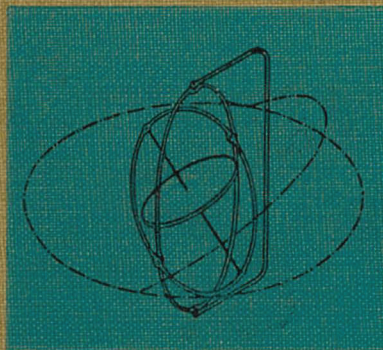


McCLURE

THEORY OF

McCLURE



THEORY OF
INERTIAL GUIDANCE

THEORY OF
INERTIAL
GUIDANCE

1960

PRENTICE-HALL, INC.

Englewood Cliffs, New Jersey

Dedicated to my parents:

Connie Lisle McClure

Gertrude Littlepage McClure

THEORY OF INERTIAL GUIDANCE

Connie L. McClure

© 1960 by PRENTICE-HALL, Inc.,
Englewood Cliffs, New Jersey

*All rights reserved. No part of this book may be
reproduced in any form, by mimeograph
or any other means, without permission in writing
from the publishers.*

L. C. Cat. No. 60-11072

Printed in the United States of America
91397-C

the preliminary theory of stabilized platforms, such preliminary description concerning the inertial and rotating angular stabilization of a gimballed orthogonal triad of reference axes. The preliminary theory of stabilized platforms is a logical prerequisite for the following chapter pertaining to the "single channel" or vertical coplanar dynamics of the leveling and indicating mechanisms (Schuler and velocity damped), and to the description of certain components, anomalies, and effects conveniently discussed within the single channel mode of thought. Remaining chapters are devoted to the spatial analytical description and simulation of inertial systems. Additional topics concerning terrestrial celestial navigational variables and computations, control system stability, and miscellaneous mathematical recollections are presented as appendices.

The work represents an endeavor to record a firmly grounded treatment of the fundamental analytical and mechanical rudiments of inertial guidance, being, hopefully, a treatment not philosophically restricted to the immediate era of technology.

Connie L. McClure

ACKNOWLEDGMENTS

PREPARATION of this work was partially facilitated by the sympathetic attitude and stimulating environment accorded the writer by the Owego facility of the Federal Systems Division of the International Business Machines Corporation. With respect to the many individuals who extended kind encouragement and indirect assistance the writer specifically acknowledges his gratitude to Mr. J. E. Lundberg, Mr. J. S. Renfro, Mr. R. D. Clements, and Mr. G. A. Malmros. Also, the writer acknowledges the indirect contributions of his close friend and sometime colleague, Dr. Cornelius T. Leondes, of the University of California at Los Angeles.

CONTENTS

chapter one	
Introduction	1
1.1 Some description concerning the cosmic environment of the terrestrial geoid, 1;	
1.2 Certain properties of the terrestrial geoid, 11;	
1.3 Terrestrial navigation and guidance, 15;	
1.4 Conceptual foundations of inertial systems and inertial guidance, 26	
chapter two	
Cartesian Transformations and the Vector Concept	31
2.1 Introduction, 31;	
2.2 Transformation of the coordinates of points, 35;	
2.3 Transformation of projections, definitions of a vector, 37	
chapter three	
Some Fundamentals of Vector Analysis	43
3.1 Certain preliminaries, 43;	
3.2 Elementary vector algebra, 46;	
3.3 Other rudimentary definitions and operations, 54	
chapter four	
Kinematic Fundamentals	57
4.1 Preliminary designations, 57;	
4.2 Compound relative motion, 59;	
4.3 Total kinetic acceleration in a rotating system, 64;	
4.4 Coplanar transport acceleration, 70;	
4.5 Comments concerning subsequent applications, 74	
chapter five	
Certain Concepts of Rigid Body Dynamics	75
5.1 Introduction, motion of a mass point, 75;	
5.2 Inertial moment of momentum of a mass point, 78;	
5.3 Inertial moment of kinetic force	

applied to a mass point, 82; 5.4 Some fundamentals of classical rigid body dynamics, 83; 5.5 Euler's dynamical equations, 89; 5.6 Rudimentary concepts relating to inertial guidance, 95

chapter six

Some Gyrodynamics 96

6.1 Precession, 96; 6.2 Euler's angles, Euler's kinematical equations, 101; 6.3 The classical top and gyroscope, 103

chapter seven

Preliminary Theory of Stabilized Platforms 116

7.1 Introduction, 116; 7.2 Kinematic configuration of a reference platform, 117; 7.3 Uniaxial inertial and rotating stabilization, 119; 7.4 Ideal stabilization of a two-gimbal platform, 147; 7.5 Stabilization degradation attending impressed oscillations (platform dynamic drift), 155; 7.6 Certain other basic features of platform theory and operation, 163

chapter eight

Single-Channel Descriptions of Terrestrial Inertial Systems 167

8.1 A description of the spatial dynamical environment of the terrestrial leveling mechanism, 167; 8.2 The ideal single-channel Schuler system, 182; 8.3 Effect of idealized gravitational deflections on the single-channel Schuler system, 191; 8.4 Effects of certain component anomalies in the single-channel Schuler system, 196; 8.5 Certain dynamically-induced anomalies described against the single-channel background (mass unbalance and anisoelastic torques, 209; 8.6 Single-channel description of some rectilinear motion sensor dynamics, 215; 8.7 Diagrammatic and implemental variations in the ideal single-channel Schuler system, 230; 8.8 Single-channel description of velocity-damped terrestrial inertial system, 232

chapter nine

Other Single-Channel Descriptions 248

9.1 Introduction, 248; 9.2 An ideal third order velocity-damped system, 248; 9.3 Single-channel description of a particular variable coefficient velocity-damped system, 250; 9.4 An ideal single-channel non-linear velocity-damped system, 252

chapter ten

Terrestrial Inertial Guidance 255

10.1 Introduction, 255; 10.2 Pilot-autopilot participation in the guidance mechanism, 257; 10.3 A spatial description of the terrestrial Schuler system, 261; 10.4 Terrestrial inertial guidance with spatial dynamics described directly in geospherical coordinates, 273; 10.5 Other terrestrial inertial guidance systems, 282; 10.6 Simulative evaluation of terrestrial inertial guidance systems, 291

Bibliography 303

Appendices

- I CERTAIN CELESTIAL COORDINATES AND DEFINITIONS, 307
- II DIRECTION COSINES IN TERMS OF EULER'S ANGLES, 312
- III CERTAIN ELEMENTARY SYSTEMATIC STABILITY CRITERIA, 314
- IV RESOLVER CHARACTERIZATION AND TANGENT SOLVER SERVODYNAMICS, 316
- V EXEMPLARY APPLICATION OF FREQUENCY TECHNIQUES IN THE ANALYSIS AND COMPENSATION OF A UNIAXIAL STABILIZATION SYSTEM, 322
- VI A NOTE CONCERNING THE CONSTANT k_1 OCCURRING IN THE SINGLE-CHANNEL VELOCITY-DAMPED INERTIAL SYSTEM, 335

Index 337

CHAPTER SEVEN PRELIMINARY THEORY OF STABILIZED PLATFORMS

7.1 | Introduction

In theory, and under an assumption of sufficient data processing capacity, an inertial-guidance system may be conceived which employs inertial-motion sensors rigidly affixed to the transporting vehicle. That is, with sufficient computer capability and an exact knowledge and incorporation of initial conditions, an inertial-guidance system with craft-oriented sensors may be designed (in the mathematical sense, at least) to generate guidance intelligence, even in the presence of the undesired motion detections due to local perturbations. However, to date, the more popular approach to inertial-guidance implementation involves a *reference platform* which is partially or completely isolated from the local angular motion of the transporting vehicle. Such reference bases, angularly isolated from local craft rotations, are usually called *stabilized platforms*.

A stabilized platform, being stabilized in accordance with some specific implementation philosophy, provides a base upon which to mount

angular and rectilinear motion sensors. Signals from these sensors induce the stabilization, and in so doing, the same signals may also contribute to generation of guidance intelligence. It will be seen that a reference platform may be angularly stabilized with respect to *inertial reference*, or may be angularly stabilized with respect to a *rotating reference frame*. Stabilization with respect to rotating reference, as involved in the maintenance of level attitude of a platform as the transporting craft moves over the Earth, is an essential requirement in systems which slave a platform axis to a terrestrial vertical. In this connection, one may consider a reference frame with orthogonal axes east, north, and vertical, which moves over the Earth with its origin centered on a reference platform carried by a moving vehicle. The hypothetical axes of such a reference frame rotate about each other with respect to inertial space as they maintain the terrestrial attitude (east, north, and vertical) under motion relative to the Earth. This is a prime example of a useful rotating reference system. Clearly, then, if the three orthogonal axes of a stabilized platform may be slaved to these rotating reference axes (east, north, and vertical), irrespective of craft perturbations, terrestrial direction of motion may be identified, and the system provides one essential feature of inertial guidance.

This chapter concerns the configuration and preliminary theory of certain typical stabilized platforms. After introduction of a mechanical configuration, it is convenient to progress *from* a consideration of the *inertial* and *rotating* stabilization of a reference line perpendicular to a *single axis*, to a consideration of the *inertial* and *rotating* stabilization of three orthogonal platform axes.

7.2 | Kinematic Configuration of a Reference Platform

Strictly speaking, the term *stabilized platform* implies only a central element, the rotational behavior of which is isolated from that of the transporting vehicle. Stabilization of a platform with respect to inertial directions, or with respect to rotating reference, may be achieved by employment of three relative-motion drive systems which respectively position the platform and the elements of a gimbal system of the nature indicated by Fig. 7-1. The platform and gimbal servo-drive systems accept signals from sensors, usually mounted on the central element (the platform), and induce relative angular motions in accord with the nature of the stabilization desired. In the case of platform stabilization with respect to *inertial* directions, only *angular-motion sensors* are involved in the multi-closed loop process. On the other hand, stabilization with respect to rotating reference usually involves both angular and rectilinear motion sensors and signals.

In the typical arrangement depicted by Fig. 7-1, where the central element or platform is denoted by the right-handed orthogonal frame ϕ_P , with axes P^1, P^2, P^3 , the platform is driven about P^3 with respect to the pitch (inner) gimbal. That is, the physical rotational axis of the platform is the P^3 axis, which is nominally vertical and is mounted in the pitch gimbal at upper and lower bearing junctions. Orthogonal axes P^1 and P^2 are perpendicular to P^3 and fixed in the physical platform to

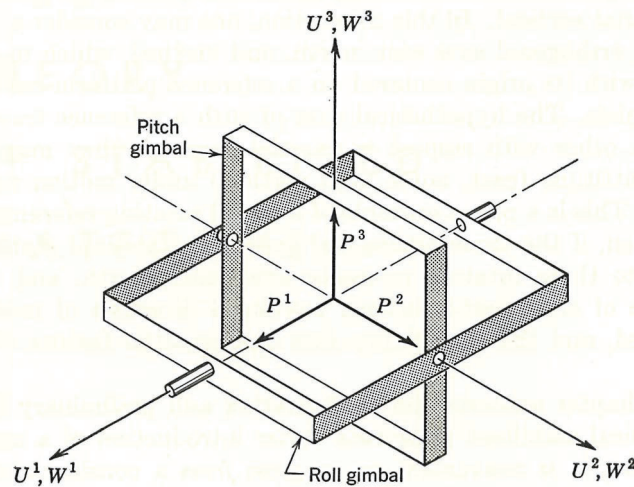


Figure 7-1

define its P^1P^2 plane, which is nominally horizontal under stabilization. The pitch gimbal is driven about W^2 with respect to the roll gimbal, while the roll gimbal is simultaneously driven about U^1 (a longitudinal axis) with respect to the vehicle. Under suitable closed-loop processing of motion-sensor signals, the arrangement provides the essential kinematic elements for inertial or rotating orientation stabilization. In the simplest sense, rotation of the vehicle disturbs the platform (either completely or with some residual rotation) through rotations which are sensed by the sensors mounted thereon. The sensors provide signals employed to drive platform and gimbals in opposition to the residual disturbances, so as to achieve stabilization. Though it is only the central element which is, in the strict sense, the stabilized platform, in common parlance the term *stabilized platform* has come to mean the entire assembly, including gimbals and servo-drive systems.

While the common particular arrangement of Fig. 7-1 provides a basic kinematic configuration for stabilization, it will later be seen that employment of two gimbals in platform stabilization entails certain *craft-attitude restrictions*. Anticipating subsequent discussion, one may easily

observe that the horizontal stabilization of the P^1P^2 platform plane of Fig. 7-1 introduces indeterminacy when the U^1 (craft) axis is in a vertical attitude (craft in vertical dive or climb). Under such attitude, a P^3 axis angular sensor, which is normally an azimuth-rotation sensor, senses roll motion while the other angular motion sensors perpendicular to P^3 , normally the two used to achieve horizontal stabilization, can *not* sense roll of the craft. Stabilization singularities attendant to such situations are sometimes rectified by employment of a third or so-called *redundant* stabilization gimbal.

Angular motion sensing required in platform stabilization may be implemented by utilization of either a pair of *two degrees of freedom* gyros, similar to the classical gyroscope of the last chapter, or by a triad of the *single degree of freedom* type, yet to be introduced. This latter device, the *single degree of freedom* gyro, will be discussed in the next section, which concerns the single-axis or coplanar stabilization of a reference line.

