

Global Positioning System: Theory and Applications

Volume I

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made on a signal waveform. Pulses of radio frequency energy can be modulated with a special pulse code sequence that has a distinct beginning (or epoch). The GPS civilian [coarse acquisition (C/A)] signal repeats such a sequence every millisecond. The C/A code is a random-like or pseudonoise (PN) binary sequence of 1023 chips. The pseudorange $t_u - t_s$ can then be recovered in a special delay lock loop receiver designed to track and to detect the code from that satellite, as shown in Fig. 5. The measurement relies on the unique code properties of each satellite signal. These properties enable the receiver to measure pseudorange to each satellite separately. The delay lock technique creates an internal replica of the known modulation sequence and adjusts the internal epoch until it exactly matches the received signal in delay. This matching is performed by cross-correlating the received and internal signals and finding the start time that maximizes the correlator output. The satellite clock time at the time of transmission is then subtracted from the user clock time to recover the measured pseudorange.

If the user receiver clock t_u and the satellite clock are both synchronized to GPS reference time,* then range $= D = c(t_u - t_s)$. For purposes of this discussion, atmospheric and other propagation path delay perturbations are neglected. Pseudorange is the same measurement when the user receiver clock t_u has an unknown and possibly time varying clock bias b_u expressed in seconds. The pseudorange is then $\rho = D + cb_u$, as shown in Fig. 5. This technique is known as *code ranging* to distinguish it from carrier phase measurements.

Most GPS receivers can also reconstruct the GPS radio frequency carrier at 1575.42 MHz and use this sinusoid as a ranging signal. This measurement is very precise (typically subcentimeter or a fraction of the 19-cm wavelength), but its *accuracy* is limited by the difficulty of resolving which cycle is being received (called the cycle ambiguity or $n\lambda$ problem). Carrier phase accuracy corresponds to the equivalent carrier phase noise expressed in distance which would be on the order of 2 mm if the carrier phase could be measured to 1% of the wavelength. However, the initial value of the carrier phase is completely ambiguous, and we must resort to the use of various phase differencing techniques. Nonetheless, it is possible to measure changes in phase both very precisely and without significant cycle slipping over many seconds. This carrier phase measurement can then allow us to make very precise position measurements. The use of carrier phase measurements for various surveying and aircraft landing applications is discussed in later chapters in this volume. Table 1 gives a rough estimate of the measurement accuracy, bias, and precision for GPS carrier and code measurements.

a. Relating pseudorange accuracy and positioning accuracy—dilution of precision (DOP). Positioning accuracy reflects the final capability of most GPS receivers. Although it is related to *ranging* accuracy, they are not the same. The relationship between them is a function of the geometry of the selected satellites; that is, the directions of arrival of the satellite signals. To achieve a positioning accuracy requirement of 10 m, the ranging accuracy and geometry must both combine to acceptable values. For example, if each individual pseudorange measurement has a statistically independent error of zero mean with the same rms

*All time intervals can be expressed in equivalent distance in meters by multiplying by the speed of light c .

Table 1 Rough order of magnitude estimates of GPS code and carrier phase measurements in meters^a

Measurement	Rough measurement accuracy	
	Bias	Precision
Code	≈ 5 m	≈ 2 m
Carrier phase	$n\lambda$	≈ 0.002 m

^aThere is assumed to be no selective availability (SA) degradation. The value of n in the carrier phase ambiguity must be determined.

value of σ (caused by all effects), then the rms position errors are given by the following:

Position error = DOP * σ , where DOP is a multiplier determined by the geometry and is typically between 1 and 100.

The quantity DOP is calculated from the unit vectors to each of the satellites, as shown in Chapters 5 and 11, this volume. Generally, if the DOP rises above six, the satellite geometry is not very good. There are several measures of positioning accuracy. For the current nominal constellation, the worldwide median position dilution of precision (PDOP) (50th percentile) is approximately 2.5. As another example, the horizontal error (in the x and y coordinates) is given by rms error horizontal = HDOP (horizontal dilution of precision) * σ , where rms denotes the root mean square error. The vertical error (in the z coordinate) is found from rms error vertical = VDOP (vertical dilution of precision) * σ .

The speed of light is approximately $c \approx 3$ m/ns and $1/c \approx 3.3$ ns/m. If the satellite geometry produces an HDOP of 3 (and a horizontal error less than 10 m is required), then $\text{HDOP} * \sigma = 3\sigma \leq 10 \text{ m} * 3.3 \text{ ns/m}$. Thus, the required ranging accuracy is then $\sigma \leq 11$ ns. In a similar manner, the rms position error (in three dimensions x , y , and z) is estimated by the relationship rms position error = PDOP * σ .

Table 2 summarizes some of the key definitions of parameters in the GPS user position calculations and error sources.

III. GPS Space Segment

A. GPS Orbit Configuration and Multiple Access

The discussions of the basic concepts for GPS and analysis of the geometric dilution of precision have shown that the user must make measurements on four or more satellites simultaneously to provide real-time three-dimensional navigation. Thus, the satellite orbital constellation must provide a user anywhere in the world simultaneous access to measure pseudorange to four or more satellites at any time, 24 h a day. Furthermore, as shown later in the DOP analysis, the satellites should be widely spaced in angle. Measurements on each of the four or more satellites must be made simultaneously or nearly simultaneously without mutual interference if we are to solve for position. This capability is termed

fix, which can start from any location. Such techniques are beyond this text and are not a part of error analysis.) This calculation is developed in this chapter.

