
ELECTROMAGNETIC WAVES

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RESONATORS

8.1 INTRODUCTION

In each of the preceding chapters, we have discussed electromagnetic wave propagation in systems that are unbounded in at least one dimension, such as unbounded media, interfaces, transmission lines, and waveguides. In this chapter, we consider structures that are constrained in **all** dimensions such that electromagnetic energy cannot propagate but is instead trapped in one place, coupling only weakly to the external world. Such resonators exhibit strong frequency-dependent behavior within narrow frequency bands centered about discrete frequencies called *resonances*. Resonators can be used not only for energy storage, frequency measurement, and filtering, but also for impedance matching and the production of high-field strengths with limited input signals.

Resonators are easily made by closing the ends of any waveguide structure, such as a *TEM* line, confining electromagnetic radiation in all three dimensions. We recall that the parallel-plate waveguide, confined in one dimension and possessing one guidance condition, produced a one-dimensional standing wave pattern in the plane transverse to the direction of wave propagation. Likewise, rectangular and circular waveguides, confined in two dimensions, gave rise to two guidance conditions and exhibited two-dimensional standing-wave patterns in the plane perpendicular to wave motion. Therefore, we expect that a resonator will have three guidance conditions and will generate a standing-wave pattern in all three dimensions. Since each guidance condition constrains one component of the wave vector $\vec{k}$, the three resonator guidance conditions together limit the magnitude of $\vec{k}$. Since $\omega = |\vec{k}|/\sqrt{\mu\epsilon}$, it follows that ω is limited to discrete values called *resonant*

frequencies. By contrast, waveguide modes may limit k_x and/or k_y but do not constrain k_z , where $\hat{z}$ is the direction of the waveguide axis. Therefore, k_z and ω may have any of a continuum of values in a waveguide.

Most common resonators are constructed by closing the ends of a rectangular or circular waveguide with conducting walls, forming a *cavity resonator*, or by reactively terminating the ends of a *TEM* transmission line, forming a *TEM resonator*. We shall discuss both of these structures in Sections 8.2 and 8.5, respectively. However, any structure that can trap oscillatory electromagnetic energy is a resonator that will possess an essentially infinite set of resonant frequencies.¹ Even a misshapen lump of dielectric in free space can be considered to be a resonator, although if its stored energy leaks away too rapidly, the sharp-frequency behavior of the system becomes “smeared” over a band of frequencies centered about each resonance.

Nonideal resonators lose energy gradually, either externally through coupling to the outside world, or internally by heat generation. After discussing ideal cavity resonators in Section 8.2, these slightly lossy resonators are considered in Section 8.3 together with methods for characterizing their nonideal resonance behavior using parameters such as the resonator quality factor Q . Each specific resonance can be closely modeled with a discrete element resonator circuit, treated in Section 8.4, while transmission line resonators, introduced in Section 8.5, provide a simple vehicle for understanding the transient behavior of resonators. Because exact analysis of nonideal resonators is difficult, Section 8.6 describes how realistic structures can often be approximated by perturbing their ideal counterparts, while Section 8.7 treats perturbations in a more general way. Sections 8.4 and 8.8 describe how resonators can be coupled to the external world in a wide variety of practical applications, many of which are described in Section 8.9. For example, resonators can be used as narrow-band filters that either pass or block energy within a bandwidth $\Delta\omega$ of a resonance, where $\Delta\omega$ is proportional to the rate at which the resonator loses energy at that frequency. Resonators can also function as transformers that couple energy into mismatched loads. In fact, some of the matching structures studied in Chapter 6 can be viewed as resonators with bandwidths varying inversely with the degree of mismatch.

8.2 CAVITY RESONATORS

We first discuss the *rectangular cavity resonator*, constructed from a waveguide of rectangular cross-section having a width a and height b ($a \geq b$) and shown in Figure 7.12. The waveguide is closed by two perfectly conducting plates located at

1. At very high frequencies, waves interact with media nonclassically; i.e., Maxwell's equations are not sufficient to describe wave behavior when quantum mechanical effects become important.

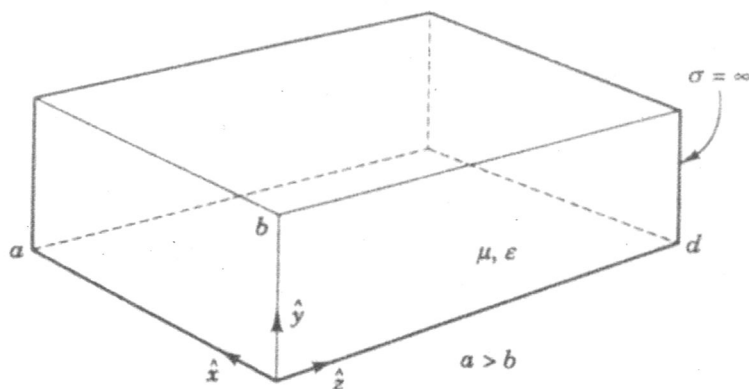


Figure 8.1 Rectangular waveguide resonator.

$z = 0$ and $z = d$ ($d \geq a$), as illustrated in Figure 8.1, forming a rectangular parallelepiped. The convention that $d \geq a \geq b$ leads to unambiguous resonator mode designations, although care must be taken in reading the literature as not all texts use the same ordering of resonator side lengths. Since both TE and TM modes propagate in rectangular waveguides, TE and TM modes also exist in rectangular cavity resonators and are superpositions of positive and negative traveling TE or TM waveguide modes, respectively. Superimposed waves propagating in both directions are needed to satisfy the boundary conditions at the ends of the resonator:

$$E_x = E_y = 0 \quad \text{at } z = 0, z = d \quad (8.2.1)$$

We may develop expressions for the TE electromagnetic fields within such a resonator by considering the electric field of a linear combination of $+\hat{z}$ - and $-\hat{z}$ -traveling TE_{mn} rectangular waveguide modes given by (7.4.13) and (7.4.14):

$$\begin{aligned} E_x &= \frac{k_y}{k_0} E_0 \cos k_x x \sin k_y y (A e^{-jk_z z} + B e^{jk_z z}) \\ E_y &= -\frac{k_x}{k_0} E_0 \sin k_x x \cos k_y y (C e^{-jk_z z} + D e^{jk_z z}) \\ E_z &= 0 \end{aligned} \quad (8.2.2)$$

where

$$k_x = m\pi/a \quad (8.2.3)$$

$$k_y = n\pi/b \quad (8.2.4)$$

$$k_z = \sqrt{\omega^2 \mu \epsilon - (m\pi/a)^2 - (n\pi/b)^2} \quad (8.2.5)$$

as was determined in (7.4.19).

By forcing the electric fields in (8.2.1) to equal zero at the $z = 0$ end of the waveguide, we find that $A = -B$ and $C = -D$, so both the E_x and E_y compo-

nents of the TE_{mn} electric field are proportional to $\sin k_z z$. The additional constraint that the electric field in (8.2.1) is zero at the $z = d$ end of the waveguide forces $\sin k_z d = 0$, or

$$k_z d = p\pi \quad p = 0, 1, 2, \dots \quad (8.2.6)$$

By requiring that $\nabla \cdot \bar{E} = 0$, we find that $A = C \equiv 1$ so that the arbitrary constant E_0 will have dimensions of V/m. Thus, the electric field of the TE_{mnp} mode is

$$\begin{aligned} E_x &= \frac{k_y}{k_0} E_0 \cos k_x x \sin k_y y \sin k_z z \\ E_y &= -\frac{k_x}{k_0} E_0 \sin k_x x \cos k_y y \sin k_z z \\ E_z &= 0 \end{aligned} \quad (8.2.7)$$

The magnetic field of the TE_{mnp} mode is determined from Faraday's law (1.4.4)

$$\begin{aligned} H_x &= \frac{jk_x k_z}{\eta k_0^2} E_0 \sin k_x x \cos k_y y \cos k_z z \\ H_y &= \frac{jk_y k_z}{\eta k_0^2} E_0 \cos k_x x \sin k_y y \cos k_z z \\ H_z &= \frac{j(k_x^2 + k_y^2)}{\eta k_0^2} E_0 \cos k_x x \cos k_y y \sin k_z z \end{aligned} \quad (8.2.8)$$

where the three guidance conditions are

$$\begin{aligned} k_x &= m\pi/a \\ k_y &= n\pi/b \\ k_z &= p\pi/d \end{aligned} \quad (8.2.9)$$

from (8.2.3), (8.2.4) and (8.2.6). A quick check of these expressions for the magnetic field shows that they too satisfy the boundary condition that $H_z = 0$ at the ends of the rectangular cavity resonator.

The dispersion relation, obtained by substituting $\bar{E}$ or $\bar{H}$ into the wave equation (1.4.9) and using (8.2.9), is:

$$\omega^2 \mu \epsilon = k_x^2 + k_y^2 + k_z^2 = (m\pi/a)^2 + (n\pi/b)^2 + (p\pi/d)^2 \quad (8.2.10)$$

This key result means that a discrete, infinite set of TE resonances may be excited in the cavity. The resonant frequency of the TE_{mnp} mode is thus

$$\omega_{mnp} = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \quad (8.2.11)$$

and is the **only** frequency for which that mode exists. If the dimensions of the resonator are such that two or more modes have exactly the same resonant frequency, then they are said to be *degenerate modes*. In our notation, the integers m and n are associated with the width and height of the waveguide (a and b), consistent with our previous conventions describing rectangular waveguide modes. The length d is associated with the integer p . Since it is not always clear which dimension of a cavity corresponds to its length, we choose $d \geq a \geq b$ as noted earlier.

A similar derivation yields the TM_{mnp} modes, where expressions for the magnetic field components can be determined from the TM_{mn} rectangular waveguide modes (7.4.16) and (7.4.17), with the z -dependence of H_x and H_y proportional to $\cos k_z z$ in order to match the boundary condition $E_y = E_x = 0$ at $z = 0$ and $z = d$, corresponding to

$$\frac{\partial H_x}{\partial z} = \frac{\partial H_y}{\partial z} = 0 \quad \text{at } z = 0, z = d \quad (8.2.12)$$

Both (8.2.1) and (8.2.12) are equivalent ways of stating that the tangential electric fields at the end walls are zero. The TM_{mnp} magnetic fields are thus

$$\begin{aligned} H_x &= \frac{k_y}{k_0} H_0 \sin k_x x \cos k_y y \cos k_z z \\ H_y &= -\frac{k_x}{k_0} H_0 \cos k_x x \sin k_y y \cos k_z z \\ H_z &= 0 \end{aligned} \quad (8.2.13)$$

and the TM_{mnp} electric field is then found from Ampere's law (1.4.5):

$$\begin{aligned} E_x &= \frac{j\eta k_x k_z}{k_0^2} H_0 \cos k_x x \sin k_y y \sin k_z z \\ E_y &= \frac{j\eta k_y k_z}{k_0^2} H_0 \sin k_x x \cos k_y y \sin k_z z \\ E_z &= \frac{j\eta(k_x^2 + k_y^2)}{k_0^2} H_0 \sin k_x x \sin k_y y \cos k_z z \end{aligned} \quad (8.2.14)$$

Note that because the TM_{mnp} modes have the same guidance conditions (8.2.9) as the TE_{mnp} modes, they must also have the **same** resonant frequencies given by (8.2.11).