7.3 | Uniaxial Inertial and Rotating Stabilization

Both the inertial and rotating stabilizations of the axial triads associated with a reference platform are most conveniently developed by departing from the preliminary descriptions pertaining to the corresponding one-axis stabilizations. Properties revealed by one-axis developments are of *more* than introductory value, since the inertial and rotating stabilizations of a *reference line perpendicular to a single axis fixed in inertial space* reveal dynamic properties which are immediately adjustable to moving spatial stabilization descriptions. In this connection, it may be mentioned that single-axis angular stabilization descriptions, and also extensions pertaining to the *single channel* characterizations of complete inertial guidance systems, are extensively employed in preliminary engineering and evaluation. A study of uniaxial angular stabilization is thus significant both with respect to the spatial stabilization of reference platforms and subsequent *single-channel* discussions of complete inertial systems.

Of first consideration in a description of single-axis angular stabilization (inertial and rotating) is the angular-motion sensor and its nominal disposition with respect to the geometry of the stabilization situation. An angular-motion sensor of extensive past and current utilization is that known as the "*single degree of freedom*" integrating *gyroscope*. Significance of the word *integrating* will be revealed later. Reference to this device may involve the abbreviation "s.d.f. gyro." More often, where unambiguous, such a device will simply be called the "gyro." Figure 7-2 depicts the configuration of a practical s.d.f. instrument gyro along with an inertial reference frame ϕ_I (axes I^1, I^2, I^3) and some of the

basic elements of the attendant single-axis stabilization system. The line to be stabilized (either inertially or with respect to a rotating reference line) is perpendicular to I^3 , and is thus always either in or parallel to the I^1I^2 plane. As indicated presently, this stabilized reference line may conveniently be the precessional axis of the gyro, or it may lie along the I^1I^2 plane projection of the gyro spin axis. In *undisturbed initial* attitude,

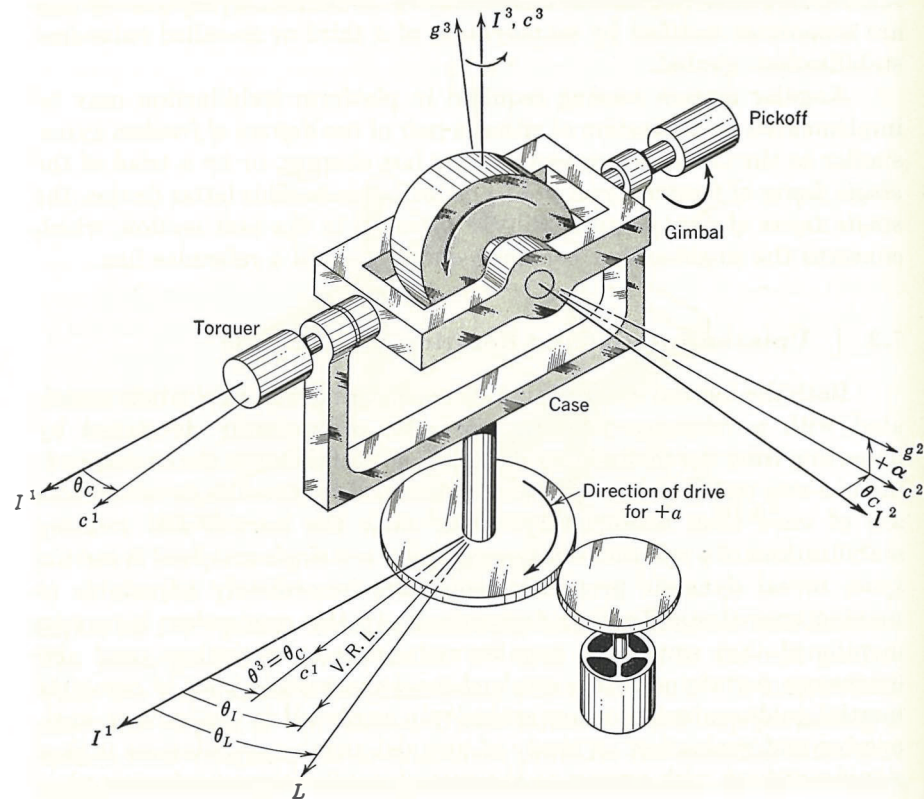


Figure 7-2

the gyro spin axis is in or parallel to the I^1I^2 plane, and, as indicated in Fig. 7-2, may be initially along I^2 . A reference frame ϕ_a (axes g^1, g^2, g^3) is associated with the precessing members of the gyro, so that g^1 is along the precessional axis and g^2 is coincident with the spin axis of the wheel. Another reference frame, ϕ_c (axes c^1, c^2, c^3), is associated with the gyro case or yoke such that c^1 is along the precessional axis, while c^3 is along the axis of base rotation, being here, in the description of single-axis stabilization, also coincident with I^3 .

Consider first just the s.d.f. gyro, without regard to its inclusion in a single-axis stabilization system. As is depicted by Fig. 7-2, the gyro

consists of the case (or yoke, as drawn), the gimbal and wheel (or rotor), and the pickoff and torquer. The pickoff is fixed to the case, and supplies an output signal proportional to the precession (angle), α , while the torquer, also fixed to the case, can supply a torque to the gimbal which is proportional to an input signal. It is at the present assumed that all bearings are frictionless. The wheel is assumed to be of homogeneous polar moment of inertia, J_w (about the spin axis), and to rotate at a constant spin angular velocity, ω_w , such that the spin angular momentum of magnitude $h_w = J_w \omega_w$ is and remains constant. In an actual unit, this condition may be achieved by utilizing a gyro wheel which is the rotor of a suitable speed regulated motor. This spin angular momentum is characterized by a constant vector projection along the spin axis, g^2 . If no motion inducing torque is applied to the gyro (a component of which lies along I^1 or I^3), the spin axis remains fixed in inertial space and, thus, does not depart from its initial coincidence with I^2 . However, if an angular velocity, ω^3 , about I^3 , is impressed on the gyro, then, as predicted by both the free-body and gyroscope developments of the last chapter, the gimbal and wheel system experience a precessional torque, τ_g^1 , about c^1 (and g^1), which can induce a precession α . See again Fig. 7-2.

The configuration of the s.d.f. gyro permits a circumvention of the nonlinear differential equations of the last chapter by admitting to description in terms of the behavior of *just the spin angular momentum vector* $\bar{h}_w = J_w \omega_w \hat{j}_g$, where $\hat{j}_g$ is the unit vector along g^2 . This description is particularly applicable for small values of α , and involves the condition that the free axial reciprocity of precession is restrained by the precessional axis bearings of the gyro. That is, in an s.d.f. gyro, the reciprocal precession of g^1 , due to the generation of a $\frac{d\alpha}{dt}$, can not occur, because the related torque is reacted by the gimbal bearings. In this connection, recall Section 6.1 of Chapter 6.

As the disturbing inertial rotation, θ^3 (associated with the disturbing inertial angular velocity ω^3), is applied *about the case axis* c^3 (which remains coincident with the inertial axis I^3 in the uniaxial input situation under discussion), it is desirable to express the wheel angular momentum, $\bar{h}_w$, in terms of projections measured along the case axes c^1 , c^2 , and c^3 . Thus

$$(\bar{h}_w)_c = h_{wc}^1 \hat{i}_c + h_{wc}^2 \hat{j}_c + h_{wc}^3 \hat{k}_c \quad 7-1$$

where, by inspection

$$h_{wc}^1 = 0 \quad h_{wc}^2 = h_w \cos \alpha \quad h_{wc}^3 = h_w \sin \alpha \quad 7-2$$

$$h_w = J_w \omega_w$$

It is now recalled, from Chapter 5, that the vector expression relating

torque and angular momentum (or moment of momentum) is

$$\bar{\tau}_{uB} = \left(\frac{d\bar{h}_w}{dt} \right)_I = \left(\frac{d\bar{h}_w}{dt} \right)_c + \bar{\omega} \times (\bar{h}_w)_c \quad 7-3$$

which involves another application of the theorem of Coriolis, here expressed in terms of the current notation. The vector

$$\bar{\omega} = \omega_c^1 \hat{i}_c + \omega_c^2 \hat{j}_{ic} + \omega_c^3 \hat{k}_c \quad 7-4$$

is the inertial angular velocity of the case (or yoke) expressed in terms of projections along its own rotating axes c^i . These projections are here just

$$\omega_c^1 = 0 \quad \omega_c^2 = 0 \quad \omega_c^3 = \omega^3 \quad 7-5$$

Employment of Equations 7-1, 7-2, 7-4, and 7-5 in an explicit expansion of Vector Expression 7-3 gives the *applied torques* which may be reacted about the case axes, c^i . Expansion and differentiation reveals these τ_{aG}^i to be

$$\tau_{aG}^1 = -\omega^3 h_w \cos \alpha \quad 7-6$$

$$\tau_{aG}^2 = -h_w \frac{d\alpha}{dt} \sin \alpha \quad 7-7$$

$$\tau_{aG}^3 = h_w \frac{d\alpha}{dt} \cos \alpha \quad 7-8$$

being respectively measured about c^1 , c^2 , and c^3 . The applied torque τ_{aG}^1 is reacted by the precessional torque τ_G^1 such that

$$\tau_G^1 = -\tau_{aG}^1 = \omega^3 h_w \cos \alpha \quad 7-9$$

while τ_{aG}^2 and τ_{aG}^3 are actually applied by the gimbal bearings to react

$$\tau_G^2 = h_w \frac{d\alpha}{dt} \sin \alpha \quad \text{and} \quad \tau_G^3 = -h_w \frac{d\alpha}{dt} \cos \alpha$$

Precession is thus again revealed to be such that the spin momentum vector rotates toward the input angular velocity vector. See again Fig. 7-2.

Now the precessional torque, τ_G^1 , is reacted by the inertia and viscous torques associated with the motion of the gimbal-wheel system about c^1 (nonviscous frictional torque about c^1 is presently assumed to be zero), and also by any torque which is intentionally applied by the torquer. That is, inertia, viscous, and torquer torques *are collectively the applied torque* τ_{aG}^1 . If *no* torque is applied by the torquer, the equation of motion characterizing precessional behavior, $\alpha = \alpha(t)$, may be written

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} - \omega^3 h_w \cos \alpha \quad 7-10$$

where J_α is the polar moment of inertia of the precessing system, measured

about c^1 , and f_α is the viscous damping coefficient measured about the same axis. For cgs units, f_α is seen to have the dimensions

$$(\text{dyne})(\text{cm})(\text{sec}) = \frac{(\text{gm})(\text{cm})^2}{(\text{sec})}$$

For small values of α , which must be maintained in a stabilization system, Equation 7-10 is essentially

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} = h_w \omega^3 = h_w \frac{d\theta^3}{dt} = h_w \frac{d\theta_c}{dt} \quad 7-11$$

$$\omega^3 = \frac{d\theta^3}{dt} = \frac{d\theta_c}{dt}$$

being exact when $\alpha = 0$. It is understood that ω^3 is the *net* inertial angular velocity impressed about I^3 (and c^3) and that it reflects the *difference* between base rotation and opposition restitution induced through stabilization based on the motion detection signal α (precession). This will be emphasized presently. Expression 7-11 is the basic equation characterizing the s.d.f. gyro when employed as an angular motion sensor in an *uniaxial inertial* angular stabilization system. On the other hand, stabilization of a reference line (say c^1) with respect to a radial line perpendicular to I^3 which *rotates* about I^3 (in or parallel to the $I^1 I^2$ plane) will be seen to *involve an application of gyro-torquer torque*. Thus, the expression for *rotating stabilization*, which corresponds to Expression 7-11 for inertial angular stabilization, is

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} + k_\tau \tau_R(t) = h_w \omega^3 \quad 7-12$$

where k_τ is a dimensional constant which gives the torquer output time function, $k_\tau \tau_R(t)$, the dimensions of a torque. The necessity for torquer torque will be established later in this section.

Interesting preliminaries to inertial and rotating uniaxial angular stabilization are provided by certain elementary functional properties of the s.d.f. gyro sensor. First, for motion which begins from null initial conditions $\alpha(0) = \dot{\alpha}(0) = \theta^3(0) = 0$, a Laplace transformation* of Expression 7-11 gives

$$A(s) = \frac{\frac{h_w}{f_\alpha}}{\frac{J_\alpha}{f_\alpha}s + 1} \Theta(s) \quad 7-13$$

$$A(s) = \mathcal{L}[\alpha(t)] \quad \Theta(s) = \mathcal{L}[\theta^3(t)]$$

* It is assumed that the reader entertains a rudimentary knowledge of the operational calculus based on the unilateral Laplace transform. See bibliography for salient literature.

which, as h_w is usually large, indicates that for large values of the damping coefficient, f_α , the precession is essentially proportional to the inertial input rotation $\theta^3(t)$ (or $\theta_c(t) = (I^1, c^1)$). On the other hand, if $f_\alpha = 0$

$$A(s) = \frac{h_w}{J_\alpha s} \Theta(s) \quad 7-14$$

which indicates that, with *no* damping, $\alpha(t)$ is proportional to the time integral of $\theta^3(t)$. Yet another characteristic is that relating to the situation where there is *no* input rotation ($\theta^3(t) = 0$), but a signal is applied to the torquer. In this case, gyroscopic torque is replaced by torquer torque, and the equation of motion about the c^1 axis becomes

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} = k_\tau \tau_R(t) \quad 7-15$$

which, for null initial conditions, may be transformed to give

$$A(s) = \frac{\frac{k_\tau}{f_\alpha}}{s \left(\frac{J_\alpha}{f_\alpha} s + 1 \right)} T(s) \quad 7-16$$

$$T(s) = \mathfrak{L}[\tau_R(t)]$$

which indicates that, if the case is fixed in inertial space about I^3 and f_α is large, then the induced precession, $\alpha = \alpha(t)$, is essentially proportional to the time integral of the input torquer signal, $\tau_R(t)$. It may now also be commented that these properties, particularly Equation 7-14, account for the term *integrating gyro*, mentioned with the introduction of the s.d.f. gyro under discussion. Opposed to this functional designation (integrating gyro), an s.d.f. gyro with *elastic torsional restraint* about the precessional axis can employ precession as a direct measure of input angular velocity, ω^3 , and is called a s.d.f. *rate gyro*. This is easily understood by contemplation of the equation

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} + k_\alpha \alpha = h_w \omega^3 \quad 7-17$$

where k_α is a torsional elastic constant, example dimensions of which may be (dyne)(cm) per radian. If $f_\alpha = 0$ and J_α is relatively small, Equation 7-17 essentially gives

$$\alpha = \frac{h_w}{k_\alpha} \omega^3 \quad 7-18$$

indicating a precession proportional to input angular velocity measured about I^3 (or c^3).

