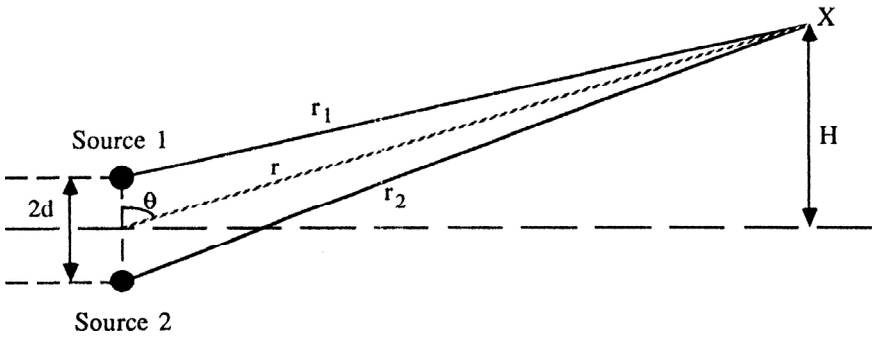



**Fig. 2.8.** Point dipole model – two point monopoles in free space separated by some small distance $2d$.

sum of the two separate velocity potentials, i.e.

$$\phi(r, t) = \phi_1(r, t) + \phi_2(r, t). \tag{2.98}$$

The separate distances from each of the point sources to the point, $X$, are given by

$$r_1^2 = (r^2 + d^2 - 2rd\cos\theta), \quad \text{and} \quad r_2^2 = (r^2 + d^2 + 2rd\cos\theta). \tag{2.99}$$

In the far field, $r \gg d$, thus the above geometrical relationships can be simplified to

$$r_1 \approx r - d\cos\theta, \quad \text{and} \quad r_2 \approx r + d\cos\theta. \tag{2.100}$$

The velocity potential for a point monopole can be obtained from equation (2.83) by letting $ka \to 0$. It is

$$\boldsymbol{\phi}(r, t) = -\frac{\mathbf{Q}(t)}{4\pi r}\,\mathrm{e}^{-\mathrm{i}kr}. \tag{2.101}$$

The combined velocity potential at $X$ due to both point monopoles is thus

$$\boldsymbol{\phi}(r, t) = -\frac{\mathbf{Q_1}(t)}{4\pi r_1}\,\mathrm{e}^{-\mathrm{i}kr_1} - \frac{\mathbf{Q_2}(t)}{4\pi r_2}\,\mathrm{e}^{-\mathrm{i}kr_2}. \tag{2.102}$$

In the far-field, $r \approx r_1 \approx r_2$ and the denominators in the above equation can be replaced by $r$. The numerators, however, should not be replaced because the phase difference between the two oscillating monopoles has to be accounted for. By substituting equation (2.100) into equation (2.102), the combined velocity potential approximates to

$$\boldsymbol{\phi}(r, \theta, t) \approx -\frac{\mathrm{e}^{-\mathrm{i}kr}}{4\pi r}\{\mathbf{Q_1}(t)\,\mathrm{e}^{\mathrm{i}kd\cos\theta} + \mathbf{Q_2}(t)\,\mathrm{e}^{-\mathrm{i}kd\cos\theta}\}. \tag{2.103}$$

For a point dipole, the two monopole sources are of equal source strength but are out of phase by 180°. Thus

$$\mathbf{Q_1}(t) = Q_\mathrm{p}\,\mathrm{e}^{\mathrm{i}(\omega t + \pi/2)}, \tag{2.104}$$

and

$$\mathbf{Q_2}(t) = Q_\mathrm{p}\,\mathrm{e}^{\mathrm{i}(\omega t - \pi/2)}. \tag{2.105}$$



*Sony v. Jawbone*
U.S. Patent No. 8,503,691
**Sony Ex. 1016 (Part 2 of 2)**

Hence,

$$\boldsymbol{\phi}(r,\theta,t) = -\frac{Q_{\mathrm{p}}\,\mathrm{e}^{\mathrm{i}(\omega t-kr)}}{4\pi r}\left\{\mathrm{e}^{\mathrm{i}(kd\cos\theta+\pi/2)} + \mathrm{e}^{-\mathrm{i}(kd\cos\theta+\pi/2)}\right\}, \tag{2.106}$$

or

$$\boldsymbol{\phi}(r,\theta,t) = \frac{Q_{\mathrm{p}}\,\mathrm{e}^{\mathrm{i}(\omega t-kr)}}{4\pi r} 2\sin(kd\cos\theta). \tag{2.107}$$

For $kd \ll 1$ this approximates to

$$\boldsymbol{\phi}(r,\theta,t) = \frac{Q_{\mathrm{p}}\,\mathrm{e}^{\mathrm{i}(\omega t-kr)}}{4\pi r} 2kd\cos\theta. \tag{2.108}$$

The sound intensity of a dipole can now be estimated either by following the same procedures adopted in the previous sub-section on monopoles or by recognising that the far-field dipole velocity potential is equivalent to the monopole velocity potential with an additional term which is a function of the polar angle $\theta$ – i.e. a directivity factor $(-2kd\cos\theta)$. It is

$$I(r,\theta) = \frac{Q_{\mathrm{p}}^2 k^2 \rho_0 c}{32\pi^2 r^2}(2kd\cos\theta)^2. \tag{2.109}$$

Thus,

$$I(r,\theta) = \frac{Q_{\mathrm{p}}^2 k^4 d^2 \rho_0 c}{8\pi^2 r^2}\cos^2\theta = \frac{Q_{\mathrm{rms}}^2 k^4 d^2 \rho_0 c}{4\pi^2 r^2}\cos^2\theta. \tag{2.110}$$

It is important to point out at this stage that the dipole velocity potential described by equation (2.108) is a far-field approximation. In reality a dipole velocity potential has two terms – one which is associated with the near-field components, and one which is associated with the far-field components. The complete dipole velocity potential can be described in terms of a dipole source strength, $\mathbf{Q_d}$ (see Reynolds[2.5], or Dowling and Ffowcs Williams[2.6]) with units of $\mathrm{m}^4\ \mathrm{s}^{-1}$, where

$$\boldsymbol{\phi}(r,\theta,t) = \frac{\cos\theta}{4\pi}\left\{\frac{1}{cr}\frac{\partial \mathbf{Q_d}(t-r/c)}{\partial t} + \frac{\mathbf{Q_d}(t-r/c)}{r^2}\right\}. \tag{2.111}$$

The radiated sound field has a $\cos\theta$ dependence – i.e. at 90° to the dipole axis the sound fields cancel each other. Also, the near-field component varies with the inverse square of distance whilst the far-field component varies with the inverse of distance. This point will be discussed again later on in this chapter in relation to flow noise.

The far-field sound power radiated by the dipole can be evaluated by integrating equation (2.110) over an arbitrary spherical surface at a given radius, $r$. The polar angular dependence has to be accounted for in this integration. The sound power radiated is

$$\Pi = \frac{Q_{\mathrm{rms}}^2 k^4 d^2 \rho_0 c}{3\pi}. \tag{2.112}$$

A comparison can be made between the sound radiating efficiencies of monopoles and dipoles by comparing equations (2.89) and (2.112). The ratio of the sound power radiated by a dipole to that radiated by a monopole is

$$\frac{\Pi_D}{\Pi_M} = \frac{4k^2d^2}{3} \sim \left\{\frac{d}{\lambda}\right\}^2. \tag{2.113}$$

$\Pi_D$ is the sound power radiated by the dipole, $\Pi_M$ is the sound power radiated by the monopole and $k = 2\pi/\lambda$, where $\lambda$ is the wavelength. It can be seen from the equation that at low frequencies (long wavelengths), the dipole is a very much less efficient radiator of noise than at high frequencies (shorter wavelengths) when compared with a monopole of the same source strength.

A dipole is produced by the fluctuating pressure forces acting along the axis between the two sources. These pressure forces are generated by the out-of-phase oscillations of the two monopoles in close proximity to each other. It is therefore useful to describe the sound power radiated by a dipole in terms of the fluctuating force (along the dipole axis) which is a function of the source strength and the source separation. The derivation of the radiated sound power in terms of the fluctuating force is available in Reynolds[2.5]. It is estimated by evaluating the r.m.s. fluctuating force, $F_{rms}$, on a spherical surface containing the dipole, and is

$$\Pi = \frac{3F_{rms}^2k^2}{4\pi\rho_0 c}. \tag{2.114}$$

The sound power radiated by aerodynamic dipole sources can be estimated from either equation (2.112) or (2.114) by using the same procedures developed in the previous sub-section for aerodynamic monopoles. The source strength, $Q_{rms}$, scales as $(2d)^2U$, where $2d$ is the separation between the two monopole sources in the region of fluid flow, and $U$ is the mean flow velocity. The fluctuating force, $F_{rms}$, scales as $\rho_0U^2d^2$. The characteristic frequency in the flow scales as $U/(2d)$, and this has to equal the frequency of aerodynamic sound generation for dimensional consistency. Thus,

$$\Pi = \frac{Q_{rms}^2\omega^4d^2\rho_0 c}{3\pi c^4} \approx \frac{(2d)^4U^2U^4d^2\rho_0 c}{(2d)^43\pi c^4} \approx \frac{\rho_0 d^2U^6}{3\pi c^3}. \tag{2.115}$$

Equation (2.115) suggests that the sound power radiated by an aerodynamically generated dipole scales with the sixth power of the flow velocity (a monopole scales with the fourth power of the flow velocity).

The directivity patterns of monopoles and dipoles in free space are schematically illustrated in Figure 2.9. Whilst the monopoles are omni-directional, the dipoles have a typical 'figure 8' sound radiation pattern and there is no sound radiation perpendicular to the dipole axis.

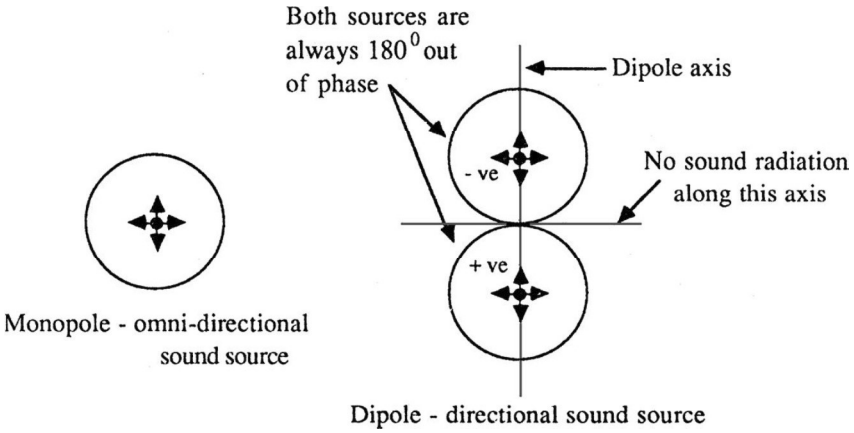



**Fig. 2.9.** Directivity patterns of monopoles and dipoles in free space.

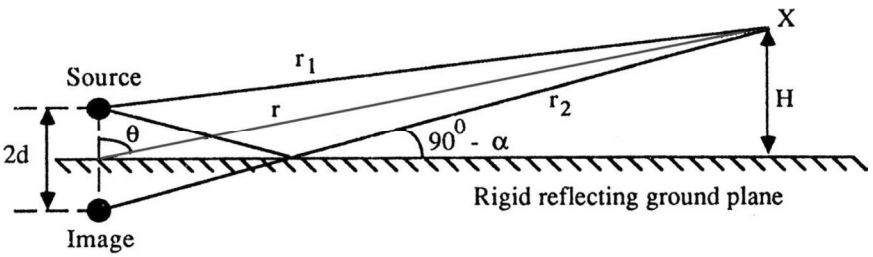
### 2.3.3 Monopoles near a rigid, reflecting, ground plane

Quite often in practice, most industrial noise sources are mounted on a ground plane or in close proximity to it. In the far-field, they can often be approximated as single point sources. The effects of the ground plane have to be accounted for though. These effects are particularly pronounced when the sound source is less than one acoustic wavelength ($\lambda$) from the ground plane. The analysis which follows is not to be confused with the concept of directivity which the reader might be familiar with. The directional effects of floors, intersecting walls, corners etc. on the sound radiation characteristics of an omni-directional noise source are well known and documented – they are reviewed in chapter 4 in this book. This section relates to a point which is often omitted in the literature on noise control engineering – that the sound power of a source can be affected by rigid, reflecting planes. This effect is in addition to the conventional and well known directivity effects.

Consider the case of a monopole near a rigid, reflecting, ground plane as illustrated in Figure 2.10. At some point in the far-field, the sound pressure will be the sum of two sound waves – i.e. a direct and a reflected wave. The reflected wave can be modelled by an image monopole below the reflecting surface. The problem thus reduces to that of two interfering monopoles. In practice, the ground plane will have some finite reflection coefficient (not all the sound will be necessarily reflected) and there will be some finite phase difference between the two waves. If one assumes that the ground plane is a hard reflecting surface, as an upper limit, then the reflection coefficient is unity and the phase difference between the two waves is zero. The problem thus reduces to two in-phase monopoles of equal source strength (a dipole was modelled in the previous sub-section as two out-of-phase monopoles of equal source strength).

The combined velocity potential at the observer position (some point, $X$, in space) can be obtained from equation (2.103) with $\mathbf{Q_1}(t)$ and $\mathbf{Q_2}(t)$ being of equal strength and phase. It is

$$\boldsymbol{\Phi}(r, \theta, t) = -\frac{Q_{\mathrm{p}}\, \mathrm{e}^{\mathrm{i}(\omega t - kr)}}{4\pi r} 2\cos(kd\cos\theta). \tag{2.116}$$




$$\Gamma = \frac{Z_s \cos\alpha - \rho_0 c}{Z_s \cos\alpha + \rho_0 c}$$

$\Gamma$ - reflection coefficient
$Z_s$ - specific acoustic impedance of reflecting surface
$\alpha$ - oblique angle of incidence
$\rho_0 c$ - characteristic acoustic impedance

**Fig. 2.10.** Monopole near a rigid, reflecting ground plane.

When $d \ll \lambda$, $kd \ll 1$ and the above equation simplifies to

$$\phi(r,t) = -\frac{2Q_p\, e^{i(\omega t - kr)}}{4\pi r}. \tag{2.117}$$

Equation (2.117) is simply double the far-field velocity potential for a monopole sound source (see equation 2.83)! The hard, reflecting ground plane has resulted in a doubling of the velocity potential. This result is very important. It is often overlooked by noise control engineers when estimating noise radiation levels from sound sources. This doubling of the velocity potential produces a fourfold increase in the sound intensity (equations 2.61, 2.85 and 2.86 with the appropriate velocity potential). In turn, there is a twofold increase in the radiated sound power because the intensity has only got to be integrated over half space (the other half is baffled by the rigid ground plane). The sound intensity is

$$I(r) = \frac{Q_{rms}^2 k^2 \rho_0 c}{4\pi^2 r^2}, \tag{2.118}$$

and the radiated sound power is

$$\Pi = 2\pi r^2 I(r) = \frac{Q_{rms}^2 k^2 \rho_0 c}{2\pi}. \tag{2.119}$$

The interesting result to come out of this approximate (i.e. limit) analysis is that the sound power of the monopole has been doubled. This is essentially because whilst the strength, $Q_{rms}$, and the surface vibrational velocity of the source have not changed (from when it is radiating into free space), the reflecting plane has produced a velocity potential, or a pressure, doubling. So, instead of having a constant sound power, the source has a constant volume velocity. These concepts of constant volume velocity sources, as opposed to the more commonly referred to constant power sources, will be discussed from an engineering noise control point of view in chapter 4. They can

be regarded as an upper limit – in practice the effects on non-perfect reflection from the ground plane will reduce the effect of the image source.

