

MACHINE DESIGN

An Integrated
Approach

THIRD EDITION



Robert L. Norton

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An Integrated Approach

Robert L. Norton

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ABOUT THE BOOK

This textbook presents an integrated approach to the design of machine elements by uniting machine-element topics with a series of case studies that demonstrate the interrelationships between force, stress and failure analysis in real-world design. While emphasizing the design aspects of the subject, the book also presents a thorough and complete treatment of engineering mechanics topics and provides a good balance between synthesis and analysis. It is based on the author's extensive experience in the practice of machine design in industry.

The text is divided into two parts. The first part presents the fundamentals of design, materials, force, stress, strain, deflection analysis, and failure theories, including an introduction to fracture mechanics. The second part treats the design of specific elements such as springs, gears, shafts, fasteners, etc., but does so in the context of their use in integrated assemblies. The case studies provide the vehicle for this, with different elements of the same assembly being addressed in various chapters. The main emphasis is on designing for high-cycle fatigue loading of machine parts and assemblies. A new chapter in this edition presents an introduction to Finite Element Analysis in which common pitfalls of this technique's application are addressed.

The machine-design subject matter is presented in an up-to-date manner using computer-aided-design techniques. Most of the 75 examples and 26 case-study analyses are solved with equation solvers and over 250 computer files are provided on the accompanying CD-ROM that allow the book's examples, case studies, and other problems to be solved in Microsoft® Excel, MATLAB®, Mathcad®, and TK Solver™. A number of custom-written programs for kinematic, dynamic, and stress analysis are also included. The book's website, accessible via the Instructor Support tab at <http://www.prenhall.com> provides Microsoft® Powerpoint® compatible files of all the book's figures and tables. Additional supplements are available at the author's website at <http://www.designofmachinery.com>.

WHAT USERS SAY ABOUT THE BOOK

This book is one of the best-written on any engineering subject that I have come across. We switched to the book because we felt that the students would get a lot out of reading the text.

—Professor Ed Howard, Milwaukee School of Engineering

(The) writing is clear and meaningful, (and the) text is much more accessible to students because of the practical nature of (its) case studies and problem sets.

—Professor Douglas Walcerz, York College of Pennsylvania

I'm doing my machine design homework now, and . . . your book is the single best text I have ever used in my educational career. It is straightforward, easy to understand and has good examples.

—Josh, student, Colorado School of Mines

ABOUT THE AUTHOR

Robert L. Norton is Professor of Mechanical Engineering at Worcester Polytechnic Institute, Worcester, MA, where he teaches machine design, kinematics, and dynamics of machinery. He has over 45 years experience in the design and manufacturing of products, devices, and machines and is the inventor or co-inventor on 13 U.S. patents.

He has also written the books *Design of Machinery: An Introduction to the Synthesis and Analysis of Mechanisms and Machines* and the *Cam Design and Manufacturing Handbook*.

Professor Norton is a registered professional engineer in Massachusetts, a member of the Society of Automotive Engineers, a Fellow of the American Society of Mechanical Engineers, and recipient of the 2002 ASME Machine Design Award.

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MACHINE DESIGN

An Integrated Approach

Third Edition

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Donald N. Zwiep

A gentleman and a leader,
without whose faith and foresight,
this book would never have been written.

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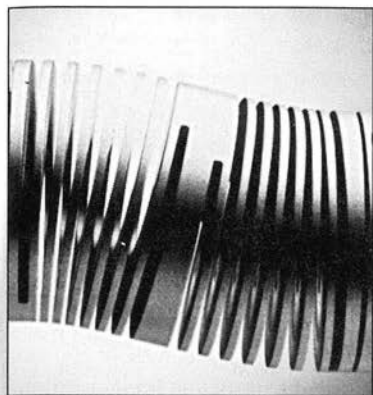
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SHAFTS, KEYS, AND COUPLINGS

*The greater our knowledge increases, the
greater our ignorance unfolds.*

JOHN F. KENNEDY

9.0 INTRODUCTION

Transmission shafts, or just shafts, are used in virtually every piece of rotating machinery to transmit rotary motion and torque from one location to another. Thus, the machine designer is often faced with the task of designing shafts. This chapter will explore some of the common problems encountered in that endeavor. Table 9-0 shows the variables used in this chapter and notes the equations or sections in which they occur.

At a minimum, a shaft typically transmits torque from the driving device (motor, or engine) through the machine. Sometimes shafts will carry gears, sheaves (pulleys), or sprockets, which transmit the rotary motion via mating gears, belts, or chains from shaft to shaft. The shaft may be an integral part of the driver, such as a motor shaft or engine crankshaft, or it may be a freestanding shaft connected to its neighbor by a coupling of some design. Automated production machinery often has line shafts that extend the length of the machine (10m or more) and carry the power to all the workstations. Shafts are carried in bearings, in a simply supported (straddle-mounted) configuration, cantilevered, or overhung, depending on the machine configuration. The pros and cons of these mounting and coupling arrangements will also be discussed.

9.1 SHAFT LOADS

The loading on rotating transmission shafts is principally one of two types: torsion due to the transmitted torque or bending from transverse loads at gears, sheaves, and sprockets. These loads often occur in combination, since, for example, the transmitted torque may be associated with forces at the teeth of gears or sprockets attached to the shafts.

Table 9-0 Variables Used in This Chapter

Symbol	Variable	ips units	SI units	See
A	area	in^2	m^2	various
c	distance to outer fiber	in	m	Sect. 9.6
C_f	coefficient of fluctuation	none	none	Eq. 9.19
d	diameter	in	m	various
e	eccentricity of a shaft or disk	in	m	Eq. 9.26
E	Young's modulus	psi	Pa	various
E_k, E_p	kinetic energy, potential energy	in-lb	joule	Eq. 9.25
F	force or load	lb	N	various
F_l	fluctuation (in speed)	rad/sec	rad/sec	Eq. 9.19
f_n	natural frequency	Hz	Hz	Eq. 9.24
g	gravitational acceleration	in/sec^2	m/sec^2	Sect. 9.13, 9.14
G	shear modulus, modulus of rigidity	psi	Pa	various
I, J	2nd moment, polar 2nd moment of area	in^4	m^4	Sects. 9.6, 9.14
I_m, I_s	mass moment of inertia about axis	lb-in-sec^2	N-m-sec^2	Sect. 9.13
k	spring rate or spring constant	lb/in	N/m	Sect. 9.14
K_f, K_{fm}	fatigue stress-concentration factors	none	none	Sects. 9.6, 9.10
K_t, K_{ts}	geometric stress-concentration factors	none	none	Sect. 9.10
l	length	in	m	various
m	mass	$\text{lb-sec}^2/\text{in}$	kg	Sect. 9.13, 9.14
M	moment, moment function	lb-in	N-m	various
N_f	safety factor in fatigue	none	none	Eq. 9.5–9.8
N_y	safety factor in yielding	none	none	Ex. 9-7
P	power	hp	watts	Eq. 9.1
p	pressure	psi	N/m^2	Sect. 9.12
r	radius	in	m	various
S_e, S_f	corrected endurance limit, fatigue strength	psi	Pa	Eqs. 9.5–9.8
S_{ut}, S_y	ultimate tensile strength, yield strength	psi	Pa	Eqs. 9.5–9.8
T	torque	lb-in	N-m	Eq. 9.1
W	weight	lb	N	various
α	angular acceleration	rad/sec^2	rad/sec^2	Eq. 9.18
δ	deflection	in	m	various
ν	Poisson's ratio	none	none	various
θ	angular deflection or beam slope	rad	rad	various
γ	weight density	lb/in^3	N/m^3	Eq. 9.23
σ	normal stress (with various subscripts)	psi	Pa	various
σ'	von Mises stress (with various subscripts)	psi	Pa	various
τ	shear stress (with various subscripts)	psi	Pa	various
ω	angular velocity	rad/sec	rad/sec	Eq. 9.1
ω_n	natural frequency	rad/sec	rad/sec	Sect. 9.14
ζ	damping ratio	none	none	Sect. 9.14

The character of both the torque and bending loads may be either steady (constant) or may vary with time. Steady and time-varying torque and bending loads can also occur in any combination on the same shaft.

If the shaft is stationary (nonrotating) and the sheaves or gears rotate with respect to it (on bearings), then it becomes a statically loaded member as long as the applied loads are steady with time. However, such a nonrotating shaft is not a transmission shaft, since it is not transmitting any torque. It is merely an axle, or round beam, and can be designed as such. This chapter is concerned with rotating, transmission shafts and their design for fatigue loading.

Note that a rotating shaft subjected to a steady, transverse-bending load will experience a fully reversed stress state as shown in Figure 9-1a. Any one stress element on the shaft surface goes from tension to compression each cycle as the shaft turns. Thus, even for steady bending loads, a rotating shaft must be designed against fatigue failure. If either or both the torque and transverse loads vary with time, the fatigue loading becomes more complex, but the fatigue-design principles remain the same, as outlined in Chapter 6. The torque, for example, could be repeated or fluctuating as shown in Figures 9-1b and c, as could the bending loads.

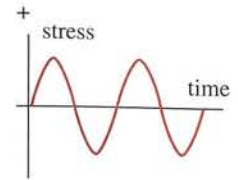
We will deal primarily with the general case, which allows for the possibility of both steady and time-varying components in both bending and torsion loads. If either load lacks a steady or time-varying component in a given case, it will merely force a term in the general equations to zero and simplify the calculation.

9.2 ATTACHMENTS AND STRESS CONCENTRATIONS

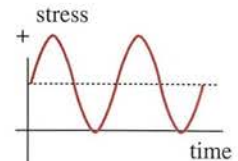
While it is sometimes possible to design useful transmission shafts that have no changes in section diameter over their length, it is most common for shafts to have a number of steps or shoulders where the diameter changes to accommodate attached elements such as bearings, sprockets, gears, etc., as shown in Figure 9-2, which also shows a collection of features commonly used to attach or locate elements on a shaft. Steps or shoulders are necessary to provide accurate and consistent axial location of the attached elements as well as to create the proper diameter to fit standard parts such as bearings.

Keys, snap rings, or cross-pins are often used to secure attached elements to the shaft in order to transmit the required torque or to capture the part axially. Keys require a groove in both shaft and part and may need a setscrew to prevent axial motion. Snap rings groove the shaft, and cross-pins create a hole through the shaft. Each of these changes in contour will contribute some stress concentration and this must be accounted for in the fatigue-stress calculations for the shaft. Use generous radii where possible, and techniques such as those shown in Figures 4-37 (p. 188), 4-38 (p. 189), and 9-2 (at the sheave and snap ring) to reduce the effects of these stress concentrations.

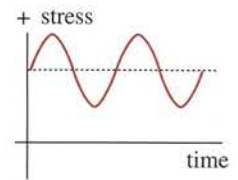
Keys and pins can be avoided by using friction to attach elements (gears, sprockets) to a shaft. Many designs of **clamp collars** (keyless fits^{*}) are available, which squeeze the outside diameter (OD) of the shaft with high compressive force to clamp something to it, as shown on the sprocket hub in Figure 9-2 and in Figure 9-34 (p. 559). The hub has a gently tapered bore, and a matching taper on this type of clamp collar is forced into the space between hub and shaft by tightening the bolts. Axial slits in the tapered portion of the collar allow it to change diameter and squeeze the shaft, creating sufficient friction to transmit the torque. Another type of clamp collar, called a **split collar**, uses a screw to close a radial slit and clamp the collar to the shaft. **Press and shrink fits** are also used for this purpose and will be discussed in a later section of this



(a) Fully reversed



(b) Repeated



(c) Fluctuating

FIGURE 9-1

Time-Varying Stresses

^{*} See the ANSI/AGMA Standard 9003-A91, *Flexible Couplings—Keyless Fits*.

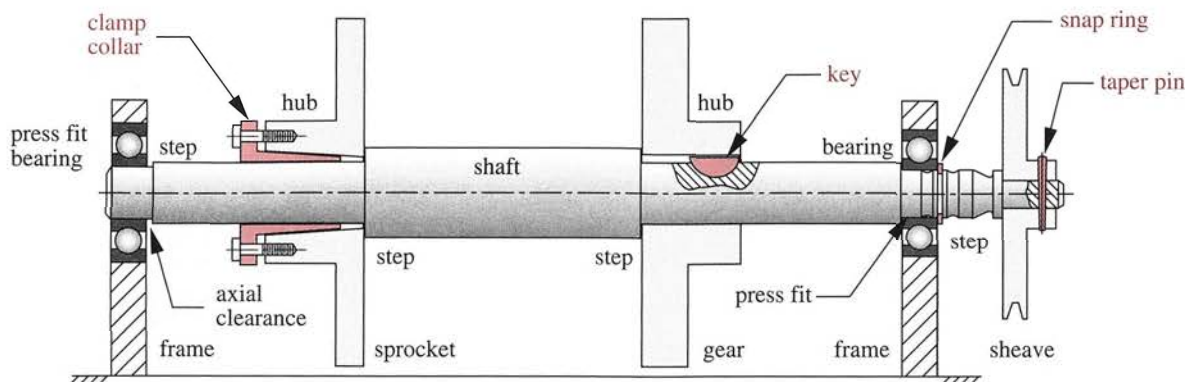


FIGURE 9-2

Various Methods to Attach Rotating Elements to Shafts

chapter. However, as we will see, these friction couplings also create stress concentrations in the shaft and can cause fretting corrosion as described in Section 7-6 (p. 430).

A standard taper pin is sometimes used to couple elements to shafts as seen in the sheave of Figure 9-2. The hole is reamed to match the standardized pin-taper and the purchased pin is driven into place. The shallow taper locks it by friction. It must be driven out for disassembly. This technique should be used with caution in locations of large bending moment, as it weakens the shaft as well as creating stress concentration.

Rolling-element bearings as shown in Figure 9-2 are intended to have their inner and outer races be press-fitted to shaft and housing, respectively. This requires close-tolerance machining of the shaft diameter and requires a step shoulder to provide a stop for the press fit and for axial location. Thus, one must start with a larger stock shaft diameter than the bearing inside diameter (*ID*) and machine the shaft to fit the selected bearing whose sizes are standardized (and are metric). A snap ring is sometimes used to guarantee no axial movement of the shaft versus the bearing as shown at the sheave end of the shaft in Figure 9-2. Snap rings are commercially available in a variety of styles and require that a small, close-tolerance groove of specific dimension be machined in the shaft. Note in Figure 9-2 how the axial location of the shaft is achieved by capturing only one of the bearings (the right one) axially. The other bearing at the left-hand end has axial clearance between it and the step. This is to prevent axial stresses being generated by thermal expansion of the shaft between the two bearings.

So, it appears that we cannot escape the problems of stress concentration in practical machinery. In the case of shafts, we need to provide steps, snap rings, or other means to accurately locate components axially on the shaft, and we have to key, pin, or squeeze the shaft to transmit the torque.

Each of these methods of attachment has its own advantages and disadvantages. A key is simple to install and sizes are standardized to the shaft diameter. It provides accurate phasing* and is easily disassembled and repaired. It may have no resistance to axial movement and it does not always provide a truly tight torque coupling due to the slight clearance between key and keyway. Torque reversals can cause slight backlash.

* Phasing means the relative angular locations of the various elements attached to the shaft.

A taper pin creates a truly tight torque coupling and locates axially as well as radially with phasing but weakens the shaft. It can be disassembled with slightly more difficulty than a key. A clamp collar is easy to install but has no repeatable phasing. This is only a disadvantage if timing of the shaft rotation to other shafts in the system is required. It allows easy (though inaccurate) adjustment of phasing if desired. Press fits are semipermanent connections that require special equipment to disassemble. They do not provide repeatable phasing.

9.3 SHAFT MATERIALS

In order to minimize deflections, steel is the logical choice for a shaft material because of its high modulus of elasticity, though cast or nodular iron is sometimes also used, especially if gears or other attachments are integrally cast with the shaft. Bronze or stainless steel is sometimes used for marine or other corrosive environments. Where the shaft also serves as the journal, running against a sleeve bearing, hardness can become an issue. Through- or case-hardened steel may be the material of choice for the shaft in these cases. See Chapter 10 for a discussion of desired relative hardness and material combinations for shafts and bearings. Rolling-element bearings do not need hardened shafts.

Most machine shafts are made from low- to medium-carbon steel, either cold rolled or hot rolled, though alloy steels are also used where their higher strengths are needed. Cold-rolled steel is more often used for smaller-diameter shafts (< about 3-in dia) and hot-rolled used for larger sizes. The same alloy when cold rolled has higher mechanical properties than if hot rolled due to the cold working, but this comes at the cost of residual tensile stresses in the surface. Machining for keyways, grooves, or steps relieves these residual stresses locally and can cause warping. Hot-rolled bars must be machined all over to remove the carburized outer layer, whereas portions of a cold-rolled surface can be left as-rolled except where machining to size is needed for bearings, etc. Prehardened (30HRC) and ground precision (straight) steel shafting can be purchased in small sizes and can be machined with carbide tools. Full-hard, ground, precision shafting (60HRC) is also available but cannot be machined.

9.4 SHAFT POWER

The power transmitted through a shaft can be found from first principles. In any rotating system, instantaneous power is the product of torque and angular velocity,

$$P = T\omega \quad (9.1a)$$

where ω must be expressed in radians per unit time. Whatever the base units used for calculations, power is usually converted to units of horsepower (hp) in any English system or to kilowatts (kW) in any metric system. (See Table 1-5 on p. 24 for conversion factors.) Both torque and angular velocity can be time varying, though much of rotating machinery is designed to operate at constant or near-constant speeds for large blocks of time. In such cases, the torque will often vary with time. The average power is found from

$$P_{avg} = T_{avg}\omega_{avg} \quad (9.1b)$$

9.5 SHAFT LOADS

The most general shaft-loading case is that of a fluctuating torque and a fluctuating moment in combination. There can be axial loads as well, if the shaft axis is vertical or is fitted with helical or worm gears having an axial force component. (A shaft should be designed to minimize the portion of its length subjected to axial loads by taking them to ground through suitable thrust bearings as close to the source of the load as possible.) Both torque and moment can be time varying, as shown in Figure 9-1, and can have both mean and alternating components.

The combination of a bending moment and a torque on a rotating shaft creates multiaxial stresses. The issues discussed in Section 6.12 (p. 372) on multiaxial stresses in fatigue are then germane. If the loadings are asynchronous, random, or misphased, then it will be a *complex multiaxial* stress case. But, even if the moment and torque are in phase (or 180° out of phase), it may still be a complex multiaxial stress case. The critical factor in determining whether it has simple or complex multiaxial stresses is the direction of the principal alternating stress on a given shaft element. If its direction is constant with time, then it is considered a simple multiaxial stress case. If it varies with time, then it is a complex multiaxial stress case. Most rotating shafts loaded in both bending and torsion will be in the complex category. While the direction of the alternating bending stress component will tend to be constant, the torsional component's direction varies as the element rotates around the shaft. Combining them on the Mohr's circle will show that the result is an alternating principal stress of varying direction. One exception to this is the case of a constant torque superposed on a time-varying moment. Since the constant torque has no alternating component to change the direction of the principal alternating stress, this becomes a simple multiaxial stress case. However even this exception cannot be taken if there are stress concentrations present, such as holes or keyways in the shaft, since they will introduce local biaxial stresses and require a complex multiaxial fatigue analysis.

Assume that the bending moment function over the length of the shaft is known or is calculable from the given data and that it has both a mean component M_m and an alternating component M_a . Likewise, assume the torque on the shaft is known or calculable from given data and has both mean and alternating components, T_m and T_a . Then the general approach follows that outlined in the list labeled **Design Steps for Fluctuating Stresses** in Section 6.11 (p. 356) in combination with the multiaxial stress issues addressed in Section 6.12 (p. 372). Any locations along the length of the shaft that appear to have large moments and/or torques (especially if in combination with stress concentrations) need to be examined for possible stress failure and the cross-sectional dimensions or material properties adjusted accordingly.

9.6 SHAFT STRESSES

With the understanding that the following equations will have to be calculated for a multiplicity of points on the shaft and their combined multiaxial effects also considered, we must first find the applied stresses at all points of interest. The largest alternating and mean bending stresses are at the outside surface and are found from

$$\sigma_a = k_f \frac{M_a c}{I} \qquad \sigma_m = k_{fm} \frac{M_m c}{I} \qquad (9.2a)$$

where k_f and k_{fm} are the bending-fatigue stress-concentration factors for the alternating and mean components, respectively (see equations 6.11 and 6.17 on pp. 339 and 360). Since the typical shaft is a solid-round cross section,* we can substitute for c and I :

$$c = r = \frac{d}{2} \qquad I = \frac{\pi d^4}{64} \qquad (9.2b)$$

giving

$$\sigma_a = k_f \frac{32M_a}{\pi d^3} \qquad \sigma_m = k_{fm} \frac{32M_m}{\pi d^3} \qquad (9.2c)$$

where d is the local shaft diameter at the section of interest.

The alternating and mean torsional shear stresses are found from

$$\tau_a = k_{fs} \frac{T_a r}{J} \qquad \tau_m = k_{fsm} \frac{T_m r}{J} \qquad (9.3a)$$

where k_{fs} and k_{fsm} are the torsional fatigue stress-concentration factors for the alternating and mean components, respectively (see equation 6.11 on p. 339 for k_{fs} and use the applied shear stresses and shear yield strength in equation 6.17 on p. 360 to get k_{fsm}). For a solid-round cross section,* we can substitute for r and J :

$$r = \frac{d}{2} \qquad J = \frac{\pi d^4}{32} \qquad (9.3b)$$

giving

$$\tau_a = k_{fs} \frac{16T_a}{\pi d^3} \qquad \tau_m = k_{fsm} \frac{16T_m}{\pi d^3} \qquad (9.3c)$$

A tensile axial load F_z , if any is present, will typically have only a mean component (such as the weight of the components) and can be found from

$$\sigma_{m_{axial}} = k_{fm} \frac{F_z}{A} = k_{fm} \frac{4F_z}{\pi d^2} \qquad (9.4)$$

9.7 SHAFT FAILURE IN COMBINED LOADING

Extensive studies of fatigue failure of both ductile steels and brittle cast irons in combined bending and torsion were done originally in England in the 1930s by Davies^[3] and Gough and Pollard.^[5] These early results are shown in Figure 9-3, which is taken from the ANSI/ASME Standard B106.1M-1985 on the *Design of Transmission Shafting*. Data from later research is also included on these plots.^{[2],[4]} The combination of torsion and bending on ductile materials in fatigue was found to generally follow the elliptical relationship as defined by the equations in the figure. Cast brittle materials (not shown) were found to fail based on the maximum principal stress. These findings are similar to those for combined torsional and bending stresses in fully reversed loading shown in Figure 6-15 (p. 317).

* For a hollow shaft, substitute the appropriate expressions for I and J .

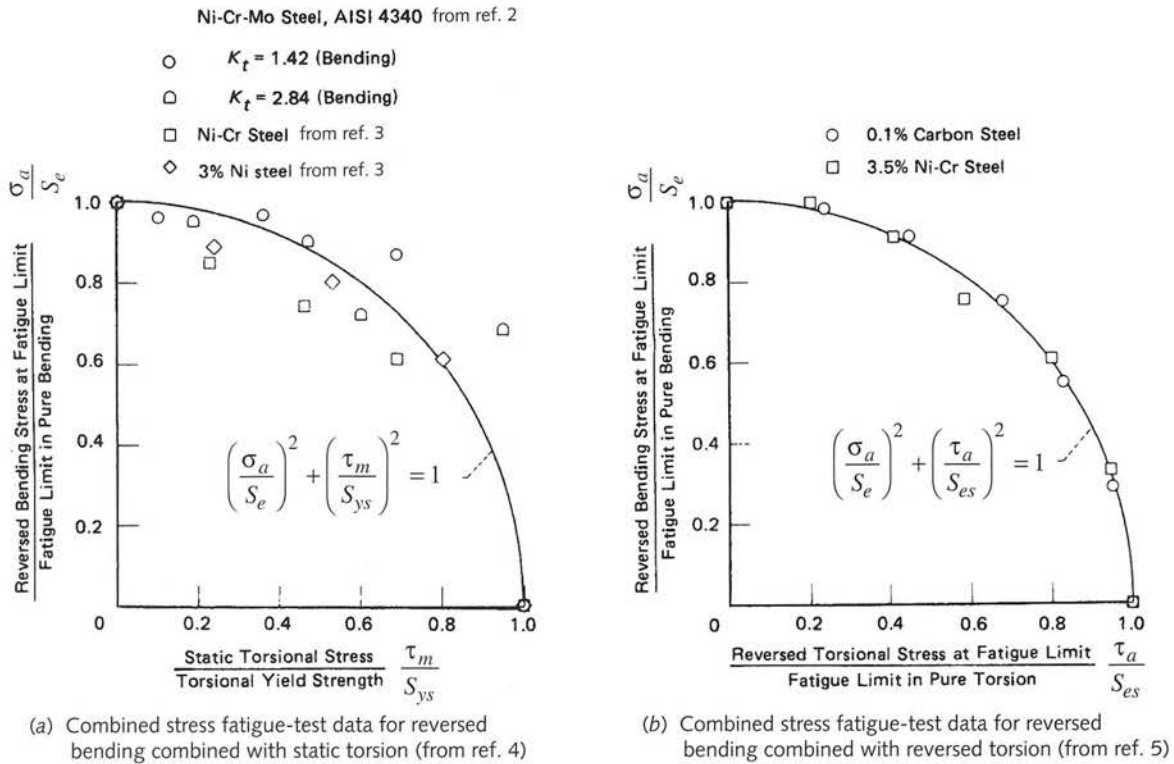


FIGURE 9-3

Results of Fatigue Tests of Steel Specimens Subjected to Combined Bending and Torsion (From *Design of Transmission Shafting*, American Society of Mechanical Engineers, New York, ANSI/ASME Standard B106.1M-1985, with permission)

9.8 SHAFT DESIGN

Both stresses and deflections need to be considered in shaft design. Often, deflection can be the critical factor, since excessive deflections will cause rapid wear of shaft bearings. Gears, belts, or chains driven from the shaft can also suffer from misalignment introduced by shaft deflections. Note that the stresses in a shaft can be calculated locally for various points along the shaft based on known loads and assumed cross sections. But, the deflection calculations require that the entire shaft geometry be defined. So, a shaft is typically first designed using stress considerations and then the deflections calculated once the geometry is completely defined. The relationship between the shaft's natural frequencies (in both torsion and bending) and the frequency content of the force- and torque-time functions can also be critical. If the forcing functions are close in frequency to the shaft's natural frequencies, resonance can create vibrations, high stresses, and large deflections.

General Considerations

Some general rules of thumb for shaft design can be stated as follows:

- 1 To minimize both deflections and stresses, the shaft length should be kept as short as possible and overhangs minimized.
- 2 A cantilever beam will have a larger deflection than a simply supported (straddle mounted) one for the same length, load, and cross section, so straddle mounting should be used unless a cantilever shaft is dictated by design constraints. (*Figure 9-2 shows a situation in which an overhung or cantilevered section of shaft is required for serviceability. The sheave on the right-hand end of the shaft carries an endless V-belt. If the sheave were mounted between the bearings, then the shaft assembly would have to be disassembled to change a belt, which is undesirable. In such cases, the cantilevered shaft can be the lesser of the evils.*)
- 3 A hollow shaft has a better stiffness/mass ratio (specific stiffness) and higher natural frequencies than a comparably stiff or strong solid shaft, but will be more expensive and larger in diameter.
- 4 Try to locate stress-raisers away from regions of large bending moment if possible and minimize their effects with generous radii and reliefs.
- 5 If minimizing deflection is the primary concern, then low-carbon steel may be the preferred material, since its stiffness is as high as that of more expensive steels and a shaft designed for low deflection will tend to have low stresses.
- 6 Deflections at gears carried on the shaft should not exceed about 0.005 in and the relative slope between the gear axes should be less than about 0.03° .^[1]
- 7 If plain (sleeve) bearings are used, the shaft deflection across the bearing length should be less than the oil-film thickness in the bearing.^[1]
- 8 If non-self-aligning rolling element bearings are used, the shaft's slope at the bearings should be kept to less than about 0.04° .^[1]
- 9 If axial thrust loads are present, they should be taken to ground through a single thrust bearing per load direction. Do not split axial loads between thrust bearings, as thermal expansion of the shaft can overload the bearings.
- 10 The first natural frequency of the shaft should be at least three times the highest forcing frequency expected in service, and preferably much more. (A factor of 10x or more is preferred, but this is often difficult to achieve in mechanical systems).

