

As an example of how A_n might be found, consider an air-filled *TEM* transmission line of length ℓ initially open-circuited at both ends and charged to V_0 volts with zero current everywhere on the line. At $t = 0^+$, both ends of the resonator are suddenly short-circuited and the structure becomes the resonator of Figure 8.12 with voltage and current governed by (8.6.1) and (8.6.2). To evaluate A_n , we set $V(z, t = 0^+) = V_0$ and $I(z, t = 0^+) = 0$. This latter boundary condition can be written as

$$I(z, t = 0^+) = \operatorname{Re} \left\{ \sum_{n=0}^{\infty} jY_0 A_n \cos k_n z \right\} = 0 \quad (8.6.3)$$

where (8.6.3) implies that A_n is a real coefficient for all n ; that is, $\operatorname{Im}\{A_n\} = 0$. Applying the voltage boundary condition at $t = 0^+$ to (8.6.1) yields

$$V(z, t = 0^+) = \operatorname{Re} \left\{ \sum_{n=0}^{\infty} A_n \sin k_n z \right\} = V_0 \quad (8.6.4)$$

but since $A_n \sin k_n z$ is purely real here, the $\operatorname{Re}\{\}$ operator in (8.6.4) may be dropped. Therefore,

$$V_0 = \sum_{n=0}^{\infty} A_n \sin k_n z, \quad k_n = n\pi/\ell \quad (8.6.5)$$

The orthogonality relation for sinusoids discussed in Section 7.3 may now be used to determine A_n . Recalling from (7.3.10) that

$$\int_0^{\ell} \frac{2}{\ell} \sin\left(\frac{n\pi z}{\ell}\right) \sin\left(\frac{m\pi z}{\ell}\right) dz = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad (8.6.6)$$

we multiply both sides of (8.6.5) by $(2/\ell) \sin k_m z$ and integrate from 0 to ℓ , yielding

$$\begin{aligned} \int_0^{\ell} \frac{2V_0}{\ell} \sin k_m z dz &= \sum_{n=0}^{\infty} A_n \int_0^{\ell} \frac{2}{\ell} \sin\left(\frac{n\pi z}{\ell}\right) \sin\left(\frac{m\pi z}{\ell}\right) dz \\ &= \sum_{n=0}^{\infty} A_n \delta_{nm} \\ &= A_m \end{aligned} \quad (8.6.7)$$

Therefore,

$$A_m = \int_0^{\ell} \frac{2V_0}{\ell} \sin k_m z dz = \frac{2V_0}{m\pi} [1 - (-1)^m] \quad (8.6.8)$$

because $\cos k_m \ell = (-1)^m$. We can thus re-express A_n as

$$A_n = \begin{cases} 4V_0/n\pi, & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad (8.6.9)$$

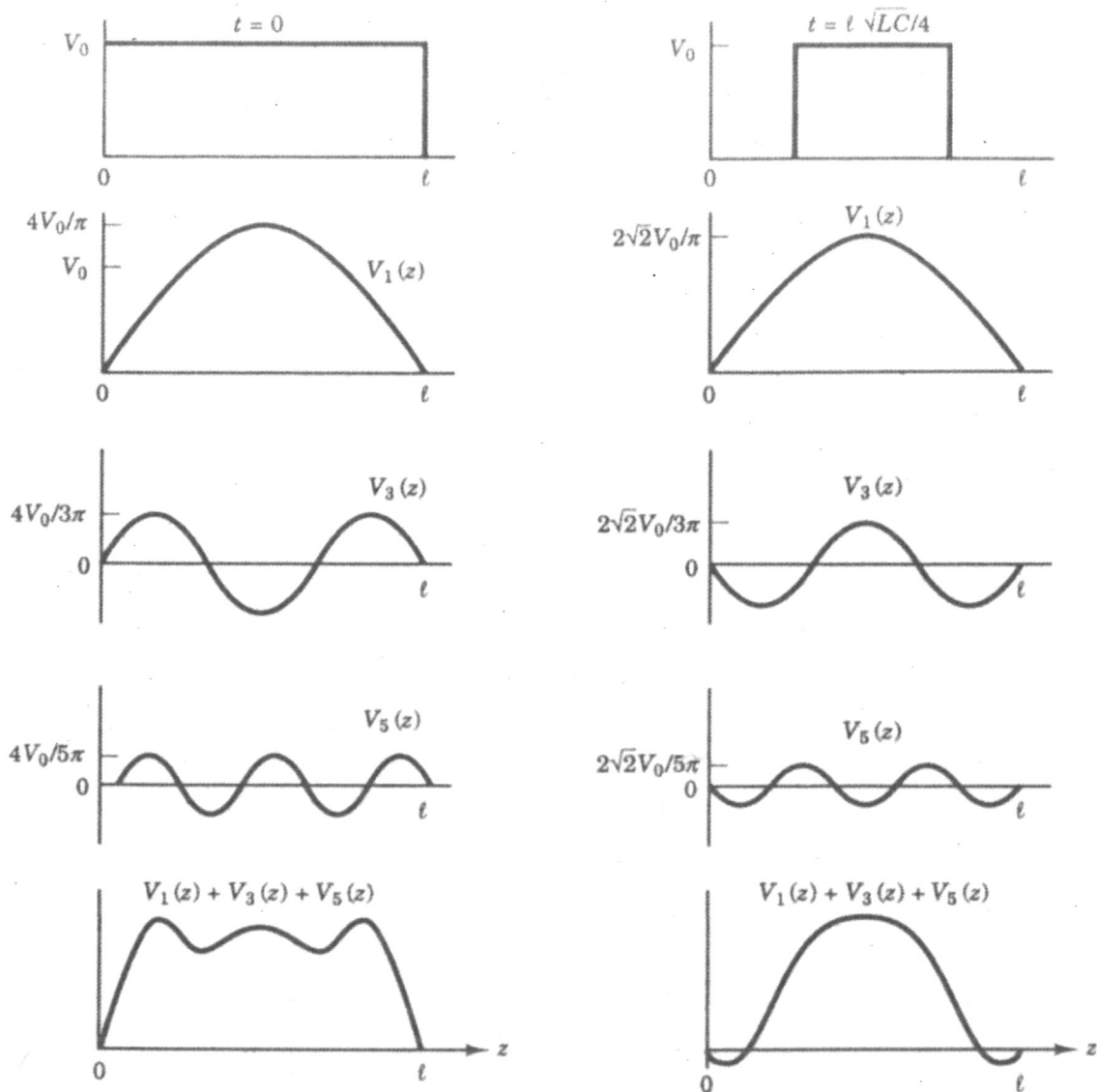


Figure 8.14 Construction of voltage $V(z, t)$ from a summation of modes.

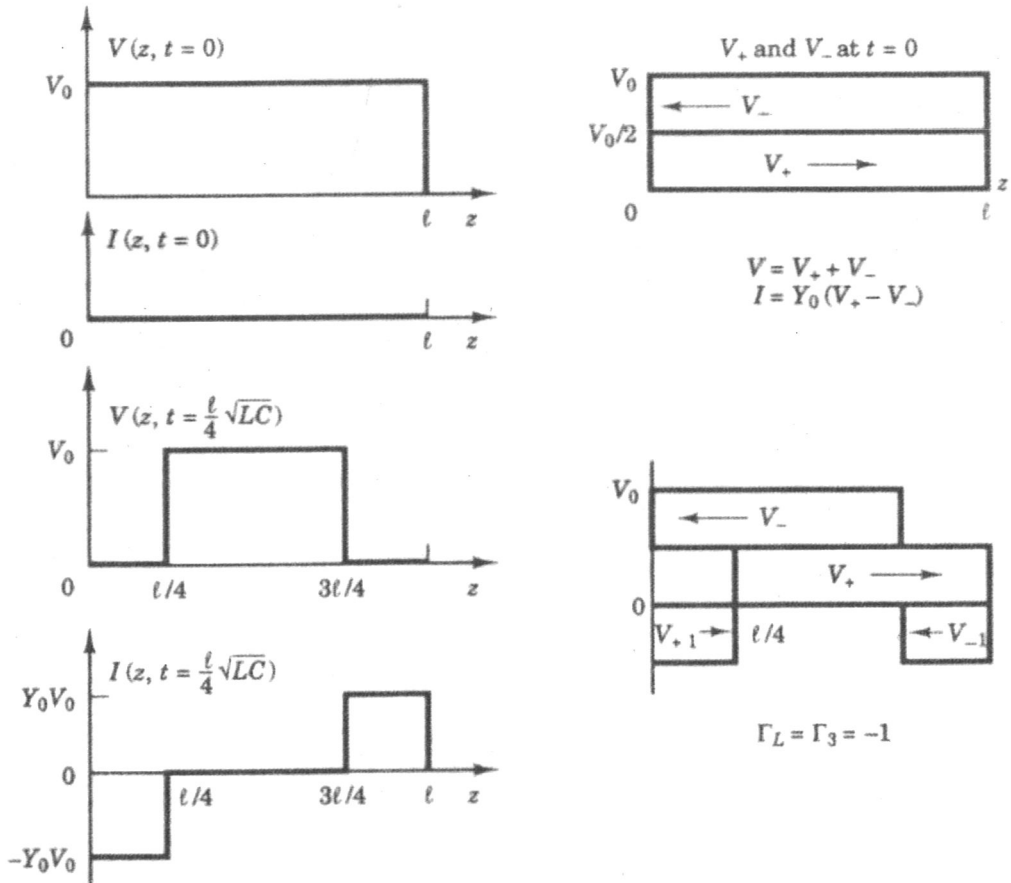
so the voltage on this line finally becomes

$$\begin{aligned}
 V(z, t) &= \operatorname{Re} \left\{ \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \sin\left(\frac{n\pi z}{\ell}\right) e^{j\omega_n t} \right\} \\
 &= \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \sin\left(\frac{n\pi z}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell}\right)
 \end{aligned} \quad (8.6.10)$$

The expression for $V(z, t)$ given by (8.6.10) is illustrated in Figure 8.14 at $t = 0^+$ and at $t = \ell/4c$, together with the contributions from $V_n(z, t) = (4V_0/n\pi) \sin(n\pi z/\ell) \cos(n\pi ct/\ell)$ for $n = 1, 3, 5$.

In the limit where the summation includes all of the odd modes, the resulting voltage $V(z, t)$ becomes a simple traveling step function that can be determined in an alternative manner using the methods of time-domain analysis presented in Sections 5.8 and 6.8. We recall that the boundary conditions at $t = 0^+$ determine the initial forward- and backward-propagating voltage pulses, both of magnitude $V_0/2$, as shown in Figure 8.15. Each pulse propagates independently, and the short-circuit boundary conditions at $z = 0$ and $z = \ell$ imply that the reflection coefficients Γ_L and Γ_S at each end of the resonator are both equal to -1 . Figure 8.15 plots the voltage $V(z, t)$ at several instants of time. (A further explanation of time-domain methods can be found in Section 6.8.) The energy oscillates between its electric and magnetic forms. At $t = 0, \ell/c, 2\ell/c, \dots$ the current I is zero everywhere and the energy is purely electric, whereas at $t = \ell/2c, 3\ell/2c, \dots$ the voltage is zero everywhere and the energy is purely magnetic.

Either method (modal decomposition or time-domain analysis) may be used to analyze a *TEM* resonator. While the time-domain method is more straightforward and intuitive for simple pulse shapes on a dispersionless *TEM* line, the



frequency-domain method of decomposing voltage and current into a set of orthogonal modes is more general and gives additional insight into resonator behavior.

This direct approach to solving problems where the initial conditions are specified is outlined below:

- (1) Expand the field expressions in terms of modes, where the resonant frequencies ω_n are determined by the spatial boundary conditions of the resonator.
- (2) Set these sums of modes with unknown coefficients equal to the initial conditions for $V(z, t)$ and $I(z, t)$.
- (3) Solve for the complex amplitude of every mode by using an appropriate orthogonality relation that singles out each mode individually.
- (4) Solve for $V(z, t)$ and $I(z, t)$ by summing over all modes, each at its own characteristic frequency $s_n = j\omega_n - \alpha_n$.

8.7* PERTURBATION OF RESONATORS

We have shown in Sections 8.2 and 8.5 that a lossless cavity or *TEM*-line resonator possesses a (possibly infinite) number of undamped resonant frequencies ω_n . In Sections 8.3–8.5, we saw how small dissipative perturbations could broaden these resonances and even shifted them slightly from ω_0 to $\sqrt{\omega_0^2 - \alpha_0^2}$ [see (8.4.5)]. In Section 8.5 we also saw how a reactive element could be added to a *TEM* resonator to produce a new set of resonant frequencies shifted somewhat from the set produced without the reactive element. (The example used was an inductor at the end of a shorted *TEM* line.) In each case, it is convenient to think of the unperturbed resonant frequency $s_n = j\omega_n$ altered to a new resonant frequency $s'_n = j\omega'_n - \alpha_n$ where $\omega'_n \approx \omega_n$ and $\alpha_n \approx \omega_n/2Q_n \approx P_n/2W_{Tn}$. In general, dissipative losses have their greatest effect on α_n , while reactive (lossless) perturbations principally affect ω_n . In this section, we develop a more rigorous approach to calculating changes in the resonant frequency s_n as small losses and reactances are introduced. These generalizations will also permit us to calculate the effects of denting the walls of cavity resonators or introducing small lumps of dielectric, magnetic, or conducting materials into a cavity.

We shall begin our derivation of an expression for frequency perturbation by applying Maxwell's equations simultaneously to a perfectly conducting resonant cavity and its almost-identical (but lossy) twin. Consider a perfectly conducting closed cavity filled with a medium having permittivity $\epsilon_A(\bar{r})$ and permeability $\mu_A(\bar{r})$. The cavity is resonant at frequency $s = j\omega_0$ and satisfies Maxwell's equations

$$\nabla \times \bar{E}_A = -j\omega_0\mu_A\bar{H}_A \quad (8.7.1)$$

$$\nabla \times \bar{H}_A = j\omega_0\epsilon_A\bar{E}_A \quad (8.7.2)$$

where $\bar{J}_A$ is zero inside the cavity since $\sigma_A = 0$ and no currents have been applied. If we now alter this cavity slightly, either by adding loss to the walls or medium and/or changing the cavity shape, the new cavity satisfies the equations

$$\nabla \times \bar{E}_B = -s\mu_B \bar{H}_B \quad (8.7.3)$$

$$\nabla \times \bar{H}_B = s\epsilon_B \bar{E}_B + \bar{J}_B \quad (8.7.4)$$

where $\bar{E}_B$ and $\bar{H}_B$ are the perturbed fields with values very close to $\bar{E}_A$ and $\bar{H}_A$, respectively, and $\bar{J}_B$ is a perturbed current introduced by adding loss to the cavity, inserting a modulated electron beam, and so on. As before, the perturbed time-harmonic fields are proportional to e^{st} where s is generally a complex frequency. The real part of s is proportional to the rate at which energy is lost from the resonator and the imaginary part of s is the frequency at which the fields resonate.

