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FUNDAMENTALS OF GLOBAL POSITIONING SYSTEM RECEIVERS

A SOFTWARE APPROACH

SECOND EDITION



JAMES BAO-YEN TSUI

Fundamentals of Global Positioning System Receivers A Software Approach

SECOND EDITION

JAMES BAO-YEN TSUI



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Basic GPS Concept

2.1 INTRODUCTION

This chapter will introduce the basic concept of how a GPS receiver determines its position. In order to better understand the concept, GPS performance requirements will be discussed first. These requirements determine the arrangement of the satellite constellation. From the satellite constellation, the user position can be solved. However, the equations required for solving the user position turn out to be nonlinear simultaneous equations. In addition, some practical considerations (i.e., the inaccuracy of the user clock) will be included in these equations. These equations are solved through a linearization and iteration method. The solution is in a Cartesian coordinate system and the result will be converted into a spherical coordinate system. However, the earth is not a perfect sphere; therefore, once the user position is found, the shape of the earth must be taken into consideration. The user position is then translated into the earth-based coordinate system. Finally, the selection of satellites to obtain better user position accuracy and the dilution of precision will be discussed.

2.2 GPS PERFORMANCE REQUIREMENTS⁽¹⁾

Some of the performance requirements are listed below:

1. The user position root mean square (rms) error should be 10–30 m.
2. It should be applicable to real-time navigation for all users including the high-dynamics user, such as in high-speed aircraft with flexible maneuverability.

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3. It should have worldwide coverage. Thus, in order to cover the polar regions the satellites must be in inclined orbits.
4. The transmitted signals should tolerate, to some degree, intentional and unintentional interference. For example, the harmonics from some narrow-band signals should not disturb its operation. Intentional jamming of GPS signals is a serious concern for military applications.
5. It cannot require that every GPS receiver utilize a highly accurate clock such as those based on atomic standards.
6. When the receiver is first turned on, it should take minutes rather than hours to find the user position.
7. The size of the receiving antenna should be small. The signal attenuation through space should be kept reasonably small.

These requirements combining with the availability of the frequency band allocation determines the carrier frequency of the GPS to be in the L band (1–2 GHz) of the microwave range.

2.3 BASIC GPS CONCEPT

The position of a certain point in space can be found from distance measured from this point to some known positions in space. Let us use some examples to illustrate this point. In Figure 2.1, the user position is on the x -axis; this is a one-dimensional case. If the satellite position S_1 and the distance to the satellite x_1 are both known, the user position can be at two places, either to the left or right of S_1 . In order to determine the user position, the distance to another satellite with known position must be measured. In this figure, the positions of S_2 and x_2 uniquely determine the user position U .

Figure 2.2 shows a two-dimensional case. In order to determine the user position, three satellites and three distances are required. The trace of a point with constant distance to a fixed point is a circle in the two-dimensional case. Two satellites and two distances give two possible solutions because two circles intersect at two points. A third circle is needed to uniquely determine the user position.

For similar reasons one might decide that in a three-dimensional case four satellites and four distances are needed. The equal-distance trace to a fixed point is a sphere in a three-dimensional case. Two spheres intersect to make a circle. This circle intersects another sphere to produce two points. In order to determine which point is the user position, one more satellite is needed.

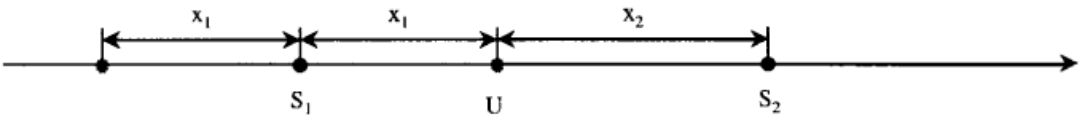


FIGURE 2.1 One-dimensional user position.

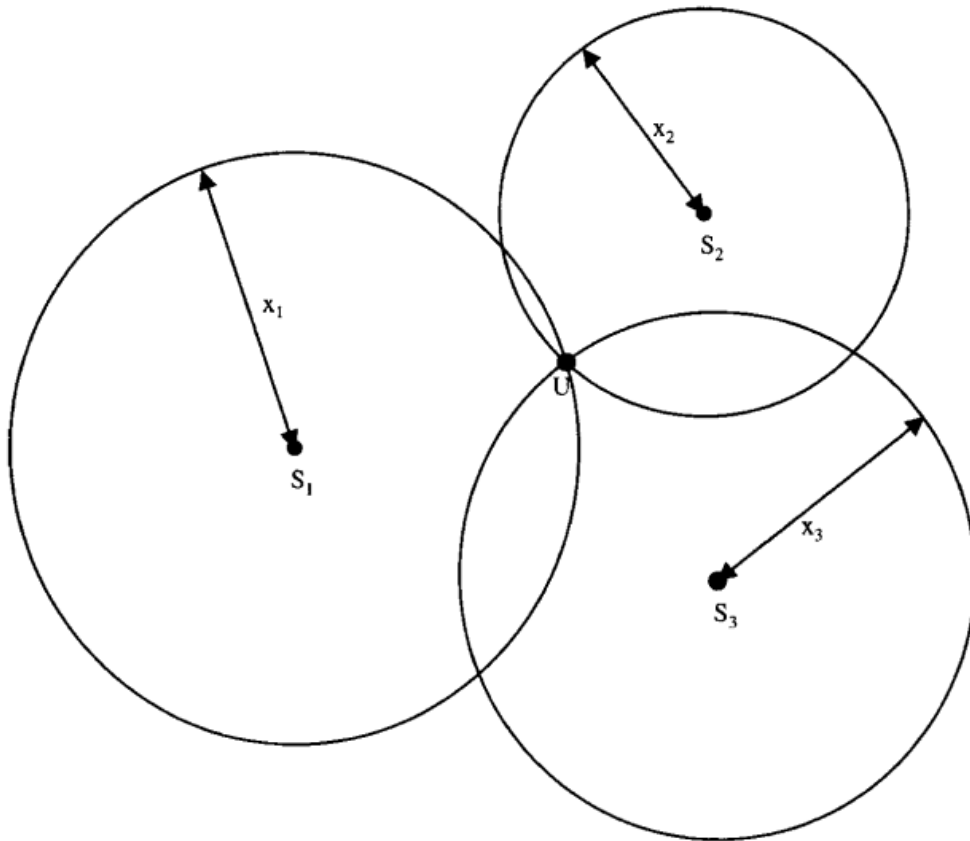


FIGURE 2.2 Two-dimensional user position.

In GPS the position of the satellite is known from the ephemeris data transmitted by the satellite. One can measure the distance from the receiver to the satellite. Therefore, the position of the receiver can be determined.

