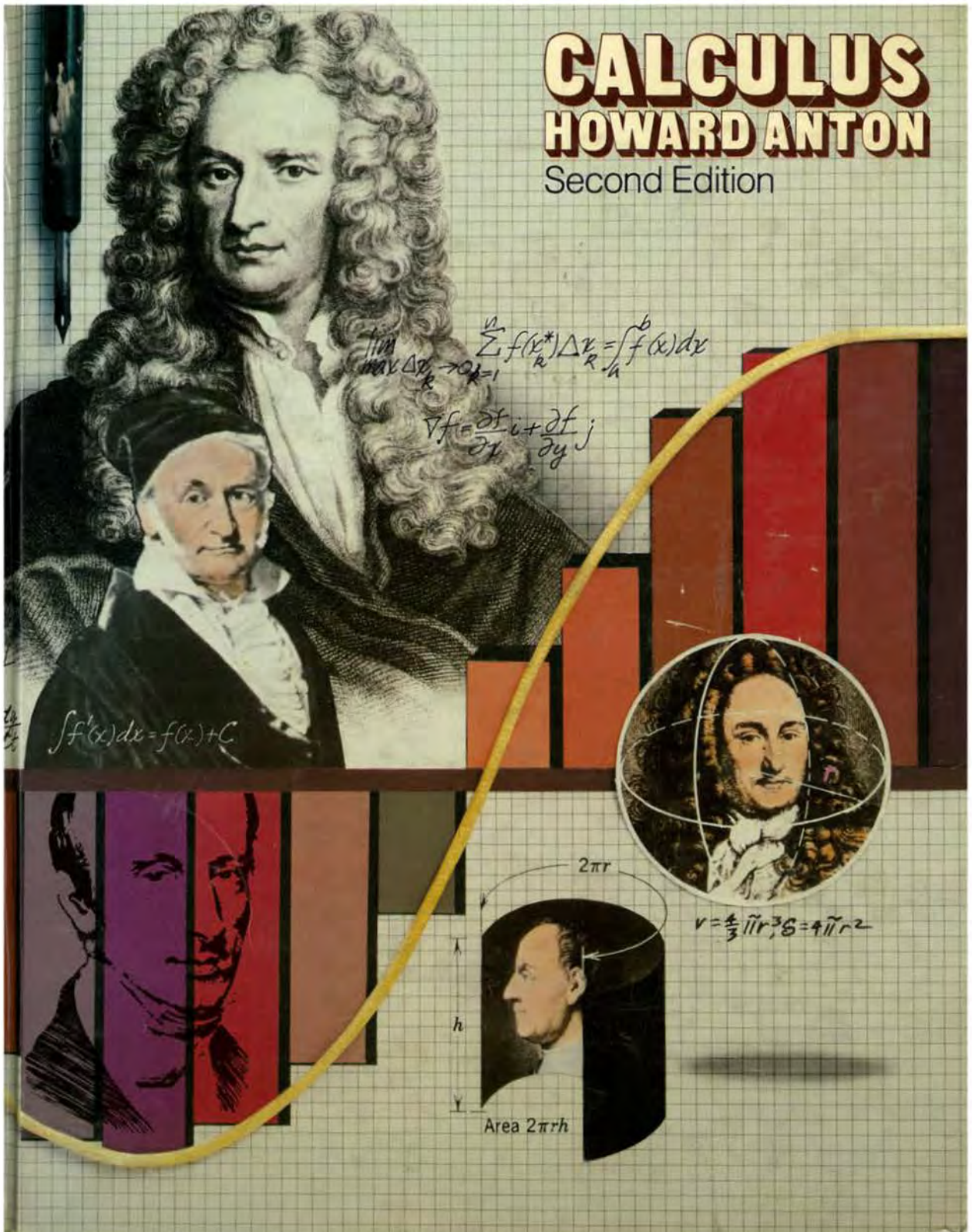


CALCULUS

HOWARD ANTON

Second Edition



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10 9

- (a) $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-y^2-z^2}} f(x, y, z) \, dx \, dy \, dz$
- (b) $\int_0^4 \int_0^2 \int_0^{z/2} f(x, y, z) \, dy \, dz \, dx$
- (c) $\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} f(x, y, z) \, dx \, dz \, dy.$

