The Characteristics of Model-Following Systems as Synthesized by Optimal Control

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Summary—Synthesis through linear optimal-control theory is applied to two types of model-following systems. Although the two system concepts are fundamentally different, the optimal-control theory in both cases generates constant feedback gains which modify the plant dynamics so that the resultant system response approaches that of the model. Each synthesis method produces an optimal-control law that has a closed-form solution and minimizes an integral quadratic performance index involving both system errors and control motions.

The first type of model-following system is generated by incorporating the model only in the performance index. The integral square error between the model and the system is minimized by one term in the performance index. The basic idea in this system is to alter, by the feedback, the coefficients in the equations of motion for the plant so that these coefficients approach those of the model. The feedback control obtained from this method depends on the solution of the matrix Riccati equation, and also on a product of constant matrices. For an infinite weight on the error term in the performance index, the solution of the Riccati equation approaches zero.

The second type of model-following system incorporates the model as part of the control system, and places it ahead of the plant as a prefilter. The difference between the model output and the plant output is minimized in the performance index. The resulting system has not only feedback gains for the plant state variables but also feedforward gains for the model state variables. Evaluation shows that the feedback gains are independent of the model, and are exactly those which would be obtained for the normal regulator problem. The feedforward gains depend on both plant and model.

LIST OF SYMBOLS AND ABBREVIATIONS

Vectors

- $\bar{x}$ state vector
- $\bar{y}$ output vector
- $\bar{\eta}$ model state vector
- $\bar{u}$ control vector
- $\bar{\lambda}$ system adjoint vector
- x_i element of a vector

Matrices

- [0] null matrix
- I identity matrix
- F system matrix
- G input matrix
- H output matrix
- L model system matrix
- Q error weighting matrix
- R control weighting matrix
- $P(t)$ time-varying matrix: solution of the Riccati differential equation
- P constant matrix: solution of the steady-state Riccati equation
- K_R control matrix from the Riccati equation
- K_m control matrix resulting from product of matrices
- K total control matrix for model-in-the-performance-index method or the control matrix for the model-in-the-system method
- K_{ff} feedforward control matrix
- f_{ij} element of the F matrix

Variables

- ϕ bank angle, rad
- $\dot{\phi}$ bank angle rate, rad/sec
- β sideslip angle, rad
- r yawing velocity, rad/sec
- ΔV incremental airspeed, ft/sec
- Δn_z rigid-body acceleration in z direction, ft/sec^2
- h altitude, ft
- α angle of attack, rad
- θ pitch angle, rad
- q pitch rate, rad/sec
- δ_r rudder angle, rad
- δ_a aileron angle, rad
- δ_e elevator control deflection, rad
- δ_t throttle change $= T/T_{\max}$

Scalars

- $\mathcal{H}$ Hamiltonian function of the dynamic system
- V performance index

Subscripts and Superscripts

- $(\ )^0$ optimal $-u^0$
- $(\dot{\ })$ time derivative
- (t) running time variable
- $(\ )'$ (prime) transpose of a matrix
- $(\hat{\ })$ (hat value) equivalent matrix in the model-in-the-performance-index method

Matrix Abbreviations

- Q/R refers to the ratio of the elements in the Q matrix to those in the R matrix—increasing this ratio means more weight on the error term in the performance index
- $Q(\ )$ abbreviation referring to the power of ten for each diagonal element, *e.g.*,

$$Q(2, 2, 1, 2) = \begin{bmatrix} 10^2 & 0 & 0 & 0 \\ 0 & 10^2 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10^2 \end{bmatrix}.$$

In this paper, $Q(0)$ is equal to 0.

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I. Introduction

LINEAR OPTIMAL-CONTROL THEORY has had limited use as a practical synthesis technique for multivariable control system design because the quadratic performance index does not always express the real design objective. Other criteria on the system transient response may be more meaningful. However, if the criteria can be formulated as a linear model, application of the quadratic performance index can be significantly extended by incorporating into the index the error between the system and the model. In particular, the introduction of the model may facilitate selecting relative weights on the state variables. The main advantage in using optimal-control theory for synthesizing a multivariable system is the fact that a total design is produced. Also, high-order systems can be designed including actuator and sensor dynamics.

The purpose of this paper is twofold. First it is intended to clearly indicate the characteristics of each model technique by a brief development of the theory including simple second-order examples, and second, it is desired to show that optimal-control theory can be used in the design of a practical system.

The paper is the result of research performed at the Flight Research Department of Cornell Aeronautical Laboratory and performed in conjunction with the author's doctoral thesis for Yale University. This paper describes two model-following techniques and their application to practical systems. The first technique, which incorporates the model only in the performance index, was developed by Dr. R. E. Kalman [1]. The second technique, which incorporates the model-in-the-system has been used by Dr. Kalman, the Flight Research Department at CAL [8], and F. T. Smith [2]. Also, the optimal-control theory that is considered in this paper was developed by Dr. Kalman.

Although these are primarily model-following optimal synthesis techniques, they also would be useful as the basis for adaptive systems providing the calculations can be done in real time. Each method represents a basic type of adaptive control system. The model-in-the-performance-index method depends on identifying the plant and then adjusting the plant feedback to match the model. The model-in-the-system method is basically similar to high-gain feedback methods which eliminate the plant dynamics, but the feedforward loops that are generated modify the concept significantly. The implementation of these methods for real-time computation has not been considered in this paper, that is, it is only the synthesis technique which is discussed.

II. Model-in-the-Performance-Index

The model-in-the-performance-index method uses the feedback gains generated by the optimal-control theory to alter the open-loop plant coefficients such that they approach the coefficients of the model. The model, however, is included only in the quadratic performance index, not in the physical system. The plant may be expressed in standard form as

$$\dot{\bar{x}} = F\bar{x} + G\bar{u}, \tag{1}$$

where $\bar{x}$ is the state vector, $\bar{u}$ is the control vector, and F and G are respectively the plant and input matrices. The dynamics of some ideal system, which will be called the model, may be written in the form,

$$\dot{\bar{\eta}} = L\bar{\eta}. \tag{2}$$

Since it is desired to have the plant dynamics approach those of the model, the performance index is set up so that one term will consist of the error between the model and the plant derivatives.

The performance index is defined as

$$2V = \int_0^T \{(\dot{\bar{y}} - L\bar{y})'Q(\dot{\bar{y}} - L\bar{y}) + \bar{u}'R\bar{u}\}\,dt, \tag{3}$$

where $\bar{y}$ is the output vector and is defined as

$$\bar{y} = H\bar{x}. \tag{4}$$

The output vector $\bar{y}$ might not be the same as the state vector $\bar{x}$ if, for example, not all of the state variables are to be compared with the model variables. If two different vectors are included in the performance index, $\bar{x}$ and $\bar{\eta}$, where $\bar{\eta}$ may now be considered as an input, the problem would become what Kalman refers to as the servo problem [3]. With only one state vector in the performance index, the system is optimized for an initial condition response and is called the regulator problem. For this reason, the transformation L is taken on $\bar{x}$ rather than $\bar{\eta}$.

If (1) and (4) are substituted into (3),

$$2V = \int_0^T \{(HF\bar{x} + HG\bar{u} - LH\bar{x})'Q(HF\bar{x} + HG\bar{u} - LH\bar{x}) + \bar{u}'R\bar{u}\}\,dt. \tag{5}$$

The fact that the first term in the performance index which is an error term now includes the control vector $\bar{u}$ will be later shown to produce interesting results for this method.