The s.d.f. gyro of the so-called integrating type is now employed as the angular motion sensor in uniaxial angular stabilization. Discussion of stabilization of a reference line, c^1 , with respect to a rotating reference line, L , is preceded by a development of *inertial* uniaxial angular stabilization.

Uniaxial Inertial Angular Stabilization. Suppose that the s.d.f. gyro of the integrating type, discussed above, is mounted on a base or vehicle in such a manner that the gyro may be driven in rotation about c^3 with respect to the vehicle, while the vehicle (entire assembly) experiences inertial rotation about I^3 (c^3 remains coincident with or parallel to I^3). The drive may be direct and concentric, or, as indicated by Fig. 7-2, it may involve a gear train between gyro base and drive-motor shaft. The latter arrangement, indicated by Fig. 7-2, constitutes the basic configuration of a common type gyro-controlled uniaxial angular stabilization system.

It is first understood that, for the hypothetical uniaxial situation involving *frictionless* mounting bearings and concentric drive (a drive motor rotor on gyro base and the field pieces fixed to the vehicle), the vehicle angular motion can apply no torque to the gyro base unless the field is energized. Consequently, for this hypothetical situation *only*, it is concluded that the inertia of the gyro element would angularly self-stabilize it in inertial space (about I^3). On the other hand, if there is gearing such that offset rotating elements *not* concentric with c^3 are angularly accelerated (by virtue of meshing with a gyro base gear centered on c^3 while the vehicle is rotated about the gyro), then *some torque* is applied to the gyro element (about c^3). Thus the gyro element is *not* quite self-stable, even though there is *no* friction. Unless there is irreversible motion transmission between drive motor and gyro base (such as the unlikely case where the motor drives the gyro through a low-lead worm drive), there is theoretically some inertial self-stabilization experienced by the gyro element under any change of its angular momentum about I^3 (and c^3). This is often ignored, and is quite apart from the implemented stabilization to be discussed. Philosophically, however, it might be said that uniaxial *inertial* angular stabilization *not* involving rigid initial application of vehicle rotation to the gyro is really a matter of correcting the instantaneous error in the inertial self-stabilization of c^1 .

Elemental features of uniaxial *inertial* angular stabilization are developed in terms of the rotations depicted by Fig. 7-3, which is a plan view of the I^1I^2 plane and duplicates part of Fig. 7-2. Ideal uniaxial *inertial* angular stabilization requires that the gyro output (precessional) axis *torquer be inoperative*. At some time t , the vehicle is in angular motion about I^3 (and c^3) and has been displaced by the input rotation θ_I . First,

if the gyro wheel is *not* rotating, then, in the presence of friction or offset gearing, self-stabilization is *not* complete and the c^1 axis is in angular motion but *lags* behind the vehicle reference line (V.R.L.) and exhibits

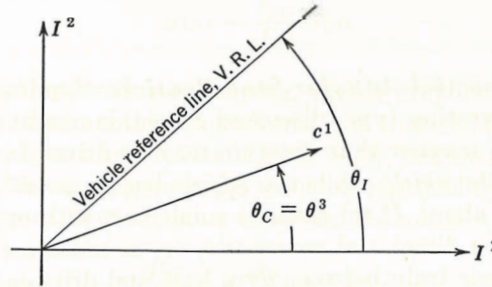


Figure 7-3

the inertial orientation $\theta_C = \theta^3$. In particular, if there is *only* friction of a *viscous* nature between gyro base and vehicle, which acts *only* about the c^3 axis, then:

$$f_R \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right) = J_G \frac{d^2\theta_C}{dt^2} \quad 7-19$$

where f_R is the rotational damping coefficient between gyro base and vehicle, and J_G is the c^3 axis polar moment of inertia of the gyro element (J_G is actually a variable if the gyro gimbal precesses, not at the moment possible as the wheel is not rotating). If, for example:

$$\theta_I(t) = \omega_k t, \quad \frac{d\theta_I}{dt} = \omega_k, \quad t > 0$$

then, for null initial conditions, it is easy to show that

$$\theta_C(t) = \omega_k t - \frac{J_G \omega_k}{f_R} + \frac{J_G \omega_k}{f_R} e^{-f_R t / J_G}$$

from which the steady-state lag is seen to be

$$(\theta_I - \theta_C)_{t \rightarrow \infty} = \frac{J_G \omega_k}{f_R}$$

Self-stabilization is thus effective here only to the extent that c^1 lags behind V.R.L. in acquiring the same angular velocity about I^3 (in the steady state).

Suppose now that the gyro wheel is rotating (the inertial angular stabilization system is now in operation) and has the constant angular momentum h_w about g^2 . At the time t , the vehicle has experienced the

positive inertial rotation θ_I (about I^3) and has, through friction, offset gearing, or combinations thereof, applied a torque to the gyro base such that the gyro reference line c^1 (here the precessional axis) has started to follow the vehicle. However, as soon as c^1 starts to rotate about I^3 (and c^3), this inertial rotation is detected through precession of the I^2 axis, as measured by the angle α . A pickoff signal proportional to α is amplified by a linear amplifier and employed to energize the stabilization motor, in such a manner that it applies opposition rotation to the gyro base relative to the rotating vehicle, so as to attempt stabilization of c^1 with respect to I^1 . Consequently, the instantaneous inertial orientation of the c^1 axis relative to I^1 , being $\theta_C = \theta^3$, is now the difference of two opposing rotations: a disturbance, denoted by θ_D , and a restoration, denoted by θ_S . The disturbance, θ_D , is equal to θ_I if gearing and friction are sufficient to transmit all vehicle rotation to the gyro base when the stabilization system is *not* operating. This is the case where self-stabilization has *no* direct effect. On the other hand, if there *is* effective self-stabilization, θ_D is then less than θ_I . In either event, the significant point is that, irrespective of the specific mechanism by which rotation of the vehicle applies torque to the gyro base, *the gyro detects only the net inertial angular velocity* (difference between disturbance and restoration)

$$\omega^3 = \frac{d\theta^3}{dt} = \frac{d\theta_C}{dt}$$

through the generation of the precessional torque τ_G^1 , which the system attempts to null by the counter-rotation of the gyro base relative to the vehicle. That is, there is no precession, α , to demand a torque between vehicle and gyro base, unless the gyro base has an inertial angular velocity ω^3 . This angular velocity detection dominates the stabilization process, which is reckoned in terms of the difference between the disturbance (the rotation of c^1 if there were no stabilization) and the restoration, θ_S . In other words, no matter how rotation of the vehicle (measured by rotation of V.R.L.) applies torque to the gyro base, the instantaneous orientation of the base reference line, c^1 , is

$$\theta_C = \theta_D - \theta_S \quad 7-20$$

or: c^1 is where it would have been without implemented stabilization *minus* the restoration (stabilization) rotation, θ_S , which is induced by the stabilization motor torque generated only when ω^3 has a nonzero value.

Torque generated by the stabilization motor is ideally assumed to be

$$\tau_M = k_A k_M k_P \alpha = k_\sigma \alpha \quad 7-21$$

where k_P is the precession pickoff constant of proportionality which may

have the dimensions

$$k_P = \frac{\text{volts}}{\text{rad}}$$

k_A is the linear amplifier gain

$$k_A = \frac{\text{volts}}{\text{volt}}$$

and k_M is a constant which characterizes an assumption that the motor delivers a torque proportional to the control voltage (amplifier output). Dimensions of k_M may be

$$k_M = \frac{(\text{dyne})(\text{cm})}{\text{volt}}$$

This motor torque, τ_M , generated about the motor rotor shaft, is reacted by the total dynamic and frictional torque associated with the generation of θ_S , *but as is reflected back to the motor shaft*. Characterization of the total stabilization torque and its reflection through a spur gear train between motor shaft and gyro base will be considered momentarily. First, it may be of interest to take note of another fundamental but usually implicit or unstated assumption in the uniaxial angular stabilization situation. This is: it is here (and almost always) assumed that the moment of inertia of the vehicle about I^3 is so large relative to that of the gyro system about I^3 (or c^3) that the inertial moment of momentum of the *vehicle* is completely unaffected by the stabilization torque generated between vehicle and gyro base. That is, it is assumed that the stabilization motor does *not* alter the nature of the *externally impressed vehicle rotation* $\theta_I = \theta_I(t)$. In view of usual relative magnitudes, this hardly need be stated.

Inertial angular stabilization is described by the closed-loop equation of inertial motion (rotation) of the c^1 axis. *Perfect* inertial angular stabilization would be specified by a response under which the system rigidly slaves c^1 to I^1 with no transient oscillations for any vehicle rotational input $\theta_I = \theta_I(t)$, about the I^3 axis. On the other hand, *ideal* inertial angular stabilization might be defined as that behavior under which c^1 settles to I^1 through a brief transient specified by the solution of a linear constant coefficient differential equation. Explicit derivation of the closed-loop uniaxial inertial angular stabilization equation is logically preceded by a brief review of *torque reflection* and a specification of the inertial rotational relations between the gear and pinion of the particular stabilization drive depicted by Fig. 7-2.

Reflection of inertia and viscous torques, $\tau_{J\theta_1}$, and $\tau_{f\theta_1}$, through a cascaded *spur* gear train of the nature depicted by Fig. 7-4, where, in the strict sense, *all axes of rotation are fixed in inertial space*, may easily

be shown to be given by the relations

$$\tau_{J\theta_1} = \left[J_1 + (k_1)^2 J_2 + (k_1 k_2)^2 J_3 + (k_1 k_2 k_3)^2 J_4 + \dots \right. \\ \left. + \left(\prod_{\substack{i=1 \\ n \geq 2}}^{n-1} k_i \right)^2 J_n \right] \frac{d^2 \theta_1}{dt^2} \quad 7-22$$

$$\tau_{f\theta_1} = \left[f_1 + (k_1)^2 f_2 + (k_1 k_2)^2 f_3 + (k_1 k_2 k_3)^2 f_4 + \dots \right. \\ \left. + \left(\prod_{\substack{i=1 \\ n \geq 2}}^{n-1} k_i \right)^2 f_n \right] \frac{d\theta_1}{dt} \quad 7-23$$

where $\tau_{J\theta_1}$ and $\tau_{f\theta_1}$ are respectively the total reflected inertia and viscous

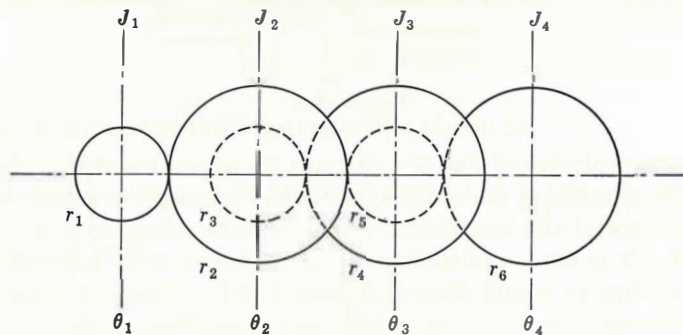


Figure 7-4

torques measured at the drive shaft of inertial rotation θ_1 , the k_i relate the contacting pitch radii through

$$k_1 = \frac{r_1}{r_2} \quad k_2 = \frac{r_3}{r_4} \quad k_3 = \frac{r_5}{r_6} \quad \dots$$

and the J_n and f_n are the individual axial polar moments of inertia and axial damping coefficients. In connection with these reflected torques, it is not difficult to appreciate that for a torque amplification train wherein

$$r_1 < r_2, \quad r_3 < r_4, \quad r_5 < r_6, \quad \dots$$

the first and second terms may contribute more reaction at the drive (motor) shaft than a much larger load (inertia and/or viscous) at the final-output shaft. From a practical point of view, Expressions 7-22 and 7-23 are almost always valid for the cascaded *spur* gear trains encountered in machine design, servo systems, etc., *without regard to the inertial state of motion of the gear axes*. However, in the study of reference line and platform stabilization, *where minute effects can become*

important, the detailed kinematic anatomy of *servo drives with orbiting axes of gear rotation* is of interest. A detailed examination of the single (external) pinion drive depicted by Fig. 7-2 provides a specific example of pinion-base gear kinematics under stabilization. In this connection, consider the rotations depicted by Fig. 7-5, which is another plan view of the $I^1 I^2$ plane.