Reynolds[2.5] performs a more rigorous analysis for the effects of a reflecting plane on a monopole and integrates the complete expressions for sound intensity to obtain an expression for the radiated sound power. For the case of perfect reflection and zero phase difference, it is

$$\Pi = \frac{Q_{\text{rms}}^2 k^2 \rho_0 c}{4\pi}\left\{1 + \frac{\sin 2kd}{2kd}\right\}. \tag{2.120}$$

For small $kd$ (i.e. the source is located less than an acoustic wavelength from the surface) this reduces to equation (2.119) and for large $kd$ (i.e. the source is located many acoustic wavelengths from the surface) it reduces to a monopole in free space.

Another interesting and somewhat important point to be noted is that at very large distances (typically of the order of several hundred metres) a critical oblique angle of incidence is reached beyond which the reflection coefficient changes sign (Bies[2.7]). When this happens, the source and its image are out of phase rather than in phase – i.e. the source is now a dipole. This has important consequences for the far-field noise radiation. Referring back to equation (2.110) for a dipole far-field sound intensity, the angle $\cos\theta$ can be approximated by $H/r$, where $H$ is the distance from the measurement point to the ground plane (see Figure 2.10). Thus the intensity at the point of interest in the far-field is

$$I(r) = \frac{Q_{\text{rms}}^2 H^2 k^4 d^2 \rho_0 c}{4\pi^2 r^4}. \tag{2.121}$$

The important practical point to come out of this is that the intensity of the single sound source now scales as $r^{-4}$ rather than $r^{-2}$. It will be seen in chapter 4 (where decibels and other noise measurement units are defined) that this produces a 12 dB decay of sound pressure level per doubling of distance rather than the usual 6 dB per doubling of distance for a point source. Bies[2.7] discusses this phenomenon in some detail.

### 2.3.4 Sound radiation from a vibrating piston mounted in a rigid baffle

The sound radiation from a vibrating piston mounted in an infinite baffle is a classical problem – one which is covered in numerous books on fundamental acoustics (e.g. Kinsler *et al.*[2.1], Reynolds[2.5], Ford[2.8]). The vibrating piston can be either a vibrating surface or a vibrating layer of air. The primary assumption in the analysis (one which is not strictly correct in practice for 'real' surfaces) is that all parts of the piston vibrate in phase and with the same amplitude. Its relavance to engineering noise control is that it serves as an introduction to the sound radiation from different types of surfaces, e.g. loudspeakers, open ends of flanged pipes, plates and shells etc.

Consider a flat, circular piston of radius $z$ which is mounted in an infinite, rigid baffle as illustrated in Figure 2.11. The noise radiated by the vibrating piston can be modelled

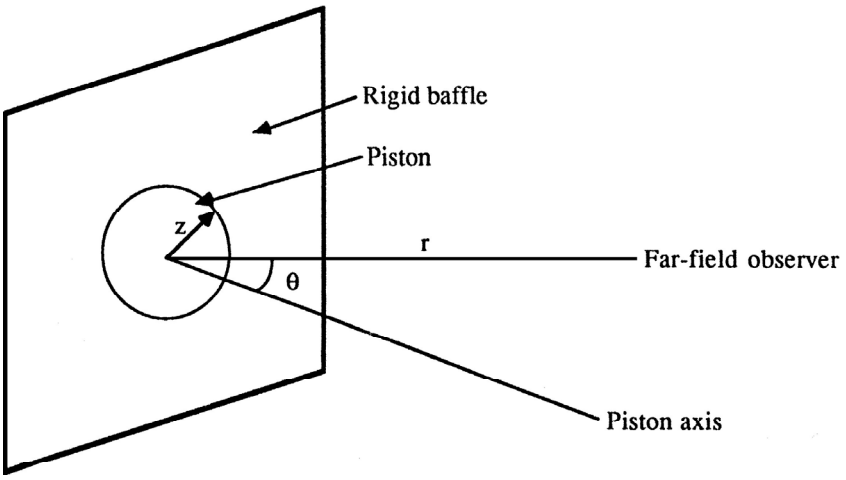



**Fig. 2.11.** Piston mounted in a rigid baffle.

in terms of numerous point monopoles (monopoles where $ka \ll 1$) radiating together. Each of the monopoles is, however, radiating from a rigid, reflecting, ground plane and not from free space. The sound pressure due to any one of the baffled monopoles is therefore twice that of an equivalent monopole in free space. It is

$$\mathbf{p}(r, t) = \frac{\mathrm{i}k\rho_0 c}{2\pi r} Q_\mathrm{p}\, \mathrm{e}^{\mathrm{i}(\omega t - kr)}. \tag{2.122}$$

In this equation, $Q_\mathrm{p}$ represents the source strength of the elemental monopole on the piston surface and it is equal to $U_\mathrm{p}\,\delta S$, where $U_\mathrm{p}$ is the peak surface velocity of the monopole and $\delta S$ is the elemental surface area. The total acoustic pressure fluctuations due to the vibrating piston is simply the resultant pressure due to all the point monopoles vibrating in phase and it is obtained by integration over the whole surface area. It is [2.1,2.5,2.8]

$$\mathbf{p}(r, \theta, t) = \frac{\mathrm{i}k\rho_0 c\pi z^2 U_\mathrm{p}\, \mathrm{e}^{\mathrm{i}(\omega t - kr)}}{2\pi r} \left\{ \frac{2J_1(kz \sin\theta)}{kz \sin\theta} \right\}. \tag{2.123}$$

In the above equation, $U_\mathrm{p}\, \mathrm{e}^{\mathrm{i}\omega t}$ is the surface velocity of the piston (i.e. each of the monopoles has the same surface velocity and phase). The radiated sound pressure has a similar form to that of a monopole in a reflecting ground plane with the exception of the term in brackets which is a directivity factor. $J_1$ is the first-order Bessel function and it can be readily evaluated from tables.

The corresponding sound intensity in the far-field [2.3]

$$I(r, \theta) = \frac{\rho_0 c k^2 U_\mathrm{rms}^2 \pi^2 z^2}{4\pi^2 r^2} \left\{ \frac{2J_1(kz \sin\theta)}{kz \sin\theta} \right\}^2. \tag{2.124}$$

Once again, the sound intensity has a similar form to that of a monopole in a reflecting ground plane with the exception of the term in brackets which is a directivity factor. The form of the directivity factor is presented in Figure 2.12. It is quite clear that the sound

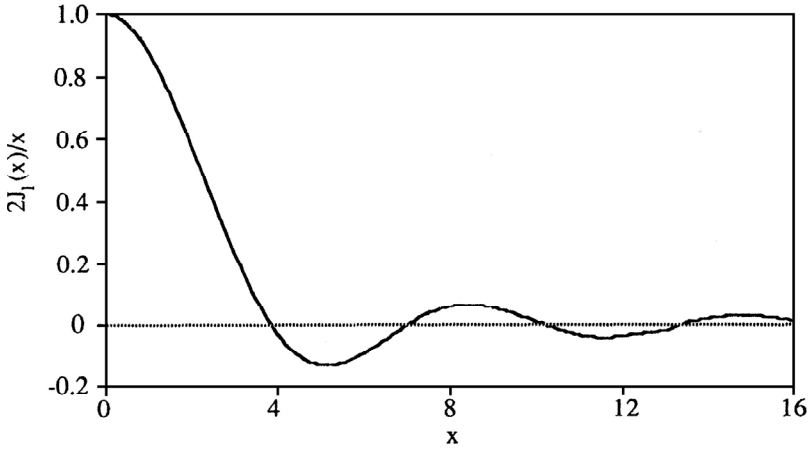



**Fig. 2.12.** Functional form of the directivity factor for a circular piston in a rigid baffle.

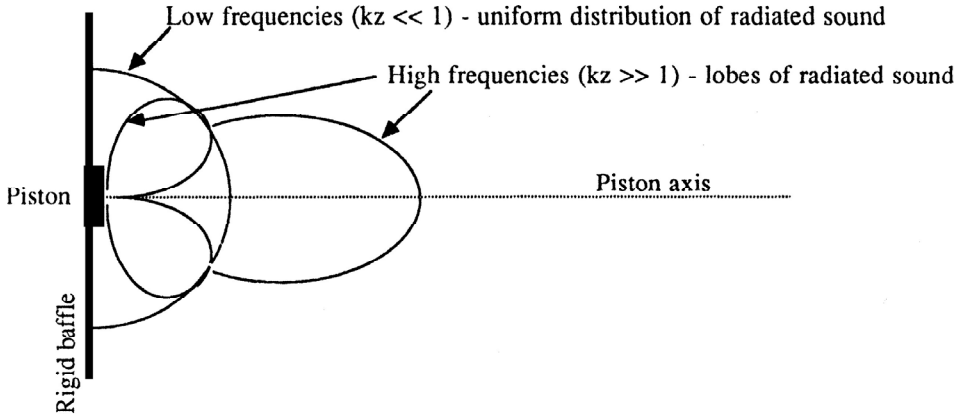



**Fig. 2.13.** Low and high frequency sound radiation patterns for a circular piston in a rigid baffle.

radiation from the piston is quite directional (except along the axis) and that it increases with frequency. There are several pressure nodes, and this results in a beam pattern of sound radiation. This is schematically illustrated in Figure 2.13. At low frequencies ($kz \ll 1$) the intensity distribution is approximately constant, whereas at high frequencies there are several nodal points and corresponding lobes of radiated sound. Hence, low frequency loudspeakers can be large and still remain omni-directional whereas high frequency loudspeakers need to be small to be relatively omni-directional. The concepts of directivity will be discussed again in chapter 4. This example illustrates how a series of omni-directional sound sources can become directional when combined.

The preceding discussion has been restricted to the acoustic far-field. Now consider an observation point on the piston surface itself. In chapter 1, the concepts of mechanical impedance (force/velocity at point of application of the force) were introduced. In acoustics, when a structure radiates sound due to its vibration, another impedance term has to be included with the mechanical impedance. It is the radiation impedance of the fluid (air) in proximity to the vibrating surface – i.e. the fluid loads the vibrating surface and this alters its vibrational response. The total sound pressure at any arbitrary element on the piston surface is a sum of the pressure due to the vibrating element itself and

the radiated pressures from all the other elements on the piston. The piston velocity, $\mathbf{U} = U_{\mathrm{p}}\,\mathrm{e}^{\mathrm{i}\omega t}$ is thus given by

$$\mathbf{U} = \frac{\mathbf{F_m}}{\mathbf{Z_m} + \mathbf{Z_r}}, \tag{2.125}$$

where $\mathbf{F_m}$ is the applied mechanical force, $\mathbf{Z_m}$ is the mechanical impedance of the piston, and $\mathbf{Z_r}$ is its radiation impedance. $\mathbf{F_m}$ is not to be confused with the force on the piston due to the acoustic pressure fluctuations, $\mathbf{F_p}$. The mechanical impedance is associated with the mechanical driving force and the radiation impedance is associated with the acoustic driving force. It can be seen from equation (2.125) that the radiation impedance 'fluid-loads' the surface vibrations of the piston. Fluid loading concepts are very relevant when analysing the vibrational characteristics of structures immersed in fluids – e.g. piping systems filled with liquids, submarines etc. The radiation impedance is thus given by

$$\mathbf{Z_r} = \frac{\mathbf{F_p}}{\mathbf{U}}. \tag{2.126}$$

The radiation impedance of a vibrating surface is sometimes defined as the ratio of the sound pressure averaged over the surface to the volume velocity through it (units of N s m$^{-5}$ as opposed to N s m$^{-1}$ for mechanical impedance) – i.e. some books would define $\mathbf{Z_r}$ as $(F_{\mathrm{p}}/\pi z^2)/\pi z^2 U$ or $F_{\mathrm{p}}/(\pi z^2)^2 U$. In this book, $\mathbf{Z_r}$ is defined, for convenience, in similar units to the mechanical impedance since the radiating surface area, $\pi z^2$, is common to both variables (pressure and volume velocity).

The radiation impedance of a piston can be obtained by integrating the elemental pressure distribution over the surface area of the piston to obtain the total sound pressure at a point and subsequently integrating this again over the surface to obtain the force, $\mathbf{F_p}$. The radiation impedance is thus obtained and given by[2.1,2.5]

$$\mathbf{Z_r} = \rho_0 c \pi z^2 \{R_1(2kz) + \mathrm{i}X_1(2kz)\}, \tag{2.127}$$

where

$$R_1(x) = \frac{x^2}{2\cdot 4} - \frac{x^4}{2\cdot 4^2\cdot 6} + \frac{x^6}{2\cdot 4^2\cdot 6^2\cdot 8} - \cdots, \tag{2.128}$$

and

$$X_1(x) = \frac{4}{\pi}\left\{\frac{x}{3} - \frac{x^3}{3^2\cdot 5} + \frac{x^5}{3^2\cdot 5^2\cdot 7} - \cdots\right\}. \tag{2.129}$$

The resistive function, $R_1(x)$, and the reactive function, $X_1(x)$, are plotted in Figure 2.14. The resistive part is real and is due to the radiated sound pressure. The imaginary part is a mass loading term due to the fluid (air or liquid) in proximity to the piston. This mass loading term can become significant when the structure is radiating into liquids.