For GPS, the underlying coordinate system is currently the 1984 World Geodetic System (WGS-84), which is an accepted worldwide geodetic coordinate system. It is expected that this will be replaced with a more accurate version as satellite geodesy improves. It is usual and convenient for the receiver to perform initial calculations in an Earth-centered, Earth-fixed (ECEF) Cartesian coordinate system. These coordinates can then be converted to any other required reference. For *error analysis*, it is usual to consider a local coordinate frame centered at the user and oriented East, North, and Up. This is convenient because many users have differing sensitivities to vertical errors and horizontal errors.

B. Derivation of the Fundamental Error Equation

This section develops the GPS error equation, beginning with the fundamental measurements and proceeding through analysis of the effects of various error sources. It lays the groundwork for the essential understanding of the dilution of precision (DOP) concept.

1. Ideal Measurement

The "true" or ideal measurement is the GPS signal arrival time. This is equal to the signal transmission time delayed by the vacuum transit time and corrected for the true additional delays caused by the ionosphere and troposphere:

$$t_A = t_T + D/c + T + I \quad (1)$$

where t_A = true arrival time (s); t_T = true transmit time (s); D = true range (m); c = vacuum speed of light (m/s); T = true tropospheric delay (s); and I = true ionospheric delay (s).

2. Measured Arrival Time

The measured arrival time reflects the user's clock bias and other measurement errors:

$$t_{Au} = t_A + b_u + v \quad (2)$$

where t_{Au} = arrival time measured by the user (s); b_u = user clock bias estimate (s); and v = receiver noise, multipath, interchannel error (different for each satellite (s)).

3. Satellite Transmission Time

The satellite clock correction transmitted by the satellite can also be in error (the dominant error may be due to selective availability SA):

$$t_{Ts} = t_T + B \quad (3)$$

where t_{Ts} = value of transmission time in the current satellite message (s); and B = true error in satellite's transmission time (includes SA).

4. True Range

The true range is the absolute value of the vector difference between the true satellite position and the true user position:

$$D = |\bar{r}_s - \bar{r}_u| = \bar{\mathbf{I}}_s \cdot [\bar{r}_s - \bar{r}_u] \quad (4)$$

where $\bar{r}_s$ = true satellite position; $\bar{r}_u$ = true user position; and $\bar{\mathbf{I}}_s$ = true unit vector from user to satellite.

The right-hand expression in Eq. (4), which uses the vector dot product, is a convenient way to calculate the range in the later user equation. In this calculation, the *estimated* user position can be used to find the unit vector from user to satellite. This unit vector need not be exact. Even *errors of several hundred meters* in user or satellite location have a very small effect on the dot product that uses this unit vector. Such a position error would produce angular errors of a few arc-seconds (about 10^{-5} rad) in the unit vector. Because the dot product *error* would be proportional to the cosine of this angle, it is of the order (angle)²/2 or about $10^{-10}/2$ times the range. Because the range is 2×10^7 m, the effect would be less than 1 mm.

5. Pseudorange

The user receiver actually “measures” the “pseudorange” ρ , given by the following:

$$\rho = c \cdot (t_{Au} - t_{Ts}) \quad (5)$$

This is called the pseudorange because it is linearly a function of the range to the satellite, but it is also corrupted by the user's clock bias, which must be estimated and removed. In addition, it must be corrected for the estimated satellite time bias and for variations in the speed of transmission.

Substituting Eqs. (3), (2), and (1) into Eq. (5) gives the following result:

$$\rho = D + c \cdot (b_u - B) + c \cdot (T + I + v) \quad (6)$$

Using Eq. (4) in this expression gives the following:

$$\rho = \bar{\mathbf{I}}_s \cdot [\bar{r}_s - \bar{r}_u] + c \cdot (b_u - B) + c \cdot (T + I + v) \quad (7)$$

To account for the estimated value ($\hat{}$) and the estimate error (Δ), each of the above terms is to be broken into two parts as follows:

$$\begin{aligned} \bar{r}_s &= \hat{\bar{r}}_s - \Delta \bar{r}_s, \text{ where } \hat{\bar{r}}_s &&= \text{satellite position reported in the transmitted message (m)} \\ \bar{r}_u &= \hat{\bar{r}}_u - \Delta \bar{r}_u, \text{ where } \hat{\bar{r}}_u &&= \text{user estimated position (m)} \\ \bar{\mathbf{I}}_s &= \hat{\bar{\mathbf{I}}}_s - \Delta \bar{\mathbf{I}}_s, \text{ where } \hat{\bar{\mathbf{I}}}_s &&= \text{unit vector from user to the satellite estimated from } \hat{\bar{r}}_s \text{ and } \hat{\bar{r}}_u \\ b_u &= \hat{b}_u - \Delta b_u, \text{ where } \hat{b}_u &&= \text{user clock bias estimate common to a set of simultaneous measurements (s)} \\ B &= \hat{B} - \Delta B - S, \text{ where } \hat{B} &&= \text{satellite transmitted clock bias (s)} \\ &&&\Delta B &&= \text{the “natural” satellite clock error; that is, the error in control system prediction (s)} \end{aligned}$$

$$\begin{aligned} T = \hat{T} - \Delta T & \quad S = \text{error in transmit time due to SA (s)} \\ I = \hat{I} - \Delta I & \quad \hat{T} = \text{estimated (or modeled) tropospheric delay (s)} \\ & \quad \hat{I} = \text{estimated (or modeled) ionospheric delay (s)} \end{aligned}$$

