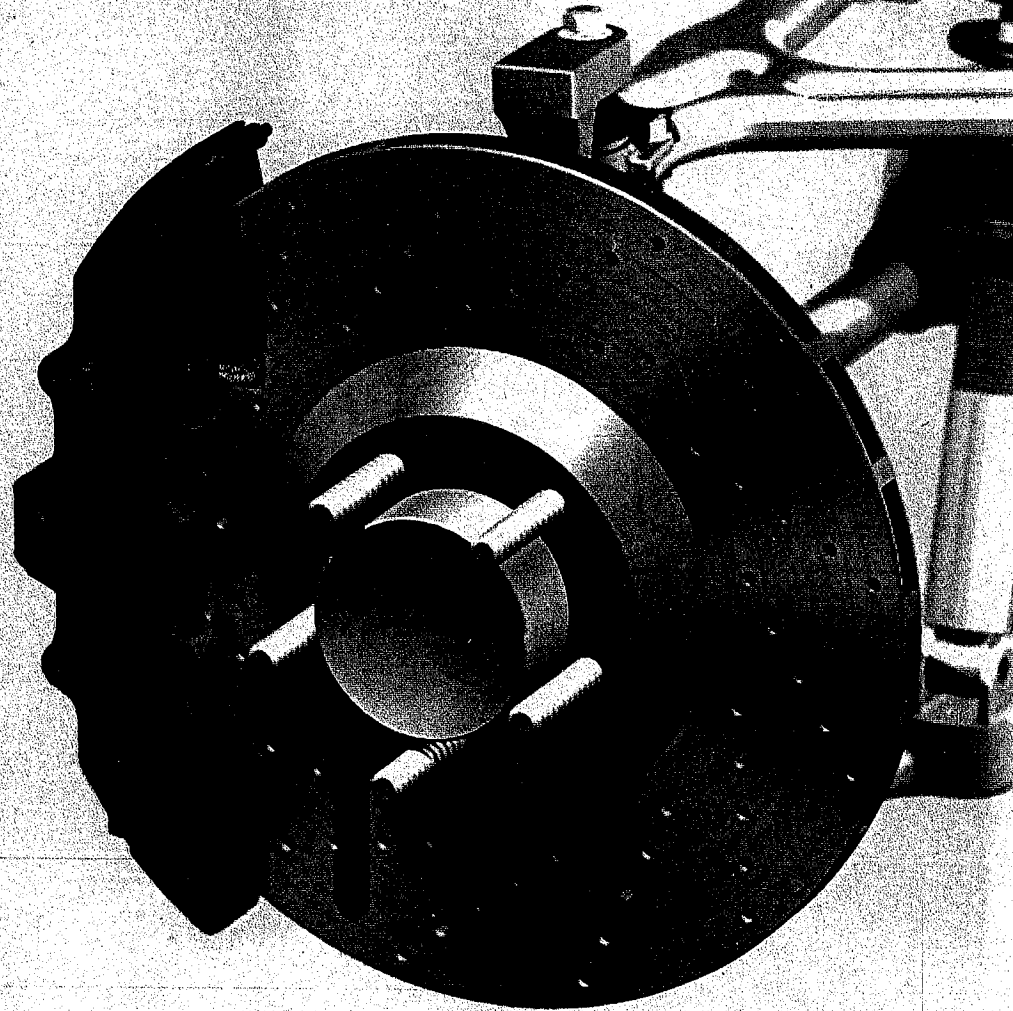


Richard G.
Budynas

J. Keith
Nisbett



Tenth Edition

Shigley's

Mechanical Engineering Design

Loco Cookers, EX1018
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Loco Crazy Good Cookers, Inc. v. North Atlantic Imports, LLC



SHIGLEY'S MECHANICAL ENGINEERING DESIGN, TENTH EDITION

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3

Load and Stress Analysis

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One of the main objectives of this book is to describe how specific machine components function and how to design or specify them so that they function safely without failing structurally. Although earlier discussion has described structural strength in terms of load or stress versus strength, failure of function for structural reasons may arise from other factors such as excessive deformations or deflections.

Here it is assumed that the reader has completed basic courses in statics of rigid bodies and mechanics of materials and is quite familiar with the analysis of loads, and the stresses and deformations associated with the basic load states of simple prismatic elements. In this chapter and Chap. 4 we will review and extend these topics briefly. Complete derivations will not be presented here, and the reader is urged to return to basic textbooks and notes on these subjects.

This chapter begins with a review of equilibrium and free-body diagrams associated with load-carrying components. One must understand the nature of forces before attempting to perform an extensive stress or deflection analysis of a mechanical component. An extremely useful tool in handling discontinuous loading of structures employs *Macaulay* or *singularity functions*. Singularity functions are described in Sec. 3-3 as applied to the shear forces and bending moments in beams. In Chap. 4, the use of singularity functions will be expanded to show their real power in handling deflections of complex geometry and statically indeterminate problems.

Machine components transmit forces and motion from one point to another. The transmission of force can be envisioned as a flow or force distribution that can be further visualized by isolating internal surfaces within the component. Force distributed over a surface leads to the concept of stress, stress components, and stress transformations (Mohr's circle) for all possible surfaces at a point.

The remainder of the chapter is devoted to the stresses associated with the basic loading of prismatic elements, such as uniform loading, bending, and torsion, and topics with major design ramifications such as stress concentrations, thin- and thick-walled pressurized cylinders, rotating rings, press and shrink fits, thermal stresses, curved beams, and contact stresses.

3-1 Equilibrium and Free-Body Diagrams

Equilibrium

The word *system* will be used to denote any *isolated* part or portion of a machine or structure—including all of it if desired—that we wish to study. A system, under this definition, may consist of a particle, several particles, a part of a rigid body, an entire rigid body, or even several rigid bodies.

If we assume that the system to be studied is motionless or, at most, has constant velocity, then the system has zero acceleration. Under this condition the system is said to be in *equilibrium*. The phrase *static equilibrium* is also used to imply that the system is *at rest*. For equilibrium, the forces and moments acting on the system balance such that

$$\sum \mathbf{F} = 0 \quad (3-1)$$

$$\sum \mathbf{M} = 0 \quad (3-2)$$

which states that *the sum of all force* and the *sum of all moment vectors* acting upon a system in equilibrium is zero.

Free-Body Diagrams

We can greatly simplify the analysis of a very complex structure or machine by successively isolating each element and studying and analyzing it by the use of *free-body diagrams*. When all the members have been treated in this manner, the knowledge obtained can be assembled to yield information concerning the behavior of the total system. Thus, free-body diagramming is essentially a means of breaking a complicated problem into manageable segments, analyzing these simple problems, and then, usually, putting the information together again.

Using free-body diagrams for force analysis serves the following important purposes:

- The diagram establishes the directions of reference axes, provides a place to record the dimensions of the subsystem and the magnitudes and directions of the known forces, and helps in assuming the directions of unknown forces.
- The diagram simplifies your thinking because it provides a place to store one thought while proceeding to the next.
- The diagram provides a means of communicating your thoughts clearly and unambiguously to other people.
- Careful and complete construction of the diagram clarifies fuzzy thinking by bringing out various points that are not always apparent in the statement or in the geometry of the total problem. Thus, the diagram aids in understanding all facets of the problem.
- The diagram helps in the planning of a logical attack on the problem and in setting up the mathematical relations.
- The diagram helps in recording progress in the solution and in illustrating the methods used.
- The diagram allows others to follow your reasoning, showing *all* forces.

EXAMPLE 3-1

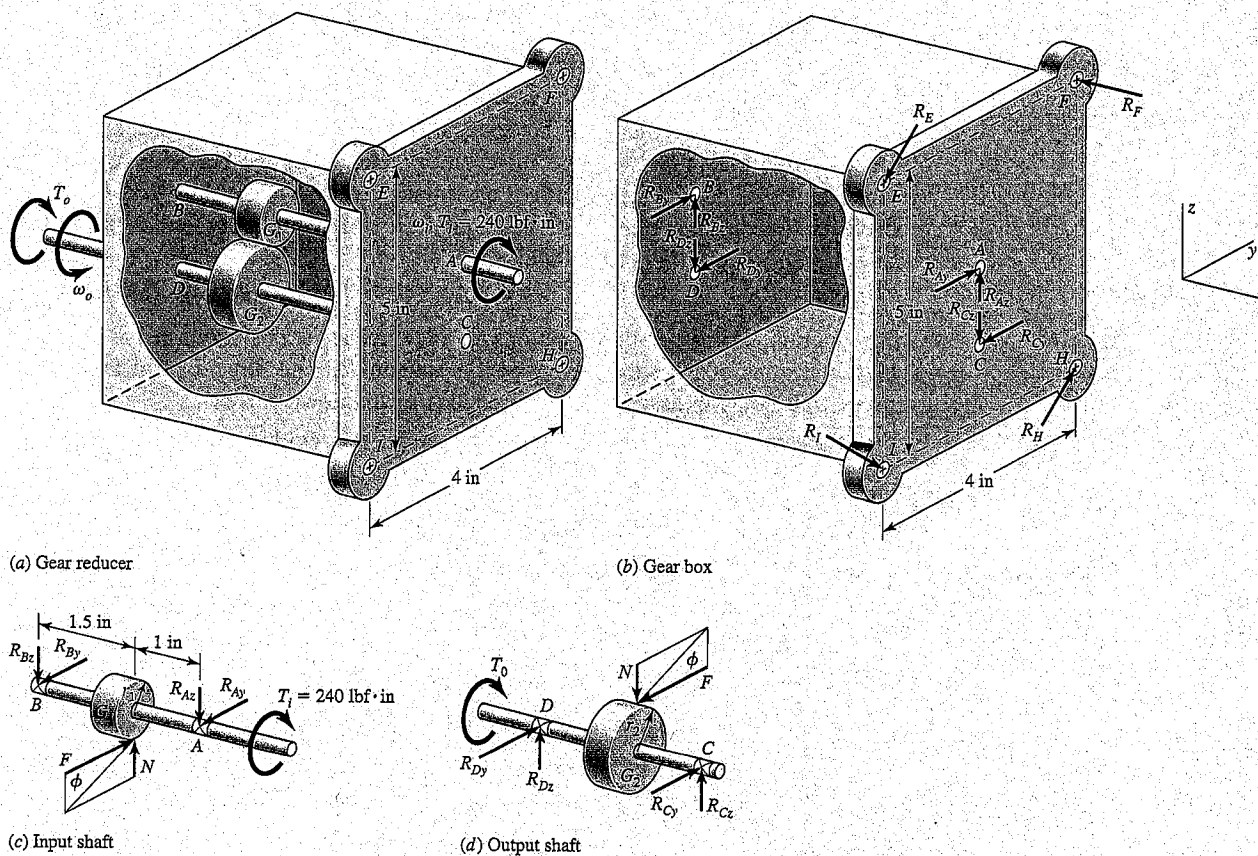
Figure 3-1*a* shows a simplified rendition of a gear reducer where the input and output shafts AB and CD are rotating at constant speeds ω_i and ω_o , respectively. The input and output torques (torsional moments) are $T_i = 240 \text{ lbf} \cdot \text{in}$ and T_o , respectively. The shafts are supported in the housing by bearings at A , B , C , and D . The pitch radii of gears G_1 and G_2 are $r_1 = 0.75 \text{ in}$ and $r_2 = 1.5 \text{ in}$, respectively. Draw the free-body diagrams of each member and determine the net reaction forces and moments at all points.

Solution

First, we will list all simplifying assumptions.

- 1 Gears G_1 and G_2 are simple spur gears with a standard pressure angle $\phi = 20^\circ$ (see Sec. 13-5).
- 2 The bearings are self-aligning and the shafts can be considered to be simply supported.
- 3 The weight of each member is negligible.
- 4 Friction is negligible.
- 5 The mounting bolts at E , F , H , and I are the same size.

The separate free-body diagrams of the members are shown in Figs. 3-1*b-d*. Note that Newton's third law, called *the law of action and reaction*, is used extensively where each member mates. The force transmitted between the spur gears is not tangential but at the pressure angle ϕ . Thus, $N = F \tan \phi$.

**Figure 3-1**

(a) Gear reducer; (b-d) free-body diagrams. Diagrams are not drawn to scale.

Summing moments about the x axis of shaft AB in Fig. 3-1d gives

$$\sum M_x = F(0.75) - 240 = 0$$

$$F = 320 \text{ lbf}$$

The normal force is $N = 320 \tan 20^\circ = 116.5 \text{ lbf}$.

Using the equilibrium equations for Figs. 3-1c and d, the reader should verify that: $R_{Ay} = 192 \text{ lbf}$, $R_{Az} = 69.9 \text{ lbf}$, $R_{By} = 128 \text{ lbf}$, $R_{Bz} = 46.6 \text{ lbf}$, $R_{Cy} = 192 \text{ lbf}$, $R_{Cx} = 69.9 \text{ lbf}$, $R_{Dy} = 128 \text{ lbf}$, $R_{Dz} = 46.6 \text{ lbf}$, and $T_o = 480 \text{ lbf} \cdot \text{in}$. The direction of the output torque T_o is opposite ω_o because it is the resistive load on the system opposing the motion ω_o .

Note in Fig. 3-1b the net force from the bearing reactions is zero whereas the net moment about the x axis is $(1.5 + 0.75)(192) + (1.5 + 0.75)(128) = 720 \text{ lbf} \cdot \text{in}$. This value is the same as $T_i + T_o = 240 + 480 = 720 \text{ lbf} \cdot \text{in}$, as shown in Fig. 3-1a. The reaction forces R_E, R_F, R_H , and R_I from the mounting bolts cannot be determined from the equilibrium equations as there are too many unknowns. Only three equations are available, $\sum F_y = \sum F_z = \sum M_x = 0$. In case you were wondering about assumption 5, here is where we will use it (see Sec. 8-12). The gear box tends to rotate about the x axis because of a pure torsional moment of $720 \text{ lbf} \cdot \text{in}$. The bolt forces must

provide an equal but opposite torsional moment. The center of rotation relative to the bolts lies at the centroid of the bolt cross-sectional areas. Thus if the bolt areas are equal, the center of rotation is at the center of the four bolts, a distance of $\sqrt{(4/2)^2 + (5/2)^2} = 3.202$ in from each bolt; the bolt forces are equal ($R_E = R_F = R_H = R_I = R$), and each bolt force is perpendicular to the line from the bolt to the center of rotation. This gives a net torque from the four bolts of $4R(3.202) = 720$. Thus, $R_E = R_F = R_H = R_I = 56.22$ lbf.

3-2 Shear Force and Bending Moments in Beams

Figure 3-2a shows a beam supported by reactions R_1 and R_2 and loaded by the concentrated forces F_1 , F_2 , and F_3 . If the beam is cut at some section located at $x = x_1$ and the left-hand portion is removed as a free body, an *internal shear force* V and *bending moment* M must act on the cut surface to ensure equilibrium (see Fig. 3-2b). The shear force is obtained by summing the forces on the isolated section. The bending moment is the sum of the moments of the forces to the left of the section taken about an axis through the isolated section. The sign conventions used for bending moment and shear force in this book are shown in Fig. 3-3. Shear force and bending moment are related by the equation

$$V = \frac{dM}{dx} \quad (3-3)$$

Sometimes the bending is caused by a distributed load $q(x)$, as shown in Fig. 3-4; $q(x)$ is called the *load intensity* with units of force per unit length and is

Figure 3-2

Free-body diagram of simply-supported beam with V and M shown in positive directions (established by the conventions shown in Fig. 3-3).

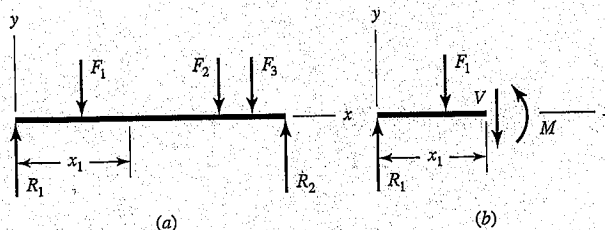


Figure 3-3

Sign conventions for bending and shear.

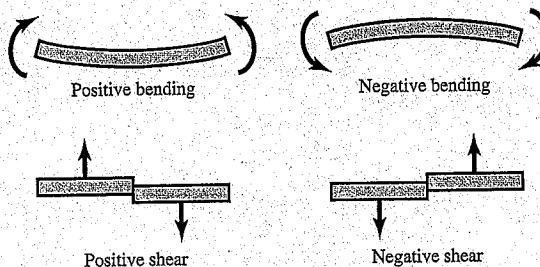
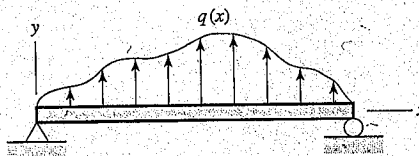


Figure 3-4

Distributed load on beam.



positive in the positive y direction. It can be shown that differentiating Eq. (3-3) results in

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q \quad (3-4)$$

Normally the applied distributed load is directed downward and labeled w (e.g., see Fig. 3-6). In this case, $w = -q$.

Equations (3-3) and (3-4) reveal additional relations if they are integrated. Thus, if we integrate between, say, x_A and x_B , we obtain

$$\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q \, dx \quad (3-5)$$

which states that *the change in shear force from A to B is equal to the area of the loading diagram between x_A and x_B .*

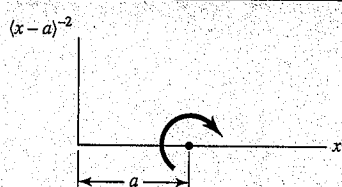
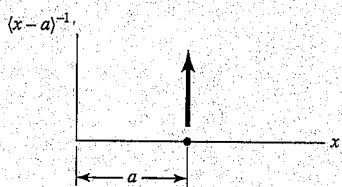
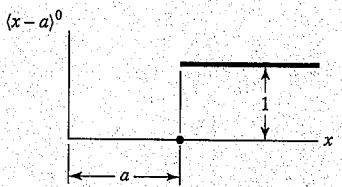
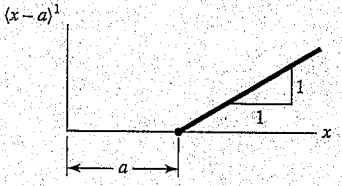
In a similar manner,

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V \, dx \quad (3-6)$$

which states that *the change in moment from A to B is equal to the area of the shear-force diagram between x_A and x_B .*

Table 3-1

Singularity (Macaulay[†])
Functions

Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x - a \rangle^{-2} = 0 \quad x \neq a$ $\langle x - a \rangle^{-2} = \pm \infty \quad x = a$ $\int \langle x - a \rangle^{-2} dx = \langle x - a \rangle^{-1}$
Concentrated force (unit impulse)		$\langle x - a \rangle^{-1} = 0 \quad x \neq a$ $\langle x - a \rangle^{-1} = +\infty \quad x = a$ $\int \langle x - a \rangle^{-1} dx = \langle x - a \rangle^0$
Unit step		$\langle x - a \rangle^0 = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}$ $\int \langle x - a \rangle^0 dx = \langle x - a \rangle^1$
Ramp		$\langle x - a \rangle^1 = \begin{cases} 0 & x < a \\ x - a & x \geq a \end{cases}$ $\int \langle x - a \rangle^1 dx = \frac{\langle x - a \rangle^2}{2}$

[†]W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics*, vol. 48, pp. 129-130, 1919.

3-3 Singularity Functions

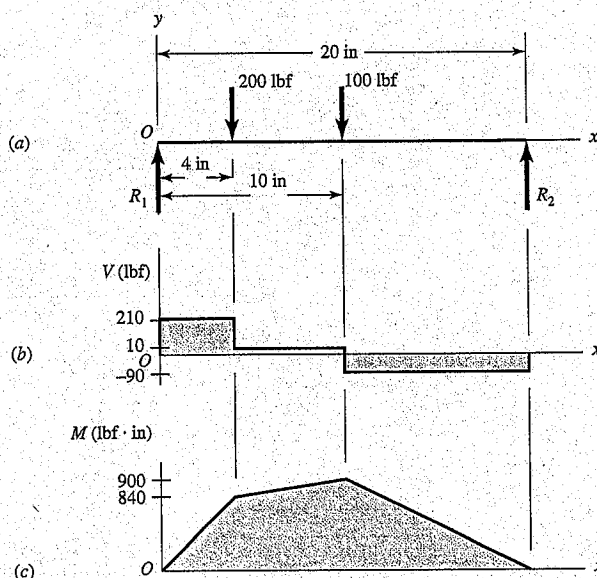
The four singularity functions defined in Table 3-1, using the *angle brackets* $\langle \rangle$, constitute a useful and easy means of integrating across discontinuities. By their use, general expressions for shear force and bending moment in beams can be written when the beam is loaded by concentrated moments or forces. As shown in the table, the concentrated moment and force functions are zero for all values of x not equal to a . The functions are undefined for values of $x = a$. Note that the unit step and ramp functions are zero only for values of x that are less than a . The integration properties shown in the table constitute a part of the mathematical definition too. The first two integrations of $q(x)$ for $V(x)$ and $M(x)$ do not require constants of integration provided *all* loads on the beam are accounted for in $q(x)$. The examples that follow show how these functions are used.

EXAMPLE 3-2

Derive the loading, shear-force, and bending-moment relations for the beam of Fig. 3-5a.

Figure 3-5

- (a) Loading diagram for a simply-supported beam.
(b) Shear-force diagram.
(c) Bending-moment diagram.



Solution Using Table 3-1 and $q(x)$ for the loading function, we find

Answer
$$q = R_1 \langle x \rangle^{-1} - 200 \langle x - 4 \rangle^{-1} - 100 \langle x - 10 \rangle^{-1} + R_2 \langle x - 20 \rangle^{-1} \quad (1)$$

Integrating successively gives

Answer
$$V = \int q \, dx = R_1 \langle x \rangle^0 - 200 \langle x - 4 \rangle^0 - 100 \langle x - 10 \rangle^0 + R_2 \langle x - 20 \rangle^0 \quad (2)$$

Answer
$$M = \int V \, dx = R_1 \langle x \rangle^1 - 200 \langle x - 4 \rangle^1 - 100 \langle x - 10 \rangle^1 + R_2 \langle x - 20 \rangle^1 \quad (3)$$

Note that $V = M = 0$ at $x = 0^-$.

