McGRAW-HILL

SERIES IN

MECHANICAL

ENGINEERING

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 1 of 143

KINEMATIC SYNTHESIS OF LINKAGES

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 3 of 143

McGraw-Hill Series in MECHANICAL ENGINEERING

ROBERT M. DRAKE, JR. & STEPHEN J. KLINE

Consulting Editors

BEGGS, *Mechanism*
CAMBEL AND JENNINGS, *Gas Dynamics*
CSANADY, *Theory of Turbomachines*
DURELLI, PHILLIPS, AND TSAO, *Introduction to the Theoretical and Experimental Analysis of Stress and Strain*
ECKERT, *Introduction to Heat and Mass Transfer*
ECKERT AND DRAKE, *Heat and Mass Transfer*
GRÖBER, ERK, AND GRIGULL, *Fundamentals of Heat Transfer*
HAM, CRANE, AND ROGERS, *Mechanics of Machinery*
HARTENBERG AND DENAVIT, *Kinematic Synthesis of Linkages*
HARTMAN, *Dynamics of Machinery*
HINZE, *Turbulence*
JACOBSEN AND AYRE, *Engineering Vibrations*
PHELAN, *Fundamentals of Mechanical Design*
RAVEN, *Automatic Control Engineering*
SABERSKY, *Elements of Engineering Thermodynamics*
SCHENCK, *Theories of Engineering Experimentation*
SCHLICHTING, *Boundary Layer Theory*
SHIGLEY, *Dynamic Analysis of Machines*
SHIGLEY, *Kinematic Analysis of Mechanisms*
SHIGLEY, *Mechanical Engineering Design*
SHIGLEY, *Theory of Machines*
SPALDING AND COLE, *Engineering Thermodynamics*
STOECKER, *Refrigeration and Air Conditioning*
WILCOCK AND BOOSER, *Bearing Design and Application*

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 4 of 143

KINEMATIC SYNTHESIS OF LINKAGES

RICHARD S. HARTENBERG
Member of ASME, VDI, I.MECH.E.

JACQUES DENAVIT
Member of ASME, VDI

Department of Mechanical Engineering
and Astronautical Sciences
Northwestern University

McGraw-Hill Book Company

New York *San Francisco* *Toronto* *London*

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 5 of 143

Kinematic Synthesis of Linkages

Copyright © 1964 by McGraw-Hill, Inc. All Rights Reserved.
Printed in the United States of America. This book, or parts thereof, may not be reproduced in any form without permission of the publishers.
Library of Congress Catalog Card Number 64-23251

26910

PREFACE

In the early 1940s, the idea of kinematic synthesis of linkages was, in a manner of speaking, mentioned for the first time in this country. After a decade of gestation, original articles on the subject began to appear in America. Now, with the passing of the second decade, it is appropriate to attempt a consolidated story of some of the many efforts of European and American kinematicians.

Conventional texts give an adequate presentation of the area of planar kinematic analysis. In these books, first contact is established with fundamental concepts, and known planar mechanisms are investigated for the displacements, velocities, and accelerations of their parts. In analysis, mechanisms are put on the psychiatrist's couch, so to speak, and are probed for their performance traits. The important and complementary area, that called kinematic synthesis—where mechanisms are created to meet certain motion specifications—is touched upon only by a consideration of cams and possibly a glimpse of the simpler aspects of planar linkage synthesis.

In the design of a mechanism for a given application, a decision must first be reached regarding the *type* of mechanism to be employed, as, for example, deciding between a cam or a linkage. The *number* of links and connections required to give the desired degree of freedom must then be determined. Finally, the required *dimensions* needed to bring about a particular motion must be deduced. In the broadest sense, kinematic synthesis thus consists of the three interrelated areas of type, number, and dimensional synthesis.

9211
ANNEX 3

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 7 of 143

Of these areas, type synthesis is based largely on nonkinematic considerations. It depends, of course, upon the designer's kinematic experience, but, beyond that, other factors, such as machine-design problems and economic considerations, must be taken into account. While recognizing the problem of type synthesis, we make no effort to cope with it and go directly to linkages, with the avoidance of cams and gears. The present text focuses on the dimensional synthesis of linkages, an area amenable to mathematical treatments. Number synthesis, not a very real problem for the simple linkages that can be devised by synthetic procedures, is given a brief discussion.

This text is devoted to the synthesis of planar and spatial linkages and considers both basic approaches, the geometric and algebraic, allowing the reader to choose whichever method best suits his needs or available tools. The geometric methods are quick and have the advantage of staying close to the physical problem. They may have adequate accuracy for many tasks but tend to become tedious under repeated use. Algebraic methods give any desired degree of accuracy and through their functional relations indicate the relative importance of any parameter. They also make possible the study of errors in dimensions, as those associated with tolerance or wear. And of course the algebraic methods lead to the use of digital computers, especially convenient if the same calculations have to be repeated often for incremental inputs or parameter changes.

Recent years have seen the publication of a wealth of new material relating to synthesis, both in this country and abroad. In addition, there has been a growing awareness of the very considerable European effort, especially the German. This material is difficult to follow because of the scattered sources, terseness of presentation, and language barriers. The present text is an attempt to present a well-balanced and intelligible account of some of the geometric and algebraic procedures, filling in as necessary, making comparisons, and elaborating on the implications to give a well-rounded picture.

The bulk of the available European material seems to divide into two broad classes. One makes use of rather sophisticated geometry and mathematics, while, in the other, descriptions of techniques predominate over the scrutiny of principles. This book attempts a middle-of-the-road presentation having "rigor" without "mortis," integrating the European and American contributions.

The book studies, compares, and applies various procedures for handling the several types of problems constituting the synthesis of linkages. The start is made by reviewing or establishing definitions, concepts, and notations, with recapitulation of familiar methods of velocity

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 8 of 143

and acceleration analysis. An account is given of type, number, and dimensional synthesis. This includes a discussion of the Chebyshev polynomials and their application to the determination of an optimal spacing of accuracy points. The properties and uses of four-bar coupler curves are discussed, including their equation, the implications of their geometry (cusps, symmetry, and crunodes), and the Roberts-Chebyshev theorem, with its extensions to the slider-crank mechanism and six-bar linkages. The Euler-Savary equation and the cubic of stationary curvature are derived analytically, by using complex numbers, the ambiguities and passage to the limit of the direct geometric approach being thus avoided. Two chapters are devoted to the geometric methods of synthesis with three and four accuracy points. Numerous sample problems are given here; some deliberately contain pitfalls to show what can be done when trouble arises. In the presentation of algebraic methods of synthesis using displacement equations or complex numbers, an attempt is made to include the latest developments on synthesis with five accuracy points and the use of digital computers in the solution of advanced problems.

The last chapter, Synthesis of Spatial Linkages, is an original contribution; this material is to be found nowhere else. Previous work of the authors on the analysis of spatial mechanisms by matrix algebra is summarized. This is followed by several sections on the synthesis of spatial linkages making use of displacement equations. Here again a number of examples are given.

In addition to the material which properly belongs under the title of kinematic synthesis, complementary material has been added to make the book more readable and self-contained. The reader is introduced to the history of kinematics, and the development of concepts and procedures from the great personalities of the past is traced. The chapter devoted to the construction of kinematic models is of the greatest aid in establishing a rapport between the static drawing of one phase and the many other positions a mechanism will assume. Models give immediate information about displacement relations, unfavorable transmission angles, dead points, binding, and friction. A review of complex numbers, determinants, and solutions of linear equations is included as an appendix. These tools, rust-covered for many readers (the voice of experience), are resharpened in an effort to make the book a self-sufficient unit suited for self-study.

Our spiritual guide and close friend was the late Professor Rudolf Beyer of Munich. His works and infectious enthusiasm have been the inspiration for this book and the source of strength for overcoming the many difficulties and discouragements inherent in the process that

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 9 of 143

creates any book, especially one dealing with the development of deductive processes.

The contributions of many others—friends, colleagues, and students in overwhelming numbers—cannot be acknowledged individually, much as we would like to. We remain grateful to our benefactors even though their kindnesses and help must go unrecorded.

It is a last pleasant duty to acknowledge the suggestions of the reviewers of the manuscript.

Richard S. Hartenberg
Jacques Denavit

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 10 of 143

CONTENTS

Preface v

Chapter 1: AN OUTLINE OF KINEMATICS TO 1900 1

Chapter 2: CONCEPTS AND NOTATIONS RELATED TO MECHANISMS 28

2-1	MECHANISMS	28
2-2	CLASSIFICATION SYSTEMS	29
2-3	RIGID AND RESISTANT BODIES	30
2-4	MOTION--RELATIVE AND ABSOLUTE	31
2-5	CONNECTIONS, AND THE PARTICULAR MOTIONS THEY PERMIT	31
2-6	LOWER-PAIR CONNECTORS	32
2-7	HIGHER-PAIR CONNECTORS	38
2-8	FOUR-BAR LINKAGES	38
2-9	SLIDER-CRANK MECHANISMS	44
2-10	TRANSMISSION DEVIATION, AND PRESSURE ANGLES	46
2-11	PLANAR AND SPATIAL: THE MOTIONS AND THE MECHANISMS	49
2-12	KINEMATIC CHAINS	52
2-13	INVERSION	55
2-14	EXPANSION OF REVOLUTE PAIRS AND OTHER DISGUISES	57
2-15	PRISMATIC PAIR AS THE LIMIT OF A REVOLUTE PAIR	59
2-16	EQUIVALENT LINKAGES	61
2-17	SYMBOLIC NOTATIONS	64

Chapter 3: KINEMATIC MODELS 68

3-1	THE MEANING OF MODELS	68
3-2	HISTORICAL SURVEY	70
3-3	PLANAR KINEMATIC MODELS	75
3-4	SPATIAL KINEMATIC MODELS	84

Chapter 4: KINEMATIC ANALYSIS OF PLANAR MOTION 88

4-1	INTRODUCTION	88
4-2	COINCIDENT POINTS	89
4-3	NOTATION	91
4-4	LINEAR AND ANGULAR VELOCITIES	92
4-5	RELATIVE VELOCITY AND VELOCITY DIFFERENCE	97
4-6	INSTANTANEOUS CENTERS OF VELOCITY	103

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 11 of 143

4-7	POLES AND CENTRODES	109
4-8	ACCELERATION	117
4-9	RELATIVE ACCELERATION AND CORIOLIS ACCELERATION	120
4.10	ACCELERATION DIFFERENCE	124

Chapter 5: TYPE, NUMBER, AND DIMENSIONAL SYNTHESIS 130

5-1	KINEMATIC SYNTHESIS	130
5-2	TYPE SYNTHESIS	132
5-3	NUMBER SYNTHESIS	132
5-4	DIMENSIONAL SYNTHESIS	138
5-5	SPACING OF ACCURACY POINTS	140
5-6	CHEBYSHEV POLYNOMIALS	144

Chapter 6: FOUR-BAR COUPLER-POINT CURVES 148

6-1	THE FOUR-BAR LINKAGE	148
6-2	EQUATION OF COUPLER CURVES	150
6-3	DOUBLE POINTS AND SYMMETRY	161
6-4	THE ROBERTS-CHEBYSHEV THEOREM	168
6-5	EXTENSIONS OF THE ROBERTS-CHEBYSHEV THEOREM	176
6-6	STRAIGHT-LINE MECHANISMS—APPROXIMATE AND EXACT	179
	APPENDIX: ATLAS OF FOUR-BAR COUPLER CURVES	186

Chapter 7: THE EULER-SAVARY EQUATION AND THE CUBIC OF STATIONARY CURVATURE 194

7-1	THE EULER-SAVARY EQUATION AND THE INFLECTION CIRCLE	194
7-2	THE CUBIC OF STATIONARY CURVATURE	204
7-3	EXAMPLE: FOUR-BAR LINKAGE TO REPLACE CIRCULAR GEAR SEGMENTS FOR SMALL ROTATIONS	210

Chapter 8: GEOMETRIC METHODS OF SYNTHESIS WITH THREE ACCURACY POINTS 215

8-1	INTRODUCTION	215
8-2	POLES OF THE FOUR-BAR LINKAGE	216
8-3	RELATIVE POLES OF THE FOUR-BAR LINKAGE	220
8-4	EXAMPLE: LOGARITHMIC-FUNCTION GENERATOR	225
8-5	EXAMPLE: SINUSOIDAL-FUNCTION GENERATOR	231
8-6	POLES OF THE SLIDER-CRANK MECHANISM	236
8-7	RELATIVE POLES OF THE SLIDER-CRANK MECHANISM	237
8-8	EXAMPLE: RECTILINEAR RECORDER MECHANISM	244

Chapter 9: GEOMETRIC METHODS OF SYNTHESIS WITH FOUR ACCURACY POINTS 249

9-1	POLE TRIANGLES	249
9-2	FOUR POSITIONS OF A PLANE	255
9-3	PROOF OF THEOREM IV	267
9-4	EXAMPLE: SINUSOIDAL-FUNCTION GENERATOR	270
9-5	EXAMPLE: RECTILINEAR RECORDER MECHANISM	274
9-6	EXAMPLE: ACTUATION OF PRESSURE CYLINDER	279
9-7	EXAMPLE: APPROXIMATE DWELL LINKAGE	286

Chapter 10: ALGEBRAIC METHODS OF SYNTHESIS USING DISPLACEMENT EQUATIONS 295

10-1 DISPLACEMENT EQUATION OF THE FOUR-BAR LINKAGE 295
10-2 CRANK AND FOLLOWER SYNTHESIS: THREE ACCURACY POINTS 297
10-3 EXAMPLES: FOUR-BAR FUNCTION GENERATORS WITH THREE ACCURACY POINTS 298
10-4 CRANK AND FOLLOWER SYNTHESIS: ANGULAR VELOCITIES AND ACCELERATIONS 303
10-5 GENERALIZATION OF THE SYNTHESIS METHOD BY LINEAR EQUATIONS 305
10-6 SYNTHESIS OF THE SLIDER-CRANK MECHANISM WITH THREE ACCURACY POINTS 305
10-7 SYNTHESIS OF THE SLIDER-CRANK MECHANISM WITH FOUR ACCURACY POINTS 308
10-8 CRANK AND FOLLOWER SYNTHESIS: FIVE ACCURACY POINTS 311
10-9 ANALYSIS OF MECHANICAL ERRORS IN LINKAGES 315
10-10 MECHANICAL ERRORS IN FOUR-BAR LINKAGES 316
10-11 GEOMETRIC INTERPRETATION OF THE ERROR DENOMINATOR 319

Chapter 11: ALGEBRAIC METHODS OF SYNTHESIS USING COMPLEX NUMBERS 321

11-1 VELOCITY AND ACCELERATION SYNTHESIS BY COMPLEX NUMBERS 321
11-2 COUPLER-CURVE SYNTHESIS: FIVE ACCURACY POINTS 327
11-3 REDUCTION OF THE FIRST PAIR OF COMPATIBILITY EQUATIONS 337

Chapter 12: SYNTHESIS OF SPATIAL LINKAGES 343

12-1 INTRODUCTION TO THE STUDY OF SPATIAL LINKAGES 343
12-2 DISPLACEMENT ANALYSIS OF THE TWO-REVOLUTE TWO-SPHERIC-PAIR MECHANISM 344
12-3 AN OUTLINE OF THE MATRIX METHOD OF ANALYSIS 347
12-4 SYNTHESIS OF FOUR-REVOLUTE SPHERICAL MECHANISMS 356
12-5 SYNTHESIS OF TWO-REVOLUTE TWO-SPHERIC-PAIR MECHANISMS 358
12-6 FUNCTION GENERATORS FOR SYMMETRICAL FUNCTIONS 362

Appendix: MATHEMATICAL TRACTS 369

A-1 COMPLEX NUMBERS 370
A-2 LINEAR SYSTEMS OF ALGEBRAIC EQUATIONS—I: DEFINITIONS 380
A-3 DETERMINANTS 380
A-4 LINEAR SYSTEMS OF ALGEBRAIC EQUATIONS—II: RANK AND SOLUTION 389

Problems 397
Index 431

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 13 of 143

1 AN OUTLINE OF KINEMATICS TO 1900

The history of kinematics, the story of the development of the geometry of motion, is composed of evolvements in machines, mechanisms, and mathematics. The story begins with the random growth of machines and mechanisms under the pressure of necessity. This was a period of invention and establishment of basic forms, but there was neither unity nor plan. Much later, when existing algebraic speculations on the generation of curves were timidly applied to physical problems, the concept of kinematics, a science, was affirmed and the ancient art of mechanisms complemented. More recently, mathematics has been applied with deliberation and firmness. Although no general relationship linking all mechanisms has been discovered, modicums of unity and plan have been recognized for some areas. More significant is the ability to solve a wide variety of problems; some require the use of sophisticated procedures feasible only because of high-speed computers.

Machine and mechanism are words giving two different images. The term machine is associated with the use and transformation of force, and although motion in varying degree is encountered in a machine, the idea of force dominates. Mechanism, on the other hand, definitely conjures up the idea of motion, and while

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 15 of 143

forces do exist, they are relatively small and unimportant compared with the exploitation of motion. However, machine and mechanism inhabit the same body.

Machines are spoken of early in history, but what was considered a machine two thousand years ago differs considerably from our present ideas. From earliest times man has found his own powers inadequate for the tasks he set himself, among them the moving of heavy weights. This was a major concern of the ancients, as we may judge from their imposing monuments of large stones. Vitruvius,[b] a military engineer writing about 28 B.C., defined a machine as "a combination of timber fastened together, chiefly efficacious in moving great weights." About a century later, Hero of Alexandria[b] summarized the practice of his day by naming the "five simple machines" for "moving a given weight by a given force" as the lever, windlass, screw for power, wedge, and tackle block (pulley). Until nearly the end of the nineteenth century it was held that these "five mechanical powers" were the building blocks from which all more complex assemblages were constructed.

Today we no longer consider a machine to be constituted of a single body such as a lever; indeed, the use of a lever involves several bodies, or objects capable of exerting force. Today's thoughts on machines involve assemblages made from six basic mechanical components, identified about 1875 by the German kinematician Franz Reuleaux.[b] These parts are (1) the eye-bar type of link, called crank[1] in kinematics; (2) the wheel, including gears; (3) the cam in its many forms; (4) the screw for communicating motion and force; (5) the intermittent-motion devices called ratchets for want of a better name; and (6) the tension-compression organs, or parts having "one-way rigidity," as belts (chains) and hydraulic lines. It is from such parts that machines —and mechanisms—are built.

Each of these components was invented in antiquity and put to use well before the beginning of the Christian era, either singly or in combination. Alexandria (in Egypt, but nevertheless a Greek town) became the greatest center of learning of the ancient world under the Ptolemies in the third century B.C. It was here that Greek science, philosophy, and technology were nurtured and further developed; Euclid wrote his imperishable "Elements" in Alexandria.

Mechanical components had been given application mainly in war machinery—"throwing engines" for tossing projectiles of various sorts—man always having found money for research on and development of armament. The mobilization of technology and science appears to

[b] The superscript *b* indicates a biographical note at the end of the chapter.

[1] The name is misleading but ineradicable; the resemblance to what is commonly called a crank today is not immediately evident. Lever is sometimes used.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 16 of 143

be as ancient as warfare. Archimedes, who wished to be remembered as a mathematician (he was the Newton of his day), is better known as an engineer because of the "engines" with which he thwarted the Roman siege of Syracuse (214–212 B.C.).

The labors of daily living were only lightly touched by applications of mechanics. The muscle power available from the bulk of the populace gave little stimulus to the construction of either laborsaving devices or production machinery except for the never-ending tasks of raising water and grinding grain. For the transmission of power, the complexity rarely exceeded the use of more than two gears to proportion input and output shaft speeds, as between water wheel and millstone, noted by Vitruvius. Mechanisms were more sophisticated (Fig. 1-1). We may note that power and complexity were not in the same device.

Many small and ingenious devices appeared in the temple automata designed to excite wonder in the ignorant people of pre-Christian Alexandria. Here pneumatics as well as "hard" mechanics were applied to mechanisms that would open temple doors after the kindling of sacrificial fires, vend liquids on the dropping of a coin, produce stage effects with moving scenery, and the like. There was a clever employment of speed and force relations, and in some cases regulators were fitted to control the motion.

With the Roman take-over of Egypt and Alexandria at the time of Julius Caesar, Cleopatra, and Christ, science almost came to a halt. The Roman Empire produced little new in the way of things mechanical but accomplished much in the shrewd exploitation of its Alexandrian inheritance by way of bigger and better machines. The Roman effort went into the construction of buildings and roads: the principal Roman invention was hydraulic cement. Following the collapse of the Western empire, the center of effort shifted to Constantinople. Before long—it was 642—the Arabs, activated and disciplined by their prophet Mohammed, conquered Egypt. From this start, scientific and technological prowess was distinctly Arabian for the next centuries. The Arabs not only had acted as transmitters of the Alexandrian or Greek works to the Western world but also had produced refinements and substantial innovations of their own.

It is wrong to assume, in the light of recent studies, that Europe was devoid of ideas during the "Dark Ages," the span between, say, A.D. 500 and 1000, during which the Arabs rose to greatness. The records of these centuries are indeed scanty if we look for machines and mechanisms. The social and economic structure, so different from the Roman, provoked a tremendous change in agricultural methods. The nonmechanical details are fascinating miniatures we cannot take the time to consider. At any rate, agricultural surplus led to rapid urbani-

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 17 of 143

zation. Artisans and merchants rose in the new cities. The later medievalists were ready to explore mechanical devices suited to "production" and to exploit the "natural powers" of wind and water to complement (and in many cases supplant) muscle power. These new powers were to spawn a multitude of machines with increasingly complex mechanisms.

Documents of the twelfth and thirteenth centuries containing machine sketches are few. However, refinements are to be seen (Fig. 1-2), and the designers' enterprise is limited by available materials

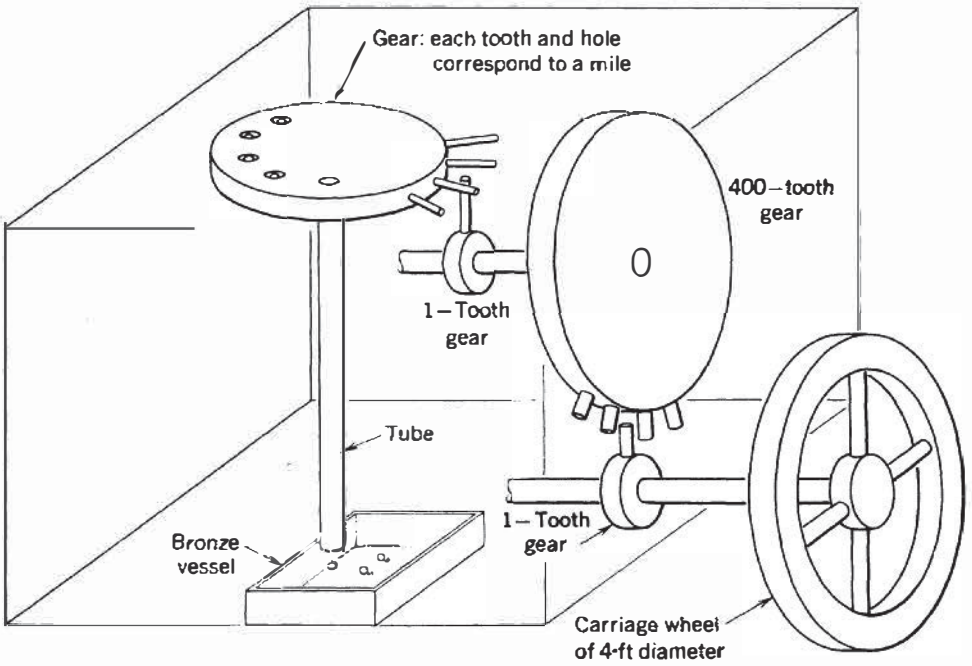


FIGURE 1-1 Schematic of an odometer for a carriage, as described about 28 B.C. by Vitruvius in "Ten Books on Architecture," translated by M. H. Morgan, Harvard University Press, Cambridge, Mass., 1926.

Vitruvius, military engineer under Augustus Caesar, was no innovator but only a reporter of his past and present. He ascribes this device, "which enables us, while sitting in a carriage or sailing by sea, to know how many miles of the journey we have accomplished," to Ctesibius, a Greek of probably the second century B.C. Vitruvius' description is verbal and not pictorial: the wheel diameter and the numbers of teeth in the gears are specified. One complete revolution of the 400-tooth gear represents 5,000 ft (called a mile). With each such revolution, a small stone in a hole of the topmost gear lined up with the tube leading to the bronze vessel (gating device!): "Hence, every stone, making a ringing sound as it falls, will give warning that we have gone one mile." On board ship, the same principle "was applied by fitting a paddle wheel to the axle" projecting through the hull.

A century later Hero described a more sophisticated form with four worm-wheel reductions between input axle and output "stone wheel"; intermediate shafts carried dials to indicate parts of a mile.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 18 of 143

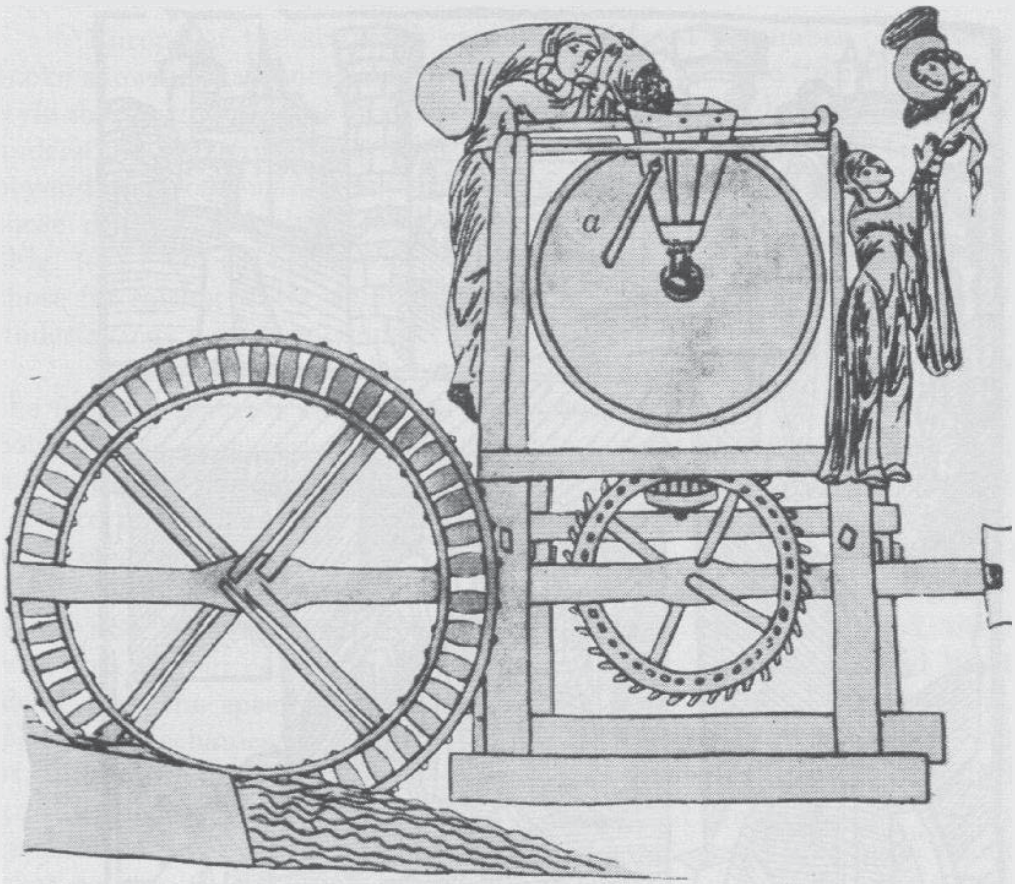


FIGURE 1-2 Water wheel and grain mill, about 1160. Mechanically, this geared [illegible] about the same as that described by Vitruvius some 1,200 years earlier; the hopper agitator (*a*) is an added refinement. (*From Herrad's "Hortus deliciarum," circa* 1175.)

Herrad of Landsperg, abbess of the cloister Hohenburg in Alsace, put together a manuscript of 324 parchment pages whose text was illustrated by 636 colored pen drawings. It was a compendium of the scientific knowledge of the times, written for her students. This work was burned during the bombardment of Strasbourg in 1870; regrettably, only a few illustrations had been copied. The projection is reminiscent of the older Egyptian and Arabic, in which all important components are turned into the same plane.

and tools rather than by lack of imagination. Windmills come into their own, and the path to industrialization that is soon to distinguish Europe from the rest of the world is open.

Leonardo da Vinci belonged to the fifteenth century, and while his notebooks are to be marveled at for the richness of the mechanical thought they contain, they cannot properly be used as exhibits for this period. The notebooks did not become publicly available until the later part of the eighteenth century; nor is there a clear record of which of the many ideas were reduced to practice during Da Vinci's lifetime.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 19 of 143



FIGURE 1-3 Tri phammer in a smithy, an example of the extension of the water w heel's utility by the conversion of continuous rotation into periodic "up-and-down" motion. At upper left is Pythagoras, legendary discoverer of musical theory, "noting" the sound of the hammer (*From Spechtshart's "Flores musicae," Strasbourg,* 1488.)

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 20 of 143

Europe of the sixteenth century possessed a number of sketch-books showing machines—and hence mechanisms. The "picture-book" style that originated was to continue for centuries, for drawings could be understood and transformed into hardware even though the text was beyond the craftsman's abilities. Some books were for specialists, as those dealing exclusively with military devices or mining machinery (Fig. 1-4). Others had a wider scope, the machines being classified into those for raising water, milling grain, sawing timber, and so on. Some undertakings were spectacular (Figs. 1-6 and 1-7).

With Leupold,[b] in the early eighteenth century, there is perhaps the first reference to the idea of *modifying motion* rather than just the construction of machinery; he speaks of cranks and cams and how, say, a continuous circular motion may be transformed into a "back-and-forward motion." Here was the earliest attempt at a systematic treatment of mechanisms.

Two great figures appear in the eighteenth century, Euler[b] (1707–1783) and Watt[b] (1736–1819). Although their lives overlapped, there was no apparent contact between the two men. Both Euler and Watt dealt with the specific and concrete, but along vastly different paths. Euler's "Mechanica sive motus scientia analytice exposita" (1736–1742) is, to quote Lagrange, "the first great work in which analysis is applied to the science of movement." The concept that planar motion may in general be described by the translation of a point and a rotation about that point appears in "Mechanica." This thought, extended to velocity and acceleration, is the root of analysis, graphically shown by the vector-polygon methods. The associated velocity and acceleration equations came after Euler but because of their heritage are sometimes given the generic title of Euler's theorems in the German literature. The fundamental idea of kinematic analysis thus stems from Euler, as do many other astute observations.

Watt, instrument maker and engineer, a contrast to Euler, was concerned with the synthesis of movement. Generally speaking, mechanism designers before Watt had confined their attentions to the motions of links attached to the frame. We may call them the first and last links; one having been given an input motion, the other produced a transformed motion called the output. It was Watt who focused on the motion of a point on the *intermediate* link of the four-bar mechanism: here was a new kind of output with the year 1784. The application of this brilliant thought allowed Watt to build a double-acting steam engine; the earlier chain connecting piston and beam (Fig. 1-8) was now replaced by a linkage able to transmit force in two directions instead of only one (Fig. 1-9). Watt had discovered coupler-point motion, although its definition in these terms lay well in the future. It was a singular achievement, a pivotal point along the road of synthesis.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 21 of 143

Of the two discoveries, Euler's was to lie unnoticed by kinematicians for a century or so. Watt's was pursued in desultory fashion by engine builders and mathematicians. Linkages were devised to compete with or supersede Watt's mechanism; that certain of these linkages were actually related developed only after a generalized treatment of coupler-point motion, known for a time as "three-bar motion," was given a century later by Samuel Roberts.[b]

A further perception is to be noticed in an observation[1] of Euler's in 1775:

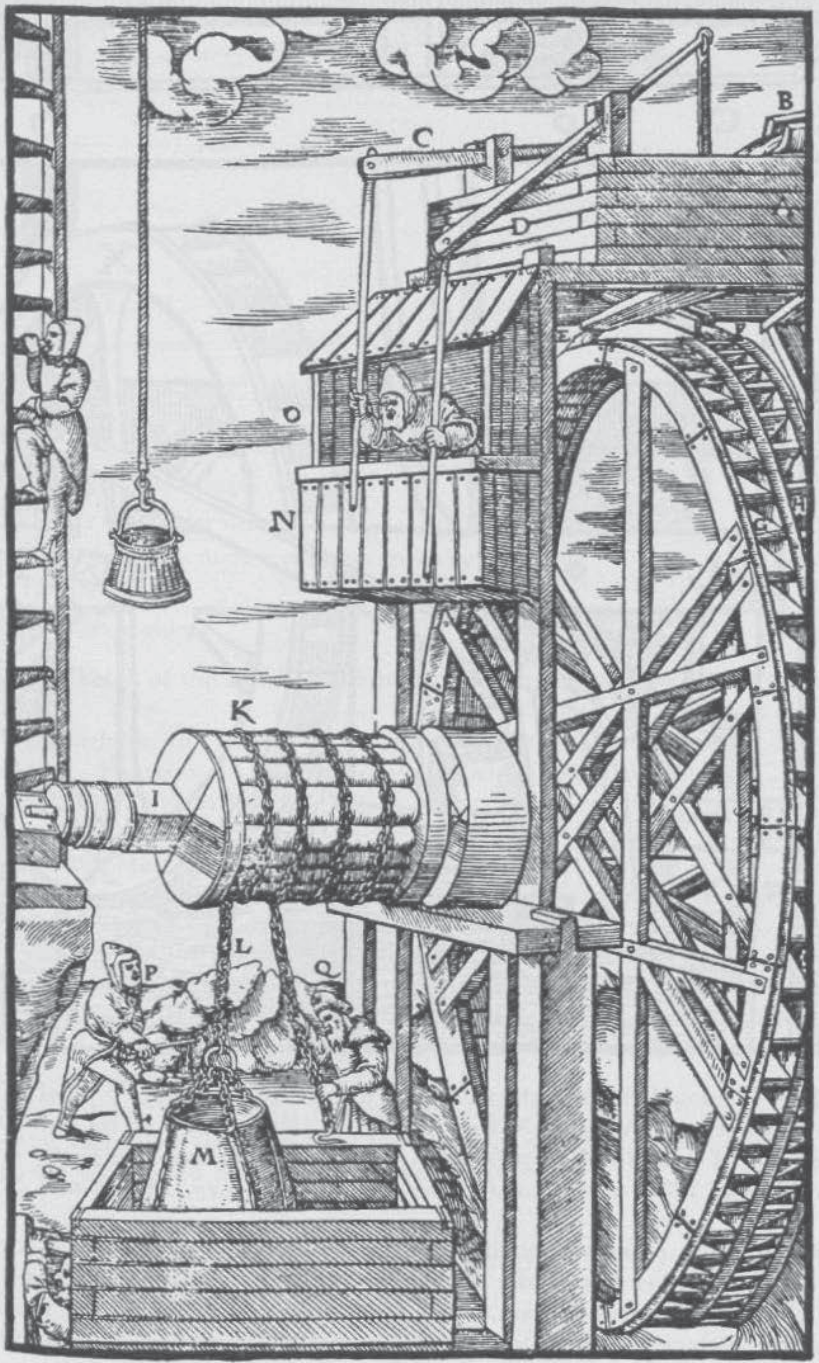
> The investigation of the motion of a rigid body may be conveniently separated into two parts, the one geometrical, the other mechanical. In the first part, the transference of the body from a given position to any other position must be investigated without respect to the causes of motion, and must be represented by analytical formulae, which will define the position of each point of the body. This investigation will therefore be referable solely to geometry, or rather to stereotomy.
>
> It is clear that by the separation of this part of the question from the other, which belongs properly to Mechanics, the determination of the motion from dynamical principles will be made much easier than if the two parts were undertaken conjointly.

Euler's remarks were directed at the rigid bodies of classical mechanics, not specifically to machine parts; yet we see here the beginning of the separation of the general problem of dynamics into kinematics and kinetics. We note in passing that his contemporaries D'Alembert and Kant were of the same mind, treating motion in a purely geometric

[1] "Novi commentarii Academiae Petrop.," vol. XX, 1775; also in "Theoria motus corporum," 1790. The translation is by Willis, "Principles of Mechanism," 2d ed., p. viii, 1870.

FIGURE 1-4 Mine hoist of the sixteenth century. This hoist is reversible, the direction of rotation being controlled by the water-admission valves attached to the levers *C* and *D*. Note that the wooden axle *I* is of square cross section, making it easy to transmit and take off torque without keys or splines. The axle is supported by gudgeons (iron pins of small diameter, one of which is to be seen at the left) and is locally reinforced by shrunk-on iron bands to prevent breakout. The use of chain was typical, vegetable fibers being unsuited to the damp of mines. From Agricola's "De re metallica," Basel, 1556, translated and annotated by Mr. and Mrs. Herbert C. Hoover long before Mr. Hoover became President of the United States. (*Mining Magazine, London*, 1912; *also reprint by Dover Publications, Inc., New York*, 1956.)

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 22 of 143



M Bag. N—Hanging cage. [illegible] the machine. P, Q—Men emptying bags.

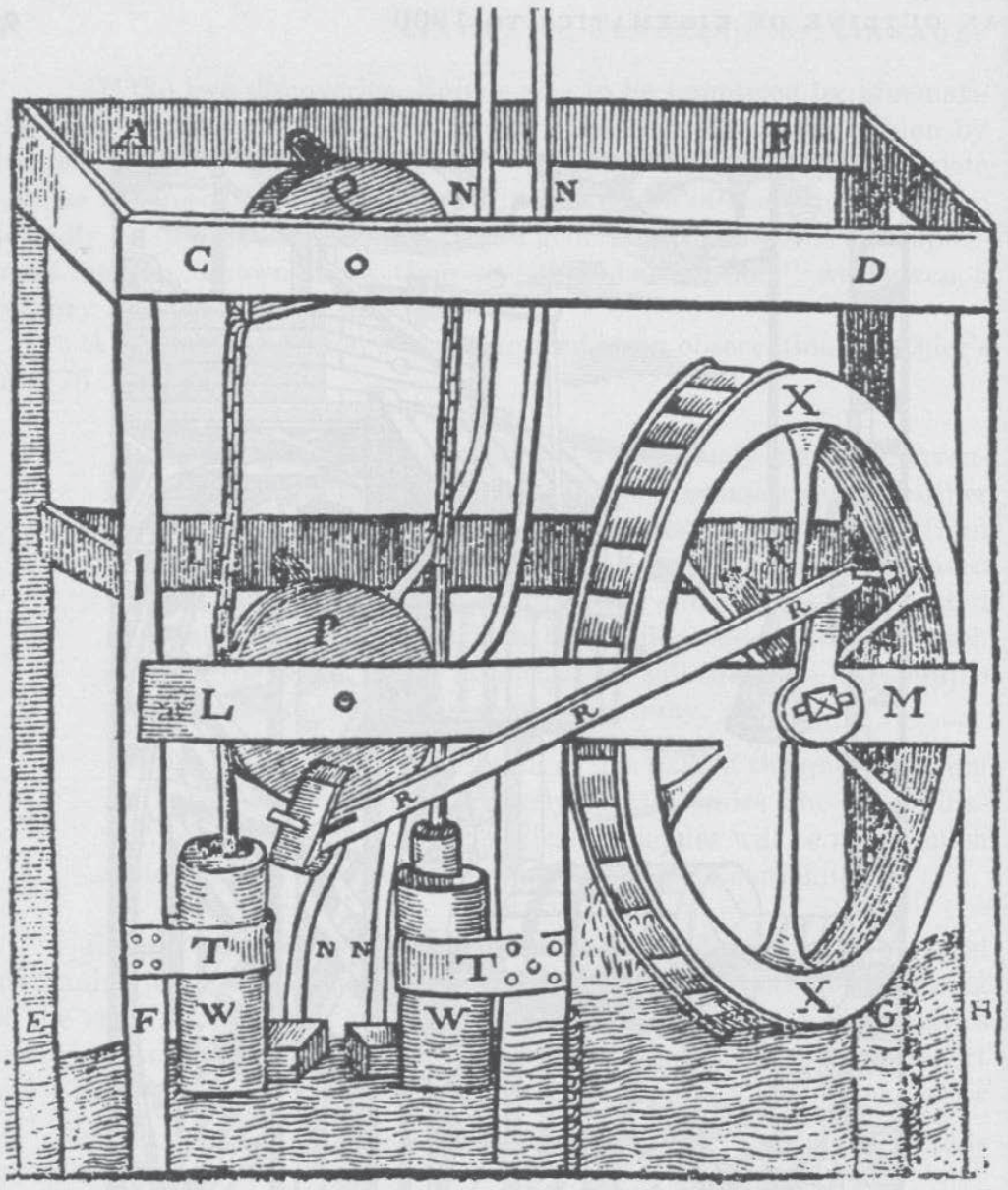


FIGURE 1-5 Londonr Br idger Waterworks, as dr awn from memory byrJohn Bate for his "The Mysteries of Art and Nature" (1635). (*From Notes on the London Bridge Waterworks in "The Collected Papers of Rhys Jenkins," Newcomen Society, University Press, Cambridge*, 1936; *also Rhys Jenkins in The Antiquary, August and September*, 1895.))

We discern a crank-rocker linkage, the crank being on the water-wheel axle. Coupler *R*, fitting "over the handle of the main wheele," gives "the half or 3 quarter wheeles" *P* and *Q* a rocking motion. The pump rods get a straight-line motion from two "chaines" fastened to the wheels. The pumps *W* fed into pipes *V*, which carried the water to a releva[illegible] tank "neare and adioyninge unto the engin."

About 1582 one Peter Morice, Morris, or Moris installed "a most artificial forcier," in this case a water-wheel-driven forcer pump, under one of the northern arches of London Bridge. The fire of 1633 destroyed superposed buildings and enabled Bate to see the machinery. His drawing is judged to be a fairly accurate representation. The whole of the waterworks was destroyed by the fire of 1666.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 24 of 143

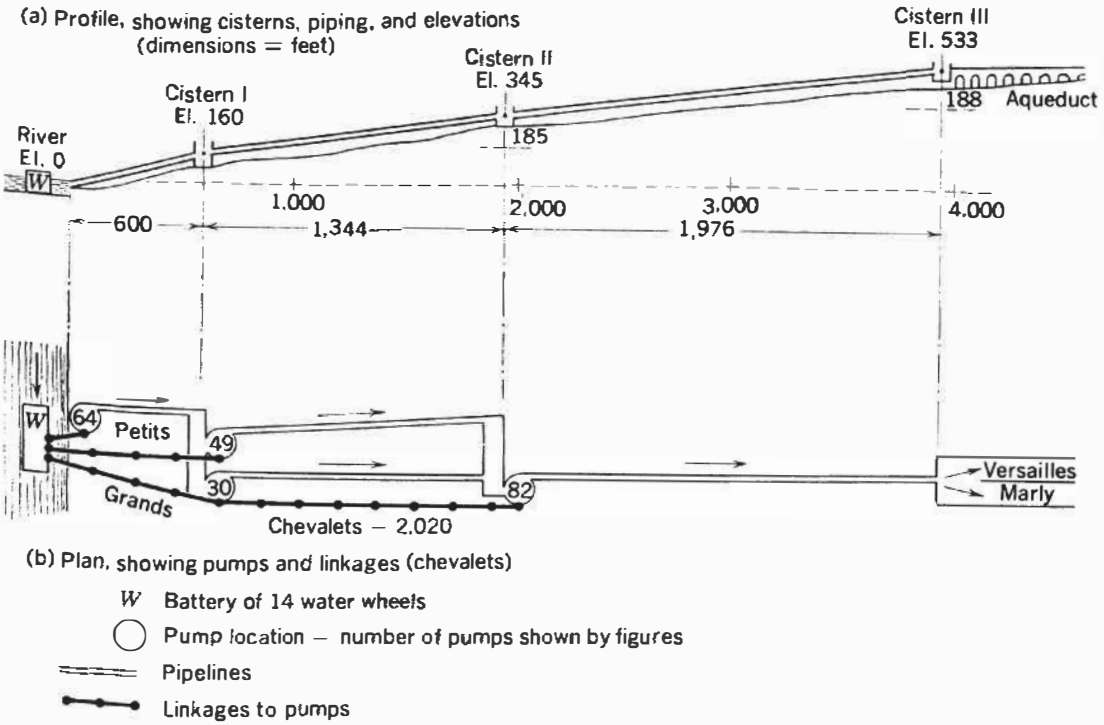


FIGURE 1-6 Sketch of the Marly machine, a pumping installation set to work in 1683 near Paris.

This machine, the largest of its day, was a wonder of the world for the next century or so. Fourteen 40-ft-diameter water wheels were exposed to 1,200 hp from the river Seine to drive 225 pumps lifting water to a reservoir 533 ft above the river, establishing a substantial gravity head for the fountains, cascades, and 60-ft-high jets at Versailles and Marly-le-Roi.

The pumping was done in three stages. The first consisted of pumps at the river, shoving water to hillside reservoirs at elevation 160. Pumps here were driven by seven 600-ft linkages, and moved the water to higher reservoirs at elevation 345. Pumps at this station were actuated by thirteen 2,020-ft-long linkages, depositing the water at the 533-ft level of the aqueduct. The linkages were of the parallelogram type and consisted in part of 64,000 ft (about 12 miles!) of iron bars. Most of the power was used to overcome friction and inertia.

What made the machine unusual was the number of linkages, the power handled, and the noise; the lengths of the linkages were modest even for those days, but the power was not to be exceeded for two centuries.

manner, i.e., disregarding the causes of motion. This is what L. N. M. Carnot[b] later called "geometric" motion.

By the close of the eighteenth century, Monge[b] had proposed a course on elements of machines for the École Polytechnique:

> By these elements are to be understood the means by which the directions of motion are changed; those by which the progressive

> motion in a right line, rotative motion and reciprocating motion, are made each to reproduce the others. The most complicated machines being merely the result of a combination of some of these elements, it is necessary that a complete enumeration of them should be drawn up.

The suggested enumeration was prepared by Monge's long-time colleague Hachette. This system classified mechanisms by capability, i.e., the nature of the motion transformation a mechanism effected between input and output. It was presented in chart form when the course finally materialized in 1806. The chart depicted those mechanisms able to do a certain job, as transforming "circular continuous motion into rectilinear alternating motion"; various motion combinations were tabulated. This chart has been most durable, for today's compendiums of mechanisms are its direct descendants. Hachette's book "Traité élémentaire des machines" did not appear until 1811; a few years later Hachette's course —and book—formed part of West Point's curriculum. It would have been the first kinematics text had it not been for "Essai sur la composition des machines" of 1808, brought out by Lanz and Bétancourt, also of the École Polytechnique. This text was based on the Monge-Hachette scheme and, because of publication prior to Hachette's book, erroneously receives credit for the pioneer work done by Hachette.

Two more names and two new but related classification systems intervene before the actual coining of the word kinematics. The men involved were Borgnis, an Italian engineer and professor of mechanics at the University of Pavia; and De Coriolis, chief engineer of roads and

---

FIGURE 1-7 Moving the Vatican obelisk, 1586. (*From Fontana, "Della trasportatione dell'obelisko Vaticano," Rome,* 1590.) The picture is by courtesy of the Burndy Library.

A 361-ton obelisk, cut in Egypt in the thirteenth century B.C., brought to Rome about A.D. 41, and placed in the Circus of Nero, was to become an architectural feature of the new cathedral of St. Peter begun in the early sixteenth century. To bring the obelisk into position in front of the cathedral required a move of 275 yd. Many plans for doing this were submitted, that of Fontana being chosen. Very briefly, it involved encasing the obelisk in oak planking, to give beam strength, and the use of a timber tower, block and tackle, levers, rollers, etc., with 800 men and 74 to 140 horses deployed at the capstans (the figure shows four).

