

$$P\{e|q, n\} = \frac{n}{q} P_e^{(0)} + \frac{q-n}{q} P_e^{(1)}. \quad (1)$$

Now, from previous work, we have

$$P_e^{(0)} = \frac{1}{2} \left\{ 1 - Q(\sqrt{a+b}, \sqrt{a-b}) + Q(\sqrt{a-b}, \sqrt{a+b}) \right\} \quad (2)$$

where the necessary integral in [3] has been expressed in terms of tabulated Q functions [6] defined by

$$Q(\alpha, \beta) = 1 - \int_0^\beta x I_0(\alpha x) \exp[-(x^2 + \alpha^2)/2] dx. \quad (3)$$

The quantities a and b in (2) are given by

$$a = \frac{E}{N_0} \left[\frac{E_I}{E} \left(1 + \frac{(2n-q-1)^2}{q-1} \right) + q \frac{E_P}{E} \right] \quad (4)$$

$$b = \frac{2E}{N_0} \sqrt{\frac{E_P E_I}{E^2}} \left(\sqrt{q-1} - \frac{2n-q-1}{\sqrt{q-1}} \right). \quad (5)$$

By symmetry, the probability $P_e^{(1)}$ is given by the same expression except that in (4) and (5) the expression $2n-q-1$ (which represents the surplus of 0's over 1's in the preceding $q-1$ reference digits given we are decoding a 0) is replaced by $q-1-2n$ which is the surplus of 1's over 0's given we are decoding a 1.

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Julian J. Busgang (A'52-M'55-SM'58), for a photograph and biography, please see page 419 of the June, 1968, issue of this TRANSACTIONS.

Myron Leiter (S'55-M'67), for a photograph and biography, please see page 419 of the June, 1968, issue of this TRANSACTIONS.

A Theoretical Study of Performance of an Orthogonal Multiplexing Data Transmission Scheme

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ABSTRACT: This paper presents a theoretical analysis of the performance of an orthogonal multiplexing data transmission scheme (parallel transmission scheme) subject to a number of degrading factors normally encountered by a practical operating system. The factors considered jointly are sampling time error, carrier phase offset, and nonideal phase characteristics of transmitting and receiving filters. Performance is measured by the familiar criterion of eye opening of the received data signal. A closed-form expression for the eye opening is obtained. It is shown that the lengthy nonlinear functions in the solution can be closely approximated by

simple piecewise linear functions for parameter values of interest. The optimum settings of the sampling time and the carrier phase are determined for given filter phase distortion. Also, considering all factors, simple formulas are developed for computing interchannel interferences, intersymbol interference, and the resulting eye opening. Simple relationships between the eye opening and filter phase distortion are explored, and a concept of *parametric eye* is introduced to aid in filter design. An illustrative example is considered which shows that the impairment of the system performance is not excessive for a reasonable range of system design parameters.

I. INTRODUCTION

SIGNAL DESIGN techniques for transmitting data by frequency division orthogonal multiplexing have been studied by a number of authors [1]-[5]. In [1]-[3], time-limited orthogonal signals have been used which have widely spread frequency spectra (e.g., a $\sin x/x$ spectrum). Consequently, when these signals are transmitted through

an essentially bandlimited transmission medium (e.g., a telephone message channel), certain portions of the signal spectrum will be cut off and interference will result. The use of bandlimited orthogonal signals which do not have this disadvantage has been proposed recently by Chang [4] and Becker [5]. A system using such signals has the following properties.

1) The maximum signaling rate for the given channel (Nyquist rate) can be approached without the use of sharp cutoff filters. This is probably the most important property in high-speed data transmission.

2) With the entire frequency band divided into a number of narrow channels, the frequency division multiplexing system (parallel system) is less sensitive to wide-band impulse noise interference than is the ordinary time division multiplexing system (serial system).

3) For the same reason, the parallel system is less sensitive to channel distortion. This has been shown quantitatively by Saltzberg [6].

4) As in the serial system [7], the parallel system provides maximum signal-to-noise ratio for analog data transmission and minimum probability of error for digital data transmission in the presence of bandlimited white Gaussian noise under the constraint of no intersymbol or interchannel interference.

5) In serial transmission, certain types of noise (such as time variable tone interference) may cause the entire system to be disabled. This can be avoided in the parallel system by adaptively dropping out only the channels affected. Thus, like a special serial transmission scheme [8], the flexibility of the parallel system enables it to adapt to a variety of noise environments.

Previous studies have not considered jointly practical design factors such as timing error, carrier phase offset, and nonideal filter characteristics. It was not known prior to this study what the combined effect of these factors would be on parallel system performance. This paper presents an analysis of the performance of a parallel system considering jointly timing error, carrier phase offset, and nonideal filter characteristics. Performance is measured by the familiar criterion of binary eye opening [7] of the received data signal. Closed-form analytical results are obtained.

Section II describes the mathematical model of this study, including a discussion of all other system parameters of interest. In Section III, we consider for given nonideal filter characteristics the optimum setting of the carrier phase and the sampling time, and the corresponding eye opening. Simple relations are derived to aid in filter design. In Section IV, eye opening is expressed as a function of all the factors considered. An illustrative example is given in Section V.

II. MATHEMATICAL MODEL

Fig. 1 illustrates the mathematical model of the parallel transmission system considered. The N parallel channels operate on equally spaced center frequencies and transmit at the same data rate. The signaling intervals of the

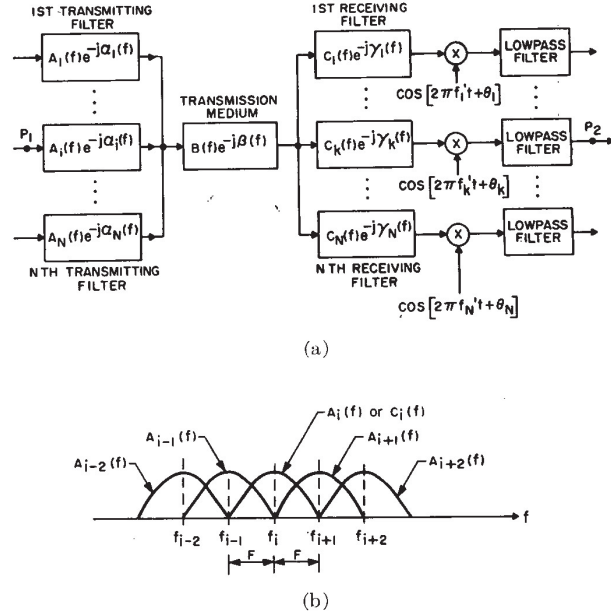


Fig. 1. Block diagram of parallel data transmission.

channels are synchronized. Let

$f_k \triangleq$ center frequency of the k th channel

$F \triangleq$ difference between two adjacent center frequencies

$f_k' \triangleq$ demodulating carrier frequency of the k th channel

$R \triangleq$ data rate per channel = $2F$ bauds¹

$T_0 \triangleq$ baud interval of each channel = $\frac{1}{2F}$ seconds. (1)

Each channel is capable of transmitting either binary or multilevel signals independent of the number of levels in other channels. In this paper we treat the case of binary signaling in each subchannel. A similar method of analysis could be developed to cover the multilevel case, but the results will become too complicated for practical use.

As it is shown in Fig. 1 (b), the transmitting filter $A_i(f)e^{-j\alpha_i(f)}$ of the i th channel is bandlimited from $f_i - F$ to $f_i + F$. (Note particularly that adjacent channels overlap in frequency.) It is also seen in Fig. 1 (b) that if the system consists of N channels, the overall bandwidth required is $(N + 1)F$. For this bandwidth, it is theoretically possible to transmit at a maximum baud rate (Nyquist rate) $R_{\max} = 2(N + 1)F$ bauds without intersymbol interference. However, this rate cannot be achieved in practice since it requires filters with perpendicular cutoffs and linear phases. In the present system, where each channel transmits at a data rate of $2F$ bauds, the overall data rate is $R = 2FN$ bauds = $(N/N + 1)R_{\max}$. Thus, as N increases, the overall data rate of the present system approaches the theoretical maximum rate $R_{\max}$.

