

Fundamentals of Noise and Vibration Analysis for Engineers

Noise and vibration affects all kinds of engineering structures, and is fast becoming an integral part of engineering courses at universities and colleges around the world. In this second edition, Michael Norton's classic text has been extensively updated to take into account recent developments in the field. Much of the new material has been provided by Denis Karczub, who joins Michael as second author for this edition.

This book treats both noise and vibration in a single volume, with particular emphasis on wave-mode duality and interactions between sound waves and solid structures. There are numerous case studies, test cases and examples for students to work through. The book is primarily intended as a text book for senior level undergraduate and graduate courses, but is also a valuable reference for practitioners and researchers in the field of noise and vibration.

Fundamentals of Noise and Vibration Analysis for Engineers

Second edition

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To
our parents,
the first author's wife Erica,
and his young daughters Caitlin and Sarah

Contents

<i>Preface</i>	<i>page xv</i>
<i>Acknowledgements</i>	<i>xvii</i>
<i>Introductory comments</i>	<i>xviii</i>

1	Mechanical vibrations: a review of some fundamentals	1
1.1	Introduction	1
1.2	Introductory wave motion concepts – an elastic continuum viewpoint	3
1.3	Introductory multiple, discrete, mass–spring–damper oscillator concepts – a macroscopic viewpoint	8
1.4	Introductory concepts on natural frequencies, modes of vibration, forced vibrations and resonance	10
1.5	The dynamics of a single oscillator – a convenient model	12
1.5.1	Undamped free vibrations	12
1.5.2	Energy concepts	15
1.5.3	Free vibrations with viscous damping	16
1.5.4	Forced vibrations: some general comments	21
1.5.5	Forced vibrations with harmonic excitation	22
1.5.6	Equivalent viscous-damping concepts – damping in real systems	30
1.5.7	Forced vibrations with periodic excitation	32
1.5.8	Forced vibrations with transient excitation	33
1.6	Forced vibrations with random excitation	37
1.6.1	Probability functions	38
1.6.2	Correlation functions	39
1.6.3	Spectral density functions	41
1.6.4	Input–output relationships for linear systems	42
1.6.5	The special case of broadband excitation of a single oscillator	50
1.6.6	A note on frequency response functions and transfer functions	52
1.7	Energy and power flow relationships	52

1.8	Multiple oscillators – a review of some general procedures	56
1.8.1	A simple two-degree-of-freedom system	56
1.8.2	A simple three-degree-of-freedom system	59
1.8.3	Forced vibrations of multiple oscillators	60
1.9	Continuous systems – a review of wave-types in strings, bars and plates	64
1.9.1	The vibrating string	64
1.9.2	Quasi-longitudinal vibrations of rods and bars	72
1.9.3	Transmission and reflection of quasi-longitudinal waves	77
1.9.4	Transverse bending vibrations of beams	79
1.9.5	A general discussion on wave-types in structures	84
1.9.6	Mode summation procedures	85
1.9.7	The response of continuous systems to random loads	91
1.9.8	Bending waves in plates	94
1.10	Relationships for the analysis of dynamic stress in beams	96
1.10.1	Dynamic stress response for flexural vibration of a thin beam	96
1.10.2	Far-field relationships between dynamic stress and structural vibration levels	100
1.10.3	Generalised relationships for the prediction of maximum dynamic stress	102
1.10.4	Properties of the non-dimensional correlation ratio	103
1.10.5	Estimates of dynamic stress based on static stress and displacement	104
1.10.6	Mean-square estimates for single-mode vibration	105
1.10.7	Relationships for a base-excited cantilever with tip mass	106
1.11	Relationships for the analysis of dynamic strain in plates	108
1.11.1	Dynamic strain response for flexural vibration of a constrained rectangular plate	109
1.11.2	Far-field relationships between dynamic stress and structural vibration levels	112
1.11.3	Generalised relationships for the prediction of maximum dynamic stress	113
1.12	Relationships for the analysis of dynamic strain in cylindrical shells	113
1.12.1	Dynamic response of cylindrical shells	114
1.12.2	Propagating and evanescent wave components	117
1.12.3	Dynamic strain concentration factors	119
1.12.4	Correlations between dynamic strain and velocity spatial maxima	119
	<i>References</i>	122
	<i>Nomenclature</i>	123

2	Sound waves: a review of some fundamentals	128
2.1	Introduction	128
2.2	The homogeneous acoustic wave equation – a classical analysis	131
2.2.1	Conservation of mass	134
2.2.2	Conservation of momentum	136
2.2.3	The thermodynamic equation of state	139
2.2.4	The linearised acoustic wave equation	140
2.2.5	The acoustic velocity potential	141
2.2.6	The propagation of plane sound waves	143
2.2.7	Sound intensity, energy density and sound power	144
2.3	Fundamental acoustic source models	146
2.3.1	Monopoles – simple spherical sound waves	147
2.3.2	Dipoles	151
2.3.3	Monopoles near a rigid, reflecting, ground plane	155
2.3.4	Sound radiation from a vibrating piston mounted in a rigid baffle	157
2.3.5	Quadrupoles – lateral and longitudinal	162
2.3.6	Cylindrical line sound sources	164
2.4	The inhomogeneous acoustic wave equation – aerodynamic sound	165
2.4.1	The inhomogeneous wave equation	167
2.4.2	Lighthill’s acoustic analogy	174
2.4.3	The effects of the presence of solid bodies in the flow	177
2.4.4	The Powell–Howe theory of vortex sound	180
2.5	Flow duct acoustics	183
	<i>References</i>	187
	<i>Nomenclature</i>	188
3	Interactions between sound waves and solid structures	193
3.1	Introduction	193
3.2	Fundamentals of fluid–structure interactions	194
3.3	Sound radiation from an infinite plate – wave/boundary matching concepts	197
3.4	Introductory radiation ratio concepts	203
3.5	Sound radiation from free bending waves in finite plate-type structures	207
3.6	Sound radiation from regions in proximity to discontinuities – point and line force excitations	216