The obelisk was lifted vertically by five large levers and tackle connected to 40 capstans, after which a long carriage was slipped under. The next step, the laying down, was handled in part by an angular brace pin-connected to the oak casing. The right, or sliding, end of this brace was controlled by further tackle. In this brace we see the use of a slider-crank mechanism, a kinematic arrangement that gave stability and materially reduced the load on the tower tackle during the most trying phase.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 26 of 143

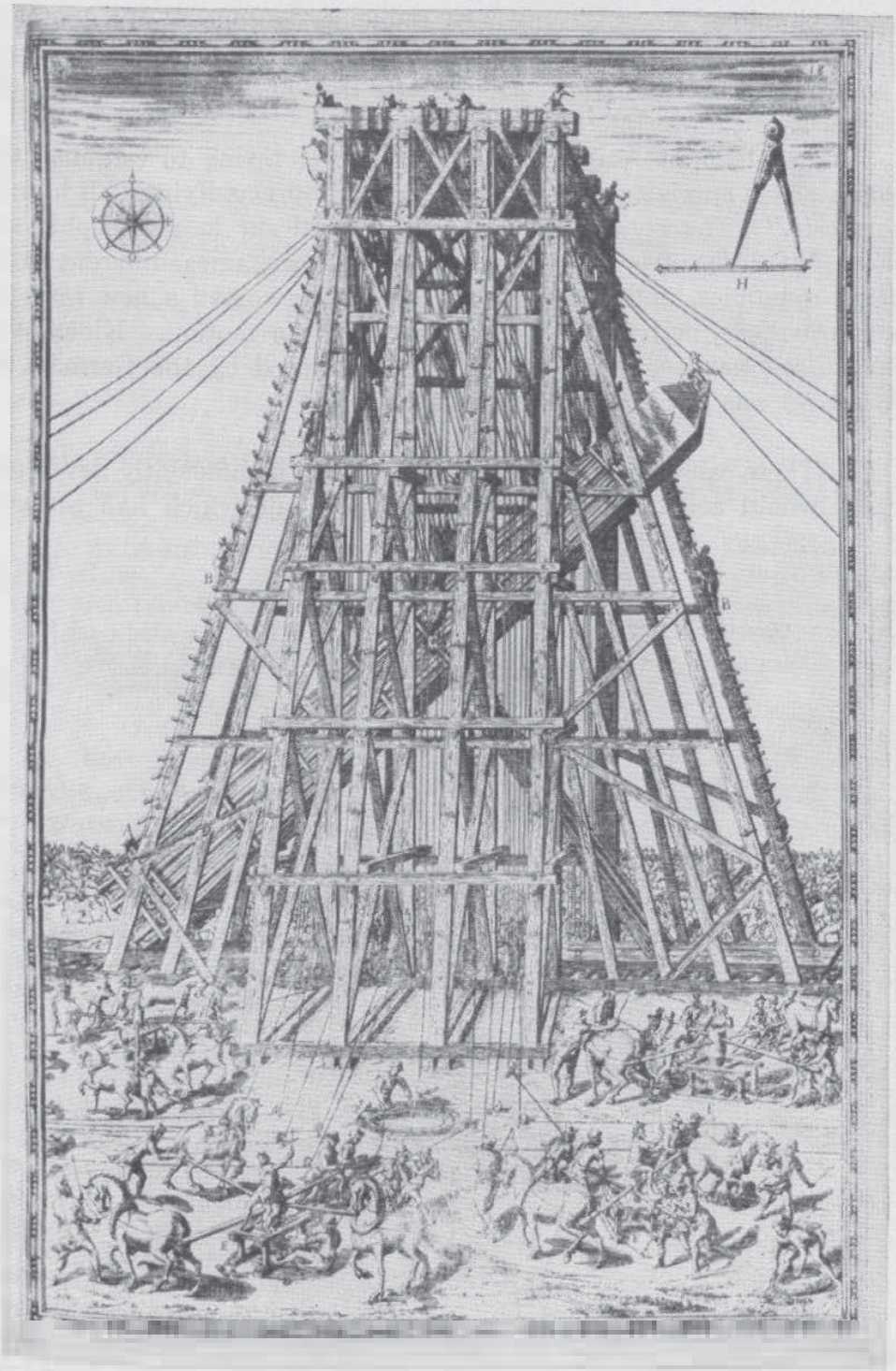


Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 27 of 143

highways, but also professor in Paris, the man of the acceleration component, who so deftly defined the product of force by distance as work. Coriolis simplified Borgnis' system by reducing the concept of a complete machine to three parts or groups of parts, viz., the receivers of energy, or *récepteurs*, the *communicateurs*, or parts transmitting motion, and the *opérateurs*, or the working parts or tools.

The physicist Ampère,[b] in the course of trying to establish the domain of his specialty, was led to defining and classifying "all human knowledge." The consequence was his "Essai sur la philosophie des sciences" (1834). To recognize an area of mechanics different from statics, dynamics, and molecular mechanics, he coined a new term; it was *cinématique*, derived from the Greek word for motion. Kinematics (*cinématique* was anglicized and was also adopted by the Germans as *Kinematik*) was given recognition as a science:

> There exist certain considerations which if sufficiently developed would constitute a complete science, but which had hitherto

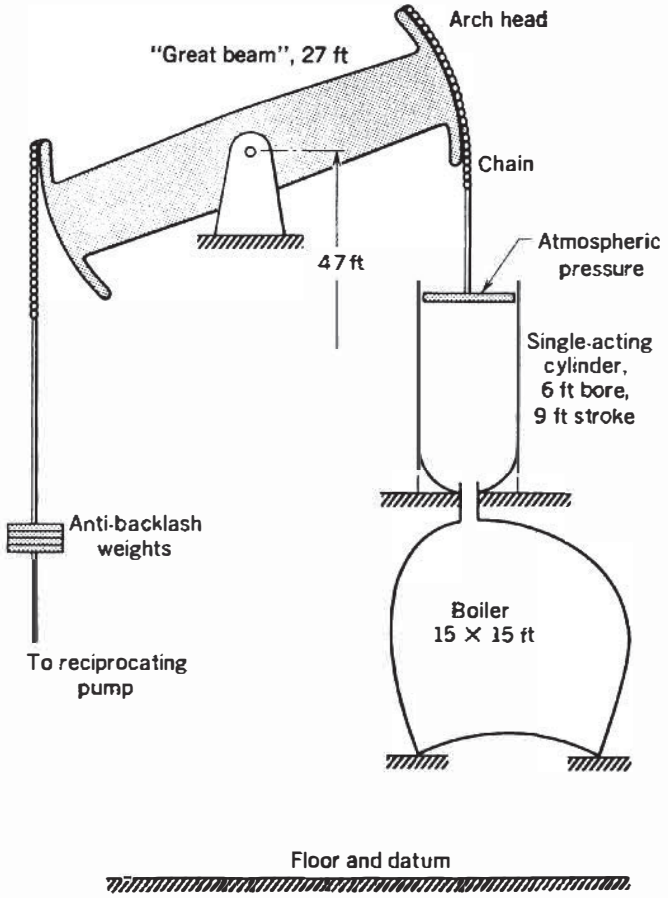


FIGURE 1-8 Schematic of a Newcomen-type atmospheric engine circa 1775, just before Watt.

The "great beam" (lever), arch heads, and chain functioned as a simple pulley or direction changer of applied force. With the piston at the top of the cylinder (having gotten there because of the weights on the pump side), steam was admitted and then condensed by a water spray. In consequence of the subatmospheric pressure in the cylinder the piston was forced down. Steam was not used to force the piston up.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 28 of 143

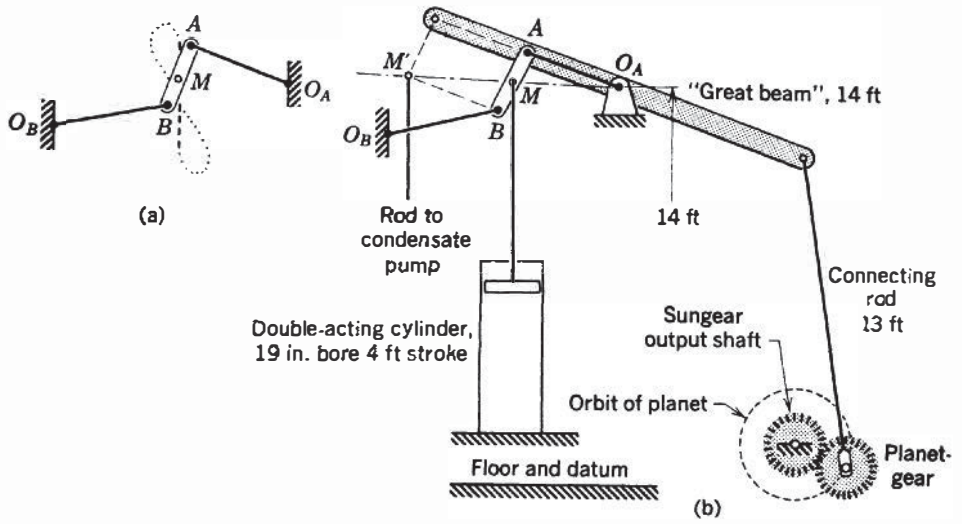


FIGURE 1-9 Schematic of Watt's rotative engine, 1784, the first engine to produce power directly on a shaft without the intervention of a water wheel fed by a reciprocating pumping engine.

At (*a*) is shown the kinematic skeleton of a four-bar linkage with coupler AB. The coupler point M describes the figure-eight-shaped coupler-point curve, possessing two segments that are nearly straight lines; the vertical segment was used for piston-rod guidance. In (*b*) the linkage is shown as part of the engine. By means of a pantograph, another point M' was given a motion similar to that of M.

> been neglected, or have formed only the subject of memoirs or special essays. This science ought to include all that can be said with respect to motion in its different kinds, independently of the forces by which it is produced. . . . It should treat in the first place of spaces passed over, and of times employed in different motions, and of the determination of velocities according to the different relations which may exist between those spaces and times.

Euler's earlier idea thus received identity as well as reiteration: kinematics was the purely geometric science of motion in the abstract. As such, kinematics was to develop in large and significant part separately from mechanics and applications of mechanics. Many mathematical contributions were reduced to mechanisms appreciably later. Forces were left as complications to be dealt with separately; we still do this under the title of kinetics of machinery.

Ampère defined the path but left the milestones to be set by others. Among the first were Chasles[b] and Poinsot.[b] They considered geometrical bodies—not points—adding method and ideas to the concept of the instantaneous center discovered by Johann Bernoulli in 1742.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 29 of 143

Across the Channel, Willis'[b] text, "Principles of Mechanism" (1841), marked a departure from the tradition of the descriptive past. Willis made a substantial contribution to the cause of synthesis—we see this now—by turning to classification. It was his idea that there should be a systematic approach to the design of mechanisms for a given purpose. He sought to augment empiricism—"that species of intuition which long familiarity with a subject usually confers. . . ."

Willis proposed that "the motions of the machine are the principal subject of contemplation"—an obvious acceptance of Ampère's views on the matter. In considering logical order, Willis' ideas were different from those of Monge and Hachette. He considered "that all the simple combinations of mechanism," i.e., from input to output, "may be distributed into three classes":

Class A directional relation and velocity ratio constant
Class B directional relation constant—velocity ratio varying
Class C directional relation changing periodically—velocity ratio either constant or varying

He counted five ways of achieving the relative motion between input and output links—rolling contact, sliding contact, wrapping connectors (belts, chains), linkwork, and reduplication (tackle of all sorts). His work had a considerable influence, for in the nearly thirty years between the first and second editions of the "Principles," the classification, nomenclature, and figures were adopted by many authors.

Willis rejected Prony's analysis of the Watt curve, saying "the algebraical equation . . . is exceedingly involved and complicated, and of no use in obtaining the required practical results, which are readily deduced by simple approximate methods, ..." He did, however, apply trigonometry to the Watt mechanism and some of its variations, but without the generalization of Prony. The (Richard) Roberts[b] four-bar linkage was likewise examined for proportions. We note, parenthetically, that this was probably the first mention of this linkage, whose actual time of origin and point of use are still to be discovered.

The second edition of Willis (1870) was substantially larger than the first. The essential principles remained unchanged, and differential calculus was introduced sparingly, although the subject of acceleration was left untouched. Older topics were enlarged upon, as, for example, the Hooke universal joint. The implications of coupler curves escaped Willis, for although he had a device capable of drawing such curves prior to 1851,[1] they receive no further comments. The Peaucellier true straight-line mechanism (1864), a linkage that caused much excitement among mathematicians, seems to have escaped Willis' notice. But the influence of Willis continued, as in the reflection by Woods and Stahl,

[1] See Fig. 3-2.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 30 of 143

"Elementary Mechanism" (1885), with a sixteenth edition in 1909, and Robinson's "Principles of Mechanism" (1896).

Turning to one of the classics of physics, we note that the first volume of the Thomson and Tait "Treatise on Natural Philosophy" (1867) devoted some 160 pages to kinematics, following the sense of Ampère, i.e., dealing with motion as totally independent of matter and force.

Rankine's[5] "A Manual of Machinery and Millwork" (1869) devoted 314 of its 578 pages to the "Geometry of Machinery, or Pure Mechanism." There is neither a general theory of machines nor a new systematic treatment of their motions, for Willis is followed more or less. The subject of pure mechanism appears as a great series of interesting but nonrelated problems. Rankine dealt with the motion of machine parts by using instantaneous centers of velocity, evolving effective methods for solving special problems.

Modern kinematics had its beginning with Reuleaux. His now classical "Theoretische Kinematik" of 1875 presented many views finding general acceptance then that are current still. They came into English with Kennedy's[6] able translation, called "Kinematics of Machines"[1] (1876). Kennedy's own book, "Mechanics of Machinery," published ten years later, although dealing with topics beyond only kinematics, made use of Reuleaux's concepts. Reuleaux's second book, "Lehrbuch der Kinematik" (1900), consolidated and extended earlier notions. This book, never translated into English, also made a lasting impression.

Reuleaux's comprehensive and orderly views mark a high point in the development of kinematics. He gave us the idea that the constrained motion of a mechanism or the kind of relative motion between parts is controlled by the form of the surfaces of contact of the adjacent parts. We may think of them as "working" surfaces of the connection. Thus, the connection between a lathe carriage and its bed is by means of working surfaces (ways) so shaped that only translational motion is possible. Each of the working surfaces he called an element. The two contacting elements of a connection constitute a kinematic pair, the properties common to different kinds of pairs allowing a certain unity of treatment.

In the matter of connections between rigid bodies, Reuleaux recognized two kinds; he called them higher and lower pairs (of elements). With higher pairs, the two elements are in contact at a point or along a line, as in a ball bearing or disk cam and follower; the relative motions of coincident points are dissimilar. Lower pairs are those for which area contact may be visualized, as in pin connections, crossheads, ball-and-

[1] Long unavailable, this classic has been reprinted by Dover Publications, Inc., New York.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 31 of 143

socket joints, and some others; the relative motions of coincident points of the elements, and hence of their links, are similar, and an exchange of elements from one link to the other does not alter the relative motion of the parts as it would with higher pairs.

The far-reaching impact of Reuleaux's classification system, in which mechanical and hydraulic devices are arranged in the six classes listed earlier, is reflected by the chapter headings of modern books on kinematics and machine design.

Reuleaux regarded a mechanism as a (kinematic) chain of connected links (or parts), one link being fixed. From this came the important proposition of the kinematic inversion of the chain, viz., that a different mechanism results every time a different link of the chain is selected as the fixed link. He also recognized the idea of the expansion of turning pairs.

Through hindsight we saw in Watt's straight-line linkage the amorphous beginning of an ordered and advanced synthetic process. Nearly a century later Reuleaux identified synthesis as a concept, as an entity that might be and must be pursued to guide the designer through

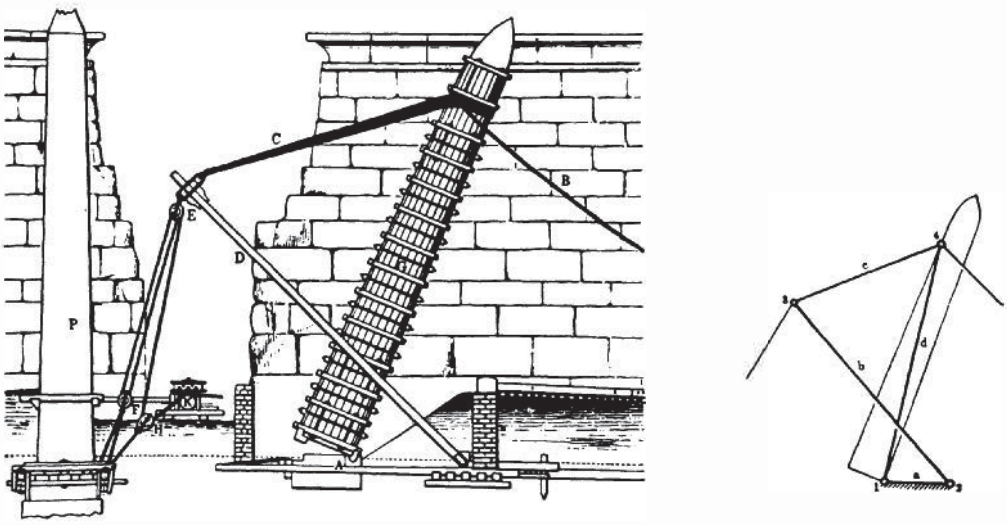


FIGURE 1-10 Lowering the obelisk of Luxor, 1831. (*From Reuleaux's "Lehrbuch der Kinematik," Brunswick,* 1900.)

This obelisk, weighing 284 tons, now stands in the Place de la Concorde in Paris, having been moved from Luxor. Like the larger Vatican obelisk, the shaft was prepared for handling by encasing it in timber. Unlike its Italian counterpart, the Luxor obelisk was tipped off its base directly, Fontana's vertical lift being unnecessary because of the kinematic arrangement—a four-bar linkage achieved with the framework *ED* (or link *b*) devised by the engineer Mimerel. The maximum load coming onto the tackle was but 43 per cent of the obelisk's weight. This rig required only 30 men and $\frac{1}{2}$ hr to lay the obelisk down.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 32 of 143

the maze of mechanisms. His views were limited to only *type* synthesis: by this is meant the determination of the type of mechanism for a given job, e.g., reaching a decision about whether to use a linkageeor a cam to produce a desired motion. He clearly defined this area, which lies ahead of the related and more recent fields of *number* and *dimensional* synthesis. The latter is the main topic of this book.

One more contribution of Reuleaux's must be mentioned, and that is the symbolic notation with which he attempted to describe kinematic chains by means of symbols instead of pictures. His high hope that the notation would be useful in discovering new mechanisms was borne out for but few situations. Incomplete and thus not wholly satisfactory, it was able to show that many mechanisms of differing physical appearance were nevertheless kinematically identical. Recent extensions of Reuleaux's kinematic notation do furnish a symbolic notation useful for a large number of problems, as will be indicated. A completely general and all-inclusive notation is still to be developed.

Geometers and algebraicists of the 1870s became interested in linkages as curve-drawing devices, not as hardware; their work has since been made part of the corpus of the kinematics of mechanisms. It was discovered that a link motion can be found to describe an algebraic curve of any order. In particular, the coupler-point motion of the four-bar linkage (then called a "three-bar motion") was studied. Samuel Roberts[5] showed that a coupler point describes a curve of the sixth order. A theorem important to synthesis and bearing the name of Roberts states the existence of three different four-bar linkages capable of drawing identical coupler curves. It is from this theorem that the relation of the Watt linkage to other approximate straight-line mechanisms is made clear. Only recently have the higher-order plane curves derivable from linkages more complex than the four-bar been examined with kinematic intent.

The period of the 1870s and 1880s was marked by substantial steps along the roads of analysis and synthesis. The former unmethodical (but hardly aimless) gropings gave way to the devising of radical methods, bringing basic premises into clear view. Thus, Aronhold in Germany and Kennedy in England greatly extended the use of instantaneous centers of velocity by recognizing the theorem of three centers. Acceleration analysis received potential aid. Kennedy, who reflected British practice, determined accelerations by graphical differentiation of velocities calculated from instant centers. In the preface of his own book (1886), he regretted not being able "to make any use . . . of the recently published graphic methods of Professor R. H. Smith, or the still more recently published (in any complete form) kinematic work of Professor Burmester . . ." because of the pressure of work and ill health.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 33 of 143

This "graphic method" of Smith's,[b] a very real break-through available in English, was presented in 1885 in a paper before the Royal Society of Edinburgh. An English text by Goodman[1] has this to say on page 142:

> Force and reciprocal diagrams are in common use by engineers for finding the forces acting on various members of a structure, but it is rare to find such diagrams used for finding the velocities of points and bars in mechanisms. We are indebted to Professor R. H. Smith for the method. (For fuller details, readers should refer to his own treatise on the subject.)

This reference is to Smith's "Graphics," published in 1889. The remarks by Kennedy and Goodman pertain to the method of velocity and acceleration polygons (or diagrams), today sometimes erroneously called the method of relative velocities (or accelerations). Smith also dwelt on the utility of the velocity and acceleration images. The great usefulness of the velocity and acceleration polygons (and the images) lies in the fact that even complicated mechanisms may be studied by simple, graphical means; analytical methods applied to such mechanisms may produce equations unsuited to hand calculations.

Burmester,[b] Mehmke, and Mohr had already introduced these concepts in Germany,[2] but Smith's was an independent, almost simultaneous discovery, available to an unresponsive English-speaking world. Regrettably, Smith took his discovery no further. In fact, Kennedy and Smith, both destined to live many years, applied their energies to tasks other than kinematic ones. And neither did their countrymen pursue Smith's fundamental idea. We note that Goodman referred only to velocity diagrams, and these he used sparingly. For acceleration studies—such as they were—he continued with the graphical differentiation which Smith had called "practically useless." The Germans, on the other hand, equally oblivious of Smith (he is not mentioned in their literature), continued to build upon the Burmester-Mehmke-Mohr foundation to fashion a very solid and distinguished edifice.

Activity in America was static, for the work was principally based on Kennedy and even older authors such as Rankine, with some lip service to Reuleaux in a vague way. A "mechanisms" course was tantamount to further exercise in mechanical drawing, for there was little

[1] "Mechanics Applied to Engineering," vol. I, 9th ed., 1926, reprinted 1943, 59 pages on mechanisms.

[2] We do not mean to belittle the German work by not dwelling on it; it is simply that the English effort is so little known or appreciated that we feel it should receive attention here.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 34 of 143

in the way of kinematic analysis. Indeed, the teaching of acceleration analysis in the United States may be said to have waited until the 1930s. This is very curious, for since the tools were available in the English language, the language barrier cannot be invoked as the deterrent. An accounting was called for in 1942 by Professor De Jonge;[1] his critical remarks set the stage for the new interest and effort that were to develop in America.

We have seen Reuleaux's early concern with synthesis. His work, sometimes the subject of bitter arguments, was not forgotten, and the Germans subjected the kinematics of mechanisms to much study. Concepts far beyond those of Reuleaux were added to the picture he had attempted to paint; the whole is now called synthesis of linkages.

The spirit of the word synthesis implies that it is the combining of parts to produce a whole; it is design, the creation of something new. Kinematically, it is the conversion of a motion idea into hardware. In this sense, synthesis is the opposite of analysis, in which existing mechanisms are examined for what they do or can cause to happen. However, analysis and synthesis, although opposites in the broad sense, are closely related, for the observations of analysis often serve as springboards for synthesis.

The whole problem of synthesis is taken to consist of three facets. For purposes of discussion they may be separated, even though there are varying degrees of interrelation. In the course of a synthesis or a design of a motion-converting device or mechanism, it is necessary to reach three fundamental decisions, each known as a kind of synthesis.

The first is *type* synthesis. Here the type of mechanism—cam, linkage, elliptical gear train, and so on—is decided upon. It is with this phase of synthesis that the name of Reuleaux is peculiarly associated.

The second is *number* synthesis. In this the requisite numbers of links and lower or higher pairs yielding constrained motion are established. This is a case of guaranteeing that a mechanism has just the right number of parts properly connected to ensure that it will "run" as desired. Criteria of movability were presented by Grübler,[b] who dealt with planar situations in 1883 and 1885, coming to the spatial chains in 1917.

The third and last kind is *dimensional* synthesis, or the determination of the proportions (lengths) of the links needed to accomplish the specified motion transformation. In Germany, Burmester was in accord with Reuleaux's fundamental concepts and most of his nomenclature and definitions. Making extensive use of mathematical principles (mostly geometrical), and considering displacement, velocity, and

[1] A. E. R. de Jonge, What Is Wrong with "Kinematics" and "Mechanisms"? *Mech. Eng.*, vol. 64, pp. 273–278, 747–751, April, 1942.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 35 of 143

acceleration, Burmester's "Lehrbuch der Kinematik" (1888) developed geometric methods that furthered analysis and showed the way to synthesis. The Russian mathematician Chebyshev[5] made signal contributions to synthesis, one of which is recognized by his sharing the theorem of the triple generation of coupler curves with Samuel Roberts; another is the four-bar linkage bearing his name. It is interesting to note that the widely used Chebyshev polynomials for approximating a desired curve were developed from a concern with the Watt linkage.

Our review has come to the end of the nineteenth century. This marks a logical terminus for our survey: it is the end of the formative period, of an era in which many principles had been uncovered, analytical methods established, and the road to synthesis opened. Subsequent growth of these ideas, no longer nurtured only in Europe, has been of such scope that no short overview is possible. Indeed, part of the mission of this book is to discuss certain of the advances, especially those relating to synthesis.

The full story of the kinematics of mechanisms, doing justice to the many who practiced the art of mechanisms and contributed to the science of kinematics, is yet to be written.

Provocative and illuminating studies, supported by extensive bibliographies, have been made by De Jonge,[1] Freudenstein,[2] and Ferguson.[3]

BIOGRAPHICAL NOTES

André Marie *Ampère* (1775–1836), professor of mathematical analysis and mechanics at the École Polytechnique, gained distinction in establishing the relation between electricity and magnetism. His mathematical theory not only explained existing phenomena but also predicted many new ones. A theorist who was also an experimentalist, Ampère was honored in having the unit of electric current strength named after himself.

Ludwig *Burmester* (1840–1927), professor of descriptive geometry and kinematics at Munich, wrote on perspective, optics, groundwater flow, and other topics that lent themselves to geometric investigation. His extraordinary "Lehrbuch der Kinematik" (1888) went to almost a thousand pages to form a systematic and comprehensive treatment of theoretical kinematics. It correlated

[1] A. E. R. de Jonge, A Brief Account of Modern Kinematics, *Trans. ASME*, vol. 65, pp. 663–683, August, 1943n

[2] F. Freudenstein, Trends in the Kinematics of Mechanisms, *Appl. Mech. Revs.*, vol. 12, pp. 587–590, September, 1959.

[3] E. S. Ferguson, Kinematics of Mechanisms from the Time of Watt, *U.S. Natl. Museum Bull.* 228, *Paper* 27, pp. 185–230, 1962 (obtainable from the Superintendent of Documents, Government Printing Office, Washington, D.C.).

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 36 of 143

the known mathematical methods with Burmester's own brilliant additions, and is still a classic.

The father of the thermodynamicist Sadi, L. N. M. *Carnot* (1753–1823) was a figure in the French Revolution: he performed the gigantic task of raising 14 armies when the power of united Europe was launched against France in 1793. Carnot aided Monge in the formation of the École Polytechnique and was a noteworthy contributor to mechanics, the art of fortification, and mathematics, especially calculus and projective geometry.

Both Michel *Chasles* (1793–1880) and Louis *Poinsot* (1777–1859) were graduates of the École Polytechnique. They shared with Moebius the most important nineteenth-century advances made in geometrical mechanics. Poinsot was the first to introduce the idea of a couple into mechanics. Chasles's history of geometry is still a standard work. A collector of autographs, Chasles achieved another distinction by buying over 27,000 "original letters"—among them 622 "written" by Newton and one from Cleopatra to Julius Caesar—from a clever forger before haling him into court.

Pafnutij *Chebyshev* (1821–1894), professor of mathematics at St. Petersburg, was one of the most important mathematicians of the nineteenth century. Within a wide range of interests he made significant contributions to the theories of probability, congruences and integrals, quadratic forms, prime numbers, the construction of maps, and other topics. Chebyshev's consuming interest in kinematics produced numerous developments in linkages, especially in their synthesis: he is revered in Russia as the founder of modern kinematics.

(Variations in the transliteration of the name Chebyshev stem from the nationality of the translator: one finds Tschebytschew, Tchebycheff, and Chebyshev, among others. We use the Library of Congress version.)

Leonhard *Euler* (1707–1783), versatile Swiss genius and favorite pupil of Johann Bernoulli, received when nineteen a prize from the Paris Academy of Sciences for a dissertation on the masting of ships. His astonishingly productive life was spent abroad as professor of mathematics at the Academies of Science in Russia (1727–1741 and 1766–1783) and Berlin (1741–1766). He lost the sight of one eye at the age of twenty-eight, becoming totally blind 17 years before death.

Euler left his mark upon a great many physical and mathematical problems. He greatly enriched mathematics in an endless number of ways, made an extensive study of optics, and, living in a period when theology and physics were still greatly intermingled, indulged in metaphysical and philosophical speculations. Our great debt to Euler lies in his treatment of mechanics: he laid the foundations (built upon so brilliantly by Lagrange) of analytical ("algebraical") mechanics. Newton's mechanics are purely geometrical, theorems being deduced from initial assumptions by geometrical methods; Euler applied algebra to geometry, producing the conditions of a proposition or figure.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 37 of 143

Martin *Grübler* (1851–1935) taught at the technical universities of Zurich, Riga, and Berlin before succeeding O. Mohr in the chair of technical mechanics at Dresden in 1900. His interests in mechanics were varied, and included, in addition to kinematics, stress analysis of rotating disks, turbine theory, structures, and aerodynamics.

Hero of Alexandria is placed in the first century A.D. because of his mention of an astronomical event now calculated to have occurred in A.D. 62. Hero's works became known to Europeans through an Arabian translation that transmitted illustrations. Much of what Hero reported on was appreciably older work, and there is no telling which parts of the works are his. It is supposed that Hero was probably an Egyptian because of his strong bias toward application, although his mathematical powers were considerable.

Alexander Blackie William *Kennedy* (1847–1928) was an engineer of many talents. A graduate in mining, he designed compound steam engines for early twin-screw ships and taught mechanics at a number of institutions before being appointed professor of engineering at University College (London) in 1874, whose first engineering laboratory he established. With respect to the theorem of three instantaneous centers he recognized Aronhold's priority but claimed independent discovery. He was knighted in 1905 and became a consulting electrical engineer, holding important government positions during World War I. Interests different from engineering are reflected in his "Ypres to Verdun" (1921) and "Petra: Its History and Monuments" (1925). The latter, a definitive monograph on an ancient caravan trade center important in the time of Christ, stemmed from explorations undertaken at the age of seventy-five.

Jacob *Leupold* (1674–1727) wrote a nine-volume series of books which constituted a "display of the mechanical sciences" and which were published during the period 1724–1739. In all, they contained some 492 copper engravings detailing the early eighteenth-century practice in machines for many purposes, instruments, water supply, and bridge building. As design texts, these books were directed at "artists, craftsmen and such people who have no knowledge of languages or other subjects, and therefore have no opportunity to . . . search out . . . the information from many sources . . . needed to maintain, build and use such machines" (as those displayed).

Gaspard *Monge* (1746–1818) is best known to engineers as the inventor of descriptive geometry. Not an engineer, but an inspiring teacher well acquainted with the engineering science and craftsmanship of his time, Monge developed a rational system of drawing, to aid in the planning of fortifications and the setting out of stonecutting and carpentry, that was treated as a military secret for nearly thirty years, until 1794. Monge was one of the principal founders of the École Polytechnique, the first polytechnical school in Europe, and he included Lagrange and Fourier among its teachers. For a short time minister of marine under Napoleon I, Monge was the only supporter of Robert Fulton's submarine venture. Together with Berthollet and Fourier, Monge directed the geodesic survey and

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 38 of 143

description of Egypt when attached to Napoleon's Egyptian expedition. He contributed much to the superior technical education that enabled the France of after the Revolution to catch up with Britain in industry. As an analyst, Monge belonged to the bright period of French mathematics further illuminated by Laplace and Legendre.

William John Macquorn *Rankine* (1820–1872), educated at Edinburgh as a physicist, occupied the chair of civil engineering and mechanics at the University of Glasgow. One of the pioneers of modern thermodynamics, Rankine wrote the first formal treatise on the subject. Rankine's short life was marked by prodigious energy; his works dealt with civil engineering, shipbuilding, machinery and millwork, steam engines, and applied mechanics. His "Applied Mechanics" was the first text in English by that name; it had a useful life of half a century and is still worth looking at if for no other reason than the philosophy of the introduction, which someone has condensed to "Theory is the distilled essence of practice." Rankine presented the theory of his day in such a way that it became practice.

Franz *Reuleaux* (1829–1905), German-born, was named professor at the Polytechnicum in Zurich in 1856. Here he developed machine design and, with perceptive awareness, defined concepts fundamental to a study of kinematics. From 1864 until his retirement in 1895 he was in Berlin: he helped establish the Technische Hochschule-Charlottenburg while making significant contributions in all areas of mechanical engineering, the emphasis of his efforts being laid on design work and kinematics. Reuleaux traveled widely as his government's representative to five world exhibitions during the period 1867–1881. His sharp criticisms of the poor quality of his country's industrial products shook the German economy. The creation of a uniform patent law (1877) for all the 21 states comprising the German Empire was the result of his efforts. He has been characterized as the last "poly"-technologist because of inspirational pioneering in many fields: he led in replacing empiricism with rationalism.

Richard *Roberts* (1789–1864), engineer of Manchester, began his career as a patternmaker under John Wilkinson, the ironmaster whose cylinder boring mill was so essential to the success of the Watt steam engine. Roberts was responsible for many innovations and inventions; among the latter are the back-geared lathe and planing mill (1817). Possessed of no worldly prudence and little business acumen, he died in financial straits despite his many successes.

Samuel *Roberts* (1827–1913), F.R.S., was a barrister who turned to mathematics and thus enriched the study of algebraic curves. He belonged to the generation of mathematicians who investigated "link motion"; their number included Arthur Cayley, A. B. Kempe, Linguine, and Koenigs. Roberts, fond of chess, angling, and philosophy, indulged in lathe work (a fashionable hobby then called "turning") and liked to construct electrical machinery and other pieces of scientific apparatus. He is not to be confused with the earlier Roberts, namely, Richard, of Manchester.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 39 of 143

Robert Henry *Smith* (1825–1916), born and educated in Edinburgh, assumed the post of organizer and professor at the Imperial University of Japan at the age of twenty-three. He was one of the group of foreign scholars of the 1870s whose mission was to introduce modern science and establish an educational system. His contributions are still warmly remembered by the Japanese. In 1880 he became the first professor of engineering (civil, mechanical, and electrical) at Mason College, the earliest antecedent of the present Birmingham University.

Pollio Marcus *Vitruvius* lived in the first century B.C., functioning as architect and engineer under Octavianus, the successor to Julius Caesar. Vitruvius' fame rests on a single book, "De Architectura," a comprehensive treatise important to the history of art and technology because of its reporting of classical antiquity. Architecture, to Vitruvius, included astronomy, horology, and mechanical and military engineering.

With the decline of Roman power, a Latin book was for centuries of no interest to comatose Europe. "De Architectura" was also neglected by the Arabs. Slightly known to the medievalists, it was lifted to glory by Renaissance scholars. A first printed edition appeared in 1486; later editors, starting in 1511, added their ideas of illustrations, the original figures not having survived even in manuscript form.

James *Watt* (1736–1819), F.R.S., sometime "mathematical instrument maker to the University" of Glasgow, led an uncertain life in his earlier years, adding to his income by making surveys and reports in connection with canal, river, and harbor works. The separate condenser, patented in 1769 and added to current steam engines, more than doubled the thermal efficiency; this, with other improvements, made power available to the drowned mines of coal-scarce regions. The windmill flyball governor (for regulating the gap between millstones) was adapted by Watt as an engine speed regulator, giving the first closed-loop servomechanism. Watt put numbers to the concept of horsepower and invented the engine indicator and a letter-copying device based on the ink-transfer process. His insistence on the use of the slide rule in his design rooms did much to popularize what for a long time was known as the "Soho sliding rule."

Robert *Willis* (1800–1875), engineer and archaeologist, was an ordained deacon and priest, as were most of his academic colleagues. He served as Jacksonian professor of natural and experimental philosophy (physics) at the University of Cambridge from 1837 until his death. Willis' "Principles of Mechanism" (1841) was the most complete work on the subject which had yet appeared: he was the first to present gear theory as part of a systematic treatment, advocated the involute form with a pressure angle of $14\frac{1}{2}°$, and invented and named the first odontograph.

Elected Fellow of the Royal Society for work done in physiology (mechanism of the larynx, vowel sounds), the gifted Willis continued and extended his Cambridge predecessor's lectures on "mechanical philosophy."

Willis' reputation as an engineer continued to grow, and touched upon other fields, among them structures and astronomy. He accomplished much

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 40 of 143

original research in architectural history, becoming an authority on medieval Latin and the construction of ancient buildings.

BIBLIOGRAPHY

Armytage, W. H. G.: "A Social History of Engineering," Faber & Faber, Ltd., London, 1961.

Derry, T. K., and Trevor I. Williams: "A Short History of Technology," Oxford University Press, New York, 1960.

Finch, James Kip: "The Story of Engineering," Anchor Book A 214, Doubleday & Company, Inc., Garden City, New York, 1960.

Usher, Abbott Payson: "A History of Mechanical Inventions," rev. ed., Harvard University Press, Cambridge, Mass., 1954.

White, Lynn, Jr.: "Medieval Technology and Social Change," Oxford University Press, New York, 1962.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 41 of 143

2 CONCEPTS AND NOTATIONS RELATED TO MECHANISMS

2-1 MECHANISMS

A mechanism is a device to transform one motion into another. If the device also transmits substantial forces, it is a machine, which means that all machines are mechanisms in spirit. If forces are associated with the conversion of the energy of high-temperature fluids (as steam or gas) to shaft power, then the aggregate *may* be called an engine. At any rate, it is recognized that the parts comprising the device—mechanism, machine, or engine—must be resistant to deformation, i.e., the parts must approximate rigid bodies. We may then say that a mechanism is an assemblage of rigid or resistant bodies connected together for the purpose of transforming motion.

Two general groups of mechanisms exist, since the motion may go from either uniform to uniform or from uniform to nonuniform. Circular gears, chains, belts, and the like, comprise most of the uniform motion converters; their many problems will not be discussed in this book. Nonuniform conversions are made with noncircular gears, cams, ratchets, and linkages, both planar

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 42 of 143

and spatial. This book will direct itself exclusively to the design of linkages to meet certain motion-conversion specifications.

2-2 CLASSIFICATION SYSTEMS

Classification implies some sort of systematic grouping of factors that will show the relation of one thing to a group of things: it is a search for order. A complete system of mechanisms would show the genealogy of motion; i.e., it would show relations and lines of common descent when they exist. Individual mechanisms would no longer be separate mysteries; each would be part of a larger group having somewhat similar characteristics modified by personal idiosyncrasies. Furthermore, an important property of a classification system would be the aid it could furnish a designer in finding the forms and arrangements best suited to satisfying certain specifications. Thus far no completely unified and general classification scheme for all mechanisms has been found, although several attempts have been made.

In the work to date, two groups may be recognized: we distinguish between the functional and the structural. The functional classification system considers the *complete* mechanism needed to transform a given motion into another, as the conversion of uniform rotation into reciprocation. The complete mechanism is like a "black box," with provision for accepting one kind of motion at the input and producing another kind of motion at the output. Since it is in the nature of things to be able to accomplish a given task in more than one way, there would be a number of black boxes to choose from—their "insides," or "works," availability, and cost would be different, but the overall effect, i.e., motion transformation, would be the same. Monge's scheme belongs to this group. The shortcoming here is that no general principles exist to guide the problem of transforming a given motion into another. All that can be done is to run down a list of complete mechanisms that will do the specified job and choose from among the several the one that best fits additional specifications of available space and manufacture. There is really no unity, and the system is a mere collection of mechanisms.

The second, or structural, classification system deals with the nature of the *parts*, considering them from the standpoint of their relative motions. Willis followed this idea by regarding how the motion transformation between input and output members was achieved—as by rolling or sliding contacts, flexible connectors, barlike links, or tackle. Reuleaux was much more intimate: he considered, not the big input-output span, but only the immediate *connection between parts*. Here the shapes of the surfaces in contact—the working surfaces—impose particular and unique motion restrictions; i.e., they allow only a particular

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 43 of 143

type of relative motion, as rotation about a single axis, rectilinear translation, rotation about a point, etc.

Even though none of the known classification systems answers all needs, each system has some merit that can be exploited without profit for particular areas of interest.

2-3 RIGID AND RESISTANT BODIES

A *rigid body* is a material body in which the distance between any two points is invariable. Thus, if A, B, and C are three points of a rigid body (Fig. 2-1), the three distances AB, BC, AC remain constant no matter how the rigid body is moved as from position 1 to position 2. If three noncollinear points A, B, C on a rigid body are known, any other point D may be identified by specifying the three distances DA, DB, DC. Even if the point D lies outside the apparent boundaries of the rigid body, it may still be considered as a point of the body as long as the three distances DA, DB, and DC are invariable: when the rigid body moves from position 1 to position 2, point D moves with it.

When a rigid body is moved from an initial position 1 to a final position 2, it is said to have been given a displacement. As in the case of a point, the displacement of a rigid body depends only on the initial and final positions and is independent of the way in which it is carried out; i.e., the actual path followed during the displacement is of no consequence. As a rigid body moves from an initial to a final position, each of its points traces a particular path; at each instant of time a point has a particular velocity and a particular acceleration.

To be honest about it, there is no such thing as a rigid body, since all known materials deform somewhat under any stress no matter how

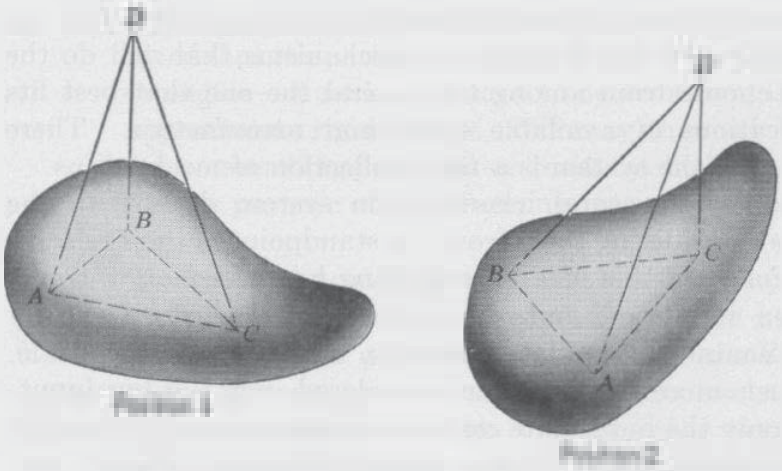


FIGURE 2-1 Two positions of a rigid body.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 44 of 143

small. If we assume, as we must at least in the beginning, that the geometry of the body does not change as it moves—that any two points always bear the same relationship to each other—then we use "rigid" in the mathematician's sense, overlooking the inevitable small deformations that are held to acceptable values by proper cross sections of the bodies. Realistically, we might better speak of resistant bodies, bodies so proportioned that their deformations are acceptably small. Under resistant bodies we may also include belts, chains, and hydraulic lines, for these have a one-way resistance to deformation. As a matter of fact, elastic deformations may become troublesome when machinery operates at high speed, but they cannot be established until the kinematic work is finished and the parts have acquired mass by having been given physical shapes and cross sections.

2-4 MOTION—RELATIVE AND ABSOLUTE

The very fact that motion exists implies reference frames of some sort on the moving parts. If the reference frame of one machine part moves with respect to the reference frame of another, we speak of relative motion. When the reference frame of one part is fixed with respect to the earth and the motions of the other parts are referred to it, then these particular relative motions are termed absolute motions: absolute motion is thus a special case of relative motion, the case in which the absolute-motion reference frame has no motion.[1] Of course, any rigid body may be chosen as a reference to which the motion of other bodies is referred; the operation of an aircraft landing gear is better referred to the fuselage than to the distant earth.

2-5 CONNECTIONS, AND THE PARTICULAR MOTIONS THEY PERMIT

A mechanism has been defined as a number of rigid bodies so connected that each moves with respect to another. The clue to the nature of a mechanism lies in how the parts are connected and what kind of relative motion the connection allows. In kinematics, a connection is a joint between two members permitting a particular kind of motion. We should note that the term connection, in fields different from kinematics, may mean an immovable connection, as a structural joint or splice, shrink fit, and the like.

Considerations based on the relative motions permitted by various mechanical connectors lead to a recognition of three broad classes of

[1] That the earth itself is in motion around the sun has no bearing on any problems to be discussed here.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 45 of 143

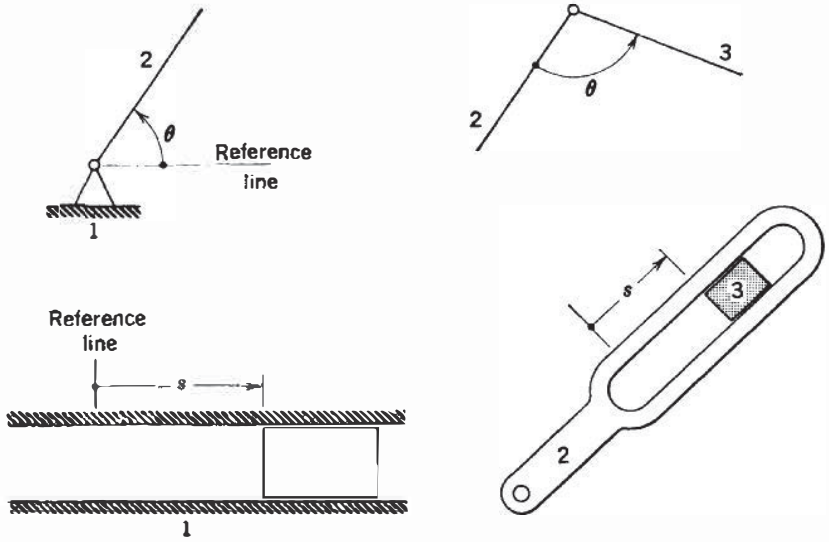


FIGURE 2-2 Conventions for showing pair variables.

connectors. These are the lower-pair connectors, the higher-pair connectors, and the wrapping connectors. The last are comprised of belts and chains with their one-way rigidity and will not be discussed here.

2-6 LOWER-PAIR CONNECTORS

Consider two links in parallel planes, connected in such a way that one may turn with respect to the other. The upper sketches in Fig. 2-2 show the two links in a conventional way: the small circle denotes the ability to turn, and the center of the circle represents the axis of rotation about which the angle of rotation is measured. Nothing has been indicated about the physical make-up of the joint: it could be made in many ways, as using a simple pin, a ball or roller bearing, etc. However, no matter how constructed, an angle such as θ is always the definitive variable of motion and will be called the motion variable of such a joint.

Another situation is that of a crosshead in its guide. Here only (rectilinear) translation is possible, and the relative motion is described by means of a linear displacement such as s measured from some convenient origin. The actual form of the crosshead and guide sections, as laid out by the machine designer, may take any of a large number of shapes, having in common the property of restricting the relative motion to translation only. No matter what the form, the relative motion is described by a single variable such as s.

With the intuitive background of these two examples we may make a systematic approach to the problems of movable connections. Consider a set of x, y, and z axes fixed to ground link 1 (Fig. 2-3). Link 2,

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 46 of 143

which is somehow suspended in space, also carries a set of rectangular axes u, v, and w, with origin at A. The two sets of axes are parallel. If link 2 is moved to any other location, we could describe this by saying that A's new location is at x_2, y_2, z_2: each axial displacement of A, that is, $x_2 - x_1 = \Delta x$, $y_2 - y_1 = \Delta y$, and $z_2 - z_1 = \Delta z$, represents a motion of translation. The vector sum of these displacements, $\Delta x + \Delta y + \Delta z = \Delta s$, is the real displacement of A. It is convenient to keep track of the motion of A by means of three orthogonal motions (translations), which may be taken in any order.