The derivation of the control which will minimize the performance index of (5) is developed in [3] and is included in Appendix A. The form of the optimal control for this performance index is

$$\bar{u}^0 = -(G'H'QHG + R)^{-1} \cdot \{G'\bar{\lambda} + G'H'Q(HF - LH)\}\bar{x}. \tag{6}$$

The canonical equations are:

$$\frac{\partial \mathcal{H}}{\partial \bar{x}} = -\dot{\bar{\lambda}} = \hat{H}'\hat{Q}\hat{H}\bar{x} + \hat{F}'\bar{\lambda} \tag{7}$$

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and

$$\frac{\partial \mathcal{H}}{\partial \bar{\lambda}} = \dot{\bar{x}} = \hat{F}\bar{x} + \hat{G}\bar{u}. \tag{8}$$

Eqs. (7) and (8) with the "hat" matrices are the same form as the canonical equations of the regulator problem. These equations may be solved for the adjoint vector $\bar{\lambda}$, and the control of (6). The adjoint vector $\bar{\lambda}$ depends on the solution of the matrix Riccati equation for $P(t)$ and is expressed as

$$\bar{\lambda} = P(t)\bar{x}. \tag{9}$$

The Riccati equation for the "hat" matrices has the form

$$\dot{P}(t) - P(t)\hat{G}\hat{R}^{-1}\hat{G}'P(t) + P(t)\hat{F} + \hat{F}'P(t) + \hat{H}'\hat{Q}\hat{H} = [0]. \tag{10}$$

The control matrix resulting from the Riccati equation is then

$$K_R(t) = \hat{R}^{-1}\hat{G}'P(t). \tag{11}$$

The control from the product of matrices is

$$K_m = \hat{R}^{-1}\hat{G}'H'Q(HF - LH). \tag{12}$$

The total control of (6) may be expressed as

$$\bar{u}^0 = -\{K_R(t) + K_m\}\bar{x} = -K(t)\bar{x}. \tag{13}$$

Since it is desirable to have constant feedback gains in airplane flight control systems, only the steady-state value of the Riccati equation will be used in this work. This eliminates the need to define the origin of time and program and time-varying gains. Constant gains are obtained by letting the upper limit of integration in the performance index approach infinity and solving the algebraic rather than the differential Riccati equation. This algebraic solution is denoted by omitting the indication of time variation on the $P(t)$ and $K(t)$ matrices. In practice this solution may be obtained analytically for single-input systems. For multi-input systems a digital program is used to solve the matrix Riccati differential equation, and only the steady-state value is used. (See Kalman and Englar [4] and Shed [5].)

The use of steady-state control values assumes that the period of regulation or control will approach infinity. In practice, of course, this is not true and so there would be some loss of optimality. It is also true, however, that in designing a flight control system for an airplane it would be impossible to know in advance the exact control inputs and disturbances for which the control system was to be optimized. Under these considerations, the loss of optimality from the regulator problem would seem to be more of an academic question than a practical loss.

The simplicity of the control as well as the way in which this method works may be shown by considering

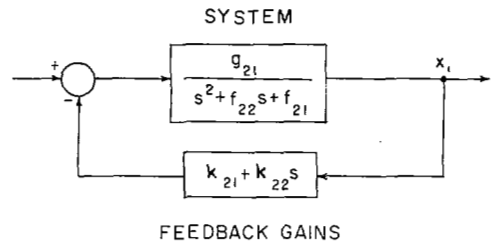


Fig. 1

a second-order, single-input example. (See Fig. 1.) A second-order system that is to follow the same order model will be considered. The matrices will then have the form,

$$F = \begin{bmatrix} 0 & 1 \\ -f_{21} & -f_{22} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ g_{21} \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \tag{14}$$
$$L = \begin{bmatrix} 0 & 1 \\ -l_{21} & -l_{22} \end{bmatrix}.$$

If these matrices are substituted into the "hat" value formulas of (56),

$$\hat{R} = G'H'QHG + R = q_{22}g_{21}^2 + r, \tag{15}$$

$$\hat{F} = F - G\hat{R}^{-1}G'H'Q(HF - LH),$$
$$= \begin{bmatrix} 0 & 1 \\ -f_{21} - \dfrac{g_{21}^2(l_{21}-f_{21})q_{22}}{q_{22}g_{21}^2+r} & -f_{22} - \dfrac{g_{21}^2 q_{22}(l_{22}-f_{22})}{q_{22}g_{21}^2+r} \end{bmatrix}, \tag{16}$$

$$\hat{G} = G = \begin{bmatrix} 0 \\ g_{21} \end{bmatrix}, \tag{17}$$

$$\hat{H} = HF - LH = \begin{bmatrix} 0 & 0 \\ (l_{21} - f_{21}) & (l_{22} - f_{22}) \end{bmatrix}, \tag{18}$$

and

$$\hat{Q} = Q - QHG\hat{R}^{-1}G'H'Q$$
$$= \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} - \dfrac{g_{21}^2 q_{22}^2}{q_{22}g_{21}^2 + r} \end{bmatrix}. \tag{19}$$

The control can be found from

$$u^0 = -\{(\hat{R}^{-1}G'P + \hat{R}^{-1}G'H'Q(HF - LH)\}\bar{x}. \tag{20}$$

The first term depends upon the steady-state solution of the Riccati equation, K_R, while the second term, K_m, may be evaluated directly from (12) as

$$K_m = \begin{bmatrix} \dfrac{g_{21}q_{22}(l_{21} - f_{21})}{q_{22}g_{21}^2 + r} & \dfrac{g_{21}q_{22}(l_{22} - f_{22})}{q_{22}g_{21}^2 + r} \end{bmatrix}. \tag{21}$$

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For this single-input case, the steady-state Riccati equation may be evaluated by letting $P(t)=0$ in (10), and solving the individual algebraic equations. If these equations are evaluated for P_{21} and P_{22}, the results are

$$P_{21} = -\frac{\hat{f}_{21}}{\hat{g}_{21}^2}\hat{r} + \sqrt{\left(\frac{\hat{f}_{21}\hat{r}}{\hat{g}_{21}^2}\right)^2 + \frac{\hat{r}}{\hat{g}_{21}^2}\hat{q}_{22}(l_{21}-f_{21})^2} \quad (22)$$

and

$$P_{22} = -\frac{\hat{f}_{22}}{\hat{g}_{21}^2}\hat{r} + \sqrt{\left(\frac{\hat{f}_{22}\hat{r}}{\hat{g}_{21}^2}\right)^2 + \frac{2P_{21}\hat{r}}{\hat{g}_{21}^2} + \frac{r}{\hat{g}_{21}^2}\hat{q}_{22}(l_{22}-f_{22})^2}. \quad (23)$$

Since P is a symmetric matrix $P_{21}=P_{12}$, therefore, only one of these equations must be solved. Also, P_{11} need not be found, as it has no effect on the control. This is due to the fact that the control depends on the product of G' and P in (20), and the g_{11} element which would multiply P_{11} is zero. This is true for any single-input system, *i.e.*, an nth order system will only have n feedback control terms, rather than the $n\times n$ terms from the P matrix.

The normal procedure for using optimal-control theory to obtain feedback gains is to vary the ratio of the weight on the error to the weight on the control (Q/R) until a satisfactory result is obtained. If too much weight is placed on the control, the system would not be fast enough, and there would be little improvement over the open-loop plant dynamics. If too much weight is placed on the error term, the resulting feedback terms might well be out of the range of physical realizability for the particular problem at hand. For this method, however, it is significant to observe the effect of a large weight on the error term. For the example being considered, this amounts to making $q_{22}g_{21}^2/r \gg 1$. R in this case is a scalar r, and q_{11} does not appear. For high q_{22}/r, the matrix R becomes approximately

$$\hat{R} \cong q_{22}g_{21}^2. \quad (24)$$

This effect in the $\hat{R}$ matrix affects the other matrices as follows:

$$\hat{F} \cong \begin{bmatrix} 0 & 1 \\ -l_{21} & -l_{22} \end{bmatrix} \quad (25)$$

$$\hat{Q} \cong \begin{bmatrix} q_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (26)$$

$$K_m \cong \begin{bmatrix} \dfrac{(l_{21}-f_{21})}{g_{21}} & \dfrac{(l_{22}-f_{22})}{g_{21}} \end{bmatrix}. \quad (27)$$

Eq. (25) shows that the equivalent system matrix becomes the model, and the q_{22} term which is the only Q element that has any effect, becomes zero. Eqs. (22) and (23) show that if $\hat{q}_{22}=0$, P_{21} and P_{22} are zero. Thus as the weight on the error term becomes greater, the control terms from the Riccati equation approach zero. For large q_{22}/r, then, the total control is simply the K_m of (27). This effect has been found to be true for multi-input systems as well, and may be verified analytically by assuming R of (3) to be the null matrix. The result of evaluating the control for this performance index is that the term $K_R(t)$ in (13) will be zero. The way in which the model is matched may be seen by evaluating the closed-loop system as

$$F - GK_m = \begin{bmatrix} 0 & 1 \\ -f_{21} & -f_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ g_{21} \end{bmatrix}\begin{bmatrix} \dfrac{g_{21}q_{22}(l_{21}-f_{21})}{q_{22}g_{21}^2+r} & \dfrac{g_{21}q_{22}(l_{22}-f_{22})}{q_{22}g_{21}^2+r} \end{bmatrix}, \quad (28)$$

which, for $q_{22}g_{21}^2 \gg r$, reduces to

$$F - GK_m = \begin{bmatrix} 0 & 1 \\ -l_{21} & -l_{22} \end{bmatrix}. \quad (29)$$

This is the exact expression for the model, and therefore the model has clearly been matched for $q_{22}g_{21}^2 \gg r$. It might also be pointed out that although the product GK_m contains $G\hat{R}^{-1}G'H'Q(HF-LH)$ all terms cancel out for large q_{22}/r except the $(HF-LH)$. In other words, the control simply depends on the difference between the system coefficients and the model coefficients. This cancellation has also been noticed in higher-order systems.