Equation (7) can then be modified to account for the estimated values:

$$\rho_j = (\hat{\mathbf{I}}_{sj} - \Delta \bar{\mathbf{I}}_{sj}) \cdot (\hat{\mathbf{r}}_{sj} - \Delta \bar{\mathbf{r}}_{sj} - \hat{\mathbf{r}}_u + \Delta \bar{\mathbf{r}}_u) + c \cdot (\hat{b}_u - \Delta b_u - \hat{B}_j + \Delta B_j + S_j) + c \cdot (\hat{I}_j - \Delta I_j + \hat{T}_j - \Delta T_j + v_j) \quad (8)$$

where the j subscript is the satellite number and has been added to point out quantities unique to each satellite.

In preparation for conversion to matrix form, Eq. (8) can be rewritten as follows:

$$\begin{aligned} \underbrace{\hat{\mathbf{I}}_{sj} \cdot \hat{\mathbf{r}}_u - c \cdot \hat{b}_u}_{(b)} - \underbrace{\hat{\mathbf{I}}_{sj} \cdot \Delta \bar{\mathbf{r}}_u + c \cdot \Delta b_u}_{(d)} &= \underbrace{\hat{\mathbf{I}}_{sj} \cdot \hat{\mathbf{r}}_{sj} - \rho_j + c \cdot [\hat{I}_j + \hat{T}_j - \hat{B}_j]}_{(b)} \\ &+ \underbrace{-\hat{\mathbf{I}}_{sj} \cdot \Delta \bar{\mathbf{r}}_{sj} - \Delta \bar{\mathbf{I}}_{sj} \cdot (\hat{\mathbf{r}}_{sj} - \hat{\mathbf{r}}_u) + c \cdot (\Delta B_j + S_j) - c \cdot (\Delta I_j + \Delta T_j) + c \cdot v_j}_{(c)} \\ &+ \text{higher order terms} \end{aligned} \quad (9)$$

The large dots represent the dot product. Note that the terms underlined as (a) are the user's position and clock errors to be solved, the terms (b) are estimated or measured by the user, and the terms (c) are unknown errors that produce the solution errors given by (d). The right-hand portion of (b) includes ρ_{sj} , or corrected pseudorange, as: $\rho_{sj} = \rho_j - c \cdot [\hat{I}_j + \hat{T}_j - \hat{B}_j]$. This variable is used below.

Next, define the following matrices for K satellites in view (note that $K = 4$ is normally the minimum number of measurements):

$$\begin{aligned} \mathbf{G}_{K \times 4} &\triangleq \begin{bmatrix} \hat{\mathbf{I}}_{s1}^T & 1 \\ \hat{\mathbf{I}}_{s2}^T & 1 \\ \vdots & \vdots \\ \hat{\mathbf{I}}_{sK}^T & 1 \end{bmatrix} & \mathbf{A}_{K \times 3K} &\triangleq \begin{bmatrix} \hat{\mathbf{I}}_{s1}^T & & 0 \\ & \hat{\mathbf{I}}_{s2}^T & \\ & & \ddots \\ 0 & & \hat{\mathbf{I}}_{sK}^T \end{bmatrix} \\ \hat{\mathbf{x}}_{4 \times 1} &\triangleq \begin{bmatrix} \hat{\mathbf{r}}_u \\ -c \cdot \hat{b}_u \end{bmatrix} & \Delta \bar{\mathbf{x}}_{4 \times 1} &\triangleq \begin{bmatrix} \Delta \bar{\mathbf{r}}_u \\ -c \cdot \Delta b_u \end{bmatrix} & \bar{\mathbf{R}}_{3K \times 1} &\triangleq \begin{bmatrix} \hat{\mathbf{r}}_{s1} \\ \hat{\mathbf{r}}_{s2} \\ \vdots \\ \hat{\mathbf{r}}_{sK} \end{bmatrix} \end{aligned}$$

$$\Delta \bar{\mathbf{R}}_{3K \times 1} \triangleq \begin{bmatrix} \Delta \bar{r}_{s1} \\ \Delta \bar{r}_{s2} \\ \vdots \\ \Delta \bar{r}_{sK} \end{bmatrix}, \quad -\hat{\bar{\rho}}_{cK \times 1} \triangleq \begin{bmatrix} -\rho_1 + c \cdot (\hat{I}_1 + \hat{T}_1 - \hat{B}_1) \\ -\rho_2 + c \cdot (\hat{I}_2 + \hat{T}_2 - \hat{B}_2) \\ \vdots \\ -\rho_K + c \cdot (\hat{I}_K + \hat{T}_K - \hat{B}_K) \end{bmatrix} = \begin{bmatrix} -\rho_{c1} \\ -\rho_{c2} \\ \vdots \\ -\rho_{cK} \end{bmatrix}$$

$$\epsilon_{K \times 3K} \triangleq \begin{bmatrix} \Delta \bar{\mathbf{I}}_1^T & & 0 \\ & \Delta \bar{\mathbf{I}}_2^T & \\ & & \ddots \\ 0 & & \Delta \bar{\mathbf{I}}_K^T \end{bmatrix}, \quad \bar{\mathbf{P}}_{3K \times 1} \triangleq \begin{bmatrix} \bar{r}_u \\ \bar{r}_u \\ \vdots \\ \bar{r}_u \end{bmatrix}$$

and $\Delta \bar{\mathbf{B}}$, $\bar{\mathbf{S}}$, $\Delta \bar{\mathbf{I}}$, $\Delta \bar{\mathbf{T}}$, $\bar{\mathbf{v}}$ are all obvious. The error in the unit vector to the satellite is $\Delta \bar{\mathbf{I}}_j$. The matrix transpose is a convenient matrix notation for the dot product.

