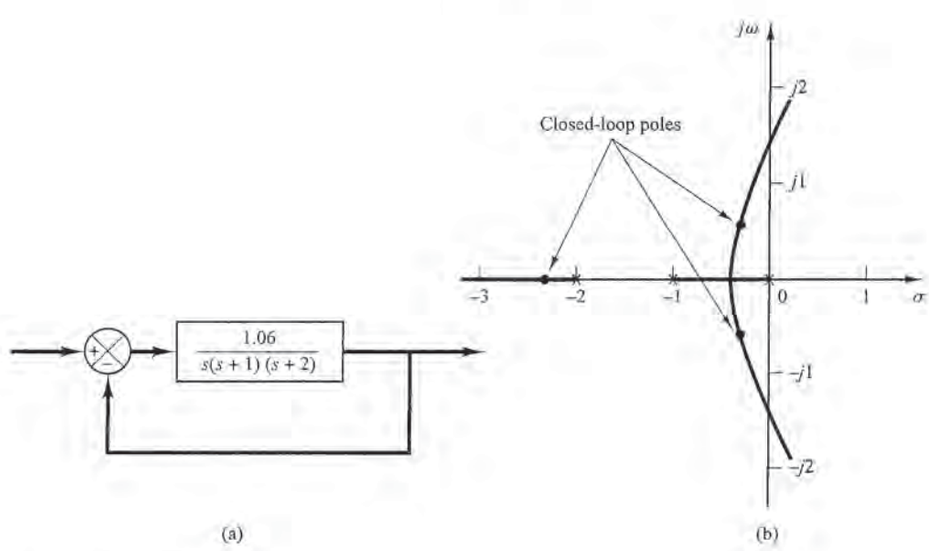


**Figure 7–13**
(a) Control system;
(b) root-locus plot.





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The dominant closed-loop poles are

$$s = -0.3307 \pm j0.5864$$

The damping ratio of the dominant closed-loop poles is $\zeta = 0.491$. The undamped natural frequency of the dominant closed-loop poles is 0.673 rad/sec. The static velocity error constant is 0.53 $\text{sec}^{-1}$.

It is desired to increase the static velocity error constant $K_v$ to about 5 $\text{sec}^{-1}$ without appreciably changing the location of the dominant closed-loop poles.

To meet this specification, let us insert a lag compensator as given by Equation (7–2) in cascade with the given feedforward transfer function. To increase the static velocity error constant by a factor of about 10, let us choose $\beta = 10$ and place the zero and pole of the lag compensator at $s = -0.05$ and $s = -0.005$, respectively. The transfer function of the lag compensator becomes

$$G_c(s) = \hat{K}_c \frac{s + 0.05}{s + 0.005}$$

The angle contribution of this lag network near a dominant closed-loop pole is about 4°. Because this angle contribution is not very small, there is a small change in the new root locus near the desired dominant closed-loop poles.

The open-loop transfer function of the compensated system then becomes

$$G_c(s)G(s) = \hat{K}_c \frac{s + 0.05}{s + 0.005} \frac{1.06}{s(s + 1)(s + 2)}$$

$$= \frac{K(s + 0.05)}{s(s + 0.005)(s + 1)(s + 2)}$$

where

$$K = 1.06\hat{K}_c$$

The block diagram of the compensated system is shown in Figure 7–14. The root-locus plot for the compensated system near the dominant closed-loop poles is shown in Figure 7–15(a), together with the original root-locus plot. Figure 7–15(b) shows the root-locus plot of the compensated system near the origin. The MATLAB program to generate the root-locus plots shown in Figures 7–15(a) and (b) is given in MATLAB Program 7–2.

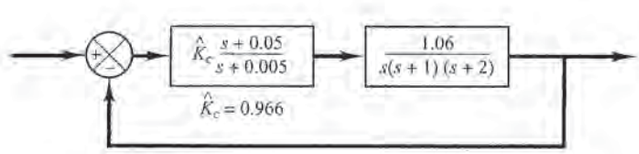


**Figure 7–14** Compensated system.





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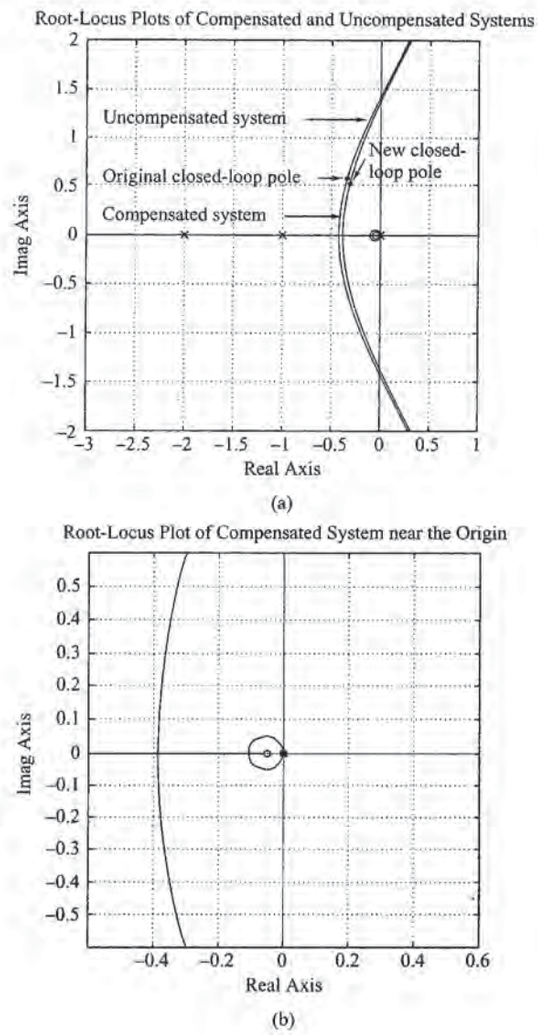


**Figure 7–15**
(a) Root-locus plots of the compensated system and uncompensated system; (b) root-locus plot of compensated system near the origin.

If the damping ratio of the new dominant closed-loop poles is kept the same, then the poles are obtained from the new root-locus plot as follows:

$$s_1 = -0.31 + j0.55, \qquad s_2 = -0.31 - j0.55$$

The open-loop gain $K$ is

$$K = \left| \frac{s(s + 0.005)(s + 1)(s + 2)}{s + 0.05} \right|_{s=-0.31+j0.55}$$
$$= 1.0235$$





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**MATLAB Program 7–2**

```
% --------- Root Locus Plot ---------

% ***** Root locus plots of the compensated system and
% uncompensated system *****

% ***** Enter the numerators and denominators of the
% compensated and uncompensated systems *****

numc = [0  0  0  1  0.05];
denc = [1  3.005  2.015  0.01  0];
num = [0  0  0  1.06];
den = [1  3  2  0];

% ***** Enter rlocus command. Plot the root loci of both
% systems *****

rlocus(numc,denc)
hold
Current plot held
rlocus(num,den)
v = [-3  1  -2  2]; axis(v); axis('square')
grid
text(-2.8,0.2,'Compensated system')
text(-2.8,1.2,'Uncompensated system')
text(-2.8,0.58,'Original closed-loop pole')
text(-0.1,0.85,'New closed-')
text(-0.1,0.62,'loop pole')
title('Root-Locus Plots of Compensated and Uncompensated Systems')

hold
Current plot released

% ***** Plot root loci of the compensated system near the origin *****

rlocus(numc,denc)
v = [-0.6  0.6  -0.6  0.6]; axis(v); axis('square')
grid
title('Root-Locus Plot of Compensated System near the Origin')
```

Then the lag compensator gain $\hat{K}_c$ is determined as

$$\hat{K}_c = \frac{K}{1.06} = \frac{1.0235}{1.06} = 0.9656$$

Thus the transfer function of the lag compensator designed is

$$G_c(s) = 0.9656\frac{s + 0.05}{s + 0.005} = 9.656\frac{20s + 1}{200s + 1} \tag{7–3}$$





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Then the compensated system has the following open-loop transfer function:

$$G_1(s) = \frac{1.0235(s + 0.05)}{s(s + 0.005)(s + 1)(s + 2)}$$

$$= \frac{5.12(20s + 1)}{s(200s + 1)(s + 1)(0.5s + 1)}$$

The static velocity error constant $K_v$ is

$$K_v = \lim_{s \to 0} sG_1(s) = 5.12 \text{ sec}^{-1}$$

In the compensated system, the static velocity error constant has increased to 5.12 $\text{sec}^{-1}$, or $5.12/0.53 = 9.66$ times the original value. (The steady-state error with ramp inputs has decreased to about 10% of that of the original system.) We have essentially accomplished the design objective of increasing the static velocity error constant to about 5 $\text{sec}^{-1}$.

Note that, since the pole and zero of the lag compensator are placed close together and are located very near the origin, their effect on the shape of the original root loci has been small. Except for the presence of a small closed root locus near the origin, the root loci of the compensated and the uncompensated systems are very similar to each other. However, the value of the static velocity error constant of the compensated system is 9.66 times greater than that of the uncompensated system.

The two other closed-loop poles for the compensated system are found as follows:

$$s_3 = -2.326, \qquad s_4 = -0.0549$$

The addition of the lag compensator increases the order of the system from 3 to 4, adding one additional closed-loop pole close to the zero of the lag compensator. (The added closed-loop pole at $s = -0.0549$ is close to the zero at $s = -0.05$.) Such a pair of a zero and pole creates a long tail of small amplitude in the transient response, as we will see later in the unit-step response. Since the pole at $s = -2.326$ is very far from the $j\omega$ axis compared with the dominant closed-loop poles, the effect of this pole on the transient response is also small. Therefore, we may consider the closed-loop poles at $s = -0.31 \pm j0.55$ to be the dominant closed-loop poles.

The undamped natural frequency of the dominant closed-loop poles of the compensated system is 0.631 rad/sec. This value is about 6% less than the original value, 0.673 rad/sec. This implies that the transient response of the compensated system is slower than that of the original system. The response will take a longer time to settle down. The maximum overshoot in the step response will increase in the compensated system. If such adverse effects can be tolerated, the lag compensation as discussed here presents a satisfactory solution to the given design problem.

Next, we shall compare the unit-ramp responses of the compensated system against the uncompensated system and verify that the steady-state performance is much better in the compensated system than the uncompensated system.

To obtain the unit-ramp response with MATLAB, we use the step command for the system $C(s)/[sR(s)]$. Since $C(s)/[sR(s)]$ for the compensated system is

$$\frac{C(s)}{sR(s)} = \frac{1.0235(s + 0.05)}{s[s(s + 0.005)(s + 1)(s + 2) + 1.0235(s + 0.05)]}$$

$$= \frac{1.0235s + 0.0512}{s^5 + 3.005s^4 + 2.015s^3 + 1.0335s^2 + 0.0512s}$$





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we have

$$\text{numc} = [0\ \ 0\ \ 0\ \ 0\ \ 1.0235\ \ 0.0512]$$
$$\text{denc} = [1\ \ 3.005\ \ 2.015\ \ 1.0335\ \ 0.0512\ \ 0]$$

Also, $C(s)/[sR(s)]$ for the uncompensated system is

$$\frac{C(s)}{sR(s)} = \frac{1.06}{s[s(s+1)(s+2)+1.06]}$$
$$= \frac{1.06}{s^4 + 3s^3 + 2s^2 + 1.06s}$$

Hence,

$$\text{num} = [0\ \ 0\ \ 0\ \ 0\ \ 1.06]$$
$$\text{den} = [1\ \ 3\ \ 2\ \ 1.06\ \ 0]$$

MATLAB Program 7–3 produces the plot of the unit-ramp response curves. Figure 7–16 shows the result. Clearly, the compensated system shows much smaller steady-state error (one-tenth of the original steady-state error) in following the unit-ramp input.

**MATLAB Program 7–3**

```
% --------- Unit ramp response ----------

% ***** Unit-ramp responses of compensated system and
% uncompensated system *****

% ***** Unit-ramp response will be obtained as the unit-step
% response of C(s)/[sR(s)] *****
% ***** Enter the numerators and denominators of C1(s)/[sR(s)]
% and C2(s)/[sR(s)], where C1(s) and C2(s) are Laplace
% transforms of the outputs of the compensated and un-
% compensated systems, respectively. *****

numc = [0 0 0 0 1.0235 0.0512];
denc = [1 3.005 2.015 1.0335 0.0512 0];
num = [0 0 0 0 1.06];
den = [1 3 2 1.06 0];

% ***** Specify the time range (such as t= 0:0.1:50) and enter
% step command and plot command. *****

t = 0:0.1:50;
[c1,x1,t] = step(numc,denc,t);
[c2,x2,t] = step(num,den,t);
plot(t,c1,'-',t,c2,'.',t,t,'--')
grid
text(2.2,27,'Compensated system');
text(26,21.3,'Uncompensated system');
title('Unit-Ramp Responses of Compensated and Uncompensated Systems')
xlabel('t Sec');
ylabel('Outputs c1 and c2')
```





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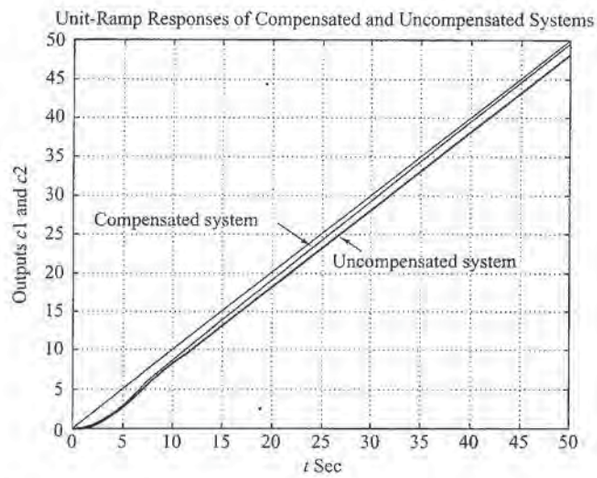


**Figure 7–16**
Unit-ramp responses of compensated and uncompensated systems. [The compensator is given by Equation (7–3).]

MATLAB Program 7–4 gives the unit-step response curves of the compensated and uncompensated systems. The unit-step response curves are shown in Figure 7–17. Notice that the lag-compensated system exhibits a larger maximum overshoot and slower response than the original uncompensated system. Notice that a pair of the pole at $s = -0.0549$ and zero at

**MATLAB Program 7–4**

```
% ---------- Unit-step response ----------

% ***** Unit-step responses of compensated system and
% uncompensated system *****
% ***** Enter the numerators and denominators of the
% compensated and uncompensated systems *****

numc = [0  0  0  1.0235  0.0512];
denc = [1  3.005  2.015  1.0335  0.0512];
num = [0  0  0  1.06];
den = [1  3  2  1.06];

% ***** Specify the time range (such as t = 0:0.1:40) and enter
% step command and plot command. *****

t = 0:0.1:40;
[c1,x1,t] = step(numc,denc,t);
[c2,x2,t] = step(num,den,t);
plot(t,c1,'-',t,c2,'.')
grid
text(13,1.12,'Compensated system')
text(13.6,0.88,'Uncompensated system')
title('Unit-Step Responses of Compensated and Uncompensated Systems')
xlabel('t Sec')
ylabel('Outputs c1 and c2')
```





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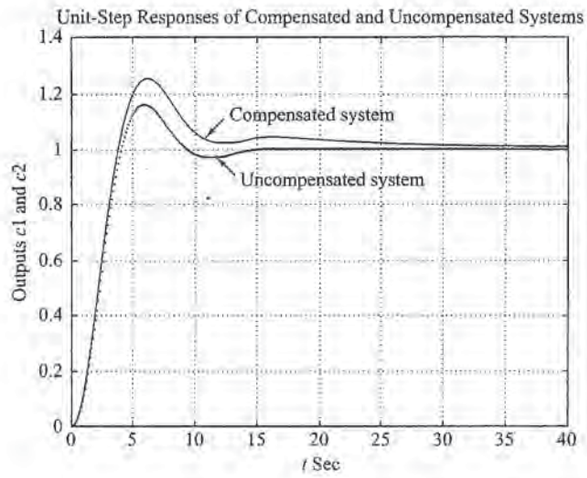


**Figure 7–17**
Unit-step responses of compensated and uncompensated systems. [The compensator is given by Equation (7–3).]

$s = -0.05$ generates a long tail of small amplitude in the transient response. If a larger maximum overshoot and a slower response are not desired, we need to use a lag–lead compensator as presented in Section 7–5.

**Comments.** It is noted that under certain circumstances, however, both lead compensator and lag compensator may satisfy the given specifications (both transient-response specifications and steady-state specifications.) Then either compensation may be used.

## 7–5 LAG–LEAD COMPENSATION

Lead compensation basically speeds up the response and increases the stability of the system. Lag compensation improves the steady-state accuracy of the system, but reduces the speed of the response.

If improvements in both transient response and steady-state response are desired, then both a lead compensator and a lag compensator may be used simultaneously. Rather than introducing both a lead compensator and a lag compensator as separate elements, however, it is economical to use a single lag–lead compensator.

Lag–lead compensation combines the advantages of lag and lead compensations. Since the lag–lead compensator possesses two poles and two zeros, such a compensation increases the order of the system by 2, unless cancellation of pole(s) and zero(s) occurs in the compensated system.

**Electronic Lag–Lead Compensator Using Operational Amplifiers.** Figure 7–18 shows an electronic lag–lead compensator using operational amplifiers. The transfer





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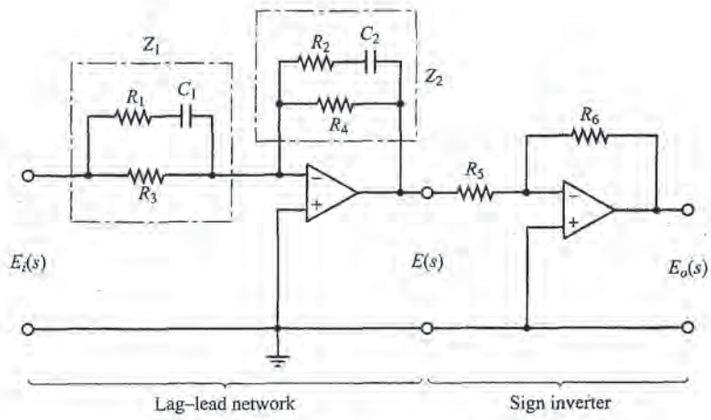


**Figure 7–18**
Lag–lead compensator.

function for this compensator may be obtained as follows: The complex impedance $Z_1$ is given by

$$\frac{1}{Z_1} = \frac{1}{R_1 + \dfrac{1}{C_1 s}} + \frac{1}{R_3}$$

or

$$Z_1 = \frac{(R_1 C_1 s + 1)R_3}{(R_1 + R_3)C_1 s + 1}$$

Similarly, complex impedance $Z_2$ is given by

$$Z_2 = \frac{(R_2 C_2 s + 1)R_4}{(R_2 + R_4)C_2 s + 1}$$

Hence, we have

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_4}{R_3}\frac{(R_1 + R_3)C_1 s + 1}{R_1 C_1 s + 1} \cdot \frac{R_2 C_2 s + 1}{(R_2 + R_4)C_2 s + 1}$$

The sign inverter has the transfer function

$$\frac{E_o(s)}{E(s)} = -\frac{R_6}{R_5}$$

Thus the transfer function of the compensator shown in Figure 7–18 is

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} = \frac{R_4 R_6}{R_3 R_5}\left[\frac{(R_1 + R_3)C_1 s + 1}{R_1 C_1 s + 1}\right]\left[\frac{R_2 C_2 s + 1}{(R_2 + R_4)C_2 s + 1}\right] \quad (7\text{–}4)$$





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Let us define

$$T_1 = (R_1 + R_3)C_1, \qquad \frac{T_1}{\gamma} = R_1C_1, \qquad T_2 = R_2C_2, \qquad \beta T_2 = (R_2 + R_4)C_2$$

Then Equation (7–4) becomes

$$\frac{E_o(s)}{E_i(s)} = K_c\frac{\beta}{\gamma}\left(\frac{T_1s + 1}{\frac{T_1}{\gamma}s + 1}\right)\left(\frac{T_2s + 1}{\beta T_2s + 1}\right) = K_c\frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\gamma}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \tag{7–5}$$

where

$$\gamma = \frac{R_1 + R_3}{R_1} > 1, \qquad \beta = \frac{R_2 + R_4}{R_2} > 1, \qquad K_c = \frac{R_2R_4R_6}{R_1R_3R_5}\frac{R_1 + R_3}{R_2 + R_4}$$

Note that $\gamma$ is often chosen to be equal to $\beta$.

**Lag–lead Compensation Techniques Based on the Root-Locus Approach.** Consider the system shown in Figure 7–19. Assume that we use the lag–lead compensator:

$$G_c(s) = K_c\frac{\beta}{\gamma}\frac{(T_1s + 1)(T_2s + 1)}{\left(\frac{T_1}{\gamma}s + 1\right)(\beta T_2s + 1)} = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right) \tag{7–6}$$

where $\beta > 1$ and $\gamma > 1$. (Consider $K_c$ to belong to the lead portion of the lag–lead compensator.)