We find that the resonator field expressions (8.2.7), (8.2.8), (8.2.13) and (8.2.14) are zero for certain combinations of m , n , and p , meaning that these *trivial modes* can never be excited. For the rectangular cavity resonator, the trivial modes are TE_{00p} , TE_{m00} , TM_{m0p} , and TM_{0np} . The nontrivial mode having the lowest useful resonant frequency is thus the TE_{101} mode, the fields for which are shown in Figure 8.2(a)–(b). The distribution of resonant frequencies for a typical

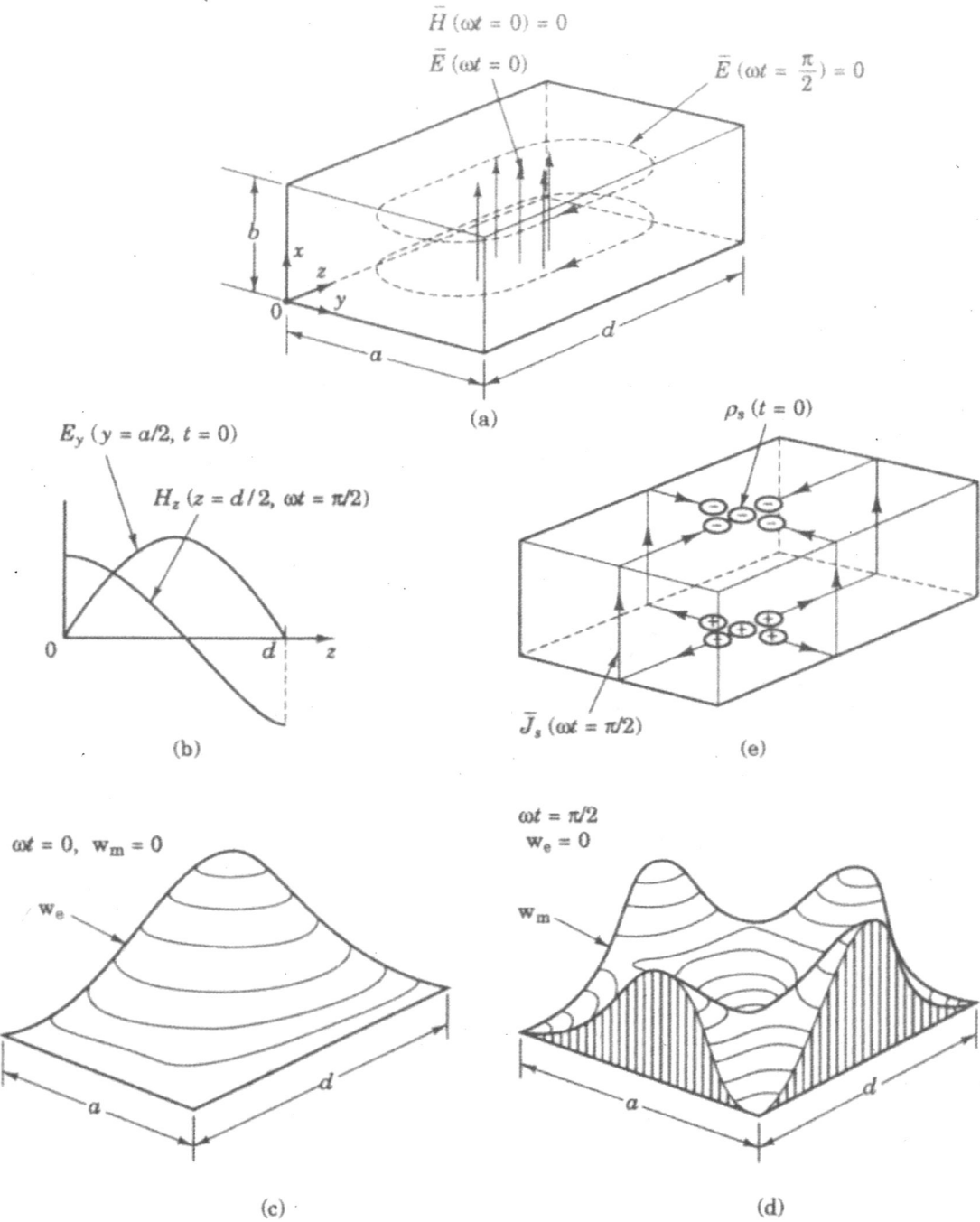


Figure 8.2 Fields, surface currents and charges, and energy densities in a rectangular waveguide resonator.

TABLE 8.1 Resonant Frequencies less than 30 GHz for a Rectangular Cavity 2 cm $\times$ 1 cm $\times$ 3 cm. Superscripts 1–8 represent Degenerate Modes

<i>mnp</i>	<i>TE</i> Modes (GHz)	<i>TM</i> Modes (GHz)
101	9.01	—
102	12.50	—
011	15.81 ¹	—
201	15.81 ¹	—
110	—	16.77 ²
103	16.77 ²	—
111	17.50 ³	17.50 ³
012	18.03 ⁴	—
202	18.03 ⁴	—
112	19.53 ⁵	19.53 ⁵
210	—	21.21 ⁶
013	21.21 ⁶	—
203	21.21 ⁶	—
211	21.79 ⁷	21.79 ⁷
301	23.05	—
212	23.45 ⁸	23.45 ⁸
302	24.62	—
310	—	27.04

cavity resonator 1 $\times$ 2 $\times$ 3 cm is given in Table 8.1. Note that many modes are degenerate (e.g., TE_{011} and TE_{201} have exactly the same frequency, as do TM_{110} and TM_{103} , and so on.)

The expression for the TE electric fields in (8.2.7) is purely real (if E_0 is real), implying that they reach maximum strength at $t = 0$ and vanish one-quarter cycle later. The TE magnetic fields in (8.2.8) are purely imaginary and are therefore zero at $t = 0$; one-quarter of a cycle later they reach maximum amplitude. Thus at $t = 0$, all of the energy in the cavity is stored in the form of electric energy concentrated near the center of the cavity, as suggested by Figure 8.2(c). When the magnetic energy peaks one-quarter cycle later, it is concentrated at the walls, illustrated in Figure 8.2(d). Thus, during every cycle there are two complete interchanges between purely electric and purely magnetic energy storage. When the energy is purely electric, the boundary conditions are satisfied by surface charges distributed appropriately at the top and bottom surfaces of the cavity. When the energy is purely magnetic, the boundary conditions are matched by surface currents flowing in the walls, reversing the surface charge distribution in preparation for the instant one-quarter cycle later when the energy will again be stored electrically but the

fields will point in the opposite direction. These charge and current distributions are pictured in Figure 8.2(e).

The electromagnetic fields and surface currents for four of the rectangular cavity resonator modes are sketched in Figure 8.3. The fields for all other resonator modes are simply combinations of an appropriate number of "unit cells" of one of these modes (or TE_{011} , which is not shown because it is merely a rotated version of TE_{101}). For example, the TE_{232} mode consists of twelve TE_{111} unit cells stacked in a $2 \times 3 \times 2$ array. The appropriate directions of the fields in adjacent cells may be established by noting that the magnetic field lines must circulate without terminating ($\nabla \cdot \vec{B} = 0$), and the electric field lines may terminate only on surface charges since $\nabla \cdot \vec{D} = \rho$ and ρ is zero inside the cavity. The boundary condition $\hat{n} \times \vec{H} = \vec{J}_s$ applies everywhere on the resonator surface, relating the direction of the surface current $\vec{J}_s$ and the adjacent magnetic field. With these observations, the field patterns are easily determined for any rectangular resonator mode.

The electromagnetic field distributions pictured in Figure 8.3 are not all truly distinct: TE_{101} , TE_{011} , and TM_{110} are identical except for a rotation of axes. The TE_{111} and TM_{111} are related by duality. In particular, the TE_{111} fields of (8.2.7) and (8.2.8) are transformed into the TM_{111} fields of (8.2.13) and (8.2.14) and vice versa, simply by making the substitutions $\vec{E} \rightarrow \vec{H}$, $\vec{H} \rightarrow -\vec{E}$, $\mu \leftrightarrow \epsilon$, and $x \rightarrow x - a/2$, $y \rightarrow y - b/2$, $z \rightarrow z - d/2$. It is necessary to shift the coordinate systems because the resonator boundary conditions are not truly dual. The tangential electric fields and normal magnetic fields must be zero at the perfectly conducting surfaces of the resonator regardless of whether the fields are TE or TM . Thus only two truly distinct topologies are represented in Figure 8.3, those corresponding to the $\{101\}$ and $\{111\}$ sets of modes.

The electric and magnetic fields are 90° out of phase not only for the TE modes, as noted earlier, but also for the TM modes: $\vec{S} = \vec{E} \times \vec{H}^*$ is purely imaginary. Therefore, the time-averaged Poynting power $\langle \vec{S} \rangle = \frac{1}{2} \text{Re} \{ \vec{S} \}$ is zero and there is no net power flow in the resonator; the waves are trapped in the cavity and cannot propagate.

If there is no loss in the cavity, the Poynting theorem can be used to prove the obvious fact that the total amount of energy in the cavity does not change with time. We recall the time-dependent Poynting theorem from (1.6.8):

$$\begin{aligned}
 - \int_V \vec{E}(t) \cdot \vec{J}(t) dv &= \frac{d}{dt} \int_V \left(\frac{1}{2} \mu |\vec{H}(t)|^2 + \frac{1}{2} \epsilon |\vec{E}(t)|^2 \right) dv \\
 &+ \oint_A \left(\vec{E}(t) \times \vec{H}(t) \right) \cdot \hat{n} da
 \end{aligned}
 \tag{8.2.15}$$

The electric field and current are never nonzero at the same location within the cavity, since $\vec{E}$ is zero in the perfectly conducting walls where $\vec{J}$ flows, and $\vec{J}$ is zero in the lossless dielectric where $\vec{E}$ is nonzero. Therefore, $\vec{E}(t) \cdot \vec{J}(t)$ is always

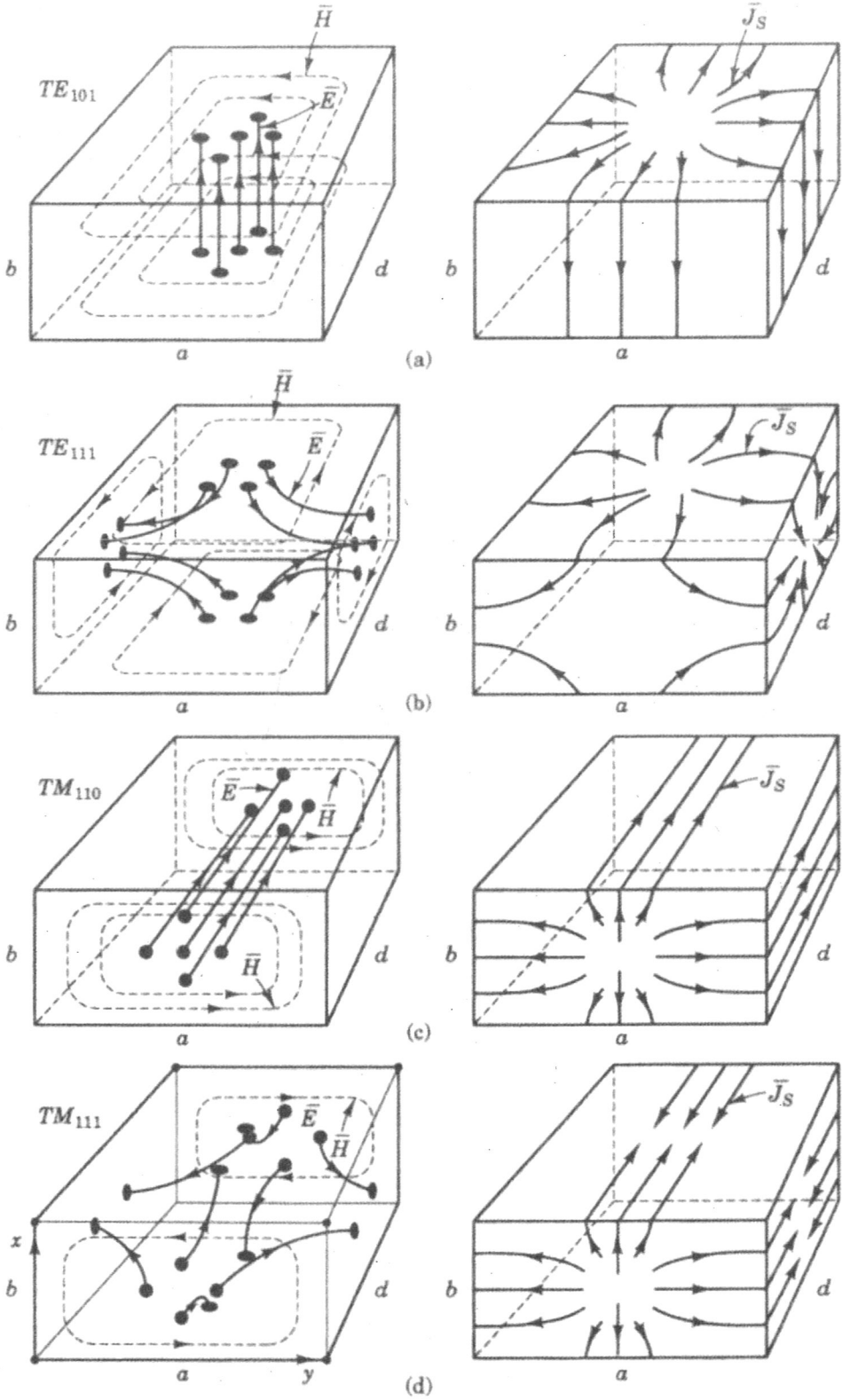


Figure 8.3 Four basic modes of a rectangular resonator.

zero for the lossless resonator cavity. The integral of the Poynting vector $\bar{E}(t) \times \bar{H}(t)$ over the resonator surface is also zero because $\bar{E}$ is everywhere perpendicular to the cavity walls so $\bar{E} \times \bar{H}$ is parallel to them. Therefore, (8.2.15) reduces to

$$0 = \frac{d}{dt} \int_V \left(\frac{1}{2} \mu |\bar{H}(t)|^2 + \frac{1}{2} \epsilon |\bar{E}(t)|^2 \right) dv = \frac{d}{dt} (w_m(t) + w_e(t)) \quad (8.2.16)$$

and we see, as expected, that the total stored energy is constant even though it alternates between storage in electric and magnetic forms at intervals of one-quarter of a cycle, as suggested earlier in Figure 8.2 for the TE_{101} mode.