The reactions R_1 and R_2 can be found by taking a summation of moments and forces as usual, or they can be found by noting that the shear force and bending

moment must be zero everywhere except in the region $0 \leq x \leq 20$ in. This means that Eq. (2) should give $V = 0$ at x slightly larger than 20 in. Thus

$$R_1 - 200 - 100 + R_2 = 0 \quad (4)$$

Since the bending moment should also be zero in the same region, we have, from Eq. (3),

$$R_1(20) - 200(20 - 4) - 100(20 - 10) = 0 \quad (5)$$

Equations (4) and (5) yield the reactions $R_1 = 210$ lbf and $R_2 = 90$ lbf.

The reader should verify that substitution of the values of R_1 and R_2 into Eqs. (2) and (3) yield Figs. 3-5b and c.

EXAMPLE 3-3

Figure 3-6a shows the loading diagram for a beam cantilevered at A with a uniform load of 20 lbf/in acting on the portion $3 \text{ in} \leq x \leq 7 \text{ in}$, and a concentrated counter-clockwise moment of 240 lbf · in at $x = 10$ in. Derive the shear-force and bending-moment relations, and the support reactions M_1 and R_1 .

Solution Following the procedure of Example 3-2, we find the load intensity function to be

$$q = -M_1 \langle x \rangle^{-2} + R_1 \langle x \rangle^{-1} - 20 \langle x - 3 \rangle^0 + 20 \langle x - 7 \rangle^0 - 240 \langle x - 10 \rangle^{-2} \quad (1)$$

Note that the $20 \langle x - 7 \rangle^0$ term was necessary to “turn off” the uniform load at C. Integrating successively gives

$$V = -M_1 \langle x \rangle^{-1} + R_1 \langle x \rangle^0 - 20 \langle x - 3 \rangle^1 + 20 \langle x - 7 \rangle^1 - 240 \langle x - 10 \rangle^{-1} \quad (2)$$

$$M = -M_1 \langle x \rangle^0 + R_1 \langle x \rangle^1 - 10 \langle x - 3 \rangle^2 + 10 \langle x - 7 \rangle^2 - 240 \langle x - 10 \rangle^0 \quad (3)$$

The reactions are found by making x slightly larger than 10 in, where both V and M are zero in this region. Noting that $\langle 10 \rangle^{-1} = 0$, Eq. (2) will then give

$$-M_1(0) + R_1(1) - 20(10 - 3) + 20(10 - 7) - 240(0) = 0$$

Figure 3-6

- (a) Loading diagram for a beam cantilevered at A.
(b) Shear-force diagram.
(c) Bending-moment diagram.

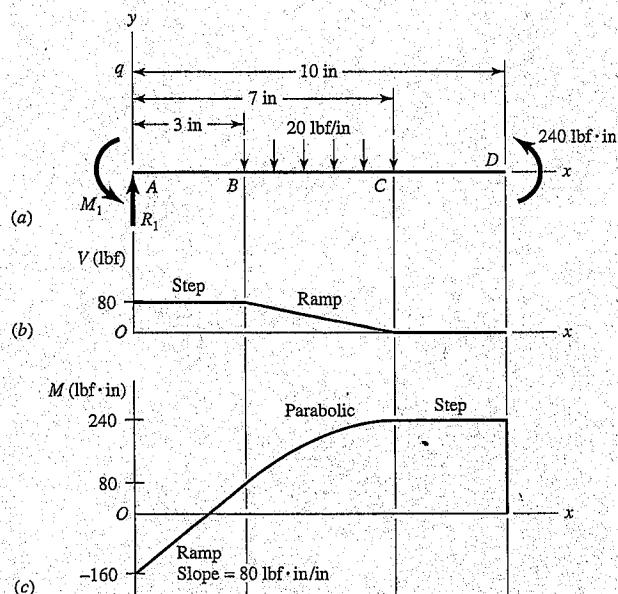


Figure
Stress co
normal t

Answer which yields $R_1 = 80$ lbf.
From Eq. (3) we get

$$-M_1(1) + 80(10) - 10(10 - 3)^2 + 10(10 - 7)^2 - 240(1) = 0$$

Answer which yields $M_1 = 160$ lbf · in.

Figures 3-6b and c show the shear-force and bending-moment diagrams. Note that the impulse terms in Eq. (2), $-M(x)^{-1}$ and $-240(x - 10)^{-1}$, are physically not forces and are not shown in the V diagram. Also note that both the M_1 and 240 lbf · in moments are counterclockwise and negative singularity functions; however, by the convention shown in Fig. 3-2 the M_1 and 240 lbf · in are negative and positive bending moments, respectively, which is reflected in Fig. 3-6c.

3-4 Stress

When an internal surface is isolated as in Fig. 3-2b, the net force and moment acting on the surface manifest themselves as force distributions across the entire area. The force distribution acting at a point on the surface is unique and will have components in the normal and tangential directions called *normal stress* and *tangential shear stress*, respectively. Normal and shear stresses are labeled by the Greek symbols σ and τ , respectively. If the direction of σ is outward from the surface it is considered to be a *tensile stress* and is a positive normal stress. If σ is into the surface it is a *compressive stress* and commonly considered to be a negative quantity. The units of stress in U.S. Customary units are pounds per square inch (psi). For SI units, stress is in newtons per square meter (N/m^2); $1 \text{ N/m}^2 = 1$ pascal (Pa).

3-5 Cartesian Stress Components

The Cartesian stress components are established by defining three mutually orthogonal surfaces at a point within the body. The normals to each surface will establish the x , y , z Cartesian axes. In general, each surface will have a normal and shear stress. The shear stress may have components along two Cartesian axes. For example, Fig. 3-7 shows an infinitesimal surface area isolation at a point Q within a body where the surface normal is the x direction. The normal stress is labeled σ_x . The symbol σ indicates a normal stress and the subscript x indicates the direction of the surface normal. The net shear stress acting on the surface is $(\tau_x)_{\text{net}}$ which can be resolved into components in the y and z directions, labeled as τ_{xy} and τ_{xz} , respectively (see Fig. 3-7).

Figure 3-7

Stress components on surface normal to x direction.

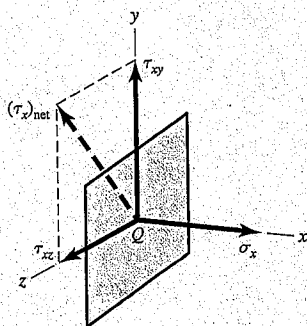
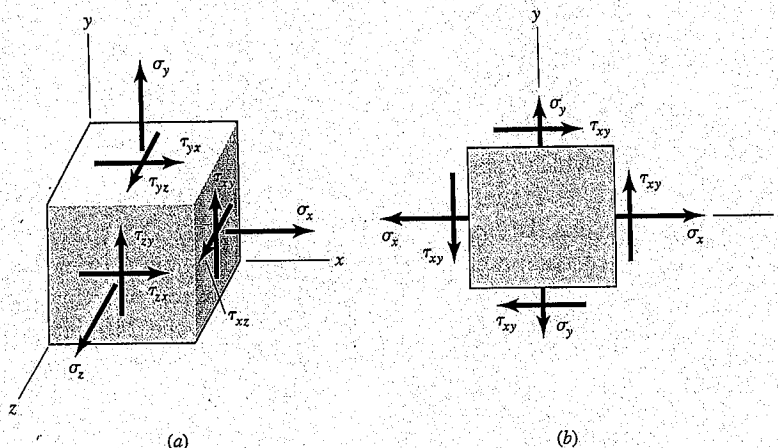


Figure 3-8

(a) General three-dimensional stress. (b) Plane stress with "cross-shears" equal.



Note that double subscripts are necessary for the shear. The first subscript indicates the direction of the surface normal whereas the second subscript is the direction of the shear stress.

The state of stress at a point described by three mutually perpendicular surfaces is shown in Fig. 3-8a. It can be shown through coordinate transformation that this is sufficient to determine the state of stress on *any* surface intersecting the point. As the dimensions of the cube in Fig. 3-8a approach zero, the stresses on the hidden faces become equal and opposite to those on the opposing visible faces. Thus, in general, a complete state of stress is defined by nine stress components, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx},$ and τ_{zy} .

For equilibrium, in most cases, "cross-shears" are equal, hence

$$\tau_{yx} = \tau_{xy} \quad \tau_{yz} = \tau_{zy} \quad \tau_{zx} = \tau_{xz} \quad (3-7)$$

This reduces the number of stress components for most three-dimensional states of stress from nine to six quantities, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$ and τ_{zx} .

A very common state of stress occurs when the stresses on one surface are zero. When this occurs the state of stress is called *plane stress*. Figure 3-8b shows a state of plane stress, arbitrarily assuming that the normal for the stress-free surface is the z direction such that $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. It is important to note that the element in Fig. 3-8b is still a three-dimensional cube. Also, here it is assumed that the cross-shears are equal such that $\tau_{yx} = \tau_{xy}$, and $\tau_{yz} = \tau_{zy} = \tau_{zx} = \tau_{xz} = 0$.

3-6 Mohr's Circle for Plane Stress

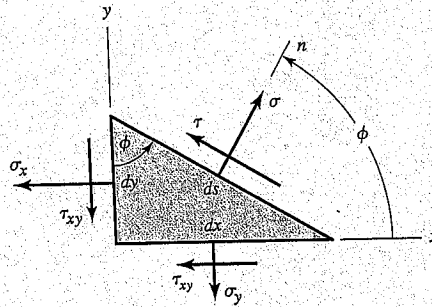
Suppose the $dx \, dy \, dz$ element of Fig. 3-8b is cut by an oblique plane with a normal n at an arbitrary angle ϕ counterclockwise from the x axis as shown in Fig. 3-9. Here, we are concerned with the stresses σ and τ that act upon this oblique plane. By summing the forces caused by all the stress components to zero, the stresses σ and τ are found to be

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi \quad (3-8)$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi \quad (3-9)$$

Equations (3-8) and (3-9) are called the *plane-stress transformation equations*.

Figure 3-9



Differentiating Eq. (3-8) with respect to ϕ and setting the result equal to zero maximizes σ and gives

$$\tan 2\phi_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (3-10)$$

Equation (3-10) defines two particular values for the angle $2\phi_p$, one of which defines the maximum normal stress σ_1 and the other, the minimum normal stress σ_2 . These two stresses are called the *principal stresses*, and their corresponding directions, the *principal directions*. The angle between the two principal directions is 90° . It is important to note that Eq. (3-10) can be written in the form

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\phi_p - \tau_{xy} \cos 2\phi_p = 0 \quad (a)$$

Comparing this with Eq. (3-9), we see that $\tau = 0$, meaning that the perpendicular surfaces containing principal stresses have zero shear stresses.

In a similar manner, we differentiate Eq. (3-9), set the result equal to zero, and obtain

$$\tan 2\phi_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad (3-11)$$

Equation (3-11) defines the two values of $2\phi_s$ at which the shear stress τ reaches an extreme value. The angle between the two surfaces containing the maximum shear stresses is 90° . Equation (3-11) can also be written as

$$\frac{\sigma_x - \sigma_y}{2} \cos 2\phi_p + \tau_{xy} \sin 2\phi_p = 0 \quad (b)$$

Substituting this into Eq. (3-8) yields

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \quad (3-12)$$

Equation (3-12) tells us that the two surfaces containing the maximum shear stresses also contain equal normal stresses of $(\sigma_x + \sigma_y)/2$.

Comparing Eqs. (3-10) and (3-11), we see that $\tan 2\phi_s$ is the negative reciprocal of $\tan 2\phi_p$. This means that $2\phi_s$ and $2\phi_p$ are angles 90° apart, and thus the angles between the surfaces containing the maximum shear stresses and the surfaces containing the principal stresses are $\pm 45^\circ$.

Formulas for the two principal stresses can be obtained by substituting the angle $2\phi_p$ from Eq. (3-10) in Eq. (3-8). The result is

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (3-13)$$

In a similar manner the two extreme-value shear stresses are found to be

$$\tau_1, \tau_2 = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (3-14)$$

Your particular attention is called to the fact that an extreme value of the shear stress *may not be the same as the actual maximum value*. See Sec. 3-7.

It is important to note that the equations given to this point are quite sufficient for performing any plane stress transformation. However, extreme care must be exercised when applying them. For example, say you are attempting to determine the principal state of stress for a problem where $\sigma_x = 14$ MPa, $\sigma_y = -10$ MPa, and $\tau_{xy} = -16$ MPa. Equation (3-10) yields $\phi_p = -26.57^\circ$ and 63.43° , which locate the principal stress surfaces, whereas, Eq. (3-13) gives $\sigma_1 = 22$ MPa and $\sigma_2 = -18$ MPa for the principal stresses. If all we wanted was the principal stresses, we would be finished. However, what if we wanted to draw the element containing the principal stresses properly oriented relative to the x, y axes? Well, we have two values of ϕ_p and two values for the principal stresses. How do we know which value of ϕ_p corresponds to which value of the principal stress? To clear this up we would need to substitute one of the values of ϕ_p into Eq. (3-8) to determine the normal stress corresponding to that angle.

A graphical method for expressing the relations developed in this section, called *Mohr's circle diagram*, is a very effective means of visualizing the stress state at a point and keeping track of the directions of the various components associated with plane stress. Equations (3-8) and (3-9) can be shown to be a set of parametric equations for σ and τ , where the parameter is 2ϕ . The parametric relationship between σ and τ is that of a circle plotted in the σ, τ plane, where the center of the circle is located at $C = (\sigma, \tau) = [(\sigma_x + \sigma_y)/2, 0]$ and has a radius of $R = \sqrt{[(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2}$. A problem arises in the sign of the shear stress. The transformation equations are based on a positive ϕ being counterclockwise, as shown in Fig. 3-9. If a positive τ were plotted above the σ axis, points would rotate clockwise on the circle 2ϕ in the opposite direction of rotation on the element. It would be convenient if the rotations were in the same direction. One could solve the problem easily by plotting positive τ below the axis. However, the classical approach to Mohr's circle uses a different convention for the shear stress.

Mohr's Circle Shear Convention

This convention is followed in drawing Mohr's circle:

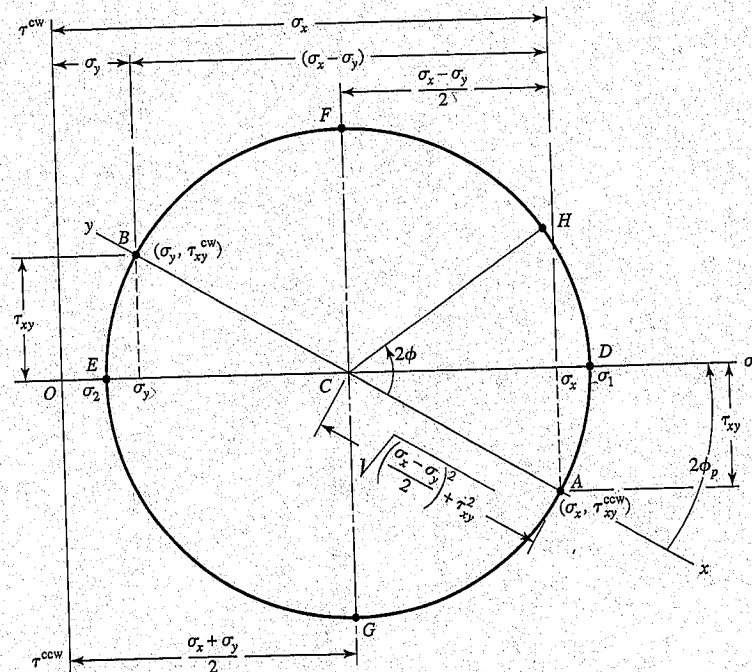
- Shear stresses tending to rotate the element clockwise (cw) are plotted *above* the σ axis.
- Shear stresses tending to rotate the element counterclockwise (ccw) are plotted *below* the σ axis.

For example, consider the right face of the element in Fig. 3-8b. By Mohr's circle convention the shear stress shown is plotted *below* the σ axis because it tends to rotate the element counterclockwise. The shear stress on the top face of the element is plotted *above* the σ axis because it tends to rotate the element clockwise.

In Fig. 3-10 we create a coordinate system with normal stresses plotted along the abscissa and shear stresses plotted as the ordinates. On the abscissa, tensile (positive) normal stresses are plotted to the right of the origin O and compressive (negative) normal stresses to the left. On the ordinate, clockwise (cw) shear stresses are plotted up; counterclockwise (ccw) shear stresses are plotted down.

Figure 3-10

Mohr's circle diagram.



Using the stress state of Fig. 3-8b, we plot Mohr's circle, Fig. 3-10, by first looking at the right surface of the element containing σ_x to establish the sign of σ_x and the cw or ccw direction of the shear stress. The right face is called the x face where $\phi = 0^\circ$. If σ_x is positive and the shear stress τ_{xy} is ccw as shown in Fig. 3-8b, we can establish point A with coordinates $(\sigma_x, \tau_{xy}^{ccw})$ in Fig. 3-10. Next, we look at the top y face, where $\phi = 90^\circ$, which contains σ_y , and repeat the process to obtain point B with coordinates $(\sigma_y, \tau_{xy}^{cw})$ as shown in Fig. 3-10. The two states of stress for the element are $\Delta\phi = 90^\circ$ from each other on the element so they will be $2\Delta\phi = 180^\circ$ from each other on Mohr's circle. Points A and B are the same vertical distance from the σ axis. Thus, AB must be on the diameter of the circle, and the center of the circle C is where AB intersects the σ axis. With points A and B on the circle, and center C, the complete circle can then be drawn. Note that the extended ends of line AB are labeled x and y as references to the normals to the surfaces for which points A and B represent the stresses.

The entire Mohr's circle represents the state of stress at a *single* point in a structure. Each point on the circle represents the stress state for a *specific* surface intersecting the point in the structure. Each pair of points on the circle 180° apart represent the state of stress on an element whose surfaces are 90° apart. Once the circle is drawn, the states of stress can be visualized for various surfaces intersecting the point being analyzed. For example, the principal stresses σ_1 and σ_2 are points D and E, respectively, and their values obviously agree with Eq. (3-13). We also see that the shear stresses are zero on the surfaces containing σ_1 and σ_2 . The two extreme-value shear stresses, one clockwise and one counterclockwise, occur at F and G with magnitudes equal to the radius of the circle. The surfaces at F and G each also contain normal stresses of $(\sigma_x + \sigma_y)/2$ as noted earlier in Eq. (3-12). Finally, the state of stress on an arbitrary surface located at an angle ϕ counterclockwise from the x face is point H.

At one time, Mohr's circle was used graphically where it was drawn to scale very accurately and values were measured by using a scale and protractor. Here, we are strictly using Mohr's circle as a visualization aid and will use a semigraphical approach, calculating values from the properties of the circle. This is illustrated by the following example.

EXAMPLE 3-4 A plane stress element has $\sigma_x = 80$ MPa, $\sigma_y = 0$ MPa, and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3-11a.

(a) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the xy coordinates. Draw another stress element to show τ_1 and τ_2 , find the corresponding normal stresses, and label the drawing completely.

(b) Repeat part a using the transformation equations only.

Solution (a) In the semigraphical approach used here, we first make an approximate freehand sketch of Mohr's circle and then use the geometry of the figure to obtain the desired information.

Draw the σ and τ axes first (Fig. 3-11b) and from the x face locate $\sigma_x = 80$ MPa along the σ axis. On the x face of the element, we see that the shear stress is 50 MPa in the cw direction. Thus, for the x face, this establishes point A (80, 50^{cw}) MPa. Corresponding to the y face, the stress is $\sigma = 0$ and $\tau = 50$ MPa in the ccw direction. This locates point B (0, 50^{ccw}) MPa. The line AB forms the diameter of the required circle, which can now be drawn. The intersection of the circle with the σ axis defines σ_1 and σ_2 as shown. Now, noting the triangle ACD, indicate on the sketch the length of the legs AD and CD as 50 and 40 MPa, respectively. The length of the hypotenuse AC is

Answer
$$\tau_1 = \sqrt{(50)^2 + (40)^2} = 64.0 \text{ MPa}$$

and this should be labeled on the sketch too. Since intersection C is 40 MPa from the origin, the principal stresses are now found to be

Answer
$$\sigma_1 = 40 + 64 = 104 \text{ MPa} \quad \text{and} \quad \sigma_2 = 40 - 64 = -24 \text{ MPa}$$

The angle 2ϕ from the x axis cw to σ_1 is

Answer
$$2\phi_p = \tan^{-1} \frac{50}{40} = 51.3^\circ$$

To draw the principal stress element (Fig. 3-11c), sketch the x and y axes parallel to the original axes. The angle ϕ_p on the stress element must be measured in the *same* direction as is the angle $2\phi_p$ on the Mohr circle. Thus, from x measure 25.7° (half of 51.3°) clockwise to locate the σ_1 axis. The σ_2 axis is 90° from the σ_1 axis and the stress element can now be completed and labeled as shown. Note that there are *no* shear stresses on this element.

The two maximum shear stresses occur at points E and F in Fig. 3-11b. The two normal stresses corresponding to these shear stresses are each 40 MPa, as indicated. Point E is 38.7° ccw from point A on Mohr's circle. Therefore, in Fig. 3-11d, draw a stress element oriented 19.3° (half of 38.7°) ccw from x . The element should then be labeled with magnitudes and directions as shown.