In designing lag–lead compensators, we consider two cases where $\gamma \neq \beta$ and $\gamma = \beta$.

Case 1. $\gamma \neq \beta$. In this case, the design process is a combination of the design of the lead compensator and that of the lag compensator. The design procedure for the lag–lead compensator follows:

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.
2. Using the uncompensated open-loop transfer function $G(s)$, determine the angle deficiency $\phi$ if the dominant closed-loop poles are to be at the desired location. The phase-lead portion of the lag–lead compensator must contribute this angle $\phi$.
3. Assuming that we later choose $T_2$ sufficiently large so that the magnitude of the lag portion

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right|$$

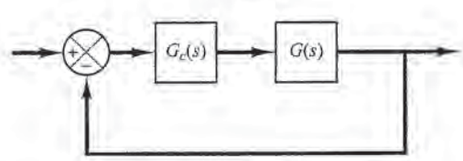


**Figure 7–19**
Control system.





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is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles, choose the values of $T_1$ and $\gamma$ from the requirement that

$$\angle \frac{s_1 + \dfrac{1}{T_1}}{s_1 + \dfrac{\gamma}{T_1}} = \phi$$

The choice of $T_1$ and $\gamma$ is not unique. (Infinitely many sets of $T_1$ and $\gamma$ are possible.) Then determine the value of $K_c$ from the magnitude condition:

$$\left| K_c \frac{s_1 + \dfrac{1}{T_1}}{s_1 + \dfrac{\gamma}{T_1}} G(s_1) \right| = 1$$

**4.** If the static velocity error constant $K_v$ is specified, determine the value of $\beta$ to satisfy the requirement for $K_v$. The static velocity error constant $K_v$ is given by

$$\begin{aligned} K_v &= \lim_{s \to 0} sG_c(s)G(s) \\ &= \lim_{s \to 0} sK_c \left( \frac{s + \dfrac{1}{T_1}}{s + \dfrac{\gamma}{T_1}} \right) \left( \frac{s + \dfrac{1}{T_2}}{s + \dfrac{1}{\beta T_2}} \right) G(s) \\ &= \lim_{s \to 0} sK_c \frac{\beta}{\gamma} G(s) \end{aligned}$$

where $K_c$ and $\gamma$ are aleady determined in step 3. Hence, given the value of $K_v$, the value of $\beta$ can be determined from this last equation. Then, using the value of $\beta$ thus determined, choose the value of $T_2$ such that

$$\left| \frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} \right| \doteq 1$$

$$-5^\circ < \angle \frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} < 0^\circ$$

(The preceding design procedure is illustrated in Example 7–3.)

Case 2. $\gamma = \beta$. If $\gamma = \beta$ is required in Equation (7–6), then the preceeding design procedure for the lag–lead compensator may be modified as follows:

**1.** From the given performance specifications, determine the desired location for the dominant closed-loop poles.





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2. The lag–lead compensator given by Equation (7–6) is modified to

$$G_c(s) = K_c \frac{(T_1 s + 1)(T_2 s + 1)}{\left(\frac{T_1}{\beta} s + 1\right)(\beta T_2 s + 1)} = K_c \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \tag{7–7}$$

where $\beta > 1$. The open-loop transfer function of the compensated system is $G_c(s)G(s)$. If the static velocity error constant $K_c$ is specified, determine the value of constant $K_c$ from the following equation:

$$\begin{aligned} K_v &= \lim_{s \to 0} sG_c(s)G(s) \\ &= \lim_{s \to 0} sK_c G(s) \end{aligned}$$

3. To have the dominant closed-loop poles at the desired location, calculate the angle contribution $\phi$ needed from the phase lead portion of the lag–lead compensator.

4. For the lag–lead compensator, we later choose $T_2$ sufficiently large so that

$$\left| \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} \right|$$

is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles. Determine the values of $T_1$ and $\beta$ from the magnitude and angle conditions:

$$\left| K_c \left( \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} \right) G(s_1) \right| = 1$$

$$\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} = \phi$$

5. Using the value of $\beta$ just determined, choose $T_2$ so that

$$\left| \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} \right| \doteqdot 1$$

$$-5^\circ < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

The value of $\beta T_2$, the largest time constant of the lag–lead compensator, should not be too large to be physically realized. (An example of the design of the lag–lead compensator when $\gamma = \beta$ is given in Example 7–4.)





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**EXAMPLE 7–3** Consider the control system shown in Figure 7–20. The feedforward transfer function is

$$G(s) = \frac{4}{s(s + 0.5)}$$

This system has closed-loop poles at

$$s = -0.2500 \pm j1.9843$$

The damping ratio is 0.125, the undamped natural frequency is 2 rad/sec, and the static velocity error constant is 8 $\sec^{-1}$.

It is desired to make the damping ratio of the dominant closed-loop poles equal to 0.5 and to increase the undamped natural frequency to 5 rad/sec and the static velocity error constant to 80 $\sec^{-1}$. Design an appropriate compensator to meet all the performance specifications.

Let us assume that we use a lag–lead compensator having the transfer function

$$G_c(s) = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right) \qquad (\gamma > 1, \beta > 1)$$

where $\gamma$ is not equal to $\beta$. Then the compensated system will have the transfer function

$$G_c(s)G(s) = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)G(s)$$

From the performance specifications, the dominant closed-loop poles must be at

$$s = -2.50 \pm j4.33$$

Since

$$\left.\angle\frac{4}{s(s + 0.5)}\right|_{s=-2.50+j4.33} = -235°$$

the phase lead portion of the lag–lead compensator must contribute 55° so that the root locus passes through the desired location of the dominant closed-loop poles.

To design the phase lead portion of the compensator, we first determine the location of the zero and pole that will give 55° contribution. There are many possible choices, but we shall here choose the zero at $s = -0.5$ so that this zero will cancel the pole at $s = -0.5$ of the plant. Once the zero is chosen, the pole can be located such that the angle contribution is 55°. By simple calculation or graphical analysis, the pole must be located at $s = -5.021$. Thus, the phase lead portion of the lag–lead compensator becomes

$$K_c\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} = K_c\frac{s + 0.5}{s + 5.021}$$

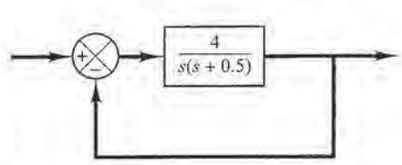


**Figure 7–20**
Control system.





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Thus

$$T_1 = 2, \qquad \gamma = \frac{5.021}{0.5} = 10.04$$

Next we determine the value of $K_c$ from the magnitude condition:

$$\left| K_c \frac{s + 0.5}{s + 5.021} \frac{4}{s(s + 0.5)} \right|_{s=-2.5+j4.33} = 1$$

Hence,

$$K_c = \left| \frac{(s + 5.021)s}{4} \right|_{s=-2.5+j4.33} = 6.26$$

The phase lag portion of the compensator can be designed as follows: First the value of $\beta$ is determined to satisfy the requirement on the static velocity error constant:

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = \lim_{s \to 0} sK_c \frac{\beta}{\gamma} G(s)$$

$$= \lim_{s \to 0} s(6.26) \frac{\beta}{10.04} \frac{4}{s(s + 0.5)} = 4.988\beta = 80$$

Hence, $\beta$ is determined as

$$\beta = 16.04$$

Finally, we choose the value of $T_2$ large enough so that

$$\left| \frac{s + \dfrac{1}{T_2}}{s + \dfrac{1}{16.04T_2}} \right|_{s=-2.5+j4.33} \doteqdot 1$$

and

$$-5^\circ < \angle \left| \frac{s + \dfrac{1}{T_2}}{s + \dfrac{1}{16.04T_2}} \right|_{s=-2.5+j4.33} < 0^\circ$$

Since $T_2 \doteqdot 5$ (or any number greater than 5) satisfies the above two requirements, we may choose

$$T_2 = 5$$

Now the transfer function of the designed lag–lead compensator is given by

$$G_c(s) = (6.26)\left( \frac{s + \dfrac{1}{2}}{s + \dfrac{10.04}{2}} \right)\left( \frac{s + \dfrac{1}{5}}{s + \dfrac{1}{16.04 \times 5}} \right)$$

$$= 6.26\left( \frac{s + 0.5}{s + 5.02} \right)\left( \frac{s + 0.2}{s + 0.01247} \right)$$

$$= \frac{10(2s + 1)(5s + 1)}{(0.1992s + 1)(80.19s + 1)}$$





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The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{25.04(s + 0.2)}{s(s + 5.02)(s + 0.01247)}$$

Because of the cancellation of the $(s + 0.5)$ terms, the compensated system is a third-order system. (Mathematically, this cancellation is exact, but practically such cancellation will not be exact because some approximations are usually involved in deriving the mathematical model of the system and, as a result, the time constants are not precise.) The root-locus plot of the compensated system is shown in Figure 7–21(a). An enlarged view of the root-locus plot near the origin is shown in Figure 7–21(b). Because the angle contribution of the phase lag portion of the lag–lead

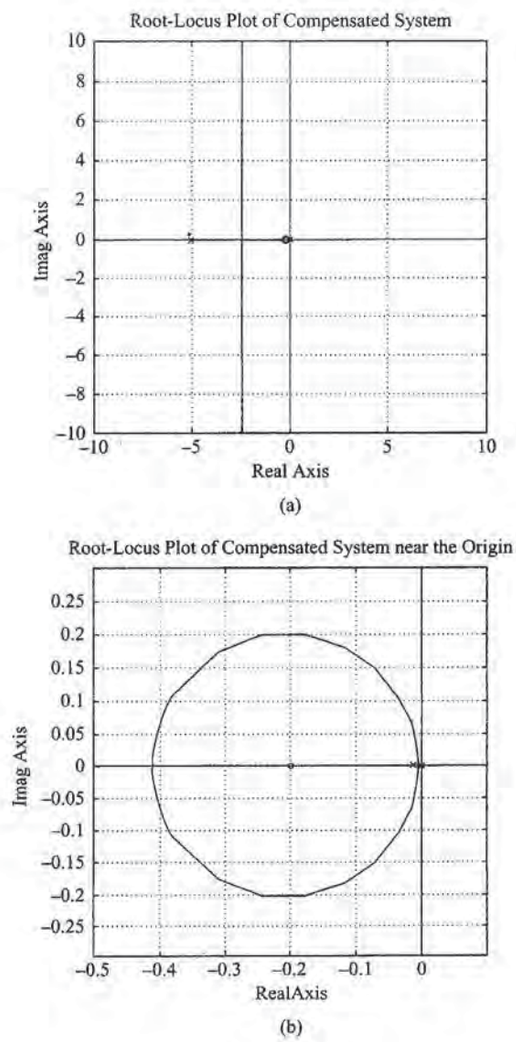


**Figure 7–21**
(a) Root-locus plot of the compensated system; (b) root-locus plot near the origin.





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compensator is quite small, there is only a small change in the location of the dominant closed-loop poles from the desired location, $s = -2.5 \pm j4.33$. In fact, the new closed-loop poles are located at $s = -2.4123 \pm j4.2756$. (The new damping ratio is $\zeta = 0.491$.) Therefore, the compensated system meets all the required performance specifications. The third closed-loop pole of the compensated system is located at $s = -0.2078$. Since this closed-loop pole is very close to the zero at $s = -0.2$, the effect of this pole on the response is small. (Note that, in general, if a pole and a zero lie close to each other on the negative real axis near the origin, then such a pole-zero combination will yield a long tail of small amplitude in the transient response.)

The unit-step response curves and unit-ramp response curves before and after compensation are shown in Figure 7–22.

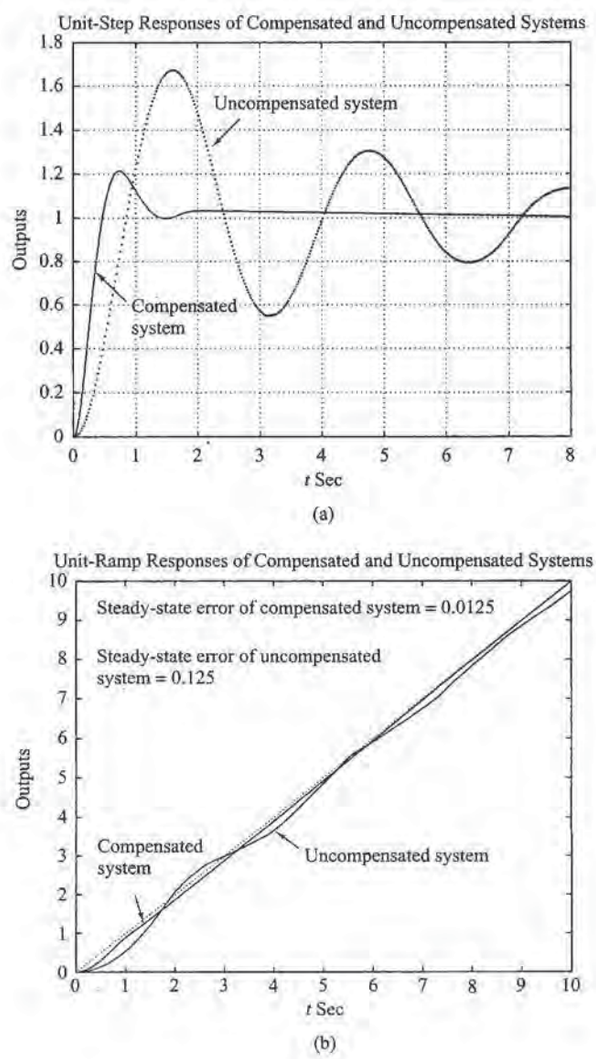


**Figure 7–22**
Transient response curves for the compensated system and uncompensated system. (a) Unit-step response curves; (b) unit-ramp response curves.





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**EXAMPLE 7–4** Consider the control system of Example 7–3. Suppose that we use a lag–lead compensator of the form given by Equation (7–7), or

$$G_c(s) = K_c \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \qquad (\beta > 1)$$

Assuming the specifications are the same as those given in Example 7–3, design a compensator $G_c(s)$.

The desired locations for the dominant closed-loop poles are at

$$s = -2.50 \pm j4.33$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = K_c \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)} \cdot \frac{4}{s(s + 0.5)}$$

Since the requirement on the static velocity error constant $K_v$ is 80 sec$^{-1}$, we have

$$K_v = \lim_{s \to 0} sG_c(s)G(s) = \lim_{s \to 0} K_c \frac{4}{0.5} = 8K_c = 80$$

Thus

$$K_c = 10$$

The time constant $T_1$ and the value of $\beta$ are determined from

$$\left|\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right| \left|\frac{40}{s(s + 0.5)}\right|_{s=-2.5+j4.33} = \left|\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right| \frac{8}{4.77} = 1$$

$$\left/\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right|_{s=-2.5+j4.33} = 55°$$

(The angle deficiency of 55° was obtained in Example 7–3.) Referring to Figure 7–23, we can easily locate points $A$ and $B$ such that

$$\angle APB = 55°, \qquad \frac{\overline{PA}}{\overline{PB}} = \frac{4.77}{8}$$

(Use a graphical approach or a trigonometric approach.) The result is

$$\overline{AO} = 2.38, \qquad \overline{BO} = 8.34$$

or

$$T_1 = \frac{1}{2.38} = 0.420, \qquad \beta = 8.34T_1 = 3.503$$





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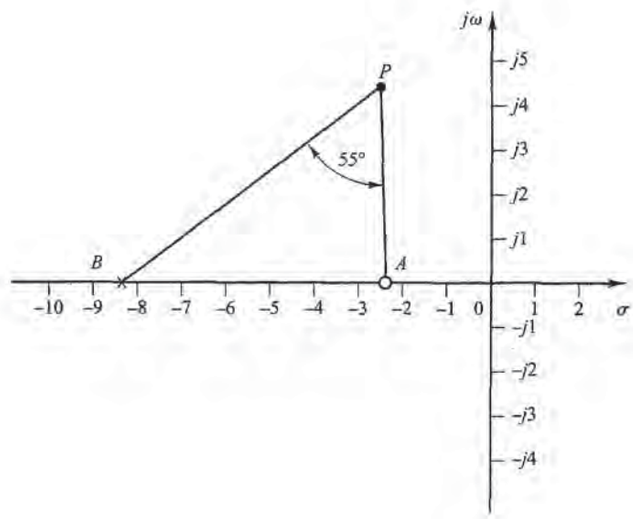


**Figure 7–23**
Determination of the desired pole-zero location.

The phase lead portion of the lag–lead network thus becomes

$$10\left(\frac{s + 2.38}{s + 8.34}\right)$$

For the phase lag portion, we may choose

$$T_2 = 10$$

Then

$$\frac{1}{\beta T_2} = \frac{1}{3.503 \times 10} = 0.0285$$

Thus, the lag–lead compensator becomes

$$G_c(s) = (10)\left(\frac{s + 2.38}{s + 8.34}\right)\left(\frac{s + 0.1}{s + 0.0285}\right)$$

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{40(s + 2.38)(s + 0.1)}{(s + 8.34)(s + 0.0285)s(s + 0.5)}$$

No cancellation occurs in this case, and the compensated system is of fourth order. Because the angle contribution of the phase lag portion of the lag–lead network is quite small, the dominant closed-loop poles are located very near the desired location. In fact, the dominant closed-loop poles are located at $s = -2.4539 \pm j4.3099$. The two other closed-loop poles are located at

$$s = -0.1003; \qquad s = -3.8604$$

Since the closed-loop pole at $s = -0.1003$ is very close to a zero at $s = -0.1$, they almost cancel each other. Thus, the effect of this closed-loop pole is very small. The remaining closed-loop pole ($s = -3.8604$) does not quite cancel the zero at $s = -2.4$. The effect of this zero is to cause a





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larger overshoot in the step response than a similar system without such a zero. The unit-step response curves of the compensated and uncompensated systems are shown in Figure 7–24(a). The unit-ramp response curves for both systems are depicted in Figure 7–24(b).

The maximum overshoot in the step response of the compensated system is approximately 38%. (This is much larger than the maximum overshoot of 21% in the design presented in Example 7–3.) It is possible to decrease the maximum overshoot by a small amount from 38%, but not to 20% if $\gamma = \beta$ is required, as in this example. Note that by not requiring $\gamma = \beta$, we have an additional parameter to play with and thus can reduce the maximum overshoot.

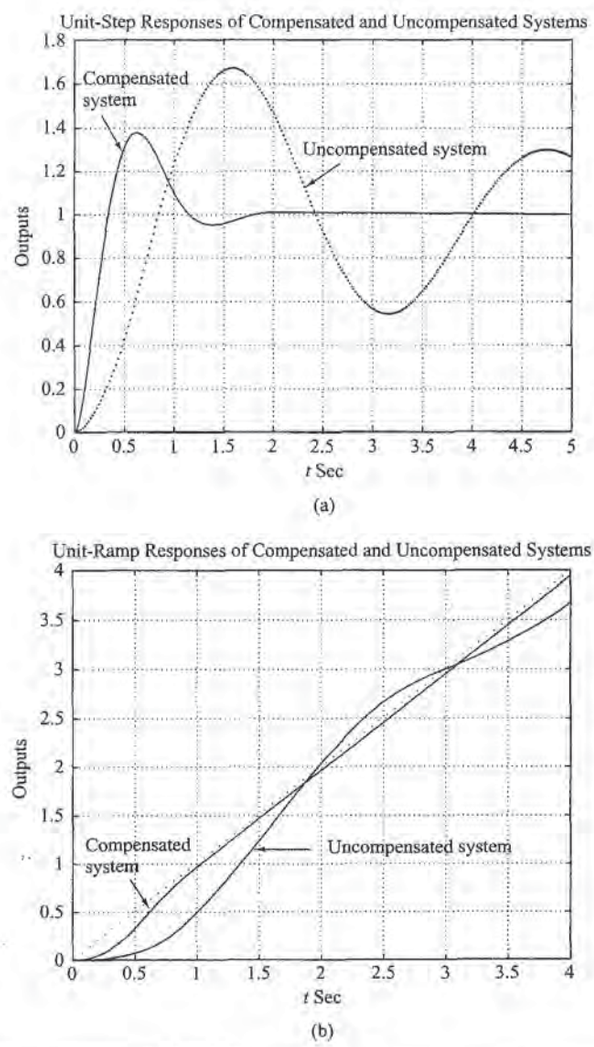


**Figure 7–24**
(a) Unit-step response curves for the compensated and uncompensated systems;
(b) unit-ramp response curves for both systems.





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## 7–6 PARALLEL COMPENSATION

Thus far we have presented series compensation techniques using lead, lag, or lag–lead compensators. In this section we discuss parallel compensation technique. Because in the parallel compensation design the controller (or compensator) is in a minor loop, the design may seem to be more complicated than in the series compensation case. It is, however, not complicated if we rewrite the characteristic equation to be of the same form as the characteristic equation for the series compensated system. In this section we present a simple design problem involving parallel compensation.