¹ The data rate in bauds is equal to the number of symbols (binary or multilevel) transmitted in one second. The time interval between the transmission of two successive symbols is termed the baud interval.

Intersymbol and interchannel interferences are also eliminated, yet rectangular filtering is not required. Furthermore, the system has the other advantages given in Section I. A method of designing the signals of this system and the principle of operation have been given in [4]. The method is briefly summarized in the following for convenience.

For a given overall frequency band, the system designer may have some freedom in choosing the parameter N or F . After F is chosen, the center frequency f_k ($k = 1, \dots, N$) is determined by

$$f_k = \left(h + k - \frac{1}{2}\right)F \quad \text{Hz} \quad (2)$$

where h is an arbitrary positive integer including zero (note that h is the same for all k). Clearly h should be chosen so that the channels fall in the available frequency band. The demodulating carrier frequency f_k' ($k = 1, \dots, N$) is determined by

$$f_k' = f_k - \frac{F}{2} \quad \text{Hz}. \quad (3)$$

For a given transmission medium $B(f)e^{-j\beta(f)}$ and for the high data rate R , the transmitting filters $A_i(f)e^{-j\alpha_i(f)}$ ($i = 1, \dots, N$) can be shaped so that the signals at the receiving filter inputs are orthogonal. The requirement on $A_i(f)$ ($i = 1, \dots, N$) is

$$\begin{aligned} A_i^2(f)B^2(f) &= G_i + Q_i(f) > 0, & \text{for } f_i - F < f < f_i + F \\ &= 0, & \text{for } f < f_i - F, \\ & & f > f_i + F \end{aligned}$$

where G_i is an arbitrary constant and $Q_i(f)$ is a shaping function having odd symmetries about $f_i + F/2$ and $f_i - F/2$. Furthermore, the function $[G_i + Q_i(f)] \cdot [G_{i+1} + Q_{i+1}(f)]$ must have even symmetry about $f_i + F/2$. The requirement on $\alpha_i(f)$ ($i = 1, \dots, N$) is

$$\begin{aligned} \alpha_i(f) - \alpha_{i+1}(f) &= \pm \frac{\pi}{2} + \nu_i(f), \\ \text{for } f_i < f < f_i + F \quad (i = 1, 2, \dots, N-1) \end{aligned}$$

where $\nu_i(f)$ is an arbitrary phase function with odd symmetry about $f_i + F/2$. When the above conditions are satisfied, the signals at the receiving filter inputs are orthogonal. The receiving filters perform correlation detection to eliminate intersymbol and interchannel interferences.

In this paper, we choose $A_i(f)$ to satisfy both the orthogonality conditions above and an additional practical consideration. Since a parallel system usually consists of a number of channels, and filters are used in each channel, individual filter cost must be kept down. Therefore, sharp rolloff amplitude characteristics should be avoided. We choose to use the familiar sinusoidal rolloff characteristic for $A_i(f)$, i.e.,

$$A_i(f) = \cos \frac{\pi}{2} \frac{f - f_i}{F}, \quad \text{for } f_i - F \leq f \leq f_i + F. \quad (4)$$

This amplitude characteristic [sketched in Fig. 1 (b)] meets the orthogonality requirements above. Furthermore, the gradual rolloff from center frequency f_i to cutoff frequencies $f_i \pm F$ can be realized closely and economically.

The orthogonality conditions allow ripples to be present in the phase characteristics $\alpha_i(f)$. Such ripples are usually small and difficult to predict; therefore, they will not be included in this analysis. The phase characteristic $\alpha_i(f)$ is therefore assumed linear. Any linear phase can be specified by two parameters, bias ϵ and normalized slope ρ ; hence, $\alpha_i(f)$ can be written in the form²

$$\begin{aligned} \alpha_i(f) &= \epsilon_i + \frac{L}{F} \frac{\pi}{2} + \rho_i (f - f_i), & \text{for } f_i - F \leq f \leq f_i + F \\ &= \text{arbitrary}, & \text{otherwise} \end{aligned} \quad (5)$$

where ϵ_i and ρ_i ($i = 1, \dots, N$) are the phase parameters to be considered, and L is an arbitrary odd integer. We shall see that when the phase parameters satisfy the orthogonality conditions there is no interchannel or intersymbol interference. We shall also determine the system performance when the orthogonality conditions are not satisfied.

In parallel transmission the entire frequency band is divided into N narrow bands; therefore, it is reasonable to assume that with some equalization the amplitude and delay characteristics of the transmission medium are flat over each narrow band, i.e.,

$$B(f)e^{-j\beta(f)} = \frac{1}{F} \quad (6)$$

over each band, where the gain factor $1/F$ is introduced to normalize the maximum eye opening to unity, and a fixed delay is neglected since it simply changes the time origin.

For correlation detection, $C_i(f)$ should be proportional to $A_i(f)$ because $B(f)$ is assumed to be flat over each band. The proportional constant can be neglected, and we have

$$C_i(f) = A_i(f). \quad (7)$$

As for the transmitting filters, we assume that the receiving filter phase characteristics are linear. Therefore, $\gamma_i(f)$ ($i = 1, \dots, N$) can be written as

$$\gamma_i(f) = \epsilon_i' + \frac{L}{F} \frac{\pi}{2} + \rho_i' (f - f_i) \quad (8)$$

where ϵ_i' and ρ_i' ($i = 1, \dots, N$) are phase parameters to be considered. Note that ϵ_i' and ρ_i' of the receiving filter may differ from ϵ_i and ρ_i of the transmitting filter.

² Unless otherwise specified, phase angles in this paper are all in radians.

The demodulating carrier phase of the k th channel is denoted θ_k . The optimum setting of θ_k and the effect of its offset will be determined.

The low-pass filters can be easily realized. As will be shown, they need only to maintain a constant gain and linear phase in the narrow frequency range $0 < f < 3F/2$. Moreover, they may cut off gradually since the unwanted demodulation components are usually located at frequencies much higher than $3F/2$.

As in serial data transmission systems, the output of the low-pass filter should be sampled at the correct time instants to recover the information symbols. The optimum setting of the sampling time and the effect of its offset are set forth in this paper.

We define $h_{ik}(t)$ as the impulse response from point P_1 to point P_2 in Fig. 1 (a). It can be seen from Fig. 1 (a) that

$$h_{ik}(t) = 2 \operatorname{Re} \int_{-W}^W \frac{1}{2} e^{-j\theta_k} A_i(f + f_k') B(f + f_k') C_k(f + f_k') \cdot \exp \{ -j[\alpha_i(f + f_k') + \beta(f + f_k') + \gamma_k(f + f_k')] \} \cdot e^{j2\pi ft} df \quad (9)$$

where W is the cutoff frequency of the low-pass filter and "Re" denotes real part.

As clearly illustrated in Fig. 1 (b), $C_i(f)$ overlaps with only $A_{i-1}(f)$, $A_i(f)$, and $A_{i+1}(f)$; hence, $h_{ik}(t)$ is identically zero if $i \neq k-1, k$, or $k+1$. Therefore, the binary eye opening $E_k(t)$ at point P_2 [Fig. 1 (a)] can be written as

$$E_k(t) = h_{kk}(t) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |h_{kk}(t - nT_0)| - \sum_i \sum_{n=-\infty}^{\infty} |h_{ik}(t - nT_0)|$$

where

$$\begin{aligned} i &= k+1, & \text{for } k &= 1 \\ &= k+1 \text{ and } k-1, & \text{for } k &= 2, 3, \dots, N-1 \\ &= k-1, & \text{for } k &= N. \end{aligned} \quad (10)$$

We shall consider the interior channels ($k = 2, \dots, N-1$) because their eye openings are smaller than those of the two end channels (an end channel has only one adjacent interfering channel but an interior channel has two). The results can be easily extended to the two end channels.