We now use the vector identity $\nabla \cdot (\bar{A} \times \bar{B}) = (\nabla \times \bar{A}) \cdot \bar{B} - (\nabla \times \bar{B}) \cdot \bar{A}$ to compute

$$\begin{aligned} \nabla \cdot (\bar{E}_A^* \times \bar{H}_B + \bar{E}_B \times \bar{H}_A^*) &= (\nabla \times \bar{E}_A^*) \cdot \bar{H}_B - (\nabla \times \bar{H}_B) \cdot \bar{E}_A^* \\ &\quad + (\nabla \times \bar{E}_B) \cdot \bar{H}_A^* - (\nabla \times \bar{H}_A^*) \cdot \bar{E}_B \\ &= j\omega_0\mu_A \bar{H}_A^* \cdot \bar{H}_B - s\epsilon_B \bar{E}_B \cdot \bar{E}_A^* - \bar{J}_B \cdot \bar{E}_A^* \\ &\quad - s\mu_B \bar{H}_B \cdot \bar{H}_A^* + j\omega_0\epsilon_A \bar{E}_A^* \cdot \bar{E}_B \end{aligned} \quad (8.7.5)$$

where (8.7.1)–(8.7.4) have been used. If we integrate (8.7.5) over the volume of the lossless resonator V_A , we can use Gauss's divergence theorem (4.1.1) to convert the left side of (8.7.5) to a closed-surface integral over the area of the unperturbed resonator A_A :

$$\begin{aligned} \oint_{A_A} (\bar{E}_A^* \times \bar{H}_B + \bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A da &= (j\omega_0\mu_A - s\mu_B) \int_{V_A} (\bar{H}_A^* \cdot \bar{H}_B) dv \\ &\quad + (j\omega_0\epsilon_A - s\epsilon_B) \int_{V_A} (\bar{E}_A^* \cdot \bar{E}_B) dv - \int_{V_A} (\bar{J}_B \cdot \bar{E}_A^*) dv \end{aligned} \quad (8.7.6)$$

Because the electric field $\bar{E}_A$ is perpendicular to the surface of the lossless resonator, $\hat{n}_A \times \bar{E}_A = 0$ and the first term on the left side of (8.7.6) vanishes. We can then solve (8.7.6) for $s - j\omega_0$, yielding

$$\begin{aligned} s - j\omega_0 &= - \left[\oint_{A_A} (\bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A da + \int_{V_A} (\bar{J}_B \cdot \bar{E}_A^*) dv + \right. \\ &\quad \left. j\omega_0(\mu_A - \mu_B) \int_{V_A} (\bar{H}_A^* \cdot \bar{H}_B) dv + j\omega_0(\epsilon_A - \epsilon_B) \int_{V_A} (\bar{E}_A^* \cdot \bar{E}_B) dv \right] / \\ &\quad \left[\int_{V_A} (\mu_B \bar{H}_A^* \cdot \bar{H}_B + \epsilon_B \bar{E}_A^* \cdot \bar{E}_B) dv \right] \end{aligned} \quad (8.7.7)$$

where (8.7.7) is still an exact equation. We now make the observation that since $\mu_B \simeq \mu_A$, $\epsilon_B \simeq \epsilon_A$, $\bar{E}_B \simeq \bar{E}_A$, and $\bar{H}_B \simeq \bar{H}_A$, the denominator of the right side of

(8.7.7) may be written:

$$\begin{aligned} \int_{V_A} (\mu_B \bar{H}_A^* \cdot \bar{H}_B + \epsilon_B \bar{E}_A^* \cdot \bar{E}_B) dv &\simeq 4 \int_{V_A} \left(\frac{1}{4} \mu_A |\bar{H}_A|^2 + \frac{1}{4} \epsilon_A |\bar{E}_A|^2 \right) dv \\ &\simeq 4 [\langle w_m \rangle + \langle w_e \rangle] \\ &\simeq 4w_T \end{aligned} \quad (8.7.8)$$

where (8.2.19) and (8.2.20) have been substituted. Note that we cannot approximate $\epsilon_B \simeq \epsilon_A$ and $\mu_B \simeq \mu_A$ in the numerator of (8.7.7) or two terms will completely vanish. Combining (8.7.7) and (8.7.8) yields the approximate difference between the perturbed and unperturbed frequencies:

$$\begin{aligned} s - j\omega_0 &\simeq -\frac{1}{4w_T} \left[\oint_{A_A} (\bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A da + \int_{V_A} (\bar{J}_B \cdot \bar{E}_A^*) dv \right. \\ &\quad \left. + j\omega_0(\mu_A - \mu_B) \int_{V_A} (\bar{H}_A^* \cdot \bar{H}_B) dv + j\omega_0(\epsilon_A - \epsilon_B) \int_{V_A} (\bar{E}_A^* \cdot \bar{E}_B) dv \right] \end{aligned} \quad (8.7.9)$$

The first two terms on the right side of (8.7.9) are proportional to the radiated and dissipated (real) power, respectively, so we see that

$$\text{Re}\{s - j\omega_0\} = \text{Re}\{j\omega' - \alpha - j\omega_0\} = -\alpha = -\frac{1}{2w_T} \text{Re}\{P\} \quad (8.7.10)$$

As discussed in the beginning of this section, $\alpha_n \simeq P_n/2w_{Tn}$, which is certainly consistent with (8.7.10). The second two terms on the right side of (8.7.9) correspond to changes in reactive or stored (imaginary) power and thus reflect a shift in the actual resonant frequency $\text{Im}\{s - j\omega_0\} = \omega' - \omega_0$. Because $\epsilon_B \simeq \epsilon_A$ and $\mu_B \simeq \mu_A$, the shift in frequency is generally small, as it must be if the perturbation approximation is to be valid. Equation (8.7.9) is called the *resonator perturbation equation*.

We now apply (8.7.9) to the resonator of Figure 8.16(a), which has a thin, flat dielectric ϵ_B introduced at the conducting wall of a lossless resonator with fields $\bar{E}_A$, $\bar{H}_A$ such that the surface of the dielectric is approximately perpendicular to $\bar{E}_A$. At the dielectric boundary, the normal displacement field $\bar{D}$ must be continuous and thus $\epsilon_B \bar{E}_B \simeq \epsilon_A \bar{E}_A$ inside the dielectric. We assume that the field inside the dielectric is almost all normally directed because the dielectric layer is so thin and flat. The surface integral $\oint_{A_A} (\bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A da$ therefore vanishes because $\bar{E}_B$ is parallel to $\bar{E}_A$ and $\hat{n}_A \times \bar{E}_A = 0$ at the surface of the perfectly conducting resonator. Likewise, $\int_{V_A} (\bar{J}_B \cdot \bar{E}_A^*) dv = 0$ because no current $\bar{J}_B$ is produced by the dielectric. Because $\mu_B = \mu_A$ and $\epsilon_B = \epsilon_A$ everywhere except where the dielectric is located, the only nonvanishing term in (8.7.9) is

$$s - j\omega_0 \simeq \frac{j\omega_0}{4w_T} (\epsilon_A - \epsilon_B) \int_{V_A} (\bar{E}_A^* \cdot \bar{E}_B) dv$$

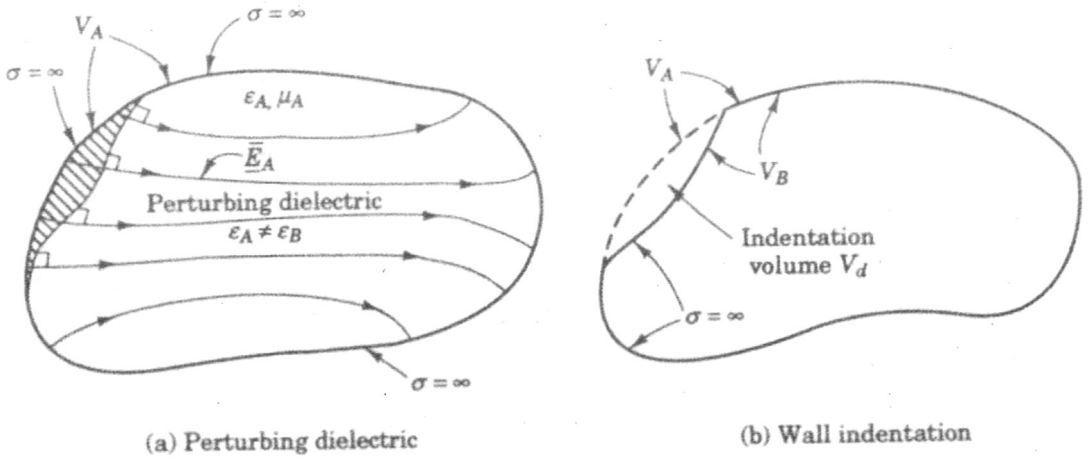


Figure 8.16 Gentle perturbations of cavity resonators.

$$\simeq \frac{j\omega_0}{4w_T} (\epsilon_A - \epsilon_B) \frac{\epsilon_A}{\epsilon_B} \int_{V_D} |\bar{E}_A|^2 dv \quad (8.7.11)$$

where V_D is the volume of the dielectric. Because (8.7.11) is purely imaginary, only the resonant frequency of the cavity shifts. If $\epsilon_B > \epsilon_A$, then the resonant frequency of the cavity is reduced slightly below ω_0 without affecting the rate of decay of energy in the resonator.

If the cavity wall were instead indented, removing the volume V_D as shown in Figure 8.16(b), this perturbation could be modeled by allowing the dielectric constant ϵ_B in Figure 8.16(a) to approach infinity so that $\bar{E}_B \rightarrow 0$, and (8.7.11) would become

$$s - j\omega_0 \simeq -\frac{j\omega_0\epsilon_A}{4w_T} \int_{V_D} |\bar{E}_A|^2 dv \quad (8.7.12)$$

But this frequency shift is not exactly correct, because μ_B must accurately model the "perfect conductor" in which $\bar{B}_B = 0$. Since $\bar{H}_B \simeq \bar{H}_A$ as a result of tangential magnetic field continuity at the dielectric surface, μ_B must equal zero so that $\bar{B}_B = \mu_B \bar{H}_B = 0$. The resonator perturbation equation (8.7.9) thus includes an additional term that vanished when $\mu_B \simeq \mu_A$ for the dielectric perturbation, and the frequency shift is more accurately represented by

$$s - j\omega_0 \simeq \frac{j\omega_0}{w_T} \int_{V_D} \left(\frac{1}{4} \mu_A |\bar{H}_A|^2 - \frac{1}{4} \epsilon_A |\bar{E}_A|^2 \right) dv \quad (8.7.13)$$

or

$$\frac{\Delta\omega}{\omega_0} = \frac{\langle \Delta w_m \rangle - \langle \Delta w_e \rangle}{w_T} \quad (8.7.14)$$

where the only contributions to (8.7.13) and (8.7.14) are from the last two terms of (8.7.9). If the cavity indentation occurs at a place where the unperturbed stored electric energy is larger than the unperturbed stored magnetic energy, then the

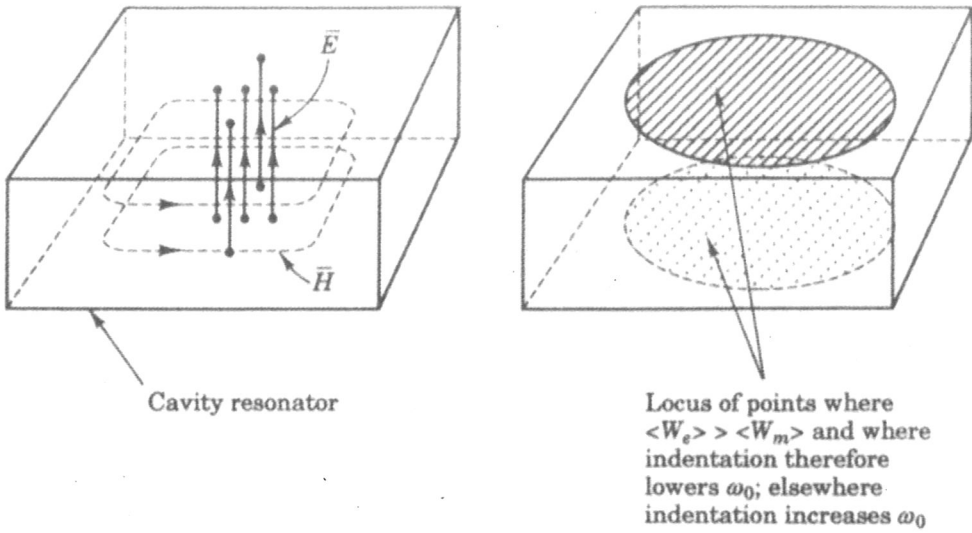


Figure 8.17 Cavity perturbation by indentation.

resonant frequency is reduced, and vice versa. Negative indentations (outward protrusions of the cavity) of course have the opposite effect. Figure 8.17 shows the locus of points where indentation would lower ω_0 for the TE_{101} mode.

The reduction of the cavity resonant frequency caused by indenting the top and bottom walls can also be understood by referring to Figure 8.3, where it is clear that such indentation would increase the capacitance C by decreasing the space between the top and bottom capacitor plates in the equivalent LC resonator, thereby reducing $\omega_0 = 1/\sqrt{LC}$.

Yet another way to understand cavity indentation is to assume that the cavity contains a fixed number of photons n each having energy $w = h\omega_0/2\pi$, where h is Planck's constant ($h = 6.625 \times 10^{-34}$ J · s) so that the total energy $w_T n$ in the cavity is $nw = nh\omega_0/2\pi$. The electric field lines of the TE_{101} mode pull at the top and bottom walls of the cavity with an average force density $\frac{1}{4}\epsilon |\bar{E}|^2$ (N/m²), and if the cavity yields slightly, the photons do work and collectively lose energy Δw_T . Since the number of photons is conserved for processes slow compared to ω_0^{-1} , ω_0 is necessarily reduced. The energy shift Δw_T and the frequency shift $\Delta\omega$ are therefore directly proportional:

$$\frac{nh\Delta\omega}{2\pi} = \Delta w_T$$

or

$$\Delta\omega = \frac{2\pi \Delta w_T}{nh} = \frac{\omega_0 \Delta w_T}{w_T} \quad (8.7.15)$$

The energy shift Δw_T also depends on the average outwardly directed magnetic forces, $\frac{1}{4}\mu |\bar{H}|^2$ N/m². At places where this force density exceeds the inward

force density of electric origin, indentation would increase ω_0 , and at places where these two forces balance, such indentation would have little effect.

Example 8.7.1

A cavity resonator has slightly lossy walls of conductivity σ . Find a simple expression for the complex resonant frequency s for this resonator, where ω_0 would be the resonant frequency if $\sigma = \infty$.

Solution: We consider V_A in (8.7.9) to exclude the walls where $\bar{J}_B$ is nonzero, and thus only the $(\bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A$ term requires evaluation. The electric field $\bar{E}_B$ parallel to the wall is given by

$$\bar{E}_B \cong \eta_w \hat{n}_A \times \bar{H}_A$$

where $\hat{n}_A$ is the surface normal vector and the wall impedance follows from (3.4.15)

$$\eta_w \cong \sqrt{\frac{\omega_0 \mu}{2\sigma}} (1 + j)$$

with $\sigma \gg \omega_0 \mu / 2$ required for the perturbation method to be useful.