In the above discussion, the distance measured from the user to the satellite is assumed to be very accurate and there is no bias error. However, the distance measured between the receiver and the satellite has a constant unknown bias, because the user clock usually is different from the GPS clock. In order to resolve this bias error one more satellite is required. Therefore, in order to find the user position five satellites are needed.

If one uses four satellites and the measured distance with bias error to measure a user position, two possible solutions can be obtained. Theoretically, one cannot determine the user position. However, one of the solutions is close to the earth's surface and the other one is in space. Since the user position is usually close to the surface of the earth, it can be uniquely determined. Therefore, the general statement is that four satellites can be used to determine a user position, even though the distance measured has a bias error.

The method of solving the user position discussed in Sections 2.5 and 2.6 is through iteration. The initial position is often selected at the center of the earth. The iteration method will converge on the correct solution rather than the one in space. In the following discussion four satellites are considered the minimum number required in finding the user position.

2.4 BASIC EQUATIONS FOR FINDING USER POSITION

In this section the basic equations for determining the user position will be presented. Assume that the distance measured is accurate and under this condition three satellites are sufficient. In Figure 2.3, there are three known points at locations r_1 or (x_1, y_1, z_1) , r_2 or (x_2, y_2, z_2) , and r_3 or (x_3, y_3, z_3) , and an unknown point at r_u or (x_u, y_u, z_u) . If the distances between the three known points to the unknown point can be measured as ρ_1 , ρ_2 , and ρ_3 , these distances can be written as

$$\begin{aligned}\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} \\ \rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} \\ \rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2}\end{aligned}\quad (2.1)$$

Because there are three unknowns and three equations, the values of x_u , y_u , and z_u can be determined from these equations. Theoretically, there should be two sets of solutions as they are second-order equations. These equations can be solved relatively easily with linearization and an iterative approach. The solution of these equations will be discussed later in Section 2.6.

In GPS operation, the positions of the satellites are given. This information can be obtained from the data transmitted from the satellites and will be discussed in Chapter 5. The distances from the user (the unknown position) to the

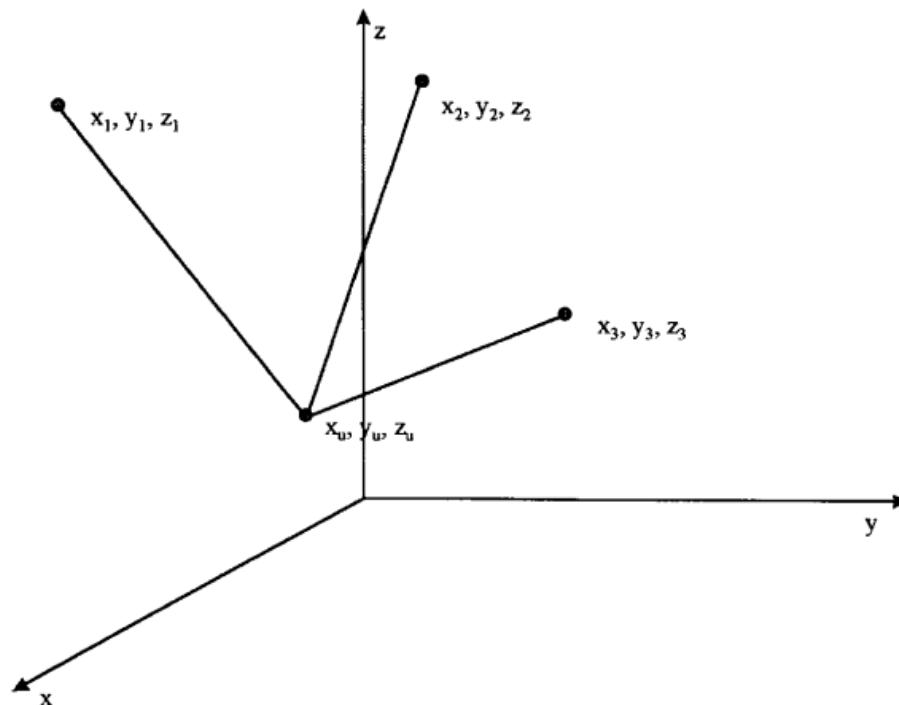


FIGURE 2.3 Use three known positions to find one unknown position.

satellites must be measured simultaneously at a certain time instance. Each satellite transmits a signal with a time reference associated with it. By measuring the time of the signal traveling from the satellite to the user the distance between the user and the satellite can be found. The distance measurement is discussed in the next section.

2.5 MEASUREMENT OF PSEUDORANGE⁽²⁾

Every satellite sends a signal at a certain time t_{si} . The receiver will receive the signal at a later time t_u . The distance between the user and the satellite i is

$$\rho_{iT} = c(t_u - t_{si}) \quad (2.2)$$

where c is the speed of light, ρ_{iT} is often referred to as the true value of pseudorange from user to satellite i , t_{si} is referred to as the true time of transmission from satellite i , t_u is the true time of reception.

From a practical point of view it is difficult, if not impossible, to obtain the correct time from the satellite or the user. The actual satellite clock time t'_{si} and actual user clock time t'_u are related to the true time as

$$\begin{aligned} t'_{si} &= t_{si} + \Delta b_i \\ t'_u &= t_u + b_{ut} \end{aligned} \quad (2.3)$$

where Δb_i is the satellite clock error, b_{ut} is the user clock bias error. Besides the clock error, there are other factors affecting the pseudorange measurement. The measured pseudorange ρ_i can be written as⁽²⁾

$$\rho_i = \rho_{iT} + \Delta D_i - c(\Delta b_i - b_{ut}) + c(\Delta T_i + \Delta I_i + v_i + \Delta v_i) \quad (2.4)$$

where ΔD_i is the satellite position error effect on range, ΔT_i is the tropospheric delay error, ΔI_i is the ionospheric delay error, v_i is the receiver measurement noise error, Δv_i is the relativistic time correction.

Some of these errors can be corrected; for example, the tropospheric delay can be modeled and the ionospheric error can be corrected in a two-frequency receiver. The errors will cause inaccuracy of the user position. However, the user clock error cannot be corrected through received information. Thus, it will remain as an unknown. As a result, Equation (2.1) must be modified as

$$\begin{aligned} \rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + b_u \\ \rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + b_u \\ \rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + b_u \\ \rho_4 &= \sqrt{(x_4 - x_u)^2 + (y_4 - y_u)^2 + (z_4 - z_u)^2} + b_u \end{aligned} \quad (2.5)$$

where b_u is the user clock bias error expressed in distance, which is related to the quantity b_{ut} by $b_u = cb_{ut}$. In Equation (2.5), four equations are needed to solve for four unknowns x_u , y_u , z_u , and b_u . Thus, in a GPS receiver, a minimum of four satellites is required to solve for the user position. The actual measurement of the pseudorange will be discussed in Chapter 9.