III. ANALYSIS OF EYE OPENING

As can be seen from (1) to (10), the eye opening of the k th channel $E_k(t)$ is a function of a large number of parameters, namely: ϵ_{k-1} , ρ_{k-1} , ϵ_k , ρ_k , ϵ_{k+1} , ρ_{k+1} , ϵ_k' , ρ_k' , L , F , sampling time t , and demodulating carrier phase θ_k . Furthermore, it can be seen from the absolute signs in (10) that the problem of evaluating the eye opening is nonlinear. The first step in solving the nonlinear problem is to break the right-hand side of (10) into components as follows:

$$E_k(t) = D_k(t) - D_{k-1}(t) - D_{k+1}(t) \quad (11)$$

where

$$D_k(t) \triangleq h_{kk}(t) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |h_{kk}(t - nT_0)| \quad (12)$$

$$D_{k-1}(t) \triangleq \sum_{n=-\infty}^{\infty} |h_{(k-1)k}(t - nT_0)| \quad (13)$$

$$D_{k+1}(t) \triangleq \sum_{n=-\infty}^{\infty} |h_{(k+1)k}(t - nT_0)|. \quad (14)$$

The three components $D_k(t)$, $D_{k-1}(t)$, and $D_{k+1}(t)$ are evaluated separately. Note that $D_{k-1}(t)$ and $D_{k+1}(t)$ represent, respectively, interchannel interference from the lower and the upper adjacent channel, and that $1 - D_k(t)$ represents intersymbol interference. As indicated after (6), the signals are normalized such that $E_k(t)$ has a maximum possible value of unity (maximum eye opening equals unity).

A. Analysis of $D_k(t)$

For convenience, the quantity

$$\delta_k \triangleq \theta_k + \epsilon_k' - \frac{1}{2} \rho_k' - L \frac{\pi}{2} \quad (15)$$

will be used in the sequel instead of the demodulating carrier phase θ_k . A constant time delay of LT_0 seconds in $h_{(k-1)k}(t)$, $h_{kk}(t)$, and $h_{(k+1)k}(t)$ will be neglected since it does not vary with k .

It can be shown from (3), (5), (8), and (9) that

$$h_{kk}(t + \tau_k) = \operatorname{Re} \int_{-W}^W A_k(f + f_k') B(f + f_k') C_k(f + f_k') \cdot e^{-j\beta(f + f_k')} e^{-j\Phi_k} e^{j2\pi ft} df \quad (16)$$

where

$$\tau_k \triangleq \frac{\rho_k + \rho_k'}{2\pi F} \quad \text{seconds}$$

$$\Phi_k \triangleq \delta_k + \epsilon_k - \frac{1}{2} \rho_k \quad \text{radians.}$$

From (3), (4), (6), and (7)

$$\begin{aligned} &A_k(f + f_k') B(f + f_k') C_k(f + f_k') e^{-j\beta(f + f_k')} \\ &= \frac{1}{2F} + \frac{1}{2F} \sin \frac{\pi f}{F}, \quad \text{for } -\frac{F}{2} \leq f \leq \frac{3F}{2} \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (17)$$

It can be seen from (16) and (17) that the cutoff frequency W of the low-pass filter needs only be larger than $3F/2$. Substituting (17) into (16) and evaluating the integral from $-F/2$ to $3F/2$ give

$$h_{kk}(t + \tau_k) = \frac{\cos(\pi Ft - \Phi_k) \sin 2\pi Ft}{2\pi Ft(1 - 4F^2 t^2)}. \quad (18)$$

If we write $h_{kk}(t)$ from (18), τ_k would appear in every term in the right-hand side of the equation. This is inconvenient; therefore, we shall use $h_{kk}(t + \tau_k)$ and compute

$D_k(t + \tau_k)$. A change of variable then gives $D_k(t)$. From (12)

$$D_k(t + \tau_k) \triangleq h_{kk}(t + \tau_k) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |h_{kk}(t + \tau_k - nT_0)|. \quad (19)$$

Because of the absolute signs in (19), the functional form of $D_k(t + \tau_k)$ depends on the values of t and Φ_k . The evaluation of $D_k(t + \tau_k)$ is a nonlinear problem.

The method of nonlinear analysis used here is to divide t and Φ_k into appropriate disjoint regions such that the absolute signs can be resolved in each region according to n . The steps are very lengthy and therefore not included here. The resulting closed-form expression of $D_k(t + \tau_k)$ is long. For the sake of clarity it is given in Appendix I together with other long formulas.

B. Analysis of $D_{k-1}(t)$ and $D_{k+1}(t)$

By resolving the absolute signs as we did in evaluating $D_k(t + \tau_k)$, we obtain after lengthy steps the closed-form expressions of $D_{k-1}(t + \tau_{k-1})$ and $D_{k+1}(t + \tau_{k+1})$. These long formulas are also given in Appendix I for the sake of clarity.

The abbreviations τ_{k-1} , Φ_{k-1} , τ_{k+1} , and Φ_{k+1} defined in Appendix I will be used in the following sections.

C. Optimum Settings

We have derived closed-form equations for $D_{k-1}(t)$, $D_k(t)$, and $D_{k+1}(t)$ as functions of the 12 parameters ϵ_{k-1} to θ_k . However, it is inconvenient to compute the eye opening from the lengthy functions in these equations. For design purposes, it is found that these lengthy functions can be closely approximated by simple functions for parameter values of interest. It can be shown from (42) that the absolute maximum eye opening of unity is obtained only if $\Phi_k = 0$ and that the maximum attainable eye opening decreases monotonically as $|\Phi_k|$ increases. Hence, the system should be designed to operate near $\Phi_k = 0$ and it is sufficient to consider a large range of Φ_k such as from -0.35 to 0.35 radians (this is equivalent to a 40-degree range, sufficiently large for all practical purposes). Furthermore, we are not interested in very large degradations such as $D_{k-1}(t) > 0.4$ (more than 40-percent reduction in eye opening due to interference of the lower adjacent channel alone). Based on such considerations, we shall consider in the sequel a parameter region which can be specified by

$$\begin{aligned} |\Phi_k| &\leq 0.35 \text{ radians} \\ D_{k-1}(t) &\leq 0.4 \\ D_k(t) &\geq 0.65 \\ D_{k+1}(t) &\leq 0.4. \end{aligned} \quad (20)$$

It can be shown that (43) can be approximated in this region by

$$D_{k-1}(t) = 1.15 \left| \delta_k + \epsilon_{k-1} + \frac{\rho_{k-1}}{2} \right|. \quad (21)$$

The maximum approximation error is ± 0.01 (one percent of the full eye opening which equals unity). Note particularly that t does not appear in the right-hand side of (21); therefore, the interchannel interference from the lower adjacent channel is very nearly independent of time. This is a rather interesting result.

It can be shown that (44) can be approximated in the region of (20) by

$$D_{k+1}(t) = 1.15 \left| \pi \frac{t}{T_0} - \delta_k - \epsilon_{k+1} + \frac{\rho_{k+1}}{2} - \rho_k' \right|. \quad (22)$$

The maximum approximation error is only ± 0.01 .

Note that $D_{k-1}(t)$ and $D_{k+1}(t)$ are very different in form. A physical explanation of this is given in Appendix III.

