AN INTRODUCTION TO THE ANALYSIS AND PROCESSING OF SIGNALS

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6

Signal Comparison

6.1 Introduction

Although we have so far concentrated on the problem of describing or analysing individual signals, the foregoing chapters have in fact laid much of the groundwork for a discussion of the ways in which signals may be compared with one another.

Any of the measures of signal properties so far described may be used as a basis for comparison. For example, if we have two random signals, we may compare their amplitude distributions, mean values, variances, autocorrelation functions, and so on. Although it is sometimes adequate to make a subjective assessment of the similarities of two signals, such as that they have 'approximately equal mean values', it is often useful to have a definite quantitative measure of shared properties. Fortunately, the description of orthogonal functions and their applications given in section 2.3 forms a good introduction to the derivation of useful quantitative measures.

In section 2.3.1, where the analogy between vectors and signals was discussed, we saw how a signal $f_1(t)$ could be approximated over any interval $t_1 < t < t_2$ by another signal or waveform $f_2(t)$, together with an error term $f_e(t)$

$$f_1(t) = C_{12} . f_2(t) + f_e(t)$$

We then adjusted the coefficient C_{12} to achieve some desired error criterion—normally to minimise the average squared value of $f_e(t)$ in the interval $t_1 < t < t_2$ (which is the same as minimising the 'error power'). This particular criterion allowed the appropriate value of C_{12} to be expressed quite simply as

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) . f_2(t) . \mathrm{d}t}{\int_{t_1}^{t_2} f_2^2(t) . \mathrm{d}t}$$

Thus if the product of $f_1(t)$ and $f_2(t)$ in the stipulated interval is zero, we say that there is no component of $f_2(t)$ in $f_1(t)$; in other words the signals are orthogonal. Conversely, a large value of C_{12} denotes the fact that $f_1(t)$ and $f_2(t)$ have much in common. We have therefore defined a quantitative approach towards comparison of the two signa.. $f_1(t)$ and $f_2(t)$.

Although this general concept could be used to compare various aspects of two signals–for example, their amplitude probability density functions–by far the most common practice is to use it, as in the above discussion, to compare the time functions describing their actual waveforms. As a starting point, let us assume that

$$f_1(t) = A_1 \cos \omega_1 t \qquad \text{and} \qquad f_2(t) = A_2 \cos \omega_2 t$$

and let us use as a measure of their relationship the average product over a long interval T_0, given by

$$\lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t) \cdot f_2(t) \cdot \mathrm{d}t$$

(this measure is no different in principle from C_{12} defined above; we have merely chosen not to normalise by the mean square value of $f_2(t)$, and to take an averaged version of the result over a long time interval). Because of the orthogonality of cosines of different frequency, our measure will be zero unless $\omega_1 = \omega_2$. If we now change $f_2(t)$ to $A_2 \sin \omega_2 t$, the result will be zero for every value of ω_2 including $\omega_2 = \omega_1$, because of the orthogonality of sine and cosine waveforms. Hence our comparative measure of $f_1(t)$ and $f_2(t)$ will only be nonzero by virtue of shared cosine, or sine, components at particular frequencies; unfortunately, however, as it stands it will yield no information about which frequencies are common to the two waveforms or about their relative amplitudes. This limitation may be resolved by use of the rather more comprehensive measure known as the cross-correlation function.

6.2 The cross-correlation function

The cross-correlation function relating two signals $f_1(t)$ and $f_2(t)$ may be defined as

$$r_{xy}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t) \cdot f_2(t + \tau) \cdot \mathrm{d}t$$

where τ is a time-shift imposed upon one of the signals. Two important points may be noted at once: firstly, the cross-correlation function (CCF) and the auto-correlation function (ACF) described in the last chapter are very similar, the only difference being that the CCF applies to two different signals, whereas the ACF relates a signal to a shifted version of itself: and secondly, the definition of the CCF ties in closely with the above discussion on orthogonality and signal comparison. However, instead of deriving a single index (such as the coefficient

C_{12}) to describe the similarity between two signals, we now define a variable r_{xy} which is a continuous function of the imposed time-shift τ. When the signals to be compared are of limited duration it is appropriate to use a 'finite' version of the cross-correlation function, given by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} f_1(t) \cdot f_2(t+\tau) \cdot \mathrm{d}t$$

This exactly parallels the use of the 'finite' version of the ACF mentioned in section 5.4.2.