III. Model-in-the-Performance-Index Applied to Airplane Lateral Equations

This particular synthesis technique was applied to practical systems, involving the airplane lateral equations of motion. The following lateral equations were used:

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\beta} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot{\beta}}\dfrac{g}{V} & N_p & \left(\begin{matrix} N_\beta + \\ N_{\dot{\beta}}Y_\beta \end{matrix}\right) & N_r - N_{\dot{\beta}} \end{bmatrix}\begin{bmatrix} \phi \\ \dot{\phi} \\ \beta \\ r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L_{\delta_r} & L_{\delta_a} \\ Y_{\delta_r} & 0 \\ \left(\begin{matrix} N_{\delta_r} + \\ N_{\dot{\beta}}Y_{\delta_r} \end{matrix}\right) & N_{\delta_a} \end{bmatrix}\begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix},$$

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where

ϕ = bank angle
β = sideslip angle
r = yaw rate
δ_r = rudder deflection
δ_a = aileron deflection.

These equations have been transformed to decouple the inertia terms.

This example was considered to see whether a control system designed by this optimal synthesis method would be physically realizable. The B-26 airplane, which has been mechanized by the Flight Research Department of the Cornell Aeronautical Laboratory as a variable stability airplane, was chosen for this example. Not only did the system have to follow the model well, but also the feedback gains could not exceed certain limiting values. Furthermore, the variables that were fed back had to be compatible with those that are measured in the actual system.

The matrices for the B-26 were

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2.93 & -4.75 & -.78 \\ .086 & 0 & -.11 & -1.0 \\ 0 & -.042 & 2.59 & -.39 \end{bmatrix}.$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ .035 & 0 \\ -2.53 & .31 \end{bmatrix}.$$

The model was

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -73.14 & 3.18 \\ .086 & 0 & -.11 & -1 \\ .0086 & .086 & 8.95 & -.49 \end{bmatrix}.$$

This model would result approximately in the following response parameters:

τ_R^{-1} = inverse of roll mode time constant = 1 sec^{-1}.
τ_S^{-1} = inverse of spiral mode time constant = 0 sec^{-1}.
ζ_d = Dutch roll damping ratio = 0.1.
ω_{n_d} = Dutch roll natural frequency = 3 rad/sec.
$|\phi/\beta| = 8$.

This model was also chosen so that only the parameters that could be physically changed in the airplane differed in the model from those of the system.

The results of trying various weighting schemes for this system indicated that the best feedback gains were produced by weighting q_{22} and q_{44} heavily and q_{11} and q_{33} not at all. This is due to the fact that the first and third rows of the model and the system matrices were identical. The value of the R matrix for this example, as well as the examples that follow, was the identity matrix. Since the control K_m (12) depends on the difference between the system matrix and the model matrix $(F-L)$, the product $\hat{G}'H'Q(HF-LH)$ is independent of both q_{11} and q_{33}. The responses to an initial sideslip angle (β) and an initial bank angle rate ($\dot{\phi}$) are shown in Figs. 2 and 3, respectively. Closed-loop responses are plotted for three different (Q) weighting matrices as well as the model and the open-loop system responses. The closed-loop system matrices, the eigenvalues of these matrices, and the control for the best weighting scheme are listed in Table I.

The results of this example indicate that this method is feasible for designing a control system for an actual system. The magnitude of the feedback gains for the case where $Q(0, 6, 0, 6)$ were perfectly reasonable and although $\dot{\beta}$ was fed back in the actual system, it is possible through a simple transformation to convert the optimal-control feedback gains to the cockpit gains of the variable-stability airplane.

IV. Model-in-the-System

It was shown in the previous method that feedback could be placed around the plant so that it would behave like the model. With the model in the system, however, no effort is made to make the system response similar to that of the model for the same input. Instead, the model is included in the physical system as a prefilter ahead of the plant. The optimal-control theory generates constant feedback gains around each of the plant state variables which improve the plant dynamics, and constant feedforward gains from each of the model state variables which provide a lead network. If enough weight is placed on the error term in the performance index, the feedback system around the plant will be fast enough to follow the output of the model. The model is incorporated into the optimal-control theory by writing it as an uncontrollable part of the state equation. A suitable choice for the H matrix allows the model variables to be compared with the plant variables. This amounts to defining the output vector as the difference between the model variables and the state variables, and minimizing this difference in the performance index.

The equations of the plant are

$$\dot{\bar{x}}_p = F_p\bar{x}_p + G_p\bar{u}_p. \tag{30}$$

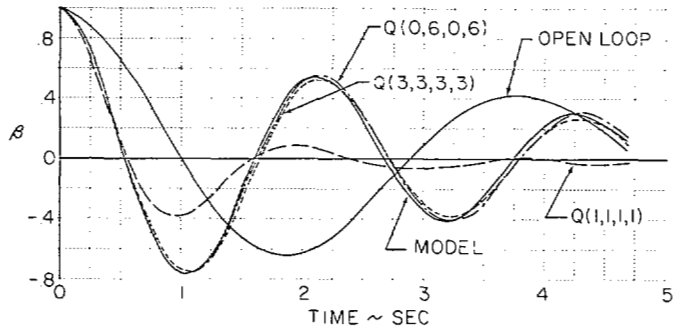
The model dynamics are defined as

$$\dot{\bar{x}}_m = L\bar{x}_m. \tag{31}$$

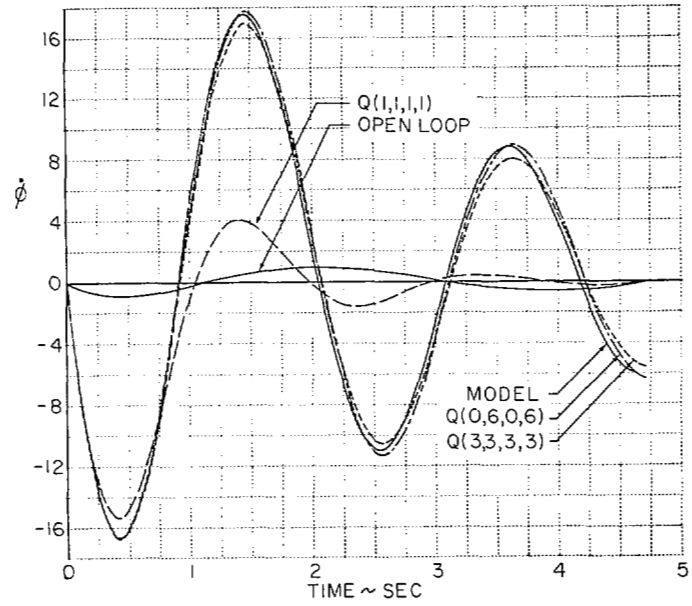
The state equation is now defined as a combination of the plant and the model equations

$$\begin{bmatrix} \dot{\bar{x}}_m \\ \dot{\bar{x}}_p \end{bmatrix} = \begin{bmatrix} L & [0] \\ [0] & F_p \end{bmatrix} \begin{bmatrix} \bar{x}_m \\ \bar{x}_p \end{bmatrix} + \begin{bmatrix} [0] \\ G_p \end{bmatrix} [\bar{u}_p]. \tag{32}$$