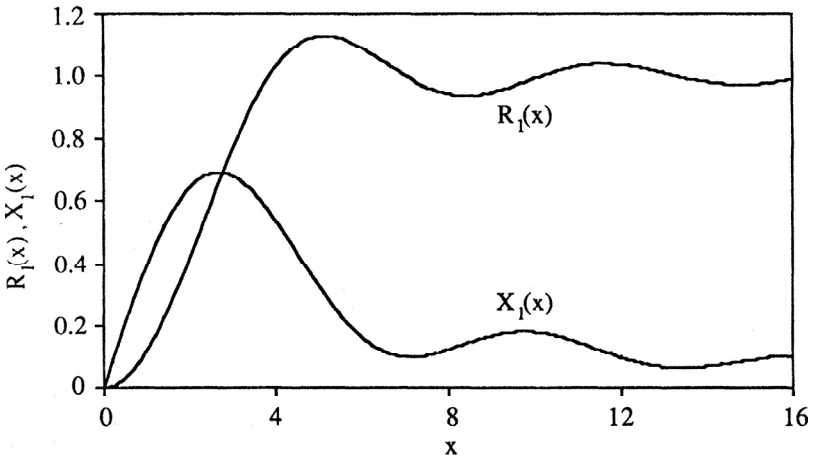



**Fig. 2.14.** Resistive and reactive functions for the radiation impendance of a circular piston.

The mechanical properties of the piston also affect its sound radiation properties. The equation of motion of the piston is (from Newton's second law)

$$M\frac{\partial \mathbf{U}}{\partial t} + C_{\mathrm{v}}\mathbf{U} + K_{\mathrm{s}}\int \mathbf{U}\,\mathrm{d}t = \mathbf{F_m} - \mathbf{F_p}, \tag{2.130}$$

where $M$ is the piston mass, $C_{\mathrm{v}}$ is its damping, and $K_{\mathrm{s}}$ is its stiffness. Thus equation (2.125) can be re-written as

$$\mathbf{U} = \frac{\mathbf{F_m}}{C_{\mathrm{v}} + \mathrm{i}(M\omega - K_{\mathrm{s}}/\omega) + \rho_0 c\pi z^2\{R_1(2kz) + \mathrm{i}X_1(2kz)\}}. \tag{2.131}$$

Equation (2.131) clearly illustrates how the vibrational velocity of the piston is a function of (i) its structural damping, (ii) its mass, (iii) its stiffness, (iv) the acoustic radiation resistance, and (v) the acoustic radiation reactance. As mentioned earlier, the acoustic radiation resistance is due to the radiated sound pressure and the acoustic radiation reactance is due to mass loading of the piston by the fluid. For low frequencies, $2kz \ll 1$, thus

$$R_1(2kz) \approx \frac{k^2z^2}{2}, \tag{2.132}$$

and

$$X_1(2kz) \approx \frac{8kz}{3\pi}. \tag{2.133}$$

For high frequencies, $2kz \gg 1$, thus

$$R_1(2kz) \approx 1, \tag{2.134}$$

and

$$X_1(2kz) \approx \frac{2}{\pi kz}. \tag{2.135}$$

The effects of mass loading can now be estimated. The amplitude of the fluid loaded mass is

$$M_{\mathrm{r}} = \frac{X_{\mathrm{r}}}{\omega} = \frac{\rho_0 c\pi z^2 X_1(2kz)}{\omega} = \frac{\rho_0 \pi z^2 X_1(2kz)}{k}. \tag{2.136}$$

Thus at low frequencies ($2kz \ll 1$)

$$M_{\mathrm{r}} = \frac{8\rho_0 z^3}{3}, \tag{2.137}$$

and at high frequencies ($2kz \gg 1$)

$$M_{\mathrm{r}} = \frac{2\rho_0 z}{k^2}. \tag{2.138}$$

The sound power radiated by the piston can now be estimated from the real part of the radiation impedance – i.e. the acoustic radiation resistance. It could also be obtained by integrating the far-field sound intensity (equation 2.124). From equation (1.162) the real power (rate of energy flow) of the piston is

$$\Pi = \tfrac{1}{2} U_{\mathrm{p}}^2 \,\mathrm{Re}\,[\mathbf{Z_m} + \mathbf{Z_r}]. \tag{2.139}$$

Thus,

$$\Pi = \tfrac{1}{2} U_{\mathrm{p}}^2 \{C_{\mathrm{v}} + \rho_0 c\pi z^2 R_1(2kz)\}, \tag{2.140}$$

where the first term inside the brackets represents the mechanical power that is dissipated and the second term represents the sound power that is radiated into the surrounding medium. Low and high frequency estimates of the radiated sound power can be readily obtained by substituting equations (2.132) and (2.134) into equation (2.140).

### 2.3.5 Quadrupoles – lateral and longitudinal

Monopoles were modelled in sub-section 2.3.1 as single oscillating spheres, and dipoles were modelled in sub-section 2.3.2 as two equal spheres oscillating out of phase. A natural extension to these acoustic source models is two dipole sources in close proximity to each other and oscillating 180° out of phase with each other. Such a sound source is called a quadrupole. Whereas a dipole has one axis (i.e. the fluctuating pressure forces act along the axis between the two sources), a quadrupole has two. The two dipoles oscillating out of phase with each other results in no nett addition or subtraction of mass away from the source, and no resultant force – i.e. there is no physical mechanism available for the mass or the momentum to vary. The quadrupole does, however, apply a stress to the medium and it is this fluctuating stress that generates the sound (monopoles generate sound via fluctuating surface velocities or the addition/subtraction of mass from a source region, and dipoles generate sound via fluctuating forces). In gas flows, for instance, quadrupoles are generated by the viscous stresses within the gas.

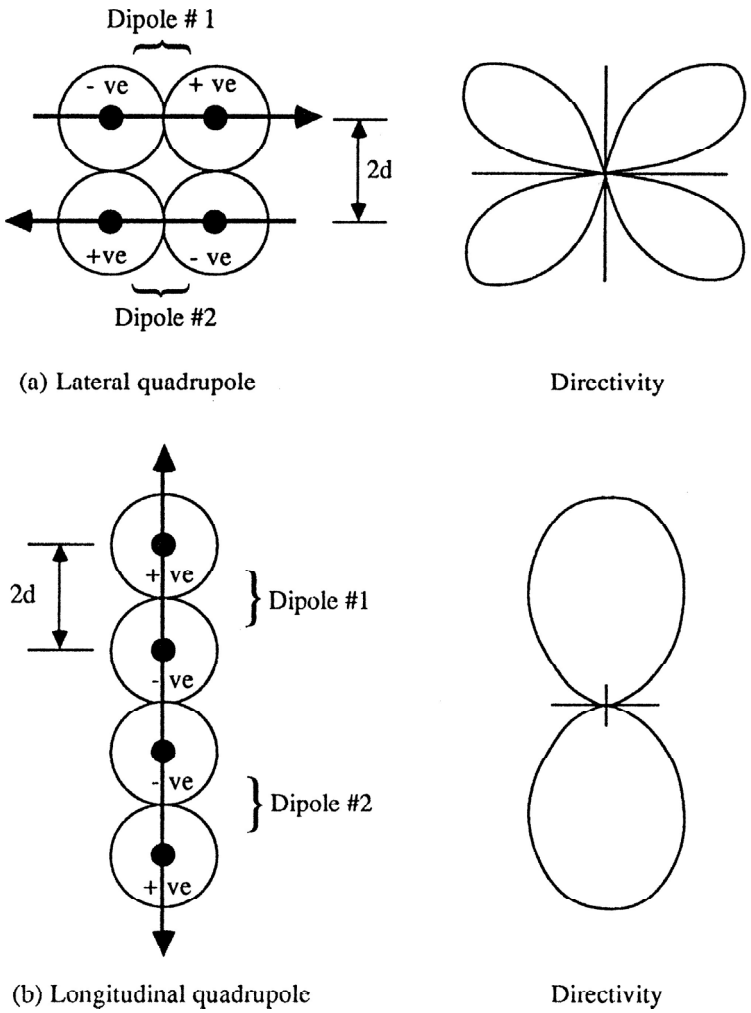



**Fig. 2.15.** Lateral and longitudinal quadrupoles and their associated directivity patterns.

The relationships for the intensity and sound power radiated by a dipole were obtained by considering the interference between two point monopoles in close proximity to each other. The relationships for the intensity and sound power radiated by a quadrupole can be obtained in a similar manner by considering the interference between two point dipoles in close proximity to each other or by expressing the quadrupole velocity potential as a space derivative of the monopole velocity potential. The mathematical manipulations required to obtain the answers are fairly extensive and are not presented here. The interested reader is referred to Pierce[2.4] or Reynolds[2.5]. Two possible combinations of quadrupoles exist. The first is when the two dipole axes do not lie on the same line, and the second is when they both lie on the same line. The former is termed a lateral quadrupole and the latter is termed a longitudinal quadrupole. The two types of quadrupoles and their associated directivity patterns are illustrated in Figure 2.15.

The sound power radiated by a lateral quadrupole is[2.4,2.5]

$$\Pi = \frac{4Q_{\text{rms}}^2 \rho_0 c d^4 k^6}{15\pi}, \tag{2.141}$$

and that radiated by a longitudinal quadrupole is

$$\Pi = \frac{4Q_{\text{rms}}^2 \rho_0 c d^4 k^6}{5\pi}. \tag{2.142}$$

In the above equation, $Q_{\text{rms}}$ is the source strength of one of the four monopoles that makes up the quadrupole, and $2d$ is the separation distance between any two monopoles.

The sound power radiated by an aerodynamic quadrupole can be estimated by using the same procedures that were used for monopoles and dipoles. The source strength, $Q_{\text{rms}}$, scales as $(2d)^2 U$, where $U$ is the mean flow velocity, and the characteristic frequency in the flow scales as $U/(2d)$. Thus, for a lateral quadrupole,

$$\Pi = \frac{4Q_{\text{rms}}^2 \rho_0 c d^4 \omega^6}{15\pi c^6} \approx \frac{\rho_0 d^2 U^8}{15\pi c^5}, \tag{2.143}$$

and for a longitudinal quadrupole

$$\Pi = \frac{4Q_{\text{rms}}^2 \rho_0 c d^4 \omega^6}{5\pi c^6} \approx \frac{\rho_0 d^2 U^8}{5\pi c^5}. \tag{2.144}$$

The equations suggest that the sound power generated by both lateral and longitudinal quadrupoles scale with the eighth power of the flow velocity (a monopole scales with the fourth power of the flow velocity and a dipole scales with the sixth power of the flow velocity).

A comparison can now be made between the sound radiating efficiencies of monopoles and quadrupoles by comparing equations (2.89) and (2.141). The ratio of the sound power radiated by a quadrupole to that radiated by a monopole is

$$\frac{\Pi_{\text{Q}}}{\Pi_{\text{M}}} \sim d^4 k^4 \sim \left\{\frac{d}{\lambda}\right\}^4. \tag{2.145}$$

Comparisons between this equation and equation (2.113) show that monopoles are the most efficient radiators of sound whereas quadrupoles are the least efficient. This important point is also deduced from the scaling relationships with flow velocity for aerodynamic type sources.

### 2.3.6 Cylindrical line sound sources

These types of sound sources are of some practical importance in that long lengths of pipeline, trains and traffic noise can often be modelled in terms of infinite or finite cylinders. For the case of the infinite cylinder, the sound radiation problem becomes two-dimensional. The wave equation has to be described in cylindrical co-ordinates and the velocity potential turns out to be a function of Hankel functions (Hankel functions are complex Bessel functions). Kinsler *et al.*[2.1], Pierce[2.4], Reynolds[2.5], Dowling and Ffowcs Williams[2.6], and Norton and Bull[2.9] all provide detailed discussions on the solutions to a range of analytical problems relating to cylindrical radiation. Pipe flow noise and vibration is discussed in chapter 7 in this book as a special case study.

An important feature of cylindrical radiation is that the sound intensity varies with $r^{-1}$ instead of $r^{-2}$ for monopole sources or $r^{-4}$ for dipoles. This suggests a 3dB decrease per doubling of distance. For an infinitely long cylinder of radius $a$, with a uniform pulsating harmonic surface velocity $\mathbf{U_r} = U_p\, e^{i\omega t}$, the far-field sound intensity is approximated by[2.10]

$$I(r) = U_{rms}^2 \rho_0 c \frac{\pi k a^2}{2r}. \tag{2.146}$$

Infinite and finite cylindrical line sources will also be discussed again in chapter 4 when considering the far-field propagation of sound waves.

## 2.4 The inhomogeneous acoustic wave equation – aerodynamic sound

So far in this chapter, the solutions to the wave equation have been sought in regions of space that do not contain any sources of sound. As pointed out in the introduction (section 2.1), this is generally the case in most industrial engineering type applications. Quite often, even when the source of sound is aerodynamically generated, one is only concerned with regions of space exterior to the source region, and classical acoustics is adequate to describe the behaviour of the wave field (e.g. external sound radiation from gas flow in a pipeline). When the source of sound is due to some vibrating body (structure-borne sound) it is generally readily identified. If, however, specific information is required about the source region itself, or the source is within the region of space of interest, then the classical, homogeneous, wave equation is inadequate. This is because different source types can in fact produce the same pressure distribution in the wave field exterior to the source region – this is an important point and it will be illustrated later on in this section.

In situations involving flow, and turbulence in particular, the identification of the sound sources is somewhat more difficult – they convect and interact with the fluid flow. Typical examples of instances where the inhomogeneous wave equation is required include unsteady duct flows, fan noise, jet noise, noise from aerofoils, noise from wall cavities, boundary layer noise etc. In these and other examples, in order to understand the mechanisms by which the sound is generated within the flow, and in order to understand the principles of the interactions of the flow with the acoustics one needs to obtain some basic information about the sources of sound. Quite often, this is easier said than done! The field of aerodynamic sound is a relatively new and complex one, and it is the subject of continued research. Its main impetus has been the jet aircraft industry.

Lighthill's[2.11,2.12] work which was published in the early 1950s provided the first general theory of aerodynamic sound. Lighthill reformulated the equations of fluid motion so as to include the source functions which drive the sound wave field. It was postulated that all the non-linearities in the motion of matter act as sources of sound,

and they were grouped as non-linear forcing terms on the right hand side of the wave equation. The sound radiated is thus estimated as if it were in a uniform medium with zero mean flow – i.e. convection effects, turbulence etc. are all incorporated within the source function on the right hand side. Lighthill's now famous 'acoustic analogy' states that the sources of sound in a fluid motion are simply the difference between the exact equations of fluid motion and the acoustical approximations.