An alternative way to prove the equality of the time-averaged electric and magnetic stored energies uses the complex Poynting theorem, which we recall from (1.6.30) is:

$$-\frac{1}{2} \int_V (\bar{E} \cdot \bar{J}^*) dv = 2j\omega [\langle w_m \rangle - \langle w_e \rangle] + \frac{1}{2} \oint_A (\bar{E} \times \bar{H}^*) \cdot \hat{n} da \quad (8.2.17)$$

The desired result follows directly because $\bar{E} \cdot \bar{J}^*$ and the surface integral of $\bar{E} \times \bar{H}^*$ are both zero for the same reasons noted earlier, leaving

$$\langle w_e \rangle = \langle w_m \rangle \quad (8.2.18)$$

Finally, we can prove the equality in (8.2.18) by direct evaluation of these time-averaged stored energies, using

$$\langle w_e \rangle = \frac{1}{4} \int_V \epsilon |\bar{E}|^2 dv \quad (8.2.19)$$

$$\langle w_m \rangle = \frac{1}{4} \int_V \mu |\bar{H}|^2 dv \quad (8.2.20)$$

For a general TE_{mnp} mode where m , n , and p are all nonzero integers,

$$\begin{aligned} \langle w_e \rangle &= \frac{1}{4} \int_V \epsilon (|E_x|^2 + |E_y|^2 + |E_z|^2) dv \\ &= \frac{\epsilon}{4} \int_0^d \int_0^b \int_0^a \left(\frac{k_y^2}{k_0^2} |E_0|^2 \cos^2 k_x x \sin^2 k_y y \sin^2 k_z z \right. \\ &\quad \left. + \frac{k_x^2}{k_0^2} |E_0|^2 \sin^2 k_x x \cos^2 k_y y \sin^2 k_z z \right) dx dy dz \end{aligned} \quad (8.2.21)$$

Because each squared sinusoid in (8.2.21) is integrated over an integral number of half periods and has average value equal to 0.5, the integral along any resonator

dimension is just one-half its length.² Therefore, (8.2.21) becomes

$$\langle w_e \rangle = \frac{\epsilon}{4} \frac{(k_x^2 + k_y^2)}{k_0^2} |E_0|^2 \frac{abd}{8} \quad (8.2.22)$$

for the TE_{mnp} modes ($m, n, p > 0$). The time-averaged stored magnetic energy for the same mode is similarly computed to be

$$\begin{aligned} \langle w_m \rangle &= \frac{1}{4} \int_V \mu (|H_x|^2 + |H_y|^2 + |H_z|^2) dv \\ &= \frac{\mu}{4} \left| \frac{E_0}{\omega \mu k_0} \right|^2 (k_x^2 k_z^2 + k_y^2 k_z^2 + (k_x^2 + k_y^2)^2) \frac{abd}{8} \\ &= \frac{\mu}{4} \left| \frac{E_0}{\omega \mu k_0} \right|^2 (k_x^2 + k_y^2) (k_x^2 + k_y^2 + k_z^2) \frac{abd}{8} \\ &= \frac{\epsilon}{4} |E_0|^2 \frac{(k_x^2 + k_y^2)}{k_0^2} \frac{abd}{8} \end{aligned} \quad (8.2.23)$$

where the dispersion relation $\omega^2 \mu \epsilon = k_x^2 + k_y^2 + k_z^2$ has been used to simplify (8.2.23). As expected, the time-averaged electric and magnetic energies are equal for such TE modes, as may be observed by comparing (8.2.22) and (8.2.23). Expressions for the time-averaged electric and magnetic energy stored in the TE_{m0p} and TE_{0np} modes are found in a similar fashion

$$\begin{aligned} \langle w_e \rangle = \langle w_m \rangle &= \frac{\epsilon}{4} |E_0|^2 \frac{k_x^2}{k_0^2} \frac{abd}{4} \quad (TE_{m0p} \text{ mode}) \\ \langle w_e \rangle = \langle w_m \rangle &= \frac{\epsilon}{4} |E_0|^2 \frac{k_y^2}{k_0^2} \frac{abd}{4} \quad (TE_{0np} \text{ mode}) \end{aligned} \quad (8.2.24)$$

as are the stored energies of the TM modes.

The stored energy distribution for the TE_{101} mode can also be related to a simple discrete element LC resonator, from which it evolves morphologically as suggested in Figure 8.4. The magnetic energy associated with the inductor L is equivalent to the excess magnetic energy near the current-carrying walls, while the electric energy associated with the capacitor C is equivalent to the excess electric energy in the high electric field zone in the center of the resonator (see

2. For example, the identity $\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$ may be used to show that

$$\int_0^a \cos^2 \left(\frac{m\pi x}{a} \right) dx = \int_0^a \left[\frac{1}{2} + \frac{1}{2} \cos \left(\frac{2m\pi x}{a} \right) \right] dx = \begin{cases} a/2, & m \neq 0 \\ a, & m = 0 \end{cases}$$

and

$$\int_0^a \sin^2 \left(\frac{m\pi x}{a} \right) dx = \int_0^a \left[1 - \cos^2 \left(\frac{m\pi x}{a} \right) \right] dx = \begin{cases} a/2, & m \neq 0 \\ 0, & m = 0 \end{cases}$$

with similar results for the y and z integrations.

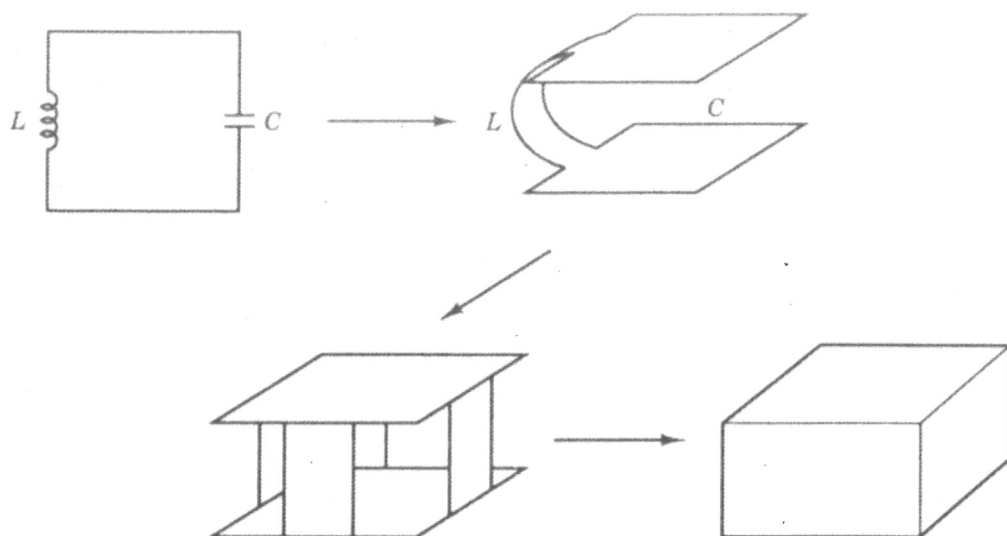


Figure 8.4 Evolution of a cavity resonator from an LC resonator.

also Figure 8.2). By spreading the current over a large wall area, the resistive losses associated with typical inductor coils are greatly reduced, and by closing the energy cavity, radiation losses are prevented. Thus cavity resonators are desirable in very high frequency applications (above $\simeq 1$ GHz), where skin-depth effects can introduce substantial series resistance in typical inductors. Unlike a discrete LC resonator with a fixed resonance at $\omega = 1/\sqrt{LC}$, a cavity has a large number of allowed resonant frequencies given by (8.2.11).

Sometimes it is important to know how densely the resonant modes of a cavity resonator are spaced, or how many modes exist within some frequency band or below some maximum frequency. From (8.2.11) we find that the resonant frequency f_{mnp} is

$$\begin{aligned} f_{mnp} = \frac{\omega_{mnp}}{2\pi} &= \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{nc}{2b}\right)^2 + \left(\frac{pc}{2d}\right)^2} \\ &= \sqrt{f_x^2 + f_y^2 + f_z^2} \end{aligned} \quad (8.2.25)$$

for cavities enclosing vacuum, where $f_x = mc/2a$, $f_y = nc/2b$, and $f_z = pc/2d$. Equation (8.2.25) suggests a simple geometric relationship between the resonator frequency and the mode numbers m , n , and p , as illustrated in Figure 8.5. The resonator frequency is simply the square root of the sum of the squares of its three components, each associated with its own modal number. For each triplet of mode numbers $\{mnp\}$, there exists both a TE and a TM resonator mode with identical frequencies given by (8.2.25). To determine how many resonant frequencies exist below some maximum frequency f_0 , we need only count the number of grid points within a spherical segment of radius f_0 . Only one-eighth of the sphere is

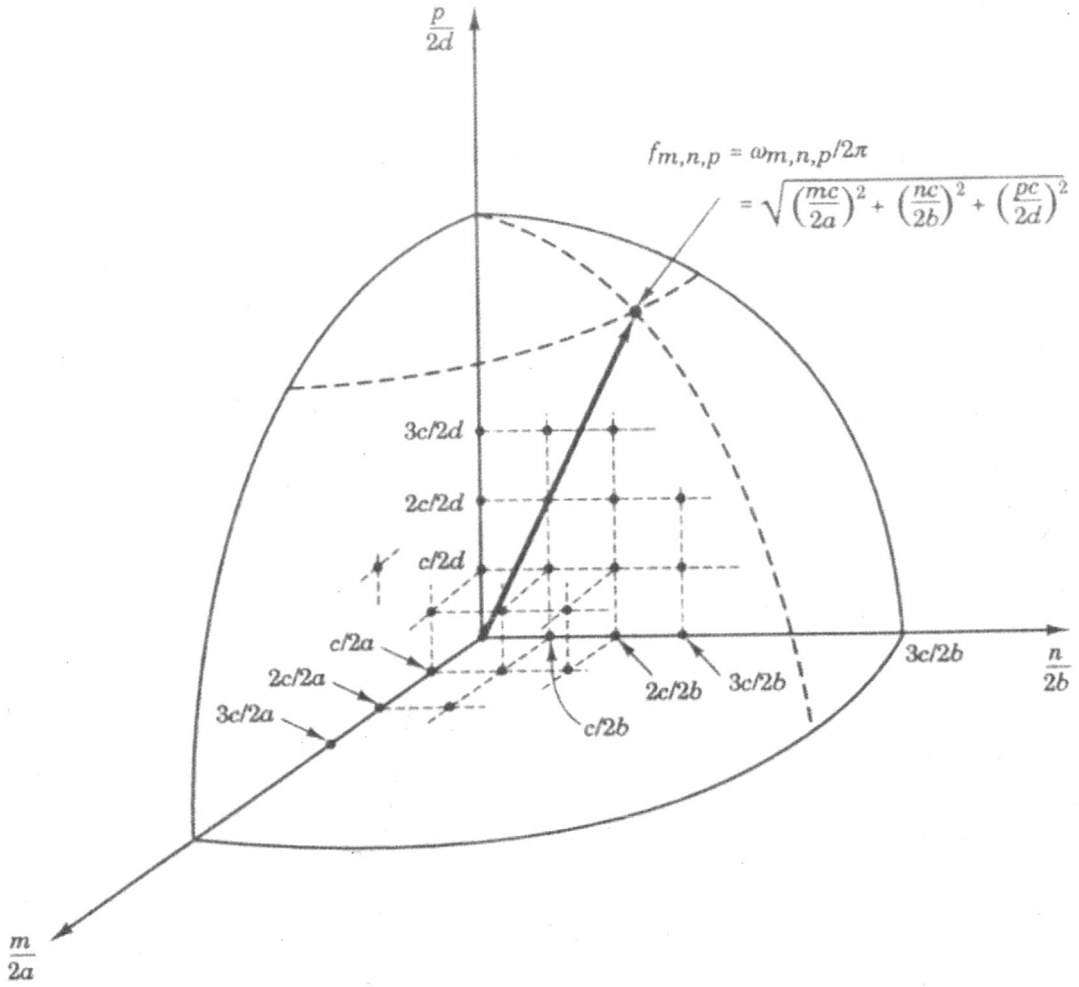


Figure 8.5 Frequency dependence of mode density for a rectangular resonator.

relevant since f_x , f_y , and f_z must all be positive. If the volume of the unit cell $\Delta f_x \Delta f_y \Delta f_z = c^3/8abd$ associated with each grid point is much smaller than the volume of the eighth sphere $(1/8)(4\pi f_0^3/3)$, then the number of resonator modes with frequencies less than f_0 is approximately

$$n_0 \simeq 2 \times \frac{\pi f_0^3/6}{c^3/8abd} = \frac{8\pi V}{3\lambda_0^3} \quad (8.2.26)$$

where the factor of two accounts for the two modes (*TE* and *TM*) that exist for each grid point. The volume of the resonator is $V = abd$ and $\lambda_0 = c/f_0$. Equation (8.2.26) is a good approximation when λ_0^3 is much less than V . The number of resonances δn between f_0 and $f_0 + \delta f$ can be found by differentiating (8.2.26), yielding

$$\delta n \simeq \left(\frac{8\pi V}{c\lambda_0^2} \right) \delta f \quad (8.2.27)$$

Thus, the number of resonances within a bandwidth δf about f_0 is proportional to $1/\lambda_0^2$ and to f_0^2 .