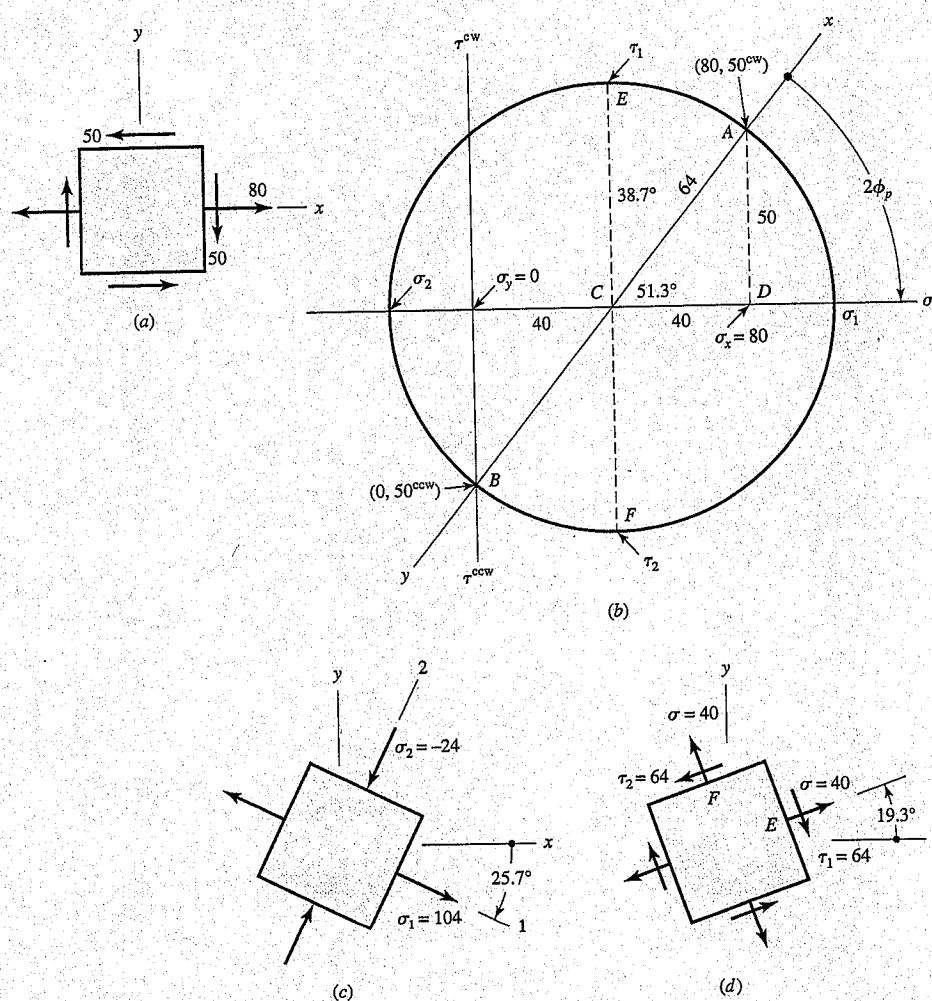
In constructing these stress elements it is important to indicate the x and y directions of the original reference system. This completes the link between the original machine element and the orientation of its principal stresses.

Figure

All stress

Figure 3-11

All stresses in MPa.



Answer

(b) The transformation equations are programmable. From Eq. (3-10),

$$\phi_p = \frac{1}{2} \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) = \frac{1}{2} \tan^{-1} \left(\frac{2(-50)}{80} \right) = -25.7^\circ, 64.3^\circ$$

From Eq. (3-8), for the first angle $\phi_p = -25.7^\circ$,

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(-25.7)] + (-50) \sin[2(-25.7)] = 104.03 \text{ MPa}$$

The shear on this surface is obtained from Eq. (3-9) as

$$\tau = -\frac{80 - 0}{2} \sin[2(-25.7)] + (-50) \cos[2(-25.7)] = 0 \text{ MPa}$$

which confirms that 104.03 MPa is a principal stress. From Eq. (3-8), for $\phi_p = 64.3^\circ$,

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(64.3)] + (-50) \sin[2(64.3)] = -24.03 \text{ MPa}$$

Answer Substituting $\phi_p = 64.3^\circ$ into Eq. (3-9) again yields $\tau = 0$, indicating that -24.03 MPa is also a principal stress. Once the principal stresses are calculated they can be ordered such that $\sigma_1 \geq \sigma_2$. Thus, $\sigma_1 = 104.03$ MPa and $\sigma_2 = -24.03$ MPa.

Since for $\sigma_1 = 104.03$ MPa, $\phi_p = -25.7^\circ$, and since ϕ is defined positive ccw in the transformation equations, we rotate *clockwise* 25.7° for the surface containing σ_1 . We see in Fig. 3-11c that this totally agrees with the semigraphical method.

To determine τ_1 and τ_2 , we first use Eq. (3-11) to calculate ϕ_s :

$$\phi_s = \frac{1}{2} \tan^{-1} \left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \right) = \frac{1}{2} \tan^{-1} \left(-\frac{80}{2(-50)} \right) = 19.3^\circ, 109.3^\circ$$

For $\phi_s = 19.3^\circ$, Eqs. (3-8) and (3-9) yield

$$\begin{aligned} \sigma &= \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(19.3)] + (-50) \sin[2(19.3)] = 40.0 \text{ MPa} \\ \tau &= -\frac{80 - 0}{2} \sin[2(19.3)] + (-50) \cos[2(19.3)] = -64.0 \text{ MPa} \end{aligned}$$

Remember that Eqs. (3-8) and (3-9) are *coordinate* transformation equations. Imagine that we are rotating the x, y axes 19.3° counterclockwise and y will now point up and to the left. So a negative shear stress on the rotated x face will point down and to the right as shown in Fig. 3-11d. Thus again, results agree with the semigraphical method.

For $\phi_s = 109.3^\circ$, Eqs. (3-8) and (3-9) give $\sigma = 40.0$ MPa and $\tau = +64.0$ MPa. Using the same logic for the coordinate transformation we find that results again agree with Fig. 3-11d.

3-7 General Three-Dimensional Stress

As in the case of plane stress, a particular orientation of a stress element occurs in space for which all shear-stress components are zero. When an element has this particular orientation, the normals to the faces are mutually orthogonal and correspond to the principal directions, and the normal stresses associated with these faces are the principal stresses. Since there are three faces, there are three principal directions and three principal stresses σ_1 , σ_2 , and σ_3 . For plane stress, the stress-free surface contains the third principal stress which is zero.

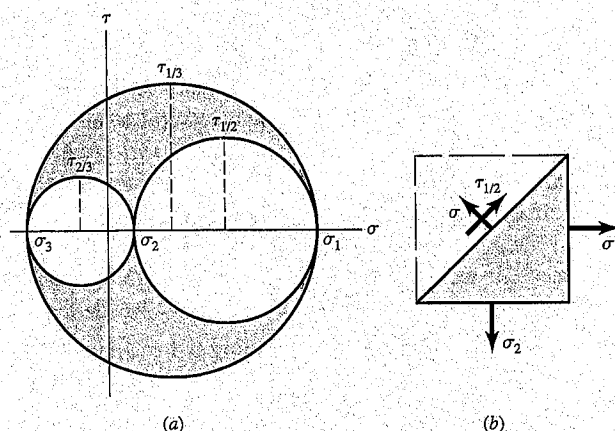
In our studies of plane stress we were able to specify any stress state σ_x , σ_y , and τ_{xy} and find the principal stresses and principal directions. But six components of stress are required to specify a general state of stress in three dimensions, and the problem of determining the principal stresses and directions is more difficult. In design, three-dimensional transformations are rarely performed since most maximum stress states occur under plane stress conditions. One notable exception is contact stress, which is not a case of plane stress, where the three principal stresses are given in Sec. 3-19. In fact, *all* states of stress are truly three-dimensional, where they might be described one- or two-dimensionally with respect to *specific* coordinate axes. Here it is most important to understand the relationship among the *three*

Figure

Mohr's circle
dimensionless

Figure 3-12

Mohr's circles for three-dimensional stress.



principal stresses. The process in finding the three principal stresses from the six stress components σ_x , σ_y , σ_z , τ_{xy} , τ_{yz} , and τ_{zx} , involves finding the roots of the cubic equation¹

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \quad (3-15)$$

In plotting Mohr's circles for three-dimensional stress, the principal normal stresses are ordered so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Then the result appears as in Fig. 3-12a. The stress coordinates σ , τ for any arbitrarily located plane will always lie on the boundaries or within the shaded area.

Figure 3-12a also shows the three *principal shear stresses* $\tau_{1/2}$, $\tau_{2/3}$, and $\tau_{1/3}$.² Each of these occurs on the two planes, one of which is shown in Fig. 3-12b. The figure shows that the principal shear stresses are given by the equations

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \quad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \quad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \quad (3-16)$$

Of course, $\tau_{\max} = \tau_{1/3}$ when the normal principal stresses are ordered ($\sigma_1 > \sigma_2 > \sigma_3$), so always order your principal stresses. Do this in any computer code you generate and you'll always generate $\tau_{\max}$.

3-8 Elastic Strain

Normal strain ϵ is defined and discussed in Sec. 2-1 for the tensile specimen and is given by Eq. (2-2) as $\epsilon = \delta/l$, where δ is the total elongation of the bar within the length l . Hooke's law for the tensile specimen is given by Eq. (2-3) as

$$\sigma = E\epsilon \quad (3-17)$$

where the constant E is called *Young's modulus* or the *modulus of elasticity*.

¹For development of this equation and further elaboration of three-dimensional stress transformations see: Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 46-78.

²Note the difference between this notation and that for a shear stress, say, τ_{xy} . The use of the shilling mark is not accepted practice, but it is used here to emphasize the distinction.

When a material is placed in tension, there exists not only an axial strain, but also negative strain (contraction) perpendicular to the axial strain. Assuming a linear, homogeneous, isotropic material, this lateral strain is proportional to the axial strain. If the axial direction is x , then the lateral strains are $\epsilon_y = \epsilon_z = -\nu\epsilon_x$. The constant of proportionality ν is called *Poisson's ratio*, which is about 0.3 for most structural metals. See Table A-5 for values of ν for common materials.

If the axial stress is in the x direction, then from Eq. (3-17)

$$\epsilon_x = \frac{\sigma_x}{E} \quad \epsilon_y = \epsilon_z = -\nu \frac{\sigma_x}{E} \quad (3-18)$$

For a stress element undergoing σ_x , σ_y , and σ_z simultaneously, the normal strains are given by

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \quad (3-19)$$

Shear strain γ is the change in a right angle of a stress element when subjected to pure shear stress, and Hooke's law for shear is given by

$$\tau = G\gamma \quad (3-20)$$

where the constant G is the *shear modulus of elasticity* or *modulus of rigidity*.

It can be shown for a linear, isotropic, homogeneous material, the three elastic constants are related to each other by

$$E = 2G(1 + \nu) \quad (3-21)$$

3-9 Uniformly Distributed Stresses

The assumption of a uniform distribution of stress is frequently made in design. The result is then often called *pure tension*, *pure compression*, or *pure shear*, depending upon how the external load is applied to the body under study. The word *simple* is sometimes used instead of *pure* to indicate that there are no other complicating effects. The tension rod is typical. Here a tension load F is applied through pins at the ends of the bar. The assumption of uniform stress means that if we cut the bar at a section remote from the ends and remove one piece, we can replace its effect by applying a uniformly distributed force of magnitude σA to the cut end. So the stress σ is said to be uniformly distributed. It is calculated from the equation

$$\sigma = \frac{F}{A} \quad (3-22)$$

This assumption of uniform stress distribution requires that:

- The bar be straight and of a homogeneous material
- The line of action of the force contains the centroid of the section
- The section be taken remote from the ends and from any discontinuity or abrupt change in cross section

Figure
Straight
bending.

For simple compression, Eq. (3-22) is applicable with F normally being considered a negative quantity. Also, a slender bar in compression may fail by buckling, and this possibility must be eliminated from consideration before Eq. (3-22) is used.³

Another type of loading that assumes a uniformly distributed stress is known as *direct shear*. This occurs when there is a shearing action with no bending. An example is the action on a piece of sheet metal caused by the two blades of tin snips. Bolts and pins that are loaded in shear often have direct shear. Think of a cantilever beam with a force pushing down on it. Now move the force all the way up to the wall so there is no bending moment, just a force trying to shear the beam off the wall. This is direct shear. Direct shear is usually assumed to be uniform across the cross section, and is given by

$$\tau = \frac{V}{A} \quad (3-23)$$

where V is the shear force and A is the area of the cross section that is being sheared. The assumption of uniform stress is not accurate, particularly in the vicinity where the force is applied, but the assumption generally gives acceptable results.

3-10 Normal Stresses for Beams in Bending

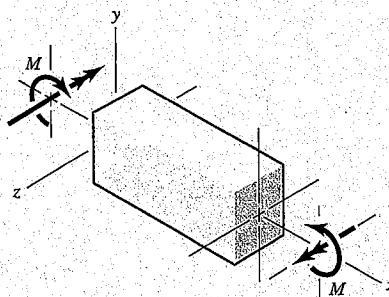
The equations for the normal bending stresses in straight beams are based on the following assumptions.

- The beam is subjected to pure bending. This means that the shear force is zero, and that no torsion or axial loads are present (for most engineering applications it is assumed that these loads affect the bending stresses minimally).
- The material is isotropic and homogeneous.
- The material obeys Hooke's law.
- The beam is initially straight with a cross section that is constant throughout the beam length.
- The beam has an axis of symmetry in the plane of bending.
- The proportions of the beam are such that it would fail by bending rather than by crushing, wrinkling, or sidewise buckling.
- Plane cross sections of the beam remain plane during bending.

In Fig. 3-13 we visualize a portion of a straight beam acted upon by a positive bending moment M shown by the curved arrow showing the physical action of the moment together with a straight, double-headed, arrow indicating the moment vector. The x axis is coincident with the *neutral axis* of the section, and the xz plane, which

Figure 3-13

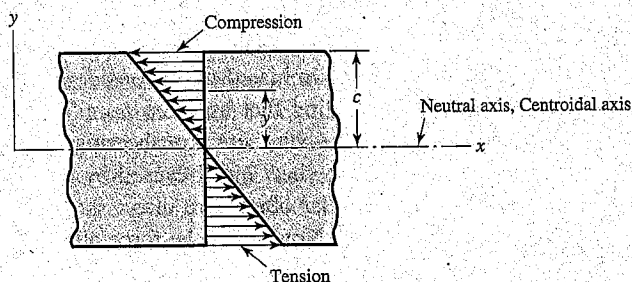
Straight beam in positive bending.



³See Sec. 4-11.

Figure 3-14

Bending stresses according to Eq. (3-24).



contains the neutral axes of all cross sections, is called the *neutral plane*. Elements of the beam coincident with this plane have zero stress. The location of the neutral axis with respect to the cross section is coincident with the *centroidal axis* of the cross section.

The bending stress varies linearly with the distance from the neutral axis, y , and is given by

$$\sigma_x = -\frac{My}{I} \quad (3-24)$$

where I is the *second-area moment* about the z axis. That is,

$$I = \int y^2 dA \quad (3-25)$$

The stress distribution given by Eq. (3-24) is shown in Fig. 3-14. The maximum magnitude of the bending stress will occur where y has the greatest magnitude. Designating $\sigma_{\max}$ as the maximum *magnitude* of the bending stress, and c as the maximum *magnitude* of y

$$\sigma_{\max} = \frac{Mc}{I} \quad (3-26a)$$

Equation (3-24) can still be used to ascertain whether $\sigma_{\max}$ is tensile or compressive.

Equation (3-26a) is often written as

$$\sigma_{\max} = \frac{M}{Z} \quad (3-26b)$$

where $Z = I/c$ is called the *section modulus*.

EXAMPLE 3-5

A beam having a T section with the dimensions shown in Fig. 3-15 is subjected to a bending moment of $1600 \text{ N} \cdot \text{m}$, about the negative z axis, that causes tension at the top surface. Locate the neutral axis and find the maximum tensile and compressive bending stresses.

Solution

Dividing the T section into two rectangles, numbered 1 and 2, the total area is $A = 12(75) + 12(88) = 1956 \text{ mm}^2$. Summing the area moments of these rectangles about the top edge, where the moment arms of areas 1 and 2 are 6 mm and $(12 + 88/2) = 56 \text{ mm}$ respectively, we have

$$1956c_1 = 12(75)(6) + 12(88)(56)$$

and hence $c_1 = 32.99 \text{ mm}$. Therefore $c_2 = 100 - 32.99 = 67.01 \text{ mm}$.

Next we calculate the second moment of area of each rectangle about its own centroidal axis. Using Table A-18, we find for the top rectangle

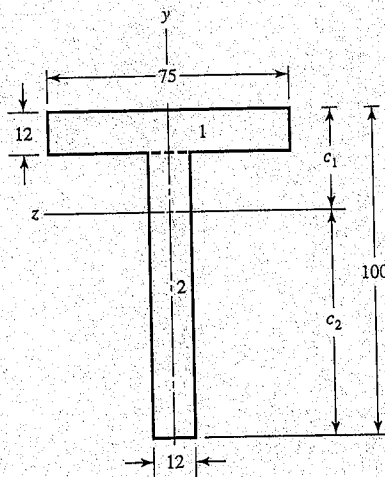
$$I_1 = \frac{1}{12}bh^3 = \frac{1}{12}(75)12^3 = 1.080 \times 10^4 \text{ mm}^4$$

Figure 3

Dimensions i

Figure 3-15

Dimensions in millimeters.



For the bottom rectangle, we have

$$I_2 = \frac{1}{12}(12)88^3 = 6.815 \times 10^5 \text{ mm}^4$$

We now employ the *parallel-axis theorem* to obtain the second moment of area of the composite figure about its own centroidal axis. This theorem states

$$I_z = I_{ca} + Ad^2$$

where I_{ca} is the second moment of area about its own centroidal axis and I_z is the second moment of area about any parallel axis a distance d removed. For the top rectangle, the distance is

$$d_1 = 32.99 - 6 = 26.99 \text{ mm}$$

and for the bottom rectangle,

$$d_2 = 67.01 - \frac{88}{2} = 23.01 \text{ mm}$$

Using the parallel-axis theorem for both rectangles, we now find that

$$\begin{aligned} I &= [1.080 \times 10^4 + 12(75)26.99^2] + [6.815 \times 10^5 + 12(88)23.01^2] \\ &= 1.907 \times 10^6 \text{ mm}^4 \end{aligned}$$

Finally, the maximum tensile stress, which occurs at the top surface, is found to be

$$\text{Answer} \quad \sigma = \frac{Mc_1}{I} = \frac{1600(32.99)10^{-3}}{1.907(10^{-6})} = 27.68(10^6) \text{ Pa} = 27.68 \text{ MPa}$$

Similarly, the maximum compressive stress at the lower surface is found to be

$$\text{Answer} \quad \sigma = -\frac{Mc_2}{I} = -\frac{1600(67.01)10^{-3}}{1.907(10^{-6})} = -56.22(10^6) \text{ Pa} = -56.22 \text{ MPa}$$

Two-Plane Bending

Quite often, in mechanical design, bending occurs in both xy and xz planes. Considering cross sections with one or two planes of symmetry only, the bending stresses are given by

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \quad (3-27)$$

where the first term on the right side of the equation is identical to Eq. (3-24), M_y is the bending moment in the xz plane (moment vector in y direction), z is the distance from the neutral y axis, and I_y is the second area moment about the y axis.

For *noncircular* cross sections, Eq. (3-27) is the superposition of stresses caused by the two bending moment components. The maximum tensile and compressive bending stresses occur where the summation gives the greatest positive and negative stresses, respectively. For solid *circular* cross sections, all lateral axes are the same and the plane containing the moment corresponding to the vector sum of M_z and M_y contains the maximum bending stresses. For a beam of diameter d the maximum distance from the neutral axis is $d/2$, and from Table A-18, $I = \pi d^4/64$. The maximum bending stress for a solid circular cross section is then

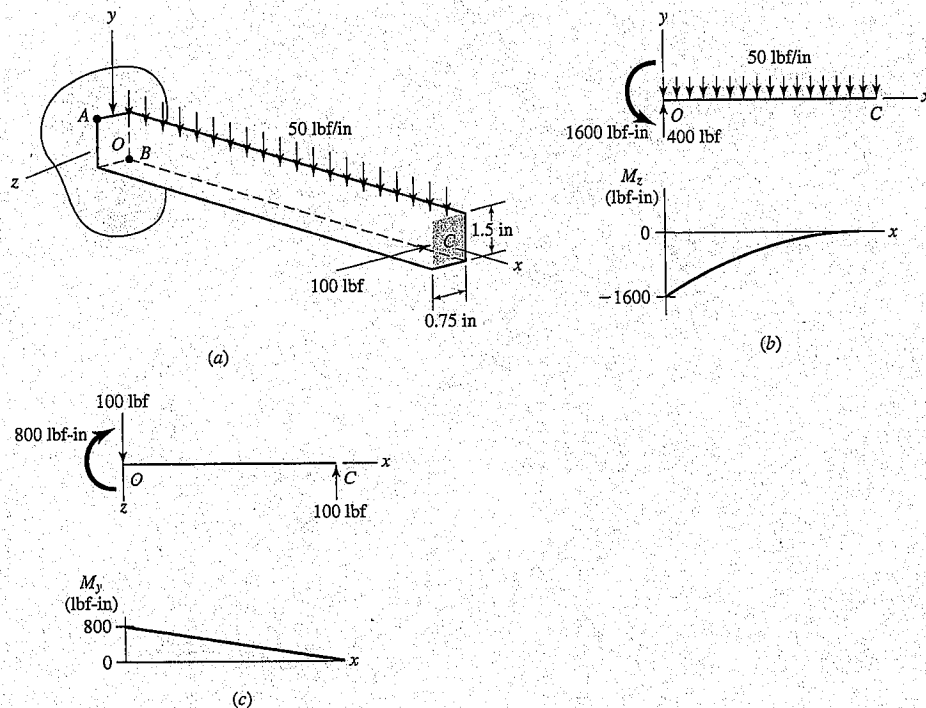
$$\sigma_m = \frac{Mc}{I} = \frac{(M_y^2 + M_z^2)^{1/2}(d/2)}{\pi d^4/64} = \frac{32}{\pi d^3}(M_y^2 + M_z^2)^{1/2} \quad (3-28)$$

EXAMPLE 3-6

As shown in Fig. 3-16a, beam OC is loaded in the xy plane by a uniform load of 50 lbf/in, and in the xz plane by a concentrated force of 100 lbf at end C . The beam is 8 in long.