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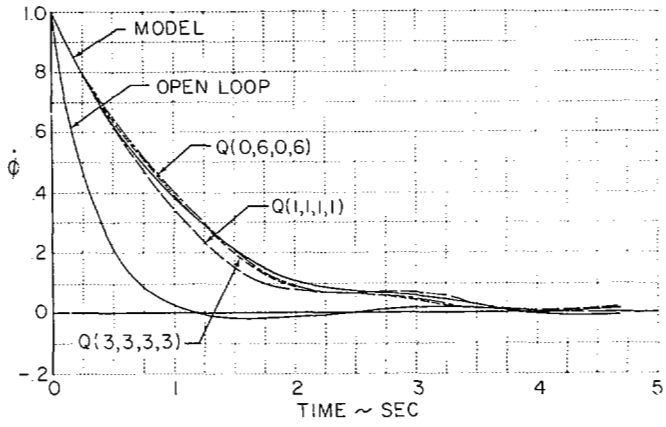


(a)

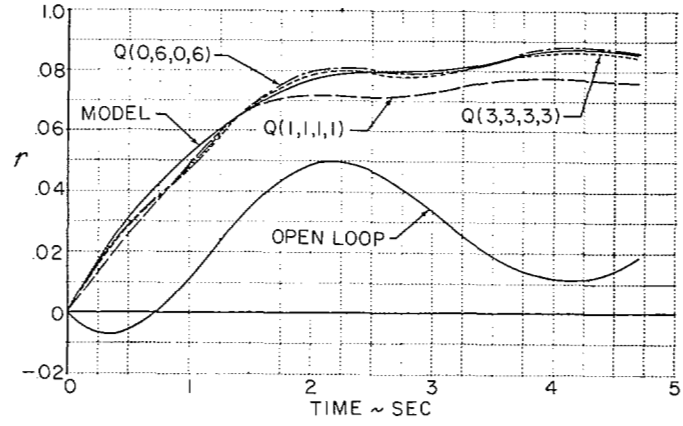


(b)

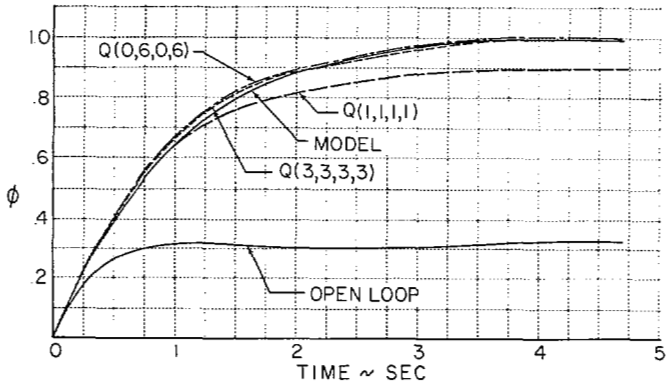
Fig. 2—B-26 model-in-the-performance-index responses to initial sideslip angle.



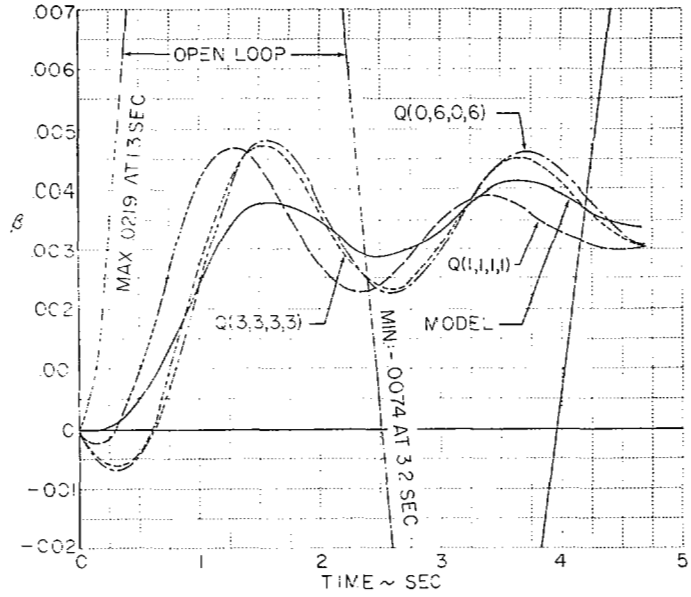
(a)



(b)



(c)



(d)

Fig. 3—B-26 model-in-the-performance-index responses to initial bank angle rate ($\dot{\phi}$).

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TABLE I

MODEL-IN-THE-PERFORMANCE-INDEX FOR B-26

		Open-Loop System	Model	$Q(0, 6, 0, 6)$
Closed-Loop Matrix $F-GK$ (F) for the Model and Open-Loop System		$\begin{matrix}0 & 1 & 0 & 0\\ 0 & -2.93 & -4.75 & .78\\ .086 & 0 & -.11 & -1.0\\ 0 & -.042 & 2.59 & -.39\end{matrix}$	$\begin{matrix}0 & 1 & 0 & 0\\ 0 & -1 & -73.14 & 3.18\\ .086 & 0 & -.11 & -1\\ .0086 & .086 & 8.95 & -.49\end{matrix}$	$\begin{matrix}0 & 1 & 0 & 0\\ 0 & -1 & -73.1 & 3.18\\ .086 & -.00389 & -.123 & -1.0\\ .0086 & .086 & 8.95 & -.49\end{matrix}$
Closed-Loop Roots (Open-Loop for Model and System)		τ_R^{-1} -2.98 sec^{-1} τ_S^{-1} .00173 sec^{-1} ζ_d .08 ω_{n_d} 1.68 rad/sec	-1.065 sec^{-1} .000275 sec^{-1} .0976 2.95 rad/sec	-1.05 sec^{-1} .000154 sec^{-1} .0944 2.92 rad/sec
Feedback Control Matrices	K_R	0	0	$\begin{matrix}.256\times10^{-5} & -.955\times10^{-7} & .154\times10^{-4} & -.304\times10^{-4}\\ .51\times10^{-7} & -.39\times10^{-7} & .784\times10^{-6} & -.615\times10^{-6}\end{matrix}$
	K	0	0	$\begin{matrix}.0034 & .111 & .371 & .0356\\ 0 & .494 & -17.5 & .614\end{matrix}$

This can be represented as

$$\dot{\bar{x}} = F\bar{x} + G\bar{u}. \tag{33}$$

If both the plant and model vectors are of the order n, and it is desired to compare all the state variables with all the model variables, then a typical choice for the H matrix would be

$$H = [I \quad -I], \qquad \text{an } n\times 2n \text{ matrix.} \tag{34}$$

This defines the output vector which is to be minimized in the performance index as

$$\bar{y} = [I \quad -I]\begin{bmatrix}\bar{x}_m\\ \bar{x}_p\end{bmatrix} = \bar{x}_m - \bar{x}_p. \tag{35}$$

The model has now been incorporated into the standard form of the regulator problem. The performance index is defined as

$$2V = \int_0^T \{(\bar{y}'Q\bar{y} + \bar{u}'R\bar{u})\}\,dt. \tag{36}$$

The development of the problem is now identically the same as that of the regulator problem of Kalman [3]. The control which minimizes the performance index of (36) subject to the constraints of the differential equations of (32) may be evaluated by the method of Appendix A. The result is:

$$\bar{u}^0 = R^{-1}G'P(t)\bar{x} \tag{37}$$

where $P(t)$ is the time-varying matrix solution of the Riccati equation. As in the previous method, only the steady-state value is used. It should be pointed out that the complexity of the computation is now twice what it would be for the normal regulator problem. This is because the order of the system matrix has been doubled, and therefore $P(t)$ is a $2n\times 2n$ matrix, assuming the plant and the model are of the same order. The type of system that this method produces may be seen by considering a second-order example with the matrices

$$F_p = \begin{bmatrix}0 & 1\\ -f_{21} & -f_{22}\end{bmatrix}, \quad G_p = \begin{bmatrix}0\\ g_{21}\end{bmatrix}, \quad Q = \begin{bmatrix}q_{11} & 0\\ 0 & q_{22}\end{bmatrix}$$

$$R = r, \qquad L = \begin{bmatrix}0 & 1\\ -l_{21} & -l_{22}\end{bmatrix}$$

$$F = \begin{bmatrix}L & [0]\\ [0] & F_p\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -l_{21} & -l_{22} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -f_{21} & -f_{22}\end{bmatrix}$$

$$G = \begin{bmatrix}0\\ 0\\ 0\\ g_{21}\end{bmatrix}, \quad H = [I \quad -I] = \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}. \tag{38}$$