Therefore, (8.7.9) becomes

$$\begin{aligned} s - j\omega_0 &\cong -\frac{1}{4w_T} \oint_{A_A} (\bar{E}_B \times \bar{H}_A^*) \cdot \hat{n}_A da \\ &\cong -\frac{1}{4w_T} \sqrt{\frac{\omega_0 \mu}{2\sigma}} (1 + j) \oint_{A_A} |\bar{H}_A|^2 da \end{aligned}$$

where A_A is the area of the inside surface of the resonator. The real and imaginary parts are both perturbed equally by the highly conducting but lossy walls, and the resonant frequency is reduced.

Example 8.7.2

A small dielectric sphere having radius R much smaller than a wavelength and permittivity ϵ_B is introduced into a cavity resonant at frequency ω_0 . What is the percentage change in the resonant frequency as a result of the perturbation? How does the frequency change if the sphere is perfectly conducting as opposed to perfectly insulating?

Solution: Without the sphere, the cavity has an electric field $\bar{E}_A$ assumed to be roughly uniform over the small region where the sphere will be introduced. Because the sphere is much smaller than a wavelength in radius, we can find the nearfields of the sphere quasistatically, i.e., by solving Laplace's equation rather than the more complicated wave equation. The electric potential is thus

$$\begin{aligned} \Phi_B &= -E_A r \cos \theta + \frac{C_1 R^3}{r^2} \cos \theta & r > R \\ \Phi_B &= C_2 r \cos \theta & r < R \end{aligned} \quad (1)$$

where Φ_B satisfies $\nabla^2 \Phi_B = 0$ in spherical coordinates. Far away from the sphere, $\Phi_B \rightarrow -E_A r \cos \theta = -E_A z$, which is the potential of the uniform field $\bar{E}_A = \hat{z} E_A$ without the sphere. (We assume that the sphere is sufficiently small that the electric field is approximately uniform in its vicinity and orient the $\hat{z}$ -axis in the direction of the field.) The constant coefficients C_1 and C_2 are found by enforcing continuity of tangential $\bar{E}$ and normal

$\bar{D}$ at $r = R$

$$\begin{aligned} -\frac{1}{r} \frac{\partial \Phi_B(r > R)}{\partial \theta} \bigg|_{r=R} &= -\frac{1}{r} \frac{\partial \Phi_B(r < R)}{\partial \theta} \bigg|_{r=R} \rightarrow -E_A + C_1 = C_2 \\ -\epsilon_A \frac{\partial \Phi_B(r > R)}{\partial r} \bigg|_{r=R} &= -\epsilon_B \frac{\partial \Phi_B(r < R)}{\partial r} \bigg|_{r=R} \rightarrow \epsilon_A(E_A + 2C_1) = -\epsilon_B C_2 \end{aligned}$$

which leads to $C_1 = -E_A(\epsilon_A - \epsilon_B)/(2\epsilon_A + \epsilon_B)$ and $C_2 = -3\epsilon_A E_A/(2\epsilon_A + \epsilon_B)$. If we neglect the $1/r^2$ field outside the sphere, then

$$\begin{aligned} \bar{E}_B &\approx \frac{3\epsilon_A}{2\epsilon_A + \epsilon_B} \bar{E}_A, & r < R \\ \bar{E}_B &\approx \bar{E}_A, & r > R \end{aligned}$$

We also neglect contributions to $\Delta\omega$ from changes in magnetic energy. The resonator perturbation equation (8.7.9) thus becomes

$$\begin{aligned} s - j\omega &\simeq \frac{j\omega_0}{4w_T} (\epsilon_A - \epsilon_B) \int_{V_{\text{sphere}}} (\bar{E}_A^* \cdot \bar{E}_B) dv \\ &= \frac{j\omega}{4w_T} \frac{3\epsilon_A(\epsilon_A - \epsilon_B)}{2\epsilon_A + \epsilon_B} |\bar{E}_A(\bar{r})|^2 \frac{4\pi R^3}{3} \end{aligned}$$

where $\bar{r}$ is the coordinate where the sphere is located. Since $\text{Im}\{s - j\omega\} = \omega' - \omega_0 = \Delta\omega$, we can also write the frequency shift as

$$\frac{\Delta\omega}{\omega_0} \simeq -\frac{\epsilon_A(\epsilon_B - \epsilon_A)}{2\epsilon_A + \epsilon_B} \frac{\pi R^3 |\bar{E}_A(\bar{r})|^2}{w_T}$$

By monitoring the frequency change $\Delta\omega$ caused by a spherical dielectric probe as a function of $\bar{r}$, it is possible to experimentally determine the approximate intensity distribution of the electric field associated with any given resonant frequency ω_0 .

If the sphere is perfectly conducting, $\epsilon_B \rightarrow \infty$, and the frequency shift becomes

$$\frac{\Delta\omega}{\omega_0} \bigg|_{\epsilon_B \rightarrow \infty} \simeq -\frac{\epsilon_A \pi R^3 |\bar{E}_A(\bar{r})|^2}{w_T} = -3 \frac{\langle \Delta w_e \rangle}{w_T}$$

where $\langle \Delta w_e \rangle = \frac{1}{4} \epsilon_A |\bar{E}_A|^2 \cdot \frac{4}{3} \pi R^3$. If the sphere is insulating, $\epsilon_B = 0$ and

$$\frac{\Delta\omega}{\omega_0} \bigg|_{\epsilon_B = 0} \simeq +\frac{3}{2} \frac{\langle \Delta w_e \rangle}{w_T}$$

When the sphere has a dielectric constant greater than that of the empty cavity, it increases the overall cavity capacitance and thereby lowers the resonant frequency ($\Delta\omega < 0$). Conversely, the insulating probe lowers the capacitance of the cavity so the resonant frequency increases and $\Delta\omega$ is positive. Note that a dielectric sphere would be pulled into the high-field regions of a cavity because the system energy is then less and because of forces on the dielectric surface charges. System energy reductions also correspond to lowered resonant frequencies.

Example 8.7.3

A rectangular cavity resonant in the TE_{101} mode has a metal tuning screw of volume V_s located at the position of maximum field strength $\bar{E}_M$, as illustrated in Figure 8.18.

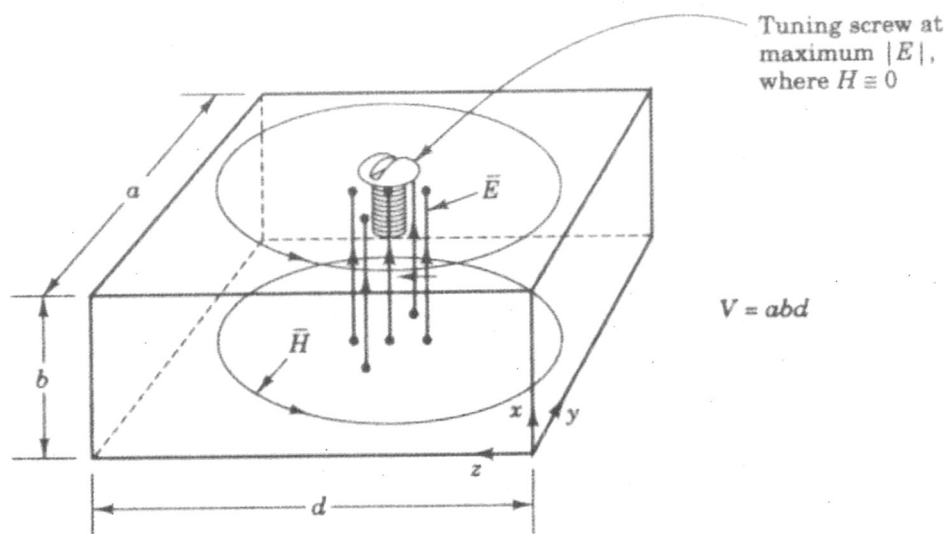


Figure 8.18 Spherical probe of a cavity resonator in Example 8.7.2.

This screw lowers the unperturbed resonant frequency f_0 by Δf . Approximately what is $\Delta f/f_0$?

Solution: If this screw is perfectly conducting ($\epsilon_B \rightarrow \infty$), then we may approximate it by a hemispherical protrusion of radius R and volume $V_s = 2\pi R^3/3$. The quasistatic approach to finding the fields near the screw is developed in Example 8.7.2, where we showed that

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} \cong -\frac{3\langle\Delta w_e\rangle}{w_T}$$

and $\langle\Delta w_e\rangle \cong \frac{1}{4}\epsilon_A |\bar{E}_M|^2 V_s$. From (8.2.24), we find the average electric and magnetic fields in the cavity to be

$$w_T = \langle w_e \rangle + \langle w_m \rangle = \frac{1}{8}\epsilon_A |\bar{E}_M|^2 V$$

where $\bar{E}_M = \hat{y}(k_x/k_0)E_0$ is the maximum field strength of the TE_{101} mode and V is the resonator volume. Therefore,

$$\frac{\Delta f}{f_0} \cong -\frac{3\left(\frac{1}{4}\epsilon_A |\bar{E}_M|^2 V_s\right)}{\frac{1}{8}\epsilon_A |\bar{E}_M|^2 V} \cong -\frac{6V_s}{V}$$

For a perturbation approach to be successful, $V_s \ll V$, which in turn implies that the frequency shift $\Delta f/f_0$ is small.

8.8* RESONATOR COUPLING TO EXTERNAL CIRCUITS

The energy storage and frequency-sensitive properties of resonators make them very useful in a wide variety of applications, requiring that they be coupled to the external world. In this section, we shall derive the input characteristics of an arbitrary resonator connected to a lossless TEM line. At special places on this line, the

resonator will look just like a series or parallel RLC resonator circuit. Of course, the resonator must be operating near one of its resonances for a simple equivalent circuit to be a useful model, and each resonance will be characterized by a different equivalent circuit. From a general description of resonator impedance that will rely heavily on the energy theorem, we shall specialize to a TEM -line resonator and will make use of the Smith chart to describe its behavior near resonance. Section 8.9 will discuss specific examples and applications of resonator circuits.

We consider an arbitrary high- Q resonator connected via a lossless TEM line to the outside world, as shown in Figure 8.19. The energy theorem, presented in Section 1.7, is a useful and general way to understand the frequency behavior of this resonator near a specific resonance.¹¹ We recall that

$$\frac{1}{4} \oint_{A'} \left(\frac{\partial \bar{E}}{\partial \omega} \times \bar{H}^* + \bar{E}^* \times \frac{\partial \bar{H}}{\partial \omega} \right) \cdot \hat{n} da = -j w_T$$

$$- \frac{1}{4} \int_V \left(\frac{\partial \bar{E}}{\partial \omega} \cdot \bar{J}^* + \bar{E}^* \cdot \frac{\partial \bar{J}}{\partial \omega} \right) dv$$
(8.8.1)

where w_T is the total energy stored in the resonator. The surface integral A' surrounds the resonator of volume V and cuts across its TEM feed line as shown in

11. The following algebra may be avoided, if desired, by moving directly to the final equations for the impedance of an arbitrary resonator in (8.8.10) and (8.8.11).

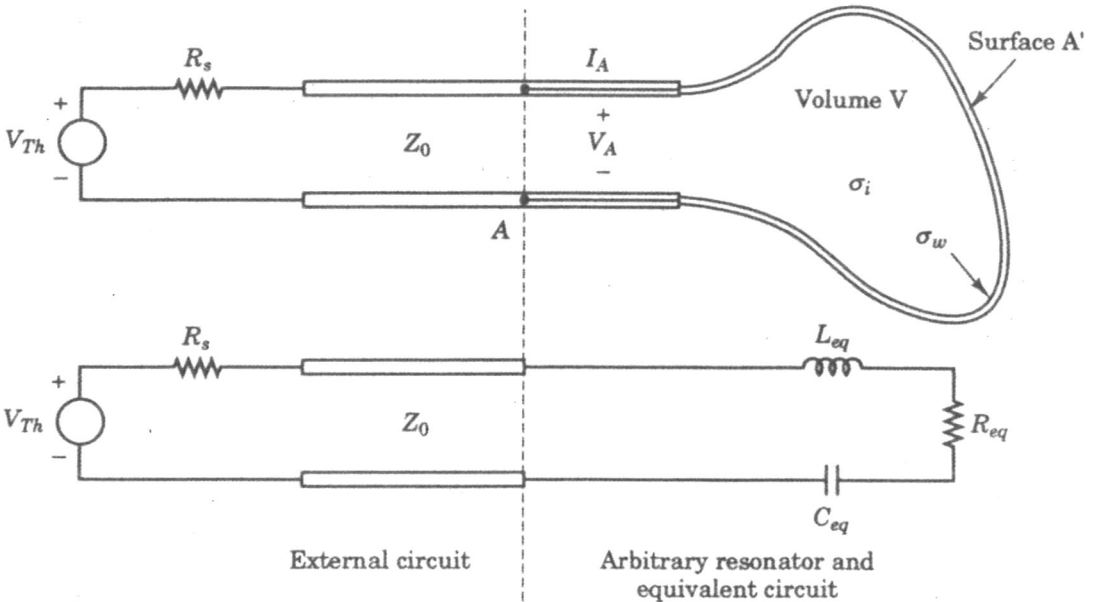


Figure 8.19 Coupling from a TEM line to an arbitrary resonator.

Figure 8.19. The point on the *TEM* line where this happens is arbitrary and indicated by the subscript *A*. The resonator may contain a slightly lossy dielectric with conductivity σ_i ($\sigma_i/\omega\epsilon \ll 1$) and/or have slightly lossy walls with conductivity σ_w ($\sigma_w/\omega\epsilon \gg 1$), but the *TEM* line feed is assumed to be lossless.

Because the walls of the resonator are almost perfectly conducting, the electric field is approximately normal to these walls so $\partial \bar{E}/\partial \omega \times \bar{H}^*$ and $\bar{E}^* \times \partial \bar{H}/\partial \omega$ are both parallel to the resonator surface. Therefore, the surface integral in (8.8.1) vanishes **except** where the surface *A'* crosses the feed line at *A*. Here, $\partial \bar{E}/\partial \omega \times \bar{H}^*$ and $\bar{E}^* \times \partial \bar{H}/\partial \omega$ point **along** the transmission line axis since $\bar{E}$ and $\bar{H}$ lie in planes perpendicular to this axis. (See the discussion in Section 5.2 and 5.3.) We have suggested in Section 5.3 that $\int_{A'} (\bar{E} \times \bar{H}^*) \cdot \hat{n} da$ is the total power VI^* (W) on the *TEM* line. A similar argument gives the following expression for the left side of (8.8.1) in terms of the voltage and current at *A*¹²

$$\frac{1}{4} \oint_{A'} \left(\frac{\partial \bar{E}}{\partial \omega} \times \bar{H}^* + \bar{E}^* \times \frac{\partial \bar{H}}{\partial \omega} \right) \cdot \hat{n} da = -\frac{1}{4} \left(\frac{\partial V_A}{\partial \omega} I_A^* + V_A^* \frac{\partial I_A}{\partial \omega} \right) \quad (8.8.2)$$

The negative sign arises because the surface normal points in the opposite direction to the power flowing into the resonator.