27. Use a triple integral to find the volume of the tetrahedron with vertices $(0, 0, 0)$, $(a, a, 0)$, $(a, 0, 0)$ and $(a, 0, a)$.
28. Let G be the rectangular box defined by the inequalities $a \leq x \leq b$, $c \leq y \leq d$, $k \leq z \leq l$. Show that

$$\begin{aligned} \iiint_G f(x)g(y)h(z) \, dV \\ = \left[\int_a^b f(x) \, dx \right] \left[\int_c^d g(y) \, dy \right] \left[\int_k^l h(z) \, dz \right] \end{aligned}$$

29. Use the result of Exercise 28 to evaluate

(a) $\iiint_G xy^2 \sin z \, dV$, where G is the set of points satisfying

$$-1 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq \frac{\pi}{2}$$

(b) $\iiint_G e^{2x+y-z} \, dV$, where G is the set of points satisfying

$$0 \leq x \leq 1, 0 \leq y \leq \ln 3, 0 \leq z \leq \ln 2$$

30. Use a triple integral to find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

17.6 CENTROIDS, CENTERS OF GRAVITY, THEOREM OF PAPPUS



The thickness of a lamina is negligible

Figure 17.6.1

Suppose that a physical body is acted on by a gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of the gravitational field on the body consists of a large number of forces distributed over the entire body. However, it is a physical fact that a body subjected to gravity behaves as if the entire force of gravity acts at a single point. This point is called the **center of gravity** or **center of mass** of the body. In this section we will show how double and triple integrals can be used to locate centers of gravity.

To begin, let us consider an idealized flat object that is sufficiently thin to be viewed as a two-dimensional solid (Figure 17.6.1). Such a solid is called a **lamina**. A lamina is called **homogeneous** if its composition and structure is uniform throughout, and is called **inhomogeneous** otherwise.

For a **homogeneous** lamina of mass M and area A , we define the **density** δ of the lamina by

$$\delta = \frac{M}{A}$$

Thus, δ measures the mass per unit area.

For an inhomogeneous lamina the composition may vary from point to point. If the lamina is placed in the xy -plane, then the “density” of the

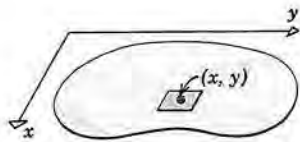


Figure 17.6.2

lamina at a general point (x, y) can be specified by a function $\delta(x, y)$, called the **density function** for the lamina. Informally, the density function can be visualized as follows. Construct a small rectangle centered at (x, y) and let ΔM and ΔA be the mass and area of the portion of lamina enclosed by this rectangle (Figure 17.6.2). If the ratio $\Delta M/\Delta A$ tends toward a limiting value as the dimensions of the rectangle tend to zero, then this limit is considered to be the density of the lamina at (x, y) . Symbolically,

$$\delta(x, y) = \lim \frac{\Delta M}{\Delta A} \quad (1)$$

From this relationship we obtain the approximation

$$\Delta M \approx \delta(x, y) \Delta A \quad (2)$$

which relates the mass and area of a small rectangular portion of lamina centered at (x, y) . It is assumed that as the dimensions of the rectangle tend to zero, the error in this approximation also tends to zero.

We can now obtain a formula for the mass of a lamina.