In addition to the motion *of* A, there may also be a motion *about* A, as when link 2 has turned about the point A. This turning is conveniently described by separate rotations about each of the axes u, v, and w, or, what amounts to the same thing, about the x, y, and z axes. The sequence of rotations is important for spatial mechanisms. With regard to the motion *about* A, this may come either before or after the consideration of the motion *of* A; the order of these two operations is immaterial.

We see from the foregoing that completely to define the position of a link requires the knowledge of six variables, three giving the translation of a point, and three giving the rotation of the link about that point. Each of these motion variables is also said to be associated with a degree of freedom, i.e., each is identified with a motion of either translation or rotation. To form a clearer picture of all this, we shall examine in detail the possible motions of link 2 with respect to link 1 by observing the nature of the connection:

1. If link 2 is permitted only a rotation about its w axis (the other

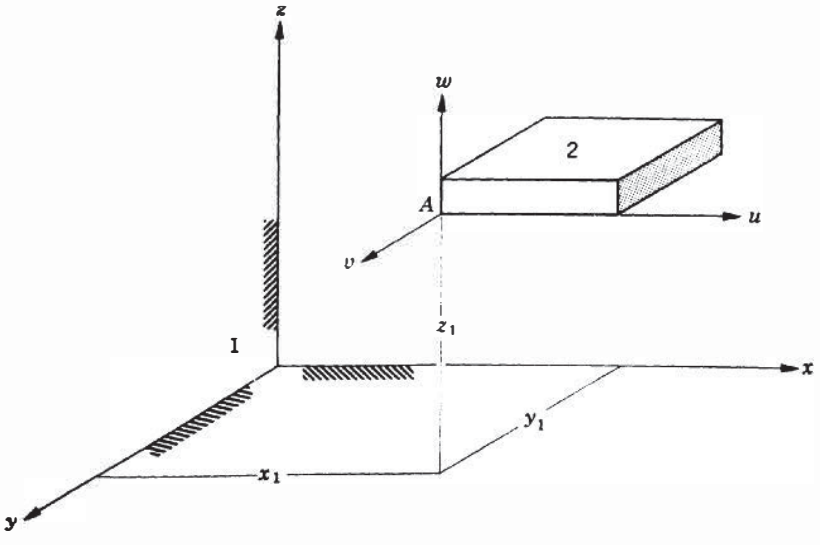


FIGURE 2-3 Reference frames for moving link.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 47 of 143

five possible motions being suppressed), a variable sufficient to describe the relative motion is an angle θ, measured in a plane perpendicular to the w axis. Following Reuleaux, we designate this connection as a revolute and give it the symbol R. The degree of freedom f of this connection is expressed by $f = 1$.

2. Were only translation along the w axis permitted (again with the other five possible motions suppressed), the two links would remain parallel to each other and the variable describing the relative motion would be the perpendicular distance, say s, between planes xy and uv. This type of motion, a rectilinear translation, is commonly associated with a crosshead and its guide. It is a prismatic connection with the symbol P, and degree of freedom $f = 1$.

3. Were both rotation about and translation along the w axis permitted, two independent variables—one for the translation, the other for the rotation—would be needed to describe the relative motion. Such a connection is called cylindric: it is the motion of a shaft in a journal bearing if there is no axial restraint. In symbolic notation we write C; the degree of freedom $f = 2$.

4. Suppose the w axis to be threaded, as a bolt, and the corner A tapped as a nut. As link 2 turned, it would remain parallel to link 1, although undergoing a translation along the w axis. Since the angle of rotation θ and the translation s are related by the (constant) lead L of the screw, there is but one variable for the relative motion of the links. For the screw connection we write S_L; the degree of freedom is $f = 1$.

5. Suppose that link 2 lay directly on the xy plane of link 1 ($z_2 = 0$) and were allowed to slide on that plane. We would then recognize three possible motions—two translations, and a rotation about the w axis. Such a planar connection (it occurs rarely) would have the symbol F (think of flat) and of course has three variables, which means that the degree of freedom $f = 3$.

6. To suppose again, assume that there is a ball-and-socket joint at A, thus joining links 1 and 2 with a spheric connection. We immediately recognize the complete suppression of any linear motions: the only possible motion of link 2 with respect to link 1 is spherical motion, which is to say that all points of link 2 move in concentric spheres referred to the center of the ball. The motion is best described by successive rotations about the three mutually perpendicular axes: the sequence of the rotations is important. This connection has three variables, whence $f = 3$. The symbol will be G (for globular).

The foregoing six types of connections, when reduced to simple forms of construction, are shown in Fig. 2-4. The common denominator of these connections appears to be the area contact between links. Each of the identical surfaces of contact, the working surfaces, is called an

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 48 of 143

element: taken together, the two elements constitute a *pair*, one element lying on one link, the second element lying on the second link. It is apparent that the relative motion of the two links is the relative motion of the pair elements and that this relative motion is defined by the variable of the connection, which we now call the *pair variable.*

The particular forms chosen for the screw, revolute, and prismatic pairs illustrated in Fig. 2-4 follow Reuleaux's suggestion that the revolute and prismatic pairs may be considered as special limiting cases of the screw pair, with the lead either zero or infinity. This observation will be put to use in designing a more complete symbolic notation to describe mechanisms in which all connections are made by lower pairs. With S_L representing a screw of lead L, then the symbols for the revolute and prismatic pairs follow logically as S_0 and S_∞, i.e., screw pairs S_L with $L = 0$ or $L = \infty$, respectively.

These six element pairs, whose appearance in their simple constructional form is dominated by area contact, were called *lower pairs* by Reuleaux; the term came into the English language with Professor Kennedy's translation of 1876. It is unfortunate that this apparent dominance of area contact has often made area contact the criterion for lower pairs: *the real concept of lower pairs lies in the particular kind of relative motion permitted the connected links;* the particular motion of each pair is defined by an obviously associated pair variable or by a simple grouping of functionally unrelated pair variables.

A moment's reflection will show that identical relative motions are possible with various joint constructions having neither area contact nor geometrically identical elements. For example, a turning connection or revolute may be constructed with a ball or pivot bearing, and for small angles of rotation a knife-edge or a flexure pivot may even be considered: no area contact in the first, no elements in the second. However, the relative motion permitted the connected parts is a rotation, defined by an angle such as θ. A typewriter carriage moves along its ways on small rollers with a motion of rectilinear translation: this means a prismatic pair without area contact; a distance s will be the pair variable. A ball spline is similar. A ball bushing is a practical way of constructing a low-friction cylindric pair, just as a ball-bearing screw is a screw pair of minimal friction. The various constructional forms have advantages for the machine designer, but they do not change the geometric relation of the connected links. They are merely different ways of physically achieving a specific kind of relative motion between the connected parts.

For all but the planar pair (symbol F) we may speak of a hollow element and a full element. Thus, for the revolute pair R of Fig. 2-4*a*, the bearing surface of link 2 is the hollow element, written R^-; and the surface of the shaft, which lies on link 1, is the full element, written R^+.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 49 of 143

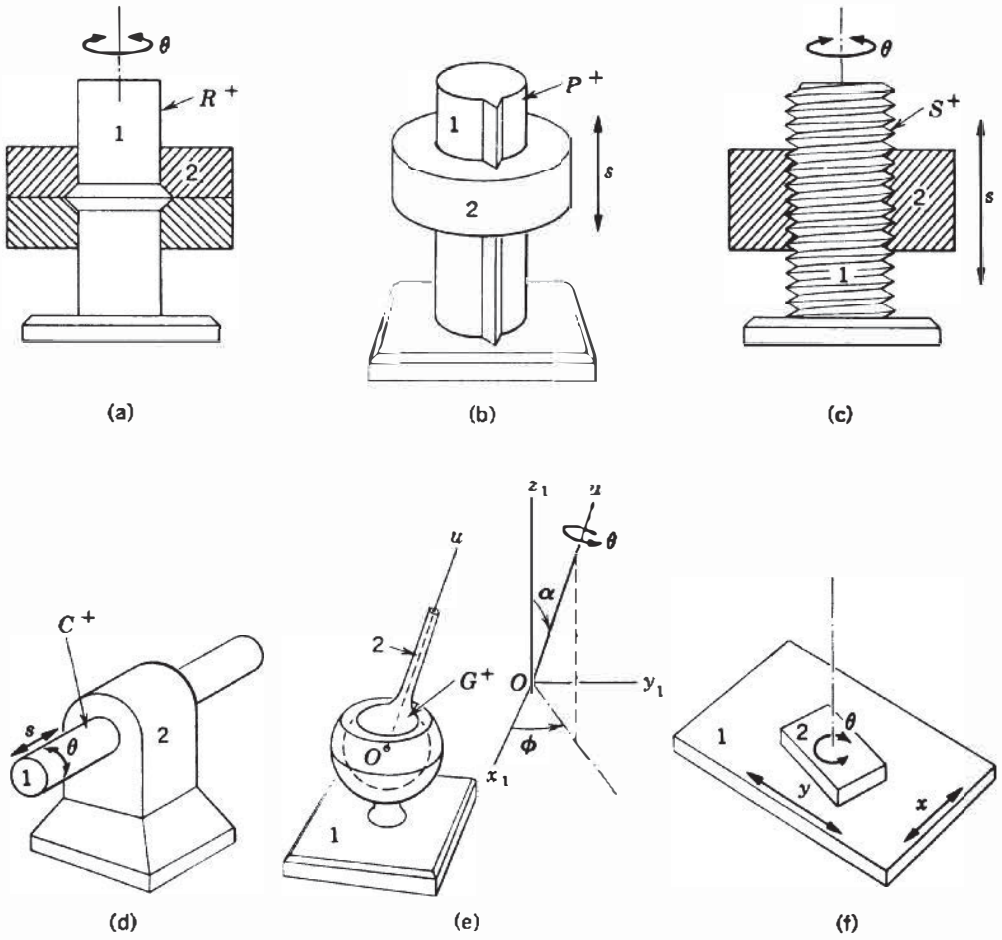


FIGURE 2-4 (*a*) *Revolute pair* (turning pair), $f = 1$. The relative motion is rotation about the axis and is defined by a single variable θ. (*b*) *Prismatic pair*, $f = 1$. The relative motion is translation and is defined by a single variable s. (c) *Screw pair*, $f = 1$. The relative motion is helical and is defined by either the rotation θ or translation s related through $\Delta\theta/2\pi = \Delta s/L$, where L is the lead of the screw (advance per revolution). (*d*) *Cylindric pair*, $f = 2$. The relative motion is a combination of a rotation θ about an axis and a translation s parallel to the same axis; there is no relation between θ and s. (*e*) *Spheric pair* (ball-and-socket joint), $f = 3$. The relative motion is spherical and is defined by three variables: two angles α and ϕ to define the direction Ou and the angle θ of rotation about Ou. (*f*) *Planar pair*, $f = 3$. The relative motion is planar and is defined in terms of two translations x and y and a rotation θ.

Since the hollow and full elements, when visualized as areas, not only are geometrically identical but the hollow element is "wrapped around" the full element, the five pairs are also known as *wrapping pairs*. We should note that the hollow and full elements of the wrapping pairs may be interchanged without affecting their relative motion. Thus, the relative motion between links 1 and 2 of the first five pairs of Fig. 2-4 will be the

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 50 of 143

same no matter whether link 1 or link 2 is the moving link. The word *wrapping* must be viewed with caution; inconsistent as it may seem, usage denotes chains and belts as wrapping *connectors*, but not as wrapping pairs.

Machine parts as actually constructed may have interrupted, or noncontinuous, elements (Fig. 2-5*a* and *b*). This feature does not change the character of the motion but is useful to the machine designer in distributing loads and stresses.

The same machine part may also carry elements of different pairs along the same axis (Fig. 2-5*c*). In the figure links 3 and 4 connect to link 2; each is a separate connection and must be so treated. There are two distinct coaxial revolutes, R_3 and R_4. The full elements R_3^+ and R_4^+ both lie on link 2, and the pair variables θ_3 and θ_4 are measured from the same reference line of link 2. The difference between θ_4 and θ_3 is the pair variable between links 4 and 3.

For mechanical convenience an intermediate pin (Fig. 2-5*d*) is generally used to fashion a revolute connection. Although the actual

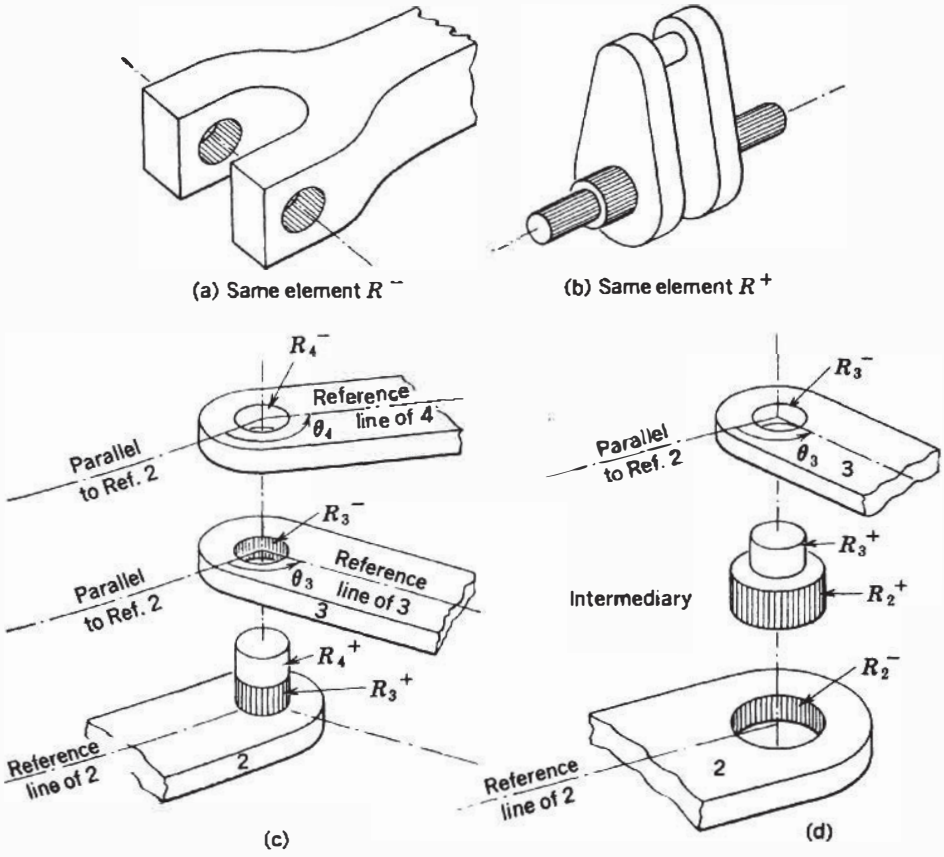


FIGURE 2-5 Interrupted and coaxial elements.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 51 of 143

number of elements has been augmented by the introduction of the intermediary, the intermediary does not have to be counted if its own motion is not of interest, since its presence does not affect the relative motion between links 2 and 3.

2-7 HIGHER-PAIR CONNECTORS

We have seen that the connection between parts may be considered in terms of pair elements, the contacting surfaces of the parts. We identified lower-pair connectors as those permitting six different kinds of specific motions.

A second type of connection must be described for the sake of completeness: it is what Reuleaux called the higher-pair connection. In the higher pairs the surface elements are so shaped that only line or point contacts are possible between elements. Point contact is found in ball bearings, as well as between the teeth of helical gears on nonparallel shafts. Line contact is characteristic of cams, roller bearings, and most gears. The relative motion of the elements of higher pairs is generally quite complicated. The involved functional relationship between translation and rotation allows no succinct statement, and an infinite number of higher pairs exist. Under the circumstances, higher pairs do not follow a simple classification scheme, as do the lower pairs. Nor are they conveniently described by means of symbols.

Higher-pair connections may on occasion be replaced by a combination of lower pairs, to reduce unit contact pressures and allow take-up for wear. Thus, the pin element of link 2 riding in link 4 (Fig. 2-6) is not a very practical construction; the same relative motion between links 2 and 4 is retained on interposing link 3. We note that the two degrees of freedom of the higher-pair connection (translation and rotation) are maintained with the substitution of the two lower pairs and that another link has been added.

2-8 FOUR-BAR LINKAGES

A versatile example of mechanism is known as the *four-bar linkage* (Fig. 2-7). It consists of four rigid members: the *frame*, or fixed member,

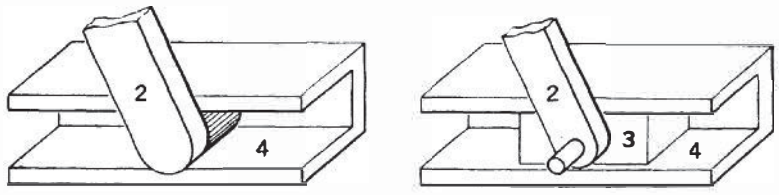


FIGURE 2-6 Substitution of two lower pairs for a higher pair; note the additional link.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 52 of 143

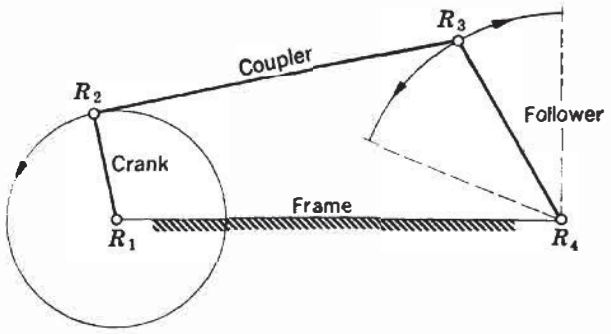


FIGURE 2-7 Four-bar linkage. With the link proportions shown, a continuous rotation of the crank is converted into an oscillation of the follower.

which is assumed stationary, and to which are pivoted the *crank* and *follower*, whose intermediary is aptly termed *coupler*. These members are connected by four revolute pairs, R_1, R_2, R_3, R_4, allowing relative rotation between adjacent members; all four revolute axes are parallel. The word linkage implies that all connections in the mechanism are lower pairs (here they all happen to be revolutes).

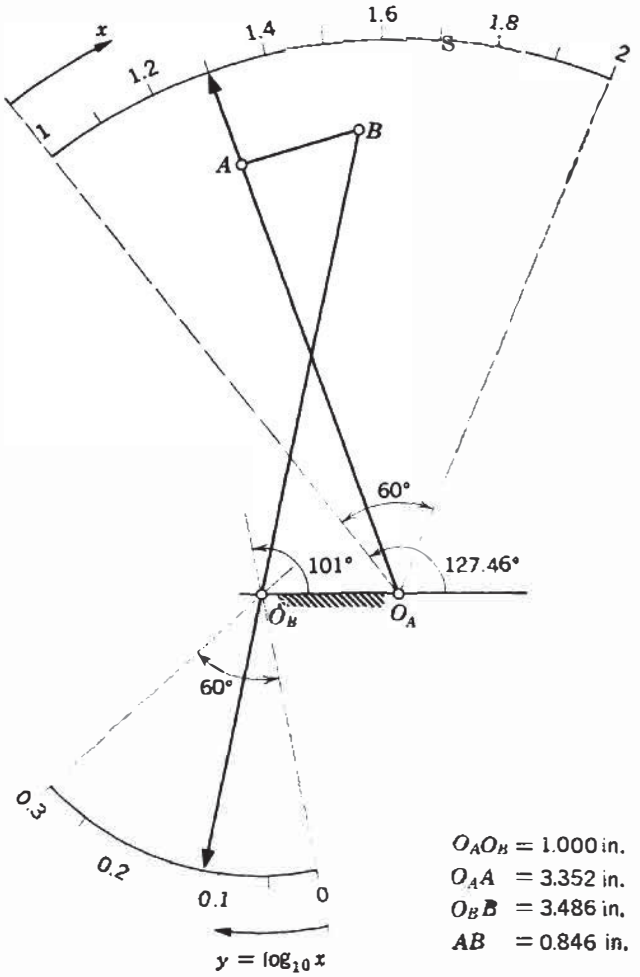


FIGURE 2-8 Four-bar linkage as generator of the logarithmic function $y = \log x$ over specified ranges of the input variable x and the arcs of oscillation.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 53 of 143

Older usage, when the four-bar linkage was known as a quadric-crank mechanism, implied that a "crank" could or could not rotate continuously, depending on its position in the mechanism. It will be convenient to use the word crank to designate (1) the input link, whether or not it is able to rotate completely (continuously in the same direction), or (2) a continuously rotating link which may or may not be the input.

As part of an instrument, the four-bar linkage may be used for scale conversion. Such devices are called function generators. To convert, for example, a linear scale into a logarithmic scale, the linkage shown in Fig. 2-8 may be used with an error which is less than 0.0037° for a 60° range of rotation of both crank and follower.

A point on the coupler of a four-bar linkage is called a *coupler point;* and its path when the crank is rotated is known as a *coupler-point curve* (or coupler curve) (Fig. 2-9), and the number of such curves is infinite. However, by proper choice of link proportions and coupler-point locations—this is one of the problems of synthesis—useful curves may be found. A curve's usefulness depends (1) on the particular shape of a segment—does it, for example, approximate a straight line or a circular arc?—or (2) on a peculiar shape of either the whole curve or parts of it. The coupler point, because of its motion characteristic, is now the output of the linkage. The coupler curve of the four-bar linkage

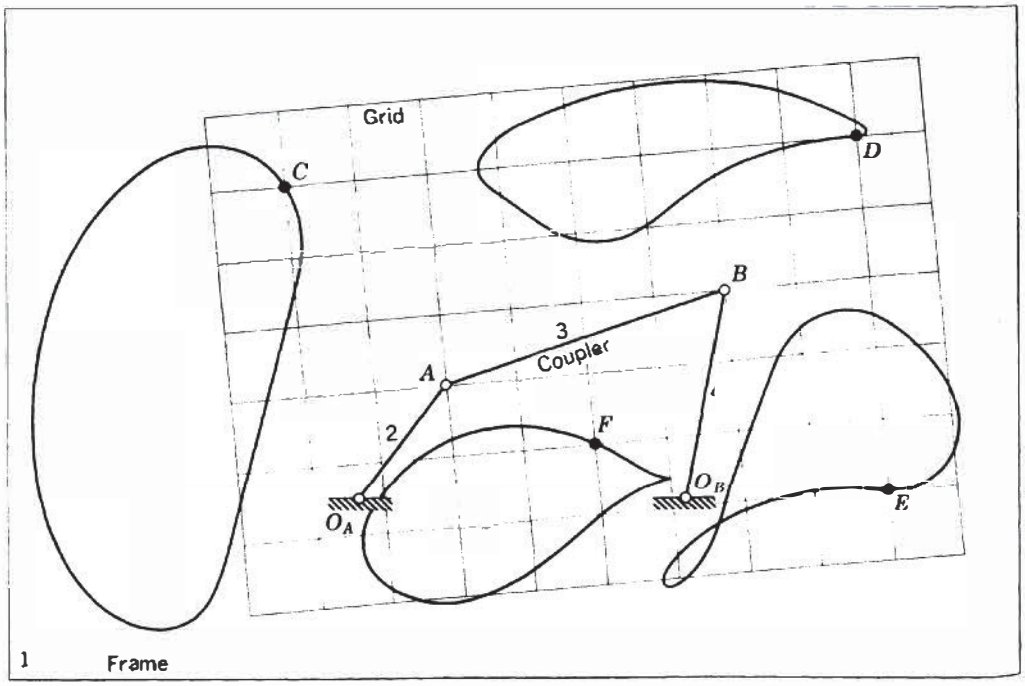


FIGURE 2-9 Four-bar coupler-point curves. The transparent grid is part of the coupler plane, link 3. The curves are traced on the plane of link 1, the frame.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 54 of 143

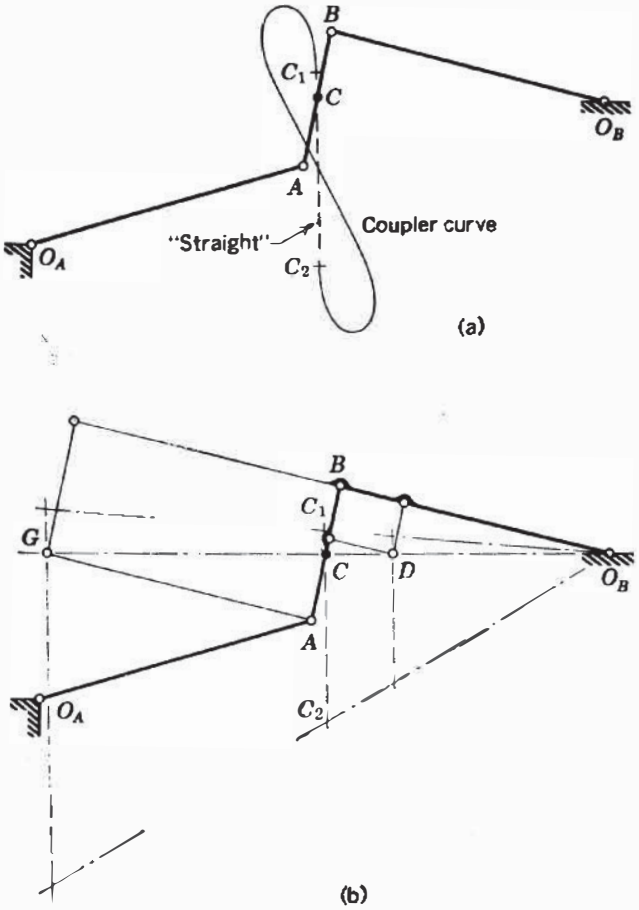


FIGURE 2-10 (*a*) Watt linkage, giving approximate straight-line motion ("parallel motion") at *C*; (*b*) pantographs, to give similar outputs at *G* and *D*.

is of the sixth order.[1] The classical and best-known coupler-point linkage isthe "straight-line," or "parallel-motion," linkage devised by Watt in 1784[2] to guide the upper end of a piston rod along a good-enough straight line. This was before the invention of the planer (1817); without the planer it was impractical to make straight surfaces 4 ft long as needed for the crosshead guides of the earliest doubleacting engines. Linkwork was simple. Watt's linkage, in about the original proportions, is shown in Fig. 2-10. The coupler point *C* traces a curve of figure-eight shape. Watt used only the middle portion, which in this case is a very good approximation to a straight line. The addition of a pantograph allowed further exploitation of the "straight" coupler-curve segment. The Watt linkage is currently used for axle and differential suspensions of some high-performance cars.

[1] Special configurations of the four-bar linkage may generate coupler curves of the fourth or second order; e.g., the coupler curves of a parallelogram linkage are circles, i.e., of order 2.

[2] In referring to his many inventions, Watt remarked that this was the one of which he was proudest.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 55 of 143

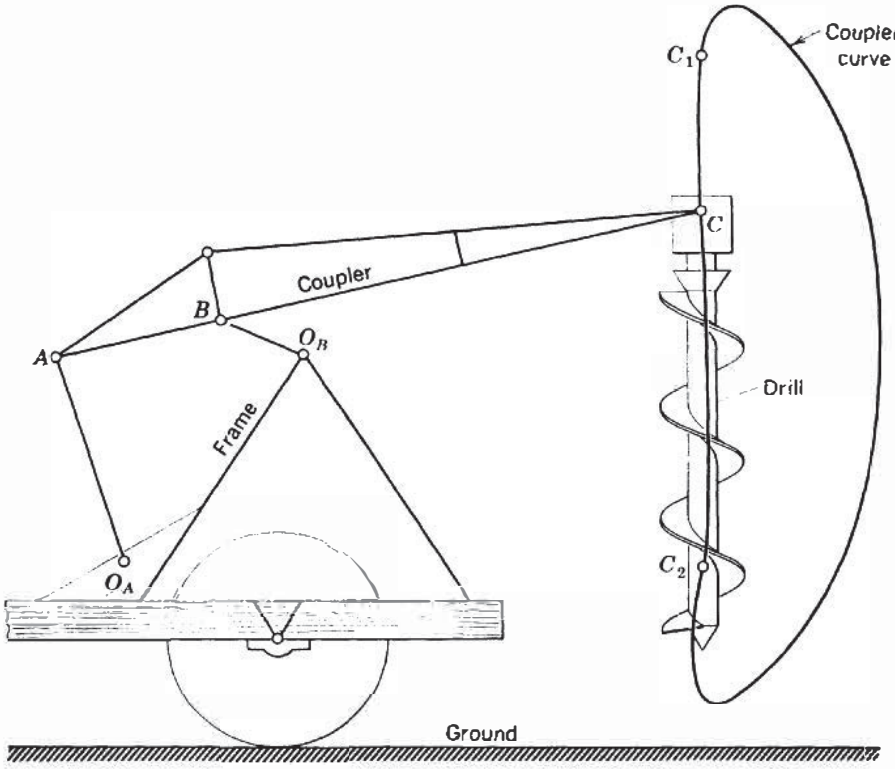


FIGURE 2-11 Portable posthole borer, or setter. (*From Volmer, VDI Ber.*, vol. 12, 1956.)

The Watt rotative engines retained the "great beam," or "lever," of the Newcomen engines, which made them just as bulky (see Figs. 1-8 and 1-9): their utility lay in the fact that they were rotative, giving power directly to a shaft. The famous "lap"[1] engine of 1787 had a great beam of about 15 ft length pinned to a connecting rod over 13 ft long; the engine was rated at 10 hp when running at 25 strokes per minute. Because of the one-to-one gearset of the sun-and-planet this gave a shaft speed of 50 rpm. Engines such as this required enormous enginehouses, whose cost sometimes equaled that of the engines.

Although the direct-connected engine, now called the slider-crank type, was explored around 1800 in association with early high-pressure[2]

[1] The engine was used for driving the machinery for lapping, or polishing, steel ornaments. It was taken out of service in 1858 and is now in the Science Museum, London.

[2] The adjective *high* is always associated with an age of development, and it is the age that sets any number. In 1800 a high-pressure engine was one dispensing with the vacuum, working with positive steam pressures of 2 or 3 atm, and exhausting at atmospheric pressure. Around 1860 a speed of 125 rpm was exciting enough to evoke the mistrust of old-timers accustomed to half that speed.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 56 of 143

engines, the slow beam engines dominated the power scene for many decades. Much effort was expended in searching for less bulky linkages that would produce straight-line guidance for piston-rod ends, for engine-building tradition demanded vertical cylinders. Of these we may mention the side-lever engines, used principally in boats and ships, and the grasshopper, or Evans, linkwork of small stationary engines. The latter is of particular interest, since it is related to the Watt linkage in a not directly obvious manner, as we shall see later.

Another example of the use of an approximately straight-line segment is the posthole borer shown in Fig. 2-11, where point C is a coupler point of the four-bar linkage O_AABO_B. The path of C approximates the straight vertical segment C_1C_2. This device, of German design, is capable of boring a vertical hole 6 ft deep.

Coupler-point curves having segments approximating circular arcs can be invoked to produce linkages having a dwell or two sufficiently complete for many practical purposes (Fig. 2-12). The figure shows a

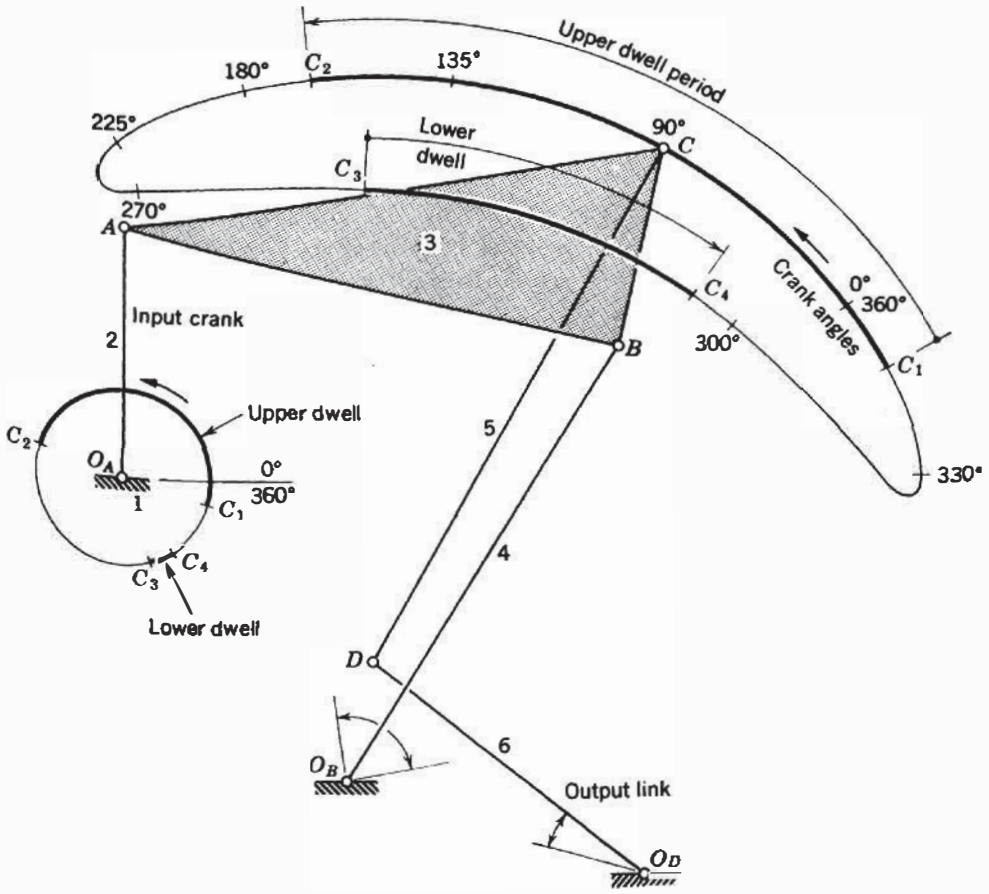


FIGURE 2-12 Double-dwell linkage.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 57 of 143

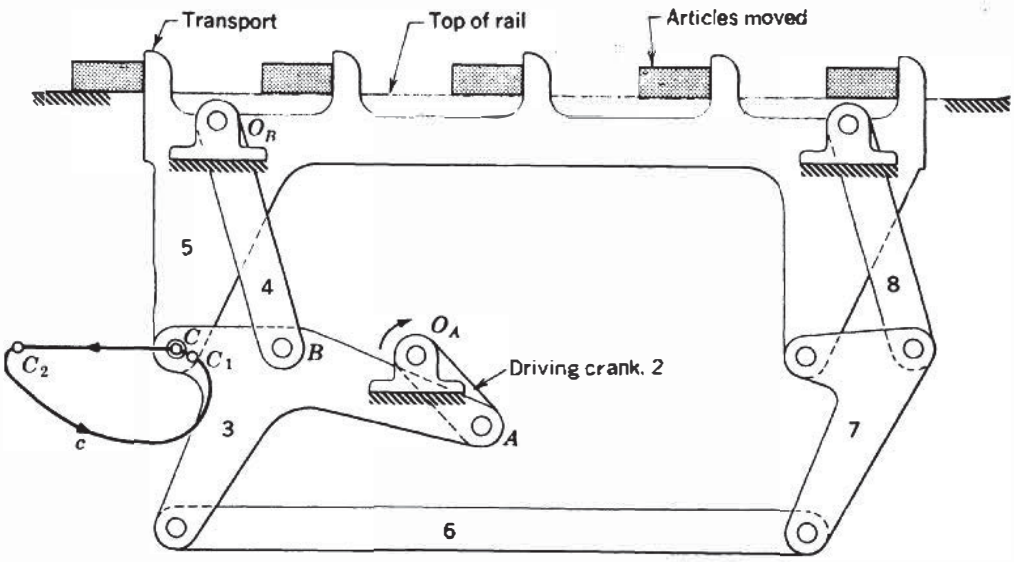


FIGURE 2-13 Transport mechanism. (*From C. W. Ham, E. J. Crane, and W. L. Rogers, "Mechanics of Machinery," 4th ed., p. 439, McGraw-Hill Book Company, New York,* 1958.)

coupler curve having two nearly circular segments C_1C_2 and C_3C_4 of nearly the "same" radius. Two bars—a dyad—are added at the coupler point C. The second bar, link 6, is the output link: it will be at dwell (rest) while the coupler point is traversing the "circular" arcs C_1C_2 and C_3C_4. It must be noted that the dwells are only as good as the approximation between the actual curves and circular arcs; the dwells will not be quite complete but nevertheless will be adequate for many applications. If the output oscillation of link 6 is followed through the complete excursion of C around the curve, it will be seen to have the characteristic of the oscillating follower of a rise-dwell-drop-dwell cam.

As another example of the use of a whole coupler curve, we may consider the piece of materials-handling equipment shown in Fig. 2-13. Here point C on the coupler of the four-bar linkage O_AABO_B describes the path c as crank 2 rotates through 360°. This motion is communicated to the transport member 5 by means of parallelogram linkages; member 5 moves horizontally for the line segment C_1C_2, then drops out of the way, to reappear later at the right, rising nearly vertically before moving to the left. A similar mechanism is used for the film transport in some motion-picture cameras (Fig. 2-14).

2-9 SLIDER-CRANK MECHANISMS

Another versatile linkage is the slider-crank mechanism (Fig. 2-15), familiar from reciprocating engines and pumps. Here the translation

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 58 of 143

of the piston is transformed into rotation of the crank shaft, or vice versa. Like the four-bar linkage, this mechanism consists of four rigid members, *frame* (fixed member), *crank*, *coupler* (connecting rod), and *follower* (slider). It differs from the four-bar linkage only in that one revolute pair R_4 has been replaced by a prismatic pair P_4, and many of the properties and applications of the four-bar linkage may be transposed to the slider-crank mechanism. The slider-crank mechanism is properly a linkage, since both revolute and prismatic pairs are lower pairs.

The dimensions of the slider-crank mechanism as used in reciprocating engines are approximately those shown in Fig. 2-15, and the path of the center of rotation of revolute R_2, the wrist pin, usually goes through the center of the main bearings, revolute R_1. When this is the case, the mechanism is called a *central* slider crank; it is otherwise an *offset*, or *eccentric*, slider crank (see Fig. 2-16).

The points of the coupler of a slider-crank mechanism are, as in

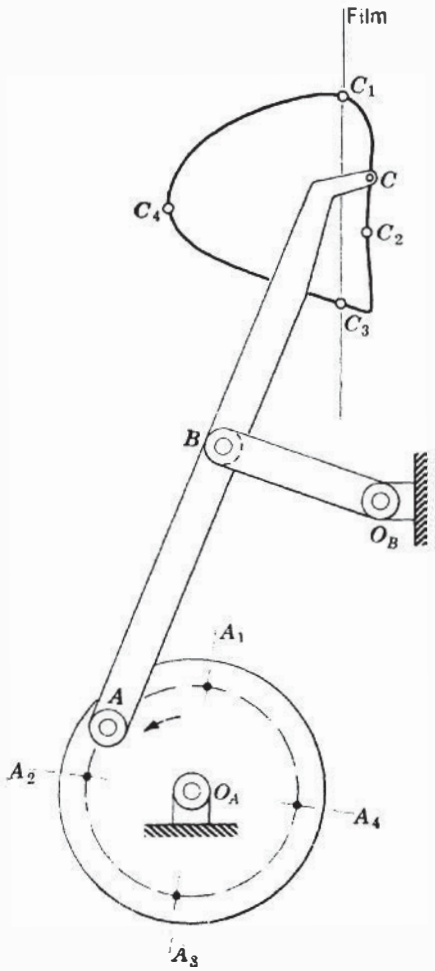


FIGURE 2-14 Film feed. (*From "Die Wissenschaftliche und Angewandte Photographie," ed. by Kurt Michel, vol. 3,* Harald Weise, *"Die Kinematographische Kamera," p. 202, Springer-Verlag OHG, Vienna, 1955.*)

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 59 of 143

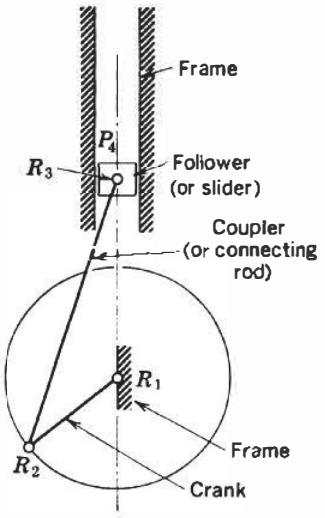


FIGURE 2-15 Central slider-crank mechanism, the conventional case.

the case of the four-bar linkage, also called coupler points, and their paths as the crank is rotated are coupler-point curves, but of the fourth order (Fig. 2-17). As with the four-bar linkage, slider-crank coupler curves may also be put to work. One application is shown in Fig. 2-18.

2-10 TRANSMISSION, DEVIATION, AND PRESSURE ANGLES

It would be useful to have a measure, criterion, or index of how well a mechanism might "run" while it is still in kinematic skeleton form on the drawing board. "Run" is a term that more formally means the effectiveness with which motion is imparted to the output link; it implies smooth operation, in which a maximum force component is available to produce a torque or a force, whatever the case might be, in an output member. Generally speaking, torque and force are not at all compatible with only the kinematics and statics of a given situation. As is well known, the magnitude of the dynamic forces may be several times as large as the static forces and may in addition possess quite different directions. Even a test on a kinematic model will check out only an approximation of the static forces and will tell nothing about the dynamic forces.

However, some evaluation of the state of affairs is better than

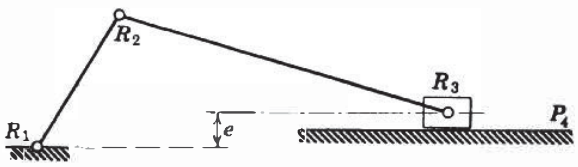


FIGURE 2-16 Offset, or eccentric, slider-crank mechanism.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 60 of 143

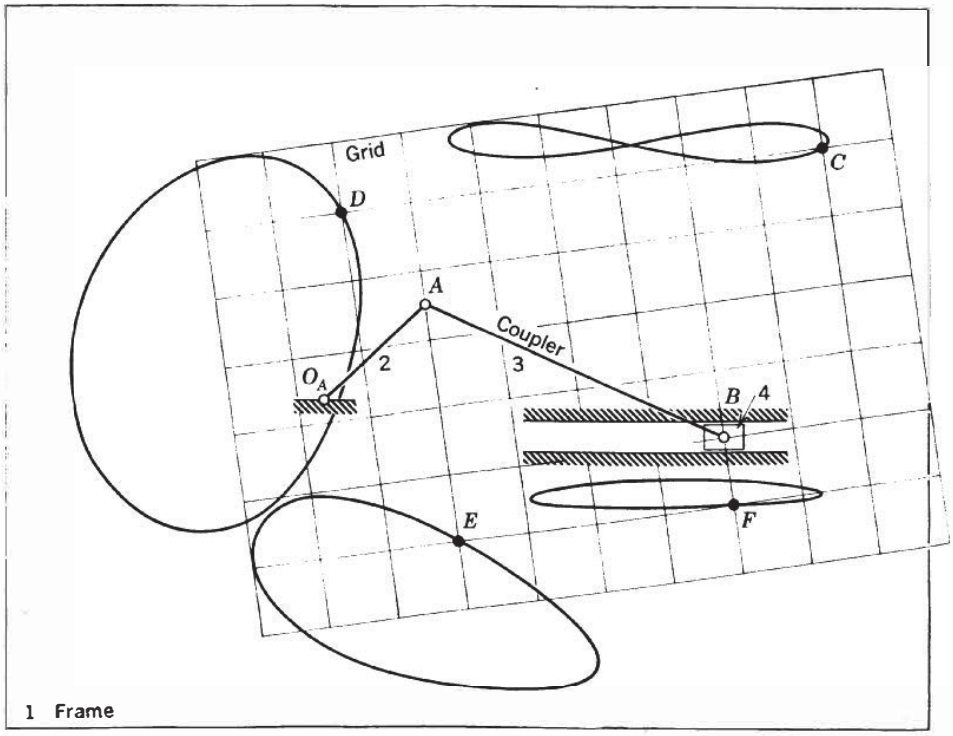


FIGURE 2-17 Slider-crank coupler-point curves. The transparent grid is part of the coupler plane, link 3. The curves are traced on the plane of link 1, the frame.

none. Alt[1] defined the aptness of motion transference from the driving link (not the input link of the mechanism) to the output link in terms of the transmission angle: the transmission angle γ is the smaller angle between the direction of the velocity difference vector $\mathbf{v}_{B,A}$ of the driving link and the direction of the absolute velocity vector $\mathbf{v}_B$ of the output link, both taken at the point of connection. This is shown in Fig. 2-19*a* for a four-bar linkage. Since the velocity vectors are perpendicular to their respective links, the transmission angle is also given by the angle between link centerlines, *b*. Clearly the optimum value of γ is 90°; the recommended tolerance is about $\pm 50°$. Alt[2] recognized that this kinematically determined transmission angle does not reflect the action of gravity or dynamic forces. Thus, in a single-cylinder piston engine the transmission angle (which is here between the connecting rod and crank) becomes zero at the dead-center positions, requiring the dynamic action of a flywheel to further the motion. Linkages with more than four bars have peculiar difficulties.

[1] H. Alt, *Werkstattstech.*, vol. 26, pp. 61–64, 1932.
[2] H. Alt, Getriebetechnik, *VDI Tagungsheft*, vol. 1, p. 197, 1953.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 61 of 143

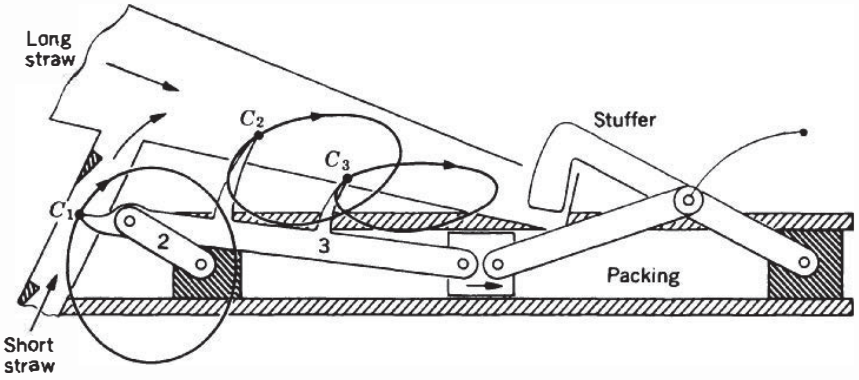


FIGURE 2-18 Straw packer making use of the coupler curves of a slider-crank mechanism. The crank is link 2; the coupler is link 3, with coupler points C_1, C_2, and C_3. (*After Kurt Rauh, "Praktische Getriebelehre," 2d rev. ed., vol. 1, fig. 299, Springer-Verlag OHG, Berlin, 1951.*)

Another approach was taken by A. Bock,[1] who suggested working with the directions of the static force and velocity at the point of connection, terming the angle between the directions the deviation angle δ. The deviation angle is shown in Fig. 2-19*b*; its optimum value is 0°. When the driving link (in this case link 3) is a two-force member, $\gamma + \delta = 90°$. This relation fails when the driving link has more than two forces acting on it, as may be seen from Fig. 2-20 (adapted from Bock). Nerge[2] inclines toward Bock's view, although he does not name the angle. The pressure angle of a disk cam with roller follower—the angle between the common normal at the point of contact and the follower motion of the roller center—is recognized as the deviation angle (Fig. 2-21).

[1] A. Bock, *VDI Ber.*, vol. 29, p. 158, 1958.
[2] G. Nerge, *VDI Ber.*, vol. 29, p. 157, 1958.

