

Understanding GPS

Principles and Applications

Second Edition



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Second Edition

Elliott D. Kaplan
Christopher J. Hegarty

Editors



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BOSTON | LONDON
artechhouse.com

Library of Congress Cataloging-in-Publication Data

Understanding GPS: principles and applications/[editors], Elliott Kaplan,
Christopher Hegarty.—2nd ed.

p. cm.

Includes bibliographical references.

ISBN 1-58053-894-0 (alk. paper)

1. Global Positioning System. I. Kaplan, Elliott D. II. Hegarty, C. (Christopher J.)

G109.5K36 2006

623.89'3—dc22

2005056270

British Library Cataloguing in Publication Data

Kaplan, Elliott D.

Understanding GPS: principles and applications.—2nd ed.

1. Global positioning system

I. Title II. Hegarty, Christopher J.

629'.045

ISBN-10: 1-58053-894-0

Cover design by Igor Valdman

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International Standard Book Number: 1-58053-894-0

10 9 8 7 6 5 4 3 2 1



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Table 7.3 GPS Precise Positioning Service Typical UERE Budget

<i>Segment Source</i>	<i>Error Source</i>	<i>1σ Error (m)</i>
Space/control	Broadcast clock	1.1
	Broadcast ephemeris	0.8
User	Residual ionospheric delay	0.1
	Residual tropospheric delay	0.2
	Receiver noise and resolution	0.1
	Multipath	0.2
System UERE	Total (RSS)	1.4

Table 7.4 GPS Standard Positioning Service Typical UERE Budget

<i>Segment Source</i>	<i>Error Source</i>	<i>1σ Error (m)</i>
Space/control	Broadcast clock	1.1
	L1 P(Y)-L1 C/A group delay	0.3
	Broadcast ephemeris	0.8
User	Ionospheric delay	7.0*
	Tropospheric delay	0.2
	Receiver noise and resolution	0.1
	Multipath	0.2
System UERE	Total (RSS)	7.1*

*Note that residual ionospheric errors tend to be highly correlated among satellites resulting in position errors being far less than predicted using $DOP \cdot UERE$ (see discussion in Section 7.3.2).

7.3 PVT Estimation Concepts

Chapter 2 described some basic techniques for estimating the PVT of a possibly mobile GPS receiver. This section discusses a variety of additional concepts regarding PVT estimation, beginning with an expanded description of the role of geometry in GPS PVT accuracy and a number of accuracy metrics that are commonly used. This section also describes a number of advanced PVT estimation techniques including the use of WLS, additional estimated parameters (beyond the user x, y, z position coordinates and clock offset), and Kalman filtering.

7.3.1 Satellite Geometry and Dilution of Precision in GPS

As motivation for the concept of dilution of precision as it applies to GPS, consider once again the foghorn example introduced in Section 2.1.1. In this example, a user locates his or her position from ranging measurements from two foghorns. The assumptions are that the user has a synchronized time base relative to the foghorns and has knowledge of the location of the foghorns and their transmission times. The user measures the TOA of each foghorn signal and computes a propagation time that determines the user's range from each foghorn. The user locates his or her position from the intersection of the range rings determined from the TOA measurements.

In the presence of measurement errors, the range rings used to compute the user's location will be in error and result in error in the computed position. The concept of dilution of precision is the idea that the position error that results from measurement errors depends on the user/foghorn relative geometry. Graphically, these ideas are illustrated in Figure 7.5. Two geometries are indicated. In Figure 7.5(a),