Design for Fully Reversed Bending and Steady Torsion

This loading case is a subset of the general case of fluctuating bending and fluctuating torsion, and, because of the absence of an alternating component of torsional stress, is considered to be a simple multiaxial fatigue case. (The presence of local stress concentrations can cause complex multiaxial stresses, however.) This simple loading case has been experimentally investigated, and data exist for failure of parts so loaded as shown in Figure 9-3. The ASME has defined an approach for the design of shafts loaded in this manner.

THE ASME METHOD An ANSI/ASME Standard for the *Design of Transmission Shafting* is published as B106.1M-1985. This standard presents a simplified approach to the design of shafts. The ASME approach *assumes that the loading is fully reversed bending (zero mean bending component) and steady torque (zero alternating torque component)* at a level that creates stresses below the torsional yield strength of the material. The standard makes the case that many machine shafts are in this category. It

uses the elliptical curve of Figure 9-3 fitted through the bending endurance strength on the σ_a axis and the **tensile yield strength** on the σ_m axis as the failure envelope. The tensile yield strength is substituted for the torsional yield strength by using the von Mises relationship of equation 5.9 (p. 247). The derivation of the ASME shaft equation is as follows.

Starting with the relationship for the failure envelope shown in Figure 9-3a:

$$\left(\frac{\sigma_a}{S_e}\right)^2 + \left(\frac{\tau_m}{S_{ys}}\right)^2 = 1 \quad (9.5a)$$

introduce a safety factor N_f

$$\left(N_f \frac{\sigma_a}{S_e}\right)^2 + \left(N_f \frac{\tau_m}{S_{ys}}\right)^2 = 1 \quad (9.5b)$$

Recall the von Mises relationship for S_{ys} from equation 5.9 (p. 247):

$$S_{ys} = S_y / \sqrt{3} \quad (9.5c)$$

and substitute it in equation 9.5b.

$$\left(N_f \frac{\sigma_a}{S_e}\right)^2 + \left(N_f \sqrt{3} \frac{\tau_m}{S_y}\right)^2 = 1 \quad (9.5d)$$

Substitute the expressions for σ_a and τ_m from equations 9.2c and 9.3c, respectively:

$$\left[\left(k_f \frac{32M_a}{\pi d^3}\right)\left(\frac{N_f}{S_e}\right)\right]^2 + \left[\left(k_{fsm} \frac{16T_m}{\pi d^3}\right)\left(\frac{N_f \sqrt{3}}{S_y}\right)\right]^2 = 1 \quad (9.5e)$$

which can be rearranged to solve for the shaft diameter d as

$$d = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_e}\right)^2 + \frac{3}{4} \left(k_{fsm} \frac{T_m}{S_y}\right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} \quad (9.6a)$$

The notation used in equation 9.6 is slightly different than that of the ANSI/ASME standard in order to remain consistent with the notation used in this text. The standard uses the approach of reducing the fatigue strength S_f by the fatigue-stress-concentration factor k_f rather than using k_f as a stress increaser as is done consistently in this text. In most cases (including this one) the result is the same. Also, the ASME standard assumes the stress concentration for mean stress k_{fsm} to be 1 in all cases, which gives

$$d = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_f}\right)^2 + \frac{3}{4} \left(\frac{T_m}{S_y}\right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} \quad (9.6b)$$

The ASME has abandoned this shaft design standard, so it has only historical value now. If used at all, equation 9.6 should be applied only to situations where the loads are as it assumes them to be, namely constant torque and fully reversed moment. **The ASME standard gives nonconservative results if either of the loading components that it assumes to be zero is in fact nonzero in a given case. We recommend using the more general approach of equation 9.8 (see below) for shaft design as it covers all loading situations.**

Figure 9-4 shows the Gough elliptical failure line of Figure 9-3 superposed on the Gerber, Soderberg, and modified-Goodman lines. Note that the elliptical line closely matches the Gerber line at the left-hand end but diverges to intersect the yield strength on the mean stress axis. The elliptical line has the advantage of accounting for possible yielding without needing to introduce an additional constraint involving the yield line. However, the Gough elliptical line, while a good fit to the failure data, is less conservative than the combination of Goodman line and yield line used as a failure envelope.

Design for Fluctuating Bending and Fluctuating Torsion

When the torque is not constant, its alternating component will create a complex multiaxial stress state in the shaft. Then the approach described in Section 6.12 (p. 372), which computes the von Mises components of the alternating and mean stresses using equations 6.22 (p. 375), can be used. A rotating shaft in combined bending and torsion has a biaxial stress state, which allows the two-dimensional version of equation 6.22b to be used.

$$\begin{aligned}\sigma'_a &= \sqrt{\sigma_a^2 + 3\tau_a^2} \\ \sigma'_m &= \sqrt{(\sigma_m + \sigma_{m_{axial}})^2 + 3\tau_m^2}\end{aligned}\tag{9.7a}$$

These von Mises stresses can now be entered into a modified-Goodman diagram (MGD) for a chosen material to find a safety factor, or equations 6.18 (pp. 363–364) can be applied without drawing the MGD.

For design purposes, where the diameter of the shaft is the desired quantity to be found, equations 9.2, 9.3, and 9.6 as presented require iteration to find a value for d , given some known loads and assumed material properties. This is not a great difficulty if an equation solver with iterative ability is used. However, a hand-calculator solution is cumbersome with the equations in this form. If a particular failure case is assumed for the MGD, the equations can be manipulated to provide a design equation (similar to equation 9.6) for shaft diameter d at the section of interest. If the failure model used is Case 3 from Section 6.11 (p. 356), which assumes that the mean and alternating loads maintain a constant ratio,* failure will occur at point R in Figure 6-46c (p. 362). The safety factor as defined in equation 6.18e (p. 364) is then

$$\frac{1}{N_f} = \frac{\sigma'_a}{S_f} + \frac{\sigma'_m}{S_{ut}}\tag{9.7b}$$

where N_f is the desired safety factor, S_f is the corrected fatigue strength at the selected cycle life (from equation 6.10 on p. 333), and S_{ut} is the ultimate tensile strength of the material.

* Note that this assumption is also implicit in equation 9.6 for the ASME method.

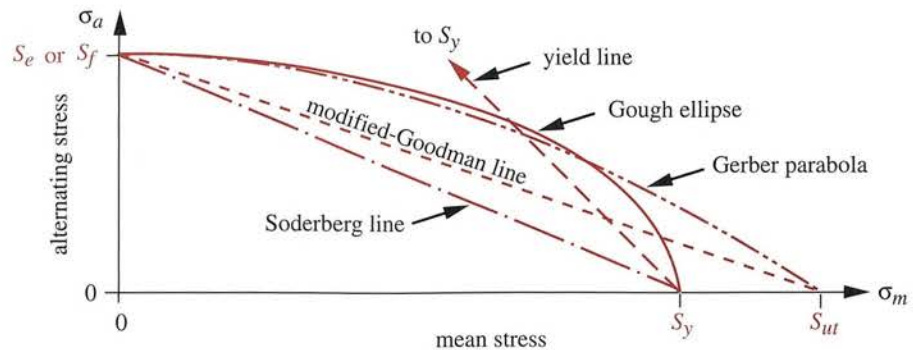


FIGURE 9-4

Elliptical Failure Line Using Yield Strength Shown with Other Failure Lines for Fluctuating Stresses

If we now also assume that the axial load on the shaft is zero and substitute equations 9.2c, 9.3c, and 9.7a into equation 9.7b we get

$$d = \left\{ \frac{32N_f}{\pi} \left[\frac{\sqrt{(k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2}}{S_f} + \frac{\sqrt{(k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2}}{S_{ut}} \right] \right\}^{\frac{1}{3}} \quad (9.8)$$

which can be used as a design equation to find a shaft diameter for any combination of bending and torsional loading with the assumptions noted above of zero axial load and a constant ratio between alternating and mean values of load over time.

EXAMPLE 9-1

Shaft Design for Steady Torsion and Fully Reversed Bending

- Problem** Design a shaft to support the attachments shown in Figure 9-5 with a minimum design safety factor of 2.5.
- Given** A preliminary design of the shaft configuration is shown in Figure 9-5. It must transmit 2 hp at 1 725 rpm. The torque and the force on the gear are both constant with time.
- Assumptions** There are no applied axial loads. Steel will be used for infinite life. Assume a stress-concentration factor of 3.5 for the step radii in bending, 2 for step radii in torsion, and 4 at the keyways.* Since the torque is steady and the bending moment fully reversed, the ASME method of equation 9.6 can be used, and it will be compared to the general method using equation 9.8.
- Solution** See Figures 9-5 through 9-8.

- 1 First determine the transmitted torque from the given power and angular-velocity using equation 9.1 (p. 507).

* See R. E. Peterson, *Stress Concentration Factors*, John Wiley, 1974, Figures 72, 79, and 183, which show these numbers as approximate maxima for these contours and loadings. Since we are creating a preliminary design at this stage and have not yet defined the shaft geometry in detail, it is not fruitful to try to define these factors any more accurately. This can be done later and the design refined accordingly.

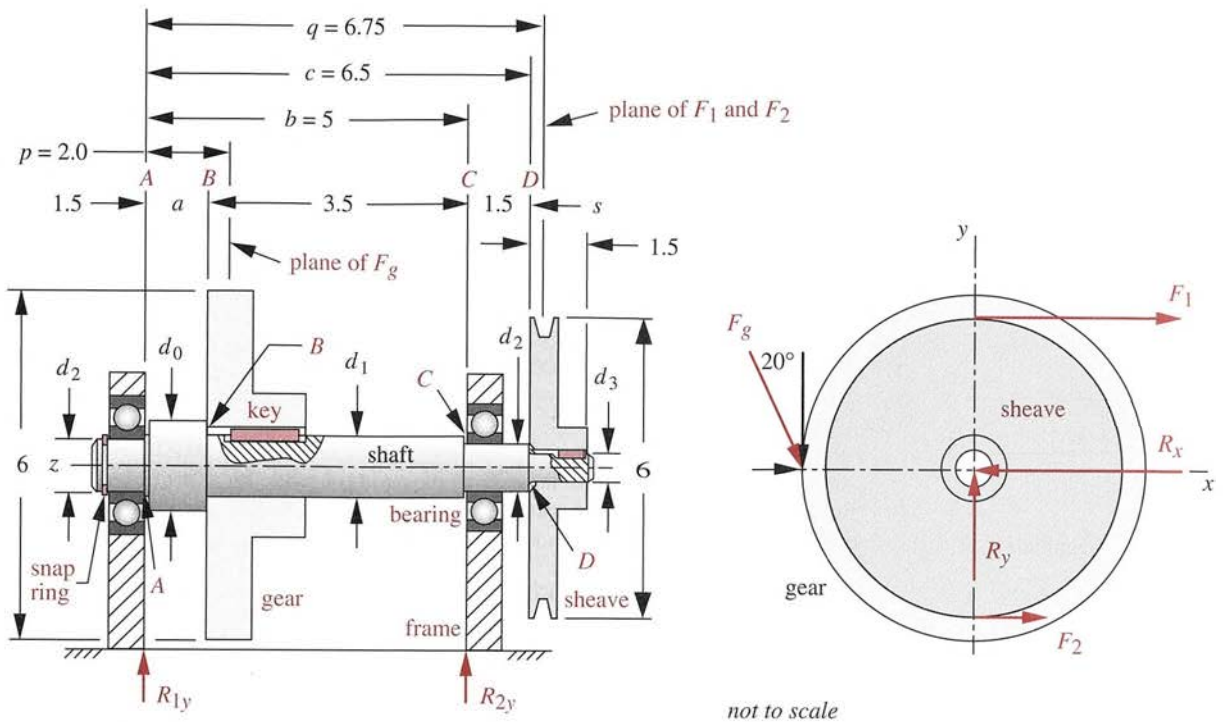


FIGURE 9-5

Geometry of a Preliminary Design for Examples 9-1 through 9-3

$$T = \frac{P}{\omega} = \frac{2 \text{ hp} \left(6600 \frac{\text{in} \cdot \text{lb/sec}}{\text{hp}} \right)}{1725 \text{ rpm} \left(\frac{2\pi}{60} \frac{\text{rad/sec}}{\text{rpm}} \right)} = 73.1 \text{ lb} \cdot \text{in} \quad (a)$$

This torque exists only over the portion of shaft between the sheave and the gear and is uniform in magnitude over that length as shown in Figure 9-6.

- The tangential forces on sheave and gear are found from the torque and their respective radii. A V-belt has tension on both sides, and the ratio between the force F_1 on the tight side and F_2 on the “slack” side is usually assumed to be about 5. The net force associated with the driving torque is $F_n = F_1 - F_2$, but the force that bends the shaft is $F_s = F_1 + F_2$. Combining these relationships gives $F_s = 1.5F_n$. Looking from the sheave end:

$$F_n = \frac{T}{r} = \frac{73.1 \text{ lb} \cdot \text{in}}{3 \text{ in}} = 24.36 \hat{i} \text{ lb}$$

$$F_s = 1.5F_n = 36.54 \hat{i} \text{ lb}$$

- The tangential force at the spur-gear tooth is

$$F_{g\text{tangential}} = \frac{T}{r} = \frac{73.1 \text{ lb} \cdot \text{in}}{3 \text{ in}} = -24.36 \hat{j} \text{ lb} \quad (c)$$

The spur gear has a 20° pressure angle as shown, which means that there will also be a radial component of force at the gear tooth of

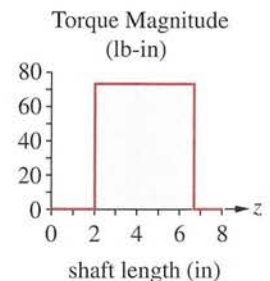


FIGURE 9-6

Torque in Example 9-1

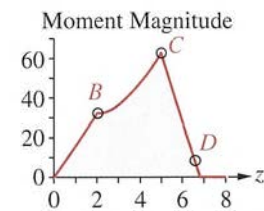
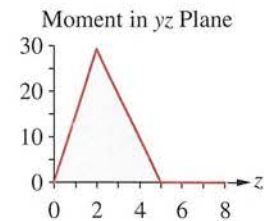
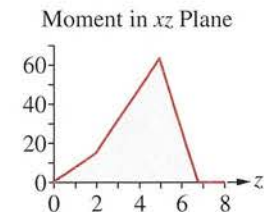
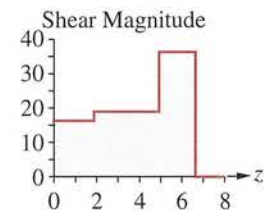
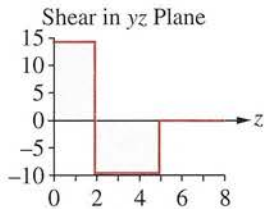
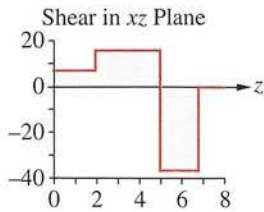


FIGURE 9-7

Loading in Example 9-1

$$F_{g_{\text{radial}}} = F_{g_{\text{tangential}}} \tan(20^\circ) = 8.87 \hat{i} \text{ lb} \quad (d)$$

- 4 We will consider the gear and sheave forces to be concentrated at their centers. Solve for the reaction forces in the xz and yz planes using $\Sigma F_x = 0$, $\Sigma M_x = 0$ and $\Sigma F_y = 0$, $\Sigma M_y = 0$ with the trial beam dimensions, $a = 1.5$, $b = 5$, and $c = 6.5$, which make $p = 2$ and $q = 6.75$.

$$\begin{aligned} \Sigma M_A &= R_2 b + F_g p + F_s q = 0 \\ R_2 &= -\frac{1}{b}(F_g p + F_s q) = -\frac{1}{5}(2F_g + 6.75F_s) = -0.40F_g - 1.35F_s \end{aligned} \quad (e)$$

$$\begin{aligned} \Sigma F &= R_1 + F_g + F_s + R_2 = 0 \\ R_1 &= -F_g - F_s - R_2 = -F_g - F_s - (-0.40F_g - 1.35F_s) = -0.60F_g + 0.35F_s \end{aligned} \quad (f)$$

Equations (e) and (f) can be solved for R_1 and R_2 in each plane, using the appropriate components of the applied loads F_g and F_s .

$$\begin{aligned} R_{1_x} &= -0.60F_{g_x} + 0.35F_{s_x} = -0.60(8.87) + 0.35(36.54) = 7.47 \text{ lb} \\ R_{1_y} &= -0.60F_{g_y} + 0.35F_{s_y} = -0.60(-24.36) + 0.35(0) = 14.61 \text{ lb} \end{aligned} \quad (g)$$

$$\begin{aligned} R_{2_x} &= -0.40F_{g_x} - 1.35F_{s_x} = -0.40(8.87) - 1.35(36.54) = -52.87 \text{ lb} \\ R_{2_y} &= -0.40F_{g_y} - 1.35F_{s_y} = -0.40(-24.36) - 1.35(0) = 9.74 \text{ lb} \end{aligned}$$

- 5 The shear load and bending moment acting on the shaft can now be found. Write an equation for the loading function q using singularity functions, integrate it to get the shear function V , and integrate again for the moment function M .

$$q = R_1 \langle z-0 \rangle^{-1} + F_g \langle z-2 \rangle^{-1} + R_2 \langle z-5 \rangle^{-1} + F_s \langle z-6.75 \rangle^{-1} \quad (h)$$

$$V = R_1 \langle z-0 \rangle^0 + F_g \langle z-2 \rangle^0 + R_2 \langle z-5 \rangle^0 + F_s \langle z-6.75 \rangle^0 \quad (i)$$

$$M = R_1 \langle z-0 \rangle^1 + F_g \langle z-2 \rangle^1 + R_2 \langle z-5 \rangle^1 + F_s \langle z-6.75 \rangle^1 \quad (j)$$

Recall that the integration constants C_1 and C_2 are zero when we include the reaction forces in the equation.

- 6 Substitute the values of the loads and reaction forces for each coordinate direction into equations (h), (i), and (j) and evaluate them for all values of z along the shaft axis. Then combine the moment-function components in the xz and yz planes (using the Pythagorean theorem) to find the maximum magnitude of the moment function.

The shear and moment distributions over the shaft length are shown in Figure 9-7. The applied torque is uniform over the portion of shaft between points B and D as shown in Figure 9-6. Within that length, there are three locations of concern where a moment occurs in combination with a stress concentration: point B at the step and keyway under the gear ($M_B = \pm 33 \text{ lb-in}$), point C at the right bearing where there is a step with a small radius to fit the bearing ($M_C = \pm 63 \text{ lb-in}$), and point D at the sheave step ($M_D = \pm 9 \text{ lb-in}$). Note that because of its high stress concentration, the snapping groove used for axial location has been placed at the end of the shaft where the moment and torque are both zero.

- 7 A trial material needs to be selected for the computations. We will first try an inexpensive, low-carbon, cold-rolled steel such as SAE 1020 with $S_{ut} = 65$ kpsi and $S_y = 38$ kpsi. Though not exceptionally strong, this material has low notch sensitivity, which will be an advantage given the large stress concentrations. Calculate the uncorrected endurance strength using equation 6.5 (p. 324):

$$S_e = 0.5S_{ut} = 0.5(65\,000) = 32\,500 \text{ psi} \quad (k)$$

This must be reduced by various factors to account for differences between the part and the test specimen.

$$S_e = C_{load} C_{size} C_{surf} C_{temp} C_{reliab} S_e' \\ S_e = (1)(1)(0.84)(1)(1)(32\,500) = 27\,300 \text{ psi} \quad (l)$$

The loading is bending and torsion, so C_{load} is 1. Since we don't yet know the part size, we will temporarily assume $C_{size} = 1$ and adjust it later. C_{surf} is chosen for a machined finish from either Figure 6-26 (p. 328) or equation 6.7e (p. 329). The temperature is not elevated, so $C_{temp} = 1$, and we assume 50% reliability at this preliminary design stage with $C_{reliab} = 1$.

- 8 The material's notch sensitivity is found from either equation 6.13 (p. 339) or Figure 6-36 (pp. 340–341) and is $q = 0.5$ in bending and $q = 0.57$ in torsion, assuming a notch radius of 0.01 in.
- 9 The fatigue stress-concentration factor is found from equation 6.11b (p. 339) using the assumed geometric stress-concentration factor noted above. For the bending stress in the step at point C:

$$K_f = 1 + q(K_t - 1) = 1 + 0.5(3.5 - 1) = 2.25 \quad (m)$$

The stress concentration for a step loaded in torsion is less than for the same geometry loaded in bending:

$$K_{fs} = 1 + q(K_{ts} - 1) = 1 + 0.57(2 - 1) = 1.57 \quad (n)$$

From equation 6.17 (p. 360) we find that in this case, the same factor should be used on the mean torsional stress component:

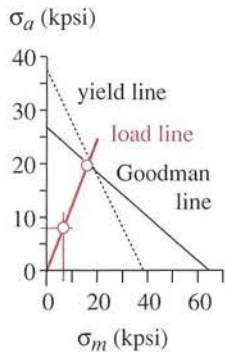
$$K_{fsm} = K_{fs} = 1.57 \quad (o)$$

- 10 The shaft diameter at point C can now be found from equation 9.6 using the moment magnitude at that point of 63.9 in-lb.

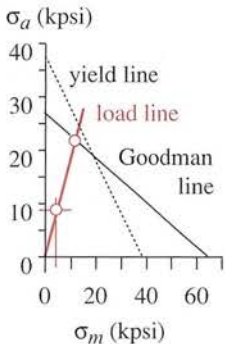
$$d_2 = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_f} \right)^2 + \frac{3}{4} \left(k_{fsm} \frac{T_m}{S_y} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} \\ = \left\{ \frac{32(2.5)}{\pi} \left[\left(2.25 \frac{63.9}{27\,300} \right)^2 + \frac{3}{4} \left(1.57 \frac{73.1}{38\,000} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} = 0.531 \text{ in} \quad (p)$$

If k_{fsm} is set to 1 as ASME recommends, then equation 9.6 gives $d = 0.520$ in. If the more general equation 9.8 is used, the result is $d = 0.557$ in. Note that the ASME

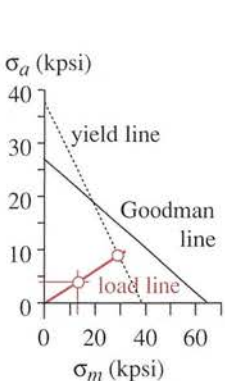
method is less conservative than equation 9.8 as it gives smaller shaft diameters for the same safety factor. A modified-Goodman diagram for this stress element is shown in Figure 9-8b. It predicts failure from fatigue.



(a) Stresses at point B



(b) Stresses at point C



(c) Stresses at point D

FIGURE 9-8

Modified-Goodman Diagrams for Three Points on Shaft in Example 9-1

- 11 At point B, under the gear, the moment is less, but the fatigue stress-concentration factors K_f and K_{fs} are greater so should also be calculated. At B:

$$K_f = 1 + q(K_t - 1) = 1 + 0.5(4 - 1) = 2.50 \quad (q)$$

$$K_{fs} = 1 + q(K_t - 1) = 1 + 0.57(4 - 1) = 2.70$$

- 12 The minimum recommended diameter at point B from equation 9.6 is

$$d_1 = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_f} \right)^2 + \frac{3}{4} \left(k_{fsm} \frac{T_m}{S_y} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}}$$

$$= \left\{ \frac{32(2.5)}{\pi} \left[\left(2.50 \frac{32.8}{27\,300} \right)^2 + \frac{3}{4} \left(2.71 \frac{73.1}{38\,000} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} = 0.517 \text{ in} \quad (r)$$

If k_{fsm} is set to 1 as ASME recommends, then equation 9.6 gives $d = 0.444$ in. If the more general equation 9.8 is used, the result is $d = 0.524$ in. Again, the ASME method is nonconservative compared to equation 9.8. A modified-Goodman diagram for this stress element is shown in Figure 9-8a. It predicts failure from fatigue.

- 13 Another location of possible failure is the step against which the sheave seats, at point D. The moment is lower than at C, being about 9.1 lb-in. (See Figure 9-7.) However, the shaft will be stepped smaller there and will have the same order of stress concentration as at point C. (The keyway for the sheave is in a region of zero moment and so will be ignored.) Using those data in equation 9.6 (p. 512) for point D:

$$d_3 = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_f} \right)^2 + \frac{3}{4} \left(k_{fsm} \frac{T_m}{S_y} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}}$$

$$= \left\{ \frac{32(2.5)}{\pi} \left[\left(2.25 \frac{9.1}{27\,300} \right)^2 + \frac{3}{4} \left(1.57 \frac{73.1}{38\,000} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} = 0.411 \text{ in} \quad (s)$$

If k_{fsm} is set to 1 as ASME recommends, then equation 9.6 gives $d = 0.360$ in. If the more general equation 9.8 is used, the result is $d = 0.387$ in. A modified-Goodman diagram for this stress element is shown in Figure 9-8c (p. 518). It predicts a yielding failure.

- 14 From these preliminary calculations, we can determine reasonable sizes for the four step diameters, d_0 , d_1 , d_2 , d_3 of Figure 9-5 (p. 515). The next largest standard ball-bearing diameter to the $d_2 = 0.531$ in calculated for point C is 15 mm or 0.591 in. Selecting this value for d_2 , we set $d_3 = 0.50$ in, and $d_1 = 0.625$ in. The stock size d_0 is then 0.75 in, left as-rolled for the outside diameter at the gear flange. These dimen-

sions will give safety factors that meet or exceed the specification. The stresses and safety factors at all three points should now be recalculated using more accurate strength-reduction (e.g., C_{size}) and stress-concentration factors based on the final dimensions.*

EXAMPLE 9-2

Shaft Design for Repeated Torsion with Repeated Bending

Problem Design a shaft to support the attachments shown in Figure 9-5 (p. 515) with a minimum design safety factor of 2.5.

Given The torque and the moment on the shaft are both varying with time in repeated fashion, i.e., their alternating and mean components are of equal magnitude. The mean and alternating components of torque are both 73 lb-in, making the peak torque twice the mean value of Example 9-1. The mean and alternating components of moment are equal in magnitude. Figure 9-9 shows the peak moment and peak torque, which are each twice the value of their fully reversed counterparts of Figure 9-5 and Example 9-1 due to the presence of the mean moment.

Assumptions There are no applied axial loads. Steel will be used for infinite life. Assume a stress-concentration factor of 3.5 for the step radii in bending, 2 for the step radii in torsion, and 4 at the keyways. Since the torsional load is not steady and the bending moment is not fully reversed, the ASME method of equation 9.6 (p. 512) should not be used.

Solution See Figures 9-5 (p. 515), 9-9, 9-10 and Table 9-1.

- For comparison purposes, we will keep all factors except the loading configuration the same as in the previous example. The same low-carbon, cold-rolled steel SAE 1020, which has $S_{ut} = 65$ kpsi, $S_y = 38$ kpsi, and a corrected $S_e = 27.3$ kpsi, is used. Its notch sensitivity is 0.5.
- There are three points of interest, labeled B, C, and D in Figure 9-5 (p. 515). The fatigue stress-concentration factors are assumed to be the same at C and D and are larger at B. See Example 9-1 (p. 514) for their calculation.
- The required shaft diameter at point C can be found from equation 9.8 (p. 514).

$$d_2 = \left\{ \frac{32N_f}{\pi} \left[\frac{\sqrt{(k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2}}{S_f} + \frac{\sqrt{(k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2}}{S_{ut}} \right] \right\}^{\frac{1}{3}}$$

$$= \left\{ \frac{32(2.5)}{\pi} \left[\frac{\sqrt{[2.25(64)]^2 + \frac{3}{4}[1.57(73.1)]^2}}{27\,300} + \frac{\sqrt{[2.25(64)]^2 + \frac{3}{4}[1.57(73.1)]^2}}{65\,000} \right] \right\}^{\frac{1}{3}}$$

$$d_2 = 0.614$$

(a)

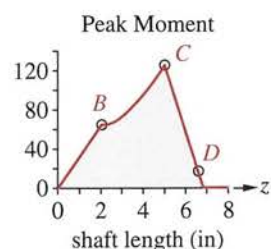
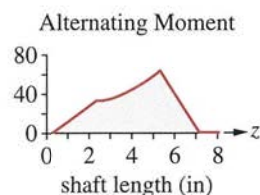
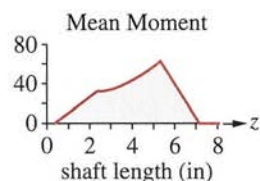
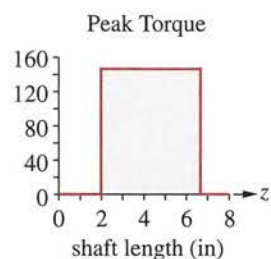
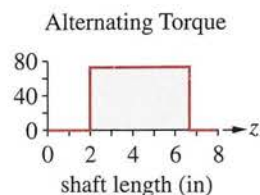
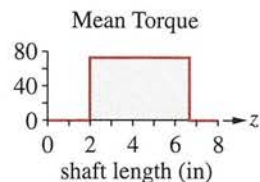
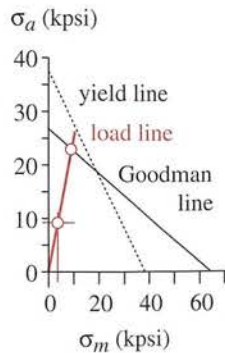


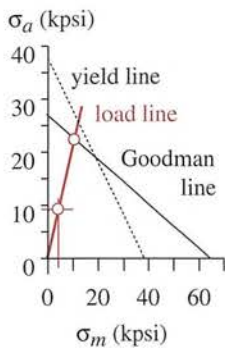
FIGURE 9-9

Torque and Moments in Example 9-2 (lb-in)

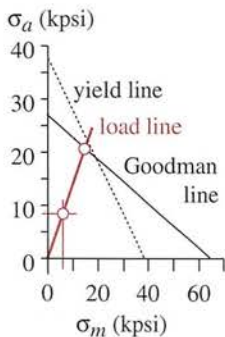
* Files EX09-01a, EX09-01b, EX09-01c, and EX09-01d are on the CD-ROM.