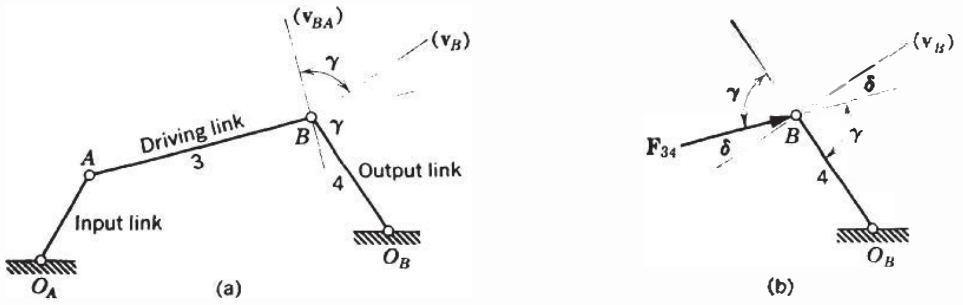


FIGURE 2-19 (*a*) Transmission angle γ of a four-bar linkage; (*b*) transmission angle γ and deviation angle δ of a four-bar linkage.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 62 of 143

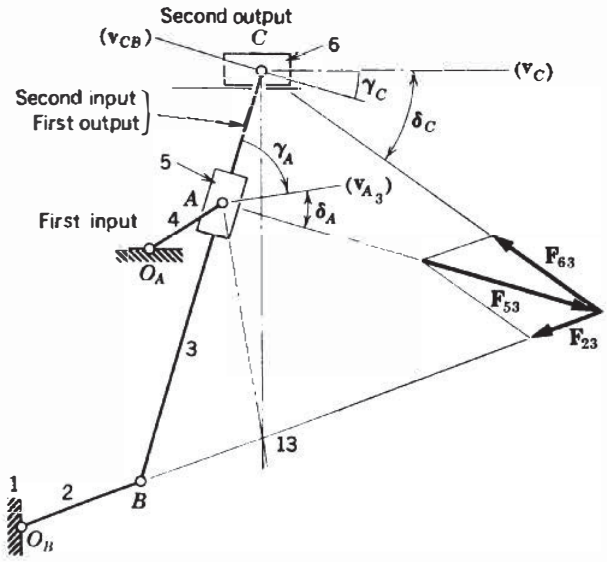


FIGURE 2-20 Transmission and deviation angles of a six-link mechanism.

2-11 PLANAR AND SPATIAL: THE MOTIONS AND THE MECHANISMS

Planar and spatial motions of bodies are distinguished from each other by noting the motions of *all* particles of the bodies. A body is said to have planar motion if *all* its particles move in parallel planes, i.e., when the true paths of *all* its particles can be represented on a single plane parallel to the planes of the moving particles. A body rotating about a *fixed* axis, for example, has planar motion, and any plane perpendicular to the axis may be considered as the plane of motion, for the true paths of all particles can be projected into this plane. Other bodies may be referred to this plane, provided that their motions are a combination of rotations about axes that are parallel to the fixed axis and translations along axes perpendicular to the fixed axis. A mechanism whose links have planar motions all parallel to the same plane is called a

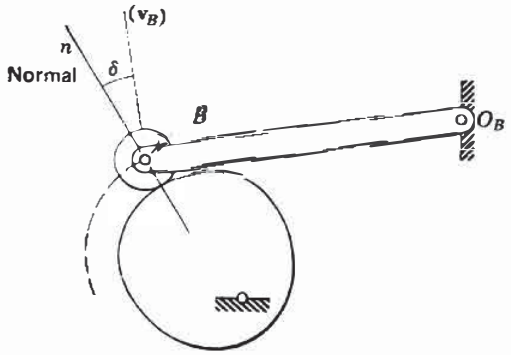


FIGURE 2-21 Disk cam, in which pressure and deviation angles are identical.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 63 of 143

planar mechanism: the true paths of all particles of all links may be shown in one plane, "the plane of the paper." The four-bar linkage, the slider-crank mechanism, gears on parallel shafts, the disk cam with reciprocating follower, and so on, are typical examples.

A body has spatial motion if all its particles do not move in parallel planes. A screw turning in its nut, and hence also moving axially, has spatial motion, since the angle of rotation and the axial translation cannot be depicted on the same plane: any particle of the screw describes a helical path in space. A mechanism which is not planar is said to be spatial. A spatial mechanism may contain but one link with spatial motion (as a screw); or it may have a number of links whose planar motions are not parallel to a common plane. Among the characteristics of spatial mechanisms are the presence of nonparallel axes of rotation and cylindric and ball-and-socket joints. The Hooke universal joint is a familiar spatial linkage; it is also a representative of the special case of spherical mechanisms.

The Hooke coupling is commonly called a universal joint because of its ability to transmit motion between two intersecting but noncollinear shafts. It should be remarked that there are a number of universal joints and that the Hooke type is but one of the lot. In continental Europe it is known as the Cardan (also Kardan) joint. As it happens, neither Cardan nor Hooke invented it; Hooke's name is associated with it since he put it to use in the seventeenth century.

A recognizable Hooke joint is shown in Fig. 2-22*a*. The two shafts misaligned by an angle α are represented by the revolutes R_1 and R_2. The central cross 4 carries the revolutes R_4 and R_3, whose axes are at right angles. Furthermore, the axis of R_1 is perpendicular to that of R_4, and the axes of R_3 and R_2 are also perpendicular. Lastly, all four revolute axes intersect at a common and fixed point O; it is this mutual intersection of all revolute axes at a fixed point that declares this spatial mechanism to be also a spherical mechanism. We may go one step further and remark that the Hooke joint is itself a special case of a spherical mechanism by reason of the three right angles.

The Hooke joint is shown in one schematic form in Fig. 2-22*b*. We recognize that all particles of link 4 (no matter what its physical shape might be) move on spherical surfaces centered at the fixed point O, that is, all particles of link 4 move on concentric spheres whose center is the fixed point O. Such a motion is specifically called spherical, to distinguish it from less well regulated spatial motions that would occur if revolute axes did not intersect at a common point. Links 1 and 3, considered individually, have planar motion; the path of any particle is a circle lying in a plane perpendicular to the particle's axis of rotation. However, since the axes of R_1 and R_2 possess a common point at O, we

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 64 of 143

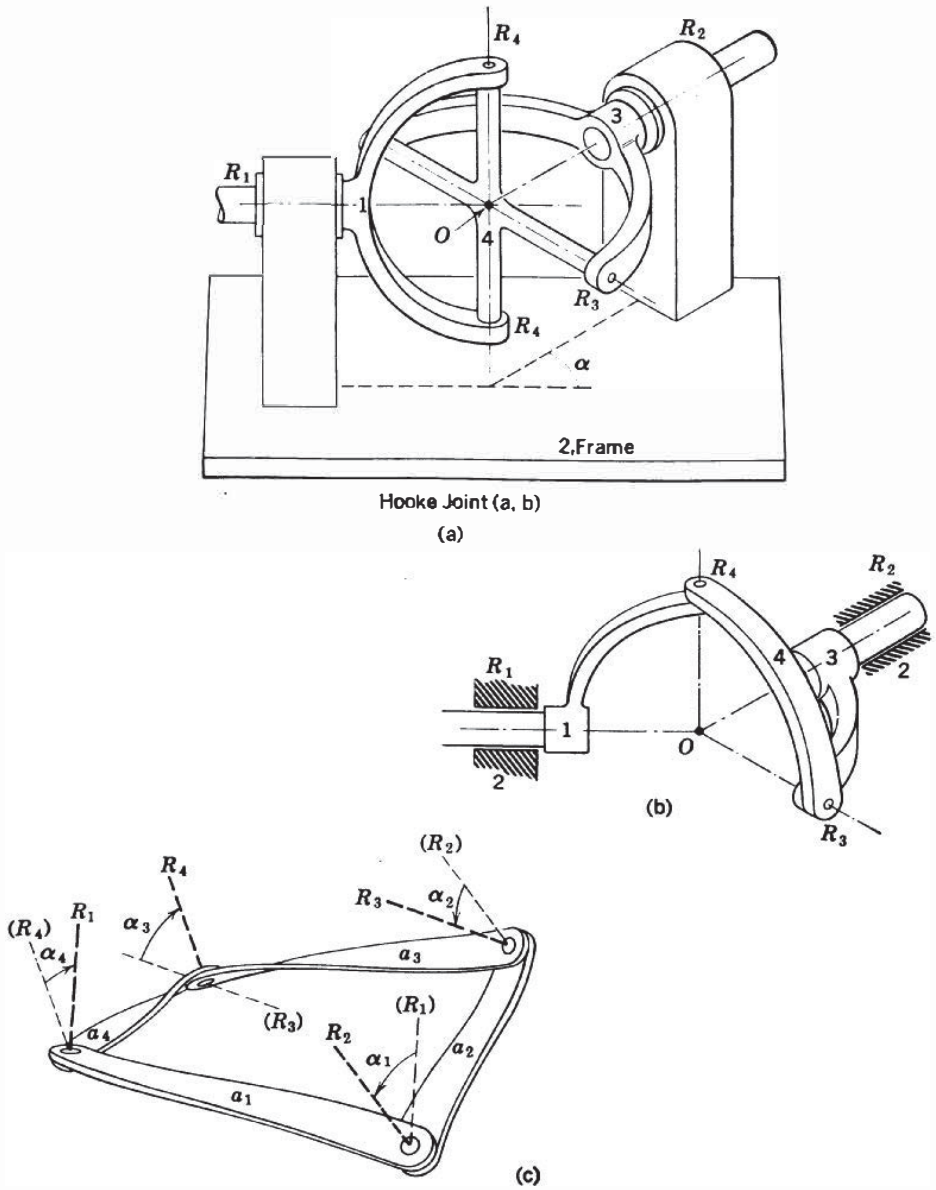


FIGURE 2-22 Four-revolute spatial mechanisms.

can also imagine the particles of links 1 and 3 to move on spheres centered at O. The simplest case of a spherical mechanism would involve two bevel gears; the simplest spatial mechanism would be a worm-and-wheel or two crossed helical gears.

The Hooke joint is a spherical four-bar linkage; like the planar four-bar, it has four revolute connections. The difference between the two lies in the orientation of the revolute axes. In a spherical four-bar,

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 65 of 143

the definitive parameters are the four angles between axes; in a planar four-bar, the parameters are the four link lengths. One other four-revolute linkage, a spatial mechanism also, exists: it is the Bennett mechanism (Fig. 2-22*c*). In this, the opposite links have the same lengths and the same angles of twist, but the lengths and the twists are related.

For other and more complicated spatial mechanisms, see Chap. 12.

2-12 KINEMATIC CHAINS

A material body with two or more kinematic elements is called a *link*. Each element represents a place of contact with, or connection to, another link. A link carrying two elements is a *binary link;* if there are three elements, it is a *ternary link,* if four, a *quaternary link;* and so on. A planar four-bar linkage, a spherical four-bar, and a Bennett mechanism are each composed of four binary links. A cam in contact with only a single follower would also be a binary link.

The bell crank, link 2 in Fig. 2-23*a*, is a ternary link, for it connects, or "contacts," links 1, 3, and 4. Link 3 of the transport mechanism of Fig. 2-13, reproduced here as Fig. 2-23*b*, is a quaternary link,

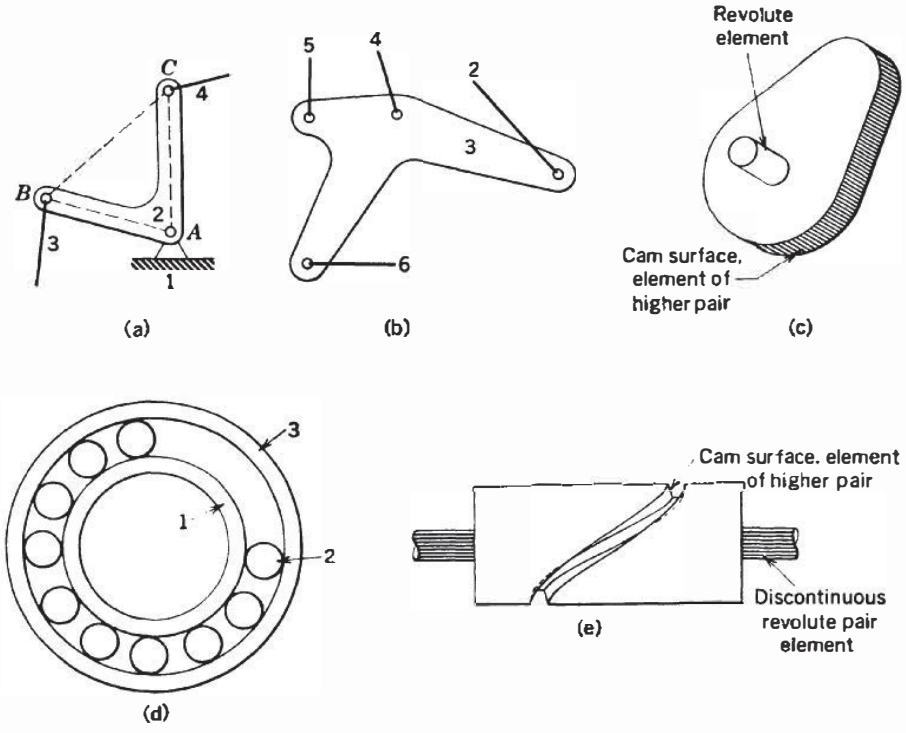


FIGURE 2-23 Examples of links.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 66 of 143

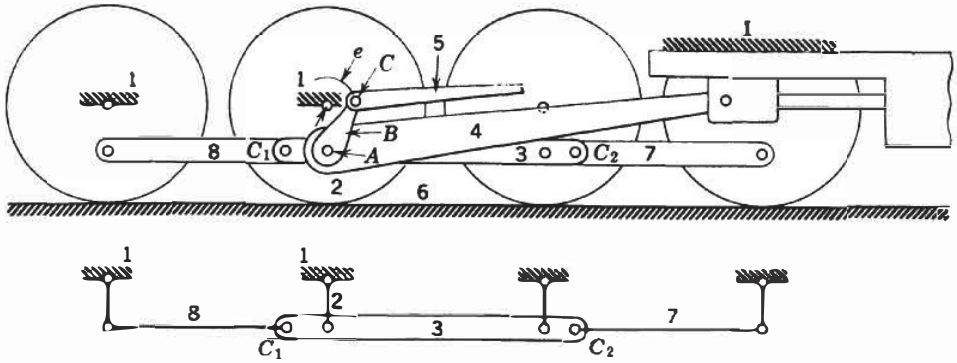


FIGURE 2-24 Example of quintary link. The wheel, link 2 (which includes B), connects with links 1, 3, 4, 5, and 6. Note that C_1 and C_2 are coupler points of a parallelogram linkage and describe circles of crank radius.

connecting as it does with the (four) links 2, 4, 5, and 6. Other examples of links, with higher-pair elements, are shown in Fig. 2-23*c*, *d*, and *e*.

The locomotive wheel of Fig. 2-24 presents an interesting situation: it not only is a quaternary link, or quintary when wheel-rail contact is considered, but also shows the construction of a short-throw crank. Wheel, axle, crankpin A, and "eccentric link" B constitute link 2. The axle is in contact with the frame (link 1); the crankpin carries the connecting rod (link 4) and side, or parallel, rod (link 3). In addition, the eccentric link B connects to the eccentric rod (link 5). This last connecting point, C, describes a circle of radius e about the center of the wheel.

Although the above definition of a link is very general and includes the possibility of one-way-rigid links such as bands, ropes, belts, and fluids, the present text will be concerned only with rigid links.

A kinematic *chain* is an assemblage of parts, or links, connected by pairs. Geometric considerations sometimes preclude motion of the chain after closure, in which case the chain is called a *structure*, and this may be statically determinate or indeterminate (Fig. 2-25). A chain is *closed* when all pairs are complete because of mated, or connected, elements, as in Fig. 2-26. Incomplete pairs indicate an *open* chain (Fig. 2-27). A *simple-closed* chain is composed of only binary links, each link connecting to but two others, as in Fig. 2-26*a* and *b*. *Compound-closed* chains contain ternary and higher-order links, each connecting to more

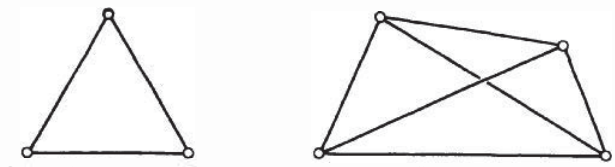


FIGURE 2-25 Nonmovable chains or structures.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 67 of 143

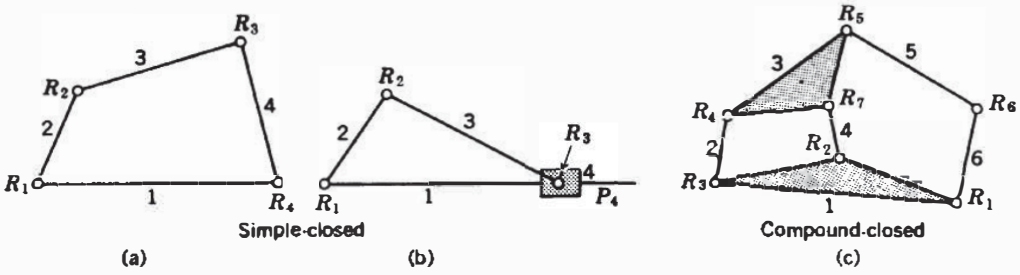


FIGURE 2-26 Movable closed chains.

than two other links, as in Fig. 2-26*c*, where each of the ternary links 1 and 3 connects to three other links.

With the aid of the notion of kinematic chain, a mechanism, considered earlier as a motion-transforming device, may now be given a new, more accurate, and perhaps more restricted definitione a mechanism is a movable closed kinematic chain with one of its links stationary.[1]

[1] Although we have gone from chain to mechanism by selecting a fixed link, no input link—the source of the motion to be transformed—has been designated. The Germans have a word for a mechanism in which the driving link has been selectede it is *Getriebe* and has caused much confusion. The German *Kette* means chain, and *Mechanismus* is the equivalent of mechanism, but there is no accepted translation for *Getriebe*, although drive and train have been used. In consequence the untranslatable *Getriebe* appears as mechanism when used as a noun. In adjective form, e.g., *Getriebelehre* (*Lehre* = theory or science of), it is translated as kinematics or mechanisms.

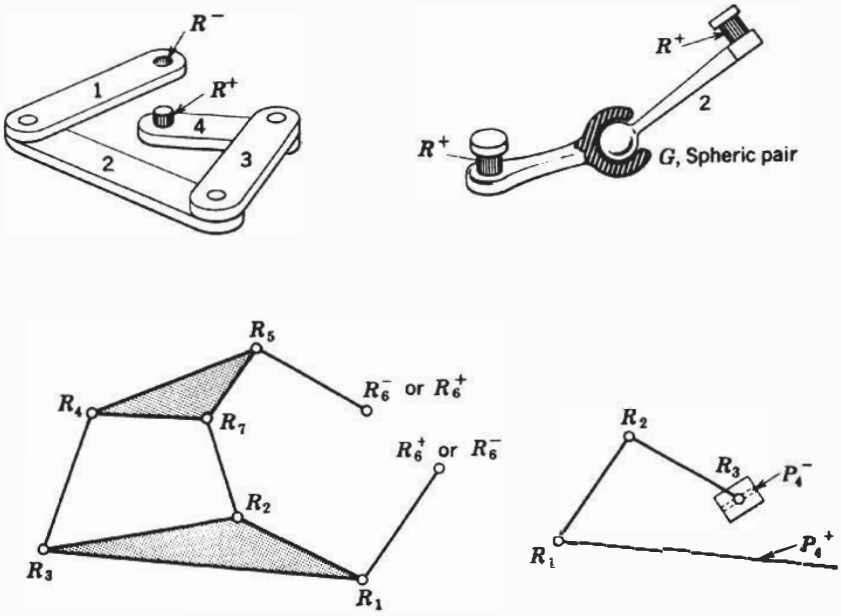


FIGURE 2-27 Open chains.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 68 of 143

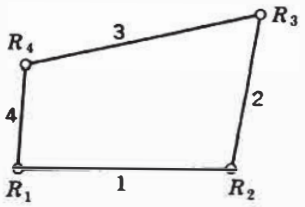


FIGURE 2-28 Four-revolute planar chain meeting Grashof condition (see Sec. 3-3).

Thus, in Fig. 2-26, the chain at (*a*) becomes a mechanism, a four-bar linkage, when one of its links (link 1, for example) is made stationary to form the frame of the mechanism. The chain at (*b*) gives a slider-crank mechanism when link 1 is made the frame.

As noted earlier, a mechanism (either planar or spatial) in which all connections are lower pairs is called a *linkage*. Linkage and lower-pair mechanism are therefore taken to be synonymous.

2-13 INVERSION

A mechanism is derived from a closed kinematic chain by making one of its links stationary; by choosing different links as the stationary link or frame, the same closed chain will yield as many distinct[1] mechanisms as it has links. The chain of Fig. 2-28, for example, yields four different four-bar linkages, as shown in Fig. 2-29. The four-bar linkage *a*, sometimes called the crank-and-rocker mechanism, gives an oscillation of the follower 2 for a continuous rotation of the crank 4. At *b*, a rotation of link 3 with constant angular velocity gives link 1 a continuous rotation with variable angular velocity. This is the drag-link mechanism; double-crank mechanism is also descriptive. At *c*, the situation is similar to that of *a*, but with different motion characteristics. At *d*, links 1 and 3 can rotate only through angles less than 360°: this is the double-rocker mechanism.

The process of fixing different links of a chain to create different mechanisms is called *kinematic inversion*. The four-bar mechanisms *a*, *b*, *c*, and *d* (Fig. 2-29) are the four inversions of the four-bar chain (Fig. 2-28).

The mechanism shown in Fig. 2-30*a*, called the Scotch yoke, consists of four rigid links connected by two revolute pairs R_1 and R_2 and two prismatic pairs P_3 and P_4. Crank rotation at constant angular velocity gives the yoke a translation that is a sinusoidal function of time. An inversion in which link 2, the crank, is chosen as the fixed link yields the mechanism of Fig. 2-30*b*. This, known as the Oldham coupling, transmits rotation between two parallel shafts with an angular velocity

[1] Distinct refers to the input-output relations of links attached to the frame; the relative motions of all links remain the same.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 69 of 143

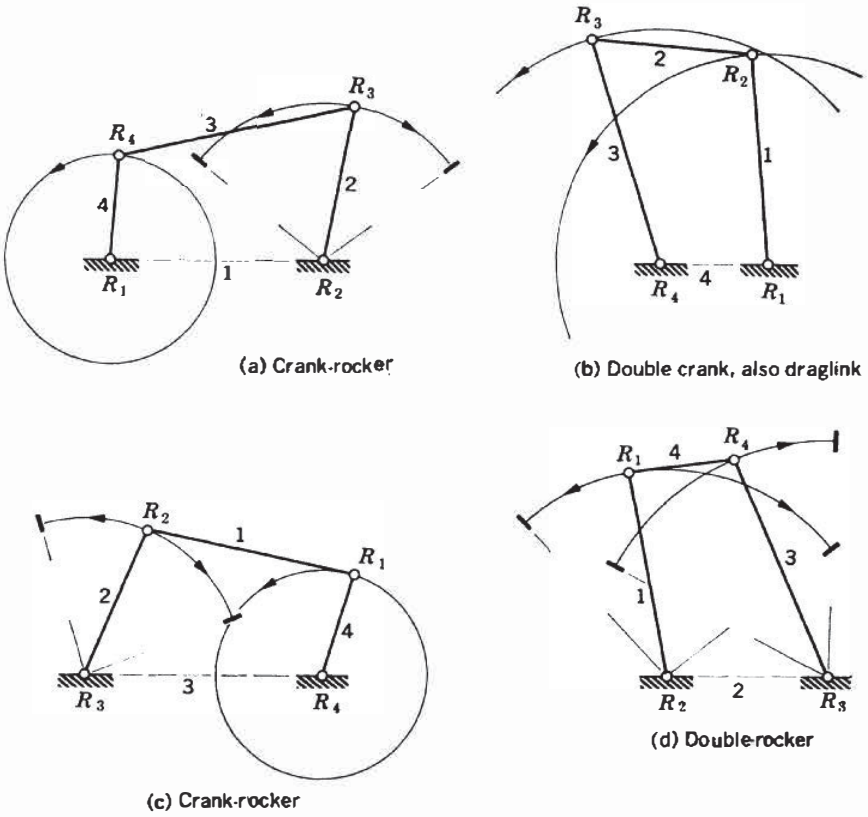


FIGURE 2-29 Inversions of the four-bar linkage.

ratio of unity. Another inversion, in which link 4 (yoke) is made the fixed link, yields the elliptic trammel (Fig. 2-30*c*); here the center point C of link 2 traces a circle, with all other points describing ellipses. A fourth and last inversion, in which link 3 (block) is fixed, gives a Scotch yoke different from the first.

As a further example of inversion we discuss the slider-crank mechanism (Fig. 2-31*a*). When crank 2 is fixed, mechanism *b* gives link 3 a continuous rotation of variable angular velocity when link 1 rotates at constant angular velocity. This motion transformation is similar to that of the drag-link inversion of the four-bar linkage; it has been applied to many machine tools, where it is called the Whitworth quick-return mechanism. With connecting rod 3 made the fixed link, mechanism *c* has found application in oscillating-cylinder steam engines[1]

[1] For example, the *Great Eastern*, built in 1858, had a four-cylinder oscillating engine (74-in. bore, 14-ft stroke) developing 3,410 hp at 11 rpm with steam at 24 psig. This engine, sitting low in the hull, drove overhead crankshafts to which 56-ft.-diameter paddle wheels were directly connected. In addition, there was a conventional four-cylinder horizontally opposed engine of 4,890 hp (and 39 rpm) for a 24-ft screw. If all this failed, six masts could spread $1\frac{1}{3}$ acres of sail!

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 70 of 143

and pumps. Mechanism *d* finds use as a hand pump: link 1 is oriented in the vertical, and link 2 is extended to form the pump handle.

Among the earliest aircraft power plants was the rotary engine. Spectacularly successful in 1910, its useful life extended well into World War I. This engine (Fig. 2-32) was the same inversion as the Whitworth mechanism: the crankshaft (link 2) was bolted to the fuselage to become the fixed link. The propeller was bolted to the crankcase (link 1), and this assembly, complete with cylinders, revolved.

2-14 EXPANSION OF REVOLUTE PAIRS AND OTHER DISGUISES

Too often the physical shape of the connection between links is such that the true character and function of the connection are not immediately apparent. The reason for the disguise may stem from practical design considerations such as strength requirements, ease of manufacture, or space limitation, all of which obscure the nature of the kinematic elements of the connection, although the relative motions of the links remain unaffected. The situation is simply that the center

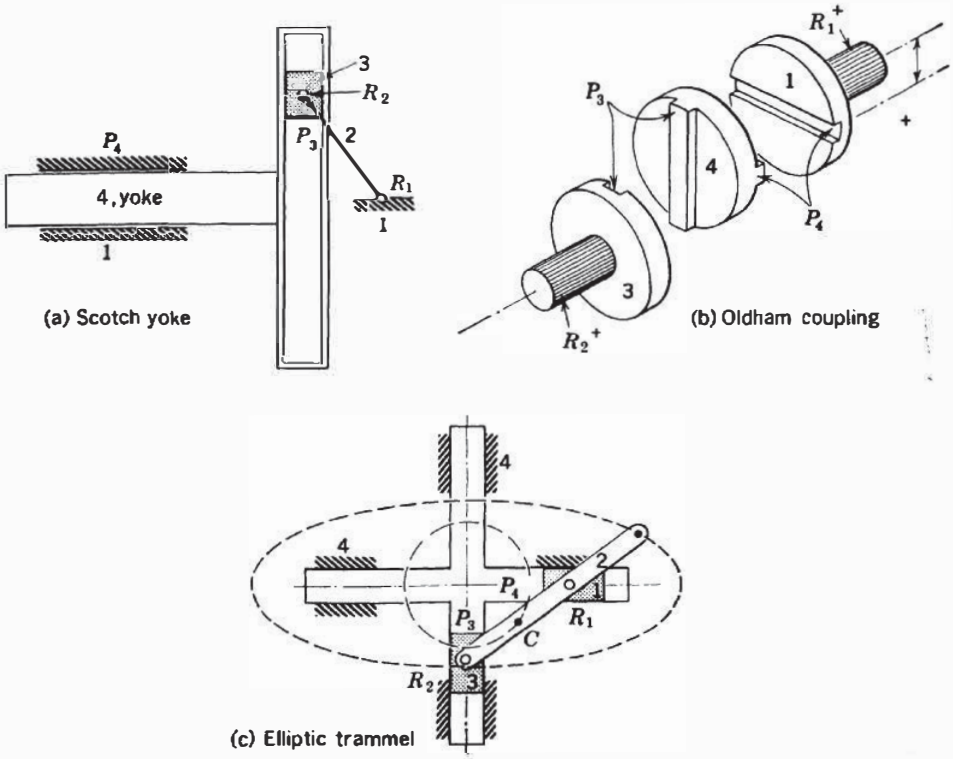


FIGURE 2-30 Inversions of the Scotch yoke.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 71 of 143

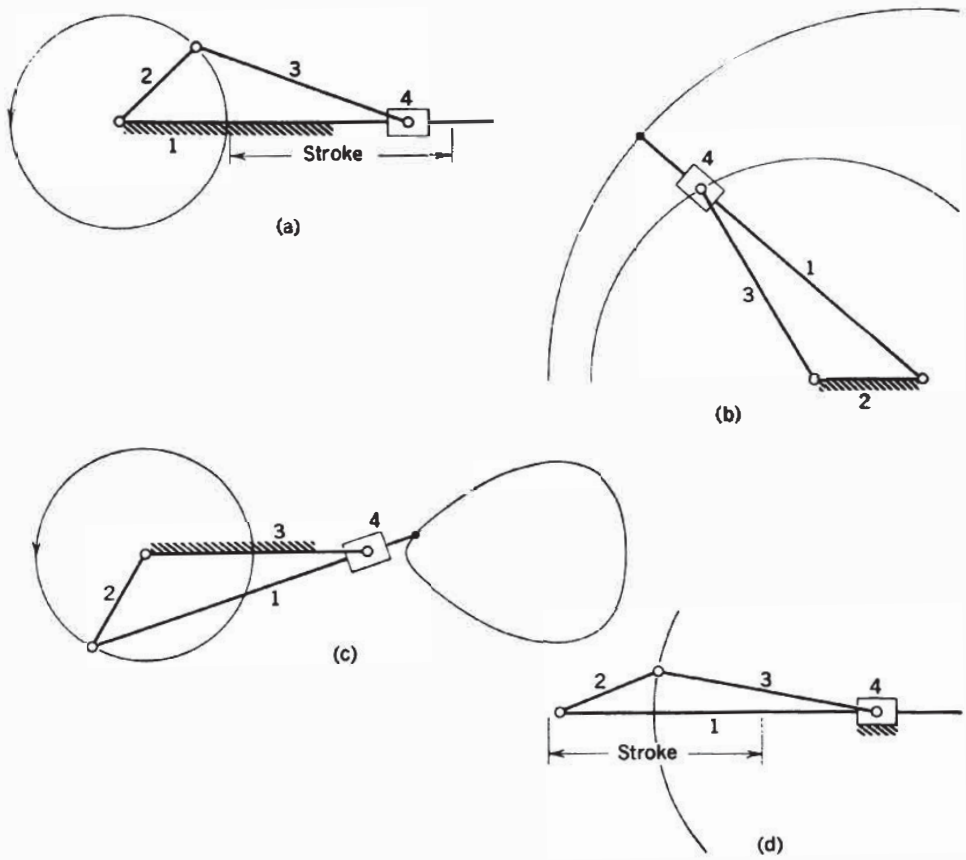


FIGURE 2-31 The slider-crank chain inversions.

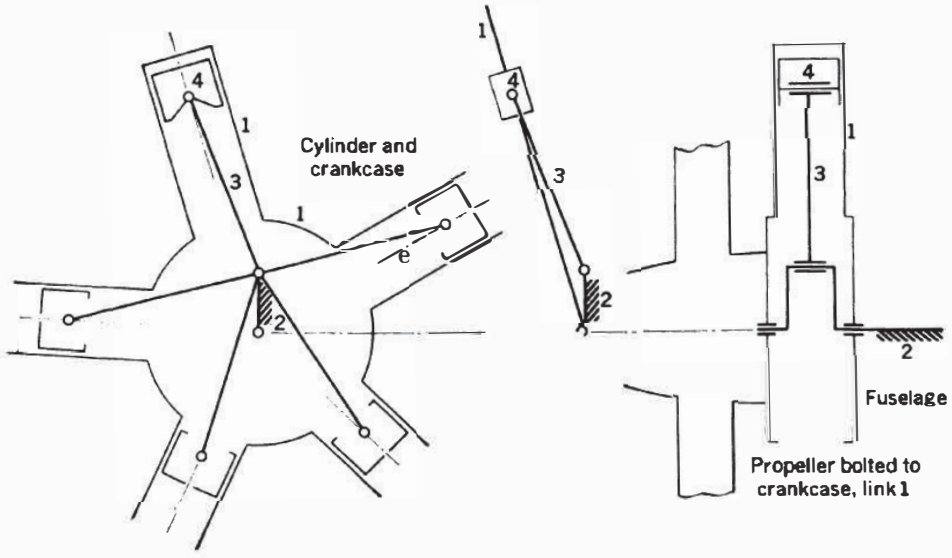


FIGURE 2-32 Rotary aircraft engine, a slider-crank inversion with link 2 fixed.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 72 of 143

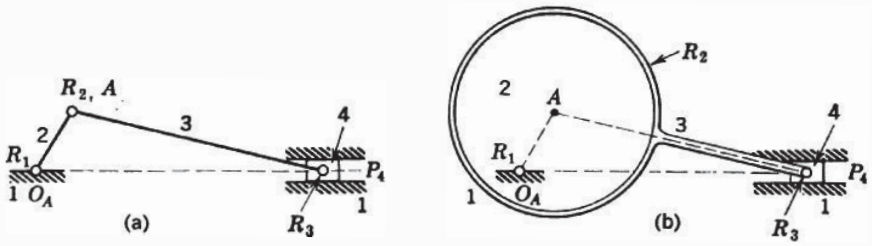


FIGURE 2-33 Expansion of revolute pair.

of what is kinematically a revolute pair is not directly discernible. A case in point is the common eccentric (Fig. 2-33*b*)ę examination shows it to be an oversized, or "expanded," crankpin, evident on a comparison with the slider-crank mechanism of Fig. 2-33*a*. Inspection identifies the center A of revolute R_2 in Fig. 2-33*b* and finds the crank of length $O_A A$. Revolute 2 in this expanded form is known as an "eccentric," and the shaft constituting revolute R_1 may now be continuous instead of interrupted by a crank. Obviously the physical diameters of the revolutes R_1 and R_2 are of no kinematic importance: what is significant is that the centers O_A and A are the same distance apart in the two mechanisms. The trick lies in discovering the disguised center of the "misproportioned" revolute R_2. Variations in form and size of a turning pair that do not alter the relative motion between connected members constitute an expansion of a revolute pair.

Further examples may be found. A four-bar linkage is shown in Fig. 2-34*a* in its familiar form; at *b* revolute pair R_3 has been expanded, as was R_2 in the slider crank, and the coupler has now become block 3. For a complete revolution of crank 2 block 3 traverses only the small arc E_1E_2 of the element of revolute R_3 on member 4. The motion of block 3 would still be described by means of an angle referred to B. Link 4 may be given a still different physical form as at *c* without altering the relative motions of the links. It should be observed that the "curved slider" is a form of revolute; the motion of block 3 in its guide is a rotation defined by an angle, not by a linear distance. We note that the space requirements of forms *a*, *b*, and *c* are quite different.

2-15 PRISMATIC PAIR AS THE LIMIT OF A REVOLUTE PAIR

Although differing in appearance from the "pin connection" of Fig. 2-34*a*, the "curved slider" of Fig. 2-34*c* remains a revolute pair as long as its radius of curvature is finite. The center of curvature B is part of the moving plane 4 from which the physical shape of link 4 has been cut. If, however, the radius of curvature of a revolute pair becomes

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 73 of 143

infinite, i.e., when its center of rotation goes to infinity, then, *and only then*, does the revolute pair become a prismatic pair, i.e., the pair variable changes from an angle to a linear distance.

This transition from a revolute to a prismatic pair is shown in Fig. 2-35. By expansion of the revolute pair R_4, the four-bar linkage shown at *a* takes the form *b*. Suppose now that the center of rotation O_B of the revolute R_4 is moved down along the vertical by increasing the lengths of members 1 and 4 as shown at *c*. By expansion of the revolute R_4, this new four-bar linkage takes the form *d*, in which the radius of curvature of curved slider R_4 is greater than at *b*. Moving the center O_B farther down simply increases the radius of curvature of R_4. At the limit, when O_B is at infinity on the vertical and members 1 and 4 have become infinitely long as at *e*, the four-bar linkage becomes a slider-crank mechanism *f*. The curved slider has now become straight, yielding a prismatic pair. Thus, a prismatic pair may be considered as a revolute pair whose center is at infinity in the direction perpendicular to the generatrix. Having arrived at this stage, O_B may be located at infinity, either "up" or "down."

In a previous section, the prismatic pair was considered as a limiting case of a screw pair with an infinite lead. The prismatic pair is here considered as a revolute pair with its center of rotation at infinity. These two interpretations of the prismatic pair, however, should not be

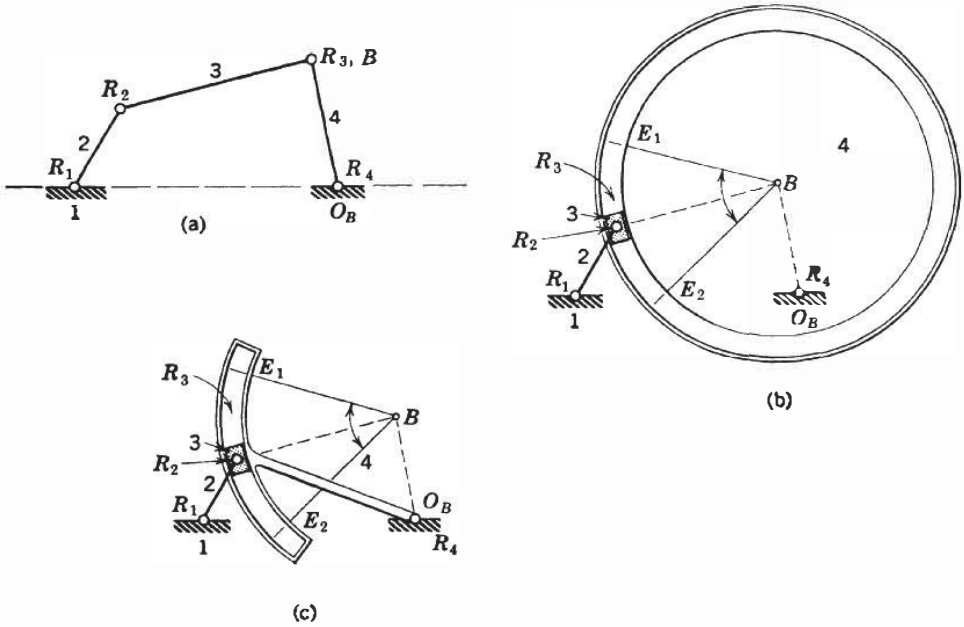


FIGURE 2-34 Expansion of revolute pair.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 74 of 143

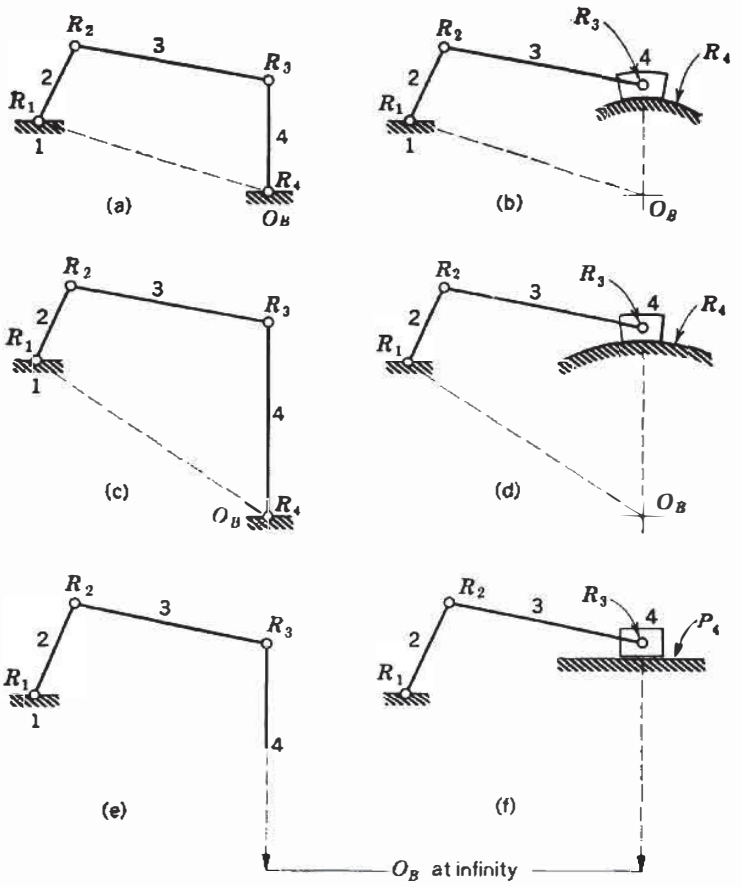


FIGURE 2-35 Prismatic pair as a limiting case of revolute pair with center at infinity.

considered as conflicting, for each view may serve a different purpose; it is a case of the end justifying the means—making use of the most convenient and legitimate argument for differing purposes. As noted before, the first interpretation is convenient in setting up a symbolic notation for lower-pair mechanisms. The second interpretation is put to use in the synthesis of planar mechanisms, allowing many of the properties of the four-bar linkage, when carried through the limit process shown in Fig. 2-35, also to become properties of the slider-crank mechanism.

2-16 EQUIVALENT LINKAGES

The complete kinematic analysis of a mechanism includes, among other things, the determination of velocities and accelerations. Difficulties may be encountered when links are connected by a higher pair,

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 75 of 143

as shown in Fig. 2-36*a*, where the relative motion between the two profiles consists in rolling coupled in uncertain fashion with sliding. On proceeding in the usual manner with vector equations (see Chap. 4), it would be necessary to apply the Coriolis theorem and to know the curvature of the path traced by a point of one link with respect to the other. If no easily recognized path is found, it may be difficult, or at the very least tedious, to establish the desired path curvature.

The equivalent linkage replaces the higher pair with properly disposed lower pairs. These will, for the instantaneous phase under consideration, give correct values of velocities and accelerations. Let

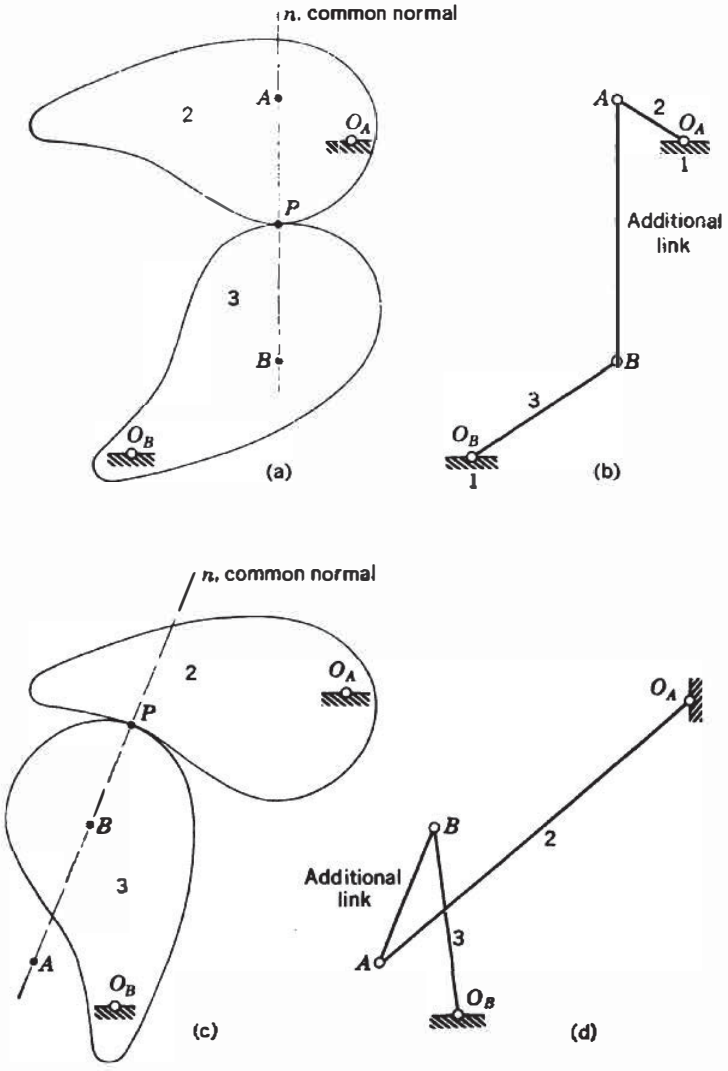


FIGURE 2-36 Equivalent linkages.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 76 of 143

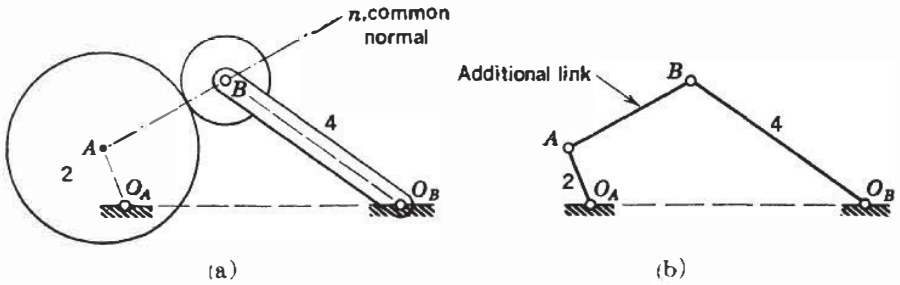


FIGURE 2-37 Cam with constant curvature profile and its invariant equivalent linkage.

A and B be the centers of curvature of the profiles of the higher pair at their point of contact. Because of the properties of the center of curvature, the distance AB will remain constant for three infinitesimally close positions of the cam and follower, and for these three positions the same correlation in displacements between links 2 and 3 may be obtained by means of the four-bar linkage O_AABO_B shown at b. This linkage is called an *equivalent linkage* of the higher-pair mechanism shown at a.

This equivalence, however, is valid only for three infinitesimally close positions of the mechanism. When the links are rotated through a finite angle, as at c, the dimensions of the equivalent linkage d are different. In other words, an equivalent linkage is generally valid only for a given instant or phase; it does not ordinarily apply to a complete cycle.

In general, two mechanisms are said to be equivalent if they give the same correlation between three infinitesimally close positions of their input and output members. The velocities and accelerations of the points of a moving link are defined by considering three infinitesimally close positions of the link. Such positions are preserved in equivalent linkages, which may therefore be used to evaluate velocities and accelerations of more complicated mechanisms. Equivalent linkages, however, cannot be used to evaluate displacements or the time rate of change of acceleration, sometimes called the jerk, or pulse, because these quantities can be defined only in terms of more than three infinitesimally close positions of a link.

In some few instances an equivalent linkage may be found that will duplicate the motion transformation between input and output links of a given mechanism throughout the motion cycle. Consider, for example, the cam mechanism shown in Fig. 2-37a, the cam profile being circular, with center at A. The roller has its center at B. The distance AB is now constant throughout the cycle, whence the equivalent linkage O_AABO_B shown at b is valid for all instants (positions or phases). The motion transformation between links 2 and 4 of both mechanisms is the same throughout the complete cycle.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 77 of 143

2-17 SYMBOLIC NOTATIONS

Language, either written or spoken, is a means of communicating an idea or thought. In written form, it involves the use of signs now so highly conventionalized that their original pictorial origins are not evident. Any system of signs may be called a language, e.g., the language of mathematics, in which the symbols stand for operations whose word descriptions are tedious and unhappy. Mathematical notation is the most powerful shorthand known: not only is it precise, but, more important, it is manipulative. Physics, chemistry, and electronics also have highly developed notations by means of which symbols other than only the mathematical allow the statement of ideas and complex situations in a compact manner.