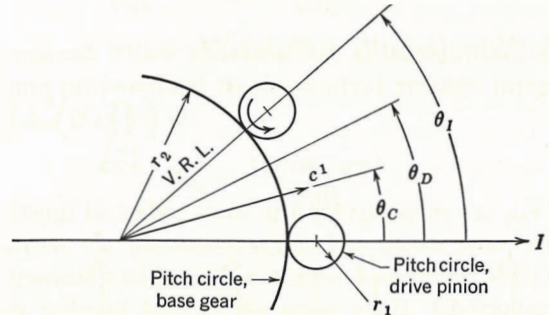


Figure 7-5

In contemplation of Fig. 7-5, it must be remembered that the drive pinion axis of rotation is carried by the vehicle through the vehicle rotation, θ_I , irrespective of the location of the gyro base reference line c^1 (to be stabilized). It is also recalled that $\theta_D = \theta_I$ if there is no self-stabilization effect, and that c^1 would then lie along V.R.L. when the implemented stabilization system is not in operation. In relating the inertial rotations of pinion and gyro base gear, it must be remembered that the inertial rotation of the pinion consists of two parts: an inertial rotation due to the mutual arc length "rolled off" by rolling around the base gear, and an inertial rotation, θ_I , due to orbiting about the base gear. For example: if pinion is locked to base gear, such that c^1 rigidly remains along V.R.L., the pinion still experiences the inertial rotation θ_I about its own orbiting axis of rotation. When not locked together, the arc length "rolled off" between gear and pinion is

$$S = r_1 \theta_{1R} = r_2 \theta_{2R}$$

where r_1 and r_2 are the radii of pinion and gear respectively, and θ_{1R} and θ_{2R} are the respective rotations due to rolling against each other. Suppose first that the base gear is fixed in inertial space, i.e., there is perfect inertial stabilization wherein c^1 stays rigidly along I^1 . The pinion center is still carried to θ_I by the rotation of the vehicle, and the total inertial rotation of the pinion is the "roll off" rotation plus the orbital rotation. Thus

$$\theta_1 = \theta_{1R} + \theta_I = \frac{r_2}{r_1} \theta_I + \theta_I = \left(1 + \frac{r_2}{r_1}\right) \theta_I \quad 7-24$$

For this simple drive, this is thus the total *net* inertial rotation (ignoring dynamic oscillations) of the drive pinion when *ideal* inertial stabilization of c^1 is achieved.

Continuing the discussion of the previous paragraph, suppose now that the reference line c^1 is carried to θ_D , while the pinion center is still, of course, carried to θ_I . Roll-off rotation of the pinion is clearly

$$\theta_{1R} = \frac{r_2}{r_1} (\theta_I - \theta_D)$$

while the total inertial rotation of the pinion is

$$\theta_1 = \theta_{1R} + \theta_I = \frac{r_2}{r_1} (\theta_I - \theta_D) + \theta_I = \left(1 + \frac{r_2}{r_1}\right) \theta_I - \frac{r_2}{r_1} \theta_D \quad 7-25$$

Finally, suppose that c^1 is now driven back to θ_C . Roll-off rotation is

$$\theta_{1R} = \frac{r_2}{r_1} (\theta_I - \theta_C)$$

while the total *net* inertial rotation of the pinion is

$$\theta_1 = \theta_{1R} + \theta_I = \frac{r_2}{r_1} (\theta_I - \theta_C) + \theta_I = \left(1 + \frac{r_2}{r_1}\right) \theta_I - \frac{r_2}{r_1} \theta_C \quad 7-26$$

If the ratio of radii

$$\frac{r_2}{r_1} = \frac{\theta_{1R}}{\theta_{2R}} = n \quad 7-27$$

is defined as the rotation ratio, n , of pinion to gear (verbally valid only when both axes are fixed in inertial space), Expression 7-26 may be written

$$\theta_1 = (1 + n)\theta_I - n\theta_C \quad 7-28$$

Inertial rotation of the drive pinion, Expression 7-28, is involved in the following description of the closed-loop inertial stabilization process implemented by the specific system which is depicted by Fig. 7-2.

The stabilization motor torque, $\tau_M = k_A k_M k_P \alpha = k_\sigma \alpha$, is dynamically reacted by the inertia and viscous torques induced in those elements which the motor drives. Inertia torque reacted by motor torque, in the example system under discussion, Fig. 7-2, is that associated with changing the inertial angular momentum of the gyro element, measured about I^3 (and c^3), plus that associated with changing the inertial angular momentum of the motor shaft (with pinion). Viscous torque reacted by the motor is proportional to the relative angular velocity between gyro base and vehicle, *reflected to the motor shaft*, plus any viscous and/or electrical damping measured about the motor shaft. An elementary development of these torques (similar to an elementary development for

fixed gear axes which gives Expression 7-22 and 7-23) is an essential step in the following derivation of the stabilization equation.

Reference to Figs. 7-2, 7-3, and 7-5 indicates that the instantaneous inertial angular momentum of the gyro element (gyro and gyro base with base gear) about I^3 is

$$\left. \begin{aligned} h_G &= J_G \left(\frac{d\theta_I}{dt} - \frac{d\theta_S}{dt} \right) = J_G \frac{d\theta_C}{dt} \\ \text{or} \quad h_G &= J_G \left(\frac{d\theta_D}{dt} - \frac{d\theta_S}{dt} \right) = J_G \frac{d\theta_C}{dt} \end{aligned} \right\} \quad 7-29$$

being proportional to the *net* inertial angular velocity, $\frac{d\theta_C}{dt}$ (about I^3), irrespective of whether $\theta_C = \theta_I - \theta_S$ or $\theta_C = \theta_D - \theta_S$, that is, irrespective of the presence of some self-stabilization effect. However, it must be remembered that, in either case, the pinion axis is still carried to θ_I and that the inertial rotation of the pinion remains

$$\theta_I = (1 + n)\theta_I - n\theta_C \quad 7-28$$

Stabilization is characterized by a differential equation which relates the inertial rotation, $\theta_C(t)$, of the stabilized line, c^1 , and the inertial rotation of the vehicle, $\theta_I(t)$, said equation being thus written in terms of rotations about I^3 (or c^3). It is again recalled that, in *inertial* angular stabilization, $\theta_C(t)$ should settle to zero, irrespective of the forcing rotation $\theta_I(t)$, as revealed by solutions of the system differential equation. To maintain a clear identification of the directions of the base motion torques involved in the stabilization mechanism, all base motion torques are referred to the rotation convention (right-handed) defined about I^3 . With this convention, the torque applied by the motor, *in a sense to achieve stabilization* (accomplished by the correct polarity of electrical input to the motor), is reacted about I^3 by the inertia of the gyro element such that

$$\tau_{IG} = - \frac{d}{dt} (h_G) = - J_G \frac{d^2\theta_C}{dt^2}$$

which is reflected to the motor shaft as

$$\tau_{MG} = - \frac{r_1}{r_2} J_G \frac{d^2\theta_C}{dt^2} = - \frac{J_G}{n} \frac{d^2\theta_C}{dt^2} \quad 7-30$$

being a *positive torque* about the pinion axis but *retaining the negative sign* by virtue of being expressed in terms of θ_C . It should also be noted that expression in terms of θ_C likewise eliminates the appearance of the square of the gear ratio n at this point. Relation 7-30 is easily established by considering the pitch-line action and reaction (forces) between base gear and pinion. The torque τ_{MG} is just a part of the inertia torque reacted by

the motor torque. There is also the torque due to the change of inertial angular momentum of the motor rotor (with pinion) measured about its own (orbiting) axis. By employment of Equation 7-28, the inertial angular acceleration of the pinion *about its own axis* is

$$\frac{d^2\theta_1}{dt^2} = (1 + n) \frac{d^2\theta_I}{dt^2} - n \frac{d^2\theta_C}{dt^2}$$

which corresponds to a positive torque about the pinion axis which is

$$\tau_{MP} = J_M \frac{d^2\theta_1}{dt^2} = J_M(1 + n) \frac{d^2\theta_I}{dt^2} - J_M n \frac{d^2\theta_C}{dt^2} \quad 7-31$$

where J_M is the polar moment of inertia of motor rotor and pinion. The total inertia torque reacted by the motor is thus here

$$\tau_{MI} = \tau_{MG} + \tau_{MP} = - \left(\frac{J_G + (n)^2 J_M}{n} \right) \frac{d^2\theta_C}{dt^2} + J_M(1 + n) \frac{d^2\theta_I}{dt^2} \quad 7-32$$

being expressed in terms of rotational derivatives involving rotations occurring about I^3 . Viscous torque about I^3 (and c^3) is proportional to the *relative* angular velocity between vehicle and gyro base. This is characterized by

$$f_G \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right)$$

where f_G is the damping coefficient effective between vehicle and gyro base. This torque tends to increase θ_C and is consequently opposed by a *positive* torque about the pinion axis which is

$$\tau_{MfG} = \frac{f_G}{n} \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right) \quad 7-33$$

being retained in terms of angular velocities measured about I^3 , again eliminating the appearance of a second power of the ratio n . Viscous torque originating about the motor rotor is proportional to the angular velocity of the rotor *with respect to its stator*. This is

$$\tau_{MfM} = f_M \frac{r_2}{r_1} \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right) = f_M n \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right) \quad 7-34$$

where f_M is the damping coefficient effective between motor rotor and stator. The quantity

$$n \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right)$$

is the relative angular velocity between rotor and stator and reflects the fact that the motor orbits about the base gear. Total torque reacted by the motor is thus here

$$\tau_M = \tau_{MI} + \tau_{MfG} + \tau_{MfM} = k_A k_M k_P \alpha = k_o \alpha$$

which is explicitly

$$\begin{aligned} \left(\frac{J_G + (n)^2 J_M}{n} \right) \frac{d^2 \theta_C}{dt^2} + \left(\frac{f_G + (n)^2 f_M}{n} \right) \frac{d \theta_C}{dt} + k_\sigma \alpha \\ = J_M (1 + n) \frac{d^2 \theta_I}{dt^2} + \left(\frac{f_G + (n)^2 f_M}{n} \right) \frac{d \theta_I}{dt} \end{aligned} \quad 7-35$$

The precession, α , of the gyro spin axis is now recalled to be specified by Expression 7-11 (being the case of *inertial* angular stabilization, wherein the gyro output axis torquer is not energized).

$$J_\alpha \frac{d^2 \alpha}{dt^2} + f_\alpha \frac{d \alpha}{dt} = h_w \frac{d \theta_C}{dt} \quad 7-11$$

Elimination of α between Equation 7-11 and 7-35 (by differentiation and substitution or operationally) then gives the closed-loop system stabilization equation. This is found to be

$$\begin{aligned} \frac{J_\alpha}{n} (J_G + (n)^2 J_M) \frac{d^3 \theta_C}{dt^3} + \left[\frac{J_\alpha}{n} (f_G + (n)^2 f_M) + \frac{f_\alpha}{n} (J_G + (n)^2 J_M) \right] \frac{d^2 \theta_C}{dt^2} \\ + \frac{f_\alpha}{n} (f_G + (n)^2 f_M) \frac{d \theta_C}{dt} + k_\sigma h_w \theta_C = J_\alpha J_M (1 + n) \frac{d^3 \theta_I}{dt^3} \\ + \left[\frac{J_\alpha}{n} (f_G + (n)^2 f_M) + f_\alpha J_M (1 + n) \right] \frac{d^2 \theta_I}{dt^2} \\ + \frac{f_\alpha}{n} (f_G + (n)^2 f_M) \frac{d \theta_I}{dt} \end{aligned} \quad 7-36$$

Expression 7-36 is the general performance equation for the uniaxial *inertial* angular stabilization system defined by the stipulations and assumptions employed in the derivation. Its validity is restricted to the single-pinion configuration of Fig. 7-2, and is further suppressed by the exclusion of nonviscous friction and other component nonlinearities. In contemplation of this system and its equation, Equation 7-36, it should be realized that the derivation and description would have been further complicated by the employment of a more extensive gear train. Inclusion of additional reduction stages between motor shaft and gyro base gear further complicates a completely explicit description (by the inclusion of the reflected dynamic contributions of all intermediate gears) of angular stabilization systems. The current explicit description of the single-pinion drive of Fig. 7-2 should enhance an appreciation of such complications. Discussion presented presently will concern a somewhat less explicit description of the uniaxial inertial angular stabilization mechanism. Other subsequent discussion will relate to effects attending certain static and dynamically induced anomalies. Of more immediate concern is the major performance of the current uniaxial inertial stabilization system, as revealed by its explicit linear characterization, Expression 7-36.

Suppose first that there is no viscous damping anywhere in the system. That is, assume that

$$f_\alpha = f_G = f_M = 0 \quad 7-37$$

Under this stipulation, Equation 7-36 reduces to just

$$\frac{J_\alpha}{n} (J_G + (n)^2 J_M) \frac{d^3 \theta_C}{dt^3} + k_s h_w \theta_C = J_\alpha J_M (1 + n) \frac{d^3 \theta_I}{dt^3} \quad 7-38$$

which is of the form

$$A_1 \frac{d^3 \theta_C}{dt^3} + A_2 \theta_C = A_3 \frac{d^3 \theta_I}{dt^3}$$

and which, for null initial conditions, gives the Laplace transform*

$$\Theta_C(s) = a_0 \frac{s^3}{s^3 + k} \Theta_I(s) \quad 7-39$$

$$a_0 = \frac{A_3}{A_1} \quad k = \frac{A_2}{A_1}$$

where the response transform $\Theta_C(s)$ would be inverted for specified disturbance transforms $\Theta_I(s) = \mathcal{L}[\theta_I(t)]$ to describe the stabilization behavior, $\theta_C(t)$, of the gyro reference line, c^1 . However, without regard to the specification of any forcing rotation of the vehicle, $\theta_1(t)$, or its transform, $\Theta_I(s)$, it may be shown that

$$(s^3 + k) = [s + (k)^{1/3}][s^2 - (k)^{1/3}s + (k)^{2/3}]$$

where the quadratic factor has a *negative* coefficient. Consequently, the system is systematically *unstable* in the absence of all damping.