(a) Stresses at point B



(b) Stresses at point C



(c) Stresses at point D

Compare this to the value of 0.557 from the same equation in the prior example where the loads were steady.

- 4 At point B the required diameter from equation 9.8 is

$$d_1 = \left\{ \frac{32(2.5)}{\pi} \left[\frac{\sqrt{[2.5(32.8)]^2 + \frac{3}{4}[2.71(73.1)]^2}}{27\,300} + \frac{\sqrt{[2.5(32.8)]^2 + \frac{3}{4}[2.71(73.1)]^2}}{65\,000} \right] \right\}^{\frac{1}{3}}$$

$$d_1 = 0.632 \quad (b)$$

Compare this to the value of 0.517 from the same equation in the prior example where the loads were steady.

- 5 And at point D:

$$d_3 = \left\{ \frac{32(2.5)}{\pi} \left[\frac{\sqrt{[2.25(9.1)]^2 + \frac{3}{4}[1.57(73.1)]^2}}{27\,300} + \frac{\sqrt{[2.25(9.1)]^2 + \frac{3}{4}[1.57(73.1)]^2}}{65\,000} \right] \right\}^{\frac{1}{3}}$$

$$d_3 = 0.512 \quad (c)$$

Compare this to the value of $d_3 = 0.411$ from the same equation in the prior example where the loads were steady.

- 6 The presence of repeated stresses requires a larger shaft to maintain the same safety factor. We need the next larger standard bearing at C, which is a 17 mm (0.669 in) inside diameter. Selecting this value for d_2 , we try values of $d_3 = 0.531$ in, which is a smaller standard inch size, and $d_1 = 0.750$ in, which is a larger standard inch size. The stock size is now 0.875 in and is left as-rolled for the outside diameter at the gear flange. These dimensions will give safety factors that meet or exceed the specification as shown in the modified-Goodman diagrams of Figure 9-10. The stresses and safety factors at all three points should now be recalculated using more accurate strength-reduction and stress-concentration factors based on the final dimensions.

- 7 Table 9-1 compares the results from Examples 9-1 and 9-2 to show the differences in shaft diameter needed for steady loading or fluctuating loading. Note that the peak load in Example 9-1 is half that of Example 9-2. The final safety factors are larger than the design minimum due to the need to size-up the shaft to fit an available stock bearing size. Files EX09-02a, EX09-02b, EX09-02c, and EX09-02d are on the CD-ROM.

9.9 SHAFT DEFLECTION

A shaft is a beam that deflects transversely and is also a torsion bar that deflects torsionally. Both modes of deflection need to be analyzed. The principles of deflection analysis were reviewed in Chapter 4 and will not be detailed again here. Section 4.10 (p. 160) developed an approach for calculating beam deflections using singularity functions and Section 4.12 (p. 173) investigated torsional deflection.

FIGURE 9-10

Modified-Goodman Diagrams for Three Points on Shaft in Example 9-2

Table 9-1 Comparison of Shaft Design Results from Examples 9-1 and 9-2Minimum Diameters Give $N_f = 2.5$ at Each Point

Design	Max Alt Torque	Max Mean Torque	Max Alt Moment	Max Mean Moment	d_0 (in) nom	d_1 (in) min / nom	d_2 (in) min / nom	d_3 (in) min / nom	S_f at C
Ex. 9-1	0	73.1	63.9	0	0.750	0.517 / 0.625	0.557 / 0.591	0.411 / 0.500	2.5 / 3.0
Ex. 9-2	73.1	73.1	63.9	63.9	0.875	0.632 / 0.750	0.614 / 0.669	0.512 / 0.531	2.5 / 3.8

Shafts as Beams

The methods of Section 4.10 are directly applicable. The only complicating factor is the usual presence of steps in a shaft that change the cross-sectional properties along its length. The integration of the M/EI function becomes much more complicated due to the fact that both I and M are now functions of the dimension along the shaft-beam. Rather than do an analytical integration as was done in Section 4.10 for the case of constant I , we will use a numerical integration technique such as Simpson's rule or the trapezoidal rule to form the slope and deflection functions from the M/EI function. This will be demonstrated in an example. If the transverse loads and moment are time varying, then the absolute maximum magnitudes should be used to calculate the deflections. The deflection function will depend on the loading and the beam boundary conditions, i.e., whether simply supported, cantilevered, or overhung.

Shafts as Torsion Bars

The methods of Section 4.12 are directly applicable, particularly equation 4.24 (p. 174), since the only practical shaft cross section is circular. The angular deflection θ (in radians) for a shaft of length l , shear modulus G , polar moment of inertia J , with torque T is

$$\theta = \frac{Tl}{GJ} \quad (9.9a)$$

from which we can form the expression for the torsional spring constant:

$$k_t = \frac{T}{\theta} = \frac{GJ}{l} \quad (9.9b)$$

If the shaft is stepped, the changing cross sections complicate the torsional deflection and spring constant calculation due to the changing polar moment of inertia J .

Any collection of adjacent, different-diameter sections of shaft can be considered as a set of springs in series since their deflections add and the torque passes through unchanged. An effective spring constant or an effective J can be computed for any segment of shaft in order to find the relative deflection between its ends. For a segment of a shaft containing three sections of differing cross sections J_1 , J_2 , and J_3 with corresponding lengths l_1 , l_2 , l_3 , the total deflection is merely the sum of the deflections of each section subjected to the same torque. We assume that the material is consistent throughout,

$$\theta = \theta_1 + \theta_2 + \theta_3 = \frac{T}{G} \left(\frac{l_1}{J_1} + \frac{l_2}{J_2} + \frac{l_3}{J_3} \right) \quad (9.9c)$$

The effective spring constant k_{eff} of a three-segment stepped shaft is

$$\frac{1}{k_{t\,eff}} = \frac{1}{k_{t_1}} + \frac{1}{k_{t_2}} + \frac{1}{k_{t_3}} \quad (9.9d)$$

These expressions can be extended to any number of segments of a stepped shaft.

EXAMPLE 9-3

Designing a Stepped Shaft to Minimize Deflection

Problem Design the same shaft as in Example 9-2 to have a maximum bending deflection of 0.002 in and a maximum angular deflection of 0.5° between sheave and gear.

Given The loading is the same as in Example 9-2. The peak torque is 146 lb-in. Figure 9-9 shows the distribution of the peak moment over the shaft length. The values are 65.6 lb-in at point *B*, 127.9 lb-in at point *C*, and 18.3 lb-in at point *D*.

Assumptions The lengths will remain the same as in previous example, but diameters can be changed to stiffen the shaft if necessary. The material is the same as in Example 9-2.

Solution See Figures 9-5 (p. 515), and 9-11 to 9-13.

- 1 The torsional deflection is found from equations 9.9. The lengths of each segment are (from Figure 9-5 on p. 515): $AB = 1.5$ in, $BC = 3.5$ in, and $CD = 1.5$ in. The polar area moments of inertia are first calculated for each segment of different diameter.

$$\begin{aligned} \text{from A to B:} \quad J &= \frac{\pi d_{AB}^4}{32} = \frac{\pi(0.875)^4}{32} = 0.0575 \text{ in}^4 \\ \text{from B to C:} \quad J &= \frac{\pi d_{BC}^4}{32} = \frac{\pi(0.750)^4}{32} = 0.0311 \text{ in}^4 \\ \text{from C to D:} \quad J &= \frac{\pi d_{CD}^4}{32} = \frac{\pi(0.669)^4}{32} = 0.0197 \text{ in}^4 \end{aligned} \quad (a)$$

and used in equation 9.9c (p. 522).

$$\begin{aligned} \theta &= \frac{T}{G} \left(\frac{l_1}{J_1} + \frac{l_2}{J_2} + \frac{l_3}{J_3} \right) \\ &= \frac{146}{1.2E7} \left(\frac{1.5}{0.0575} + \frac{3.5}{0.0311} + \frac{1.5}{0.0197} \right) = 0.15 \text{ deg} \end{aligned} \quad (b)$$

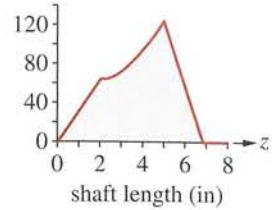
This deflection is within the requested specification.

- 2 The moment function for this shaft was derived using singularity functions as equation (j) in Example 9-1 (p. 514). It must now be divided by the product of E and

the area moment of inertia I at each point along the shaft axis. While E is constant, the value of I changes with each diametral change in the stepped shaft.

$$\frac{M}{EI} = \frac{1}{EI} [R_1 \langle z-0 \rangle^1 + F_g \langle z-1.5 \rangle^1 + R_2 \langle z-5 \rangle^1 + F_s \langle z-6.5 \rangle^1] \quad (c)$$

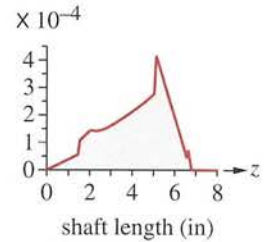
Figure 9-11a shows the moment function for this shaft as derived in the previous examples and Figure 9-11b shows the M/EI function for the section diameters defined in Example 9-2.



(a) Moment magnitude

$$\theta = \int \frac{M}{EI} dz + C_3 \quad (d)$$

$$\delta = \iint \frac{M}{EI} dz + C_3 z + C_4 \quad (e)$$



(b) Moment / EI

- 3 The bending deflection is found by integrating the moment function twice.

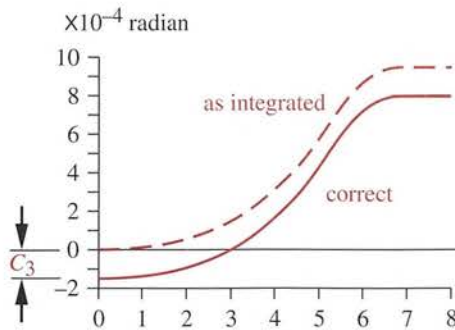
- 4 The first integration of the M/EI function from equation (c) gives the beam slope and the second integration gives the deflection function. In previous discussions of beam deflection (see Section 4.10 on p. 160 and Examples 4-4 to 4-7 on pp. 162 to 169) the cross section I of the beam was constant across its length. In a stepped shaft, I is a function of the shaft length. This makes the analytical integration of the M/EI function much more complicated. A simpler approach is to numerically integrate the function twice using a trapezoidal or Simpson's rule. This numerical integration must be done for each coordinate direction to obtain the x and y components of deflection. These are then combined vectorially to get the deflection-magnitude and phase-angle functions over the shaft length.

- 5 Since the shaft deflection is zero at $z = 0$, $C_4 = 0$. The other constant of integration C_3 can be determined numerically. Figure 9-12a shows the beam slope in the y direction as integrated by a trapezoidal rule, and also shows the corrected slope function. The integrated result is shifted up by the integration constant C_3 . However, we do not know where the proper zero crossover is for this function, so we cannot determine C_3 from the beam-slope function.
- 6 The as-integrated deflection function in Figure 9-12b does not equal zero at the second support. Since the deflection is really zero there, the error in this integrated

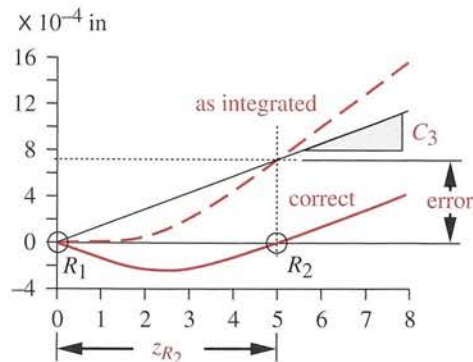
FIGURE 9-11

Moment and Moment / EI Functions in Example 9-3

9



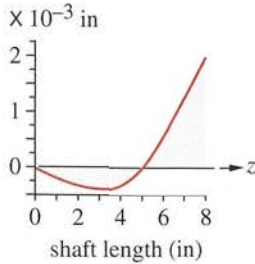
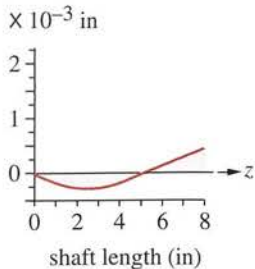
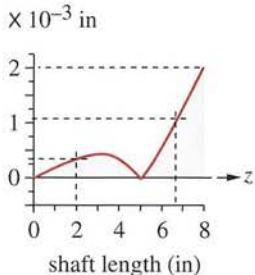
(a) Beam slope



(b) Beam deflection

FIGURE 9-12

Numerical Integration of Moment Function and Finding the Integration Constant C_3

x Deflection**y Deflection****Deflection Magnitude****FIGURE 9-13**

Deflection Functions for
Example 9-3

function can be used to determine the constant of integration C_3 . A line is drawn in Figure 9-12b from the origin to the point on the curve at $z = 5$ where the function should be zero. The slope of this straight line is the constant C_3 for the y direction, which is found from

$$C_{3y} = \frac{\text{error}_y}{z_{R2}} = \frac{0.0007}{5.0} = 0.00014 \text{ in} \quad (f)$$

The constant for the x direction is found in similar manner. The functions are then recalculated using the correct values of C_3 .

- 7 These deflection functions are plotted in Figure 9-13 for the shaft diameters of $d_0 = 0.875$, $d_1 = 0.750$, $d_2 = 0.669$, $d_3 = 0.531$ from Example 9-2. The magnitude of the deflection at the gear is 0.000 3 in, which is well within the requested specification. At the sheave the deflection is 0.001 in, also within the specification. The deflection at the right-hand end of the shaft is 0.002 in. Files EX09-03a and EX09-02b are on the CD-ROM.

9.10 KEYS AND KEYWAYS

The ASME defines a **key** as “a demountable machinery part which, when assembled into keyseats, provides a positive means for transmitting torque between the shaft and hub.” Keys are standardized as to size and shape in several styles.* A **parallel key** is square or rectangular in cross section and of constant height and width over its length. (See Figure 9-14a.) A **tapered key** is of constant width but its height varies with a linear taper of 1/8 in per foot, and it is driven into a tapered slot in the hub until it locks. It may either have no head or have a **gib head** to facilitate removal. (See Figure 9-14b.) A **Woodruff key** is semicircular in plan and of constant width. It fits in a semicircular keyseat milled in the shaft with a standard circular cutter. (See Figure 9-14c.) The tapered key serves to lock the hub axially on the shaft, but the parallel or Woodruff keys require some other means for axial fixation. Retaining rings or clamp collars are sometimes used for this purpose.

Parallel Keys

Parallel keys are the most commonly used. The ANSI and ISO standards each define particular key cross-sectional sizes and keyseat depths as a function of shaft diameter at the keyseat. A partial reproduction of this information is provided in Table 9-2 for the lower range of shaft diameters. Consult the respective standards for larger shaft sizes. Square keys are recommended for shafts up to 6.5-in dia (US), or 25 mm-dia (ISO), and rectangular keys for larger diameters. The parallel key is placed with half of its height in the shaft and half in the hub, as shown in Figure 9-14a.

Parallel keys are typically made from standard cold-rolled bar stock, which is conventionally “negatively toleranced,” meaning it will never be larger than its nominal dimension, only smaller. For example, a nominal 1/4-in square bar will have a tolerance on width and height of +0.000, −0.002 in. Thus, the keyseat can be cut with a standard 1/4-in milling cutter and the bar stock key will fit with slight clearance. A special keystock is also available, which is positively toleranced (e.g., 0.250 +0.002, −0.000).

* ANSI standard B17.1-1967, *Keys and Keyseats*, and B17.2-1967, *Woodruff Keys and Keyseats*, available from the American Society of Mechanical Engineers, 345 East 47th St., New York, N.Y. 10017.

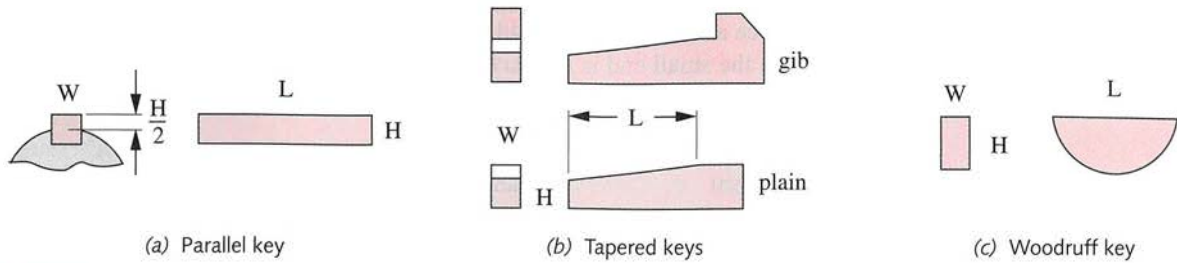


FIGURE 9-14

Various Styles of Keys

It is used when a closer fit between key and keyseat is desired and may require machining of the keystock to final dimension.

The key fit can be of concern when the torque loading is alternating from positive to negative each cycle. When the torque changes sign, any clearance between key and keyway will be suddenly taken up with resulting impact and high stresses. This is termed **backlash**. A setscrew in the hub, placed at 90° from the key, can both hold the hub axially and stabilize the key against backlash. The ANSI Standard also defines the size of setscrew to be used with each size key as shown in Table 9-2. Key length should be less than about 1.5 times the shaft diameter to avoid excessive twisting with shaft deflection. If more strength is needed, two keys can be used, oriented at 90° and 180° for example.

Tapered Keys

The width of a tapered key for a given shaft diameter is the same as for a parallel key as shown in Table 9-2. The taper and gib-head size are defined in the standard. The

Table 9-2 Standard Key and Setscrew Sizes for US and Metric Sized Shafts

Shaft Diameter (in)	Nominal Key Width (in)	Setscrew Dia. (in)	Shaft Diameter (mm)	Key Width x Height (mm)
$0.312 < d \leq 0.437$	0.093	#10	$8 < d \leq 10$	3 x 3
$0.437 < d \leq 0.562$	0.125	#10	$10 < d \leq 12$	4 x 4
$0.562 < d \leq 0.875$	0.187	0.250	$12 < d \leq 17$	5 x 5
$0.875 < d \leq 1.250$	0.250	0.312	$17 < d \leq 22$	6 x 6
$1.250 < d \leq 1.375$	0.312	0.375	$22 < d \leq 30$	8 x 7
$1.375 < d \leq 1.750$	0.375	0.375	$30 < d \leq 38$	10 x 8
$1.750 < d \leq 2.250$	0.500	0.500	$38 < d \leq 44$	12 x 8
$2.250 < d \leq 2.750$	0.625	0.500	$44 < d \leq 50$	14 x 9
$2.750 < d \leq 3.250$	0.750	0.625	$50 < d \leq 58$	16 x 10
$3.250 < d \leq 3.750$	0.875	0.750	$58 < d \leq 65$	18 x 11
$3.750 < d \leq 4.500$	1.000	0.750	$65 < d \leq 75$	20 x 12
$4.500 < d \leq 5.500$	1.250	0.875	$75 < d \leq 85$	22 x 14
$5.500 < d \leq 6.500$	1.500	1.000	$85 < d \leq 95$	25 x 14

taper is a locking one, which means that the friction force between the surfaces holds the key in place axially. The gib head is optional and provides a surface for prying the key out when the small end is not accessible. Tapered keys tend to create eccentricity between hub and shaft, as they drive all the radial clearance to one side.

Woodruff Keys

Woodruff keys are used on smaller shafts. They are self-aligning, so are preferred for tapered shafts. The penetration of a Woodruff key into the hub is the same as that of a square key, i.e., half the key width. The semicircular shape creates a deeper keyseat in the shaft, which resists key-rolling, but weakens the shaft compared to a square or tapered keyseat. Woodruff key widths as a function of shaft diameter are essentially the same as those for square keys shown in Table 9-2. The other dimensions of the Woodruff key are defined in the ANSI Standard, and keyseat cutters are readily available to match these dimensions. Table 9-3 reproduces a sample of the key-size specifications from the standard. Each key size is given a key number, which encodes its dimensions. The ANSI Standard states: "The last two digits give the nominal key diameter in eighths of an inch and the digits preceding the last two give the nominal width in thirty-seconds of an inch." For example, the key number 808 defines a key size of $8/32 \times 8/8$ or $1/4$ wide $\times$ 1-in dia. See reference 6 for complete dimensional information on keys.

Stresses in Keys

There are two modes of failure in keys: shear and bearing. A shear failure occurs when the key is sheared across its width at the interface between the shaft and hub. Bearing failure occurs by crushing either side in compression.

Table 9-3 ANSI Standard Sizes of Woodruff Keys

A Partial List: See the Standard for Complete Information and Figure 9-14c for Labels

Key Number	Nominal Key Size W x L	Height H
202	0.062 x 0.250	0.106
303	0.093 x 0.375	0.170
404	0.125 x 0.500	0.200
605	0.187 x 0.625	0.250
806	0.250 x 0.750	0.312
707	0.218 x 0.875	0.375
608	0.187 x 1.000	0.437
808	0.250 x 1.000	0.437
1208	0.375 x 1.000	0.437
610	0.187 x 1.250	0.545
810	0.250 x 1.250	0.545
1210	0.187 x 1.250	0.545
812	0.250 x 1.500	0.592
1212	0.375 x 1.500	0.592

SHEAR FAILURE The average stress due to direct shear was defined in equation 4.9, repeated here:

$$\tau_{xy} = \frac{F}{A_{shear}} \quad (9.10)$$

where F is the applied force and A_{shear} is the shear area being cut. In this case A_{shear} is the product of the key's width and length. The force on the key can be found from the quotient of the shaft torque and the shaft radius. If the shaft torque is constant with time, the force will be also and the safety factor can be found by comparing the shear stress to the shear yield strength of the material. If the shaft torque is time varying, then a fatigue failure of the key in shear is possible. The approach then is to compute the mean and alternating shear-stress components and use them to compute the mean and alternating von Mises stresses. These can then be used in a modified-Goodman diagram to find the safety factor as described in Section 6.13 (p. 377).

BEARING FAILURE The average bearing stress is defined as

$$\sigma_x = \frac{F}{A_{bearing}} \quad (9.11)$$

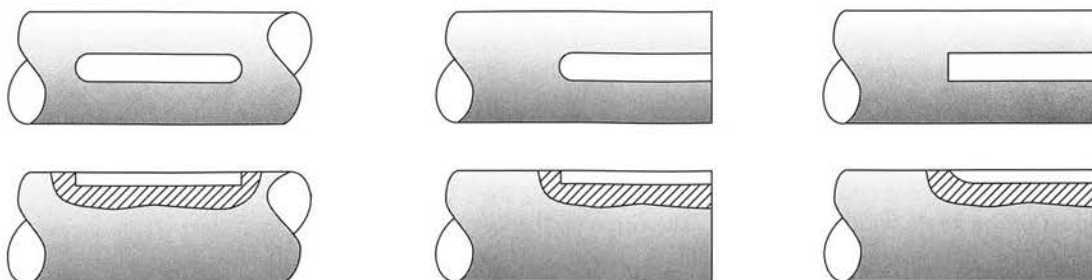
where F is the applied force and the bearing area is the area of contact between the key side and the shaft or the hub. For a square key this will be its half-height times its length. A Woodruff key has a different bearing area in the hub than in the shaft. The hub's Woodruff bearing area is much smaller and will fail first. The bearing stress should be calculated using the maximum applied force, whether constant or time varying. Since compressive stresses do not cause fatigue failures, bearing stresses can be considered static. The safety factor is found by comparing the maximum bearing stress to the material yield strength in compression.

Key Materials

Because keys are loaded in shear, ductile materials are used. Soft, low-carbon steel is the most common choice unless a corrosive environment requires a brass or stainless steel key. Square or rectangular keys are often made from cold-rolled bar stock and merely cut to length. The special keystock mentioned above is used when a closer fit is required between key and keyway. Tapered and Woodruff keys are also usually made from soft, cold-rolled steel.

Key Design

Only a few design variables are available when sizing a key. The shaft diameter at the keyseat determines the key width. The key height (or its penetration into the hub) is also determined by the key width. This leaves only the length of the key and the number of keys used per hub as design variables. A straight or tapered key can be as long as the hub allows. A Woodruff key can be had in a range of diameters for a given width, which effectively determines its length of engagement in the hub. Of course, as the Woodruff key diameter is increased, it further weakens the shaft with its deeper keyseat. If a single key cannot handle the torque at reasonable stresses, an additional key can be added, rotated 90° from the first.



(a) End-milled keyway—double-ended

(b) End-milled keyway—single-ended

(c) Sled-runner keyway—single-ended

FIGURE 9-15

Various Styles of Keyways in Shafts

It is common to size the key so that it will fail before the keyseat or other location in the shaft fails in the event of an overload. The key then acts like a shear pin in an outboard motor to protect the more expensive elements from damage. A key is inexpensive and relatively easy to replace if the keyseat is undamaged. This is one reason to use only soft, ductile materials for the key, having lower strength than that of the shaft so that a bearing failure will selectively affect the key rather than the keyway if the system sees an overload beyond its design range.

Stress Concentrations in Keyways

Since keys have relatively sharp corners (< 0.02 -in radius), keyseats must also. This causes significant stress concentrations. The keyway is broached in the hub and runs through its length, but the keyway must be milled into the shaft and has one or two ends. If an end-mill is used, the keyway will look like Figure 9-15a and will have sharp corners in the side view at one or both ends as well as along each side. If, instead, a sled-runner keyway is cut as shown in Figure 9-15c, the sharp corner at the end is eliminated and the stress concentration reduced. A Woodruff keyseat in the shaft also has a large radius in the side view but it (and every keyseat) suffers from sharp corners on the sides.

Peterson^[7] shows experimentally derived stress-concentration curves for end-milled keyseats in shafts under either bending or torsional loading. These are reproduced in Figure 9-16. These factors range from about 2 to about 4 depending on the ratio of the corner radius to the shaft diameter. Curve-fits to Figure 9-16 have been done and functions created for these curves so that the stress-concentration factor can be determined “on the fly” during a shaft-design computation. See the file SHFTDES for example. These factors should be applied to the bending and shear stresses in the shaft at the keyway location as was done in Examples 9-1 and 9-2.

EXAMPLE 9-4

Designing Shaft Keys

Problem

Design the keys for the shaft in Examples 9-2 (p. 519) and 9-3 (p. 522) and refine the estimate of the shaft’s safety factors based on the

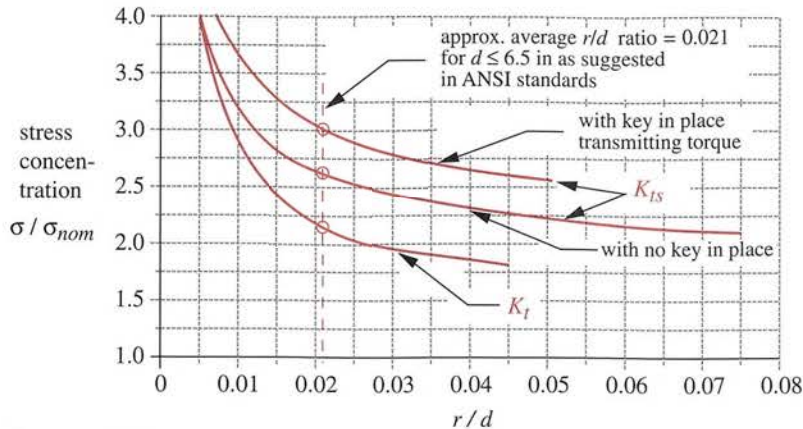


FIGURE 9-16

Stress-Concentration Factors for an End-Milled Keyseat in Bending (K_t) and Torsion (K_{ts}) Source: R. E. Peterson, *Stress Concentration Factors*, 1974, Figures 182 and 183, pp. 266–267, reprinted by permission of John Wiley & Sons, Inc.

preliminary design dimensions from that earlier example in conjunction with refined stress-concentration factors.