The main problem which has limited researchers for many years since is that the source function (i.e. the difference between the exact equations of fluid motion and the acoustical approximations) is generally very difficult to evaluate. This is primarily because sufficient information about the fluid flow is not always readily available. Lighthill, in his acoustic analogy, reduced the problem of aerodynamically generated sound to an analogous classical acoustical problem. In that form, whilst the wave equation is exact, its main limitation is that it does not shed any light on any subsequent interaction between the aerodynamically generated sound and the fluid flow which caused it in the first place. It has been shown that Lighthill's analogy allows for a precise analysis of the sound field for low Mach number flows (i.e. compact source regions where the typical source dimension is significantly smaller than the corresponding acoustic wavelength). However, at higher flow speeds (including supersonic flows), it has been shown that 'excess noise' is often present due to flow–acoustic interactions. There have been several attempts to reformulate Lighthill's source terms, firstly to include the effects of solid boundaries (i.e. obstacles in the flow), secondly to minimise the requirements for a detailed description of the fluid flow in the source region (as far as noise radiation is concerned), and thirdly to shed some light on the effects that the interaction between the flow field and the sound field has on the radiated sound.

Curle[2.13], Powell[2.14] and Ffowcs Williams and Hall[2.15] (also see Dowling and Ffowcs Williams[2.6]) extended Lighthill's theory to incorporate the effects of solid bodies. Most of this work is now generally available in the research literature. However, only a few textbooks are currently available on the subject of aerodynamic sound. The reader is referred to three recent books – the first by Goldstein[2.16], the second by Dowling and Ffowcs Williams[2.6], and the third by Blake[2.17]. Blake's book (two volumes) in particular is the most recent comprehensive publication on aerodynamic sound; it is an up-to-date research monograph on the mechanics of flow-induced sound and vibration and it deals with general concepts, elementary sources and complex flow–structure interactions. Ffowcs Williams[2.18] also provides a comprehensive review article on aerodynamic sound in the *Annual Review of Fluid Mechanics*.

A major advance in the field of aerodynamic sound occurred in 1975 when Howe[2.19,2.20] rearranged the exact equations of conservation of mass and momentum in terms of entropy and vorticity variations. He subsequently described the interactions between flow and sound fields in terms of fluid vorticity and acoustical particle velocities associated with the generated sound field. The concepts relating to the generation

of aerodynamic sound by the motion of vortices in an unsteady fluid flow relate back to Powell[2.21]. In his review paper, Ffowcs Williams[2.18] points out that Howe's[2.19] identification of acoustic sources within vortical regions of isentropic flow is in fact a formalisation of the steps originally taken by Powell[2.21]. It is now widely recognised amongst researchers in the field that Lighthill's theory of aerodynamic sound (1952) and the Powell–Howe theory of vortex sound (1964, 1975) are the two singularly most important advances in the field of aerodynamic sound to date. Howe's contribution, in particular, is a successful attempt to separately describe both the flow–acoustic interactions and the mean flow effects.

Whilst it is felt that it is necessary for noise control engineers to be aware of the existence of the inhomogeneous wave equation, and its relevance to the control of aerodynamically generated sound, a detailed discussion of the topic is well beyond the scope of this textbook. In the remaining sections of this chapter, the inhomogeneous acoustic wave equation is derived, and the basic solutions for some simple acoustic source processes are discussed, such as to provide the reader with sufficient background information to pursue the matter further if required. Lighthill's acoustic analogy is derived and discussed in relation to fluid dynamically generated monopoles, dipoles and quadrupoles, but some of the more recent advanced work in the area is only qualitatively discussed.

### 2.4.1 The inhomogeneous wave equation

In section 2.3, solutions to the homogeneous wave equation were obtained for a range of simple sources. These solutions were obtained for far-field approximations and the instantaneous values were limited to constant frequencies – i.e. the solutions for instantaneous pressure, particle velocity etc. were obtained in terms of single frequency components using harmonic waves and complex algebra in very much the same way that instantaneous values of displacement, velocity, and acceleration were obtained in terms of single frequency components using harmonic waves and complex algebra in chapter 1. The resulting equations for mean-square pressure, intensity etc. were steady-state, time-averaged values. The procedures adopted in section 2.3 cannot be used if one requires instantaneous solutions for the acoustic variables (for all frequencies) and that the source be accounted for in the wave field. A somewhat different approach has to be taken – i.e. the initial time-dependent conditions have to be considered to describe the transient behaviour of the source.

Reconsider the simple spherical sound (monopole) of section 2.3.1. The solution for the acoustic pressure (equation 2.85) is valid everywhere except at the origin where a singularity occurs. It was stated at the outset that the homogeneous wave equation is only valid in the wave field and that it is not valid in the source field. The singularity at the origin of a simple spherical source is thus consistent with this argument!

A spherical sound source is a fluctuating source of mass at the origin. Its instantaneous source strength ($m^3\ s^{-1}$), $Q(t)$ is given by

$$Q(t) = 4\pi a^2 u_a(t), \tag{2.147}$$

where $u_a(t)$ is the instantaneous normal surface velocity, and $a$ is the radius. The mass flux per unit time ($kg\ s^{-1}$) through the volume is

$$Q(t) = 4\pi a^2 \rho u_a(t), \tag{2.148}$$

where $\rho$ is the density of the fluid. The rate of change of mass flux ($kg\ s^{-2}$) is thus

$$Q'(t) = 4\pi a^2 \frac{\partial(\rho u_a)}{\partial t}. \tag{2.149}$$

From the momentum equation (equation 2.11) neglecting the viscous stress terms

$$\frac{\partial(\rho u_a)}{\partial t} = -\frac{\partial P}{\partial x}, \tag{2.150}$$

and since $p = c^2\rho'$ (equations 2.29, 2.30 and 2.35),

$$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x} = c^2 \frac{\partial \rho'}{\partial x}. \tag{2.151}$$

Thus, in spherical (radial) co-ordinates

$$Q'(t) = -4\pi a^2 c^2 \frac{\partial \rho'}{\partial r}. \tag{2.152}$$

It was shown in section 2.3 (and in chapter 1) that $\{G(ct - r)\}/r$ satisfies the wave radiation condition and that it is a solution to the spherical, homogeneous wave equation. It is convenient in this section to represent the function as $\{G(t - r/c)\}/r$ instead. It is straightforward to show that it is also a general solution to the homogeneous wave equation. The term $r/c$ is the retardation time – i.e. the time it takes the sound wave to travel from the source to the observer. The concepts of retardation times are important in aerodynamic sound – this will become evident later on in this section. Thus,

$$\rho'(r, t) = \frac{1}{r} G\left(t - \frac{r}{c}\right) \tag{2.153}$$

is a solution to the homogeneous wave equation (in terms of the density fluctuations) for outward travelling waves. Hence, the rate of change of mass flux, $Q'(t)$, can be estimated by evaluating $\partial\rho'/\partial r$ at $r = a$ from equation (2.153) for $a \to 0$. Thus,

$$Q'(t) = 4\pi c^2 G(t), \tag{2.154}$$

and hence

$$G\left(t - \frac{r}{c}\right) = \frac{1}{4\pi c^2} Q'\left(t - \frac{r}{c}\right). \tag{2.155}$$

Thus, the solution to the homogeneous wave equation (in terms of the density fluctuations) for outward travelling waves is (from equation 2.153)

$$\rho'(r,t) = \frac{1}{4\pi c^2 r} Q'\left(t - \frac{r}{c}\right). \tag{2.156}$$

The acoustic pressure fluctuations are thus given by

$$p(r,t) = \frac{1}{4\pi r} Q'\left(t - \frac{r}{c}\right). \tag{2.157}$$

Equation (2.157) is consistent with equation (2.85) and

$$Q'\left(t - \frac{r}{c}\right) = \mathrm{Re}[\mathrm{i}k\rho_0 c\mathbf{Q}(t)\,\mathrm{e}^{-\mathrm{i}kr}]. \tag{2.158}$$

The exponential factor $\mathrm{e}^{-\mathrm{i}kr}$ in the above equation is analogous to the time shift $r/c$, since $\mathbf{Q}(t) = Q_\mathrm{p}\,\mathrm{e}^{\mathrm{i}\omega t}$.

Equation (2.157) is: (i) only valid in a region exterior to the source because of the singularity at the origin, (ii) a function of the rate of change of mass flux, and (iii) a function of the retardation time. It can be extended to a volume region in space where there is some rate of flux of mass per unit volume, $q(\vec{y}, t)$, with units of kg m$^{-3}$ s$^{-1}$. Note that the vector $\vec{y}$ represents spatial positions which are within the source region. The equation of conservation of mass (equation 2.5) is now

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho\vec{u} = q(\vec{y}, t). \tag{2.159}$$

It can be combined in the usual manner with the momentum equation to yield the wave equation, which is now inhomogeneous with the time derivative of $q$ (i.e. $q'(\vec{y}, t)$) as the forcing term (note that $q'$ has units of kg m$^{-3}$ s$^{-2}$). It is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = q'(\vec{y}, t). \tag{2.160}$$

Equation (2.160) is the inhomogeneous wave equation. The function $q'$ is only non-zero within the source region. In regions exterior to the source, it vanishes, and the wave equation reverts to being homogeneous.

One can readily see from equation (2.160) that in the wave field the pressure distributions can be the same for different forcing functions since $q'(\vec{y}, t)$ is zero outside the source region – i.e. different source distributions of equal strength will produce the same wave field. Information obtained in the wave field (from the homogeneous wave equation) will therefore not provide any information about the source distribution.

A general solution to equation (2.160) for the pressure in the sound field (exterior to the source region) can be obtained by an integration of terms similar to equation (2.157)

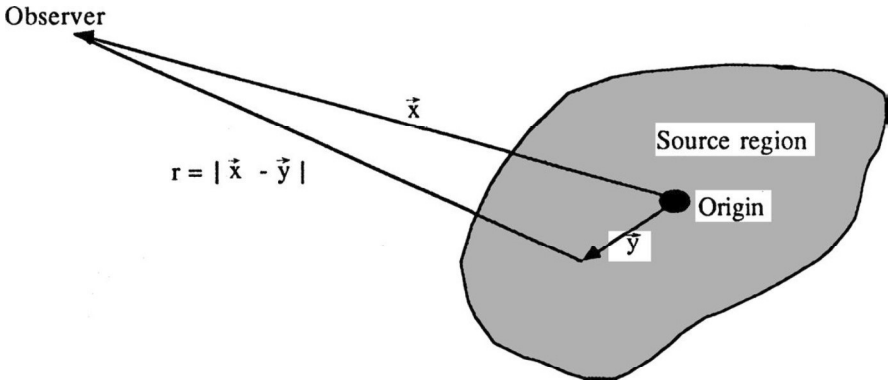



**Fig. 2.16.** Schematic illustration of source and observer positions.

over the whole source region. It is

$$p(\vec{x},t)=\int_V \frac{q'\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)}{4\pi|\vec{x}-\vec{y}|}\mathrm{d}^3\vec{y}. \tag{2.161}$$

The term $\vec{y}$ represents position vectors within the source region; the term $\vec{x}$ represents position vectors in the sound field which is exterior to the source region; and $|\vec{x}-\vec{y}|$ represents the distance between the source and the observer. This point is illustrated in Figure 2.16. Equation (2.161) is an important general equation in the field of aerodynamic noise. It can be re-written as

$$p(\vec{x},t)=\frac{1}{4\pi}\frac{\partial}{\partial t}\int_V \frac{q\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)}{|\vec{x}-\vec{y}|}\mathrm{d}^3\vec{y}, \tag{2.162}$$

where it can be clearly seen that the acoustic field, in this instance, is due to the rate of change of mass flux per unit volume – i.e. steady mass flux will not produce any sound; it is the unsteady mass flux that generates the sound!

Equation (2.161) is the formal solution to the inhomogeneous wave equation for a given arbitrary source function and it can take on many forms (e.g. equation 2.162 for mass flux). Equation (2.162) is thus the solution to the inhomogeneous wave equation for a monopole type sound source. It will be seen later on in this chapter that equation (2.161) can also be used to describe dipoles and quadrupoles. It has been derived heuristically (by comparison with equation 2.157) and it is instructive also to outline its derivation in a more formal sense.

The general solution to the inhomogeneous wave equation can be obtained by superposition of the acoustic fields due to simple point sources. In order to do this, a function called a Green's function needs to be introduced. Green's functions satisfy the inhomogeneous wave equation, and if the medium surrounding the source is stationary (zero mean flow) and is not restricted by any boundaries, they are commonly referred to as free space Green's functions. Green's functions are also used in classical acoustics,

and in fact were used earlier on in this chapter (this will become obvious shortly!). The reason why free space Green's functions are so relevant to aerodynamic sound is that Lighthill simply reduced the problem of predicting sound in a moving medium to that of the equivalent classical acoustical problem in a stationary fluid! – this too will be demonstrated in a short while. The free space Green's function is described by the transform pair[2.4,2.16]

$$\mathbf{G}_\omega(\vec{y}, \omega|\vec{x}, \omega) = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}, \tag{2.163}$$

and

$$G_t(\vec{y}, \tau|\vec{x}, t) = \frac{1}{4\pi r}\delta(\tau - t + r/c), \tag{2.164}$$

where

$$G_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{G}_\omega \mathrm{e}^{\mathrm{i}(\omega t - \tau)}\,\mathrm{d}\omega. \tag{2.165}$$

From equation (2.163) one can readily see how the Green's function was used (inadvertently) earlier on in this chapter. The Green's function in equation (2.163) can be interpreted as the solution to the wave equation in free space for an applied unit, time-harmonic, point source, and the Green's function in equation (2.164) can be interpreted as the solution to the wave equation in free space for a unit, impulsive, point source. An important property of the Green's function is that it is symmetrical – i.e. it remains the same when the source and the receiver position are interchanged. This property, commonly known as reciprocity, has important applications in noise and vibration control, and it will be discussed in some detail in chapters 3 and 6.

The Green's function for an impulsive, unit, point source located at $\vec{y}$ and set off at time $\tau$ satisfies the inhomogeneous wave equation – i.e.