Consider now the system as is characterized by the nonzero values of all its parameters. Equation 7-36 may be written

$$A_1 \frac{d^3 \theta_C}{dt^3} + A_2 \frac{d^2 \theta_C}{dt^2} + A_3 \frac{d \theta_C}{dt} + A_4 \theta_C = A_5 \frac{d^3 \theta_I}{dt^3} + A_6 \frac{d^2 \theta_I}{dt^2} + A_7 \frac{d \theta_I}{dt} \quad 7-40$$

where $A_1, A_2, A_3, \dots$ respectively represent the parametric coefficients of Equation 7-36. Suppose that the vehicle rotation is specified by the ramp function

$$\theta_I(t) = \omega_k t \quad t > 0$$

from which

$$\frac{d \theta_I}{dt} = \omega_k \quad 7-41$$

That is, the vehicle experiences a rotation of constant angular velocity,

* Direct superscripts on the Laplace variable, s , are exponents.

ω_k , about I^3 , which is applied at $t = 0+$. Let the initial values of all variables be zero (which *eliminates* the inclusion of the second and third derivatives of $\theta_I(t)$ as being the unit impulse multiplied by ω_k and the doublet impulse multiplied by ω_k , respectively). As the transform of ω_k is

$$\mathcal{L}[\omega_k] = \frac{\omega_k}{s}$$

transformation of Equation 7-40 then gives

$$\Theta_C(s) = \frac{A_7 \omega_k}{s(A_1 s^3 + A_2 s^2 + A_3 s + A_4)} \quad 7-42$$

Application of the Laplace transform final value theorem

$$\lim_{t \rightarrow \infty} \theta_C(t) = \lim_{s \rightarrow 0} s \Theta_C(s)$$

then reveals that

$$\theta_C(\infty) = \lim_{t \rightarrow \infty} \theta_C(t) = \frac{A_7 \omega_k}{A_4} = \frac{f_\alpha(f_G + (n)^2 f_M) \omega_k}{n k_A k_M k_P h_w}, (\text{rad}) \quad 7-43$$

which indicates that, while not remaining rigidly along I^1 , c^1 does settle to a constant angular displacement, $\theta_C(\infty)$, from I^1 , and thus experiences uniaxial inertial stabilization, though not, by definition, ideal. However, system servo stability demands that

$$A_2 A_3 > A_1 A_4$$

which is explicitly

$$[J_\alpha(f_G + (n)^2 f_M) + f_\alpha(J_G + (n)^2 J_M)] > \frac{h_w J_\alpha k_\alpha n (J_G + (n)^2 J_M)}{f_\alpha(f_G + (n)^2 f_M)} \quad 7-44$$

being a condition which may not be practically achievable for realistic values of the parameters involved. Acceptable performance in angular stabilization often necessitates the inclusion of a signal-processing *compensating network*, and consequently a rederivation of the system equation. A particular example of system compensation is incorporated in the following further description of the uniaxial inertial angular stabilization mechanism.

Consider now a system of the same general configuration as that of Fig. 7-2, but one wherein the drive gear reduction, of overall ratio n , may involve any number of stages. Assume that the only damping in the drive system (not including the gyro) is that which occurs about the motor shaft, and that there is no other friction between vehicle and gyro base. The total torque reacted by the drive motor is now

$$\tau_M = J_\Sigma \frac{d^2 \theta_1}{dt^2} + f_M \frac{d\theta_R}{dt} \quad 7-45$$

where J_Σ is the total inertia of the driven members, as reflected to the motor shaft of inertial rotation θ_I , f_M is the damping coefficient between motor rotor and stator, and θ_R is the rotation of motor rotor relative to the stator. The *inertial* rotation of the motor rotor (not now the pinion) is still

$$\theta_I = \theta_I + n(\theta_I - \theta_C) = (1 + n)\theta_I - n\theta_C$$

while the rotor rotation relative to the stator is

$$\theta_R = n\theta_I - n\theta_C$$

Equation 7-45 may thus be written

$$\tau_M = J_\Sigma(1 + n) \frac{d^2\theta_I}{dt^2} + f_M n \frac{d\theta_I}{dt} - J_\Sigma n \frac{d^2\theta_C}{dt^2} - f_M n \frac{d\theta_C}{dt} \quad 7-46$$

With respect to the previous single-pinion system, Equation 7-36, it may be noted that the reflected inertia as specified by Expression 7-22 is for *that* system

$$J_\Sigma = \left(J_M + \frac{J_G}{(n)^2} \right)$$

so that

$$J_\Sigma n = \frac{J_G + (n)^2 J_M}{n}$$

as was independently derived for the coefficient of the first term of Expression 7-35. Suppose now that the pickoff voltage $e_P = k_P \alpha$, which is the signal proportional to the precession α , specified by the equation

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} = h_w \frac{d\theta_C}{dt} \quad 7-11$$

is *not* fed directly to the linear amplifier, but is rather the input to a compensating filter of the d-c configuration depicted by Fig. 7-6. The con-

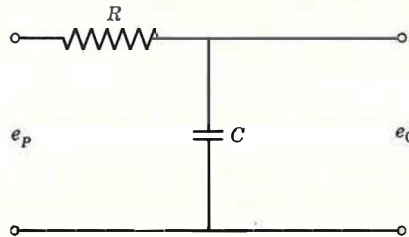


Figure 7-6

stant-coefficient description of this circuit is easily found to be

$$e_P = RC \frac{de_0}{dt} + e_0 \quad 7-47$$

The voltage $e_0(t)$ is now the input to the amplifier of output $e_A = k_A e_0(t)$. Stabilization motor torque is thus characterized by $\tau_M = k_A k_M e_0(t)$, which with Equations 7-11 and 7-47, is found to be related to the instantaneous inertial displacement of c^1 , measured by $\theta_C(t)$, through the equation

$$J_\alpha RC \frac{d^2 \tau_M}{dt^2} + (J_\alpha + f_\alpha RC) \frac{d\tau_M}{dt} + f_\alpha \tau_M = k_\sigma h_w \theta_C(t) \quad 7-48$$

Elimination of τ_M between Equations 7-46 and 7-48 then gives the system equation

$$\begin{aligned} J_\alpha J_\Sigma n RC \frac{d^4 \theta_C}{dt^4} + [n RC (J_\alpha f_M + J_\Sigma f_\alpha) + J_\alpha J_\Sigma n] \frac{d^3 \theta_C}{dt^3} \\ + [n (J_\alpha f_M + J_\Sigma f_\alpha) + f_\alpha f_M n RC] \frac{d^2 \theta_C}{dt^2} \\ + f_\alpha f_M n \frac{d\theta_C}{dt} + k_\sigma h_w \theta_C = J_\alpha J_\Sigma (1 + n) RC \frac{d^4 \theta_I}{dt^4} \\ + [J_\alpha f_M n RC + J_\alpha J_\Sigma (1 + n) + J_\Sigma f_\alpha (1 + n) RC] \frac{d^3 \theta_I}{dt^3} \\ + (J_\alpha f_M n + f_\alpha f_M n RC + J_\Sigma f_\alpha (1 + n)) \frac{d^2 \theta_I}{dt^2} \\ + f_\alpha f_M n \frac{d\theta_I}{dt} \end{aligned} \quad 7-49$$

which is of the form

$$\begin{aligned} A_1 \frac{d^4 \theta_C}{dt^4} + A_2 \frac{d^3 \theta_C}{dt^3} + A_3 \frac{d^2 \theta_C}{dt^2} + A_4 \frac{d\theta_C}{dt} + A_5 \theta_C = A_6 \frac{d^4 \theta_I}{dt^4} \\ + A_7 \frac{d^3 \theta_I}{dt^3} + A_8 \frac{d^2 \theta_I}{dt^2} + A_4 \frac{d\theta_I}{dt} \end{aligned} \quad 7-50$$

Of first consideration, in the discussion of the system characterized by Equation 7-50, is the question of systematic stability. Systematic stability here demands that*

$$A_2 A_3 A_4 > (A_2)^2 A_5 + A_1 (A_4)^2 \quad 7-51$$

which might now be more conveniently achieved by virtue of the additional latitude of coefficient adjustment which attends variation of the electrical parameters, R and C .

It is now observed that, for practical values of the ratio n (say $n > 100$), the orbital rotation, θ_I , of the motor rotor (not now the pinion) is of small contribution to its *total* inertial rotation. If this orbital rotation is *ignored*, then all of the coefficient factors which are $(1 + n)$ in

* See Appendix III.

Equation 7-49 are replaced by just n , and Equation 7-50 becomes

$$\begin{aligned} A_1 \frac{d^4 \theta_C}{dt^4} + A_2 \frac{d^3 \theta_C}{dt^3} + A_3 \frac{d^2 \theta_C}{dt^2} + A_4 \frac{d \theta_C}{dt} + A_5 \theta_C \\ = A_1 \frac{d^4 \theta_I}{dt^4} + A_2 \frac{d^3 \theta_I}{dt^3} + A_3 \frac{d^2 \theta_I}{dt^2} + A_4 \frac{d \theta_I}{dt} \end{aligned} \quad 7-52$$

A new dependent variable, θ_α , is introduced such that

$$\theta_\alpha(t) = \theta_I(t) - \theta_C(t) \quad 7-53$$

which, for the case where $\theta_D = \theta_I$ (no self-stabilization effect), is observed to be synonymous with the θ_S (restoration) of prior discussion. Equation 7-52 thus becomes

$$A_1 \frac{d^4 \theta_\alpha}{dt^4} + A_2 \frac{d^3 \theta_\alpha}{dt^3} + A_3 \frac{d^2 \theta_\alpha}{dt^2} + A_4 \frac{d \theta_\alpha}{dt} + A_5 \theta_\alpha = A_5 \theta_I \quad 7-54$$

Clearly, if $\theta_\alpha(t)$ tends to $\theta_I(t)$ (in magnitude) then $\theta_C(t)$ tends to zero and the c^1 axis experiences ideal uniaxial inertial angular stabilization. For null initial conditions, the Laplace transform of Equation 7-54 is

$$\Theta_\alpha(s) = \frac{A_5}{A_1 s^4 + A_2 s^3 + A_3 s^2 + A_4 s + A_5} \Theta_I(s) \quad 7-55$$

If Condition 7-51 is realized, then the final value theorem indicates that, for a step-function disturbance (vehicle rotation), $\theta_I(t) = k1(t)$, ($t > 0$), the variable $\theta_\alpha(t)$ approaches $\theta_I(t)$ with increasing time. Consequently, for the step-function disturbance, $\theta_C \rightarrow 0$ and ideal inertial angular stabilization is achieved. Speed of inertial stabilization response (measured by the time to reduce $\theta_C(t)$ to a specified value) depends on the particular (stable) distribution of coefficient values. Coefficient selection is a linear servo design problem in the elementary sense. If the vehicle rotation is the ramp function $\theta_I(t) = \omega_k t$, ($t > 0$), of constant angular velocity ω_k , then

$$\mathcal{L}[\theta_I(t)] = \Theta_I(s) = \frac{\omega_k}{s^2}$$

and an inversion of Equation 7-55 for this forcing transform reveals that

$$\theta_\alpha(t) = \omega_k t - \frac{A_4}{A_5} \omega_k + T_\omega(t)$$

where $T_\omega(t)$ represents transient terms which depend on the roots of the denominator of Equation 7-55 and which decay with time if the system is stable. It then follows that

$$\theta_C(t) = \theta_I(t) - \theta_\alpha(t) = \omega_k t - \left(\omega_k t - \frac{A_4}{A_5} \omega_k + T_\omega(t) \right) = \frac{A_4}{A_5} \omega_k - T_\omega(t)$$