17.6.1 DEFINITION
Mass of a Lamina

If a lamina with density function $\delta(x, y)$ occupies a region R in the xy -plane, then its total mass M is defined to be

$$M = \iint_R \delta(x, y) dA \quad (3)$$

To motivate this result, let a rectangle enclosing the region R be divided into subrectangles by lines parallel to the coordinate axes, and exclude from consideration all those rectangles that contain any points outside of R . Let

$$\Delta A_1, \Delta A_2, \dots, \Delta A_n$$

be the areas of the remaining rectangles and let

$$\Delta M_1, \Delta M_2, \dots, \Delta M_n$$

be the masses of the portions of lamina enclosed by these rectangles. We can approximate the total mass of the lamina by adding the masses of these pieces

$$M = \text{total mass} \approx \sum_{k=1}^n \Delta M_k \quad (4)$$

If we now let (x_k^*, y_k^*) be the center of the k th rectangle, then it follows from (2) that

$$\Delta M_k \approx \delta(x_k^*, y_k^*) \Delta A_k$$

Substituting this in (4) yields

$$M \approx \sum_{k=1}^n \delta(x_k^*, y_k^*) \Delta A_k \quad (5)$$

If we now increase n in such a way that the dimensions of the rectangles tend to zero, then the errors in our approximations will diminish so that

$$M = \lim_{n \rightarrow \infty} \sum_{k=1}^n \delta(x_k^*, y_k^*) \Delta A_k = \iint_R \delta(x, y) dA$$

► **Example 1** A triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ has density function $\delta(x, y) = xy$. Find its total mass.

Solution. Referring to (3) and Figure 17.6.3, the mass M of the lamina is

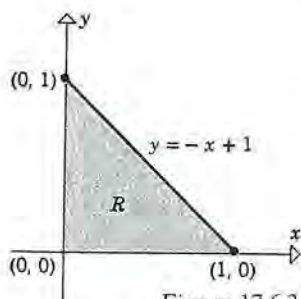


Figure 17.6.3

$$\begin{aligned} M &= \iint_R \delta(x, y) dA = \iint_R xy dA = \int_0^1 \int_0^{-x+1} xy dy dx \\ &= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2} x^3 - x^2 + \frac{1}{2} x \right] dx \\ &= \frac{1}{24} \text{ (units of mass)} \end{aligned}$$

Let us now consider the following problem.

17.6.2 PROBLEM Suppose a lamina with density function $\delta(x, y)$ occupies a region R in a horizontal xy -plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity.

To motivate the solution, consider what happens if we try to balance the lamina on a knife edge parallel to the x -axis. Suppose the lamina in Figure 17.6.4 is placed on a knife edge along a line $y = c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity $(\bar{x}, \bar{y})$, the lamina will be rotationally unstable and the force of gravity will cause a rotation about $y = c$. Similarly, the lamina will undergo a rotation if placed on a knife edge along $y = d$. However, if the knife edge runs along the line $y = \bar{y}$ through the center of

gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife edge along the line $x = \bar{x}$ through the center of gravity.

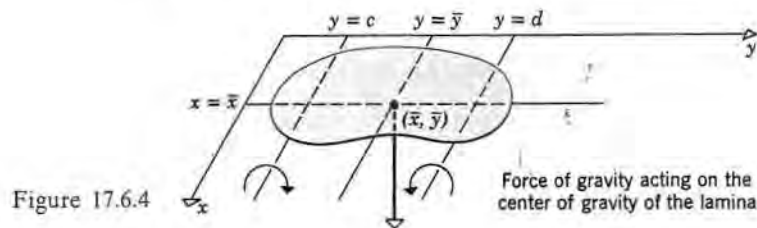


Figure 17.6.4

This discussion suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the x -axis and the other parallel to the y -axis. In order to find these lines of balance, we will need some preliminary results about rotations.

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for a mass to produce rotation is proportional not only to the magnitude of the mass but also the distance between the mass and the fulcrum. To be precise, if a point-mass m is located on a coordinate axis at a point x , then the tendency for that mass to produce a rotation about a point a on the axis is measured by the following quantity, called the **moment of m about a** ,

$$\left[\begin{array}{c} \text{moment of } m \\ \text{about } a \end{array} \right] = m(x - a)$$

The number $x - a$ is called the **lever arm**. Depending on whether the mass is to the right or left of a , the lever arm is either the distance between x and a or the negative of this distance (Figure 17.6.5). Positive lever arms result in positive moments and clockwise rotations, while negative lever arms result in negative moments and counterclockwise rotations.