8.3 LOSS IN RESONATORS

As the energy in a resonator oscillates back and forth between its electric and magnetic forms, some of this energy typically heats the resonator or escapes to the outside environment. If the total electromagnetic energy of the n^{th} mode w_{Tn} decays over time, then the resonator is said to be lossy, and w_{Tn} has the approximate form³

$$w_{Tn}(t) = w_{Tn}(t = 0) e^{-\omega_n t / Q_n} \quad (8.3.1)$$

where ω_n is the resonant frequency of the n^{th} mode, and the *quality factor* Q_n is a dimensionless constant measuring how well the resonator stores energy. This quality factor is defined by (8.3.1) as the average number of radians over which the stored energy w_{Tn} decays by a factor of $1/e$. This relationship is also suggested in Figure 8.6. Note that we are considering only the total energy loss with **time** here, not attenuation in space, irrelevant in resonators in which no time-averaged power flows.

The quality factor Q_n typically differs for each mode, with larger Q -values associated with higher-quality resonances and smaller losses; Q becomes infinite in the absence of loss. (Ideal, lossless, rectangular resonators were described in Section 8.2, where it was shown that the total stored energy did not change in time.)

The quality factor Q_n may also be obtained by computing P_n , the power lost by the n^{th} mode, averaged over one cycle. From (8.3.1), we see that

$$P_n \simeq -\frac{dw_{Tn}}{dt} = \frac{\omega_n w_{Tn}}{Q_n} \quad (8.3.2)$$

Rearranging (8.3.2) gives

$$Q_n = \frac{w_{Tn}}{P_n / \omega_n} = \frac{\text{total energy of the } n^{\text{th}} \text{ mode}}{\text{average energy lost per radian of the } n^{\text{th}} \text{ mode}} \quad (8.3.3)$$

Although the average dissipated power decreases exponentially with time, $P_n(t)$ may have a sinusoidal component peaking twice per cycle. If the loss is dominated by current flowing in the resonator walls, the dissipated power will peak when the magnetic energy is maximum and will be zero when the wall currents become zero. Alternatively, if the loss is associated exclusively with dielectric conduction within the cavity, it will peak twice per cycle when the electric energy storage is a maximum, exactly out of phase with any loss associated with wall

3. In this and subsequent sections, the single index n denotes a specific resonator mode. In the case of a rectangular cavity resonator, the $n = 1$ mode might actually represent the TE_{101} mode while the TE_{102} mode might be designated by $n = 2$.

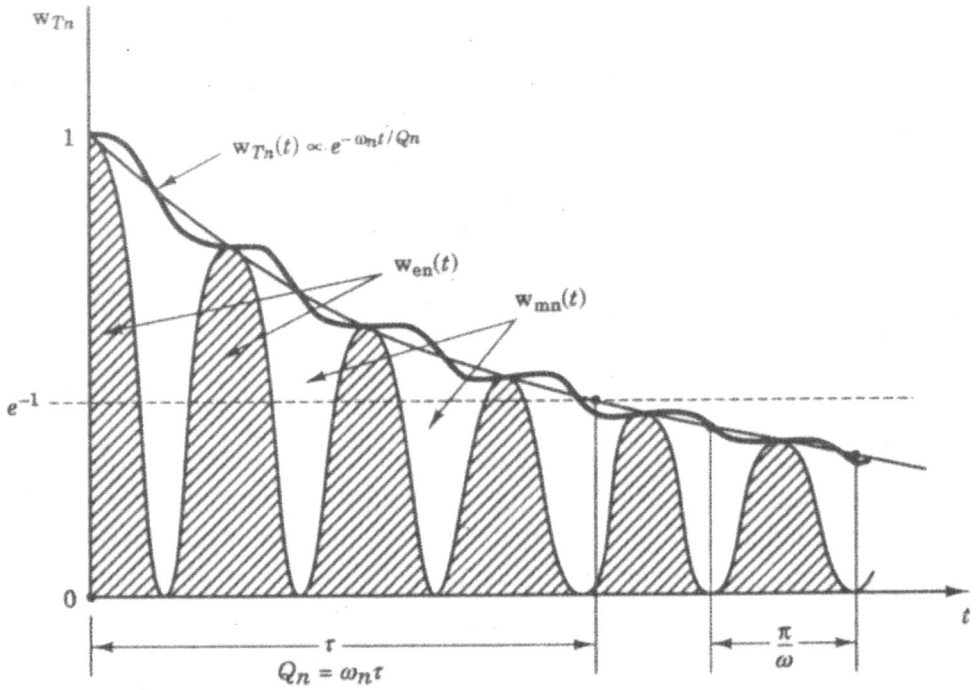


Figure 8.6 Energy of a lossy resonator as a function of time.

currents. This power ripple effect is illustrated in Figure 8.6 for the case where wall losses dominate.⁴

Up until now, we have tacitly assumed that ω_n is a real frequency. In the presence of loss, however, we will have to represent resonances by complex frequencies, commonly denoted by s_n , where

$$s_n = j\omega_n - \alpha_n \quad (8.3.4)$$

and both ω_n and α_n are now explicitly real quantities. In the absence of loss, $\alpha_n = 0$ so that $e^{s_n t} = e^{j\omega_n t}$ as usual. The time dependence of the electric and

4. The power ripple effect is caused by a slight additional phase shift between electric and magnetic energy storage. To first order, we can represent the energies as

$$w_e(t) \simeq w_{Tn}(t=0) e^{-\alpha_n t / Q_n} \cos^2(\omega_n t + \phi_n)$$

$$w_m(t) \simeq w_{Tn}(t=0) e^{-\alpha_n t / Q_n} \sin^2(\omega_n t + \phi_n + \delta_n)$$

where $\delta_n \ll 1$. The total energy of the lossy resonator is thus

$$w_{Tn}(t) \simeq w_{Tn}(t=0) e^{-\alpha_n t / Q_n} [1 + \delta_n \sin 2(\omega_n t + \phi_n)]$$

where terms of order δ_n^2 have been neglected. The ripples can now be seen to vary about the exponentially decaying envelope at twice the resonant frequency of the mode.

magnetic field is then

$$\begin{aligned}\bar{E}_n(\bar{r}, t) &= \text{Re} \left\{ \bar{E}_n(\bar{r}) e^{s_n t} \right\} = e^{-\alpha_n t} \text{Re} \left\{ \bar{E}_n(\bar{r}) e^{j\omega_n t} \right\} \\ \bar{H}_n(\bar{r}, t) &= \text{Re} \left\{ \bar{H}_n(\bar{r}) e^{s_n t} \right\} = e^{-\alpha_n t} \text{Re} \left\{ \bar{H}_n(\bar{r}) e^{j\omega_n t} \right\}\end{aligned}\quad (8.3.5)$$

Because the average electric and magnetic field energies are proportional to the field intensities $|\bar{E}_n(\bar{r}, t)|^2$ and $|\bar{H}_n(\bar{r}, t)|^2$, respectively, we see from (8.3.5) that both are proportional to $e^{-2\alpha_n t}$ and that the total resonator energy is just

$$w_{Tn}(t) = w_{Tn}(t=0) e^{-2\alpha_n t} \quad (8.3.6)$$

By comparing (8.3.1) and (8.3.6), we observe an important relationship between α_n and Q_n :

$$Q_n = \frac{\omega_n}{2\alpha_n} \quad (8.3.7)$$

When more than one mode of a resonator is excited, the average decay rate of each mode is usually independent of the amplitudes of the other modes. This independence requires that the interacting modes be orthogonal in space or time; usually the frequencies are simply different. In unusual cases, two degenerate modes with the same frequency can superimpose coherently at a resistive element and thus interact. If two resonant frequencies are close but not exactly the same, they may interact at the beat frequency (the difference between the two resonances). The likelihood that two modes might interact is increased as the loss is concentrated at one point in the resonator instead of being distributed throughout.

To determine the Q of a particular resonance, we need simply to evaluate (8.3.3). Unfortunately, exact computation is difficult because power dissipation depends explicitly on the electric and magnetic fields, but the field distributions also depend on how much power is dissipated. Although exact solutions can be found in very special cases, a more general approach to determining electromagnetic losses involves using the perturbation methods discussed in Section 8.7. We have already used perturbation techniques informally in Sections 4.2, 4.4 and 7.4 where we calculated power dissipation at a lossy planar interface and in lossy waveguides, and we shall continue with this informal approach for the remainder of this section.

The essence of all *perturbation methods* is to find a quantity that differs from its lossless value by only a small percentage, and then to use the lossless (unperturbed) value of that quantity in all subsequent calculations, with often quite acceptable accuracy. Because the **percentage** change of the quantity must be small, its unperturbed value cannot be zero.

To illustrate these perturbation methods, we first calculate the quality factor for a rectangular resonator having lossless (perfectly conducting) walls but filled with a slightly lossy dielectric with conductivity σ_i , where the loss tangent $\sigma_i/\omega\epsilon$ is much less than unity. The exact fields in this resonator will be represented by the phasors $\bar{E}_n$, $\bar{H}_n$ and so on, while the fields that would arise in an identically

shaped, perfectly insulating resonator are $\bar{E}_n^{(0)}$, $\bar{H}_n^{(0)}$, given by (8.2.7) and (8.2.8) for the TE_{mnp} modes and (8.2.13) and (8.2.14) for the TM_{mnp} modes. If σ_i is sufficiently small,⁵ we expect that the actual and lossless fields can be used somewhat interchangeably. The total energy stored in the n^{th} mode is then

$$w_{Tn} = 2\langle w_e \rangle = \frac{1}{2} \int_V \epsilon |\bar{E}_n|^2 dv \simeq \frac{1}{2} \int_V \epsilon |\bar{E}_n^{(0)}|^2 dv \quad (8.3.8)$$

where V is the resonator volume and $\bar{E}_n$ has been replaced by $\bar{E}_n^{(0)}$. The average power dissipated in one cycle in the n^{th} mode is

$$P_n = \frac{1}{2} \int_V \bar{E}_n \cdot \bar{J}_n^* dv \quad (8.3.9)$$

where we may choose to express $\bar{J}_n = \sigma_i \bar{E}_n \simeq \sigma_i \bar{E}_n^{(0)}$ or $\bar{E}_n = \bar{J}_n / \sigma_i \simeq \bar{J}_n^{(0)} / \sigma_i$. Because the current inside the cavity is zero if the cavity is perfectly insulating ($\sigma_i = 0$), $\bar{J}_n^{(0)}$ is zero and is thus an unacceptable quantity on which to use perturbation techniques; the change from $\bar{J}_n^{(0)} = 0$ to $\bar{J}_n \neq 0$ is an infinite percentage change! In contrast, $\bar{E}_n^{(0)}$ and $\bar{E}_n$ differ only by a small percentage. We therefore represent P_n by

$$P_n \simeq \frac{1}{2} \int_V \sigma_i |\bar{E}_n^{(0)}|^2 dv \quad (8.3.10)$$

which necessarily vanishes as $\sigma_i \rightarrow 0$; no power is dissipated in a lossless resonator. Equation (8.3.10) is combined with (8.3.8) and (8.3.3) to find Q_n :

$$Q_n \simeq \frac{\omega_n w_{Tn}}{P_n} \simeq \frac{\omega_n \frac{1}{2} \int_V \epsilon |\bar{E}_n^{(0)}|^2 dv}{\frac{1}{2} \int_V \sigma_i |\bar{E}_n^{(0)}|^2 dv} = \frac{\omega_n \epsilon}{\sigma_i} \quad (8.3.11)$$

This remarkably simple result does not require any knowledge of the resonator fields (except insofar as ω_n must be calculated) because both energy storage and dissipation are proportional to $|\bar{E}_n|^2$. Therefore, the time it takes for the energy of the n^{th} mode to decay by $1/e$ is the same for all modes, since $\tau_n = Q_n / \omega_n = \epsilon / \sigma_i$ is just the charge relaxation time of the lossy dielectric! Note that Q_n becomes infinite as σ_i vanishes, and that our earlier assumption that $\sigma_i / \omega \epsilon \ll 1$ automatically guarantees that Q_n will be a large number, as it must be if perturbation techniques are to succeed. Often, we calculate Q assuming perturbation methods to be valid. If Q is sufficiently large at the end of the calculation, we can be fairly sure that our original assumption was justified.

5. Note that if σ_i is large instead, the current $\bar{J}_n \simeq \sigma_i \bar{E}_n^{(0)}$ will be nonnegligible. This current will give rise to a magnetic field correction by Ampere's law, which will in turn lead to an electric field correction by Faraday's law and to an additional current by Ohm's law. This secondary current will in turn produce new $\bar{E}$ and $\bar{H}$ fields, and this process will continue indefinitely. Only when loss is small can these additional higher-order self-consistent field correction terms be neglected.

A more usual loss mechanism in a cavity resonator is power dissipation in the resonator walls, caused by an imperfectly conducting cavity closure. In order for perturbation techniques to be applicable to this case, we consider resonator walls, each with large but finite conductivity σ_w , where $\sigma_w/\omega\epsilon \gg 1$. Once again, the total unperturbed energy in the resonator is given by (8.3.8) and the average power dissipated in the n^{th} mode is given by (8.3.9), where this time $\frac{1}{2} \int_V \bar{E}_n \cdot \bar{J}_n^* dv$ must be computed within the resonator walls.