Figure 3-16

(a) Beam loaded in two planes; (b) loading and bending-moment diagrams in xy plane; (c) loading and bending-moment diagrams in xz plane.



(a) For the cross section shown determine the maximum tensile and compressive bending stresses and where they act.

(b) If the cross section was a solid circular rod of diameter, $d = 1.25$ in, determine the magnitude of the maximum bending stress.

Solution (a) The reactions at O and the bending-moment diagrams in the xy and xz planes are shown in Figs. 3-16*b* and *c*, respectively. The maximum moments in both planes occur at O where

$$(M_z)_O = -\frac{1}{2}(50)8^2 = -1600 \text{ lbf-in} \quad (M_y)_O = 100(8) = 800 \text{ lbf-in}$$

The second moments of area in both planes are

$$I_z = \frac{1}{12}(0.75)1.5^3 = 0.2109 \text{ in}^4 \quad I_y = \frac{1}{12}(1.5)0.75^3 = 0.05273 \text{ in}^4$$

The maximum tensile stress occurs at point A , shown in Fig. 3-16*a*, where the maximum tensile stress is due to both moments. At A , $y_A = 0.75$ in and $z_A = 0.375$ in. Thus, from Eq. (3-27)

Answer
$$(\sigma_x)_A = -\frac{-1600(0.75)}{0.2109} + \frac{800(0.375)}{0.05273} = 11\,380 \text{ psi} = 11.38 \text{ kpsi}$$

The maximum compressive bending stress occurs at point B where, $y_B = -0.75$ in and $z_B = -0.375$ in. Thus

Answer
$$(\sigma_x)_B = -\frac{-1600(-0.75)}{0.2109} + \frac{800(-0.375)}{0.05273} = -11\,380 \text{ psi} = -11.38 \text{ kpsi}$$

(b) For a solid circular cross section of diameter, $d = 1.25$ in, the maximum bending stress at end O is given by Eq. (3-28) as

Answer
$$\sigma_m = \frac{32}{\pi(1.25)^3} [800^2 + (-1600)^2]^{1/2} = 9329 \text{ psi} = 9.329 \text{ kpsi}$$

Beams with Asymmetrical Sections⁴

The bending stress equations, given by Eqs. (3-24) and (3-27), can also be applied to beams having asymmetrical cross sections, provided the planes of bending coincide with the *area principal axes* of the section. The method for determining the orientation of the area principal axes and the values of the corresponding *principal second-area moments* can be found in any statics book. If a section has an axis of symmetry, that axis and its perpendicular axis are the area principal axes.

For example, consider a beam in bending, using an equal leg angle as shown in Table A-6. Equation (3-27) cannot be used if the bending moments are resolved about axis 1-1 and/or axis 2-2. However, Eq. (3-27) can be used if the moments are resolved about axis 3-3 and its perpendicular axis (let us call it, say, axis 4-4). Note, for this

⁴For further discussion, see Sec. 5.3, Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999.

cross section, axis 4-4 is an axis of symmetry. Table A-6 is a standard table, and for brevity, does not directly give all the information needed to use it. The orientation of the area principal axes and the values of I_{2-2} , I_{3-3} , and I_{4-4} are not given because they can be determined as follows. Since the legs are equal, the principal axes are oriented $\pm 45^\circ$ from axis 1-1, and $I_{2-2} = I_{1-1}$. The second-area moment I_{3-3} is given by

$$I_{3-3} = A(k_{3-3})^2 \quad (a)$$

where k_{3-3} is called the *radius of gyration*. The sum of the second-area moments for a cross section is invariant, so $I_{1-1} + I_{2-2} = I_{3-3} + I_{4-4}$. Thus, I_{4-4} is given by

$$I_{4-4} = 2 I_{1-1} - I_{3-3} \quad (b)$$

where $I_{2-2} = I_{1-1}$. For example, consider a $3 \times 3 \times \frac{1}{4}$ angle. Using Table A-6 and Eqs. (a) and (b), $I_{3-3} = 1.44 (0.592)^2 = 0.505 \text{ in}^4$, and $I_{4-4} = 2 (1.24) - 0.505 = 1.98 \text{ in}^4$.

3-11 Shear Stresses for Beams in Bending

Most beams have both shear forces and bending moments present. It is only occasionally that we encounter beams subjected to pure bending, that is to say, beams having zero shear force. The flexure formula is developed on the assumption of pure bending. This is done, however, to eliminate the complicating effects of shear force in the development of the formula. For engineering purposes, the flexure formula is valid no matter whether a shear force is present or not. For this reason, we shall utilize the same normal bending-stress distribution [Eqs. (3-24) and (3-26)] when shear forces are also present.

In Fig. 3-17a we show a beam segment of constant cross section subjected to a shear force V and a bending moment M at x . Because of external loading and V , the shear force and bending moment change with respect to x . At $x + dx$ the shear force and bending moment are $V + dV$ and $M + dM$, respectively. Considering forces in the x direction only, Fig. 3-17b shows the stress distribution σ_x due to the bending moments. If dM is positive, with the bending moment increasing, the stresses on the right face, for a given value of y , are larger in magnitude than the stresses on the left face. If we further isolate the element by making a slice at $y = y_1$ (see Fig. 3-17b), the net force in the x direction will be directed to the left with a value of

$$\int_{y_1}^c \frac{(dM)y}{I} dA$$

as shown in the rotated view of Fig. 3-17c. For equilibrium, a shear force on the bottom face, directed to the right, is required. This shear force gives rise to a shear stress τ , where, if assumed uniform, the force is $\tau b dx$. Thus

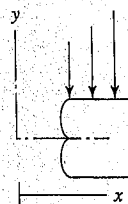
$$\tau b dx = \int_{y_1}^c \frac{(dM)y}{I} dA \quad (a)$$

The term dM/I can be removed from within the integral and $b dx$ placed on the right side of the equation; then, from Eq. (3-3) with $V = dM/dx$, Eq. (a) becomes

$$\tau = \frac{V}{Ib} \int_{y_1}^c y dA \quad (3-29)$$

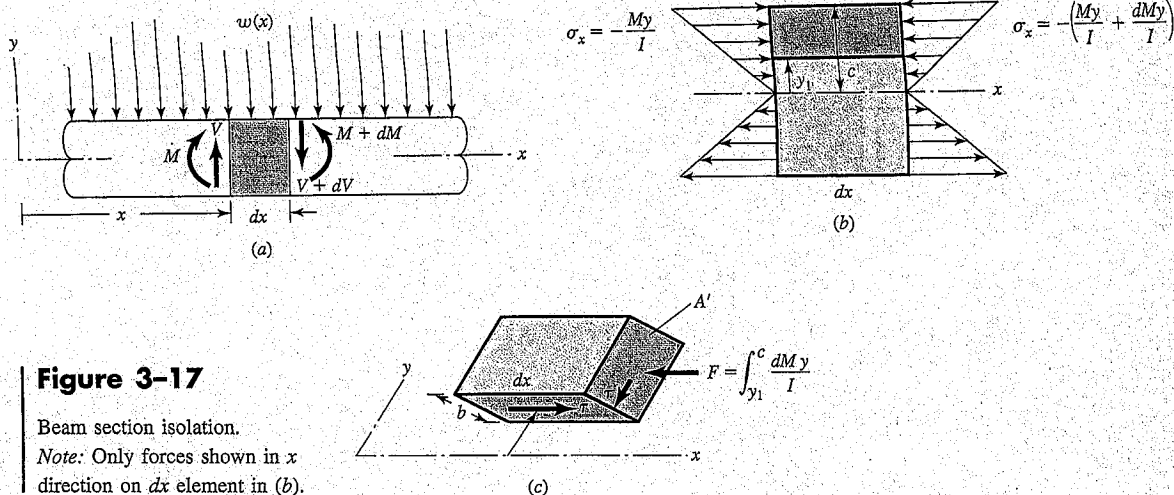
In this equation, the integral is the first moment of the area A' with respect to the neutral axis (see Fig. 3-17c). This integral is usually designated as Q . Thus

$$Q = \int_{y_1}^c y dA = \bar{y}' A' \quad (3-30)$$



Figure

Beam section
Note: Only
direction on



where, for the isolated area y_1 to c , $\bar{y}'$ is the distance in the y direction from the neutral plane to the centroid of the area A' . With this, Eq. (3-29) can be written as

$$\tau = \frac{VQ}{Ib} \quad (3-31)$$

This stress is known as the *transverse shear stress*. It is always accompanied with bending stress.

In using this equation, note that b is the width of the section at $y = y_1$. Also, I is the second moment of area of the entire section about the neutral axis.

Because cross shears are equal, and area A' is *finite*, the shear stress τ given by Eq. (3-31) and shown on area A' in Fig. 3-17c occurs only at $y = y_1$. The shear stress on the lateral area varies with y , normally maximum at $y = 0$ (where $\bar{y}'A'$ is maximum) and zero at the outer fibers of the beam where $A' = 0$.

The shear stress distribution in a beam depends on how Q/b varies as a function of y_1 . Here we will show how to determine the shear stress distribution for a beam with a rectangular cross section and provide results of maximum values of shear stress for other standard cross sections. Figure 3-18 shows a portion of a beam with a rectangular cross section, subjected to a shear force V and a bending moment M . As a result of the bending moment, a normal stress σ is developed on a cross section such as $A-A$, which is in compression above the neutral axis and in tension below. To investigate the shear stress at a distance y_1 above the neutral axis, we select an element of area dA at a distance y above the neutral axis. Then, $dA = b dy$, and so Eq. (3-30) becomes

$$Q = \int_{y_1}^c y dA = b \int_{y_1}^c y dy = \frac{by^2}{2} \Big|_{y_1}^c = \frac{b}{2}(c^2 - y_1^2) \quad (b)$$

Substituting this value for Q into Eq. (3-31) gives

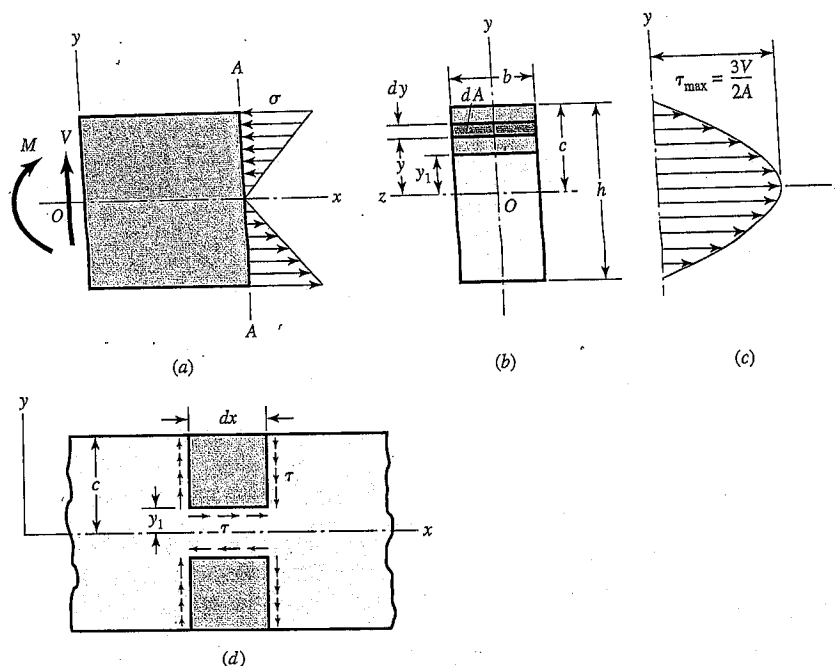
$$\tau = \frac{V}{2I}(c^2 - y_1^2) \quad (3-32)$$

This is the general equation for shear stress in a rectangular beam. To learn something about it, let us make some substitutions. From Table A-18, the second moment of area for a rectangular section is $I = bh^3/12$; substituting $h = 2c$ and $A = bh = 2bc$ gives

$$I = \frac{Ac^2}{3} \quad (c)$$

Figure 3-18

Transverse shear stresses in a rectangular beam.



If we now use this value of I for Eq. (3-32) and rearrange, we get

$$\tau = \frac{3V}{2A} \left(1 - \frac{y_1^2}{c^2} \right) \quad (3-33)$$

We note that the maximum shear stress exists when $y_1 = 0$, which is at the bending neutral axis. Thus

$$\tau_{\max} = \frac{3V}{2A} \quad (3-34)$$

for a rectangular section. As we move away from the neutral axis, the shear stress decreases parabolically until it is zero at the outer surfaces where $y_1 = \pm c$, as shown in Fig. 3-18c. Horizontal shear stress is always accompanied by vertical shear stress of the same magnitude, and so the distribution can be diagrammed as shown in Fig. 3-18d. Figure 3-18c shows that the shear τ on the vertical surfaces varies with y . We are almost always interested in the horizontal shear, τ in Fig. 3-18d, which is nearly uniform over dx with constant $y = y_1$. The maximum horizontal shear occurs where the vertical shear is largest. This is usually at the neutral axis but may not be if the width b is smaller somewhere else. Furthermore, if the section is such that b can be minimized on a plane not horizontal, then the horizontal shear stress occurs on an inclined plane. For example, with tubing, the horizontal shear stress occurs on a radial plane and the corresponding “vertical shear” is not vertical, but tangential.

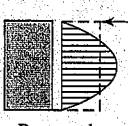
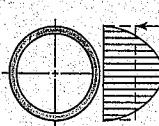
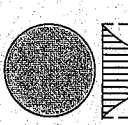
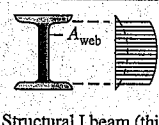
The distributions of transverse shear stresses for several commonly used cross sections are shown in Table 3-2. The profiles represent the VQ/Ib relationship, which is a function of the distance y from the neutral axis. For each profile, the formula for the maximum value at the neutral axis is given. Note that the expression given for the I beam is a commonly used approximation that is reasonable for a standard I beam with a thin web. Also, the profile for the I beam is idealized. In reality the transition from the web to the flange is quite complex locally, and not simply a step change.

Table 3

Formula
Transverse
from VQ/Ib

Table 3-2

 Formulas for Maximum Transverse Shear Stress from VQ/Ib

Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{\max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{\max} = \frac{2V}{A}$
 Circular	$\tau_{\max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{\max} \approx \frac{V}{A_{\text{web}}}$

It is significant to observe that the transverse shear stress in each of these common cross sections is maximum on the neutral axis, and zero on the outer surfaces. Since this is exactly the opposite of where the bending and torsional stresses have their maximum and minimum values, the transverse shear stress is often not critical from a design perspective.

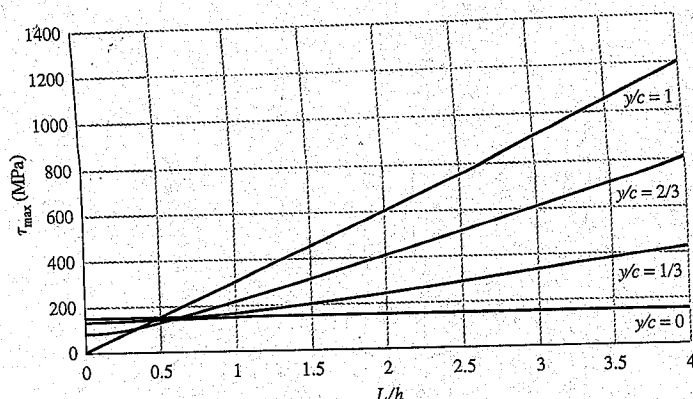
Let us examine the significance of the transverse shear stress, using as an example a cantilever beam of length L , with rectangular cross section $b \times h$, loaded at the free end with a transverse force F . At the wall, where the bending moment is the largest, at a distance y from the neutral axis, a stress element will include both bending stress and transverse shear stress. In Sec. 5-4 it will be shown that a good measure of the combined effects of multiple stresses on a stress element is the maximum shear stress. Inserting the bending stress (My/I) and the transverse shear stress (VQ/Ib) into the maximum shear stress equation, Eq. (3-14), we obtain a general equation for the maximum shear stress in a cantilever beam with a rectangular cross section. This equation can then be normalized with respect to L/h and y/c , where c is the distance from the neutral axis to the outer surface ($h/2$), to give

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{3F}{2bh} \sqrt{4(L/h)^2(y/c)^2 + [1 - (y/c)^2]^2} \quad (d)$$

To investigate the significance of transverse shear stress, we plot $\tau_{\max}$ as a function of L/h for several values of y/c , as shown in Fig. 3-19. Since F and b appear only as linear multipliers outside the radical, they will only serve to scale the plot in the vertical direction without changing any of the relationships. Notice that at the neutral axis where $y/c = 0$, $\tau_{\max}$ is constant for any length beam, since the bending stress is zero at the neutral axis and the transverse shear stress is independent of L . On the other hand, on the outer surface where $y/c = 1$, $\tau_{\max}$ increases linearly with L/h because of the bending moment. For y/c between zero and one, $\tau_{\max}$ is nonlinear for low values of L/h , but behaves linearly as L/h increases, displaying the dominance of the bending stress as the moment arm increases. We can see from the graph that the critical stress element (the largest value of $\tau_{\max}$) will always be either on the outer surface ($y/c = 1$) or at the neutral axis ($y/c = 0$), and never between. Thus, for the rectangular cross section, the transition between these two locations occurs at $L/h = 0.5$ where the line for $y/c = 1$ crosses the horizontal line for $y/c = 0$. The critical stress element is either on the outer surface where the transverse shear is zero, or if L/h is small enough, it is on the neutral axis where the bending stress is zero.

Figure 3-19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.



The conclusions drawn from Fig. 3-19 are generally similar for any cross section that does not increase in width farther away from the neutral axis. This notably includes solid round cross sections, but not I beams or channels. Care must be taken with I beams and channels that have thin webs that extend far enough from the neutral axis that the bending and shear may both be significant on the same stress element (See Ex. 3-7). For any common cross section beam, if the beam length to height ratio is greater than 10, the transverse shear stress is generally considered negligible compared to the bending stress at any point within the cross section.

EXAMPLE 3-7

A simply supported beam, 12 in long, is to support a load of 488 lbf acting 3 in from the left support, as shown in Fig. 3-20a. The beam is an I beam with the cross-sectional dimensions shown. To simplify the calculations, assume a cross section with square corners, as shown in Fig. 3-20c. Points of interest are labeled (a, b, c, and d) at distances y from the neutral axis of 0 in, 1.240⁻ in, 1.240⁺ in, and 1.5 in (Fig. 3-20c). At the critical axial location along the beam, find the following information.

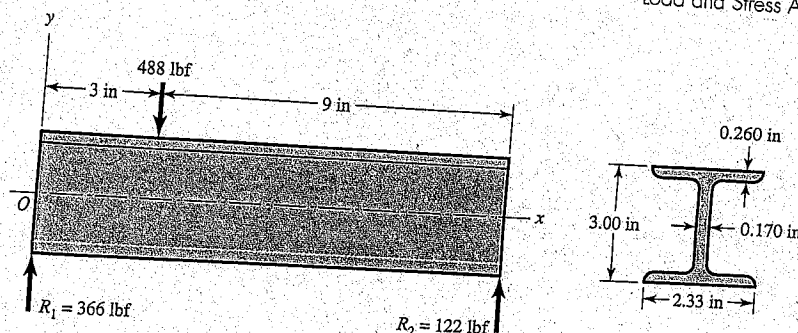
- Determine the profile of the distribution of the transverse shear stress, obtaining values at each of the points of interest.
- Determine the bending stresses at the points of interest.
- Determine the maximum shear stresses at the points of interest, and compare them.

Solution

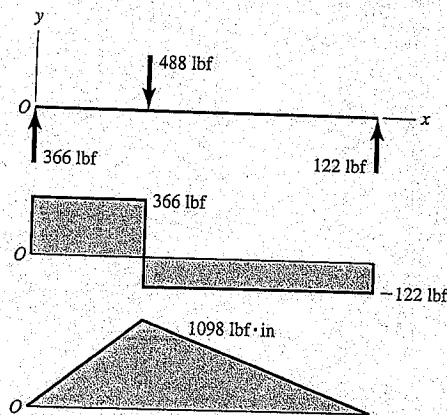
First, we note that the transverse shear stress is not likely to be negligible in this case since the beam length to height ratio is much less than 10, and since the thin web and wide flange will allow the transverse shear to be large. The loading, shear-force, and bending-moment diagrams are shown in Fig. 3-20b. The critical axial location is at $x = 3^-$ where the shear force and the bending moment are both maximum.

(a) We obtain the area moment of inertia I by evaluating I for a solid 3.0-in $\times$ 2.33-in rectangular area, and then subtracting the two rectangular areas that are not part of the cross section.

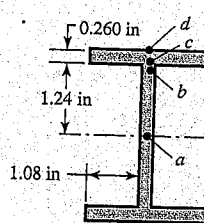
$$I = \frac{(2.33)(3.00)^3}{12} - 2 \left[\frac{(1.08)(2.48)^3}{12} \right] = 2.50 \text{ in}^4$$



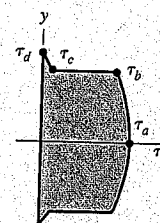
(a)



(b)



(c)



(d)

Finding Q at each point of interest using Eq. (3-30) gives

$$Q_a = \left(\sum \bar{y}'A' \right) \Big|_{y=0}^{y=1.5} = \left(1.24 + \frac{0.260}{2} \right) [(2.33)(0.260)] + \left(\frac{1.24}{2} \right) [(1.24)(0.170)] = 0.961 \text{ in}^3$$

$$Q_b = Q_c = \left(\sum \bar{y}'A' \right) \Big|_{y=1.24}^{y=1.5} = \left(1.24 + \frac{0.260}{2} \right) [(2.33)(0.260)] = 0.830 \text{ in}^3$$

$$Q_d = \left(\sum \bar{y}'A' \right) \Big|_{y=1.5}^{y=1.5} = (1.5)(0) = 0 \text{ in}^3$$

Applying Eq. (3-31) at each point of interest, with V and I constant for each point, and b equal to the width of the cross section at each point, shows that the magnitudes of the transverse shear stresses are

Answer

$$\tau_a = \frac{VQ_a}{Ib_a} = \frac{(366)(0.961)}{(2.50)(0.170)} = 828 \text{ psi}$$

$$\tau_b = \frac{VQ_b}{Ib_b} = \frac{(366)(0.830)}{(2.50)(0.170)} = 715 \text{ psi}$$

$$\tau_c = \frac{VQ_c}{Ib_c} = \frac{(366)(0.830)}{(2.50)(2.33)} = 52.2 \text{ psi}$$

$$\tau_d = \frac{VQ_d}{Ib_d} = \frac{(366)(0)}{(2.50)(2.33)} = 0 \text{ psi}$$

The magnitude of the idealized transverse shear stress profile through the beam depth will be as shown in Fig. 3-20*d*.