3.7	Radiation ratios of finite structural elements	221
3.8	Some specific engineering-type applications of the reciprocity principle	227
3.9	Sound transmission through panels and partitions	230
3.9.1	Sound transmission through single panels	232
3.9.2	Sound transmission through double-leaf panels	241
3.10	The effects of fluid loading on vibrating structures	244
3.11	Impact noise	247
	<i>References</i>	249
	<i>Nomenclature</i>	250

4 Noise and vibration measurement and control procedures 254

4.1	Introduction	254
4.2	Noise and vibration measurement units – levels, decibels and spectra	256
4.2.1	Objective noise measurement scales	256
4.2.2	Subjective noise measurement scales	257
4.2.3	Vibration measurement scales	259
4.2.4	Addition and subtraction of decibels	261
4.2.5	Frequency analysis bandwidths	263
4.3	Noise and vibration measurement instrumentation	267
4.3.1	Noise measurement instrumentation	267
4.3.2	Vibration measurement instrumentation	270
4.4	Relationships for the measurement of free-field sound propagation	273
4.5	The directional characteristics of sound sources	278
4.6	Sound power models – constant power and constant volume sources	279
4.7	The measurement of sound power	282
4.7.1	Free-field techniques	282
4.7.2	Reverberant-field techniques	283
4.7.3	Semi-reverberant-field techniques	287
4.7.4	Sound intensity techniques	290
4.8	Some general comments on industrial noise and vibration control	294
4.8.1	Basic sources of industrial noise and vibration	294
4.8.2	Basic industrial noise and vibration control methods	295
4.8.3	The economic factor	299
4.9	Sound transmission from one room to another	301
4.10	Acoustic enclosures	304
4.11	Acoustic barriers	308
4.12	Sound-absorbing materials	313
4.13	Vibration control procedures	320

4.13.1 Low frequency vibration isolation – single-degree-of-freedom systems	322
4.13.2 Low frequency vibration isolation – multiple-degree-of-freedom systems	325
4.13.3 Vibration isolation in the audio-frequency range	327
4.13.4 Vibration isolation materials	330
4.13.5 Dynamic absorption	332
4.13.6 Damping materials	334
<i>References</i>	335
<i>Nomenclature</i>	336

5 The analysis of noise and vibration signals 342

5.1 Introduction	342
5.2 Deterministic and random signals	344
5.3 Fundamental signal analysis techniques	347
5.3.1 Signal magnitude analysis	347
5.3.2 Time domain analysis	351
5.3.3 Frequency domain analysis	352
5.3.4 Dual signal analysis	355
5.4 Analogue signal analysis	365
5.5 Digital signal analysis	366
5.6 Statistical errors associated with signal analysis	370
5.6.1 Random and bias errors	370
5.6.2 Aliasing	372
5.6.3 Windowing	374
5.7 Measurement noise errors associated with signal analysis	377
<i>References</i>	380
<i>Nomenclature</i>	380

6 Statistical energy analysis of noise and vibration 383

6.1 Introduction	383
6.2 The basic concepts of statistical energy analysis	384
6.3 Energy flow relationships	387
6.3.1 Basic energy flow concepts	388
6.3.2 Some general comments	389
6.3.3 The two subsystem model	391

6.3.4	<i>In-situ</i> estimation procedures	393
6.3.5	Multiple subsystems	395
6.4	Modal densities	397
6.4.1	Modal densities of structural elements	397
6.4.2	Modal densities of acoustic volumes	400
6.4.3	Modal density measurement techniques	401
6.5	Internal loss factors	407
6.5.1	Loss factors of structural elements	408
6.5.2	Acoustic radiation loss factors	410
6.5.3	Internal loss factor measurement techniques	412
6.6	Coupling loss factors	417
6.6.1	Structure–structure coupling loss factors	417
6.6.2	Structure–acoustic volume coupling loss factors	419
6.6.3	Acoustic volume–acoustic volume coupling loss factors	420
6.6.4	Coupling loss factor measurement techniques	421
6.7	Examples of the application of S.E.A. to coupled systems	423
6.7.1	A beam–plate–room volume coupled system	424
6.7.2	Two rooms coupled by a partition	427
6.8	Non-conservative coupling – coupling damping	430
6.9	The estimation of sound radiation from coupled structures using total loss factor concepts	431
6.10	Relationships between dynamic stress and strain and structural vibration levels	433
	<i>References</i>	435
	<i>Nomenclature</i>	437

7 Pipe flow noise and vibration: a case study 441

7.1	Introduction	441
7.2	General description of the effects of flow disturbances on pipeline noise and vibration	443
7.3	The sound field inside a cylindrical shell	446
7.4	Response of a cylindrical shell to internal flow	451
7.4.1	General formalism of the vibrational response and sound radiation	451
7.4.2	Natural frequencies of cylindrical shells	454
7.4.3	The internal wall pressure field	455
7.4.4	The joint acceptance function	458
7.4.5	Radiation ratios	460

7.5 Coincidence – vibrational response and sound radiation due to higher order acoustic modes	461
7.6 Other pipe flow noise sources	467
7.7 Prediction of vibrational response and sound radiation characteristics	471
7.8 Some general design guidelines	477
7.9 A vibration damper for the reduction of pipe flow noise and vibration	479
<i>References</i>	481
<i>Nomenclature</i>	483