We again must decide whether to approximate $\bar{E}_n \cdot \bar{J}_n^*$ as $\sigma_w |\bar{E}_n^{(0)}|^2$ or $|\bar{J}_n^{(0)}|^2/\sigma_w$. Because the electric field inside perfectly conducting resonator walls is zero, the electric field is not a "slightly" perturbed quantity even if $\bar{E}_n$ is small. On the other hand, $\bar{J}_n^{(0)}$ is not zero in the walls of a perfect conductor because the discontinuity in the tangential magnetic field at the cavity surface gives rise to a surface current, and $\bar{J}_n$ differs only slightly from $\bar{J}_n^{(0)}$ if the wall conductivity is large. We thus approximate the dissipated power as

$$P_n \simeq \frac{1}{2} \int_{V'} \frac{|\bar{J}_n^{(0)}|^2}{\sigma_w} dv \quad (8.3.12)$$

which also has the required property that $P_n \rightarrow 0$ as $\sigma_w \rightarrow \infty$. The volume V' is the volume of the conducting walls of the resonator in which the surface current flows, **not** the entire resonator volume V .

Because the resonator surface is an excellent conductor, most of the current flows within a skin depth $\delta = \sqrt{2/\omega_n \mu \sigma_w}$ of the surface. We can therefore make the usual approximation

$$\bar{J}_n^{(0)} \simeq \bar{J}_{sn}^{(0)}/\delta \quad (8.3.13)$$

which was discussed in Sections 4.2 and 4.4. For computational purposes, all of the current is assumed to flow uniformly within the thickness δ . The mathematical justification for this approximation formed the basis of Example 4.2.2.

Combining (8.3.12) and (8.3.13) yields

$$\begin{aligned} P_n &\simeq \frac{1}{2\sigma_w} \int_0^\delta \int_{A'} \left| \frac{\bar{J}_{sn}^{(0)}}{\delta} \right|^2 dv \\ &= \frac{1}{2\sigma_w \delta} \int_{A'} |\bar{J}_{sn}^{(0)}|^2 da \end{aligned} \quad (8.3.14)$$

where the volume integral V' is separated into an area integral over the resonator surface A' and an integral over the skin depth δ . We can rewrite $\bar{J}_{sn}^{(0)}$ in terms of the unperturbed magnetic field $\bar{H}_n^{(0)}$ by using Ampere's boundary condition $\bar{J}_{sn}^{(0)} = \hat{n} \times \bar{H}_n^{(0)}$. Equation (8.3.14) finally becomes

$$P_n \simeq \frac{1}{2\sigma_w \delta} \int_{A'} |\hat{n} \times \bar{H}_n^{(0)}(\text{cavity surface})|^2 da \quad (8.3.15)$$

where $\hat{n}$ is the normal to the cavity surface A' pointing into the cavity.

We shall use (8.3.15) to find Q_n for a rectangular resonator with lossy walls ($\sigma_w \neq \infty$) and a lossless dielectric ($\sigma_i = 0$). For simplicity, only the TE_{101} mode will be considered. We start by finding $P_{TE_{101}}$ in each of the six cavity walls; by symmetry the power dissipated in each pair of opposite walls is the same. Therefore,

$$\begin{aligned} P_{TE_{101}} \simeq & 2 \times \left[\frac{1}{2\sigma_w \delta} \int_0^d \int_0^b \left| \hat{x} \times \overline{H}_{TE_{101}}^{(0)}(x=0) \right|^2 dy dz \right. \\ & + \frac{1}{2\sigma_w \delta} \int_0^d \int_0^a \left| \hat{y} \times \overline{H}_{TE_{101}}^{(0)}(y=0) \right|^2 dx dz \\ & \left. + \frac{1}{2\sigma_w \delta} \int_0^b \int_0^a \left| \hat{z} \times \overline{H}_{TE_{101}}^{(0)}(z=0) \right|^2 dx dy \right] \end{aligned} \quad (8.3.16)$$

where

$$\overline{H}_{TE_{101}} = \hat{x} \frac{j\pi^2 E_0}{ad\eta k_0^2} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi z}{d}\right) + \hat{z} \frac{j\pi^2 E_0}{a^2\eta k_0^2} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi z}{d}\right) \quad (8.3.17)$$

from (8.2.8) and (8.2.9). Combining (8.3.16) and (8.3.17) yields

$$\begin{aligned} P_{TE_{101}} \simeq & \frac{1}{\sigma_w \delta} \left[\int_0^d \int_0^b \left| \frac{\pi^2 E_0}{a^2\eta k_0^2} \right|^2 \sin^2\left(\frac{\pi z}{d}\right) dy dz \right. \\ & + \int_0^d \int_0^a \left| \frac{\pi^2 E_0}{ad\eta k_0^2} \right|^2 \sin^2\left(\frac{\pi x}{a}\right) \cos^2\left(\frac{\pi z}{d}\right) dx dz \\ & + \int_0^d \int_0^a \left| \frac{\pi^2 E_0}{a^2\eta k_0^2} \right|^2 \cos^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi z}{d}\right) dx dz \\ & \left. + \int_0^b \int_0^a \left| \frac{\pi^2 E_0}{ad\eta k_0^2} \right|^2 \sin^2\left(\frac{\pi x}{a}\right) dx dy \right] \\ = & \frac{1}{\sigma_w \delta} \left| \frac{\pi^2 E_0}{2a^2 d \eta k_0^2} \right|^2 [ad(a^2 + d^2) + 2b(a^3 + d^3)] \end{aligned} \quad (8.3.18)$$

From (8.2.24), we find that the total stored energy of the TE_{101} mode is

$$W_{TE_{101}} \simeq \frac{\epsilon \pi^2 |E_0|^2 abd}{8k_0^2 a^2} \quad (8.3.19)$$

which means that (8.3.18) and (8.3.19) may be substituted into (8.3.3) to yield

$$Q_{TE_{101}} \simeq \frac{\omega_{TE_{101}} \langle W_{TE_{101}} \rangle}{P_{TE_{101}}} \simeq \frac{abd}{\delta} \left[\frac{a^2 + d^2}{ad(a^2 + d^2) + 2b(a^3 + d^3)} \right] \quad (8.3.20)$$

where we have simplified (8.3.20) by using the dispersion relation, resonator $(\pi/a)^2 + (\pi/d)^2 = \omega_{TE_{101}}^2 \mu \epsilon$ for the TE_{101} mode.

Note that Q depends upon cavity geometry and is inversely proportional to the skin depth $\delta = \sqrt{2/\omega_n \mu \sigma_w}$. Therefore, $Q \propto \sqrt{\omega_n \sigma_w}$ becomes infinite as the walls of the resonator become lossless or as the frequency of the n^{th} mode becomes infinite.

If the resonator has **both** lossy walls and a lossy dielectric, we can easily recompute the new Q for this system if we know the Q -factors for each type of loss separately. The resonant frequency ω_n and the total energy stored in the resonator w_{Tn} are the same for any type of small loss, and the total power dissipated is just the sum of the powers dissipated in the dielectric and walls. Therefore, if $Q_{ni} = \omega_n w_{Tn} / P_{ni}$ is the quality factor for a lossy dielectric resonator, and $Q_{nw} = \omega_n w_{Tn} / P_{nw}$ is the quality factor for a lossy wall resonator, then the total Q_n for a resonator with both lossy walls and dielectric is

$$Q_n \simeq \frac{\omega_n w_{Tn}}{P_{ni} + P_{nw}} = \left(\frac{1}{Q_{ni}} + \frac{1}{Q_{nw}} \right)^{-1} \quad (8.3.21)$$

Note that the total Q_n is less than either Q_{ni} or Q_{nw} because two types of resonator loss mechanisms are present. In practice, losses in most cavity resonators result primarily from imperfectly conducting cavity walls.

Example 8.3.1

A certain microwave oven manufacturer wants to sell the smallest oven possible that will be able to cook a 10-cm hamburger oriented in any direction. The operating frequency of the oven is 1200 MHz.⁶ What oven cavity dimensions should he use such that at least one resonance falls at or below 1200 MHz?

Solution: The resonant frequencies of the rectangular oven are

$$f_{mnp} = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

where $d \geq a \geq b$. It is clear that the smaller the values of a , b , and d we choose, the larger the minimum resonant frequency will be, so

$$f_{\min} = f_{TE_{101}} = \frac{c}{2} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2} = 1200 \text{ MHz} \quad (1)$$

We also have the constraint that the area $A = ad$ is to be as small as possible, so we can rewrite (1) as a function of the variables A and a and solve for A . Minimizing A by setting $\partial A / \partial a = 0$ yields the intuitive result that the area (and hence the volume) of the resonator is smallest when $a = d$. Substituting $a = d$ into (1) and solving for a yields

$$a = d = 17.7 \text{ cm}$$

and therefore $b = 10.0 \text{ cm}$ to comply with the requirement that the minimum dimension be

6. Actual microwave ovens operate at the government-allowed 2450 MHz, but this is a nonstandard model.

at least 10 cm. The total volume of the oven is then

$$V = (17.7 \text{ cm})^2 \times 10.0 \text{ cm} = 3125 \text{ cm}^3$$

Note that only the TE_{101} mode can resonate in this oven; all of the higher-order modes are cut off. This is actually not a very good design for a microwave oven, because the fields are strong only at the center of the oven. If the oven can support a large number of modes having a variety of field distributions and the source frequency and/or oven geometry varies (e.g., from moving fan blades or a rotating carousel), then food will tend to be more evenly cooked.

Example 8.3.2

The microwave oven described in Example 8.3.1 has dimensions $17.7 \times 17.7 \times 10.0$ cm and is resonant in the TE_{101} mode at 1200 MHz. A hamburger of dimensions $8.0 \times 8.0 \times 1.0$ cm having complex permittivity $\epsilon = \epsilon_0(4 - j0.1)$ is placed in the center of the oven. The wall conductivity is 4×10^6 S/m. What is the resonator Q resulting from the hamburger alone? If the oven is empty and the 400 W transmitter power is dissipated in the oven walls, what is the peak electric field strength in the cavity? Will the oven arc?

Solution: In order to determine whether perturbation techniques may be used for this problem, we first compute the hamburger loss tangent $\sigma_b/\omega\epsilon$. Because $\epsilon - j\sigma_b/\omega = 4\epsilon_0 - j0.1\epsilon_0$, $\epsilon = 4\epsilon_0$ and $\sigma_b/\omega = 0.1\epsilon_0$ so $\sigma_b/\omega\epsilon = 0.025$, sufficiently less than unity so that the burger just slightly perturbs the cavity fields that exist when it is not present. Therefore, we can assume that the electric field in the hamburger is approximately uniform and nearly equal to the unperturbed electric field at the center of the oven. From (8.2.7), we find that this electric field in the burger is

$$\bar{E}_b^{(0)} \simeq \hat{y} \frac{\pi E_0}{k_0 a} \quad (\text{V/m}) \quad (1)$$

where (8.2.7) has been evaluated at $x = a/2$, $y = b/2$, and $z = d/2$. The power dissipated in the burger is then given by (8.3.10)

$$P_b \simeq \frac{1}{2} \int_V \sigma_b |\bar{E}_b^{(0)}|^2 dv \simeq \frac{\sigma_b}{2} \left(\frac{\pi}{k_0 a} \right)^2 |E_0|^2 V_b \quad (\text{W}) \quad (2)$$

where $V_b = 64 \text{ cm}^3$ is the volume of the meat. The total stored electromagnetic energy in the oven is given by (8.2.24) for the TE_{101} mode:

$$w_T \simeq 2 \langle w_e \rangle = \frac{\epsilon}{8} \left(\frac{\pi}{k_0 a} \right)^2 |E_0|^2 abd \quad (\text{J}) \quad (3)$$

Combining (2) and (3) yields a value for Q :

$$Q_b \simeq \frac{\omega w_T}{P_b} = \frac{\omega \epsilon}{4 \sigma_b} \left(\frac{abd}{V_b} \right) = \frac{1}{4 \times 0.025} \left(\frac{3125}{64} \right) \simeq 488$$

If the microwave power is instead completely dissipated in the oven walls, we can use (8.3.18) directly to solve for the field at the center of the oven, $|\bar{E}_b| \simeq \pi |E_0|/k_0 a$ from (1). Because only the TE_{101} mode is excited, the electric field at the center of the oven is also

the maximum field. We rewrite (8.3.18) as

$$P = \frac{1}{\sigma_w \delta} \left| \frac{\pi^2 E_b}{\eta a k_0^2} \right|^2 \left(1 + \frac{2b}{a} \right) \quad (\text{W})$$

where we have substituted $d = a$, and thus

$$|\overline{E}_b| = \left(\frac{\omega \mu a}{\pi} \right)^2 \sqrt{\frac{2 P \sigma_w \delta}{1 + 2b/a}} \quad (\text{V/m}) \quad (4)$$

Since $\omega = 2\pi \times 1200 \times 10^6 = 7.54 \times 10^9$ rad/s and $\delta = \sqrt{2/\omega \mu \sigma_w} = 7.26 \times 10^{-6}$ m, we can substitute these parameters into (4) to yield

$$|E_b| = 2.98 \times 10^5 \text{ V/m}$$

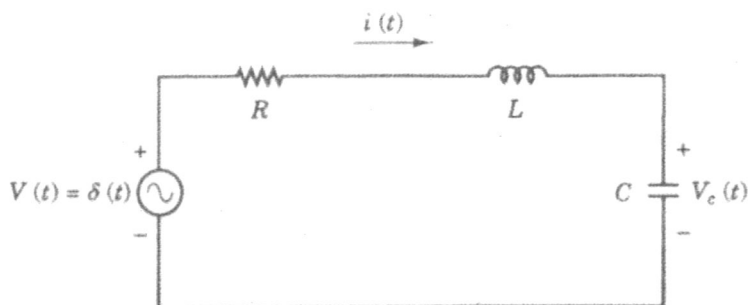
which is an insufficiently large field to cause arcing except near sharp metallic objects (which should never be placed in a microwave oven). Arcing occurs at field strengths of approximately 3×10^6 V/m, depending on humidity and other factors. If the walls were perfectly conducting and 400 W could be continuously supplied, the electric field intensity at the center of the empty oven would in time become infinitely large. In reality, the oven would arc first, dissipating power by other means.