The second term on the right side of (8.8.1) is nonzero only in the lossy regions of the resonator because $\bar{E}$ and $\bar{J}$ are never simultaneously nonzero in a lossless resonator or *TEM* line. As usual, we try a perturbation approach to find this term. Inside the resonator, the conduction current $\bar{J}$ is approximately $\sigma_i \bar{E}^{(0)}$, while within the resonator walls of volume V_w , the electric field $\bar{E}$ is nearly $\bar{J}^{(0)}/\sigma_i$. Dropping the superscript (0) hereafter, we find

12. The surface integral in (8.8.2) consists of two orthogonal line integrals proportional to current and voltage:

$$\begin{aligned} & \frac{1}{4} \oint_{A'} \left(\frac{\partial \bar{E}}{\partial \omega} \times \bar{H}^* + \bar{E}^* \times \frac{\partial \bar{H}}{\partial \omega} \right) \cdot \hat{n} da \\ &= \frac{1}{4} \int_C \left(\frac{\partial \bar{E}}{\partial \omega} \cdot d\bar{\ell} \right) \oint_{C'} (\bar{H}^* \cdot d\bar{s}) - \frac{1}{4} \int_C (\bar{E}^* \cdot d\bar{\ell}) \oint_{C'} \left(\frac{\partial \bar{H}^*}{\partial \omega} \cdot d\bar{s} \right) \\ &= -\frac{1}{4} \frac{\partial V_A}{\partial \omega} I_A^* + \frac{1}{4} V_A^* \frac{\partial I_A}{\partial \omega} \end{aligned}$$

where $da = d\ell ds$.

$$\begin{aligned}
\frac{1}{4} \int_V \left(\frac{\partial \bar{E}}{\partial \omega} \cdot \bar{J}^* + \bar{E}^* \cdot \frac{\partial \bar{J}}{\partial \omega} \right) dv &= \frac{1}{4} \int_V \left(\sigma_i^* \frac{\partial \bar{E}}{\partial \omega} \cdot \bar{E}^* + \sigma_i \bar{E}^* \cdot \frac{\partial \bar{E}}{\partial \omega} \right) dv \\
&\quad + \frac{1}{4} \int_{V_w} \left(\frac{1}{\sigma_w} \frac{\partial \bar{J}}{\partial \omega} \cdot \bar{J}^* + \frac{1}{\sigma_w^*} \bar{J}^* \cdot \frac{\partial \bar{J}}{\partial \omega} \right) dv \\
&= \frac{\sigma_i}{2} \int_V \left(\bar{E}^* \cdot \frac{\partial \bar{E}}{\partial \omega} \right) dv + \frac{1}{2\sigma_w} \int_{V_w} \left(\bar{J}^* \cdot \frac{\partial \bar{J}}{\partial \omega} \right) dv
\end{aligned} \tag{8.8.3}$$

The conductivities σ_i and σ_w are assumed to be real.

Because Maxwell's equations are linear, V_A and I_A are both proportional to $\bar{E}$, $\bar{H}$, and $\bar{J}$ inside the resonator where the various proportionality constants are frequency-dependent.¹³ Therefore, we can relate the volume integrals of $\bar{E}^* \cdot \partial \bar{E} / \partial \omega$ and $\bar{J}^* \cdot \partial \bar{J} / \partial \omega$ in (8.8.3) to the voltage and current at A :

$$\frac{1}{2} \sigma_i \int_V \left(\bar{E}^* \cdot \frac{\partial \bar{E}}{\partial \omega} \right) dv \equiv \frac{1}{2} G(\omega) V_A^* \frac{\partial V_A}{\partial \omega} \tag{8.8.4}$$

$$\frac{1}{2\sigma_w} \int_{V_w} \left(\bar{J}^* \cdot \frac{\partial \bar{J}}{\partial \omega} \right) dv \equiv \frac{1}{2} R(\omega) I_A^* \frac{\partial I_A}{\partial \omega} \tag{8.8.5}$$

where $G(\omega) \propto \sigma_i$ and $R(\omega) \propto 1/\sigma_w$. These relations (8.8.4) and (8.8.5) **define** the values of $R(\omega)$ and $G(\omega)$, which may be complex. If we now substitute (8.8.2)–(8.8.5) into the energy theorem (8.8.1), we are left with

$$\frac{1}{4} \left(\frac{\partial V_A}{\partial \omega} I_A^* + V_A^* \frac{\partial I_A}{\partial \omega} \right) = j\omega_T + \frac{1}{2} R I_A^* \frac{\partial I_A}{\partial \omega} + \frac{1}{2} G V_A^* \frac{\partial V_A}{\partial \omega} \tag{8.8.6}$$

We define the impedance looking into the resonator at A to be $Z_A \equiv V_A / I_A$, which means that we can rewrite the left side of (8.8.6) as

$$\begin{aligned}
\frac{1}{4} \left(\frac{\partial Z_A}{\partial \omega} |I_A|^2 + Z_A I_A^* \frac{\partial I_A}{\partial \omega} + Z_A^* I_A^* \frac{\partial I_A}{\partial \omega} \right) \\
= \frac{1}{4} \frac{\partial Z_A}{\partial \omega} |I_A|^2 + \frac{1}{2} R_A I_A^* \frac{\partial I_A}{\partial \omega}
\end{aligned} \tag{8.8.7}$$

where Z_A has a resistive part R_A and a reactive part X_A ($Z_A = R_A + jX_A$) and so $Z_A + Z_A^* = 2R_A$. We now substitute (8.8.7) into (8.8.6) and separate the resulting

13. We exclude the singular case of an ideal lossless resonator where V_A or I_A is zero.

equation into its real and imaginary parts:

$$\begin{aligned} \text{Re: } \frac{1}{4}(\partial R_A/\partial\omega) |I_A|^2 &= \frac{1}{2} \text{Re} \{ (R - R_A) I_A^* \partial I_A/\partial\omega \} + \\ &\quad \frac{1}{2} \text{Re} \{ G V_A^* \partial V_A/\partial\omega \} \\ \text{Im: } \frac{1}{4} \partial X_A/\partial\omega |I_A|^2 &= w_T + \frac{1}{2} \text{Im} \{ (R - R_A) I_A^* \partial I_A/\partial\omega \} + \\ &\quad \frac{1}{2} \text{Im} \{ G V_A^* \partial V_A/\partial\omega \} \end{aligned} \quad (8.8.8)$$

Up until now, the location A on the TEM feed line has been somewhat arbitrary, but we now choose A such that the real part of (8.8.8) vanishes near resonance. This implies that $\partial R_A/\partial\omega$ is zero and

$$R_A(\omega_0) = \frac{\text{Re} \{ G V_A^* (\partial V_A/\partial\omega) \} + \text{Re} \{ R I_A^* (\partial I_A/\partial\omega) \}}{\text{Re} \{ I_A^* (\partial I_A/\partial\omega) \}} \bigg|_{\omega=\omega_0} \quad (8.8.9)$$

This choice of terminal location substantially simplifies the expressions for R_A and X_A near resonance, as we shall see. Note that if $\sigma_i = G = 0$, then $R_A(\omega_0) = R$.¹⁴

Near resonance, the second and third terms on the right side of (8.8.8) can be neglected because the total energy stored in the resonator must be much larger than terms proportional to dissipated power if the resonator is to have a large Q .¹⁵ Equation (8.8.8) thus simplifies considerably, yielding equations for both the real and imaginary parts of the resonator impedance $Z_A = R_A + jX_A$:

$$\frac{\partial R_A}{\partial\omega} \bigg|_{\omega \approx \omega_0} = 0 \quad (8.8.10)$$

$$\frac{\partial X_A}{\partial\omega} \bigg|_{\omega \approx \omega_0} = \frac{4w_T}{|I_A|^2} \quad (8.8.11)$$

Equations (8.8.10) and (8.8.11) are **only** true near resonance and at the location A where (8.8.9) is valid.

At a resonance ω_0 , the impedance $R_A + jX_A$ is purely resistive, since maximum power is dissipated in the resonator at resonance, and we find

$$\begin{aligned} R_A(\omega_0) &= R_{eq} \\ X_A(\omega_0) &= 0 \end{aligned} \quad (8.8.12)$$

14. It is almost always possible to find a location on the TEM line where (8.8.9) is satisfied, as we see by appealing to a Smith chart. A useful resonator cannot dissipate much power if it is to have a sharp frequency response near resonance, so its reflection coefficient off-resonance must be close to unity; most of the power fed to it becomes trapped. The off-resonance input impedance directly at the entrance to the resonator must therefore lie very near the $|\Gamma| = 1$ circle on the Smith chart. This circle with $|\Gamma| \approx 1$ intersects almost every circle of constant real part near the point $\Gamma = 1$. Only those small circles near $\Gamma = 1$ that have resistances greater than $\text{Re} \{ (1 + \Gamma)/(1 - \Gamma) \}$ cannot be reached at some point on the TEM feed line (by rotating about the origin of the Smith chart with constant radius $|\Gamma| \approx 1$). If (8.8.9) can not be satisfied at some position A , then we generally do not have a useful resonator.

15. Note that if $G = 0$ so that $R_A = R$, these terms vanish automatically.

The equivalent resistance R_{eq} of the resonator is just $R_A(\omega_0)$, given by (8.8.9). Combining (8.8.10)–(8.8.12) thus yields a very general expression for the impedance Z_A near resonance

$$Z_A(\omega \approx \omega_0) = R_{eq} + j \frac{4\langle w_T \rangle}{|I_{A0}|^2} (\omega - \omega_0) \quad (8.8.13)$$

where I_{A0} is the current at A where $\omega \approx \omega_0$ and $\langle w_T \rangle$ is the total stored energy at resonance. Equation (8.8.13) is the input impedance of a series resonator circuit!¹⁶

Because the real and imaginary parts of $Z_A(\omega \approx \omega_0)$ are equal at the half-power frequencies $\omega = \omega_0 \pm \alpha_0$,¹⁷ we substitute $\omega - \omega_0 = \alpha_0$ into (8.8.13), yielding

$$R_{eq} = \frac{4\langle w_T \rangle}{|I_{A0}|^2} \alpha_0 \quad (8.8.14)$$

which means that the internal quality factor of this resonator is

$$Q_I = \frac{\omega_0 \langle w_T \rangle}{P_I} = \frac{\omega_0 \langle w_T \rangle}{\frac{1}{2} R_{eq} |I_{A0}|^2} = \frac{\omega_0}{2\alpha_0} = \frac{\omega_0}{\Delta\omega} \quad (8.8.15)$$

We have thus demonstrated the connection between Q_I , ω_0 , and $\alpha_0 = \Delta\omega/2$ for an **arbitrary** resonator.

It is customary to express the reactance $X(\omega)$ in (8.8.13) as the sum of the reactances of an equivalent capacitor C_{eq} and inductor L_{eq} , so

$$jX(\omega) = j \frac{4\langle w_T \rangle}{|I_{A0}|^2} (\omega - \omega_0) \equiv j\omega L_{eq} + \frac{1}{j\omega C_{eq}} \quad (8.8.16)$$

16. The impedance of an RLC resonator is

$$Z(\omega) = R + j\omega L + \frac{1}{j\omega C}$$

which can be represented near resonance ($\omega = \omega_0 + \delta\omega$) as

$$\begin{aligned} Z(\omega_0 + \delta\omega) &= R + j(\omega_0 + \delta\omega)L + \frac{1}{j(\omega_0 + \delta\omega)C} \\ &\approx R + j\omega_0 L - \frac{1}{j\omega_0 C} + j\delta\omega L + \frac{j}{\omega_0^2 C} \delta\omega \\ &= R + 2jL\delta\omega \end{aligned}$$

where $\omega_0 = 1/\sqrt{LC}$. Since $4\omega_0 \langle w_T \rangle / |I_{A0}|^2 = 2L$, we see that (8.8.13) also characterizes this discrete-element series resonator.

17. If the real and imaginary parts of Z_A are equal, we see from (8.8.13) that $Z_A \approx R_{eq}(1 + j)$, which means that the power dissipated in the resonator is $V_S^2/4R_{eq}$ in contrast to the power dissipation at resonance ($V_S^2/2R_{eq}$) when $Z_A = R_{eq}$. Therefore, we are justified in calling the frequency for which $Z_A = R_{eq}(1 + j)$ the half-power frequency.

Because the reactance must be zero at $\omega = \omega_0$ from (8.8.13), we see at once that

$$\omega_0 = \frac{1}{\sqrt{L_{eq}C_{eq}}} \quad (8.8.17)$$

If we take the derivative of both sides of (8.8.16) and evaluate the result at resonance, we find that

$$\left. \frac{\partial X(\omega)}{\partial \omega} \right|_{\omega=\omega_0} = \frac{4\langle w_T \rangle}{|I_{A0}|^2} = L_{eq} + \frac{1}{\omega_0^2 C_{eq}} = 2L_{eq} \quad (8.8.18)$$

so we finally arrive at

$$\begin{aligned} L_{eq} &= \frac{2\langle w_T \rangle}{|I_{A0}|^2} = \frac{R_{eq}Q_l}{\omega_0} \\ C_{eq} &= \frac{1}{\omega_0^2 L_{eq}} = \frac{|I_{A0}|^2}{2\omega_0^2 \langle w_T \rangle} = \frac{1}{\omega_0 R_{eq}Q_l} \end{aligned} \quad (8.8.19)$$

for the values of the three equivalent series circuit elements.

We have therefore shown that **any resonator structure can be replaced by an equivalent series RLC circuit as seen from a special location on the external TEM line where (8.8.9) is valid.** This location is called the *series resonance plane*. If we move a distance $\lambda/4$ away from this plane, we rotate halfway around the Smith chart so impedances become admittances and vice versa. The equivalent series RLC circuit thus becomes an equivalent parallel GLC circuit with $G_{eq} = 1/R_{eq}$ and the same values for C_{eq} and L_{eq}

$$Y_A = G_{eq} + j\omega C_{eq} + \frac{1}{j\omega L_{eq}} \quad (8.8.20)$$

where (8.8.20) is valid only at the *parallel resonance plane* for ω_0 near resonance. We can check that (8.8.20) is correct by multiplying $Z_A Y_A$; the result should equal unity. The series and parallel resonance planes therefore alternate and are spaced $\lambda/4$ apart. It is of course possible to define a frequency-dependent impedance from **any** location on the TEM feed line, but only the series and parallel resonance planes have simple interpretations.

When analyzing or designing cavity, TEM, or other resonators, it is often more convenient not to evaluate L_{eq} or C_{eq} but to work directly in terms of Q_l and Q_E . Consider the impedance $Z_A(\omega)$ near resonance for the arbitrary resonator characterized by (8.8.13), where we now assume it is coupled to a matched (Thevenin or Norton) source of impedance Z_0 . The normalized impedance of the resonator $Z_{An}(\omega)$ is then

$$Z_{An}(\omega) = \frac{R_{eq}}{Z_0} + j \frac{4w_T}{Z_0 |I_{A0}|^2} (\omega - \omega_0) \quad (8.8.21)$$

But $R_{eq}/Z_0 = Q_E/E_l$ from (8.4.22), and we have shown in (8.8.14) that $4w_T/$

$|I_{A0}|^2 = R_{eq}/\alpha_0 = 2R_{eq}Q_I/\omega_0$. Therefore, we can rewrite (8.8.21) as

$$Z_{An} = \frac{Q_E}{Q_I} + 2jQ_E \frac{(\omega - \omega_0)}{\omega_0} \quad (8.8.22)$$

which is simply a function of Q_I , Q_E , ω , and ω_0 . Note that $R_{An} = 1$ if $Q_I = Q_E$ and the resonator is matched or critically coupled.