8 Noise and vibration as a diagnostic tool 488

8.1 Introduction	488
8.2 Some general comments on noise and vibration as a diagnostic tool	489
8.3 Review of available signal analysis techniques	493
8.3.1 Conventional magnitude and time domain analysis techniques	494
8.3.2 Conventional frequency domain analysis techniques	501
8.3.3 Cepstrum analysis techniques	503
8.3.4 Sound intensity analysis techniques	504
8.3.5 Other advanced signal analysis techniques	507
8.3.6 New techniques in condition monitoring	511
8.4 Source identification and fault detection from noise and vibration signals	513
8.4.1 Gears	514
8.4.2 Rotors and shafts	516
8.4.3 Bearings	518
8.4.4 Fans and blowers	523
8.4.5 Furnaces and burners	525
8.4.6 Punch presses	527
8.4.7 Pumps	528
8.4.8 Electrical equipment	530
8.4.9 Source ranking in complex machinery	532
8.4.10 Structural components	536
8.4.11 Vibration severity guides	539
8.5 Some specific test cases	541
8.5.1 Cabin noise source identification on a load–haul–dump vehicle	541
8.5.2 Noise and vibration source identification on a large induction motor	547
8.5.3 Identification of rolling-contact bearing damage	550
8.5.4 Flow-induced noise and vibration associated with a gas pipeline	554

8.5.5 Flow-induced noise and vibration associated with a racing sloop (yacht)	557
8.6 Performance monitoring	557
8.7 Integrated condition monitoring design concepts	559
References	562
Nomenclature	563
Problems	566
Appendix 1: Relevant engineering noise and vibration control journals	599
Appendix 2: Typical sound transmission loss values and sound absorption coefficients for some common building materials	600
Appendix 3: Units and conversion factors	603
Appendix 4: Physical properties of some common substances	605
Answers to problems	607
Index	621

2 Sound waves: a review of some fundamentals

2.1 Introduction

Sound is a pressure wave that propagates through an elastic medium at some characteristic speed. It is the molecular transfer of motional energy and cannot therefore pass through a vacuum. For this wave motion to exist, the medium has to possess inertia and elasticity. Whilst vibration relates to such wave motion in structural elements, noise relates to such wave motion in fluids (gases and liquids). Two fundamental mechanisms are responsible for sound generation. They are:

- (i) the vibration of solid bodies resulting in the generation and radiation of sound energy – these sound waves are generally referred to as structure-borne sound;
- (ii) flow-induced noise resulting from pressure fluctuations induced by turbulence and unsteady flows – these sound waves are generally referred to as aerodynamic sound.

With structure-borne sound, the regions of interest are generally in a fluid (usually air) at some distance from the vibrating structure. Here, the sound waves propagate through the stationary fluid (the fluid has a finite particle velocity due to the sound wave, but a zero mean velocity) from a readily identifiable source to the receiver. The region of interest does not therefore contain any sources of sound energy – i.e. the sources which generated the acoustic disturbance are external to it. A simple example is a vibrating electric motor. Classical acoustical theory (analysis of the homogeneous wave equation) can be used for the analysis of sound waves generated by these types of sources. The solution for the acoustic pressure fluctuation, p , describes the wave field external to the source. This wave field can be modelled in terms of combinations of simple sound sources. If required, the source can be accounted for in the wave field by considering the initial, time-dependent conditions.

With aerodynamic sound, the sources of sound are not so readily identifiable and the regions of interest can be either within the fluid flow itself or external to it. When the regions of interest are within the fluid flow, they contain sources of sound energy because the sources are continuously being generated or convected with the flow (e.g. turbulence, vortices, etc.). These aerodynamic sources therefore have to be included in the wave equation for any subsequent analysis of the sound waves in order that they can

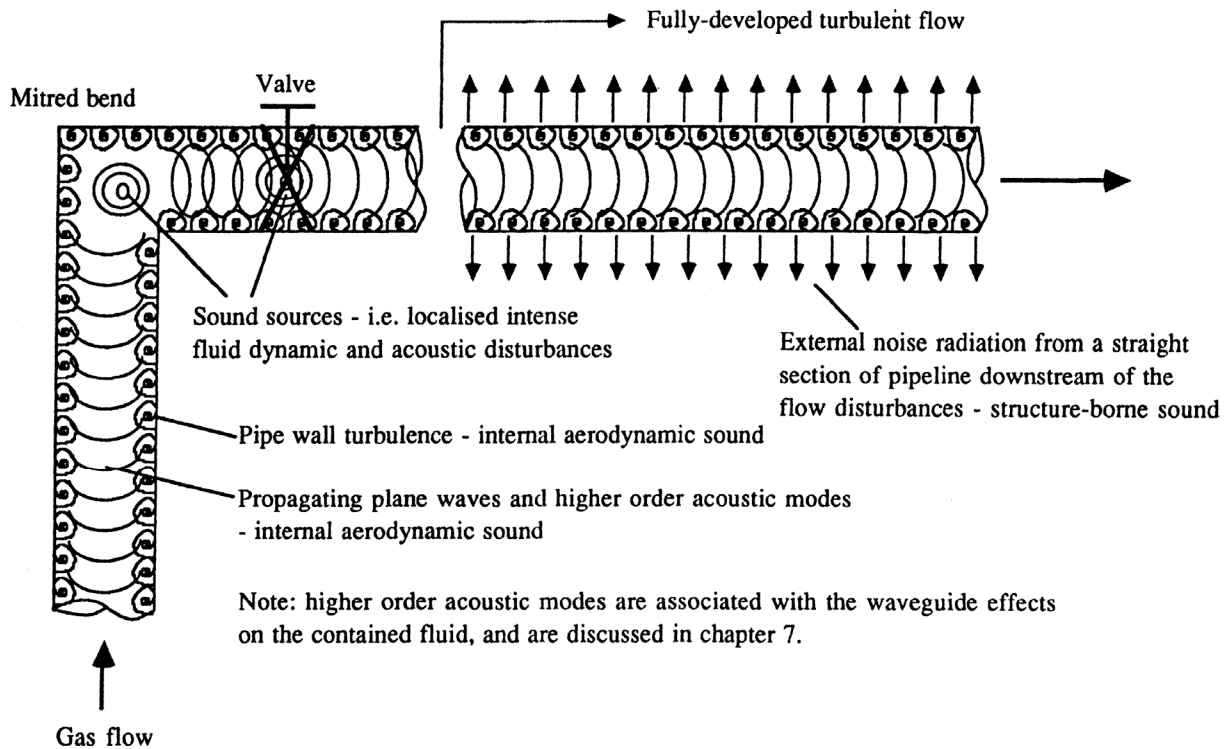


Fig. 2.1. A schematic model of internal aerodynamic sound and external structure-borne sound in a gas pipeline.

be correctly identified. The wave equation is now inhomogeneous (because it includes these source terms) and its solution is somewhat different to that of the homogeneous wave equation in that it now describes both the source and the wave fields.