As $T_\omega(t) \rightarrow 0$ with increasing time, the gyro reference line, c^1 , settles to the steady-state value

$$\theta_C(\infty) = \frac{A_4}{A_5} \omega_k, \quad (\text{rad})$$

and is inertially stable to within a constant angular displacement. On the other hand, if the vehicle rotation is given by the constant *angular acceleration* function $\theta_I(t) = \frac{1}{2}a_\theta(t)^2$, ($t > 0$), then

$$\mathcal{L}[\theta_I(t)] = \Theta_I(s) = \frac{a_\theta}{s^3}$$

and inversion of Equation 7-55 leads to

$$\theta_C(t) = a_\theta \frac{A_4}{A_5} t - a_\theta \left[\left(\frac{A_4}{A_5} \right)^2 - \frac{A_3}{A_5} \right] - T_a(t)$$

where $T_a(t)$ represents the transient terms. It is thus seen that the reference line c^1 drifts away from inertial stabilization in a manner which is essentially a linear function of time, when the vehicle disturbance is a continuously applied constant angular acceleration. It has thus been revealed that, under this reduced description, where the relatively small orbital rotation of the motor rotor is ignored, ideal inertial angular stabilization is *not* completely achieved for the vehicle rotation functions

$$\theta_I(t) = \omega_k t, \quad \theta_I(t) = \frac{1}{2}a_\theta(t)^2$$

However, without additional qualification and interpretation, the unbounded application of these vehicle disturbances (particularly the second) imposes performance requirements on the system which are, in general, unrealistically severe. Failure to achieve complete ideal stabilization under the unbounded application of these forcing functions does not impair the utility of the uniaxial stabilization mechanism. Rather, as in the generally related servomechanism technology, the *ideal* response to the step disturbance $\theta_I(t) = k1(t)$ is significant. In this connection, it may be noted that an explicit inversion of Equation 7-55 for $\theta_I(t) = k1(t)$ and an underdamped stable distribution of system coefficients may be achieved by writing

$$\begin{aligned} \Theta_\alpha(s) &= \frac{kA_5}{s(A_1s^4 + A_2s^3 + A_3s^2 + A_4s + A_5)} \\ &= \frac{k \frac{\alpha_3\delta_3}{\alpha_1\delta_1}}{s \left(s^2 + \frac{\alpha_2}{\alpha_1}s + \frac{\alpha_3}{\alpha_1} \right) \left(s^2 + \frac{\delta_2}{\delta_1}s + \frac{\delta_3}{\delta_1} \right)} \\ \Theta_\alpha(s) &= \frac{k(\omega_1)^2(\omega_2)^2}{s(s^2 + 2\zeta_1\omega_1s + (\omega_1)^2)(s^2 + 2\zeta_2\omega_2s + (\omega_2)^2)} \quad 7-56 \end{aligned}$$

where

$$A_1 = \alpha_1 \delta_1 \quad A_2 = \alpha_1 \delta_2 + \alpha_2 \delta_1$$

$$A_3 = \alpha_1 \delta_3 + \alpha_2 \delta_2 + \alpha_3 \delta_1 \quad A_4 = \alpha_2 \delta_3 + \alpha_3 \delta_2 \quad A_5 = \alpha_3 \delta_3$$

and

$$\zeta_1 = \frac{\alpha_2}{2 \sqrt{\alpha_1 \alpha_3}} \quad \zeta_2 = \frac{\delta_2}{2 \sqrt{\delta_1 \delta_3}} \quad \omega_1 = \sqrt{\frac{\alpha_3}{\alpha_1}} \quad \omega_2 = \sqrt{\frac{\delta_3}{\delta_1}}$$

whereupon $\bullet_\alpha(s)$ may be inverted by the straightforward but often tedious procedure of partial fractions and transform identifications. In relation to this, it is recalled that for *real* coefficients the complex roots of

$$F(s) = A_1 s^4 + A_2 s^3 + A_3 s^2 + A_4 s + A_5 = 0$$

must occur in *conjugate pairs*. Consequently, $F(s) = 0$ must have either four real roots, two real and two complex roots, or four complex roots. Desirable performance usually dictates that the stable distribution of coefficients provide $F(s) = 0$ with two pairs of conjugate complex roots (under stability, these roots will have negative real parts). If so, these conjugate pairs of roots are then the roots of the quadratic factors which appear in the denominator of Equation 7-56. It is also recalled that *multiple* real roots correspond to time functions involving positive integral powers of t in product with negative exponentials, and consequently to *critical damping*. For example:

$$\mathcal{L}[te^{-\beta t}] = \frac{1}{(s + \beta)^2}$$

Equation 7-56 is expressed in the standard control system form, involving a product of two (elementary) underdamped quadratics, and thus the two *damping ratios*, ζ_1 and ζ_2 , and the two *natural frequencies*, ω_1 and ω_2 .

Uniaxial inertial angular stabilization, as described immediately above, where the orbital rotation of the motor rotor of a particular system was ignored, was seen to resolve into a control-system or servomechanism study of elementary configuration. Further investigation of the same system with the inclusion of the complete inertial rotation of the rotor, as characterized by Equation 7-49, would reveal a synonymous situation. In general, the design of a uniaxial inertial angular stabilization system which must conform to specified performance requirements falls directly into the domain of control systems engineering. Angular motion detection, as supplied by a s.d.f. gyro, is just another case of error detection which may be employed in connection with the well-developed linear techniques of the servomechanisms art to implement a null seeking system and thus stabilization. Specific sophistications relating to design,

compensation, and general analysis and synthesis are adequately treated in the abundant literature.*†

As was indicated by preliminary remarks of this section, the uniaxial inertial angular stabilization of a reference line (stabilized in a plane normal to the axis around which the vehicle is disturbed) describes a mechanism which may be extended to the inertial angular stabilization of an orthogonal triad of lines, and thus to the angular spatial stabilization of a reference platform. A simultaneous extension of the uniaxial mechanism to *three* orthogonal lines is physically achieved by the employment of a platform gimbal system, one configuration of which is indicated by Fig. 7-1. Rudimentary features of platform stabilization were reviewed in Section 7.2. With the inclusion of *axial cross effects*, the uniaxial inertial angular stabilization mechanism is the essential feature of the inertial angular spatial stabilization of a platform which employs an orthogonal triad of angular motion sensors. Moreover, the angular stabilization of a reference line with respect to a *rotating reference line*, considered in the following subsection, is an immediate extension of uniaxial inertial angular stabilization (rotating stabilization is achieved by employment of the gyro output axis torquer), which is in turn extended to the *rotating spatial stabilization* of a platform triad. Thus, uniaxial inertial angular stabilization is preliminary to uniaxial rotating stabilization, and both are respectively extended to the mechanisms of inertial and rotating *platform stabilization*.

Uniaxial Rotating Stabilization (Variable Reference Orientation). Kinematically opposed to the uniaxial angular stabilization of a reference line with respect to an inertial direction, considered in the previous subsection, is the uniaxial angular stabilization of a reference line with respect to a coplanar *rotating* reference line. As depicted by Fig. 7-2, the reference line c^1 is now to be stabilized with respect to the rotating line L , which is coplanar with or parallel to the plane of c^1 , and which is perpendicular to and rotates about I^3 (or c^3). Perfect uniaxial rotating angular stabilization would be realized when c^1 could be rigidly slaved to L (or its projection) with $\theta_c \equiv \theta_L$ irrespective of the angular motion of the vehicle reference line, V.R.L., located by θ_r . As extended to an orthogonal triad of platform reference axes, rotating angular stabilization, or variable reference orientation, is a common requirement in terrestrial vehicular inertial systems. As described presently, implementation of uniaxial rotating angular stabilization involves the application of an externally generated signal to the gyro precessional axis *torquer*,

*See Bibliography.

† See Appendix V for an exemplary application of frequency techniques in analysis and compensation.

under the condition that the system would otherwise achieve inertial angular stabilization. Subsequent developments pertaining to platform stabilization under curvilinear terrestrial motion will indicate that torquing signals may be generated by suitable integration and contraction of curvilinear acceleration signals.

As mentioned above, the requirement that c^1 be stabilized with respect to the rotating reference line, L , inertially located by $\theta_L(t)$, may be satisfied by the application of a suitable torquer signal, $\tau_R(t)$, to the gyro precessional axis torquer. It is recalled that, if there is no net inertial angular velocity, ω^3 , impressed about the gyro input axis, $c^3(I^3)$, and there is no torquer torque, $k\tau_R(t)$, then the precession, α , remains zero and the stabilization drive motor receives no command signal. Consequently, $\theta_C(t)$ retains zero value (ideally). Application of a torquer torque, $k\tau_R(t)$, then tends to generate a *negative* α , and thus causes the motor to drive the gyro base with respect to the vehicle so as to generate a *positive* θ_C . That is, the drive motor must respond to the signal generated by α , whether α be induced by gyroscopic precession or by torquer torque, or by a combination of both. See again Fig. 7-2.

Continuing the qualitative discussion initiated in the last paragraph, if residual vehicle rotation and torquer signal are applied simultaneously, then the following simultaneous actions tend to occur: positive residual rotation transmitted to (about) the gyro input axis generates a positive α which commands the drive motor to drive c^1 back toward inertial reference. While this tends to occur, the torquer torque opposes the gyroscopic torque (see Equation 7-12) and attempts to drive α in the negative direction. If α goes negative, it then supplies the motor a command to drive c^1 positively with respect to the vehicle. If α remains negative, c^1 continues to rotate positively with respect to the vehicle reference line, V.R.L., providing one revelation of system capability to follow a rotating reference. However, the gyro is meanwhile experiencing a positive input angular velocity, and thus generates a gyroscopic torque which opposes the torquer torque. Therefore, if the drive motor torque is directly proportional to α (no filter compensation), dynamic descent to a constant α results in an eventual constant torque between vehicle and gyro base which is reacted in accord with the specific frictional properties of the drive system. The simplest example, here, is that where there is only viscous friction between vehicle and gyro base. If this is the case, then the constant torque will maintain a constant relative angular velocity between gyro base and vehicle, while each has a constant but different inertial angular velocity about I^3 . When the dynamic situation is such that torquer and gyroscopic torques are balanced at $\alpha = 0$, there is no drive motor torque and, disregarding self-stabilization effects, c^1 remains fixed relative to the vehicle. On the other hand, under some other

dynamic situation (depending on system and state of motion), self-stabilization effect in the gyro element may establish some small value of α to achieve stabilization of c^1 relative to the vehicle, or relative to the rotating line, L . In general, if α settles to a positive value, the inertial rotation of the vehicle tends to lead that of c^1 . Conversely, a negative steady-state value of α indicates that c^1 is moving ahead of V.R.L. It should perhaps be mentioned that, in actual practice, α is usually confined to small values ($\pm$ a degree or so) by employment of high control loop gain.

Consider again the uniaxial angular stabilization system of the general configuration depicted by Fig. 7-2, which was previously discussed with respect to the *inertial* stabilization of the gyro reference line, c^1 . Assume again that the vehicle experiences the inertial disturbance rotation $\theta_I = \theta_I(t)$, but that now, by virtue of a torquer torque, c^1 is to be slaved to the *rotating reference line*, L , instantaneously located by $\theta_L = \theta_L(t)$. This stabilization of c^1 with respect to L ($\theta_C(t) \rightarrow \theta_L(t)$) should, of course, be independent of the vehicle disturbance, $\theta_I(t)$. The problem is to determine the torquer torque

$$\tau_T = k_T \tau_R(t) \quad 7-57$$

such that the desired rotating stabilization ($\theta_C(t) \rightarrow \theta_L(t)$) is achieved. As α is restricted to small magnitudes, the equation of motion about the gyro precessional axis is assumed to be characterized by Expression 7-12, previously established.

$$J_\alpha \frac{d^2\alpha}{dt^2} + f_\alpha \frac{d\alpha}{dt} + k_T \tau_R(t) = h_w \frac{d\theta_C}{dt} \quad 7-12$$

Torque generated by the stabilization motor is presently assumed to be directly proportional to the precession, α , as specified by Expression 7-21.

$$\tau_M = k_A k_M k_P \alpha = k_\alpha \alpha \quad 7-21$$

That is, there is presently no compensating filter in the system. The motor torque, τ_M , is assumed to be dynamically reacted such that

$$\tau_M = J_\Sigma \frac{d^2\theta_1}{dt^2} + f_\Sigma \frac{d\theta_R}{dt} + \tau_{\Sigma s} \quad 7-58$$

where J_Σ is the total inertia of the driven members reflected to the motor rotor shaft, θ_1 is the inertial rotation of the motor rotor shaft

$$\theta_1 = \theta_I + n(\theta_I - \theta_C) = (1 + n)\theta_I - n\theta_C,$$

n is the overall gear ratio, f_Σ is the total reflected damping coefficient effective at the rotor shaft, $\frac{d\theta_R}{dt}$ is the relative angular velocity of the rotor

shaft

$$\frac{d\theta_R}{dt} = n \left(\frac{d\theta_I}{dt} - \frac{d\theta_C}{dt} \right),$$

and τ_{zs} is the total Coulomb frictional torque (friction which is independent of relative velocity) reflected to the motor rotor shaft. In contemplation of the last two terms of Equation 7-58, it is understood that frictional torques are *relative* and *not inertial* effects, and thus may be reflected through the gear train (Equations 7-22 and 7-23) as though all gear axes were fixed in inertial space. Equations 7-21 and 7-58 give

$$J_z(1+n) \frac{d^2\theta_I}{dt^2} + f_{zn} \frac{d\theta_I}{dt} + \tau_{zs} = J_z n \frac{d^2\theta_C}{dt^2} + f_{zn} \frac{d\theta_C}{dt} + k_\sigma \alpha \quad 7-59$$

whereupon, for null initial conditions, an operational elimination of α between Equations 7-12 and 7-59 gives the Laplace domain expression

$$\begin{aligned} \Theta_C(s) = & \frac{s \{ J_\alpha J_z (1+n) s^2 + [J_\alpha f_{zn} + J_z f_\alpha (1+n)] s + f_\alpha f_{zn} \}}{J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_{zn} s + k_\sigma h_w} \Theta_I(s) \\ & + \frac{J_\alpha \tau_{zs}}{J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_{zn} s + k_\sigma h_w} \\ & + \frac{f_\alpha \tau_{zs}}{s(J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_{zn} s + k_\sigma h_w)} \\ & + \frac{k_\sigma k_\tau}{s(J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_{zn} s + k_\sigma h_w)} T_R(s) \quad 7-60 \end{aligned}$$

where the Coulomb torque is treated as a constant, and $T_R(s)$ is the Laplace transform of the currently unknown torquer signal, $\tau_R(t)$. Expression 7-60 is a general uniaxial angular stabilization equation, in that it characterizes both inertial and rotating stabilization, and also some effect of nonviscous friction. The first term characterizes the inertial stabilization behavior of c^1 and, except for a generalization of reflected effects, is almost synonymous with previous descriptions of uniaxial inertial angular stabilization. Effect in the behavior of c^1 which attends the presence of the constant reflected frictional torque, τ_{zs} , is indicated by the second and third terms. The fourth term is that which describes the effect of torquer torque, $\tau_T = k_\tau \tau_R(t)$, and is the expression of interest in relation to the stabilization of c^1 with respect to the rotating reference line, L .