Equation (8) then becomes the following (neglecting higher-order terms):

$$\begin{aligned} \mathbf{G} \cdot \hat{\bar{\mathbf{x}}} - \mathbf{G} \cdot \Delta \bar{\mathbf{x}} &= \mathbf{A} \cdot \bar{\mathbf{R}} - \hat{\bar{\rho}}_c - \mathbf{A} \cdot \Delta \bar{\mathbf{R}} + c \\ &\cdot (\Delta \bar{\mathbf{B}} + \bar{\mathbf{S}} - \Delta \bar{\mathbf{I}} - \Delta \bar{\mathbf{T}} + \bar{\mathbf{v}}) + \epsilon \cdot (\bar{\mathbf{R}} - \bar{\mathbf{P}}) \end{aligned} \quad (10)$$

The user does not know the last terms of Eq. (10), which are the errors, and calculates position based on the following:

$$\mathbf{G} \cdot \hat{\bar{\mathbf{x}}} = \mathbf{A} \cdot \bar{\mathbf{R}} - \hat{\bar{\rho}}_c$$

to find the following for $K = 4$:

$$\hat{\bar{\mathbf{x}}} = \mathbf{G}^{-1}(\mathbf{A} \cdot \bar{\mathbf{R}} - \hat{\bar{\rho}}_c) \quad (11a)$$

and the following for $K > 4$:

$$\hat{\bar{\mathbf{x}}} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T (\mathbf{A} \cdot \bar{\mathbf{R}} - \hat{\bar{\rho}}_c) \quad (11b)$$

using the generalized matrix inverse (or pseudoinverse) of $\mathbf{G}$.

These are the fundamental position calculations. Note that $\mathbf{G}$, the geometry matrix, is constructed from the set of approximate directions to the satellites, as is the matrix $\mathbf{A}$. The vector $\bar{\mathbf{R}}$ is constructed from the location of the satellites that has been transmitted and $\hat{\bar{\rho}}_c$ is the corrected pseudorange to each satellite.

Inserting Eq. (11a) back into Eq. (10) cancels appropriate terms and leaves the fundamental error equation*:

$$\mathbf{G} \cdot \Delta \bar{\mathbf{x}} = c \cdot (-\Delta \bar{\mathbf{B}} - \bar{\mathbf{S}} + \Delta \bar{\mathbf{I}} + \Delta \bar{\mathbf{T}} - \bar{\mathbf{v}}) - \epsilon \cdot (\bar{\mathbf{R}} - \bar{\mathbf{P}}) + \mathbf{A} \cdot \Delta \bar{\mathbf{R}} \equiv \Delta \bar{\mathbf{p}} \quad (12)$$

Thus, the right-hand side consists of all the ranging and calculation errors expressed in meters as we have defined them. Distance and time can be equated by recalling that light travels one meter in about three ns.

* The situation for the generalized inverse, Eq. (11b), is somewhat more complicated and is not presented here. The same expression for $\Delta \bar{\mathbf{x}}$ is obtained. The same generalized development can then be used.

$$\text{for } K = 4, \quad \Delta \bar{x} = \mathbf{G}^{-1} \Delta \bar{p}_e \quad (13a)$$

$$\text{for } K > 4, \quad \Delta \bar{x} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \Delta \bar{p}_e \quad (13b)$$

where $\Delta \bar{x}$ is the positioning error in meters.

III. Geometric Dilution of Precision

It is intuitively obvious that satellite geometry can affect the accuracy of Eq. (13). This section develops the quantitative tools to understand the "dilution" of precision caused by various satellite geometrical configurations.

A. Derivation of the Geometric Dilution of Precision Equation

Now the well-known geometric dilution of precision (GDOP) equation can be easily derived. The covariance of position (m) is calculated as follows:

$$\text{cov}(\text{position}) = E(\Delta \bar{x} \cdot \Delta \bar{x}^T) = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \cdot E[(\Delta \bar{p} \cdot \Delta \bar{p}^T)] \cdot \mathbf{G} (\mathbf{G}^T \mathbf{G})^{-1}, K > 4 \quad (14a)$$

$$\text{cov}(\text{position}) = \mathbf{G}^{-1} \cdot E[\Delta \bar{p} \cdot \Delta \bar{p}^T] \cdot \mathbf{G}^{-T}, K = 4 \quad (14b)$$

where E is the expectation operator. Because the $\mathbf{G}$ matrix does not have a random component, it has been brought outside the expectation operation.