It is very important to be aware of and to understand the difference between the homogeneous and the inhomogeneous wave equations for the propagation of sound waves in fluids. The vast majority of engineering noise and vibration control relates to sources which can be readily identified, and regions of interest which are outside the source region – in these cases the homogeneous wave equation is sufficient to describe the wave field and the subsequent noise radiation. Most machinery noise, for instance, is associated with the vibration of solid bodies. Engineers should, however, be aware of the existence of the inhomogeneous wave equation and of the instances when it has to be used in place of the more familiar (and easier to solve!) homogeneous wave equation.

A good industrial example that combines both types of noise generation mechanisms is high speed gas flow in a pipeline. Such pipelines are typically found in oil refineries and liquid-natural-gas plants. A typical section of pipeline is illustrated in Figure 2.1. Inside the pipe there are pressure fluctuations which are caused by turbulence, sound waves generated at the flow discontinuities (e.g. bends, valves, etc.), and vortices which are convected downstream of some bluff body such as a butterfly valve splitter-plate. These pressure fluctuations result in internal pipe flow noise which is aerodynamic in

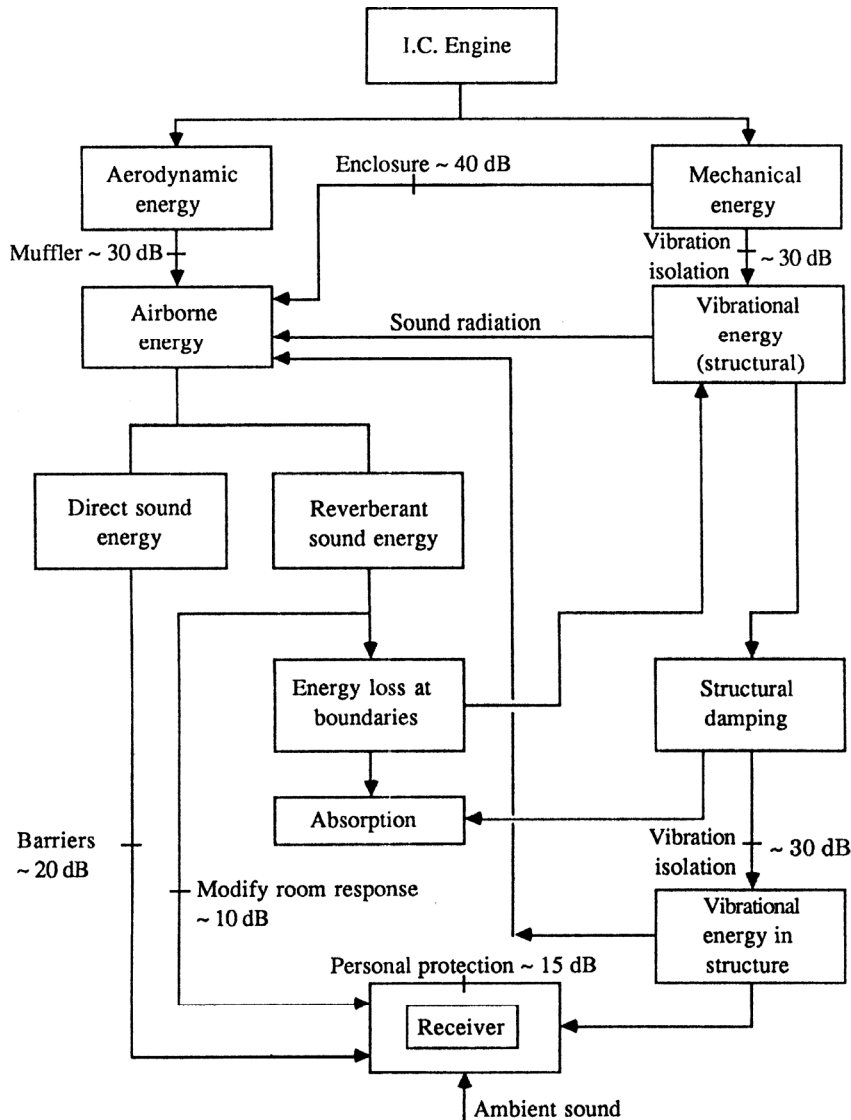


Fig. 2.2. Typical noise and vibrations paths for a machinery source. (Adapted from Pickles^{2,10}.)

nature – the source of sound is distributed along the whole length of the inside of the pipe. If an analysis of the sound sources within the pipe is required, the inhomogeneous wave equation would have to be used. This internal aerodynamic noise excites the structure internally and the vibrating structure subsequently radiates noise to the surrounding external medium. The source of sound is not in the region of space under analysis for the external noise radiation, and this problem can therefore be handled with the homogeneous wave equation. A knowledge of the internal source field (the wall pressure fluctuations) allows for a prediction of the external sound radiation; the converse is, however, not true. A point which is sometimes overlooked is that a description of the external wave field does not contain sufficient information for the source to be identified, but, once the source has been identified and described, the sound field can be predicted. Pipe flow noise is discussed in chapter 7 as a case study.