A similar expression for the normalized admittance of a parallel resonance is found by duality, where $R_{eq} \rightarrow G_{eq}$, $Z \rightarrow Y$ and $I \rightarrow V$:

$$Y_{An}(\omega) = \frac{G_{eq}}{Y_0} + j \frac{4W_T}{Y_0|V_{A0}|^2}(\omega - \omega_0) \quad (8.8.23)$$

We then rewrite this admittance in terms of the internal and external Q s in (8.4.23), giving

$$Y_{An} = \frac{Q_E}{Q_I} + 2jQ_E \frac{(\omega - \omega_0)}{\omega_0} \quad (8.8.24)$$

which is identical in form to (8.8.22). Again $G_{An} = 1$ if $Q_I = Q_E$.

Example 8.8.1

A resonator consists of an air-filled TEM line of length ℓ and characteristic impedance Z_0 terminated by a very small resistance $R_L \ll Z_0$. This TEM line is connected in series to a reactance jX , which is in turn connected in series to an external TEM line having Thevenin impedance Z_0 as illustrated in Figure 8.20. If this resonator is to be critically

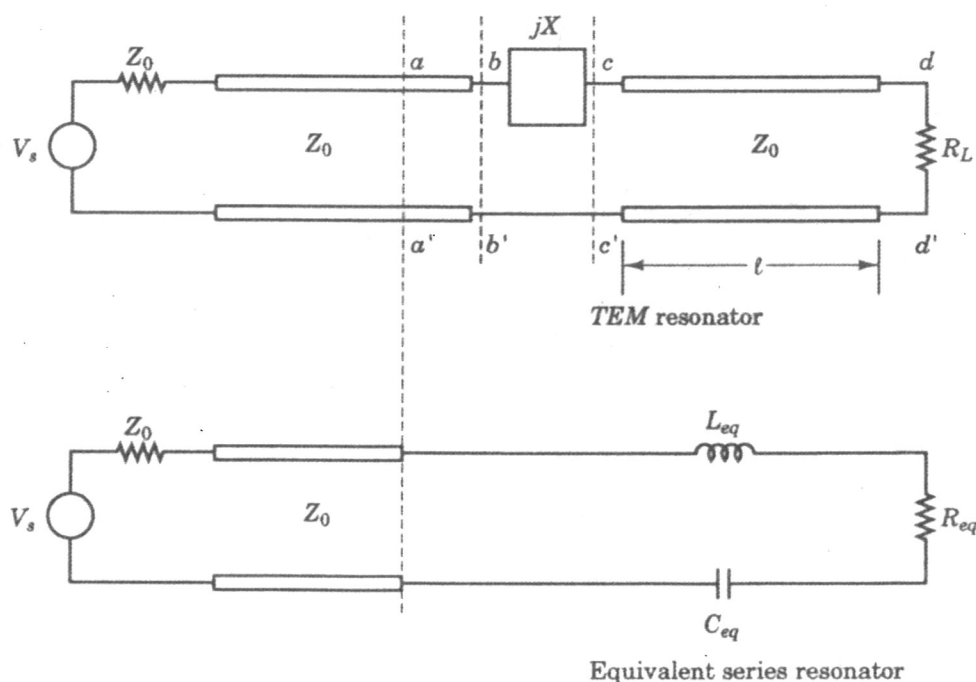


Figure 8.20 TEM resonator coupled to TEM line.

coupled to the external circuit, what values of ℓ and X are necessary? What values of R_{eq} , L_{eq} and C_{eq} are needed to model this resonator as a discrete-element series resonator with frequency ω_0 , and where is the series resonance plane?

Solution: If the resonator is to be critically coupled, the power dissipated in the external circuit (with impedance Z_0) must be the same as that dissipated in the resonator; i.e., $R_{eq} = Z_0$ and the normalized impedances $Z_{n,aa'}(\omega_0) = Z_{n,bb'}(\omega_0) = 1$ where the planes aa' and bb' are given in Figure 8.20. Therefore, $Z_{n,cc'}(\omega_0) = 1 - jX_n$ ($X_n = X/Z_0$) and $Z_{n,cc'}$ lies on the unit resistance circle on the Smith chart at resonance. We find ℓ by rotating the normalized load impedance $Z_{n,dd'} = R_L/Z_0$ about the Smith chart center to the unit circle as shown in Figure 8.21(a). Two possible values of ℓ are possible for any frequency ω_0 depending on whether X is capacitive or inductive. Because $R_L/Z_0 \ll 1$, $\ell \simeq \lambda/4$ for the lowest resonant frequency mode. We now use the analytic expression for the normalized

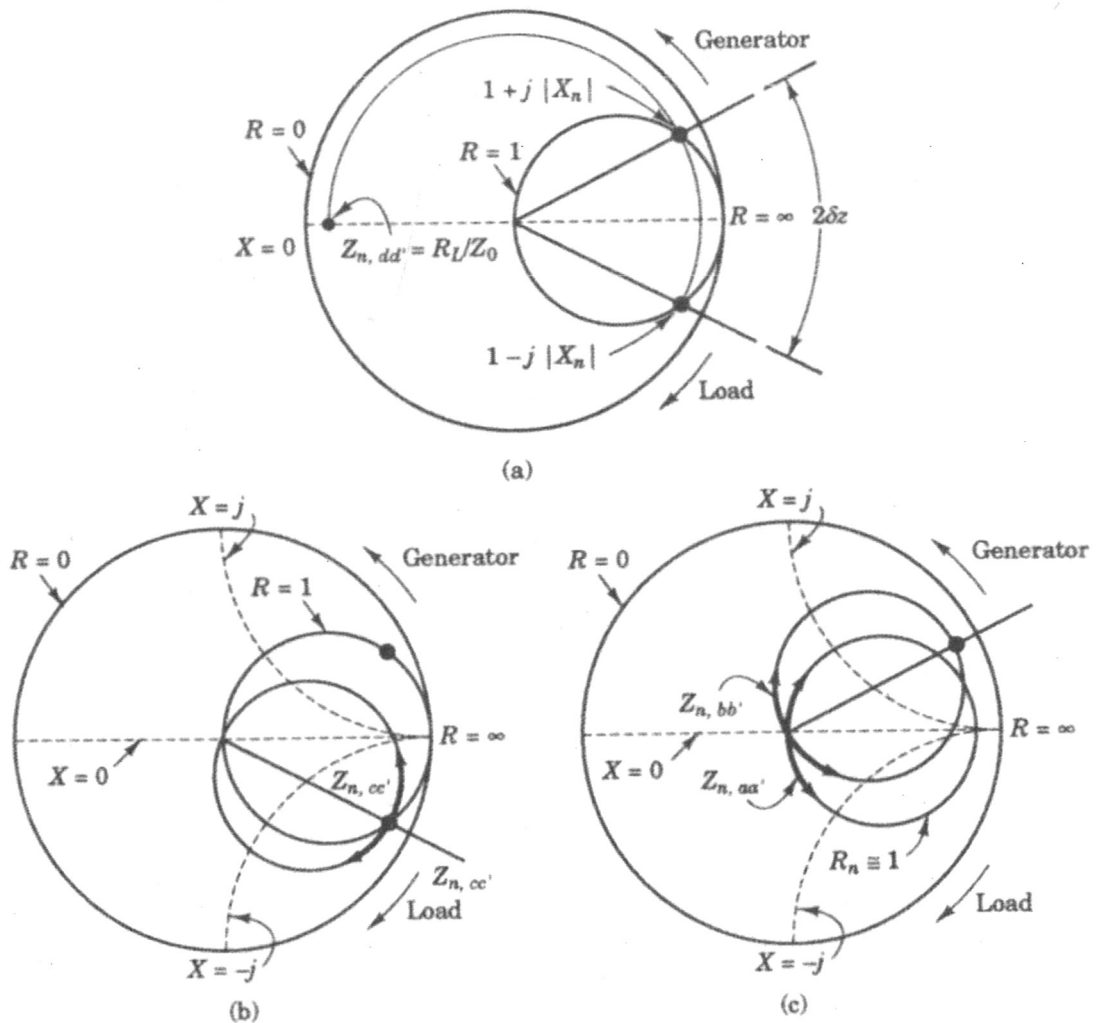


Figure 8.21 A critically coupled resonator and the Smith chart.

impedance $Z_{n,cc'}(\omega_0)$ in (6.1.5), given that the normalized load is R_L/Z_0 ,

$$Z_{n,cc'}(\omega_0) = \frac{(R_L/Z_0) + j \tan k_0 \ell}{1 + j(R_L/Z_0) \tan k_0 \ell} = 1 - jX_n \quad (1)$$

from which we find that $\tan k_0 \ell = \pm \sqrt{Z_0/R_L}$ and $X_n \approx \mp \sqrt{Z_0/R_L}$ after equating real and imaginary parts of (1). Because $\tan k_0 \ell$ is a very large number, ℓ is close to $\lambda/4$ as expected. If we arbitrarily choose ℓ to be slightly larger than a quarter wavelength ($\ell = \lambda/4 + \delta z$), we find that $\tan k_0(\lambda/4 + \delta z) = -\cot k_0 \delta z \approx (k_0 \delta z)^{-1} = -\sqrt{Z_0/R_L}$. In this case, the lumped reactance connecting the two lines is positive: $X = Z_0 X_n = Z_0 \sqrt{Z_0/R_L}$. The possible design choices for ℓ and X are thus

$$\ell = \frac{\lambda}{4} \pm \frac{\lambda}{2\pi} \sqrt{\frac{R_L}{Z_0}}$$

and

$$X = \pm Z_0 \sqrt{\frac{Z_0}{R_L}} \quad (2)$$

The values of ℓ and X above perfectly match the resonator to the external *TEM* line at resonance; off-resonance, the resonator is either undercoupled or overcoupled. Because $Z_{n,cc'}(\omega) = |\Gamma_L| e^{2j\omega\ell/c}$, it is clear that $Z_{n,cc'}(\omega)$ lies along a circle centered at $\Gamma = 0$ with radius Γ_L . As ω increases (decreases) past resonance, the impedance is rotated in the clockwise (counterclockwise) direction an additional amount past the $\text{Re}\{Z_n\} = 1$ circle. For small changes in ω about ω_0 , this path lies approximately along another circle as shown in Figure 8.21(b) for the case where X and δz are both positive. With some algebra, it is possible to show the locus of points representing $Z_{n,b'b'}(\omega_0 \pm \Delta\omega)$ in Figure 8.21(c). These points can be made to coincide with the $\text{Re}\{Z_n\} = 1$ circle by rotating it δz meters toward the load when X is inductive. From this last rotation, however, we obtain the impedance of a matched series *RLC* circuit having $Z_n = 1 + 2jQ\Delta\omega/\omega_0$ [see (8.8.22)]. The series resonance plane at aa' is therefore $\delta z = (\lambda/2\pi)\sqrt{R_L/Z_0}$ meters from bb' when X is positive, and $\lambda/2 - \delta z$ meters from bb' when X is negative.

The resonator Q can be calculated by finding the power dissipated in R_L ($R_L|I_0|^2/2$) and the energy stored in the *TEM* line ($L|I_0|^2\ell/4$).

$$Q_I = \frac{\omega_0 \langle W_T \rangle}{P} = \frac{\omega_0 L \ell}{2R_L} = \frac{\pi Z_0}{4R_L} \quad (3)$$

because $\omega_0 = ck_0 = 2\pi c/\lambda$, $\ell \approx \lambda/4$, $c = 1/\sqrt{LC}$, and $Z_0 = \sqrt{L/C}$. Since the resonator is critically coupled, we have already noted that $R_{eq} = Z_0$ and $Q_E = Q_I$. We find L_{eq} and C_{eq} from (8.8.19):

$$\begin{aligned} L_{eq} &= \frac{R_{eq} Q_I}{\omega_0} = \frac{\pi Z_0^2}{4R_L \omega_0} \\ C_{eq} &= \frac{1}{\omega_0 R_{eq} Q_I} = \frac{4R_L}{\pi \omega_0 Z_0^2} \end{aligned} \quad (4)$$

8.9* DESIGN AND APPLICATIONS OF RESONATORS

Resonators are widely used for signal filtering, impedance matching, energy storage, and frequency measurement. They can be implemented in discrete-element circuits, *TEM* transmission lines, waveguides of all types, and cavities, as well as being incorporated on partially reflecting surfaces or distributed throughout a medium. Resonance can also be an unwanted side effect of some electrical systems. For example, any of the impedance matching circuits discussed in Chapter 6 behave as resonators if forced to couple badly mismatched impedances—the resulting narrow-band behavior can pose problems for some applications.

Designing resonators for specific applications is a straightforward exercise, provided that inaccuracies introduced by the perturbation techniques used to compute losses are acceptable. If the resonator Q is large, perturbation methods are usually successful. Most resonators can be characterized by a few simple parameters: resonant frequency, half-power bandwidth, impedance at resonance, and possibly the ability to accommodate a load impedance mismatch at resonance. Related parameters include the resonator energy decay time, maximum energy storage for a given input power, and power dissipation.

Resonators match loads because they permit field strengths to build internally until large enough to dissipate the amount of power equal to the power available from the source. When matching low impedance loads, the resonator fields increase until the **current** flowing through the load is adequate to ensure that all the incident power is dissipated rather than reflected. If no power is reflected, then the load is matched. Resonators match high impedance loads similarly, except that the load is placed so that the **voltage** or electric fields across it are sufficiently large. The same argument also makes it clear that the circuits needed to match badly mismatched loads will behave as resonators within limited bandwidths because the field strengths and therefore the stored energy must be large. But since energy storage is directly proportional to the resonator Q for fixed input power [see (8.8.15)], badly mismatched circuits require increasingly narrow-band matching circuits.