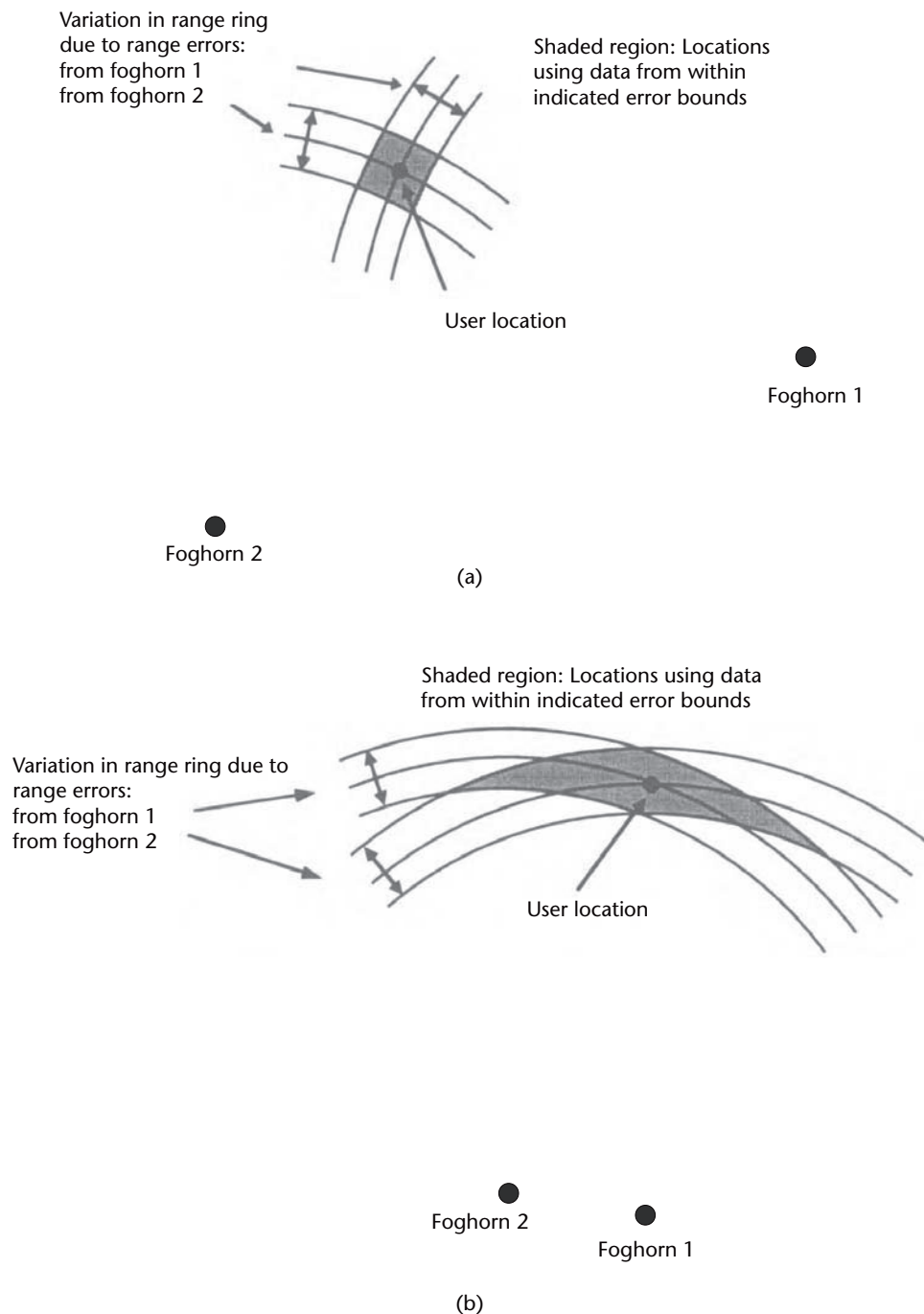


Figure 7.5 Relative geometry and dilution of precision: (a) geometry with low DOP, and (b) geometry with high DOP.

the foghorns are located approximately at right angles with respect to the user location. In Figure 7.5(b), the angle between the foghorns as viewed from the user is much smaller. In both cases, portions of the error-free range rings are indicated and intersect at the user's location. Additional ring segments that illustrate the variation in range ring position resulting from ranging errors to the foghorns are included. The error range illustrated in both figures is the same. The shaded regions indicate the set of locations that can be obtained if one uses ranging measurements within the illustrated error bounds. The accuracy of the computed location is very different for the two cases. With the same measurement error variation, geometry in Figure 7.5(b) gives considerably more error in the computed user's location than in Figure 7.5(a), as is evident from comparison of the shaded regions. Geometry in Figure 7.5(b) is said to have a larger dilution of precision than geometry in Figure 7.5(a). For comparable measurement errors, geometry in Figure 7.5(b) results in larger errors in the computed location.

A formal derivation of the DOP relations in GPS begins with the linearization of the pseudorange equations given in Section 2.4.2. The linearization is the Jacobian relating changes in the user position and time bias to changes in the pseudorange values. This relationship is inverted in accordance with the solution algorithm and is used to relate the covariance of the user position and time bias to the covariance of the pseudorange errors. The DOP parameters are defined as geometry factors that relate parameters of the user position and time bias errors to those of the pseudorange errors.

The offset $\Delta \mathbf{x}$ in the user's position and time bias relative to the linearization point is related to the offset in the error-free pseudorange values $\Delta \mathbf{p}$ by the relation

$$\mathbf{H}\Delta \mathbf{x} = \Delta \mathbf{p} \quad (7.30)$$

The vector $\Delta \mathbf{x}$ has four components. The first three are the position offset of the user from the linearization point; the fourth is the offset of the user time bias from the bias assumed in the linearization point. $\Delta \mathbf{p}$ is the vector offset of the error-free pseudorange values corresponding to the user's actual position and the pseudorange values that correspond to the linearization point. $\mathbf{H}$ is the $n \times 4$ matrix

$$\mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & 1 \end{bmatrix} \quad (7.31)$$

and the $\mathbf{a}_i = (a_{xi}, a_{yi}, a_{zi})$ are the unit vectors pointing from the linearization point to the location of the i th satellite. If $n = 4$ and data from just four satellites are being used, and if the linearization point is close to the user's location, the user's location and time offset are obtained by solving (7.30) for $\Delta \mathbf{x}$ (i.e., if the linearization point is close enough to the user position, iteration is not required). One obtains

$$\Delta \mathbf{x} = \mathbf{H}^{-1} \Delta \mathbf{p} \quad (7.32)$$

and the offset of the user's position from the linearization point is expressed as a linear function of $\Delta\mathbf{p}$. In the case of $n > 4$, the method of least squares can be used to solve (7.30) for $\Delta\mathbf{x}$ (see Appendix A). The least square result can be obtained formally by multiplying both sides of (7.30) on the left by the matrix transpose of $\mathbf{H}$ obtaining $\mathbf{H}^T\mathbf{H}\Delta\mathbf{x} = \mathbf{H}^T\Delta\mathbf{p}$. The matrix combination $\mathbf{H}^T\mathbf{H}$ is a square 4×4 matrix, and one can solve for $\Delta\mathbf{x}$ by multiplying both sides by the inverse, $(\mathbf{H}^T\mathbf{H})^{-1}$. (The matrix will be invertible provided the tips of the unit vectors $\mathbf{a}_i$ do not all lie in a plane.) One obtains

$$\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\mathbf{p} \quad (7.33)$$

which is the least square formulation for $\Delta\mathbf{x}$ as a function of $\Delta\mathbf{p}$. We observe that if $n = 4$, $(\mathbf{H}^T\mathbf{H})^{-1} = \mathbf{H}^{-1}(\mathbf{H}^T)^{-1}$ and (7.33) reduces to (7.32).