Figure 17.6.5

Suppose masses $m_1, m_2, \dots, m_n$ are located at points $x_1, x_2, \dots, x_n$ on a coordinate axis and a fulcrum is positioned at the point a (Figure 17.6.6). Depending on whether the sum of the moments about a ,

$$\sum_{k=1}^n m_k(x_k - a) = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a)$$

is positive, negative, or zero, the axis will rotate clockwise about a , rotate counterclockwise about a , or balance perfectly. In the last case, the system of masses is said to be in *equilibrium*.

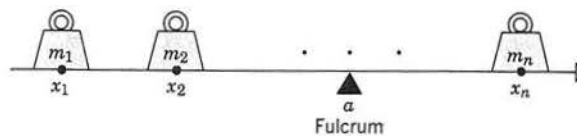
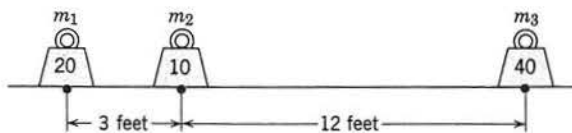


Figure 17.6.6

► **Example 2** Where should a fulcrum be placed so that the following system of masses is in equilibrium?



Solution. Introduce an x -axis with origin at the mass m_1 , and let a be the unknown coordinate of the fulcrum (Figure 17.6.7). (Our choice of location for the origin is arbitrary. Any convenient location will suffice.)



Figure 17.6.7

We will determine a so that the total moment about a is zero. The sum of the moments about a is

$$\text{total moment} = 20(0 - a) + 10(3 - a) + 40(15 - a) = 630 - 70a$$

For equilibrium, a should be chosen so that

$$630 - 70a = 0$$

or $a = 9$. Thus, the fulcrum should be located 9 feet to the right of mass m_1 .

The ideas we have discussed can be extended to masses distributed in two-dimensional space. If a mass m is located at a point (x, y) in a horizontal xy -plane, then we define the *moment of m about the line $x = a$* and the *moment of m about the line $y = c$* by

$$\left[\begin{array}{l} \text{moment of } m \\ \text{about the} \\ \text{line } x = a \end{array} \right] = m(x - a) \quad \left(\begin{array}{l} \text{mass times} \\ \text{lever arm} \end{array} \right) \quad (6)$$

$$\left[\begin{array}{l} \text{moment of } m \\ \text{about the} \\ \text{line } y = c \end{array} \right] = m(y - c) \quad \left(\begin{array}{l} \text{mass times} \\ \text{lever arm} \end{array} \right) \quad (7)$$

(See Figure 17.6.8.)

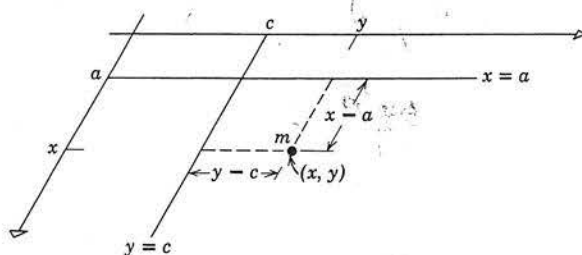


Figure 17.6.8

If a number of masses are distributed in the xy -plane and if we imagine the xy -plane to be a weightless sheet, then this sheet will balance on the line $x = a$ (or $y = c$) if the sum of the moments about that line is zero.

Suppose now that we are dealing not with isolated point masses, but with a lamina in the xy -plane. How should we define the moments about the lines $x = a$ and $y = c$? To motivate the appropriate definitions, assume the lamina occupies the region R in the xy -plane and has density function $\delta(x, y)$. Following a now familiar procedure, let R be partitioned into subregions and let R_k be a typical rectangle interior to R . Let ΔA_k be the area of R_k , let ΔM_k be the mass of that portion of the lamina enclosed by R_k , and let (x_k^*, y_k^*) be the center of the rectangle R_k . From (2), the mass ΔM_k may be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*) \Delta A_k$$