Resonators can also be used for temporary energy storage. For example, energy can be built up slowly and then released in a burst by throwing a switch, or it can be introduced suddenly into a resonator and then released slowly. A typical application is a device that powers an arc discharge. Here, the resonator energy builds slowly until the spark gap breaks down and the resonator discharges its power through the low-impedance plasma at the desired resonator frequency. Another example is a very large enclosed superconducting cavity that stores energy in its dominant mode for later use. Large cavities have low resonant frequencies, and superconducting cavities have very high Q s, so energy can be stored for hours or days, depending on the cavity size. If the superconducting walls of a cavity could tolerate magnetic fields up to one tesla (10,000 gauss), then the peak energy storage density would be $\frac{1}{4}\mu_0|\overline{B}|^2$ or $10^7/16\pi$ J/m³. Fifteen cubic meters of

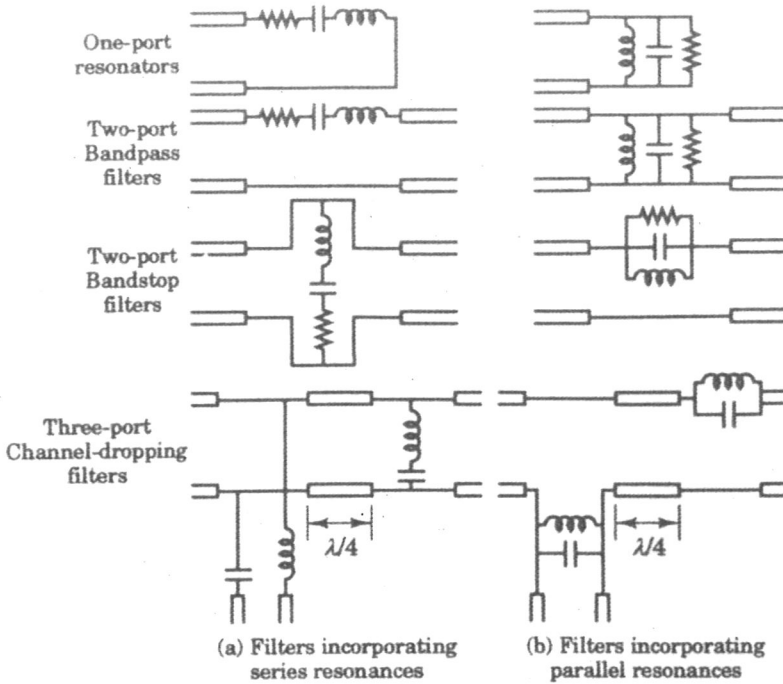


Figure 8.22 Topologies of resonant filters.

vacuum in such a cavity would rival the storage capacity of a car battery, roughly 1 kWh. Unfortunately, the corresponding electric fields of $\sim 3 \times 10^8$ V/m would cause dielectric breakdown and arcing in air, limiting use of this technique.

The major use of resonators is for filtering signals. Filters can pass or block narrow bands of frequencies, or can shunt narrow bands to side circuits while permitting the rest of the band to pass unaltered from one circuit to another. These three circuit types are called *band-pass filters*, *band-stop filters*, and *channel-dropping filters*, respectively, and examples are discussed later in this section. Bandpass and band-stop filters can be implemented in one- or two-port configurations, while channel-dropping filters need two or three ports. A series or parallel *RLC* circuit can be a one-port resonator, as it can be driven by a single *TEM* line. If two *TEM* lines are connected to the resonator, then the filter has two ports. Some of these configurations are illustrated in Figure 8.22.

Channel-dropping filters are usually implemented in three-port configurations, with the desired narrow band extracted from one port while the remainder of the power passes out the other output power. The input port is typically approximately matched over the entire band. Channel-dropping filters are often cascaded to provide many output ports, one for each frequency band of interest. They are employed when the simpler alternative of connecting the output ports in series or parallel transfers power too inefficiently. They can also be implemented by combining lossless band-stop or band-pass filters with three-port ferrite isolators, which

pass energy from the input port to the output port but pass energy coming from the output port into a side port instead. This combination of an isolator and a band-pass filter forms a three-port circuit, which also can be cascaded to form a larger circuit with one input and multiple outputs covering each frequency band of interest.

All of these filters can be designed using discrete-element circuits, *TEM* lines, or waveguides and cavities. Alternatively, they may be implemented by depositing specially shaped resonant structures on planar interfaces to manipulate uniform plane waves. Resonant beam-splitters, for example, can be designed to selectively reflect a desired frequency band in a specific direction.

Cavity resonators can be coupled to waveguides by cutting holes in their top, side, or end walls, where the degree of coupling is determined by the size, shape, and position of the holes. If the holes are small, then coupling occurs predominantly via the magnetic fields that leak between the two structures, and the associated wall currents that also pass through the holes. For larger holes, the role of the electric field increases, and additional directivity may be imparted to the emerging waves in the second waveguide.

A cavity resonator attached to the wall of a waveguide by a small connecting hole can have a profound effect on the radiation passing through the waveguide. If the cavity Q_I is sufficiently large compared to Q_E , the fields in the cavity can build in strength so that despite the small size of the hole, they can radiate in both directions down the waveguide, largely canceling the forward-propagating wave by reflecting it losslessly toward the source. This can occur inadvertently when a section of oversized waveguide connects two smaller guides. The middle section may support modes that cannot escape through the small guides, which thus become resonant. If dimensional imperfections couple these trapped resonant modes to the propagating waveguide mode, sharp partial nulls may result in the pass-band of the waveguide.

Cavities can also be coupled to coaxial cables and other *TEM* structures by soldering one conductor to the outside of the cavity and bringing the other wire into the cavity as an antenna. Several antenna configurations are possible, including a short-dipole antenna (a short wire protruding into the cavity plus its reflection in the conducting wall through which it enters), or a small loop antenna, usually constructed by soldering the wire to the cavity wall. The loop then can intercept the magnetic fields associated with the cavity modes to which coupling is desired. If the joint between the open-ended *TEM* line and cavity is capable of rotation, then the loop can be oriented to intercept only the desired amount of magnetic flux. Figure 8.23 illustrates some of these coupling techniques.

To design a resonator circuit to perform any of these functions, it is necessary only to select the desired configuration, resonant frequency, bandwidth, and impedance mismatch at resonance before and after the resonator is in place. Other

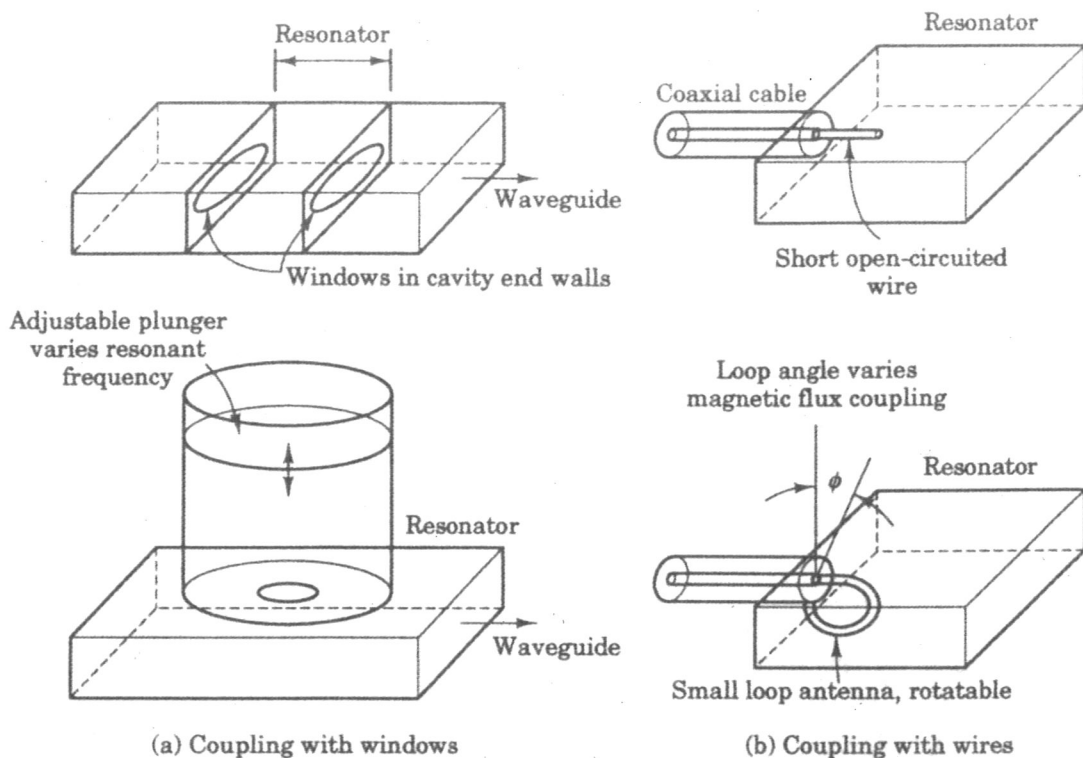


Figure 8.23 Coupling to cavity resonators.

considerations such as tunability, power-handling capability, broad-band behavior, and size limitations may also affect design decisions. A few key equations, summarized in Table 8.2, are the most useful for resonator design. Subsequent examples should help clarify how these formulas are used.

Example 8.9.1

Design a one-port RLC resonator that absorbs all incident power at 1 MHz from a $100\text{-}\Omega$ TEM line, but resembles a short circuit otherwise. The total resonant circuit also is to have a half-power bandwidth of 10 kHz.

Solution: We choose a parallel RLC resonator (also called a GLC resonator) since its admittance $Y_n(\omega)$ given by (8.8.24) is equal to Q_E/Q_I at resonance and is large for $\omega \neq \omega_0$. Therefore, this resonator resembles a short circuit off resonance. (A series resonator resembles an open circuit.) For the parallel RLC circuit to match the TEM line, $Y_n(\omega_0) = 1$, which means that $Q_I = Q_E$ and $R = 1/G = Z_0 = 100\text{ }\Omega$. From (8.4.13), we find the loaded Q

$$Q_L = \frac{f_0}{\Delta f_0} = \frac{1\text{ MHz}}{10\text{ kHz}} = 100$$

TABLE 8.2 Resonator Design Equations.

Equation	Application	Eq. No.
$\omega_n = (n + 1/2) \pi v / \ell$	TEM line open at one end; closed at the other	(8.5.4)
$\omega_n = n \pi v / \ell$	TEM line open- or short-circuited at both ends	(8.5.4)
$Q_n = \omega_n \langle w_{Tn} \rangle / P_n$	All resonators	(8.3.3), (8.3.7)
$Q_L^{-1} = Q_I^{-1} + Q_E^{-1}$	All resonators	(8.4.21)
$= \omega_n / 2\alpha_n = \omega_n / \Delta\omega_n$		(8.4.13)
$\omega_0 = 1 / \sqrt{L_{eq} C_{eq}}$	Series, parallel RLC circuits	(8.4.6), (8.8.17)
$Q_I = \omega_0 L_{eq} / R_{eq}$	Series circuits	(8.4.8), (8.8.19)
$Q_I = \omega_0 C_{eq} / G_{eq}$	Parallel circuits	(8.4.16)
$Z_n = Q_E / Q_I + 2j Q_E (\omega - \omega_0) / \omega_0$	Series circuits	(8.8.22)
$Y_n = Q_E / Q_I + 2j Q_E (\omega - \omega_0) / \omega_0$	Parallel circuits	(8.8.24)

so $Q_I = Q_E = 2Q_L = 200$ from (8.4.21). We find C and L from (8.8.19) with the duality-inspired substitutions $L \leftrightarrow C$ and $R \rightarrow G$:

$$C = \frac{G Q_I}{\omega_0} = \frac{0.01 \times 200}{2\pi \times 10^6} = 0.32 \mu\text{F}$$

$$L = \frac{1}{\omega_0 G Q_I} = \frac{1}{2\pi \times 10^6 \times 0.01 \times 200} = 80 \text{ nH}$$

Example 8.9.2

Design a two-port band-stop series RLC filter that passes only 90 percent of its input power at 1 MHz and that has a half-power bandwidth equal to 10 kHz. This filter is to be connected between two 100- Ω TEM lines so that the out-of-band power passes down the lines unperturbed. Such a circuit can be used to determine frequency by varying C and L and noting at what point the output power briefly diminishes by about 10 percent.

Solution: The series band-stop configuration shown in Figure 8.22(a) is the basis for this circuit design. Off-resonance, the RLC resonator is essentially open-circuited so the two-port appears to be a 100- Ω Thevenin circuit driving a 100- Ω load. All power is transferred to the load and the system is matched:

$$P(\omega \neq \omega_0) = \frac{|V_{Th}|^2}{8Z_0} \quad (1)$$

On resonance, the RLC circuit can be replaced by the resistance R , and the power dissi-

pated in the second *TEM* line is

$$P(\omega_0) = \frac{Z_0 |V_{Th}|^2}{2(2Z_0 + Z_0^2/R)^2} \quad (2)$$

where (2) results from calculating the power dissipated in Z_0 in the parallel Z_0, R combination, which is connected in series to another Z_0 impedance. Since $P(\omega_0) = 0.9P(\omega \neq \omega_0)$, we find that $R = 924 \Omega$. The loaded Q is found from (8.4.13)

$$Q_L = \frac{f_0}{\Delta f} = 100$$

which is related to the inductance of the series circuit by

$$L = \frac{RQ_I}{\omega_0} = \frac{R_L Q_L}{\omega_0} = (R + Z_0/2) \frac{Q_L}{\omega_0} = 0.016H$$

so $C = 1.63$ pF. Because loaded Q is specified, the total resistive load $R_L = R + Z_0/2$ must be used in the calculation of the resonator parameters.

Example 8.9.3

Repeat Example 8.9.1 using an air-filled *TEM* resonator instead of a discrete-element *RLC* configuration.

Solution: One approach is to use the solution of Example 8.9.1 to find R_{eq} , L_{eq} , and C_{eq} for the equivalent parallel *TEM* resonator circuit, and use (8.4.23) to determine Q_E and Q_I . The *TEM* resonator may then be designed using the perturbation techniques described in Sections 8.5 and 8.7. A more direct approach is taken here, bypassing the unnecessary intermediate step of finding the equivalent circuit elements. If the *TEM* line is air-filled and either shorted or opened at both ends, then its length ℓ must be a multiple of a half wavelength. Because the wavelength is long, we choose the lowest nonzero mode ($\ell = 150$ m) and short-circuit the *TEM* line at both ends to minimize radiation losses. (If this line length were still unacceptably long, we could use a quarter-wave resonator 75 m long, which is shorted at one end and opened at the other.) For the resonator to be critically coupled at resonance, we need $Q_I = Q_E = 2Q_L = 200$ since $Q_L = 100$ to produce a 10-kHz half-power bandwidth at 1 MHz.

To relate these Q s to the resonator requires some decisions about its configuration. Assume that the *TEM* line is lossless, so the only loss in the resonator is the resistor R_L , which shunts the line at a distance d_2 from one end, as illustrated in Figure 8.24. Assume the resonator is fed by a parallel *TEM* line of impedance Z_0 at a distance d_1 from its other end. If we choose R_L equal to Z_0 , then we need only compute Q and d once, since $Q = Q_I = Q_E$ and $d = d_1 = d_2$. From (8.5.12), we see that

$$Q_I = \frac{\pi R_L}{2Z_0 \sin^2(\pi d_2/\ell)}$$

which means that $d_2 = 4.24$ m.