No real symbolic notation is widely used in kinematics. Practically all that exists is a stylized representation of connections, useful for drawing skeleton diagrams. These diagrams sort out the basic geometry of a problem and are therefore indispensable, but they lead no further, for they do not represent a shorthand of the whole mechanism —they are only a quick picture of it, and no manipulation is possible.

Notwithstanding the controversial nature of the subject, the authors have been led to reconsider the matter of symbolic notations and devise an extension of Reuleaux's system to supplement the use of skeleton diagrams. Of this new symbolic notation, particularly convenient when applied to lower-pair mechanisms (both planar and spatial), an elementary form[1] will be presented in this section. It provides a concise qualitative method for identifying linkages by the number and nature of their pairs. This elementary form will be elaborated upon in Chap. 12 to make it quantitative by including the geometric relations between the pair elements of each link. The quantitative form of the symbolic notation allows an interpretation in terms of matrix algebra to make it manipulative; it is particularly useful for the study of spatial linkages. However, the complete form is not necessary for the planar studies forming the major parts of this book.

1. *Symbols for lower pairs* Each of the six lower pairs described in Sec. 2-6 will be represented by its symbol:

Spheric pair	G (think of *Globe*—sphere)
Planar pair	F (think of *Flat*)
Cylinder pair	C
Screw pair with lead L	S_L
Revolute pair	R
Prismatic pair	P

[1] This form is in the spirit of what Reuleaux called the "contracted formulae" (Kennedy, "Reuleaux' Kinematics of Machinery," p. 263).

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 78 of 143

As already remarked in Sec. 2-6, a revolute pair may be considered as a screw pair with a lead equal to zero; similarly, a prismatic pair is a screw pair with an infinite lead. The symbols for the revolute and prismatic pairs may therefore be written respectively as S_0 and S_∞ without the introduction of any new symbols. However, because the revolute and prismatic pairs are very common, it is convenient to give them their particular symbols of R and P.

2. *Symbolic description of simple-closed chains* A simple-closed chain consists of only binary links; i.e., each link is connected to two, and only two, other links. The chain is completely described if each connection is designated by its proper symbol, the connections being labeled in sequence, clockwise or counterclockwise. One connection is arbitrarily chosen as the starting point and the others noted in sequence while going around the loop in either sense. Thus, the symbolic descriptions of the chains shown in Fig. 2-26 might be written

For *a*:

$$R_1R_2R_3R_4 \qquad R_1R_4R_3R_2 \qquad R_2R_3R_4R_1 \qquad \text{or} \qquad R_2R_1R_4R_3 \qquad \text{etc.}$$

For *b*:

$$R_1R_2R_3P_4 \qquad R_2R_3P_4R_1 \qquad \text{etc.}$$

3. *Symbolic description of compound-closed chains* A compound-closed chain consists of a combination of simple-closed chains, possible because some of the links connect to more than two other links, i.e., some of the links are ternary, quaternary, etc. Such a chain is completely described if each connection is designated by its proper symbol (as in the case of simple-closed chains) and enough simple-closed chains are described to include all the connections (pairs). This method is similar to the analysis of electrical circuits, in which the voltage equation is written for each independent loop of the circuit. Examination of the compound-closed chain of Fig. 2-26 discloses three different simple-closed chains:

$$R_1R_2R_3R_4 \qquad R_1R_2R_5R_6R_7 \qquad \text{and} \qquad R_4R_3R_5R_6R_7$$

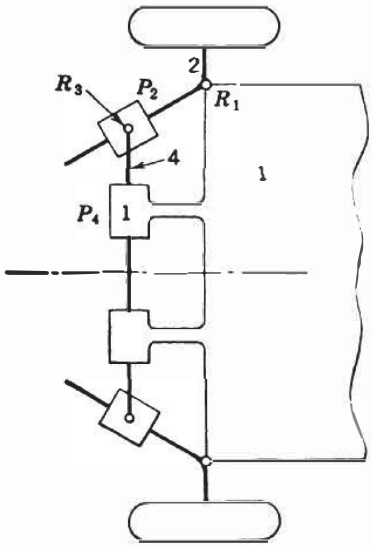
Any two of the above chains, however, are sufficient to describe the compound-closed chain, since they include all the pairs. The compound-closed chain of Fig. 2-26 may therefore be written as

$$R_1R_2R_3R_4$$
$$R_1R_2R_5R_6R_7$$

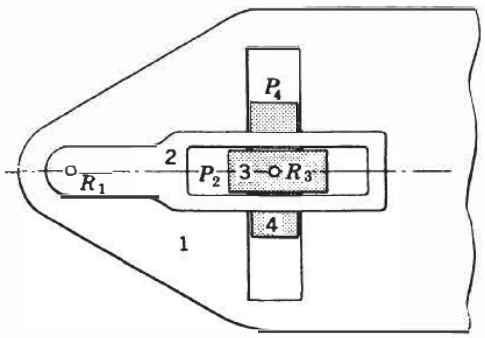
Two other combinations of simple-closed chains would also serve.

The utility of the symbolic notation is shown in rather striking fashion by Fig. 2-38. At first glance, and perhaps even at the second,

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 79 of 143



(a) Davis automobile steering gear



(b) Rapson's slide, a marine steering gear: the rudder post is at R_1, and block 4 is actuated by chains or wire ropes

FIGURE 2-38 Two kinematically identical mechanisms $R_1P_2R_3P_4$.

Rapson's slide and the Davis steering arrangement evidence no kinship. Identification and ordering of the kinematic pairs show the two mechanisms to be the same chain. This is all nicely obscured by the different shapes of the corresponding pieces of hardware (in part due to the exchange of hollow and full elements of the prismatic pairs) and, of course, the unrelated areas of application. Reuleaux, who was the first to appreciate the need for and the niceties of a symbolic notation by creating one, applied it to showing the kinematic similarities existing in the many rotary steam engines known to him.[1]

Mechanisms with higher pairs are subject to symbolic representation only in terms of their equivalent linkages. For certain cases, such as that of Fig. 2-37, a single invariant equivalent linkage may be found that will duplicate the motion transformation between input and output links of the mechanism for the complete cycle. Here the symbolic representation of the cam mechanism is given by that of its invariant equivalent linkage, a planar four-bar.

The foregoing symbolic descriptions are qualitative rather than quantitative: they identify the nature and number of the pairs involved in a mechanism and the order in which the pairs appear but give no information about angles and distances between pair axes. Such a description does not differentiate, for example, between a planar four-bar linkage, spherical four-bar, Hooke joint, and Bennett mechanism, for they are all written $R_1R_2R_3R_4$; further qualification must be supplied

[1] *Ibid.*, pp. 342–384.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 80 of 143

by an adjective. An idea of how the necessary quantitative information is added to the symbolic notation will be given in Chap. 12; until then it must be conveyed in words as part of the context.

BIBLIOGRAPHY

Kennedy, Alex. B. W.: "Reuleaux' Kinematics of Machinery," Macmillan & Co., Ltd., London, 1876. Reprinted by Dover Publications, Inc., New York, 1963.

———: "The Mechanics of Machinery," Macmillan & Co., Ltd., London, 1886.

Reuleaux, F.: "Theoretische Kinematik," Friedrich Vieweg und Sohn, Brunswick, Germany, 1875.

———: "Lehrbuch der Kinematik," Friedrich Vieweg und Sohn, Brunswick, Germany, 1900.

Willis, Robert: "Principles of Mechanism," 1st ed., Parker, London, 1841.

———: "Principles of Mechanism," 2d ed., Longmans, Green & Co., Ltd., London, 1870.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 81 of 143

3 KINEMATIC MODELS

3-1 THE MEANING OF MODELS

The term *model* evokes many interpretations if considered in detail. In the large sense of the word, however, there is the implication that a model—whatever its form—establishes communication and furnishes an understanding transcending the written word. Model has nothing to do with size: it has to do only with perceptive form.

Models of varied kinds have been used for millennia. Depending on their purpose, models may express ideas of design, confirm or negate an assumption or hypothesis, guide analytical reasoning away from a false path, obtain fundamental data, and so on.

In the study of models we shall be principally concerned with hardware types suitable for kinematics. These represent a language of a very special kind, quite distinct from the conventional symbolic notations that have been devised for convenience and manipulative procedures. Models are a visual language. The figures of a geometer are models: with them the relations of points, lines, angles, and planes are kept track of. But ordinary geometry is static, one diagram being sufficient. Kinematics is geometry in motion: there is a new diagram for each instant. Displacement, the first aspect of motion, demands a movable geometric figure capable of displaying (1) the changing

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 82 of 143

relations of angles and line lengths, (2) the position changes of points, and (3) the curves traced by the points. It is one thing to show a mechanism in a given position, and quite another to show several phases, separating each phase in the welter of lines. In fact, the time spent in making a model may very well be less than that required to pursue graphically a motion through the several phases judged to be critical or of interest. Good and bad transmission angles will discover themselves, unexpected change points may appear, and the like. A model certainly furnishes the quickest way to the display of coupler-point curves.

While Lagrange's epoch-making "Méchanique analytique" was further distinguished by a total lack of figures, most persons require diagrams and sketches to define the geometrical relations according to which their work proceeds. Kinematic models able to demonstrate the displacement relations (among other things) consequent to any position are the kinematic equivalents of the geometer's diagrams and the mechanicist's figures.

With a spatial mechanism the drafting problem may verge on the unspeakable because of the maze of construction lines and the need for auxiliary views. Spatial models, more difficult to construct than their planar counterparts, are nevertheless well worth the trouble, for situations defying the imagination are put on display.

Kinematic models fall into three categories. The first group is composed of models that the instructor demonstrates to the class: all members of the class may or may not follow the motions as they come and go. The models of the second group are those which are available for personal manipulation and study ("fussing"). The third and most rewarding category embraces all models, no matter how crude or rude, made by the interested individual. This group, and we denote it as the "do-it-yourself" variety, is the most valuable of the lot. The virtue and merit come from the fact that the man who builds his own models—who nurses them along and makes them work—has lived with the mechanism and learned its moods. Because of this intimacy the model builder learns much more than the model viewer or model fusser.

Fortunately, a great number of planar models of even considerable complication can be made on a desk top from materials and office equipment at hand: no machine-shop facilities are required. Cardboard strips, thumbtacks, scissors, knife, adhesive, etc., are the matériel and tools. Spatial models, having links subjected to bending and torsion and connections such as ball-and-socket joints and cylindrical pairs, require metal[1] and somewhat more in the way of a "shop" facility, but no high-grade shop with elaborate tools. The model maker, in using whatever

[1] It is possible to fold torsionally and flexurally stiff links from cardboard, but metal strips, if available, are more satisfactory.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 83 of 143

material is convenient, will have to supply ingenuity, patience, and time. However, after gaining a little experience, the time expended on a model will be reduced and a simple model produced with little effort. Another way of putting this is to say that not only can standard situations be investigated—those presented in a book or paper, say—but different proportions of the links (lengths) may also be looked at. All this is much more satisfying and instructive than being bound to a prepared model of invariant proportions (e.g., the first and second categories mentioned earlier) or trying to follow link motions on a drawing. This is not to say that drawing-board work can be dispensed with; on the contrary, the drawings become much more meaningful. Models augment drawings; they do not supplant them.

It is of these do-it-yourself models that we shall speak after having considered something of the general background of kinematic model work.

3-2 HISTORICAL SURVEY

Geometric models of various sorts have an ancient history. For example, Archimedes (third century B.C.) weighed models of parabolic segments before developing the theorem that the parabolic area is two-thirds of the area of the circumscribing parallelogram. Archimedes is also known to have made a planetarium of some kind in which the motions of the sun, moon, and planets were displayed. We note that this early kinematic model was concerned with showing the relative motions of *points*. More elaborate devices of the same nature, but operating by wheelwork and stemming from the eighteenth century, are known as orreries.

Nonkinematic models of the past and present are legion. Architects and builders of medieval times made models of their cathedral structures to plan the erection of the stonework. For centuries ships were built by the scaling up of dimensions taken from hand-carved hulls. In the elaborate "Admiralty models," details such as interior framing, decks, and most other structural features were included. For the seventeenth century one may read that it had become the "custom of the times" to develop large-scale undertakings from the miniature. To do more than mention the models used by all phases of the aircraft industry would be to belabor the point.

An early, perhaps first, discussion of kinematic models stems from Robert Willis (1800–1875), the kinematician of Cambridge. Willis was the successor to the Rev. William Farish, a remarkable man. Farish (1759–1837) occupied the chair of chemistry before being elected to the office of Jacksonian professor of natural and experimental philoso-

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 84 of 143

phy in the year 1813. We read in Willis (1851) that Farish "soon after commenced a Course of Lectures on Arts and Manufactures, which he repeated yearly until his death in 1837. The plan of this Course included the exhibition of almost all the more important machines which were then in use in the manufactures of Britain. This led him to conceive the possibility of devising a system of mechanical apparatus consisting of the separate parts of which machines are made, so adapted to each other, that they might admit to being put together at pleasure in the form of any machine that might be required."[1] Willis goes on to say that by 1837, when he himself became Jacksonian professor, "it appeared . . . that his [Farish's] idea of a Protean mechanism was capable of being carried out in a different and more complete manner, so as to be of greater practical utility, and of a more extensive application to philosophical apparatus in general."

Few details are known about the Farish "system," but the pieces were chiefly made of metal, some of the steel shafting being of $\frac{3}{4}$-in. octagon section. Willis said:

> The entire plan . . . had the merit of great simplicity and ingenuity, but the machines were apt to appear somewhat embarrassed and complicated by the variety of clamps and junctions of the framework. It is to be regretted that the ingenious inventor of this mechanism did not draw up a detailed account of his system. The general appearance of his models may be gathered from the sketch of an optical grinding engine appended to his Paper on Isometrical Perspective. . . .

This is Fig. 3-1. In this paper Farish proposed the now widely used isometric perspective of 30°, derived from sighting down the diagonal of a cube. He coined the name *isometric*, since the three perpendicular dimensions of the cube were equal.

Building on the experience of his predecessor, and taking advantage "of the facilities which the improved state of machine-making afforded," Willis devised a rather complete system, or kit, of parts,[2] giving all dimensions, naming the pieces, and so on. He recommended that each setup, once achieved and in operating order, be sketched in isometric projection for the record.

The various parts of the Willis kit were also substantial, since

[1] These lectures became part of the curriculum designed to liberalize education in the classics!

[2] It seems that these kit parts have not been preserved. Such models as are ascribed to Willis—and a few still exist at Cambridge—are made of materials and components quite different from those described so glowingly in 1851.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 85 of 143

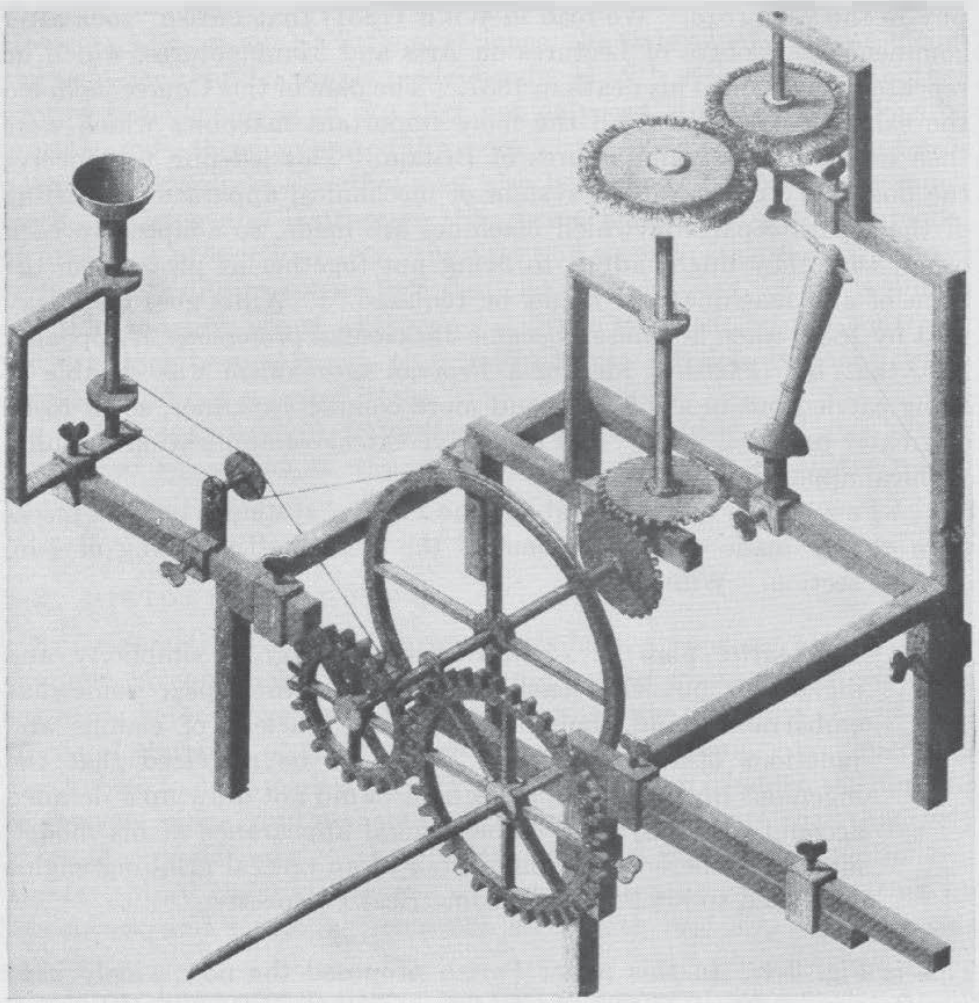


FIGURE 3-1 Optical-grinding engine model built from Farish's kit of components. This is perhaps the first machine drawing rendered in 30° isometric. [*From William Farish, On Isometrical Perspective* (1820), *Cambridge Phil. Soc. Trans., vol.* 1, 1822.]

the apparatuses, intended for lecture-hall use before large audiences, needed generous proportions for visibility. The shafting diameter varied from $\frac{1}{2}$ to 1 in.; the tables and framework (in lengths up to 10 ft) were of $1\frac{1}{4}$- by $2\frac{1}{2}$-in. wood; cast iron was used for brackets, etc., the connections of the framing being made with $\frac{3}{8}$-in. bolts.

Willis found that his extended system accomplished four purposes: (1) the construction of models of mechanisms such as discussed in his "Principles of Mechanism" (1841); (2) the construction of complete machines; (3) "the construction and arrangement of apparatus for Experimental Philosophy in general"; (4) "the trial of new combinations and original research."

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 86 of 143

We see from the Farish-Willis efforts that the first formal kinematic models were assembled as needed from a kit of parts and taken down after use; no permanent display of models was maintained.

Some idea of Willis' system of components may be gained from Fig. 3-2. Here, in Willis' fig. 47, parts have been assembled into "an arrangement for the exhibition of trial of various regulators for mecha-

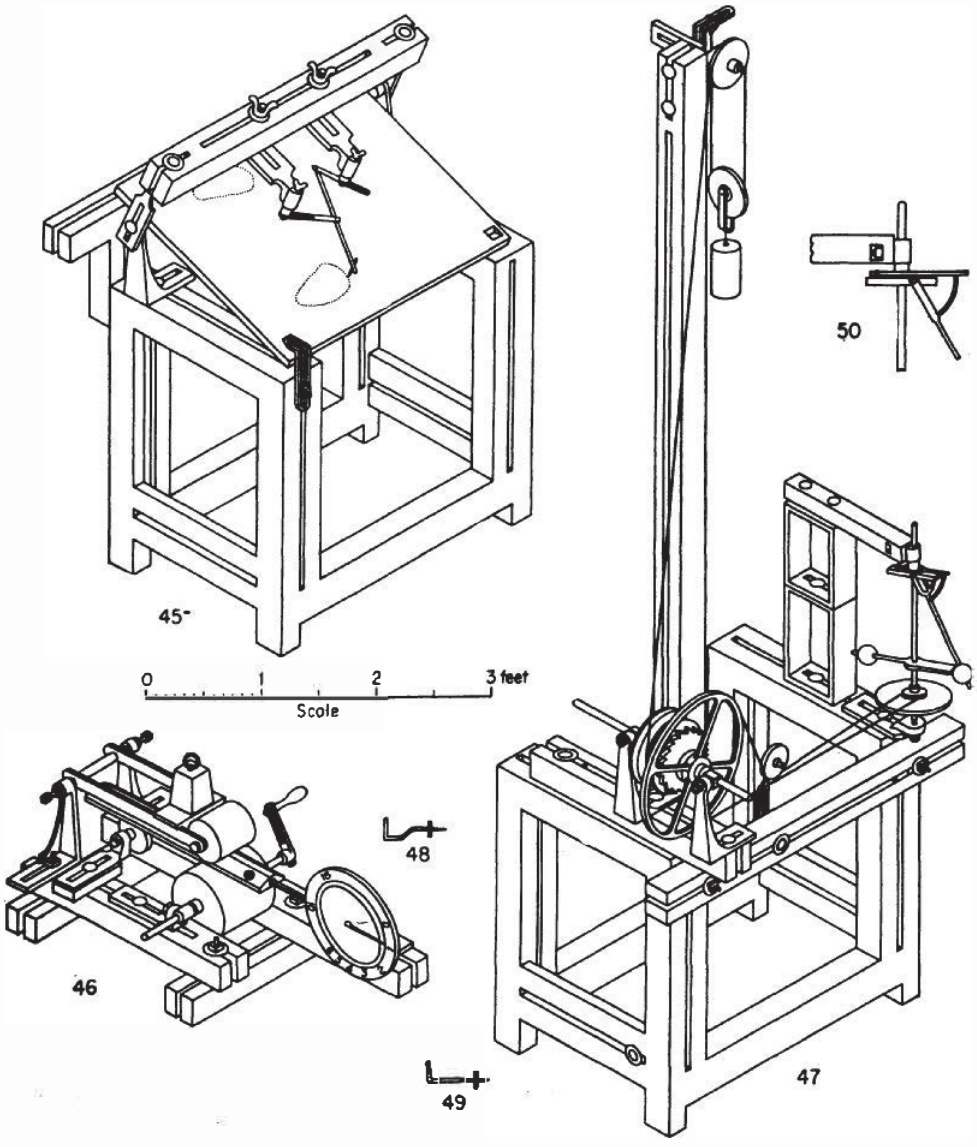


FIGURE 3-2 Models built from Willis' kit of components. Their identification is taken from Willis (1851): fig. 45, parallel-motion curve machine; fig. 46, friction engine; fig. 47, equatorial clock; figs. 48, 49, hook bolts; fig. 50, detail of pendulum.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 87 of 143

nism moved by weight"; and "fig. 46 is a machine to elucidate the laws of friction."

The eye-catching upper-left figure must also be described by Willis:

> Figure 45 is a machine to describe the curves that belong to parallel motions[1] under various proportions of the radius rods and link, and different positions of the describing point upon the latter. . . . A parallel motion consists of two radius rods connected by a link: a pencil is attached to this link, and when the rods are made to turn about their fixed centres of motion, the pencil describes a curve, which, under certain proportions between the lengths of rods, link, etc., possesses the property that a part of its length is so nearly rectilinear, that in practice it may be employed as if it were a true straight line.

Willis has just described what is today called the four-bar linkage and the coupler-point curves that it can draw. We note his emphasis on the "nearly rectilinear" portions of the curve; today we speak of approximate straight-line segments. In Willis' time there was a greater interest in such parallel-motion linkages because of the steam engine. As noted earlier, Watt had shown the way in 1784 when he devised a four-bar linkage whose coupler point guided the upper end of his piston rod along an approximate straight line, good enough for his purpose. In Watt's day, long before the invention of the planer (1817), pin-connected linkages were readily made, while sliding crossheads were not.

[1] Willis (1841 ed., p. 399) explains:

> A parallel motion is a term somewhat awkwardly applied to a combination of jointed rods, the purpose of which is to cause a point to describe a straight line by communicating to it simultaneously two or more motions in circular arcs, the deviations of these motions from rectilinearity being made as nearly as possible to counteract each other.
>
> The rectilinear motion so produced is not strictly accurate, but by properly proportioning the parts of the contrivance, the errors are rendered so slight that they may be neglected.

Kempe (p. 49; see Bibliography) says:

> I have been more than once asked to get rid of the objectionable term "parallel motion." I do not know how it came to be employed, and it certainly does not express what is intended. The apparatus does not give "parallel motion," but approximate "rectilinear motion." The expression, however, has now become crystallized, and I for one cannot undertake to find a solvent.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 88 of 143

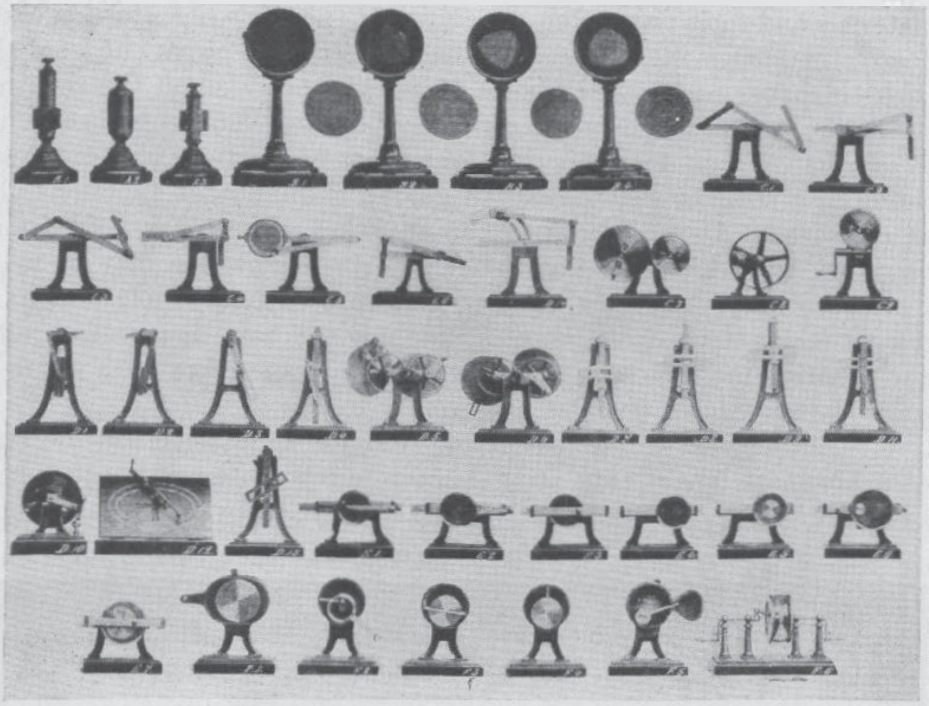


FIGURE 3-3 A page from the 1907 catalogue of kinematic models made to Reuleaux's designs. Some 368 were available. (*Courtesy of the Department of Machine Design, Cornell University.*)

The burgeoning interest in engines gave rise to a number of "parallel motions" different from Watt's

We must recognize another great model activity. It is that of Reuleaux (1829-1905) in Berlin and after the 1870s. Reuleaux's were all permanent models, sturdily constructed of iron and brass, each showing a particular kinematic situation (Fig. 3-3). The magnificent Berlin collection of several hundred models was destroyed in World War II Smaller collections of Reuleaux models still exist; the principal one is at Cornell University and numbers 266 items

The new Technical University of Berlin has not attempted to re-create its lost collection. Instead, planar models, individually mounted on panels perhaps 7 by 14 in., have been assembled in groups of 21 in wall cabinets. Such arrays are brought into motion by a push button, each panel's models being driven by a single electric motor.

3-3 PLANAR KINEMATIC MODELS

As can be imagined, the problems attendant upon the construction of planar models are less trying than those of spatial models. The

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 89 of 143

materialsanditoolspresentnoidifficulties. Theibasiclinkistockiisicardboard; the models illnstratcd were made of strips or pieces cut from the backs of paper pads. The choice of stock depends upon the required stiffness. A phototrimmer is a convenient but not indispensable adjunct toscissors. Thumbtacksare used for turning connections between links; riding in a straight slot (crosshead guide), a thumbtack functions also as crosshead. Depending upon the situation, thumbtack points may be up or down. When a point sticks up, the connection may be made morei secure byi pressingi ai circulari punchingi (ordinaryi holei punch) of manilai folderi stocki overtheipoint, clippingitheihazardousextralength, andisecuringtheipunchingitoithcirremainderiofitheipointiwithia dabiof quick-drying cement (such as Duco)

Although it will be instructive to build some models in the form of a kinematic chain (Fig. 3-4), most interest centers in mechanisms, i.e.,ichainsiwith oneifixedilink. Thisilink isifixediby mounting iti on a

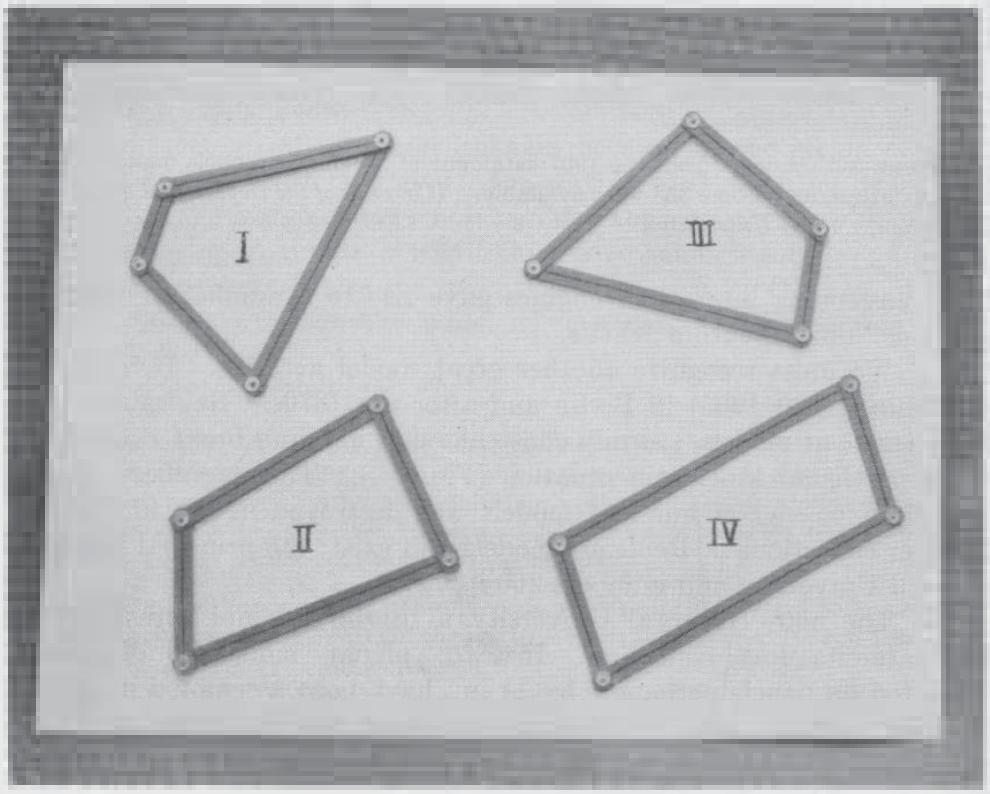


FIGURE 3-4 Four-barichainstoillustratetheGrashoficondition. (I) $l + s < p + q$ the "Grashofifour-bar" linkwhichitheshortestlink rotatesicompletelyiwithrespectto itsneighbors. (II) $l + s > p + q$,ionlydouble-rockermechanisms possible. (III) $l + s = p + q$, pathological case of change points. (IV) $l + s = p + q$, case III in parallelogram form.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 90 of 143

"breadboard" or base of some sort. A very convenient base is a piece of $\frac{1}{2}$-in.-thick insulating board, cut to $8\frac{1}{2}$- by 11-in. size. If this is not available, several thicknesses of carton stock will also serve. Such materials accept thumbtacks readily and also permit their easy retraction. A piece of paper, on which the frame points and other pertinent data have been entered, may be stapled over the base so that coupler curves may be drawn on it, etc.

The previous remarks give some idea of what may be involved in the construction of planar models. The problems incident to various situations—sired by necessity and born out of trouble—cannot be discussed in detail, but solutions will present themselves. There will be annoyances, such as buckling links, the need for reconnection in order to have one link pass over another, sliders that stick or are too loose, and so on. The need for some debugging should always be anticipated. A little patience and a light touch will help. The model will be a pretty thing to see when it runs; not the least of the pleasure is to note the amount of quantitative information that is furnished, in addition to the basic qualitative data.

The four-bar linkage and the study of some of its moods are ideal for first models. The four-bar is perhaps the most widely used linkage, for despite its apparent simplicity—only three moving parts—it manages to conceal a great sophistication. To start with, there are of course as many mechanisms as there are links, a different link being fixed to define each case. In addition, the link proportions (lengths) or parameters control the extent of the relative motion between any two connected links. We may have a particular interest in the motions of the two links connected to the frame. Confining our attention to the four-bar linkage, we shall call a link able to rotate continuously in the same direction a crank; if it can only oscillate, a rocker. The remaining moving link is the coupler.

Possible combinations of frame-connected links yield crank-rocker, double-crank, and double-rocker mechanisms: we shall call these different kinds. Criteria for the various situations may be studied with the aid of the Grashof relation.

Let us identify the longest link as l, the shortest as s, and the remaining two as p and q. The following relations, stated without proof, are valid:

1. A Grashof four-bar is one in which the sum of the lengths of the longest and shortest links is less than the sum of the lengths of the other two links, that is, $l + s < p + q$. Then *a, b.* Two different crank-rocker mechanisms are possible. In each case, the shortest link will be the crank, the frame being either adjacent link.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 91 of 143

c. One double-crank (drag-link) mechanism results when the shortest link is the frame.

d. One double-rocker mechanism is formed when the link opposite the shortest is the frame.

2. If $l + s > p + q$, then only double-rocker mechanisms result; there will be four, depending on which link is made the frame.
3. If $l + s = p + q$, the four possible mechanisms are like those of (1), all, however, suffering from a condition known as change point. At the change point the centerlines of all links become collinear, whence the cranks may change direction of rotation unless given guidance.
4. *a.* The parallelogram linkage is a special case of (3); all four mechanisms are double cranks if they are controlled through the change points.

   b. The deltoid linkage is the other possible special case of (3). In this, two equal short links are connected to two equal longer links, and in that order. With a long side as frame, a crank rocker is possible; the frame as short side may give a double-crank mechanism, in which the short rotating link makes two turns to the longer link's one.[1] The uncertainties of these mechanisms are related to getting past the change points.

Mechanisms to the foregoing proportions may be investigated on the drawing board. Greater satisfaction is derived from a set of models, for which some convenient dimensions are suggested:

I Links 1.5, 3, 4, and 5 in. long
II Links 2.5, 3, 4, and 5 in. long
III Links 2, 3, 4, and 5 in. long

Linkages made to the above dimensions are shown in Fig. 3-4. In addition, there is a parallelogram linkage. The links may be connected in any order, for the relations of $l + s$ and $p + q$ are invariant with respect to the link sequence. The models shown can be held in the hand, although it may be desirable to thumbtack the link chosen as fixed to the breadboard.

Another example involves the most spectacular feature of the four-bar linkage, the coupler-point curve, familiarly called the coupler curve. A coupler point is any point of the coupler plane. We recall that any actual coupler is only a discrete part of the plane—as much as may be useful or necessary. Only two points of the coupler plane have rotational motion; all other points are said to have planar motion, treated as a combination of translation and rotation. The coupler

[1] This is what the Germans call a Galloway mechanism, after the English engineer who patented it in 1844.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 92 of 143

curvesareitraced byiaicouplerpoint on the plane ofithe frame link; in our case this is the link tacked to the breadboard. These curves, of thesixth order in the general case, may have very curious shapes indeed Portions of these curves—segments of particular shape—can often be exploited to good use, as we shall see. Special four-bar linkage configurations will give fourth- and second-order curves

Theibest-knowniexampleiof theiuseiofithcicouplericurveiisiini the mechanismdevised byiJamesWatt,theiso-calledparallelimotioniready

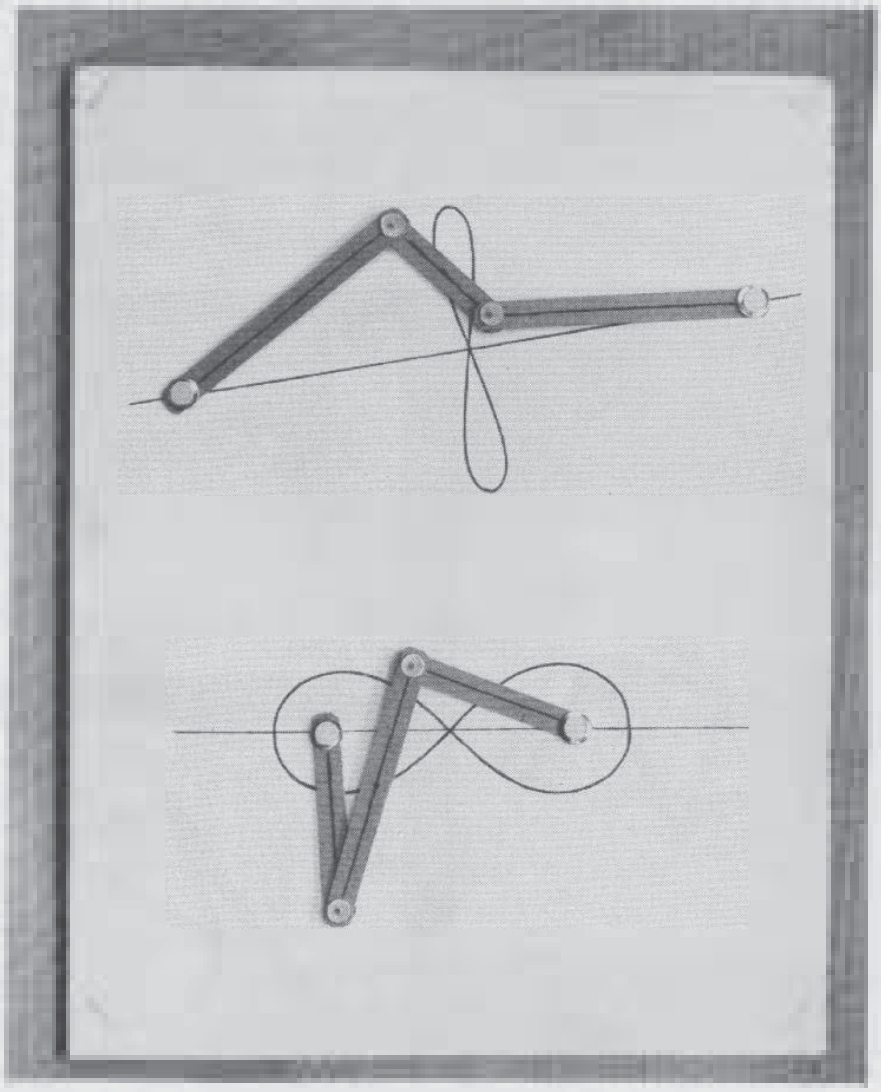


FIGURE 3-5 Coupler-point curves of two symmetrical four-bar mechanisms. (*Upper*) Approximate straight-line mechanism of Watt's proportions. (*Lower*) With links in the proportion of $1:\sqrt{2}$, the coupler point describes Bernoulli's lemniscate.

noted. Its action may be followed by preparing three links, two of them 3 in. long, and one of $1\frac{1}{2}$ in. The last will be the coupler and should have a small hole for a pencil point at midlength: this will be the coupler point to be investigated. Mounting the long links about 6.5 in. apart on the base (Fig. 3-5) will define the frame. A pencil in the coupler point will trace the coupler-point curve when the linkage is moved through a full cycle.

The figure-eight curve has two segments that are approximate straight lines. Watt used one of these segments as a good-enough rectilinear guide, thus obviating the need for a crosshead. The symmetry of the curve is a consequence of the symmetry of the linkage. A change in proportions, such that the frame and coupler links are $\sqrt{2}$ times as long as the equal crank and follower, gives a linkage drawing the lemniscate of Bernoulli from the midpoint of the coupler (Fig. 3-5, bottom). This is a fourth-order curve.

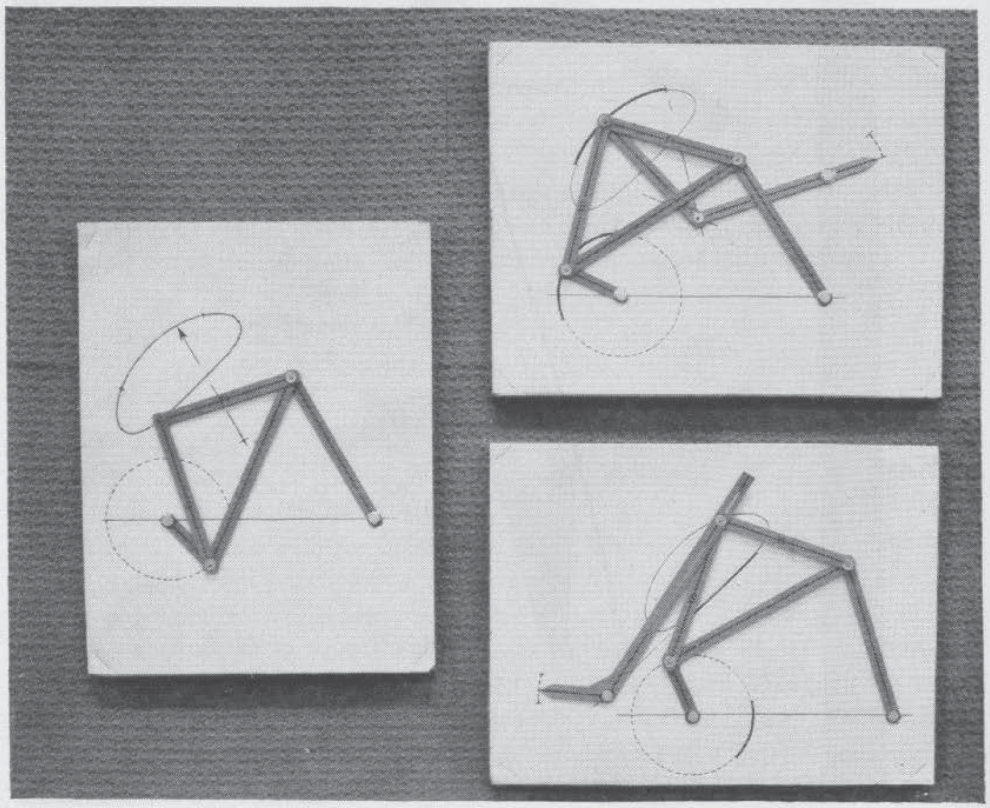


FIGURE 3-6 Approximate dwell linkages derived from a four-bar coupler curve. (*Left*) The coupler curve and its approximately circular segment, with location of center. (*Upper right*) Output link with dwell (to right). (*Lower right*) A dwell linkage taking advantage of the approximately straight segment by means of a slider.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 94 of 143

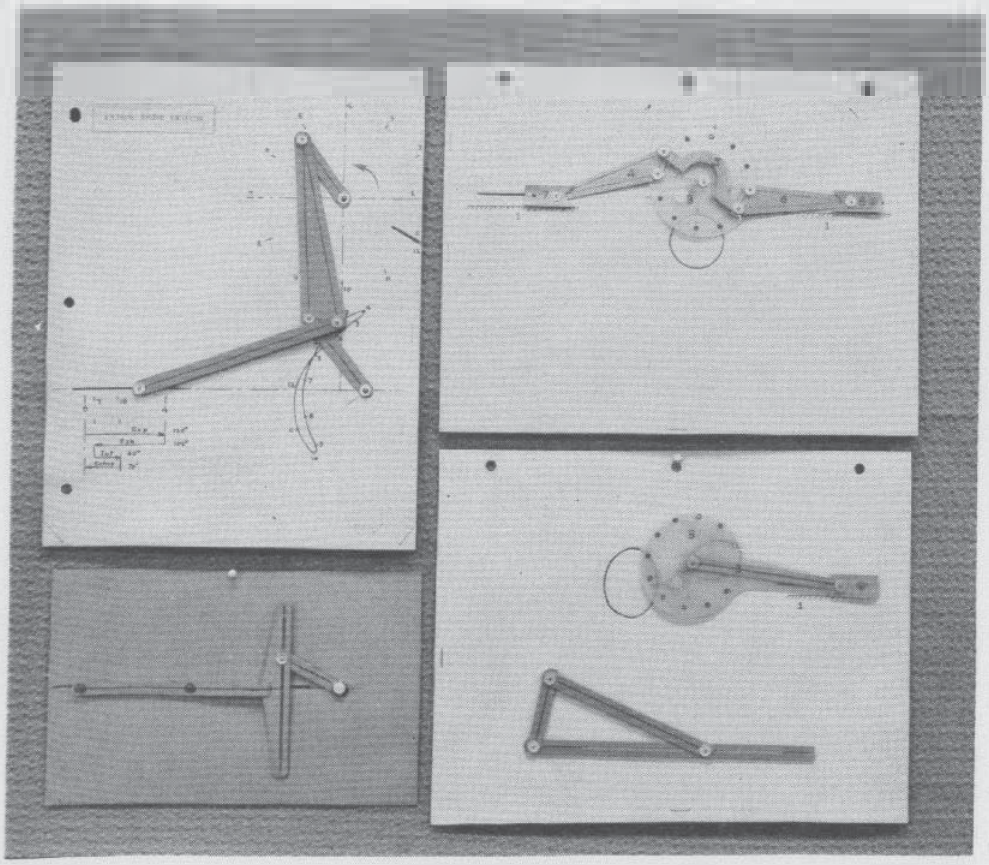
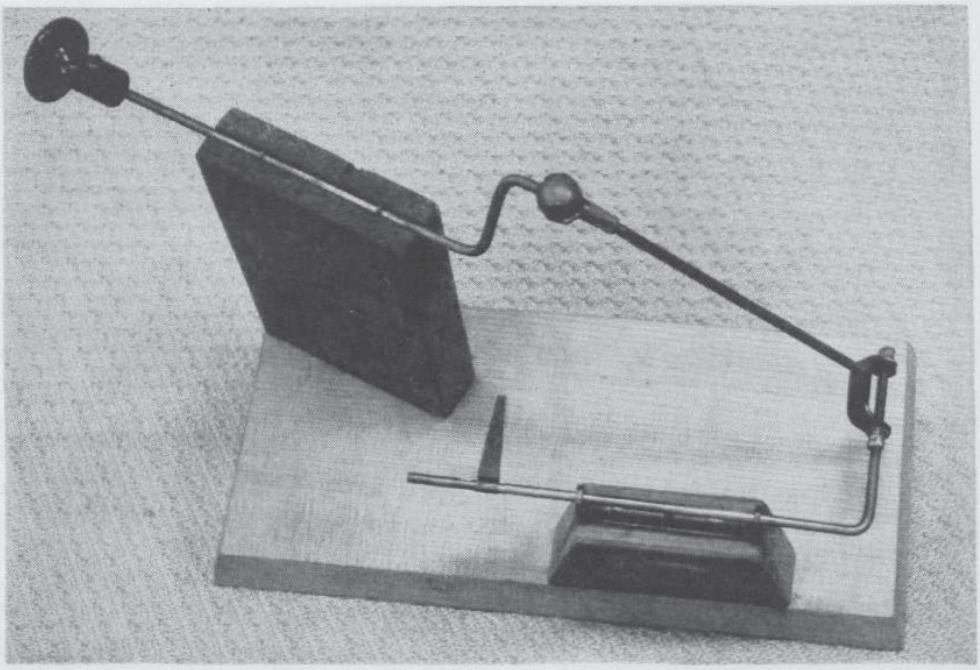


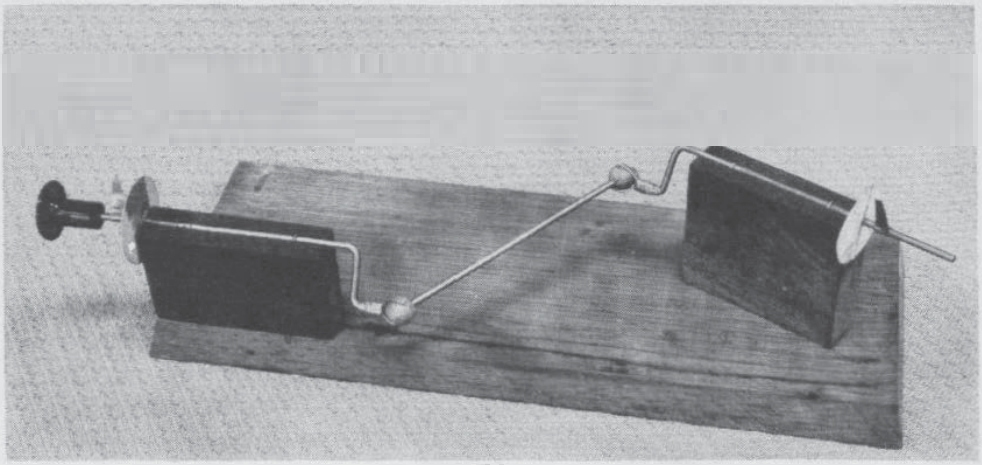
FIGURE 3-7 Typical models suitable for $8\frac{1}{2}$- by 11-in. notebook or file folder. (*Upper left*) Atkinson engine mechanism—four piston strokes per crankshaft revolution. (*Lower left*) Simple harmonic motion. (*Upper right*) Swashplate engine, in which all rods connecting to ball joints describe circular paths. (*Lower right*) Conventional radial engine in which only the center of the mother rod 3 describes a circular path; all other rod ends trace complex coupler paths to give the piston-connected rollers.