**Basic Principle for Designing Parallel Compensated System.** Referring to Figure 7–25(a), the closed-loop transfer function for the system with series compensation is

$$\frac{C}{R} = \frac{G_c G}{1 + G_c G H}$$

The characteristic equation is

$$1 + G_c G H = 0$$

Given $G$ and $H$, the design problem becomes that of determining the compensator $G_c$ that satisfies the given specification.

The closed-loop transfer function for the system with parallel compensation [Figure 7–25(b)] is

$$\frac{C}{R} = \frac{G_1 G_2}{1 + G_2 G_c + G_1 G_2 H}$$

The characteristic equation is

$$1 + G_1 G_2 H + G_2 G_c = 0$$

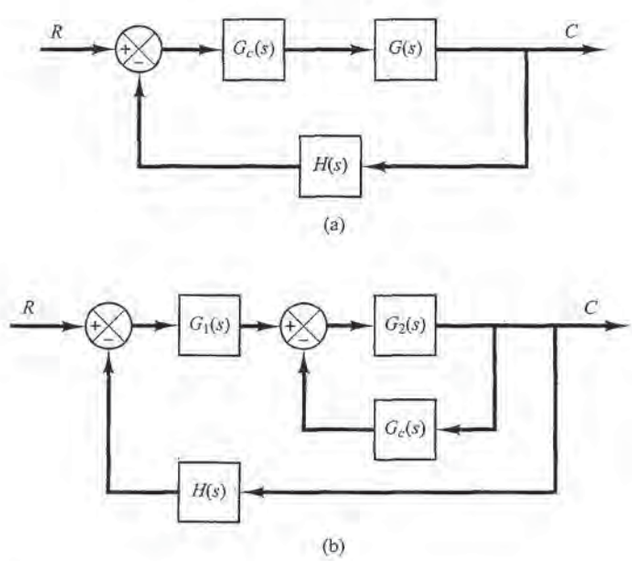


**Figure 7–25** (a) Series compensation; (b) parallel or feedback compensation.





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By dividing this characteristic equation by the sum of the terms that do not involve $G_c$, we obtain

$$1 + \frac{G_c G_2}{1 + G_1 G_2 H} = 0 \tag{7–8}$$

If we define

$$G_f = \frac{G_2}{1 + G_1 G_2 H}$$

then Equation (7–8) becomes

$$1 + G_c G_f = 0$$

Since $G_f$ is a fixed transfer function, the design of $G_c$ becomes the same as the case of series compensation. Hence the same design approach applies to the parallel compensated system.

**Velocity Feedback Systems.** A velocity feedback system (tachometer feedback system) is an example of parallel compensated systems. The controller (or compensator) in such a system is a gain element. The gain of the feedback element in a minor loop must be determined properly so that the entire system satisfies the given design specifications. The characteristic of such a velocity-feedback system is that the variable parameter does not appear as a multiplying factor in the open-loop transfer function, so that direct application of the root-locus design technique is not possible. However, by rewriting the characteristic equation such that the variable parameter appears as a multiplying factor, then the root-locus approach to the design is possible.

An example of control system design using parallel compensation technique is presented in Example 7–5.

**EXAMPLE 7–5** Consider the system shown in Figure 7–26. Draw a root-locus diagram. Then determine the value of $k$ such that the damping ratio of the dominant closed-loop poles is 0.4.

Here the system involves velocity feedback. The open-loop transfer function is

$$\text{Open-loop transfer function} = \frac{20}{s(s+1)(s+4) + 20ks}$$

Notice that the adjustable variable $k$ does not appear as a multiplying factor. The characteristic equation for the system is

$$s^3 + 5s^2 + 4s + 20ks + 20 = 0 \tag{7–9}$$

Define

$$20k = K$$

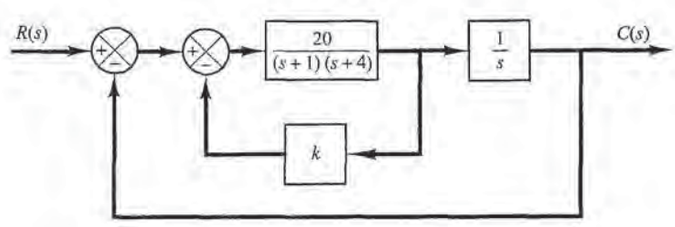


**Figure 7–26**
Control system.





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Then Equation (7–9) becomes

$$s^3 + 5s^2 + 4s + Ks + 20 = 0 \tag{7–10}$$

Dividing both sides of Equation (7–10) by the sum of the terms that do not contain $K$, we get

$$1 + \frac{Ks}{s^3 + 5s^2 + 4s + 20} = 0$$

or

$$1 + \frac{Ks}{(s + j2)(s - j2)(s + 5)} = 0 \tag{7–11}$$

Equation (7–11) is of the form of Equation (6–11).

We shall now sketch the root loci of the system given by Equation (7–11). Notice that the open-loop poles are located at $s = j2$, $s = -j2$, $s = -5$, and the open-loop zero is located at $s = 0$. The root locus exists on the real axis between 0 and −5. Since

$$\lim_{s\to\infty} \frac{Ks}{(s + j2)(s - j2)(s + 5)} = \lim_{s\to\infty} \frac{K}{s^2}$$

we have

$$\text{Angles of asymptote} = \frac{\pm 180°(2k + 1)}{2} = \pm 90°$$

The intersection of the asymptotes with the real axis can be found from

$$\lim_{s\to\infty} \frac{Ks}{s^3 + 5s^2 + 4s + 20} = \lim_{s\to\infty} \frac{K}{s^2 + 5s + \cdots} = \lim_{s\to\infty} \frac{K}{(s + 2.5)^2}$$

as

$$s = -2.5$$

The angle of departure (angle $\theta$) from the pole at $s = j2$ is obtained as follows:

$$\theta = 180° - 90° - 21.8° + 90° = 158.2°$$

Thus, the angle of departure from the pole $s = j2$ is 158.2°. Figure 7–27 shows a root-locus plot for the system. Notice that two branches of the root locus originate from the poles at $s = \pm j2$ and terminate on the zeros at infinity. The remaining one branch originates from the pole at $s = -5$ and terminates on the zero at $s = 0$.

Note that the closed-loop poles with $\zeta = 0.4$ must lie on straight lines passing through the origin and making the angles ±66.42° with the negative real axis. In the present case, there are two intersections of the root-locus branch in the upper half $s$ plane and the straight line of angle 66.42°. Thus, two values of $K$ will give the damping ratio $\zeta$ of the closed-loop poles equal to 0.4. At point $P$, the value of $K$ is

$$K = \left|\frac{(s + j2)(s - j2)(s + 5)}{s}\right|_{s=-1.0490+j2.4065} = 8.9801$$

Hence

$$k = \frac{K}{20} = 0.4490 \qquad \text{at point } P$$

At point $Q$, the value of $K$ is

$$K = \left|\frac{(s + j2)(s - j2)(s + 5)}{s}\right|_{s=-2.1589+j4.9652} = 28.260$$





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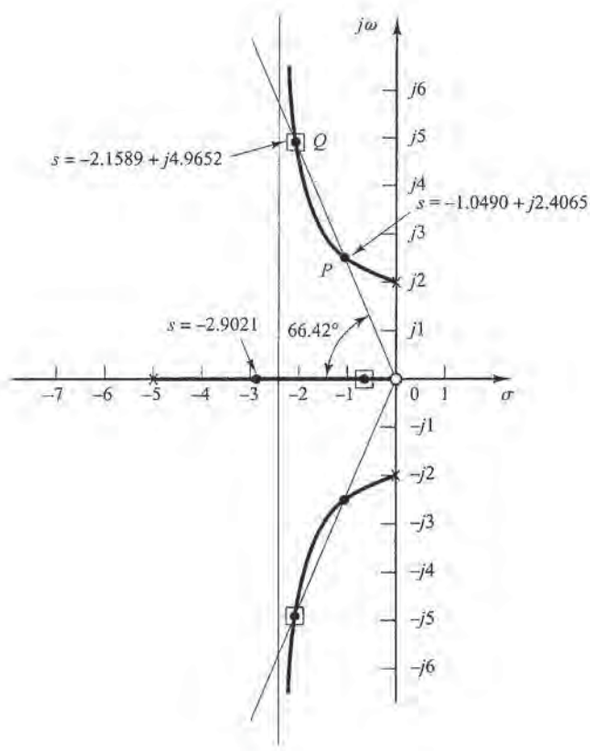


**Figure 7–27**
Root-locus plot for the system shown in Figure 7–26.

Hence

$$k = \frac{K}{20} = 1.4130 \qquad \text{at point } Q$$

Thus, we have two solutions for this problem. For $k = 0.4490$, the three closed-loop poles are located at

$$s = -1.0490 + j2.4065, \qquad s = -1.0490 - j2.4065, \qquad s = -2.9021$$

For $k = 1.4130$, the three closed-loop poles are located at

$$s = -2.1589 + j4.9652, \qquad s = -2.1589 - j4.9652, \qquad s = -0.6823$$

It is important to point out that the zero at the origin is the open-loop zero, but not the closed-loop zero. This is evident, because the original system shown in Figure 7–26 does not have a closed-loop zero, since

$$\frac{G(s)}{R(s)} = \frac{20}{s(s+1)(s+4) + 20(1+ks)}$$

The open-loop zero at $s = 0$ was introduced in the process of modifying the characteristic equation such that the adjustable variable $K = 20k$ was to appear as a multiplying factor.





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We have obtained two different values of $k$ to satisfy the requirement that the damping ratio of the dominant closed-loop poles be equal to 0.4. The closed-loop transfer function with $k = 0.4490$ is given by

$$\frac{C(s)}{R(s)} = \frac{20}{s^3 + 5s^2 + 12.98s + 20}$$

$$= \frac{20}{(s + 1.0490 + j2.4065)(s + 1.0490 - j2.4065)(s + 2.9021)}$$

The closed-loop transfer function with $k = 1.4130$ is given by

$$\frac{C(s)}{R(s)} = \frac{20}{s^3 + 5s^2 + 32.26s + 20}$$

$$= \frac{20}{(s + 2.1589 + j4.9652)(s + 2.1589 - j4.9652)(s + 0.6823)}$$

Notice that the system with $k = 0.4490$ has a pair of dominant complex-conjugate closed-loop poles, while in the system with $k = 1.4130$ the real closed-loop pole at $s = -0.6823$ is dominant, and the complex-conjugate closed-loop poles are not dominant. In this case, the response characteristic is primarily determined by the real closed-loop pole.

Let us compare the unit-step responses of both systems. MATLAB Program 7–5 may be used for plotting the unit-step response curves in one diagram. The resulting unit-step response curves $[c_1(t)$ for $k = 0.4490$ and $c_2(t)$ for $k = 1.4130]$ are shown in Figure 7–28.

From Figure 7–28 we notice that the response of the system with $k = 0.4490$ is oscillatory. (The effect of the closed-loop pole at $s = -2.9021$ on the unit-step response is small.) For the

**MATLAB Program 7–5**

```
% --------- Unit-step response ---------

% ***** Enter numerators and denominators of systems with
% k = 0.4490 and k = 1.4130, respectively. *****

num1 = [0 0 0 20];
den1 = [1 5 12.98 20];
num2 = [0 0 0 20];
den2 = [1 5 32.26 20];
t = 0:0.1:10;
[c1,x1,t] = step(num1,den1,t);
[c2,x2,t] = step(num2,den2,t);
plot(t,c1,t,c2)
text(2.5,1.12,'k = 0.4490')
text(3.7,0.85,'k = 1.4130')
grid
title('Unit-step Responses of Two Systems')
xlabel('t Sec')
ylabel('Outputs c1 and c2')
```





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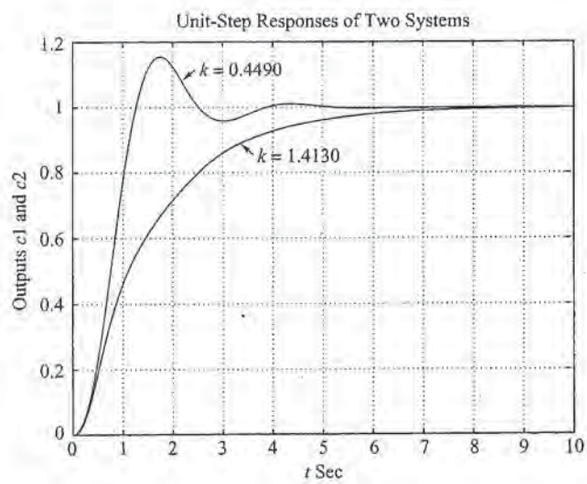


**Figure 7–28**
Unit-step response curves for the system shown in Figure 7–26 when the damping ratio $\zeta$ of the dominant closed-loop poles is set equal to 0.4. (Two possible values of $k$ give the damping ratio $\zeta$ equal to 0.4.)

system with $k = 1.4130$, the oscillations due to the closed-loop poles at $s = -2.1589 \pm j4.9652$ damp out much faster than purely exponential response due to the closed-loop pole at $s = -0.6823$.

The system with $k = 0.4490$ (which exhibits a faster response with relatively small overshoot) has a much better response characteristic than the system with $k = 1.4130$ (which exhibits a slow overdamped response). Therefore, we should choose $k = 0.4490$ for the present system.

## EXAMPLE PROBLEMS AND SOLUTIONS

**A–7–1.** Obtain the transfer function of the mechanical system shown in Figure 7–29. Assume that the displacement $x_i$ is the input and displacement $x_o$ is the output of the system.

**Solution.** From the diagram we obtain the following equations of motion:

$$b_2(\dot{x}_i - \dot{x}_o) = b_1(\dot{x}_o - \dot{y})$$
$$b_1(\dot{x}_o - \dot{y}) = ky$$

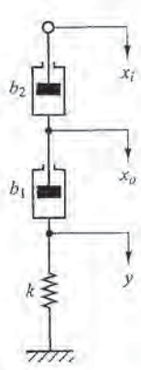


**Figure 7–29**
Mechanical system.

Taking the Laplace transforms of these two equations, assuming zero initial conditions, and then eliminating $Y(s)$, we obtain

$$\frac{X_o(s)}{X_i(s)} = \frac{b_2}{b_1 + b_2} \frac{\frac{b_1}{k}s + 1}{\frac{b_2}{b_1 + b_2}\frac{b_1}{k}s + 1}$$

This is the transfer function between $X_o(s)$ and $X_i(s)$. By defining

$$\frac{b_1}{k} = T, \qquad \frac{b_2}{b_1 + b_2} = \alpha < 1$$





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we obtain

$$\frac{X_o(s)}{X_i(s)} = \alpha \frac{Ts + 1}{\alpha Ts + 1} = \frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\alpha T}}$$

This mechanical system is a mechanical lead network.

**A–7–2.** Obtain the transfer function of the mechanical system shown in Figure 7–30. Assume that the displacement $x_i$ is the input and displacement $x_o$ is the output.

**Solution.** The equations of motion for this system are

$$b_2(\dot{x}_i - \dot{x}_o) + k_2(x_i - x_o) = b_1(\dot{x}_o - \dot{y})$$

$$b_1(\dot{x}_o - \dot{y}) = k_1 y$$

By taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$b_2[sX_i(s) - sX_o(s)] + k_2[X_i(s) - X_o(s)] = b_1[sX_o(s) - sY(s)]$$

$$b_1[sX_o(s) - sY(s)] = k_1 Y(s)$$

If we eliminate $Y(s)$ from the last two equations, the transfer function $X_o(s)/X_i(s)$ can be obtained as

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\dfrac{b_1}{k_1}s + 1\right)\left(\dfrac{b_2}{k_2}s + 1\right)}{\left(\dfrac{b_1}{k_1}s + 1\right)\left(\dfrac{b_2}{k_2}s + 1\right) + \dfrac{b_1}{k_2}s}$$

Define

$$T_1 = \frac{b_1}{k_1}, \qquad T_2 = \frac{b_2}{k_2}, \qquad \frac{b_1}{k_1} + \frac{b_2}{k_2} + \frac{b_1}{k_2} = \frac{T_1}{\beta} + \beta T_2 \qquad (\beta > 1)$$

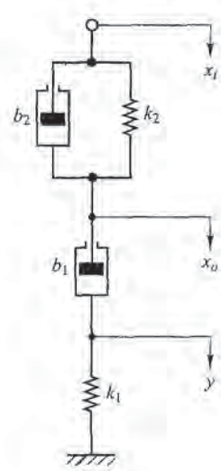


**Figure 7–30**
Mechanical system.





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Then $X_o(s)/X_i(s)$ can be simplified as

$$\frac{X_o(s)}{X_i(s)} = \frac{(T_1 s + 1)(T_2 s + 1)}{\left(\frac{T_1}{\beta} s + 1\right)(\beta T_2 s + 1)} = \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)}$$

From this transfer function we see that this mechanical system is a mechanical lag–lead network.

**A–7–3.** Consider the electrical network shown in Figure 7–31. Derive the transfer function of the network. (As usual in the derivation of the transfer function of any four-terminal network, we assume that the source impedance that the network sees is zero and that the output load impedance is infinite.)

**Solution.** Using the symbols defined in Figure 7–31, we find that the complex impedances $Z_1$ and $Z_2$ are

$$Z_1 = \frac{R_1}{R_1 C s + 1}, \qquad Z_2 = R_2$$

The transfer function between the output $E_o(s)$ and the input $E_i(s)$ is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{R_1 + R_2} \frac{R_1 C s + 1}{\frac{R_1 R_2}{R_1 + R_2} C s + 1}$$

Define

$$R_1 C = T, \qquad \frac{R_2}{R_1 + R_2} = \alpha < 1$$

Then the transfer function becomes

$$\frac{E_o(s)}{E_i(s)} = \alpha \frac{Ts + 1}{\alpha T s + 1} = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

Since $\alpha$ is less than 1, this network is a lead network.

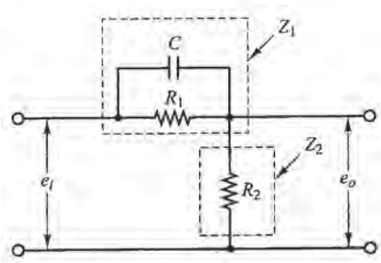


**Figure 7–31**
Electrical network.





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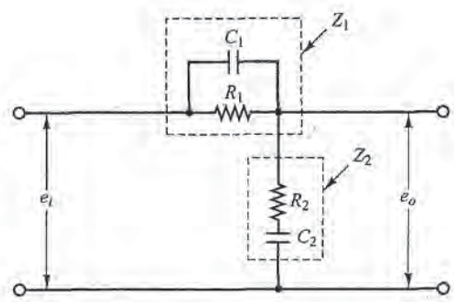


**Figure 7–32**
Electrical network.

**A–7–4.** Obtain the transfer function of the network shown in Figure 7–32.

**Solution.** The complex impedances $Z_1$ and $Z_2$ are

$$Z_1 = \frac{R_1}{R_1C_1s + 1}, \qquad Z_2 = R_2 + \frac{1}{C_2s}$$

The transfer function between $E_o(s)$ and $E_i(s)$ is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s}$$

The denominator of this transfer function can be factored into two real terms. Let us define

$$R_1C_1 = T_1, \qquad R_2C_2 = T_2, \qquad R_1C_1 + R_2C_2 + R_1C_2 = \frac{T_1}{\beta} + \beta T_2 \qquad (\beta > 1)$$

Then $E_o(s)/E_i(s)$ can be simplified to

$$\frac{E_o(s)}{E_i(s)} = \frac{(T_1s + 1)(T_2s + 1)}{\left(\frac{T_1}{\beta}s + 1\right)(\beta T_2s + 1)} = \frac{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{\beta}{T_1}\right)\left(s + \frac{1}{\beta T_2}\right)}$$

This is a lag–lead network.

**A–7–5.** A control system with

$$G(s) = \frac{K}{s^2(s + 1)}, \qquad H(s) = 1$$

is unstable for all positive values of gain $K$.