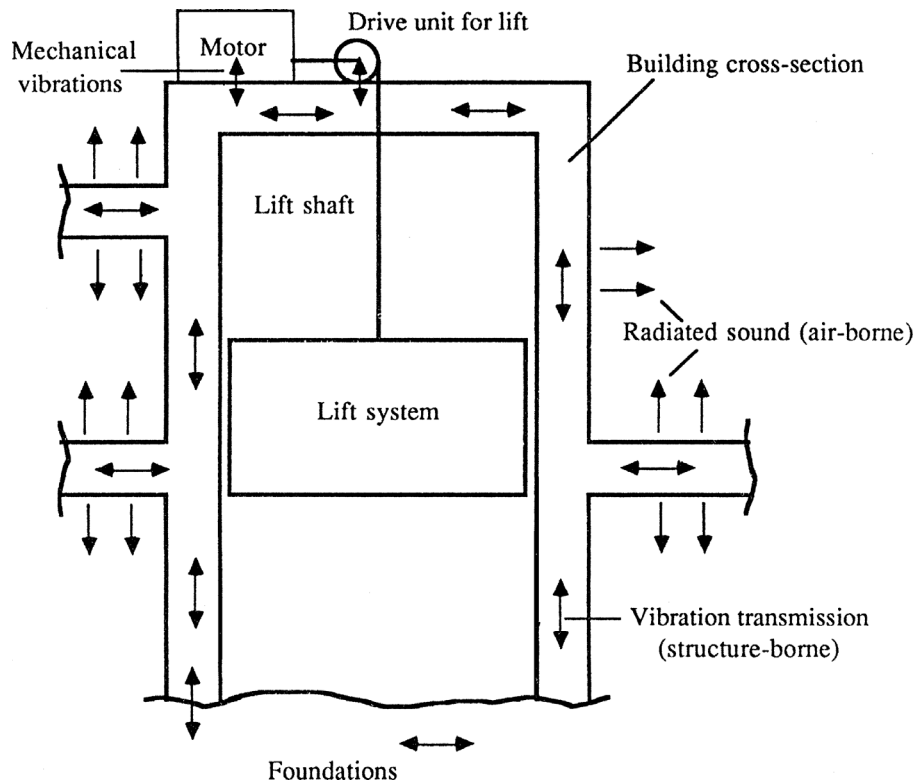


Fig. 2.3. Schematic illustration of structure-borne sound in a building.

Typical machinery noise control problems in industry involve (i) a source, (ii) a path, and (iii) a receiver. There is always interaction and feedback between the three, and there are generally several possible noise and vibration energy transmission paths for a typical machinery noise source. An internal combustion engine, for instance, generates both aerodynamic and mechanical energy, each with several possible transmission paths. This is illustrated schematically in Figure 2.2. The two sources of sound energy are (i) the aerodynamic energy associated with the combustion process and the exhaust system, and (ii) mechanical vibration energy associated with the various functional requirements of the engine. Source modification to reduce the aerodynamic noise component would require changes in the combustion process itself or in the design of the exhaust system. Source modification to reduce the mechanical vibration energy would require a re-design of the moving parts of the engine itself. Various options are open for the reduction of path noise. These include muffling the exhaust noise, structural modification such as adding mass, stiffness or damping to the various radiating panels, providing anti-vibration mounts, enclosing the engine, and providing acoustic barriers. Finally, the receiver could be provided with personal protection such as an enclosure or hearing protectors.

A specific example of structure-borne sound is the vibration and stop-start shocks that can emanate from a lift if it is not properly isolated. This is illustrated in Figure 2.3. In cases such as these, the vibrations are transmitted throughout the building – the waves are carried for large distances without being significantly attenuated. There are

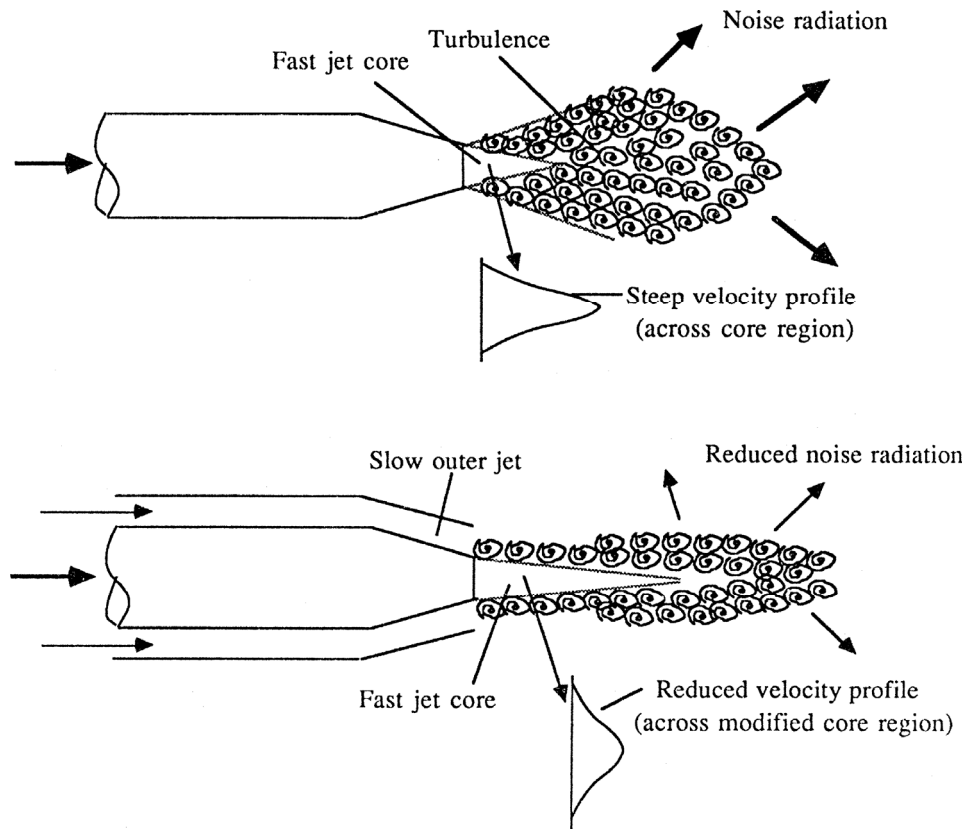


Fig. 2.4. Schematic illustration of aerodynamic sound emanating from a jet nozzle.

no sources of sound in the ambient air in the building, and any acoustic analysis would only require usage of the homogeneous wave equation. The problem could be overcome by isolating the winding machinery from the rest of the structure or by separating the lift shaft and the winding machinery from the remainder of the building.