Given

The loading is the same as in Example 9-2. The peak torque is 146 lb-in. Figure 9-9 (p. 519) shows the distribution of the peak moment over the shaft length. The values are 65.6 lb-in at point *B* and 18.3 lb-in at point *D*. The preliminary shaft diameters at the keys are $d_1 = 0.750$ in at *B* and $d_3 = 0.531$ in at *D*. See Figure 9-5 (p. 515) for labels.

Assumptions

Use square, parallel keys with end-milled keyways. The shaft material is the same as in Example 9-3. A low-carbon steel, SAE 1010, will be used for the keys. Its $S_{ut} = 53$ kpsi and $S_y = 44$ kpsi. S_e is calculated to be 22 990 psi. See Figure 9-16 for stress-concentration factors.

Solution

See Figure 9-5 (p. 515).

- 1 There are two locations with keys on this shaft, at points *B* and *D*. The design diameters chosen for these sections in Example 9-3 were $d_1 = 0.750$ at *B* and $d_3 = 0.531$ in at *D*. Table 9-2 (p. 525) shows that the standard key width for d_1 is 0.187 in and for d_3 is 0.125 in. The key length can be adjusted for each location.
- 2 At point *B*, the mean and alternating components of force on the key are found from the torque component divided by the shaft radius at that point.

$$F_a = \frac{T_a}{r} = \frac{73.1}{0.375} = 194.67 \text{ lb}$$

$$F_m = \frac{T_m}{r} = \frac{73.1}{0.375} = 194.67 \text{ lb} \quad (a)$$

- 3 Assume a key length of 0.5 in and calculate the alternating and mean shear stress components from

$$\tau_a = \frac{F_a}{A_{\text{shear}}} = \frac{194.67}{0.187(0.500)} = 2082 \text{ psi}$$

$$\tau_m = \frac{F_m}{A_{\text{shear}}} = \frac{194.67}{0.187(0.500)} = 2082 \text{ psi} \quad (b)$$

- 4 To find a safety factor for shear fatigue of the key, compute the von Mises equivalent stresses for each of these components from equation 5.7d (p. 245),

$$\begin{aligned}\sigma'_a &= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} = \sqrt{3(2\,082)^2} = 3\,606 \text{ psi} \\ \sigma'_m &= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} = \sqrt{3(2\,082)^2} = 3\,606 \text{ psi}\end{aligned}\quad (c)$$

and use them in equation 6.18e (p. 364) to determine the fatigue safety factor:

$$N_f = \frac{1}{\frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}} = \frac{1}{\frac{3\,606}{22\,990} + \frac{3\,606}{53\,000}} = 4.4 \quad (d)$$

- 5 The bearing stress on the key is compression and thus can be considered a static load. It is calculated using the maximum force on the key:

$$\sigma_{max} = \frac{F_m + F_a}{A_{bearing}} = \frac{194.67 + 194.67}{0.093\,5(0.500)} = 8\,328 \text{ psi} \quad (e)$$

- 6 Calculate the safety factor for bearing failure from:

$$N_s = \frac{S_y}{\sigma_{max}} = \frac{44\,000}{8\,328} = 5.3 \quad (f)$$

- 7 At point *D*, the force on the key is

$$\begin{aligned}F_a &= \frac{T_a}{r} = \frac{73.1}{0.266} = 275 \text{ lb} \\ F_m &= \frac{T_m}{r} = \frac{73.1}{0.266} = 275 \text{ lb}\end{aligned}\quad (g)$$

- 8 Assume a key length of 0.50 in and calculate the alternating and mean shear stress components from

$$\begin{aligned}\tau_a &= \frac{F_a}{A_{shear}} = \frac{275}{0.125(0.50)} = 4\,400 \text{ psi} \\ \tau_m &= \frac{F_m}{A_{shear}} = \frac{275}{0.125(0.50)} = 4\,400 \text{ psi}\end{aligned}\quad (h)$$

- 9 Compute the von Mises equivalent stresses for each of these components from equation 5.7d,

$$\begin{aligned}\sigma'_a &= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} = \sqrt{3(4\,399)^2} = 7\,620 \text{ psi} \\ \sigma'_m &= \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} = \sqrt{3(4\,399)^2} = 7\,620 \text{ psi}\end{aligned}\quad (i)$$

and use them in equation 6.18e:

$$N_f = \frac{1}{\frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}} = \frac{1}{\frac{7\,620}{22\,990} + \frac{7\,620}{53\,000}} = 2.1 \quad (j)$$

- 10 The bearing stress on the key is calculated using the maximum force on the key:

$$\sigma_{max} = \frac{F_m + F_a}{A_{bearing}} = \frac{275 + 275}{0.0625(0.50)} = 17\,600 \text{ psi} \quad (k)$$

- 11 Calculate the safety factor for bearing failure from

$$N_s = \frac{S_y}{\sigma_{max}} = \frac{44\,000}{17\,600} = 2.5 \quad (l)$$

- 12 The safety factors for the shaft at these locations can now be recalculated using a stress-concentration factor at the keyways that takes into account the actual shaft diameter and actual notch radii. Our previous design calculation in Example 9-2 used a worst-case assumption for these values. Figure 9-16 shows the stress-concentration functions for end-milled keyways in both bending and torsion. To use these charts we must calculate the r/d ratio of the end-mill radius versus the shaft diameter. Assume a radius on the end mill of 0.010 in. The r/d ratios for the two points are then

$$\begin{aligned} \text{for point } B: \quad \frac{r}{d} &= \frac{0.010}{0.750} = 0.0133 \\ \text{for point } D: \quad \frac{r}{d} &= \frac{0.010}{0.531} = 0.0188 \end{aligned} \quad (m)$$

The corresponding stress-concentration factors are read from Figure 9-16 as

$$\begin{aligned} \text{for point } B: \quad K_t &= 2.5 & K_{ts} &= 2.9 \\ \text{for point } D: \quad K_t &= 2.2 & K_{ts} &= 2.7 \end{aligned} \quad (n)$$

- 13 These are used in equations (m), (n), and (o) of Example 9-1 to obtain the fatigue stress-concentration factors, which for a material notch sensitivity $q = 0.5$ are

$$\begin{aligned} \text{for point } B: \quad K_f &= 1.75 & K_{fs} &= 2.09 \\ \text{for point } D: \quad K_f &= 1.60 & K_{fs} &= 1.97 \\ \text{for both points:} \quad K_{fm} &= K_f & K_{fsm} &= K_{fs} \end{aligned} \quad (o)$$

- 14 The new safety factors are then calculated using equation 9.8 (p. 514) with the data from equations (b) and (c) from Example 9-2 with the design values for shaft diameter and the above stress-concentration values inserted. For point B:

$$0.75 = \left\{ \frac{32(N_f)}{\pi} \left[\frac{\sqrt{[1.75(32.8)]^2 + \frac{3}{4}[2.09(73.1)]^2}}{27\,300} + \frac{\sqrt{[1.75(32.8)]^2 + \frac{3}{4}[2.09(73.1)]^2}}{65\,000} \right] \right\}^{\frac{1}{3}}$$

$$N_f = 5.5 \quad (p)$$

for point D:

$$0.531 = \left\{ \frac{32(N_f)}{\pi} \left[\frac{\sqrt{[1.60(9.1)]^2 + \frac{3}{4}[1.97(73.1)]^2}}{27\,100} + \frac{\sqrt{[1.60(9.1)]^2 + \frac{3}{4}[1.97(73.1)]^2}}{65\,000} \right] \right\}^{\frac{1}{3}}$$

$$N_f = 2.2 \quad (q)$$

At point *B* the safety factor is greater than the specified value of 2.5. At point *D* it is lower. Increasing the diameter at *D* to 0.562 in gives a safety factor of 2.7. Then the safety factors for key failure (4.4 at *B* and 2.1 at *D*) are lower than those for shaft failure, which is desirable, since the keys will then fail before the shafts in an overload situation. This is now a viable and acceptable design.*

9.11 SPLINES

When more torque must be transmitted than can be handled by keys, splines can be used instead. Splines are essentially “built-in keys” formed by contouring the outside of the shaft and inside of the hub with toothlike forms. Early splines had teeth of square cross section but these have been supplanted by involute spline teeth, as shown in Figure 9-17. The involute tooth form is universally used on gears, and the same cutting techniques are used to manufacture splines. In addition to its manufacturing advantage, the involute tooth has less stress concentration than a square tooth and is stronger. The SAE defines standards for both square and involute spline tooth forms and ANSI publishes involute spline standards.† The standard involute spline has a pressure angle of 30° and half the depth of a standard gear tooth. The tooth size is defined by a fraction whose numerator is the diametral pitch (which defines tooth width—see Chapter 11 for more information on these terms) and whose denominator controls tooth depth (and is always double the numerator). Standard diametral pitches are 2.5, 3, 4, 5, 6, 8, 10, 12, 16, 20, 24, 32, 40, and 48. Standard splines can have from 6 to 50 teeth. Splines may have either a flat root or a filleted root, both shapes being shown in Figure 9-17. See reference 8 for complete dimensional information on standard splines.

Some advantages of splines are maximum strength at the root of the tooth, accuracy of tooth form due to the use of standard cutters, and good machined surface finish from the standard gear-cutting (hobbing) process, which eliminates the need for grinding. A major advantage of splines over keys is their ability (with proper clearances) to accommodate large axial movements between shaft and hub while simultaneously transmitting torque. They are used to connect the transmission output shaft to the driveshaft

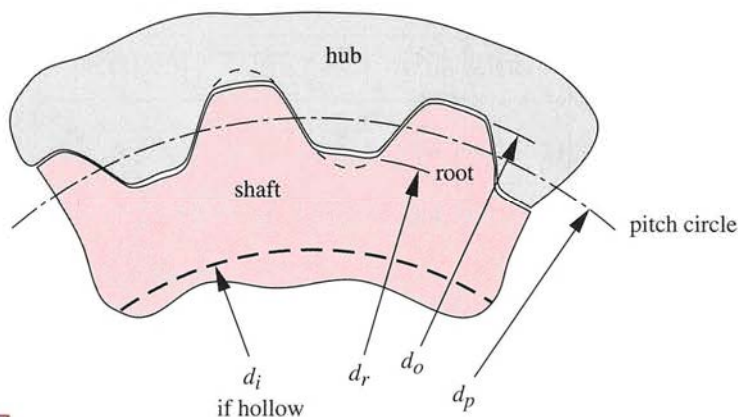


FIGURE 9-17
Involute Spline Geometry

* Files EX09-04A, EX09-04B, EX09-04C, and EX09-04D are on the CD-ROM.

† ANSI Standard B92.1 and B92.2M, American National Standards Institute, 11 West 42nd St., New York, N.Y. 10036

in automobiles and trucks where the suspension movement causes axial motion between the members. They are also used within nonautomatic, nonsynchromesh truck transmissions to couple the axially shiftable gears to their shafts. In addition, engine torque is usually passed into the transmission through a spline that connects the engine clutch to the transmission input shaft and allows the axial motion necessary to disengage the clutch from the flywheel.

The loading on a spline is typically pure torsion, either steady or fluctuating. While it is possible to have bending loads superposed, good design practice will minimize the bending moments by proper placement of bearings and by keeping a cantilevered spline as short as possible. As with keys, two failure modes are possible, bearing or shear. Shear failure is usually the limiting mode. Unlike keys, many teeth are available to share the load to some degree. Ideally, the spline length l needs to be only as long as is required to develop a combined tooth shear strength equal to the torsional shear strength of the shaft itself. If the spline were made perfectly with no variation in either tooth thickness or spacing, all teeth would share the load equally. However, the realities of manufacturing tolerances prohibit this ideal condition. The SAE states that “actual practice has shown that due to inaccuracies in spacing and tooth form, the equivalent of about 25% of teeth are in contact, so that a good approximate formula for a splined shaft (length) is”

$$l \cong \frac{d_r^3(1 - d_i^4/d_r^4)}{d_p^2} \quad (9.12)$$

where d_r is the root diameter of the external spline, d_i is the internal diameter (if any) of a hollow shaft, and d_p is the pitch diameter of the spline, which is approximately at mid-tooth. The variable l represents the actual engaged length of the spline teeth with one another and should be considered as the minimum value needed to develop the strength in the teeth of an equivalent-diameter shaft.

The shear stress is calculated at the pitch diameter of the splines, where the shear area is

$$A_{shear} = \frac{\pi d_p l}{2} \quad (9.13a)$$

The shear stress can be calculated using the SAE assumption that only 25% of the teeth are actually sharing the load at any one time by considering only 1/4 of the shear area to be stressed:

$$\tau \cong \frac{4F}{A_{shear}} = \frac{4T}{r_p A_{shear}} = \frac{8T}{d_p A_{shear}} = \frac{16T}{\pi d_p^2 l} \quad (9.13b)$$

where T is the torque on the shaft. Any bending stresses on the spline must also be calculated and properly combined with this shear stress. If the loading is pure torsion and static, then the shear stress of equation 9.13b is compared to the shear yield strength of the material to obtain a safety factor. If the loads are fluctuating or if bending is present, then the applied stresses should be converted to equivalent von Mises tensile stresses and compared to the appropriate strength criteria using the modified-Goodman diagram.

9.12 INTERFERENCE FITS

Another common means of coupling a hub to a shaft is to use a **press** or **shrink fit**, also called an **interference fit**. A press fit is obtained by machining the hole in the hub to a slightly smaller diameter than that of the shaft as shown in Figure 9-18. The two parts are then forced together slowly in a press, preferably with oil lubricant applied to the joint. The elastic deflection of both shaft and hub act to create large normal and frictional forces between the parts. The friction force transmits the shaft torque to the hub and resists axial motion as well. The American Gear Manufacturers Association (AGMA) publishes a standard AGMA 9003-A91, *Flexible Couplings—Keyless Fits* which defines formulas for the calculation of interference fits.

Only relatively small parts can be press-fitted without exceeding the force capacity of a typical shop press. For larger parts, a **shrink fit** can be made by heating the hub to expand its inside diameter and/or an **expansion fit** can be made by cooling the shaft to reduce its diameter. The hot and cold parts can be slipped together with little axial force, and when they equilibrate to room temperature, their dimensional change creates the desired interference for frictional contact. Another method is to hydraulically expand the hub with pressurized oil delivered through passageways in the shaft or hub. This technique can be used to remove a hub as well.

The amount of interference needed to create a tight joint varies with the diameter of the shaft. Approximately 0.001 to 0.002 units of diametral interference per unit of shaft diameter is typical (the rule of thousandths), the smaller amount being used with larger shaft diameters. For example, the interference for a diameter of 2 in would be about 0.004 in, but a diameter of 8 in would receive only about 0.009 to 0.010 in of interference. Another (and simpler) machinist's rule of thumb is to use 0.001 in of interference for diameters up to 1 in, and 0.002 in for diameters from 1 to 4 in.

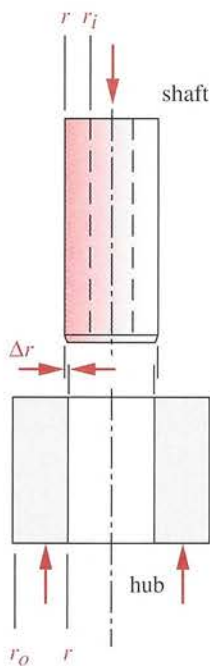


FIGURE 9-18
An Interference Fit

Stresses in Interference Fits

An interference fit creates the same stress state in the shaft as would a uniform external pressure on its surface. The hub experiences the same stresses as a thick-walled cylinder subjected to internal pressure. The equations for stresses in a thick-walled cylinder were presented in Section 4.17 (p. 200) and depend on the applied pressures and the radii of the elements. The pressure p created by the press fit can be found from the deformation of the materials caused by the interference.

$$p = \frac{0.5\delta}{\frac{r}{E_o} \left(\frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{r}{E_i} \left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)} \quad (9.14a)$$

where $\delta = 2\Delta r$ is the total diametral interference between the two parts, r is the nominal radius of the interface between the parts, r_i is the inside radius (if any) of a hollow shaft, and r_o is the outside radius of the hub as shown in Figure 9-18. E and ν are the Young's moduli and Poisson's ratios of the two parts, respectively.

The torque that can be transmitted by the interference fit can be defined in terms of the pressure p at the interface, which creates a friction force at the shaft radius.

$$T = 2\pi r^2 \mu p l \quad (9.14b)$$

where l is the length of the hub engagement, r is the shaft radius, and μ is the coefficient of friction between shaft and hub. The AGMA standard suggests a value of $0.12 \leq \mu \leq 0.15$ for hydraulically expanded hubs and $0.15 \leq \mu \leq 0.20$ for shrink or press-fit hubs. AGMA assumes (and recommends) a surface finish of $32 \mu\text{in rms}$ ($1.6 \mu\text{m } R_a$), which requires a ground finish on both diameters. Equations 9.14a and 9.14b can be combined to give an expression that defines the torque obtainable from a particular deformation, coefficient of friction, and geometry.

$$T = \frac{\pi l r \mu \delta}{\frac{1}{E_o} \left(\frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{1}{E_i} \left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)} \quad (9.14c)$$

The pressure p is used in equations 4.47 (p. 201) to find the radial and tangential stresses in each part. For the shaft:

$$\sigma_{t_{shaft}} = -p \frac{r^2 + r_i^2}{r^2 - r_i^2} \quad (9.15a)$$

$$\sigma_{r_{shaft}} = -p \quad (9.15b)$$

where r_i is the inside radius of a hollow shaft. If the shaft is solid, r_i will be zero.

For the hub:

$$\sigma_{t_{hub}} = p \frac{r_o^2 + r^2}{r_o^2 - r^2} \quad (9.16a)$$

$$\sigma_{r_{hub}} = -p \quad (9.16b)$$

These stresses need to be kept below the yield strengths of the materials to maintain the fit. If the materials yield, the hub will become loose on the shaft.

Stress Concentration in Interference Fits

Even though there may be no disruption of the smooth surface of the press-fit shaft by shoulders or keyways, an interference fit nevertheless creates stress concentrations in the shaft and hub at the ends of the hub due to the abrupt transition from uncompressed to compressed material. Figure 9-19a shows a photoelastic study of a press-fit hub on a shaft. The fringes show stress concentration at the corners. Figure 9-19b shows how the stress concentration can be reduced by providing circumferential relief grooves in the faces of the hub close to the shaft diameter. These grooves make the material at the edge of the hub more compliant enabling it to deflect away from the shaft and reduce the stress locally. This approach is similar to the techniques for stress-concentration reduction shown in Figure 4-38 (p. 189).

Figure 9-20 shows curves of stress-concentration factors for interference fits between hubs and shafts developed from the photoelastic study of Figure 9-19a. The values on the abscissa are ratios of hub length to shaft diameter. These geometric stress-concentration

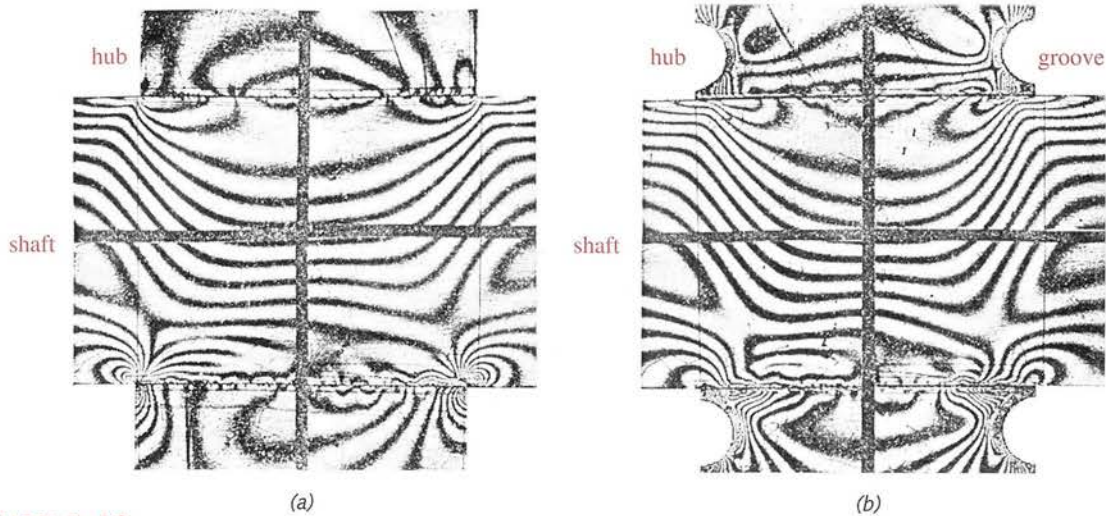


FIGURE 9-19

Photoelastic Stress Analysis of (a) A Plain Press-fit Assembly and (b) A Grooved-Hub Press-fit Assembly Source: R. E. Peterson and A. M. Wahl, "Fatigue of Shafts at Fitted Members, with a Related Photoelastic Analysis," *ASME J. App. Mech.*, vol. 57, p. A1, 1935.

factors are applied in the same manner as before. For static loading they need to be used to determine whether local yielding will compromise the interference fit. For dynamic loading, they are modified by the material's notch sensitivity to get a fatigue stress-concentration factor to use in equation 9.8 (p. 514) for shaft design.

Fretting Corrosion

This problem was discussed in Chapter 7. Interference fits are the primary victims of fretting problems. Though the fretting mechanism is not yet fully understood, certain precautions are known to help reduce its severity. See Section 7.6 (p. 430) for more details.

$$p/\sigma = \frac{\text{nominal press-fit pressure}}{\text{nominal bending stress}}$$

$$l/d = \frac{\text{length of hub}}{\text{diameter of shaft}}$$

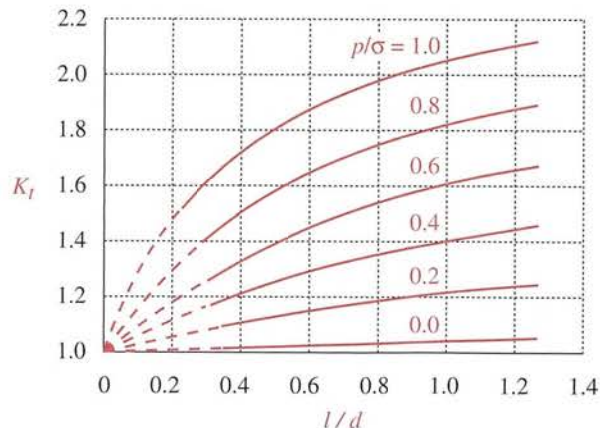


FIGURE 9-20

Stress Concentration in a Press-Fit or Shrink-Fit Hub on a Shaft Source: R. E. Peterson and A. M. Wahl, "Fatigue of Shafts at Fitted Members, with a Related Photoelastic Analysis," *ASME J. App. Mech.*, vol. 57, p. A73, 1935.

EXAMPLE 9-5**Designing an Interference Fit**

Problem Redesign the attachment of the gear to the shaft in Figure 9-5 (p. 515) to make it an interference fit rather than a keyed connection. Define the shaft and gear hole dimensions and their tolerances for a press fit.

Given The loading is the same as in Example 9-2 (p. 519). The peak torque at the gear (point B) is 146 lb-in. Use the nominal shaft diameters of $d_0 = 0.875$ in, $d_1 = 0.750$ in. See Figure 9-5 for labels. The gear hub diameter is 3 in and its length is 1.5 in.

Assumptions The shaft material is the same as in Example 9-2. Class 40 gray cast iron is used for the gear with $S_{ut} = 42$ kpsi and $E = 14$ Mpsi. See Figure 9-20 for stress-concentration factors. The 0.750 shaft diameter will be increased slightly to 0.780 nominal, where it press fits into the gear hub to allow the gear to slip over the rest of the shaft at assembly.

Solution See Figure 9-5 and Table 9-4.

- 1 The nominal diameter of the shaft at the gear hub is 0.780 in. Based on the rule of thousandths, a reasonable diametral interference would be 0.0015 in. From this assumption, the bearing pressure after pressing can be found from equation 9.14 (pp. 534–535).

$$\begin{aligned}
 p &= \frac{0.5\delta}{\frac{r}{E_o} \left(\frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{r}{E_i} \left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)} \\
 &= \frac{0.5(0.0015)}{\frac{0.390}{1.4E7} \left(\frac{1.5^2 + 0.390^2}{1.5^2 - 0.390^2} + 0.28 \right) + \frac{0.390}{3.0E7} \left(\frac{0.390^2 + 0}{0.390^2 - 0} - 0.28 \right)} \quad (a) \\
 p &= 15\,288 \text{ psi}
 \end{aligned}$$

- 2 The stresses in the shaft after pressing are found from equations 9.15 (p. 535).

$$\sigma_{t_{\text{shaft}}} = -p \frac{r^2 + r_i^2}{r^2 - r_i^2} = -15\,288 \frac{0.390^2 + 0}{0.390^2 - 0} = -15\,288 \text{ psi} \quad (b)$$

$$\sigma_{r_{\text{shaft}}} = -p = -15\,288 \text{ psi} \quad (c)$$

- 3 The stresses in the hub after pressing are found from equations 9.16 (p. 535).

$$\sigma_{t_{\text{hub}}} = p \frac{r_o^2 + r^2}{r_o^2 - r^2} = 15\,288 \frac{1.5^2 + 0.390^2}{1.5^2 - 0.390^2} = 17\,505 \text{ psi} \quad (d)$$

$$\sigma_{r_{\text{hub}}} = -p = -15\,288 \text{ psi} \quad (e)$$

- 4 To find the stress-concentration factor we need the hub-length to shaft-diameter ratio, l/d :

$$\frac{l}{d} = \frac{1.500}{0.780} = 1.923 \quad (f)$$

and the ratio of press-fit pressure to nominal bending stress:

$$\sigma = \frac{Mc}{I} = \frac{65.6(0.390)64}{\pi(0.780^4)} = 1\,408 \text{ psi} \quad (g)$$

$$\frac{p}{\sigma} = \frac{15\,288}{1\,408} = 10.9 \quad (h)$$

- 5 Taking these values to Figure 9-20, we find that they are off the chart. We will assume an approximate value for the stress-concentration factor of:

$$K_t \cong 2.4 \quad (i)$$

- 6 The safety factors against failure during press fitting can now be found:

$$N_{s_{shaft}} = \frac{S_y}{K_t \sigma_{t_{shaft}}} = \frac{-38\,000}{2.4(-15\,288)} = 1.0 \quad (j)$$

$$N_{s_{hub}} = \frac{S_{ut}}{K_t \sigma_{t_{hub}}} = \frac{42\,000}{2.4(17\,505)} = 1.0 \quad (k)$$

- 7 Both parts are at failure with this much interference. The computations were repeated for a range of interference values from 0.0008 to 0.0015 in, and those results are shown in Table 9-4. The safety factors range from 2.0 to 1.0.
- 8 We would like a tolerance range of at least 0.0002 in on each part or a total variation in interference of at least 0.0004 in for a set of mass-produced parts. So we choose to set the range of interference from 0.0008 to 0.0013 in, a total range of 0.0005 in.
- 9 The part dimensions are then established as

$$hub \text{ dia} = 0.7799 + 0.0003 / -0.0000 = \frac{0.7802}{0.7799} \text{ in} \quad (l)$$

$$shaft \text{ dia} = 0.7812 + 0.0000 / -0.0002 = \frac{0.7812}{0.7810} \text{ in}$$

giving a range of interference of

$$\begin{aligned} \text{min interference} &= 0.7810 - 0.7802 = 0.0008 \text{ in} \\ \text{max interference} &= 0.7812 - 0.7799 = 0.0013 \text{ in} \end{aligned} \quad (m)$$

- 10 What torque will this pressed joint transmit at its minimum interference, assuming $\mu = 0.15$? From equation 9.14c (p. 535):

$$\begin{aligned} T &= \frac{\pi l r \mu \delta}{\frac{1}{E_o} \left(\frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{1}{E_i} \left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)} \\ T &= \frac{\pi(1.5)(0.375)(0.15)(0.0008)}{\frac{0.375}{1.4E7} \left(\frac{1.5^2 + 0.390^2}{1.5^2 - 0.390^2} + 0.28 \right) + \frac{0.375}{3.0E7} \left(\frac{0.390^2 + 0}{0.390^2 - 0} - 0.28 \right)} \\ T &= 1\,753 \text{ in-lb} \end{aligned} \quad (n)$$

* See file EX09-05 on the CD-ROM.