Plot the root loci of the system. By using this plot, show that this system can be stabilized by adding a zero on the negative real axis or by modifying $G(s)$ to $G_1(s)$, where

$$G_1(s) = \frac{K(s + \alpha)}{s^2(s + 1)} \qquad (0 \le \alpha < 1)$$

**Solution.** A root-locus plot for the system with

$$G(s) = \frac{K}{s^2(s + 1)}, \qquad H(s) = 1$$





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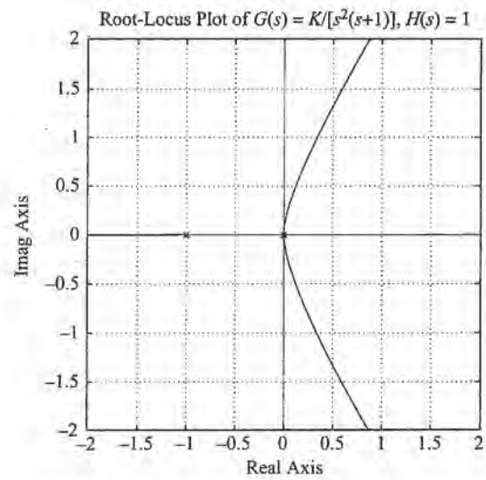
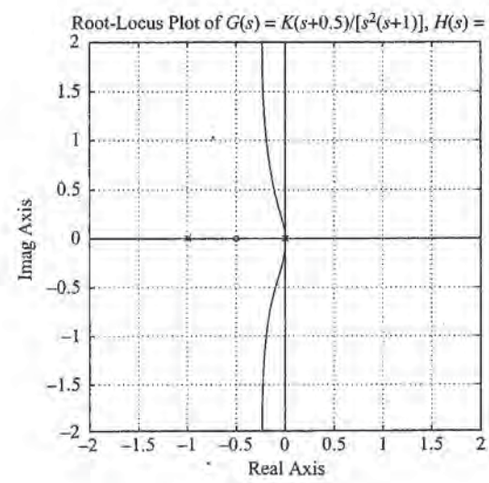



(a)




(b)

**Figure 7–33**
(a) Root-locus plot of the system with $G(s) = K/[s^2(s+1)]$ and $H(s) = 1$; (b) root-locus plot of the system with $G_1(s) = K(s+\alpha)/[s^2(s+1)]$ and $H(s) = 1$, where $\alpha = 0.5$.

is shown in Figure 7–33(a). Since two branches lie in the right half-plane, the system is unstable for any value of $K > 0$.

Addition of a zero to the transfer function $G(s)$ bends the right half-plane branches to the left and brings all root-locus branches to the left half-plane, as shown in the root-locus plot in Figure 7–33(b). Thus, the system with

$$G_1 = \frac{K(s+\alpha)}{s^2(s+1)}, \qquad H(s) = 1 \qquad (0 \le \alpha < 1)$$

is stable for all $K > 0$.





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**A–7–6.** Consider a system with an unstable plant as shown in Figure 7–34(a). Using the root-locus approach, design a proportional-plus-derivative controller (that is, determine the values of $K_p$ and $T_d$) such that the damping ratio $\zeta$ of the closed-loop system is 0.7 and the undamped natural frequency $\omega_n$ is 0.5 rad/sec.

**Solution.** Note that the open-loop transfer function involves two poles at $s = 1.085$ and $s = -1.085$ and one zero at $s = -1/T_d$, which is unknown at this point.

Since the desired closed-loop poles must have $\omega_n = 0.5$ rad/sec and $\zeta = 0.7$, they must be located at

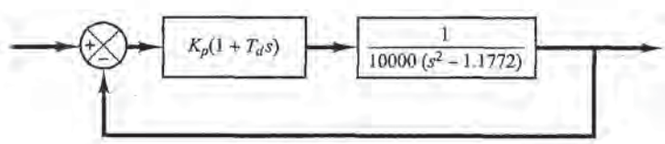
$$s = 0.5\,\underline{/180° \pm 45.573°}$$

($\zeta = 0.7$ corresponds to a line having an angle of 45.573° with the negative real axis.) Hence, the desired closed-loop poles are at

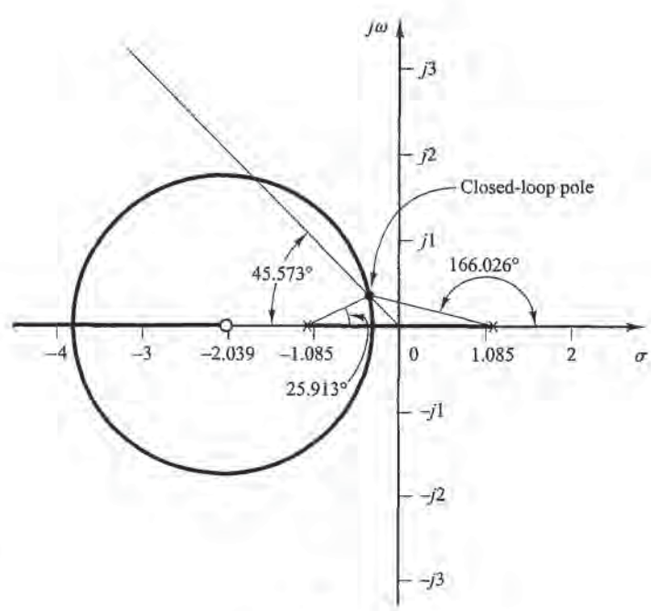
$$s = -0.35 \pm j0.357$$

The open-loop poles and the desired closed-loop pole in the upper half-plane are located in the diagram shown in Figure 7–34(b). The angle deficiency at point $s = -0.35 + j0.357$ is

$$-166.026° - 25.913° + 180° = -11.939°$$




(a)




(b)

**Figure 7–34**
(a) PD control of an unstable plant;
(b) root-locus diagram for the system.





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This means that the zero at $s = -1/T_d$ must contribute 11.939°, which, in turn, determines the location of the zero as follows:

$$s = -\frac{1}{T_d} = -2.039$$

Hence, we have

$$K_p(1 + T_d s) = K_p T_d\left(\frac{1}{T_d} + s\right) = K_p T_d(s + 2.039) \tag{7-12}$$

The value of $T_d$ is

$$T_d = \frac{1}{2.039} = 0.4904$$

The value of gain $K_p$ can be determined from the magnitude condition as follows:

$$\left| K_p T_d \frac{s + 2.039}{10000(s^2 - 1.1772)} \right|_{s=-0.35+j0.357} = 1$$

or

$$K_p T_d = 6999.5$$

Hence,

$$K_p = \frac{6999.5}{0.4904} = 14{,}273$$

By substituting the numerical values of $T_d$ and $K_p$ into Equation (7-12), we obtain

$$K_p(1 + T_d s) = 14{,}273(1 + 0.4904s) = 6999.5(s + 2.039)$$

which gives the transfer function of the desired proportional-plus-derivative controller.

**A-7-7.** Consider the control system shown in Figure 7-35. Design a lag compensator $G_c(s)$ such that the static velocity error constant $K_v$ is 50 sec$^{-1}$ without appreciably changing the location of the original closed-loop poles, which are at $s = -2 \pm j\sqrt{6}$.

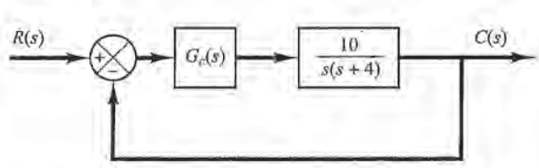


**Figure 7-35**
Control system.





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**Solution.** Assume that the transfer function of the lag compensator is

$$G_c(s) = \hat{K}_c \frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\beta T}} \qquad (\beta > 1)$$

Since $K_v$ is specified as 50 $\text{sec}^{-1}$, we have

$$K_v = \lim_{s \to 0} sG_c(s) \frac{10}{s(s+4)} = \hat{K}_c \beta 2.5 = 50$$

Thus

$$\hat{K}_c \beta = 20$$

Now choose $\hat{K}_c = 1$. Then

$$\beta = 20$$

Choose $T = 10$. Then the lag compensator can be given by

$$G_c(s) = \frac{s + 0.1}{s + 0.005}$$

The angle contribution of the lag compensator at the closed-loop pole $s = -2 + j\sqrt{6}$ is

$$\underline{/G_c(s)}\Big|_{s=-2+j\sqrt{6}} = \tan^{-1}\frac{\sqrt{6}}{-1.9} - \tan^{-1}\frac{\sqrt{6}}{-1.995}$$

$$= -1.3616^\circ$$

which is small. Thus the change in the location of the dominant closed-loop poles is very small.

The open-loop transfer function of the system becomes

$$G_c(s)G(s) = \frac{s + 0.1}{s + 0.005}\frac{10}{s(s+4)}$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{10s + 1}{s^3 + 4.005s^2 + 10.02s + 1}$$

To compare the transient-response characteristics before and after the compensation, the unit-step and unit-ramp responses of the compensated and uncompensated systems are shown in Figures 7–36(a) and (b), respectively. The steady-state error in the unit-ramp response is shown in Figure 7–36(c).





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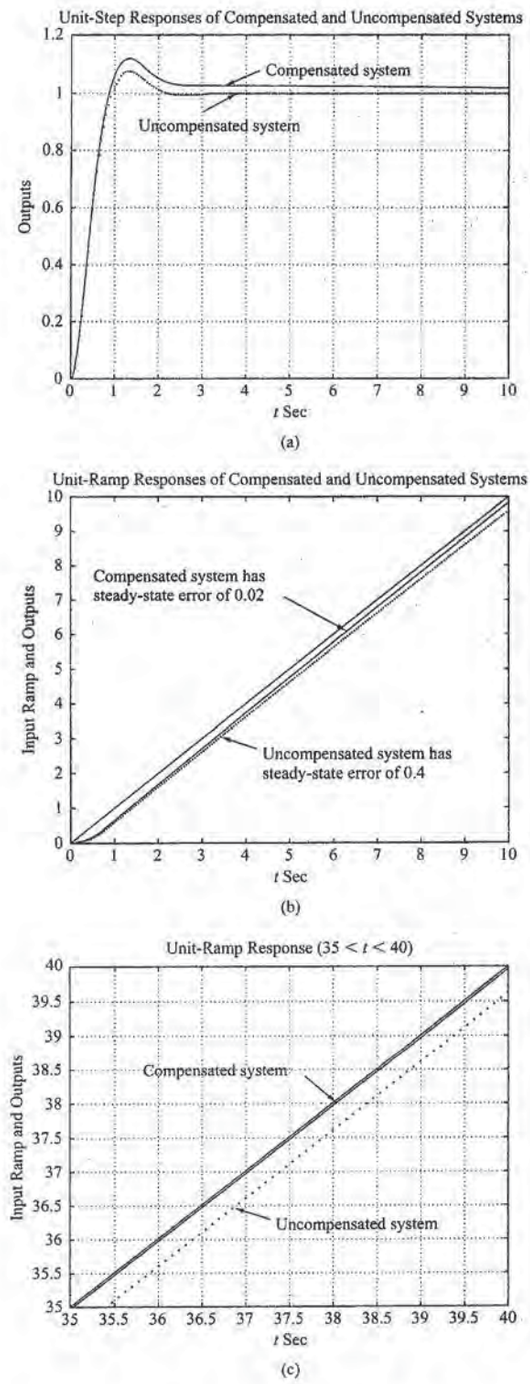


**Figure 7–36**
(a) Unit-step responses of the compensated and uncompensated systems; (b) unit-ramp responses of both systems; (c) unit-ramp responses showing steady-state errors.





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**A–7–8.** Consider a unity-feedback control system whose feedforward transfer function is given by

$$G(s) = \frac{10}{s(s+2)(s+8)}$$

Design a compensator such that the dominant closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$ and the static velocity error constant $K_v$ is equal to 80 $\text{sec}^{-1}$.

**Solution.** The static velocity error constant of the uncompensated system is $K_v = \frac{10}{16} = 0.625$. Since $K_v = 80$ is required, we need to increase the open-loop gain by 128. (This implies that we need a lag compensator.) The root-locus plot of the uncompensated system reveals that it is not possible to bring the dominant closed-loop poles to $-2 \pm j2\sqrt{3}$ by just a gain adjustment alone. See Figure 7–37. (This means that we also need a lead compensator.) Therefore, we shall employ a lag–lead compensator.

Let us assume that the transfer function of the lag–lead compensator to be

$$G_c(s) = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)$$

where $K_c = 128$. This is because

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sK_cG(s) = K_c\frac{10}{16} = 80$$

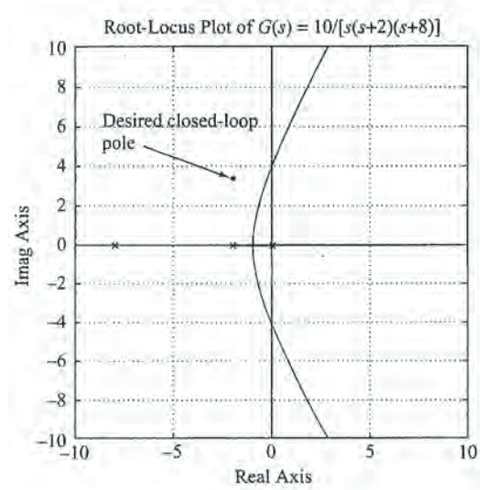


**Figure 7–37**
Root-locus plot of $G(s) = 10/[s(s+2)(s+8)]$.





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and we obtain $K_c = 128$. The angle deficiency at the desired closed-loop pole $s = -2 + j2\sqrt{3}$ is

$$\text{Angle deficiency} = 120° + 90° + 30° - 180° = 60°$$

The lead portion of the lag–lead compensator must contribute this angle. To choose $T_1$ we may use the graphical method presented in Section 7–5.

The lead portion must satisfy the following conditions:

$$\left| 128\left(\frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}}\right) G(s_1) \right|_{s_1=-2+j2\sqrt{3}} = 1$$

and

$$\left\angle \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} \right|_{s_1=-2+j2\sqrt{3}} = 60°$$

The first condition can be simplified as

$$\left| \frac{s_1 + \frac{1}{T_1}}{s_1 + \frac{\beta}{T_1}} \right|_{s_1=-2+j2\sqrt{3}} = \frac{1}{13.3333}$$

By using the same approach as used in Section 7–5, the zero $(s = 1/T_1)$ and pole $(s = \beta/T_1)$ can be determined as follows:

$$\frac{1}{T_1} = 3.70, \qquad \frac{\beta}{T_1} = 53.35$$

See Figure 7–38. The value of $\beta$ is thus determined as

$$\beta = 14.419$$

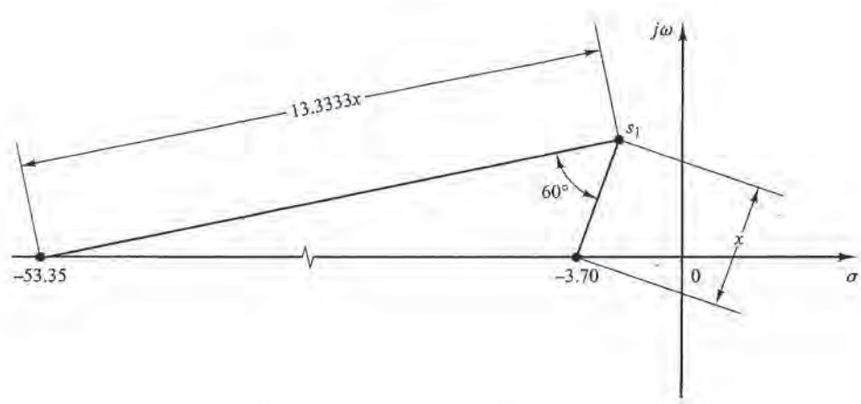


**Figure 7–38** Graphical determination of the zero and pole of the lead portion of the compensator.





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For the lag portion of the compensator, we may choose

$$\frac{1}{\beta T_2} = 0.01$$

Then

$$\frac{1}{T_2} = 0.1442$$

Noting that

$$\left|\frac{s_1 + 0.1442}{s_1 + 0.01}\right|_{s_1=-2+j2\sqrt{3}} = 0.9837$$

$$\left/\frac{s_1 + 0.1442}{s_1 + 0.01}\right|_{s_1=-2+j2\sqrt{3}} = -1.697°$$

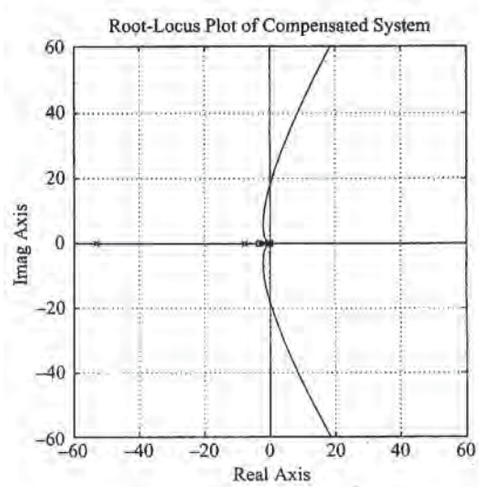
the angle contribution of the lag portion is $-1.697°$ and the magnitude contribution is 0.9837. This means that the dominant closed-loop poles lie close to the desired location $s = -2 \pm j2\sqrt{3}$. Thus the compensator designed,

$$G_c(s) = 128\left(\frac{s + 3.70}{s + 53.35}\right)\left(\frac{s + 0.1442}{s + 0.01}\right)$$

is acceptable. The feedforward transfer function of the compensated system becomes

$$G_c(s)G(s) = \frac{1280(s + 3.7)(s + 0.1442)}{s(s + 53.35)(s + 0.01)(s + 2)(s + 8)}$$

A root-locus plot of the compensated system is shown in Figure 7–39(a). An enlarged root-locus plot near the origin is shown in Figure 7–39(b).




(a)

**Figure 7–39**
(a) Root-locus plot of compensated system; (b) root-locus plot near the origin.





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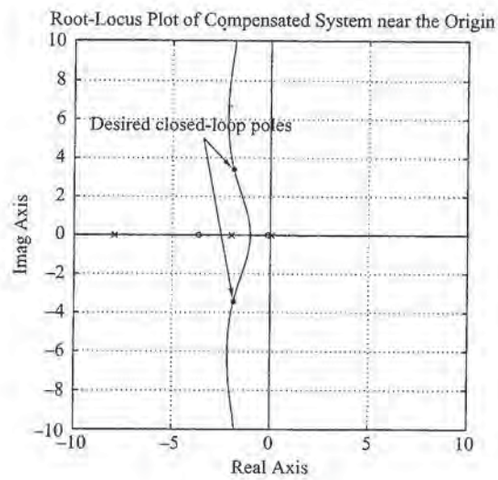
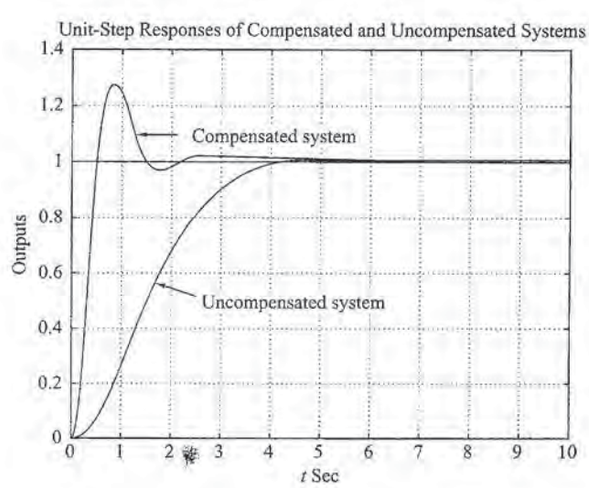



(b)

**Figure 7–39**
(*Continued*)

To verify the improved system performance of the compensated system, see the unit-step responses and unit-ramp responses of the compensated and uncompensated systems shown in Figures 7–40 (a) and (b), respectively.




(a)

**Figure 7–40**
(a) Unit step responses of compensated and uncompensated systems; (b) unit-ramp responses of both systems.





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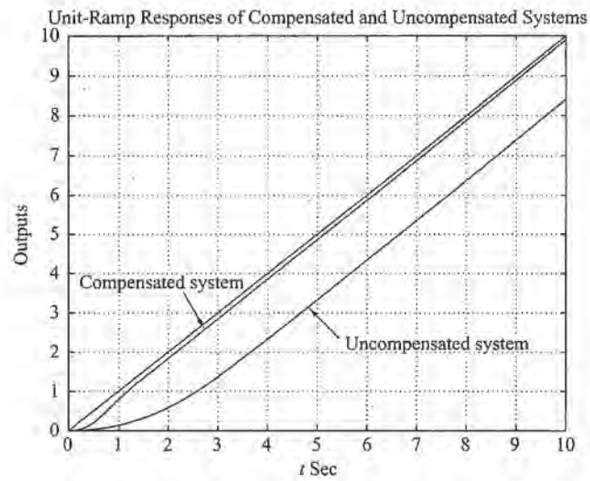


(b)

**Figure 7–40**
(*Continued*)

**A–7–9.** Consider the system shown in Figure 7–41. Design a lag–lead compensator such that the static velocity error constant $K_v$ is 50 sec$^{-1}$ and the damping ratio $\zeta$ of the dominant closed-loop poles is 0.5. (Choose the zero of the lead portion of the lag–lead compensator to cancel the pole at $s = -1$ of the plant.) Determine all closed-loop poles of the compensated system.