Coupler-point curves may also have segments approximating circular arcs. If linkage proportions are propitious, a six-bar mechanism having an approximate dwell in the output motion may be devised. The idea may be demonstrated by preparing a coupler of 5 in., a crank of $1\frac{1}{2}$ in., and a follower of 4 in., spacing the last two 5 in. apart on the base with the crank to the left. Two additional strips are needed, a strip m $3\frac{3}{4}$ in. long and a strip n $3\frac{3}{8}$ in. long. One end of m is placed on the thumbtack connecting crank and coupler and one end of n on the tack between follower and coupler. The two free ends of m and n should be to the outside of the linkage and connected with the point of a pencil. The pencil point is then a coupler point whose curve should be traced (Fig. 3-6, left).

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 95 of 143



(a)



(b)

The upper portion of the curve contains a nearly circular arc portion. The "center" of this arc is determined with a pair of dividers, the arc length being restricted to about 2 in.

This center is labeled R, and a link is made to this length. One end of the link is stuck over a tack inserted at the coupler point; the other end is pinned to the board at R. This gives a five-bar linkage, movable after a fashion over the arc length, and binding to either side

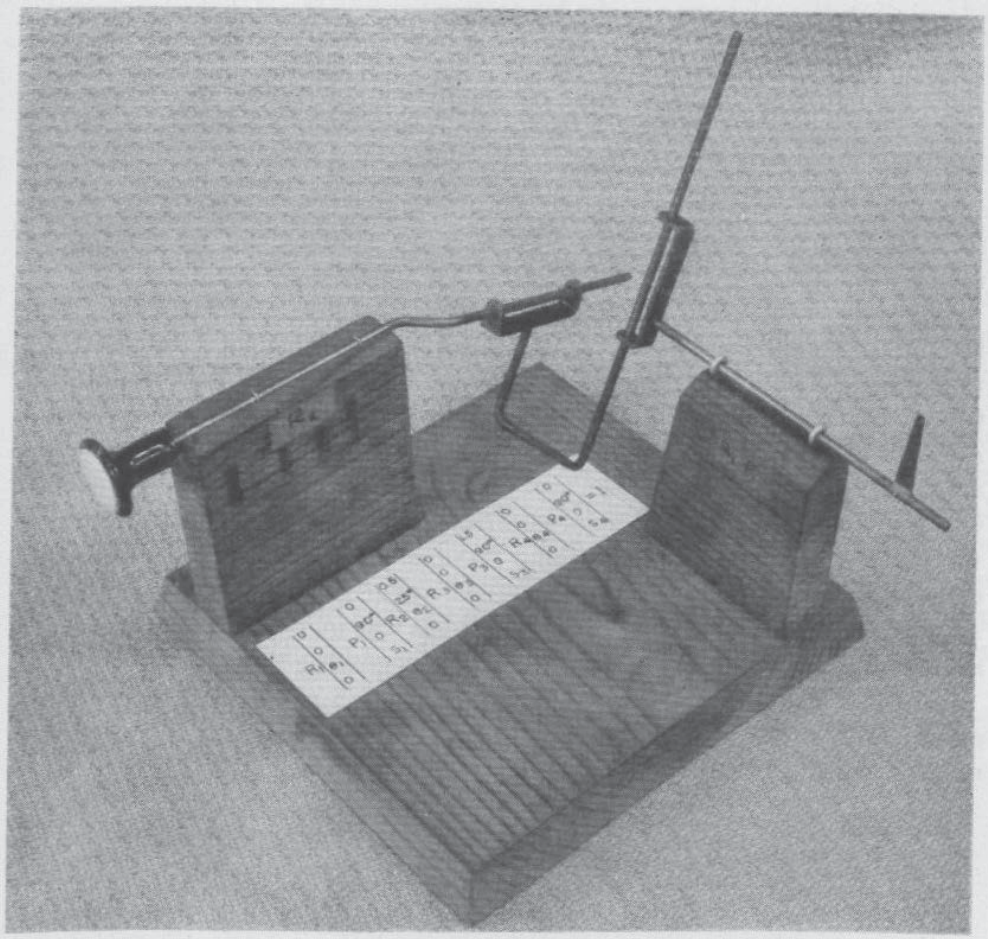


FIGURE 3-8 Examples of spatial mechanisms with revolute, cylindrical, and spherical pairs. The inputs are in all cases continuous-rotation at the knob. The links are of $\frac{1}{8}$-in. brazing rod. (*a*) An *RGRC* mechanism. The output is combined translation and oscillation. Size of base $4\frac{1}{4}$ by $8\frac{3}{4}$ in. (*b*) An *RGGR* mechanism. The output is oscillation. Size of base 5 by 13 in. (*c*) An *RCCC* mechanism. The output is combined translation and oscillation. Size of base 7 by 7 in.

of it. The movability[1] (theoretically impossible) accrues from the play in the connections and deformation of the links, since the arc is not truly circular.

A sixth link, say 4 in. long, is prepared. The tack at *R* is reversed and one end of the new link is connected to the radius link. The other end of the new link is pinned to the board at some convenient location; its motion will be the output of the mechanism. One moving the mechanism through a full cycle, the output link will be seen to have an approxi-

[1] Movability is discussed in Sec. 5-3.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 97 of 143

mate dwell while the coupler moves over the approximate circular arc and the crank rotates continuously. This dwell period, like the approximate straight-line motion, may be good enough for many purposes. Since four-bar coupler points generally describe sixth-order curves, their curves will possess neither truly straight lines nor truly circular arcs—but they may be satisfactory approximations. A different dwell mechanism may be constructed by utilizing the "straight" portion of the curve.

Other examples of planar mechanisms are shown in Fig. 3-7. The method of coping with sliders is indicated, and it must be admitted that the simple-harmonic-motion device has an action that leaves something to be desired. The Atkinson engine model (upper left) unravels the mystery of how four piston strokes per crankshaft revolution may be achieved by having the piston rod driving along a coupler-point curve. Other situations that also seem confusing when seen as only drawings, such as the Nordberg radial-engine linkage (upper right), and even the conventional radial engine (coupler curve again!), are clarified by models. A moving link of each of the last two models was made of Mylar to permit a view of the curves traced by points of the moving link.

The last three models were constructed on pieces of manila folder cut to $8\frac{1}{2}$ by 11 in.; their format allows them to be carried in notebooks and stored in file folders.

3-4 SPATIAL KINEMATIC MODELS

To put it somewhat inelegantly, spatial linkages are those taking motion around a corner, whereas planar linkages merely transform motion in a plane. In leaving the plane and adding an effective third linear dimension to the links, the forces at the connections of the now nonflat links give rise to bending moments and torques. These moments and torques require that the links be stiffer to maintain the geometrical truth of the connections. Generally speaking, cardboard links will no longer do unless they are formed to give substantial moduli of inertia; and neither is the cardboard convenient for twisted links or the fashioning of cylindric and spheric pairs. In consequence, it is usually best to use metal, whose choice of form—strip, rod, or tube—depends on the method of fabrication and the nature of the connections between links. Spatial mechanisms of less than seven links will contain cylindric and spheric connections, both more difficult to make than the turning and sliding pairs of planar mechanisms, and metal links lend themselves to this sort of thing. The models of Fig. 3-8 show some solutions to various problems.

It is evident that such spatial models do require more of a shop

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 98 of 143

than is needed before the construction of planar models. But it is also obvious that the necessary shop equipment is minimal.

The bases were cut from packing-case lumber. The shafting is brazing rod; the flat strips of the cylindric pairs happened to be brass. It was convenient to make some fastenings via threaded rod ends and tapped holes. The balls of the spherical joints are made of plastic beads known as Pop-it, available at small expense. Paper clips soldered to the cranks form the sockets. The input shafts are constrained to motion of only rotation by two staples sitting into grooves in the shafts. Screw eyes supporting the output links permit the simultaneous translation and rotation of the rod constituting an element of the cylindric output pair.

Some spatial mechanisms whose links are connected by only turning pairs, such as the Bennett, Bricard, and Goldberg, may be made of, say, $\frac{3}{8}$-in. sheet-metal strips suitably twisted and connected by metal eyelets (Figs. 3-9 and 3-10). Eyelet kits and setting tool (a trade name is Dritz) may be found in hardware stores and notions counters of dime stores. The axes of the turning pairs have been emphasized by soldering wires into the eyelets. The turning pairs of spherical linkages allow a

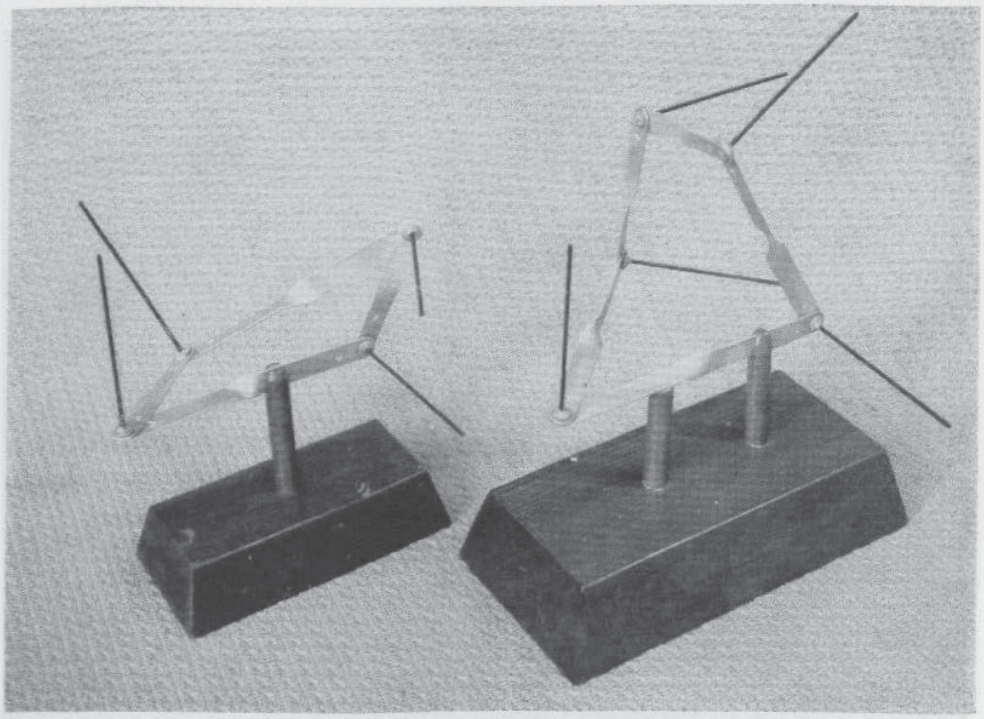


FIGURE 3-9 (*Left*) A Bennett mechanism, with link ratio of 1:2 and twist angles of 30 and 90°. (*Right*) A Goldberg five-bar mechanism. Aluminum strips of $\frac{3}{8}$-in. width, $\frac{1}{16}$-in.-diameter brazing rod, base sizes $4\frac{1}{2}$ by $2\frac{1}{4}$ and $6\frac{1}{2}$ by $3\frac{1}{2}$ in.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 99 of 143

simple cardboard construction (Figs. 12-9 and 12-13). It is well, after laying out the central angles of the linkage that define the sectors, to cut the sectors free and hinge them with masking tape (not cellophane tape). This provides a free motion, without the bias of just a fold, and allows laying in wires to call out the axes of rotation.

Spherical joints of higher quality than the bead and paper-clip type are available in a variety of sizes, grades, angular motion, and rod ends. Their use is shown in Fig. 12-11.

For very special models, such as those needed for quantitative verification, commercially available components may often be used to good advantage. The utilization of parts such as bearing supports, bearings, shafting, accurately divided dials, and verniers leaves only such things as cranks for local manufacture (Fig. 12-16).

The foregoing materials and methods relating to model construction have been presented as suggestions. Certainly other materials and techniques will give equally satisfactory, or even better, results. The principal idea is to build models and note the motion characteristics they are capable of demonstrating.

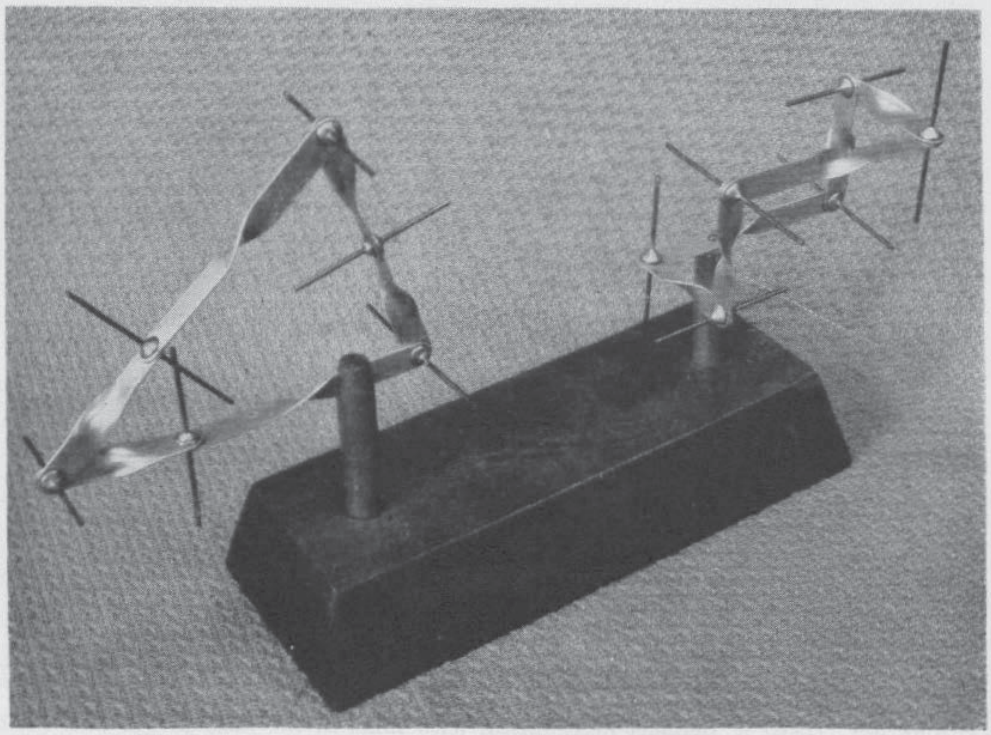


FIGURE 3-10. Bricard six-bar mechanism, shown in two different phases. Size of base $3\frac{1}{2}$ by 8 in.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 100 of 143

BIBLIOGRAPHY

Cundy, H. Martyn, and A. P. Rollett: "Mathematical Models," Oxford University Press, New York, 1954, 1961.

Farish, William: On Isometrical Perspective (1820), *Cambridge Phil. Soc. Trans.*, vol. 1, 1822.

Grashof, F.: "Theoretische Maschinenlehre," vol. 2, Voss, Hamburg, 1883.

Hartenberg, R. S.: Die Modellsprache in der Getriebetechnik, *VDI Ber.*, vol. 29, 1958.

Kempe, A.: "How to Draw a Straight Line," Macmillan & Co., Ltd., London, 1877. Also in E. W. Hobson et al., "Squaring the Circle," Chelsea Publishing Company, New York, 1953.

Willis, Robert: "A System of Apparatus for the Use of Lecturers in Mechanical Philosophy, Especially in Those Branches Which Are Connected with Mechanism," John Weale, London, 1851.

———: "Principles of Mechanism," 1st ed., Parker, London, 1841.

———: "Principles of Mechanism," 2d ed., Longmans, Green & Co., Ltd., London, 1870.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 101 of 143

4 KINEMATIC ANALYSIS OF PLANAR MOTION

4-1 INTRODUCTION

Kinematics, the study of geometry in motion, covers two broad and interrelated areas that are subject to separate study. An existing or specified mechanism or hypothetical situation may be investigated for whatever characteristics and properties it possesses; this is formally called analysis. The inverse of such a procedure is synthesis: in this, a mechanism is created meeting the specification of certain desirable characteristics and properties. Many of the operations forming a synthesis are directly related to or make use of procedures stemming from analysis.

This chapter presents some generally useful aspects of velocity and acceleration analysis for planar mechanisms, based on geometrical (graphical) procedures. No previous acquaintance with kinematic analysis is assumed; this allows a treatment in closed form, but due attention is given to details and concepts deemed important even to the mature reader who may have fallen into careless habits. Sophisticated procedures and special methods for particular types of mechanisms are not a direct part of the material to be discussed.

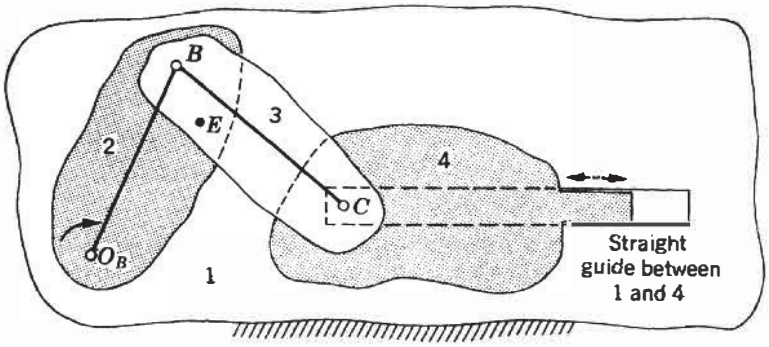
Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 102 of 143

4-2 COINCIDENT POINTS

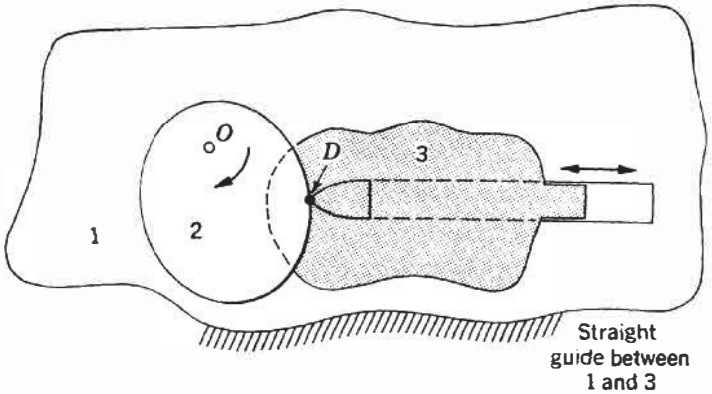
The properties of coincident points are best discussed by considering a mechanism to consist of as many superposed planes as there are links. Each plane is of infinite extent, and any physical link or machine member is then only a portion of a plane. The superposed planes of two mechanisms are visualized in Fig. 4-1.

The relative motion of one plane with respect to its neighbor is dictated by the nature of the connection or contact between planes. Thus, a revolute or pin connection will allow only rotation, as for example at O_B, B, and C in Fig. 4-1*a*. A prismatic pair or rectilinear slide, as between links 1 and 4 in Fig. 4-1*a*, limits the relative motion to rectilinear translation. Turning to the cam mechanism (Fig. 4-1*b*), we find a motion between link 2 (cam) and 3 (follower) that is a combination of rolling and sliding. The latter would also occur if two links were connected by a pin riding in a curved slot.

Consider the pin located at O_B (Fig. 4-1*a*). Whether it is large



(a) The four planes of a slider-crank mechanism



(b) The three planes of a cam and follower mechanism

FIGURE 4-1 Coincident points.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 103 of 143

or small, roller bearing or not will have nothing to do with its kinematic function of limiting motion to only rotation: kinematically it is only the axis about which rotation between links 1 and 2 takes place that is important. This axis pierces both planes at O_B; and this point, common to both planes, is identified equally well as O_{B1} or O_{B2}, points on links 1 and 2 at O_B, but clearly coincident or superposed. Each of these coincident points is firmly attached to its own link at the axis of rotation, superposition being always maintained. We shall call these permanent coincident points permanent centers.

Consider next point E, shown on link 3 (Fig. 4-1*a*). We shall call it E_3. Directly under it at this moment—or this position of the mechanism—lies point E_2 of link 2, and under this E_1 of link 1. These three superposed points are fixed to their links at the locations at which we see them; with motion of the links, they retain their positions *on their respective links*, but not with respect to each other, each going with its link. Motion of link 2 will cause separation of the coincident points. Each no longer superposed point will trace a curve on each of the planes different from its own. Sometimes the shapes of the curves are obvious, but not always We see that E_1 will trace a circular arc (about $\bullet_B$) on plane 2; E_2 will trace a circular arc (about $\bullet_B$) on plane 1. These two curves are traced in opposite directions; i.e., if link 2 is rotating counterclockwise, E_1's trace on plane 2 will be developing in the opposite sense.

E_2 will also trace a circular arc on plane 3. E_3 will trace a circular arc on plane 2 and some sort of curve on plane 1. The shape of this curve—actually a coupler-point curve—is usually difficult to imagine. It is of the fourth order, since it is derived from a slider-crank mechanism (the corresponding curve of a four-bar linkage is of the sixth order). Point E_1 will trace a different fourth-order curve on plane 3.

It should be remarked that while our sketch does not attempt to show it, there is also a point E_4 busy tracing curves. Thus, E_4 makes a straight-line trace on plane 1, a circular arc on plane 3, and some kind of curve on plane 2. And, of course, points E_1, E_2, and E_3 leave their traces on plane 4.

As drawn, Fig. 4-1*a* evidences three coincident points at C. Points C_3 and C_4 of the pin axis are identical in their actions with respect to plane 1, tracing straight coincident lines on it. Meanwhile C_1 is tracing a straight line on plane 4 and a fourth-order curve on plane 3.

The motion between cam profile and follower point (D of Fig. 4-1*b*) is a combination of roll and slide, as we have noted earlier. An observer stationed at D_3 would notice the cam rolling about D_3 and simultaneously sliding by it; i.e., the point D_2 (fixed to the cam profile) would trace some curve on the observer's plane 3. D_3's trace on plane 2 is the cam profile itself. Here, too, the originally superposed points move away from each other.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 104 of 143

Coincident points of moving systems are thus seen to be of two kinds: (1) those which remain permanently coincident, occurring only along a permanent axis of rotation; and (2) those which separate on motion, having been coincident only instantaneously, as at the moment of a particular configuration.

4-3 NOTATION

The discussion of physical events is dependent upon a notation of some sort, preferably associative to convey meaning. In the absence of a universally accepted notation, we shall frequently use the symbols noted below and shall also improvise as necessary. The context will serve to distinguish the meanings of symbols having several qualities.

$\mathbf{a}$	total acceleration (vector)
$\mathbf{a}^n$, $\mathbf{a}^t$	normal and tangential components (vectors)
$\mathbf{a}^r$, $\mathbf{a}^\theta$	radial and transverse components (vectors)
$\mathbf{a}^{cor}$	Coriolis component (vector)
$a = \lvert\mathbf{a}\rvert$	magnitude of acceleration (scalar)
$\mathbf{a}_{CB}$	acceleration difference of points C and B, defined by $\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{CB}$
$A, B, C, \ldots$	points of a linkage, usually joints
O	point of zero motion
$P, Q, R, \ldots$	points on links other than joints
$\mathbf{PQ}$	vector representing the directed distance PQ
$\mathbf{v}$	total velocity (vector)
$\mathbf{v}^r$, $\mathbf{v}^\theta$	radial and transverse components (vectors)
$v = \lvert\mathbf{v}\rvert$	magnitude of velocity (scalar)
$\mathbf{v}_{CB}$	velocity difference between points C and B, defined by $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$
$1, 2, 3, \ldots$	links comprising the chain or mechanism
ϑ	("tetta"), angle defining the ϑ line of velocity distribution
$\boldsymbol{\rho}$	position vector, variable
$\rho = \lvert\boldsymbol{\rho}\rvert$	magnitude of the position vector
r	radius of a circle, fixed
$O_BB, BC, \ldots$	distances
θ	angle of position vector with respect to reference line; ρ and θ are polar coordinates defining the position of a point
x, y	rectangular position coordinates
ω	angular velocity
α	angular acceleration

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 105 of 143

Consistent units are implied, with radians for angular measure unless otherwise specified.

4-4 LINEAR AND ANGULAR VELOCITIES

Motion refers to an event requiring a certain amount of time for its completion. Motion includes at least three distinctly different yet related characteristics—displacement, velocity, and acceleration. To these may be added higher derivatives, such as jerk and those following. To describe the motion of anything, we need a reference system to which we refer the motion or from which we observe it. In kinematics, any arbitrary reference system will do, but, of the lot, one may be much more convenient than the others. The motions of the various links of a mechanism are generally referred to a link called the *frame*, and the motions with respect to it, i.e., relative to it, are called *absolute* motions. Motions referred to a link other than the frame are called *relative* motions. We see that absolute motion is merely a special case of relative motion.

When only planar motion is considered, as in this chapter, we may speak of a reference plane. Following the selection of a convenient origin O and rectangular axes Ox and Oy fixed in the reference plane, the position of a point may be defined by cartesian (x, y) or polar (ρ, θ) coordinates; the angular position of a line may be given by an angle such as θ between the axis Ox and the line.

Displacement is defined as the difference between the position coordinates of the final and initial locations. Dependence on only position coordinates makes displacement a vector quantity; it is the most direct route from here to there. Distance, on the other hand, depends upon the actual route or path; it is not a vector, but only a magnitude. For example, a man (point) driving from Chicago to New York suffers a displacement expressible by the differences in the latitudes and longitudes of the two cities. The distance traveled depends upon the highways chosen and how often the man got lost and had to retrace or correct his route. Had the man been able to go "as the crow flies," his displacement and distance would have been the same. On the man's completion of a round trip, his displacement was zero (same position coordinates after as before), but the distance was the odometer reading of his car.

The same ideas apply to the angular motion of a line. If a line rotates directly to 270°, then both angular displacement and distance will be 270°. If the line oscillates on the way to the 270° mark, the angular distance will be greater than the 270° displacement. When the line rotates continuously in the same direction, the angular displacement will be zero on passing its original position; the angular distance is of

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 106 of 143

course 360ℓ. With continuous rotation the angular distance mounts by 360ℓ for each revolution, while the displacement can never exceed 360°.

Linear velocity is defined as the time rate of change of position of a point. Since this change of position or displacement is a vector quantity, velocity is also a vector quantity, having the sense of the displacement being taken on.

Angular velocity is the time rate of change of angular position of a line; it is a positive or negative number for planar mechanisms.

Speed is the time rate of covering distance; it is not a vector, but only a number, for it has all the varied directions of the path. Our Chicago–New York driver would have had an average velocity given by dividing the displacement by the trip time, whereas his average speed would have been the result of dividing the actual distance by the time. The speedometer showed the instantaneous speed, which was also the magnitude of the instantaneous velocity. Angular speed is usually given in rpm; when there is a designation of direction, as "1,800 rpm, clockwise when facing the output shaft," we speak of angular velocity.

We consider the details of linear velocity—or what happens when a point of any plane changes its position—with the aid of Fig. 4-2. The plane of the paper is the reference plane on which the point B of the moving link k traces the path s shown; the point changes its location from s_i to s_f in a time interval Δt.

We are at liberty to select an arbitrary origin and reference line for the polar coordinates ρ and θ. We shall consider the path s to be traced by the head of the bound vector $\boldsymbol{\varrho}$ as the time t varies. The vectors $\boldsymbol{\varrho}(t)$ and $\boldsymbol{\varrho}(t + \Delta t)$ for positions s_i and s_f will also be called position vectors. For analytical reasons they must of course be functions of time t and must possess at least a second derivative. We also establish two sets of rectangular directions at s_i. They are the r (radial) and θ (transverse) directions, related to the *reference coordinate system;* the t (tangential) and n (normal) directions related to the *path.*

The finite vector change $\Delta\boldsymbol{\varrho}(t) = \boldsymbol{\varrho}(t + \Delta t) - \boldsymbol{\varrho}(t)$ corresponding to a small but finite time interval Δt is directed along the chord $s_i s_f$. If we form $\Delta\boldsymbol{\varrho}(t)/\Delta t$, we recognize this as the average velocity vector $\mathbf{v}_{B,\mathrm{av}}$, also lying along the chord. This says that we get from s_i to s_f by proceeding along the chord $\Delta\boldsymbol{\varrho}(t)$ at a constant (i.e., average) rate (speed or magnitude of velocity) numerically equal to $|\Delta\boldsymbol{\varrho}(t)|/\Delta t$.

As the time interval Δt approaches zero, we have

$$\lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\varrho}(t)}{\Delta t} = \mathbf{v}_B(t)$$

the velocity of the point B at the instant t defining the position s_i. The vector $\mathbf{v}_B(t)$ becomes tangent to the curve at s_i. To find the magnitude

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 107 of 143

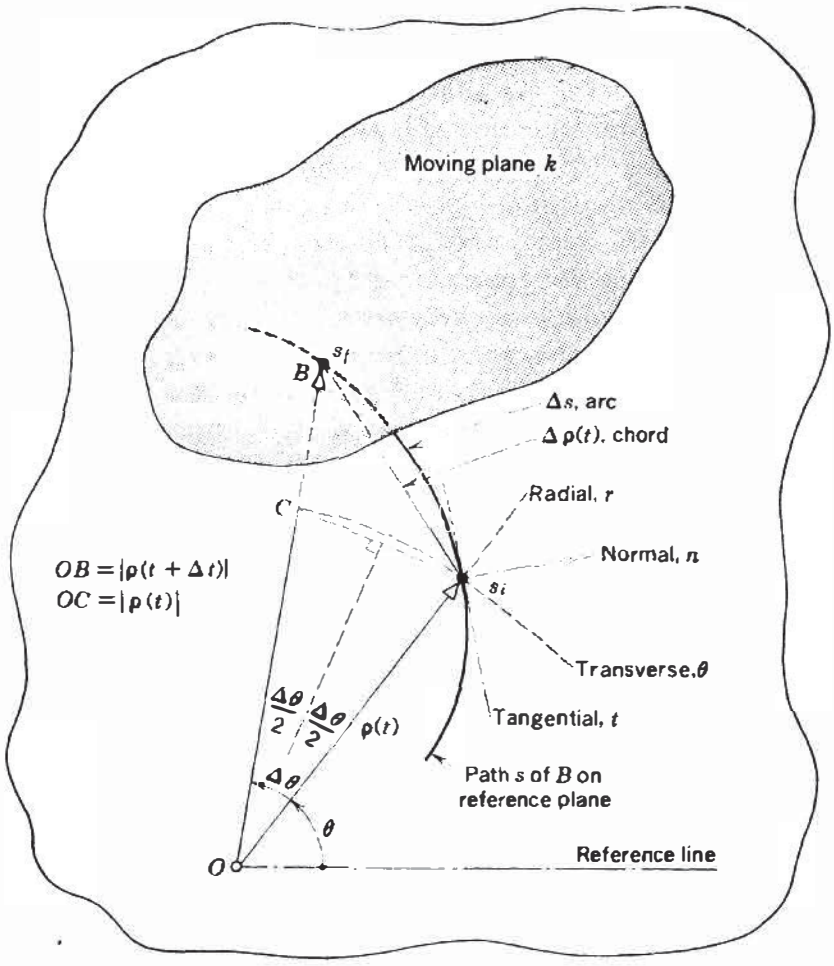


FIGURE 4-2 Velocity of a point of a plane moving with respect to a reference plane.

of $\mathbf{v}_B(t)$, we note that $\Delta\rho(t) \neq 0$ for small $\Delta t > 0$. We may then write

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{|\Delta\rho(t)|} \frac{|\Delta\rho(t)|}{\Delta t} = \frac{\Delta s}{|\Delta\rho(t)|} \left| \frac{\Delta\rho(t)}{\Delta t} \right|$$

In this, $|\Delta\rho(t)|$ is the length of the chord, and the ratio $\Delta s/|\Delta\rho(t)|$ = arc/chord tends to 1, whence in the limit

$$|\mathbf{v}_B(t)| = \frac{ds}{dt} = \left| \frac{d\rho(t)}{dt} \right|$$

The velocity vector $\mathbf{v}_B$ thus has the magnitude $|\mathbf{v}_B| = ds/dt$, where s is the arc length along the path.

Inspection of the figure shows that $\mathbf{v}_B$ has no component in the direction n of the normal to the path, since it is directed along the tan-

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 108 of 143

gent, lying wholly in it. However, components are seen to exist for the radial r and transverse θ directions. We shall investigate these.

We take the vector $\Delta\boldsymbol{\rho}(t)$ to be composed of two other vectors, namely, $\Delta\boldsymbol{\rho}(t) = \mathbf{s_iC} + \mathbf{Cs_f}$. On dividing by Δt,

$$\frac{\Delta\boldsymbol{\rho}(t)}{\Delta t} = \frac{\mathbf{s_iC}}{\Delta t} + \frac{\mathbf{Cs_f}}{\Delta t} = \mathbf{v}_{B,\mathrm{av}}$$

That is, $\mathbf{v}_{B,\mathrm{av}}$ is composed of two velocity vectors each having a different direction. This situation is maintained to the limit,

$$\lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{\rho}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\mathbf{s_iC}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\mathbf{Cs_f}}{\Delta t} = \mathbf{v}_B$$

The vector $\mathbf{s_iC}/\Delta t$ has a direction perpendicular to the bisector of the angle $s_iOs_f = \Delta\theta$. and its magnitude is

$$\frac{s_iC}{\Delta t} = \left|\frac{2\rho(t)\,\sin\,(\Delta\theta/2)}{\Delta t}\right| = \left|\boldsymbol{\rho}(t)\,\frac{\Delta\theta}{\Delta t}\right|$$

the sine of a small angle being equal to itself (expressed in radians). With decreasing Δt, $\Delta\theta$ also decreases, and, in the limit $\Delta t \to 0$, s_f coincides with s_i, and the bisector of the angle coincides with Os_i. The vector $\mathbf{s_iC}/\Delta t$ thus becomes perpendicular to the position vector $\boldsymbol{\rho}(t)$, and its magnitude is $|\boldsymbol{\rho}(t)\,\Delta\theta/\Delta t| = |\boldsymbol{\rho}\,d\theta/dt|$. This component of the velocity is denoted as $\mathbf{v}_B{}^{\theta}$.

The second component vector $\mathbf{Cs_f}/\Delta t$ has the direction of the line Os_f; its magnitude is the increment in magnitude (i.e., length) of the position vector during the time Δt, namely, $|\boldsymbol{\rho}(t+\Delta t)| - |\boldsymbol{\rho}(t)|$. As Δt is taken smaller and smaller, $\mathbf{Os_f}$ approaches $\mathbf{Os_i}$, whence the vector $\mathbf{Cs_f}/\Delta t$ lies in the direction of the position vector $\boldsymbol{\rho}(t)$ of location s_i; its magnitude is the rate of change of length of the position vector, or $d\rho/dt$. This radial component of velocity is written $\mathbf{v}_B{}^{\rho}$.

In summary, the velocity $\mathbf{v}_B$ of a point at a given instant is composed of two rectangular components,

$$\mathbf{v}_B = \mathbf{v}_B{}^{\theta} + \mathbf{v}_B{}^{\rho}$$

where the vectors $\mathbf{v}_B{}^{\theta}$ and $\mathbf{v}_B{}^{\rho}$ are as follows:

Transverse component $\mathbf{v}_B{}^{\theta}$
- Magnitude: $\rho\,d\theta/dt$
- Direction: perpendicular to position vector $\boldsymbol{\rho}$

Radial component $\mathbf{v}_B{}^{\rho}$
- Magnitude: $d\rho/dt$
- Direction: along position vector $\boldsymbol{\rho}$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 109 of 143

As Δt approaches zero, the direction of the vector $\Delta\rho$ approaches the tangent to the path, which means that there is never a normal component, namely, $\mathbf{v}_B{}^t = \mathbf{v}_B$ and $\mathbf{v}_B{}^n = 0$.

It is important to note again that the velocity of a moving point is with respect to a *plane* (not to a point in the plane); the existence of a velocity requires a path. For example, an aircraft's velocity is reckoned with respect to the earth, not with respect to a point such as a city.

A particular case in which a moving point describes a circular arc about an already specified point of the reference plane occurs so often that special comment is necessary (Fig. 4-3*a*). Since the origin has been arbitrarily (but by specification) placed at the center of the circular path,

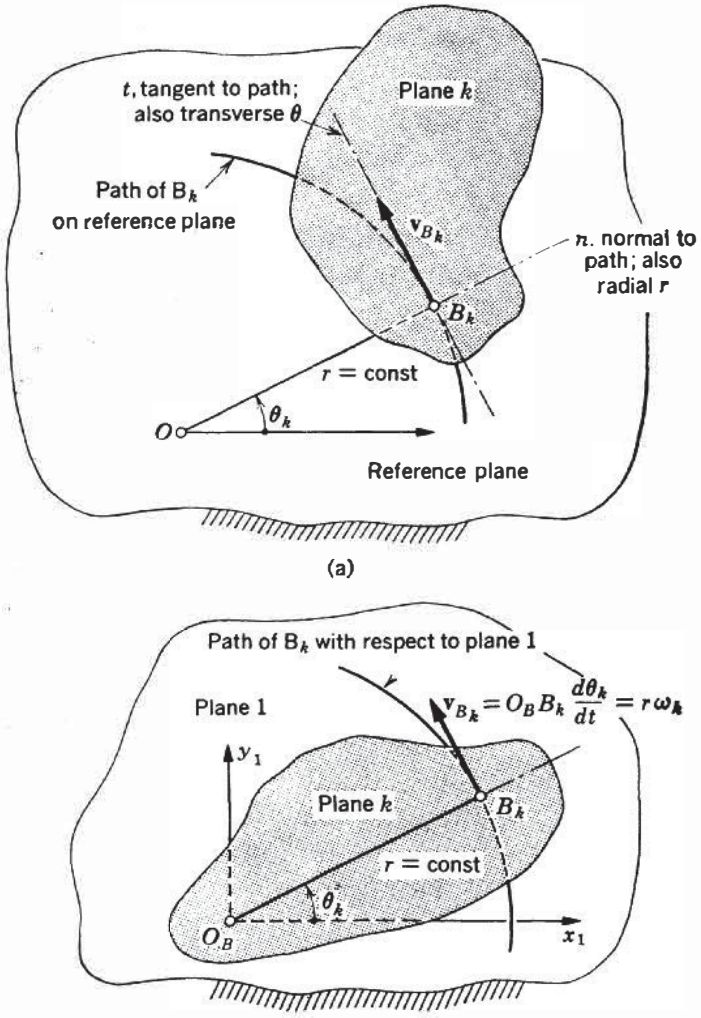


FIGURE 4-3 Point of moving plane k describing circular-arc path on reference plane 1.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 110 of 143

$\rho = r = \text{const}$, whence the velocity reduces to

$$\mathbf{v}_{B_k} = \mathbf{v}_{B_k}{}^{\theta} = r\frac{d\theta_k}{dt}$$

As before, $\mathbf{v}_{B_k}$ lies on the tangent to the circle (path). The r and θ directions, although identifiable, are usually suppressed in favor of the n and t directions, with which they coincide.

Thus far we have discussed the motion of a point with respect to a reference plane in general terms. We had no mechanism for guiding the point on the path. We may now consider a specific and very common type of guidance, namely, that occasioned by a pin connection.

The physical situation is sketched in Fig. 4-3*b*, in which plane k (link k) is pin-connected to the reference plane 1 (link 1) at O_B. Link point B_k then describes a circular arc on plane 1 from whose reference system x_1y_1 the θ_k of the moving link may be measured.o Then

$$|\mathbf{v}_{B_k}| = r\frac{d\theta_k}{dt} = r\omega_k = (O_BB)\omega_k$$

We remark again that the above equation is a consequence of two circumstances, (1) the circular path, i.e., a link of constant length, and (2) the fortuitous specification of the origin O_B. But even with all this, the velocity of B_k is with respect to the reference *plane;* we observe that O_B was convenient for calculation, the velocity being defined by one vector component instead of the two that would have resulted from the specification of any other origin.

4-5 RELATIVE VELOCITY AND VELOCITY DIFFERENCE

The concept of relative velocity is best understood by considering a plane 2 moving with respect to a reference plane 1. A point[1] called B moving over plane 2 has an *absolute velocity* referred to plane 1, denoted by $\mathbf{v}_B$ (Fig. 4-4). Seen by an observer moving with plane 2, the point B has a *relative velocity* with respect to plane 2 denoted by $\mathbf{v}_{B/2}$. Finally, the point B_2 coincident with B at the instant considered, but moving with plane 2, has an absolute velocity $\mathbf{v}_{B_2}$.

The relation connecting the three velocities—the absolute velocity $\mathbf{v}_B$, the relative velocity $\mathbf{v}_{B/2}$, and the absolute velocity of the coincident point $\mathbf{v}_{B_2}$—is obtained by considering small displacements as shown in Fig. 4-5. Here, plane 2 moves from position 2 to position 2′ during the time interval Δt. At the beginning of the interval, B_2 and B are coinci-

[1] This point B may be either (1) a point of another plane or (2) a point defined by a certain geometry, e.g., the intersection of two moving lines.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 111 of 143

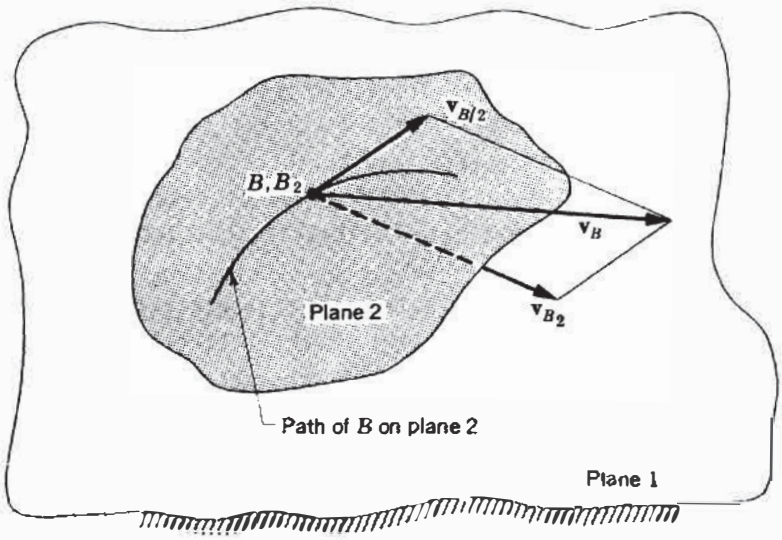


FIGURE 4-4 Relative velocity.

dent; during the interval Δt, B_2 moves with plane 2 to position B_2'. Point B, however, moves with respect to plane 2 and at the end of the interval will occupy position B' distinct from B_2'. Now, from the definition of velocity,

$$\mathbf{v}_B = \lim_{\Delta t \to 0} \frac{\mathbf{BB'}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\mathbf{B_2B_2'}}{\Delta t} + \frac{\mathbf{B_2'B'}}{\Delta t} \right)$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{B_2B_2'}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{B_2'B'}}{\Delta t}$$

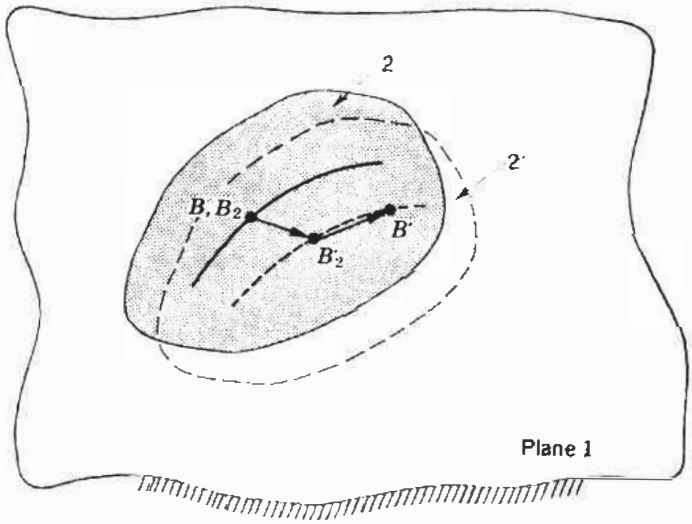


FIGURE 4-5 Proof of relative-velocity theorem.

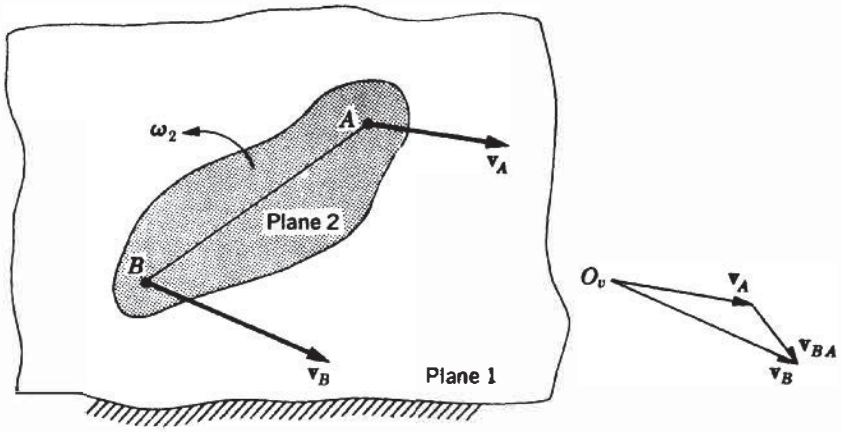


FIGURE 4-6 Velocity difference of two points on the same link.

But the first limit in the second equation above is simply the velocity of point B_2, and the second limit is the velocity of B as seen by an observer moving with plane 2, i.e., the relative velocity of B, whence

$$\mathbf{v}_B = \mathbf{v}_{B_2} + \mathbf{v}_{B/2} \tag{4-1}$$

The above equation thus relates the absolute velocities of two coincident points B_2 and B moving independently of one another. The relative velocity $\mathbf{v}_{B/2}$ is the velocity that point B would appear to have to any observer moving with plane 2.