A lengthy study shows that (42) can be approximated in the region of (20) by

$$\begin{aligned} D_k(t) &= 1 + 2.4 \frac{t}{T_0} - 1.2 \left(\delta_k + \epsilon_k + \frac{\rho_k}{7} + \frac{\rho_k'}{1.56} \right), \\ &\text{for } 0 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0.35, \quad \frac{t}{T_0} \leq \eta_2 \\ &\text{or } -0.35 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0, \quad \frac{t}{T_0} \leq \eta_3 \\ &= 1 + 1.45 \frac{t}{T_0} - 1.2(\delta_k + \epsilon_k - 0.11\rho_k + 0.39\rho_k'), \\ &\text{for } 0 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0.35, \quad \eta_2 \leq \frac{t}{T_0} \leq \eta_3 \\ &= 1 - 1.45 \frac{t}{T_0} + 1.2(\delta_k + \epsilon_k - 0.11\rho_k + 0.39\rho_k'), \\ &\text{for } -0.35 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0, \quad \eta_3 \leq \frac{t}{T_0} \leq \eta_2 \\ &= 1 - 2.4 \frac{t}{T_0} + 1.2 \left(\delta_k + \epsilon_k + \frac{\rho_k}{7} + \frac{\rho_k'}{1.56} \right), \\ &\text{for } 0 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0.35, \quad \eta_3 \leq \frac{t}{T_0} \\ &\text{or } -0.35 \leq \delta_k + \epsilon_k - \frac{\rho_k}{2} \leq 0, \quad \eta_2 \leq \frac{t}{T_0} \end{aligned} \quad (23)$$

where

$$\begin{aligned} \eta_2 &\triangleq \frac{1}{\pi} (\rho_k + \rho_k') \\ \eta_3 &\triangleq \frac{2}{\pi} \left(\delta_k + \epsilon_k + \frac{\rho_k}{2} \right). \end{aligned}$$

The approximate $D_k(t)$ in (23) is less than the exact $D_k(t)$ in (42). The maximum approximation error is -0.03 .

The sampling time and the demodulating carrier phase can be controlled in a system. Naturally one is interested in the optimum setting of the sampling time and the demodulating carrier phase and the resulting system performance. The following theorem provides formulas for determining these quantities. (As mentioned in (14), δ_k is used instead of the demodulating carrier phase θ_k .)

Theorem 1

Denote t^* and δ_k^* as the setting of t and δ_k that jointly maximize the eye opening $E_k(t)$, and denote E_k^* , D_{k-1}^* , D_k^* , and D_{k+1}^* as, respectively, the value of $E_k(t)$, $D_{k-1}(t)$, $D_k(t)$, and $D_{k+1}(t)$ at t^* and δ_k^* . If the parameter values are in the region of (20) and if the close approximations (21)–(23) are used, then

$$t^* = \frac{T_0}{\pi} \left(-\epsilon_{k-1} - \frac{\rho_{k-1}}{2} + \epsilon_{k+1} - \frac{\rho_{k+1}}{2} + \rho_k' \right) \quad (24)$$

$$\delta_k^* = -\epsilon_{k-1} - \frac{\rho_{k-1}}{2} \quad (25)$$

$$D_{k-1}^* = D_{k+1}^* = 0 \text{ (force interchannel interferences to be zero)} \quad (26)$$

$$\begin{aligned} E_k^* &= D_k^* \\ &= 1 + 0.369 Z_k - 0.231 Z_{k+1}, \\ &\quad \text{for } Z_{k+1} \geq 0, Z_k \leq -Z_{k+1} \\ &\quad \text{or } Z_{k+1} \leq 0, Z_k \leq Z_{k+1} \\ &= 1 + 0.218 Z_k - 0.382 Z_{k+1}, \\ &\quad \text{for } Z_{k+1} \geq 0, -Z_{k+1} \leq Z_k \leq Z_{k+1} \\ &= 1 - 0.218 Z_k + 0.382 Z_{k+1}, \\ &\quad \text{for } Z_{k+1} \leq 0, Z_{k+1} \leq Z_k \leq -Z_{k+1} \\ &= 1 - 0.369 Z_k + 0.231 Z_{k+1}, \\ &\quad \text{for } Z_{k+1} \geq 0, Z_{k+1} \leq Z_k \\ &\quad \text{or } Z_{k+1} \leq 0, -Z_{k+1} \leq Z_k \end{aligned} \quad (27)$$

where

$$\begin{aligned} Z_i &\triangleq 2\epsilon_i - 2\epsilon_{i-1} - \rho_i - \rho_{i-1} \\ i &= k, k+1. \end{aligned}$$

Theorem 1 is proven in Appendix II. As explicitly stated, Theorem 1 is developed for the parameter region specified in (20). It is easily seen from (26) and (25) that the parameters are in this region (hence Theorem 1 can be used) if

$$\begin{aligned} E_k^* &\geq 0.65 \\ \left| \frac{Z_k}{2} \right| &\triangleq \left| \epsilon_k - \epsilon_{k-1} - \frac{\rho_k}{2} - \frac{\rho_{k-1}}{2} \right| \leq 0.35 \text{ radians.} \end{aligned} \quad (28)$$

As discussed previously, these two conditions should be easily satisfied.

D. Uses of the Theorem

For a given filter design, the values of ϵ_{k-1} , ρ_{k-1} , ϵ_k , ρ_k , ϵ_{k+1} , ρ_{k+1} , ϵ_k' , ρ_k' , L , and F are known. Now we can easily determine the optimum sampling time t^* from (24), the optimum demodulating carrier phase θ_k^* from (25) and (15), and, most importantly, the maximum eye opening E_k^* from (27). The quantity $1 - E_k^*$ is the closure of the eye and can be used in determining whether the filter design is satisfactory.

A simple result should be emphasized. As shown in Appendix II, $D_{k-1}(t)$ is reduced to zero by setting δ_k to δ_k^* in (25), and then $D_{k+1}(t)$ is reduced to zero by setting t to t^* in (24). Thus, an important result is that the demodulating carrier phase should first be set to eliminate interchannel interference from the lower adjacent channel, and then the sampling time should be set to eliminate interference from the upper adjacent channel.

Next we consider a simple relationship between the filter phase parameters ϵ_i and ρ_i . It is seen from (27) that E_k^* depends only on Z_k and Z_{k+1} . This means that the eye opening of the second channel E_2^* depends only on Z_2 and Z_3 , the eye opening of the third channel E_3^* depends only on Z_3 and Z_4 , etc. Therefore, the eye openings E_2^* , E_3^* , $\dots$, E_{N-1}^* depend on a single set of parameters Z_i ($i = 2, 3, \dots, N$). It is seen from (27) that $E_k^* \leq 1$ ($k = 2, 3, \dots, N-1$) as expected and that the equality sign holds if and only if

$$Z_i = 0 \quad (i = 2, 3, \dots, N). \quad (29)$$

Thus, to obtain the absolute maximum eye opening of unity (i.e., to completely eliminate intersymbol and interchannel interference), the filters must be designed to satisfy (29). When (29) is not satisfied, we use the theorem to determine the eye opening. Letting $0 < y < 0.35$ be an arbitrary constant, it can be shown from (27) that the requirement $E_k^* \geq 1 - y$ is met if and only if Z_k and Z_{k+1} are in the closed region $P_1P_2P_3P_4P_5P_6$ in Fig. 2. Since this region contains the region $P_2P_3P_5P_6$, we have the following corollary.