**Solution.** Let us employ the lag–lead compensator given by

$$G_c(s) = K_c\left(\frac{s + \dfrac{1}{T_1}}{s + \dfrac{\beta}{T_1}}\right)\left(\frac{s + \dfrac{1}{T_2}}{s + \dfrac{1}{\beta T_2}}\right) = k_c\frac{(T_1 s + 1)(T_2 s + 1)}{\left(\dfrac{T_1}{\beta}s + 1\right)(\beta T_2 s + 1)}$$

where $\beta > 1$. Then

$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(s) \\ &= \lim_{s\to 0} s\frac{K_c(T_1 s + 1)(T_2 s + 1)}{\left(\dfrac{T_1}{\beta}s + 1\right)(\beta T_2 s + 1)}\frac{1}{s(s+1)(s+5)} \\ &= \frac{K_c}{5} \end{aligned}$$

The specification that $K_v = 50$ sec$^{-1}$ determines the value of $K_c$, or

$$K_c = 250$$

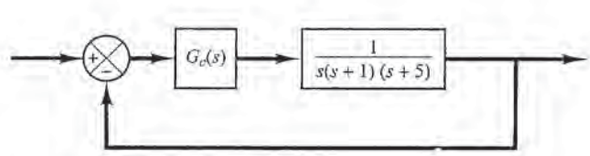


**Figure 7–41**
Control system.





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We now choose $T_1 = 1$ so that $s + (1/T_1)$ will cancel the $(s + 1)$ term of the plant. The lead portion then becomes

$$\frac{s+1}{s+\beta}$$

For the lag portion of the lag–lead compensator we require

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right| \doteqdot 1, \qquad -5^\circ < \angle\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

where $s = s_1$ is one of the dominant closed-loop poles. For $s = s_1$, the open-loop transfer function becomes

$$G_c(s_1)G(s_1) \doteqdot K_c\left(\frac{s_1+1}{s_1+\beta}\right)\frac{1}{s_1(s_1+1)(s_1+5)} = K_c\frac{1}{s_1(s_1+\beta)(s_1+5)}$$

Noting that at $s = s_1$ the magnitude and angle conditions are satisfied, we have

$$\left|K_c\frac{1}{s_1(s_1+\beta)(s_1+5)}\right| = 1 \tag{7–13}$$

$$\angle K_c\frac{1}{s_1(s_1+\beta)(s_1+5)} = \pm 180^\circ(2k+1) \tag{7–14}$$

where $k = 0, 1, 2, \ldots$. In Equations (7–13) and (7–14), $\beta$ and $s_1$ are unknowns. Since the damping ratio $\zeta$ of the dominant closed-loop poles is specified as 0.5, the closed-loop pole $s = s_1$ can be written as

$$s_1 = -x + j\sqrt{3}x$$

where $x$ is as yet undetermined.

Notice that the magnitude condition, Equation (7–13), can be rewritten as

$$\left|\frac{K_c}{(-x+j\sqrt{3}x)(-x+\beta+j\sqrt{3}x)(-x+5+j\sqrt{3}x)}\right| = 1$$

Noting that $K_c = 250$, we have

$$x\sqrt{(\beta - x)^2 + 3x^2}\,\sqrt{(5-x)^2 + 3x^2} = 125 \tag{7–15}$$

The angle condition, Equation (7–14), can be rewritten as

$$\angle K_c\frac{1}{(-x+j\sqrt{3}x)(-x+\beta+j\sqrt{3}x)(-x+5+j\sqrt{3}x)}$$

$$= -120^\circ - \tan^{-1}\left(\frac{\sqrt{3}x}{-x+\beta}\right) - \tan^{-1}\left(\frac{\sqrt{3}x}{-x+5}\right) = -180^\circ$$

or

$$\tan^{-1}\left(\frac{\sqrt{3}x}{-x+\beta}\right) + \tan^{-1}\left(\frac{\sqrt{3}x}{-x+5}\right) = 60^\circ \tag{7–16}$$





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We need to solve Equations (7–15) and (7–16) for $\beta$ and $x$. By several trial-and-error calculations, it can be found that

$$\beta = 16.025, \qquad x = 1.9054$$

Thus

$$s_1 = -1.9054 + j\sqrt{3}\,(1.9054) = -1.9054 + j3.3002$$

The lag portion of the lag–lead compensator can be determined as follows: Noting that the pole and zero of the lag portion of the compensator must be located near the origin, we may choose

$$\frac{1}{\beta T_2} = 0.01$$

That is,

$$\frac{1}{T_2} = 0.16025 \qquad \text{or} \qquad T_2 = 6.25$$

With the choice of $T_2 = 6.25$, we find

$$\left|\frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}}\right| = \left|\frac{-1.9054 + j3.3002 + 0.16025}{-1.9054 + j3.3002 + 0.01}\right|$$

$$= \left|\frac{-1.74515 + j3.3002}{-1.89054 + j3.3002}\right| = 0.98 \doteq 1 \tag{7–17}$$

and

$$\angle\frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} = \angle\frac{-1.9054 + j3.3002 + 0.16025}{-1.9054 + j3.3002 + 0.01}$$

$$= \tan^{-1}\left(\frac{3.3002}{-1.74515}\right) - \tan^{-1}\left(\frac{3.3002}{-1.89054}\right) = -1.937^\circ \tag{7–18}$$

Since

$$-5^\circ < -1.937^\circ < 0^\circ$$

our choice of $T_2 = 6.25$ is acceptable. Then the lag–lead compensator just designed can be written as

$$G_c(s) = 250\left(\frac{s + 1}{s + 16.025}\right)\left(\frac{s + 0.16025}{s + 0.01}\right)$$

Therefore, the compensated system has the following open-loop transfer function:

$$G_c(s)G(s) = \frac{250(s + 0.16025)}{s(s + 0.01)(s + 5)(s + 16.025)}$$

A root-locus plot of the compensated system is shown in Figure 7–42(a). An enlarged root-locus plot near the origin is shown in Figure 7–42(b).





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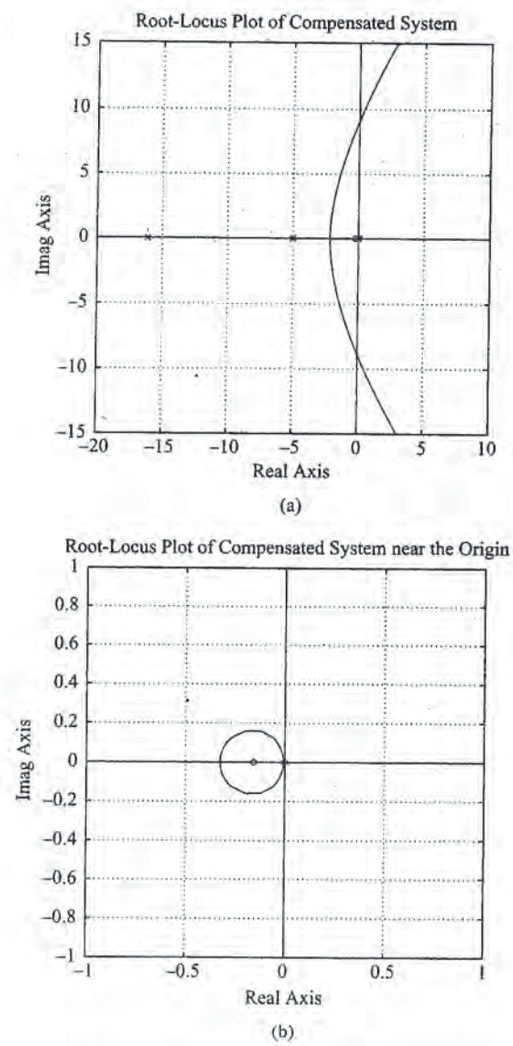


**Figure 7–42**
(a) Root-locus plot of compensated system; (b) root-locus plot near the origin.

The closed loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{250(s + 0.16025)}{s(s + 0.01)(s + 5)(s + 16.025) + 250(s + 0.16025)}$$

The closed-loop poles are located at

$$s = -1.8308 \pm j3.2359$$
$$s = -0.1684$$
$$s = -17.205$$





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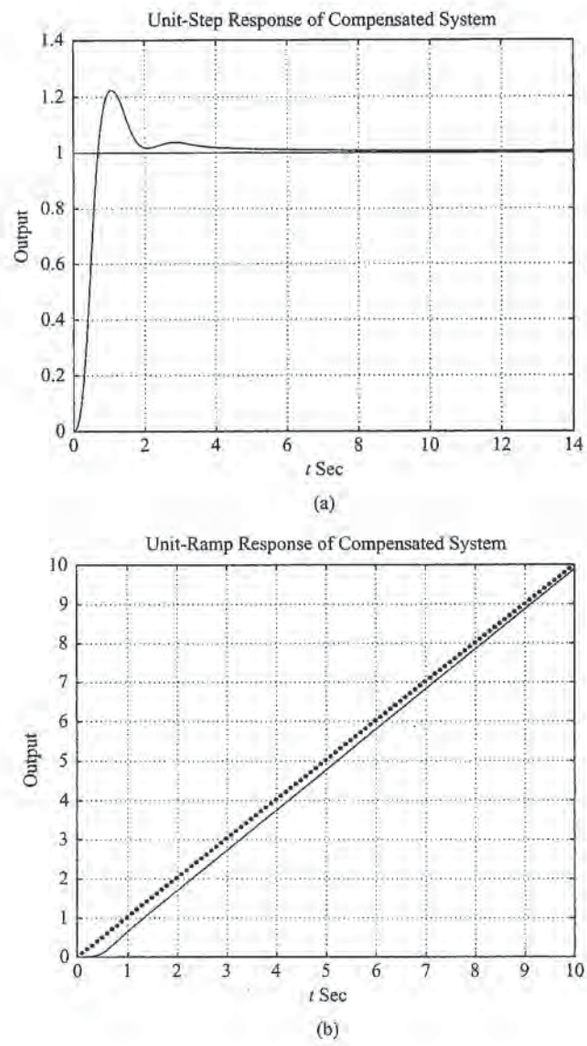


**Figure 7–43**
(a) Unit-step response of the compensated system; (b) unit-ramp response of the compensated system.

Notice that the dominant closed-loop poles $s = -1.8308 \pm j3.2359$ differ from the dominant closed-loop poles $s = \pm s_1$ assumed in the computation of $\beta$ and $T_2$. Small deviations of the dominant closed-loop poles $s = -1.8308 \pm j3.2359$ from $s = \pm s_1 = -1.9054 \pm j3.3002$ are due to the approximations involved in determining the lag portion of the compensator [See Equations (7–17) and (7–18)].

Figures 7–43(a) and (b) show the unit-step response and unit-ramp response of the designed system, respectively. Note that the closed-loop pole at $s = -0.1684$ almost cancels the zero at $s = -0.16025$. However, this pair of closed-loop pole and zero located near the origin produces a long tail of small amplitude. Since the closed-loop pole at $s = -17.205$ is located





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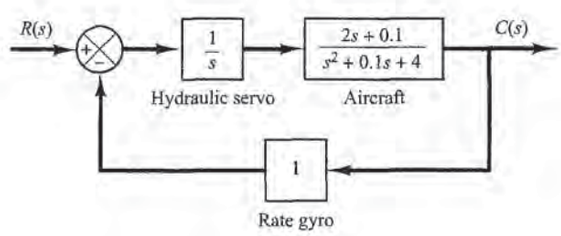
very much farther to the left compared to the closed-loop poles at $s = -1.8308 \pm j3.2359$, the effect of this real pole on the system response is very small. Therefore, the closed-loop poles at $s = -1.8308 \pm j3.2359$ are indeed dominant closed-loop poles that determine the response characteristics of the closed-loop system. In the unit-ramp response, the steady-state error in following the unit-ramp input eventually becomes $1/K_v = \frac{1}{50} = 0.02$.

**A–7–10.** Figure 7–44(a) is a block diagram of a model for an attitude-rate control system. The closed-loop transfer function for this system is

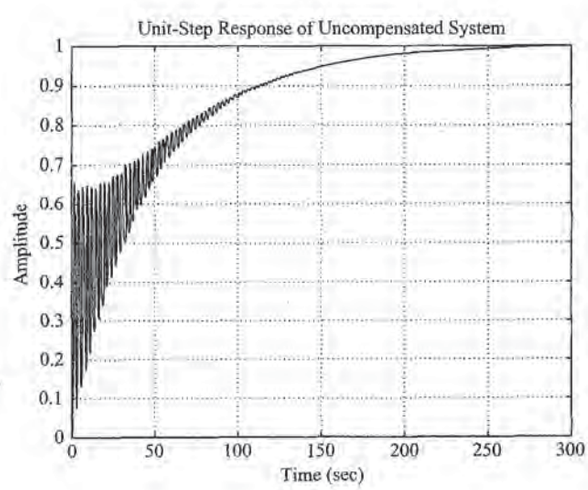
$$\frac{C(s)}{R(s)} = \frac{2s + 0.1}{s^3 + 0.1s^2 + 6s + 0.1}$$
$$= \frac{2(s + 0.05)}{(s + 0.0417 + j2.4489)(s + 0.0417 - j2.4489)(s + 0.0167)}$$

The unit-step response of this system is shown in Figure 7–44(b). The response shows high-frequency oscillations at the beginning of the response due to the poles at $s = -0.0417 \pm j2.4489$. The response is dominated by the pole at $s = -0.0167$. The settling time is approximately 240 sec.

It is desired to speed up the response and also eliminate the oscillatory mode at the beginning of the response. Design a suitable compensator such that the dominant closed-loop poles are at $s = -2 \pm j2\sqrt{3}$.




(a)




(b)

**Figure 7–44**
(a) Attitude-rate control system; (b) unit-step response.





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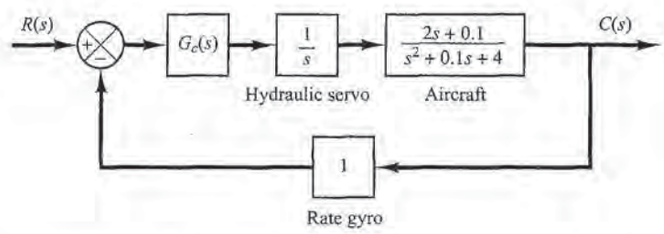


**Figure 7–45**
Compensated attitude-rate control system.

**Solution.** Figure 7–45 shows a block diagram for the compensated system. Note that the open-loop zero at $s = -0.05$ and the open-loop pole at $s = 0$ generate a closed-loop pole between $s = 0$ and $s = -0.05$. Such a closed-loop pole becomes a dominant closed-loop pole and make the response quite slow. Hence, it is necessary to replace this zero by a zero that is located far away from the $j\omega$ axis, for example, a zero at $s = -4$.

We now choose the compensator in the following form:

$$G_c(s) = \hat{G}_c(s)\frac{s+4}{2s+0.1}$$

Then the open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = \hat{G}_c(s)\frac{s+4}{2s+0.1}\frac{1}{s}\frac{2s+0.1}{s^2+0.1s+4}$$

$$= \hat{G}_c(s)\frac{s+4}{s(s^2+0.1s+4)}$$

To determine $\hat{G}_c(s)$ by the root-locus method, we need to find the angle deficiency at the desired closed-loop pole $s = -2 + j2\sqrt{3}$. The angle deficiency can be found as follows:

$$\text{Angle deficiency} = -143.088° - 120° - 109.642° + 60° + 180°$$

$$= -132.73°$$

Hence, the lead compensator $\hat{G}_c(s)$ must provide 132.73°. Since the angle deficiency is 132.73°, we need two lead compensators, each providing 66.365°. Thus $G_c(s)$ will have the following form:

$$G_c(s) = K_c\left(\frac{s+s_z}{s+s_p}\right)^2$$

Suppose that we choose two zeros at $s = -2$. Then the two poles of the lead compensators can be obtained from

$$\frac{3.4641}{s_p - 2} = \tan(90° - 66.365°) = 0.4376169$$

or

$$s_p = 2 + \frac{3.4641}{0.4376169}$$

$$= 9.9158$$

Hence,

$$\hat{G}_c(s) = K_c\left(\frac{s+2}{s+9.9158}\right)^2$$





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The entire compensator $G_c(s)$ for the system becomes

$$G_c(s) = \hat{G}_c(s)\frac{s+4}{2s+0.1} = K_c\frac{(s+2)^2}{(s+9.9158)^2}\frac{s+4}{2s+0.1}$$

The value of $K_c$ can be determined from the magnitude condition. Since the open-loop transfer function is

$$G_c(s)G(s) = K_c\frac{(s+2)^2(s+4)}{(s+9.9158)^2 s(s^2+0.1s+4)}$$

the magnitude condition becomes

$$\left|K_c\frac{(s+2)^2(s+4)}{(s+9.9158)^2 s(s^2+0.1s+4)}\right|_{s=-2+j2\sqrt{3}} = 1$$

Hence,

$$K_c = \left|\frac{(s+9.9158)^2 s(s^2+0.1s+4)}{(s+2)^2(s+4)}\right|_{s=-2+j2\sqrt{3}}$$
$$= 88.0227$$

Thus the compensator $G_c(s)$ becomes

$$G_c(s) = 88.0227\frac{(s+2)^2(s+4)}{(s+9.9158)^2(2s+0.1)}$$

The open-loop transfer function is given by

$$G_c(s)G(s) = \frac{88.0227(s+2)^2(s+4)}{(s+9.9158)^2 s(s^2+0.1s+4)}$$

A root-locus plot for the compensated system is shown in Figure 7–46. The closed-loop poles for the compensated system are indicated in the plot. The closed-loop poles, the roots of the characteristic equation

$$(s+9.9158)^2 s(s^2+0.1s+4) + 88.0227(s+2)^2(s+4) = 0$$

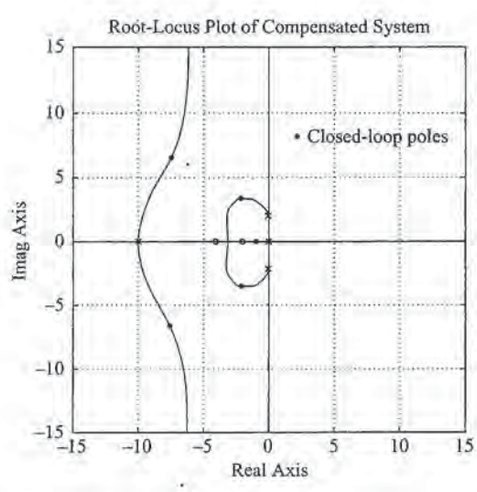


**Figure 7–46**
Root-locus plot of the compensated system.





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are as follows:

$$s = -2.0000 \pm j3.4641$$
$$s = -7.5224 \pm j6.5326$$
$$s = -0.8868$$

Now that the compensator has been designed, we shall examine the transient response characteristics with MATLAB. The closed-loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{88.0227(s+2)^2(s+4)}{(s+9.9158)^2 s(s^2+0.1s+4) + 88.0227(s+2)^2(s+4)}$$

Figures 7–47(a) and (b) show the plots of the unit-step response and unit-ramp response of the compensated system. These response curves show that the designed system is acceptable.

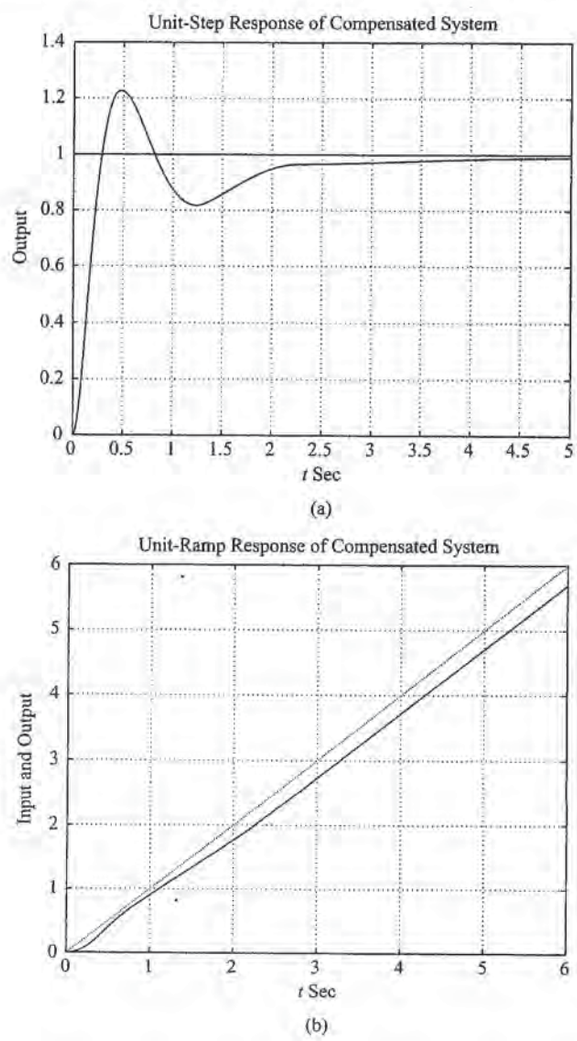


**Figure 7–47**
(a) Unit-step response of the compensated system; (b) unit-ramp response of the compensated system. (Problem **A–7–10**.)