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right\} G_t(\vec{y}, \tau|\vec{x}, t) = \delta(\tau - t)\,\delta(\vec{x} - \vec{y}). \tag{2.166}$$

The function $G_t$ must vanish for $t < \tau$ (this is referred to as the causality condition), and it must exhibit outgoing wave behaviour (this is referred to as the radiation condition). Thus, for some source function $f(\vec{y}, \tau)$,

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right\} f(\vec{y}, \tau) G_t(\vec{y}, \tau|\vec{x}, t) = f(\vec{y}, \tau)\,\delta(\tau - t)\,\delta(\vec{x} - \vec{y}). \tag{2.167}$$

Hence,

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right\} \int_T \int_V f(\vec{y}, \tau)\frac{\delta(\tau - t + r/c)}{4\pi r}\,\mathrm{d}\tau\,\mathrm{d}^3\vec{y}$$
$$= \int_T \int_V f(\vec{y}, \tau)\,\delta(\tau - t)\,\delta(\vec{x} - \vec{y})\,\mathrm{d}\tau\,\mathrm{d}^3\vec{y}. \tag{2.168}$$

Thus,

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right\}\int_V \frac{f(\vec{y},t-r/c)}{4\pi r}\,\mathrm{d}^3\vec{y}=\int_V f(\vec{y},t)\,\delta(\vec{x}-\vec{y})\,\mathrm{d}^3\vec{y}. \tag{2.169}$$

Now, in terms of the pressure fluctuations, the inhomogeneous wave equation is

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right\}p=f(\vec{x},t)=\int_V f(\vec{y},t)\,\delta(\vec{x}-\vec{y})\,\mathrm{d}^3\vec{y}, \tag{2.170}$$

where $f(\vec{x},t)$ is an arbitrary source function. A general solution can be obtained for $p(\vec{x},t)$ by equating equations (2.169) and (2.170). Hence, for $r=|\vec{x}-\vec{y}|$,

$$p(\vec{x},t)=\int_V \frac{f\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)}{4\pi|\vec{x}-\vec{y}|}\,\mathrm{d}^3\vec{y}. \tag{2.171}$$

Equation (2.171) is the formal solution for the acoustic pressure fluctuations (in a region where there is zero mean flow and no solid boundaries) for an arbitrary source function, $f$. It is identical in form to equation (2.161) and is a superposition of all the waves generated by all the elements within the source region.

For a sound source that only represents mass flux into a volume, the function $f$ in equation (2.171) is replaced by the rate of change of mass flux per unit volume, $q'$ (kg m$^{-3}$ s$^{-2}$). The rate of change of mass flux through the whole volume region is

$$Q'(t)\int_V q'(\vec{y},t)\,\mathrm{d}^3\vec{y}. \tag{2.172}$$

If it is assumed that the source region is compact, small phase differences between the various source elements can be neglected – i.e. retardation times can be neglected as far as a far-field observer is concerned, and

$$p(r,t)=\frac{1}{4\pi r}\int_V q'(\vec{y},t-r/c)\,\mathrm{d}^3\vec{y}. \tag{2.173}$$

Hence,

$$p(r,t)=\frac{1}{4\pi r}Q'\left(t-\frac{r}{c}\right), \tag{2.174}$$

where $r=|\vec{x}-\vec{y}|\approx|\vec{x}|$. Equation (2.174) is identical to equation (2.157) and it represents a point monopole radiation field with uniform directivity – the amplitude of the pressure field is proportional to the nett rate of change of mass flux over the whole source region.

For a sound source that only represents some externally applied force $f_i(\vec{x},t)$, the function $q'$ is replaced by the divergence of the vector force – i.e.

$$q'(\vec{x},t)=-\mathrm{div}\,f_i(\vec{x},t)=-\frac{\partial f_i(\vec{x},t)}{\partial x_i}. \tag{2.175}$$

The function $f_i(\vec{x}, t)$ represents an external applied body force per unit volume – e.g. a solid body in the flow – $f_i$ is a vector in the $i$th direction, thus $\partial f_i/\partial x_i$ is a scalar. The momentum equation (neglecting viscous stresses) becomes

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial P}{\partial x_i} = f_i(\vec{x}, t), \tag{2.176}$$

the equation of conservation of mass remains as per equation (2.5), and the inhomogeneous wave equation therefore becomes

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\frac{\partial f_i(\vec{x}, t)}{\partial x_i} = -\text{div}\, f_i(\vec{x}, t). \tag{2.177}$$

The source term on the right hand side of equation (2.177) has to be such that it vanishes outside some finite region of space in proximity to the source. Hence, by analogy with equations (2.170) and (2.171) the solution $p(\vec{x}, t)$ is

$$p(\vec{x}, t) = -\int_V \frac{\text{div}\, f_i\left(\vec{y}, t - \dfrac{|\vec{x} - \vec{y}|}{c}\right)}{4\pi|\vec{x} - \vec{y}|}\, \mathrm{d}^3\vec{y}, \tag{2.178}$$

or

$$p(\vec{x}, t) = -\frac{\partial}{\partial x_i}\int_V \frac{f_i\left(\vec{y}, t - \dfrac{|\vec{x} - \vec{y}|}{c}\right)}{4\pi|\vec{x} - \vec{y}|}\, \mathrm{d}^3\vec{y}. \tag{2.179}$$

There is a subtle difference between equations (2.178) and (2.179). In the former equation, the source is represented as a series of monopoles of strength $\partial f_i/\partial y_i$. In the latter, it is represented as a dipole of strength $f_i$ – the force exerted on the fluid volume is equal to the rate at which the momentum changes, since the mass flux in and out of the fluid volume is the same. By retaining the solution in the form of equation (2.178), all the various retardation times between the various source elements and the observer position would have to be known. If the retardation times were neglected, the volume integral would reduce to zero and there would be no sound produced! This can be proved mathematically[2.6] but it is to be intuitively expected because there is, by definition, a zero rate of mass flux (i.e. $Q' = 0$). The various monopole elements within the source region effectively cancel each other, and if it were not for the retardation time differences between all the various source elements, the nett sound radiation would be zero. Thus, by transforming equation (2.178) into the dipole distribution of equation (2.179), the radiated sound pressure can be estimated from a knowledge of the force distribution and the requirement for a detailed knowledge of the retardation times between source and observer positions is thus avoided.

Equation (2.179) can be further simplified for engineering approximations. Assuming $r = |\vec{x} - \vec{y}|$, it can be re-written as

$$p(\vec{x}, t) = -\int_V \frac{\partial r}{\partial x_i}\frac{\partial}{\partial r}\frac{f_i\left(\vec{y}, t - \dfrac{|\vec{x} - \vec{y}|}{c}\right)}{4\pi r}\,\mathrm{d}^3\vec{y}. \tag{2.180}$$

Hence,

$$\mathrm{p}(\vec{x}, t) = \frac{1}{4\pi}\int_V \frac{\partial r}{\partial x_i}\left\{\frac{f_i}{r^2} + \frac{1}{cr}\frac{\partial f_i}{\partial t}\right\}\mathrm{d}^3\vec{y}, \tag{2.181}$$

since

$$\frac{\partial f_i}{\partial r} = -\frac{1}{c}\frac{\partial f_i}{\partial t}. \tag{2.182}$$

Now, $\partial r/\partial x_i$ can be interpreted as the dipole directivity factor – i.e. it is the cosine of the angle between the dipole axis (the line along which the fluctuating force is acting) and the distance between the source and the observer. Thus,

$$p(\vec{x}, t) = \frac{1}{4\pi}\int_V \left\{\frac{f_i}{r^2} + \frac{1}{cr}\frac{\partial f_i}{\partial t}\right\}\cos\theta_i\,\mathrm{d}^3\vec{y}. \tag{2.183}$$

The radiated pressure has both a near- and a far-field component in addition to the $\cos\theta$ angular dependence.

### 2.4.2 Lighthill's acoustic analogy

In the previous sub-section it has been illustrated that (i) for a monopole type aerodynamic sound source, the amplitude of the radiated sound pressure field is a function of the rate of change of mass flux over the whole source region, and (ii) for a dipole type aerodynamic sound source, the amplitude of the radiated sound pressure field is a function of an externally applied body force. Acoustic sources can also be generated in flow situations where the flow velocities are large enough not to comply with the linear acoustic equations – i.e. high Reynold's number flows.

Lighthill[2.11] considered an unbounded region of space, a part of which included a fluctuating fluid flow, and grouped all the non-linear terms in the fluid dynamic equations of motion together as acoustic source terms on the right hand side of the inhomogeneous wave equation (i.e. the function $f(\vec{x}, t)$ in equation 2.170). In so doing, the radiated sound pressure due to the high Reynold's number flows could be calculated by using the procedures described previously (equation 2.171) provided that information is available about the source terms. Lighthill's acoustic analogy is thus based on the fact that the sources of sound are the difference between the exact laws of fluid motion and the linearised acoustical approximations – it is the non-linearities that generate the sound!

Consider an unbounded region of space in which there is a fluctuating fluid flow. Conservation of mass states that the time rate of change of mass within a region has to

equal the nett mass flow into it. Thus, from sub-section 2.2.1 (equation 2.5) for mass conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0. \tag{2.184}$$

It is conventional and convenient to use tensor rather than vector notation in aerodynamic sound and the above equation can be re-written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0. \tag{2.185}$$

Here, the suffix $i$ can take on any value from 1 to 3. Hence a product term such as $u_i u_j$ has nine possible permuations since $i = 1, 2, 3$ and $j = 1, 2, 3$.

Conservation of momentum states that the time rate of change of momentum in a region equals the sum of all the forces acting on the region plus the nett rate of flow of momentum through the region. For non-linear flow, where viscous effects associated with the shear forces due to the interaction of fluid particles are taken into account, conservation of momentum implies that

$$\frac{\partial (\rho \vec{u})}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} + \vec{\nabla} P = \nu \nabla^2 \vec{u}, \tag{2.186}$$

where $\nu$ is the coefficient of shear viscosity. In tensor notation, this is

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial (p \delta_{ij})}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} = 0, \tag{2.187}$$

where

$$\nu \nabla^2 \vec{u} = \frac{\nu \partial^2 u_i}{\partial x_j \partial x_j}, \quad \tau_{ij} = \nu \frac{\partial u_i}{\partial x_j}, \tag{2.188}$$

$\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and $\tau_{ij}$ is the viscous shear stress. Thus,

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (p \delta_{ij} - \tau_{ij} + \rho u_i u_j)}{\partial x_j} = 0. \tag{2.189}$$

Equation (2.189) is the exact momentum equation in tensor notation and $\rho u_i u_j$ is the unsteady Reynold's shear stress.

The wave equation can now be derived in the usual manner by taking the time derivative $(\partial / \partial t)$ of the equation of conservation of mass, and the divergence $(\partial / \partial x_i)$ of the equation of conservation of momentum. Thus,

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial^2 (\rho u_i)}{\partial x_i \partial t} = 0, \tag{2.190}$$

and

$$\frac{\partial^2 (\rho u_i)}{\partial x_i \partial t} + \frac{\partial^2 (p \delta_{ij} - \tau_{ij} + \rho u_i u_j)}{\partial x_i \partial x_j} = 0. \tag{2.191}$$

Subtracting equation (2.191) from (2.190) yields

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{\partial^2 (p\delta_{ij} - \tau_{ij} + \rho u_i u_j)}{\partial x_i \partial x_j}, \tag{2.192}$$

since $\rho = \rho_0 + \rho'$, and $\rho_0$ is time invariant. Equation (2.192) can now be forced to take the form of the inhomogeneous wave equation by introducing the factor $c^2\nabla^2\rho'$ and subtracting it from both sides. Thus,

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2\nabla^2\rho' = \frac{\partial^2 (p\delta_{ij} - \tau_{ij} + \rho u_i u_j - c^2\rho'\delta_{ij})}{\partial x_i \partial x_j}, \tag{2.193}$$

since

$$c^2\nabla^2\rho' = c^2 \frac{\partial^2 \rho'\delta_{ij}}{\partial x_i \partial x_j}. \tag{2.194}$$

The Lighthill stress tensor is defined as

$$T_{ij} = p\delta_{ij} - \tau_{ij} + \rho u_i u_j - c^2\rho'\delta_{ij}; \tag{2.195}$$

hence, Lighthill's non-linear, viscous, inhomogeneous wave equation is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{2.196}$$

with $p$ replacing $c^2\rho'$ (i.e. see equations 2.29, 2.30 and 2.35).

The source term on the right hand side of Lighthill's equation represents a double divergence – i.e. there is a double tendency for the various source elements within the fluid to cancel. It was seen in the previous sub-section that a dipole is produced as a consequence of cancelling monopoles – the instantaneous dipole strength is zero and the sound radiation is due to the retardation time effects, i.e. the cancellation is not complete because of the time delays between the various sound waves reaching the observer position. The same arguments apply here and the sound radiation is once again due to the retardation time effects between the various source elements. This double divergence of a monopole field (or divergence of a dipole field) is a quadrupole. Because of the double tendency for cancellation one would qualitatively expect quadrupole sound radiation to be less efficient than dipole sound radiation, and dipole sound radiation to be less efficient than monopole sound radiation.

The solution to Lighthill's equation for the radiated sound pressure can be readily obtained from equation (2.179). It is

$$p(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\vec{y}, t - \dfrac{|\vec{x} - \vec{y}|}{c}\right)}{4\pi|\vec{x} - \vec{y}|} \mathrm{d}^3\vec{y}. \tag{2.197}$$

The qualitative arguments that were used in the previous section to justify proceeding from equation (2.178) to equation (2.179) also apply here – i.e. the formal solution

has the second derivative inside the volume integral (i.e. $\partial^2/\partial y_i \partial y_j$) and it is transformed into equation (2.197) with the derivative outside the integral. Goldstein[2.16] provides the formal mathematical procedures that have to be followed in proceeding with the transformation. This transformation procedure is the key to the correct estimation of aerodynamic sound because it is the small retardation time differences between the different source elements that in fact produce the radiated sound. The dipole and quadrupole representations of the source elements account for the cancellations that occur, and thus allow for a more accurate estimate of the radiated sound.

In aerodynamic sound predictions, the main problem is therefore to suitably identify the components within the Lighthill stress tensor, $T_{ij}$, that are dominant. The $p\delta_{ij} - c^2\rho'\delta_{ij}$ component represents the effects of heat conduction which affect the speed of sound within the fluid. For low Mach number flows this is a second-order effect. The $\tau_{ij}$ component is the viscous shear stress, and this is generally very small when compared with the Reynold's shear stress component, $\rho u_i u_j$. Hence, for low Mach number flows, it is the Reynold's shear stresses, $\rho u_i u_j$, which are dominant. For linear, inviscid flows, the wave equation is homogeneous – this is consistent with Lighthill's wave equation since $p = c^2\rho'$, $\tau_{ij} = 0$, $\rho u_i u_j$ is negligible, and therefore $T_{ij} \sim 0$.

Equation (2.197) can now be re-arranged such as to identify the quadrupole nature of the sound source. Like the dipole, one would intuitively expect a near- and a far-field radiation term. Equation (2.197) has to be differentiated in a similar manner to which equation (2.179) was differentiated to yield equation (2.183) for the dipole.