(*b*) The bending stresses at each point of interest are

Answer

$$\sigma_a = \frac{My_a}{I} = \frac{(1098)(0)}{2.50} = 0 \text{ psi}$$

$$\sigma_b = \sigma_c = -\frac{My_b}{I} = -\frac{(1098)(1.24)}{2.50} = -545 \text{ psi}$$

$$\sigma_d = -\frac{My_d}{I} = -\frac{(1098)(1.50)}{2.50} = -659 \text{ psi}$$

(*c*) Now at each point of interest, consider a stress element that includes the bending stress and the transverse shear stress. The maximum shear stress for each stress element can be determined by Mohr's circle, or analytically by Eq. (3-14) with $\sigma_y = 0$,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Thus, at each point

$$\tau_{\max,a} = \sqrt{0 + (828)^2} = 828 \text{ psi}$$

$$\tau_{\max,b} = \sqrt{\left(\frac{-545}{2}\right)^2 + (715)^2} = 765 \text{ psi}$$

$$\tau_{\max,c} = \sqrt{\left(\frac{-545}{2}\right)^2 + (52.2)^2} = 277 \text{ psi}$$

$$\tau_{\max,d} = \sqrt{\left(\frac{-659}{2}\right)^2 + 0} = 330 \text{ psi}$$

Answer

Interestingly, the critical location is at point *a* where the maximum shear stress is the largest, even though the bending stress is zero. The next critical location is at point *b* in the web, where the thin web thickness dramatically increases the transverse shear stress compared to points *c* or *d*. These results are counterintuitive, since both points *a* and *b* turn out to be more critical than point *d*, even though the bending stress is maximum at point *d*. The thin web and wide flange increase the impact of the transverse shear stress. If the beam length to height ratio were increased, the critical point would move from point *a* to point *b*, since the transverse shear stress at point *a* would remain constant, but the bending stress at point *b* would increase. The designer should be particularly alert to the possibility of the critical stress element not being on the outer surface with cross sections that get wider farther from the neutral axis, particularly in cases with thin web sections and wide flanges. For rectangular and circular cross sections, however, the maximum bending stresses at the outer surfaces will dominate, as was shown in Fig. 3-19.

| Figure

3-12 Torsion

Any moment vector that is collinear with an axis of a mechanical part is called a *torque vector*, because the moment causes the part to be twisted about that axis. A bar subjected to such a moment is said to be in *torsion*.

As shown in Fig. 3-21, the torque T applied to a bar can be designated by drawing arrows on the surface of the bar to indicate direction or by drawing torque-vector arrows along the axes of twist of the bar. Torque vectors are the hollow arrows shown on the x axis in Fig. 3-21. Note that they conform to the right-hand rule for vectors.

The *angle of twist*, in radians, for a solid round bar is

$$\theta = \frac{Tl}{GJ} \quad (3-35)$$

where T = torque

l = length

G = modulus of rigidity

J = polar second moment of area

Shear stresses develop throughout the cross section. For a round bar in torsion, these stresses are proportional to the radius ρ and are given by

$$\tau = \frac{T\rho}{J} \quad (3-36)$$

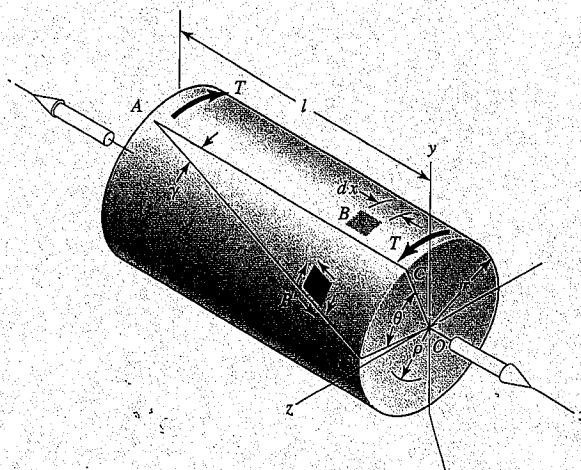
Designating r as the radius to the outer surface, we have

$$\tau_{\max} = \frac{Tr}{J} \quad (3-37)$$

The assumptions used in the analysis are:

- The bar is acted upon by a pure torque, and the sections under consideration are remote from the point of application of the load and from a change in diameter.
- The material obeys Hooke's law.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.

Figure 3-21



The last assumption depends upon the axisymmetry of the member, so it does not hold true for noncircular cross sections. Consequently, Eqs. (3-35) through (3-37) apply *only* to circular sections. For a solid round section,

$$J = \frac{\pi d^4}{32} \quad (3-38)$$

where d is the diameter of the bar. For a hollow round section,

$$J = \frac{\pi}{32} (d_o^4 - d_i^4) \quad (3-39)$$

where the subscripts o and i refer to the outside and inside diameters, respectively.

There are some applications in machinery for noncircular cross section members and shafts where a regular polygonal cross section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided. The development of equations for stress and deflection for torsional loading of noncircular cross sections can be obtained from the mathematical theory of elasticity. In general, the shear stress does not vary linearly with the distance from the axis, and depends on the specific cross section. In fact, for a rectangular section bar the shear stress is zero at the corners where the distance from the axis is the largest. The maximum shearing stress in a rectangular $b \times c$ section bar occurs in the middle of the *longest* side b and is of the magnitude

$$\tau_{\max} = \frac{T}{\alpha b c^2} \approx \frac{T}{b c^2} \left(3 + \frac{1.8}{b/c} \right) \quad (3-40)$$

where b is the width (longer side) and c is the thickness (shorter side). They can *not* be interchanged. The parameter α is a factor that is a function of the ratio b/c as shown in the following table.⁵ The angle of twist is given by

$$\theta = \frac{Tl}{\beta b c^3 G} \quad (3-41)$$

where β is a function of b/c , as shown in the table.

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Equation (3-40) is also approximately valid for equal-sided angles; these can be considered as two rectangles, each of which is capable of carrying half the torque.⁶

It is often necessary to obtain the torque T from a consideration of the power and speed of a rotating shaft. For convenience when U. S. Customary units are used, three forms of this relation are

$$H = \frac{FV}{33\,000} = \frac{2\pi Tn}{33\,000(12)} = \frac{Tn}{63\,025} \quad (3-42)$$

⁵S. Timoshenko, *Strength of Materials*, Part I, 3rd ed., D. Van Nostrand Company, New York, 1955, p. 290.
⁶For other sections see W. C. Young, R. G. Budynas, and A. M. Sadegh, *Roark's Formulas for Stress and Strain*, 8th ed., McGraw-Hill, New York, 2012.

where H = power, hp

T = torque, lbf · in

n = shaft speed, rev/min

F = force, lbf

V = velocity, ft/min

When SI units are used, the equation is

$$H = T\omega \quad (3-43)$$

where H = power, W

T = torque, N · m

ω = angular velocity, rad/s

The torque T corresponding to the power in watts is given approximately by

$$T = 9.55 \frac{H}{n} \quad (3-44)$$

where n is in revolutions per minute.

EXAMPLE 3-8

Figure 3-22 shows a crank loaded by a force $F = 300$ lbf that causes twisting and bending of a $\frac{3}{4}$ -in-diameter shaft fixed to a support at the origin of the reference system. In actuality, the support may be an inertia that we wish to rotate, but for the purposes of a stress analysis we can consider this a statics problem.

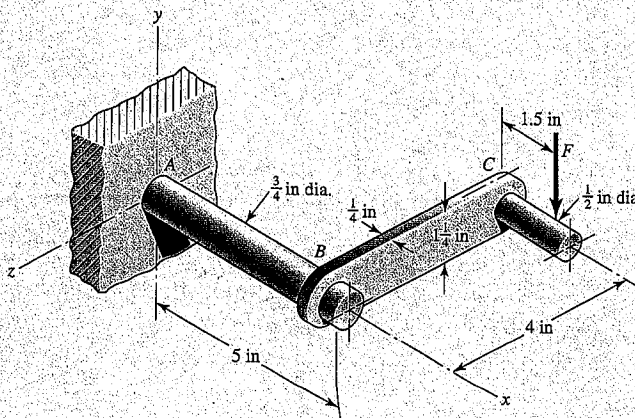
(a) Draw separate free-body diagrams of the shaft AB and the arm BC , and compute the values of all forces, moments, and torques that act. Label the directions of the coordinate axes on these diagrams.

(b) Compute the maxima of the torsional stress and the bending stress in the arm BC and indicate where these act.

(c) Locate a stress element on the top surface of the shaft at A , and calculate all the stress components that act upon this element.

(d) Determine the maximum normal and shear stresses at A .

Figure 3-22



Solution

(a) The two free-body diagrams are shown in Fig. 3-23. The results are

$$\begin{aligned} \text{At end } C \text{ of arm } BC: & \quad \mathbf{F} = -300\mathbf{j} \text{ lbf}, \mathbf{T}_C = -450\mathbf{k} \text{ lbf} \cdot \text{in} \\ \text{At end } B \text{ of arm } BC: & \quad \mathbf{F} = 300\mathbf{j} \text{ lbf}, \mathbf{M}_1 = 1200\mathbf{i} \text{ lbf} \cdot \text{in}, \mathbf{T}_1 = 450\mathbf{k} \text{ lbf} \cdot \text{in} \\ \text{At end } B \text{ of shaft } AB: & \quad \mathbf{F} = -300\mathbf{j} \text{ lbf}, \mathbf{T}_2 = -1200\mathbf{i} \text{ lbf} \cdot \text{in}, \mathbf{M}_2 = -450\mathbf{k} \text{ lbf} \cdot \text{in} \\ \text{At end } A \text{ of shaft } AB: & \quad \mathbf{F} = 300\mathbf{j} \text{ lbf}, \mathbf{M}_A = 1950\mathbf{k} \text{ lbf} \cdot \text{in}, \mathbf{T}_A = 1200\mathbf{i} \text{ lbf} \cdot \text{in} \end{aligned}$$

(b) For arm BC , the bending moment will reach a maximum near the shaft at B . If we assume this is $1200 \text{ lbf} \cdot \text{in}$, then the bending stress for a rectangular section will be

Answer

$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(1200)}{0.25(1.25)^2} = 18\,400 \text{ psi} = 18.4 \text{ kpsi}$$

Of course, this is not exactly correct, because at B the moment is actually being transferred into the shaft, probably through a weldment.

For the torsional stress, use Eq. (3-43). Thus

Answer

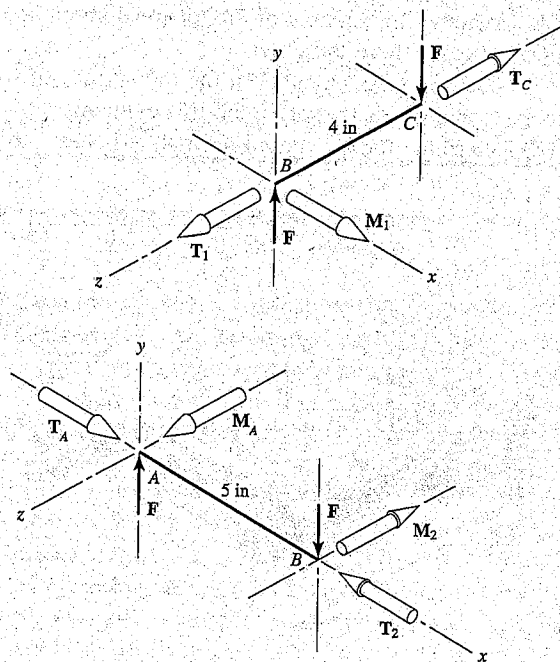
$$\tau_{\max} = \frac{T}{bc^2} \left(3 + \frac{1.8}{b/c} \right) = \frac{450}{1.25(0.25)^2} \left(3 + \frac{1.8}{1.25/0.25} \right) = 19\,400 \text{ psi} = 19.4 \text{ kpsi}$$

This stress occurs on the outer surfaces at the middle of the $1\frac{1}{4}$ -in side.

(c) For a stress element at A , the bending stress is tensile and is

Answer

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(1950)}{\pi(0.75)^3} = 47\,100 \text{ psi} = 47.1 \text{ kpsi}$$

Figure 3-23

The torsional stress is

Answer
$$\tau_{xz} = \frac{-T}{J/c} = \frac{-16T}{\pi d^3} = \frac{-16(1200)}{\pi(0.75)^3} = -14\,500 \text{ psi} = -14.5 \text{ kpsi}$$

where the reader should verify that the negative sign accounts for the direction of τ_{xz} .

(d) Point A is in a state of plane stress where the stresses are in the xz plane. Thus the principal stresses are given by Eq. (3-13) with subscripts corresponding to the x, z axes.

Answer The maximum normal stress is then given by

$$\begin{aligned}\sigma_1 &= \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} \\ &= \frac{47.1 + 0}{2} + \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 51.2 \text{ kpsi}\end{aligned}$$

Answer The maximum shear stress at A occurs on surfaces different than the surfaces containing the principal stresses or the surfaces containing the bending and torsional shear stresses. The maximum shear stress is given by Eq. (3-14), again with modified subscripts, and is given by

$$\tau_1 = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 27.7 \text{ kpsi}$$

EXAMPLE 3-9

The 1.5-in-diameter solid steel shaft shown in Fig. 3-24a is simply supported at the ends. Two pulleys are keyed to the shaft where pulley B is of diameter 4.0 in and pulley C is of diameter 8.0 in. Considering bending and torsional stresses only, determine the locations and magnitudes of the greatest tensile, compressive, and shear stresses in the shaft.

Solution

Figure 3-24b shows the net forces, reactions, and torsional moments on the shaft. Although this is a three-dimensional problem and vectors might seem appropriate, we will look at the components of the moment vector by performing a two-plane analysis. Figure 3-24c shows the loading in the xy plane, as viewed down the z axis, where bending moments are actually vectors in the z direction. Thus we label the moment diagram as M_z versus x . For the xz plane, we look down the y axis, and the moment diagram is M_y versus x as shown in Fig. 3-24d.

The net moment on a section is the vector sum of the components. That is,

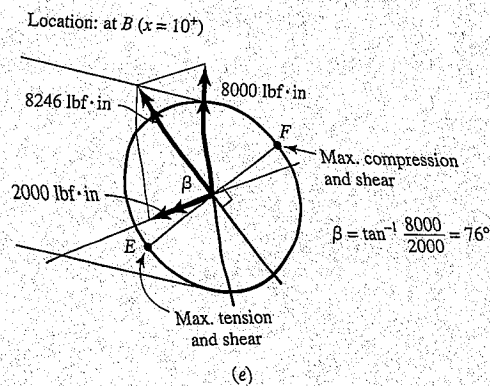
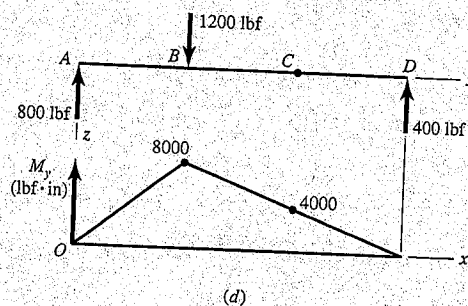
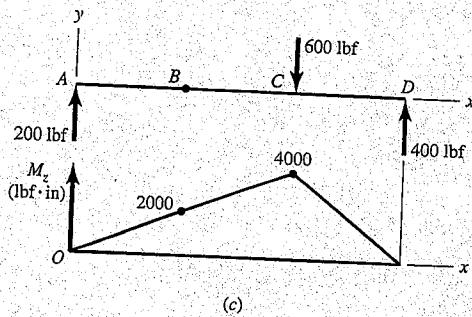
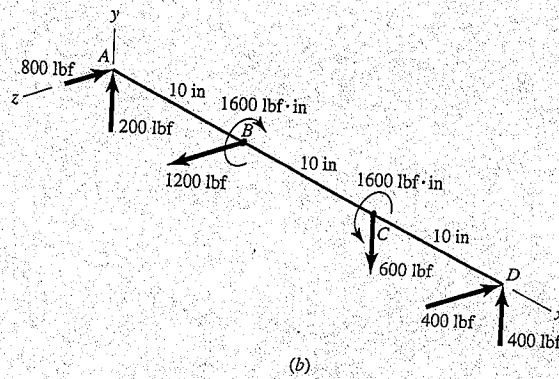
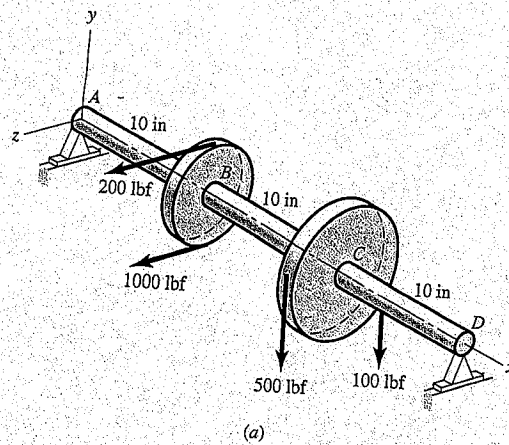
$$M = \sqrt{M_y^2 + M_z^2} \quad (1)$$

At point B,

$$M_B = \sqrt{2000^2 + 8000^2} = 8246 \text{ lbf} \cdot \text{in}$$

At point C,

$$M_C = \sqrt{4000^2 + 4000^2} = 5657 \text{ lbf} \cdot \text{in}$$



Thus the maximum bending moment is 8246 lbf · in and the maximum bending stress at pulley *B* is

$$\sigma = \frac{M d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(8246)}{\pi(1.5^3)} = 24\,890 \text{ psi} = 24.89 \text{ kpsi}$$

The maximum torsional shear stress occurs between *B* and *C* and is

$$\tau = \frac{T d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(1600)}{\pi(1.5^3)} = 2414 \text{ psi} = 2.414 \text{ kpsi}$$

The maximum bending and torsional shear stresses occur just to the right of pulley *B* at points *E* and *F* as shown in Fig. 3-24e. At point *E*, the maximum tensile stress will be σ_1 given by

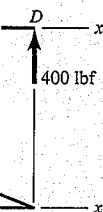
Answer
$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{24.89}{2} + \sqrt{\left(\frac{24.89}{2}\right)^2 + 2.414^2} = 25.12 \text{ kpsi}$$

At point *F*, the maximum compressive stress will be σ_2 given by

Answer
$$\sigma_2 = \frac{-\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{-24.89}{2} - \sqrt{\left(\frac{-24.89}{2}\right)^2 + 2.414^2} = -25.12 \text{ kpsi}$$

The extreme shear stress also occurs at *E* and *F* and is

Answer
$$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm 24.89}{2}\right)^2 + 2.414^2} = 12.68 \text{ kpsi}$$



Closed Thin-Walled Tubes ($t \ll r$)⁷

In closed thin-walled tubes, it can be shown that the product of shear stress times thickness of the wall τt is constant, meaning that the shear stress τ is inversely proportional to the wall thickness t . The total torque T on a tube such as depicted in Fig. 3-25 is given by

$$T = \int \tau t r \, ds = (\tau t) \int r \, ds = \tau t (2A_m) = 2A_m \tau t$$

where A_m is the area enclosed by the section median line. Solving for τ gives

$$\tau = \frac{T}{2A_m t} \quad (3-45)$$

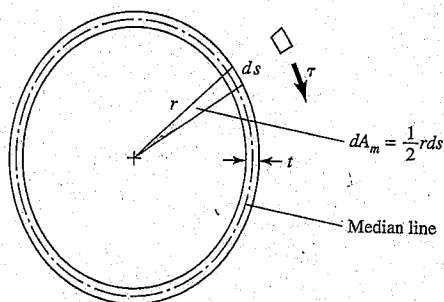
For constant wall thickness t , the angular twist (radians) per unit of length of the tube θ_1 is given by

$$\theta_1 = \frac{TL_m}{4GA_m^2 t} \quad (3-46)$$

⁷See Sec. 3-13, F. P. Beer, E. R. Johnston, and J. T. De Wolf, *Mechanics of Materials*, 5th ed., McGraw-Hill, New York, 2009.

Figure 3-25

The depicted cross section is elliptical, but the section need not be symmetrical nor of constant thickness.



where L_m is the length of the section median line. These equations presume the buckling of the tube is prevented by ribs, stiffeners, bulkheads, and so on, and that the stresses are below the proportional limit.

EXAMPLE 3-10

A welded steel tube is 40 in long, has a $\frac{1}{8}$ -in wall thickness, and a 2.5-in by 3.6-in rectangular cross section as shown in Fig. 3-26. Assume an allowable shear stress of 11 500 psi and a shear modulus of $11.5(10^6)$ psi.

- Estimate the allowable torque T .
- Estimate the angle of twist due to the torque.