2.6 SOLUTION OF USER POSITION FROM PSEUDORANGES

One common way to solve Equation (2.5) is to linearize them. The above equations can be written in a simplified form as

$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} + b_u \quad (2.6)$$

where $i = 1, 2, 3$, and 4 , and x_u , y_u , z_u , and b_u are the unknowns. The pseudorange ρ_i and the positions of the satellites x_i , y_i , z_i are known.

Differentiate this equation, and the result is

$$\begin{aligned} \delta\rho_i &= \frac{(x_i - x_u)\delta x_u + (y_i - y_u)\delta y_u + (z_i - z_u)\delta z_u}{\sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2}} + \delta b_u \\ &= \frac{(x_i - x_u)\delta x_u + (y_i - y_u)\delta y_u + (z_i - z_u)\delta z_u}{\rho_i - b_u} + \delta b_u \end{aligned} \quad (2.7)$$

In this equation, δx_u , δy_u , δz_u , and δb_u can be considered as the only unknowns. The quantities x_u , y_u , z_u , and b_u are treated as known values because one can assume some initial values for these quantities. From these initial values a new set of δx_u , δy_u , δz_u , and δb_u can be calculated. These values are used to modify the original x_u , y_u , z_u , and b_u to find another new set of solutions. This new set of x_u , y_u , z_u , and b_u can be considered again as known quantities. This process continues until the absolute values of δx_u , δy_u , δz_u , and δb_u are very small and within a certain predetermined limit. The final values of x_u , y_u , z_u , and b_u are the desired solution. This method is often referred to as the iteration method.

With δx_u , δy_u , δz_u , and δb_u as unknowns, the above equation becomes a set of linear equations. This procedure is often referred to as linearization. The above equation can be written in matrix form as

$$\begin{bmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \\ \delta\rho_4 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{bmatrix} \begin{bmatrix} \delta x_u \\ \delta y_u \\ \delta z_u \\ \delta b_u \end{bmatrix} \quad (2.8)$$

where

$$\alpha_{i1} = \frac{x_i - x_u}{\rho_i - b_u} \quad \alpha_{i2} = \frac{y_i - y_u}{\rho_i - b_u} \quad \alpha_{i3} = \frac{z_i - z_u}{\rho_i - b_u} \quad (2.9)$$

The solution of Equation (2.8) is

$$\begin{bmatrix} \delta x_u \\ \delta y_u \\ \delta z_u \\ \delta b_u \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \delta \rho_1 \\ \delta \rho_2 \\ \delta \rho_3 \\ \delta \rho_4 \end{bmatrix} \quad (2.10)$$

where $[\]^{-1}$ represents the inverse of the α matrix. This equation obviously does not provide the needed solutions directly; however, the desired solutions can be obtained from it. In order to find the desired position solution, this equation must be used repetitively in an iterative way. A quantity is often used to determine whether the desired result is reached and this quantity can be defined as

$$\delta v = \sqrt{\delta x_u^2 + \delta y_u^2 + \delta z_u^2 + \delta b_u^2} \quad (2.11)$$

When this value is less than a certain predetermined threshold, the iteration will stop. Sometimes, the clock bias b_u is not included in Equation (2.11).

The detailed steps to solve the user position will be presented in the next section. In general, a GPS receiver can receive signals from more than four satellites. The solution will include such cases as when signals from more than four satellites are obtained.

2.7 POSITION SOLUTION WITH MORE THAN FOUR SATELLITES⁽³⁾

When more than four satellites are available, a more popular approach to solve the user position is to use all the satellites. The position solution can be obtained in a similar way. If there are n satellites available where $n > 4$, Equation (2.6) can be written as

$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} + b_u \quad (2.12)$$

where $i = 1, 2, 3, \dots, n$. The only difference between this equation and Equation (2.6) is that $n > 4$.

Linearize this equation, and the result is

$$\begin{bmatrix} \delta \rho_1 \\ \delta \rho_2 \\ \delta \rho_3 \\ \delta \rho_4 \\ \vdots \\ \delta \rho_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & 1 \end{bmatrix} \begin{bmatrix} \delta x_u \\ \delta y_u \\ \delta z_u \\ \delta b_u \end{bmatrix} \quad (2.13)$$

where

$$\alpha_{i1} = \frac{x_i - x_u}{\rho_i - b_u} \quad \alpha_{i2} = \frac{y_i - y_u}{\rho_i - b_u} \quad \alpha_{i3} = \frac{z_i - z_u}{\rho_i - b_u} \quad (2.9)$$

Equation (2.13) can be written in a simplified form as

$$\delta\rho = \alpha\delta x \quad (2.14)$$

where $\delta\rho$ and δx are vectors, α is a matrix. They can be written as

$$\begin{aligned} \delta\rho &= [\delta\rho_1 \quad \delta\rho_2 \quad \cdots \quad \delta\rho_n]^T \\ \delta x &= [\delta x_u \quad \delta y_u \quad \delta z_u \quad \delta b_u]^T \\ \alpha &= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 1 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 1 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & 1 \end{bmatrix} \end{aligned} \quad (2.15)$$

where $[]^T$ represents the transpose of a matrix. Since α is not a square matrix, it cannot be inverted directly. Equation (2.13) is still a linear equation. If there are more equations than unknowns in a set of linear equations, the least-squares approach can be used to find the solutions. The pseudoinverse of the α can be used to obtain the solution. The solution is⁽³⁾

$$\delta x = [\alpha^T \alpha]^{-1} \alpha^T \delta\rho \quad (2.16)$$

From this equation, the values of δx_u , δy_u , δz_u , and δb_u can be found. In general, the least-squares approach produces a better solution than the position obtained from only four satellites, because more data are used.

The following steps summarize the above approach:

- A. Choose a nominal position and user clock bias x_{u0} , y_{u0} , z_{u0} , b_{u0} to represent the initial condition. For example, the position can be the center of the earth and the clock bias zero. In other words, all initial values are set to zero.
- B. Use Equation (2.5) or (2.6) to calculate the pseudorange ρ_i . These ρ_i values will be different from the measured values. The difference between the measured values and the calculated values is $\delta\rho_i$.
- C. Use the calculated ρ_i in Equation (2.9) to calculate α_{i1} , α_{i2} , α_{i3} .
- D. Use Equation (2.16) to find δx_u , δy_u , δz_u , δb_u .
- E. From the absolute values of δx_u , δy_u , δz_u , δb_u and from Equation (2.11) calculate δv .