It only remains to find the length of the connecting line to the first parallel resonance plane, where the resonator appears close to a short-circuit off resonance. If we consider the resonator to consist of a 4.24-m length of shorted line in parallel with a much longer piece,

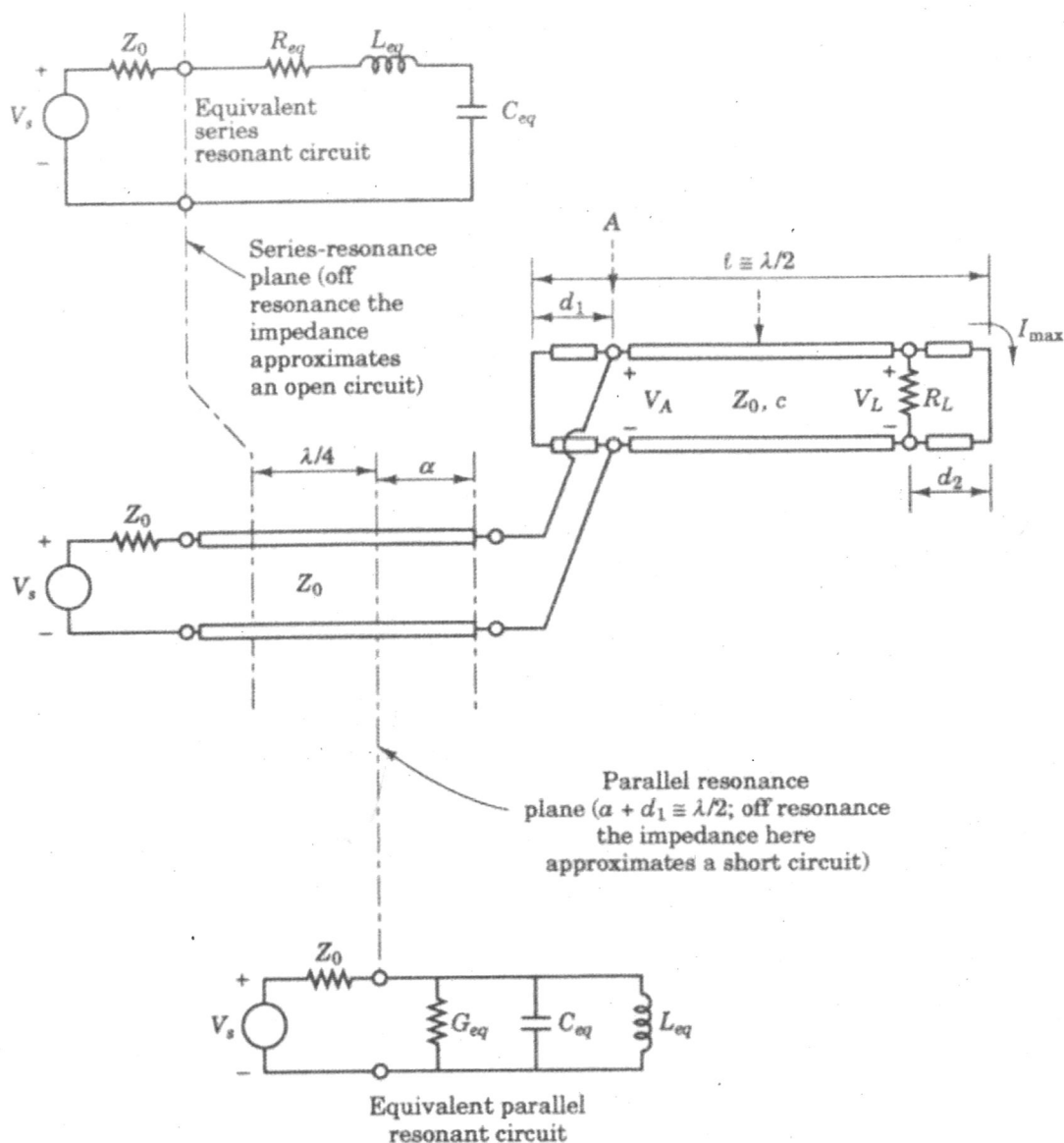


Figure 8.24 One-port TEM resonator in Example 8.9.3.

then at most frequencies well-removed from resonance, the total admittance of this parallel combination is approximately that of the short segment. Because this large admittance transforms to a short circuit $\lambda/2 = 150$ m from the end of the resonator, the parallel resonance plane is at $150 - 4.24 = 145.76$ m from the end of the external circuit. (See also Example 8.8.1 for another discussion of the locations of the series and parallel resonance planes.) Note that d and ℓ are unchanged if we wish to find the series resonance plane at which the resonator appears to be an open circuit. We merely move $\lambda/4 = 75$ m to the right or left of the parallel resonance plane.

Example 8.9.4

Use a critically coupled air-filled *TEM* resonator of the same general form described in Example 8.9.3 to show that if $R_L = 400Z_0$, the bandwidth of the matching circuit is necessarily narrow.

Solution: The power dissipated in R_L is $\sim |V_L|^2/2R_L$, where V_L is the unperturbed line voltage across the load and the load is d_2 m from the short-circuited end of the line, as illustrated above. At resonance, this power must equal the input power $|V_S|^2/2Z_0$ where V_S is the source voltage: $|V_L| = 20|V_S|$ although the peak voltage on the resonator may be even larger. To maximize the bandwidth of the matching circuit, we wish to minimize $Q_L = Q_I/2 = Q_E/2$. But the energy theorem tells us that $Q_L = \omega_0 \langle w_T \rangle / P$, and since ω_0 and P are fixed, we must minimize $\langle w_T \rangle = \frac{1}{4} C \ell |V_{\max}|^2$ to minimize Q_L which means minimizing $|V_{\max}|$. This is best done by placing the R_L at the center of the resonator so that $d_2 = 75$ m and $|V_L| = |V_{\max}|$. Substituting these values in (8.4.16) yields $Q_I = \omega_0 C \ell R_L/2 = \pi R_L/2Z_0 = 628$, which means that the half-power bandwidth of this matching circuit can be no greater than $f_0/Q_L \simeq 1.6$ kHz.

Example 8.9.5

Design a critically coupled cavity band-pass filter for a resonance frequency of 10 GHz and a Q_L of 1000. Assume it is made of coin silver. What is the insertion loss (power lost in the resonator walls) for your design?

Solution: Although we know that Q_I is greatest when we use large resonators with maximum volume-to-surface ratios, here we shall assume that the resonator is constructed simply by placing two windows across a waveguide to form a cavity resonant at 10 GHz, as illustrated in Figure 8.23. If the dimensions of this waveguide are 1×2 cm, then from (8.2.11), the unperturbed cavity length d for a 10-GHz TE_{101} resonance is 2.27 cm. Since $Q_E = Q_I = 2Q_L = 2000$ for critical coupling, the windows in the end walls that connect the lossless resonator to the input and output waveguides must have the same electrical characteristics and are normally identical. It only remains to find the size, shape, and positions of the holes in the end walls that yield the desired Q_L ; this is often done empirically by drilling them progressively larger until the desired performance is obtained. (To account for cavity wall losses, the output hole would be slightly smaller to preserve $Q_E = Q_I$.) Calculating the window shapes from first principles is difficult, although powerful electromagnetic computer software packages are making these calculations easier.

Even though we may not know the sizes of the windows, we can estimate an insertion loss by computing Q_w associated with the wall losses P_w . For silver, $\sigma \simeq 5 \times 10^7 \Omega^{-1}\text{m}^{-1}$ so the skin depth is approximately $0.712 \mu\text{m}$, and (8.3.20) can be used to find the unperturbed $Q_w \simeq 7210$. The fraction of power lost in the walls is simply $P_I/(P_w + P_I) = Q_w^{-1}/(Q_w^{-1} + Q_I^{-1})$ or about 22 percent. This is sufficiently high to suggest using a cavity with a cross-section larger than the waveguide.

Example 8.9.6

A certain infrared *TEM* resonator can be modeled as an infinite dielectric slab ($\epsilon = 9\epsilon_0$) $\ell = 10$ cm thick bounded on one side by a mirror with conductivity $\sigma = 10^7 \Omega^{-1}\text{m}^{-1}$ and on the other side by a lossless multilayer dielectric mirror designed to reflect 99 percent of the $\lambda = 100 \mu\text{m}$ radiation incident upon it. Approximately what resonator mode is being excited, and what are Q_E , Q_I , and Q_L for the mode? What is the width $\Delta\omega$ of this

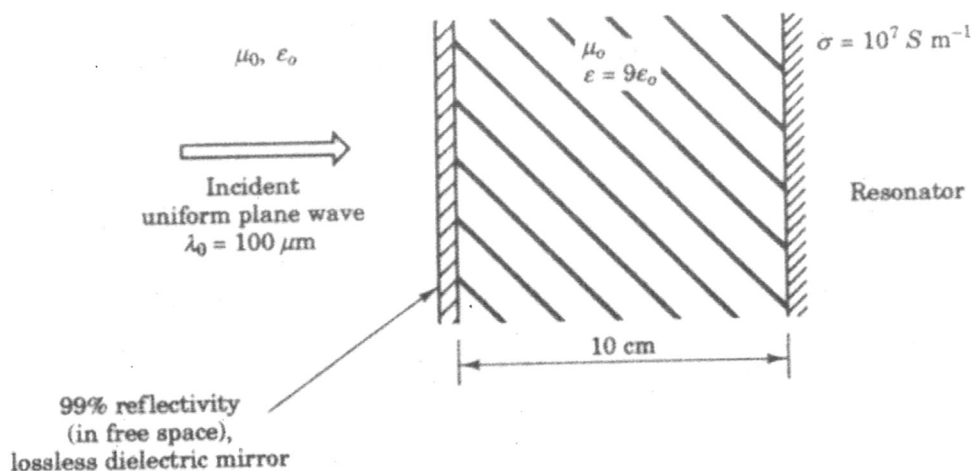


Figure 8.25 Infrared TEM resonator in Example 8.9.6.

resonance, and how does it compare to the spacing between resonances $\omega_{n+1} - \omega_n$? At resonance, what fraction of the power is absorbed by the resonator? What reflectivity is needed in the dielectric mirror to couple all incident power into the metallic mirror? This resonator is pictured in Figure 8.25.

Solution: In the dielectric, $\lambda = \lambda_0 / \sqrt{\epsilon/\epsilon_0} = \lambda_0/3$. The mode number n follows from $n\lambda/2 = \ell$, so $n \cong 2\ell/\lambda = 0.2/(\lambda_0/3) = 0.2 \times 3/10^{-4} = 6000$. From (8.4.20)

$$Q_E = \omega_0 w_T / P_E$$

where P_E is the power lost externally through the lossless mirror. The total energy w_T in this resonator can be considered to be composed of one stream moving to the right and another approximately equal one moving to the left, each at velocity $v = 1/\sqrt{9\epsilon_0\mu_0} \cong 10^8$ (m/s). Each stream conveys power $P_+ = P_-$, where $P_E = 10^{-2}P_-$. The energy in these two streams is $\langle w_T \rangle$:

$$\langle w_T \rangle = (P_+ + P_-)\ell/v \cong 2P_-10^{-1}/10^8 = 2 \times 10^{-9}P_-$$

and

$$Q_E = \frac{\omega_0 \langle w_T \rangle}{P_E} = \frac{2\pi \times 10^{12} \times 2 \times 10^{-9} P_-}{10^{-2} P_-} = 1.26 \times 10^6$$

The power per unit area P_I lost internally in the metal mirror can be found from (4.4.29)

$$P_I = \frac{|E_0|^2}{\eta^2} \sqrt{\frac{2\omega_0\mu_0}{\sigma}}$$

and the energy stored in the slab is $\langle w_T \rangle = \epsilon |E_0|^2 \ell / 4$, so

$$Q_I = \frac{\omega_0 \langle w_T \rangle}{P_I} = \ell \sqrt{\frac{\sigma \omega_0 \mu_0}{8}} = 3.14 \times 10^5$$

Therefore, $Q_L = (Q_I^{-1} + Q_E^{-1})^{-1} = 2.5 \times 10^5$ and the bandwidth is $\Delta\omega = \omega_0/Q_L = 2.5 \times 10^7$ rad/s. The frequency difference between modes is $\delta\omega \cong \omega_0/6000 = 10^9$ rad/s, which is much greater than $\Delta\omega$.

At resonance, the fraction of power absorbed can be found using a reference plane for a parallel resonance

$$\frac{Q_E}{Q_I} = \frac{G_{eq}}{Y_0} = \frac{1}{Z_n}$$

$$|\Gamma|^2 = \left| \frac{Z_n - 1}{Z_n + 1} \right|^2 = \left| \frac{Q_I - Q_E}{Q_I + Q_E} \right|^2 = 0.36$$

so only 74 percent of the incident power is absorbed at resonance. To achieve 100 percent absorption, we set $Q_I = Q_E$ by decreasing the reflectivity γ of the dielectric mirror from $\gamma = 0.99$ to

$$\gamma = 1 - \frac{4000\pi}{Q_I} = 0.96$$

8.10 SUMMARY

A *resonator* is any structure that can trap electromagnetic energy, although useful (high-quality) resonators also prevent this energy from leaking away too quickly. In accordance with the *energy theorem* (1.7.6), resonators exhibit strongly frequency-dependent behavior near one or more discrete frequencies called *resonances*. Resonators are used for energy storage, frequency measurement, filtering, impedance matching, and the production of high field strengths from modest input sources.

A *rectangular cavity resonator*, formed by closing the ends of a rectangular waveguide, can support TE_{mnp} and TM_{mnp} modes, where m , n , and p are integers indicating how many half wavelengths can fit between the resonator walls of lengths a , b , and d ($b \leq a \leq d$), respectively. Both the TE_{mnp} and TM_{mnp} modes resonate at the frequency f_{mnp} , where

$$f_{mnp} = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

Resonators can also be fabricated by terminating TEM lines with open or short circuits, or with any highly reflective (mostly reactive) impedance. For a TEM line of length ℓ short-circuited at both ends, resonances occur for $f_n = nv/2\ell$. Circuits containing both inductive and capacitive reactances typically possess one or more resonances, as do dielectric waveguides of finite length and almost any arbitrarily shaped conducting or nonconducting structure.

The *quality factor* Q of a particular resonance mode is dependent on the rate at which that mode loses energy, where for the n^{th} mode,

$$Q_n = \frac{w_{Tn}}{P_n/\omega_n} = \frac{\text{total energy of the } n^{th} \text{ mode}}{\text{average energy lost per radian of the } n^{th} \text{ mode}}$$

The *complex frequency* of the n^{th} resonance is commonly denoted s_n , where

$$s_n = j\omega_n - \alpha_n$$

and $Q_n = \omega_n / 2\alpha_n$.

We distinguish three types of quality factors: *internal*, *external*, and *loaded*, represented by Q_I , Q_E , and Q_L , respectively. The internal Q is related to power dissipation within the resonant structure, while the external Q depends on the composition and configuration of the external circuit that couples to the resonator. The loaded Q is a measure of the total power dissipated by all loss mechanisms in the resonator circuit, so that

$$\frac{1}{Q_L} = \frac{1}{Q_I} + \frac{1}{Q_E}$$

When a resonator is connected to an external circuit, the *half-power bandwidth* of the resonance is $\Delta\omega = \omega / Q_L$, so high-quality resonators have narrow bandwidths.

The loss in any particular resonator near any particular resonance can be easily approximated using *perturbation techniques*. The voltages, currents, and/or electromagnetic fields are first computed for a lossless structure at the resonant frequency of interest. The power lost per frequency cycle is then estimated by using these unperturbed fields to compute the power that would be dissipated if loss were present. Dissipation in high conductivity elements is estimated using the unperturbed currents, which do not change appreciably as loss is added, whereas unperturbed voltages must be calculated to estimate power dissipation in low-conductivity structures. Only the ratio of energy stored to power dissipated is needed to compute Q .