Corollary

For any $k = 2, 3, \dots, N-1$, the requirement

$$E_k^* \geq 1 - y \quad (30)$$

is met if Z_k and Z_{k+1} are in the region $P_2P_3P_5P_6$ which can be specified by

$$\begin{aligned} |0.382 Z_k - 0.218 Z_{k+1}| &\leq y \\ |0.218 Z_k - 0.382 Z_{k+1}| &\leq y. \end{aligned} \quad (31)$$

The region $P_2P_3P_5P_6$ is used because of its simple shape and because very little is lost in substituting $P_2P_3P_5P_6$ for $P_1P_2P_3P_4P_5P_6$ (see Fig. 2). We may call the region $P_2P_3P_5P_6$ the *parametric eye* and rewrite the corollary as

$$E_k^* \geq 1 - y$$

if Z_k and Z_{k+1} are in the parametric eye. The parametric eye is sketched in Fig. 3 for $y = 0.05$ and 0.1 . The simple

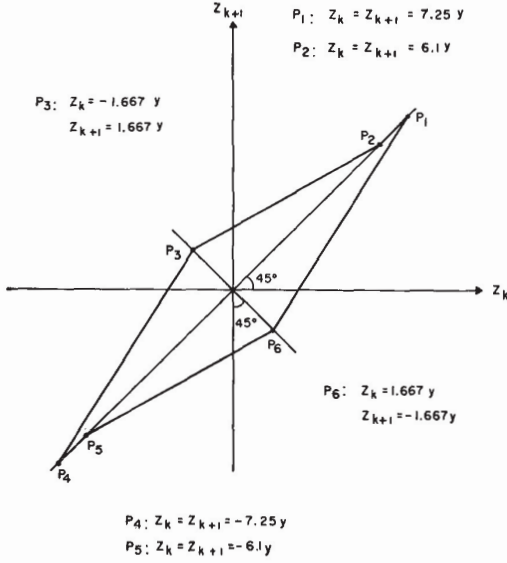


Fig. 2. $E_k^* \geq 1 - y$ if and only if Z_k and Z_{k+1} are in the closed region $P_1P_2P_3P_4P_5P_6$.

picture and rule help in visualizing the relationship between the parameters E_k^* , Z_k , and Z_{k+1} and in designing the filters. For instance, ideally we would like to have $Z_k = Z_{k+1} = 0$ (the origin). If this is not satisfied, we see from Fig. 3 that errors in the first and third quadrants are less critical than errors in the second and fourth quadrants. Note also that large errors can be tolerated along the line $Z_k = Z_{k+1}$. For example, there is only a 10 percent reduction in eye opening ($E_k^* = 0.9$) when Z_k and Z_{k+1} are as large as 0.61 radians (35 degrees).

IV. JOINT ERROR ANALYSIS

We have considered phase parameter errors, the optimum setting of the demodulating carrier phase and the sampling time, and the corresponding eye opening. Now consider the eye opening in the presence of phase parameter errors, demodulating carrier phase offset, and sampling time error. Let the actual setting of δ_k be

$$\delta_k = \delta_k^* + \tilde{\delta}_k \quad (32)$$

where δ_k^* is the optimum setting in (25) and $\tilde{\delta}_k$ is the demodulating carrier phase offset. From (45) in Appendix II the optimum sampling time is

$$t^* = \frac{T_0}{\pi} \left(\delta_k + \epsilon_{k+1} - \frac{\rho_{k+1}}{2} + \rho_k' \right). \quad (33)$$

The actual sampling time is

$$t = t^* + \tilde{t} \quad (34)$$

where $\tilde{t}$ is the sampling time error. From (32) to (34)

$$t = \frac{T_0}{\pi} \left(-\epsilon_{k-1} - \frac{\rho_{k-1}}{2} + \tilde{\delta}_k + \epsilon_{k+1} - \frac{\rho_{k+1}}{2} + \rho_k' \right) + \tilde{t}. \quad (35)$$

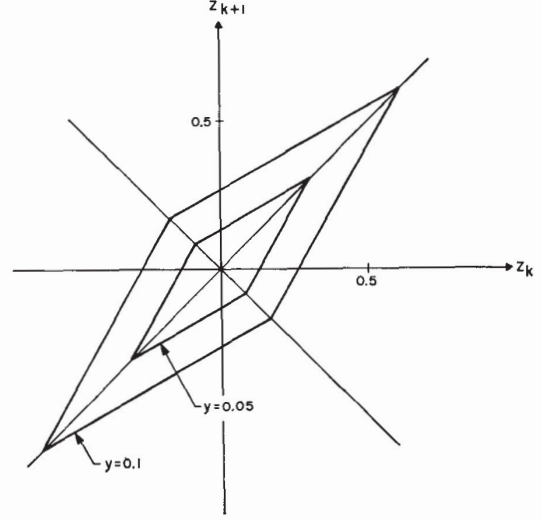


Fig. 3. Parametric eye.

Theorem 2

Denote $\tilde{D}_{k-1}$, $\tilde{D}_k$, $\tilde{D}_{k+1}$, and $\tilde{E}_k$, respectively, as the value of $D_{k-1}(t)$, $D_k(t)$, $D_{k+1}(t)$, and $E_k(t)$ when δ_k and t are set as in (32) and (35). If the parameter values are in the region of (20) and if the close approximations (21)–(23) are used, then

$$\tilde{D}_{k-1} = 1.15 |\tilde{\delta}_k| \quad (36)$$

$$\tilde{D}_{k+1} = 1.15 \left| \pi \frac{\tilde{t}}{T_0} \right| \quad (37)$$

$$\begin{aligned} \tilde{E}_k &= \tilde{D}_k - \tilde{D}_{k-1} - \tilde{D}_{k+1} \\ &= 1 + 0.369 \tilde{Z}_k - 0.231 \tilde{Z}_{k+1} - 1.15 |\tilde{\delta}_k| - 1.15 \\ &\quad \times \left| \pi \frac{\tilde{t}}{T_0} \right|, \text{ for } \tilde{Z}_{k+1} \geq 0, \tilde{Z}_k \leq -\tilde{Z}_{k+1} \\ &\quad \text{or } \tilde{Z}_{k+1} \leq 0, \tilde{Z}_k \leq \tilde{Z}_{k+1} \\ &= 1 + 0.218 \tilde{Z}_k - 0.382 \tilde{Z}_{k+1} - 1.15 |\tilde{\delta}_k| - 1.15 \\ &\quad \times \left| \pi \frac{\tilde{t}}{T_0} \right|, \text{ for } \tilde{Z}_{k+1} \geq 0, -\tilde{Z}_{k+1} \leq \tilde{Z}_k \leq \tilde{Z}_{k+1} \\ &= 1 - 0.218 \tilde{Z}_k + 0.382 \tilde{Z}_{k+1} - 1.15 |\tilde{\delta}_k| - 1.15 \\ &\quad \times \left| \pi \frac{\tilde{t}}{T_0} \right|, \text{ for } \tilde{Z}_{k+1} \leq 0, \tilde{Z}_{k+1} \leq \tilde{Z}_k \leq -\tilde{Z}_{k+1} \\ &= 1 - 0.369 \tilde{Z}_k + 0.231 \tilde{Z}_{k+1} - 1.15 |\tilde{\delta}_k| - 1.15 \\ &\quad \times \left| \pi \frac{\tilde{t}}{T_0} \right|, \text{ for } \tilde{Z}_{k+1} \geq 0, \tilde{Z}_k \geq \tilde{Z}_{k+1} \\ &\quad \text{or } \tilde{Z}_{k+1} \leq 0, \tilde{Z}_k \geq -\tilde{Z}_{k+1} \end{aligned} \quad (38)$$

where

$$\begin{aligned}\bar{Z}_k &\triangleq Z_k + 2\bar{\delta}_k \\ \bar{Z}_{k+1} &\triangleq Z_{k+1} + 2\pi \frac{\bar{l}}{T_0}\end{aligned}$$

Theorem 2 can be proven by substituting (32) and (35) into (21)–(23) and rearranging the results. As stated explicitly, Theorem 2 is developed for parameter values in the region of (20). The parameter values are in this region (hence Theorem 2 can be used) if

$$\begin{aligned}\left| \epsilon_k - \epsilon_{k-1} - \frac{\rho_k}{2} - \frac{\rho_{k-1}}{2} + \bar{\delta}_k \right| &\leq 0.35 \text{ radians} \\ \bar{D}_{k-1} &\leq 0.4 \\ \bar{D}_k &\geq 0.65 \\ \bar{D}_{k+1} &\leq 0.4.\end{aligned}\quad (39)$$

As discussed previously, these conditions should be easily satisfied.