If all ranging errors have the same variance $[\sigma_R^2]$ (m^2) and are uncorrelated zero mean ($E[\Delta p_i \Delta p_j] = 0, i \neq j$), then the expectation in Eqs. (14a and 14b) becomes $\sigma_R^2 \cdot U$, where U is the 4×4 identity matrix. Then both Eqs. (14) collapse to the following:

$$\text{cov}(\text{position}) = \sigma_R^2 \cdot [\mathbf{G}^T \mathbf{G}]^{-1}$$

Therefore, $[\mathbf{G}^T \mathbf{G}]^{-1}$ is the matrix of multipliers of ranging variance to give position variance. It is known as the GDOP or geometric dilution of precision matrix. If the position coordinates are the ordered right-hand set, east, north and up, then the square root of the ordered diagonal terms from the upper left are: east DOP, north DOP, vertical DOP, and time DOP. Note that σ_R^2 is expressed in m^2 rather than s^2 .

$\text{cov}(\text{Position}) =$

$$\sigma_R^2 (\text{m}^2) \begin{bmatrix} (\text{East DOP})^2 & & & \text{covariance terms} \\ & (\text{North DOP})^2 & & \\ & & (\text{Vertical DOP})^2 & \\ \text{covariance terms} & & & (\text{Time DOP})^2 \end{bmatrix}$$

The scalar GDOP is defined to be the square root of the trace of the GDOP matrix. Also HDOP (horizontal) $= \sqrt{(\text{NorthDOP})^2 + (\text{EastDOP})^2}$, and PDOP-(position) $= \sqrt{(\text{HDOP})^2 + (\text{VDOP})^2}$.

For satellites constrained to be above a minimum elevation angle* (greater than 0 deg), the best GDOP for $K = 4$ is obtained when one satellite is overhead and the other three are equally spaced at the minimum elevation angles around the horizon. In later chapters, we explore the power of pseudolites that provide ranging signals *transmitted from the ground*. These have negative elevation angles for an aircraft and can significantly improve geometry.

B. Power of the GDOP Concept

The concept of GDOP is a powerful tool for GPS. All receivers use some algorithm based on GDOP to select the best set of satellites to track among the group of up to 11 satellites in view. *Positioning* accuracy can then be estimated as the *ranging* accuracy multiplied by a dilution factor. This dilution factor (DOP) depends solely on geometry.

Typically, variations in geometry are far greater than variations in ranging accuracy for the nominal satellite constellation. The GDOP concept also quantizes the effect when the nominal satellites are not in view. Examples include local terrain shading, satellite outages, and user shading caused by vehicle extensions such as aircraft wings, etc. During these circumstances, the GDOP calculation for those satellites still being tracked will give the multiplier on ranging accuracies to yield positioning accuracies.

C. Example Calculations

To gain insight into the GDOP concept, some sample calculations are useful. They reveal the tradeoff of accuracy with satellite location. We define satellite direction as azimuth (Az—measured 360 deg clockwise from true north) and elevation (E—measured up from local horizontal—0–90 deg). This can be translated into the east, north, up coordinate frame, and the G matrix becomes:

$$\begin{bmatrix} (\cos(E_1) \sin(Az_1)) & (\cos(E_1) \cos(Az_1)) & \sin(E_1) & 1 \\ (\cos(E_2) \sin(Az_2)) & (\cos(E_2) \cos(Az_2)) & \sin(E_2) & 1 \\ (\cos(E_3) \sin(Az_3)) & (\cos(E_3) \cos(Az_3)) & \sin(E_3) & 1 \\ (\cos(E_4) \sin(Az_4)) & (\cos(E_4) \cos(Az_4)) & \sin(E_4) & 1 \end{bmatrix}$$

Using only satellites as ranging sources, the best accuracy† is found with three satellites equally spaced on the horizon, at minimum elevation angle, and one satellite directly overhead (See Table 1).

* Elevation angle is the angle of the satellite above the local horizontal.

† Note that the lower satellite elevation angles tend to have the greater *ranging errors*. The statement is that this configuration offers the best *geometry*. Usually geometry has a larger effect on accuracy than ranging errors.

Table 1 Satellite location

	Sat 1	Sat 2	Sat 3	Sat 4
Elevation, deg	5	5	5	90
Azimuth, deg	0	120	240	0

The following example, for 5-deg minimum satellite elevation, illustrates the technique. Using the best satellite geometry previously described, and the formulas for GDOP we get the following:

$$\mathbf{G} = \begin{bmatrix} 0.000 & 0.996 & 0.087 & 1.000 \\ 0.863 & -0.498 & 0.087 & 1.000 \\ -0.863 & -0.498 & 0.087 & 1.000 \\ 0.000 & 0.000 & 1.000 & 1.000 \end{bmatrix}$$

Recall that the position and time solution ($\mathbf{x}$) is simply given by $\mathbf{G}^{-1}$ (for the four-measurement case*) times the column of corrected pseudoranges ρ_c . $\mathbf{G}$ inverse for the above $\mathbf{G}$ is listed below. Note that each of the first three rows sums to exactly zero. This implies that any common bias in ranging measurements will not affect the position solution. Note also that the last row sums to one. This is the row that gives the time solution (or bias in the local clock). This implies that any common error shows up solely as an error in the local clock. To see these two results, simply multiply on the right by a column whose elements are all the same value. These two results are always true. They arise from inverting the $\mathbf{G}$ matrix, which always has a constant last column of ones.

$$\mathbf{G}^{-1} = \begin{bmatrix} 0.000 & 0.580 & -0.580 & 0.000 \\ 0.670 & -0.335 & -0.335 & 0.000 \\ -0.365 & -0.365 & -0.365 & 1.095 \\ 0.365 & 0.365 & 0.365 & -0.095 \end{bmatrix}$$

The GDOP matrix for this example $(\mathbf{G}^T \mathbf{G})^{-1}$ is shown below. Note that, in this case, all off-diagonal terms are zero except for those correlating vertical errors and time. This correlation is negative, which implies that the errors tend to have opposite signs. In general, large correlations will be found between vertical errors and timing errors, and in general, off-diagonal terms will not be zero.