Solution (a) Within the section median line, the area enclosed is

$$A_m = (2.5 - 0.125)(3.6 - 0.125) = 8.253 \text{ in}^2$$

and the length of the median perimeter is

$$L_m = 2[(2.5 - 0.125) + (3.6 - 0.125)] = 11.70 \text{ in}$$

Answer From Eq. (3-45) the torque T is

$$T = 2A_m t \tau = 2(8.253)(0.125)(11\,500) = 23\,730 \text{ lbf} \cdot \text{in}$$

Answer (b) The angle of twist θ from Eq. (3-46) is

$$\theta = \theta_1 l = \frac{TL_m}{4GA_m^2 t} l = \frac{23\,730(11.70)}{4(11.5 \times 10^6)(8.253^2)(0.125)} (40) = 0.0284 \text{ rad} = 1.62^\circ$$

Figure 3-26

A rectangular steel tube produced by welding.

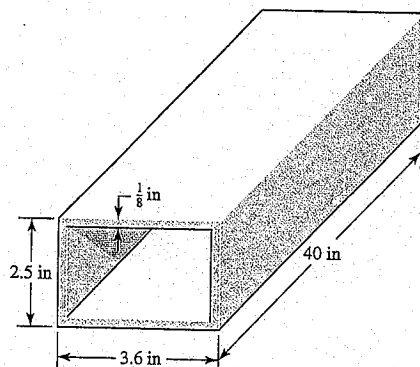


Figure
Some c

EXAMPLE 3-11

Compare the shear stress on a circular cylindrical tube with an outside diameter of 1 in and an inside diameter of 0.9 in, predicted by Eq. (3-37), to that estimated by Eq. (3-45).

Solution From Eq. (3-37),

$$\tau_{\max} = \frac{Tr}{J} = \frac{Tr}{(\pi/32)(d_o^4 - d_i^4)} = \frac{T(0.5)}{(\pi/32)(1^4 - 0.9^4)} = 14.809T$$

From Eq. (3-45),

$$\tau = \frac{T}{2A_mt} = \frac{T}{2(\pi 0.95^2/4)0.05} = 14.108T$$

Taking Eq. (3-37) as correct, the error in the thin-wall estimate is -4.7 percent.

Open Thin-Walled Sections

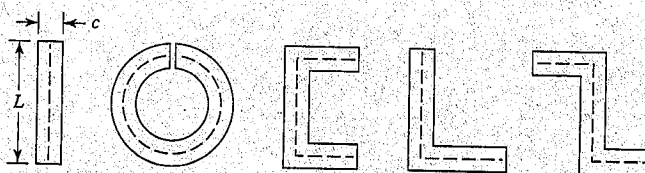
When the median wall line is not closed, the section is said to be an *open section*. Figure 3-27 presents some examples. Open sections in torsion, where the wall is thin, have relations derived from the membrane analogy theory⁸ resulting in:

$$\tau = G\theta_1 c = \frac{3T}{Lc^2} \quad (3-47)$$

where τ is the shear stress, G is the shear modulus, θ_1 is the angle of twist per unit length, T is torque, and L is the length of the median line. The wall thickness is designated c (rather than t) to remind you that you are in open sections. By studying the table that follows Eq. (3-41) you will discover that membrane theory presumes $b/c \rightarrow \infty$. Note that open thin-walled sections in torsion should be avoided in design. As indicated in Eq. (3-47), the shear stress and the angle of twist are inversely proportional to c^2 and c^3 , respectively. Thus, for small wall thickness, stress and twist can become quite large. For example, consider the thin round tube with a slit in Fig. 3-27. For a ratio of wall thickness of outside diameter of $c/d_o = 0.1$, the open section has greater magnitudes of stress and angle of twist by factors of 12.3 and 61.5, respectively, compared to a closed section of the same dimensions.

Figure 3-27

Some open thin-wall sections.



⁸See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970, Sec. 109.

EXAMPLE 3-12

A 12-in-long strip of steel is $\frac{1}{8}$ in thick and 1 in wide, as shown in Fig. 3-28. If the allowable shear stress is 11 500 psi and the shear modulus is $11.5(10^6)$ psi, find the torque corresponding to the allowable shear stress and the angle of twist, in degrees, (a) using Eq. (3-47) and (b) using Eqs. (3-40) and (3-41).

Solution (a) The length of the median line is 1 in. From Eq. (3-47),

$$T = \frac{Lc^2\tau}{3} = \frac{(1)(1/8)^2 11\,500}{3} = 59.90 \text{ lbf} \cdot \text{in}$$

$$\theta = \theta_1 l = \frac{\tau l}{Gc} = \frac{11\,500(12)}{11.5(10^6)(1/8)} = 0.0960 \text{ rad} = 5.5^\circ$$

A torsional spring rate k_t can be expressed as T/θ :

$$k_t = 59.90/0.0960 = 624 \text{ lbf} \cdot \text{in/rad}$$

(b) From Eq. (3-40),

$$T = \frac{\tau_{\max} b c^2}{3 + 1.8/(b/c)} = \frac{11\,500(1)(0.125)^2}{3 + 1.8/(1/0.125)} = 55.72 \text{ lbf} \cdot \text{in}$$

From Eq. (3-41), with $b/c = 1/0.125 = 8$,

$$\theta = \frac{Tl}{\beta b c^3 G} = \frac{55.72(12)}{0.307(1)0.125^3(11.5)10^6} = 0.0970 \text{ rad} = 5.6^\circ$$

$$k_t = 55.72/0.0970 = 574 \text{ lbf} \cdot \text{in/rad}$$

The cross section is not thin, where b should be greater than c by at least a factor of 10. In estimating the torque, Eq. (3-47) provided a value of 7.5 percent higher than Eq. (3-40), and 8.5 percent higher than the table on page 116.

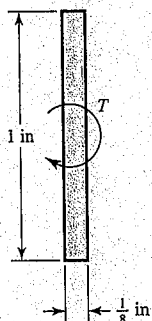


Figure 3-28

The cross-section of a thin strip of steel subjected to a torsional moment T .

3-13 Stress Concentration

In the development of the basic stress equations for tension, compression, bending, and torsion, it was assumed that no geometric irregularities occurred in the member under consideration. But it is quite difficult to design a machine without permitting some changes in the cross sections of the members. Rotating shafts must have shoulders designed on them so that the bearings can be properly seated and so that they will take thrust loads; and the shafts must have key slots machined into them for securing pulleys and gears. A bolt has a head on one end and screw threads on the other end, both of which account for abrupt changes in the cross section. Other parts require holes, oil grooves, and notches of various kinds. Any discontinuity in a machine part alters the stress distribution in the neighborhood of the discontinuity so that the elementary stress equations no longer describe the state of stress in the part at these locations. Such discontinuities are called *stress raisers*, and the regions in which they occur are called areas of *stress concentration*. Stress concentrations can also arise from some irregularity not inherent in the member, such as tool marks, holes, notches, grooves, or threads.

Figure 3

Thin plate in compression central hole. force is F = the thickness nominal stress $\sigma_0 = \frac{F}{(w -$

A *theoretical*, or *geometric*, stress-concentration factor K_t or K_{ts} is used to relate the actual maximum stress at the discontinuity to the *nominal stress*. The factors are defined by the equations

$$K_t = \frac{\sigma_{\max}}{\sigma_0} \quad K_{ts} = \frac{\tau_{\max}}{\tau_0} \quad (3-48)$$

where K_t is used for normal stresses and K_{ts} for shear stresses. The nominal stress σ_0 or τ_0 is the stress calculated by using the elementary stress equations and the net area, or net cross section. Sometimes the gross cross section is used instead, and so it is always wise to double check the source of K_t or K_{ts} before calculating the maximum stress.

The stress-concentration factor depends for its value only on the *geometry* of the part. That is, the particular material used has no effect on the value of K_t . This is why it is called a *theoretical* stress-concentration factor.

The analysis of geometric shapes to determine stress-concentration factors is a difficult problem, and not many solutions can be found. Most stress-concentration factors are found by using experimental techniques.⁹ Though the finite-element method has been used, the fact that the elements are indeed finite prevents finding the exact maximum stress. Experimental approaches generally used include photoelasticity, grid methods, brittle-coating methods, and electrical strain-gauge methods. Of course, the grid and strain-gauge methods both suffer from the same drawback as the finite-element method.

Stress-concentration factors for a variety of geometries may be found in Tables A-15 and A-16.

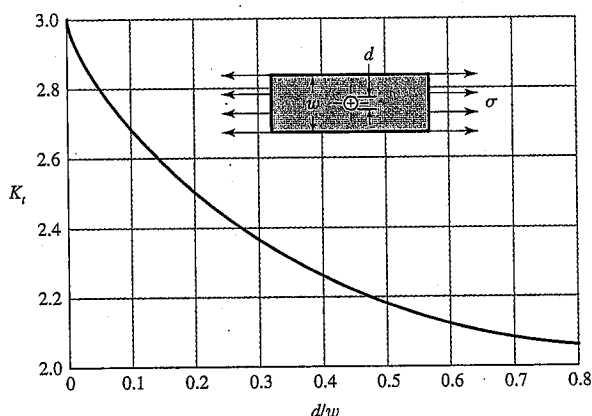
An example is shown in Fig. 3-29, that of a thin plate loaded in tension where the plate contains a centrally located hole.

In *static loading*, stress-concentration factors are applied as follows. In *ductile materials* ($\epsilon_f \geq 0.05$), the stress-concentration factor is *not* usually applied to predict the critical stress, because plastic strain in the region of the stress concentration is localized and has a strengthening effect. In *brittle materials* ($\epsilon_f < 0.05$), the geometric stress-concentration factor K_t is applied to the nominal stress before comparing it with strength. Gray cast iron has so many inherent stress raisers that the stress raisers introduced by the designer have only a modest (but additive) effect.

Figure 3-29

Thin plate in tension or simple compression with a transverse central hole. The net tensile force is $F = \sigma wt$, where t is the thickness of the plate. The nominal stress is given by

$$\sigma_0 = \frac{F}{(w-d)t} = \frac{w}{(w-d)} \sigma$$



⁹The best source book is W. D. Pilkey and D. F. Pilkey, *Peterson's Stress Concentration Factors*, 3rd ed., John Wiley & Sons, New York, 2008.

Consider a part made of a ductile material and loaded by a gradually applied static load such that the stress in an area of a stress concentration goes beyond the yield strength. The yielding will be restricted to a very small region, and the permanent deformation as well as the residual stresses after the load is released will be insignificant and normally can be tolerated. If yielding does occur, the stress distribution changes and tends toward a more uniform distribution. In the region where yielding occurs, there is little danger of fracture of a ductile material, but if the possibility of a brittle fracture exists, the stress concentration must be taken seriously. Brittle fracture is not just limited to brittle materials. Materials often thought of as being ductile can fail in a brittle manner under certain conditions, e.g., any single application or combination of cyclic loading, rapid application of static loads, loading at low temperatures, and parts containing defects in their material structures (see Sec. 5-12). The effects on a ductile material of processing, such as hardening, hydrogen embrittlement, and welding, may also accelerate failure. Thus, care should always be exercised when dealing with stress concentrations.

For *dynamic loading*, the stress concentration effect is significant for *both* ductile and brittle materials and must always be taken into account (see Sec. 6-10).

EXAMPLE 3-13

The 2-mm-thick bar shown in Fig. 3-30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?

Solution Since the material is brittle, the effect of stress concentrations near the discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, the nominal stress is

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w - d)t} = \frac{10\,000}{(40 - 4)2} = 139 \text{ MPa}$$

The theoretical stress concentration factor, from Fig. A-15-1, with $d/w = 4/40 = 0.1$, is $K_t = 2.7$. The maximum stress is

Answer $\sigma_{\max} = K_t \sigma_0 = 2.7(139) = 380 \text{ MPa}$

Similarly, for an 8-mm hole,

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w - d)t} = \frac{10\,000}{(40 - 8)2} = 156 \text{ MPa}$$

With $d/w = 8/40 = 0.2$, then $K_t = 2.5$, and the maximum stress is

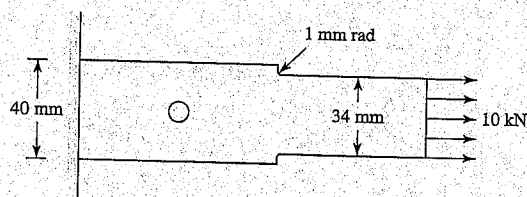
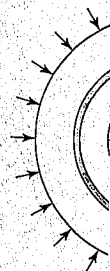


Figure 3-30



Figure

A cylinder
internal a

Answer

$$\sigma_{\max} = K_t \sigma_0 = 2.5(156) = 390 \text{ MPa}$$

Though the stress concentration is higher with the 4-mm hole, in this case the increased nominal stress with the 8-mm hole has more effect on the maximum stress.

For the fillet,

$$\sigma_0 = \frac{F}{A} = \frac{10\,000}{(34)^2} = 147 \text{ MPa}$$

From Table A-15-5, $D/d = 40/34 = 1.18$, and $r/d = 1/34 = 0.026$. Then $K_t = 2.5$.

Answer

$$\sigma_{\max} = K_t \sigma_0 = 2.5(147) = 368 \text{ MPa}$$

Answer

The crack will most likely occur with the 8-mm hole, next likely would be the 4-mm hole, and least likely at the fillet.

3-14

Stresses in Pressurized Cylinders

Cylindrical pressure vessels, hydraulic cylinders, gun barrels, and pipes carrying fluids at high pressures develop both radial and tangential stresses with values that depend upon the radius of the element under consideration. In determining the radial stress σ_r and the tangential stress σ_t , we make use of the assumption that the longitudinal elongation is constant around the circumference of the cylinder. In other words, a right section of the cylinder remains plane after stressing.

Referring to Fig. 3-31, we designate the inside radius of the cylinder by r_i , the outside radius by r_o , the internal pressure by p_i , and the external pressure by p_o . Then it can be shown that the tangential and radial stresses are¹⁰

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i) / r^2}{r_o^2 - r_i^2} \quad (3-49)$$

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i) / r^2}{r_o^2 - r_i^2}$$

As usual, positive values indicate tension and negative values, compression. For the special case of $p_o = 0$, Eq. (3-49) gives

$$\sigma_t = \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r^2} \right)$$

$$\sigma_r = \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 - \frac{r_o^2}{r^2} \right) \quad (3-50)$$

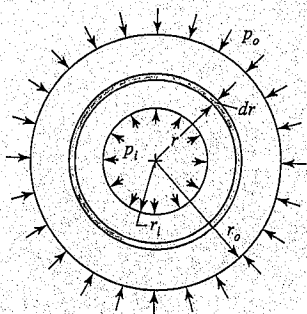


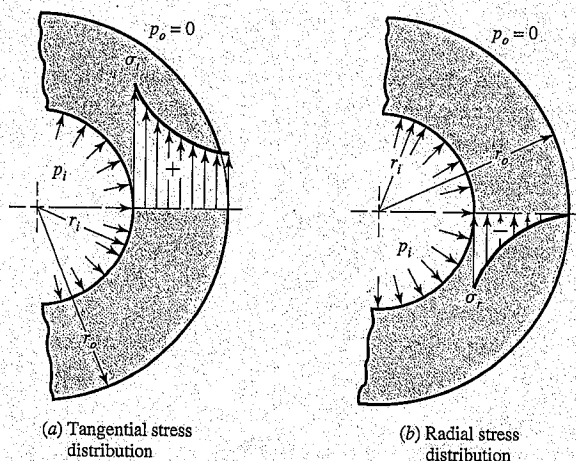
Figure 3-31

A cylinder subjected to both internal and external pressure.

¹⁰See Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 348-352.

Figure 3-32

Distribution of stresses in a thick-walled cylinder subjected to internal pressure.



The equations of set (3-50) are plotted in Fig. 3-32 to show the distribution of stresses over the wall thickness. It should be realized that longitudinal stresses exist when the end reactions to the internal pressure are taken by the pressure vessel itself. This stress is found to be

$$\sigma_l = \frac{p_i r_i^2}{r_o^2 - r_i^2} \quad (3-51)$$

We further note that Eqs. (3-49), (3-50), and (3-51) apply only to sections taken a significant distance from the ends and away from any areas of stress concentration.

Thin-Walled Vessels

When the wall thickness of a cylindrical pressure vessel is about one-tenth, or less, of its radius, the radial stress that results from pressurizing the vessel is quite small compared with the tangential stress. Under these conditions the tangential stress, called the *hoop stress*, can be obtained as follows: From Eq. (3-50), the average tangential stress is given by

$$(\sigma_t)_{av} = \frac{\int_{r_i}^{r_o} \sigma_t dr}{r_o - r_i} = \frac{\int_{r_i}^{r_o} \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r_i^2}\right) dr}{r_o - r_i} = \frac{p_i r_i}{r_o - r_i} = \frac{p_i d_i}{2t} \quad (3-52)$$

where d_i is the inside diameter.

For a thin-walled pressure vessel, an approximation to the maximum tangential stress is

$$(\sigma_t)_{max} = \frac{p(d_i + t)}{2t} \quad (3-53)$$

In a closed cylinder, the longitudinal stress σ_l exists because of the pressure upon the ends of the vessel. If we assume this stress is also distributed uniformly over the wall thickness, we can easily find it to be

$$\sigma_l = \frac{p d_i}{4t} \quad (3-54)$$

EXAMPLE 3-14

An aluminum-alloy pressure vessel is made of tubing having an outside diameter of 8 in and a wall thickness of $\frac{1}{4}$ in.

(a) What pressure can the cylinder carry if the permissible tangential stress is 12 kpsi and the theory for thin-walled vessels is assumed to apply?

(b) On the basis of the pressure found in part (a), compute the stress components using the theory for thick-walled cylinders.

Solution (a) Here $d_i = 8 - 2(0.25) = 7.5$ in, $r_i = 7.5/2 = 3.75$ in, and $r_o = 8/2 = 4$ in. Then $t/r_i = 0.25/3.75 = 0.067$. Since this ratio is less than 0.1, the theory for thin-walled vessels should yield safe results.

We first solve Eq. (3-53) to obtain the allowable pressure. This gives

Answer
$$p = \frac{2t(\sigma_t)_{\max}}{d_i + t} = \frac{2(0.25)(12)(10)^3}{7.5 + 0.25} = 774 \text{ psi}$$

(b) The maximum tangential stress will occur at the inside radius, and so we use $r = r_i$ in the first equation of Eq. (3-50). This gives

Answer
$$(\sigma_t)_{\max} = \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r_i^2} \right) = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 774 \frac{4^2 + 3.75^2}{4^2 - 3.75^2} = 12\,000 \text{ psi}$$

Similarly, the maximum radial stress is found, from the second equation of Eq. (3-50) to be

Answer
$$\sigma_r = -p_i = -774 \text{ psi}$$

The stresses σ_t and σ_r are principal stresses, since there is no shear on these surfaces. Note that there is no significant difference in the stresses in parts (a) and (b), and so the thin-wall theory can be considered satisfactory for this problem.

3-15 Stresses in Rotating Rings

Many rotating elements, such as flywheels and blowers, can be simplified to a rotating ring to determine the stresses. When this is done it is found that the same tangential and radial stresses exist as in the theory for thick-walled cylinders except that they are caused by inertial forces acting on all the particles of the ring. The tangential and radial stresses so found are subject to the following restrictions:

- The outside radius of the ring, or disk, is large compared with the thickness $r_o \geq 10t$.
- The thickness of the ring or disk is constant.
- The stresses are constant over the thickness.

The stresses are¹¹

$$\begin{aligned} \sigma_t &= \rho \omega^2 \left(\frac{3 + \nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1 + 3\nu}{3 + \nu} r^2 \right) \\ \sigma_r &= \rho \omega^2 \left(\frac{3 + \nu}{8} \right) \left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2 \right) \end{aligned} \quad (3-55)$$

¹¹Ibid, pp. 348-357.

where r is the radius to the stress element under consideration, ρ is the mass density, and ω is the angular velocity of the ring in radians per second. For a rotating disk, use $r_i = 0$ in these equations.

3-16 Press and Shrink Fits

When two cylindrical parts are assembled by shrinking or press fitting one part upon another, a contact pressure is created between the two parts. The stresses resulting from this pressure may easily be determined with the equations of the preceding sections.

Figure 3-33 shows two cylindrical members that have been assembled with a shrink fit. Prior to assembly, the outer radius of the inner member was larger than the inner radius of the outer member by the *radial interference* δ . After assembly, an interference contact pressure p develops between the members at the nominal radius R , causing radial stresses $\sigma_r = -p$ in each member at the contacting surfaces. This pressure is given by¹²

$$p = \frac{\delta}{R \left[\frac{1}{E_o} \left(\frac{r_o^2 + R^2}{r_o^2 - R^2} + \nu_o \right) + \frac{1}{E_i} \left(\frac{R^2 + r_i^2}{R^2 - r_i^2} - \nu_i \right) \right]} \quad (3-56)$$

where the subscripts o and i on the material properties correspond to the outer and inner members, respectively. If the two members are of the same material with $E_o = E_i = E$, and $\nu_o = \nu_i$, the relation simplifies to

$$p = \frac{E\delta}{2R^3} \left[\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{r_o^2 - r_i^2} \right] \quad (3-57)$$

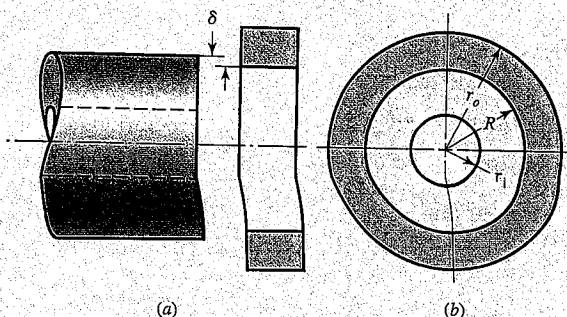
For Eqs. (3-56) or (3-57), diameters can be used in place of R , r_i , and r_o , provided δ is the diametral interference (twice the radial interference).

With p , Eq. (3-49) can be used to determine the radial and tangential stresses in each member. For the inner member, $p_o = p$ and $p_i = 0$. For the outer member, $p_o = 0$ and $p_i = p$. For example, the magnitudes of the tangential stresses at the transition radius R are maximum for both members. For the inner member

$$(\sigma_t)_i \Big|_{r=R} = -p \frac{R^2 + r_i^2}{R^2 - r_i^2} \quad (3-58)$$

Figure 3-33

Notation for press and shrink fits. (a) Unassembled parts; (b) after assembly.