The benefits of introducing a time shift between $f_1(t)$ and $f_2(t)$ are not hard to appreciate. Suppose, to take a rather extreme example, $f_1(t)$ and $f_2(t)$ are two random signals which differ only in their time origin, as illustrated in figure 6.1(a). The average of the product $[f_1(t) \cdot f_2(t)]$ over a long time interval may well

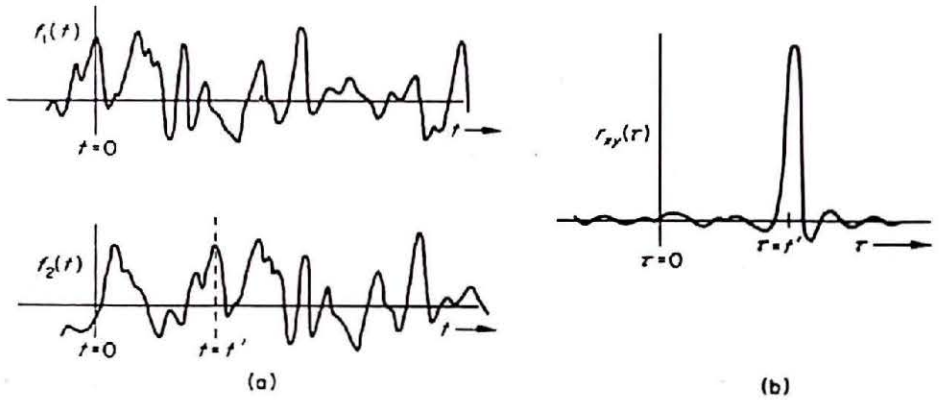


Figure 6.1 *(a) Portions of two random signals which differ only in their time origin, and (b) an estimate of their cross-correlation function*

be zero, but we would be quite wrong to infer from this that they have nothing in common; indeed, apart from a simple time shift of t' seconds, they are identical. This fact is revealed clearly by their CCF, the value of which is large when the shift τ equals t', as shown in figure 6.1(b). Similarly, a sine and cosine wave of the same frequency, which are orthogonal and therefore 'have nothing in common' as they stand, certainly do have something in common when one of them is shifted. In other words the cross-correlation function satisfies the commonsense notion that two signal waveforms with common frequency components are in an important sense alike, in spite of different timing or phase relationships.

In order to formalise the properties of the cross-correlation function we now investigate its form for the two signals

$$f_1(t) = A_1 \cos \omega t, \qquad f_2(t) = A_2 \cos(\omega t + \theta)$$

where θ represents a general value of phase shift.

We have

$$r_{xy}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A_1 A_2 \cos \omega t \,.\cos\left[\omega(t+\tau)+\theta\right].\mathrm{d}t$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{A_1 A_2}{2} \left[\cos(2\omega t + \omega\tau + \theta) + \cos(\omega\tau + \theta)\right].\mathrm{d}t$$

The first cosine term in the integrand averages out to zero over a long interval; the second is not a function of t, hence

$$r_{xy}(\tau) = \lim_{T_0 \to \infty} \frac{1}{T_0} \cdot \frac{A_1 A_2}{2} \cdot \cos(\omega\tau + \theta) \cdot \left[t\right]_{-T_0/2}^{T_0/2}$$

$$= \frac{A_1 A_2}{2} \cos(\omega\tau + \theta)$$

Thus the form of the CCF reflects the product of the amplitudes of $f_1(t)$ and $f_2(t)$, their common frequency ω, and their relative phase angle θ; it is plotted in figure 6.2. When the two signals being cross-correlated share a number of common frequencies, each gives a corresponding contribution to the CCF. In this respect the auto- and cross-correlation functions are essentially similar; however, the ACF loses all information about the phase of various frequency components in a signal, whereas the CCF retains information about the relative phases of common frequency components in the two signals which it is comparing. A consequence is that, unlike the ACF, the CCF is not normally an even function of τ.

Since the cross-correlation function is essentially a time-averaged measure of shared signal properties it is very suitable for comparing random signals. It then yields an average measure of frequency components held in common, and a practical estimate of it only approaches the theoretical one when very long portions of the signals are available; otherwise there are the inevitable 'sampling errors'