* For a greater number of satellites, $\mathbf{G}^{-1}$ is replaced with $(\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T$.

$$(G^T G)^{-1} = \begin{bmatrix} 0.672 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.672 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.600 & -0.505 \\ 0.000 & 0.000 & -0.505 & 0.409 \end{bmatrix}$$

Listed below are the results of taking the square roots of appropriate diagonal terms of the GDOP matrix above:

$$\text{HDOP (horizontal DOP)} = 1.16 \quad \text{TDOP (time DOP)} = 0.64$$

$$\text{VDOP (vertical DOP)} = 1.26 \quad \text{GDOP} = 1.83$$

$$\text{PDOP (position DOP)} = 1.72$$

A PDOP of 1.72 is very good. In fact, it is the optimum for four satellites with a minimum elevation angle of 5 deg. A more representative median (50th percentile) worldwide result for PDOP is about 2.5. With all satellites of a 24-satellite constellation available, PDOP numbers as high as six or seven will be found. Thus, geometry can affect the results by a factor of as much as five or more.

D. Impact of Elevation Angle on GDOP

Figure 1 illustrates the impact of satellite position on DOP.^{3,4} Maintaining the same symmetric spacing in azimuth, the minimum elevation angle is varied, including negative angles that could only be achieved with Earth-based transmitters (pseudolites).

The HDOP is quite flat over the whole range, which reflects the optimum azimuthal configuration. Satellites more concentrated in one-half of the sky than the other would cause greater variation in these results. Another interesting conclusion is that the lower elevation angles are of significant help for the vertical position.

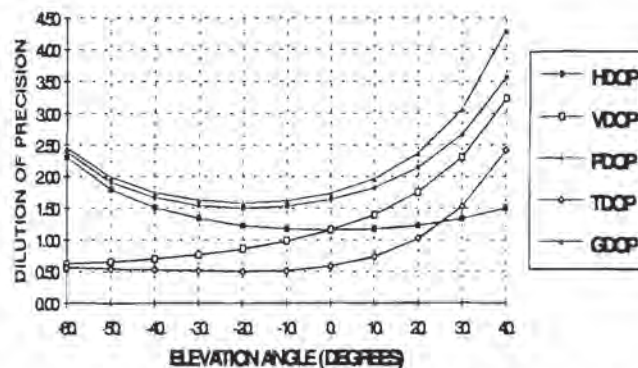


Fig. 1 Dilution of position values for symmetric satellites at various elevation angles above the horizon. The minimum GDOP is at -19.5 deg (i.e., below the local horizontal). This corresponds to locating four transmitters at the apexes of a regular tetrahedron.

IV. Ranging Errors

A. Six Classes of Errors

Ranging errors are grouped into the six following classes:

- 1) *Ephemeris data*—Errors in the transmitted location of the satellite
- 2) *Satellite clock*—Errors in the transmitted clock, including SA
- 3) *Ionosphere*—Errors in the corrections of pseudorange caused by ionospheric effects
- 4) *Troposphere*—Errors in the corrections of pseudorange caused by tropospheric effects
- 5) *Multipath*—Errors caused by reflected signals entering the receiver antenna
- 6) *Receiver*—Errors in the receiver's measurement of range caused by thermal noise, software accuracy, and interchannel biases

Each class is briefly discussed in the following sections. Representative values for these errors are used to construct an error table in a later section of this chapter. A more complete discussion of individual error sources can be found in succeeding chapters.

B. Ephemeris Errors

Ephemeris errors result when the GPS message does not transmit the correct satellite location. It is typical that the radial component of this error is the smallest: the tangential and cross-track errors may be larger by an order of magnitude. Fortunately, the larger components do not affect ranging accuracy to the same degree. This can be seen in the fundamental error Eq. (12). The $\Delta \bar{R}$ represents each satellite position error, but when dot-multiplied by the unit satellite direction vector (in the A matrix), only the projection of satellite positioning error along the line of sight creates a ranging error.

Because satellite errors reflect a position prediction, they tend to grow with time from the last control station upload. It is possible that a portion of the deliberate SA error is added to the ephemeris as well. However, the predictions are long smooth arcs, so all errors in the ephemeris tend to be slowly changing with time. Therefore, their utility in SA is quite limited.

As reported during phase one, (Bowen, 1986) in 1984,⁵ for predictions of up to 24 hours, the rms ranging error attributable to ephemeris was 2.1 m. These errors were closely correlated with the satellite clock, as we would expect. Note that these errors are the same for both the P- and C/A-codes (see Chapter 16 of this volume for a more detailed discussion of ephemeris and clock errors).