Moreover, if the rectangle R_k is small, there should be little error in assuming that the entire mass ΔM_k is concentrated at the center (x_k^*, y_k^*) of the rectangle. Thus, from (6) and (7), the moment of the mass ΔM_k about the line $x = a$ is approximately

$$(x_k^* - a) \Delta M_k \approx (x_k^* - a) \delta(x_k^*, y_k^*) \Delta A_k$$

and the moment about the line $y = c$ is approximately

$$(y_k^* - c) \Delta M_k \approx (y_k^* - c) \delta(x_k^*, y_k^*) \Delta A_k$$

Therefore, the total moment about $x = a$ produced by all the rectangular regions interior to R is approximately

$$\sum_{k=1}^n (x_k^* - a) \delta(x_k^*, y_k^*) \Delta A_k \quad (8)$$

and the total moment about $y = c$ is approximately

$$\sum_{k=1}^n (y_k^* - c) \delta(x_k^*, y_k^*) \Delta A_k \quad (9)$$

If we assume that the errors in our approximations diminish as the dimensions of the rectangles tend to zero, then (8) and (9) suggest the definitions.

$$\left[\begin{array}{l} \text{moment of} \\ \text{lamina } R \\ \text{about the} \\ \text{line } x = a \end{array} \right] = \lim_{n \rightarrow +\infty} \sum_{k=1}^n (x_k^* - a) \delta(x_k^*, y_k^*) \Delta A_k = \iint_R (x - a) \delta(x, y) dA \quad (10)$$

$$\left[\begin{array}{l} \text{moment of} \\ \text{lamina } R \\ \text{about the} \\ \text{line } y = c \end{array} \right] = \lim_{n \rightarrow +\infty} \sum_{k=1}^n (y_k^* - c) \delta(x_k^*, y_k^*) \Delta A_k = \iint_R (y - c) \delta(x, y) dA \quad (11)$$

Of special importance are the moments about the y -axis ($a = 0$) and about the x -axis ($c = 0$). These moments are denoted by

$$M_y = \left[\begin{array}{l} \text{moment of lamina } R \\ \text{about the } y\text{-axis} \end{array} \right] = \iint_R x \delta(x, y) dA \quad (12)$$

$$M_x = \left[\begin{array}{l} \text{moment of lamina } R \\ \text{about the } x\text{-axis} \end{array} \right] = \iint_R y \delta(x, y) dA \quad (13)$$

We are now in a position to obtain formulas for the center of gravity $(\bar{x}, \bar{y})$ of a lamina. Since a lamina balances on the lines $x = \bar{x}$ and $y = \bar{y}$ through the center of gravity, the moments of the lamina about these lines must be zero. Thus, from (10) and (11)

$$\iint_R (x - \bar{x}) \delta(x, y) dA = 0$$

$$\iint_R (y - \bar{y}) \delta(x, y) dA = 0$$

Since $\bar{x}$ and $\bar{y}$ are constant, these equations can be rewritten

$$\iint_R x\delta(x, y) dA = \bar{x} \iint_R \delta(x, y) dA$$

$$\iint_R y\delta(x, y) dA = \bar{y} \iint_R \delta(x, y) dA$$

from which we obtain the following formulas:

Center of gravity of a lamina

$$\bar{x} = \frac{\iint_R x\delta(x, y) dA}{\iint_R \delta(x, y) dA} \quad \bar{y} = \frac{\iint_R y\delta(x, y) dA}{\iint_R \delta(x, y) dA} \quad (14a)$$

By virtue of (3), (12), and (13), these formulas can be written as

$$\bar{x} = \frac{\left[\begin{array}{c} \text{moment about} \\ \text{the } y\text{-axis} \end{array} \right]}{\left[\begin{array}{c} \text{mass of the} \\ \text{lamina} \end{array} \right]} = \frac{M_y}{M} \quad \bar{y} = \frac{\left[\begin{array}{c} \text{moment about} \\ \text{the } x\text{-axis} \end{array} \right]}{\left[\begin{array}{c} \text{mass of the} \\ \text{lamina} \end{array} \right]} = \frac{M_x}{M} \quad (14b)$$

► **Example 3** Find the center of gravity of the triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ and density function $\delta(x, y) = xy$.