The state variables are now

$$\bar{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}, \tag{39}$$

where the first two variables are the model variables, and the second two are the system variables. The fact that the difference between the model variables and the plant variables will be minimized by the performance index can be verified by finding the output vector,

$$\bar{y} = H\bar{x}$$

or

$$\bar{y} = \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}x_1 - x_3\\ x_2 - x_4\end{bmatrix}. \tag{40}$$

The feedback and feedforward terms may be found from the steady-state Riccati equation. For this example, the Riccati equation becomes

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$$\frac{g_{21}^2}{r}\begin{bmatrix} P_{41}^2 & P_{14}P_{42} & P_{14}P_{43} & P_{14}P_{44} \\ P_{41}P_{24} & P_{24}P_{42} & P_{24}P_{43} & P_{24}P_{44} \\ P_{41}P_{34} & P_{34}P_{42} & P_{34}P_{43} & P_{34}P_{44} \\ P_{41}P_{44} & P_{44}P_{42} & P_{44}P_{43} & P_{44}^2 \end{bmatrix}$$

$$+\begin{bmatrix} -l_{21}P_{21} & -l_{21}P_{22} & -l_{21}P_{23} & -l_{21}P_{24} \\ P_{11}-l_{22}P_{21} & P_{12}-l_{22}P_{22} & P_{13}-l_{22}P_{23} & P_{14}-l_{22}P_{24} \\ P_{41} & P_{42} & P_{43} & P_{44} \\ -f_{21}P_{31}-f_{22}P_{41} & -f_{21}P_{32}-f_{22}P_{42} & -f_{21}P_{33}-f_{22}P_{43} & -f_{21}P_{34}-f_{22}P_{44} \end{bmatrix}$$

$$+\begin{bmatrix} -l_{21}P_{21} & P_{11}-l_{22}P_{21} & P_{41} & -f_{21}P_{31}-f_{22}P_{41} \\ -l_{21}P_{22} & P_{12}-l_{22}P_{22} & P_{42} & -f_{21}P_{32}-f_{22}P_{42} \\ -l_{21}P_{23} & P_{13}-l_{22}P_{23} & P_{43} & -f_{21}P_{33}-f_{22}P_{43} \\ -l_{21}P_{24} & P_{14}-l_{22}P_{24} & P_{44} & -f_{21}P_{34}-f_{22}P_{44} \end{bmatrix}$$

$$+\begin{bmatrix} q_{11} & 0 & -q_{11} & 0 \\ 0 & q_{22} & 0 & -q_{22} \\ -q_{11} & 0 & q_{11} & 0 \\ 0 & -q_{22} & 0 & q_{22} \end{bmatrix} = [0]. \tag{41}$$

Because of the structure of the G matrix, the terms P_{43}, P_{44} produce the feedback terms and the terms P_{41}, P_{42} produce the feedforward terms. Eq. (41) shows that P_{43} and P_{44} may be solved exactly as they were in the regulator problem, that is, by combining elements in the third row and third column, and fourth row and fourth column, respectively. Exactly the same P_{43} and P_{44} will be found in this example that would have been found without the model. The feedforward terms may be found by combining elements in the upper right-hand 2×2 matrices of (41). The fact that the feedback terms are completely independent of the model is an important result that is characteristic of this method. A block diagram of this system is shown in Fig. 4.

The results that were found concerning the independence of the feedback terms may be generalized to any order system and model. A system will be considered that has an nth order plant, an mth order input, and lth order model. The matrices for such a system, with the order of each matrix written above, would be

$$\overset{(l+n)\times(l+n)}{F} = \begin{bmatrix} L & [0] \\ [0] & F_p \end{bmatrix},$$

where L is an $l \times l$ and F_p is an $n \times n$

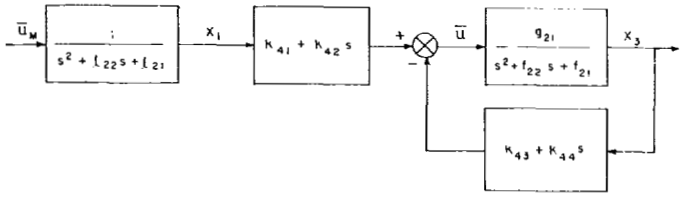


Fig. 4—Model-in-the-system method: block diagram for second-order case.

$$\overset{(l+n)\times m}{G} = \begin{bmatrix} [0] \\ G_p \end{bmatrix}, \qquad R = \overset{m\times m}{R_p}, \qquad Q = \overset{n\times n}{Q_p}$$

The p subscript denotes the plant.

H would depend on what variables were to be minimized, but if, for example, the order of the system were equal to the order of the model, *i.e.*, $n=l$,

$$\overset{l\times(l+n)}{H} = [I \; -I]$$

where I is an $l\times l$ and $-I$ is an $l\times n$. If the order of the plant were greater than the order of the model, the decision would have to be made of which model variables to compare with which of the system variables. If these generalized matrices are substituted into the steady-state Riccati equation,

$$\begin{bmatrix} P_{12}GR^{-1}G'P_{21} & P_{12}GR^{-1}G'P_{22} \\ P_{22}GR^{-1}G'P_{21} & P_{22}GR^{-1}G'P_{22} \end{bmatrix} + \begin{bmatrix} LP_{11} & LP_{12} \\ F_pP_{21} & F_pP_{22} \end{bmatrix}$$
$$\begin{bmatrix} P_{11}L & P_{12}F_p \\ P_{21}L & P_{22}F_p \end{bmatrix} + \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix} = [0]. \tag{42}$$

The closed-loop system matrix would be

$$F - GK = \begin{bmatrix} L & [0] \\ [0] & F_p \end{bmatrix} - \begin{bmatrix} [0] \\ G_p \end{bmatrix} R^{-1} \begin{bmatrix} [0] & G_p \end{bmatrix} \cdot \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{43}$$

This simplifies to

$$F - GK = \begin{bmatrix} L & [0] \\ -G_pR^{-1}G_pP_{21} & F_p - G_pR^{-1}G_pP_{22} \end{bmatrix}. \tag{44}$$

Eq. (44) indicates that the feedback terms depend on

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P_{22}. P_{22}, however, is independent of the model; that is, if this matrix and the other matrices which are multiplied by it to obtain the control were expanded for a particular system, the feedback terms would be exactly the same as they would be without the model. Eq. (42) also shows that the feedforward terms are a function of the model, the plant, and the feedback terms. This can be verified by combining the elements in the upper right and lower left corner of each matrix in (42).

V. Numerical Example—Second-Order System and Model

To illustrate how this method works, a numerical example will be considered. Eqs. (38) and (39) apply, with the following matrices.

$$F_p = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 \\ -25 & -7.07 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & -7.07 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R = 1 \quad H = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}. \qquad (45)$$

In the first case, $Q = 10I$, and the results of solving the equations in (41) are

$$K = [2.75 \quad -.735 \quad 2.32 \quad 3.63]$$

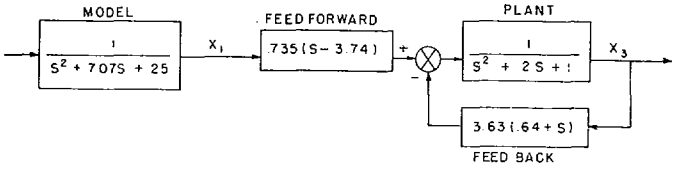


Fig. 5—Block diagram for $Q = 10I$.

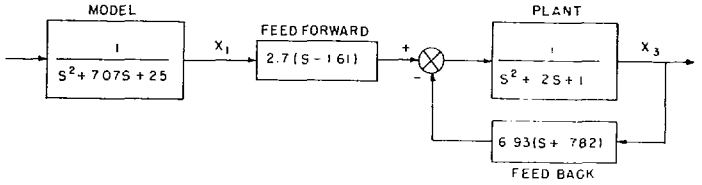


Fig. 6—Block diagram for $Q = 100I$.

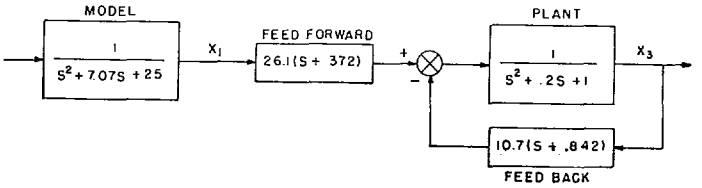


Fig. 7—Block diagram for $Q = 1000I$.