A specific example of aerodynamic noise is the formation of turbulence in the mixing region at the exhaust of a jet nozzle such as the nozzle of a jet used for cleaning machine components with compressed air. The jet noise increases with flow velocity and the strength of the turbulence is related to the relative speed of the jet in relation to the ambient air. By introducing a secondary, low velocity, air-stream, as illustrated in Figure 2.4, and thus reducing the velocity profile across the jet, significant reductions in radiated noise levels can be achieved. Hence, compound nozzles are sometimes used in industry – here, the velocity of the core jet remains the same but its noise radiating characteristics are reduced by the introduction of a slower outer stream.

2.2 The homogeneous acoustic wave equation – a classical analysis

Three methods are available for approaching problems in acoustics. They are (i) wave acoustics, (ii) ray acoustics and (iii) energy acoustics.

Wave acoustics is a description of wave propagation using either molecular or particulate models. The general preference is for the particulate model, a particle being a fluid volume large enough to contain millions of molecules and small enough such that density, pressure and temperature are constant. Ray acoustics is a description of wave propagation over large distances, e.g. the atmosphere. Families of rays are used to describe the propagation of sound waves and inhomogeneities such as temperature gradients or wind have to be accounted for. Over large distances, the ray tracing procedures are preferred because they approximate and simplify the exact wave approach. Finally, energy acoustics describes the propagation of sound waves in terms of the transfer of energy of various statistical parameters where techniques referred to as statistical energy analysis (or S.E.A.) are used.

The wave acoustics approach is probably the most fundamental and important approach to the study of all disciplines of acoustics. The ray acoustics approach generally relates to outdoor or underwater sound propagation over large distances and is therefore not directly relevant to industrial noise and vibration control. The S.E.A. approach is fast becoming popular for quick and effective answers to complex industrial noise and vibration problems. The wave acoustics approach will thus be adopted for the better part of this book, and this chapter is devoted to some of the more important fundamental principles of sound waves. The subject of ray acoustics is not discussed in this book, but the concepts and applications of statistical energy analysis techniques are discussed in chapter 6.

Sound waves in non-viscous (inviscid) fluids are simply longitudinal waves and adjacent regions of compression and rarefaction are set up – i.e. the particles oscillate to and fro in the wave propagation direction, hence the acoustic particle velocity is in the same direction as the phase velocity. The pressure change that is produced as the fluid compresses and expands is the source of the restoring force for the oscillatory motion. There are four variables that are of direct relevance to the study of sound waves. They are pressure, P , velocity, $\vec{U}$, density, ρ , and temperature, T . Pressure, density and temperature are scalar quantities whilst velocity is a vector quantity (i.e. an arrow over a symbol denotes a vector quantity). Each of the four variables has a mean and a fluctuating component. Thus,

$$P(\vec{x}, t) = P_0(\vec{x}) + p(\vec{x}, t), \quad (2.1a)$$

$$\vec{U}(\vec{x}, t) = \vec{U}_0(\vec{x}) + \vec{u}(\vec{x}, t), \quad (2.1b)$$

$$\rho(\vec{x}, t) = \rho_0(\vec{x}) + \rho'(\vec{x}, t), \quad (2.1c)$$

$$T(\vec{x}, t) = T_0(\vec{x}) + T'(\vec{x}, t). \quad (2.1d)$$

The wave equation can thus be set up in terms of any one of these four variables. In acoustics, it is the pressure fluctuations, $p(\vec{x}, t)$, that are of primary concern – i.e. noise radiation is a fluctuating pressure. Thus it is common for acousticians to solve the wave equation in terms of the pressure as a dependent variable. It is, however, quite valid to solve the wave equation in terms of any of the other three variables. Also,

generally, $\vec{U}_0(\vec{x})$ is zero (i.e. the ambient fluid is stationary) and therefore $\vec{U}(\vec{x}, t) = \vec{u}(\vec{x}, t)$.

As for wave propagation in solids, several simplifying assumptions need to be made. They are:

- (1) the fluid is an ideal gas;
- (2) the fluid is perfectly elastic – i.e. Hooke's law holds;
- (3) the fluid is homogeneous and isotropic;
- (4) the fluid is inviscid – i.e. viscous-damping and heat conduction terms are neglected;
- (5) the wave propagation through the fluid media is adiabatic and reversible;
- (6) gravitational effects are neglected – i.e. P_0 and ρ_0 are assumed to be constant;
- (7) the fluctuations are assumed to be small – i.e. the system behaves linearly.

In order to develop the acoustic wave equation, equations describing the relationships between the various acoustic variables and the interactions between the restoring forces and the deformations of the fluid are required. The first such relationship is referred to as continuity or the conservation of mass; the second relationship is referred to as Euler's force equation or the conservation of momentum; and the third relationship is referred to as the thermodynamic equation of state. In practice, sound waves are generally three-dimensional. It is, however, convenient to commence with the derivation of the above equations in one dimension and to subsequently extend the results to three dimensions.