Under an assumption of systematic stability, the current point of interest in Expression 7-60 is isolated by assuming that $\theta_I(t) = 0$, or that it is a step function or a ramp function, so that the contribution corresponding to the first term is either zero, steady-state zero, or steady-state constant, and may thus be here disregarded. That is, it is first assumed that inertial stabilization is achieved in the absence of the other terms. Further, under systematic stability, the second term, induced by

τ_{zs} , corresponds to a time function which decays to zero with increasing time, while the time function corresponding to the third term, also due to τ_{zs} , decays to a constant value. The foregoing assumptions and observations justify an isolation of the response property of current interest, being that of the stabilization of c^1 with respect to the rotating reference line, L . The Laplace transform of interest is thus

$$\Theta_C(s) = \Theta_{CL}(s) = \frac{k_\sigma k_\tau}{s(J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_z n s + k_\sigma h_w)} T_R(s) \quad 7-61$$

If the torquer input signal, $\tau_R(t)$, is made directly proportional to the exact desired angular velocity, as expressed by

$$\tau_R(t) = k_R \frac{d\theta_L}{dt} \quad 7-62$$

then, if $\theta_L(0) = 0$

$$T_R(s) = k_R s \Theta_L(s)$$

Expression 7-61 becomes

$$\Theta_{CL}(s) = \frac{k_\sigma k_R k_\tau}{J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_z n s + k_\sigma h_w} \Theta_L(s) \quad 7-63$$

where the constants k_R and k_τ must be such that

$$\tau_T = k_\tau \tau_R(t) = k_R k_\tau \frac{d\theta_L}{dt} = \text{torque, say, in (dyne)(cm)}$$

Thus, for cgs units

$$k_R k_\tau \cdot \frac{\text{rad}}{\text{sec}} = (\text{dyne})(\text{cm}) = \frac{(\text{gm})(\text{cm})^2}{(\text{sec})^2}$$

from which

$$k_R k_\tau = \frac{(\text{gm})(\text{cm})^2}{(\text{sec})^2} \cdot (\text{sec}) = \frac{(\text{gm})(\text{cm})^2}{\text{sec}}$$

which are the dimensions of angular momentum, as measured in cgs units. Thus, the product $k_R k_\tau$ has the same dimensions as h_w . If now the magnitude of $k_R k_\tau$ may be adjusted so as to remain equal to h_w

$$k_R k_\tau = h_w$$

then Expression 7-63 becomes

$$\Theta_{CL}(s) = \frac{k_\sigma h_w}{J_\alpha J_z n s^3 + n(J_\alpha f_z + J_z f_\alpha) s^2 + f_\alpha f_z n s + k_\sigma h_w} \Theta_L(s) \quad 7-64$$

which, for a stable distribution of coefficients, here assured by

$$(J_\alpha f_z + J_z f_\alpha) > \frac{J_\alpha J_z k_\sigma h_w}{f_\alpha f_z n}$$

is the transform of a control system which ideally provides *zero* steady-state position error ($\epsilon = \theta_L - \theta_{CL}$) for a step function input ($\theta_L(t) = k1(t)$), and a *constant* steady-state position error for a constant velocity (ramp function, $\theta_L(t) = \omega_k t$) input. Under the stipulated qualifications, then, a torquer torque suitably proportional to the desired angular velocity $\frac{d\theta_L}{dt}$, may be employed to implement uniaxial angular stabilization with respect to a *rotating* reference line, L , regarded as being located by the inertial angle, $\theta_L = \theta_L(t)$. Here again, as in inertial angular stabilization, design for specific performance requirements involves system compensation, rederivation of the system equation, and the other usual sophistications of the servomechanisms art. Generation of the desired angular velocity required for the torquer signal involves implementation which, in terms of the current description, is external to the uniaxial stabilization mechanism. Subsequent discussion will reveal the participation of angular stabilization in the larger mechanisms of platform leveling and inertial guidance.

7.4 | Ideal Stabilization of a Two-Gimbal Platform

Return now to the particular platform suspension achieved by the two-gimbal arrangement indicated in Fig. 7-1. Recall that the angularly stabilized central element, being that part of the system which is actually the stabilized platform, is kinematically identified by the right-handed orthogonal triad, ϕ_P , with axes P^1 , P^2 , and P^3 , and that P^3 is the axis mounted in the inner gimbal. Three s.d.f. gyros may be fixed to the platform in such a way that their respective case input axes (with respect to the uniaxial discussions, these are axes corresponding to c^3 , as opposed to the precessing input axis g^3) lie along P^1 , P^2 , and P^3 . These gyros are here the angular motion sensors fixed to the platform, which is nominally oriented with the P^3 axis upward along a terrestrial vertical of some definition (geocentric, geographic, or astronomic), and which has some location on or above the geoid defined by the location of the origin of ϕ_P . It is convenient verbally to designate these orthogonally arranged s.d.f. gyros in accord with their functions in a terrestrial leveling system. So designating, the gyros are the *azimuth gyro* (input axis nominally along a specified vertical) and *attitude gyros* I and II. The situation is as indicated by Fig. 7-7. Significantly, if the P^3 axis of ϕ_P is upward, as in the convention depicted by Fig. 7-7, then positive rotations about the vertical are counterclockwise when looking down on the P^1P^2 (nominally horizontal) plane. In this connection, it is remembered that this convention is in opposition to the definition of positive terrestrial heading, which is measured from the north toward the east.

Consider first the *inertial* stabilization of the platform (now identified by the frame ϕ_P), as it is carried over the rotating geoid (Earth) by a transporting vehicle. The initial orientation of the P^i axes, which may conveniently be assumed to be initially coincident with the respective vehicle axes U^i , defines the inertial frame with respect to which the P^i are

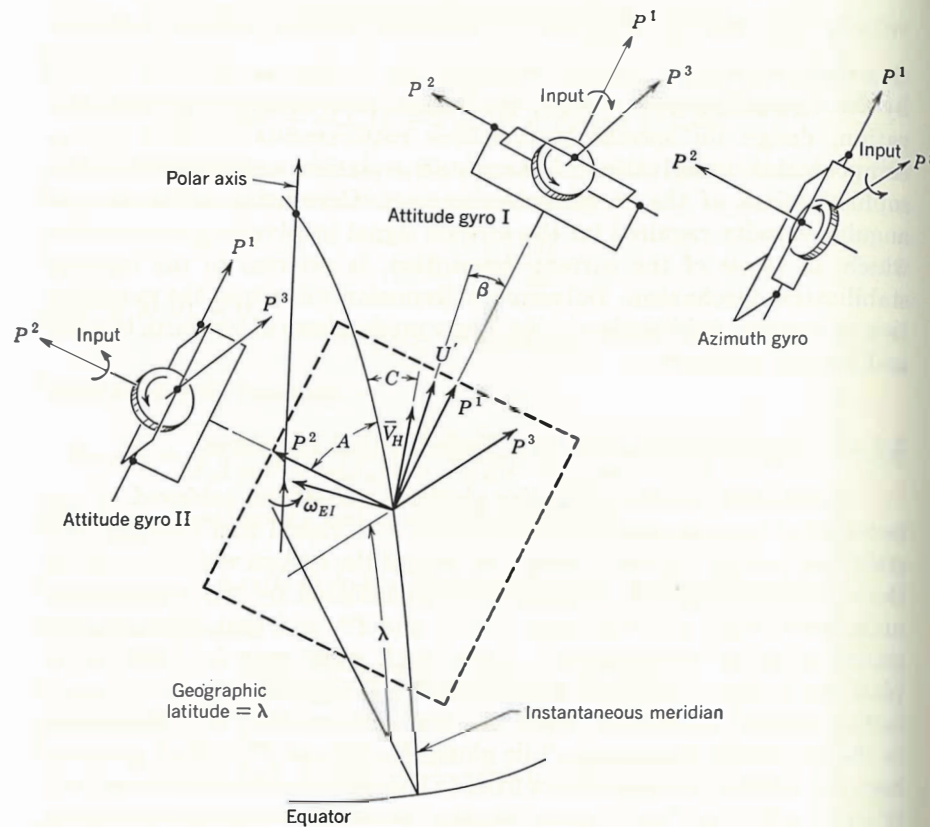


Figure 7-7

thereafter stabilized. As the system then moves over the Earth, which is in turn rotating with respect to inertial space (Earth's orbital motion is here taken to contribute only to its inertial angular velocity, as reviewed in Chapter 1), the respective gyros sense the residual inertial angular velocities impressed about their input axes, and command the stabilization motors to rotate the platform and gimbals so as to null the angular velocity of ϕ_P with respect to ϕ_I (where ϕ_I , axes I^1, I^2, I^3 , is now defined by the initial directions of the P^i). No gyro torquer signals are ideally involved in *inertial* stabilization of the platform. The specific mechanism of the multiaxial action of platform inertial stabilization, and the related

mechanism of the stabilization with respect to a rotating reference (which does involve gyro torquing signals), are somewhat more sophisticated than is immediately apparent. Details of the platform stabilization mechanism are expanded in following paragraphs. It may be noted that *ideal inertial* angular stabilization of a reference platform as *an individual mechanism*, and as *opposed to its inclusion in a guidance system*, is the same in the neighborhood of the Earth as in spatial regions far removed. That is, the inertial directional stabilization of a platform, in the ideal sense and *without* regard to *dynamically impressed* anomalies (considered subsequently), is a matter of angular-velocity detection and servo suppression. On the other hand, the situation is basically different for the rotating stabilization involved in the overall dynamic processes of a *leveling system* or some inertial guidance systems. In the latter applications, the torquing signals involved in rotating stabilization of the platform are functions of the general kinetic state of the system. Inertial sensors which measure the curvilinear kinetic state of the system (accelerometers, for example) are also influenced by the environmental gravitational field, thus interjecting the gravitational properties of the planet into the overall platform-guidance dynamics.

Angular stabilization performance, inertial or rotating, depends in part on mechanical properties and effects which attend gimbal system configuration. Continuing the discussion of the *inertial stabilization* of a platform with the gimbal suspension described in Section 7.2, depicted by Fig. 7-1, suppose now that the vehicle supporting the platform has been in relative terrestrial motion for a time t . At the time t , the vehicle axes, U^1, U^2, U^3 , have some angular orientation with respect to the perfect directions of the inertially stabilized platform axes, P^1, P^2, P^3 (perfect inertial stabilization assumed at the moment). This angular orientation of the U^i with respect to the P^i (where the U^i may have been initially coincident with the P^i is, in general, due to the rotation of the Earth, the history of the relative curvilinear terrestrial trajectory, and the local perturbations of the vehicle. Gross inertial angular velocities (not to be confused with the residual angular velocities impressed about the gyros within the operating stabilization mechanism) resolved onto the axes of ϕ_P are:

$$\left. \begin{aligned} \omega_P^1 &= a_1^1 \omega_U^1 + a_1^2 \omega_U^2 + a_1^3 \omega_U^3 \\ \omega_P^2 &= a_2^1 \omega_U^1 + a_2^2 \omega_U^2 + a_2^3 \omega_U^3 \\ \omega_P^3 &= a_3^1 \omega_U^1 + a_3^2 \omega_U^2 + a_3^3 \omega_U^3 \end{aligned} \right\} \quad 7-65$$

where the ω_U^i are the inertial angular velocities measured about the right-handed orthogonal vehicle axes, U^i (not to be confused with the gimbal system axes W^i which are *not* in general orthogonal), and the *direction*

$\cosines, a_j^i$, instantaneously measured between the axes P^i of the platform and the axes U^i of the vehicle, appear in the *transposed* order of an *inverse transformation*, as is consistent with the basic conventions of Section 2.3 (see Expressions 2-10, 2-13, and 2-14). In contemplation of this, consider again Fig. 7-1, and suppose that U^1 is in the all-positive octant of ϕ_P . Under achievement of perfect inertial stabilization, the nine direction angles corresponding to the nine a_j^i would be measurable consequences of that stabilization, and may be the basis of one type of guidance system. However, the point at the moment is the revelation that, in general, each gyro detects angular velocities measured about all three vehicle axes. Thus, the stabilization servo drives (one between vehicle and roll gimbal, one between roll gimbal and pitch gimbal, and one between pitch gimbal and platform) must, in general, accept signals* from *all three gyros* to achieve inertial stabilization of the platform. *Appropriate resolution of gyro signals* is thus an *essential feature* of platform stabilization.