This is well in excess of the peak operating torque of 146 in-lb, so it will work.*

Table 9-4 Safety Factors for Various Interferences in Example 9-5

Interference (in)	p (psi)	p/σ	K_t	$N_{s_{shaft}}$	$N_{s_{hub}}$
0.000 8	8 154	5.8	2.3	2.0	1.9
0.000 9	9 173	6.5	2.4	1.8	1.7
0.001 0	10 192	7.2	2.4	1.6	1.5
0.001 1	11 212	8.0	2.4	1.4	1.4
0.001 2	12 231	8.7	2.4	1.3	1.3
0.001 3	13 250	9.4	2.4	1.2	1.2
0.001 4	14 269	10.1	2.4	1.1	1.1
0.001 5	15 288	10.9	2.4	1.0	1.0

9.13 FLYWHEEL DESIGN*

A flywheel is used to smooth out variations in the speed of a shaft caused by torque fluctuations. Many machines have load patterns that cause the torque-time function to vary over the cycle. Piston compressors, punch presses, rock crushers, etc., all have time-varying loads. The prime mover can also introduce torque oscillations to the transmission shaft. Internal-combustion engines with one or two cylinders are an example. Other systems may have both smooth torque sources and smooth loads, such as an electrical generator driven by a steam turbine. These smooth-acting devices have no need for a flywheel. If the source of the driving torque or the load torque has a fluctuating nature, then a flywheel is usually called for.

A flywheel is an energy-storage device. It absorbs and stores kinetic energy when speeded up and returns energy to the system when needed by slowing its rotational speed. The kinetic energy E_k in a rotating system is

$$E_k = \frac{1}{2} I_m \omega^2 \quad (9.17a)$$

where I_m is the mass moment of inertia of all rotating mass on the shaft about the axis of rotation and ω is the rotational velocity. This includes the I_m of the motor rotor and anything else rotating with the shaft plus that of the flywheel.

Flywheels may be as simple as a cylindrical disk of solid material, or may be of spoked construction with a hub and rim. The latter arrangement is more efficient of material, especially for large flywheels, as it concentrates the bulk of its mass in the rim, which is at the largest radius. Since the mass moment of inertia I_m of a flywheel is proportional to mr^2 , mass at larger radius contributes much more. If we assume a solid-disk geometry with inside radius r_i and outside radius r_o , the mass moment of inertia is

$$I_m = \frac{m}{2} (r_o^2 + r_i^2) \quad (9.17b)$$

The mass of a solid circular disk of constant thickness t and having a central hole is

$$m = \frac{W}{g} = \pi \frac{\gamma}{g} (r_o^2 - r_i^2) t \quad (9.17c)$$

* Portions of this section are adapted from R. L. Norton, *Design of Machinery*, 2nd ed, McGraw-Hill, 1999, pp. 548–554, with the publisher's permission.

Substituting in equation 9.17b gives an expression for I_m in terms of the disk geometry:

$$I_m = \frac{\pi \gamma}{2g} (r_o^4 - r_i^4) t \quad (9.17d)$$

where γ is the material's weight density and g is the gravitational constant.

There are two stages to the design of a flywheel. First, the amount of energy required for the desired degree of smoothing must be found and the moment of inertia needed to absorb that energy determined. Then a flywheel geometry must be defined that both supplies that mass moment of inertia in a reasonably sized package and is safe against failure at design speeds.

Energy Variation in a Rotating System

Figure 9-21 shows a flywheel, designed as a flat circular disk, attached to a motor shaft. The motor supplies a torque magnitude T_m , which we would like to be as constant as possible, i.e., to be equal to the average torque T_{avg} . Assume that the load on the other side of the flywheel demands a torque T_l , which is time varying, as shown in Figure 9-22. This torque variation can cause the shaft speed to vary depending on the torque-speed characteristic of the driving motor. We need to determine how much I_m to add in the form of a flywheel to reduce the speed variation of the shaft to an acceptable level. Write Newton's law for the free-body diagram in Figure 9-21.

$$\begin{aligned} \sum T &= I_m \alpha \\ T_l - T_m &= I_m \alpha \end{aligned} \quad (9.18a)$$

but we want

$$T_m = T_{avg}$$

so

$$T_l - T_{avg} = I_m \alpha \quad (9.18b)$$

Substituting

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{dt} \left(\frac{d\theta}{d\theta} \right) = \omega \frac{d\omega}{d\theta}$$

gives

$$T_l - T_{avg} = I_m \omega \frac{d\omega}{d\theta}$$

$$(T_l - T_{avg}) d\theta = I_m \omega d\omega \quad (9.18c)$$

Integrating

$$\begin{aligned} \int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} (T_l - T_{avg}) d\theta &= \int_{\omega_{min}}^{\omega_{max}} I_m \omega d\omega \\ \int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} (T_l - T_{avg}) d\theta &= \frac{1}{2} I_m (\omega_{max}^2 - \omega_{min}^2) \end{aligned} \quad (9.18d)$$

The left-hand side of this expression represents the change in kinetic energy E_k between the maximum and minimum shaft ω 's and is equal to the area under the torque-time diagram of Figure 9-22 between those extreme values of ω . The right-hand side of equation 9.18c is the change in kinetic energy stored in the flywheel. The only way to extract kinetic energy from the flywheel is to slow it down, as shown in equation

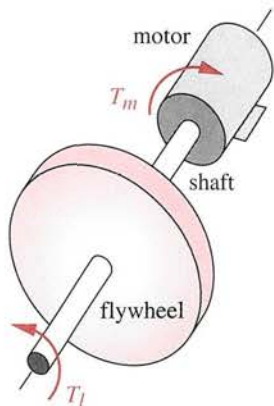


FIGURE 9-21

A Flywheel on a Transmission Shaft

9.17a. Adding kinetic energy speeds it up. It is impossible to obtain exactly constant shaft velocity in the face of changing energy demands by the load. The best we can do is to minimize the speed variation ($\omega_{max} - \omega_{min}$) by providing a flywheel with sufficiently large I_m .

EXAMPLE 9-6

Determining the Energy Variation in a Torque-Time Function

Problem Find the energy variation per cycle in a torque-time function that needs to be absorbed by a flywheel for smooth operation.

Given A torque-time function that varies over its cycle as shown in Figure 9-22. The torque varies during the 360° cycle about its average value.

Assumptions The one cycle of torque variation shown is representative of the steady-state condition. Energy delivered from the source to the load will be considered positive and energy returned from load to source, negative.

Solution

- 1 Calculate the average value of the torque-time function over one cycle using numerical integration. In this case it is 7 020 lb-in. (Note that in some cases the average value may be zero.)
- 2 Note that the integration on the left-hand side of equation 9.18c is done with respect to the average line of the torque function, not with respect to the θ axis. (From the definition of the average, the sum of positive area above an average line is equal to the sum of negative area below that line.) The integration limits in equation 9.18 are

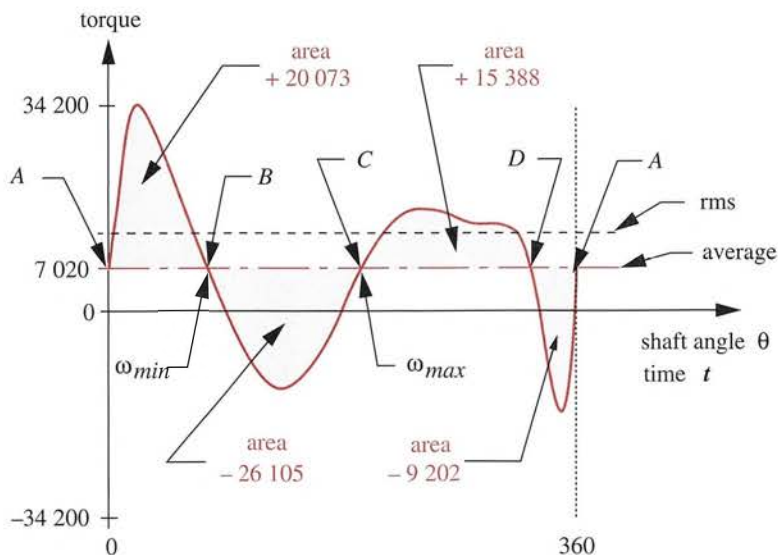


FIGURE 9-22

Integrating the Pulses Above and Below the Average Value in the Torque-Time Function

Table 9-5 Accumulation of Energy Pulses Under a Torque-Time Curve

From	$\Delta \text{Area} = \Delta E$	Accumulated Sum = E	Min & Max
A to B	+20 073	+20 073	ω_{min} @ B
B to C	-26 105	-6 032	ω_{max} @ C
C to D	+15 388	+9 356	
D to A	-9 202	+154	
Total $\Delta \text{Energy} = E @ \omega_{max} - E @ \omega_{min}$			
$= (-6\ 032) - (+20\ 073) = -26\ 105 \text{ in-lb}$			

from the shaft angle θ , at which the shaft ω is a minimum to the shaft angle θ , at which ω is a maximum.

- 3 The minimum ω will occur after the maximum positive energy has been delivered from the motor to the load, i.e., at a point θ where the summation of positive energy (area) in the torque pulses is at its largest positive value.
- 4 The maximum ω will occur after the maximum negative energy has been returned to the load, i.e., at a point θ where the summation of energy (area) in the torque pulses is at its largest negative value.
- 5 To find these locations in θ corresponding to the maximum and minimum ω 's and thus find the amount of energy needed to be stored in the flywheel, we need to numerically integrate each pulse of this function from crossover to crossover with the average line. The crossover points have been labeled A, B, C, and D and the areas between them are shown in Figure 9-22.
- 6 The remaining task is to accumulate these pulse areas beginning at an arbitrary crossover (in this case point A) and proceeding pulse by pulse across the cycle. Table 9-5 shows this process and the result.
- 7 Note in Table 9-5 that the minimum shaft speed occurs after the largest accumulated positive energy pulse (+20 073 in-lb) has been delivered from the driveshaft to the system. This delivery of energy slows the motor down. The maximum shaft speed occurs after the largest accumulated negative energy pulse (-6 032 in-lb) has been received back from the load by the shaft. This return of stored energy will tend to speed up the motor. The total energy variation is the algebraic difference between these two extreme values, which in this example is -26 105 in-lb. This energy coming back from the load needs to be absorbed by the flywheel and later returned to the system within each cycle to smooth the variations in shaft speed. See files EX09-07a and EX09-07b that also contain the calculations for Example 9-6.

Determining the Flywheel Inertia

We must now determine how large a flywheel is needed to absorb this energy with an acceptable change in speed. The change in shaft speed during a cycle is called its fluctuation Fl and is equal to

$$Fl = \omega_{max} - \omega_{min} \quad (9.19a)$$

We can normalize this to a dimensionless ratio by dividing it by the average shaft speed. This ratio is called the coefficient of fluctuation C_f .

$$C_f = \frac{(\omega_{max} - \omega_{min})}{\omega_{avg}} \quad (9.19b)$$

This coefficient of fluctuation is a design parameter to be chosen by the designer. It is typically set to a value between 0.01 and 0.05 for precision machinery, and as high as 0.20 for crushing or hammering machinery, which correspond to a 1 to 5% fluctuation in shaft speed. The smaller this chosen value, the larger the flywheel will have to be. This presents a design trade-off. A larger flywheel will add more cost and weight to the system, which are factors that have to be considered against the smoothness of operation desired.

We found the required change in kinetic energy E_k by integrating the torque curve,

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} (T_l - T_{avg}) d\theta = E_k \quad (9.20a)$$

and can now set it equal to the right-hand side of equation 9.18c:

$$E_k = \frac{1}{2} I_m (\omega_{max}^2 - \omega_{min}^2) \quad (9.20b)$$

Factoring this expression:

$$E_k = \frac{1}{2} I_m (\omega_{max} + \omega_{min})(\omega_{max} - \omega_{min}) \quad (9.20c)$$

If the torque-time function were a pure harmonic, then its average value could be expressed exactly as

$$\omega_{avg} = \frac{(\omega_{max} + \omega_{min})}{2} \quad (9.21)$$

Our torque functions will seldom be pure harmonics, but the error introduced by using this expression as an approximation of the average is acceptable. We can now substitute equations 9.19b and 9.21 into equation 9.20c to get an expression for the mass moment of inertia I_s needed in the entire rotating system in order to obtain the selected coefficient of fluctuation.

$$E_k = \frac{1}{2} I_s (2\omega_{avg})(C_f \omega_{avg})$$

$$I_s = \frac{E_k}{C_f \omega_{avg}^2} \quad (9.22)$$

Equation 9.22 can be used to design the physical flywheel by choosing a desired coefficient of fluctuation C_f , and using the value of E_k from the numerical integration of the torque curve (see Table 9-5 for an example) and the average shaft ω to compute the needed system I_s . The physical flywheel's mass moment of inertia I_m is then set equal to the required system I_s . But, if the moments of inertia of the other rotating elements on the same shaft (such as the motor) are known, the physical flywheel's required I_m can be reduced by those amounts.

The most efficient flywheel design in terms of maximizing I_m for minimum material used is one in which the mass is concentrated in its rim and its hub is supported on spokes, like a carriage wheel. This puts the majority of the mass at the largest radius possible and minimizes the weight for a given I_m . Even if a flat, solid circular disk flywheel design is chosen, either for simplicity of manufacture or to obtain a flat surface for other functions (such as an automobile clutch), the design should be done with an eye to reducing weight and thus cost. Since, in general, $I_m = mr^2$, a thin disk of large diameter will need fewer pounds of material to obtain a given I_m than will a thicker disk of smaller diameter. Dense materials such as cast iron and steel are the obvious choices for a flywheel. Aluminum is seldom used. Though many metals (lead, gold, silver, platinum) are more dense than iron and steel, one can seldom get the accounting department's approval to use them in a flywheel.

Figure 9-23 shows the change in the torque of Figure 9-22 after the addition of a flywheel sized to provide a coefficient of fluctuation of 0.05. The oscillation in torque about the unchanged average value is now 5%, much less than what it was without the flywheel. Note that the peak value is now 87 instead of 372 lb-in. A much smaller-horsepower motor can now be used because the flywheel is available to absorb the energy returned from the load during the cycle.

Stresses in Flywheels

As a flywheel spins, the centrifugal force acts upon its distributed mass and attempts to pull it apart. These centrifugal forces are similar to those caused by an internal pressure in a cylinder. Thus, the stress state in a spinning flywheel is analogous to a thick-walled cylinder under internal pressure (see Section 4.17 on p. 200). The tangential stress of a solid-disk flywheel as a function of its radius r is

$$\sigma_t = \frac{\gamma}{g} \omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right) \quad (9.23a)$$

and the radial stress is

$$\sigma_r = \frac{\gamma}{g} \omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2 \right) \quad (9.23b)$$

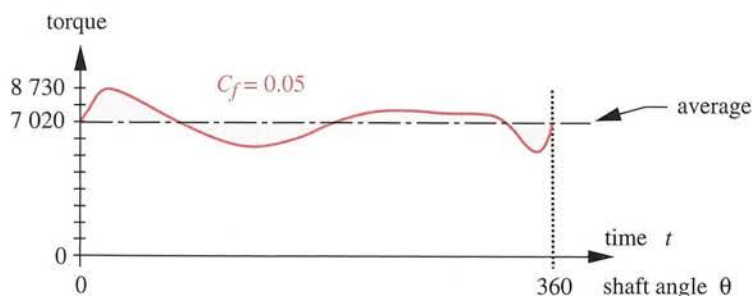


FIGURE 9-23

Torque-Time Function of Figure 9-22 After Adding a Flywheel with $C_f = 0.05$

where γ = material weight density, ω = angular velocity in rad/sec, ν = Poisson's ratio, r is the radius to a point of interest, and r_i , r_o are inside and outside radii of the solid-disk flywheel.

Figure 9-24 shows how these stresses vary over the radius of the flywheel. The tangential stress is a maximum at the inner radius. The radial stress is zero at inside and outside radii and peaks at an interior point but is everywhere less than the tangential stress. The point of most interest is then at the inside radius. The tangential tensile stress at that point is what fails a flywheel, and when it fractures at that point, it typically fragments and explodes with extremely dangerous results. Since the forces causing the stress are a function of rotational speed, there will always be some speed that will fail the flywheel. A maximum safe operating speed should be calculated for a flywheel and some means taken to preclude its operation at higher speeds, such as a speed control or governor. A safety factor against overspeeding can be determined as the quotient of the speed that will cause yielding over the operating speed, $N_{os} = \omega / \omega_{yield}$.

Failure Criteria

If the flywheel spends most of its life operating at essentially constant speed, then it can be considered to be statically loaded and the yield strength used as a failure criterion. The number of start-stop cycles in its operating regime will determine whether a fatigue-loading situation needs to be considered. Each runup to operational speed and rundown to zero constitutes a fluctuating stress cycle. If the number of these start-stop cycles is large enough over the projected life of the system, then fatigue-failure criteria should be applied. A low-cycle fatigue regime may require a strain-based fatigue failure analysis rather than a stress-based one, particularly if there exists the possibility of any transient overloads that may cause the local stresses to exceed the yield stress at stress concentrations.

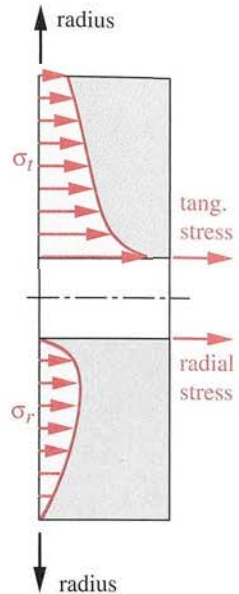


FIGURE 9-24

Stress Distributions Along the Radius of a Spinning Flywheel

EXAMPLE 9-7

Designing a Solid-Disk Flywheel

- Problem** Design a suitable flywheel for the system of Example 9-6. An overspeed safety factor of at least 2 is desired.
- Given** An input torque-time function which varies over its cycle as shown in Figure 9-22 (p. 541). The torque is varying during the 360° cycle about its average value, and the energy variation per cycle is 26 105 in-lb as shown in Table 9-5 (p. 542). The shaft $\omega = 800$ rad/sec.
- Assumptions** The one cycle of torque variation shown is representative of the steady-state condition. The desired coefficient of fluctuation is 0.05. The system is in continuous operation with minimal start-stop cycles. A steel of 62 kpsi yield strength will be used. No keyway will be used in order to reduce stress concentrations. Instead, a tapered locking hub will frictionally couple it to the shaft and the hub will be axially bolted to the flywheel.

Solution

See Figure 9-25 and Table 9-6.

- 1 We know the amount of energy needed from the torque-time diagram from Example 9-6 and have defined the shaft ω and a desired coefficient of fluctuation. From these data we can determine a required mass moment of inertia for the system I_s using equation 9.22 (p. 543).

$$I_s = \frac{E}{C_f \omega_{avg}^2} = \frac{26\,105}{0.05(800)^2} = 0.816 \text{ lb} \cdot \text{in} \cdot \text{sec}^2 \quad (a)$$

The flywheel need only supply a portion of this amount if other rotating masses such as the motor armature are present. However, we will assume in this example that the flywheel will provide all of the required inertia, making $I_m = I_s$.

- 2 The flywheel dimensions for this moment of inertia can be defined from equation 9.17d. We will assume a steel material with $\gamma = 0.28 \text{ lb/in}^3$ and an inside radius $r_i = 1 \text{ in}$.

$$\begin{aligned} I_m &= \frac{\pi \gamma}{2g} (r_o^4 - r_i^4) t = 0.816 \\ 0.816 &= \frac{\pi}{2} \left(\frac{0.28}{386} \right) (r_o^4 - 1^4) t \\ 716.14 &= (r_o^4 - 1) t \end{aligned} \quad (b)$$

So, the required I_m can be obtained with an infinity of combinations of flywheel radius r_o and thickness t for the assumed data.

- 3 The best solution to equation (b) will be one that balances the conflicting factors of flywheel size, weight, stresses, and safety factor. Consider two possible designs, one with a small thickness t and the other with a large t . The thin flywheel will be larger in diameter but considerably lighter than the thick one due to the nonlinearity of the terms involving r_o . But, as r_o increases, so will the stresses, because the mass at larger radius exerts more centrifugal force on the material.
- 4 To get a value of r_o that is consistent with any desired safety factor, the tangential stress equation 9.23a can be back-solved with assumed values of $\sigma_t = S_y / N_y$, r_i , and material parameters ν and γ ,

$$\begin{aligned} \sigma_t &= \frac{\gamma}{g} \omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right) = \frac{S_y}{N_y} \\ \frac{62\,000}{N_y} &= \frac{0.28}{386} (800)^2 \left(\frac{3+0.28}{8} \right) \left(1 + r_o^2 + \frac{r_o^2}{1} - \frac{1+3(0.28)}{3+0.28} (1) \right) \\ r_o^2 &= \frac{162.9}{N_y} - 0.535 \end{aligned} \quad (c)$$

That value of r_o can then be used in equation (b) to find the flywheel thickness. For a design safety factor against yielding of 2.5 and the values assumed in equation (c), we get $r_o = 8.06 \text{ in}$ and $t = 0.172 \text{ in}$.

- 5 With the flywheel geometry now defined, the rotational speed at which yielding will begin can be computed from equation 9.17d using the yield strength for the stress value.

Table 9-6 Data for Example 9-7 with $I_m = 0.816 \text{ lb-in-sec}^2$

Thickness (in)	Diameter (in)	r/t ratio	Weight (lb)	Stress (psi)	Safety Factor Yielding	Safety Factor Overspeed
0.125	17.40	69.6	8.20	28 896	2.1	1.5
0.250	14.63	29.3	11.60	20 459	3.0	1.7
0.375	13.22	17.6	14.10	16 722	3.7	1.9
0.500	12.31	12.3	16.20	14 494	4.3	2.1
0.625	11.64	9.3	18.10	12 974	4.8	2.2
0.750	11.12	7.4	19.70	11 852	5.2	2.3
0.875	10.70	6.1	21.30	10 980	5.6	2.4
1.000	10.35	5.2	22.70	10 277	6.0	2.5
1.125	10.05	4.5	24.00	9 695	6.4	2.5
1.250	9.79	3.9	25.20	9 202	6.7	2.6

$$\sigma_t = S_y = \frac{\gamma}{g} \omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right)$$

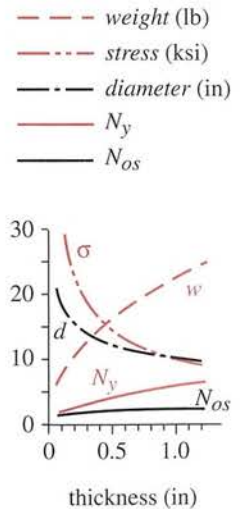
$$62\,000 = \frac{0.28}{g} \omega_{\text{yield}}^2 \left(\frac{3+0.28}{8} \right) \left(1 + 8.06^2 + \frac{8.06^2}{1} - \frac{1+3(0.28)}{3+0.28} (1) \right) \quad (d)$$

$$\omega_{\text{yield}} = 1\,265 \text{ rad/sec}$$

Since this operating speed causes failure, a safety factor against overspeeding can be computed from

$$N_{os} = \frac{\omega_{\text{yield}}}{\omega} = \frac{1\,265 \text{ rad/sec}}{800 \text{ rad/sec}} = 1.6 \quad (e)$$

- 6 To show the variation of parameters with flywheel geometry, this set of equations was solved for a list of possible thicknesses t chosen over a reasonable design range of 0.125 to 1.25 in. Table 9-6 shows the resulting data and Figure 9-25 plots the trends. Note that the weight *increases* as the outside diameter *decreases* and the thickness *increases*. The maximum tangential stress at the inside radius *decreases* with *decreasing* r_o , the safety factor against yielding *increases* from 2.1 to 6.7, and the safety factor for overspeed varies from 1.5 to 2.6 over this range of thicknesses.
- 7 The final design chosen is $t = 0.438$ in and $r_o = 6.36$ in because it has a reasonable mix of parameter values (size, weight) and provides a safety factor of 2 against overspeeding. In other words, the flywheel could be run at up to twice its design speed before it would yield. The safety factor against yielding at the lower design speed will then always be higher and is now 4. Choosing larger safety factors will exact a weight penalty, as can be seen in Figure 9-25. See files EX09-07a and EX09-07b that also contain the calculations for Example 9-6.

**FIGURE 9-25**

Variation of Weight, Stress, Safety Factors, and Diameter with Flywheel Thickness in Example 9-7

9.14 CRITICAL SPEEDS OF SHAFTS

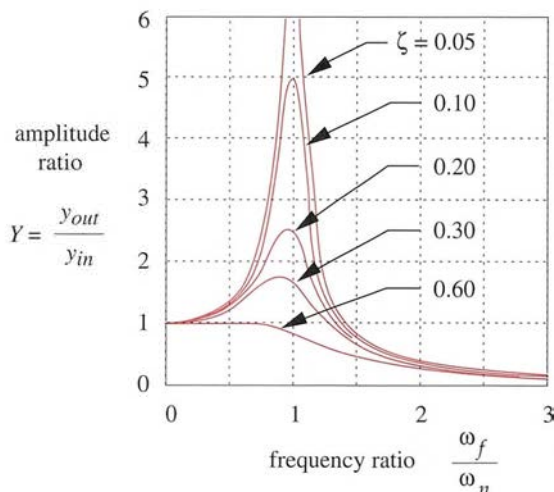
All systems containing energy-storage elements will possess a set of natural frequencies at which the system will vibrate with potentially large amplitudes. Any moving

mass stores kinetic energy and any spring stores potential energy. All machine elements are made of elastic materials and thus can act as springs. All elements have mass, and if they also have a velocity, they will store kinetic energy. When a dynamic system vibrates, a transfer of energy from potential to kinetic to potential, etc., repeatedly takes place within the system. Shafts meet these criteria, rotating with some velocity and deflecting both in torsion and in bending.

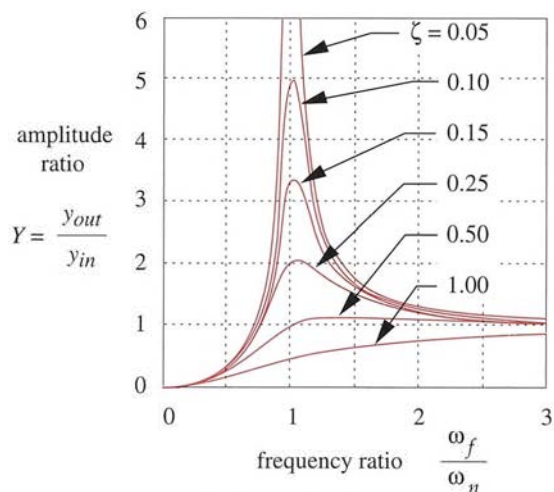
If a shaft, or any element for that matter, is subjected to a time-varying load, it will vibrate. Even if it receives only a transient load, such as a hammer blow, it will vibrate at its natural frequencies, just as a bell rings when struck. This is called a **free vibration**. Such a transient or free vibration will eventually die out due to the damping present in the system. If the time-varying loading is sustained, as, for example, in a sinusoidal manner, the shaft or other element will continue to vibrate at the driving function's **forcing frequency**. If the forcing frequency happens to coincide with one of the element's natural frequencies, then the amplitude of the vibratory response will be much larger than the amplitude of the driving function. The element is then said to be in **resonance**.

Figure 9-26a shows the amplitude response of a forced vibration, and Figure 9-26b a self-excited vibration, as a function of the ratio of the forcing frequency to the system's natural frequency ω_f / ω_n . When this ratio is 1, the system is in resonance and the amplitude of the response approaches infinity in the absence of damping. The amplitude responses in Figure 9-26 are shown as a dimensionless ratio of the output to input amplitudes. Any damping, shown as a damping ratio ζ , reduces the amplitude ratio at resonance. A natural frequency is also called a **critical frequency** or **critical speed**. *Exciting a system at or near its critical (resonant) frequencies must be avoided, since the resulting deflections will often cause stresses large enough to quickly fail the part.*

A system consisting of discrete lumps of mass connected with discrete spring elements can be considered to have a finite number of natural frequencies equivalent to its number of kinematic degrees of freedom. But a continuous system such as a beam



(a) Externally forced vibration response



(b) Self-excited vibration response

FIGURE 9-26

Response of a Single-Degree-of-Freedom System to Varying Forcing or Self-Excitation Frequencies

or shaft has an infinite number of particles, each of which is capable of elastic motion versus its neighboring particles. Thus a continuous system has an infinity of natural frequencies. In either case, the lowest, or fundamental, natural frequency is usually of most interest.

The natural frequencies of vibration of a system can be expressed either as a circular frequency ω_n with units of rad/sec or rpm, or as a linear frequency f_n with units of hertz (Hz). They are the same frequencies expressed in different units. The general expression for the fundamental natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{rad/sec} \quad (9.24a)$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{Hz} \quad (9.24b)$$

where k is the spring constant of the system and m is its mass. The natural frequencies are a physical property of the system; once built, it retains them essentially unchanged unless it loses or gains mass or stiffness during its useful life. Equations 9.24 define the undamped natural frequency. Damping reduces the natural frequency slightly. Shafts, beams, and most machine parts tend to be lightly damped, so the undamped value can be used with little error.