This produced the system of Fig. 5.

The results of increasing the weight to $100I$ and $1000I$ are shown in Figs. 6 and 7 respectively. Bode diagrams for these three systems are plotted in Fig. 8. Fig. 8 shows the model response and the over-all system response, including the model, for various Q's. The plant poles tend to move apart which tends to make the system faster. The zero from the feedforward moves such that more lead is added to the over-all system. It may also be noted that the dc gain decreases as Q becomes large, until when Q is $1000I$, there is an appreciable drop in dc gain between the output of the model and the output of the plant. This might have been expected, as there is nothing in the performance index that would insure that the dc gains of the plant and the model would be the same. Transient responses to an initial condition for the three Q cases are shown in Figs. 9 and 10.

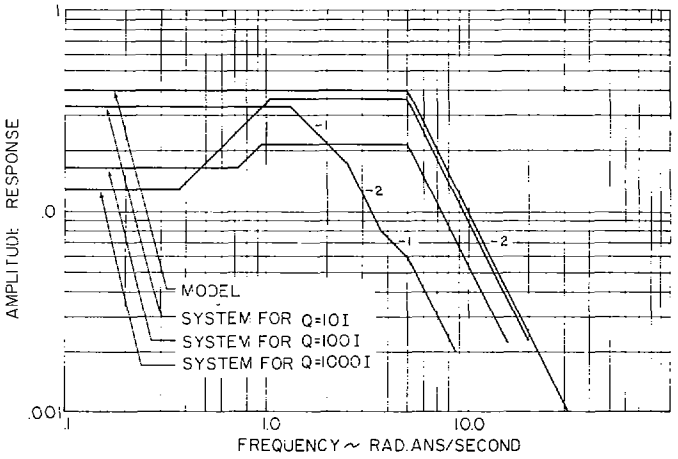


Fig. 8—Model-in-the-system: amplitude response for model and over-all system for different error weights.

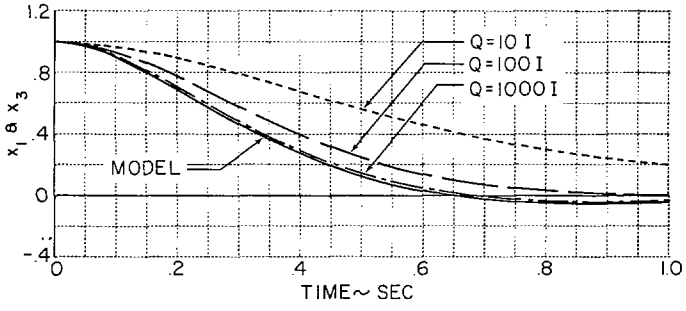


Fig. 9—Model-in-the-system: second-order system and model.

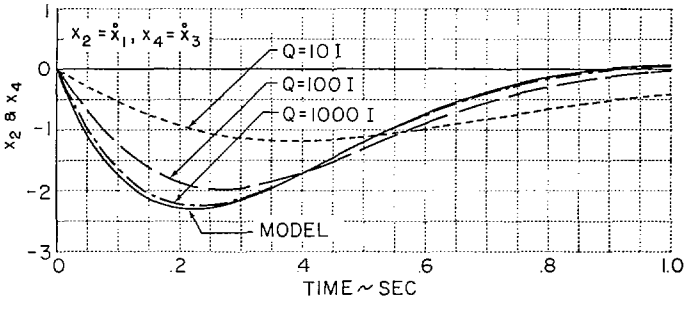


Fig. 10—Model-in-the-system second-order system and model.

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VI. Model-in-the-System Technique Applied to Airplane Lateral Equations

The model-in-the-system technique was applied to the same B-26 variable stability airplane system discussed in Section III. The purpose of doing this was to get a comparison between the two methods as well as to evaluate this particular method. Since the two methods are basically different, it is not appropriate to compare the two systems for the same error weighting. It is interesting, however, to see how well the two systems match the model for approximately the same control magnitudes. Because of the magnitude of the feedback control terms would approach infinity for an infinite weight on the error, it was also interesting to see how well this method would work for control magnitudes that did not exceed the limits of the physical system.

The state equation for this method, (32), was obtained by combining the F, L and G matrices for the B-26 example in Section III. The closed-loop system matrix, the roots of this matrix, and the control matrices are listed in Table II. The open-loop system matrix and the model matrix would be the same as Table I. The responses to an initial sideslip angle (β) are shown in Fig. 11.

It was found that although selecting the Q matrix was more of a trial-and-error procedure than it was for the model in the performance index method, the resultant system performed well. Runs were tried for $Q(1, 1, 1, 1)$, $Q(2, 2, 2, 2)$ and $Q(3, 3, 3, 3)$. In the cases where $Q(2, 2, 2, 2)$, $Q(3, 3, 3, 3)$, the model followed the system almost identically. The case where $Q(3, 3, 3, 3)$ was not plotted since it was impossible to discriminate between the model and the system responses. The control resulting from $Q(1, 1, 1, 1)$ proved to be a reasonable design. The elements of the control matrix were not too large, with the exception of the K_{22} feedback term. This term can be reduced from -2.73 to -1.9 by making $q_{22}=5$ rather than 10. A comparison of the feedback gains of this method (Table II) with those of the model-in-the performance-index method (Table I) shows that although the model-in-the-system method might at first appear as a high-gain system, the actual feedback gains obtained (Table II) are not as high as would be expected.

TABLE II
B-26 Model-in-the-System Results

	$Q(1, 1, 1, 1)$			
Closed-Loop Matrix $F-GK$	0 −12.3 .0941 .390	1 −13.6 .00777 .242	0 −2.06 −.192 8.27	0 1.04 −.886 −8.68
Closed-Loop Roots	−1.00	−1.26	−7.66	−12.7
Feedback Control Matrices K	−2.32 −3.15	−.222 −2.73	2.33 .687	−3.27 .067
Feedforward Control Matrices K_{ff}	.127 3.01	.393 3.14	−4.10 −16.6	4.58 2.15

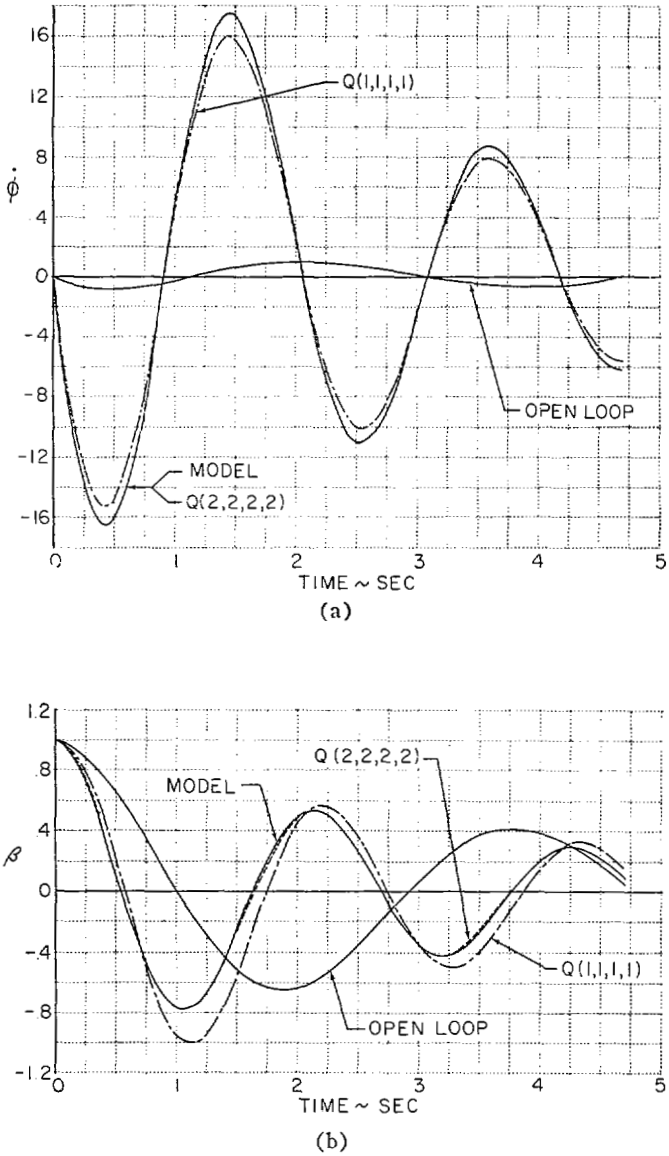


Fig. 11—B-26 model-in-the-system responses to an initial sideslip angle.