Quite a different situation will now be considered (Fig. 4-6)t again plane 2 moves with respect to the reference plane 1, but now plane 2 contains two distinct points A and B with absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$. The *velocity difference* between these two points is defined as $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$, from which

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{4-2}$$

The velocity difference $\mathbf{v}_{BA}$ will now be related to the distance AB and the angular velocity ω_2 of plane 2. Again, velocities will be considered as limits of vanishingly small displacements, as shown in Fig. 4-7. The displacement of plane 2 during the time interval Δt is considered to consist of two steps: a translation from AB to $A'B''$, followed by a rotation about A' from $A'B''$ to $A'B'$. Then,

$$\mathbf{v}_B = \lim_{\Delta t \to 0} \frac{\mathbf{BB'}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\mathbf{BB''}}{\Delta t} + \frac{\mathbf{B''B'}}{\Delta t} \right)$$
$$= \lim_{\Delta t \to 0} \frac{\mathbf{AA'}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{B''B'}}{\Delta t}$$

since $\mathbf{BB''} = \mathbf{AA'}$. The first limit in the last equation above is $\mathbf{v}_A$, the

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 113 of 143

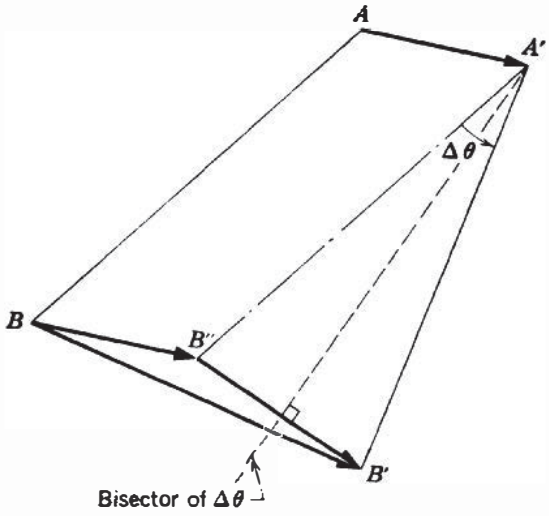


FIGURE 4-7 Determination of the velocity difference of A and B.

velocity of point A; the second limit may therefore be identified with the velocity difference of the two points,

$$\mathbf{v}_{BA} = \lim_{\Delta t \to 0} \frac{\mathbf{B''B'}}{\Delta t}$$

The vector $\mathbf{B''B'}$ has a direction perpendicular to the bisector of the angle $B''A'B' = \Delta\theta$ and a magnitude $B''B' = (AB)\,\Delta\theta$, $\Delta\theta$ being a small angle. As Δt approaches zero, the bisector approaches the direction AB, and $B''B/\Delta t = AB/(\Delta\theta/\Delta t)$ approaches $AB/(d\theta/dt)$. Noting that $\Delta\theta$ is the angle of rotation of plane 2 during the time interval Δt, we conclude that the velocity difference $\mathbf{v}_{BA}$ is perpendicular to AB and has a magnitude equal to $(AB)\omega_2$.

Because of the similarity between Eqs. (4-1) and (4-2), the velocity difference is often called the "relative velocity" of B with respect to A. It should be noted, however, that the two situations described by these equations are different, and it is misleading to identify relative velocity ($\mathbf{v}_{B/2}$) and velocity difference ($\mathbf{v}_{BA}$) only on the basis of a similarity in equations. We have already noticed that both velocity and displacement are vectors and can be referred only to a plane (for planar motion) in which a reference direction such as Ox has been defined. A velocity cannot be referred to a point, since its direction cannot be specified. The difference between relative velocity and velocity difference may be further explained in terms of an example. A person (point B) pacing the interior of a railroad car (plane 2) riding through the night would be able, by simple observation, to determine his velocity with respect to the car ($\mathbf{v}_{B/2}$), for this is independent of the motion of the car. It would be impossible for him, however, to establish the velocity difference between the two ends of the car, for this velocity

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 114 of 143

difference is the difference of two (absolute) velocities referred to the earth, which the person cannot see; the velocity difference depends on the motion of the car with respect to the earth.

Velocity analysis—the determination of the linear velocities of points and the angular velocities of links of mechanisms—is conveniently handled by the method of vector polygons. To establish the fundamentals of velocity analysis based on vector polygons, we shall examine the velocity condition of the four-bar linkage of Fig. 4-8*a*, the angular velocity ω_2 of link 2 being known. This will involve finding the linear velocities of the several points, the angular velocities of links 3 and 4, and the use of the velocity image.

Suppose that in our awkwardness we attempt first the velocity of D. We would write the equation $\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{DB}$ and proceed to evaluate the three terms as best we can. Each of the vector terms has two endowments, magnitude and direction, each of which may be treated as an "unknown" in the same sense as the unknowns of algebraic equations. Algebraic equations are scalars, and one such equation cannot be solved if the number of unknowns exceeds one. A vector equation (such as we have) may be viewed as consisting of two scalar equations, whence the limit of unknowns is two per vector equation. In algebra,

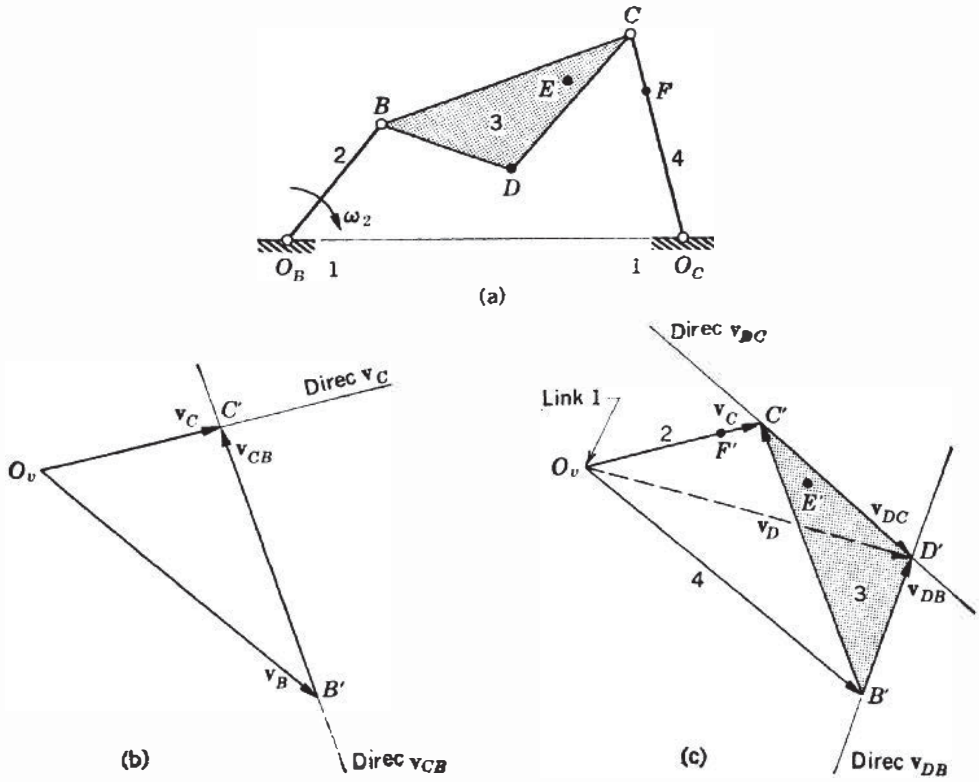


FIGURE 4-8 Velocity analysis of the four-bar linkage.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 115 of 143

as many simultaneous equations as there are unknowns are needed for solution; with vector equations, only half as many equations as unknowns are needed.

To return to the evaluation of the vectors, we shall examine each in turn for (1) direction and (2) magnitude and shall hope that not more than two unknowns exist for the entire equation:

Term $\mathbf{v}_D$ both magnitude and direction are unknown
Term $\mathbf{v}_B$ completely known in magnitude $(O_BB)\omega_2$ and direction
Term $\mathbf{v}_{DB}$ only direction is known

This information is conveniently summarized on the equation itself by a system of checks (knowns) and x's (unknowns) as follows:

$$\overset{x_1x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_3}{\mathbf{v}_{DB}} \tag{4-3}$$

With three unknowns the situation is hopeless, whence another tactic—another point—must be looked at, with hope for better luck.

Turning to the alternative point C, we write $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$. Termwise examination allows the equation to be marked as

$$\overset{x_1\checkmark}{\mathbf{v}_C} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_2}{\mathbf{v}_{CB}} \tag{4-4}$$

Here x_1 and x_2 indicate unknown velocity magnitudes. But since the unknowns total only two, the vector equation can be solved. We choose an origin O_v and from it lay off the completely known $\mathbf{v}_B$ to some convenient scale (Fig. 4-8*b*). The equation instructs us to add $\mathbf{v}_{CB}$ to $\mathbf{v}_B$; all that can be done now is to lay off the direction of $\mathbf{v}_{CB}$. Only $\mathbf{v}_C$ is left; it is an absolute velocity and hence originates from O_v, going out to meet the "other side" of the equation. On laying off the direction of $\mathbf{v}_C$ the line intersects the $\mathbf{v}_{CB}$ direction line. Since the tail of the vector $\mathbf{v}_{CB}$ must start from the head of $\mathbf{v}_B$—the cue comes from the equation—the sense (as different from the direction) and magnitude of each of the partially known vectors are established; i.e., vectors $\mathbf{v}_{CB}$ and $\mathbf{v}_C$ are completely defined. The angular velocity of link 3 is given by $\omega_3 = v_{CB}/BC$. Similarly, $\omega_4 = v_C/O_CC$. The senses of the angular velocities are deduced from those of the linear velocities.

With $\mathbf{v}_C$ at hand, the velocity $\mathbf{v}_D$ becomes approachable, for we may involve $\mathbf{v}_D$ in two equations, which, after being checked off, show an acceptable four unknowns:

$$\overset{x_1x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_B} + \overset{\checkmark x_3}{\mathbf{v}_{DB}}$$

$$\overset{x_1x_2}{\mathbf{v}_D} = \overset{\checkmark\checkmark}{\mathbf{v}_C} + \overset{\checkmark x_4}{\mathbf{v}_{DC}}$$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 116 of 143

For clarity, the first vector polygon has been redrawn (Fig. 4-8*c*) and the solution of the last two equations added. Common to both equations, $\mathbf{v}_D$ is shown by the broken-line vector $\mathbf{O}_v\mathbf{D}'$. The shaded figure $B'C'D'$ is known as the velocity image of link 3; it is geometrically similar to the figure BCD that is link 3. The line $B'C'$ stands at right angles to the line BC, the image having been rotated 90° in the sense of ω_3. Note also that D' is on the side of $B'C'$ corresponding to the location of ◗ with respect to BC.

The velocity image provides a great convenience, for the velocity of any other point of link 3, such as E, may be found without recourse to further equations. It is necessary only to locate the corresponding E' on the image, and the vector $\mathbf{O}_v\mathbf{E}'$ (not shown) will be $\mathbf{v}_E$.

It follows now, of course, that the velocity image of a link may be drawn directly after establishing the velocity of two points of the link, e.g., the points B and C. Had we done this, there would have been no need for the simultaneous solution of the two vector equations relating point ◗ to points B and C. However, there are situations in which the procedure of simultaneous solution is necessary.

4-6 INSTANTANEOUS CENTERS OF VELOCITY

By instantaneous center is understood a pair of coincident points having either zero relative velocity or acceleration at the moment of observation. The instantaneous centers of velocity and acceleration do not occur at the same point except in trivial situations. That two points of different planes have momentarily equal velocities or accelerations provides a means for taking the known motion of one plane into another.

While the position of the velocity center is in many cases intuitive or is found without too much difficulty, the same cannot be said for the acceleration center. As an analytical device, the instantaneous center of velocity has wide application, and the following remarks will be directed to the velocity center, sometimes abbreviated simply to IC.

Consider a plane 2 moving with respect to a reference plane 1, and assume that the velocities of two points A_2 and B_2 are known, as in Fig. 4-9*a*. Note that, while the velocity of one point, say A_2, could be chosen arbitrarily, the velocity of the other point B_2 has to be such that the velocity difference $\mathbf{v}_{B_2A_2} = \mathbf{v}_{B_2} - \mathbf{v}_{A_2}$ is perpendicular to the line A_2B_2. Having chosen an origin O_v, we can draw a velocity diagram for the motion of plane 2 with respect to plane 1, as shown in Fig. 4-9*b*. Now the velocity of any point C_2 of plane 2 may be found by locating its image C_2' on the velocity diagram, similar triangles being used as shown in the preceding section.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 117 of 143

Since there is a one-to-one correspondence between the points of plane 2 and their images on the velocity diagram, we may also solve the reverse problem by the same method and find which point C_2 of plane 2 has a specified velocity $\mathbf{v}_{C_2}$. The specified velocity locates the image C_2' such that $O_v C_2' = \mathbf{v}_{C_2}$, and a triangle $A_2B_2C_2$ similar to $A_2'B_2'C_2'$ is constructed over the line segment A_2B_2 to locate C_2. In particular, we may find by this method the point O_2 of plane 2 having zero velocity at the instant considered. This point is the instantaneous center of velocities of plane 2 with respect to plane 1; its image must be the origin O_v of the velocity diagram. The instantaneous center O_2 is therefore located as shown in Fig. 4-9, such that triangles $A_2B_2O_2$ and $A_2'B_2'O_v$ are similar. Note that this means that O_2 is the intersection of the lines A_2u and B_2v, respectively perpendicular to the velocities at points A_2 and B_2. Note further, by comparing plane 2 with the velocity polygon, that the velocity of any point such as C_2 is perpendicular to the radius vector O_2C_2 from the IC to the point, and the magnitude of the velocity is $v_{C_2} = \omega_2(O_2C_2)$.

In a mechanism consisting of a number of links, an IC relates to two links; with a mechanism of n links the total number N of possible ICs is the number of combinations of n things taken two at a time, or $N = n(n-1)/2$. Thus a four-link mechanism will have an N of 6; for a six-link mechanism, $N = 15$; an eight-link mechanism will have $N = 28$; and so on.

Clearly some sort of bookkeeping system is needed when the

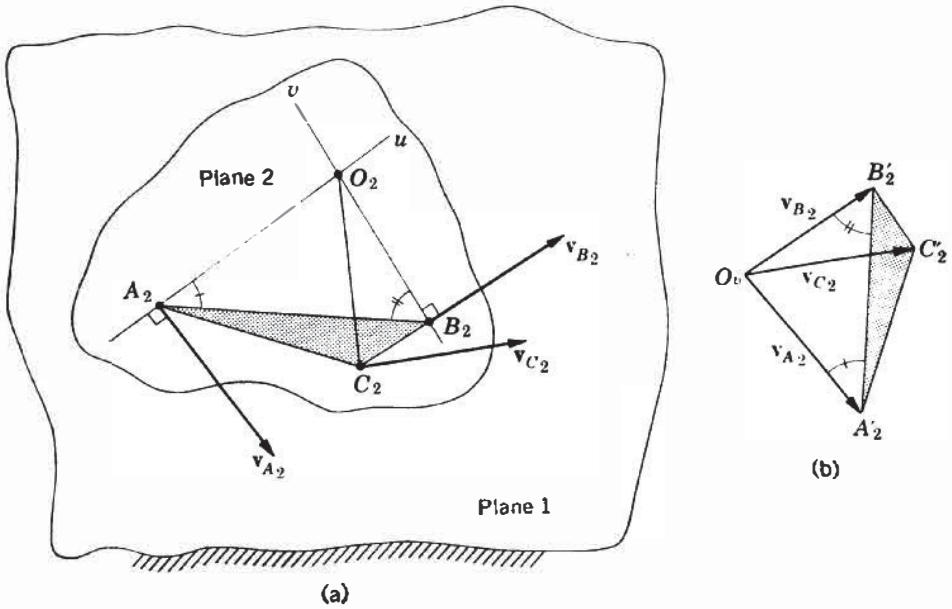


FIGURE 4-9 Existence and properties of the instantaneous center of two planes in relative motion.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 118 of 143

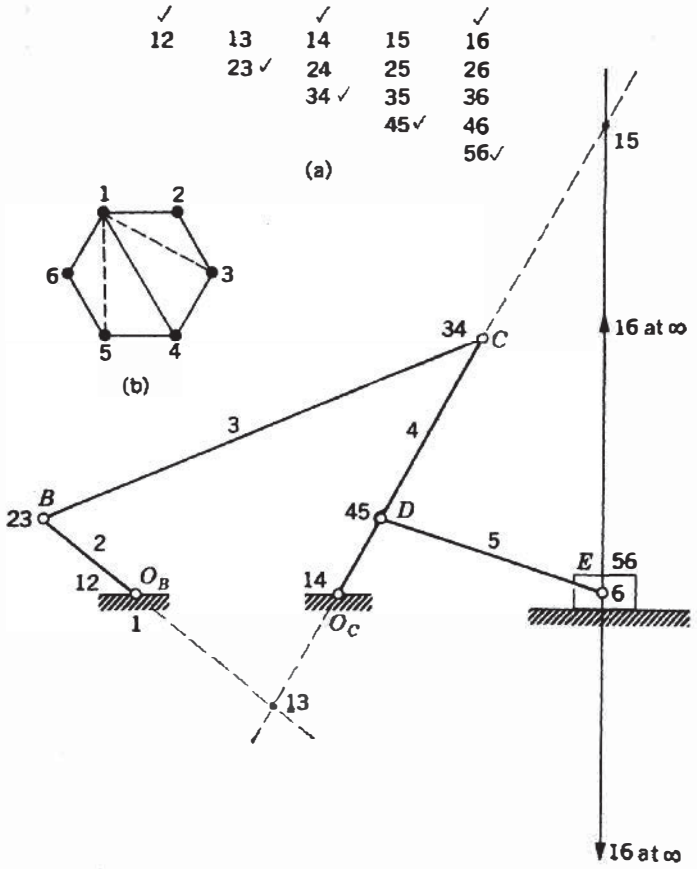


FIGURE 4-10 Determination of ICs of a six-link mechanism.

number of links exceeds four. A six-link mechanism (Fig. 4-10) will serve as example. The ICs may be tabulated as in Fig. 4-10*a*, each link with every other, and checked off as they are found. However, the circle diagram of Fig. 4-10*b* has greater utility. In this, each dot represents a link, and a line between any two dots represents the IC of the two links (dots) so connected. Starting with the dots ordered around a circle (the latter for the sake of neatness), the obvious ICs are entered in the diagram by drawing full lines between appropriate dots.

The obvious ICs are those associated with permanent connections between the links. The term permanent means pin connections and also includes the infinitely distant IC of a slider, located on a normal to the direction of rectilinear translation. This idea is based on the philosophy that such translation is regarded as the limiting case of rotation about an infinitely distant permanent center of rotation, a center common to the two links bearing the kinematic elements of the sliding pair.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 119 of 143

In our example (Fig. 4-10) the obvious or directly available ICs are 12,* 23, 34, 45, 14, 56, and 16, the last being involved with translation. The normal on which IC 16 lies is drawn through IC 56 because of the implication of the theorem of three centers, to be stated below. Actually, any line parallel to it will point to IC 16.

The remaining centers are found from deductions based on the theorem of three centers:† *The ICs of any three links (connected or not) having planar motion lie on the same straight line.* (We leave the proof as an exercise.) It is in the application of this theorem that the circle diagram is more aggressive than the tabular array in suggesting what to do next. Since a point such as an IC is defined by the intersection of at least two lines, an unknown IC is specified if it is a line in the circle diagram that is the common side of two triangles. Thus, IC 13 (shown as a broken line in Fig. 4-10*b*) is rendered by the intersection of the line 12-23 with the line 14-34. Similarly, IC 15 is at the intersection of 14-45 and 56-16. The remaining ICs are found in like fashion.

We may now distinguish between *absolute* and *relative* ICs. Absolute ICs are those carrying the frame numeral or digit; they are the top row of the tabular array, Fig. 4-10*a*, characterized by the zero velocity of their coincident points. All other ICs are called relative; their coincident points have velocities different from zero, being in motion with respect to the frame. The significance of these ICs is that the coincident points on the two links involved in a relative IC have equal velocities, for example, $\mathbf{v}_{(23)_2} = \mathbf{v}_{(23)_3}$, $\mathbf{v}_{(34)_3} = \mathbf{v}_{(34)_4}$, etc.

It will be noticed that permanent ICs are distributed among both the absolute and the relative ICs. The use of the IC as a medium for determining both velocity distribution in a given link and motion transmission between links will be discussed in terms of a four-bar linkage (Fig. 4-11*a*). A line drawn from 13 to the tip of $\mathbf{v}_{23}$ (which is also $\mathbf{v}_{B_3}$) defines an angle ϑ_3. (The symbol ϑ is the cursive θ and may be pronounced "tetta" to distinguish it from theta.) This ϑ_3 line gives the velocity distribution along any radial line of link 3 emanating from 13, whence the velocity of any other point, such as $\mathbf{v}_{C_3} = \mathbf{v}_{34}$, may be found by duplicating ϑ_3 on a radial line from 13 passing through the point C_3. Also, $\omega_3 = v_{23}/(13\text{-}23) = v_{34}/(13\text{-}34)$, and $\omega_4 = v_{34}/(14\text{-}34)$.

Action at a relative center such as IC 24 may be visualized as in Fig. 4-11*b*. IC 24, as remarked, lies at the intersection of lines 23-34 and 12-14. The sketch shows planes 2 and 4 extended to cover the

* Read "IC one-two." If the smaller digit is always placed first, possible dualities will be avoided

† This is known as the Aronhold theorem in Germany, and sometimes as the Kennedy theorem in the United States, after the independent discoverers, Aronhold, 1872, and Kennedy, 1886.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 120 of 143

center 24; each plane, of course, retains its absolute rotation about its own center, as 2 about IC 12, etc. From $\mathbf{v}_{B_4} = \mathbf{v}_{23}$ the ϑ_2 line for link 2 is found and then set off on the radial line 14-24. The perpendicular to it at IC 24 defines $\mathbf{v}_{(24)_2}$; but since this is at the center 24, the coincident point of plane 4 has a velocity precisely equal to that of its mate on plane 2, namely, $\mathbf{v}_{(24)_2} = \mathbf{v}_{(24)_4} = \mathbf{v}_{24}$. In this manner a known velocity of plane 2 is transferred to plane 4.

The ϑ_4 line may now be established and then reconstructed with

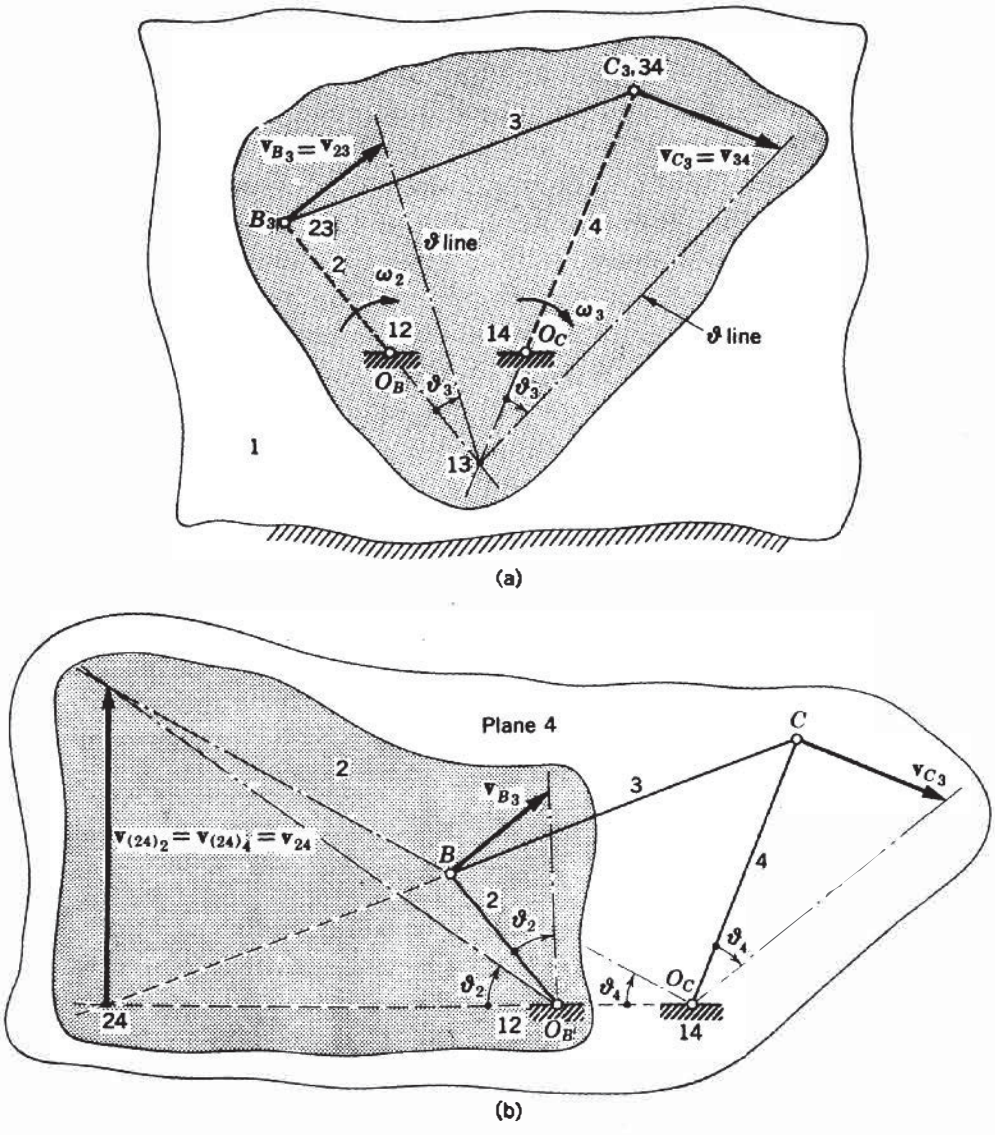


FIGURE 4-11 Velocity distributions and ϑ lines of a four-bar linkage. (*a*) Coupler and frame, absolute IC 13; (*b*) crank and follower, relative IC 24.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 121 of 143

respect to line 14-34 to define $\mathbf{v}_{C_4} = \mathbf{v}_{34}$. From this $\omega_4 = v_{34}/(14\text{-}34)$. We should note that we were able to get from link 2 to link 4 without involving the intermediate link 3.

The case of IC 24 is typical of all relative IC situations. The action at such an IC can always be visualized (if the need arises) by considering the pertinent planes to be extended far enough to cover the center, after which the ϑ lines are brought into play.

Relative centers allow one to go directly from one link to any other, provided that the IC of the two links is at hand. For example, if we had an eight-link mechanism we could make the direct transition from link 2 to link 8 via IC 28. However, IC 28 might be troublesome to find: it could be the last to disclose its location [$N = (8 \times 7)/2 = 28$].

Four-bar and slider-crank linkages were at one time called indirect contact mechanisms, since the coupler link 3 is the intermediary of links 2 and 4, one of which is the input link, with the other the output. Mechanisms in which the input and output links are in immediate connection on curved or camlike surfaces, as in Fig. 4-12*a*, are now often called "direct-contact" mechanisms. Other examples include meshed gear teeth and many conventional cam and follower setups.

For the purposes of this discussion, the plane of link 2 has been extended beyond the limits of the profile it carries, and link 3 lies over it. At the point of contact, B, the two curves have a common tangent t and normal n. The point B is comprised of two coincident points B_2 and B_3. The path that B_3 would describe on plane 2 is not a simple curve; as the figure shows, it does not coincide with the profile of link 2, and its shape and curvature cannot be found without some trouble. Similarly, B_2's path on plane 3 is not an obvious curve.

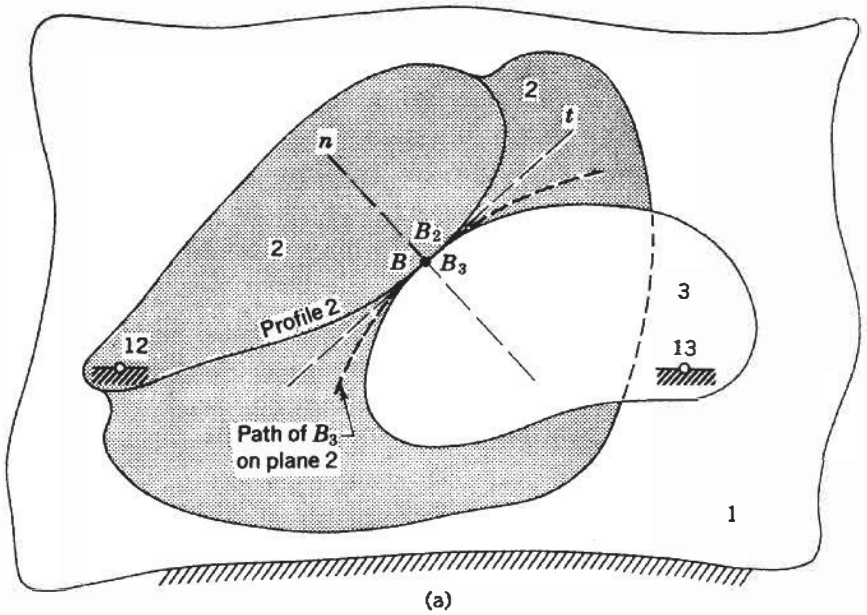
As links 2 and 3 move while remaining in contact along their profiles, the instantaneous contact point B runs along the profiles and thus moves with respect to both links 2 and 3. By application of the theorem of relative velocities [Eq. (4-1)], the velocity of this point may be written in two different ways,

$$\mathbf{v}_B = \mathbf{v}_{B/2} + \mathbf{v}_{B_2} \qquad \text{and} \qquad \mathbf{v}_B = \mathbf{v}_{B/3} + \mathbf{v}_{B_3}$$

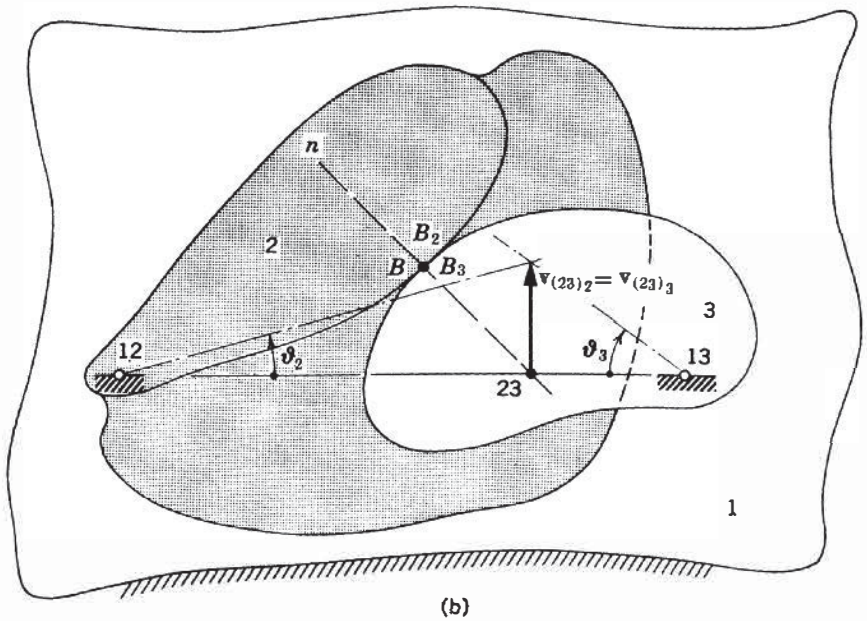
from which $$\mathbf{v}_{B/2} - \mathbf{v}_{B/3} = \mathbf{v}_{B_3} - \mathbf{v}_{B_2} = \mathbf{v}_{B_3/2} = -\mathbf{v}_{B_2/3}$$

Since B moves along the profiles of links 2 and 3, both $\mathbf{v}_{B/2}$ and $\mathbf{v}_{B/3}$ are along the common tangent t and from the last equalities above we conclude that the relative velocities $\mathbf{v}_{B_3/2}$ and $\mathbf{v}_{B_2/3}$ are also along this tangent. With $\mathbf{v}_{B_3/2}$ along the common tangent t, the IC 23 must lie on the common normal n, and with the aid of the theorem of three centers we can now locate 23 as the intersection of 12-13 with the normal n as shown in Fig. 4-12*b*. The figure also shows the ϑ lines of the velocity distribution of links 2 and 3 with $\mathbf{v}_{(23)_2} = \mathbf{v}_{(23)_3}$.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 122 of 143



(a)



(b)

FIGURE 4-12 ICs of direct-contact mechanisms.

4-7 POLES AND CENTRODES

The instantaneous center of velocity, IC, of a plane 2 moving with respect to a reference plane 1 was defined as the point of plane 2 having zero velocity at the instant considered. The location and properties of this point were deduced by velocity images. The purpose of

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 123 of 143

the present section is to reconsider instantaneous centers of velocity from a different point of view, which will be useful for further developments in Chap. 8. This new approach is based on the awareness that every displacement of a moving plane over a reference plane is equivalent to either a unique translation or a unique rotation. For a rotation in which a finite angle $\Delta\theta$ is involved, the center of rotation is called a *pole*. A succession of positions of a body may then be likened to a succession of finite rotations about properly selected poles. The use of the pole furnishes a powerful tool for synthesis as developed in Chaps. 8 and 9.

Let AB of Fig. 4-13 represent a line in a moving link. Successive positions are shown as A_1B_1 and A_2B_2. In Fig. 4-13*a* the link is moved from position 1 to position 2 by turning about the pole P_{12}. This pole

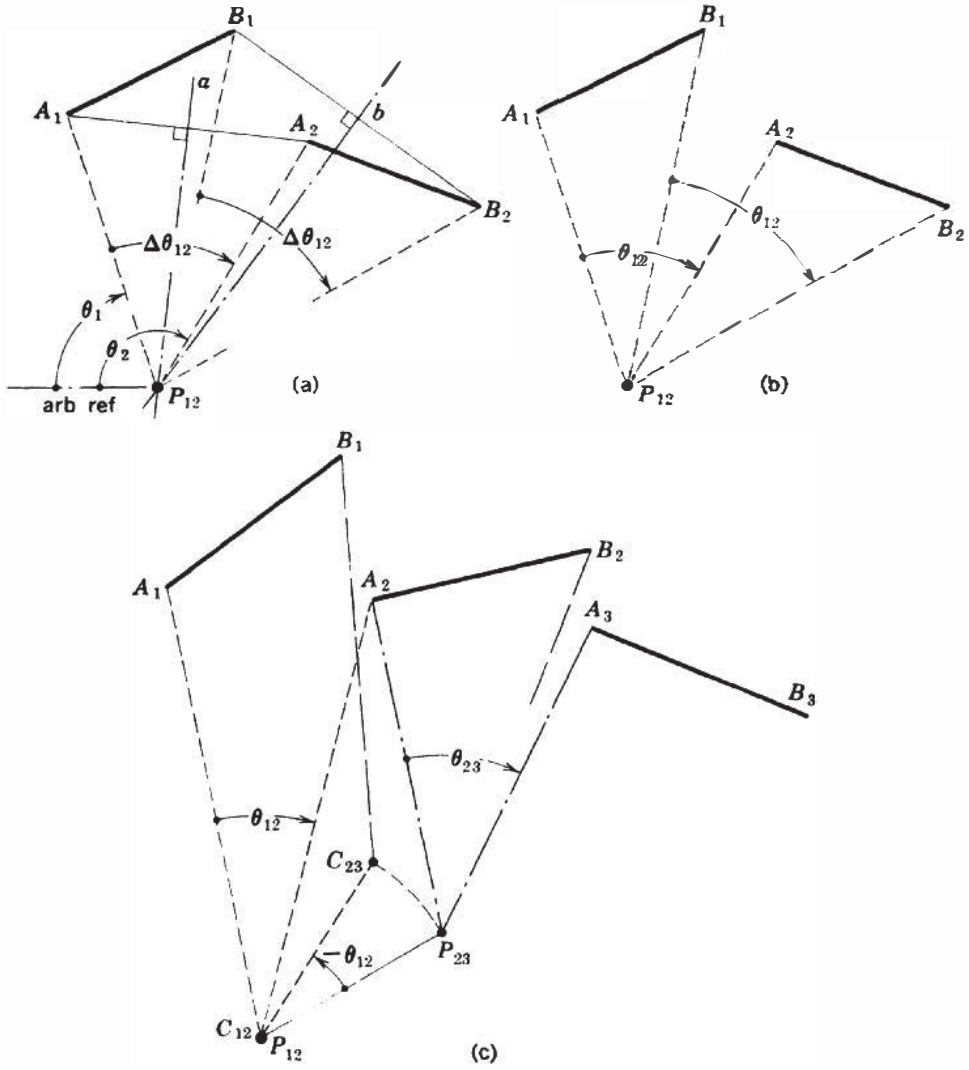


FIGURE 4-13 The poles of finite displacements.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 124 of 143

was defined by the intersection of the midnormals a and b, that is, the perpendicular bisectors of the distances A_1A_2 and B_1B_2. The A and B rays have been rotated through the finite angle $\Delta\theta_{12}$, the increment in angle between θ_1 and θ_2. The same situation, but without construction lines and by now superfluous identification, is shown in Fig. 4-13*b*. Note that $\Delta\theta_{12}$, the finite angle of rotation between positions 1 and 2, is written simply as θ_{12}. It is understood that the pole was found from the midnormals' intersection.

In Fig. 4-13*c* a third position A_3B_3 is shown, the pole P_{23} having been determined from the midnormals to A_2A_3 and B_2B_3. For convenience, the pole P_{12} is also labeled C_{12}. Suppose the line $P_{12}P_{23}$ to be rotated by an amount $-\theta_{12}$, thus defining point C_{23}, and that $C_{23}B_1$ is drawn. The line $C_{12}C_{23}$ may now be considered to be part of the moving plane containing AB. When A_1B_1 is swung into A_2B_2 about P_{12}, $C_{12}C_{23}$ swings down to become coincident with $P_{12}P_{23}$; figures $C_{12}A_2B_1C_{23}$ and $P_{12}A_2B_2P_{23}$ are identical, and so are, of course, the triangles $P_{12}A_1B_1$ and $P_{12}A_2B_2$. With the move to position 3, the triangles $P_{23}A_2B_2$ and $P_{23}A_3B_3$ are identical.

The foregoing allows us to take a different view of things. Although we are really concerned with moving AB from position 1 (A_1B_1) to position 2 (A_2B_2), and although we have no idea of the mechanism that is to do this, we begin to see how it might be accomplished: the clue lies with the lines $C_{12}C_{23}$ and $P_{12}P_{23}$. $P_{12}P_{23}$ was established from the poles; these poles are fixed in the reference plane, and so is the line $P_{12}P_{23}$. $C_{12}C_{23}$ was found by a rotation of $-\theta_{12}$; it is a line now attached to the moving plane carrying AB. We are thus able to imagine going from position 1 to position 2 by rolling the line $C_{12}C_{23}$ of the moving plane into congruence with the fixed line $P_{12}P_{23}$. We still have the moving plane rotating, but we think now of two lines—one moving, one fixed—providing the action by coming into congruence and effecting the position change of AB by a rolling action.

Figure 4-14 shows the moving plane in six successive positions; the poles were determined from the intersections of the appropriate midnormals. By joining the poles we obtain the beginnings of a polygon; it is incomplete, since AB (or the moving plane) does not return to its original position. This polygon is called the fixed polygon, for it is in the reference plane.

Another polygon, but one that moves because it is attached to the moving plane AB, must next be determined. The point C_{23} is located as before; and, as we have seen, when $C_{12}C_{23}$ swings down to coincide with $P_{12}P_{23}$, the link moves into position A_2B_2. The line $C_{23}C_{34}$ has the length of $P_{23}P_{34}$ and makes an angle $(\widehat{23} + \theta_{23})$ with $C_{12}C_{23}$. Note that the θ_{23} added to $\widehat{23}$ is of opposite sense to the finite

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 125 of 143

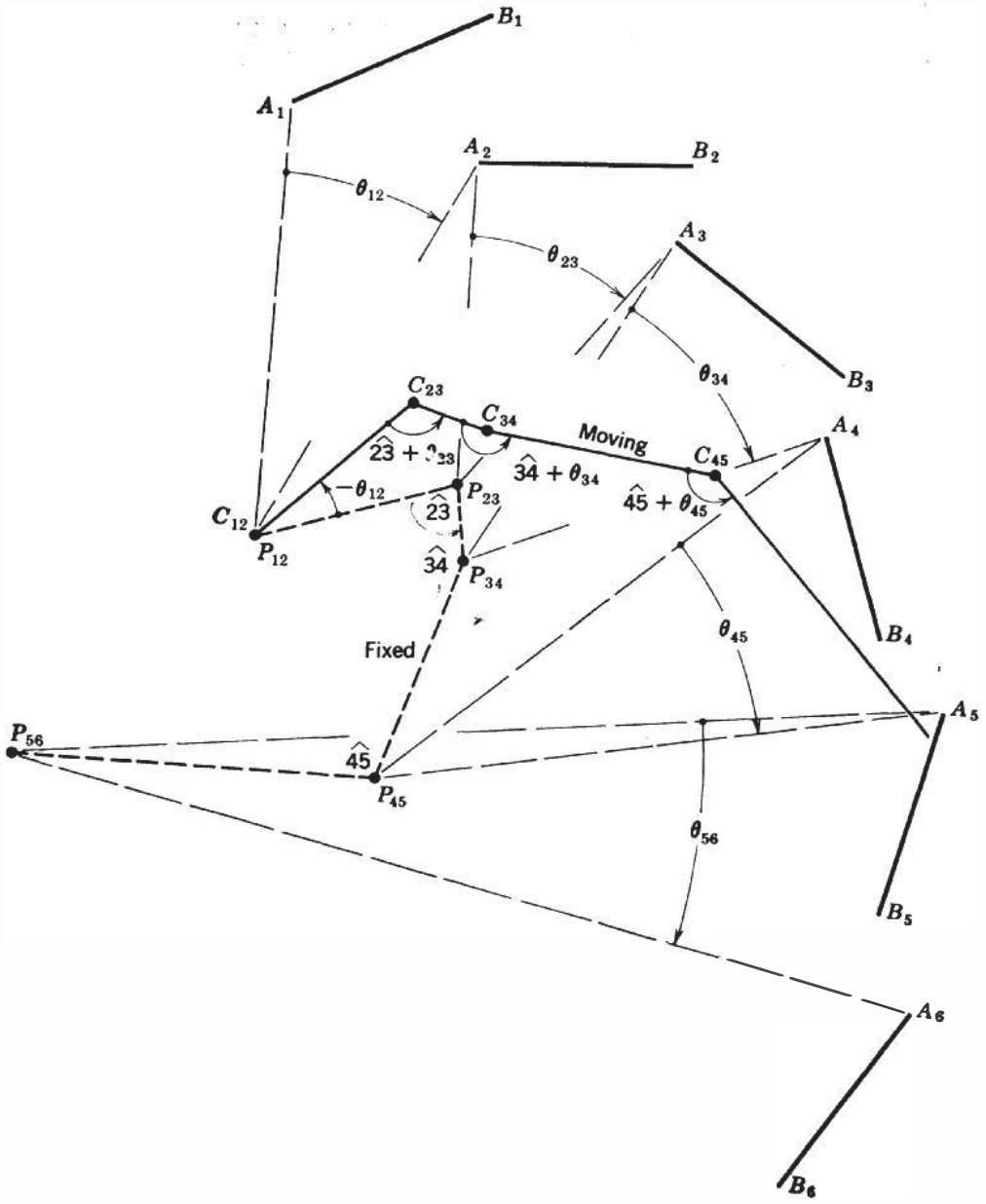


FIGURE 4-14 Poles, fixed and moving polygons for finite rotations.

rotation θ_{23} of the link between positions 2 and 3. This process is continued; thus $C_{34}C_{45} = P_{34}P_{45}$, and the angle with its predecessor line is $(\widehat{34} + \theta_{34})$, and so on. The moving polygon is created in this manner; when it rolls around the corners of the fixed polygon, the link AB will assume the successive positions. This is easily demonstrated by drawing A_1B_1 and the moving polygon on a transparent overlay and rolling the latter's polygon on the fixed polygon. We remark that these polygons are independent of the actual method of moving the link AB; all that they manage to do is to position the link for a succession of finite rotations.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 126 of 143

Ordinarily the motion of a body is continuous. The six positions of AB may be likened to six flashlight photographs of a continuously moving link, the finite angles $\Delta\theta = \theta$ corresponding to finite times Δt between successive photographs. With our poles we have been able to account for the positions at the beginning and end of any interval. Were we to think of photographing the procedure at shorter and shorter intervals, we should have to provide more poles, until at the end, when Δt between pictures approached zero, the infinity of individual poles would have to be called instantaneous centers. The polygons would no longer have corners because of the infinitesimal spacing of the defining points; instead there would be smooth curves with the special name of *centrode*. The centrodes are actually the loci of the instantaneous centers and will be identified by the symbol π (lowercase p of the Greek alphabet; think of *path*). These curves, or centrodes, retain the roll property of the polygons from which they sprang. By rolling the centrode attached to the body about the centrode fixed to the reference plane, the motion of the link is faithfully represented. Any point on either is the instantaneous center for any virtual motion. We shall distinguish between the curves by calling them the moving and fixed centrodes; they are also known as the body (moving) and space (fixed) centrodes.

Figure 4-15 shows a line AB of a moving plane 2 somehow guided continuously along the two paths shown. The velocity directions (but

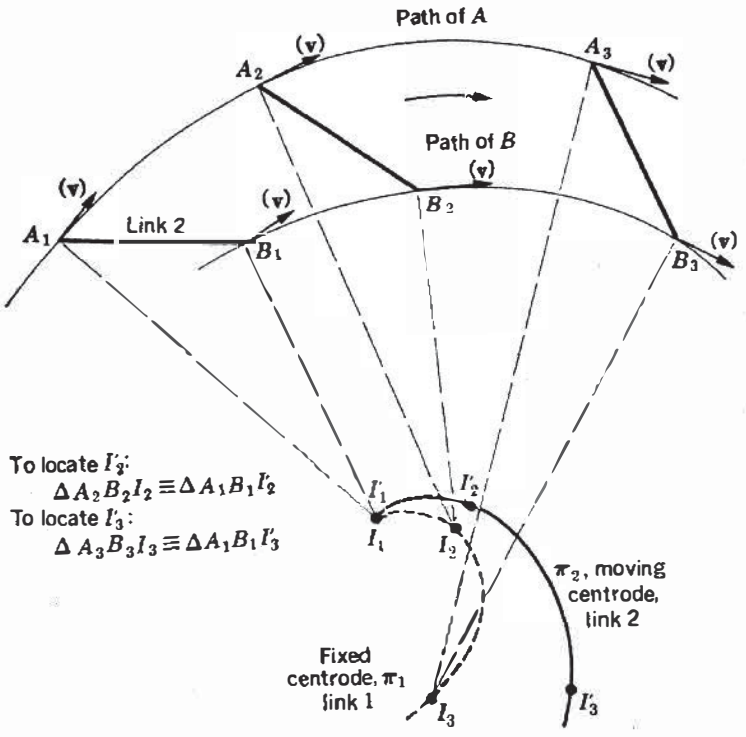


FIGURE 4-15 Determination of moving centrode.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 127 of 143

not their magnitudes) are indicated by (v) at A and B for successive positions; these directions are both tangents. The intersections of the normals from the pair of velocities at each of the three positions define the instantaneous centers for each, namely, I_1, I_2, and I_3. A curve drawn through these points is then the locus of the instantaneous centers fixed to the frame link 1: this is the fixed centrode, labeled π_1 for the link (plane) to which it is attached.

It is considerably more trouble to establish the moving centrode π_2. The use of a transparent overlay on which the line AB has been drawn will prove convenient. Upon laying this over A_2B_2 (position 2), I_2 is marked on the overlay. The overlay's line AB is then laid over A_1B_1, and I_2' is located by pricking through the overlay's point I_2. By repetition of this operation at A_3B_3, at which I_3 is picked up, and from which the overlay is moved back to A_1B_1, I_3' is found. We may say that the triangle $A_nB_nI_n$ must always be returned to A_1B_1 to locate I_n', the point on the moving centrode. The arc lengths, such as $I_1'I_2'$ of the moving centrode π_2, are of course equal to the corresponding arc lengths I_1I_2, just as the mating side of the former polygons were of equal length. Without this equality of the vis-à-vis arcs no rolling would be possible. It will be instructive to make an overlay having on it A_1B_1 and the moving centrode π_2; when π_2 is rolled over π_1, the various positions of AB will be indicated.

A four-bar linkage (double-crank) is shown in Fig. 4-16, with portions of the fixed π_1 and moving π_3 centrodes. Successive positions of the coupler are indicated by the number of primes on each I; thus, I''' means position 3. We may now imagine removing the actual physical guides (links 2 and 4) between the coupler (link 3) and the frame (link 1) and achieving the proper displacement of the coupler by rolling π_3 on π_1. The proper velocity is dependent on supplying the rolling motion with the proper angular velocity. The two centrodes amount to connecting the planes 3 and 1 by a kinematic pair having but one degree of freedom, viz., roll. The elements of this pair are the centrodes π_3 and π_1.