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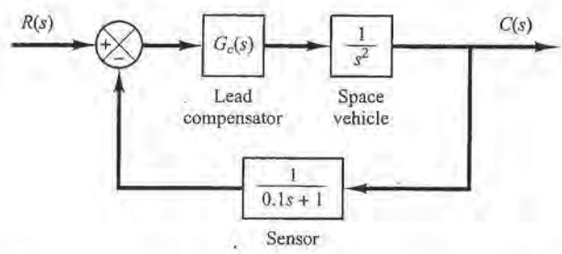


**Figure 7–48**
Space vehicle control system.

**A–7–11.** Consider the model for a space vehicle control system shown in Figure 7–48. Design a lead compensator $G_c(s)$ such that the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$ of the dominant closed-loop poles are 0.5 and 2 rad/sec, respectively.

**Solution.**
*First Attempt:* Assume the lead compensator $G_c(s)$ to be

$$G_c(s) = K_c\left(\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}\right) \qquad (0 < \alpha < 1)$$

From the given specifications, $\zeta = 0.5$ and $\omega_n = 2$ rad/sec, the dominant closed-loop poles must be located at

$$s = -1 \pm j\sqrt{3}$$

We first calculate the angle deficiency at this closed-loop pole.

$$\begin{aligned}\text{Angle deficiency} &= -120° - 120° - 10.8934° + 180° \\ &= -70.8934°\end{aligned}$$

This angle deficiency must be compensated by the lead compensator. There are many ways to determine the locations of the pole and zero of the lead network. Let us choose the zero of the compensator at $s = -1$. Then, referring to Figure 7–49, we have the following equation:

$$\frac{1.73205}{x-1} = \tan(90° - 70.8934°) = 0.34641$$

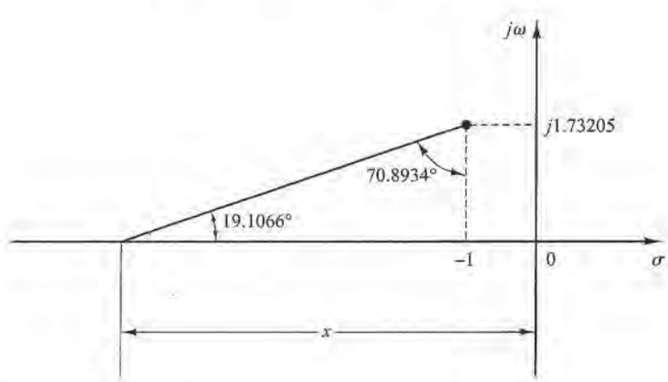


**Figure 7–49**
Determination of the pole of the lead network.





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or

$$x = 1 + \frac{1.73205}{0.34641} = 6$$

Hence,

$$G_c(s) = K_c \frac{s+1}{s+6}$$

The value of $K_c$ can be determined from the magnitude condition

$$K_c \left| \frac{s+1}{s+6} \frac{1}{s^2} \frac{1}{0.1s+1} \right|_{s=-1+j\sqrt{3}} = 1$$

as follows:

$$K_c = \left| \frac{(s+6)s^2(0.1s+1)}{s+1} \right|_{s=-1+j\sqrt{3}} = 11.2000$$

Thus

$$G_c(s) = 11.2 \frac{s+1}{s+6}$$

Since the open-loop transfer function becomes

$$G_c(s)G(s)H(s) = 11.2 \frac{s+1}{(s+6)s^2(0.1s+1)}$$
$$= \frac{11.2(s+1)}{0.1s^4 + 1.6s^3 + 6s^2}$$

a root-locus plot of the compensated system can be obtained easily with MATLAB by entering num and den and using rlocus command. The result is shown in Figure 7–50.

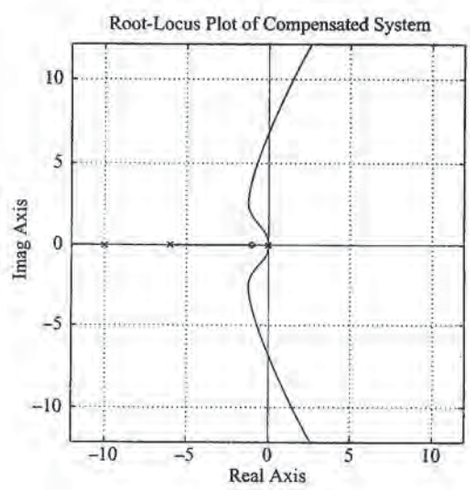


**Figure 7–50**
Root-locus plot of the compensated system.





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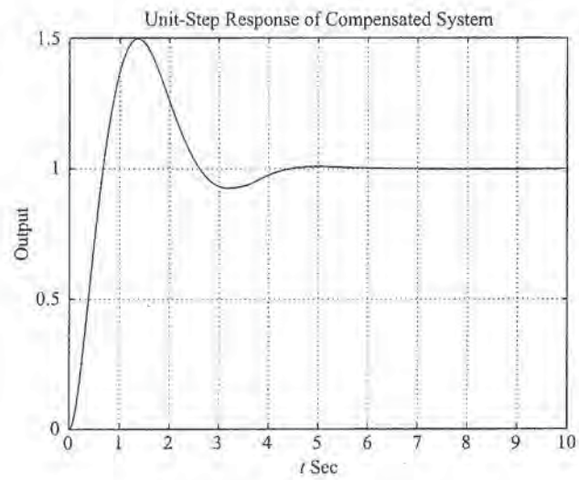


**Figure 7–51**
Unit-step response of the compensated system.

The closed-loop transfer function for the compensated system becomes

$$\frac{C(s)}{R(s)} = \frac{11.2(s+1)(0.1s+1)}{(s+6)s^2(0.1s+1) + 11.2(s+1)}$$

Figure 7–51 shows the unit-step response curve. Even though the damping ratio of the dominant closed-loop poles is 0.5, the amount of overshoot is very much higher than expected. A closer look at the root-locus plot reveals that the presence of the zero at $s = -1$ is increasing the amount of the maximum overshoot. [In general, if a closed-loop zero or zeros (compensator zero or zeros) lie to the right of the dominant pair of the complex poles, then the dominant poles are no longer dominant.] If large maximum overshoot cannot be tolerated, the compensator zero(s) should be shifted so that the zero(s) will almost cancel the real closed-loop pole(s).

In the current design, it is desirable to modify the compensator and make the maximum overshoot smaller. This can be done by modifying the lead compensator, as presented in the following second attempt.

*Second Attempt:* To modify the shape of the root loci, we may use two lead networks, each contributing half the necessary lead angle, which is $70.8934°/2 = 35.4467°$. Let us choose the location of the zeros at $s = -3$. (This is an arbitrary choice. Other choices such as $s = -2.5$ and $s = -4$ may be made.)

Once we choose two zeros at $s = -3$, the necessary location of the poles can be determined as shown in Figure 7–52, or

$$\frac{1.73205}{y-1} = \tan(40.89334° - 35.4467°)$$
$$= \tan 5.4466° = 0.09535$$

which yields

$$y = 1 + \frac{1.73205}{0.09535} = 19.1652$$





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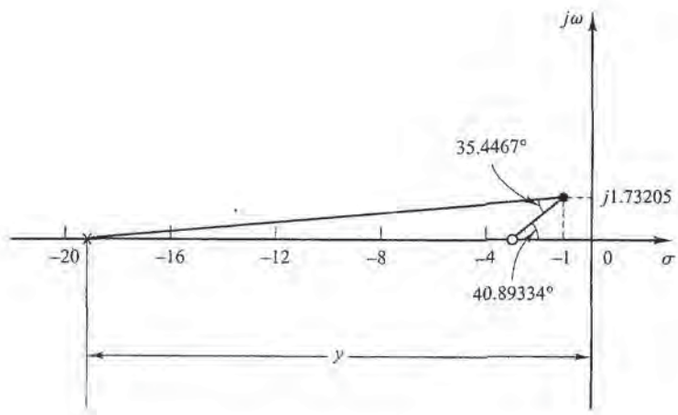


**Figure 7–52**
Determination of the pole of the lead network.

Hence, the lead compensator will have the following transfer function:

$$G_c(s) = K_c\left(\frac{s+3}{s+19.1652}\right)^2$$

The value of $K_c$ can be determined from the magnitude condition as follows:

$$\left|K_c\left(\frac{s+3}{s+19.1652}\right)^2 \frac{1}{s^2}\frac{1}{0.1s+1}\right|_{s=-1+j\sqrt{3}} = 1$$

or

$$K_c = 174.3864$$

Then the lead compensator just designed is

$$G_c(s) = 174.3864\left(\frac{s+3}{s+19.1652}\right)^2$$

Then the open-loop transfer function becomes

$$G_c(s)G(s)H(s) = 174.3864\left(\frac{s+3}{s+19.1652}\right)^2 \frac{1}{s^2}\frac{1}{0.1s+1}$$

A root-locus plot for the compensated system is shown in Figure 7–53(a). Notice that there is no closed-loop zero near the origin. An expanded view of the root-locus plot near the origin is shown in Figure 7–53(b).

The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{174.3864(s+3)^2(0.1s+1)}{(s+19.1652)^2 s^2(0.1s+1) + 174.3864(s+3)^2}$$

The closed-loop poles are found as follows:

$$s = -1 \pm j1.73205$$

$$s = -9.1847 \pm j7.4814$$

$$s = -27.9606$$





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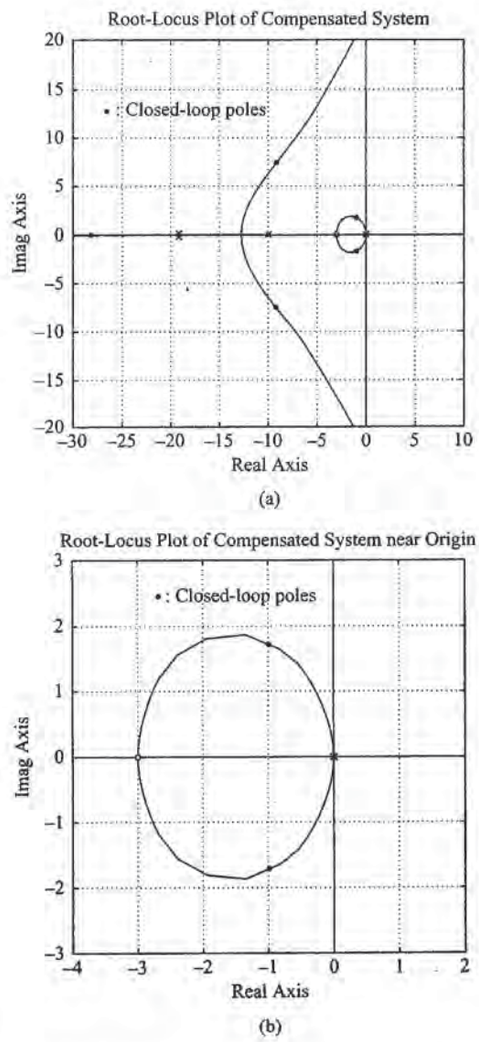


**Figure 7–53**
(a) Root-locus plot of compensated system; (b) root-locus plot near the origin.

Figures 7–54(a) and (b) show the unit-step response and unit-ramp response of the compensated system. The unit-step response curve is reasonable and the unit-ramp response looks acceptable. Notice that in the unit-ramp response the output leads the input by a small amount. This is because the system has a feedback transfer function $1/(0.1s + 1)$. If the feedback signal versus $t$ is plotted, together with the unit-ramp input, the former will not lead the input ramp at steady state. See Figure 7–54(c).





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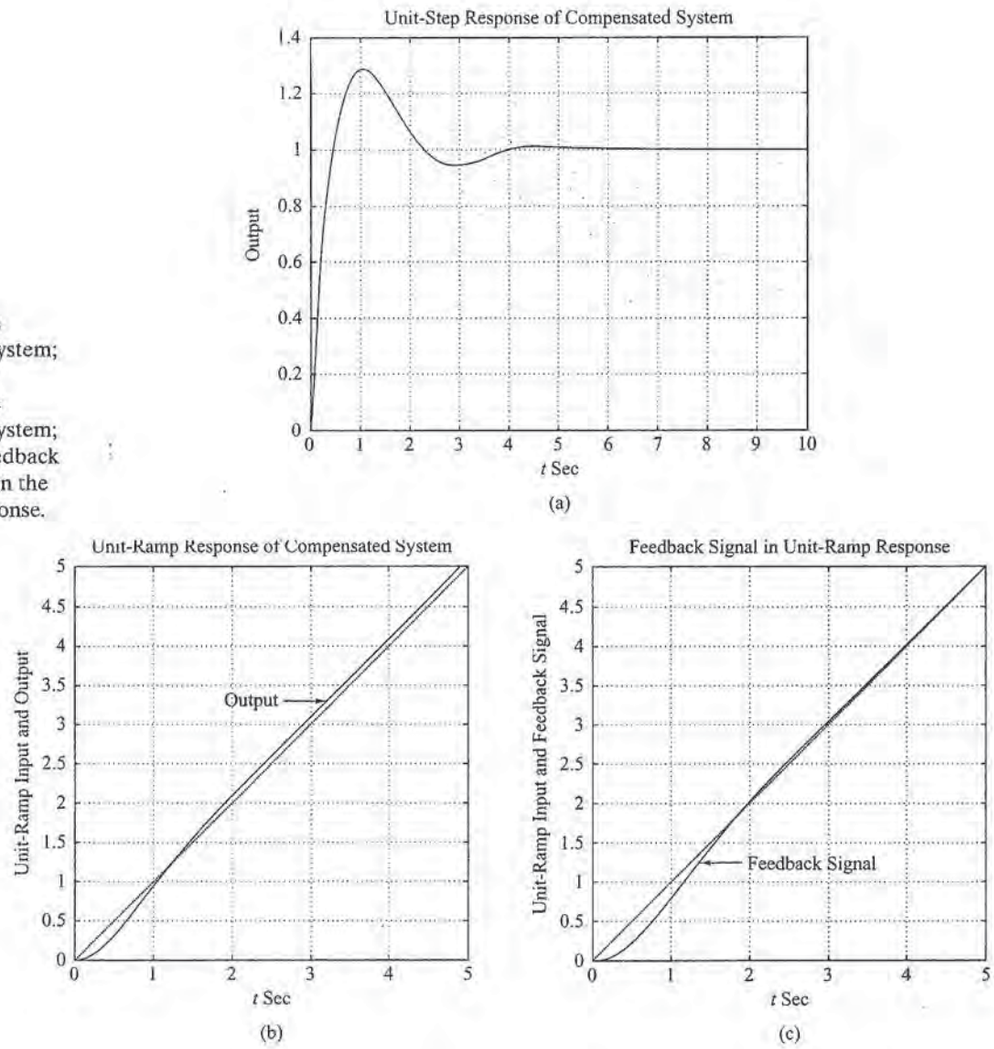


**Figure 7–54**
(a) unit-step response of the compensated system; (b) unit-ramp response of the compensated system; (c) a plot of feedback signal versus $t$ in the unit-ramp response.

**A–7–12.** Consider the system shown in Figure 7–55(a). Determine the value of $a$ such that the damping ratio $\zeta$ of the dominant closed poles is 0.5.

**Solution.** The characteristic equation is

$$1 + \frac{10(s + a)}{s(s + 1)(s + 8)} = 0$$

The variable $a$ is not a multiplying factor. Hence, we need to modify the characteristic equation. Since the characteristic equation can be written as

$$s^3 + 9s^2 + 18s + 10a = 0$$





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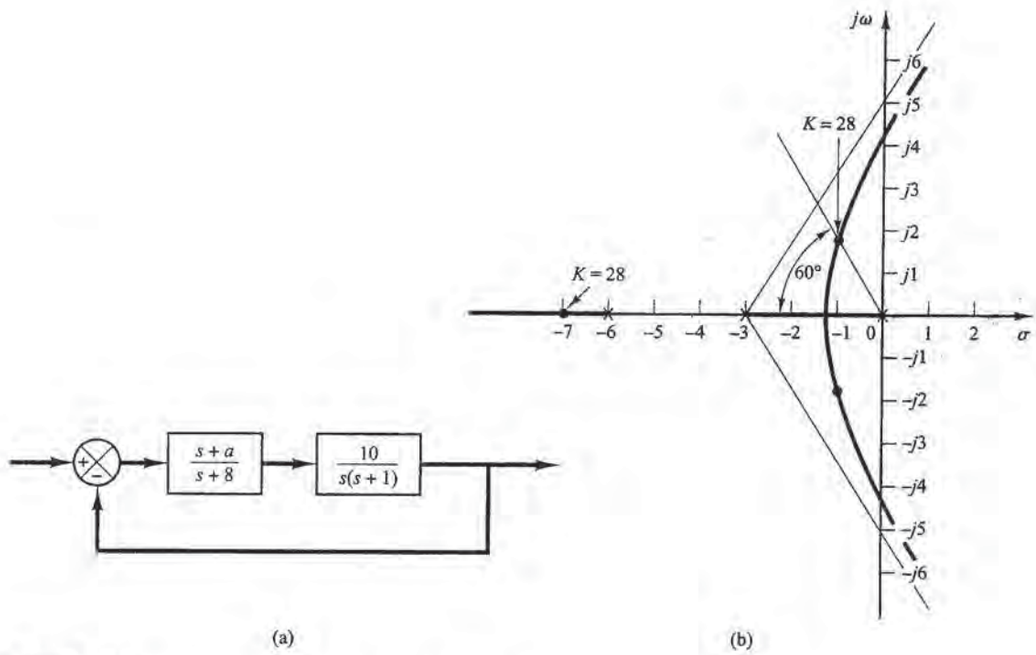


**Figure 7–55**
(a) Control system; (b) root-locus plot, where $K = 10a$.

we rewrite this equation such that $a$ appears as a multiplying factor as follows:

$$1 + \frac{10a}{s(s^2 + 9s + 18)} = 0$$

Define

$$10a = K$$

Then the characteristic equation becomes

$$1 + \frac{K}{s(s^2 + 9s + 18)} = 0$$

Notice that the characteristic equation is in a suitable form for the construction of the root loci.

This system involves three poles and no zero. The three poles are at $s = 0$, $s = -3$, and $s = -6$. A root-locus branch exists on the real axis between points $s = 0$ and $s = -3$. Also, another branch exists between points $s = -6$ and $s = -\infty$.

The asymptotes for the root loci are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

The intersection of the asymptotes and the real axis is obtained from

$$s = -\frac{0 + 3 + 6}{3} = -3$$





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The breakaway and break-in points can be determined from $dK/ds = 0$, where

$$K = -(s^3 + 9s^2 + 18s)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 18s + 18) = 0$$

which yields

$$s^2 + 6s + 6 = 0$$

or

$$s = -1.268, \qquad s = -4.732$$

Point $s = -1.268$ is on a root-locus branch. Hence, point $s = -1.268$ is an actual breakaway point. But point $s = -4.732$ is not on the root locus and therefore is neither a breakaway nor break-in point.

Next we shall find points where root-locus branches cross the imaginary axis. We substitute $s = j\omega$ in the characteristic equation, which is

$$s^3 + 9s^2 + 18s + K = 0$$

as follows:

$$(j\omega)^3 + 9(j\omega)^2 + 18(j\omega) + K = 0$$

or

$$(K - 9\omega^2) + j\omega(18 - \omega^2) = 0$$

from which we get

$$\omega = \pm 3\sqrt{2}, \qquad K = 9\omega^2 = 162 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

The crossing points are at $\omega = \pm 3\sqrt{2}$ and the corresponding value of gain $K$ is 162. Also, a root-locus branch touches the imaginary axis at $\omega = 0$. Figure 7–55(b) shows a sketch of the root loci for the system.

Since the damping ratio of the dominant closed-loop poles is specified as 0.5, the desired closed-loop pole in the upper-half $s$ plane is located at the intersection of the root-locus branch in the upper-half $s$ plane and a straight line having an angle of 60° with the negative real axis. The desired dominant closed-loop poles are located at

$$s = -1 + j1.732, \qquad s = -1 - j1.732$$

At these points, the value of gain $K$ is 28. Hence,

$$a = \frac{K}{10} = 2.8$$

Since the system involves two or more poles than zeros (in fact, three poles and no zero), the third pole can be located on the negative real axis from the fact that the sum of the three closed-loop poles is −9. Hence, the third pole is found to be at

$$s = -9 - (-1 + j1.732) - (-1 - j1.732)$$

or

$$s = -7$$





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**A–7–13.** Consider the system shown in Figure 7–56(a). Sketch the root loci of the system as the velocity feedback gain $k$ varies from zero to infinity. Determine the value of $k$ such that the closed-loop poles have the damping ratio $\zeta$ of 0.7.