From Theorem 2, $\bar{E}_k$ can be easily computed for given phase parameter errors, demodulating carrier phase offset, and timing error. In certain cases, one may wish to plot $\bar{E}_k$ versus $\bar{\delta}_k$ or $\bar{l}/T_0$. Note that (38) is linear between the five breakpoints:

$$\begin{aligned}\bar{\delta}_k = 0, \quad \frac{\bar{l}}{T_0} = 0, \quad \bar{Z}_{k+1} = 0 \\ \bar{Z}_k = -\bar{Z}_{k+1}, \quad \bar{Z}_k = \bar{Z}_{k+1}.\end{aligned}$$

Hence to plot $\bar{E}_k$ versus $\bar{\delta}_k$ or $\bar{l}/T_0$, it is only necessary to determine the breakpoints and the end points and connect them with straight lines. For instance, to plot $\bar{E}_k$ versus $\bar{l}/T_0$ for a fixed $\bar{\delta}_k$ over the range $-\tau \leq \bar{l}/T_0 \leq \tau$, compute $\bar{E}_k$ for the two end points $\bar{l}/T_0 = -\tau, \tau$ and for the last four breakpoints (the first breakpoint no longer exists for $\bar{\delta}_k$ is fixed). Then we connect these points with straight lines to obtain the curve. (See the example in the next section and Fig. 5.)

V. AN EXAMPLE

The phase parameter errors can be controlled by specifications. A standard way of specifying the tolerances of the parameters is by using upper and lower bounds. For instance, $\rho_k = 0 \pm b$ or equivalently $|\rho_k| \leq b$. We consider a reasonable specification as follows:

$$\begin{aligned}\epsilon_i - \epsilon_{i-1} &= 0 \pm 3 \text{ degrees} \\ \rho_i &= 0 \pm 4 \text{ degrees}\end{aligned}\quad (40)$$

and examine what performance can be expected. Other values of the bounds can be considered in a similar manner.

Note that when the parameters are at the centers of the specified ranges, the absolute maximum eye opening of

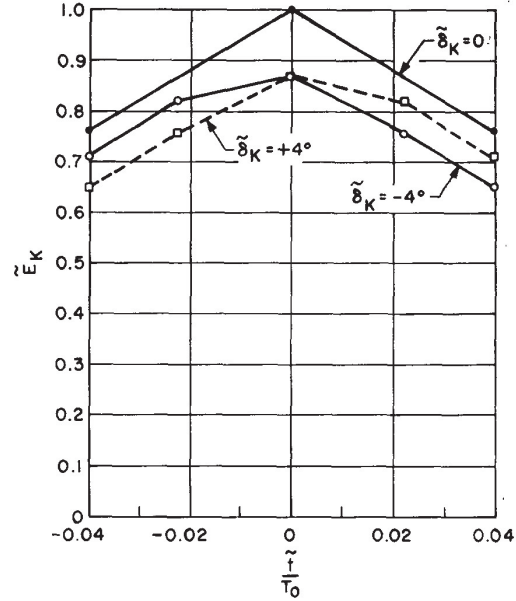


Fig. 4. Eye opening $\bar{E}_k$ versus timing error $\bar{l}$ and demodulating carrier phase offset $\bar{\delta}_k$, for the case $Z_k = Z_{k+1} = 0$.

unity is obtained. Also note that phase angles in this paper are all in radians; degrees are used in the above specifications because they are familiar to engineers. They must be converted into radians in computations.

Theorems 1 and 2 will be used in the following since the parameter values considered easily satisfy (28) and (39). For instance, in the following

$$\left| \epsilon_k - \epsilon_{k-1} - \frac{\rho_k}{2} - \frac{\rho_{k-1}}{2} + \bar{\delta}_k \right| \leq 0.192 \text{ radians}.$$

This easily satisfies the first inequality in (39).

Consider the worst combination of phase parameter errors which minimizes E_k^* . It can be easily shown from (27) and (40) that this worst combination is

$$\begin{aligned}\epsilon_k - \epsilon_{k-1} &= -3 \text{ degrees} \\ \epsilon_{k+1} - \epsilon_k &= 3 \text{ degrees} \\ \rho_{k-1} &= 4 \text{ degrees} \\ \rho_k &= -4 \text{ degrees} \\ \rho_{k+1} &= -4 \text{ degrees}.\end{aligned}\quad (41)$$

To obtain this combination we have to set all the errors to the limits of their six- or eight-degree ranges, and to choose the signs such that the errors all reduce the eye. Clearly this worst combination rarely takes place. From (41) and the second equality of (27), the minimum E_k^* for this worst combination is

$$E_k^*|_{\min} = 0.884.$$

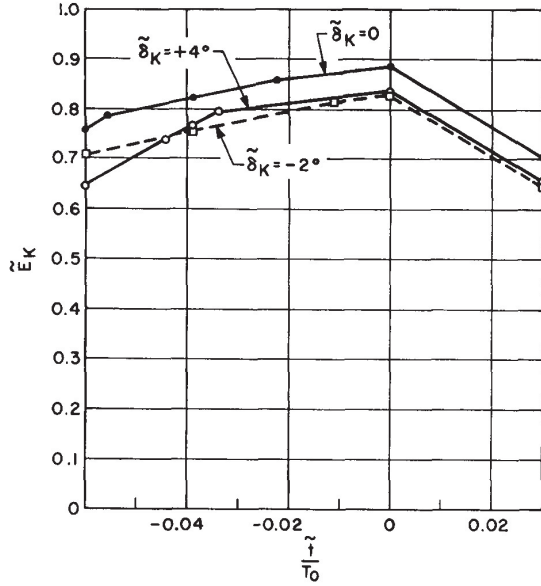


Fig. 5. Eye opening $\tilde{E}_k$ versus timing error $\tilde{l}$ and demodulating carrier phase offset $\tilde{\delta}_k$, for the worst combination of filter phase parameter errors in (41).

Next consider demodulating carrier phase offset and timing error for two sets of phase parameter values:

- 1) The ϵ_i and ρ_i are such that $Z_k = Z_{k+1} = 0$.
- 2) The ϵ_i and ρ_i are as specified in (41).

In the first case there is no reduction in eye opening due to phase parameters. In the second case the worst combination of phase parameter errors is assumed. Using (38) of Theorem 2, we can easily compute the eye opening $\tilde{E}_k$ versus $\tilde{\delta}_k$ and $\tilde{l}$. The results are plotted in Figs. 4 and 5 for Cases 1 and 2, respectively. Each curve shows $\tilde{E}_k$ versus $\tilde{l}/T_0$ for a fixed $\tilde{\delta}_k$.

It is easily seen from Fig. 4 that if the demodulating carrier phase is allowed to vary over an eight-degree range ($\tilde{\delta}_k$ from -4° to 4°), and if the sampling time is allowed to vary over a seven-percent range of the baud interval (for instance, $\tilde{l}/T_0$ from -0.04 to 0.03 when $\tilde{\delta}_k = -4^\circ$), the eye opening will be larger than 0.7. From Fig. 5 we see that if the demodulating carrier phase varies over a six-degree range ($\tilde{\delta}_k$ from -2° to 4°), and if the sampling time varies over a seven-percent range of the baud interval (for instance, $\tilde{l}/T_0$ from -0.05 to 0.02 when $\tilde{\delta}_k = -2^\circ$), the eye opening will be larger than 0.7. From these we conclude that if

- 1) the phase parameters vary over six- or eight-degree ranges as specified in (40),
- 2) the demodulating carrier phase varies over a six-degree range,
- 3) and the sampling time varies over a seven-percent range of the baud interval,

then the eye opening will be larger than 0.7.