The pseudorange measurements are not error-free and can be viewed as a linear combination of three terms,

$$\Delta\mathbf{p} = \mathbf{p}_T - \mathbf{p}_L + d\mathbf{p} \quad (7.34)$$

where $\mathbf{p}_T$ is the vector of error-free pseudorange values, $\mathbf{p}_L$ is the vector of pseudorange values computed at the linearization point, and $d\mathbf{p}$ represents the net error in the pseudorange values. Similarly, $\Delta\mathbf{x}$ can be expressed as

$$\Delta\mathbf{x} = \mathbf{x}_T - \mathbf{x}_L + d\mathbf{x} \quad (7.35)$$

where $\mathbf{x}_T$ is the error-free position and time, $\mathbf{x}_L$ is the position and time defined as the linearization point, and $d\mathbf{x}$ is the error in the position and time estimate. Substituting (7.34) and (7.35) into (7.33) and using the relation $\mathbf{x}_T - \mathbf{x}_L = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T(\mathbf{p}_T - \mathbf{p}_L)$ —this follows from the relation $\mathbf{H}(\mathbf{x}_T - \mathbf{x}_L) = (\mathbf{p}_T - \mathbf{p}_L)$, which is a restatement of (7.30)—one obtains

$$d\mathbf{x} = \left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T \right] d\mathbf{p} = \mathbf{K}d\mathbf{p} \quad (7.36)$$

The matrix $\mathbf{K}$ is defined by the expression in brackets. Equation (7.36) gives the functional relationship between the errors in the pseudorange values and the induced errors in the computed position and time bias. It is valid provided that the linearization point is sufficiently close to the user's location and that the pseudorange errors are sufficiently small so that the error in performing the linearization can be ignored.

Equation (7.36) is the fundamental relationship between pseudorange errors and computed position and time bias errors. The matrix $(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$, which is sometimes called the least-squares solution matrix, is a $4 \times n$ matrix and depends only on the relative geometry of the user and the satellites participating in the least square solution computation. In many applications, the user/satellite geometry can be considered fixed, and (7.36) yields a linear relationship between the pseudorange errors and the induced position and time bias errors.

The pseudorange errors are considered to be random variables, and (7.36) expresses $d\mathbf{x}$ as a random variable functionally related to $d\mathbf{p}$. The error vector $d\mathbf{p}$ is usually assumed to have components that are jointly Gaussian and to be zero mean. With the geometry considered fixed, it follows that $d\mathbf{x}$ is also Gaussian and zero mean. The covariance of $d\mathbf{x}$ is obtained by forming the product $d\mathbf{x}d\mathbf{x}^T$ and computing an expected value. By definition, one obtains

$$\text{cov}(d\mathbf{x}) = E[d\mathbf{x}d\mathbf{x}^T] \quad (7.37)$$

where $\text{cov}(d\mathbf{x}) = E[d\mathbf{x}d\mathbf{x}^T]$ denotes the covariance of $d\mathbf{x}$ and E represents the expectation operator. Substituting from (7.36) and viewing the geometry as fixed, one obtains

$$\begin{aligned} \text{cov}(d\mathbf{x}) &= E[\mathbf{K}d\mathbf{p}k\mathbf{p}^T\mathbf{K}^T] = E\left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T d\mathbf{p}d\mathbf{p}^T\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\right] \\ &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T \text{cov}(d\mathbf{p})\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1} \end{aligned} \quad (7.38)$$

Note that in this computation, $(\mathbf{H}^T\mathbf{H})^{-1}$ is symmetric. [This follows from an application of the general matrix relations $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, which are valid whenever the indicated operations are defined.] The usual assumption is that the components of $d\mathbf{p}$ are identically distributed and independent and have a variance equal to the square of the satellite UERE. With these assumptions, the covariance of $d\mathbf{p}$ is a scalar multiple of the identity

$$\text{cov}(d\mathbf{p}) = \mathbf{I}_{n \times n} \sigma_{\text{URE}}^2 \quad (7.39)$$

where $\mathbf{I}_{n \times n}$ is the $n \times n$ identity matrix. Substitution into (7.38) yields

$$\text{cov}(d\mathbf{x}) = (\mathbf{H}^T\mathbf{H})^{-1} \sigma_{\text{URE}}^2 \quad (7.40)$$