Solution. The lamina is shown in Figure 17.6.3. In Example 1 we found the mass of the lamina to be

$$M = \iint_R \delta(x, y) dA = \iint_R xy dA = \frac{1}{24}$$

The moment of the lamina about the y -axis is

$$\begin{aligned} M_y &= \iint_R x\delta(x, y) dA = \iint_R x^2y dA = \int_0^1 \int_0^{-x+1} x^2y dy dx \\ &= \int_0^1 \left[\frac{1}{2}x^2y^2 \right]_{y=0}^{-x+1} dx = \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2 \right) dx = \frac{1}{60} \end{aligned}$$

and the moment about the x -axis is

$$\begin{aligned} M_x &= \iint_R y\delta(x, y) dA = \iint_R xy^2 dA = \int_0^1 \int_0^{-x+1} xy^2 dy dx \\ &= \int_0^1 \left[\frac{1}{3}xy^3 \right]_{y=0}^{-x+1} dx = \int_0^1 \left(-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x \right) dx = \frac{1}{60} \end{aligned}$$

From (14b),

$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

so the center of gravity is $(2/5, 2/5)$. ◀

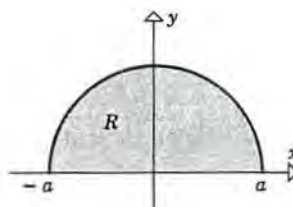
In the special case of a *homogeneous* lamina the center of gravity is called the **centroid of the lamina** or sometimes the **centroid of the region R** . Because the density function δ is constant for a homogeneous lamina, the factor δ may be moved through the integral signs in (14a) and canceled. Thus, the centroid $(\bar{x}, \bar{y})$ of a region R is given by the formulas

Centroid of a region R

$$\bar{x} = \frac{\iint_R x dA}{\iint_R dA} = \frac{1}{\text{area of } R} \iint_R x dA \quad (15a)$$

$$\bar{y} = \frac{\iint_R y dA}{\iint_R dA} = \frac{1}{\text{area of } R} \iint_R y dA \quad (15b)$$

► **Example 4** Find the centroid of the following semicircular region:



Solution. By symmetry, $\bar{x} = 0$ since the y -axis is obviously a line of balance. From (15b),

$$\bar{y} = \frac{\iint_R y \, dA}{\text{area of } R} = \frac{\iint_R y \, dA}{\frac{1}{2}\pi a^2} \quad (16)$$

The integral in (16) is best evaluated in polar coordinates:

$$\begin{aligned} \iint_R y \, dA &= \int_0^\pi \int_0^a (r \sin \theta) r \, dr \, d\theta = \int_0^\pi \left[\frac{1}{3} r^3 \sin \theta \right]_{r=0}^a d\theta \\ &= \frac{1}{3} a^3 \int_0^\pi \sin \theta \, d\theta = \frac{2}{3} a^3 \end{aligned}$$

Thus, from (16)

$$\bar{y} = \frac{\frac{2}{3} a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}$$

so the centroid is $\left(0, \frac{4a}{3\pi}\right)$.

For three-dimensional solids the formulas for moments, centers of gravity, and centroids are similar to the formulas for laminas. If a solid G of mass M and volume V is homogeneous in composition, we define the **density** δ of the solid to be

$$\delta = \frac{M}{V}$$

which measures the mass per unit volume. If the solid G is not homogeneous, then the density at a general point (x, y, z) is specified by a **density function** $\delta(x, y, z)$ that may be viewed as a limit

$$\delta(x, y, z) = \lim \frac{\Delta M}{\Delta V}$$

where ΔM and ΔV represent the mass and volume of a rectangular parallel-piped, centered at (x, y, z) , whose dimensions tend to zero (Figure 17.6.9).