The procedures can be found in Goldstein[2.16] or Richards and Mead[2.22]. Thus, equation (2.197) can be differentiated to yield

$$p(\vec{x}, t) = \frac{1}{4\pi} \int_V \frac{\partial^2 r}{\partial x_i \partial x_j} \left\{ \frac{1}{c^2 r} \frac{\partial^2 T_{ij}}{\partial t^2} + \frac{2}{cr^2} \frac{\partial T_{ij}}{\partial t} + \frac{2}{r^3} T_{ij} \right\} \mathrm{d}^3\vec{y}. \tag{2.198}$$

The term in brackets represents the near- and far-field radiation terms – i.e. there are two near-field terms ($r^{-2}$ and $r^{-3}$) and one far-field term. As for the dipole, the $\partial^2 r/\partial x_i \partial x_j$ term is a directivity factor. For a longitudinal quadrupole, $i = j$ and $\partial^2 r/\partial x_i \partial x_j \approx \cos^2\theta$; for a lateral quadrupole $i \not= j$ and $\partial^2 r/\partial x_i \partial x_j \approx \cos\theta \sin\theta$.

### 2.4.3 The effects of the presence of solid bodies in the flow

Solid body interactions with flows are an important part of aerodynamic sound generation. Turbulent flows interacting with solid bodies produce increased sound over free space turbulence – e.g. a butterfly valve arrangement in an industrial gas pipeline, aircraft engine turbine blades etc. The main reason for this is that the presence of solid bodies allows for the existence of monopoles (due to the fluctuating motion or dilatations of the solid body resulting from the unsteady flow) and dipoles (due to the fluctuating forces on the solid body resulting from the unsteady flow), both of which are more efficient sound radiators than quadrupoles.

Lighthill's inhomogeneous wave equation is exact and it is therefore possible to use it to estimate the effects of the presence of solid bodies in the flow. Dowling and Ffowcs Williams[2.6] and Goldstein[2.16] derive the necessary fundamental equations. In summary, the procedure is as follows:

(i) it is assumed that the observer and the solid body are clearly separated;
(ii) the quadrupoles contained within the interior of the solid body are transformed into boundary sources over the surface of the solid body – this procedure converts the interior quadrupole source field into an equivalent monopole (fluctuating motion) and dipole (fluctuating force) field on the surface of the solid body;
(iii) the equations are solved for the density or pressure fluctuations and added to the solution for free turbulence to obtain the complete solution in some region exterior to the solid body.

For sound volume, $V$, bounded by a solid body of surface, $S$, the far-field radiated sound pressure is given by

$$p(\vec{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)}{4\pi|\vec{x}-\vec{y}|}\,\mathrm{d}^3\vec{y} - \frac{\partial}{\partial x_i}\int_S \frac{f_i\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)}{4\pi|\vec{x}-\vec{y}|}\,\mathrm{d}S(\vec{y}) + \frac{\partial}{\partial t}\int_S \frac{\rho\vec{u}\left(\vec{y},t-\dfrac{|\vec{x}-\vec{y}|}{c}\right)\cdot\vec{n}}{4\pi|\vec{x}-\vec{y}|}\,\mathrm{d}S(\vec{y}). \tag{2.199}$$

The first term in the equation represents the free turbulence component; the second term represents the component due to fluctuating body forces; and the third term represents fluctuating motions of the solid body resulting from the unsteady flow. In the third term, $\vec{u}$ is the surface velocity and $\vec{n}$ is the unit normal vector. The equation can be used to develop useful dimensional parametric relationships for aerodynamic sound. Only low subsonic Mach number flows will be analysed here – the reader is referred to Dowling and Ffowcs Williams[2.6], and Goldstein[2.16] for the analyses relating to supersonic flows.

If the free turbulence component is the dominant term (e.g. flow exhausting from a jet nozzle), then the turbulent eddies scale with the mean flow, $U$, and a boundary layer thickness or a jet nozzle diameter, $D$. The time scale of the sound field is thus $D/U$. Thus, the sound wavelength scale is $\lambda = cD/U = D/M$, where $c$ is the speed of sound and $M$ is the Mach number of the flow. Thus,

$$M = \frac{D}{\lambda} = \frac{\text{typical source dimension}}{\text{acoustical wavelength}}. \tag{2.200}$$

An important point is now made. For low Mach number flows, $\lambda \gg D$ and the source region is acoustically compact. This means that the retardation times between the different points within the source region can be neglected. This allows for a considerable simplification in the analysis. Alternatively, for high subsonic and supersonic flows, $D \sim \lambda$ or $D \gg \lambda$ and the retardation times are critical to the analysis.

The various terms associated with the free turbulence component can now be suitably scaled. $\partial/\partial x_i$ scales with $1/\lambda$ or $M/D$; the volume integral scales with $D^3$; and the Lighthill stress tensor $T_{ij}$ scales with $\rho U^2$ since it is the Reynold's shear stresses which are dominant. Thus, from the free turbulence component of equation (2.199)

$$p(\vec{x},t) \sim \left\{\frac{M}{D}\right\}^2 \frac{D^3 \rho U^2}{4\pi r} = \frac{\rho D}{4\pi r c^2} U^4, \tag{2.201}$$

and

$$I(r) \sim \frac{p^2(\vec{x},t)}{(\rho c)_0} = \frac{\rho^2 D^2}{16\pi^2 r^2 c^4 (\rho c)_0} U^8. \tag{2.202}$$

Equation (2.202) suggests that the radiated sound power scales with the eighth power of the flow velocity – this is Lighthill's famous $U^8$ law for turbulence generated sound. It is similar to the $U^8$ relationship that was derived in section 2.3.5 for quadrupoles. It should be noted that the term $(\rho c)_0$ is the density and speed of sound at the observer position. In principle, it could be different from the density, $\rho$, and speed of sound, $c$, in the source region, particularly if hot gases etc. were involved.

The total fluctuating force scales as $\rho U^2 D^2$, thus from the fluctuating force component of equation (2.199)

$$p(\vec{x},t) \sim \left\{\frac{M}{D}\right\} \frac{D^2 \rho U^2}{4\pi r} = \frac{\rho D}{4\pi c r} U^3, \tag{2.203}$$

and

$$I(r) \sim \frac{p^2(\vec{x},t)}{(\rho c)_0} = \frac{\rho^2 D^2}{16\pi^2 c^2 r^2 (\rho c)_0} U^6. \tag{2.204}$$

Equation (2.203) suggests that the radiated sound power scales with the sixth power of the flow velocity. It is similar to the $U^6$ relationship that was derived in section 2.3.2 for dipoles.

Finally the sound power radiation associated with the third term (fluctuating motions of the solid bodies which produce volume dilatations) in equation (2.199) can be evaluated. Here, $\partial/\partial t$ scales as the acoustic frequency, i.e. $U/D$, and the surface integral scales as $D^2$. Thus

$$p(\vec{x},t) \sim \frac{U}{D} \frac{\rho D^2 U}{4\pi r} = \frac{\rho D}{4\pi r} U^2, \tag{2.205}$$

and

$$I(r) \sim \frac{p^2(\vec{x},t)}{(\rho c)_0} = \frac{\rho^2 D^2}{16\pi^2 r^2 (\rho c)_0} U^4. \tag{2.206}$$

Equation (2.206) suggests that the radiated sound power scales with the fourth power of the flow velocity. It is similar to the $U^4$ relationship that was derived in section 2.3.1 for monopoles.

Several important observations can be made in relation to equations (2.201)–(2.206). Firstly, the radiated sound intensity (or the mean-square sound pressure) is proportional to the inverse square of distance; secondly, the intensity is proportional to the square of the typical source dimension, e.g. boundary layer thickness or jet nozzle diameter; thirdly, the velocity dependence ranges from $U^8$ for quadrupoles to $U^4$ for monopoles.

### 2.4.4 The Powell–Howe theory of vortex sound

Lighthill's theory of aerodynamic sound has been extensively used, both in the subsonic and supersonic flow regimes, for jet noise prediction studies, boundary layer noise studies, unsteady flows over stationary and moving solid bodies, etc. When the complete details of the flow are unavailable and the only available parameters are the intensities and length scales, Lighthill's stress tensor, $T_{ij}$, is a very powerful tool. If, however, detailed knowledge of the flow–acoustic interactions is required, then the source terms in Lighthill's equation have to be re-defined.

In 1964, Powell[2.21] postulated that the origin of aerodynamic sound might be attributed to the process of forming eddies or vortices – the very action that causes the formation of vortices, simultaneously gives rise to the sound radiation. Howe[2.19] reformulated Lighthill's theory for low Mach number flows in terms of Powell's concept of vortex sound and associated the aerodynamic sound sources with certain regions in the flow where the total (i.e. mean plus unsteady) vorticity vector, $\vec{\omega}$, is non-vanishing. It should be noted here that, whilst vorticity is a necessary condition for an aerodynamic sound source, it is not a sufficient condition.

From fluid dynamics, when the flow contains both rotational and irrotational components, the total velocity, $\vec{U}$, is given by equation (2.48) – i.e.

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\psi}, \tag{2.207}$$

where $\phi$ is the irrotational velocity potential, and $\vec{\psi}$ is the rotational, vector velocity potential. In turbulent flow regimes, $\vec{v} = \vec{\nabla} \times \vec{\psi}$ is the total (mean plus unsteady) rotational, incompressible component of the velocity field, and $\vec{u} = \vec{\nabla}\phi$ is the irrotational unsteady component. The irrotational, unsteady component is only non-zero when the fluid is compressible and it is therefore associated with the acoustic particle velocity. The vorticity is thus only related to the vector velocity potential and it is

$$\vec{\omega} = \vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}). \tag{2.208}$$

For vortical regions located in free space, Howe[2.19] re-formulated Lighthill's wave equation to yield

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_0 \vec{\nabla} \cdot (\vec{\omega} \times \vec{v}). \tag{2.209}$$

The term $\rho_0(\vec{\omega} \times \vec{v})$ is referred to in the literature as the Lamb vector and it is the

unsteady vortical lifting force. Howe went on to generalise the equation, with the stagnation enthalpy rather than the density as the acoustic variable, to account for interactions with solid bodies etc. His classical analysis resulted in the following analogy with the Lighthill stress tensor –

$$\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = \rho_0 \vec{\nabla} \cdot \{(\omega \times \vec{v}) - T\vec{\nabla} S\}, \tag{2.210}$$

where $T$ is the temperature and $S$ is the entropy. The equation illustrates that the sources of aerodynamic sound are contained in the regions of flow where the vorticity vector and the entropy-gradient vector are non-zero. Howe's analysis has been successfully applied to a range of complex problems of sound generation in inhomogeneous flows[2.19,2.20,2.23]. It is useful to note that it is not necessarily restricted to potential flows; in fact distributed vorticity, as implied by equation (2.211) below, negates potential flow. Also, viscosity does not disable the theory. Crighton[2.24] points out, however, that the Powell–Howe vortex theory, whilst associating the sound sources with regions of vorticity, is still non-locally and non-linearly related to the vorticity. No universal procedures are yet available to linearly relate the sound sources to vorticity. Whilst the Lighthill theory allows for accurate general results when the stress tensor, $T_{ij}$, can be readily evaluated, the Powell–Howe theory allows for specific problems relating to flow–acoustic interactions to be solved.

Howe[2.20,2.23] subsequently established a general momentum balance relationship describing the rate of 'dissipation' of sound energy by the generation of vorticity in the presence of low sub-sonic mean flows. The term 'dissipation' is used loosely as it can refer to both negative dissipation (generation of sound) and positive dissipation (absorption of sound) – i.e. the generation of sound by vorticity produces negative dissipation, and the generation of vorticity by sound produces positive dissipation. Howe's dissipation formula is

$$\Pi = \rho_0 \int_V \{(\vec{\omega} \times \vec{v}) \cdot \vec{u}\} \, \mathrm{d}V, \tag{2.211}$$

where $\vec{\omega}$ is the total (i.e. mean plus unsteady) vorticity vector, $\vec{v}$ is the total (mean plus unsteady) rotational, incompressible component of the velocity field – it is the vorticity convection velocity – and $\vec{u}$ is the acoustic particle velocity. This relationship is especially useful in that it allows for both a qualitative and a quantitative description of different types of flow–acoustic interactions; in particular, regions of energy transfer between the sound and the flow field can be explicitly identified. Howe[2.20] illustrates the positive dissipation (sound absorption) aspect in relation to the influence of mean flows on the diffraction of sound from semi-infinite plates, and the attenuation of sound by grazing flow perforated screens. He also discusses how negative dissipation (sound generation) can occur at certain flow speeds and directions of propagation. The mean flow energy can also be converted into sound if an aerodynamic acoustic resonance is set

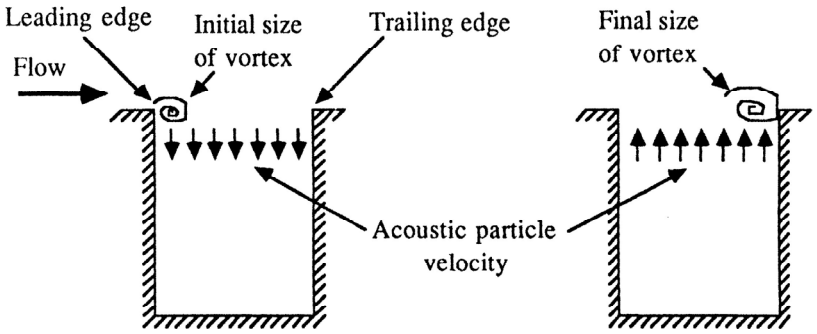



**Fig. 2.17.** Schematic illustration of vortex generation in a Helmholtz resonator. (Adapted from Stokes *et al.*[2.26].)

up, i.e. sound waves in a duct synchronising with some internal, flow-induced, vortex shedding phenomena. Welsh *et al.*[2.25–2.27] have recently studied several such types of flow–acoustic interactions and successfully applied Howe's dissipation formula to them.

Several important observations[2.25] can be made in relation to equation (2.211). Firstly, $\vec{\omega}$, $\vec{v}$ and $\vec{u}$ must have large angles between them – i.e., if the acoustic particle velocity is parallel to the vortex path, there will be no interactions between the sound and flow fields. Secondly, $\vec{\omega}$, $\vec{v}$ and $\vec{u}$ must all have large magnitudes – in regions where the acoustic particle velocity is small there will be very little absorption or generation of sound. Thirdly, because $\vec{u}$ is oscillatory, the time-averaged nett sound energy will approximate to zero (because of cancellations) unless either the magnitude or direction of the vectors changes during an oscillatory cycle. Welsh *et al.*[2.25–2.27] refer to this condition as 'imbalance'. In a recent paper[2.26], they discuss how the energy is transferred from the mean flow to sustain an aerodynamic acoustic resonance condition, for four specific examples. One of those examples, that of a Helmholtz cavity resonator, is discussed here.