Under the stated assumptions, the covariance of the errors in the computed position and time bias is just a scalar multiple of the matrix $(\mathbf{H}^T\mathbf{H})^{-1}$. The vector $d\mathbf{x}$ has four components, which represent the error in the computed value for the vector $\mathbf{x}_T = (x_u, y_u, z_u, ct_b)$. The covariance of $d\mathbf{x}$ is a 4×4 matrix and has an expanded representation

$$\text{cov}(d\mathbf{x}) = \begin{bmatrix} \sigma_{x_u}^2 & \sigma_{x_u y_u}^2 & \sigma_{x_u z_u}^2 & \sigma_{x_u ct_b}^2 \\ \sigma_{x_u y_u}^2 & \sigma_{y_u}^2 & \sigma_{y_u z_u}^2 & \sigma_{y_u ct_b}^2 \\ \sigma_{x_u z_u}^2 & \sigma_{y_u z_u}^2 & \sigma_{z_u}^2 & \sigma_{z_u ct_b}^2 \\ \sigma_{x_u ct_b}^2 & \sigma_{y_u ct_b}^2 & \sigma_{z_u ct_b}^2 & \sigma_{ct_b}^2 \end{bmatrix} \quad (7.41)$$

The components of the matrix $(\mathbf{H}^T\mathbf{H})^{-1}$ quantify how pseudorange errors translate into components of the covariance of $d\mathbf{x}$.

Dilution of precision parameters in GPS are defined in terms of the ratio of combinations of the components of $\text{cov}(d\mathbf{x})$ and σ_{URE}^2 . [It is implicitly assumed in the DOP definitions that the user/satellite geometry is considered fixed. It is also

assumed that local user coordinates are being used in the specification of $\text{cov}(d\mathbf{x})$ and $d\mathbf{x}$. The positive x -axis points east, the y -axis points north, and the z -axis points up.] The most general parameter is termed the geometric dilution of precision (GDOP) and is defined by the formula

$$\text{GDOP} = \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{t_b}^2}}{\sigma_{\text{URE}}} \quad (7.42)$$

A relationship for GDOP is obtained in terms of the components of $(\mathbf{H}^T \mathbf{H})^{-1}$ by expressing $(\mathbf{H}^T \mathbf{H})^{-1}$ in component form

$$(\mathbf{H}^T \mathbf{H})^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \quad (7.43)$$

and then substituting (7.43) and (7.41) into (7.40). A trace operation on (7.40) followed by a square root shows that GDOP can be computed as the square root of the trace of the $(\mathbf{H}^T \mathbf{H})^{-1}$ matrix:

$$\text{GDOP} = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}} \quad (7.44)$$

Equation (7.42) can be rearranged to obtain

$$\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{t_b}^2} = \text{GDOP} \times \sigma_{\text{URE}} \quad (7.45)$$

which has the form given in (7.1) The square root term on the left side gives an overall characterization of the error in the GPS solution. GDOP is the geometry factor. It represents the amplification of the standard deviation of the measurement errors onto the solution. From (7.44), GDOP is seen to be a function solely of the satellite/user geometry. The value σ_{URE} is the pseudorange error factor.

Several other DOP parameters in common use are useful to characterize the accuracy of various components of the position/time solution. These are termed position dilution of precision (PDOP), horizontal dilution of precision (HDOP), vertical dilution of precision (VDOP), and time dilution of precision (TDOP). These DOP parameters are defined in terms of the satellite UERE and elements of the covariance matrix for the position/time solution as follows:

$$\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2} = \text{PDOP} \times \sigma_{\text{URE}} \quad (7.46)$$

$$\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2} = \text{HDOP} \times \sigma_{\text{URE}} \quad (7.47)$$

$$\sigma_{z_u} = \text{VDOP} \times \sigma_{\text{URE}} \quad (7.48)$$

$$\sigma_{t_b} = \text{TDOP} \times \sigma_{\text{URE}} \quad (7.49)$$

The DOP values can be expressed in terms of the components of $(\mathbf{H}^T \mathbf{H})^{-1}$ as follows:

$$\text{PDOP} = \sqrt{D_{11} + D_{22} + D_{33}} \quad (7.50)$$

$$\text{HDOP} = \sqrt{D_{11} + D_{22}} \quad (7.51)$$

$$\text{VDOP} = \sqrt{D_{33}} \quad (7.52)$$

$$\text{TDOP} = \sqrt{D_{44}}/c \quad (7.53)$$

(In some treatments of DOP, TDOP is defined by the formula $\sigma_{ct_b} = \text{TDOP} \times \sigma_{\text{URE}}$. In this case, (7.53) takes the simpler form $\text{TDOP} = \sqrt{D_{44}}$. The variable ct_b represents a range equivalent of the time bias error, and σ_{ct_b} is its standard deviation. In the current formulation, TDOP is defined so that when multiplied by σ_{URE} , the standard deviation of the time bias error is obtained directly. This is the more relevant formulation if actual time accuracy is of interest. The linear relationship between t_b and ct_b yields the formula $c\sigma_{t_b} = \sigma_{ct_b}$ between their standard deviations, and one can easily convert between the formulations.)

7.3.2 Accuracy Metrics

The formulas derived in Section 7.3.1 allow one to compute 1-sigma horizontal, vertical, or three-dimensional position errors as a function of satellite geometry and the 1-sigma range error. They also allow one to compute 1-sigma user clock errors. It is important to recall that these formulas were derived under the assumptions that range errors are zero mean with a Gaussian distribution and that range errors are independent from satellite to satellite. Oftentimes, other metrics besides 1-sigma position errors are used to characterize GPS accuracy performance. Some common metrics are derived and discussed in this subsection.