**Solution.** The open-loop transfer function is

$$\text{Open-loop transfer function} = \frac{10}{(s + 1 + 10k)s}$$

Since $k$ is not a multilying factor, we modify the equation such that $k$ appears as a multiplying factor. Since the characteristic equation is

$$s^2 + s + 10ks + 10 = 0$$

we rewrite this equation as follows:

$$1 + \frac{10ks}{s^2 + s + 10} = 0 \tag{7–19}$$

Define

$$10k = K$$

Then Equation (7–19) becomes

$$1 + \frac{Ks}{s^2 + s + 10} = 0$$

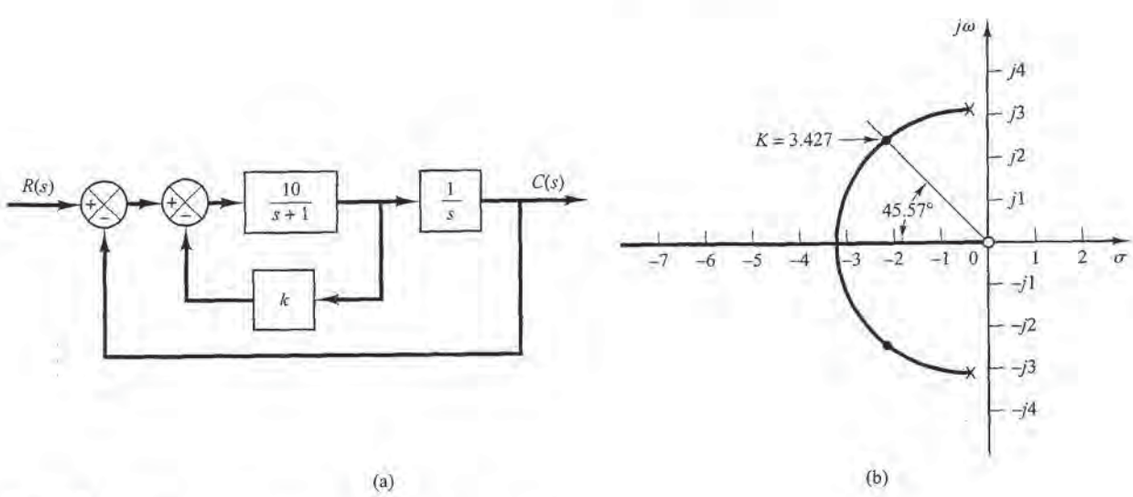


**Figure 7–56**
(a) Control system; (b) root-locus plot, where $K = 10k$.





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Notice that the system has a zero at $s = 0$ and two poles at $s = -0.5 \pm j3.1225$. Since this system involves two poles and one zero, there is a possibility that a circular root locus exists. In fact, this system has a circular root locus, as will be shown. Since the angle condition is

$$\underline{\left/ \frac{Ks}{s^2 + s + 10} \right.} = \pm 180°(2k + 1)$$

we have

$$\underline{/s} - \underline{/s + 0.5 + j3.1225} - \underline{/s + 0.5 - j3.1225} = \pm 180°(2k + 1)$$

By substituting $s = \sigma + j\omega$ into this last equation and rearranging, we obtain

$$\underline{/\sigma + 0.5 + j(\omega + 3.1225)} + \underline{/\sigma + 0.5 + j(\omega - 3.1225)} = \underline{/\sigma + j\omega} \pm 180°(2k + 1)$$

which can be rewritten as

$$\tan^{-1}\left(\frac{\omega + 3.1225}{\sigma + 0.5}\right) + \tan^{-1}\left(\frac{\omega - 3.1225}{\sigma + 0.5}\right) = \tan^{-1}\left(\frac{\omega}{\sigma}\right) \pm 180°(2k + 1)$$

Taking tangents of both sides of this last equation, we obtain

$$\frac{\dfrac{\omega + 3.1225}{\sigma + 0.5} + \dfrac{\omega - 3.1225}{\sigma + 0.5}}{1 - \left(\dfrac{\omega + 3.1225}{\sigma + 0.5}\right)\left(\dfrac{\omega - 3.1225}{\sigma + 0.5}\right)} = \frac{\omega}{\sigma}$$

which can be simplified to

$$\frac{2\omega(\sigma + 0.5)}{(\sigma + 0.5)^2 - (\omega^2 - 3.1225^2)} = \frac{\omega}{\sigma}$$

or

$$\omega(\sigma^2 - 10 + \omega^2) = 0$$

which yields

$$\omega = 0 \qquad \text{or} \qquad \sigma^2 + \omega^2 = 10$$

Notice that $\omega = 0$ corresponds to the real axis. The negative real axis (between $s = 0$ and $s = -\infty$) corresponds to $K \geq 0$, and the positive real axis corresponds to $K < 0$. The equation

$$\sigma^2 + \omega^2 = 10$$

is an equation of a circle with center at $\sigma = 0$, $\omega = 0$ with the radius equal to $\sqrt{10}$. A portion of this circle that lies to the left of the complex poles corresponds to the root locus for $K > 0$. The portion of the circle which lies to the right of the complex poles corresponds to the root locus for $K < 0$. Figure 7–56(b) shows a sketch of the root loci.

Since we require $\zeta = 0.7$ for the closed-loop poles, we find the intersection of the circular root locus and a line having an angle of 45.57° (note that $\cos 45.57° = 0.7$) with the negative real axis. The intersection is at $s = -2.214 + j2.258$. The gain $K$ corresponding to this point is 3.427. Hence, the desired value of the velocity feedback gain $k$ is

$$k = \frac{K}{10} = 0.3427$$





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## PROBLEMS

**B–7–1.** Consider the mechanical system shown in Figure 7–57. It consists of a spring and two dashpots. Obtain the transfer function of the system. The displacement $x_i$ is the input and displacement $x_o$ is the output. Is this system a mechanical lead network or lag network?

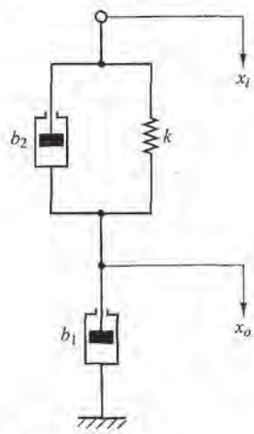


**Figure 7–57**
Mechanical system.

**B–7–2.** Obtain the transfer function of the electrical network shown in Figure 7–58. Show that it is a lag network.

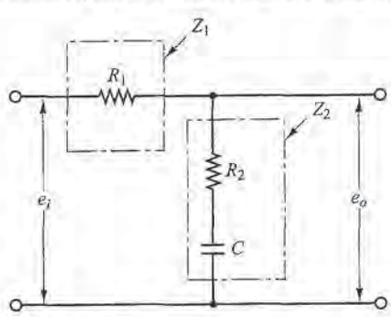


**Figure 7–58**
Electrical network.

**B–7–3.** Is the following $G_c(s)$ a lead network or lag network?

$$G_c(s) = \frac{3.5s + 1.4}{s + 2}$$

**B–7–4.** Consider the following $G_c(s)$:

$$G_c(s) = K\frac{s + b}{s + a}$$

where $a$, $b$, and $K$ are positive real numbers. What is the condition for $a$ and $b$ for $G_c(s)$ to be a lead network?

**B–7–5.** Design an op-amp controller to realize the following phase-lead transfer function:

$$G_c(s) = \frac{5(s + 1)}{s + 8}$$

**B–7–6.** Consider the system shown in Figure 7–59. Plot the root loci for the system. Determine the value of $K$ such that the damping ratio $\zeta$ of the dominant closed-loop poles is 0.5. Then determine all closed-loop poles. Plot the unit-step response curve with MATLAB.

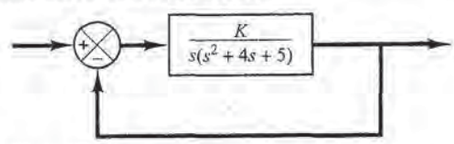


**Figure 7–59**
Control system.

**B–7–7.** Determine the values of $K$, $T_1$, and $T_2$ of the system shown in Figure 7–60 so that the dominant closed-loop poles have the damping ratio $\zeta = 0.5$ and the undamped natural frequency $\omega_n = 3$ rad/sec.

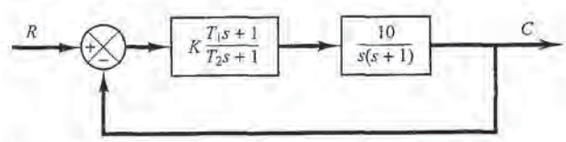


**Figure 7–60**
Control system.

**B–7–8.** Consider the control system shown in Figure 7–61. Determine the gain $K$ and time constant $T$ of the controller $G_c(s)$ such that the closed-loop poles are located at $s = -2 \pm j2$.

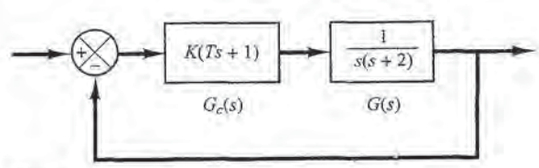


**Figure 7–61**
Control system.





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**B–7–9.** Consider the system shown in Figure 7–62. Design a lead compensator such that the dominant closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$. Plot the unit-step response curve of the designed system with MATLAB.

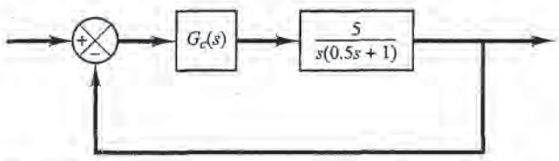


**Figure 7–62**
Control system.

**B–7–10.** Consider the system shown in Figure 7–63. Design a compensator such that the dominant closed-loop poles are located at $s = -1 \pm j1$.

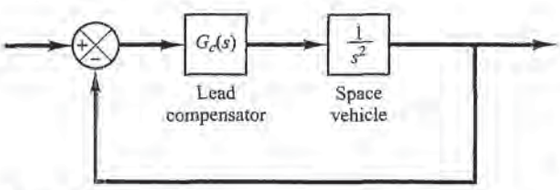


**Figure 7–63**
Control system.

**B–7–11.** Referring to the system shown in Figure 7–64, design a compensator such that the static velocity error constant $K_v$ is 20 sec$^{-1}$ without appreciably changing the original location $(s = -2 \pm j2\sqrt{3})$ of a pair of the complex-conjugate closed-loop poles.

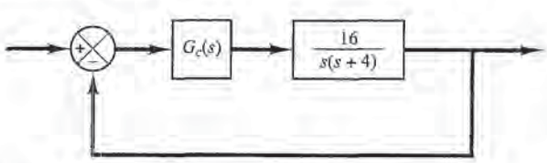


**Figure 7–64**
Control system.

**B–7–12.** Consider the angular-positional system shown in Figure 7–65. The dominant closed-loop poles are located at $s = -3.60 \pm j4.80$. The damping ratio $\zeta$ of the dominant closed-loop poles is 0.6. The static velocity error constant $K_v$ is 4.1 sec$^{-1}$, which means that for a ramp input of 360°/sec the steady-state error in following the ramp input is

$$e_v = \frac{\theta_i}{K_v} = \frac{360°/\text{sec}}{4.1\ \text{sec}^{-1}} = 87.8°$$

It is desired to decrease $e_v$ to one-tenth of the present value, or to increase the value of the static velocity error constant $K_v$ to 41 sec$^{-1}$. It is also desired to keep the damping ratio $\zeta$ of the dominant closed-loop poles at 0.6. A small change in the undamped natural frequency $\omega_n$ of the dominant closed-loop poles is permissible. Design a suitable lag compensator to increase the static velocity error constant as desired.

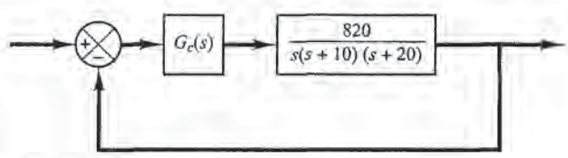


**Figure 7–65**
Angular-positional system.

**B–7–13.** Consider the control system shown in Figure 7–66. Design a compensator such that the dominant closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$ and the static velocity error constant $K_v$ is 50 sec$^{-1}$.

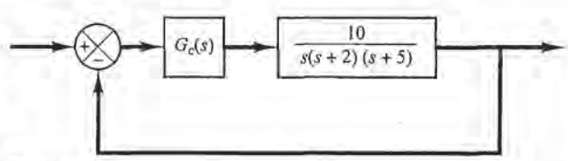


**Figure 7–66**
Control system.

**B–7–14.** Consider the control system shown in Figure 7–67. The plant is critically stable in the sense that oscillations will continue indefinitely. Design a suitable compensator such that the unit-step response will exhibit maximum overshoot of less than 40% and settling time of 5 sec or less.

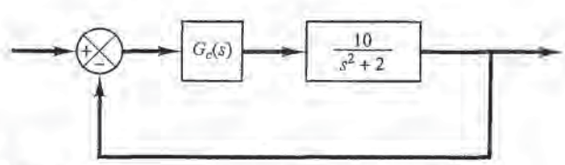


**Figure 7–67**
Control system.





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**B–7–15.** Consider the control system shown in Figure 7–68. Design a compensator such that the unit-step response curve will exhibit maximum overshoot of 30% or less and settling time of 3 sec or less.

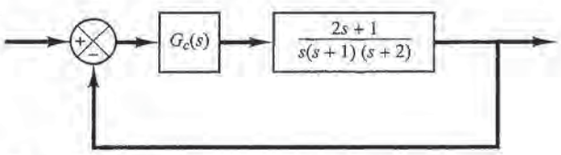


**Figure 7–68**
Control system.

**B–7–16.** Consider the control system shown in Figure 7–69. Design a compensator such that the unit-step response curve will exhibit maximum overshoot of 25% or less and settling time of 5 sec or less.

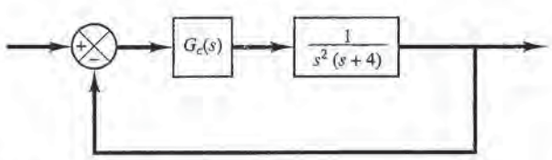


**Figure 7–69**
Control system.

**B–7–17.** Consider the system shown in Figure 7–70, which involves velocity feedback. Determine the values of the amplifier gain $K$ and the velocity feedback gain $K_h$ so that the following specifications are satisfied:

1. Damping ratio of the closed-loop poles is 0.5
2. Settling time $\leq$ 2 sec
3. Static velocity error constant $K_v \geq 50\ \text{sec}^{-1}$
4. $0 < K_h < 1$

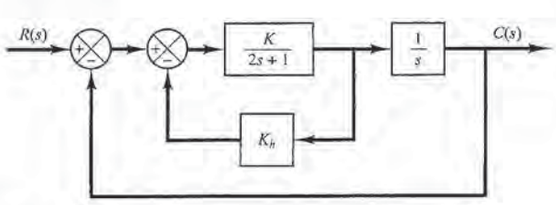


**Figure 7–70**
Control system.

**B–7–18.** Consider the system shown in Figure 7–71. Determine the values of the gain $K$ and the velocity feedback coefficient $K_h$ so that the closed-loop poles are at $s = -1 \pm j\sqrt{3}$. Then, using the determined value of $K_h$, plot the root loci.

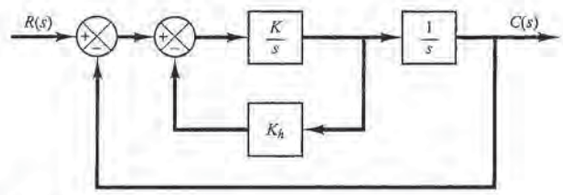


**Figure 7–71**
Control system.

**B–7–19.** Consider the system shown in Figure 7–72. The system involves velocity feedback. Determine the value of gain $K$ such that the dominant closed-loop poles have a damping ratio of 0.5. Using the gain $K$ thus determined, obtain the unit-step response of the system.

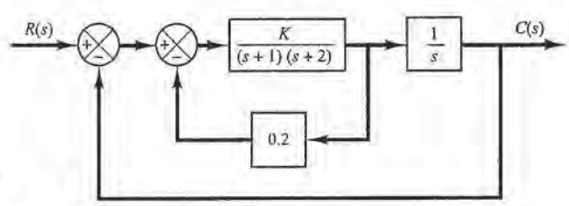


**Figure 7–72**
Control system.

**B–7–20.** Consider the system shown in Figure 7–73. Plot the root loci as $a$ varies from 0 to $\infty$. Determine the value of $a$ such that the damping ratio of the dominant closed-loop poles is 0.5.

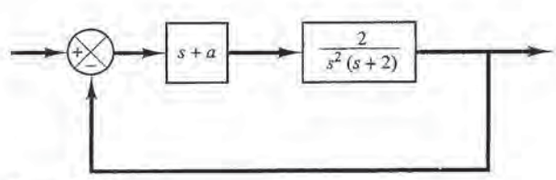


**Figure 7–73**
Control system.





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**B–7–21.** Consider the system shown in Figure 7–74. Plot the root loci as the value of $k$ varies from 0 to $\infty$. What value of $k$ will give a damping ratio of the dominant closed-loop poles equal to 0.5? Find the static velocity error constant of the system with this value of $k$.

**B–7–22.** Consider the system shown in Figure 7–75. Assuming that the value of gain $K$ varies from 0 to $\infty$, plot the root loci when $K_h = 0.1$, 0.3, and 0.5.

Compare unit-step responses of the system for the following three cases:

(1) $K = 10, \quad K_h = 0.1$

(2) $K = 10, \quad K_h = 0.3$

(3) $K = 10, \quad K_h = 0.5$

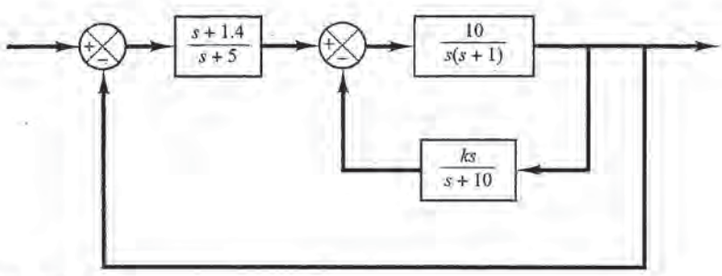


**Figure 7–74**
Control system.

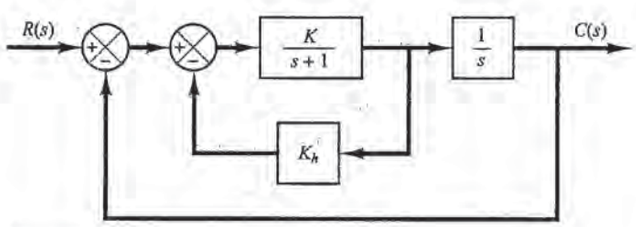


**Figure 7–75**
Control system.





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# 8

# Frequency-Response Analysis

## 8-1 INTRODUCTION

By the term *frequency response,* we mean the steady-state response of a system to a sinusoidal input. In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

In this and the next chapter we present frequency-response approaches to the analysis and design of control systems. The information we get from such analysis is different from what we get from root-locus analysis. In fact, the frequency response and root-locus approaches complement each other. One advantage of the frequency-response approach is that we can use the data obtained from measurements on the physical system without deriving its mathematical model. In many practical designs of control systems both approaches are employed. Control engineers must be familiar with both.

Frequency-response methods were developed in 1930s and 1940s by Nyquist, Bode, Nichols, and many others. The frequency-response methods are most powerful in conventional control theory. They are also indispensable to robust control theory.

The Nyquist stability criterion enables us to investigate both the absolute and relative stabilities of linear closed-loop systems from a knowledge of their open-loop frequency-response characteristics. An advantage of the frequency-response approach is that frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often the transfer functions of complicated components can be determined experimentally by frequency-response tests. In addition, the frequency-response approach has the advantages that a system may be designed so that the effects of undesirable noise are negligible and that such analysis and design can be extended to certain nonlinear control systems.





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Although the frequency response of a control system presents a qualitative picture of the transient response, the correlation between frequency and transient responses is indirect, except for the case of second-order systems. In designing a closed-loop system, we adjust the frequency-response characteristic of the open-loop transfer function by using several design criteria in order to obtain acceptable transient-response characteristics for the system.

**Obtaining Steady-State Outputs to Sinusoidal Inputs.** We shall show that the steady-state output of a transfer function system can be obtained directly from the sinusoidal transfer function, that is, the transfer function in which $s$ is replaced by $j\omega$, where $\omega$ is frequency.

Consider the stable, linear, time-invariant system shown in Figure 8–1. The input and output of the system, whose transfer function is $G(s)$, are denoted by $x(t)$ and $y(t)$, respectively. If the input $x(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

Let us assume that the input signal is given by

$$x(t) = X \sin \omega t$$

Suppose that the transfer function $G(s)$ can be written as a ratio of two polynomials in $s$; that is,

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + s_1)(s + s_2)\cdots(s + s_n)}$$

The Laplace-transformed output $Y(s)$ is then

$$Y(s) = G(s)X(s) = \frac{p(s)}{q(s)} X(s) \tag{8–1}$$

where $X(s)$ is the Laplace transform of the input $x(t)$.