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Coeffici
Temper
0-100°C

¹²Ibid, pp. 348-354.

and, for the outer member

$$(\sigma_r)_o \Big|_{r=R} = p \frac{r_o^2 + R^2}{r_o^2 - R^2} \quad (3-59)$$

Assumptions

It is assumed that both members have the same length. In the case of a hub that has been press-fitted onto a shaft, this assumption would not be true, and there would be an increased pressure at each end of the hub. It is customary to allow for this condition by employing a stress-concentration factor. The value of this factor depends upon the contact pressure and the design of the female member, but its theoretical value is seldom greater than 2.

3-17 Temperature Effects

When the temperature of an unrestrained body is uniformly increased, the body expands, and the normal strain is

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(\Delta T) \quad (3-60)$$

where α is the coefficient of thermal expansion and ΔT is the temperature change, in degrees. In this action the body experiences a simple volume increase with the components of shear strain all zero.

If a straight bar is restrained at the ends so as to prevent lengthwise expansion and then is subjected to a uniform increase in temperature, a compressive stress will develop because of the axial constraint. The stress is

$$\sigma = -\epsilon E = -\alpha(\Delta T)E \quad (3-61)$$

In a similar manner, if a uniform flat plate is restrained at the edges and also subjected to a uniform temperature rise, the compressive stress developed is given by the equation

$$\sigma = -\frac{\alpha(\Delta T)E}{1 - \nu} \quad (3-62)$$

The stresses expressed by Eqs. (3-61) and (3-62) are called *thermal stresses*. They arise because of a temperature change in a clamped or restrained member. Such stresses, for example, occur during welding, since parts to be welded must be clamped before welding. Table 3-3 lists approximate values of the coefficients of thermal expansion.

Table 3-3

Coefficients of Thermal Expansion (Linear Mean Coefficients for the Temperature Range 0-100°C)

Material	Celsius Scale ($^{\circ}\text{C}^{-1}$)	Fahrenheit Scale ($^{\circ}\text{F}^{-1}$)
Aluminum	$23.9(10)^{-6}$	$13.3(10)^{-6}$
Brass, cast	$18.7(10)^{-6}$	$10.4(10)^{-6}$
Carbon steel	$10.8(10)^{-6}$	$6.0(10)^{-6}$
Cast iron	$10.6(10)^{-6}$	$5.9(10)^{-6}$
Magnesium	$25.2(10)^{-6}$	$14.0(10)^{-6}$
Nickel steel	$13.1(10)^{-6}$	$7.3(10)^{-6}$
Stainless steel	$17.3(10)^{-6}$	$9.6(10)^{-6}$
Tungsten	$4.3(10)^{-6}$	$2.4(10)^{-6}$

3-18 Curved Beams in Bending¹³

The distribution of stress in a curved flexural member is determined by using the following assumptions:

- The cross section has an axis of symmetry in the plane of bending.
- Plane cross sections remain plane after bending.
- The modulus of elasticity is the same in tension as in compression.

We shall find that the neutral axis and the centroidal axis of a curved beam, unlike the axes of a straight beam, are not coincident and also that the stress does not vary linearly from the neutral axis. The notation shown in Fig. 3-34 is defined as follows:

r_o = radius of outer fiber

r_i = radius of inner fiber

h = depth of section

c_o = distance from neutral axis to outer fiber

c_i = distance from neutral axis to inner fiber

r_n = radius of neutral axis

r_c = radius of centroidal axis

$e = r_c - r_n$, distance from centroidal axis to neutral axis

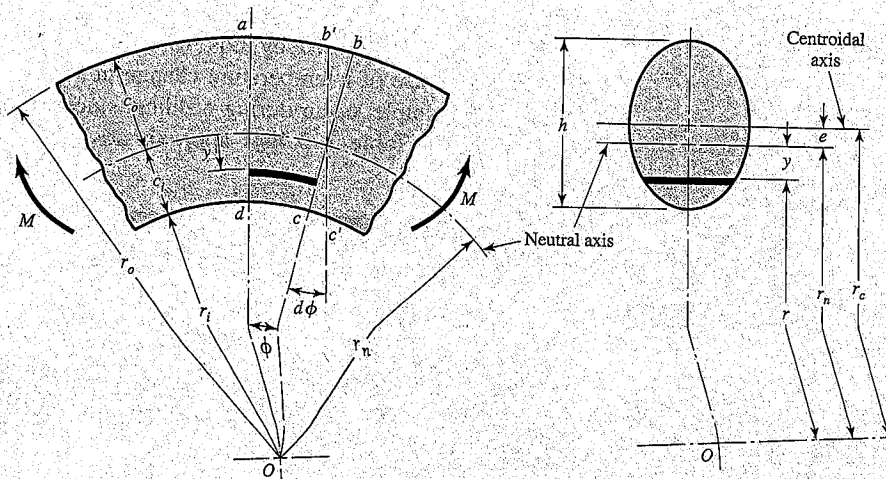
M = bending moment; positive M decreases curvature

Figure 3-34 shows that the neutral and centroidal axes are not coincident. The location of the neutral axis with respect to the center of curvature O is given by the equation

$$r_n = \frac{A}{\int \frac{dA}{r}} \quad (3-63)$$

Figure 3-34

Note that y is positive in the direction toward the center of curvature, point O .



¹³For a complete development of the relations in this section, see Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 309-317.

Furthermore, it can be shown that the stress distribution is given by

$$\sigma = \frac{My}{Ae(r_n - y)} \quad (3-64)$$

where M is positive in the direction shown in Fig. 3-34. The stress distribution given by Eq. (3-64) is *hyperbolic* and not linear as is the case for straight beams. The critical stresses occur at the inner and outer surfaces where $y = c_i$ and $y = -c_o$, respectively, and are

$$\sigma_i = \frac{Mc_i}{Aer_i} \quad \sigma_o = -\frac{Mc_o}{Aer_o} \quad (3-65)$$

These equations are valid for pure bending. In the usual and more general case, such as a crane hook, the U frame of a press, or the frame of a C clamp, the bending moment is due to a force acting at a distance from the cross section under consideration. Thus, the cross section transmits a bending moment *and* an axial force. The axial force is located at the *centroidal axis* of the section and the bending moment is then computed at this location. The tensile or compressive stress due to the axial force, from Eq. (3-22), is then added to the bending stresses given by Eqs. (3-64) and (3-65) to obtain the resultant stresses acting on the section.

EXAMPLE 3-15

Plot the distribution of stresses across section A-A of the crane hook shown in Fig. 3-35a. The cross section is rectangular, with $b = 0.75$ in and $h = 4$ in, and the load is $F = 5000$ lbf.

Solution Since $A = bh$, we have $dA = bdr$ and, from Eq. (3-63),

$$r_n = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r} dr} = \frac{h}{\ln \frac{r_o}{r_i}} \quad (1)$$

From Fig. 3-35b, we see that $r_i = 2$ in, $r_o = 6$ in, $r_c = 4$ in, and $A = 3$ in². Thus, from Eq. (1),

$$r_n = \frac{h}{\ln(r_o/r_i)} = \frac{4}{\ln \frac{6}{2}} = 3.641 \text{ in}$$

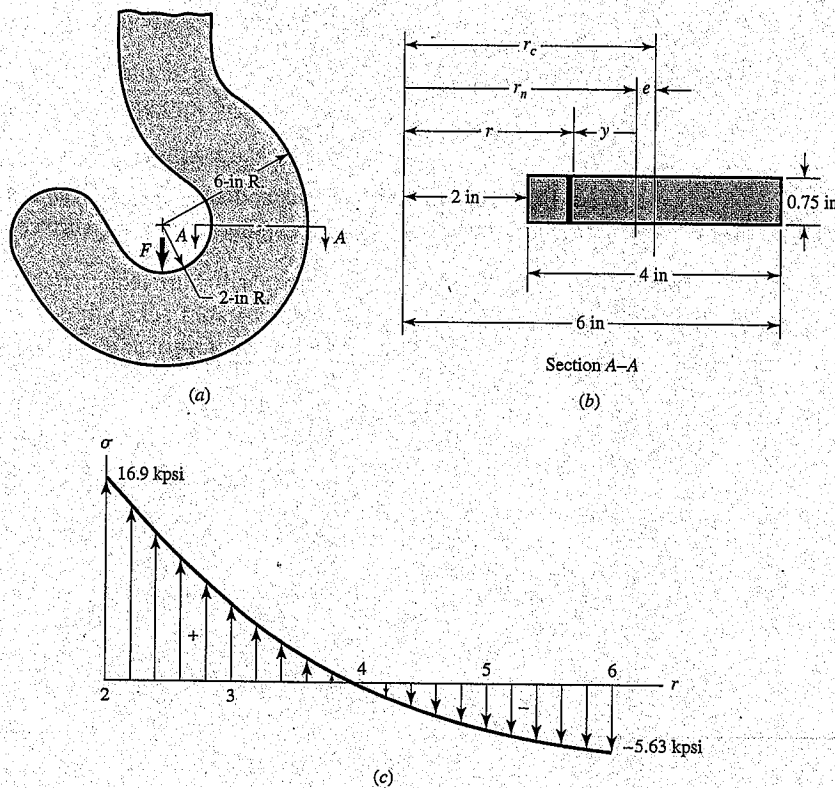
and the eccentricity is $e = r_c - r_n = 4 - 3.641 = 0.359$ in. The moment M is positive and is $M = Fr_c = 5000(4) = 20\,000$ lbf · in. Adding the axial component of stress to Eq. (3-64) gives

$$\sigma = \frac{F}{A} + \frac{My}{Ae(r_n - y)} = \frac{5000}{3} + \frac{(20\,000)(3.641 - r)}{3(0.359)r} \quad (2)$$

Substituting values of r from 2 to 6 in results in the stress distribution shown in Fig. 3-35c. The stresses at the inner and outer radii are found to be 16.9 and -5.63 kpsi, respectively, as shown.

Figure 3-35

- (a) Plan view of crane hook;
 (b) cross section and notation;
 (c) resulting stress distribution.
 There is no stress concentration.



Note in the hook example, the symmetrical rectangular cross section causes the maximum tensile stress to be 3 times greater than the maximum compressive stress. If we wanted to design the hook to use material more effectively we would use more material at the inner radius and less material at the outer radius. For this reason, trapezoidal, T, or unsymmetric I, cross sections are commonly used. Sections most frequently encountered in the stress analysis of curved beams are shown in Table 3-4.

Alternative Calculations for e

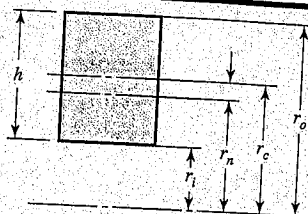
Calculating r_n and r_c mathematically and subtracting the difference can lead to large errors if not done carefully, since r_n and r_c are typically large values compared to e . Since e is in the denominator of Eqs. (3-64) and (3-65), a large error in e can lead to an inaccurate stress calculation. Furthermore, if you have a complex cross section that the tables do not handle, alternative methods for determining e are needed. For a quick and simple approximation of e , it can be shown that¹⁴

$$e \approx \frac{I}{r_c A} \quad (3-66)$$

¹⁴Ibid., pp. 317-321. Also presents a numerical method.

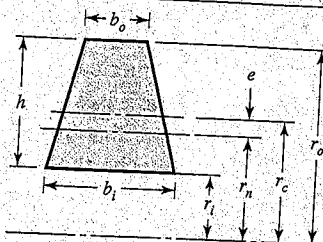
Table 3-4

Formulas for Sections
of Curved Beams



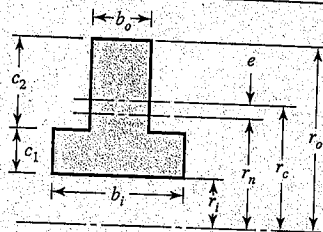
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



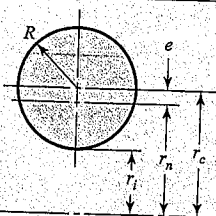
$$r_c = r_i + \frac{h}{3} \frac{b_i + 2b_o}{b_i + b_o}$$

$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h] \ln(r_o/r_i)}$$



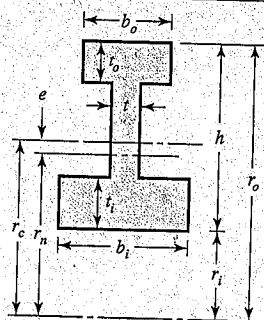
$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$



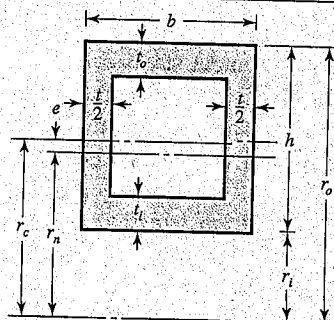
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2(r_c - \sqrt{r_c^2 - R^2})}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2t + \frac{1}{2}t^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln \frac{r_i + t}{r_i} + t \ln \frac{r_o - t_o}{r_i + t_i} + b_o \ln \frac{r_o}{r_o - t_o}}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2t + \frac{1}{2}t^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b \left(\ln \frac{r_i + t_i}{r_i} + \ln \frac{r_o}{r_o - t_o} \right) + t \ln \frac{r_o - t_o}{r_i + t_i}}$$

This approximation is good for a large curvature where e is small with $r_n \approx r_c$. Substituting Eq. (3-66) into Eq. (3-64), with $r_n - y = r$, gives

$$\sigma \approx \frac{My r_c}{I r} \quad (3-67)$$

If $r_n \approx r_c$, which it should be to use Eq. (3-67), then it is only necessary to calculate r_c and to measure y from this axis. Determining r_c for a complex cross section can be done easily by most CAD programs or numerically as shown in the before-mentioned reference. Observe that as the curvature increases, $r \rightarrow r_c$, and Eq. (3-67) becomes the straight-beam formulation, Eq. (3-24). Note that the negative sign is missing because y in Fig. 3-34 is vertically downward, opposite that for the straight-beam equation.

EXAMPLE 3-16

Consider the circular section in Table 3-4 with $r_c = 3$ in and $R = 1$ in. Determine e by using the formula from the table and approximately by using Eq. (3-66). Compare the results of the two solutions.

Solution Using the formula from Table 3-4 gives

$$r_n = \frac{R^2}{2(r_c - \sqrt{r_c^2 - R^2})} = \frac{1^2}{2(3 - \sqrt{3^2 - 1})} = 2.914 \text{ 21 in}$$

This gives an eccentricity of

$$\text{Answer} \quad e = r_c - r_n = 3 - 2.914 \text{ 21} = 0.085 \text{ 79 in}$$

The approximate method, using Eq. (3-66), yields

$$\text{Answer} \quad e \approx \frac{I}{r_c A} = \frac{\pi R^4/4}{r_c(\pi R^2)} = \frac{R^2}{4r_c} = \frac{1^2}{4(3)} = 0.083 \text{ 33 in}$$

This differs from the exact solution by -2.9 percent.

3-19 Contact Stresses

When two bodies having curved surfaces are pressed together, point or line contact changes to area contact, and the stresses developed in the two bodies are three-dimensional. Contact-stress problems arise in the contact of a wheel and a rail, in automotive valve cams and tappets, in mating gear teeth, and in the action of rolling bearings. Typical failures are seen as cracks, pits, or flaking in the surface material.

The most general case of contact stress occurs when each contacting body has a double radius of curvature; that is, when the radius in the plane of rolling is different from the radius in a perpendicular plane, both planes taken through the axis of the contacting force. Here we shall consider only the two special cases of contacting spheres and contacting cylinders.¹⁵ The results presented here are due to H. Hertz and so are frequently known as *Hertzian stresses*.

¹⁵A more comprehensive presentation of contact stresses may be found in Arthur P. Boresi and Richard J. Schmidt, *Advanced Mechanics of Materials*, 6th ed., Wiley, New York, 2003, pp. 589–623.

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Spherical Contact

Figure 3-36 shows two solid spheres of diameters d_1 and d_2 pressed together with a force F . Specifying E_1, ν_1 and E_2, ν_2 as the respective elastic constants of the two spheres, the radius a of the circular contact area is given by the equation

$$a = \sqrt[3]{\frac{3F}{8} \frac{(1 - \nu_1^2)/E_1 + (1 - \nu_2^2)/E_2}{1/d_1 + 1/d_2}} \quad (3-68)$$

The pressure distribution within the contact area of each sphere is hemispherical, as shown in Fig. 3-36b. The maximum pressure occurs at the center of the contact area and is

$$p_{\max} = \frac{3F}{2\pi a^2} \quad (3-69)$$

Equations (3-68) and (3-69) are perfectly general and also apply to the contact of a sphere and a plane surface or of a sphere and an internal spherical surface. For a plane surface, use $d = \infty$. For an internal surface, the diameter is expressed as a negative quantity.

The maximum stresses occur on the z axis, and these are principal stresses. Their values are

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -p_{\max} \left[\left(1 - \left| \frac{z}{a} \right| \tan^{-1} \frac{1}{|z/a|} \right) (1 + \nu) - \frac{1}{2 \left(1 + \frac{z^2}{a^2} \right)} \right] \quad (3-70)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{1 + \frac{z^2}{a^2}} \quad (3-71)$$

Figure 3-36

(a) Two spheres held in contact by force F ; (b) contact stress has a hemispherical distribution across contact zone of diameter $2a$.

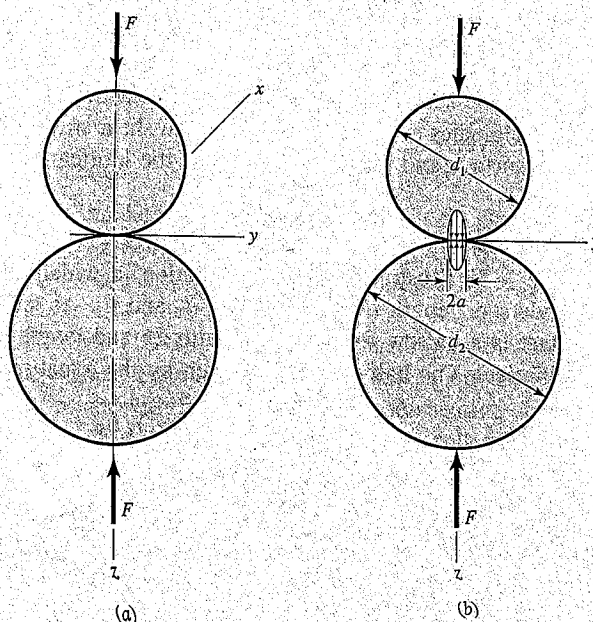
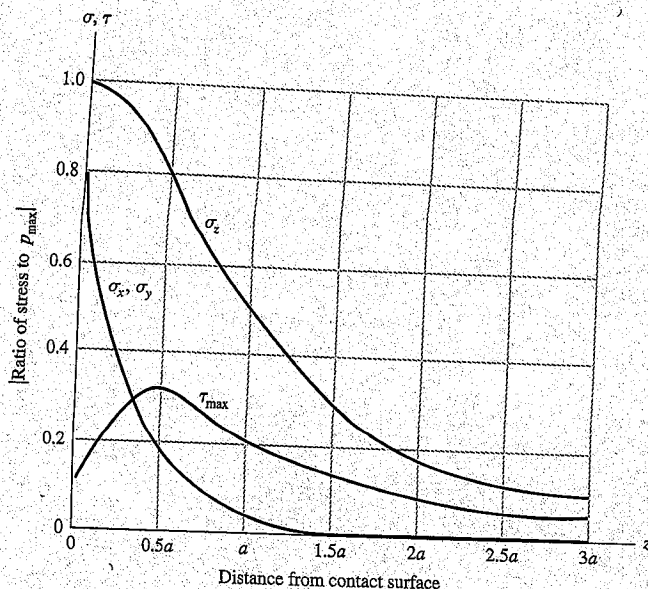


Figure 3-37

Magnitude of the stress components below the surface as a function of the maximum pressure of contacting spheres. Note that the maximum shear stress is slightly below the surface at $z = 0.48a$ and is approximately $0.3p_{\max}$. The chart is based on a Poisson ratio of 0.30. Note that the normal stresses are all compressive stresses.



These equations are valid for either sphere, but the value used for Poisson's ratio must correspond with the sphere under consideration. The equations are even more complicated when stress states off the z axis are to be determined, because here the x and y coordinates must also be included. But these are not required for design purposes, because the maxima occur on the z axis.

Mohr's circles for the stress state described by Eqs. (3-70) and (3-71) are a point and two coincident circles. Since $\sigma_1 = \sigma_2$, we have $\tau_{1/2} = 0$ and

$$\tau_{\max} = \tau_{1/3} = \tau_{2/3} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_2 - \sigma_3}{2} \quad (3-72)$$

Figure 3-37 is a plot of Eqs. (3-70), (3-71), and (3-72) for a distance to $3a$ below the surface. Note that the shear stress reaches a maximum value slightly below the surface. It is the opinion of many authorities that this maximum shear stress is responsible for the surface fatigue failure of contacting elements. The explanation is that a crack originates at the point of maximum shear stress below the surface and progresses to the surface and that the pressure of the lubricant wedges the chip loose.