2.2.1 Conservation of mass

The equation of conservation of mass (continuity) provides a relationship between the density, $\rho(\vec{x}, t)$, and the particle velocity, $\vec{u}(\vec{x}, t)$ – i.e. it relates the fluid motion to its compression.

Consider the mass flow of particles in the x -direction through an elemental, fixed, control volume, dV , as illustrated in Figure 2.5. For mass to be conserved, the time rate of change of the elemental mass has to equal the nett mass flow into the elemental volume. Because the flow is one-dimensional, the vector notation is temporarily dropped. It will be re-introduced later on when the equations are extended to three-dimensional flow. Note that

$$\vec{u} = u_x \vec{i} + u_y \vec{j} + u_z \vec{k},$$

where u_x , u_y and u_z are the particle velocities in the x -, y - and z -directions, respectively. For flow in the x -direction only:

- (i) the elemental mass is $\rho A dx$ (where $A = dy dz$);
- (ii) the mass flow into the elemental volume is $(\rho u A)_x$;
- (iii) the mass flow out of the elemental volume is $(\rho u A)_{x+dx}$.

For the conservation of mass,

$$\frac{\partial(\rho A dx)}{\partial t} = (\rho u A)_x - (\rho u A)_{x+dx}. \quad (2.2)$$

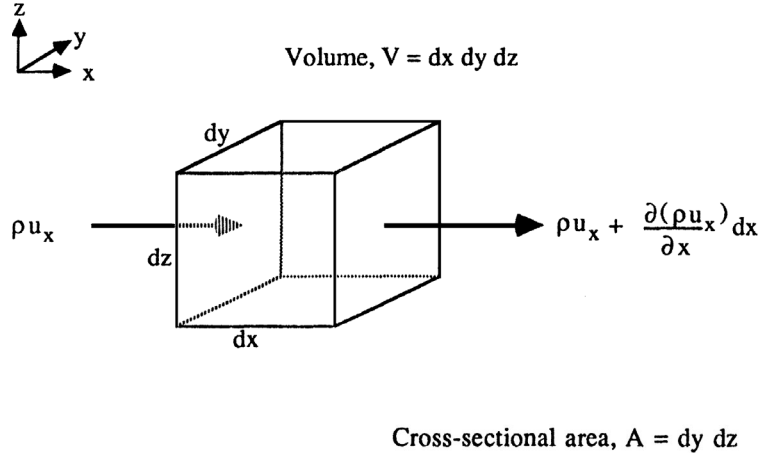


Fig. 2.5. Mass flow of particles in the x -direction through an elemental, fixed, control volume.

Using a Taylor series expansion,

$$\frac{\partial(\rho A \, dx)}{\partial t} = \left\{ (\rho u A)_x - (\rho u A)_x - \frac{\partial(\rho u A)_x}{\partial x} dx \right\}. \quad (2.3)$$

Hence,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} = 0. \quad (2.4)$$

Equation (2.4) represents the one-dimensional conservation of mass in the x -direction. It can be extended to three dimensions, and the three-dimensional equation of conservation of mass is therefore

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0, \quad (2.5)$$

where $\vec{\nabla}$ is the divergence operator, i.e.

$$\vec{\nabla} = \left\{ \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right\}. \quad (2.6)$$

Equation (2.5) is thus a vector representation for

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0. \quad (2.7)$$

The equation of conservation of mass (equations 2.5 or 2.7) is a scalar quantity. It is also non-linear because the mass flow terms involve products of two small fluctuating components ($\vec{u}$ and ρ'). These terms have second-order effects as far as the propagation of sound waves is concerned – i.e. the equation can be linearised. Substituting for $\rho(\vec{x}, t) = \rho_0(\vec{x}) + \rho'(\vec{x}, t)$ into equation (2.7) and deleting second- and higher-order terms yields

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u_x}{\partial x} + \rho_0 \frac{\partial u_y}{\partial y} + \rho_0 \frac{\partial u_z}{\partial z} = 0, \quad (2.8)$$

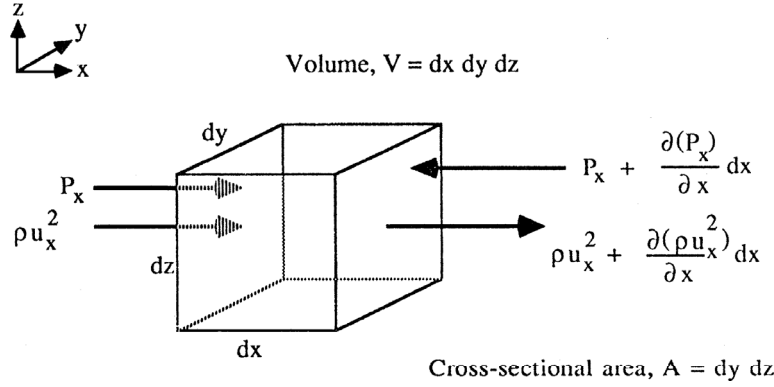


Fig. 2.6. Momentum balance in the x -direction for an elemental, fixed, control volume.

or

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u} = 0. \quad (2.9)$$

Equation (2.9) is the linearised equation of conservation of mass (continuity).

2.2.2 Conservation of momentum

The equation of conservation of momentum provides a relationship between the pressure, $P(\vec{x}, t)$, the density, $\rho(\vec{x}, t)$, and the particle velocity, $\vec{u}(\vec{x}, t)$. It can be obtained either by observing the stated law of conservation of momentum with respect to an elemental, fixed, control volume, dV , in space, or by a direct application of Newton's second law with respect to the fluid particles that move through the elemental, fixed, control volume. To a purist both procedures are identical! It is, however, instructive to consider them both.