C. Satellite Clock Errors

Fundamental to GPS is the one-way ranging that ultimately depends on satellite clock predictability. These satellite clock errors affect both the C/A- and P-code users in the same way. The error effect can be seen in the fundamental error Eq. (11) as ΔB . This effect is also independent of satellite direction, which is important when the technique of differential corrections is used. All differential stations and users measure an identical satellite clock error.

A major source of apparent clock error is SA, which is varied so as to be unpredictable over periods longer than about 10 minutes. The rms value of SA

is typically about 20 m in ranging, but this can change after providing appropriate notice, depending on need. The U.S. Air Force has guaranteed that the two-dimensional rms (2 DRMS) positioning error (approximately 90th percentile) will be kept to less than 100 m. This is now a matter of U.S. federal policy and can only be changed by order of the President of the United States.

More interesting is the underlying accuracy of the system with SA off. The ability to predict clock behavior is a measure of clock quality. GPS uses atomic clocks (cesium and rubidium oscillators),¹ which have stabilities of about 1 part in 10^{13} over a day. If a clock can be predicted to this accuracy, its error in a day ($\sim 10^5$ s) will be about 10^{-8} s or about 3.5 m. The experience reported in 1984 was 4.1 m for 24-hour predictions. *Because the standard deviations of these errors were reported to grow quadratically with time, an average error of 1–2 m for 12-hour updates is the normal expectation.*

D. Ionosphere Errors

Because of free electrons in the ionosphere, GPS signals do not travel at the vacuum speed of light as they transit this region. The modulation on the signal is *delayed* in proportion to the number of free electrons encountered and is also (to first order) proportional to the inverse of the carrier frequency squared ($1/f^2$). The phase of the radio frequency carrier is *advanced* by the same amount because of these effects. Carrier-smoothed receivers should take this into account in the design of their filters. The ionosphere is usually reasonably well-behaved and stable in the temperate zones; near the equator or magnetic poles it can fluctuate considerably. An in-depth discussion of this can be found in Chapter 12, this volume.

All users will correct the raw pseudoranges for the ionospheric delay. The simplest correction will use an internal diurnal model of these delays. The parameters can be updated using information in the GPS communications message (although the accuracy of these updates is not yet clearly established). *The effective accuracy of this modeling is about 2–5 m in ranging for users in the temperate zones.*

A second technique for *dual-frequency P-code* receivers is to measure the signal at both frequencies and directly solve for the delay. The difference between L_1 and L_2 arrival times allows a direct algebraic solution. *This dual-frequency technique should provide 1–2 m of ranging accuracy, due to the ionosphere, for a well-calibrated receiver.*

A third technique is to rely on a near real-time update. An example would be the proposed Wide Area Differential GPS system (WADGPS). *This should also produce corrections with accuracies of 1–2 m or better in the temperate zones of the world.*

E. Troposphere Errors

Another deviation from the vacuum speed of light is caused by the troposphere. Variations in temperature, pressure, and humidity all contribute to variations in the speed of light of radio waves. Both the code and carrier will have the same delays. This is described further in the chapter devoted to these effects, Chapter

13 of this volume. *For most users and circumstances, a simple model should be effectively accurate to about 1 m or better.*

F. Multipath Errors

Multipath is the error caused by reflected signals entering the front end of the receiver and masking the real correlation peak. These effects tend to be more pronounced in a static receiver near large reflecting surfaces, where 15 m or more in ranging error can be found in extreme cases. Monitor or reference stations require special care in siting to avoid unacceptable errors. The first line of defense is to use the combination of antenna cut-off angle and antenna location that minimizes this problem. A second approach is to use so-called "narrow correlator" receivers which tend to minimize the impact of multipath on range tracking accuracies. *With proper siting and antenna selection, the net impact to a moving user should be less than 1 m under most circumstances.* See Chapter 14 of this volume for further discussion of multipath errors.

G. Receiver Errors

Initially most GPS commercial receivers were sequential in that one or two tracking channels shared the burden of locking on to four or more satellites. With modern chip technology, it is common to place three or more tracking channels on a single inexpensive chip. As the size and cost have shrunk, techniques have improved and five- or six-channel receivers are common. Most modern receivers use reconstructed carrier to aid the code tracking loops. This produces a precision of better than 0.3 m. Interchannel bias is minimized with digital sampling and all-digital designs.

The limited precision of the receiver software also contributed to errors in earlier designs, which relied on 8-bit microprocessors. With ranges to the satellites of over 20 million meters, a precision of $1:10^{10}$ or better was required. Modern microprocessors now provide such precision along with the co-requisite calculation speeds. *The net result is that the receiver should contribute less than 0.5 ms error in bias and less than 0.2 m in noise.* Further information on receiver errors is available in Chapters 3, 7, 8, and 9 of this volume.

V. Standard Error Tables

These overview discussions on error sources and magnitudes, as well as the effects of satellite geometry, can be summarized with the following error tables. Each error is described as a bias (persistence of minutes or more) and a random effect that is, in effect "white" noise and exhibits little correlation between samples of range. The total error in each category is found as the root sum square (rss) of these two components.

Each *component* of error is assumed to be statistically uncorrelated with all others, so they are combined as an rss as well. The receiver is assumed to filter the measurements so that about 16 samples are effectively averaged reducing the random content by the square root of 16. Of course, averaging cannot improve the bias-type errors.