If pseudorange errors are Gaussian-distributed, (7.36) tells us that vertical position errors also have a Gaussian distribution:

$$dz = \sum_{m=1}^N K_{3,m} d\rho_m \quad (7.54)$$

where dz is the error in the vertical component of the computed position. This result is obtained by noting that a linear function of Gaussian random variables is itself a Gaussian random variable. One common measure of vertical positioning accuracy is the error magnitude that 95% of the measurements fall within, which is approximately equal to the 2-sigma value for a Gaussian random variable. Thus:

$$95\% \text{ vertical position accuracy} \approx 2\sigma_{dz} = 2 \cdot \text{VDOP} \cdot \sigma_{\text{URE}} \quad (7.55)$$

assuming that pseudorange errors are additionally zero mean and independent among satellites. Using a global average value of 1.6 for VDOP for the nominal 24-satellite GPS constellation and the URE values from Tables 7.3 and 7.4 yields 95% vertical position accuracy of 4.5m for the PPS and 22.7m for the SPS, respec-

tively. The value for the PPS is reasonably consistent with observed performance, but the value for the SPS is significantly pessimistic as compared with observed performance (see Section 7.7). The reason for this discrepancy is that the dominant SPS UERE component, residual ionospheric delay error, is highly correlated among satellites. This correlation invalidates one of the assumptions used in the derivation of the DOPs. When the Klobuchar model's estimate of vertical ionospheric delay is too high (low), the slant delay estimate for each visible satellite also tends to be too high (low). A significant component of each residual ionospheric delay error is thus common to each satellite and drops out into the user clock solution.

With regard to horizontal position errors, (7.36) can be specialized to the horizontal plane yielding

$$d\mathbf{R} = \mathbf{K}_{2 \times n} d\mathbf{p} \quad (7.56)$$

where $d\mathbf{R} = (dx, dy)^T$ is the vector component of the position error in the horizontal plane, $d\mathbf{p} = (dp_1, \dots, dp_n)^T$ is the pseudorange measurement errors, and n is the number of satellites being used in the position calculation. $\mathbf{K}_{2 \times n}$ is the upper $2 \times n$ submatrix of $\mathbf{K}$ and consists of its first two rows. For the standard least square solution technique, $\mathbf{K} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T$.

For a fixed satellite geometry, (7.56) expresses the horizontal position errors as a linear function of the pseudorange measurement errors. If the pseudorange errors are zero mean and jointly Gaussian, $d\mathbf{R}$ also has these properties. If the pseudorange errors are also uncorrelated and identically distributed with variance σ_{UERE}^2 , the covariance of the horizontal errors is given as

$$\text{cov}(d\mathbf{R}) = \left((\mathbf{H}^T \mathbf{H})^{-1} \right)_{2 \times 2} \sigma_{\text{UERE}}^2 \quad (7.57)$$

where the subscript notation denotes the upper left 2×2 submatrix of $(\mathbf{H}^T \mathbf{H})^{-1}$. The density function for $d\mathbf{R}$ is

$$f_{d\mathbf{R}}(x, y) = \frac{1}{2\pi [\det(\text{cov}(d\mathbf{R}))]^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \mathbf{u}^T [\text{cov}(d\mathbf{R})]^{-1} \mathbf{u}\right) \quad (7.58)$$

where $\mathbf{u} = (x, y)^T$ and $\det$ represents the determinant of a matrix.

The density function defines a two-dimensional bell-shaped surface. Contours of constant density are obtained by setting the exponent in parenthesis to a constant. One obtains equations of the form

$$\mathbf{u}^T [\text{cov}(d\mathbf{R})]^{-1} \mathbf{u} = m^2 \quad (7.59)$$

where the parameter m ranges over positive values. The contour curves that result form a collection of concentric ellipses when plotted in the plane. The ellipse obtained when m equals 1 is termed the 1σ ellipse and has the equation

$$\mathbf{u}^T [\text{cov}(d\mathbf{R})]^{-1} \mathbf{u} = 1 \quad (7.60)$$

(The 1σ ellipse is defined here as a specific cut through the pdf and is not to be confused with 1σ containment. The latter curve is the locus of points, one point on each ray from the origin, where the points are at a distance of 1σ for the ray's direction. In general, the 1σ containment curve is a figure-eight-shaped curve that encloses the 1σ ellipse.) If the major and minor axis of the ellipse are aligned with the x and y axes, the equation for the ellipse reduces to $x^2 / \sigma_x^2 + y^2 / \sigma_y^2 = 1$. In general, however, the off-diagonal terms in $\text{cov}(d\mathbf{R})$ are nonzero, and the elliptical contours for the density function are rotated relative to the x and y axes. Denote the major and minor axes of the 1σ ellipse by σ_L and σ_S ("long" and "short"), respectively. In general, the 1σ ellipse is contained in a box of width σ_x and height σ_y centered on the ellipse. Figure 7.6 illustrates graphically the relationship between the ellipse and the parameters σ_x , σ_y , σ_L , and σ_S .

The probability that the error lies within the elliptical contour defined for a specific value of m is $1 - e^{-m^2/2}$. In particular, the probability of being in the 1σ ellipse ($m = 1$) is 0.39; the probability of being in the 2σ ellipse ($m = 2$) is 0.86. (These values are in contrast to the one-dimensional Gaussian result that the probability of being within $\pm 1\sigma$ of the mean is 0.68.)