The usual design strategy is to keep all forcing, or self-exciting, frequencies below the first critical frequency by some comfort margin. The larger this margin, the better, but a factor of at least 3 or 4 is desirable. This keeps the amplitude-response ratio close to one or zero, as shown in Figures 9-26a and 9-26b. In some cases, the fundamental frequency of a shaft system cannot be made higher than the required rotating frequency. If the system can be accelerated rapidly enough through resonance, before the vibrations have a chance to build up amplitude, then the system can be run at a speed higher than resonance. Stationary power plants are in this category. The massiveness of the turbines and generators gives a low fundamental frequency (see equation 9.24), but they must be run at a high speed to generate the proper line frequency of AC current. Thus they operate to the right of the peak in Figure 9-14b where the amplitude ratio approaches one at high ratios of ω_f/ω_n . Their start-ups and shutdowns may be infrequent but must always be accomplished rapidly to get through the resonance peak before any damage is caused by excessive deflections. Also, sufficient driving power must be available to provide the energy absorbed at resonance by the oscillations in addition to accelerating the rotating mass. If the driver lacks sufficient power, then the system may stall in resonance, unable to increase its speed in the face of the potentially destructive vibrations. This is called the Sommerfeld effect.^[9]

There are three types of shaft vibration of concern:

- 1 Lateral vibration
- 2 Shaft whirl
- 3 Torsional vibration

The first two involve bending deflections and the last torsional deflection of the shaft.

Lateral Vibration of Shafts and Beams—Rayleigh's Method

A complete analysis of the natural frequencies of a shaft or beam is a complicated problem, especially if the geometry is complex, and is best solved with the aid of *Finite Element Analysis* software. A so-called **modal analysis** can be done on a finite element model of even complex geometries and will yield a large number of natural frequencies (in three dimensions) from the fundamental up. This is the preferred and frequently used approach when analyzing a completed or mature design in detail. However, in the early stages of design, when the part geometries are still not fully defined, a quick and easily applied method for finding at least an approximate fundamental frequency for a proposed design is very useful. **Rayleigh's method** serves that purpose. It is an energy method that gives results within a few percent of the true ω_n . It can be applied to a continuous system or to a lumped-parameter model of the system. The latter approach is usually preferred for simplicity.

RAYLEIGH'S METHOD equates the potential and kinetic energies in the system. The potential energy is in the form of strain energy in the deflected shaft and is maximum at the largest deflection. The kinetic energy is a maximum when the vibrating shaft passes through the undeflected position with maximum velocity. This method assumes that the lateral vibrating motion of the shaft is sinusoidal and that some external excitation is present to force the lateral vibration (Figure 9-26a).

To illustrate the application of this method, consider a simply-supported shaft with three disks (gears, sheaves, etc.) on it as shown in Figure 9-27. We will model this as three discrete lumps of known mass on a massless shaft. The shaft's geometry will define the bending spring constant, thus lumping all the "spring" into the shaft. The total potential energy stored at maximum deflection is the sum of the potential energies of each lumped mass:

$$E_p = \frac{g}{2} (m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3) \quad (9.25a)$$

where the deflections are all taken as positive regardless of the local shape of the deflection curve because the strain energy is not affected by the external coordinate system. The energy of the deflected shaft is ignored as small compared to the disk energy.

The total kinetic energy is the sum of the individual kinetic energies:

$$E_k = \frac{\omega_n^2}{2} (m_1 \delta_1^2 + m_2 \delta_2^2 + m_3 \delta_3^2) \quad (9.25b)$$

where the velocities are taken as positive.

Equating these gives

$$\omega_n = \sqrt{g \frac{\sum_{i=1}^n m_i \delta_i}{\sum_{i=1}^n m_i \delta_i^2}} = \sqrt{g \frac{\sum_{i=1}^n (W_i/g) \delta_i}{\sum_{i=1}^n (W_i/g) \delta_i^2}} = \sqrt{g \frac{\sum_{i=1}^n W_i \delta_i}{\sum_{i=1}^n W_i \delta_i^2}} \quad (9.25c)$$

the second version resulting from substituting $m = W/g$, where the W_i are the weight forces of the discrete lumps into which we divided the system and the δ_i are the dynamic deflections at the locations of the weights due to their vibration. The weight forces and their deflections are all taken as positive to represent the maximum stored energies.

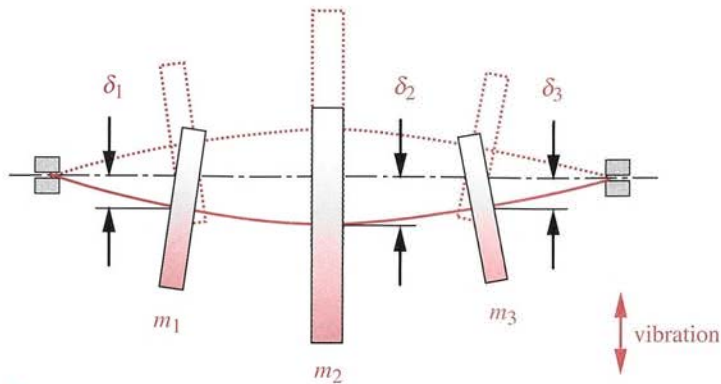


FIGURE 9-27

A Shaft in Lateral Vibration (amplitude greatly exaggerated)

The problem is that we typically do not know the dynamic deflections of the system *a priori*. Rayleigh showed that virtually any estimate of the deflection curve, provided that it reasonably represents the maximum deflection and the boundary conditions of the actual dynamic curve, will suffice. The static deflection curve due to the weights of the assumed lumps (including the weight of the shaft or not as desired) is a very suitable estimate. Note that any applied external loads are not included in this deflection calculation, only the gravitational forces. The resulting approximate ω_n will always be higher than the true fundamental frequency by a few percent regardless of the assumed deflection-curve shape. If more than one estimated deflection curve is tried, the one yielding the lowest value of ω_n should be used, as it will be the closest approximation.

Equation 9.25c can be applied to a system of any complexity by breaking it into a large number of lumps. If gears, pulleys, etc., are on the shaft, they make logical lumped masses. If the shaft mass is significant or dominant, it can be broken into discrete elements along its length with each piece providing a term in the summation.

Rayleigh's method can theoretically be used to find higher frequencies than the fundamental but to do so is difficult without a good estimate of the shape of the higher-order deflection curve. More accurate methods for approximating both the fundamental frequency and the higher frequencies exist but are somewhat more complicated to implement. Ritz modified Rayleigh's method (Rayleigh-Ritz) to allow iteration to the higher frequencies. Holzer's method is more accurate and can find multiple frequencies. See reference 10 for more information.

Shaft Whirl

Shaft whirl is a self-excited vibratory phenomenon to which all shafts are potentially subject. While it is common and recommended practice to dynamically balance all rotating elements in machinery (especially if operated at high speeds), it is not possible to achieve exact dynamic balance except by chance. (See reference 12 for a discussion of dynamic balancing.) Any residual unbalance of a rotating element causes its true mass center to be eccentric from the axis of shaft rotation. This eccentricity creates a centrifugal force that tends to deflect the shaft in the direction of the eccentricity, increasing it and thus further increasing the centrifugal force. The only resistance to this force

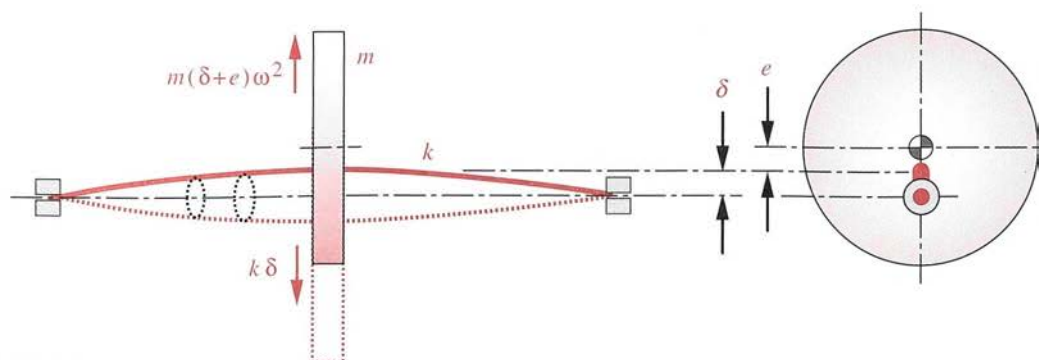


FIGURE 9-28

Shaft Whirl (amplitude greatly exaggerated)

comes from the elastic stiffness of the shaft as shown in Figure 9-28. The initial shaft eccentricity is labeled e and the dynamic deflection is δ . A free-body diagram shows the forces acting to be

$$k\delta = m(\delta + e)\omega^2 \quad (9.26a)$$

$$\delta = \frac{e\omega^2}{(k/m) - \omega^2} \quad (9.26b)$$

The dynamic deflection of the shaft from this centrifugal force causes it to whirl about its axis of rotation with points at the center of the deflected shaft describing circles about the axis. Note in equation 9.26b that the deflection becomes infinite when $\omega^2 = k/m$. As the rotational speed of the shaft approaches the fundamental natural (or critical) speed of lateral vibration, a similar resonance phenomenon to that of lateral vibration occurs. Note in equation 9.26b that when $\omega^2 = k/m$, $\delta = \infty$. Equation 9.26b can be normalized to a nondimensional form, which clearly shows the relationship:

$$\frac{\delta}{e} = \frac{(\omega/\omega_n)^2}{1 - (\omega/\omega_n)^2} \quad (9.26c)$$

Equation 9.26c and Figure 9-29 show the amplitude of shaft deflection normalized to the original eccentricity (δ/e) as a function of the ratio between the rotational frequency and critical frequency ω/ω_n . Note that at $\omega/\omega_n = 0$, there is no response, unlike the forced vibration of the previous section. This is because no centrifugal force exists unless the shaft is rotating. As the shaft speed increases, the deflection rapidly increases. If no damping is present ($\zeta = 0$), at $\omega/\omega_n = 0.707$, the deflection of the shaft is equal to the eccentricity and it becomes theoretically infinite at resonance ($\omega/\omega_n = 1$). Of course, there will always be some damping present, but if ζ is small, the deflections will be very large at resonance and can cause stresses large enough to fail the shaft.

Note what happens when the shaft speed passes through ω_n . The phase shifts 180° , which means that the deflection switches sides abruptly at resonance. At higher ratios of ω/ω_n , the deflection approaches $-e$, which means that the system is then rotating about the mass center of the eccentric mass, and the shaft centerline is eccentric. Conservation of energy makes the system want to rotate about its true mass center. In any

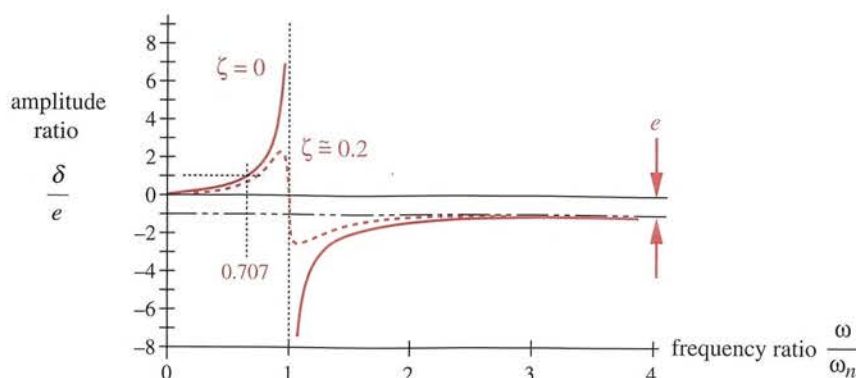


FIGURE 9-29

Amplitude Response of a Self-Excited Shaft-Whirl System as a Function of Frequency Ratio

system where the rotating elements are eccentric and large compared to the shaft this will occur. Perhaps you have observed a pivoted ceiling fan rotating with its motor center orbiting about the axis of rotation. The fan blades are usually not in perfect balance and the assembly rotates about the mass center of the blade assembly rather than about the motor/shaft centerline.

It should be clear that rotation of a system at or near its critical frequency is to be strictly avoided. The critical frequency for shaft whirl is the same as for lateral vibration and can be found using Rayleigh's method or any other suitable technique. Because this shaft-whirl vibration amplitude ratio starts at zero rather than at one (as with forced vibrations) the forcing frequency can be closer to the critical frequency than for the lateral vibration case. Keeping the operating speed below about half the shaft-whirl critical frequency will usually provide good results unless the initial eccentricity is excessive (which should not be allowed anyway).

Note the difference between shaft *lateral vibration* and *shaft whirl*. **Lateral vibration** is a *forced vibration*, requiring some outside source of energy such as vibrations from other parts of the machine to precipitate it, and the shaft then vibrates in one or more lateral planes whether or not it is rotating. **Shaft whirl** is a *self-excited vibration* caused by the shaft rotation acting on an eccentric mass. It will *always* occur when both rotation and eccentricity are present. The shaft assumes a deflected shape, which then rotates or whirls about the axis much like a jump-rope being swung by children.

Torsional Vibration

Just as a shaft can vibrate laterally, it can also vibrate torsionally and will have one or more torsional natural frequencies. The same equations that describe lateral vibrations can be used for torsional ones. The systems are analogous. Force becomes torque, mass becomes mass moment of inertia, and linear spring constant becomes torsional spring constant. Equation 9.24 (p. 549) for the circular natural frequency becomes, for a single-degree-of-freedom rotating system:

$$\omega_n = \sqrt{\frac{k_t}{I_m}} \quad \text{rad/sec} \quad (9.27a)$$

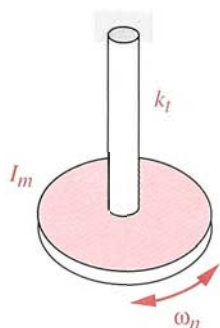


FIGURE 9-30

Disk on Axle in Torsional Vibration

The torsional spring constant k_t for a solid circular shaft is

$$k_t = \frac{GJ}{l} \quad \text{lb} \cdot \text{in}/\text{rad} \text{ or } \text{N} \cdot \text{m}/\text{rad} \quad (9.27b)$$

where G is the material's modulus of rigidity, and l is the shaft length. The polar second moment of area J of a solid circular shaft is

$$J = \frac{\pi d^4}{32} \quad \text{in}^4 \text{ or } \text{m}^4 \quad (9.27c)$$

If the shaft is stepped, then an equivalent polar second moment of area J_{eff} is found from

$$J_{\text{eff}} = \frac{l}{\sum_{i=1}^n \frac{l_i}{J_i}} \quad (9.27d)$$

where l is total shaft length, and J_i , and l_i are the polar moments and lengths of the sub-sections of shaft of differing diameters, respectively.

The mass moment of inertia of a solid circular disk about its axis of rotation is:

$$I_m = \frac{mr^2}{2} \quad \text{in} \cdot \text{lb} \cdot \text{sec}^2 \text{ or } \text{kg} \cdot \text{m}^2 \quad (9.27e)$$

where r is the disk radius and m is its mass.

These equations are sufficient to find the critical frequency of a single disk mounted on a fixed axle, as shown in Figure 9-30.

Two Disks on a Common Shaft

A more interesting problem is that of two (or more) disks displaced on a common shaft as shown in Figure 9-31. The two disks shown will oscillate torsionally at the same natural frequency, 180° out of phase. There will be a point, called a node, somewhere on the shaft, at which there will be no angular deflection. On either side of the node,

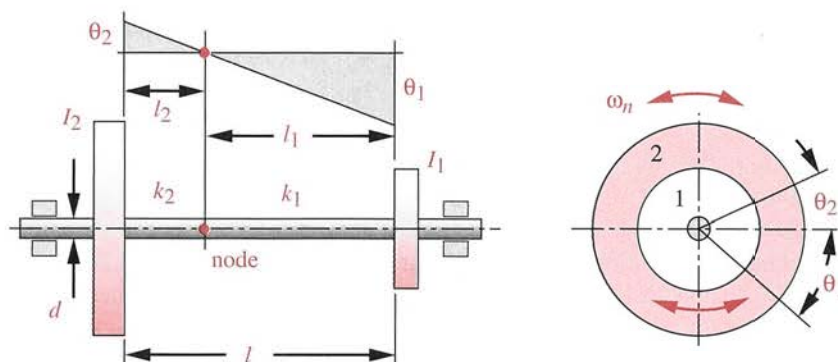


FIGURE 9-31

Torsional Vibration of Two Disks on a Common Shaft

points on the shaft rotate in opposite angular directions when vibrating. The system can be modeled as two separate, single-mass systems coupled at this stationary node. One has mass moment and spring constant I_1, k_1 and the other I_2, k_2 . Their common natural frequency is then

$$\omega_n = \sqrt{\frac{k_1}{I_1}} = \sqrt{\frac{k_2}{I_2}} \quad (9.28a)$$

The spring constants of the shaft segments are each found from $k_t = JG/l$ assuming that the J is constant across the node.

$$\sqrt{\frac{JG}{l_1 I_1}} = \sqrt{\frac{JG}{l_2 I_2}}$$

and
$$l_1 I_1 = l_2 I_2 = I_2 (l - l_1)$$

so
$$l_1 = \frac{I_2 l}{I_1 + I_2} \quad (9.28b)$$

Equation 9.28b allows the location of the node to be found. Substituting this expression into equation 9.28a gives

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_1}{I_1}} = \sqrt{\frac{JG}{l_1 I_1}} = \sqrt{\frac{JG}{l} \frac{I_1 + I_2}{I_1 I_2}} \\ \omega_n &= \sqrt{k_t \frac{I_1 + I_2}{I_1 I_2}} \quad \text{or} \quad \omega_n^2 = k_t \frac{I_1 + I_2}{I_1 I_2} \end{aligned} \quad (9.28c)$$

which defines the critical speed for torsional vibration in terms of the known mass properties of the two disks and the overall spring constant of the shaft.

The critical frequency needs to be avoided in any forcing functions applied to the shaft in order to avoid torsional resonance that will overload it. Devices attached to the shaft, such as piston engines or piston pumps, will have frequencies in their torque-time functions that correspond to the pulses of their operation multiplied by their rotational frequency. For example, a four-cylinder engine will have a strong forcing frequency component at four times its rpm. If this fourth harmonic coincides with the shaft's critical frequency, there could be a problem. When designing a shaft, the frequency characteristics of the driving and driven rotating devices attached to it must be taken into account along with their primary rotational frequency.

Multiple Disks on a Common Shaft

Two disks on a common shaft have one node and one torsional natural frequency. Three disks will have two nodes and two natural frequencies. This pattern will hold for any number of disks, assuming in all cases that the masses of the disks dominate the shaft mass, allowing it to be ignored. N disks will have $N - 1$ nodes and natural frequencies. The degree of the equation for the natural frequencies will also be $N - 1$ if we consider the variable to be ω_n^2 rather than ω_n . Note that equation 9.28c for two masses is linear with this assumption. The equation for three masses is quadratic in ω_n^2 and for four masses is a cubic in ω_n^2 .

For the three-mass case, the squares of the natural frequencies are the two roots of

$$I_1 I_2 I_3 (\omega_n^2)^2 - [k_2(I_1 I_2 + I_1 I_3) + k_1(I_2 I_3 + I_1 I_3)] \omega_n^2 + k_1 k_2 (I_1 + I_2 + I_3) = 0 \quad (9.29)$$

Higher-order polynomials can be derived for additional masses and an iterative root-finding method can be used to solve them.

Approximate methods are also available to solve for the torsional natural frequencies with any number of masses. These allow the shaft mass to be easily accounted for if desired by breaking it into discrete masses. Holzer's method is commonly used for torsional as well as for lateral vibration of shafts. See reference 10 or any vibrations text for a derivation and discussion of these methods. Space does not allow a complete treatment of them here.

Controlling Torsional Vibrations

When shafts are long and/or have a number of masses distributed along their length, torsional vibrations can be a serious design problem. Internal-combustion engine crankshafts are an example. The crank-throw geometry severely reduces the torsional stiffness, which lowers their natural frequency. This, in combination with the presence of strong higher harmonics from the cylinder explosions in the torque, can lead to early failure from torsional fatigue. The straight-eight engine popular in the 1930s through 1940s was less successful than its straight-six cousin, due in part to the problems of torsional vibrations in the long, eight-throw crankshaft. The vee-eight engine with its shorter, stiffer, four-throw crankshaft has completely supplanted the straight-eight. Even in these shorter engines, torsional crankshaft vibrations can be a problem.

Several methods can be used to counter the effects of an unwanted correspondence between the forcing frequencies and the system natural frequencies. The first line of defense is to redesign the mass and stiffness properties of the system to get the critical frequencies as far above the highest forcing frequency as possible. This usually involves increasing stiffness while removing mass, something not always easy to do. Effective use of geometry to obtain the maximum stiffness with a minimum of material is required. The term **specific stiffness** refers to the *stiffness-to-mass ratio* of an object. We want to maximize the specific stiffness to increase the natural frequencies. Finite element analysis can be very useful in refining a design's geometry to alter its natural frequencies because of the detailed information obtainable from that analysis.

Another approach is the addition of a **tuned absorber** to the system. A tuned absorber is a mass-spring combination added to the system whose presence alters the set of natural frequencies away from any dominant forcing frequencies. The system is effectively tuned away from the undesired frequencies. This approach can be quite effective in some cases and is used in linear-motion as well as in torsional systems.

A **torsional damper** is usually added to the end of an engine crankshaft to reduce its oscillations. This device, also called a Lanchester damper after its inventor, is a disk coupled to the shaft through an energy-absorbent medium such as rubber or oil. The oil-coupling provides viscous damping and the rubber has significant internal hysteresis damping. Its effect is to reduce the peak amplitude at resonance, as can be seen in Figure 9-26 (p. 548) for larger values of ζ . The reader is referred to reference 11 for more information on all of these methods.

EXAMPLE 9-8**Determining the Critical Frequencies of a Shaft**

- Problem** Find the shaft whirl and torsional critical frequencies of the shaft in Example 9-2 and compare them to its forcing frequency.
- Given** The steel shaft dimensions are 0.875-in dia for 1.5 in, 0.750-in dia for 3.5 in, 0.669-in dia for 1.5 in, and 0.531-in dia for 1.5 in. Its rotational speed is 1 725 rpm. Shaft supports are at 0 and 5 in on an 8-in-long shaft. The steel gear weighs 10 lb and acts at $z = 2$ in. It has a mass moment of inertia of 0.23 lb-in-sec². The aluminum sheave weighs 3 lb and acts at $z = 6.75$ in. It has a mass moment of inertia of 0.07 lb-in-sec².
- Assumptions** The static deflection of the shaft due to the weights of gear and sheave will be used as an estimate for Rayleigh's method, but the gear and sheave weights will be applied in the directions that give the largest static deflection. The shaft weight will be ignored.
- Solution** See Figures 9-5 (p. 515) and 9-32.

- The deflection of the stepped shaft is found by the same technique used in Example 9-3 (p. 522). In this instance, the loads are taken as just the weights of the two disks. But, we will consider the gear weight-force to act downward and the sheave weight-force to act upward, since that arrangement better represents the dynamic situation where the inertia forces act outward from the axis in whatever direction increases the deflection. If we directed both weight forces downward in this case, we would get a smaller maximum deflection and a different curve shape than that of the dynamic deflection. Figure 9-32 shows the applied weight forces and the deflection curve for this shaft. The magnitude of the deflection at the gear is $6.0E-5$ in and at the sheave is $1.25E-4$ in. These values are needed in equation 9.25c (p. 550).
- Calculate the critical frequency for shaft whirl from equation 9.25c:

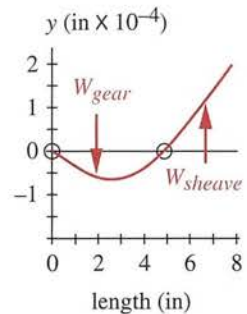
$$\omega_n = \sqrt{g \frac{\sum_{i=1}^n W_i \delta_i}{\sum_{i=1}^n W_i \delta_i^2}} = \sqrt{386 \frac{10(6.0E-5) + 3(1.25E-4)}{10(6.0E-5)^2 + 3(1.25E-4)^2}} = 2\,131 \text{ rad/sec} \quad (a)$$

Note that the magnitudes of the weight forces and their corresponding deflections are all taken as positive regardless of their vector directions in the static deflection case.

- Compare the critical whirl frequency to the forcing frequency.

$$\frac{\omega_n}{\omega_f} = \frac{30/\pi(2\,131 \text{ rad/sec})}{1\,725 \text{ rpm}} = \frac{20\,350 \text{ rpm}}{1\,725 \text{ rpm}} = 11.8 \quad (b)$$

This is a very comfortable margin. If, in this example, the shaft weight is included in both the deflection calculation and the critical-frequency calculation, the critical frequency becomes 20 849 rpm, which is 12.1 times the forcing frequency. Even with the relatively light disks on this shaft, ignoring the shaft weight does not introduce a large error. Both of these values from Rayleigh's method are higher than the actual natural frequency. The reader can examine their differences in the models supplied by simply changing the density of the shaft material from 0.28 lb/in³ (for

**FIGURE 9-32**

Static Deflection Due to Weight Forces of Disks on Shaft Oriented to Give the Maximum Deflection of a Shape Similar to the Dynamic Deflection

steel) to zero to eliminate the shaft's effect from the computation. The models include the shaft weight in the computation by dividing it into 50 increments over its length.

- 4 To find the torsional critical frequency of a stepped shaft requires that an effective spring constant for the combined stepped sections be found. The shaft portion of interest is between the sheave and gear. The spring constant of any one section is

$$k_t = \frac{GJ}{l} = \frac{G\pi d^4}{32l} \text{ lb-in/rad}$$

$$k_{t_1} = \pi \frac{11.5E6(0.75)^4}{32(3.5)} = 102\,065 \text{ lb-in/rad} \quad (c)$$

$$k_{t_2} = \pi \frac{11.5E6(0.669)^4}{32(1.5)} = 150\,769 \text{ lb-in/rad}$$

Since the shaft sections all have the same torque but different deflections (which sum to the total deflection as shown in equation 9.9b on p. 521), they act as springs in series. The effective spring constant $k_{t_{eff}}$ of the portion of stepped shaft between the two torque loads is found from equation 9.9d:

$$\frac{1}{k_{t_{eff}}} = \frac{1}{k_{t_1}} + \frac{1}{k_{t_2}} = \frac{1}{102\,065} + \frac{1}{150\,769}$$

$$k_{t_{eff}} = 60\,863 \text{ lb-in/rad} \quad (d)$$

- 5 The critical torsional frequency is found from equation 9.28c (p. 555).

$$\omega_n = \sqrt{k_{t_{eff}} \frac{I_1 + I_2}{I_1 I_2}} = \sqrt{60\,863 \frac{0.23 + 0.07}{(0.23)(0.07)}} = 1\,065 \text{ rad/sec} \quad (e)$$

- 6 Compare the critical torsional frequency to the forcing frequency.

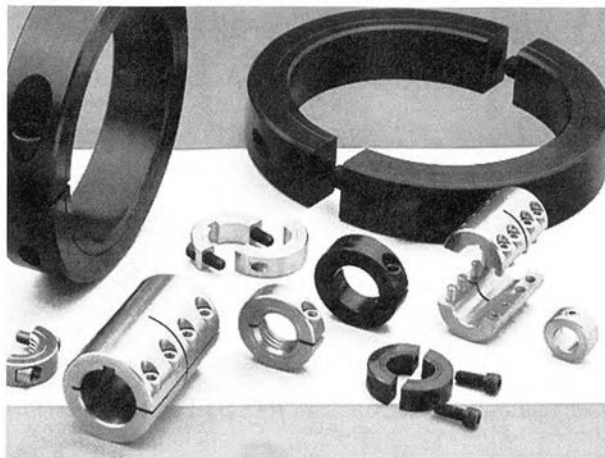
$$\frac{\omega_n}{\omega_f} = \frac{30/\pi(1\,065 \text{ rad/sec})}{1\,725 \text{ rpm}} = \frac{10\,170 \text{ rpm}}{1\,725 \text{ rpm}} = 5.9 \quad (f)$$

This is an acceptable margin.

- 7 The files EX09-08 can be found on the CD-ROM.

9.15 COUPLINGS

A wide variety of commercial shaft couplings are available, ranging from simple keyed, rigid couplings to elaborate designs that utilize gears, elastomers, or fluids to transmit the torque from one shaft to another or to other devices in the presence of various types of misalignment. Couplings can be roughly divided into two categories, rigid and compliant. Compliant in this context means that the coupling can absorb some misalignment between the two shafts and rigid implies that no misalignment is allowed between the connected shafts.