Finally, each satellite error is assumed to be uncorrelated and of zero mean, so the application of HDOP and VDOP are justified as the last step. Despite these limiting assumptions, the resulting error model has proved to be surprisingly valid. Of course, the assumptions on uncorrelated errors is almost always violated to some degree. For example, if the estimate of zenith ionosphere delay is in error, a proportional error is induced in all measurements through the obliquity calculation. Clearly, such an error would be correlated. These and other correlations have not caused serious problems in the use of this model.

A. Error Table without SA: Normal Operation for C/A Code

Table 2 assumes that SA is not operating. Consequently, the residual satellite clock error, at 2.1 m, is not the dominant error; in fact, the largest error is expected to be the mismodeling of the ionosphere, at 4.0 m. Thus, the worldwide civilian positioning error for GPS is potentially about 10 m (horizontal), as shown in Table 2.

B. Error Table with SA

A second example shows the impact of SA on these errors. Because the deliberately mismodeled clock so dominates the ranging error, all other effects could be safely ignored in the error budget. The results of Table 3 have been repeatedly corroborated by actual measurements. Note that SA is listed as a bias because it cannot be averaged to zero with a 1 s (or less) filter. Selective availability is expected to be zero mean, but only when averaged over many hours or perhaps days. Of course, such averaging is not practical for a dynamic user who only sees the satellite for a portion of the orbit. If differential corrections are used,

Table 2 Standard error model—no SA

Error source	One-sigma error, m		
	Bias	Random	Total
Ephemeris data	2.1	0.0	2.1
Satellite clock	2.0	0.7	2.1
Ionosphere	4.0	0.5	4.0
Troposphere	0.5	0.5	0.7
Multipath	1.0	1.0	1.4
Receiver measurement	<u>0.5</u>	<u>0.2</u>	<u>0.5</u>
User equivalent range error (UERE), rms ^a	5.1	1.4	5.3
Filtered UERE, rms	5.1	0.4	5.1
Vertical one-sigma errors—VDOP= 2.5			12.8
Horizontal one-sigma errors—HDOP= 2.0			10.2

^aThis is the statistical ranging error (one-sigma) that represents the total of all contributing sources. The dominant error is usually the ionosphere. A horizontal error of 10 m (one-sigma) is the expected performance for the temperate latitudes using civilian (C/A-code) receivers.

Table 3 SA error model

Error source	One-sigma error, m		
	Bias	Random	Total
Ephemeris data	2.1	0.0	2.1
Satellite clock	20.0	0.7	20.0
Ionosphere	4.0	0.5	4.0
Troposphere	0.5	0.5	0.7
Multipath	1.0	1.0	1.4
Receiver measurement	0.5	0.2	0.5
UERE, rms	20.5	1.4	20.6
Filtered UERE, rms	20.5	0.4	20.5
Vertical one-sigma errors—VDOP= 2.5			51.4
Horizontal one-sigma errors—HDOP= 2.0			41.1

they will eliminate the SA error entirely (if corrections are passed at a sufficiently high data rate) as discussed in Chapter 21, this volume.

The 41-m horizontal error is a one-sigma (σ) result; under the existing agreement between the U.S. Department of Transportation (DOT) and the U.S. Department of Defense (DOD), the 2 DRMS horizontal error is to be less than 100 m. The impact on the vertical error is probably greater, because the VDOP value usually exceeds the HDOP value.

C. Error Table for Precise Positioning Service (PPS Dual-Frequency P/Y Code)

The errors for dual-frequency P/Y code are similar to those above except that SA errors are eliminated because the authorized user can decode the magnitude as part of a classified message. An expected horizontal error is less than 10 m. The ionosphere error is reduced to 1-m bias and about 0.7 m of noise by the dual-frequency measurement. The dominant sources are the satellite ephemeris and clocks. This is illustrated in Table 4.

VI. Summary

Excluding the deliberate degradation of SA, the dominant error source for satellite ranging with single frequency receivers is usually the ionosphere. It is on the order of four meters, depending on the quality of the single-frequency model. For dual-frequency (P/Y-code) receivers (which eliminate SA) the Standard Error Model of Table 1 has one principal change (in addition to the elimination of the SA error). The ionospheric error is reduced from four meters to about one meter.

Greater variations in the errors are due to geometry, which are quantified as dilutions of precision or DOPs. While geometric dilutions of 2.5 are about the worldwide average, this factor can range up to 10 or more with poor satellite geometry. Reduced satellite availability (and consequent increases in DOP) could

Table 4 Precise error model, dual-frequency, P/Y code

Error source	One-sigma error, m		
	Bias	Random	Total
Ephemeris data	2.1	0.0	2.1
Satellite clock	2.0	0.7	2.1
Ionosphere	1.0	0.7	1.2
Troposphere	0.5	0.5	0.7
Multipath	1.0	1.0	1.4
Receiver measurement	0.5	0.2	0.5
UERE, rms	3.3	1.5	3.6
Filtered UERE, rms	3.3	0.4	3.3
Vertical one-sigma errors—VDOP = 2.5			8.3
Horizontal one-sigma errors—HDOP = 2.0			6.6

be caused by satellite outages, local terrain masking, or user antenna tilting (for example due to aircraft banking). Typical normal accuracy (one-sigma) for well-designed civil equipment under nominal operating conditions *with SA off* should be about 10 m horizontal and 13 m vertical.

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