Cylindrical Contact

Figure 3-38 illustrates a similar situation in which the contacting elements are two cylinders of length l and diameters d_1 and d_2 . As shown in Fig. 3-38b, the area of contact is a narrow rectangle of width $2b$ and length l , and the pressure distribution is elliptical. The half-width b is given by the equation

$$b = \sqrt{\frac{2F(1 - \nu_1^2)/E_1 + (1 - \nu_2^2)/E_2}{\pi l (1/d_1 + 1/d_2)}} \quad (3-73)$$

The maximum pressure is

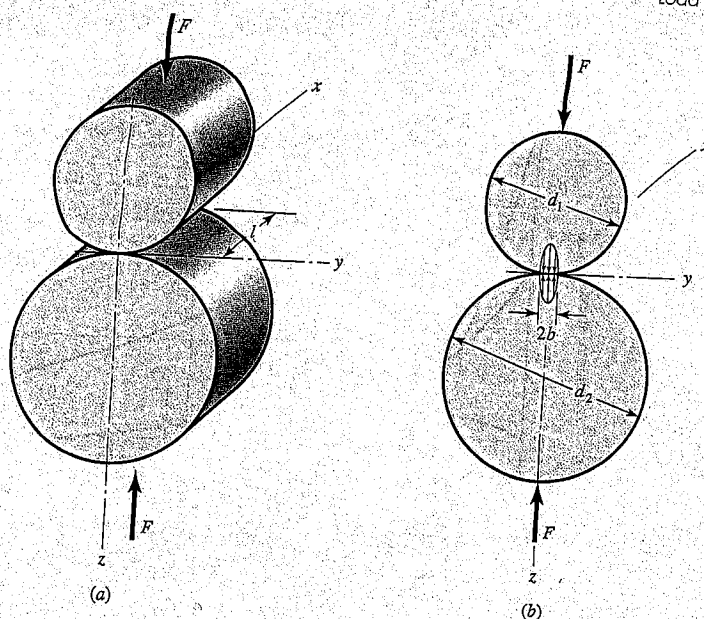
$$p_{\max} = \frac{2F}{\pi bl} \quad (3-74)$$

Figure

(a) Two rigid cylinders in contact under forces F applied along the cylinder axis.
(b) Contact area is elliptical.

Figure 3-38

- (a) Two right circular cylinders held in contact by forces F uniformly distributed along cylinder length l .
 (b) Contact stress has an elliptical distribution across the contact zone of width $2b$.



Equations (3-73) and (3-74) apply to a cylinder and a plane surface, such as a rail, by making $d = \infty$ for the plane surface. The equations also apply to the contact of a cylinder and an internal cylindrical surface; in this case d is made negative for the internal surface.

The stress state along the z axis is given by the equations

$$\sigma_x = -2\nu p_{\max} \left(\sqrt{1 + \frac{z^2}{b^2}} - \left| \frac{z}{b} \right| \right) \quad (3-75)$$

$$\sigma_y = -p_{\max} \left(\frac{1 + 2\frac{z^2}{b^2}}{\sqrt{1 + \frac{z^2}{b^2}}} - 2 \left| \frac{z}{b} \right| \right) \quad (3-76)$$

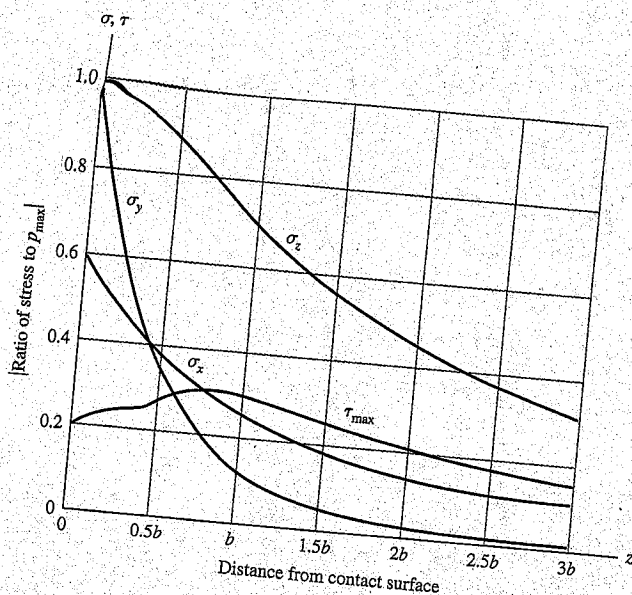
$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{\sqrt{1 + z^2/b^2}} \quad (3-77)$$

These three equations are plotted in Fig. 3-39 up to a distance of $3b$ below the surface. For $0 \leq z \leq 0.436b$, $\sigma_1 = \sigma_x$, and $\tau_{\max} = (\sigma_1 - \sigma_3)/2 = (\sigma_x - \sigma_z)/2$. For $z \geq 0.436b$, $\sigma_1 = \sigma_y$, and $\tau_{\max} = (\sigma_y - \sigma_z)/2$. A plot of $\tau_{\max}$ is also included in Fig. 3-39, where the greatest value occurs at $z/b = 0.786$ with a value of $0.300 p_{\max}$.

Hertz (1881) provided the preceding mathematical models of the stress field when the contact zone is free of shear stress. Another important contact stress case is *line of contact* with friction providing the shearing stress on the contact zone. Such shearing stresses are small with cams and rollers, but in cams with flatfaced followers, wheel-rail contact, and gear teeth, the stresses are elevated above the Hertzian field. Investigations of the effect on the stress field due to normal and shear stresses in the contact zone were begun theoretically by Lundberg (1939), and continued by Mindlin (1949), Smith-Liu (1949), and Poritsky (1949) independently. For further detail, see the reference cited in footnote 15, p. 136.

Figure 3-39

Magnitude of the stress components below the surface as a function of the maximum pressure for contacting cylinders. The largest value of $\tau_{\max}$ occurs at $z/b = 0.786$. Its maximum value is $0.30p_{\max}$. The chart is based on a Poisson ratio of 0.30. Note that all normal stresses are compressive stresses.



3-20

Summary

The ability to quantify the stress condition at a critical location in a machine element is an important skill of the engineer. Why? Whether the member fails or not is assessed by comparing the (damaging) stress at a critical location with the corresponding material strength at this location. This chapter has addressed the description of stress.

Stresses can be estimated with great precision where the geometry is sufficiently simple that theory easily provides the necessary quantitative relationships. In other cases, approximations are used. There are numerical approximations such as finite element analysis (FEA, see Chap. 19), whose results tend to converge on the true values. There are experimental measurements, strain gauging, for example, allowing inference of stresses from the measured strain conditions. Whatever the method(s), the goal is a robust description of the stress condition at a critical location.

The nature of research results and understanding in any field is that the longer we work on it, the more involved things seem to be, and new approaches are sought to help with the complications. As newer schemes are introduced, engineers, hungry for the improvement the new approach promises, begin to use the approach. Optimism usually recedes, as further experience adds concerns. Tasks that promised to extend the capabilities of the nonexpert eventually show that expertise is not optional.

In stress analysis, the computer can be helpful if the necessary equations are available. Spreadsheet analysis can quickly reduce complicated calculations for parametric studies, easily handling "what if" questions relating trade-offs (e.g., less of a costly material or more of a cheaper material). It can even give insight into optimization opportunities.

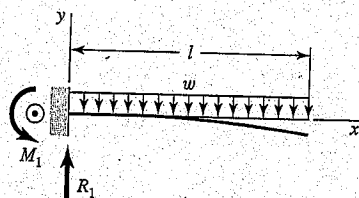
When the necessary equations are not available, then methods such as FEA are attractive, but cautions are in order. Even when you have access to a powerful FEA code, you should be near an expert while you are learning. There are nagging questions of convergence at discontinuities. Elastic analysis is much easier than elastic-plastic analysis. The results are no better than the modeling of reality that was used

Table A-9

Shear, Moment, and Deflection of Beams

(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)

(Continued)

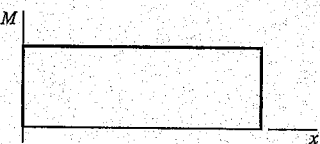
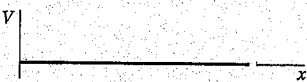
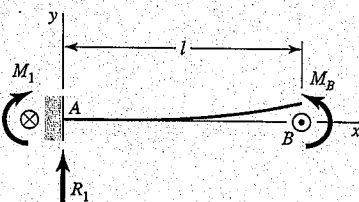
3 Cantilever—uniform load

$$R_1 = wl \quad M_1 = \frac{wl^2}{2}$$

$$V = w(l - x) \quad M = -\frac{w}{2}(l - x)^2$$

$$y = \frac{wx^2}{24EI}(4lx - x^2 - 6l^2)$$

$$y_{\max} = -\frac{wl^4}{8EI}$$

4 Cantilever—moment load

$$R_1 = V = 0 \quad M_1 = M = M_B$$

$$y = \frac{M_B x^2}{2EI} \quad y_{\max} = \frac{M_B l^2}{2EI}$$

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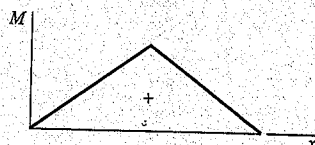
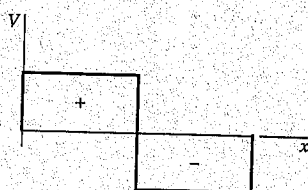
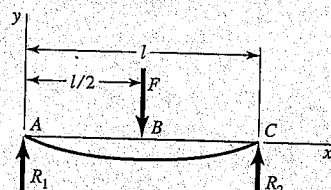
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Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)

(Continued)

5 Simple supports—center load



$$R_1 = R_2 = \frac{F}{2}$$

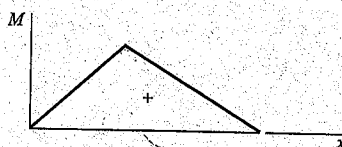
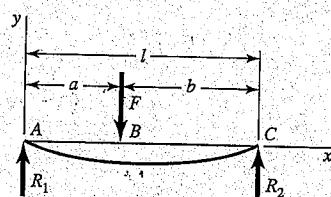
$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fx}{2} \quad M_{BC} = \frac{F}{2}(l-x)$$

$$y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)$$

$$y_{\max} = -\frac{Fl^3}{48EI}$$

6 Simple supports—intermediate load



$$R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l-x)$$

$$y_{AB} = \frac{Fbx}{6EI}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l-x)}{6EI}(x^2 + a^2 - 2lx)$$

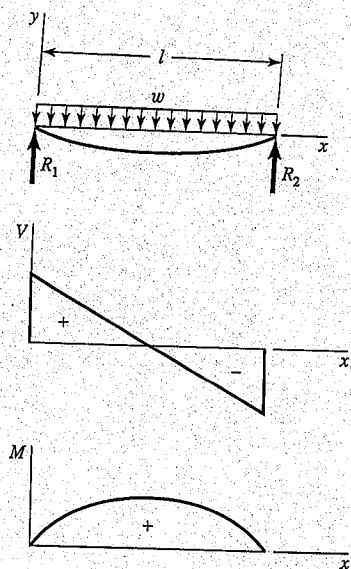
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Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)

(Continued)

7 Simple supports—uniform load



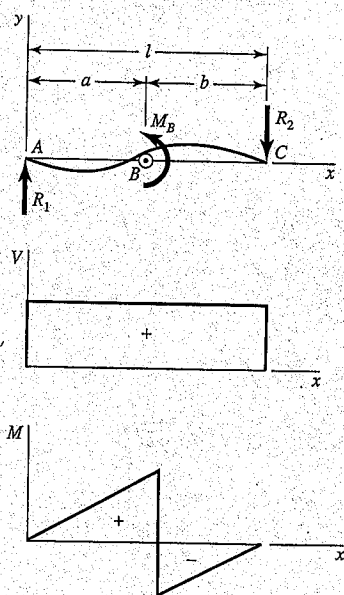
$$R_1 = R_2 = \frac{wl}{2} \quad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\max} = -\frac{5wl^4}{384EI}$$

8 Simple supports—moment load



$$R_1 = R_2 = \frac{M_B}{l} \quad V = \frac{M_B}{l}$$

$$M_{AB} = \frac{M_B x}{l} \quad M_{BC} = \frac{M_B}{l}(x - l)$$

$$y_{AB} = \frac{M_B x}{6EI}(x^2 + 3a^2 - 6al + 2l^2)$$

$$y_{BC} = \frac{M_B}{6EI}[x^3 - 3lx^2 + x(2l^2 + 3a^2) - 3a^2l]$$

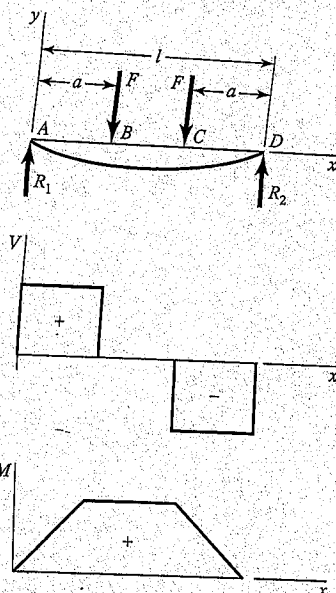
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Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)
(Continued)

9 Simple supports—twin loads



$$R_1 = R_2 = F \quad V_{AB} = F \quad V_{BC} = 0$$

$$V_{CD} = -F$$

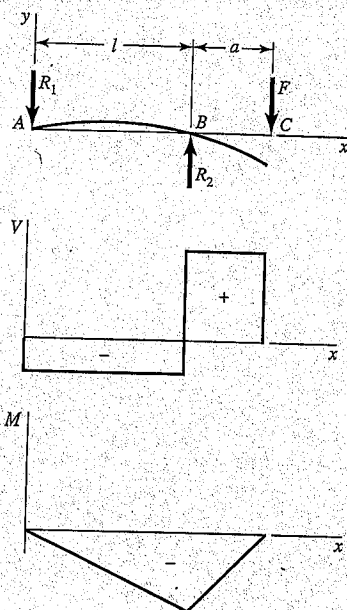
$$M_{AB} = Fx \quad M_{BC} = Fa \quad M_{CD} = F(l-x)$$

$$y_{AB} = \frac{Fx}{6EI}(x^2 + 3a^2 - 3la)$$

$$y_{BC} = \frac{Fa}{6EI}(3x^2 + a^2 - 3lx)$$

$$y_{\max} = \frac{Fa}{24EI}(4a^2 - 3l^2)$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \quad R_2 = \frac{F}{l}(l+a)$$

$$V_{AB} = -\frac{Fa}{l} \quad V_{BC} = F$$

$$M_{AB} = -\frac{Fax}{l} \quad M_{BC} = F(x-l-a)$$

$$y_{AB} = \frac{Fax}{6EI}(l^2 - x^2)$$

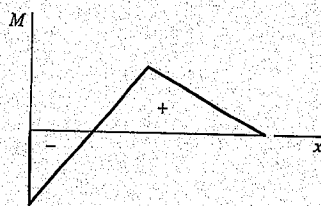
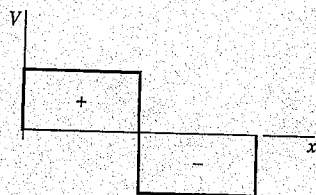
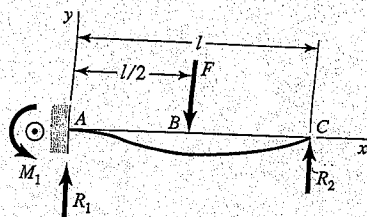
$$y_{BC} = \frac{F(x-l)}{6EI}[(x-l)^2 - a(3x-l)]$$

$$y_c = -\frac{Fa^2}{3EI}(l+a)$$

(Continued)

Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)
(Continued)

11 One fixed and one simple support—center load


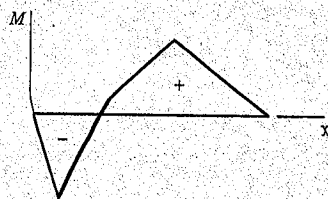
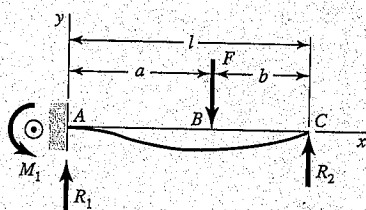
$$R_1 = \frac{11F}{16} \quad R_2 = \frac{5F}{16} \quad M_1 = \frac{3Fl}{16}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{F}{16}(11x - 3l) \quad M_{BC} = \frac{5F}{16}(l - x)$$

$$y_{AB} = \frac{Fx^2}{96EI}(11x - 9l)$$

$$y_{BC} = \frac{F(l-x)}{96EI}(5x^2 + 2l^2 - 10lx)$$

12 One fixed and one simple support—intermediate load


$$R_1 = \frac{Fb}{2l^3}(3l^2 - b^2) \quad R_2 = \frac{Fa^2}{2l^3}(3l - a)$$

$$M_1 = \frac{Fb}{2l^2}(l^2 - b^2)$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb}{2l^3}[b^2l - l^3 + x(3l^2 - b^2)]$$

$$M_{BC} = \frac{Fa^2}{2l^3}(3l^2 - 3lx - al + ax)$$

$$y_{AB} = \frac{Fbx^2}{12EI l^3}[3l(b^2 - l^2) + x(3l^2 - b^2)]$$

$$y_{BC} = y_{AB} - \frac{F(x-a)^3}{6EI}$$

Table

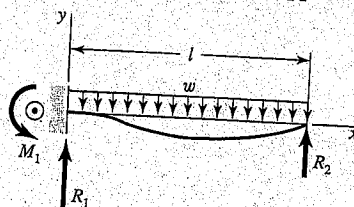
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Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)

(Continued)

13 One fixed and one simple support—uniform load

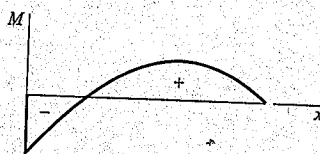
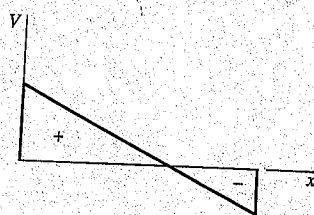


$$R_1 = \frac{5wl}{8} \quad R_2 = \frac{3wl}{8} \quad M_1 = \frac{wl^2}{8}$$

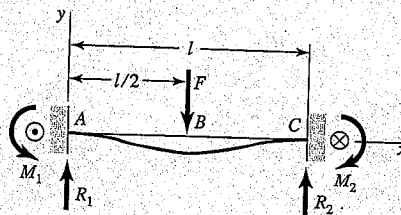
$$V = \frac{5wl}{8} - wx$$

$$M = -\frac{w}{8}(4x^2 - 5lx + l^2)$$

$$y = \frac{wx^2}{48EI}(l-x)(2x-3l)$$



14 Fixed supports—center load



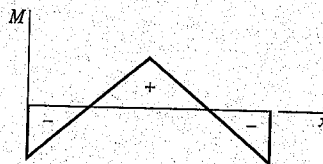
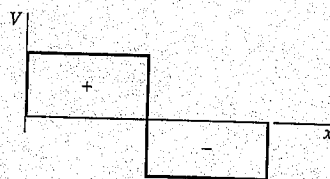
$$R_1 = R_2 = \frac{F}{2} \quad M_1 = M_2 = \frac{Fl}{8}$$

$$V_{AB} = -V_{BC} = \frac{F}{2}$$

$$M_{AB} = \frac{F}{8}(4x - l) \quad M_{BC} = \frac{F}{8}(3l - 4x)$$

$$y_{AB} = \frac{Fx^2}{48EI}(4x - 3l)$$

$$y_{\max} = -\frac{Fl^3}{192EI}$$



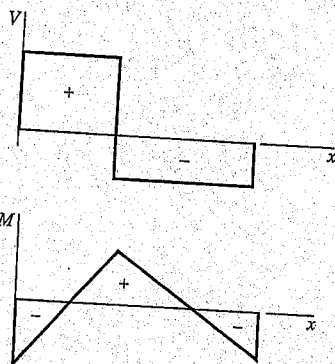
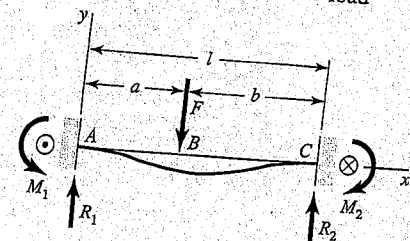
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Table A-9

Shear, Moment, and Deflection of Beams
(Note: Force and moment reactions are positive in the directions shown; equations for shear force V and bending moment M follow the sign conventions given in Sec. 3-2.)

(Continued)

15. Fixed supports—intermediate load



$$R_1 = \frac{Fb^2}{l^3}(3a + b) \quad R_2 = \frac{Fa^2}{l^3}(3b + a)$$

$$M_1 = \frac{Fb^2}{l^2} \quad M_2 = \frac{Fa^2}{l^2}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

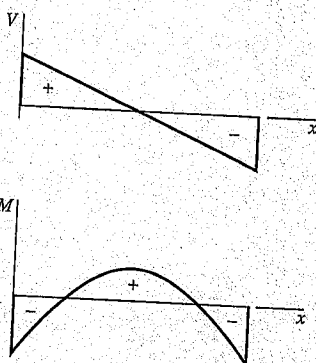
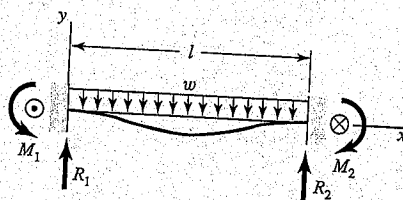
$$M_{AB} = \frac{Fb^2}{l^3}[x(3a + b) - al]$$

$$M_{BC} = M_{AB} - F(x - a)$$

$$y_{AB} = \frac{Fb^2x^2}{6EI l^3}[x(3a + b) - 3al]$$

$$y_{BC} = \frac{Fa^2(l - x)^2}{6EI l^3}[(l - x)(3b + a) - 3bl]$$

16. Fixed supports—uniform load



$$R_1 = R_2 = \frac{wl}{2} \quad M_1 = M_2 = \frac{wl^2}{12}$$

$$V = \frac{w}{2}(l - 2x)$$

$$M = \frac{w}{12}(6lx - 6x^2 - l^2)$$

$$y = -\frac{wx^2}{24EI}(l - x)^2$$

$$y_{\max} = -\frac{wl^4}{384EI}$$

Table

Cumulative

$\Phi(z_\alpha)$

z_α

0.0

0.1

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1.1

1.2

1.3

1.4

1.5

1.6

1.7

1.8

1.9

2.0

2.1

2.2

2.3

2.4

2.5

2.6

2.7

2.8

2.9