Several parameters are in common use that characterize the magnitude of the horizontal error. The *distance root mean square* (drms) is defined by the formula

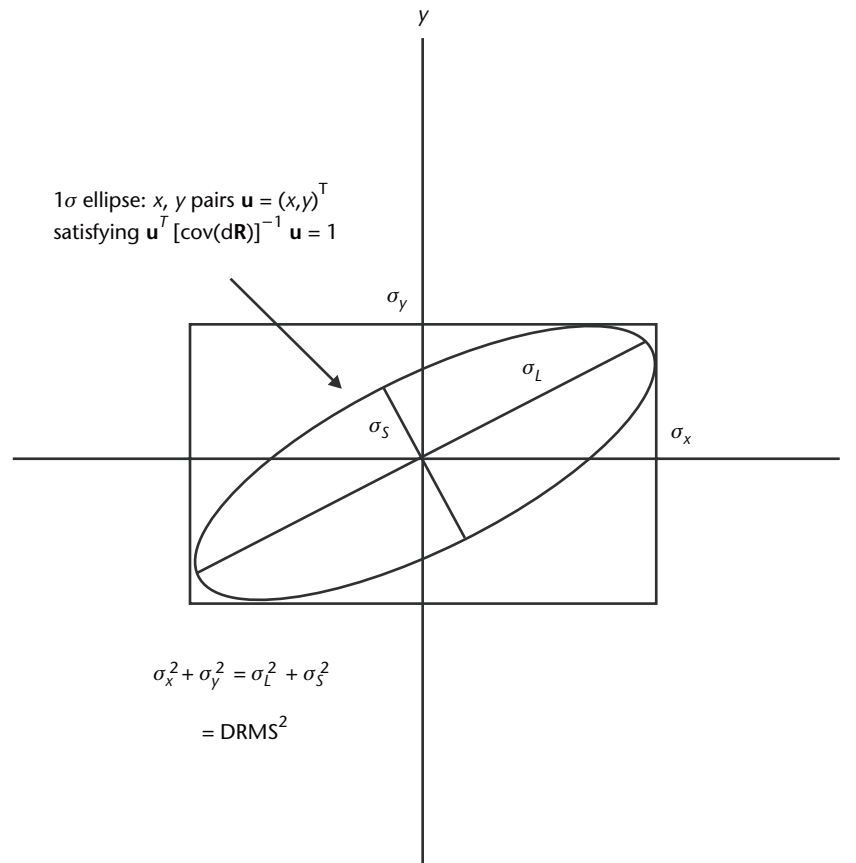


Figure 7.6 Relationship between 1σ ellipse and distribution parameters.

$$\text{drms} = \sqrt{\sigma_x^2 + \sigma_y^2} \quad (7.61)$$

For a zero-mean random variable such as $d\mathbf{R}$, one has $\text{drms} = \sqrt{E(|d\mathbf{R}|^2)}$, and the drms corresponds to the square root of the mean value of the squared error (hence, its name). From (7.47), one immediately has

$$\text{drms} = \text{HDOP} \cdot \sigma_{\text{URE}} \quad (7.62)$$

and the drms can be computed from the values of HDOP and σ_{URE} . The probability that the computed location is within a circle of radius drms from the true location depends on the ratio σ_s/σ_L for the 1σ ellipse. If the two-dimensional error distribution is close to being circular ($\sigma_s/\sigma_L \approx 1$), the probability is about 0.63; for a very elongated distribution ($\sigma_s/\sigma_L \approx 0$), the probability approaches 0.69. Two times the drms is given by

$$2 \text{ drms} = 2 \cdot \text{HDOP} \cdot \sigma_{\text{URE}} \quad (7.63)$$

and the probability that the horizontal error is within a circle of radius 2 drms ranges between 0.95 and 0.98, depending on the ratio σ_s/σ_L . The 2-drms value is commonly taken as the 95% limit for the magnitude of the horizontal error.

Another common metric for horizontal errors is *circular error probable* (CEP), defined as the radius of a circle that contains 50% of the error distributions when centered at the correct (i.e., error-free) location. Thus, the probability that the magnitude of the error is less than the CEP is precisely one-half. The CEP for a two-dimensional Gaussian random variable can be approximated by the formula

$$\text{CEP} \approx 0.59(\sigma_L + \sigma_s) \quad (7.64)$$

assuming it is zero mean. For a derivation of this and other approximations, see [33].

The CEP can also be estimated in terms of drms and, using (7.62), in terms of HDOP and σ_{URE} . This is convenient since HDOP is widely computed in GPS applications. Figure 7.7 presents curves giving the probability that the magnitude of the error satisfies $|d\mathbf{R}| \leq k \text{ drms}$ as a function of k for different values of the ratio σ_s/σ_L . (The horizontal error is assumed to have a zero-mean two-dimensional Gaussian distribution.) For k equal to 0.75, one obtains a probability in the range 0.43 to 0.54. Hence, one has the approximate relation

$$\text{CEP} \approx 0.75 \text{ drms} = 0.75 \cdot \text{HDOP} \cdot \sigma_{\text{URE}} \quad (7.65)$$

It is interesting to note that for $k = 1.23$, the probability that $|d\mathbf{R}| \leq k \text{ drms}$ is roughly 0.78, almost independent of σ_s/σ_L . The probabilities associated with several other values of k are summarized in Table 7.5.