8.4 DISCRETE-ELEMENT RESONATOR CIRCUITS

Cavity resonators exhibit an infinite number of resonances, making it difficult to model their global behavior in a simple way. The same problem exists for **all** distributed electromagnetic structures where the electric and magnetic energy is not stored exclusively in discrete elements such as inductors or capacitors, nor is power necessarily dissipated in discrete resistive elements. Fortunately, we are generally interested in the frequency response of such devices near specific, isolated resonances, for which equivalent discrete-element circuits can be found. These equivalent circuit models are attractive for several reasons—they not only enable us to view resonators with all the intuition developed from circuit theory, but they are also often easier to use when discussing the coupling of resonators to the external world. However, we should be aware that the equivalence between discrete *RLC* circuit resonators and cavity resonators is approximate and applies **only near the specific resonant frequency of the discrete-element resonator**. Each of the infinite number of cavity resonances must therefore be modeled by a separate equivalent resonator circuit. Because modeling resonators near resonance as simple *RLC* circuits is such a powerful technique, we devote this section to reviewing the behavior of these discrete circuits.

Consider the series *RLC* circuit connected to a voltage source, illustrated in Figure 8.7. Because we care only about the behavior of this circuit near resonance, we focus on the frequency domain, assuming that all of the signals in the circuit have time dependence e^{st} where s is the complex frequency

$$s = j\omega - \alpha, \quad \alpha \geq 0 \quad (8.4.1)$$

Figure 8.7 Series RLC discrete-element resonator.

The current $I(s)$ flowing through each element in the resonator is just the Laplace transform of the time-dependent current $i(t) = \text{Re}\{I(s)e^{st}\}$ and the voltage drops across the resistor, inductor, and capacitor are $V_R(s)$, $V_L(s)$, and $V_C(s)$, respectively. Each frequency-dependent voltage $V(s)$ in the circuit is also related to a corresponding time-dependent voltage $v(t) = \text{Re}\{V(s)e^{st}\}$. The complex quantities $V(s)$ and $I(s)$ are exactly analogous to the phasors $\bar{E}$ and $\bar{H}$ defined by (1.4.2) and (1.4.3). The only difference is that here we are not limited to real frequencies.

The voltage/current relations for these three elements are

$$\begin{aligned} v_R(t) &= R i(t) & V_R(s) &= R I(s) \\ v_L(t) &= L di(t)/dt & V_L(s) &= sL I(s) \\ v_C(t) &= (\int i(t) dt) / C & V_C(s) &= I(s)/sC \end{aligned} \quad (8.4.2)$$

where the first expression in each pair is the time-domain representation and the second is the frequency-domain relation between voltage and current. Kirchhoff's voltage law then states that the source voltage $V_S(s)$ is the sum of the voltage drops across each of the series elements, so

$$\begin{aligned} V_S(s) &= \left(R + sL + \frac{1}{sC} \right) I(s) \\ &= \left(\frac{s^2 + sR/L + 1/LC}{s/L} \right) I(s) \end{aligned} \quad (8.4.3)$$

Equation (8.4.3) may be factored using the quadratic formula, resulting in

$$V_S(s) = \frac{(s - s_1)(s - s_2)}{s/L} I(s) \quad (8.4.4)$$

where

$$\begin{aligned} s_{1,2} &= -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{2L}} \\ &= -\alpha_0 \pm j\sqrt{\omega_0^2 - \alpha_0^2} \end{aligned} \quad (8.4.5)$$

and we have suggestively defined the new variables

$$\omega_0 \equiv 1/\sqrt{LC} \quad (8.4.6)$$

$$\alpha_0 \equiv R/2L \quad (8.4.7)$$

For this RLC circuit to exhibit strong resonant behavior, the amount of power dissipated in the circuit must be small. Since α_0 is proportional to R , α_0 must also be small and therefore $\alpha_0 \ll \omega_0$, which means that the resonant frequency $\sqrt{\omega_0^2 - \alpha_0^2}$ can be approximated by ω_0 . We immediately see that the natural, complex frequency of this circuit is approximately $s \cong -\alpha_0 + j\omega_0$, which means that the resonant frequency in this circuit is near ω_0 and the rate at which power is dissipated is proportional to α_0 . All voltages and currents in this circuit have time-dependencies given approximately by $e^{-\alpha_0 t} e^{j\omega_0 t}$ so the stored energies are proportional to $|e^{-\alpha_0 t} e^{j\omega_0 t}|^2 = e^{-2\alpha_0 t}$. If we use the definition of Q in (8.3.3) and the definition of ω_0 and α_0 in (8.4.6) and (8.4.7), the quality factor for this resonance is

$$Q = \frac{\omega_0}{2\alpha_0} = \frac{\omega_0 L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}} \quad (8.4.8)$$

where Q becomes infinite as the lossy element R vanishes; (8.4.8) is exactly the same as (8.3.7). We may also find Q directly from its definition in (8.3.3)

$$Q = \frac{\omega_0 w_T}{P} = \frac{\omega_0 2 \langle w_m(s) \rangle}{P(s)} = \frac{\omega_0 2 \cdot \frac{1}{4} L |I(s)|^2}{\frac{1}{2} R |I(s)|^2} = \frac{\omega_0 L}{R} \quad (8.4.9)$$

where the power dissipated in the resistor is just $P(s) = \frac{1}{2} R |I(s)|^2$ and the total energy stored in this circuit is twice the magnetic energy $\langle w_m(s) \rangle = \frac{1}{4} L |I(s)|^2$.

We should now like to find the power dissipation in this resonator as a function of frequency. Using the definition of $P(s)$ and the relations between current and source voltage given by (8.4.3)–(8.4.4), we find

$$\begin{aligned} P(s) &= \frac{1}{2} R |I(s)|^2 = \frac{1}{2} R |V_S(s)|^2 \left| \frac{s/L}{(s-s_1)(s-s_2)} \right|^2 \\ &= \frac{1}{2} R |V_S(s)|^2 \left| \frac{s/L}{(s+\alpha_0-j\omega_0)(s-\alpha_0+j\omega_0)} \right|^2 \end{aligned} \quad (8.4.10)$$

which is written in terms of the source voltage $V_S(s)$ with the approximation that $\sqrt{\omega_0^2 - \alpha_0^2} \cong \omega_0$. Both the dissipated power and the complex admittance $I(s)/V_S(s)$ approach infinity whenever the complex frequency s approaches either of the *frequency-plane poles* at s_1 or s_2 . The location of these admittance poles

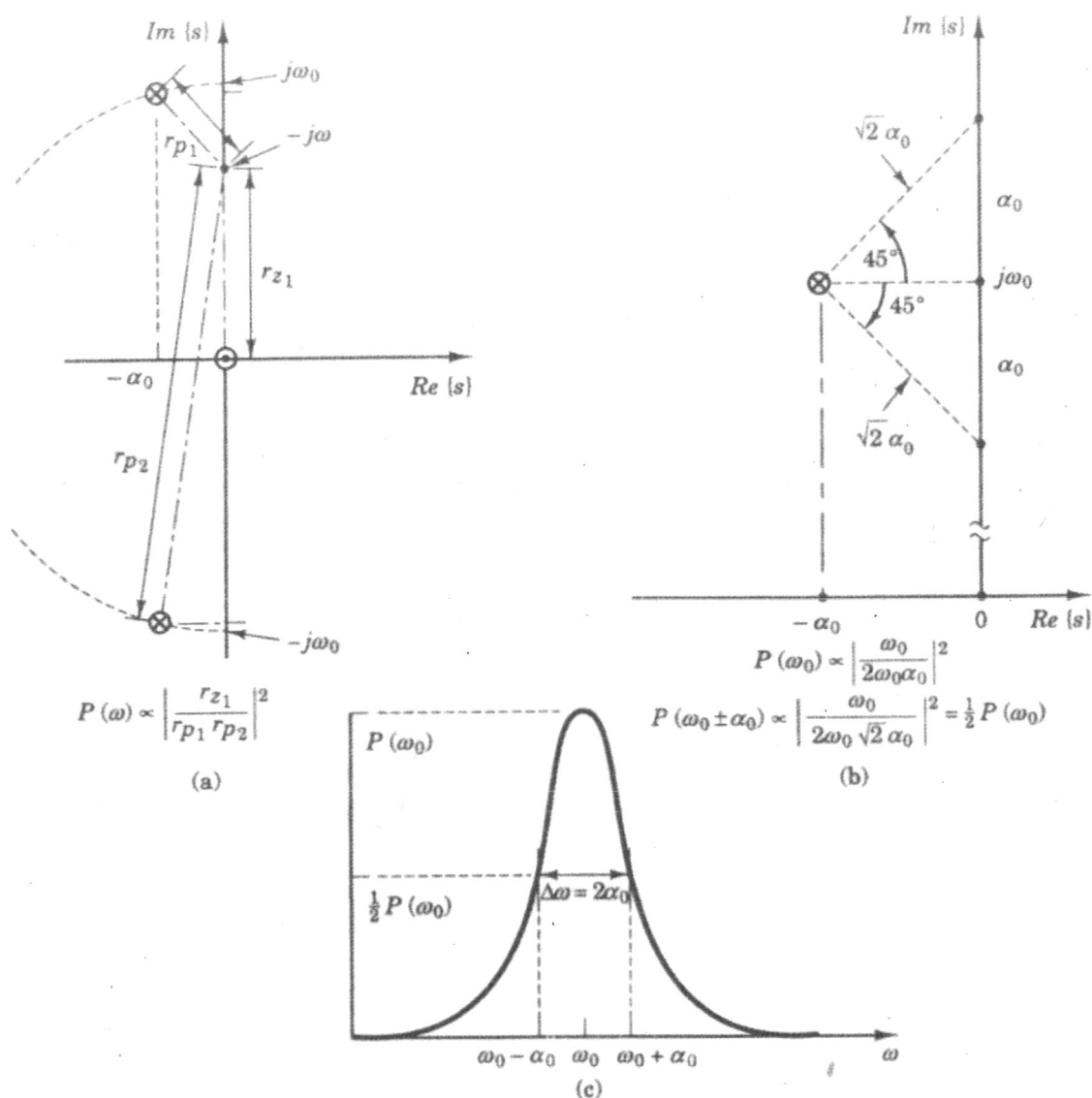


Figure 8.8 RLC resonator in the complex plane.

in the s -plane or complex frequency plane is illustrated in Figure 8.8(a). In addition to the poles at s_1 and s_2 , there is also a frequency-plane zero at $s = 0$ in both admittance and the power dissipation function (8.4.10).⁷

Because we must drive the series circuit with a real frequency ω (equivalent to a wholly imaginary complex frequency $s = j\omega$), we are confined to s values that lie along the imaginary axis in the complex frequency plane and therefore cannot get closer than the distance α_0 to either pole. The complex admittance of

7. Note that the admittance poles become impedance zeros and vice versa because impedance and admittance are reciprocal functions.

this circuit is never infinite, therefore, unless $\alpha_0 = 0$, corresponding to a perfect resonator with $R = 0$ and $Q \rightarrow \infty$.

The driving frequency which minimizes the distance to the pole at $s_1 = -\alpha_0 + j\sqrt{\omega_0^2 - \alpha_0^2}$ is $\omega = \sqrt{\omega_0^2 - \alpha_0^2} \simeq \omega_0$, which means that maximum power is dissipated in the resistor R at resonance. This power can be approximated by substituting $s = j\omega_0$ into (8.4.10), yielding

$$\begin{aligned} P(\omega_0) &= \frac{R}{2} |V_S(\omega_0)|^2 \left| \frac{j\omega_0/L}{\alpha_0(2j\omega_0 + \alpha_0)} \right|^2 \\ &\approx \frac{R}{2} |V_S(\omega_0)|^2 \left| \frac{j\omega_0/L}{\alpha_0(2j\omega_0)} \right|^2 \\ &\approx \frac{|V_S(\omega_0)|^2}{2R} \end{aligned} \quad (8.4.11)$$

where $\alpha_0 = R/2L$ has been used to simplify (8.4.11) and the factor $2j\omega_0 + \alpha_0$ has been approximated by $2j\omega_0$. The value of maximum power dissipation given by (8.4.11) is the power that would be dissipated if the reactances of the inductor and capacitor canceled each other exactly, leaving an input impedance $Z = R$. This occurs when $j\omega L + 1/j\omega C = 0$, or when $\omega = 1/\sqrt{LC} = \omega_0$, i.e., approximately at resonance.