Consider the first approach – consider the momentum flow through an elemental, fixed, control volume, dV , as illustrated in Figure 2.6. For momentum to be conserved the time rate of change of momentum contained in the fixed volume plus the nett rate of flow of momentum through the surfaces of the volume are equal to the sum of all the forces acting on the volume. Once again, because the flow is one-dimensional, the vector notation is temporarily dropped. Also, body forces are neglected – i.e. only pressure forces act on the body. For flow in the x -direction only:

- (i) the momentum of the control volume is $\rho u_x A dx$;
- (ii) the momentum flow into the control volume is $(\rho u^2 A)_x$;
- (iii) the momentum flow out of the control volume is $(\rho u^2 A)_{x+dx}$;
- (iv) the force at position x is $(PA)_x$;
- (v) the force at position $x + dx$ is $-(PA)_{x+dx}$.

For the conservation of momentum

$$\frac{\partial(\rho u_x A) dx}{\partial t} = (\rho u^2 A)_x - (\rho u^2 A)_{x+dx} + (PA)_x - (PA)_{x+dx}. \quad (2.10)$$

If a Taylor series expansion is used for $(\rho u^2 A)_{x+dx}$ and $(PA)_{x+dx}$, equation (2.10) simplifies to

$$\frac{\partial(\rho u_x)}{\partial t} = -\frac{\partial(\rho u_x^2)}{\partial x} - \frac{\partial P}{\partial x}. \quad (2.11)$$

Equation (2.11) can be re-arranged as

$$\rho \frac{\partial u_x}{\partial t} + u_x \left\{ \frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial u_x}{\partial x} \right\} + \rho u_x \frac{\partial u_x}{\partial x} + \frac{\partial P}{\partial x} = 0, \quad (2.12)$$

where the term in brackets is the continuity equation. Thus, equation (2.12) simplifies to

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} + \frac{\partial P}{\partial x} = 0. \quad (2.13)$$

Equation (2.13) represents the one-dimensional conservation of momentum in the x -direction. Similar expressions can be obtained for the y - and z -directions. The three-dimensional equation of conservation of momentum is therefore obtained by introducing the divergence operator, $\vec{\nabla}$. It is

$$\rho \left\{ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right\} + \vec{\nabla} P = 0. \quad (2.14)$$

This equation is the non-linear, inviscid momentum equation or Euler's equation.

Equation (2.14) can also be obtained by a direct application of Newton's second law with respect to the fluid particles that move through the elemental, fixed, control volume. The control volume in Figure 2.6 contains a mass, dm , of fluid at any instant in time. The nett force on the volume element is

$$d\vec{f} = \vec{a} dm, \quad (2.15)$$

from Newton's second law. It is very important to recognise that the acceleration, $\vec{a}$, is the rate of change of velocity of a given fluid particle as it moves about in space and it is not the rate of change of fluid velocity at a fixed point in space.

The particle velocity, $\vec{u}$, is a function of space and time. Thus, at some time t , a particle is at position (x, y, z) and it has a particle velocity $\vec{u}(x, y, z, t)$. At some further time, $t + dt$ the particle is at position $(x + dx, y + dy, z + dz)$ and it has a particle velocity $\vec{u}(x + dx, y + dy, z + dz, t + dt)$. The particle acceleration is

$$\vec{a} = \lim_{dt \rightarrow 0} \frac{\vec{u}(x + dx, y + dy, z + dz, t + dt) - \vec{u}(x, y, z, t)}{dt}. \quad (2.16)$$

The particle velocity at time $t + dt$ can be re-expressed as $\vec{u}(x + u_x dt, y + u_y dt, z + u_z dt, t + dt)$, where $dx = u_x dt$, $dy = u_y dt$, and $dz = u_z dt$. Thus by re-expressing

the particle velocity at time $t + dt$ and using a Taylor series expansion

$$\begin{aligned}\vec{u}(x + dx, y + dy, z + dz, t + dt) = \vec{u}(x, y, z, t) + \frac{\partial \vec{u}}{\partial x} u_x dt + \frac{\partial \vec{u}}{\partial y} u_y dt \\ + \frac{\partial \vec{u}}{\partial z} u_z dt + \frac{\partial \vec{u}}{\partial t} dt.\end{aligned}\quad (2.17)$$

Hence the particle acceleration is

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + u_x \frac{\partial \vec{u}}{\partial x} + u_y \frac{\partial \vec{u}}{\partial y} + u_z \frac{\partial \vec{u}}{\partial z}.\quad (2.18)$$

Equation (2.18) can be re-expressed in vector notation by using the vector operation $\vec{u} \cdot \vec{\nabla}$. Hence,

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u},\quad (2.19)$$

where

$$\vec{u} \cdot \vec{\nabla} = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}.\quad (2.20)$$

Equation (2.19) is the total acceleration of a fluid particle in an Eulerian frame of reference. It has both a convective part and a local time rate of change when the flow is unsteady.

Neglecting viscosity, the nett force on the elemental fluid volume in the x -direction is

$$d\vec{f} = \left\{ P - \left(P + \frac{\partial P}{\partial x} \right) dx \right\} dy dz = -\frac{\partial P}{\partial x} dV \vec{i}.\quad (2.21)$$

The complete, three-dimensional, vector force is thus

$$-\vec{\nabla} P dV = -\left\{ \frac{\partial P}{\partial x} \vec{i} + \frac{\partial P}{\partial y} \vec{j} + \frac{\partial P}{\partial z} \vec{k} \right\} dV.\quad (2.22)$$

Hence, from Newton's second law,

$$\rho dV \left\{ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right\} = -\vec{\nabla} P dV,\quad (2.23)$$

where $\rho dV = dm$. Thus,

$$\rho \left\{ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right\} + \vec{\nabla} P = 0.\quad (2.24)$$

Equation (2.24) is identical to equation (2.14) – it is the non-linear, inviscid momentum or Euler's equation. Like the equation of conservation of mass, it can be simplified by linearisation – second- and higher-order terms can be neglected