When a flowing fluid approaches a cavity, as illustrated in Figure 2.17, a vortex motion is set up at the point where the shear layer separates at the edge of the cavity. The vortex propagates across the cavity to the opposite edge before being convected away. The vortex grows across the cavity and reaches its full strength as it approaches the trailing edge of the cavity; it takes one cycle for the vortex to be generated and convected from the leading to the trailing edge. The acoustic particle velocities within the resonating cavity change sign during one cycle (because they are oscillatory), hence there is a sign change in the triple-vector product over any one cycle. Now, since the vortex has grown during this cycle, there is a nett generation of sound energy – i.e. the time-average of the Howe integral (equation 2.211) is not zero over any one cycle, and there is a significant angle between the acoustic particle velocity and the vortex convection velocity.

The above example qualitatively illustrates the application of the vortex theory of sound to the identification of both regions and mechanisms of flow-induced sound generation.

## 2.5 Flow duct acoustics

This section deals with the application of acoustics to the practical problem of one-dimensional sound transmission through a duct with mean flow based on the use of acoustic impedance and travelling wave concepts developed at the start of this chapter, taking into account the different effects of mean flow on positive and negative travelling waves. Applications of the modelling principles outlined below include (i) evaluation of acoustic resonances in a piping system, (ii) design and analysis of engine exhaust systems and (iii) design and analysis of pulsation control for reciprocating pump and compressor systems.

The transmission of low-frequency, one-dimensional sound through a duct is governed by the one-dimensional homogeneous wave equation (equation 2.50), and is described in terms of two propagating waves moving in opposite directions, where the acoustic pressure and particle velocity at a given position $x$ along the duct are given by equations (2.51) and (2.52), respectively. In the presence of gas flow through the duct, as in most cases of practical interest, the wavenumbers of the positive and negative moving waves are modified by the presence of mean flow:

$$k_+ = \frac{k}{1+M} = \frac{\omega}{c+u_0} = k_c(1-M) \tag{2.212}$$

and

$$k_- = \frac{k}{1-M} = \frac{\omega}{c-u_0} = k_c(1+M), \tag{2.213}$$

where $k$ is the wavenumber for stationary gas, $k_+$ is the wavenumber of the wave propagating in the same direction as the flow ($\mathbf{p}_+ = \mathbf{A_1}\,\mathrm{e}^{\mathrm{i}(\omega t - k_+ x)}$), $k_-$ is the wavenumber of the wave propagating in the opposite direction to the flow ($\mathbf{p}_- = \mathbf{A_2}\,\mathrm{e}^{\mathrm{i}(\omega t + k_- x)}$), $M$ is the Mach number of mean flow through the duct, $u_0$ is the mean flow velocity, and $k_c$ is defined as the convective wavenumber,

$$k_c = \frac{k}{(1-M^2)}. \tag{2.214}$$

The travelling wave solutions to the wave equation for acoustic pressure and particle velocity in the presence of mean flow in a duct are then conveniently written as

$$\mathbf{p}(x,\omega) = \left(\mathbf{A_1}\,\mathrm{e}^{-\mathrm{i}k_c x} + \mathbf{A_2}\,\mathrm{e}^{\mathrm{i}k_c x}\right)\mathrm{e}^{\mathrm{i}Mk_c x}\,\mathrm{e}^{\mathrm{i}\omega t} \tag{2.215}$$

and

$$\mathbf{u}(x,\omega) = \frac{1}{\rho_0 c}\left(\mathbf{A_1}\,\mathrm{e}^{-\mathrm{i}k_c x} - \mathbf{A_2}\,\mathrm{e}^{\mathrm{i}k_c x}\right)\mathrm{e}^{\mathrm{i}Mk_c x}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{2.216}$$

These equations describe the acoustic pressure and particle velocity at any position along a uniform duct with mean flow, provided that the wavenumber satisfies

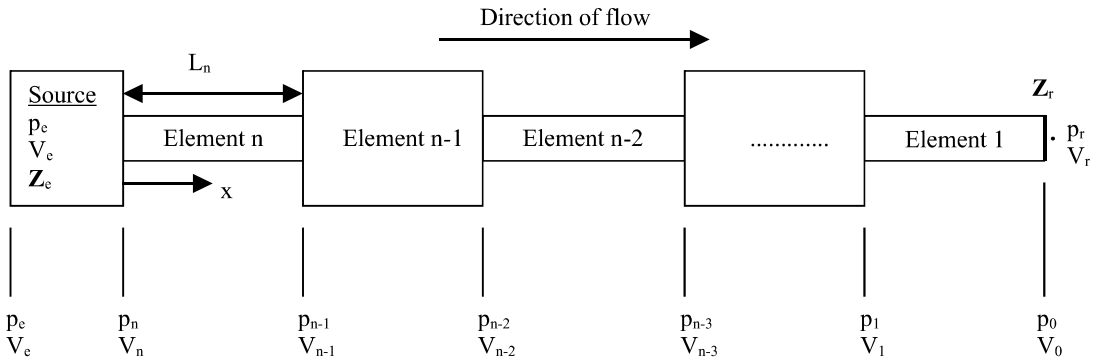



**Fig. 2.18.** Transmission matrix modelling conventions.

$ka \ll 1.84/(1 - M^2)$ for one-dimensional wave propagation, where $a$ is the radius of a circular duct.

In a system where there are multiple sections of duct, each with different diameter and length, travelling wave solutions with unique complex constants must be written separately for each section of duct, and the complex constants are solved by considering continuity of acoustic pressure and particle velocity where ducts of different diameter meet. At these discontinuities where ducts of different diameter meet, there will be partial transmission and reflection of sound waves. Advantage is taken of this in reactive silencer design to minimise the amount of sound transmitted from the source end of the duct (e.g. fan, engine or compressor) to the receiver end. The transmission of sound across each section of duct is conveniently modelled using transmission matrices, where the acoustic pressure and particle velocity at either end of a section of duct $n$ are related by

$$\begin{bmatrix} \mathbf{p_n} \\ \mathbf{u_n} \end{bmatrix} = \begin{bmatrix} \mathbf{A_n} & \mathbf{B_n} \\ \mathbf{C_n} & \mathbf{D_n} \end{bmatrix} \begin{bmatrix} \mathbf{p_{n-1}} \\ \mathbf{u_{n-1}} \end{bmatrix}. \tag{2.217}$$

A similar matrix may be written relating the acoustic parameters $[\mathbf{p_{n-1}}, \mathbf{u_{n-1}}]$ at the inlet of duct $n-1$ to the acoustic parameters $[\mathbf{p_{n-2}}, \mathbf{u_{n-2}}]$ at the outlet of duct $n-1$, and so on for each section of duct. By convention, the elements and acoustic parameters $[\mathbf{p_n}, \mathbf{u_n}]$ are numbered as illustrated in Figure 2.18. Combining the transmission matrices for ducts $n$ and $n-1$, we obtain $[\mathbf{p_n}, \mathbf{u_n}]$ in terms of $[\mathbf{p_{n-2}}, \mathbf{u_{n-2}}]$:

$$\begin{aligned} \begin{bmatrix} \mathbf{p_n} \\ \mathbf{u_n} \end{bmatrix} &= \begin{bmatrix} \mathbf{A_n} & \mathbf{B_n} \\ \mathbf{C_n} & \mathbf{D_n} \end{bmatrix} \begin{bmatrix} \mathbf{p_{n-1}} \\ \mathbf{u_{n-1}} \end{bmatrix} = \begin{bmatrix} \mathbf{A_n} & \mathbf{B_n} \\ \mathbf{C_n} & \mathbf{D_n} \end{bmatrix} \begin{bmatrix} \mathbf{A_{n-1}} & \mathbf{B_{n-1}} \\ \mathbf{C_{n-1}} & \mathbf{D_{n-1}} \end{bmatrix} \begin{bmatrix} \mathbf{p_{n-2}} \\ \mathbf{u_{n-2}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p_{n-2}} \\ \mathbf{u_{n-2}} \end{bmatrix}. \end{aligned} \tag{2.218}$$

Hence, it is possible to construct a single two-by-two (four-pole) transmission matrix relating the acoustic pressures and particle velocities at the source and receiver ends

of a complex duct system that contains several sections of duct, by writing the transmission matrices for each section of duct and multiplying these matrices together in the correct order. For convenience, volume velocity $V_n$ ($= Su_n$ where $S$ is the duct cross-sectional area) is usually used in place of particle velocity as this avoids the need to include a transmission matrix for area discontinuities.

The transmission matrix for a section of pipe with length $L_n$, as in Figure 2.18, is obtained by considering the travelling wave solutions for acoustic pressure and volume velocity at either end of the duct:

$$\mathbf{p_n} = \mathbf{A_1} + \mathbf{A_2} \qquad x = 0 \tag{2.219}$$

$$\mathbf{V_n} = \frac{S}{\rho_0 c}(\mathbf{A_1} - \mathbf{A_2}) \qquad x = 0 \tag{2.220}$$

$$\mathbf{p_{n-1}} = \left(\mathbf{A_1}\,\mathrm{e}^{-\mathrm{i}k_c L_n} + \mathbf{A_2}\,\mathrm{e}^{\mathrm{i}k_c L_n}\right)\mathrm{e}^{\mathrm{i}Mk_c L_n} \qquad x = L_n \tag{2.221}$$

$$\mathbf{V_{n-1}} = \frac{S}{\rho_0 c}\left(\mathbf{A_1}\,\mathrm{e}^{-\mathrm{i}k_c L_n} - \mathbf{A_2}\,\mathrm{e}^{\mathrm{i}k_c L_n}\right)\mathrm{e}^{\mathrm{i}Mk_c L_n} \qquad x = L_n. \tag{2.222}$$

Making use of de Moivre's theorem, and eliminating $\mathbf{A_1}$ and $\mathbf{A_2}$, we obtain

$$\mathbf{p}_1 = \left(\mathbf{p}_2 \cos(k_c L_n) - \frac{\mathrm{i}\rho_0 c}{S}\mathbf{V}_2 \sin(k_c L_n)\right)\mathrm{e}^{\mathrm{i}Mk_c L_n} \tag{2.223}$$

and

$$\mathbf{V}_1 = \left(-\frac{\mathrm{i}S}{\rho_0 c}\mathbf{p}_2 \sin(k_c L_n) + \mathbf{V}_2 \cos(k_c L_n)\right)\mathrm{e}^{\mathrm{i}Mk_c L_n}. \tag{2.224}$$

Putting these equations in matrix form and inverting, the transmission matrix equation relating acoustic pressure and volume velocity at either end of a section of duct is

$$\begin{bmatrix}\mathbf{p_n}\\ \mathbf{V_n}\end{bmatrix} = \begin{bmatrix}\mathrm{e}^{-\mathrm{i}Mk_c L_n}\cos(k_c L_n) & \mathrm{i}\mathrm{e}^{-\mathrm{i}Mk_c L_n}(\rho_0 c/S)\sin(k_c L_n)\\ \mathrm{i}\mathrm{e}^{-\mathrm{i}Mk_c L_n}(S/\rho_0 c)\sin(k_c L_n) & \mathrm{e}^{-\mathrm{i}Mk_c L_n}\cos(k_c L_n)\end{bmatrix}\begin{bmatrix}\mathbf{p_{n-1}}\\ \mathbf{V_{n-1}}\end{bmatrix}. \tag{2.225}$$

For an in-line lumped element such as an orifice plate with impedance $\mathbf{Z_a}$, or for acoustic radiation at the system outlet, volume velocity is assumed to be the same either side of the element and the transmission matrix equation is

$$\begin{bmatrix}\mathbf{p_n}\\ \mathbf{V_n}\end{bmatrix} = \begin{bmatrix}1 & \mathbf{Z_a}/S\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{p_{n-1}}\\ \mathbf{V_{n-1}}\end{bmatrix} \tag{2.226}$$

where the factor $1/S$ accounts for the use of volume velocity in place of particle velocity and the impedance $\mathbf{Z_a}$ is as defined in equation (2.57). In the case of a side-branch element as illustrated in Figure 2.19, a three-way junction is formed and the net volume velocity must balance whilst the acoustic pressure is the same for all elements

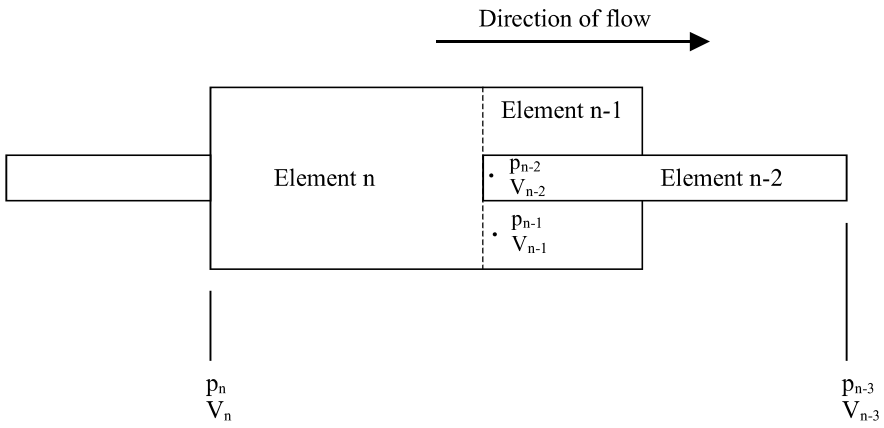



**Fig. 2.19.** Annular side-branch.

connecting the junction. The general transmission matrix equation for a side-branch element is

$$\begin{bmatrix} \mathbf{p_{n-1}} \\ \mathbf{V_{n-1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S/\mathbf{Z_b} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p_{n-2}} \\ \mathbf{V_{n-2}} \end{bmatrix}, \tag{2.227}$$

where $\mathbf{Z_b}$ is the impedance of the side branch evaluated at the inlet to the side branch. In the case of a simple annular side branch as in Figure 2.19, the specific impedance of the side branch (with co-ordinate $x = 0$ at the inlet and $x = L_\mathrm{b}$ at the closed end) at position $x$ is

$$\mathbf{Z_b}(x) = \frac{p(x)}{V(x)} = \mathrm{i}\frac{\rho_0 c}{S}\frac{\cos k(L_\mathrm{b} - x)}{\sin k(L_\mathrm{b} - x)}, \tag{2.228}$$

and hence the impedance at the inlet of the side branch (at $x = 0$) is

$$\mathbf{Z_b} = -\mathrm{i}\rho_0 c \cot(kL_\mathrm{b}). \tag{2.229}$$

The acoustic performance of a reactive silencer inserted into a duct to reduce the level of noise at the receiver end is usually described in terms of insertion loss, where insertion loss is defined as the difference in the radiated sound power resulting from insertion of the silencer into the duct. For radiation into free space, the insertion loss is given by

$$IL = 20\log_{10}\left|\frac{p_{\mathrm{rms,no\ silencer}}}{p_{\mathrm{rms,silencer}}}\right|. \tag{2.230}$$