If we move away from resonance to $\omega_0 \pm \alpha_0$, the distance to the $-\alpha_0 + j\omega_0$ pole goes from α_0 to $\sqrt{2}\alpha_0$ as seen graphically in Figure 8.9(b). Because this pole dominates (8.4.10), the dissipated power drops to one-half its peak power, shown by substituting $s = j(\omega_0 \pm \alpha_0)$ into (8.4.10):

$$\begin{aligned} P(\omega_0 \pm \alpha_0) &= \frac{R}{2} |V_S(\omega_0 \pm \alpha_0)|^2 \times \\ &\quad \left| \frac{j(\omega_0 \pm \alpha_0)/L}{[j(\omega_0 \pm \alpha_0) + \alpha_0 - j\omega_0][j(\omega_0 \pm \alpha_0) + \alpha_0 + j\omega_0]} \right|^2 \\ &\approx \frac{R}{2} |V_S(\omega_0)|^2 \left| \frac{j\omega_0/L}{(\alpha_0 \pm j\alpha_0)(2j\omega_0)} \right|^2 \approx \frac{R}{4} |V_S(\omega_0)|^2 \approx \frac{1}{2} P(\omega_0) \end{aligned} \quad (8.4.12)$$

The frequencies $\omega = \omega_0 \pm \alpha_0$ are therefore called the *half-power frequencies* of the resonance, and we define $\Delta\omega \cong 2\alpha_0$ as the *half-power bandwidth* or full width at half maximum (FWHM).

Because $\alpha_0 \ll \omega_0$, the circuit is a high-quality resonator and $P(\omega)$ is sharply peaked about ω_0 , as suggested in Figure 8.8(c). As the poles approach the imaginary axis in the complex s -plane, $2\alpha_0$ becomes smaller and the resonance peak is narrowed. As $Q \rightarrow \infty$, the resonance becomes a delta function, and **only** the resonant frequency itself can be excited, with zero driving point impedance. The width

of this resonance is closely related to its quality factor Q . From (8.4.8),

$$Q = \frac{\omega_0}{2\alpha_0} \cong \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f} \quad (8.4.13)$$

The dual circuit to a series RLC resonator driven by a voltage source is a parallel GLC resonator driven by a current source, as illustrated in Figure 8.9. The current and voltage in this resonator are related by

$$\begin{aligned} I_s(s) &= \left(G + sC + \frac{1}{sL} \right) V(s) \\ &= \left(\frac{s^2 + sG/C + 1/LC}{s/C} \right) V(s) \end{aligned} \quad (8.4.14)$$

which is completely analogous to (8.4.3) if V and I are interchanged along with $G \leftrightarrow R$ and $L \leftrightarrow C$. The resonant frequency $\omega_0 = 1/\sqrt{LC}$ is not affected when L and C are interchanged, but the value of α_0 becomes

$$\alpha_0 = \frac{G}{2C} \quad (8.4.15)$$

for the parallel circuit; the quality factor is then

$$Q = \frac{\omega_0}{2\alpha_0} = \frac{\omega_0 C}{G} = \frac{1}{G} \sqrt{\frac{C}{L}} \quad (8.4.16)$$

This value of Q can also be found from its definition

$$Q = \frac{\omega_0 w_T}{P} = \frac{\omega_0 2\langle w_e(s) \rangle}{P(s)} = \frac{\omega_0 2 \cdot \frac{1}{4} C |V(s)|^2}{\frac{1}{2} G |V(s)|^2} = \frac{\omega_0 C}{G} \quad (8.4.17)$$

which is just the dual of (8.4.9). If we recall that the Q of a cavity resonator with lossy dielectric is $\omega_0 \epsilon / \sigma_i$ from (8.3.11), we find a plausible relation between C , G , ϵ , and σ_i :

$$\frac{\epsilon}{\sigma_i} = \frac{C}{G} \quad (8.4.18)$$

A similar relationship between cavity resonators and RLC circuits is not so readily

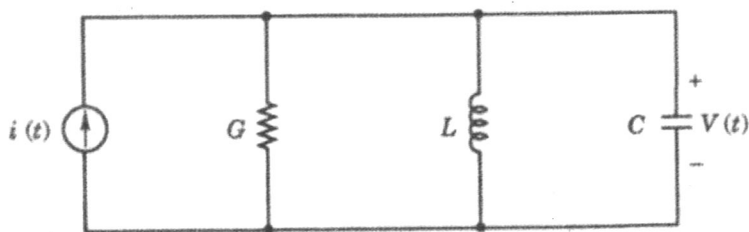


Figure 8.9 Parallel GLC resonator circuit.

found, since the specific mode and actual structure of the cavity are important in determining resistive losses.

We now describe the coupling of a series RLC or parallel GLC resonator to an external circuit, in preparation for Section 8.8 in which we shall discuss coupling by modeling the resonator by an equivalent circuit. Figure 8.10(a) depicts a discrete-element RLC resonator coupled to a Thevenin equivalent circuit modeled as a voltage source V_{Th} in series with an impedance Z_{Th} , representing **any** passive, external circuit.

Because there are now two elements in the combined circuit that dissipate power, we specify power lost in the internal circuit as P_I and power lost in the external circuit as P_E . The quality factors for the internal and external circuits are also specified explicitly as the *internal* Q , Q_I and *external* Q , Q_E

$$Q_I \equiv \frac{\omega w_T}{P_I} \quad (8.4.19)$$

$$Q_E \equiv \frac{\omega w_T}{P_E} \quad (8.4.20)$$

where ω is the resonant frequency and w_T is the total energy stored in the resonator. The total, or *loaded* Q , Q_L is the quality factor that results from both loss

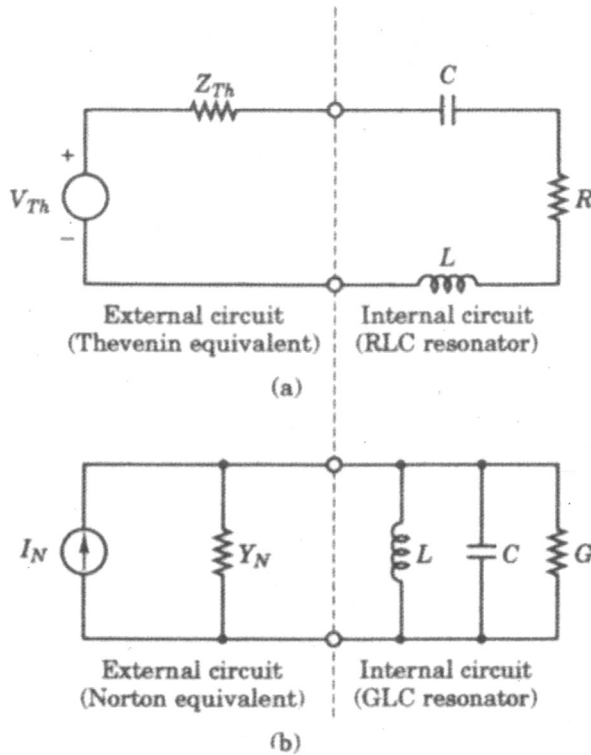


Figure 8.10 Coupling between circuit resonators and external sources.

mechanisms:

$$Q_L = \frac{\omega W_T}{P_I + P_E} = \left[\frac{1}{Q_I} + \frac{1}{Q_E} \right]^{-1} \quad (8.4.21)$$

In the resonator circuits discussed to this point, Q_E was assumed to be infinite so that Q_L and Q_I could be used interchangeably.

If the maximum possible amount of power is dissipated in the resonator from the external circuit, then the resonator is said to be *critically coupled* or *matched* to the external circuit. It is easy to show that critical coupling occurs when the resistive parts of the internal and external circuits are equal and the reactive parts of the two circuits cancel. In this case, the same amount of power is dissipated in both circuits so $P_I = P_E$, $Q_I = Q_E$, and $Q_L = Q_I/2$ (see (6.3.12)).

We can find the value of Z_{Th} necessary to critically couple both halves of the circuit in Figure 8.10(a) at resonance by noting that the impedance $Z(\omega)$ looking into the RLC circuit is $Z(\omega) = R + j\omega L + 1/j\omega C$. At the resonant frequency $\omega_0 = 1/\sqrt{LC}$, the resonator impedance is $Z(\omega_0) = R$. The Thevenin impedance that matches this resonator at resonance is therefore the resistor $R_{Th} = R$.⁸

Overcoupling occurs when more power is dissipated in the external circuit than in the resonator so that $P_E > P_I$ and $Q_E < Q_I$. *Undercoupling* is the reverse situation where $P_E < P_I$ and $Q_E > Q_I$. For the circuit in Figure 8.10(a), an overcoupled circuit is one in which $R_{Th} > R$ and an undercoupled circuit implies $R_{Th} < R$.

We can therefore define Q_I , Q_E , and Q_L for this circuit by the following relations

$$\begin{aligned} Q_I &= \omega_0 L / R \\ Q_E &= \omega_0 L / R_{Th} \\ Q_L &= \omega_0 L / (R + R_{Th}) \end{aligned} \quad (8.4.22)$$

which follow from (8.4.8) and the definition of critical coupling.

A similar coupling between an external circuit and a parallel GLC resonator is illustrated in Figure 8.10(b), where the external Thevenin equivalent circuit is replaced by a Norton equivalent circuit, consisting of a current source I_N in parallel with an admittance Y_N . Because the two circuits in Figure 8.10 are complete duals of one another, the entire discussion above can be repeated with the substitutions $V_{Th} \rightarrow I_N$, $Z_{Th} \rightarrow Y_N$, $R \rightarrow G$, and $L \rightarrow C$. Critical-coupling again occurs when the parallel GLC resonator is matched to the external circuit, or $Y_N = G_N = G$.⁹

8. Strictly speaking, it is also possible to have a reactive part of the Thevenin impedance as long as $X_{Th}(\omega_0) = 0$.

9. Or, more generally, when $Y_N = G + jB$ where $B(\omega_0) = 0$.

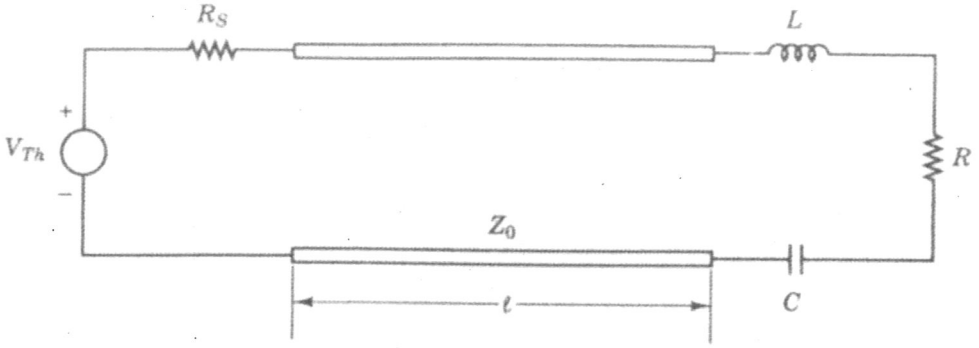


Figure 8.11 Power coupling to a RLC resonator in Example 8.4.1.

This circuit has quality factors Q_I , Q_E , and Q_L given by

$$\begin{aligned} Q_I &= \omega_0 C / G \\ Q_E &= \omega_0 C / G_{Th} \\ Q_L &= \omega_0 C / (G + G_{Th}) \end{aligned} \quad (8.4.23)$$

which result from (8.4.16) and the definition of critical coupling.

Example 8.4.1

We wish to couple the maximum amount of power (at resonance) to a series RLC resonator via a transmission line connected to a series resistor R_s , shown in Figure 8.11. Find a relationship between R_s , Z_0 , ℓ , and R . Determine R_s in terms of Z_0 and R when $\ell = \lambda/2$ and $\ell = \lambda/4$.

Solution: At resonance, the RLC circuit has a purely real impedance R . For maximum power transfer to occur, the resonator must be critically coupled and the Thevenin equivalent circuit should be simply a voltage source in series with a resistor having the value $Z_{Th} = R_{Th} = R$. We can find the Thevenin impedance of the external circuit by looking back into the TEM line and shorting the voltage source. From (6.1.14), we find

$$Z_{Th} = Z_0 \left(\frac{R_s + j Z_0 \tan k\ell}{Z_0 + j R_s \tan k\ell} \right) \quad (1)$$

which is purely real when $\tan k\ell = n\pi$ ($\ell = 0, \lambda/2, \lambda, \dots$) or when $\tan k\ell = (n + 1/2)\pi$ ($\ell = \lambda/4, 3\lambda/4, \dots$). If ℓ is a multiple of $\lambda/2$, then $Z_{Th} = R_s$ and so $R_s = R$ is necessary for critical coupling. If, instead, ℓ is an odd multiple of $\lambda/4$, $Z_{Th} = Z_0^2 / R_s$ so $R_s = Z_0^2 / R$ is the appropriate choice for the series resistance.