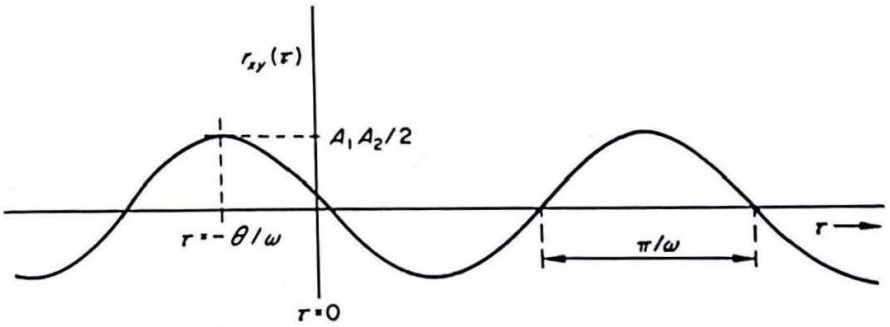


Figure 6.2 *The cross-correlation function for two cosine waves of amplitudes A_1 and A_2, frequency ω, and relative phase angle θ*

associated with any attempt to extract statistical measures from limited signal portions (see section 5.4.4). An example of a cross-correlation function applied to random signals has already been shown in figure 6.1, illustrating the great use of the CCF in showing up time delays. A further example is shown in figure 6.3, which represents an estimated CCF of two random signals with shared frequency ranges. The periodicities in the CCF reflect these shared ranges, although it is certainly difficult to extract any precise information from the figure. In cases such as this where simple time shifts are not involved, it is often more appropriate and rewarding to use the so-called cross-spectral density function.

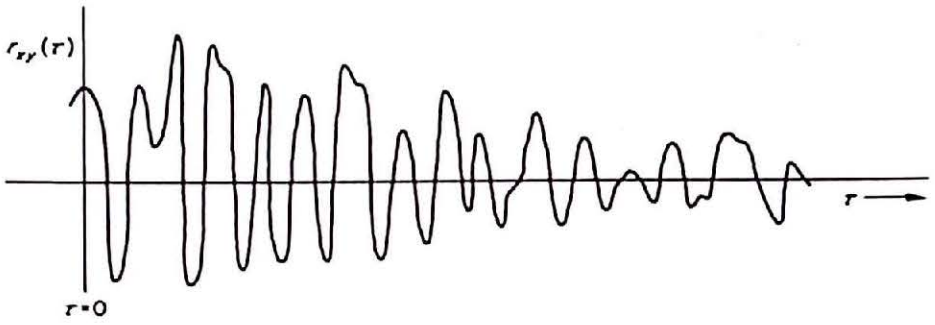


Figure 6.3 *A typical estimated cross-correlation function for two random signals with two closely spaced, shared, frequency ranges*

6.3 The cross-spectral density function

Just as the autocorrelation function of a signal has its counterpart in the frequency domain, so the cross-correlation function may be transformed into an equivalent frequency domain function—the cross-spectral density, or cross-spectrum. Formally, the relationship is expressed in terms of the following Fourier transform pair

$$P_{xy}(\omega) = \int_{-\infty}^{\infty} r_{xy}(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

and

$$r_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{xy}(\omega) \cdot e^{j\omega\tau} \cdot d\omega$$

where $P_{xy}(\omega)$ is the cross-spectral density function. As in the case of the auto-correlation and power spectral density functions used to describe an individual signal, the cross-correlation and cross-spectral density functions contain equivalent information, and which of the two is used for signal comparison in a particular case is therefore largely a matter of convenience.

We have already seen that when two signals $f_1(t) = A_1 \cos \omega t$ and $f_2(t) = A_2 \cos(\omega t + \theta)$ are cross-correlated the resulting wave has the same period in τ as $f_1(t)$ and $f_2(t)$ have in t, with an amplitude equal to $A_1 A_2/2$. Furthermore, its maximum value occurs at $\tau = -\theta/\omega$, where θ is the relative phase shift between

the two underlying signals (see figure 6.2). The corresponding spectral description therefore involves a term of magnitude $(A_1A_2/2)$ and phase angle θ, at a frequency ω radians/second. In the general case, $f_1(t)$ and $f_2(t)$ contain a number of common frequencies or frequency ranges, all of which are indicated by their cross-spectrum. A typical cross-spectrum of two periodic signals containing a number of common components is shown in figure 6.4: each cross-spectral term has a magnitude equal