VI. SUMMARY AND CONCLUSIONS

This paper presents a theoretical analysis of the performance of a parallel data transmission scheme. The performance is based on the eye opening of the received data signal, considering jointly sampling time error, demodulating carrier phase offset, and nonideal phase characteristics of all the transmitting and receiving filters. An analytic study of the nonlinear problem gives the closed-form expression of the eye opening as a function of all the parameters considered. It is shown that the lengthy closed-form expression can be closely approximated with simple piecewise linear functions for parameter values of interest. Accuracies of the approximations are given.

For given filter phase parameters, the joint optimum setting of the sampling time and demodulating carrier phase is determined, and the corresponding eye opening E_k^* found. These results are based on the close approximation mentioned above. It is shown that the demodulating carrier phase should first be set as in (25) to eliminate interchannel interference from the lower adjacent channel, and then the sampling time should be set as in (24) to eliminate interference from the upper adjacent channel. The quantity $1 - E_k^*$ is the closure of the eye due to nonideal filter phase characteristics and hence can be used in determining whether the filter design is satisfactory. It is seen that E_k^* depends only on the phase parameters of the transmitting filters. Therefore, if the sampling time and the demodulating carrier phase can be adjusted, it is not necessary to control the phase parameters of the receiving filters. It is shown that the absolute maximum eye opening of unity is obtained if and only if the transmitting filter phase parameters satisfy the simple requirement in (29). Furthermore, it is shown that $E_k^* \geq 1 - \gamma$ if and only if the transmitting filter phase parameters are in the region in Fig. 2, or, to a very close approximation, in the simple region termed *parametric eye* in Fig. 3. This clearly illustrates how the eye opening varies with the phase parameters. As discussed in Section III-D, such simple figures and rules can be of considerable value in practical filter design.

After the optimum settings are determined, sampling time error and demodulating carrier phase offset are considered. Simple formulas are given in Theorem 2 for computing the eye opening $\tilde{E}_k$ from a combination of sampling time error, demodulating carrier phase offset, and filter phase parameter errors. One can also compute the interchannel interferences $\tilde{D}_{k-1}$ and $\tilde{D}_{k+1}$ and the intersymbol interference $1 - \tilde{D}_k$ separately for comparison purposes. As can be seen from (36) and (37), the lower adjacent channel interferes (i.e., $\tilde{D}_{k-1} \neq 0$) only when there is demodulating carrier phase offset, and the upper adjacent channel interferes only when there is sampling time error.

An illustrative example is given in Section V, in which we consider some ranges of the parameter values and draw some conclusions about the performance of the system. Similar results can be obtained for other cases.

APPENDIX I

The closed-form expression of $D_k(t + \tau_k)$ is

$$\begin{aligned}
 D_k(t + \tau_k) &= \frac{1}{\pi} \left[\frac{1}{2Ft} - \frac{1}{1 + 2Ft} - r(t) \right] \\
 &\quad \times \sin 2\pi Ft \cos(\pi Ft - \Phi_k) \\
 &\quad + \frac{1}{\pi} \left[\frac{1}{2Ft} - \frac{2}{1 + 2Ft} + \frac{1}{2 + 2Ft} - r(t) \right] \\
 &\quad \times \sin 2\pi Ft \sin(\pi Ft - \Phi_k), \\
 &\quad \text{for } 0 \leq \Phi_k \leq \frac{\pi}{2}, \quad -T_0 + \frac{\Phi_k}{\pi F} \leq t \leq 0 \\
 &\quad \text{or } -\frac{\pi}{2} \leq \Phi_k \leq 0, \quad -T_0 \leq t \leq \frac{\Phi_k}{\pi F} \\
 &= \frac{1}{\pi} \left[\frac{1}{2Ft} - \frac{1}{1 + 2Ft} - r(t) \right] \\
 &\quad \times \sin 2\pi Ft \cos(\pi Ft - \Phi_k) \\
 &\quad - \frac{1}{\pi} \left[\frac{1}{2Ft} - \frac{2}{1 + 2Ft} + \frac{1}{2 + 2Ft} - r(t) \right] \\
 &\quad \times \sin 2\pi Ft \sin(\pi Ft - \Phi_k), \\
 &\quad \text{for } -\frac{\pi}{2} \leq \Phi_k \leq 0, \quad \frac{\Phi_k}{\pi F} \leq t \leq 0 \\
 &= \frac{1}{\pi} \left[\frac{1}{2Ft} + \frac{1}{1 - 2Ft} + r(t) \right] \\
 &\quad \times \sin 2\pi Ft \cos(\pi Ft - \Phi_k) \\
 &\quad + \frac{1}{\pi} \left[\frac{1}{2Ft} + \frac{2}{1 - 2Ft} - \frac{1}{2 - 2Ft} + r(t) \right] \\
 &\quad \times \sin 2\pi Ft \sin(\pi Ft - \Phi_k), \\
 &\quad \text{for } 0 \leq \Phi_k \leq \frac{\pi}{2}, \quad 0 \leq t \leq \frac{\Phi_k}{\pi F} \\
 &= \frac{1}{\pi} \left[\frac{1}{2Ft} + \frac{1}{1 - 2Ft} + r(t) \right] \\
 &\quad \times \sin 2\pi Ft \cos(\pi Ft - \Phi_k) \\
 &\quad - \frac{1}{\pi} \left[\frac{1}{2Ft} + \frac{2}{1 - 2Ft} - \frac{1}{2 - 2Ft} + r(t) \right] \\
 &\quad \times \sin 2\pi Ft \sin(\pi Ft - \Phi_k), \\
 &\quad \text{for } 0 \leq \Phi_k \leq \frac{\pi}{2}, \quad \frac{\Phi_k}{\pi F} \leq t \leq T_0 \\
 &\quad \text{or } -\frac{\pi}{2} \leq \Phi_k \leq 0, \quad 0 \leq t \leq T_0 + \frac{\Phi_k}{\pi F} \quad (42)
 \end{aligned}$$

where

$$r(t) \triangleq \sum_{n=1}^{\infty} \frac{2n(-1)^n}{n^2 - (2Ft)^2}.$$

The closed-form expression of $D_{k-1}(t + \tau_{k-1})$ is

$$\begin{aligned}
 D_{k-1}(t + \tau_{k-1}) &\triangleq \sum_{n=-\infty}^{\infty} |h_{(k-1)k}(t + \tau_{k-1} - nT_0)| \\
 &= \frac{2|\sin \Phi_{k-1}|}{\pi} \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} + \frac{\sin \pi Ft}{4Ft - (2Ft)^2} \right] \\
 &\quad \text{for } 0 \leq t \leq T_0 \\
 &= \frac{2|\sin \Phi_{k-1}|}{\pi} \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} + \frac{\sin \pi Ft}{4Ft + (2Ft)^2} \right], \\
 &\quad \text{for } -T_0 \leq t \leq 0 \quad (43)
 \end{aligned}$$

where

$$\tau_{k-1} \triangleq \frac{1}{2\pi F} (\rho_{k-1} + \rho_k') \quad \text{seconds}$$

$$\Phi_{k-1} \triangleq \delta_k + \epsilon_{k-1} + \frac{\rho_{k-1}}{2} \quad \text{radians.}$$