8.5 TRANSMISSION LINE RESONATORS

The one-dimensional TEM -line resonator is another important resonating structure, formed by terminating a section of TEM transmission line at both ends. Such resonators exhibit all the subtleties of resonator behavior without the additional complexities arising from a three-dimensional structure. Slightly less simple than

the series and parallel circuit resonators described in the previous section, the *TEM*-line resonators possess an infinite number of resonant frequencies instead of just a single resonance at $\omega = 1/\sqrt{LC}$.

We first discuss a *TEM*-line resonator formed from a piece of uniform *TEM* transmission line of length ℓ shorted at both ends, as illustrated in Figure 8.12.

The voltage and current for such a resonator are given by the usual forward- and backward-propagating wave solutions (5.3.45) and (5.3.49):

$$\begin{aligned} V(z) &= V_+ e^{-jkz} + V_- e^{jkz} \\ I(z) &= Y_0(V_+ e^{-jkz} - V_- e^{jkz}) \end{aligned} \quad (8.5.1)$$

The short-circuit boundary condition $V(z=0) = 0$ may be applied to (8.5.1) to show that $V_+ = -V_-$. Therefore,

$$\begin{aligned} V(z) &= -2jV_+ \sin kz \\ I(z) &= 2Y_0V_+ \cos kz \end{aligned} \quad (8.5.2)$$

Since the voltage in (8.5.2) is also forced to vanish at $z = \ell$, then $\sin k\ell = 0$ or

$$k\ell = n\pi, \quad n = 0, 1, 2, \dots \quad (8.5.3)$$

Therefore, the wavenumber k may take on a series of discrete values indexed by n such that $k_n = n\pi/\ell$.¹⁰ Because $k = \omega\sqrt{LC} = \omega/v$, the resonator can only sustain voltages and currents at the resonant frequencies

$$\omega_n = \frac{k_n}{\sqrt{LC}} = \frac{n\pi}{\ell\sqrt{LC}} = \frac{n\pi v}{\ell}, \quad n = 0, 1, 2, \dots \quad (8.5.4)$$

where $v = 1/\sqrt{LC}$ is the speed of wave propagation on the transmission line.

10. In this case, where both ends are shorted, we see that $\ell = n\lambda_n/2$. The same result follows for *TEM* resonators open-circuited at both ends, whereas the length ℓ of a resonator shorted at one end and open-circuited at the other is an odd multiple of $\lambda_n/4$.

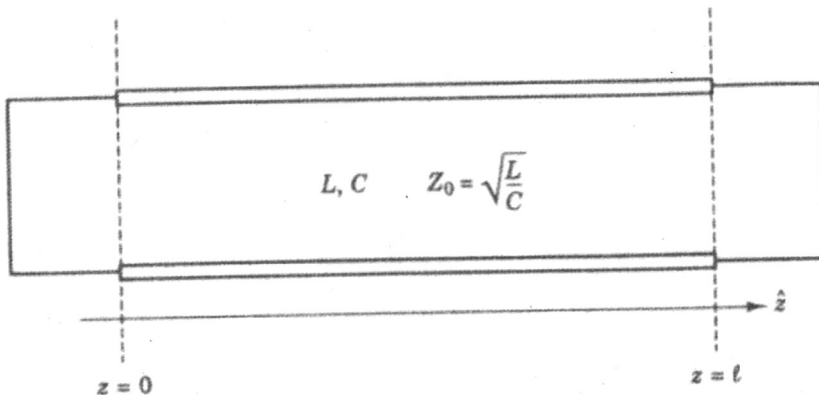


Figure 8.12 Short-circuit *TEM*-line resonator.

Note that one of these modes ($n = 0$) has zero frequency, corresponding to a steady (DC) current $I(z) = 2Y_0 V_+$ flowing in a loop and purely magnetic energy storage since $V(z)$ is everywhere zero for this mode.

We can also find the resonant frequency for *TEM*-line resonators terminated with reactive (capacitive and/or inductive) loads. Consider the case where the *TEM*-line resonator of Figure 8.7 is terminated at $z = \ell$ with an inductor L instead of a short-circuit, while the $z = 0$ end remains a short-circuit. The impedance at $z = \ell$ is then

$$Z(z = \ell) = \frac{V(z = \ell)}{I(z = \ell)} = \frac{-2jV_+ \sin k\ell}{2Y_0 V_+ \cos k\ell} = -jZ_0 \tan k\ell \quad (8.5.5)$$

using the voltage and current in (8.5.2), which satisfy $V(z = 0) = 0$. But the impedance at $z = \ell$ must also be equal to the inductor impedance $j\omega L$, resulting in the transcendental equation

$$\omega L = -Z_0 \tan \left(\frac{\omega \ell}{v} \right) \quad (8.5.6)$$

which has an infinite number of solutions ω_n . In this case, the resonant frequencies must be obtained either graphically or numerically. If $L \rightarrow 0$, corresponding to a short-circuit impedance, then $\tan(\omega \ell / v)$ must also vanish, recovering the resonant frequencies $\omega_n = n\pi v / \ell$ of (8.5.4). In the opposite limit, where $L \rightarrow \infty$ and the resonator is open-circuited at $z = \ell$, $\tan(\omega_n \ell / v) \rightarrow \infty$ and $\omega_n = (n + 1/2)\pi v / \ell$.

Once the currents and voltages are found for a *TEM*-line resonator, it is straightforward to calculate the Q of any resonance. Suppose we insert a small series resistor R into the short-circuited *TEM*-line resonator at $z = a$ as shown in Figure 8.13(a). We assume that R is small enough that the current flowing through the transmission line is not seriously perturbed from its original value. Since we have already determined in (5.3.50) that a lossy *TEM* line has the approximate impedance

$$Z \approx \sqrt{\frac{j\omega L + R}{j\omega C + G}} \quad (8.5.7)$$

we find that $R \ll \omega L$ must be true near resonance for perturbation methods to be successfully applied. Since $\omega L \approx Z_0$ near resonance, this condition is equivalent to $R \ll Z_0$. The power dissipated in the resistor is thus

$$P_n = \frac{1}{2} R |I_n(z = a)|^2 \approx \frac{1}{2} R |I_n^{(0)}(z = a)|^2$$

where $I_n^{(0)}(z = a) = -2jY_0 V_{+n} \cos k_n a$ is the unperturbed current that would exist if the resistor were not present. Therefore,

$$P_n \simeq \frac{1}{2} R |I_n^{(0)}(z = a)|^2 = \frac{1}{2} R |2Y_0 V_{+n}|^2 \cos^2 \left(\frac{n\pi a}{\ell} \right) \quad (8.5.8)$$

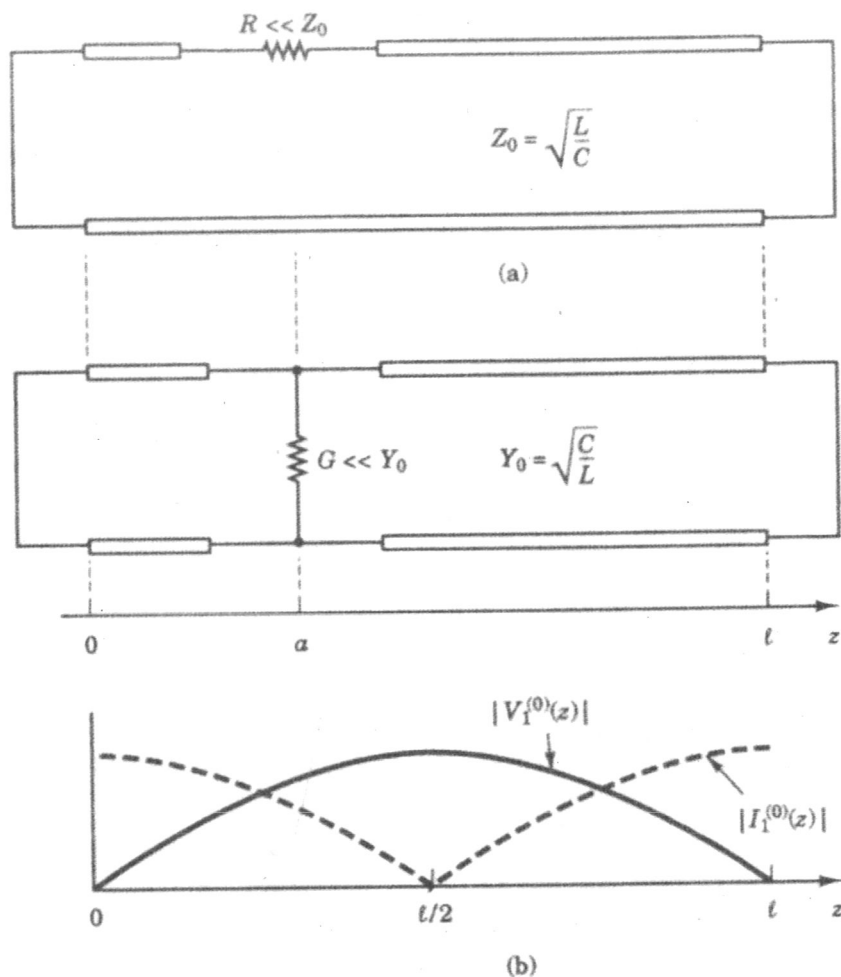


Figure 8.13 Loss in a transmission line resonator.

The total energy stored in this resonator is twice its time-averaged magnetic energy

$$\begin{aligned}
 w_{Tn} &= 2 \cdot \frac{1}{4} \int_0^\ell L |I_n^{(0)}(z)|^2 dz = \frac{L}{2} \int_0^\ell |2Y_0 V_{+n}|^2 \cos^2 k_n z dz \\
 &= \frac{\ell}{4} L |2Y_0 V_{+n}|^2
 \end{aligned} \quad (8.5.9)$$

where we have written w_{Tn} in terms of $I_n^{(0)}(z)$ to simplify the calculation of Q . An expression for the quality factor of this resistive-wall resonator results from its definition in (8.3.3):

$$Q_n \simeq \frac{\omega_n w_{Tn}}{P_n} = \frac{n\pi Z_0}{2R} \frac{1}{\cos^2(n\pi a/\ell)} \quad (8.5.10)$$

It is clear that Q is large when R/Z_0 is much less than unity. However, Q is also

large if we put the resistor at a current null on the ideal TEM line [$\cos(n\pi a/\ell) = 0$] so that no power can be dissipated. In this case Q would be very large even if $R \approx Z_0$!

We can also perturb the TEM -line resonator by connecting a small conductance G in parallel across the resonator at $z = a$, as shown in Figure 8.13(b). This resonator is the dual of the one just described. That is, we can repeat the entire discussion of the small R resonator above but interchange the roles of $I \leftrightarrow V$, $Z_0 \leftrightarrow Y_0$, and $R \leftrightarrow G$ to describe the small G resonator.

If $G \ll Y_0$, very little current will flow through the conducting shunt and the voltage will not differ much from its unperturbed value $V_n^{(0)}(z = a)$. This suggests that we write

$$P_n \approx \frac{1}{2} G |V_n^{(0)}|^2 = \frac{1}{2} G |2V_{+n}|^2 \sin^2\left(\frac{n\pi a}{\ell}\right) \quad (8.5.11)$$

using (8.5.2) and (8.5.3). The total stored energy w_{Tn} is given by (8.5.9) so the quality factor for the n^{th} mode of the conducting-shunt resonator is

$$Q_n \approx \frac{\omega_n w_{Tn}}{p_n} = \frac{n\pi Y_0}{2G} \frac{1}{\sin^2(n\pi a/\ell)} \quad (8.5.12)$$

If the shunt is placed close to either end of the resonator (at $a \approx 0$ or $a \approx \ell$), then $Q \rightarrow \infty$ even for large G because the voltage is small at either end of the resonator so very little power can be dissipated. Note that the ratios Z_0/R and Y_0/G are both important measures of resonance quality, but do not uniquely determine Q .

8.6 TRANSIENTS IN RESONATORS

Transients are significant only for limited intervals of time, having signal components at a wide range of frequencies. The more limited the time interval associated with the transient, the broader the range of frequencies that are present, as a Fourier or Laplace transform reveals. In order to understand the transient behavior of resonators, we express the voltage and current as the superposition of an infinite number of terms, each at a distinct resonant frequency of the system. As an example, we revisit the short-circuited TEM -line resonator of length ℓ with voltage and current given by (8.5.2). An arbitrary waveform is constructed by superposing the voltages and currents of each mode using (8.5.1)–(8.5.4)

$$V(z, t) = \sum_{n=0}^{\infty} V_n(z, t) = \text{Re} \left\{ \sum_{n=0}^{\infty} A_n \sin k_n z e^{j\omega_n t} \right\} \quad (8.6.1)$$

$$I(z, t) = \sum_{n=0}^{\infty} I_n(z, t) = \text{Re} \left\{ \sum_{n=0}^{\infty} jY_0 A_n \cos k_n z e^{j\omega_n t} \right\} \quad (8.6.2)$$

where $A_n = -2jV_{+n}$ is the complex coefficient of the n^{th} mode of this resonator and has yet to be determined explicitly.