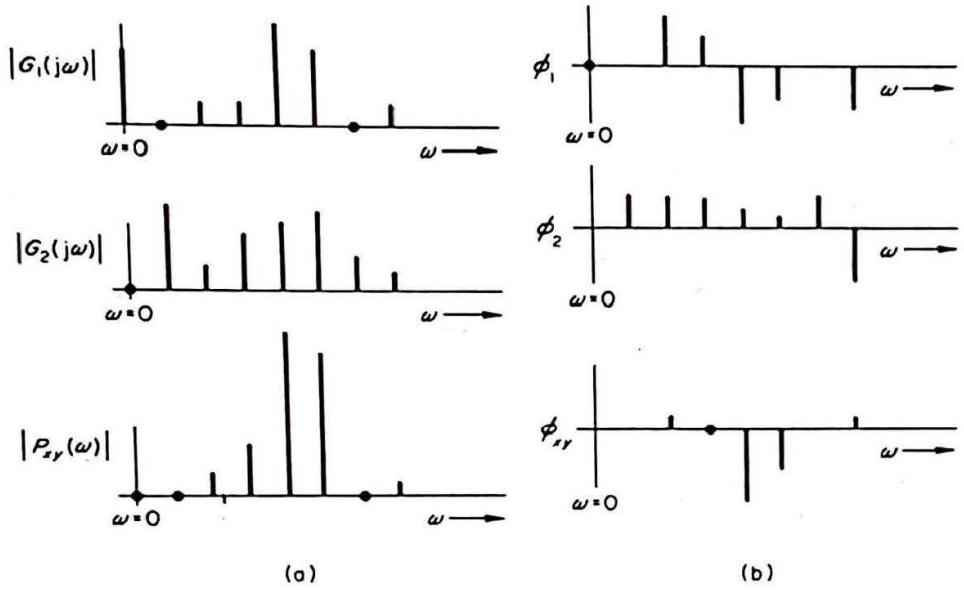


Figure 6.4 *(a) The magnitude characteristics of two signal spectra and of their cross-spectrum, and (b) the corresponding phase characteristics*

to half the product of the amplitudes of the corresponding waves in $f_1(t)$ and $f_2(t)$, and a phase equal to the difference between their individual phase angles.

Unlike the power spectrum of a single signal, the cross-spectrum relating two different signals is generally complex, involving phase as well as magnitude terms as illustrated above in figure 6.4. In the rather special case when $f_2(t)$ is essentially a delayed but otherwise unaltered version of $f_1(t)$, as in figure 6.1, the phase difference between common components is just proportional to their frequency (this point has already been covered in section 4.2), whereas their amplitudes are the same. Therefore the cross-spectrum will have a magnitude characteristic equal to that of the power spectrum of each individual signal, and a phase characteristic which is proportional to frequency.

It is sometimes convenient to split the cross-spectrum function $P_{xy}(\omega)$ into real and imaginary parts, rather than work with magnitude and phase characteristics. Hence

$$P_{xy}(\omega) = C_{xy}(\omega) + \mathrm{j}\, Q_{xy}(\omega)$$

where $C_{xy}(\omega)$ is called the 'co-spectrum' and $Q_{xy}(\omega)$ the 'quadrature-spectrum'. We therefore have

$$|P_{xy}(\omega)| = [C_{xy}^2(\omega) + Q_{xy}^2(\omega)]^{1/2}$$

and

$$\phi_{xy}(\omega) = \tan^{-1}\left[\frac{Q_{xy}(\omega)}{C_{xy}(\omega)}\right]$$

where $\phi_{xy}(\omega)$ represents the phase difference between components in the two signals at a frequency ω.

6.4 Discussion

In view of the central importance of frequency spectra in signal theory, it is not surprising that the two indices most commonly used for signal comparison—the cross-correlation and cross-spectrum functions—are essentially measures of shared spectral components. Their widespread use should not, however, obscure the fact that it is occasionally useful to compare other aspects of two signals, such as their amplitude distributions or moments. It must also be appreciated that the cross-correlation function fails to indicate any correlation in some situations where commonsense would suggest otherwise. A simple example of this is illustrated in figure 6.5. Here the signal $f_1(t)$ is a cosine wave $A\cos\omega t$, and $f_2(t)$ is simply equal to $[f_1(t)]^2$, which might very well suggest that the two signals are correlated. However, since

$$f_2(t) = [f_1(t)]^2 = A^2\cos^2\omega t = \frac{A^2}{2} + \frac{A^2}{2}\cos 2\omega t$$

$f_2(t)$ contains both a zero-frequency component and one at 2ω radians/second, but not one at ω radians/second. Therefore $f_1(t)$ and $f_2(t)$ have no shared frequencies and their cross-correlation function is zero for all values of τ. The basic reason for this is that the relationship between $f_1(t)$ and $f_2(t)$ is a so-called 'nonlinear' one, whereas the correlation functions we have described in both this and the previous chapter indicate only 'linear' signal relationships. The meaning