It will be shown that, after waiting until steady-state conditions are reached, the frequency response can be calculated by replacing $s$ in the transfer function by $j\omega$. It will also be shown that the steady-state response can be given by

$$G(j\omega) = Me^{j\phi} = M\underline{/\phi}$$

where $M$ is the amplitude ratio of the output and input sinusoids and $\phi$ is the phase shift between the input sinusoid and the output sinusoid. In the frequency-response test, the input frequency $\omega$ is varied until the entire frequency range of interest is covered.

The steady-state response of a stable, linear, time-invariant system to a sinusoidal input does not depend on the initial conditions. (Thus, we can assume the zero initial condition.) If $Y(s)$ has only distinct poles, then the partial fraction expansion of Equation (8–1) yields

$$\begin{aligned} Y(s) = G(s)X(s) &= G(s)\frac{\omega X}{s^2 + \omega^2} \\ &= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \cdots + \frac{b_n}{s + s_n} \end{aligned} \tag{8–2}$$

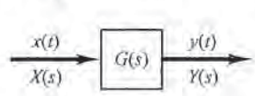


**Figure 8–1**
Stable, linear, time-invariant system.





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where $a$ and the $b_i$ (where $i = 1, 2, \ldots, n$) are constants and $\bar{a}$ is the complex conjugate of $a$. The inverse Laplace transform of Equation (8–2) gives

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{-s_1 t} + b_2 e^{-s_2 t} + \cdots + b_n e^{-s_n t} \qquad (t \geq 0) \tag{8–3}$$

For a stable system, $-s_1, -s_2, \ldots, -s_n$ have negative real parts. Therefore, as $t$ approaches infinity, the terms $e^{-s_1 t}, e^{-s_2 t}, \ldots$, and $e^{-s_n t}$ approach zero. Thus, all the terms on the right-hand side of Equation (8–3), except the first two, drop out at steady state.

If $Y(s)$ involves multiple poles $s_j$ of multiplicity $m_j$, then $y(t)$ will involve terms such as $t^{h_j}e^{-s_j t}(h_j = 0, 1, 2, \ldots, m_j - 1)$. For a stable system, the terms $t^{h_j}e^{-s_j t}$ approach zero as $t$ approaches infinity.

Thus, regardless of whether the system is of the distinct-pole type, the steady-state response becomes

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \tag{8–4}$$

where the constant $a$ can be evaluated from Equation (8–2) as follows:

$$a = G(s)\frac{\omega X}{s^2 + \omega^2}(s + j\omega)\bigg|_{s=-j\omega} = -\frac{XG(-j\omega)}{2j}$$

Note that

$$\bar{a} = G(s)\frac{\omega X}{s^2 + \omega^2}(s - j\omega)\bigg|_{s=j\omega} = \frac{XG(j\omega)}{2j}$$

Since $G(j\omega)$ is a complex quantity, it can be written in the following form:

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

where $|G(j\omega)|$ represents the magnitude and $\phi$ represents the angle of $G(j\omega)$; that is,

$$\phi = \underline{/G(j\omega)} = \tan^{-1}\left[\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right]$$

The angle $\phi$ may be negative, positive, or zero. Similarly, we obtain the following expression for $G(-j\omega)$:

$$G(-j\omega) = |G(-j\omega)|e^{-j\phi} = |G(j\omega)|e^{-j\phi}$$

Then, noting that

$$a = \frac{X|G(j\omega)|e^{-j\phi}}{2j}, \qquad \bar{a} = \frac{X|G(j\omega)|e^{j\phi}}{2j}$$

Equation (8–4) can be written

$$\begin{aligned} y_{ss}(t) &= X|G(j\omega)|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\ &= X|G(j\omega)|\sin(\omega t + \phi) \\ &= Y\sin(\omega t + \phi) \end{aligned} \tag{8–5}$$





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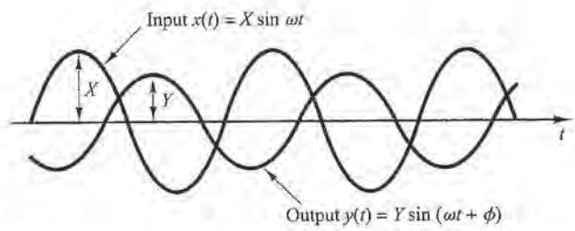


**Figure 8–2**
Input and output sinusoidal signals.

where $Y = X|G(j\omega)|$. We see that a stable, linear, time-invariant system subjected to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase of the output will, in general, be different from those of the input. In fact, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$, while the phase angle differs from that of the input by the amount $\phi = \underline{/G(j\omega)}$. An example of input and output sinusoidal signals is shown in Figure 8–2.

On the basis of this, we obtain this important result: For sinusoidal inputs,

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right| = \text{amplitude ratio of the output sinuisoid to the input sinusoid}$$

$$\underline{/G(j\omega)} = \underline{\left/\frac{Y(j\omega)}{X(j\omega)}\right.} = \text{phase shift of the output sinusoid with respect to the input sinusoid}$$

Hence, the steady-state response characteristics of a system to a sinusoidal input can be obtained directly from

$$\frac{Y(j\omega)}{X(j\omega)} = G(j\omega)$$

The function $G(j\omega)$ is called the sinusoidal transfer function. It is the ratio of $Y(j\omega)$ to $X(j\omega)$, is a complex quantity, and can be represented by the magnitude and phase angle with frequency as a parameter. The sinusoidal transfer function of any linear system is obtained by substituting $j\omega$ for $s$ in the transfer function of the system.

A positive phase angle is called phase lead, and a negative phase angle is called phase lag. A network that has phase-lead characteristics is called a lead network, while a network that has phase-lag characteristics is called a lag network.

**EXAMPLE 8–1** Consider the system shown in Figure 8–3. The transfer function $G(s)$ is

$$G(s) = \frac{K}{Ts + 1}$$

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be found as follows: Substituting $j\omega$ for $s$ in $G(s)$ yields

$$G(j\omega) = \frac{K}{jT\omega + 1}$$

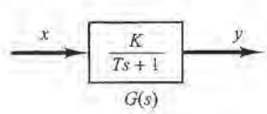


**Figure 8–3**
First-order system.





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The amplitude ratio of the output to the input is

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$

while the phase angle $\phi$ is

$$\phi = \underline{/G(j\omega)} = -\tan^{-1} T\omega$$

Thus, for the input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be obtained from Equation (8–5) as follows:

$$y_{ss}(t) = \frac{XK}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \tan^{-1} T\omega) \tag{8–6}$$

From Equation (8–6), it can be seen that for small $\omega$, the amplitude of the steady-state output $y_{ss}(t)$ is almost equal to $K$ times the amplitude of the input. The phase shift of the output is small for small $\omega$. For large $\omega$, the amplitude of the output is small and almost inversely proportional to $\omega$. The phase shift approaches $-90°$ as $\omega$ approaches infinity. This is a phase-lag network.

**EXAMPLE 8–2** Consider the network given by

$$G(s) = \frac{s + \dfrac{1}{T_1}}{s + \dfrac{1}{T_2}}$$

Determine whether this network is a lead network or lag network.

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be found as follows: Since

$$G(j\omega) = \frac{j\omega + \dfrac{1}{T_1}}{j\omega + \dfrac{1}{T_2}} = \frac{T_2(1 + T_1 j\omega)}{T_1(1 + T_2 j\omega)}$$

we have

$$|G(j\omega)| = \frac{T_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}}$$

and

$$\phi = \underline{/G(j\omega)} = \tan^{-1} T_1\omega - \tan^{-1} T_2\omega$$

Thus the steady-state output is

$$y_{ss}(t) = \frac{XT_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}} \sin(\omega t + \tan^{-1} T_1\omega - \tan^{-1} T_2\omega)$$

From this expression, we find that if $T_1 > T_2$, then $\tan^{-1} T_1\omega - \tan^{-1} T_2\omega > 0$. Thus, if $T_1 > T_2$, then the network is a lead network. If $T_1 < T_2$, then the network is a lag network.

**Presenting Frequency-Response Characteristics in Graphical Forms.** The sinusoidal transfer function, a complex function of the frequency $\omega$, is characterized by its magnitude and phase angle, with frequency as the parameter. There are three commonly used representations of sinusoidal transfer functions:





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1. Bode diagram or logarithmic plot
2. Nyquist plot or polar plot
3. Log-magnitude-versus-phase plot (Nichols plots)

We shall discuss these representations in detail in this chapter. We shall also discuss the MATLAB approach to obtain Bode diagrams, Nyquist plots, and Nichols plots.

**Outline of the Chapter.** Section 8–1 has presented introductory material on the frequency response. Section 8–2 presents Bode diagrams of various transfer-function systems. Section 8–3 discusses a computational approach to obtain Bode diagrams with MATLAB, Section 8–4 treats polar plots of sinusoidal transfer functions, and Section 8–5 discusses drawing Nyquist plots with MATLAB. Section 8–6 briefly presents log-magnitude-versus-phase plots. Section 8–7 gives a detailed account of the Nyquist stability criterion, Section 8–8 discusses the stability analysis of closed-loop systems using the Nyquist stability criterion, and Section 8–9 treats the relative stability analysis of closed-loop systems. Measures of relative stability such as phase margin and gain margin are introduced here. The correlation between the transient response and frequency response is also discussed. Section 8–10 presents a method for obtaining the closed-loop frequency response from the open-loop frequency response by use of the $M$ and $N$ circles. Use of the Nichols chart is also discussed for obtaining the closed-loop frequency response. Finally, Section 8–11 deals with the determination of the transfer function based on an experimental Bode diagram.

## 8–2 BODE DIAGRAMS

**Bode Diagrams or Logarithmic Plots.** A Bode diagram consists of two graphs: One is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20\log|G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (but in decibels) or phase angle (in degrees). (The frequency range of interest determines the number of logarithmic cycles required on the abscissa.)

The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations. Such approximation by straight-line asymptotes is sufficient if only rough information on the frequency-response characteristics is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots. Expanding the low-frequency range by use of a logarithmic scale for the frequency is highly advantageous since characteristics at low frequencies are most important in practical systems. Although it is not possible to plot the curves right down to zero frequency because of the logarithmic frequency ($\log 0 = -\infty$), this does not create a serious problem.

Note that the experimental determination of a transfer function can be made simple if frequency-response data are presented in the form of a Bode diagram.





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**Basic Factors of $G(j\omega)H(j\omega)$.** As stated earlier, the main advantage in using the logarithmic plot is the relative ease of plotting frequency-response curves. The basic factors that very frequently occur in an arbitrary transfer function $G(j\omega)H(j\omega)$ are

1. Gain $K$
2. Integral and derivative factors $(j\omega)^{\mp 1}$
3. First-order factors $(1 + j\omega T)^{\mp 1}$
4. Quadratic factors $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{\mp 1}$

Once we become familiar with the logarithmic plots of these basic factors, it is possible to utilize them in constructing a composite logarithmic plot for any general form of $G(j\omega)H(j\omega)$ by sketching the curves for each factor and adding individual curves graphically, because adding the logarithms of the gains corresponds to multiplying them together.

**The Gain $K$.** A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value. The log-magnitude curve for a constant gain $K$ is a horizontal straight line at the magnitude of $20 \log K$ decibels. The phase angle of the gain $K$ is zero. The effect of varying the gain $K$ in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

A number–decibel conversion line is given in Figure 8–4. The decibel value of any number can be obtained from this line. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20. This may be seen from the following:

$$20 \log(K \times 10) = 20 \log K + 20$$

Similarly,

$$20 \log(K \times 10^n) = 20 \log K + 20n$$

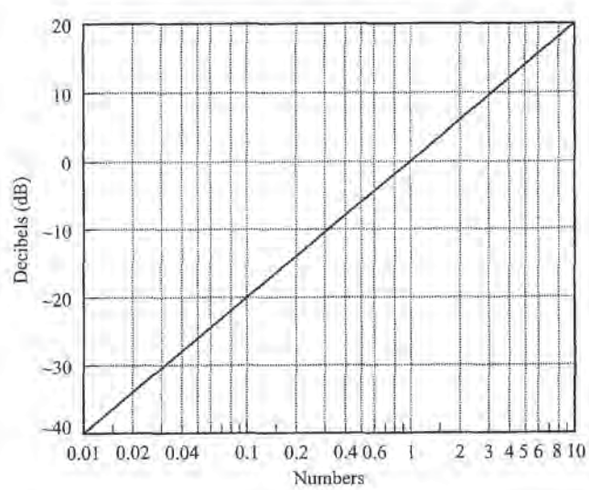


**Figure 8–4**
Number–decibel conversion line.





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Note that, when expressed in decibels, the reciprocal of a number differs from its value only in sign; that is, for the number $K$,

$$20 \log K = -20 \log \frac{1}{K}$$

**Integral and Derivative Factors $(j\omega)^{\mp 1}$.** The logarithmic magnitude of $1/j\omega$ in decibels is

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \text{ dB}$$

The phase angle of $1/j\omega$ is constant and equal to $-90°$.

In Bode diagrams, frequency ratios are expressed in terms of octaves or decades. An octave is a frequency band from $\omega_1$ to $2\omega_1$, where $\omega_1$ is any frequency value. A decade is a frequency band from $\omega_1$ to $10\omega_1$, where again $\omega_1$ is any frequency. (On the logarithmic scale of semilog paper, any given frequency ratio can be represented by the same horizontal distance. For example, the horizontal distance from $\omega = 1$ to $\omega = 10$ is equal to that from $\omega = 3$ to $\omega = 30$.)

If the log magnitude $-20 \log \omega$ dB is plotted against $\omega$ on a logarithmic scale, it is a straight line. To draw this straight line, we need to locate one point (0 dB, $\omega = 1$) on it. Since

$$(-20 \log 10\omega) \text{ dB} = (-20 \log \omega - 20) \text{ dB}$$

the slope of the line is $-20$ dB/decade (or $-6$ dB/octave).

Similarly, the log magnitude of $j\omega$ in decibels is

$$20 \log |j\omega| = 20 \log \omega \text{ dB}$$

The phase angle of $j\omega$ is constant and equal to $90°$. The log-magnitude curve is a straight line with a slope of 20 dB/decade. Figures 8–5(a) and (b) show frequency-response curves for $1/j\omega$ and $j\omega$, respectively. We can clearly see that the differences in the frequency responses of the factors $1/j\omega$ and $j\omega$ lie in the signs of the slopes of the log-magnitude curves and in the signs of the phase angles. Both log magnitudes become equal to 0 dB at $\omega = 1$.

If the transfer function contains the factor $(1/j\omega)^n$ or $(j\omega)^n$, the log magnitude becomes, respectively,

$$20 \log \left| \frac{1}{(j\omega)^n} \right| = -n \times 20 \log |j\omega| = -20n \log \omega \text{ dB}$$

or

$$20 \log |(j\omega)^n| = n \times 20 \log |j\omega| = 20n \log \omega \text{ dB}$$

The slopes of the log-magnitude curves for the factors $(1/j\omega)^n$ and $(j\omega)^n$ are thus $-20n$ dB/decade and $20n$ dB/decade, respectively. The phase angle of $(1/j\omega)^n$ is equal to $-90° \times n$ over the entire frequency range, while that of $(j\omega)^n$ is equal to $90° \times n$ over the entire frequency range. The magnitude curves will pass through the point (0 dB, $\omega = 1$).





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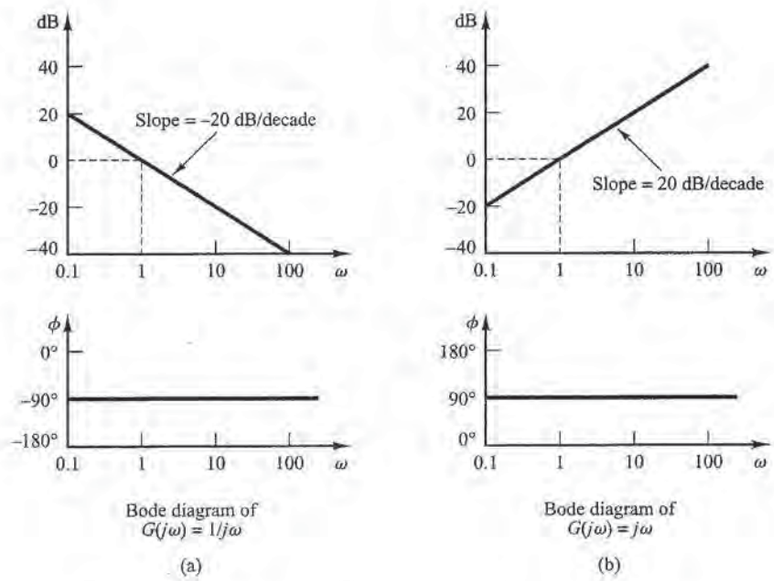


**Figure 8–5**
(a) Bode diagram of $G(j\omega) = 1/j\omega$;
(b) Bode diagram of $G(j\omega) = j\omega$.

**First-Order Factors $(1 + j\omega T)^{\mp 1}$.** The log magnitude of the first-order factor $1/(1 + j\omega T)$ is

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}$$

For low frequencies, such that $\omega \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such that $\omega \gg 1/T$,

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log \omega T \text{ dB}$$

This is an approximate expression for the high-frequency range. At $\omega = 1/T$, the log magnitude equals 0 dB; at $\omega = 10/T$, the log magnitude is −20 dB. Thus, the value of $-20 \log \omega T$ dB decreases by 20 dB for every decade of $\omega$. For $\omega \gg 1/T$, the log-magnitude curve is thus a straight line with a slope of −20 dB/decade (or −6 dB/octave).

Our analysis shows that the logarithmic representation of the frequency-response curve of the factor $1/(1 + j\omega T)$ can be approximated by two straight-line asymptotes, one a straight line at 0 dB for the frequency range $0 < \omega < 1/T$ and the other a straight line with slope −20 dB/decade (or −6 dB/octave) for the frequency range $1/T < \omega < \infty$. The exact log-magnitude curve, the asymptotes, and the exact phase-angle curve are shown in Figure 8–6.

The frequency at which the two asymptotes meet is called the *corner* frequency or *break* frequency. For the factor $1/(1 + j\omega T)$, the frequency $\omega = 1/T$ is the corner frequency since at $\omega = 1/T$ the two asymptotes have the same value. (The low-frequency asymptotic expression at $\omega = 1/T$ is 20 log 1 dB = 0 dB, and the high-frequency





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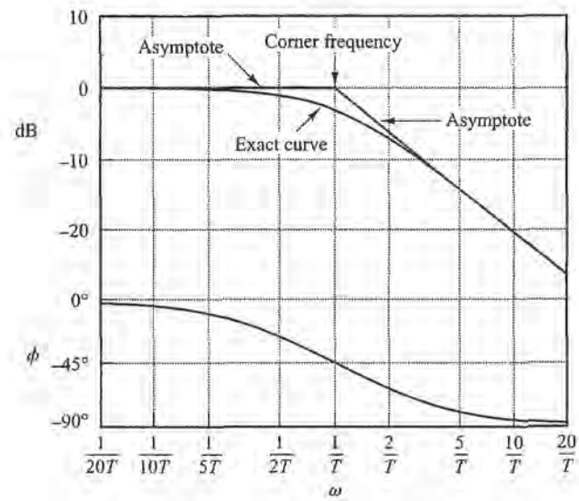


**Figure 8–6**
Log-magnitude curve, together with the asymptotes, and phase-angle curve of $1/(1 + j\omega T)$.

asymptotic expression at $\omega = 1/T$ is also 20 log 1 dB = 0 dB.) The corner frequency divides the frequency-response curve into two regions: a curve for the low-frequency region and a curve for the high-frequency region. The corner frequency is very important in sketching logarithmic frequency-response curves.

The exact phase angle $\phi$ of the factor $1/(1 + j\omega T)$ is

$$\phi = -\tan^{-1}\omega T$$

At zero frequency, the phase angle is 0°. At the corner frequency, the phase angle is

$$\phi = -\tan^{-1}\frac{T}{T} = -\tan^{-1}1 = -45°$$

At infinity, the phase angle becomes −90°. Since the phase angle is given by an inverse-tangent function, the phase angle is skew symmetric about the inflection point at $\phi = -45°$.

The error in the magnitude curve caused by the use of asymptotes can be calculated. The maximum error occurs at the corner frequency and is approximately equal to −3 dB since

$$-20\log\sqrt{1+1} + 20\log 1 = -10\log 2 = -3.03 \text{ dB}$$

The error at the frequency one octave below the corner frequency, that is, at $\omega = 1/(2T)$, is

$$-20\log\sqrt{\frac{1}{4}+1} + 20\log 1 = -20\log\frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$

The error at the frequency one octave above the corner frequency, that is, at $\omega = 2/T$, is

$$-20\log\sqrt{2^2+1} + 20\log 2 = -20\log\frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$





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