- F. Compare δv with an arbitrarily chosen threshold; if δv is greater than the threshold, the following steps will be needed.
- G. Add these values δx_u , δy_u , δz_u , δb_u to the initial chosen position x_{u0} , y_{u0} , z_{u0} , and the clock bias b_{u0} ; a new set of positions and clock bias can be obtained and they will be expressed as x_{u1} , y_{u1} , z_{u1} , b_{u1} . These values will be used as the initial position and clock bias in the following calculations.
- H. Repeat the procedure from A to G, until δv is less than the threshold. The final solution can be considered as the desired user position and clock bias, which can be expressed as x_u , y_u , z_u , b_u .

In general, the δv calculated in the above iteration method will keep decreasing rapidly. Depending on the chosen threshold, the iteration method usually can achieve the desired goal in less than 10 iterations. A computer program (p2_1) to calculate the user position is listed at the end of this chapter. In this book, some lines in the programs are too long to be listed in one line; however, it should be easily recognized.

2.8 USER POSITION IN SPHERICAL COORDINATE SYSTEM

The user position calculated from the above discussion is in a Cartesian coordinate system. It is usually desirable to convert to a spherical system and label the position in latitude, longitude, and altitude as the every-day maps use these notations. The latitude of the earth is from -90 to 90 degrees with the equator at 0 degree. The longitude is from -180 to 180 degrees with the Greenwich meridian at 0 degree. The altitude is the height above the earth's surface. If the earth is a perfect sphere, the user position can be found easily as shown in Figure 2.4. From this figure, the distance from the center of the earth to the user is

$$r = \sqrt{x_u^2 + y_u^2 + z_u^2} \quad (2.17)$$

The latitude L_c is

$$L_c = \tan^{-1} \left(\frac{z_u}{\sqrt{x_u^2 + y_u^2}} \right) \quad (2.18)$$

The longitude l is

$$l = \tan^{-1} \left(\frac{y_u}{x_u} \right) \quad (2.19)$$

The altitude h is

$$h = r - r_e \quad (2.20)$$

where r_e is the radius of an ideal spherical earth or the average radius of the earth. Since the earth is not a perfect sphere, some of these equations need to be modified.

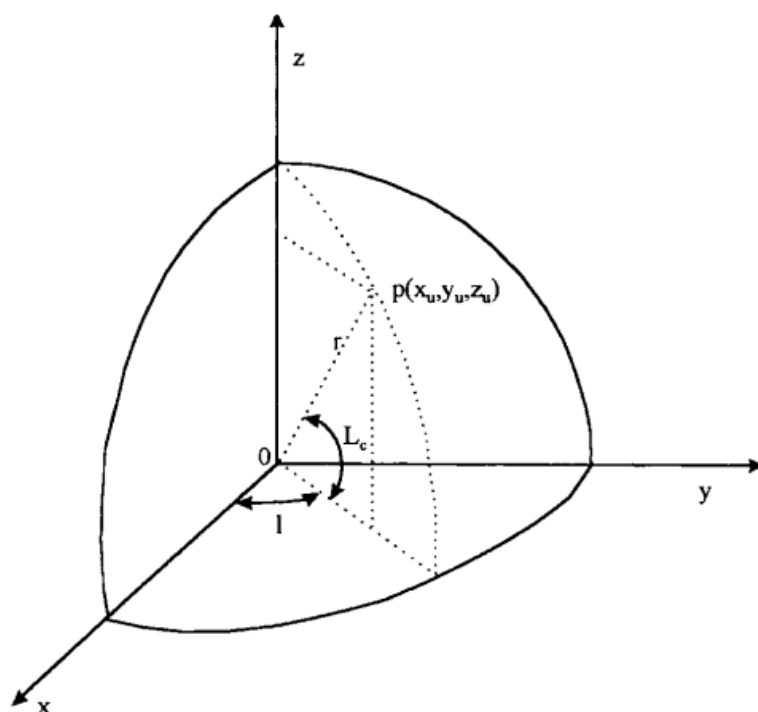


FIGURE 2.4 An octet of an ideal spherical earth.

2.9 EARTH GEOMETRY⁽⁴⁻⁶⁾

The earth is not a perfect sphere but is an ellipsoid; thus, the latitude and altitude calculated from Equations (2.18) and (2.20) must be modified. However, the longitude l calculated from Equation (2.19) also applies to the nonspherical earth. Therefore, this quantity does not need modification. Approximations will be used in the following discussion, which is based on references 4 through 6. For an ellipsoid, there are two latitudes. One is referred to as the geocentric latitude L_c , which is calculated from the previous section. The other one is the geodetic latitude L and is the one often used in every-day maps. Therefore, the geocentric latitude must be converted to the geodetic latitude. Figure 2.5 shows a cross section of the earth. In this figure the x -axis is along the equator, the y -axis is pointing inward to the paper, and the z -axis is along the north pole of the earth. Assume that the user position is on the x - z plane and this assumption does not lose generality. The geocentric latitude L_c is obtained by drawing a line from the user to the center of the earth, which is calculated from Equation (2.18).

The geodetic latitude is obtained by drawing a line perpendicular to the surface of the earth that does not pass the center of the earth. The angle between this line and the x is the geodetic latitude L . The height of the user is the distance h perpendicular and above the surface of the earth.

The following discussion is used to determine three unknown quantities from two known quantities. As shown in Figure 2.5, the two known quantities are the distance r and the geocentric latitude L_c and they are measured from the ideal spherical earth. The three unknown quantities are the geodetic latitude

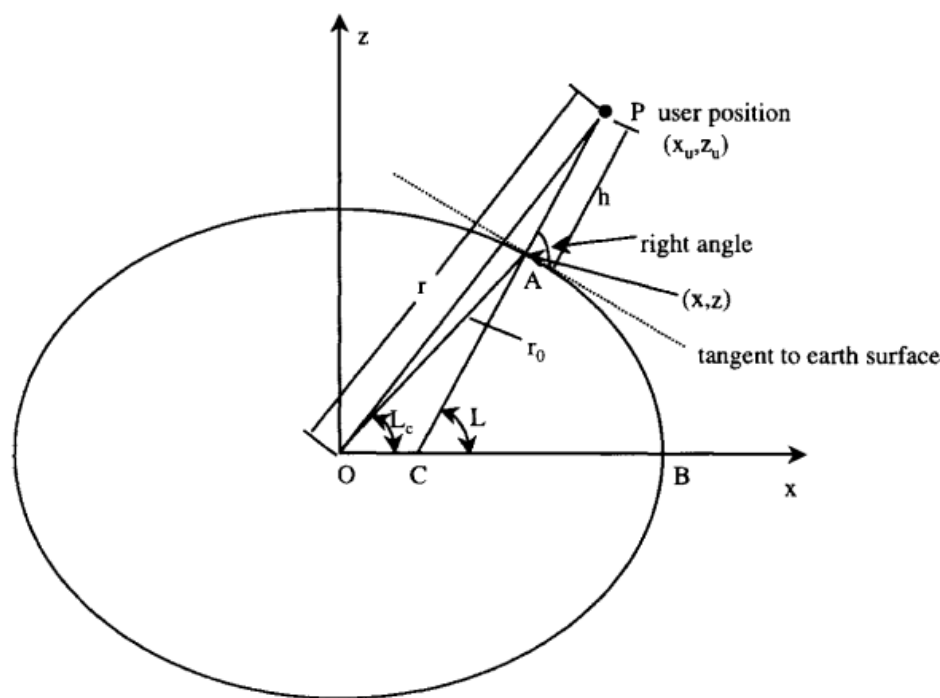


FIGURE 2.5 Geocentric and geodetic latitudes.