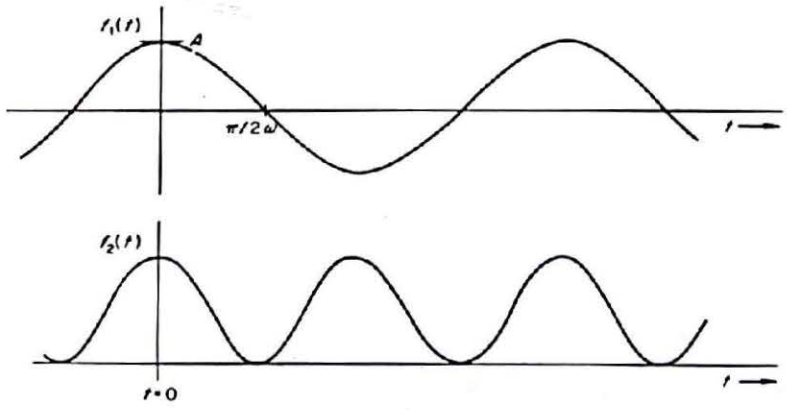


Figure 6.5

of the terms linear and nonlinear in a signal context will be explored rather more fully in the next chapter.

The practical importance of the cross-correlation and cross-spectrum functions is easier to appreciate if we consider why we might wish to compare two signals in the first place. The most likely reason is that some sort of causal relationship between them is anticipated, and we wish to define it quantitatively. For example, the signal $f_1(t)$ might represent an input or disturbing function applied to some system (such as a mechanical force exerted on a structure), and $f_2(t)$ a response (for example, a resulting mechanical displacement) measured at some point. If we are able to define the relationship between $f_1(t)$ and $f_2(t)$ by cross-correlating them, it is not hard to see that we have obtained an important measure of the properties of the system itself. Furthermore, since the cross-correlation function applies to random as well as to deterministic signals, it suggests the possibility of defining the performance of a system by disturbing it with a random input. However at this stage we are beginning to trespass heavily on the subject of signal processing, rather than signal analysis, and a fuller discussion of these points is therefore best postponed until some basic aspects of signal processing have been introduced.

Problems

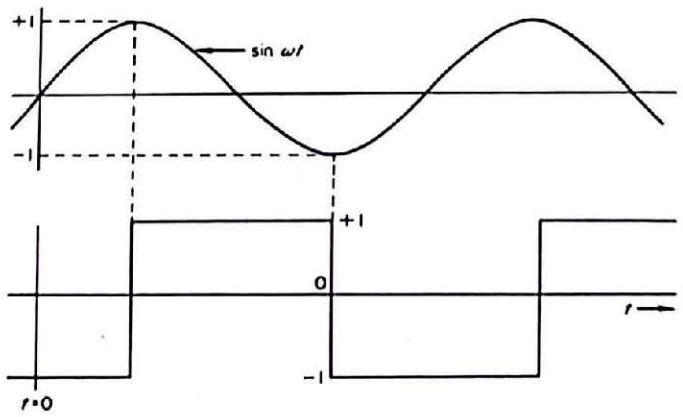
1. Two random signals have equal mean values and variances. What clues, if any, does this information give about similarities between their

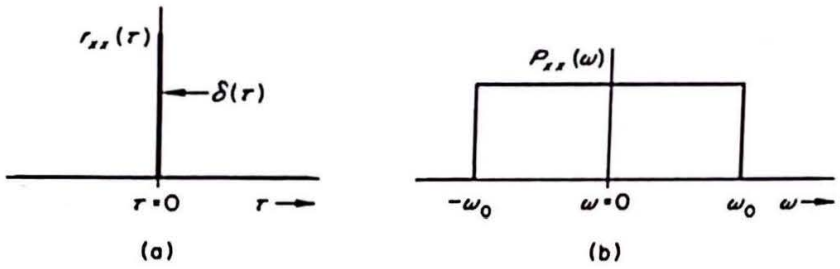
   (i) average powers,
   (ii) autocorrelation functions,
   (iii) second moments,
   (iv) amplitude distributions,

   and what clues does it give about their

   (v) cross-correlation function,
   (vi) cross-spectrum?

2. Evaluate the cross-correlation function (CCF) of the square and sinusoidal waveforms shown in the accompanying figure, and show that it reflects just



their single common frequency component. Comment on the peak magnitude and phase of the CCF.

3. A random signal is known to have the autocorrelation function shown in part (a) of the accompanying figure: a second random signal, statistically independent of the first, has the power spectrum indicated in part (b). What is the theoretical form of their cross-spectrum? What may be inferred about their cross-correlation function? Would these two measures be of any practical interest?