**FIGURE 9-33**

Various Types and Sizes of Rigid Shaft Couplings *Courtesy of Ruland Manufacturing Inc., Watertown, Mass.*

Rigid Couplings

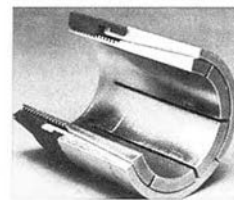
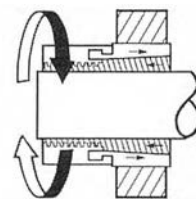
Rigid couplings lock the two shafts together allowing no relative motion between them, though some axial adjustment is possible at assembly. These are used when accuracy and fidelity of torque transmission are of paramount importance, as, for example, when the phase relationship between driver and driven device must be accurately maintained. Automated production machinery driven by long line-shafts often uses rigid couplings between shaft sections for this reason. Servomechanisms also need zero backlash connections in the drive train. The trade-off is that the alignment of the coupled shaft axes must be adjusted with precision to avoid introducing large side forces and moments when the coupling is clamped in place.

Figure 9-33 shows some examples of commercial rigid couplings. There are three general types: setscrew couplings, keyed couplings, and clamp couplings.

SETScrew COUPLINGS use a hard setscrew that digs into the shaft to transmit both torque and axial loads. These are not recommended for any but light-load applications and can loosen with vibration.

KEYED COUPLINGS use standard keys as discussed in an earlier section and can transmit substantial torque. Setscrews are often used in combination with a key, the screw being located 90° from the key. For proper holding against vibration, a cup-point setscrew is used to dig into the shaft. For better security, the shaft should be dimpled with a shallow drilled hole under the setscrew to provide a mechanical interference against axial slip rather than relying on friction.

CLAMP COUPLINGS are made in several designs, the most common being one- or two-piece split couplings that clamp around both shafts and transmit torque through friction as shown in Figure 9-33. A taper-lock coupling uses a split-tapered collet which is squeezed between shaft and the tapered coupling housing to clamp the shaft as shown in Figure 9-34.

**FIGURE 9-34**

A Trantorque Taper-Lock Coupling - Courtesy of Fenner Manheim, Manheim, Pa. 17545

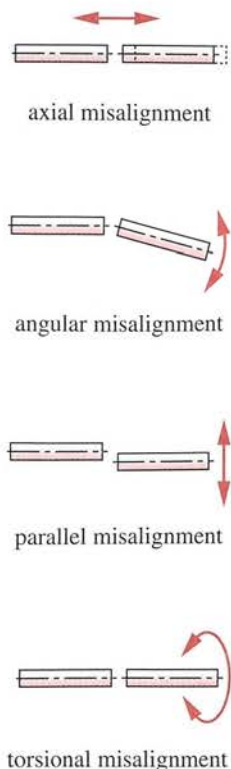


FIGURE 9-35

Types of Shaft
Misalignment



FIGURE 9-37

A Flexible Disk Coupling
Courtesy of Zero-Max,
Minneapolis, Minn. 55441

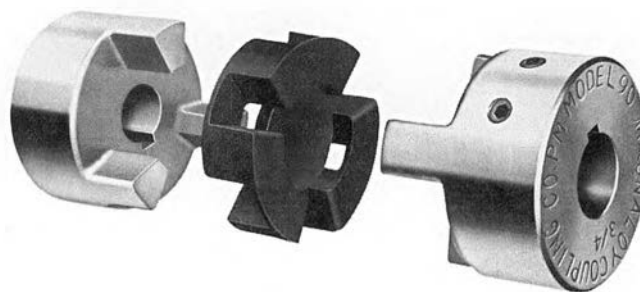


FIGURE 9-36

Exploded View of a Jaw Coupling Showing Jaws and Elastomer Insert Courtesy of Magnaloy Coupling Company, Alpena, Mich. 49707

Compliant Couplings

A shaft as a rigid body has six potential degrees of freedom (DOF) with respect to a second shaft. However, due to symmetry only four of these DOF are of concern. They are **axial**, **angular**, **parallel**, and **torsional** misalignment, as shown in Figure 9-35. These can occur singly or in combination and may be present at assembly due to manufacturing tolerances or may occur during operation due to the relative motions of the two shafts. The final driveline of an automobile has relative motion between the ends of the driveshaft. The drive end is affixed to the frame and the driven end is on the road. The frame and road are separated by the car's suspension, so the driveshaft couplings must absorb both angular and axial misalignment as the car traverses bumps.

Unless care is taken to align two adjacent shafts, there can be axial, angular, and parallel misalignment in any machinery. Torsional misalignment occurs dynamically when a driven load attempts to lead or lag the driver. If the coupling allows any torsional clearance, there will be backlash when the torque reverses sign. This is undesirable if accurate phasing is needed, as in servomechanisms. Torsional compliance in a coupling may be desirable if large shock loads or torsional vibrations must be isolated from the driver.

Numerous designs of compliant couplings are manufactured and each offers a different combination of features. The designer can usually find a suitable coupling available commercially for any application. Compliant couplings can be roughly divided into several subcategories, which are listed in Table 9-7 along with some of their characteristics. The torque ratings are not shown, as these vary widely with size and materials. Various size couplings can handle power levels from subfractional horsepower to thousands of horsepower.

JAW COUPLINGS have two (often identical) hubs with protruding jaws, as shown in Figure 9-36. These jaws overlap axially and interlock torsionally through a compliant insert of rubber or soft-metal material. The clearances allow some axial, angular, and parallel misalignment, but can also allow some undesirable backlash.

FLEXIBLE-DISK COUPLINGS are similar to jaw couplings in that their two hubs are connected by a compliant member (disk) of elastomeric or metallic-spring material, as shown in Figure 9-37. These allow axial, angular, and parallel misalignment, with some torsional compliance but little or no backlash.

GEAR AND SPLINE COUPLINGS use straight or curved external gear teeth in mesh with internal teeth, as shown in Figure 9-38. These can allow substantial axial movement between shafts and, depending on the tooth shapes and clearances, can absorb small angular and parallel misalignment as well. They have high torque capacity due to the number of teeth in mesh.

HELICAL AND BELLOW COUPLINGS are one-piece designs that use their elastic deflections to allow axial, angular, and parallel misalignment with little or no backlash. Helical couplings (Figure 9-39 and Chapter 9 title page photograph) are made from a solid metal cylinder cut with a helical slit to increase its compliance. Metal-bellows couplings (Figure 9-40) are made of thin sheet metal by welding a series of cupped washers together, by hydraulically forming a tube into the shape, or by electroplating a thick coating on a mandrel. These couplings have limited torque capability compared to other designs but offer zero backlash and high torsional stiffness in combination with axial, angular, and parallel misalignment.

LINKAGE COUPLINGS or Schmidt couplings (Figure 9-41) connect two shafts through a network of links that allow significant parallel misalignment with no side loads or torque losses and no backlash. Some designs allow small amounts of angular and axial misalignment as well. These couplings are often used where large parallel adjustments or dynamic motions are needed between shafts.

UNIVERSAL JOINTS are of two common types, the Hooke coupling (Figure 9-42), which does not have constant velocity (CV), and the Rzeppa coupling, which does. Hooke couplings are generally used in pairs to cancel their velocity error. Both types can handle very large angular misalignment, and in pairs provide large parallel offsets

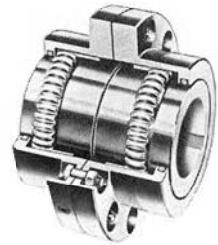


FIGURE 9-38

A Flexible Gear Coupling
Courtesy of Amerigear/Zurn
Industries, Inc., Erie, Pa. 16514

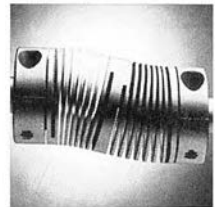


FIGURE 9-39

A Helical Coupling Courtesy of
Helical Products Co., Inc., Santa
Maria, Calif. 93456



FIGURE 9-40

Metal-Bellows Coupling
Courtesy of Senior Flexonics Inc.,
Metal Bellows Division, Sharon,
Mass. 02067

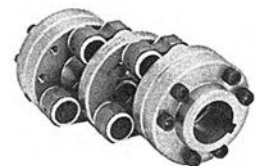


FIGURE 9-41

Schmidt Off-set Coupling
Courtesy of Zero-Max/Helland
Co., Minneapolis, Minn. 55441

Table 9-7 Characteristics of Various Types of Couplings

Class	Misalignment Tolerated				Comments
	Axial	Angular	Parallel	Torsional	
Rigid	large	none	none	none	requires accurate alignment
Jaw	slight	slight ($< 2^\circ$)	slight (3% d)	moderate	shock absorption— significant backlash
Gear	large	slight ($< 5^\circ$)	slight ($< 1/2\%$ d)	none	slight backlash—large torque capacity
Spline	large	none	none	none	slight backlash—large torque capacity
Helical	slight	large (20°)	slight ($< 1\%$ d)	none	one piece - compact—no backlash
Bellows	slight	large (17°)	moderate (20% d)	none	subject to fatigue failure
Flexible disk	slight	slight (3°)	slight (2% d)	slight to none	shock absorption—no backlash
Linkage (Schmidt)	none	slight (5°)	large (200% d)	none	no backlash—no side loads on shaft
Hooke	none	large	large (in pairs)	none	slight backlash—speed variation unless used in pairs
Rzeppa	none	large	none	none	constant velocity

**FIGURE 9-42**

A Hooke's Coupling Courtesy
of Lovejoy, Inc., Downers Grove,
Ill. 60515

as well. These are used in automobile driveshafts, a pair of Hooke couplings in a rear-drive driveshaft and Rzeppas (called CV joints) in a front-drive automobile.

The variety of couplings available makes it necessary for the designer to seek more detailed information on their capabilities from their manufacturers, who are always willing to help in the selection of the proper type of coupling for any application. Manufacturers can often supply test data on the load and alignment capabilities of their particular couplings.

9.16 CASE STUDY

We will now address the design of shafts in one of the Case Study assemblies that were defined in Chapter 8.

Designing Driveshafts for a Portable Air Compressor

The preliminary design of this device is shown in Figure 8-1 (repeated opposite). There are two shafts, input and output. The output shaft will have the larger torque in this case since it is at the slower speed of 1 500 rpm. The torque on that shaft was defined in the earlier Case Study 7A in Chapter 8 and is shown in Figure 8-3 (repeated on p. 564). Since this is a time-varying torque, the shaft must be designed for fatigue loading. In addition to the torque on the shafts, there will be side loads from the gears that apply bending moments, making a combined-loading situation. Note in Figure 8-1 that the shafts are shown as short, being only long enough to accommodate the gear and bearings. This is done to minimize the bending moments from the gear forces.

Since the gears are yet to be designed, we will have to make some assumptions about their diameters and thicknesses to do a preliminary design of the shafts. Later selection of bearings may also dictate some changes to our shaft design. This is typical of design problems, since all of their elements interact. Iteration is necessary to refine all the elements' designs.

CASE STUDY 7B

Preliminary Design of Shafts for a Compressor Drive Train

Problem	Determine reasonable sizes for the input and output shafts of the gearbox in Figure 8-1 (opposite) based on the loadings defined in Case Study 7A and specify a suitable type of coupling.
Given	The torque-time function on the output shaft is as shown in Figure 8-3 (p. 564). The required gear ratio is a 2.5:1 reduction in velocity from the input to the output shaft.
Assumptions	Try an input gear (pinion) diameter of 4 in and output gear diameter of 10 in, both of 2 in thickness and 20° pressure angle. Ball bearings of standard diameters will be used on all shafts.
Solution	See Figures 8-1, 8-3, and 9-43.

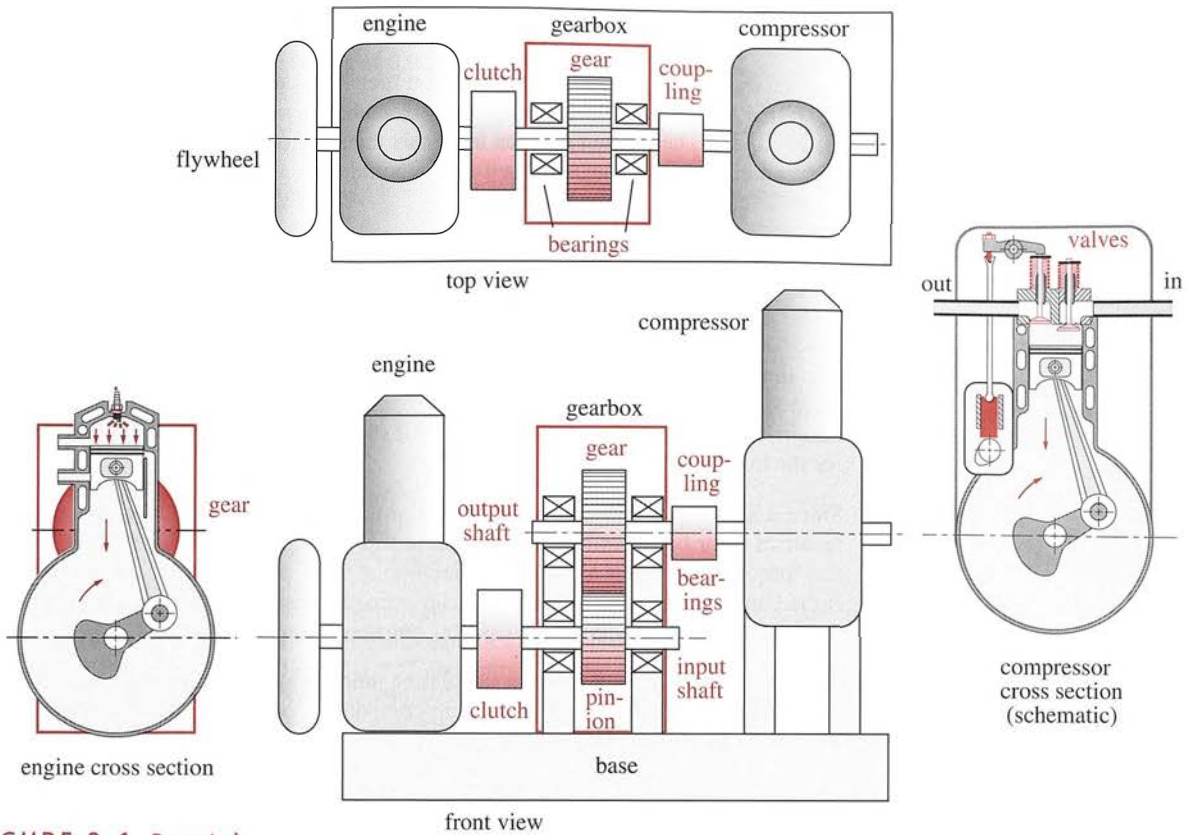


FIGURE 8-1 Repeated

Preliminary Design Schematic of Gasoline-Engine-Powered Portable Air Compressor, Gearbox, Couplings, Shafts, and Bearings

- The time-varying torque on the output shaft is defined in Figure 8-3 (repeated overleaf) as varying between -175 and $+585$ lb-in. From these data and the assumed gear diameters, we can determine the forces at the gear mesh that are felt by the shaft. Figure 9-43 shows a free-body diagram of a gearset. Because of the pressure angle ϕ between the gears, there are both radial and tangential components of force at the gear mesh. The tangential component is found from the known torque and the assumed gear radius:

$$F_{t_{\max}} = \frac{T_{\max}}{r_g} = \frac{585 \text{ lb-in}}{5 \text{ in}} = 117 \text{ lb}$$

$$F_{t_{\min}} = \frac{T_{\min}}{r_g} = \frac{-175 \text{ lb-in}}{5 \text{ in}} = -35 \text{ lb}$$

- The maximum and minimum resultant forces are found from:

$$F_{\max} = \frac{F_{t_{\max}}}{\cos \phi} = \frac{117 \text{ lb}}{\cos 20^\circ} = 124.5 \text{ lb}$$

$$F_{\min} = \frac{F_{t_{\min}}}{\cos \phi} = \frac{-35 \text{ lb}}{\cos 20^\circ} = -37.25 \text{ lb}$$

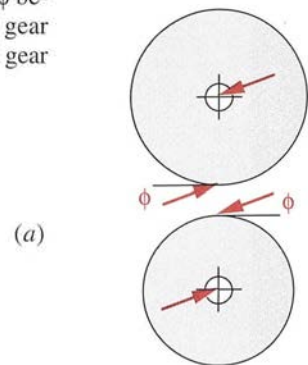


FIGURE 9-43

Forces on a Gearset

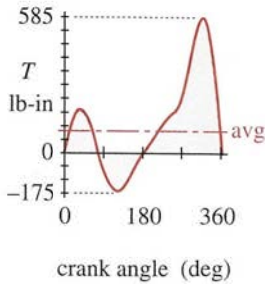


FIGURE 8-3 Repeated

Total Torque-Time Function at Crankshaft with Constant ω

Note that these forces will be the same on the input shaft, whose torque is 0.4 times that of the output shaft because of the 1:2.5 gear ratio.

- 3 The maximum and minimum moments on the shaft can now be found. We will assume that the gears will be centered between the simply supported bearings that are set 4 in apart. The bearing reaction forces are then half of the gear forces and the bending moments peak in the center with a magnitude of

$$M_{max} = F_{max} \frac{l}{4} = 124.5 \frac{4}{4} = 124.5 \text{ lb-in}$$

$$M_{min} = F_{min} \frac{l}{4} = -37.25 \frac{4}{4} = -37.25 \text{ lb-in} \quad (c)$$

- 4 The shafts will ultimately need to be stepped for the bearings at each end and either stepped or snap-ringed to provide axial location for the gears. At this stage of the design, we will assume a constant-diameter shaft in order to get an approximate size for the torque and moment loading.

Since a keyway will probably be needed at the gear, assume a stress-concentration factor of 3 for both bending and torsion at that critical location where both moment and torque components are largest. (See Figure 9-16, p. 529.) After the gears are designed and the bearings selected, we can refine the design, including stepped shoulders and using more accurate stress-concentration factors.

- 5 The loading is a combination of a fluctuating moment and a fluctuating torque that are synchronous. The mean and alternating components of both moment and torque are needed for the stress calculations.

$$M_m = \frac{M_{max} + M_{min}}{2} = \frac{124.5 - 37.25}{2} = 43.6 \text{ lb-in}$$

$$M_a = \frac{M_{max} - M_{min}}{2} = \frac{124.5 + 37.25}{2} = 80.9 \text{ lb-in} \quad (d)$$

$$T_m = \frac{T_{max} + T_{min}}{2} = \frac{585 - 175}{2} = 205 \text{ lb-in}$$

$$T_a = \frac{T_{max} - T_{min}}{2} = \frac{585 + 175}{2} = 380 \text{ lb-in} \quad (e)$$

- 6 A trial material needs to be selected for the computations. We will first try an inexpensive, low-carbon, cold-rolled steel such as SAE 1018 with $S_{ut} = 64$ kpsi and $S_y = 54$ kpsi. Though not exceptionally strong, this material has low notch sensitivity, which will be an advantage given the stress concentrations. Calculate the uncorrected endurance strength using equation 6.5 (pp. 324–326):

$$S_e' = 0.5S_{ut} = 0.5(64\,000) = 32\,000 \text{ psi} \quad (f)$$

This must be reduced by various factors to account for differences between the part and the test specimen.

$$S_e = C_{load} C_{size} C_{surf} C_{temp} C_{reliab} S_e'$$

$$S_e = (1)(1)(0.84)(1)(1)(32\,000) \approx 27\,000 \text{ psi} \quad (g)$$

The loading is bending and torsion, so C_{load} is 1. Since we don't yet know the part size, we will temporarily assume $C_{size} = 1$ and adjust it later. C_{surf} is chosen for a machined finish from either Figure 6-26 (p. 328) or equation 6.7e (p. 329). The temperature is not elevated, so $C_{temp} = 1$, and we assume 50% reliability with $C_{reliab} = 1$.

- 7 The notch sensitivity of the material is found from either equation 6.13 (p. 339) or Figure 6-36 (pp. 340–341) and for bending is $q = 0.50$, and for torsion is $q = 0.57$, both for an assumed notch radius of 0.01 in.
- 8 The bending-fatigue stress-concentration factor is found from equation 6.11b (p. 339) using the assumed geometric stress-concentration factor.

For bending stress in the keyway:

$$K_f = 1 + q(K_t - 1) = 1 + 0.50(3.0 - 1) = 2.00 \quad (h)$$

For torsional stress in the keyway:

$$K_f = 1 + q(K_t - 1) = 1 + 0.57(3.0 - 1) = 2.15 \quad (i)$$

- 9 From equation 6.17 (p. 360) we find that in this case, the same factor should be used on the mean stress components:

$$\begin{aligned} K_{fm} &= K_f = 2.00 \\ K_{fsm} &= K_{fs} = 2.15 \end{aligned} \quad (j)$$

- 10 The shaft diameter can now be found from equation 9.8 (p. 514) using an assumed safety factor of 3 to account for the uncertainties in this preliminary design. Note that the ASME equation (9.6) on p. 512 cannot be safely used in this case since it assumes constant torque. The more general, modified-Goodman line approach of equation 9.8 must be used.

$$\begin{aligned} d_{\text{output}} &= \left\{ \frac{32N_{sf}}{\pi} \left[\frac{\sqrt{(k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2}}{S_f} + \frac{\sqrt{(k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2}}{S_{ut}} \right] \right\}^{\frac{1}{3}} \\ &= \left\{ \frac{32(3)}{\pi} \left[\frac{\sqrt{[2.00(80.9)]^2 + \frac{3}{4}[2.15(380)]^2}}{27\,000} + \frac{\sqrt{[2.00(43.6)]^2 + \frac{3}{4}[2.15(205)]^2}}{64\,000} \right] \right\}^{\frac{1}{3}} \\ d_{\text{output}} &= 1.00 \text{ in} \end{aligned} \quad (k)$$

So a 1-in nominal shaft diameter seems acceptable for the output shaft.

- 11 The input shaft has the same mean and alternating bending moments as the output shaft, but its torque is only 40% of the output shaft's. The mean and alternating torques on it are 82 and 152 lb-in. When these are put in equation 9.8 with all other factors the same, a smaller shaft diameter results.

$$\begin{aligned} d_{\text{input}} &= \left\{ \frac{32(3)}{\pi} \left[\frac{\sqrt{[2.00(80.9)]^2 + \frac{3}{4}[2.15(152)]^2}}{27\,000} + \frac{\sqrt{[2.00(43.6)]^2 + \frac{3}{4}[2.15(82)]^2}}{64\,000} \right] \right\}^{\frac{1}{3}} \\ d_{\text{input}} &= 0.768 \text{ in} \end{aligned} \quad (l)$$

The input shaft can then be a nominal 0.781-in diameter, which is a stock size.

- 12 The couplings between the engine and input shaft and between the output shaft and compressor must be able to accommodate some angular and parallel misalignment due to tolerances in the mounting of the three subassemblies of engine, gearbox, and compressor. Torsional compliance in the couplings would also serve to absorb some of the shock associated with the torque reversal seen in Figure 8-3 (p. 564).

With these constraints, a jaw-type coupling with an elastomer insert might be a good choice for both couplings. The backlash inherent in these couplings would not be a problem, since the phasing of the three subassemblies is not critical in this application. A flexible-disk coupling could also be used. A rigid coupling would require more accurate mounting of the subassemblies than is otherwise warranted. See Table 9-7 (p. 561) for information on coupling characteristics.

- 13 The files CASE7B-1 and CASE7B-2 are on the CD-ROM.
-

9.17 SUMMARY

Shafts are used in all rotating machinery. Steel is the usual choice of material to obtain high stiffness for low deflections. Shafts may be of soft, low-carbon steel or of medium- to high-carbon steel for higher strength or if a hard surface finish is needed for wear resistance. Machine shafts usually have stepped shoulders for axial location of attached elements such as bearings, gears, or sprockets (sheaves). These shoulders create stress concentrations that must be considered in the stress analysis. Keyways or interference fits also create stress concentrations.

The loading on shafts is usually a combination of torsion and bending, either or both of which can be time varying. The general loading case of fluctuating torque combined with fluctuating bending requires a modified-Goodman diagram approach to its failure analysis. For the common case of the torque and bending moment being related through common forces, the modified-Goodman approach is captured in equation 9.8 (p. 514) which provides a design tool to determine a shaft diameter for known fluctuating loads, stress concentrations, material strengths, and a chosen safety factor. The ASME shaft design equation (9.6 on p. 512) only applies to cases of constant torque with a constant applied bending moment that is fully reversed due to shaft rotation. Equation 9.6 can be used only in situations that match this loading limitation. It is always better to use the general design equation 9.8 (p. 514) that accounts for fluctuating moments and torques if they are present.

Several techniques or devices such as keys, splines, and interference fits are commonly used to attach elements to shafts. Keys are standardized to the shaft diameter. Consult the ANSI standard or reference 3 for data on size ranges not reproduced in this chapter. Splines provide greater torque capacity than keys. Interference fits may be by direct press fit or by thermally expanding or shrinking one or both members. Very high stresses can be created by these techniques, possibly failing the part during assembly.

Flywheels are used when some torque or velocity smoothing is needed. The flywheel must be sized to give the desired coefficient of speed fluctuation and then checked for stress at the operating speed. The maximum stresses in a flywheel occur at the in-

side diameter. A maximum safe speed must be determined, as the stresses increase with the square of the rotational speed. When a flywheel fails while spinning, it typically flies apart and can cause serious injury.

All spinning shafts will have critical frequencies at which they will resonate with large deflections, causing failure. The fundamental lateral and torsional frequencies will be different and both must be avoided in operation by keeping the rotational speed well below the lowest critical frequency of the shaft.

A wide variety of shaft couplings are commercially available. Some types and their characteristics are briefly discussed in this chapter. The manufacturers should be consulted for more complete and definitive information.

Important Equations Used in This Chapter

See the referenced sections for information on the proper use of these equations.

Power-Torque Relationship (Section 9.4):

$$P = T\omega \quad (9.1a)$$

ASME Shaft-Design Equation (Section 9.8):

$$d = \left\{ \frac{32N_f}{\pi} \left[\left(k_f \frac{M_a}{S_f} \right)^2 + \frac{3}{4} \left(\frac{T_m}{S_y} \right)^2 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}} \quad (9.6b)$$

General Shaft-Design Equation (Section 9.8):

$$d = \left\{ \frac{32N_f}{\pi} \left[\frac{\sqrt{(k_f M_a)^2 + \frac{3}{4}(k_{fs} T_a)^2}}{S_f} + \frac{\sqrt{(k_{fm} M_m)^2 + \frac{3}{4}(k_{fsm} T_m)^2}}{S_{ut}} \right] \right\}^{\frac{1}{3}} \quad (9.8)$$

Shaft Torsional Deflection (Section 9.9):

$$\theta = \frac{Tl}{GJ} \quad (9.9a)$$

Pressure Generated by an Interference Fit (Section 9.11):

$$p = \frac{0.5\delta}{\frac{r}{E_o} \left(\frac{r_o^2 + r^2}{r_o^2 - r^2} + \nu_o \right) + \frac{r}{E_i} \left(\frac{r^2 + r_i^2}{r^2 - r_i^2} - \nu_i \right)} \quad (9.14a)$$

Tangential Stresses in Shaft and Hub of an Interference Fit (Section 9.11):

$$\sigma_{t_{shaft}} = -p \frac{r^2 + r_i^2}{r^2 - r_i^2} \quad (9.15a)$$

$$\sigma_{t_{hub}} = p \frac{r_o^2 + r^2}{r_o^2 - r^2} \quad (9.16a)$$

Energy Stored in a Spinning Flywheel (Section 9.13):

$$E_k = \frac{1}{2} I_m \omega^2 \quad (9.17a)$$

Mass Moment of Inertia of a Solid-Disk Flywheel (Section 9.13):

$$I_m = \frac{\pi \gamma}{2g} (r_o^4 - r_i^4) t \quad (9.17d)$$

Flywheel Inertia Needed for a Chosen Coefficient of Fluctuation (Section 9.13):

$$E_k = \frac{1}{2} I_s (2\omega_{avg}) (C_f \omega_{avg})$$

$$I_s = \frac{E_k}{C_f \omega_{avg}^2} \quad (9.22)$$

Tangential Stress in a Spinning Flywheel (Section 9.13):

$$\sigma_t = \frac{\gamma}{g} \omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right) \quad (9.23a)$$

Natural Frequency of Single-Degree-of-Freedom System (Section 9.14):

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{rad/sec} \quad (9.24a)$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{Hz} \quad (9.24b)$$

First Lateral Critical Frequency (Approximate) (Section 9.14):

$$\omega_n = \sqrt{g \frac{\sum_{i=1}^n m_i \delta_i}{\sum_{i=1}^n m_i \delta_i^2}} = \sqrt{g \frac{\sum_{i=1}^n (W_i/g) \delta_i}{\sum_{i=1}^n (W_i/g) \delta_i^2}} = \sqrt{g \frac{\sum_{i=1}^n W_i \delta_i}{\sum_{i=1}^n W_i \delta_i^2}} \quad (9.25c)$$

First Torsional Critical Frequency for Two Masses on Weightless Shaft (Section 9.14):

$$\omega_n = \sqrt{k_t \frac{I_1 + I_2}{I_1 I_2}} \quad \text{or} \quad \omega_n^2 = k_t \frac{I_1 + I_2}{I_1 I_2} \quad (9.28c)$$

9.18 REFERENCES

- 1 **R. C. Juvinall and K. M. Marshek**, *Fundamentals of Machine Component Design*. John Wiley & Sons: New York, p. 656, 1991.
- 2 **D. B. Kececioglu and V. R. Lalli**, "Reliability Approach to Rotating Component Design," *Technical Note TN D-7846*, NASA, 1975.