As an application of these formulations, for an average global HDOP of 1.0 and with $\sigma_{\text{URE}} = 1.4\text{m}$, estimates for the CEP, the 80% point, and the 95% point for the magnitude of the horizontal error are given as follows:

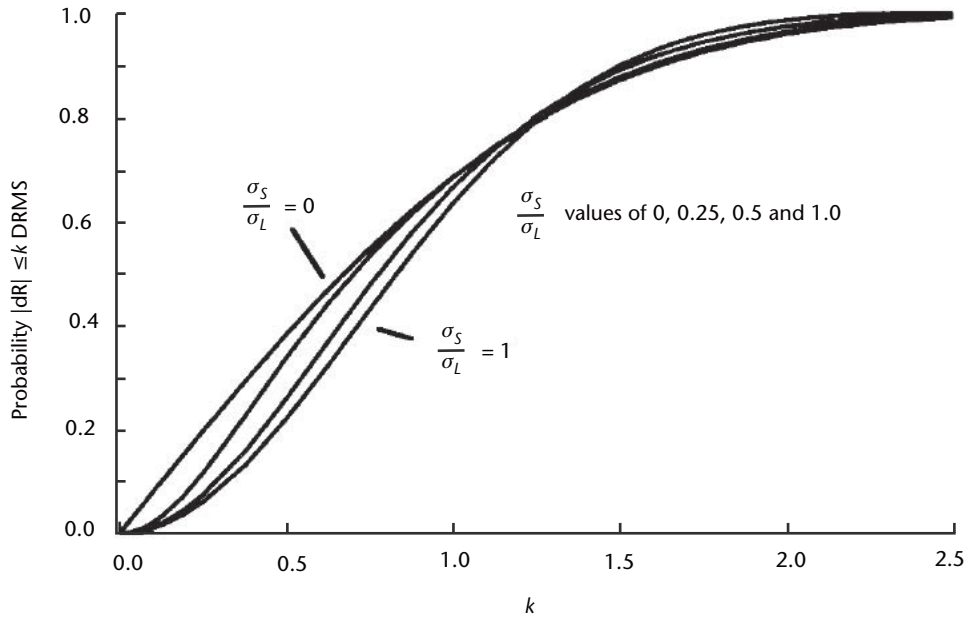


Figure 7.7 Cumulative distribution of radial error for various values of σ_S/σ_L for a two-dimensional Gaussian random variable.

Table 7.5 Approximate Formulas for the Magnitude of the Horizontal Error

Approximation Formula*	Probability Range
$CEP_{50} \approx 0.75 \text{ HDOP } \sigma_{URE}$	0.43–0.54
$CEP_{80} \approx 1.28 \text{ HDOP } \sigma_{URE}$	0.80–0.81
$CEP_{90} \approx 1.6 \text{ HDOP } \sigma_{URE}$	0.89–0.92
$CEP_{95} \approx 2.0 \text{ HDOP } \sigma_{URE}$	0.95–0.98

* CEP_n is defined as the radius of the circle that when centered at the error-free location includes xx% of the error distribution. Hence, $CEP_{50} = CEP$.

$$\begin{aligned}
 CEP_{50} &\approx 0.75 \cdot \text{HDOP} \cdot \sigma_{URE} = 0.75 \times 1 \times 1.4 = 1.1 \text{ m} \\
 CEP_{80} &\approx 1.28 \cdot \text{HDOP} \cdot \sigma_{URE} = 1.2875 \times 1 \times 1.4 = 1.8 \text{ m} \\
 CEP_{95} &\approx 2.0 \cdot \text{HDOP} \cdot \sigma_{URE} = 2.0 \times 1 \times 1.4 = 2.8 \text{ m}
 \end{aligned} \tag{7.66}$$

For applications where three-dimensional error distributions are of interest, one final commonly used metric is *spherical error probable* (SEP), which is defined as the radius of a sphere centered at the true position that contains 50% of the measured positions.

7.3.3 Weighted Least Squares (WLS)

Oftentimes, the UEREs among the visible satellites are not well described as being independent and identically distributed. In such circumstances, the least-squares

position estimate is not optimal. As derived in Appendix A, if the pseudorange errors are Gaussian and the covariance of UEREs for the visible satellites is given by the matrix $\mathbf{R}$, then the optimal solution for user position is given by the WLS estimate

$$\Delta \mathbf{x} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}^{-1} \Delta \mathbf{p} \quad (7.67)$$

(Note that, as with the ordinary least-squares solution, we are truly solving for a correction to an initial estimate of the user position and clock error.) Equation (7.67) collapses to (7.33) in the case when $\mathbf{R} = \sigma_{\text{URE}}^2 \mathbf{I}$ with $\mathbf{I}$ equal to the $n \times n$ identity matrix, as expected since this case corresponds to our original independent and identically distributed assumption. For a general matrix $\mathbf{R}$, (7.67) can be thought of as implementing an optimal weighting of pseudorange measurements based on their relative noise levels and relative importance for each estimated quantity.