L , the distance r_0 , and the height h . All three quantities are calculated from approximation methods. Before the actual calculations of the unknowns, let us introduce some basic relationships in an ellipse.

2.10 BASIC RELATIONSHIPS IN AN ELLIPSE⁽⁴⁻⁷⁾

In order to derive the relationships mentioned in the previous section, it is convenient to review the basic functions in an ellipse. Figure 2.6 shows an ellipse which can be used to represent a cross section of the earth passing through the polar axis.

Let us assume that the semi-major axis is a_e , the semi-minor axis is b_e , and the foci are separated by $2c_e$. The equation of the ellipse is

$$\frac{x^2}{a_e^2} + \frac{z^2}{b_e^2} = 1 \text{ and } a_e^2 - b_e^2 = c_e^2 \tag{2.21}$$

The eccentricity e_e is defined as

$$e_e = \frac{c_e}{a_e} = \frac{\sqrt{a_e^2 - b_e^2}}{a_e} \text{ or } \frac{b_e}{a_e} = \sqrt{1 - e_e^2} \tag{2.22}$$

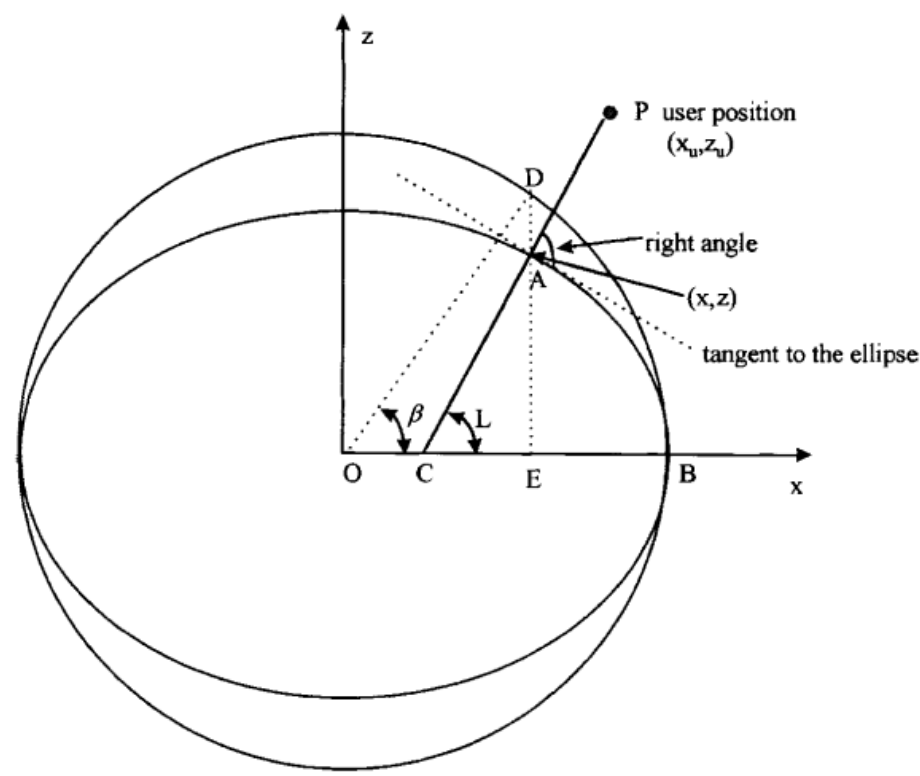


FIGURE 2.6 A basic ellipse with accessory lines.

The ellipticity e_p is defined as

$$e_p = \frac{a_e - b_e}{a_e} \tag{2.23}$$

where $a_e = 6378137 \pm 2$ m, $b_e = 6356752.3142$ m, $e_e = 0.0818191908426$, and $e_p = 0.00335281066474$.^(6,7) The value of b_e is calculated from a_e ; thus, the result has more decimal points.

From the user position P draw a line perpendicular to the ellipse that intercepts it at A and the x -axis at C . To help illustrate the following relation a circle with radius equal to the semi-major axis a_e is drawn as shown in Figure 2.6. A line is drawn from point A perpendicular to the x -axis and intercepts it at E and the circle at D . The position $A(x, y)$ can be found as

$$\begin{aligned} x &= OE = OD \cos \beta = a_e \cos \beta \\ z &= AE = DE \frac{b_e}{a_e} = (a_e \sin \beta) \frac{b_e}{a_e} = b_e \sin \beta \end{aligned} \tag{2.24}$$

The second equation can be obtained easily from the equation of a circle $x^2 + z^2 = a_e^2$ and Equation (2.21). The tangent to the ellipse at A is dz/dx . Since line CP is perpendicular to the tangent,

$$\tan L = -\frac{dx}{dz} \tag{2.25}$$

completing the square for $r_0 + h$ and taking the square root as

$$r = [(r_0 + h)^2 - 2r_0h(1 - \cos D_0)]^{1/2} = (r_0 + h) \left[1 - \frac{2hr_0(1 - \cos D_0)}{(r_0 + h)^2} \right]^{1/2} \quad (2.29)$$

Since angle D_0 is very small, it can be approximated as

$$1 - \cos D_0 \approx \frac{D_0^2}{2} \quad (2.30)$$

where D_0 is the angle expressed in radians. The r value can be written as

$$r \approx (r_0 + h) \left[1 - \frac{2hr_0 D_0^2/2}{(r_0 + h)^2} \right]^{1/2} = r_0 + h - \frac{hr_0 D_0^2}{2(r_0 + h)} \quad (2.31)$$

At latitude of 45 degrees D_0 ($\approx 1/297$ radian) becomes maximum. If D_0 is neglected, the result is

$$r \approx r_0 + h - \frac{r_0 h D_0^2}{2(r_0 + h)} \approx r_0 + h \quad (2.32)$$

Using this result, if $h = 100$ km, and $r_0 = r_e = 6368$ km (the average radius of the earth), the error term calculated is less than 0.6 m. Thus

$$h = r - r_0 \quad (2.33)$$

is a good approximation. However, in this equation the value of r_0 must be evaluated, as discussed in Section 2.12.