VII. Model-in-the-System Technique Applied to Airplane Longitudinal Equations

A final evaluation was obtained by applying the model-in-the-system method to a higher-order variable stability airplane that had proved to be a difficult system to design by conventional techniques. The longitudinal mode was considered and actuator dynamics for both the throttle and the elevator were included. The model for this example has characteristics similar to those for the Supersonic Transport.

The equations in terms of the original variables were

$$T\dot{\bar{x}} = F\bar{x} + G\bar{u} \tag{46}$$

where

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$$\bar{x} = \begin{bmatrix} \Delta V \\ \alpha \\ q \\ \theta \\ \delta_e \\ \dot{\delta}_e \\ \delta_T \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} \delta_{ec} \\ \delta_{eT} \end{bmatrix}.$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_T/V_T & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -M_{\dot{\alpha}} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \omega_n^2 & 0 \\ 0 & 1/\tau \end{bmatrix}$$

$$F = \begin{bmatrix} -D_V & -V_T D_\alpha & 0 & -g & 0 & 0 & -V_T D_{\delta_T} \\ z_V/V_T & z_\alpha & 1 & z_\theta & z_{\delta_e} & 0 & 0 \\ M_V & M_\alpha & M_q & 0 & M_{\delta_e} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\omega_n^2 & -2\zeta\omega_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/\tau \end{bmatrix},$$

These variables, however, were not the ones that would be measured in the actual system. A transformation therefore was found to convert the original variables into the ones to be measured. This transformation,

$$\bar{y} = W\bar{x} \tag{47}$$

produced a final state equation of

$$\dot{\bar{y}} = WT^{-1}FW^{-1}y + WT^{-1}G'\bar{u} \tag{48}$$

where

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-z_V}{g} & \dfrac{-V_T z_\alpha}{g} & 0 & 0 & \dfrac{-V_T z_{\delta_e}}{g} & 0 & 0 \\ 0 & -V_T & 0 & V_T & 0 & 0 & 0 \\ \dfrac{z_V - \alpha_T D_V}{V_T} & z_\alpha + \alpha_T D_\alpha & 1 & \dfrac{z_\theta + g\alpha_T}{V_T} & z_{\delta_e} & 0 & \alpha_T D_{\delta_T} \\ -D_V & -V_T D_\alpha & 0 & -g & 0 & 0 & -V_T D_{\delta_T} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\bar{y} = \begin{bmatrix} \Delta V \\ \Delta nz \\ h \\ \dot{\alpha} \\ \delta_e \\ \delta_T \end{bmatrix}$$

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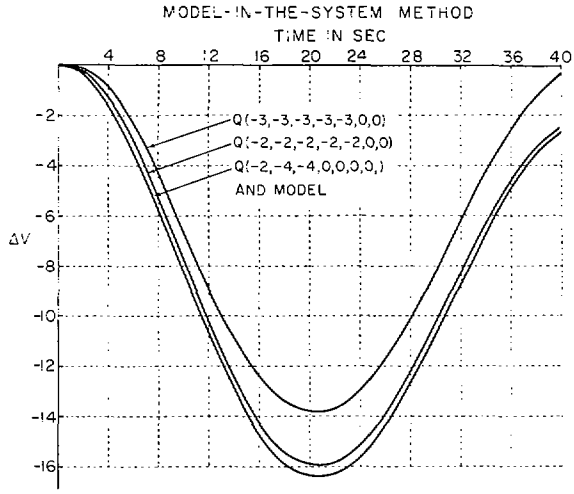


Fig. 12—Responses to an elevator step $\delta_{e_{MC}} = -1°$.

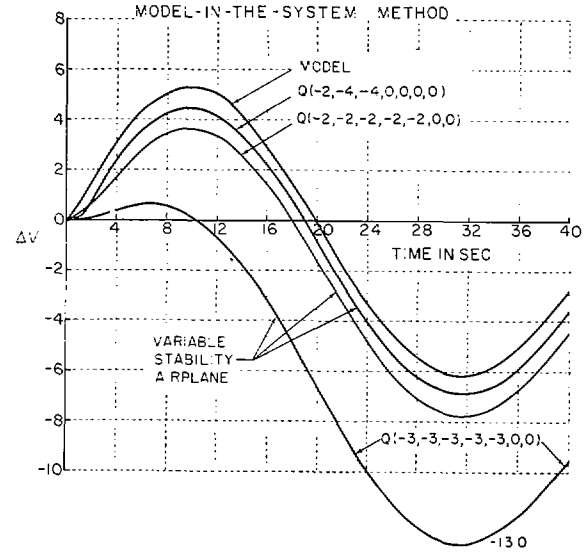


Fig. 13—Responses to a throttle step $\delta_{T_{MC}} = .1(\Delta T/T_0 = 10\%)$.

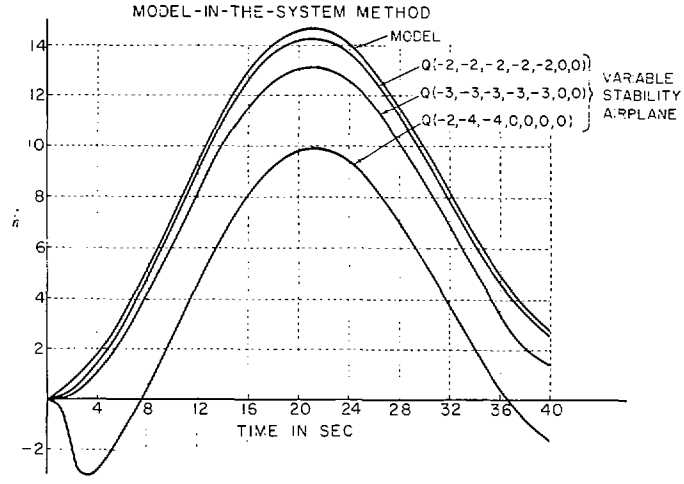


Fig. 14—Responses to a throttle step $\delta_{T_{MC}} = .1(\Delta T/T_0 = 10\%)$.

Eq. (48) for both the model and the plant numerical values was substituted into (32). The numerical values used are listed in Appendix B. The synthesis problem in this example was to obtain a system that would follow the model well with feedback gains of a reasonable magnitude. The responses to a step function on the model were used rather than initial condition responses because they produced results that were easier to interpret and compare with results that had been obtained by conventional methods. The steady-state errors for the step function response were very small for high Q cases, but became progressively worse as the weighting was decreased. The results of this example are shown in Figs. 12 through 14. The variables ΔV, n_z, and $\dot{h}$ were the hardest to match. However, the feedback gains $\delta_e/\dot{\alpha}$ and $\delta_e/\dot{V}$ were the most difficult to reduce to an acceptable value. The cases of $Q(-3, -3, -3, -3, 0, 0)$ and $Q(-2, -2, -2, -2, 0, 0)$ were tried to get an idea of the over-all performance and the resulting gains. The case $Q(-2, -2, -2, -2, 0, 0)$ resulted in a very good model-following system. However, some of the gains were too large for this case. It was desired to have the feedbacks from $\dot{\delta}_e$ and δ_r as low as possible, and so no weight was placed on these variables. Lowering the weighting to $Q(-3, -3, -3, -3, -3, 0, 0)$ produced a response which was not as good for ΔV. A final run was made in an effort to improve the ΔV response while maintaining a good Δn_z and $\dot{h}$ response. Also the variables $\dot{\alpha}$ and $\dot{V}$ were not weighted at all as the feedback gains from these variables had been too high in previous cases.

The result of this final case shows a good response in ΔV (Figs. 12 and 13). However, the $\dot{h}$ response to a throttle step suffered in this case (Fig. 14). A final design would have to represent a compromise between these three cases while still maintaining the magnitude of the feedback gains below the prescribed values. It should be noted that to just obtain the feedback gains or to vary the weighting in the Q matrix to see how a different weighting affects the magnitude of the gains, the complete system need not be used. It is possible to compute the same feedback gains by optimizing the plant only and setting the model $[F_M]$ to zero due to the fact that the feedback gains are independent of the model. In this example, this method would be a considerable advantage and would result in the computation with 14×14 matrices rather than the 28×28 matrices of the complete system. In general, it is quite easy to affect the response of a particular variable by changing the weighting with the corresponding q_{ii} term. It is not, however, always easy to make one of the feedback gains (K_{ij}) come out to be a certain value by this same technique of altering the weighting matrix.