As one example of an error covariance matrix, consider the single-frequency SPS user whose pseudorange measurement errors are dominated by residual ionospheric delays. As noted in the discussion in Section 7.3.2, residual ionospheric errors for single-frequency users are highly correlated. The covariance matrix of residual ionospheric errors can be approximated as

$$\mathbf{R} = \sigma_{iv}^2 \begin{bmatrix} m^2(el_1) & m(el_1)m(el_2) & \cdots & m(el_1)m(el_n) \\ m(el_1)m(el_2) & m^2(el_2) & & \\ \vdots & & \ddots & \\ m(el_1)m(el_n) & & & m^2(el_n) \end{bmatrix} \quad (7.68)$$

where σ_{iv}^2 is the residual vertical ionospheric delay variance, which could be approximated as some fraction of the Klobuchar vertical delay estimate. The ij th element of the matrix in (7.68) is the product of two ionospheric mapping functions, $m(el)$. For example, (7.21) could be used, corresponding to the elevation angles (el) for satellite i and j .

Another typical example of a covariance matrix is a diagonal matrix whose diagonal elements are obtained using an approximation for pseudorange error variance versus elevation angle, usually a monotonically increasing function as elevation angle decreases (e.g., see [14]). The use of such a covariance matrix in a WLS solution deweights low-elevation angle satellites that are expected to be noisier due to typical characteristics of multipath and residual tropospheric errors.

7.3.4 Additional State Variables

Thus far, we have focused on estimation of the user's (x, y, z) position coordinates and clock bias. The complete set of parameters that are estimated within a GPS receiver, often referred to as the *state* or *state vector*, may include a number of other variables. For instance, if in addition to pseudorange measurements, Doppler measurements (from an FLL or PLL) or differenced carrier-phase measurements are available, then velocity in each of the three coordinates ($\dot{x}, \dot{y}, \dot{z}$), and clock drift, $\dot{t}_u$, may also be estimated. The same least squares or WLS techniques used for position

Least Squares and Weighted Least Squares Estimates

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Let $\mathbf{x} = [x_1 x_2 \dots x_M]^T$ be a column vector containing M unknown parameters that are to be estimated and $\mathbf{y} = [y_1 y_2 \dots y_N]^T$ be a set of noisy measurements that are linearly related to $\mathbf{x}$ as described by the expression:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (\text{A.1})$$

where $\mathbf{n} = [n_1 n_2 \dots n_N]^T$ is a vector describing the errors corrupting the N measurements, and $\mathbf{H}$ is an $N \times M$ matrix describing the connection between the measurements and $\mathbf{x}$.

The *maximum likelihood* estimate of $\mathbf{x}$, denoted as $\hat{\mathbf{x}}$, is defined as (see, for example, [1]):

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x}} p(\mathbf{y}/\mathbf{x}) \quad (\text{A.2})$$

where $p(\mathbf{y}/\mathbf{x})$ is the probability density function of the measurement $\mathbf{y}$ for a fixed value of $\mathbf{x}$.

If the measurement errors, $\{n_i\}$, for $i = 1, \dots, N$, are identically Gaussian distributed with zero-mean and variance σ^2 , and furthermore if errors for different measurements are statistically independent, then (A.2) becomes:

$$\begin{aligned} \hat{\mathbf{x}} &= \arg \max_{\mathbf{x}} \frac{1}{(2\pi\sigma)^{N/2}} e^{-\frac{1}{2\sigma^2} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2} \\ &= \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \end{aligned} \quad (\text{A.3})$$

The solution to (A.3) can readily be found by first differentiating $\|\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\|^2$ with respect to $\hat{\mathbf{x}}$:

$$\frac{d}{d\hat{\mathbf{x}}} \|\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\|^2 = 2\mathbf{H}^T \mathbf{H}\hat{\mathbf{x}} - 2\mathbf{H}^T \mathbf{y} \quad (\text{A.4})$$

and then setting this quantity equal to zero to obtain:

$$\hat{\mathbf{x}} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{y} \quad (\text{A.5})$$

where it is assumed that the matrix inverse involved exists (i.e., that $\mathbf{H}^T \mathbf{H}$ is not singular).

The estimate described by (A.5) is referred to as a *least squares estimate*, since, as shown in (A.3), it results in the minimum square error between the measurement vector $\mathbf{y}$ and $\mathbf{H}\mathbf{x}$, where the latter is the expected measurement vector based upon the estimate of $\mathbf{x}$.

Next, consider the more general case where the measurement errors are still Gaussian distributed with zero-mean but are not necessarily identically distributed or independent of each other. In this case, the maximum likelihood estimate can be expressed as

$$\begin{aligned} \hat{\mathbf{x}} &= \arg \max_{\mathbf{x}} \frac{1}{(2\pi)^{N/2} |\mathbf{R}_n|^{1/2}} e^{-\frac{1}{2}(\mathbf{y}-\mathbf{H}\mathbf{x})^T \mathbf{R}_n^{-1} (\mathbf{y}-\mathbf{H}\mathbf{x})} \\ &= \arg \min_{\mathbf{x}} (\mathbf{y} - \mathbf{H}\mathbf{x})^T \mathbf{R}_n^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}) \end{aligned} \quad (\text{A.6})$$

where $\mathbf{R}_n$ is the covariance matrix associated with the measurement errors and $|\mathbf{R}_n|$ is its determinant.

Proceeding as before, (A.6) can be solved to yield:

$$\hat{\mathbf{x}} = (\mathbf{H}^T \mathbf{R}_n^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{R}_n^{-1} \mathbf{y} \quad (\text{A.7})$$

The estimate in (A.7) is referred to as a WLS solution.

Reference

- [1] Stark, H., and J. W. Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, Englewood Cliffs, NJ: Prentice-Hall, 1986.