The closed-form expression of $D_{k+1}(t + \tau_{k+1})$ is

$$\begin{aligned}
 D_{k+1}(t + \tau_{k+1}) &\triangleq \sum_{n=-\infty}^{\infty} |h_{(k+1)k}(t + \tau_{k+1} - nT_0)| \\
 &= \frac{2}{\pi} \sin(2\pi Ft - \Phi_{k+1}) \\
 &\quad \times \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} + \frac{\sin \pi Ft}{4Ft - (2Ft)^2} \right], \\
 &\quad \text{for } 0 \leq \Phi_{k+1} \leq \pi, \quad \frac{\Phi_{k+1}}{2\pi F} \leq t \leq T_0 \\
 &\quad \text{or } -\pi \leq \Phi_{k+1} \leq 0, \quad 0 \leq t \leq T_0 + \frac{\Phi_{k+1}}{2\pi F} \\
 &= -\frac{2}{\pi} \sin(2\pi Ft - \Phi_{k+1}) \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} \right. \\
 &\quad \left. + \frac{\sin \pi Ft}{4Ft - (2Ft)^2} \right], \\
 &\quad \text{for } 0 \leq \Phi_{k+1} \leq \pi, \quad 0 \leq t \leq \frac{\Phi_{k+1}}{2\pi F} \\
 &= \frac{2}{\pi} \sin(2\pi Ft - \Phi_{k+1}) \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} \right. \\
 &\quad \left. + \frac{\sin \pi Ft}{4Ft + (2Ft)^2} \right], \\
 &\quad \text{for } -\pi \leq \Phi_{k+1} \leq 0, \quad \frac{\Phi_{k+1}}{2\pi F} \leq t \leq 0 \\
 &= -\frac{2}{\pi} \sin(2\pi Ft - \Phi_{k+1}) \left[\frac{\cos \pi Ft}{1 - (2Ft)^2} \right. \\
 &\quad \left. + \frac{\sin \pi Ft}{4Ft + (2Ft)^2} \right], \\
 &\quad \text{for } 0 \leq \Phi_{k+1} \leq \pi, \quad -T_0 + \frac{\Phi_{k+1}}{2\pi F} \leq t \leq 0 \\
 &\quad \text{or } -\pi \leq \Phi_{k+1} \leq 0, \quad -T_0 \leq t \leq \frac{\Phi_{k+1}}{2\pi F} \quad (44)
 \end{aligned}$$

where

$$\tau_{k+1} \triangleq \frac{\rho_{k+1} + \rho_k'}{2\pi F} \quad \text{seconds}$$

$$\Phi_{k+1} \triangleq \delta_k + \epsilon_{k+1} - \frac{3}{2} \rho_{k+1} \quad \text{radians.}$$

APPENDIX II

Theorem 1 is proven in this appendix. It is clear from (21) to (23) that $E_k(t)$ is linear between the breakpoints

$$\begin{aligned} \delta_k + \epsilon_{k-1} + \frac{\rho_{k-1}}{2} &= 0 \\ \pi \frac{t}{T_0} - \delta_k - \epsilon_{k+1} + \frac{\rho_{k+1}}{2} - \rho_k' &= 0 \\ \frac{t}{T_0} &= \eta_2 \\ \frac{t}{T_0} &= \eta_3. \end{aligned}$$

The maximum of $E_k(t)$ must occur at a breakpoint since $E_k(t)$ is linear between the breakpoints. It can be shown that $E_k(t)$ is maximized with respect to t for all δ_k at the second breakpoint, i.e., at

$$\frac{t^*}{T_0} = \frac{1}{\pi} \left(\delta_k + \epsilon_{k+1} - \frac{\rho_{k+1}}{2} + \rho_k' \right) \quad (45)$$

where t^* denotes the optimum setting of T . Since $D_{k-1}(t)$ in (21) is independent of t , we only have to substitute (45) into (22) and (23). Substituting (45) into (22) gives

$$D_{k+1}(t^*) = 0. \quad (46)$$

Equation (46) shows that the sampling time should be set to eliminate interchannel interference from the upper adjacent channel.

Substituting (45) into (23) and rearranging the results we obtain after some steps

$$\begin{aligned} D_k(t^*) &= 1 + 0.739 \delta_k + 1.2 \epsilon_k - 0.132 \rho_k - 0.462 \\ &\quad \times \epsilon_{k+1} + 0.231 \rho_{k+1}, \quad \text{for } \xi_1 \leq \xi_2, \quad \delta_k \leq \xi_1 \\ &\quad \text{or } \xi_1 \geq \xi_2, \quad \delta_k \leq \xi_2 \\ &= 1 + 0.436 \delta_k + 1.2 \epsilon_k + 0.172 \rho_k - 0.764 \\ &\quad \times \epsilon_{k+1} + 0.382 \rho_{k+1}, \quad \text{for } \xi_1 \leq \xi_2, \quad \xi_1 \leq \delta_k \leq \xi_2 \\ &= 1 - 0.436 \delta_k - 1.2 \epsilon_k - 0.172 \rho_k + 0.764 \\ &\quad \times \epsilon_{k+1} - 0.382 \rho_{k+1}, \quad \text{for } \xi_1 \geq \xi_2, \quad \xi_2 \leq \delta_k \leq \xi_1 \\ &= 1 - 0.739 \delta_k - 1.2 \epsilon_k + 0.132 \rho_k + 0.462 \\ &\quad \times \epsilon_{k+1} - 0.231 \rho_{k+1}, \quad \text{for } \xi_1 \leq \xi_2, \quad \delta_k \geq \xi_2 \\ &\quad \text{or } \xi_1 \geq \xi_2, \quad \delta_k \geq \xi_1 \quad (47) \end{aligned}$$

where

$$\begin{aligned} \xi_1 &\triangleq \rho_k - \epsilon_{k+1} + \frac{\rho_{k+1}}{2} \\ \xi_2 &\triangleq -2\epsilon_k + \epsilon_{k+1} - \frac{\rho_{k+1}}{2}. \end{aligned}$$

It can be seen from (21), (46), and (47) that $E_k(t^*)$ is maximized with respect to δ_k if δ_k is selected such that

$$D_{k-1}(t^*) = 0$$

or if

$$\delta_k^* = -\epsilon_{k-1} - \frac{\rho_{k-1}}{2} \quad (48)$$

where δ_k^* denotes the optimum setting of δ_k . This means that the demodulating carrier phase should be set to eliminate interchannel interference from the lower adjacent channel. Substituting (48) into (45) gives (24) of Theorem 1. The maximum eye opening E_k^* at t^* and δ_k^* can be computed from (46) to (48). The result is (27). This completes the proof of Theorem 1.

APPENDIX III

An important point should be emphasized here. In some conventional parallel transmission schemes, the interferences from the two adjacent channels are the same. Thus, it might be a surprise to see from Appendix I that the two interferences $D_{k-1}(t)$ and $D_{k+1}(t)$ are very different. For instance, it is shown in Section C that $D_{k-1}(t)$ is very nearly independent of time, while $D_{k+1}(t)$ has a large variation with time. Rather than tracing the lengthy equations to explain this phenomenon, we offer a physical explanation which yields some insight. As can be seen from Fig. 1 (a), when an impulse is applied at the input of the $(k-1)$ th channel, the interference at the output of the k th receiving filter has an amplitude characteristic

$$H_{(k-1)k}(f) = A_{k-1}(f)B(f)C_k(f).$$

From (6), (7), and the shapes of the $A_i(f)$ in Fig. 1 (b), one can see that $H_{(k-1)k}(f)$ and $H_{(k+1)k}(f)$ are symmetrical about the center frequency f_k of the k th channel. In the present orthogonal multiplexing system, the demodulating carrier frequency is not equal to f_k , but is

$$f_k' = f_k - \frac{F}{2}.$$

Therefore, after demodulation, $H_{(k-1)k}(f + f_k')$ and $H_{(k+1)k}(f + f_k')$ are symmetrical about $F/2$ (not about the origin). Consequently, interchannel interference from the lower adjacent channel occupies the frequency band $|f| < F/2$, while interference from the upper adjacent channel occupies a higher frequency band $F/2 < |f| < 3F/2$. From the preceding we expect the two interferences to be very different.

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see also J.W. Smith, BSTJ 3/68 p. 273



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