VII. Conclusions

Both model-following techniques that have been evaluated in this paper would be feasible for the design of multivariable control systems. The particular method used would depend upon the application.

If the plant can be identified accurately, and it is physically possible to match the model, then the model-in-the-performance-index would probably be the more useful technique. Because the model-matching ability of this method is highly dependent on whether each coefficient of the system can be independently made to

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match each of the model coefficients, this method is somewhat limited. The fact that, with a large weight on the error term in the performance index, the feedback gains approach a limiting value makes the selection of weighting matrices much easier. Also, because the control depends predominantly on a product of constant matrices, real-time computation would be very easy. It should be noted, however, that it is not always possible to disregard the control terms from the Riccati equation. In any practical system there would be limitations on the magnitude of the control. If the weight on the error is so high that the Riccati terms are negligible, the limitations on the control may be exceeded.

The model-in-the-system method is perhaps a more general method. It is not limited directly by the dependence of the coefficients in the model and the plant.

Results indicate, however, that the control terms are not as large as might be expected because of the favorable effect of the feedforward terms. The fact that the basic feedback system, *i.e.*, feedback gains, are independent of the model or input seems also to be an advantage in this method.

It should be emphasized that these methods, like any other synthesis technique, involve to some extent a trial and error procedure, rather than an automatic way of designing a control system.

Appendix A

Derivation of the Control

The derivation of the control for a quadratic performance index has been shown by Kalman and others and is included here for completeness.

Once the performance index is defined, an $\mathcal{H}$ function is set up to introduce the system differential equations as constraints. For the performance index of (5) and the plant described by (1),

$$2\mathcal{H} = (HF\bar{x} + HG\bar{u} - LH\bar{x})'Q(HF\bar{x} + HG\bar{u} - LH\bar{x}) + \bar{u}'R\bar{u} + 2\bar{\lambda}\cdot(F\bar{x} + G\bar{u}) \quad (49)^1$$

This function evaluated for the optimal control vector u^0 is called the Hamiltonian. To find the form of the optimal control, the partial derivative of (49) is taken with respect to $\bar{u}$ and the result is equated to zero. This minimizes (49) with respect to $\bar{u}$.

$$\frac{\partial\mathcal{H}}{\partial\bar{u}} = (G'H'QHG + R)\bar{u} + G'H'Q(HF - LH)\bar{x} + G'\bar{\lambda} = [0] \quad (50)$$

or

$$\bar{u}^0 = -(G'H'QHG + R)^{-1}\cdot\{G'\bar{\lambda} + G'H'Q(HF - LH)\bar{x}\} \quad (51)$$

Pontryagin's maximum principle may be used to evaluate the necessary conditions for optimizing the performance index. These conditions are

$$\frac{\partial\mathcal{H}}{\partial\bar{x}} = -\dot{\bar{\lambda}} \quad (52)$$

$$\frac{\partial\mathcal{H}}{\partial\bar{\lambda}} = \dot{\bar{x}}. \quad (53)$$

Evaluating (52) and (53) for (49) yields the following:

$$\frac{\partial\mathcal{H}}{\partial\bar{x}} = -\dot{\bar{\lambda}} = \{(HF - LH)'Q(HF - LH) - (HF - LH)'QHG(G'H'QHG + R)^{-1}G'H'Q(HF - LH)\}\bar{x} + \{F' - (HF - LH)'Q'HG(G'H'QHG + R')^{-1}G'\}\bar{\lambda} \quad (54)$$

and

$$\frac{\partial\mathcal{H}}{\partial\bar{\lambda}} = \dot{\bar{x}} = \{F - G(G'H'QHG + R)^{-1}G'H'Q(HF - LH)\}\bar{x} - G(G'H'QHG + R)^{-1}G'\bar{\lambda} \quad (55)$$

These equations are equivalent to the canonical equations of the regulator problem. If the matrix coefficients of the canonical equations of the regulator problem are equated to the coefficients of (54) and (55), the results are

$$\hat{R} = (G'H'QHG + R).$$
$$\hat{F} = F - G\hat{R}^{-1}G'H'Q(HF - LH).$$
$$\hat{G} = G.$$
$$\hat{Q} = Q - QHG\hat{R}^{-1}G'H'Q.$$
$$\hat{H} = HF - LH. \quad (56)$$

Eqs. (54) and (55) may now be simplified to

$$\dot{\bar{x}} = \hat{F}\bar{x} - \hat{G}\hat{R}^{-1}\hat{G}'\bar{\lambda} \quad (57)$$

and

$$\dot{\bar{\lambda}} = -\hat{H}'\hat{Q}\hat{H}\bar{x} - \hat{F}'\bar{\lambda} \quad (58)$$

These two equations with the "hat" matrices are now the same as the canonical equations for the regulator problem and therefore the same solution will apply. It is assumed that

$$\bar{\lambda} = P(t)\bar{x} \quad (59)$$

where P is a symmetric time-varying matrix. The canonical equations of (59) may be written as a matrix Riccati equation from (59):

$$\dot{\bar{\lambda}} = \dot{P}(t)\bar{x} + P(t)\dot{\bar{x}}. \quad (60)$$

Substituting (57) and (58) into (60),

$$-\hat{H}'\hat{Q}\hat{H}\bar{x} - \hat{F}'\bar{\lambda} = P(t)\bar{x} + P(t)(\hat{F}\bar{x} + \hat{G}\hat{R}^{-1}\hat{G}'\bar{\lambda}). \quad (61)$$

Simplifying (61) for an arbitrary $\bar{x}$, results in

$$P - P\hat{G}\hat{R}^{-1}\hat{G}'P + \hat{F}'P + P\hat{F} + \hat{H}'\hat{Q}\hat{H} = [0]. \quad (62)$$

[1] $2\bar{\lambda}\cdot(F\bar{x}+G\bar{u})$ is a dot product in this equation.

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This is the matrix Riccati differential equation. Eq. (62) is necessary for computing the control

$$\bar{u}^0 = -\hat{R}^{-1}\hat{G}'P(t)\bar{x}. \quad (63)$$

This part of the control may be found the same as it would be for the regulator problem, it is only necessary to compute the "hat" values, and enter them in place of the normal matrices.

Appendix B

Numerical Values for Model-in-the-System Variable Stability Airplane Example

	Model	Plant
α_T	.253	.209
V_T	257	257
$M_{\dot{\alpha}}$	−.439	−.215
D_V	.038	.0296
D_α	.00884	−.0679
g	32.2	32.2
z_V	−.257	−.253
z_θ	−.0317	−.0262
M_V	0	.000366
M_α	−.741	−1.75
M_q	−.658	−.536
z_α	−.501	−.667
D_{δ_T}	−.0377	−.0329
z_{δ_e}	−.0720	−.0326
M_{δ_E}	.823	−2.66

Actuators

$\omega_n = 40$

$\zeta = .7$

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Optimal-Control Theory Applied to a Probabilistic Intercept Problem

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Summary—**The method of dynamic programming is applied to the problem of the optimum guidance of an interceptor designed to intercept a target capable of random maneuvers. A cost is associated with any final miss distance and with the interceptor maneuvers, and, because of the randomness of the target motion, the cost is also a random variable. By assuming that the cost components are quadratic functions of their arguments and by utilizing the ensemble average of the cost, a relatively simple closed-form expression for the control and for the average cost is obtained. It is found that the optimum system is stable and well damped, and that the effect of the random maneuvers is relatively small.**

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Introduction

THE OPTIMUM-GUIDANCE PROCEDURE for an interceptor designed to intercept a target capable of random maneuvers can be derived by considering the cost of missing the target and the maneuvering cost. The cost of missing the target is generally some even function of the minimum miss distance. It might range from zero in the case of a direct hit to a very large value if the miss distance is greater than the interceptor's lethal zone. The maneuvering cost can be related directly to fuel expended in making the maneuver.

For a target capable of random maneuvers, the optimum-control procedure is not completely obvious. All

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