2.12 CALCULATION OF GEODETIC LATITUDE⁽⁵⁻⁷⁾

Referring to Figure 2.7, the relation between angles L and L_c can be found from the triangle OPC . From the simple geometry it can be seen that

$$L = L_c + D \quad (2.34)$$

If the angle D can be found, the relation between L and L_c can be obtained. To find this angle, let us find the distance OC first. Combining Equations (2.24) and (2.27), the following result is obtained:

$$\begin{aligned} OC &= OE - CE = a_e \cos \beta - \frac{AE}{\tan L} = a_e \cos \beta - \frac{b_e \sin \beta}{\tan L} \\ &= a_e \cos \beta [1 - (1 - e_e^2)] = a_e e_e^2 \cos \beta = e_e^2 OE \end{aligned} \quad (2.35)$$

where β is not shown in this figure but is shown in Figure 2.6.

From the triangle OPC and the law of sine, one can write

$$\frac{\sin D}{OC} = \frac{\sin(\pi - L)}{r} \quad (2.36)$$

From Equation (2.35),

$$OC = e_e^2 OE = e_e^2 r_0 \cos L_{co} \quad (2.37)$$

but

$$L_{co} = L - D_0 \quad (2.38)$$

Therefore,

$$OC = e_e^2 r_0 \cos(L - D_0) = e_e^2 r_0 (\cos L \cos D_0 + \sin L \sin D_0) \quad (2.39)$$

From Equation (2.23), the ellipticity e_p of the earth is

$$e_p = \frac{a_e - b_e}{a_e} \quad (2.40)$$

The eccentricity and the ellipticity can be related as

$$e_e^2 = \frac{a_e^2 - b_e^2}{a_e^2} = \frac{(a_e - b_e)(a_e + b_e)}{a_e^2} = e_p \frac{(2a_e - a_e + b_e)}{a_e} = e_p(2 - e_p) \quad (2.41)$$

Substituting Equations (2.39) and (2.41) into Equation (2.36), the result is

$$\sin D = 2e_p \left(1 - \frac{e_p}{2}\right) \frac{r_0}{r_0 + h} \left(\frac{1}{2} \sin 2L \cos D_0 + \sin^2 L \sin D_0\right) \quad (2.42)$$

In the above equation the relation $r = r_0 + h$ is used. Since D and D_0 are both very small angles, the above equation can be written as

$$D = 2e_p \left(1 - \frac{e_p}{2}\right) \frac{r_0}{r_0 + h} \left(\frac{1}{2} \sin 2L + D_0 \sin^2 L\right) \quad (2.43)$$

The relations

$$\sin D \approx D; s_p \sin D_0 \approx D_0; s_p \cos D_0 \approx 1 \quad (2.44)$$

are used in obtaining the results of Equation (2.43). If the height $h = 0$, then from Figure 2.7 $D = D_0$. Using this relation Equation (2.43) can be written as

$$D_0 \left[1 - 2e_p \left(1 - \frac{e_p}{2}\right) \sin^2 L\right] = e_p \left(1 - \frac{e_p}{2}\right) \sin 2L \text{ or} \\ D_0 = e_p \sin 2L + \epsilon_1 \quad (2.45)$$

where

$$\epsilon_1 = -\frac{e_p^2}{2} \sin L + 2e_p^2 \sin 2L \sin^2 L + \dots \leq 1.6 \text{ arc-sec} \quad (2.46)$$

Substitute the approximation of $D_0 \approx e_p \sin 2L$ into Equation (2.43) and the result is

$$D = 2e_p \left(1 - \frac{e_p}{2}\right) \left(1 - \frac{h}{r_0}\right) \left(\frac{1}{2} \sin 2L + e_p \sin 2L \sin^2 L\right) \quad (2.47)$$

or

$$D = e_p \sin 2L + \epsilon \quad (2.48)$$

where

$$\epsilon = -\frac{e_p^2}{2} \sin 2L - \frac{he_p}{r_0} \sin 2L + \dots \quad (2.49)$$

This error is less than 4.5 arc-sec for $h = 30$ km. Using the approximate value of D , the relation between angle L and L_c can be found from Equation (2.34) as

$$L = L_c + e_p \sin 2L \quad (2.50)$$

This is a nonlinear equation that can be solved through the iteration method. This equation can be written in a slightly different form as

$$L_{i+1} = L_c + e_p \sin 2L_i \quad (2.51)$$

where $i = 0, 1, 2, \dots$. One can start with $L_0 = L_c$. If the difference $(L_{i+1} - L_i)$ is smaller than a predetermined threshold, the last value of L_i can be considered as the desired one. It should be noted that during the iteration method L_c is a constant that is obtained from Equation (2.18).

2.13 CALCULATION OF A POINT ON THE SURFACE OF THE EARTH⁽⁵⁾

The final step of this calculation is to find the value r_0 in Equation (2.33). This value is also shown in Figure 2.7. The point $A(x, z)$ is on the ellipse; therefore, it satisfies the following elliptic Equation (2.21). This equation is rewritten here for convenience,

$$\frac{x^2}{a_e^2} + \frac{z^2}{b_e^2} = 1 \quad (2.52)$$

where a_e and b_e are the semi-major and semi-minor axes of the earth. From Figure 2.7, the x and z values can be written as

$$\begin{aligned} x &= r_0 \cos L_{co} \\ z &= r_0 \sin L_{co} \end{aligned} \quad (2.53)$$

Substituting these relations into Equation (2.52) and solving for r_0 , the result is

$$\begin{aligned}
 r_0^2 \left(\frac{\cos^2 L_{co}}{a_e^2} + \frac{\sin^2 L_{co}}{b_e^2} \right) &= r_0^2 \left(\frac{b_e^2 \cos^2 L_{co} + a_e^2 (1 - \cos^2 L_{co})}{a_e^2 b_e^2} \right) = 1 \quad \text{or} \\
 r_0^2 &= \frac{a_e^2 b_e^2}{a_e^2 \left[1 - \left(1 - \frac{b_e^2}{a_e^2} \right) \cos L_{co} \right]} = \frac{b_e^2}{1 - e_e^2 \cos L_{co}} \quad \text{or} \\
 r_0 &= b_e \left(1 + \frac{1}{2} e_e^2 \cos^2 L_{co} + \dots \right) \quad (2.54)
 \end{aligned}$$