The following formulas generalize the corresponding results for laminas:

$$\text{mass of } G = \iiint_G \delta(x, y, z) \, dV \quad (17)$$

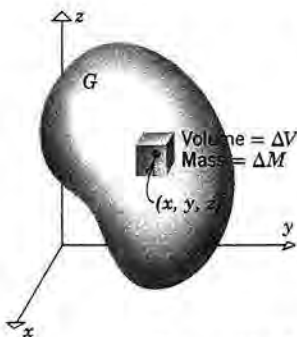


Figure 17.6.9

Center of gravity $(\bar{x}, \bar{y}, \bar{z})$ of a solid G

$$\bar{x} = \frac{\iiint_G x \delta(x, y, z) dV}{\iiint_G \delta(x, y, z) dV} = \frac{1}{\text{mass of } G} \iiint_G x \delta(x, y, z) dV \quad (18a)$$

$$\bar{y} = \frac{\iiint_G y \delta(x, y, z) dV}{\iiint_G \delta(x, y, z) dV} = \frac{1}{\text{mass of } G} \iiint_G y \delta(x, y, z) dV \quad (18b)$$

$$\bar{z} = \frac{\iiint_G z \delta(x, y, z) dV}{\iiint_G \delta(x, y, z) dV} = \frac{1}{\text{mass of } G} \iiint_G z \delta(x, y, z) dV \quad (18c)$$

Centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid G

$$\bar{x} = \frac{\iiint_G x dV}{\iiint_G dV} = \frac{\iiint_G x dV}{\text{volume of } G} \quad (19a)$$

$$\bar{y} = \frac{\iiint_G y dV}{\iiint_G dV} = \frac{\iiint_G y dV}{\text{volume of } G} \quad (19b)$$

$$\bar{z} = \frac{\iiint_G z dV}{\iiint_G dV} = \frac{\iiint_G z dV}{\text{volume of } G} \quad (19c)$$

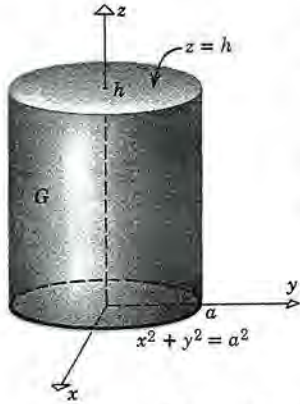


Figure 17.6.9

► **Example 5** Find the mass and the center of gravity of the adjacent cylindrical solid, assuming that the density at each point is proportional to the distance between the point and the base of the solid.

Solution. Since the density is proportional to the distance z from the base, the density function has the form $\delta(x, y, z) = kz$ where k is some (unknown) constant of proportionality. From (17) the mass of the solid is

$$\begin{aligned}
 M &= \iiint_G \delta(x, y, z) \, dV = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h kz \, dz \, dy \, dx \\
 &= k \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2}h^2 \, dy \, dx \\
 &= kh^2 \int_{-a}^a \sqrt{a^2-x^2} \, dx \\
 &= \frac{1}{2}kh^2\pi a^2 \quad \left[\begin{array}{l} \text{See Example 4} \\ \text{of Section 9.5} \end{array} \right]
 \end{aligned}$$

Without additional information, the constant k cannot be determined. However, as we shall now see, the value of k does not affect the center of gravity.

From (18),

$$\bar{z} = \frac{\iiint_G z\delta(x, y, z) \, dV}{\text{mass of } G} = \frac{\iiint_G z\delta(x, y, z) \, dV}{\frac{1}{2}kh^2\pi a^2}$$

But

$$\begin{aligned}
 \iiint_G z\delta(x, y, z) \, dV &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h z(kz) \, dz \, dy \, dx \\
 &= k \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{3}h^3 \, dy \, dx \\
 &= \frac{2}{3}kh^3 \int_{-a}^a \sqrt{a^2-x^2} \, dx \\
 &= \frac{1}{3}kh^3\pi a^2
 \end{aligned}$$

so that

$$\bar{z} = \frac{\frac{1}{3}kh^3\pi a^2}{\frac{1}{2}kh^2\pi a^2} = \frac{2}{3}h$$