- 3 **V. C. Davies, H. J. Gough, and H. V. Pollard**, "Discussion to The Strength of Metals Under Combined Alternating Stresses," *Proc. of the Inst. Mech Eng.*, **131**(3): pp. 66–69, 1935.
- 4 **S. H. Loewenthal**, Proposed Design Procedure for Transmission Shafting Under Fatigue Loading, *Technical Note TM-78927*, NASA, 1978.
- 5 **H. J. Gough and H. V. Pollard**, The Strength of Metals Under Combined Alternating Stresses. *Proc. of the Inst. Mech Eng.*, **131**(3): pp. 3–103, 1935.
- 6 **E. Oberg and F. D. Jones**, eds. *Machinery's Handbook*, 17th ed., Industrial Press Inc.: New York, pp. 867–883, 1966.
- 7 **R. C. Peterson**, *Stress-concentration factors*. John Wiley & Sons: New York, pp. 266–267, 1974.
- 8 **E. Oberg et al**, *Machinery's Handbook*, 25th ed., Industrial Press Inc.: New York, pp. 2042–2070, 1996.
- 9 **M. Dimentberg et al.**, "Passage Through Critical Speed with Limited Power by Switching System Stiffness," AMD-Vol. 192 / DE-Vol. 78, *Nonlinear and Stochastic Dynamics*, ASME 1994.
- 10 **C. R. Mischke**, *Elements of Mechanical Analysis*. Addison Wesley: Reading, Mass., pp. 317–320, 1963.
- 11 **R. M. Phelan**, *Dynamics of Machinery*. McGraw-Hill: New York, pp. 178–196, 1967.
- 12 **R. L. Norton**, *Design of Machinery*. 3ed, McGraw-Hill: New York, p. 608, 2004.

Table P9-0†

Topic/Problem Matrix

9.4 Shaft Power

9-15, 9-16

9.8 Shaft Design**9-1, 9-2, 9-3, 9-8, 9-31, 9-32, 9-33, 9-34, 9-35****9.9 Shaft Deflection**

9-4, 9-5, 9-17b, 9-18, 9-21

9.10 Keys and Keyways

9-6, 9-7, 9-9, 9-10, 9-17a, 9-19, 9-20, 9-28, 9-29, 9-30, 9-36

9.12 Interference Fits

9-11, 9-37, 9-40

9.13 Flywheel Design

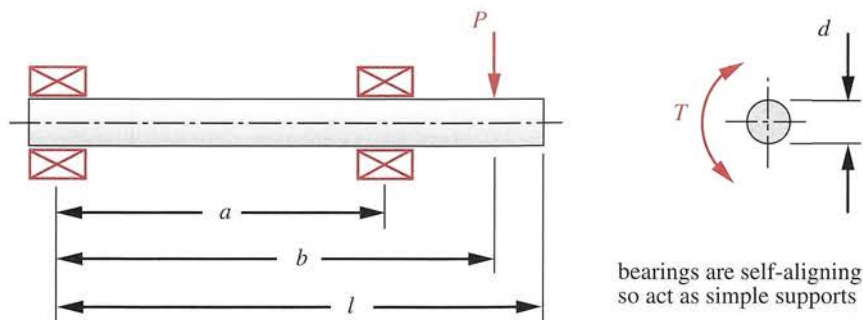
9-12, 9-38, 9-39

9.14 Critical Speeds

9-13, 9-14, 9-17c, 9-22, 9-23, 9-24, 9-25, 9-26, 9-27

9.19 PROBLEMS

- *†9-1** A simply supported shaft is shown in Figure P9-1. A constant-magnitude transverse load P is applied as the shaft rotates subject to a time-varying torque that varies from T_{min} to T_{max} . For the data in the row(s) assigned from Table P9-1, find the diameter of shaft required to obtain a safety factor of 2 in fatigue loading if the shaft is steel of $S_{ut} = 108$ kpsi and $S_y = 62$ kpsi. The dimensions are in inches, the force in pounds, and the torque in lb-in. Assume no stress concentrations are present.

**FIGURE P9-1**

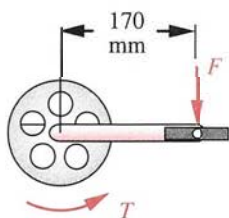
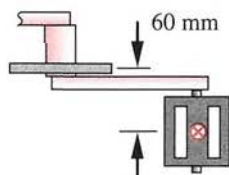
Shaft Design for Problems 9-1, 9-4, and 9-15

* Answers to these problems are provided in Appendix F.

† Problem numbers in *italics* are design problems. Problem numbers in **boldface** are extended from similar problems in earlier chapters with the same dash number.

Table P9-1 Data for Problems

Row	l	a	b	P or p	T_{min}	T_{max}
<i>a</i>	20	16	18	1 000	0	2 000
<i>b</i>	12	2	7	500	-100	600
<i>c</i>	14	4	12	750	-200	400
<i>d</i>	8	4	8	1 000	0	2 000
<i>e</i>	17	6	12	1 500	-200	500
<i>f</i>	24	16	22	750	1 000	2 000



Repeated

FIGURE P6-1

Problem 9-3

***9-2** A simply supported shaft is shown in Figure P9-2. A constant magnitude distributed unit load p is applied as the shaft rotates subject to a time-varying torque that varies from T_{min} to T_{max} . For the data in the row(s) assigned from Table P9-1, find the diameter of shaft required to obtain a safety factor of 2 in fatigue loading if the shaft is steel of $S_{ut} = 745$ MPa and $S_y = 427$ MPa. The dimensions are in cm, the distributed force in N/cm, and the torque in N-m. Assume no stress concentrations are present.

9-3 For the bicycle pedal arm assembly in Figure P6-1 (repeated here) assume a rider-applied force that ranges from 0 to 1500 N at each pedal each cycle. Design a suitable shaft to connect the two pedal arms and carry the sprocket against a step. Use the fatigue safety factor of 2 and a material with $S_{ut} = 500$ MPa. The shaft has a square detail on each end where it inserts into the pedal arms.

***9-4** Determine the maximum deflections in torsion and in bending of the shaft shown in Figure P9-1 for the data in the row(s) assigned in Table P9-1 if the steel shaft diameter is 1.75 in.

***9-5** Determine the maximum deflections in torsion and in bending of the shaft shown in Figure P9-2 for the data in the row(s) assigned in Table P9-1 if the steel shaft diameter is 4 cm.

***9-6** Determine the size of key necessary to give a safety factor of at least 2 against both shear and bearing failure for the design shown in Figure P9-3 using the data from the row(s) assigned in Table P9-1. Assume a shaft diameter of 1.75 in. The shaft is steel of $S_{ut} = 108$ kpsi and $S_y = 62$ kpsi. The key is steel of $S_{ut} = 88$ kpsi and $S_y = 52$ kpsi.

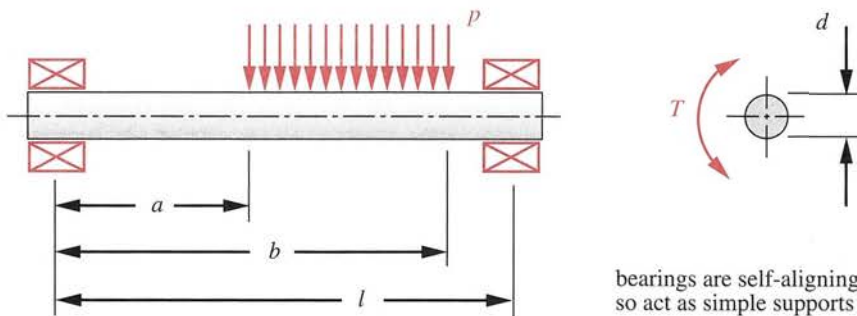


FIGURE P9-2

Shaft Design for Problems 9-2, 9-5, and 9-16

* Answers to these problems are provided in Appendix F. Problem numbers in *italics* are design problems. Problem numbers in **boldface** are extended from similar problems in earlier chapters with the same dash number.

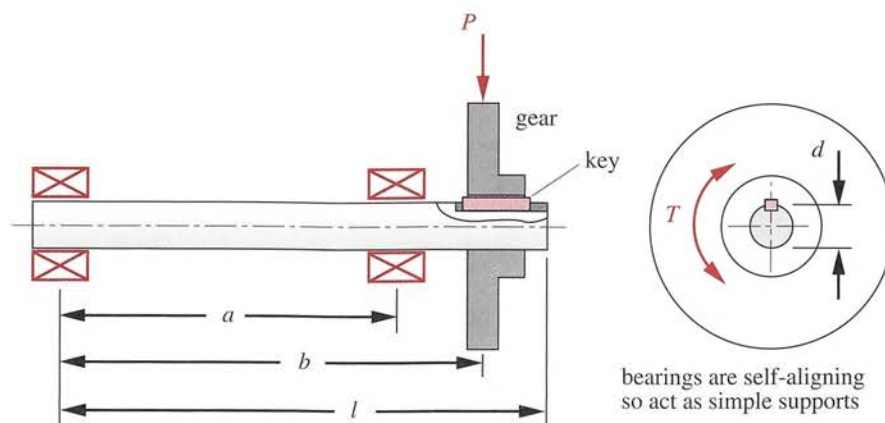


FIGURE P9-3

Shaft Design for Problems 9-6, 9-9, 9-11, and 9-12

- 9-7 Determine the size of key necessary to give a safety factor of at least 2 against both shear and bearing failure for the design shown in Figure P9-4 using the data from the row(s) assigned in Table P9-1. Assume a shaft diameter of 4 cm. The shaft is steel of $S_{ut} = 745$ MPa and $S_y = 427$ MPa. The key is steel of $S_{ut} = 600$ MPa and $S_y = 360$ MPa.
- *9-8 A paper machine processes rolls of paper having a density of 984 kg/m^3 . The paper roll is 1.50-m outside dia. (OD) $\times$ 22-cm inside diameter (ID) $\times$ 3.23-m long and is on a simply supported, hollow, steel shaft with $S_{ut} = 400$ MPa. Find the shaft ID needed to obtain a dynamic safety factor of 2 for a 10-year life if the shaft OD is 22 cm and the roll turns at 50 rpm with 1.2 hp absorbed.
- *9-9 Repeat Problem 9-1 taking the stress concentration at the keyway shown in Figure P9-3 into account.
- 9-10 Repeat Problem 9-2 taking the stress concentrations at the keyways shown in Figure P9-4 into account.
- *9-11 Determine the amount of diametral interference needed to provide a suitable interference fit for the 6-in-dia by 1-in-thick gear of Figure P9-3 using a shaft dia of

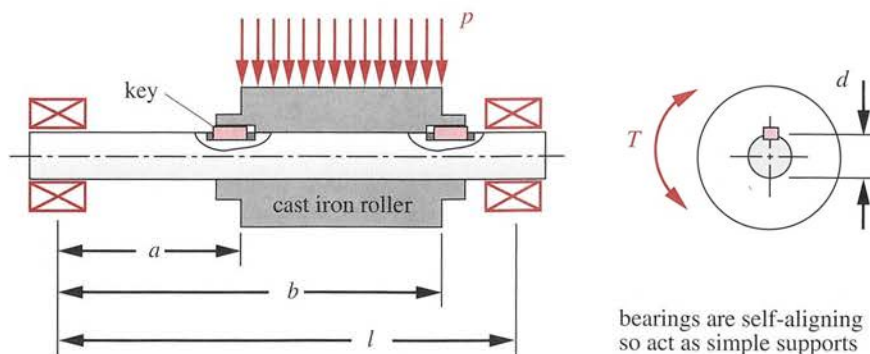


FIGURE P9-4

Shaft Design for Problems 9-7, 9-10, and 9-14

* Answers to these problems are provided in Appendix F. Problem numbers in *italics* are design problems. Problem numbers in **boldface** are extended from similar problems in earlier chapters with the same dash number.

1.75 in, such that the stresses in the hub and shaft will be safe and the torque from the assigned row(s) in Table P9-1 can be transmitted through the interference fit. Assume that $S_{ut} = 108$ kpsi and $S_y = 62$ kpsi.

- 9-12 Assume that the device shown keyed to the shaft of Figure P9-3 is a Class 50, cast iron flywheel of 20-in outside diameter and 1-in thickness. The hub is 4-in dia and 3-in thick. Determine the maximum speed at which it can safely be run using a safety factor of 2. Use the dimensions and other appropriate data from Problem 9-6. Consider the transverse force P to be zero in this case.
- *9-13 Determine the critical frequency of shaft whirl for the assembly shown in Figure P9-3 using the dimensions from the assigned row(s) of Table P9-1 and a steel shaft diameter of 2 in. Use the flywheel dimensions of Problem 9-12.
- 9-14 Determine the critical frequency of shaft whirl for the assembly shown in Figure P9-4 using the dimensions from the assigned row(s) of Table P9-1 and a steel shaft diameter of 4 cm. The cast iron roller diameter is 3 times the shaft diameter.
- *9-15 What are the maximum, minimum, and average power values for the shaft shown in Figure P9-1 for the data in the row(s) assigned in Table P9-1 if the shaft speed is 750 rpm?
- *9-16 What are the maximum, minimum, and average power values for the shaft shown in Figure P9-2 for the data in the row(s) assigned in Table P9-1 if the shaft speed is 50 rpm?
- *9-17 Figure P9-5 shows a roller assembly driven by a gear. The roller extends over 80% of length a and is centered in that dimension. The roller occupies 95% of the exposed shaft length between the bearing faces. The shaft is steel with $S_y = 427$ MPa and $S_{ut} = 745$ MPa. For the data in the assigned row(s) of Table P9-1, find:
- The safety factor against fatigue failure for a shaft diameter of 40 mm.
 - The maximum torsional deflection between gear and roller.
 - The torsional natural frequency of the shaft.
- *9-18 Figure P9-5 shows a roller assembly driven by a gear. The roller extends over 80% of length a and is centered in that dimension. The roller occupies 95% of the exposed shaft length between the bearing faces. For the data in the assigned row(s) of Table P9-1, find the maximum bending deflection of the 40-mm-dia shaft.

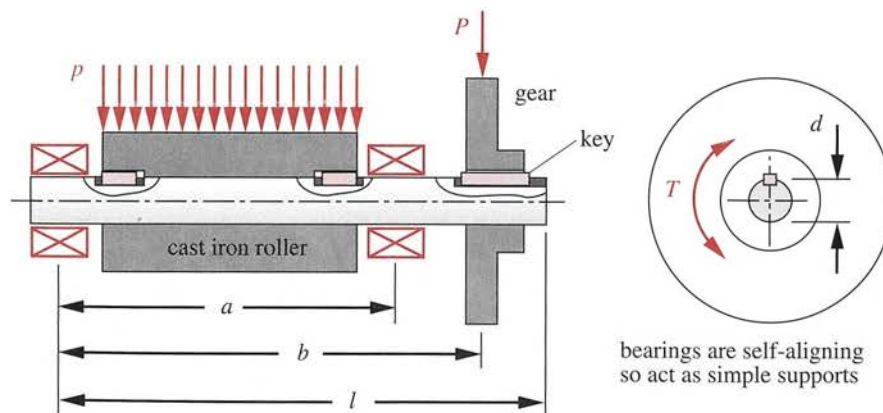
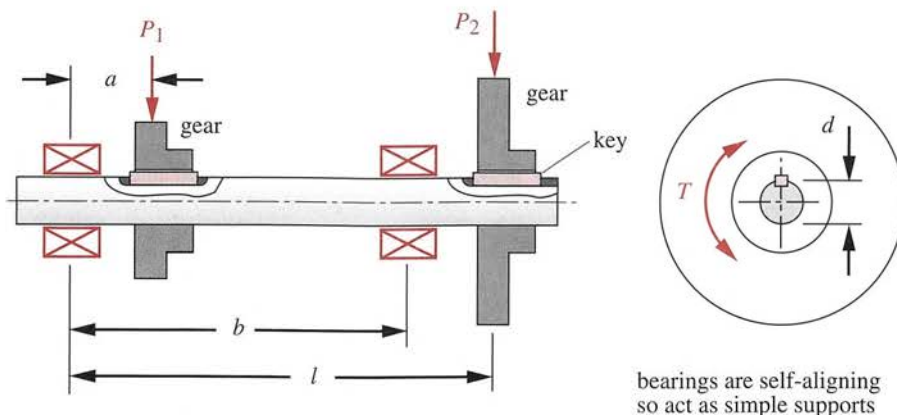


FIGURE P9-5

Shaft Design for Problems 9-17 and 9-18

* Answers to these problems are provided in Appendix F.



bearings are self-aligning
so act as simple supports

FIGURE P9-6

Shaft Design for Problem 9-19

- *9-19 Figure P9-6 shows two gears on a common shaft. Assume that the constant radial force P_1 is 40% of P_2 . For the data in the row(s) assigned from Table P9-1, find the diameter of shaft required to obtain a safety factor of 2 in fatigue loading if the shaft is steel of $S_{ut} = 108$ kpsi and $S_y = 62$ kpsi. The dimensions are inches, the force lb, and the torque lb-in.
- 9-20 A 12-in-long, solid, straight shaft is supported in self-aligning bearings at each end. A gear is attached at the middle of the shaft with a 3/8-in square steel key in a slot. The geometric stress-concentration factor in the keyslot is 2.5 and its corner radius is 0.02 in. The gear drives a fluctuating load which creates a bending moment that varies from +100 lb-in to +900 lb-in and a torque that varies from -300 lb-in to +1 500 lb-in each cycle. The material chosen is cold-drawn 4140 steel, hardened and tempered to Rockwell C45 ($S_{ut} = 180$ kpsi). Design the shaft for infinite life and determine the diameter needed for a safety factor of 1.5.
- 9-21 Figure P9-7 shows a dial assembly. The dial is 500-mm-dia by 16-mm-thick solid steel and is bolted to the top flange of the shaft. The shaft is 70 mm dia over length L_1 and 40 mm dia over length L_2 and is carried on tapered roller bearings. All shaft fillet radii are 0.5 mm. A 5-kw motor drives the shaft through a 20:1 reduction gearbox and a commercial bellows coupling. The maximum rated torque from the motor is 17.75 N-m and its stall torque is 3X its rated torque. Find the maximum stress and angular deflection in the shaft under conditions of dial locked and motor at stall.
- 9-22 For the dial assembly of Problem 9-21 and Figure P9-7, find the first torsional natural frequency of the shaft-dial assembly as felt at the motor shaft. The gearbox has a torsional stiffness of $1.56E5$ N-m/rad and the coupling has a torsional stiffness of $5.1E5$ N-m/rad. Assume the dial is empty of tooling and all ground support is infinitely stiff.
- 9-23 Repeat Problem 9-22 when the dial is fitted with 20 sets of tooling bolted to its top surface, equispaced on a 416-mm bolt circle. Each tooling assembly weighs 75.62 N. Assume all ground support is infinitely stiff.
- 9-24 Repeat Problem 9-22 when the dial assembly has a motor mount with a torsional stiffness of $2.44E5$ N-m/rad.

* Answers to these problems are provided in Appendix F. Problem numbers in *italics* are design problems.

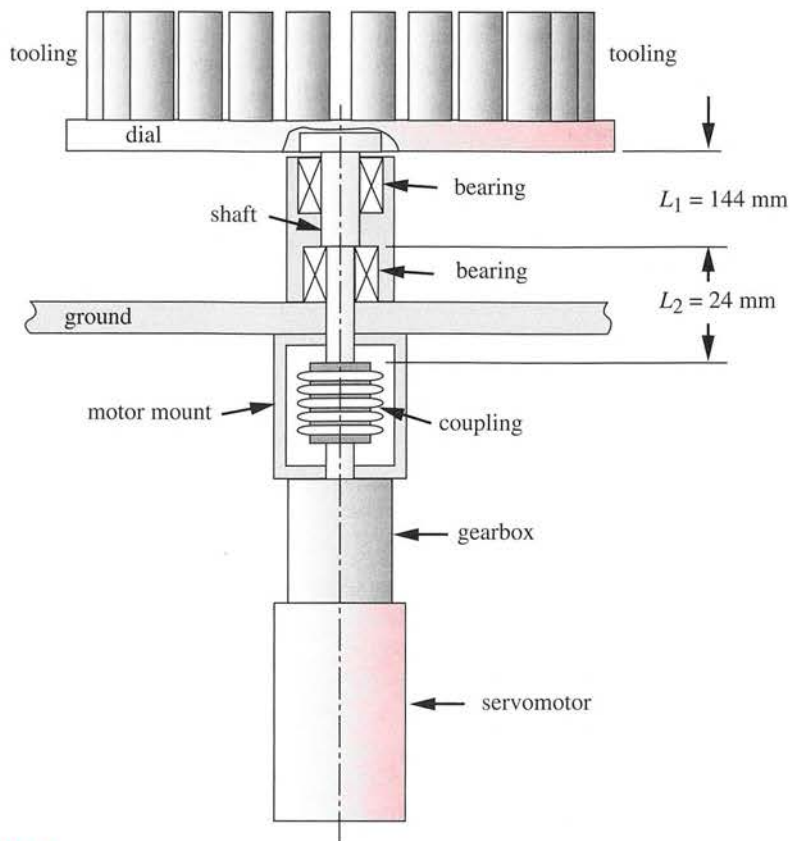


FIGURE P9-7

Shaft Design for Problems 9-21 to 9-28

- 9-25 The tooling on the dial assembly of Problem 9-21 and Figure P9-7 imparts a time-varying torque to the dial that ranges from a peak of 30% of the motor's rated torque (at the motor) to zero, 20 times per dial revolution. If the motor speed is 600 rpm, find the worst-case stress-time and deflection-time functions for the shaft. Choose a shaft material to give a safety factor of at least 3 against failure. Disregard the response of the system to the forced vibration.
- 9-26 Combining the data from Problems 9-23 and 9-25 (a) find the ratio between the torsional forcing frequency and the first torsional natural frequency of the dial assembly; (b) using the frequency ratio calculated in part (a), repeat Problem 9-25 taking into account the forced vibration response of the system if the damping ratio $\zeta = 0.20$.
- 9-27 Repeat Problem 9-25 for an impulsive impact loading of 500 N applied tangentially to the dial rim 20 times per revolution, equispaced in time.
- 9-28 For the dial assembly of Problem 9-21 and Figure P9-7, size a square key to couple the 40-mm-dia dial shaft to the coupling.
- 9-29 Repeat Problem 9-28 for the loading of Problem 9-25.
- 9-30 Repeat Problem 9-28 for the loading of Problem 9-27.

- †9-31 A simply supported shaft with overhanging load is shown in Figure P9-1. A constant-magnitude transverse force P is applied as the shaft rotates. The shaft is also subject to a constant torque of T_{max} . For the data in the row(s) assigned from Table P9-1 (ignoring T_{min}), find the diameter of shaft required to obtain a safety factor of 2.5 in fatigue loading if the shaft is steel of $S_{ut} = 118$ kpsi and $S_y = 102$ kpsi. The dimensions are in inches, the force in pounds, and the torque in lb-in. Assume no stress concentrations are present, the shaft is machined, the required reliability is 90%, and the shaft operates at room temperature.
- 9-32 Repeat Problem 9-31 taking the stress concentration at the keyway shown in Figure P9-3 into account.
- 9-33 A simply supported shaft is shown in Figure P9-2. A constant-magnitude distributed unit load p is applied as the shaft rotates. The shaft is also subject to a steady torque of T_{max} . For the data in the row(s) assigned from Table P9-1 (ignoring T_{min}), find the diameter of shaft required to obtain a safety factor of 2.5 in fatigue loading if the shaft is steel of $S_{ut} = 814$ MPa and $S_y = 703$ MPa. The dimensions are in cm, the distributed force in N/cm, and the torque in N-m. Assume no stress concentrations are present, the shaft is machined, the required reliability is 90%, and the shaft operates at room temperature.
- 9-34 Repeat Problem 9-33 taking the stress concentration at the keyways shown in Figure P9-4 into account.
- 9-35 Figure P9-8 shows the last stage in a gearbox with dual output. The gear is manufactured integrally with the shaft. The shaft is supported by two self-aligning ball bearings. Crank arms are connected to each end of the shaft. The load on the cranks produces equal fluctuating transverse forces on the shaft ends as well as equal fluctuating torques. The torque is transmitted through end-milled keyways in the crank and shaft and a parallel key that fits snugly in each keyway. The crank is located axially by a shoulder that is $L = 2$ -in from the plane in which the transverse load acts. The fillet radius to shaft diameter is $r/d = 0.05$ and the shoulder to shaft diameter ratio is $D/d = 1.2$. The shaft/gear material is SAE 4130 steel Q&T @ 1200F. The transverse force fluctuates from 1800 lb to 3700 lb and the torque fluctuates from 9400 in-lb to 19 300 in-lb. For a factor of safety of 2.5 against an infinite-life fatigue failure, determine a suitable shaft diameter, d .

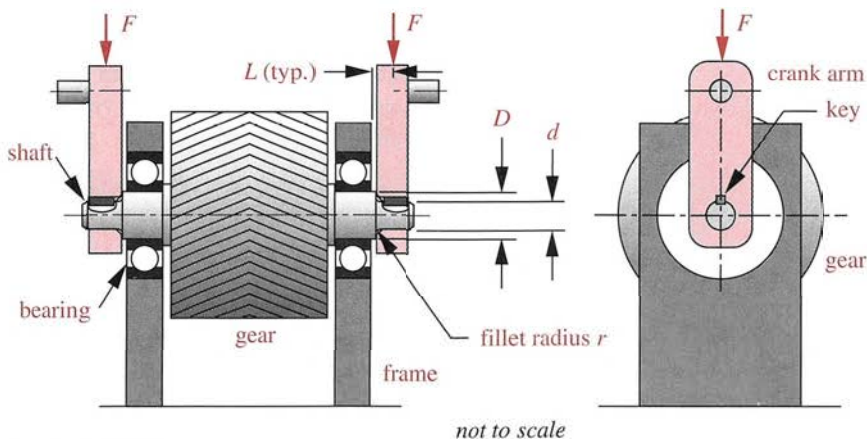


FIGURE P9-8

Problems 9-35 through 9-37

† Problem numbers in *italics* are design problems.

- 9-36 Determine the size of key necessary to give a safety factor of at least 2 against both shear and bearing failure for the crank/shaft connection of Problem 9-35. Assume a shaft diameter of 2.25 in and a key made from SAE 1040 CR steel.
- *9-37 As an alternative to the keyed connection in Problem 9-35 determine the amount of diametral interference needed to provide a suitable interference fit for the crank of Figure P9-8 using a shaft diameter of 2.25 in, such that the stresses in the hub and shaft will be safe and the maximum torque can be transmitted through the interference fit. The crank material is the same as the shaft and its length along the shaft is 2.50 in. The effective outside diameter of the crank is 6.00 in.
- *9-38 An electric motor-driven shaft that has a fluctuating load torque turns at an average speed of 1600 rpm. The load torque varies sinusoidally once per revolution of the shaft with a peak torque of 29 500 in-lb. The shaft diameter is 2.0 in. Design a suitable flywheel for this system that will provide a coefficient of fluctuation of 0.05 and an overspeed safety factor of at least 4. The flywheel is to be a solid circular disc of constant thickness made from SAE 1020 CR steel.
- 9-39 Repeat Problem 9-38 for a peak load torque of 40 500 in-lb and a torque vs. shaft position function of $Tl = T_{peak} (\sin \theta + \sin 2\theta)$.

* Answers to these problems are provided in Appendix F. Problem numbers in *italics* are design problems.

As an alternative to the keyed connection in Problem 9-6 determine the amount of diametral interference needed to provide a suitable interference fit for the gear of Figure P9-3 using the data of Problem 9-6 and the row(s) assigned in Table P9-1. The gear hub diameter is 3.50 in, its length along the shaft is 2.00 in, and it has the same properties as the shaft material.