Use Equation (2.23) to replace b_e by a_e , Equation (2.41) to replace e_e by e_p , and L to replace L_{co} because $L \approx L_{co}$, and then

$$\begin{aligned}
 r_0 &\approx a_e (1 - e_p) \left[1 + \left(e_p - \frac{e_p^2}{2} \right) \cos^2 L + \dots \right] \\
 &\approx a_e (1 - e_p) (1 + e_p - e_p \sin^2 L + \dots) \quad (2.55)
 \end{aligned}$$

In this equation the higher order of e_p is neglected. The value of r_0 can be found as

$$r_0 \approx a_e (1 - e_p \sin^2 L) \quad (2.56)$$

To solve for the latitude and altitude of the user, use Equation (2.51) to find the geodetic latitude L first. Then use Equation (2.56) to find r_0 , and finally, use Equation (2.33) to find the altitude. The result is

$$h \approx r - r_0 \approx \sqrt{x_u^2 + y_u^2 + z_u^2} - a_e (1 - e_p \sin^2 L) \quad (2.57)$$

2.14 SATELLITE SELECTION^(1,8)

A GPS receiver can simultaneously receive signals from 4 up to 11 satellites, if the receiver is on the surface of the earth. Under this condition, there are two approaches to solve the problem. The first one is to use all the satellites to calculate the user position. The other approach is to choose only four satellites from the constellation. The usual way is to utilize all the satellites to calculate the user position, because additional measurements are used. In this section and section 2.15 the selection of satellites will be presented. In order to focus on this subject only the four-satellite case will be considered.

If there are more than four satellite signals that can be received by a GPS receiver, a simple way is to choose only four satellites and utilize them to solve for the user position. Under this condition, the question is how to select the four satellites. Let us use a two-dimensional case to illustrate the situation, because it is

easier to show graphically. In order to solve a position in a two-dimensional case, three satellites are required considering the user clock bias. In this discussion, it is assumed that the user position can be uniquely determined as discussed in Section 2.3. If this assumption cannot be used, four satellites are required.

Figure 2.8a shows the results measured by three satellites on a straight line, and the user is also on this line. Figure 2.8b shows that the three satellites and the user form a quadrangle. Two circles with the same center but different radii are drawn. The solid circle represents the distance measured from the user to the satellite with bias clock error. The dotted circle represents the distance after the clock error correction. From observation, the position error in Figure 2.8a is greater than that in Figure 2.8b because in Figure 2.8a all three dotted circles are tangential to each other. It is difficult to measure the tangential point accurately. In Figure 2.8b, the three circles intersect each other and the point of intersection can be measured more accurately. Another way to view this problem is to measure the area of a triangle made by the three satellites. In Figure 2.8a the total area is close to zero, while in Figure 2.8b the total area is quite large. In general, the larger the triangle area made by the three satellites, the better the user position can be solved.

The general rule can be extended to select the four satellites in a three-dimensional case. It is desirable to maximize the volume defined by the four satellites. A tetrahedron with an equilateral base contains the maximum volume and therefore can be considered as the best selection. Under this condition, one satellite is at zenith and the other three are close to the horizon and separated by 120 degrees.⁽⁸⁾ This geometry will generate the best user position estimation. If all four satellites are close to the horizon, the volume defined by these satellites and the user is very small. Occasionally, the user position error calculated with this arrangement can be extremely large. In other words, the δv calculated from Equation (2.11) may not converge.

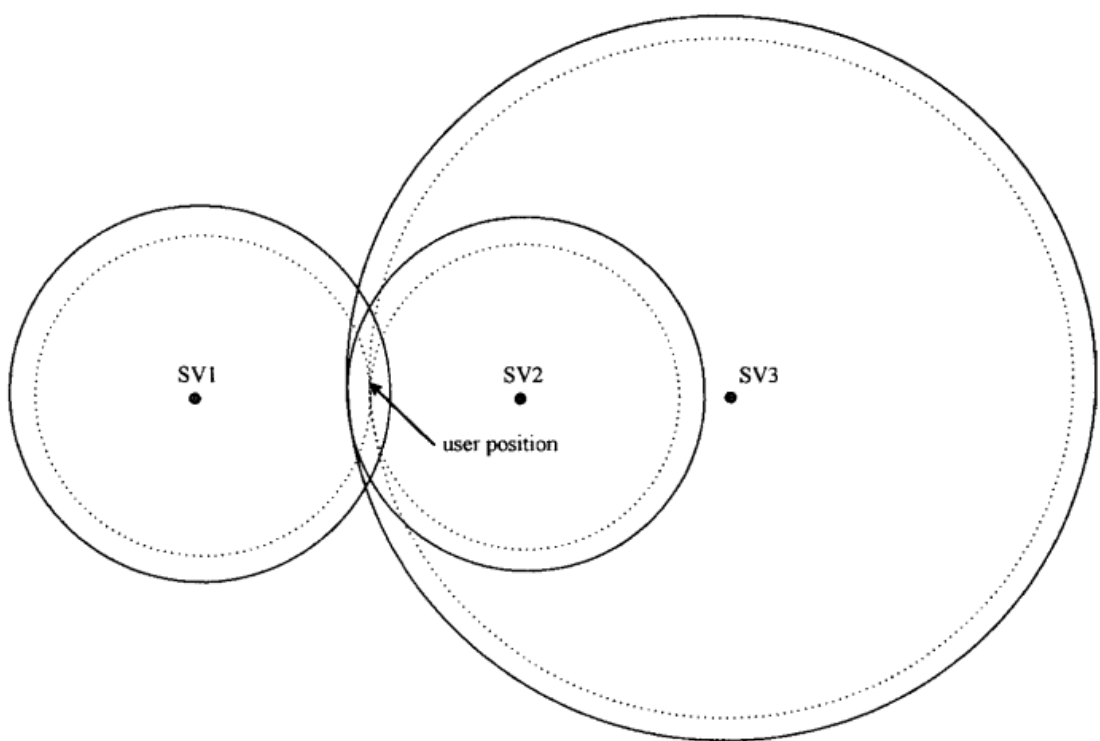
2.15 DILUTION OF PRECISION^(1,8)

The dilution of precision (DOP) is often used to measure user position accuracy. There are several different definitions of the DOP. All the different DOPs are a function of satellite geometry only. The positions of the satellites determine the DOP value. A detailed discussion can be found in reference 8. Here only the definitions and the limits of the values will be presented.