In like manner, but by the use of IC 24, two centrodes π_2 and π_4 could be developed. Since both centrodes are attached to the moving links 2 and 4, they are denoted as relative centrodes.

Centrodes simulating any planar motion other than translation may be found. With translation, the centrodes move to infinity. In the double-crank linkage of Fig. 4-16, the cranks never become parallel, whence the centrodes π_3 and π_1 will be closed curves. This is not true for crank-rocker mechanisms, for the crank and rocker become parallel at two phases, IC 13 going to infinity for each.

For some situations the centrodes are simple curves. When a wheel rolls along a rail, the IC is at the point of contact: the moving centrode is the rim of the wheel, and the rail is the fixed centrode (Fig.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 128 of 143

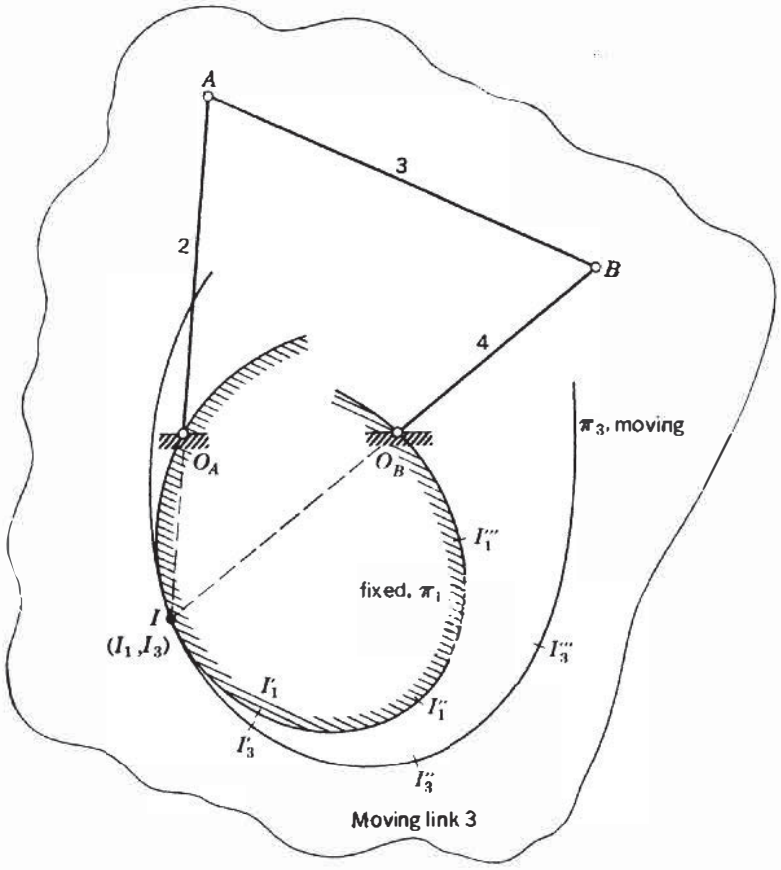


FIGURE 4-16 Centrodes of the coupler (3) with respect to the frame (1) of a four-bar linkage.

4-17*a*). With two circular gears in contact (Fig. 4-17*b*), the relative IC is always at the point of contact (pitch point) of the pitch circles; the two relative centrodes are the pitch circles. Also, as shown in Fig. 4-17*c*, displaying a crossed parallelogram linkage, both the relative centrodes are ellipses.

The direction of the common tangent (or the common normal) to a pair of centrodes at a given instant will have a particular significance in the studies of path curvature to be undertaken in Chap. 7. One method to determine these directions, without actually drawing the centrodes themselves, will now be presented as an application of the velocity analysis. The direction of the common tangent is along the IC velocity,[1] $\mathbf{v}_I$, whence our objective becomes the determination of that velocity.

Again, for the sake of simplicity, the example chosen is that of the four-bar linkage O_AABO_B, shown in Fig. 4-18*a*. In order to apply

[1] See page 196.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 129 of 143

the method of velocity diagrams, we "materialize" the point I, IC of link 3 with respect to 1, as the center of a pin riding between two forks as shown, so that for all positions of the links this point is at the IC. Assume for convenience an angular velocity $\omega_3 = 1$ rad/sec counterclockwise. With the assumed dimensions we have

$$v_A = IA\omega_3 = 2.33 \text{ fps} \qquad v_B = IB\omega_3 = 1.74 \text{ fps}$$

Then, using the theorem of relative velocities [Eq. (4-1)],

$$\mathbf{v}_I = \mathbf{v}_{I/2} + \mathbf{v}_{I_2} \qquad \text{and} \qquad \mathbf{v}_I = \mathbf{v}_{I/4} + \mathbf{v}_{I_4}$$

or

$$\underset{\checkmark x_1}{\mathbf{v}_{I/2}} + \underset{\checkmark\checkmark}{\mathbf{v}_{I_2}} = \underset{\checkmark x_2}{\mathbf{v}_{I/4}} + \underset{\checkmark\checkmark}{\mathbf{v}_{I_4}}$$

The velocities $\mathbf{v}_{I_2}$ and $\mathbf{v}_{I_4}$ may be found by using ϑ lines and the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ as shown, so that both their direction and magnitude may be checked as known. The directions of $\mathbf{v}_{I/2}$ and $\mathbf{v}_{I/4}$ are along the forks of links 2 and 4, so that their directions may be checked. The remaining two unknowns, the magnitudes of $\mathbf{v}_{I/2}$ and $\mathbf{v}_{I/4}$, are found by drawing

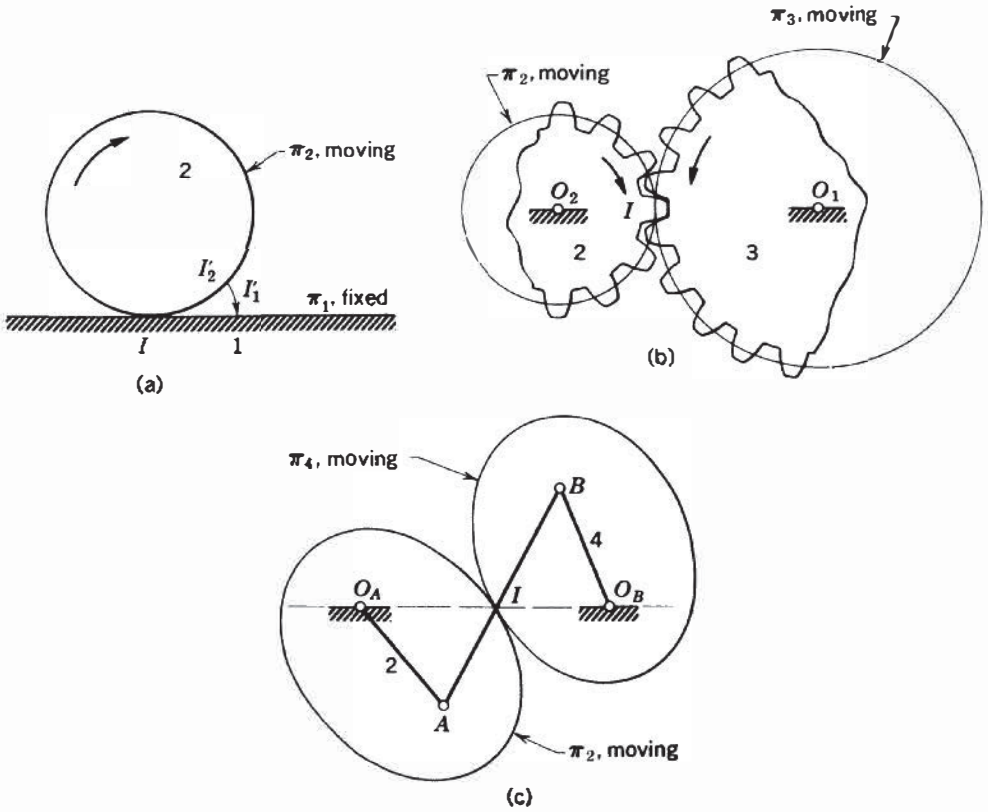


FIGURE 4-17 Situations for which centrodes are simple curves.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 130 of 143

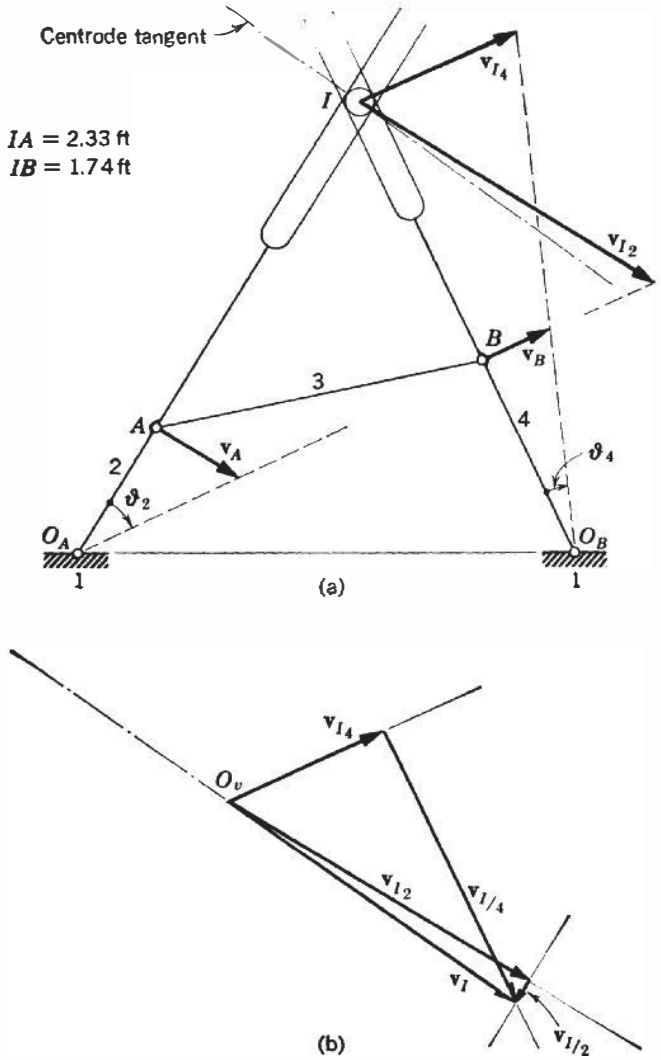


FIGURE 4-18 Determination of IC velocity. $O_AO_B = 3$ ft, $O_AA = 0.9$ ft, $AB = 2$ ft, $O_BB = 1.3$ ft.

the velocity diagram shown in Fig. 4-18*b*. An IC velocity of magnitude

$$v_I = 8.5 \text{ fps}$$

is found. Its direction is that of the common tangent to the centrodes at the instant considered.

4-8 ACCELERATION

The concept of acceleration is more difficult than the motion aspects discussed so far. Position, displacement, path, velocity, and speed are all qualities we can observe directly. We can judge the

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 131 of 143

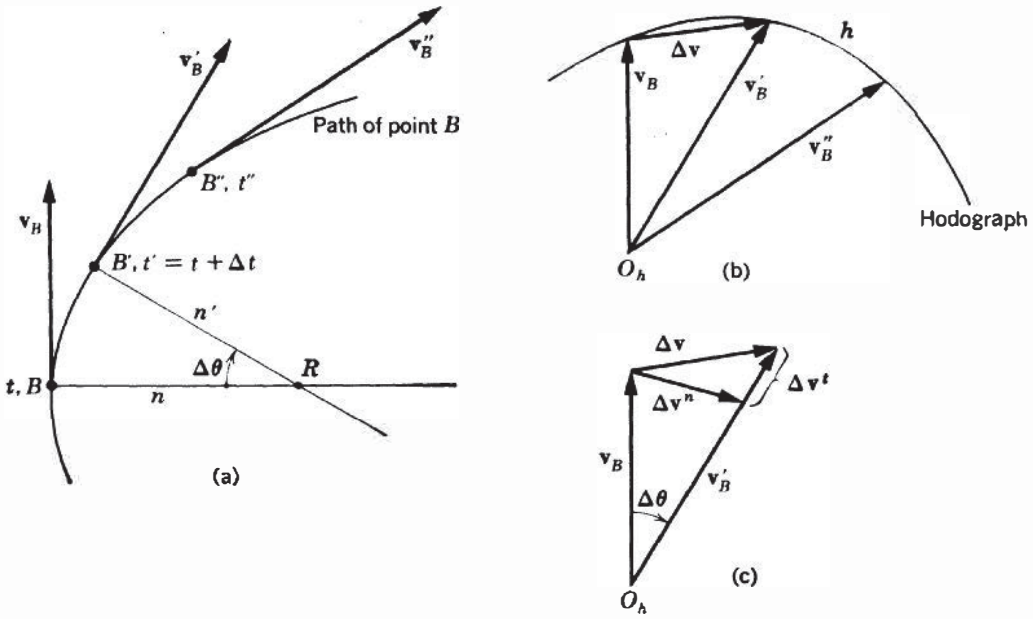


FIGURE 4-19 Definition of acceleration and hodograph.

implications of these qualities because of our considerable experience with them. For example, we are experts at integrating velocity vectors and are alive only because of our successes in doing so. The avoidance of collision between two moving bodies depends on an integration which it is hoped will produce different position coordinates of the moving bodies at all times, as in driving a car or crossing a street.

Acceleration is much more subtle and complex in its nature than the velocity change from which it comes. It may be said that our personal experience has been gained indirectly, for we observe the effect of acceleration, force, rather than the acceleration itself. For the layman there are no instruments providing numerical values of acceleration, in the sense that there are devices, as speedometers, that display information about distance and speed.

Consider (Fig. 4-19*a*) the path traced by a moving point occupying positions B and B' at instants t and $t' = t + \Delta t$. The velocities of the point at the instants considered are $\mathbf{v}_B$ and $\mathbf{v}_B' = \mathbf{v}_B + \Delta\mathbf{v}$, where $\Delta\mathbf{v}$ is the velocity increment during the time interval Δt (Fig. 4-19*b*). The average acceleration during this time interval Δt is

$$\mathbf{a}_{B,\mathrm{av}} = \frac{\Delta\mathbf{v}}{\Delta t}$$

and the instantaneous acceleration, or simply acceleration at time t, is

$$\mathbf{a}_B = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t}$$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 132 of 143

The acceleration is therefore a vector representing the instantaneous rate of change of velocity; its sense is that of the velocity being acquired.

A useful aid to the understanding of acceleration is the *hodograph*. The path traced by a moving point is shown in Fig. 4-19*a*, with several positions corresponding to times t, $t' = t + \Delta t$, etc., and their corresponding velocities. If these and all intermediate velocities are laid off from an origin O_h as shown in Fig. 4-19*b*, the curve h joining the tips of the velocity vectors is known as the hodograph. We may think of the hodograph as being generated by the tip of a rotating and stretching velocity vector. The chord between the tips of any two velocity vectors of the hodograph is the velocity increment $\Delta\mathbf{v}$ for the associated time interval Δt, and the "velocity" with which the hodograph is being traversed is the acceleration of point B at the instant considered. The acceleration of B is therefore at all times tangent to the hodograph at the corresponding point.

The velocity increment $\Delta\mathbf{v}$ may be resolved into the sum of two components, $\Delta\mathbf{v}^n$ and $\Delta\mathbf{v}^t$, as shown in Fig. 4-19*c*. On considering these two components separately, the acceleration is given by

$$\mathbf{a}_B = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}^t}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}^n}{\Delta t}$$

The direction of $\Delta\mathbf{v}^n$ is perpendicular to the bisector of the angle $\Delta\theta$ between $\mathbf{v}$ and $\mathbf{v}'$, and its magnitude is $\Delta v^n = v\,\Delta\theta$, $\Delta\theta$ being assumed to be a small angle. The angle $\Delta\theta$ also appears between the normals n and n' to the path at positions B and B' (see Fig. 4-19*a*). For a small interval Δt the distance BB' will be small, and the normals n and n' will intersect at R, close to the center of curvature of the path, so that $RB \approx RB' = \rho$. Then considering triangle RBB', $\Delta\theta = BB'/\rho = v\,\Delta t/\rho$ and $\Delta v^n = v^2\,\Delta t/\rho$. As Δt approaches zero, $\Delta\mathbf{v}^n/\Delta t$ becomes perpendicular to $\mathbf{v}$, that is, normal to the path, and its magnitude becomes v^2/ρ, where ρ is the radius of curvature of the path at B. This vector, called the *normal acceleration* and denoted as $\mathbf{a}_B{}^n$, always points toward the center of curvature of the path and is zero when the center of curvature is at infinity (rectilinear path, or inflection point).

The direction of $\Delta\mathbf{v}^t$ is along the velocity vector $\mathbf{v}'$, and its magnitude is the change in magnitude Δv of the velocity during the time interval Δt. As Δt approaches zero, the direction of $\Delta\mathbf{v}^t$ approaches that of $\mathbf{v}$; that is, it becomes tangent to the path, and its magnitude is dv/dt, the rate of change of the speed of the point along the path. This last vector gives the *tangential acceleration*, denoted as $\mathbf{a}_B{}^t$.

In summary, we have shown that the acceleration of a moving point is the sum of two rectangular components,

$$\mathbf{a}_B = \mathbf{a}_B{}^n + \mathbf{a}_B{}^t$$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 133 of 143

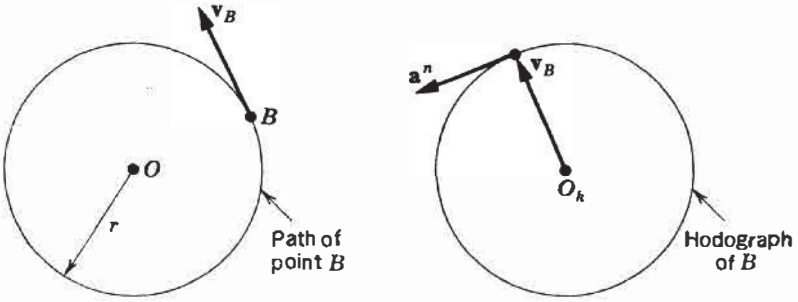


FIGURE 4-20 Hodograph of a point moving at constant speed around a circular path, $|\mathbf{v}_B| = v$ (const), $a^n = v^2/r$.

where the vectors $\mathbf{a}_B{}^n$ and $\mathbf{a}_B{}^t$ are as follows:

Normal acceleration $\mathbf{a}_B{}^n$

Magnitude: v^2/ρ

Direction: normal to path, pointing toward center of curvature

Tangential acceleration $\mathbf{a}_B{}^t$

Magnitude: dv/dt

Direction: tangential to path, pointing in direction of increasing speed

The hodographs of points having particular motions are interesting to note. If a point is moving along a straight line with constant speed, its hodograph is just a point, for its acceleration is zero. If a point moves around a circle of radius r with constant speed v, its hodograph is also a circle, but of radius v (Fig. 4-20). The acceleration of the point is then only the normal with a magnitude $a^n = v^2/r$.

4-9 RELATIVE ACCELERATION AND CORIOLIS ACCELERATION

In dealing with accelerations we shall follow the same pattern as with velocities and shall establish the distinction between relative acceleration and acceleration difference. Relative acceleration will be considered in this section, together with its necessary complement, the Coriolis acceleration component. Acceleration difference will be considered in the next section.

As in the study of relative velocity, consider a plane 2 moving with respect to a reference plane 1 (Fig. 4-21). A point[1] called B moves with respect to plane 2 with a relative velocity $\mathbf{v}_{B/2}$ and a *relative acceleration* $\mathbf{a}_{B/2}$. This relative acceleration is that which would be measured

[1] As in the case of relative velocity, this point B may be either (1) a point of another plane or (2) a point defined by a certain geometry, e.g., the intersection of two moving lines.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 134 of 143

by an observer stationed on plane 2. It is related to the velocity of B with respect to plane 2 and the curvature of the path traced by B on plane 2. Point B, however, also has an absolute velocity $\mathbf{v}_B$ and an *absolute acceleration* $\mathbf{a}_B$ referred to plane 1. Finally, the point B_2 fixed in plane 2 and coincident with B at the instant considered has velocity $\mathbf{v}_{B_2}$ and acceleration $\mathbf{a}_{B_2}$ with respect to plane 1. From the theorem on relative velocity we have that

$$\mathbf{v}_B = \mathbf{v}_{B/2} + \mathbf{v}_{B_2}$$

and one might expect something analogous to hold for acceleration. This, however, is not the case, for whenever plane 2 has an angular velocity ω_2 different from zero,

$$\mathbf{a}_B = \mathbf{a}_{B/2} + \mathbf{a}_{B_2} + \mathbf{a}^{\text{cor}} \tag{4-5}$$

where $\mathbf{a}^{\text{cor}}$ is the *Coriolis acceleration* component. There is nothing directly obvious about this component, also known as the compound supplementary acceleration. The term supplementary is descriptive, for the Coriolis component results from changes of two linear velocity vectors occasioned by the rotation ω_2 of the path that B traces upon plane 2. The Coriolis acceleration has a magnitude $2v_{B/2}\omega_2$; its *direction* is always normal to the path of the relative motion, and its *sense* is that of $\mathbf{v}_{B/2}$ after that vector has been turned 90° in the sense of the angular velocity. Different situations are shown in Fig. 4-22.

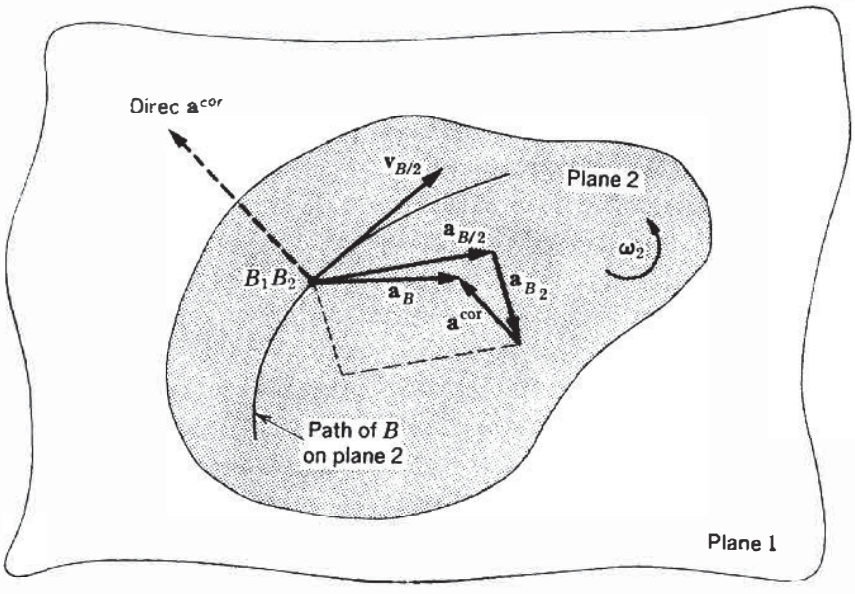


FIGURE 4-21 Relative acceleration and Coriolis acceleration.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 135 of 143

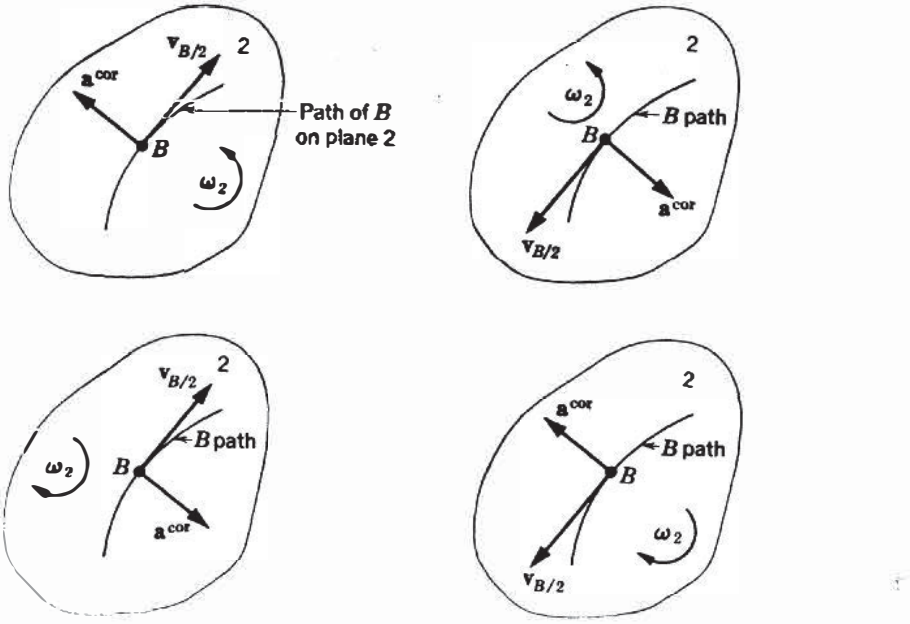


FIGURE 4-22 Direction of Coriolis acceleration for different cases.

To prove the Coriolis theorem [Eq. (4-5)] in its complete generality (even within the realm of planar motion) by the direct methods of the present chapter would be tedious and serve no useful purpose with its complexity.[1] The proof that follows is based upon a special situation in which the center of rotation is fixed. As implied above, the general case, in which the center of rotation itself is moving, will answer to the Coriolis theorem as developed here.

The situation shown in Fig. 4-23*a* represents a plane 2 rotating about O_B with respect to the fixed plane 1 with an angular velocity ω_2. The point[2] B of block 3 is guided by the slot in plane 2 and moves with respect to this plane with a velocity $\mathbf{v}_{B/2}$ and an acceleration $\mathbf{a}_{B/2}$. Position vectors and velocity components are shown in Fig. 4-23*b* for two positions of the slider corresponding to times t and $t + \Delta t$. It is clear that both the velocity components (radial and transverse) have undergone changes of magnitude and direction. These changes are best seen by considering separately what has happened to each velocity component.

The radial component is treated in Fig. 4-23*c*, the increment $\Delta\mathbf{v}_B{}^r = \mathbf{DF}$ being resolved into two components, a radial **DE** and a transverse **EF**. The diagram in Fig. 4-23*d* shows the resolution of the increment $\Delta\mathbf{v}_B{}^\theta = \mathbf{GJ}$ into its components, a radial **HJ** and a transverse **GH**.

[1] See Prob. 4-9 for a complete (planar) proof using complex number notation.

[2] This point B is associated with plane 3 and should therefore be identified as B_3. However, the subscript 3 has been suppressed to reduce unwieldiness.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 136 of 143

The total change of velocity in the radial direction is given by DE − JH. The radial component of acceleration is then

$$a_{B^r} = \lim_{\Delta t \to 0} \frac{\mathrm{DE}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\mathrm{JH}}{\Delta t}$$

By analogy with similar vector situations we may write

$$a_{B^r} = \frac{d}{dt} v_{B^r} - v_{B^\theta} \frac{d\theta_2}{dt}$$

With $v_{B^r} = d\rho/dt$, $v_{B^\theta} = \rho(d\theta_2/dt)$, and $d\theta_2/dt = \omega_2$, this magnitude of the radial component of the acceleration is

$$a_{B^r} = \frac{d^2\rho}{dt^2} - \rho\omega_2{}^2 = a_{B/2} - \rho\omega_2{}^2 \tag{4-6}$$

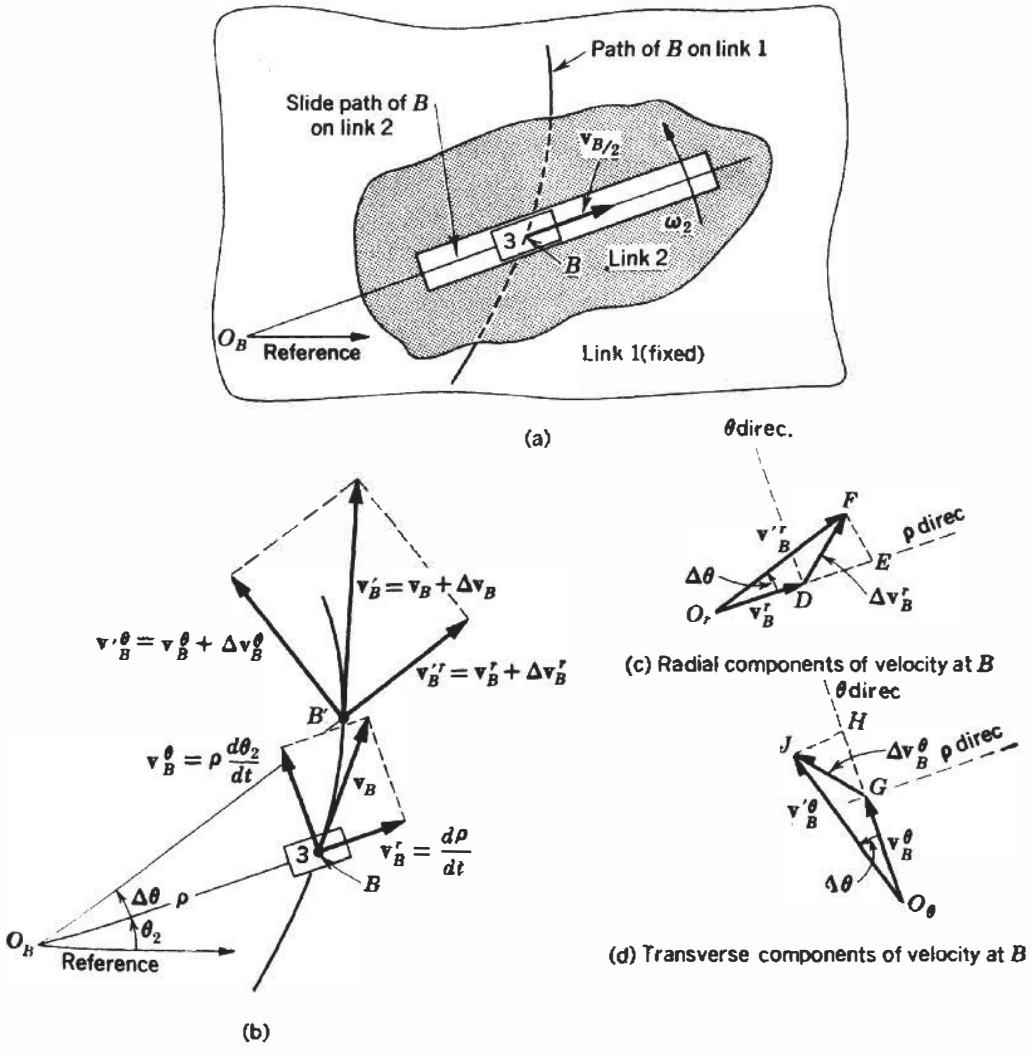


FIGURE 4-23 Proof of Coriolis theorem in a special case.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 137 of 143

For this radial, or slide-path, direction we recognize $a_{B/2}$ as the magnitude of the relative linear acceleration of B with respect to link 2; it is a measure of the speed change of the sliding velocity that would be present even if link 2 were not rotating. The $\rho\omega_2^2$ term is the normal acceleration of the coincident point B_2, obviously a consequence of rotation of link 2.

As for the transverse direction, the total velocity change is given by **EF** + **GH**, whence

$$\mathbf{a}_{B^\theta} = \lim_{\Delta t \to 0} \frac{\mathbf{EF}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\mathbf{GH}}{\Delta t}$$

or

$$\begin{aligned} a_{B^\theta} &= v_{B^r} \frac{d\theta_2}{dt} + \frac{dv_{B^\theta}}{dt} = \frac{d\rho}{dt}\frac{d\theta_2}{dt} + \frac{d}{dt}\left(\rho \frac{d\theta_2}{dt}\right) \\ &= \frac{d\rho}{dt}\omega_2 + \rho\frac{d^2\theta_2}{dt^2} + \frac{d\rho}{dt}\omega_2 \\ &= v_{B/2}\omega_2 + \rho\alpha_2 + v_{B/2}\omega_2 \end{aligned}$$

In the θ direction, which is perpendicular to the slide path, the component $\rho\alpha_2$ is the angular acceleration of point B_2. The first of the $v_{B/2}\omega_2$ terms is a result of the direction change of the radial velocity component; the second $v_{B/2}\omega_2$ has to do with the changing magnitude of the transverse velocity component. Gathering the two like terms into one, we realize that this supplementary term is compounded from a sliding velocity along a rotating path. We write

$$a_{B^\theta} = \rho\alpha_2 + 2v_{B/2}\omega_2 \tag{4-7}$$

Assembling the acceleration components to form $\mathbf{a}_B$, we have

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_{B^r} + \mathbf{a}_{B^\theta} \\ &= \mathbf{a}_{B_2} + \mathbf{a}_{B/2} + \mathbf{a}^{\mathrm{cor}} \end{aligned}$$

where we have

Coriolis acceleration $\mathbf{a}^{\mathrm{cor}}$

Magnitude: $2v_{B/2}\omega_2$

Direction: rotated 90° from $\mathbf{v}_{B/2}$ in same sense as ω_2

It will be observed that the Coriolis acceleration was derived from the **EF** and **GH** components (Fig. 4-23*c* and *d*). As remarked, these components are in the θ direction, whence they are normal to the path (and $\mathbf{v}_{B/2}$).

4-10 ACCELERATION DIFFERENCE

Two points A and B of a moving plane 2 are shown in Fig. 4-24. The reference plane is 1, and the angular velocity and acceleration of plane 2 with respect to plane 1 at the instant considered are ω_2 and α_2.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 138 of 143

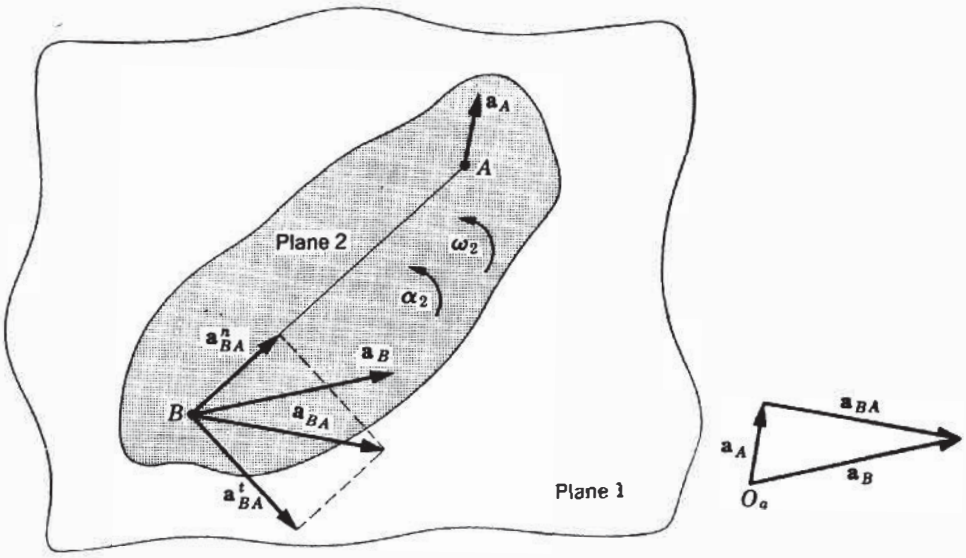


FIGURE 4-24 Acceleration difference of two points on the same moving plane.

With $\mathbf{a}_A$ and $\mathbf{a}_B$ representing the accelerations of points A and B, the acceleration difference of these two points is defined as

$$\mathbf{a}_{BA} = \mathbf{a}_B - \mathbf{a}_A$$

from which

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \tag{4-8}$$

The remarks made concerning the concepts of relative velocity and velocity difference apply equally well to the relative acceleration and acceleration difference. It is inconsistent to call $\mathbf{a}_{BA}$ a relative acceleration, as is often done, because an acceleration cannot be referred to a point but must be referred to a plane (in the case of planar motion). Furthermore, since Eq. (4-8) deals with the difference of two absolute accelerations, no component such as the Coriolis can be distinguished. We emphasize that A and B are points of the *same* moving plane 2; they are not points of two different planes, as in the preceding section.

The acceleration difference will now be related to the distance AB and the angular motion of plane 2 as described by ω_2 and α_2. Again, accelerations are considered in terms of vanishingly small velocity increments. During a small time interval Δt, points A and B are assumed to move to positions A' and B' as shown in Fig. 4-25*a*. The diagrams in Fig. 4-25*b* and *c* show the situations so far as velocities are concerned at the beginning and end of the time interval; or in equation form we have

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A \quad \text{and} \quad \mathbf{v}'_{BA} = \mathbf{v}'_B - \mathbf{v}'_A$$

But $\mathbf{v}'_A = \mathbf{v}_A + \Delta\mathbf{v}_A \qquad \mathbf{v}'_B = \mathbf{v}_B + \Delta\mathbf{v}_B \qquad \mathbf{v}'_{BA} = \mathbf{v}_{BA} + \Delta\mathbf{v}_{BA}$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 139 of 143

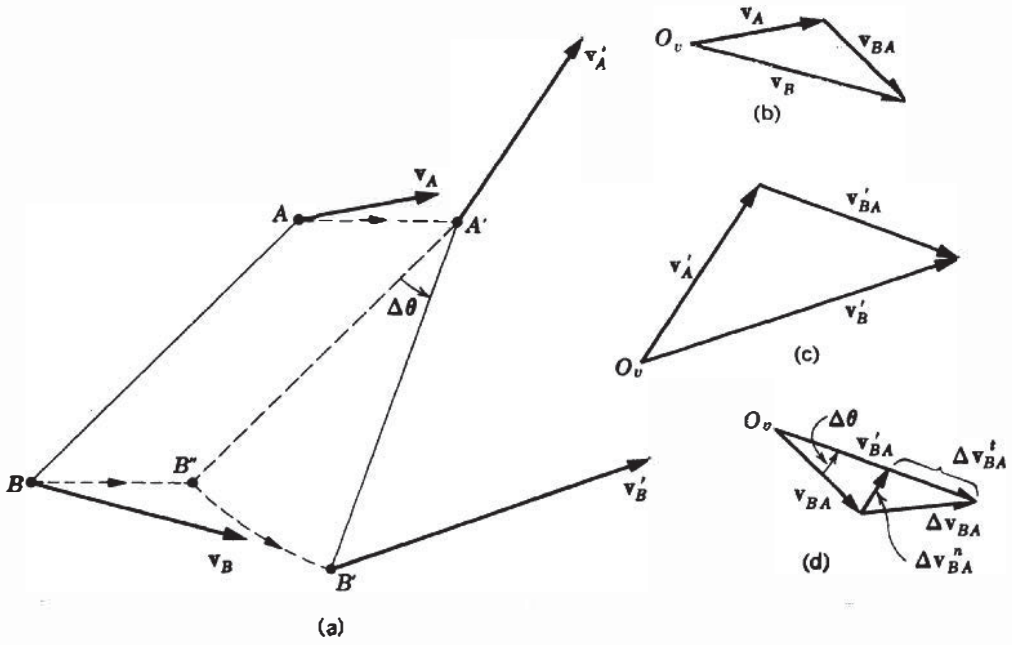


FIGURE 4-25 Determination of the acceleration difference of A and B.

where $\Delta\mathbf{v}_A$, $\Delta\mathbf{v}_B$, and $\Delta\mathbf{v}_{BA}$ are the increments in the velocity of A, the velocity of B, and the velocity difference between A and B during the time interval Δt. Then

$$\Delta\mathbf{v}_{BA} = \Delta\mathbf{v}_B - \Delta\mathbf{v}_A = (\mathbf{v}'_B - \mathbf{v}'_A) - (\mathbf{v}_B - \mathbf{v}_A)$$

The acceleration difference, which may be written as

$$\mathbf{a}_{BA} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_B}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_A}{\Delta t}$$

is then

$$\mathbf{a}_{BA} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}_{BA}}{\Delta t}$$

The velocity differences $\mathbf{v}_{BA}$ and $\mathbf{v}'_{BA}$ are shown in Fig. 4-25*d*, which also shows the increment $\Delta\mathbf{v}_{BA}$ as the sum of two components $(\Delta\mathbf{v}_{BA})^n$ and $(\Delta\mathbf{v}_{BA})^t$. The angle $\Delta\theta$ shown in that diagram is also the angle of rotation of plane 2 during Δt. Bypassing intermediate steps which are now familiar, we state the final result: the acceleration difference is the sum of two rectangular components,

$$\mathbf{a}_{BA} = \mathbf{a}_{BA}{}^n + \mathbf{a}_{BA}{}^t$$

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 140 of 143

where the vectors $\mathbf{a}_{BA}{}^n$ and $\mathbf{a}_{BA}{}^t$ are as follows (Fig. 4-24):

Normal acceleration difference $\mathbf{a}_{BA}{}^n$

Magnitude: $v_{BA}{}^2/AB$

Direction: along AB, pointing from B toward A

Tangential acceleration difference $\mathbf{a}_{BA}{}^t$

Magnitude: $(AB)\alpha_2$

Direction: rotated 90° from AB; in the same direction as α_2

As an example, we shall investigate the acceleration condition of the slider-crank mechanism of Fig. 4-26, assuming the angular velocity ω_2 and angular acceleration α_2 of link 2 to be known.

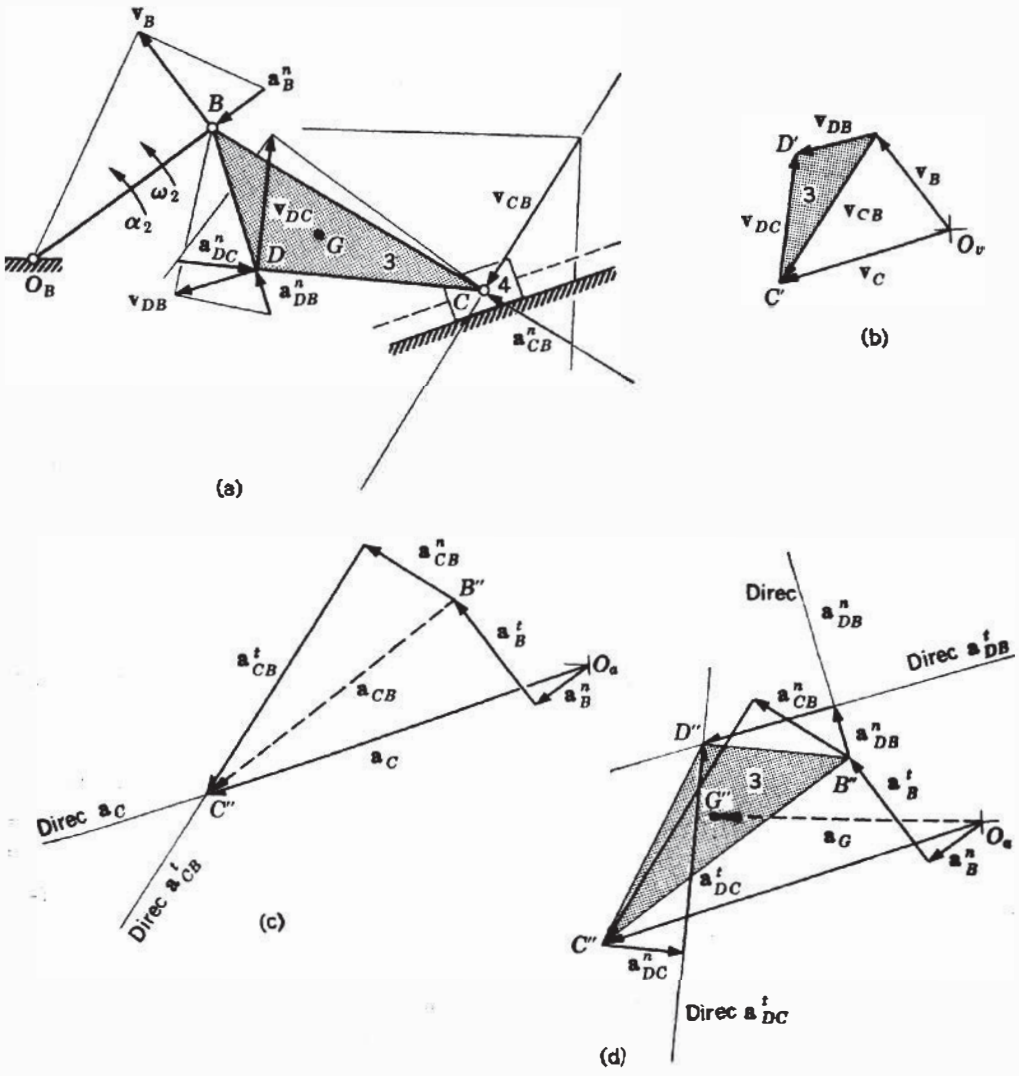


FIGURE 4-26 Acceleration analysis of the slider-crank mechanism.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 141 of 143

It will be convenient to employ scales conforming to the relation $k_a = k_v^2/k_s$.* The velocity polygon (Fig. 4-26*b*) is then constructed from the single equation $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$ and the velocity image. With point D' at hand, the velocity differences $\mathbf{v}_{DB}$ and $\mathbf{v}_{DC}$ are directly available.

Full possession of the velocity vectors allows the graphical determination of all a^n components at points B, C, and D, as shown in Fig. 4-26*a*. Only a_{B^t} needs external calculation, as by slide rule.

The diagram in Fig. 4-26*c* shows the vector polygon of the equation

$$\mathbf{a}_C = \mathbf{a}_{B^n} + \mathbf{a}_{B^t} + \mathbf{a}_{CB^n} + \mathbf{a}_{CB^t}$$

in which the magnitude of $\mathbf{a}_C$ and the direction of $\mathbf{a}_{CB^t}$ are the unknowns. After completion of the polygon, the absolute angular acceleration of link 3 is calculated from $\alpha_3 = a_{CB^t}/BC$ (clockwise).

The acceleration of D can be established from the simultaneous solution of two equations in four unknowns,

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{DB^n} + \mathbf{a}_{DB^t}$$

$$\mathbf{a}_D = \mathbf{a}_C + \mathbf{a}_{DC^n} + \mathbf{a}_{DC^t}$$

For clarity, the first vector polygon has been redrawn in Fig. 4-26*d* and the solutions of the last two equations added, defining the homologous point D''. The vector $\mathbf{O}_a\mathbf{D}''$, common to both equations, is the acceleration of D; it is not shown because of the already many lines. The shaded figure $B''C''D''$ is known as the acceleration image of link 3; it is geometrically similar to the figure BCD that is link 3. Note that D'' is on the same side of $B''C''$ as D is with respect to BC. The acceleration image has been turned through an angle $\pi - \beta$ with respect to its link in the sense of α_3; $\beta = \arctan(\alpha_3/\omega_3^2)$.

In use, the performance of the acceleration image is similar to that of the velocity image: the simultaneous solution of the two vector equations for $\mathbf{a}_D$ could have been avoided. Thus, the acceleration of point G is given by the vector $\mathbf{O}_a\mathbf{G}''$ of the acceleration diagram (Fig. 4-26*d*), since G'' is the homologous point of G.

* The scale factors are defined in consistent units, e.g.:
1 in. on drawing = a distance of k_s ft on real mechanism.
1 in. of velocity vector = k_v fps.
1 in. of acceleration vector = k_a fps^2.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 142 of 143

BIBLIOGRAPHY

Beggs, J. S.: "Mechanism," McGraw-Hill Book Company, New York, 1955.

Cowie, A.: "Kinematics and the Design of Mechanisms," International Textbook Company, Scranton, Pa., 1961.

Ham, C. W., E. J. Crane, and W. L. Rogers: "Mechanics of Machinery," 4th ed., McGraw-Hill Book Company, New York, 1958.

Hrones, J. A., and G. L. Nelson: "Analysis of the Four Bar Linkage," Massachusetts Institute of Technology Press, Cambridge, Mass., 1951.

Kennedy, A. B. W.: "The Mechanics of Machinery," Macmillan & Co., Ltd., London, 1886.

Smith, Robert H.: A New Graphic Analysis of the Kinematics of Mechanisms, *Trans. Roy. Soc. Edinburgh*, vol. 32, pp. 507–517, 1882–1885. Also in "Graphics," pp. 114–162, Longmans, Green & Co., Ltd., London, 1889.

Exhibit 1108
Bazooka v. Nuhn - IPR2024-00098
Page 143 of 143