At a specific resonance, an arbitrary resonator may be modeled by a (non-unique) *equivalent discrete-element RLC circuit*, where a different circuit may be necessary to describe each of the distinct resonant frequencies. A specific resonance can be modeled by either a series or parallel *RLC circuit*, depending on whether the complex impedance at resonance is small or large, respectively. In both cases, the impedance is purely real at resonance. If we imagine a Thevenin equivalent source driving a series *RLC resonator* for which the Thevenin resistance R_{Th} is equal to R , then all available power is transferred to the resonator at the resonant frequency $\omega_0 = 1/\sqrt{LC}$ (equivalently, $|\Gamma(\omega_0)| = 0$). Such a resonator is said to be *critically coupled* since its internal $Q_I = \omega_0 L / R$ is equal to Q_E . If the cavity or *TEM line resonator* is instead modeled by a parallel *RLC circuit* driven by an external Norton equivalent circuit having the Norton resistance R_N equal to R , the resonator is again critically coupled; in this case $Q_I = Q_E = \omega_0 C / G$. Resonators for which Q_E is less than Q_I are said to be *undercoupled*, while those for which $Q_E > Q_I$ are *overcoupled*. In either case there is imperfect power transfer at resonance.

It is often useful to tune resonators about their natural resonant frequencies

by slightly modifying their structures. As loss is added to a lossless resonator, the resonant frequency $s = j\omega_0$ typically becomes complex ($s = j\omega + \alpha$), where both the frequency shift $\omega - \omega_0$ and the quality factor $Q = \omega/2\alpha$ can be computed using the *resonator perturbation equation* (8.7.9). Resonators can also be used in nonsinusoidal steady-state configurations. For example, any closed electromagnetic structure that is energized and then allowed to behave naturally after a specific time will have energy resident in each of its many resonant modes simultaneously. Each mode oscillates with a decaying exponential behavior corresponding to its own complex resonant frequency s_n . The total voltage, current, and field distributions are then linear superpositions of those associated with the separate orthogonal modes. Since each of these modes can be analyzed individually, the behavior of the total system can also be studied.

Design of resonators is straightforward. We start with the desired resonant frequencies and bandwidths and choose between *band-pass*, *band-stop*, and *channel-dropping* resonator configurations. Once the desired mismatch at resonance is specified, perturbation techniques are used to compute the corresponding internal, external, and loaded quality factors. In some cases, the dimensions of the resonator must be altered or lower loss materials chosen in order to achieve the desired values for the internal Q . These quality factors can then be related by perturbation theory to the various resonator parameters and to its external coupling structure.

8.11 PROBLEMS

8.2.1

- What are the two lowest resonant frequencies for a perfectly conducting cubical resonator 1 cm on a side filled with dielectric having $\epsilon = 16\epsilon_0$?
- Note that several modes share each of these frequencies. List all modes corresponding to each of these two frequencies (TE_{mnp} and TM_{mnp}), where for the purposes of this question, we let m , n , p correspond to the $\hat{x}$ -, $\hat{y}$ -, and $\hat{z}$ -axes, respectively.

8.2.2 A rectangular evacuated superconducting cavity is $2 \times 3 \times 4$ m.

- What are the four lowest resonant frequencies of this cavity, and what are their mode designations (e.g., TM_{112})?
- The largest magnetic field this superconductor can stand before going "normal" is 20 Wb/m^2 (200,000 gauss, or 20 tesla). What is the maximum total energy that can be stored in this cavity at its lowest resonant frequency? Neglect electrical breakdown phenomena.
- A good car battery can store a nominal 1 kWh. Compare the kWh volume storage efficiencies (J/m^3) of these two media.
- Show all places and orientations where thin straight slots might be placed in the cavity wall without interfering significantly with the TE_{101} mode.
- Repeat part (d) for the TM_{111} mode.

- 8.3.1 (a) If the conductivity of the dielectric having $\epsilon = 4\epsilon_0$ in a 1-cm cubical resonator is $\sigma = 10^{-2}$ S/m, what is the Q of the TE_{101} mode?
 (b) What is the half-power bandwidth (Hz) of this resonance?
- 8.3.2 A rectangular perfectly conducting cavity resonator is $1 \times 2 \times 3$ cm in size.
 (a) What are the three TE_{mnp} modes with the lowest resonant frequencies, and what are these frequencies ω_{mnp} ?
 (b) Sketch the magnetic field distribution for the TE_{102} mode at some instant of time when it is nonzero. Also sketch the corresponding surface currents on the resonator walls.
 (c) If the cavity is filled with a dielectric $\epsilon = (4 - j10^{-3})\epsilon_0$, then what is (i) the decay constant $\alpha_{TE_{102}}$ at the resonant frequency $\omega_{TE_{102}}$? ($s_{TE_{102}} = j\omega_{TE_{102}} - \alpha_{TE_{102}}$), (ii) $Q_{TE_{102}}$?
- 8.4.1 A series RLC resonator has an impedance $Z(s)$. Let $R = 2 \Omega$, $C = 3$ farads, $L = 4$ henries.
 (a) Sketch the locations (quantitatively) of the poles and zeros of $Z(s)$ in the s plane.
 (b) This resonator is driven by a unit-step voltage source. What is the resulting current $I(t)$? Give an equation for $I(t)$ and then sketch it.
- 8.5.1 Find all the resonant frequencies for each of these TEM resonators:
 (a) Air-filled, open circuited at both ends, 2 m long.
 (b) Same as part (a), but short-circuited at one end.
 (c) Same as part (a), but connected in a closed loop with a single twist (i.e., the top wire is connected to the bottom wire of the other end, and vice versa).
- 8.5.2 Consider an air-filled 100- Ω TEM resonator 1 m long, which is short-circuited at both ends.
 (a) What are the resonant frequencies f_n (Hz)?
 (b) Assume $f = f_1$, the lowest nonzero resonant frequency. What is the resonator Q if one of the short circuits is replaced by a 1- Ω resistor? Here we use the unperturbed current to find I_R , and $\frac{1}{2}R|I_R|^2$ to find the approximate power dissipated, where I_R is the current through the resistor R . Why don't we use the unperturbed resistor voltage instead?
 (c) What are the Q and lowest nonzero complex resonant frequency $f_1 = j\omega_0 - \alpha$ if $R = 10,000 \Omega$? Is the dissipated power here approximately $\frac{1}{2}|V_R|^2/R$ or $\frac{1}{2}R|I_R|^2$, where V_R and I_R are the unperturbed voltage and current at the load?
- 8.5.3 A nearly lossless resonator is constructed from a 3-m TEM line characterized by ϵ_0 , μ_0 , $Z_0 = 100 \Omega$. One end is shorted and the other is an open circuit ($R_L \rightarrow \infty$).
 (a) List the two lowest nonzero resonant frequencies (Hz) for this resonator.
 (b) If the load resistor $R_L = 10^4 \Omega$ is placed across the open circuit, what is the approximate internal Q_i for this resonator when the free-space wavelength λ is 12 m? Give a numerical answer.
- 8.6.1 Consider a TEM resonator short-circuited at both ends, ℓ meters long, and characterized by L (H/m) and C (F/m).
 (a) What are all the resonant frequencies ω_n ?

- (b) Show that the total time-averaged electric and magnetic energies (W_e) and (W_m) are equal at any nonzero resonant frequency.
- (c) At $t = 0$ the voltage is everywhere zero and the current is one ampere on the right half of the line ($\ell/2 < z < \ell$) and zero elsewhere ($0 < z < \ell/2$). Decompose the current step into a sum of modes and evaluate the first two modes in this series at $t = 0$ and $t = \ell\sqrt{LC}$ (the lowest mode is DC).
- (d) These two lowest modes have resonant frequencies $\omega_0 = 0$ and ω_1 . They are shifted approximately to $s_0 = j\omega_0 - \alpha_0$ and $s_1 = j\omega_1 - \alpha_1$ by the insertion in series of a small perturbing loss $R = 10^{-2}\sqrt{L/C}$ at one end of the line. Evaluate α_0 and α_1 .

8.7.1 A rectangular cavity $1 \times 2 \times 3$ cm is filled with a dielectric having permittivity $4\epsilon_0$ and permeability μ_0 .

- (a) List the four lowest distinct resonant frequencies of the cavity and their corresponding mode designations (TE_{mnp} , TM_{mnp}).
- (b) What is the resonant frequency ω_{mnp} for this cavity corresponding to the fields sketched in Figure 8.3(d)?
- (c*) We wish to tune this cavity by appropriately hitting it from the outside with a ballpeen hammer. Indicate the approximate regions where hitting the cavity increases the fundamental resonant frequency, and the regions where the resonant frequency decreases. Compute the location of the boundary between the two regions on the 1×2 cm surface; i.e., find the places where hitting the resonator has no effect on the resonant frequency of this mode.

8.7.2 (a) If a rectangular metal waveguide 1.5×0.75 cm in cross-section is closed with perfectly conducting plates at $z = 0$ and $z = 3$ cm, the TM_{11} mode at cutoff becomes the fundamental resonator field TM_{110} . What is the resonant frequency?

- (b*) If the resonator is pushed in at the center of the front wall at $z = 0$ or the back wall at $z = 3$ cm, will the resonant frequency for TM_{110} mode become lower or higher?

8.7.3 Consider a resonator made of a section of coaxial line with length d and short-circuited with conducting plates at both ends. It is filled with dielectric having $\epsilon = 9\epsilon_0$.

- (a) What are the electric and magnetic fields for the TEM modes?
- (b) Let $d = 10$ cm. What is the lowest nonzero resonant frequency (which is associated with the TEM_1 mode)?
- (c) Plot the magnitude of the electric field and the magnitude of the magnetic field as functions of z for the TEM_2 mode.
- (d*) Suppose the TEM_2 mode is excited inside the resonator, and an inward perturbation of the outside wall, achieved by using a perfectly conducting screw, is made near the center ($z = d/2$) of the resonator. Will the resonant frequency of that mode be lowered or raised?

8.8.1* An air-filled 1-m long TEM resonator open-circuited at both ends is made of $100\text{-}\Omega$

line and is connected to a $100\text{-}\Omega$ resistor located in series with the line 1 cm from one end.

- What is the Q of the lowest nonzero resonance (ω_1) of this structure (where the line is $\sim \lambda/2$ long)? Use perturbation techniques.
- A $100\text{-}\Omega$ *TEM* line is then connected in series with this line one centimeter from the other end, connecting the resonator to an external matched load. What are the external and total Q 's (Q_I and Q_L) for this resonator at ω_1 ?
- What is the 3-dB bandwidth (Hz) of this coupled resonator, as perceived externally?
- The feed line connecting to the resonator has a length D such that the resonance is a series resonance. What value of D has this property?
- What is the impedance Z_R of this series resonator at ω_1 ?
- What is $Z_R(\omega)$?
- Find an equivalent *RLC* resonator circuit valid for $\omega \cong \omega_1$, giving numerical values for R_{eq} , L_{eq} , and C_{eq} .

8.8.2* Design a *TEM* resonator of the general form employed in Problem 8.8.1, but using different dimensions. Assume the $100\text{-}\Omega$ air-filled *TEM* line is lossless ($R = G = 0$). We want a band-pass filter centered at 100 MHz and having a half-power (3-dB) bandwidth of 100 kHz. All the power should be coupled to the $200\text{-}\Omega$ load at resonance.

- What is one possible value for ℓ , the length of the *TEM* resonator?
- If the $200\text{-}\Omega$ load R_L is connected in series with the *TEM* line d_2 meters from one end of the resonator, and the feed line ($Z_0 = 100\text{ }\Omega$) is connected in series d_1 meters from the other end, what are d_1 and d_2 ? Give numerical values using the perturbation approximation.
- What is the voltage-amplification factor for this resonator, i.e., the ratio of $|V_{\max}|$ inside the resonator to $|V|$ on the feed line at resonance?
- Approximately what is D , the length of the feed line for the resonator to appear to be a series *RLC* circuit?
- Find the values of L_{eq} , C_{eq} , and R_{eq} at this series resonance plane.

8.8.3* A *TEM* resonator consists of a 1-m long air-filled coaxial cable with $Z_0 = 50\text{ }\Omega$ that is short-circuited at both ends.

- What are the inductance L (H/m) and capacitance C (F/m) for this line?
- A $1\text{-}\Omega$ resistor is placed across one end of the lossless resonator.
 - What are the two lowest resonant frequencies (ω_1 and ω_2) of this lossless resonator?
 - What is Q_I of the lossy resonance ω_1 ?
 - This resonator is now connected to the external world by a $50\text{-}\Omega$ *TEM* transmission line, connected across the $1\text{ }\Omega$ resistor. What is Q_E for ω_1 ?
 - What is Q_L for ω_1 now?
- For the resonator of part (b-iii), approximately where is the nearest parallel-resonance reference plane on the feed line? Where is the nearest series-resonance reference plane?

- (d) Find the equivalent circuit for the series resonance ω_1 (as viewed from the series resonance reference plane); i.e., find R_{eq} , L_{eq} , and C_{eq} .
- (e) Find the equivalent circuit for the parallel resonance ω_1 (as viewed from the parallel resonance reference plane); i.e., find G_{eq} , L_{eq} , and C_{eq} .
- (f) What is $\Delta\omega$, the half-power bandwidth for the ω_1 resonance (when coupled to the given feed line)?

8.9.1* A home TV set is being bothered by a strong interfering carrier signal that is near a channel of interest. The question is how well a filter constructed from commercially available 300- Ω twin-lead TEM cable can filter out this carrier at 200 MHz. Assume the losses in twin-lead cable are dominated by resistive losses in the two parallel wires, each of which has diameter of 1 mm and a bulk conductivity of 5×10^7 (S/m); the wires are separated by 1 cm. Assume that the line is effectively air-filled and characterized by ϵ_0 , μ_0 ; its impedance is 300 Ω .

- (a) If we characterize this line by L (H/m), C (F/m), and R (Ω /m), approximately what are L , C , and R (at 200 MHz)?
- (b) Power transmitted along this line decays as $P = P_0 e^{-\alpha_0 z}$. At 200 MHz what is the decay length α_0^{-1} (meters)?
- (c) For a 200-MHz resonator formed by short-circuiting the ends of this cable, what is the internal Q_I ? For fixed frequency, does it depend on mode number n , where n is the length of this resonator in units of $\lambda/2$? Explain briefly.
- (d) Assume such a TEM resonator is used as a two-port band-stop filter (see Figure 8.22) to reject the unwanted carrier at 200 MHz, and suppose it is used to provide a parallel resonance as illustrated in Figure 8.24. To provide 10-dB of carrier rejection at 200 MHz, what must G_{eq} be?
- (e) For the value of G_{eq} found in part (d) plus the maximum value of Q_I found in part (c), (i) what is the resulting value of Q_E for this resonator? What then is the (ii) maximum value of Q_L and (iii) minimum value of $\Delta\omega$ we can achieve here for 10-dB rejection?
- (f) Repeat parts (d) and (e) for 20-dB rejection. Note the trade-off between rejection and $\Delta\omega$.
- (g) Complete the resonator design for 20-dB rejection at 200 MHz. That is, find the lengths d_1 and a for $n = 1$. Which of these answers (d_1 and a) depend significantly on mode number n ? Explain briefly.