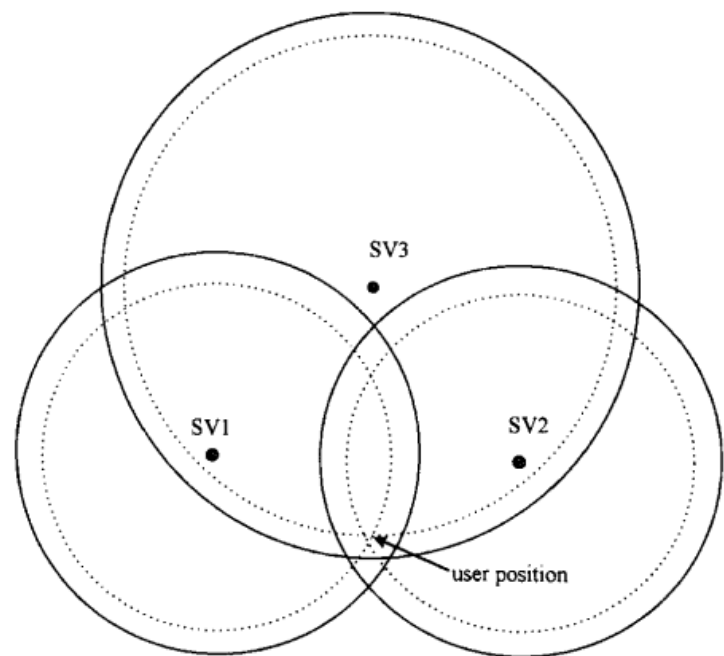
The geometrical dilution of precision (GDOP) is defined as

$$\text{GDOP} = \frac{1}{\sigma} \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + \sigma_b^2} \quad (2.58)$$

where σ is the measured rms error of the pseudorange, which has a zero mean, $\sigma_x \sigma_y \sigma_z$ are the measured rms errors of the user position in the xyz directions, and σ_b is the measured rms user clock error expressed in distance.



(a) Satellites and user form a straight line.



(b) Satellites and user form a quadrangle.

FIGURE 2.8 Three satellites are used to measure two-dimensional user position.

The position dilution of precision is defined as

$$\text{PDOP} = \frac{1}{\sigma} \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \quad (2.59)$$

The horizontal dilution of precision is defined as

$$\text{HDOP} = \frac{1}{\sigma} \sqrt{\sigma_x^2 + \sigma_y^2} \quad (2.60)$$

The vertical dilution of precision is

$$\text{VDOP} = \frac{\sigma_z}{\sigma} \quad (2.61)$$

The time dilution of precision is

$$\text{TDOP} = \frac{\sigma_b}{\sigma} \quad (2.62)$$

The smallest DOP value means the best satellite geometry for calculating user position. It is proved in reference 8 that in order to minimize the GDOP, the volume contained by the four satellites must be maximized. Assume that the

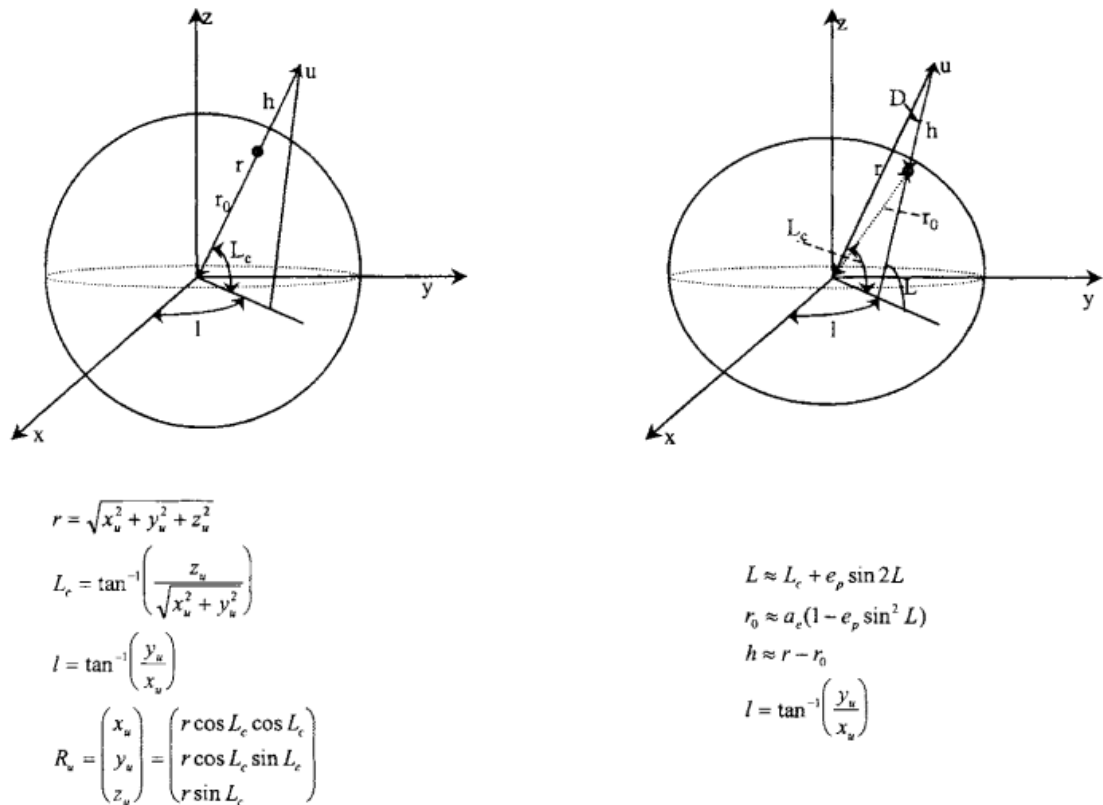


FIGURE 2.9 Relations to change from spherical to ellipsoidal earth.

four satellites form the optimum constellation. Under this condition the elevation angle is 0 degree and three of the four satellites form an equilateral triangle. The observer is at the center of the base of the tetrahedron. Under this condition, the DOP values are: $\text{GDOP} = \sqrt{3} \approx 1.73$, $\text{PDOP} = 2\sqrt{2/3} \approx 1.63$, $\text{HDOP} = \text{VDOP} = 2/\sqrt{3} \approx 1.15$, and $\text{TDOP} = 1/\sqrt{3} \approx 0.58$. These values can be considered as the minimum values (or the limits) of the DOPs. In selecting satellites, the DOP values should be as small as possible in order to generate the best user position accuracy.

2.16 SUMMARY

This chapter discusses the basic concept of solving the GPS user position. First use four or more satellites to solve the user position in terms of latitude, longitude, altitude, and the user clock bias as discussed in Section 2.5. However, the solutions obtained through this approach are for a spherical earth. Since the earth is not a perfect sphere, the latitude and altitude must be modified to reflect the ellipsoidal shape of the earth. Equations (2.51) and (2.57) are used to derive the desired values. These results are shown in Figure 2.9 as a quick reference. Finally, the selection of satellites and the DOP are discussed.

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