The insertion loss can be evaluated directly from the matrix elements **A**, **B**, **C** and **D** of the overall transmission matrix in equation (2.218). Considering the engine exhaust system muffler in Figure 2.18, with source impedance $\mathbf{Z_e}$ and tailpipe outlet radiation impedance $\mathbf{Z_r}$, the insertion loss is given by

$$IL = 20\log_{10}\left|\frac{\mathbf{AZ_r} + \mathbf{B} + \mathbf{CZ_eZ_r} + \mathbf{DZ_r}}{\mathbf{A'Z_r} + \mathbf{B'} + \mathbf{C'Z_eZ_r} + \mathbf{D'Z_r}}\right|, \tag{2.231}$$

where the primed coefficients are for the duct system without the silencer (typically, the

system without the silencer is modelled assuming a constant diameter duct in place of the silencer). The source impedance is not generally known, and is either taken as being infinite (constant volume velocity source), zero (constant pressure source) or equal to the characteristic impedance of passages within the source that are not modelled as sections of duct. For the case of infinite source impedance, dividing through by $\mathbf{Z_e}$ and taking the limit at $\mathbf{Z_e} \to \infty$, equation (2.231) simplifies to

$$IL = 20\log_{10}\left|\frac{\mathbf{CZ_r} + \mathbf{D}}{\mathbf{C'Z_r} + \mathbf{D'}}\right|. \tag{2.232}$$

The radiation impedance is better known and for zero flow is approximated by (Jones[2.28])

$$Z_r = \rho_0 c((ka)^2/4 + 0.6133jka) \quad \text{for } ka < 0.5. \tag{2.233}$$

It should be noted that the acoustic pressure tends to zero at the outlet from a duct radiating to the atmosphere, but that the volume velocity is non-zero resulting in radiation of sound to the atmosphere. The interested reader is referred to Munjal[2.29], Davies[2.30], Jones[2.28] and Prasad and Crocker[2.31,2.32] for more information pertaining to the application of transmission matrix modelling and evaluation of insertion loss for reactive silencers in internal combustion engine exhaust systems.

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## NOMENCLATURE

| | |
|---|---|
| $a$ | radius of an oscillating sphere, source dimension, circular duct radius |
| $\vec{a}$ | particle acceleration (vector quantity) |
| $A$ | surface area |

| | |
|---|---|
| $\mathbf{A}$, $\mathbf{A_1}$, $\mathbf{A_2}$ | arbitrary complex constants |
| $B$ | adiabatic bulk modulus |
| $\mathbf{B}$ | frequency-dependent complex constant |
| $c$ | speed of sound |
| $c_0$ | speed of sound for constant pressure and density |
| $C_\mathrm{v}$ | piston mechanical damping |
| $\mathbf{C}$ | frequency-dependent complex constant |
| $d$ | distance, separation between point sources, distance between source and reflecting plane etc. |
| $\mathrm{d}\vec{f}$ | incremental increase in force (vector quantity) |
| $\mathrm{d}m$ | incremental increase in mass |
| $\mathrm{d}V$ | incremental increase in volume |
| $D$ | mean sound energy density, diameter |
| $D'$ | instantaneous sound energy density |
| $\mathbf{D}$ | frequency-dependent complex constant |
| $\vec{f}$ | force (vector quantity) |
| $f(\vec{y}, t)$, $f(\vec{y}, t - r/c)$ | arbitrary source functions |
| $f_i(\vec{x}, t)$, $f_i(\vec{y}, t - r/c)$ | externally applied force ($r = \lvert\vec{x} - \vec{y}\rvert$) |
| $\mathbf{F}$ | complex force |
| $\vec{F}$ | force (vector quantity) |
| $\mathbf{F_m}$ | complex applied mechanical force |
| $\mathbf{F_p}$ | complex force on piston due to acoustic pressure fluctuations |
| $F_\mathrm{rms}$ | root-mean-square fluctuating force |
| $G$ | universal gas constant |
| $G_1$, $G_2$ | arbitrary independent functions which satisfy the wave equation |
| $\mathbf{G}_\omega(\vec{y}, \omega\lvert\vec{x}, \omega)$ | free space Green's function for a unit, time-harmonic, point source – i.e. frequency domain Green's function (complex function) |
| $G_t(\vec{y}, \tau\lvert\vec{x}, t)$ | free space Green's function for a unit, impulsive, point source – i.e. time domain Green's function |
| $H$ | height |
| $I$, $\vec{I}$ | mean sound intensity (arrow denotes vector quantity) |
| $I(r)$ | mean sound intensity as a function of radial distance |
| $I(r, \theta)$ | mean sound intensity as a function of radial and angular distance |
| $\vec{I}'$ | instantaneous sound intensity vector |
| $J_1$ | first-order Bessel function |
| $k$, $k_1$, $k_2$, etc. | wavenumbers |
| $k_+$ | wavenumber of positive moving wave |
| $k_-$ | wavenumber of negative moving wave |
| $k_\mathrm{c}$ | convective wavenumber |
| $K_\mathrm{s}$ | piston stiffness |
| $L$ | length dimension |
| $L_\mathrm{b}$ | length of side branch |
| $L_n$ | length of element $n$ |

| | |
|---|---|
| $m$ | mass |
| $M$ | molecular weight, piston mass, Mach number |
| $M_{\mathrm{r}}$ | amplitude of fluid-loaded mass |
| $\vec{n}$ | unit normal vector |
| $p$ | pressure fluctuation |
| $p_e$ | complex acoustic pressure at the source |
| $p_r$ | complex acoustic pressure radiated to atmosphere |
| $p_{\mathrm{rms}}$ | root-mean-square pressure |
| $p(r,t)$, $\mathbf{p}(r,t)$ | pressure fluctuation as a function of radial distance (bold signifies complex) |
| $p(r,\theta,t)$, $\mathbf{p}(r,\theta,t)$ | pressure fluctuation as a function of radial and angular distance (bold signifies complex) |
| $p(\vec{x},t)$, $\mathbf{p}(\vec{x},t)$ | pressure fluctuation (bold signifies complex) |
| $\mathbf{p_n}$ | complex acoustic pressure at the inlet of element $n$ |
| $\mathbf{p}^*(r,t)$, $\mathbf{p}^*(\vec{x},t)$ | complex conjugates of $\mathbf{p}(r,t)$, $\mathbf{p}(\vec{x},t)$ |
| $\mathbf{p}_+$, $\mathbf{p}_-$ | complex pressure amplitudes |
| $\hat{p}$ | pressure amplitude |
| $P(\vec{x},t)$ | total pressure |
| $P_0(\vec{x})$ | static (mean) pressure |
| $q(\vec{y},t)$ | rate of mass flux per unit volume (kg m$^{-3}$ s$^{-1}$) |
| $q'(\vec{y},t)$, $q'(\vec{y},t-r/c)$ | rate of change of mass flux per unit volume (kg m$^{-3}$ s$^{-2}$) |
| $Q_{\mathrm{p}}$ | peak source strength |
| $Q_{\mathrm{rms}}$ | root-mean-square source strength |
| $Q(t)$, $\mathbf{Q}(t)$, $\mathbf{Q}_1(t)$, etc. | source strength (m$^3$ s$^{-1}$; bold signifies complex) |
| $\mathbf{Q_d}$ | complex dipole source strength |
| $\mathbf{Q_d}(t-r/c)$ | complex dipole source strength as a function of retarded time |
| $Q(t)$ | mass flux per unit time (kg s$^{-1}$) |
| $Q'(t)$ | rate of change of mass flux (kg s$^{-2}$) |
| $r$ | radius, radial distance |
| $r_1$, $r_2$, etc. | radial distances |
| $R$ | gas constant |
| $R_1(2kz)$, $R_1(x)$ | resistive function associated with the radiation impedance of a piston |
| $S$ | surface, entropy, duct cross-sectional area |
| $t$ | time |
| $T$ | time, kinetic energy, temperature |
| $T_{ij}$, $T_{ij}(\vec{y},t-r/c)$ | Lighthill stress tensor ($r=\lvert\vec{x}-\vec{y}\rvert$) |
| $T(\vec{x},t)$ | total temperature |
| $T_0(\vec{x})$ | mean temperature |
| $T_k$ | absolute temperature |
| $T'(\vec{x},t)$ | temperature fluctuation |
| $u$ | fluid velocity fluctuation, acoustic particle velocity (scalar quantity) |
| $\vec{u}$, $\vec{\mathbf{u}}$ | fluid velocity fluctuation, acoustic particle velocity, irrotational unsteady velocity component (vector quantity, bold signifies complex) |
| $\mathbf{u_a}$ | complex normal surface velocity of an oscillating sphere |

| | |
|---|---|
| $\mathbf{u_n}$ | complex acoustic particle velocity at the inlet of element $n$ |
| $u_0$ | mean flow velocity |
| $u_i, u_j$ | fluid velocity fluctuations and/or acoustic particle velocities in tensor notation |
| $u_r$ | fluid velocity fluctuation and/or acoustic particle velocity in the $r$-direction |
| $u_x$ | fluid velocity fluctuation and/or acoustic particle velocity in the $x$-direction |
| $u_y$ | fluid velocity fluctuation and/or acoustic particle velocity in the $y$-direction |
| $u_z$ | fluid velocity fluctuation and/or acoustic particle velocity in the $z$-direction |
| $u(r, t), \mathbf{u}(r, t)$ | fluid velocity fluctuation and/or acoustic particle velocity as a function of radial distance (scalar quantity, bold signifies complex) |
| $u_a(t)$ | instantaneous normal surface velocity of an oscillating sphere |
| $u(x, t)$ | fluid velocity fluctuation, acoustic particle velocity (scalar quantity) |
| $\vec{u}(\vec{x}, t), \vec{\mathbf{u}}(\vec{x}, t)$ | fluid velocity fluctuation, acoustic particle velocity (vector quantity, bold signifies complex) |
| $\vec{\mathbf{u}}^*$ | complex conjugate of $\vec{\mathbf{u}}$ (vector quantity) |
| $\vec{\mathbf{u}}$ | surface velocity (vector quantity) |
| $U$ | potential energy, mean fluid velocity |
| $\mathbf{U}$ | complex piston surface velocity |
| $U_a$ | peak normal surface velocity of an oscillating sphere |
| $U_\mathrm{p}$ | peak piston surface velocity, peak cylindrical line source surface velocity |
| $\mathbf{U_r}$ | complex surface velocity of a cylindrical line source |
| $\vec{U}(\vec{x}, t)$ | total fluid velocity (vector quantity) |
| $\vec{U}_0(\vec{x})$ | mean fluid velocity (vector quantity) |
| $\vec{v}$ | total (mean plus unsteady) rotational incompressible component of the velocity field in a turbulent flow regime, vorticity convection velocity (vector quantity) |
| $V$ | volume |
| $V_e$ | complex acoustic volume velocity at the source |
| $V_r$ | complex acoustic volume velocity radiated to the atmosphere |
| $\mathbf{V_n}$ | complex acoustic volume velocity at the inlet of element $n$ |
| $\vec{x}$ | position vector in the $x$-direction |
| $X_1(2kz), X_1(x)$ | reactive function associated with the radiation impedance of a piston |
| $X_\mathrm{r}$ | reactive component of the radiation impedance of a piston |
| $\vec{y}$ | position vector in the $y$-direction |
| $z$ | piston radius |
| $\vec{z}$ | position vector in the $z$-direction |
| $Z_\mathrm{a}, \mathbf{Z_a}, \mathbf{Z_b}$ | specific acoustic impedance (bold signifies complex) |
| $\mathbf{Z_e}$ | complex source impedance |

| | |
|---|---|
| $\mathbf{Z_m}$ | mechanical impedance (complex function) |
| $\mathbf{Z_r}$ | radiation impedance (complex function) |
| $\mathbf{Z_s}$ | specific acoustic impedance of a reflecting surface (complex function) |
| $\alpha$ | oblique angle of incidence |
| $\gamma$ | ratio of specific heats |
| $\delta(\tau - t)$, $\delta(\tau - t + r/c)$, $\delta(\vec{x} - \vec{y})$ | unit impulse functions (delta functions) |
| $\delta_{ij}$ | unit impulse function in tensor notation |
| $\theta, \theta_i$ | angle (spherical co-ordinates) |
| $\lambda$ | wavelength |
| $\nu$ | coefficient of shear viscosity |
| $\pi$ | 3.14 . . . |
| $\Pi$ | power, sound power |
| $\Pi_{\mathrm{D}}$ | sound power radiated by a dipole |
| $\Pi_{\mathrm{M}}$ | sound power radiated by a monopole |
| $\Pi_{\mathrm{Q}}$ | sound power radiated by a quadrupole |
| $\rho$, $\rho(\vec{x}, t)$ | total fluid density |
| $\rho_0$, $\rho_0(\vec{x})$ | mean fluid density |
| $\rho'$, $\rho'(\vec{x}, t)$ | fluid density fluctuation |
| $\tau_{ij}$ | viscous shear stress (in tensor notation) |
| $\Gamma$ | reflection coefficient |
| $\phi$ | acoustic velocity potential, scalar velocity potential |
| $\phi(r, t)$, $\boldsymbol{\phi}(r, t)$ | acoustic velocity potential as a function of radial distance (bold signifies complex) |
| $\phi(r, \theta, t)$, $\boldsymbol{\phi}(r, \theta, t)$ | acoustic velocity potential as a function of radial and angular distance (bold signifies complex) |
| $\phi(\vec{x}, t)$, $\boldsymbol{\phi}(\vec{x}, t)$ | acoustic velocity potential (bold signifies complex) |
| $\vec{\psi}$ | vector velocity potential |
| $\omega$ | radian (circular) frequency |
| $\vec{\omega}$ | total vorticity (mean plus unsteady; vector quantity) |
| $\vec{\nabla}$ | divergence operator (vector quantity) |