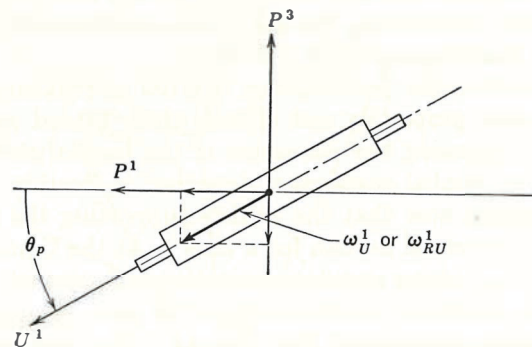


Figure 7-8

A simplified exemplification of the signal resolution requirement is readily established by consideration of Fig. 7-8. This figure depicts the P^1P^3 plane (with P^2 out of the paper) at the instant that the vehicle is pitched through the positive rotation $\theta_p = \theta^2$ (θ^2 denotes a rotation about P^2). Clearly, if the roll gimbal has any residual angular velocity, ω_{RU}^1 , measured about the roll axis (reflecting an instantaneous error in the stabilization system), this angular velocity is detected by the gyro with input axis along P^1 (attitude gyro I), and also by the gyro with input axis along P^3 (azimuth gyro). It should be noted that the azimuth and attitude gyros lose their verbal identity under *inertial* stabilization of the platform. Angular velocities impressed about the gyro case input axes,

* Either in terms of electrical signals or in terms of reflected rotational kinematic effect.

respectively along P^1 and P^3 , are instantaneously

$$\left. \begin{aligned} \omega_P^1 &= \omega_{RU}^1 \cos \theta_p \\ \omega_P^3 &= -\omega_{RU}^1 \sin \theta_p \end{aligned} \right\} \quad 7-66$$

Now the detection-stabilization mechanism must function so that the precession α_1 (measured about the P^1 gyro output axis, normal to P^1), and the precession α_3 (measured about the P^3 gyro output axis, normal to P^3), together generate signals which are combined in such a manner that the composite effect is equivalent to a direct detection of ω_{RU}^1 , as would be measured about U^1 . That is, as the roll stabilization drive which suppresses ω_{RU}^1 , by driving the roll gimbal relative to the vehicle, applies its torque about U^1 , the mechanism must, in effect, attempt an instantaneous uniaxial stabilization about U^1 , even though ω_{RU}^1 is detected in terms of components measured about (projections along) P^1 and P^3 . Neglecting for the present the cross-effects which attend the small precessions, α_1 and α_3 , of the actual input axes, g_1^3 and g_3^3 (as opposed to the case input axes, c_1^3 and c_3^3), the Laplace transforms (null initial conditions) of the angular velocities measured by the gyros with case input axes respectively along P^1 and P^3 are

$$\Omega_P^1(s) = \frac{J_\alpha s^2 + f_\alpha s}{h_w} A_1(s) \quad \Omega_P^3(s) = \frac{J_\alpha s^2 + f_\alpha s}{h_w} A_3(s)$$

where the gyros are assumed to be identical. Transforms of the voltage signals from the gyros are then respectively

$$E^1(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^1(s) \quad 7-67$$

$$E^3(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^3(s) \quad 7-68$$

where k_P is the pickoff constant, and the amplifiers of gain k_A are regarded as being associated individually with each gyro. *Resolution* of these signals is physically accomplished by employment of a *resolver* mounted so that the relative rotation between pitch and roll gimbals (which occurs about W^2 , see Fig. 7-1) is reflected as rotation between its rotor and stator. Constants of proportionality of the resolver, schematically indicated by Fig. 7-9, are assumed to be adjusted so that the voltage outputs, e_0^1 , and e_0^3 are related (ideally) to the voltage inputs, e_i^1 and e_i^3 , by the expressions

$$\left. \begin{aligned} e_0^1 &= e_i^1 \cos \theta_p - e_i^3 \sin \theta_p \\ e_0^3 &= e_i^1 \sin \theta_p + e_i^3 \cos \theta_p \end{aligned} \right\} \quad 7-69$$

If now

$$E_i^1(s) = \mathcal{L}[e_i^1(t)] \quad E_i^3(s) = \mathcal{L}[e_i^3(t)]$$

then the instantaneous output voltages supplied by the resolver are operationally characterized by the transforms

$$E_0^1(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^1(s) \cos \theta_p - \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^3(s) \sin \theta_p \quad 7-70$$

$$E_0^3(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^1(s) \sin \theta_p + \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^3(s) \cos \theta_p \quad 7-71$$

If the voltage outputs from the resolver are applied to the control field

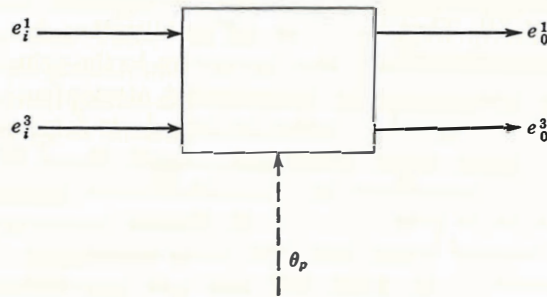


Figure 7-9

of the roll motor as a sum

$$e_M(t) = e_0^1(t) + e_0^3(t)$$

$$E_M(s) = E_0^1(s) + E_0^3(s)$$

which, in the operational domain, is explicitly

$$E_M(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^1(s) (\cos \theta_p + \sin \theta_p) + \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_P^3(s) (\cos \theta_p - \sin \theta_p) \quad 7-72$$

the resolution then implements the required mechanism. That this is ideally valid is demonstrated by substitution of the Laplace transforms of the environmental angular velocity components, Expressions 7-66, into 7-72. Thus

$$E_M(s) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_{RU}^1(s) (\cos^2 \theta_p + \sin^2 \theta_p) = \frac{k_A k_P h_w}{s(J_\alpha s + f_\alpha)} \Omega_{RU}^1(s) \quad 7-73$$

which indicates that the sum of the resolver output voltages is (ideally) a direct dynamic measure of the residual roll angular velocity, ω_{RU} . One may then proceed to an ideal (or realistic) study of the roll-stabiliza-

tion servo system in terms of an uniaxial description of the nature reviewed in Section 7.3. However, such a study still excludes the small cross-effects (discussed later) which attend the fact that small precessions of the actual gyro input axes, g_i^2 (as opposed to the case input axes c_i^2) must occur if the stabilization mechanism is to function.

The immediately preceding discussion has indicated the necessity of signal resolution in terms of an instantaneous exemplification. In general, the inertial stabilization of the platform under discussion, with the gimbal suspension of Fig. 7-1, involves the simultaneous participation of three resolvers respectively located so as to implement signal resolutions between vehicle and roll gimbal, between roll gimbal and pitch gimbal (as in the example above), and between pitch gimbal and platform. Simultaneous operation of these resolvers, in correlation with the proper adjustment of the servo parameters of the three drive systems, accommodates the resolutions required in the inertial stabilization of the platform.

A very significant utilization of the *rotating* stabilization of a platform suspended by the two-gimbal arrangement of the current discussion, Fig. 7-1 (supplemented by Fig. 7-7), is that involved in the implementation of *terrestrial vertical* indication. Such implementations are often called *leveling systems*, and are important not only because of vertical and horizontal reference indications (as is involved in bombing, etc.), but also because the mechanism of vertical indication is often an integral part of the terrestrial inertial guidance mechanism. Integration of the rotating stabilization of the leveling mechanism with the dynamics of the guidance problem will be developed in following chapters. Of current concern is a qualitative description of the specific mechanism of that particular rotating stabilization involved in maintenance of a terrestrial vertical.

Previous designations (attitude gyros I and II, azimuth gyro, etc.) have predicted that it is the P^3 axis of the platform triad, ϕ_P , which is normally slaved to a terrestrial vertical when a system of the configuration of Fig. 7-1 is so employed. It is thus anticipated, from previous discussion, that a maintenance of a vertical platform attitude by slaving P^3 to a moving vertical (of some definition), transported by the vehicle in motion relative to the geoid, involves an application of appropriate torquing signals to attitude gyros I and II. That is, the P^1P^2 plane, otherwise stabilized with respect to inertial space when no torquer signals are applied to the attitude gyros, must be properly erected, and then rotated in accord with the vehicle motion over the geoid, so that it remains horizontal and thus keeps P^3 along the corresponding vertical. Or, in other words, P^3 is stabilized with respect to a rotating reference line which radiates from somewhere near the center of the Earth (if the vertical is *geocentric*, then the rotating reference line radiates from the center of the

Earth). Vertical stabilization of P^3 does *not* dictate any particular rotational behavior of the P^1 and P^2 axes within the horizontally stabilized P^1P^2 plane. It will later be seen that the azimuth behavior of the P^1 and P^2 axes may be implemented in several different ways.

It has been demonstrated, within the uniaxial descriptions of Section 7.3, that rotating stabilization requires an application of torquer signals which are directly proportional to the angular velocity of the rotating line with respect to which the driven reference line is to be stabilized. Consequently, stabilization of P^1 and P^2 to the horizontal plane, synonymous with the stabilization of P^3 along the corresponding vertical, requires torquer signals proportional to the *angular velocities* generated about P^1 and P^2 due to the *curvilinear motion* of the vehicle over the geoid. Generation of these torquer signals may be accomplished by implementation which operates on signals provided by two orthogonal P^1P^2 plane *rectilinear accelerometers*, which complement the angular motion sensors (gyros) in completing the sensing requirements for an inertial continuance of level platform attitude. It is convenient to locate the acceleration-sensitive axes of these accelerometers directly along P^1 and P^2 , respectively. The only current stipulation relating to the physical properties of the accelerometers is that they are assumed to provide signals directly proportional to the *total* accelerations impressed along their sensitive axes. In this connection, it must be recognized that the total acceleration along any axis is the sum of the total *inertial kinetic* acceleration along that axis and any component of gravitational acceleration which may appear thereon. Signals provided by the accelerometers are corrected (in an effort to derive relative terrestrial acceleration from the total acceleration which the accelerometers must measure) and *integrated* with respect to time, to give measures of the terrestrial *linear velocity* projections along P^1 and P^2 . In this connection, it is apparent that:

- Curvilinear velocity in the P^1P^2 plane induces angular velocity about P^3 ;
- Curvilinear velocity in the P^1P^3 plane induces angular velocity about P^2 ;
- Curvilinear velocity in the P^2P^3 plane induces angular velocity about P^1 .

Thus, the terrestrial curvilinear velocity projection along P^1 induces an angular velocity about P^2 , and conversely. Explicitly, division of the relative linear velocity v_R^1 (along P^1) by the pertinent terrestrial radius provides an *angular velocity*, ω_v^2 , about P^2 , and division of the relative linear velocity v_R^2 (along P^2) by the pertinent terrestrial radius provides an *angular velocity*, ω_v^1 , about P^1 . Torquing signals applied to the output axis torquers of attitude gyros I and II are then respectively proportional to ω_v^1 and ω_v^2 (that is, respectively proportional to v_R^2 and v_R^1 , for constant terrestrial radii). Pickoff signals from the attitude gyros are

simultaneously resolved through the P^1P^2 plane projection of the angle between P^1 and U^1 and applied simultaneously to the gimbal servo drives to continue level attitude of the platform. These resolutions are performed by that resolver which operates between the pitch gimbal and platform. Usually, adequate characterizations of the closed loop system dynamics of the leveling mechanism are conveniently established by the so-called *single-channel* descriptions of a following chapter.

As previously implied, continuance of a terrestrially level platform attitude does *not* require the application of a torquer signal to the *azimuth* gyro. Clearly, the orientation of the P^1 and P^2 axes with respect to the P^1P^2 plane projection, $\bar{V}_H$, of the instantaneous relative velocity, $\bar{V}_R$, is certainly involved in the acceleration projections sensed by the linear accelerometers (the signals from which are those processed to provide the projections v_R^1 and v_R^2 of $\bar{V}_H$). However, so long as the platform-pitch gimbal resolution of pickoff signals is correct, the correct drive signals are applied to the attitude motors. As the pickoff signal resolutions occur parallel to the P^1P^2 plane, it is tacitly assumed that the platform is essentially level when the continuance of level attitude begins, that is, when the vehicle starts to move relative to the Earth. Automatic erection of the platform constitutes a different mode of system operation.

Further reference to Fig. 7-7 indicates features of the leveling situation which will be significant in subsequent discussion. Under level attitude, the P^1P^2 plane is parallel to a tangent to the surface of a terrestrial reference ellipsoid (or sphere), while the P^1 and P^2 axes are instantaneously oriented with respect to a meridian by an angle A (azimuth) and also have some orientation with respect to the horizontal velocity, $\bar{V}_H$. Time variation of A depends on the particular azimuth control employed in the system, and is a topic of later discussion. It is understood that the vehicle may experience rotational displacements or oscillations with respect to the velocity vector $\bar{V}_H$, and consequently that a projection of the U^1 axis on the P^1P^2 plane is *not* necessarily along $\bar{V}_H$. As the untorqued stabilization system drives to stabilize the platform with respect to inertial space, under ideal operation, the platform is isolated from these local disturbances, while continuing the level attitude under much slower torquing signals. Implicit in this ideal operation is the assumption that the linear accelerometers are sufficiently isolated from spurious accelerations attending the rotational perturbations.

7.5 | Stabilization Degradation Attending Impressed Oscillations (Platform Dynamic Drift)

Prior discussion of the dynamics of the precession phenomenon has revealed that the actual input axis of an s.d.f. gyro is that axis which

remains mutually normal to *both* the output axis and the spin axis, said actual input axis being otherwise identified as the $\mathfrak{g}^3$ axis in the uniaxial discussions. See again Fig. 7-2. Particular reflection of this basic property is observed in the expression

$$\tau_G^1 = \omega^3 h_w \cos \alpha \quad 7-9$$

which applies to the situation wherein the only input angular velocity is applied about the case input axis c^3 . See again Expressions 7-1 through 7-9. It is recalled that all prior platform behavior descriptions have involved the assumption that α is always small, this being an assumption justified by the tight control loop which governs the magnitude of α . Expression 7-9 is thus assumed to be essentially $\tau_G^1 = \omega^3 h_w$. Actually, however, as soon as the precession α has *any nonzero value*, the gyro is sensitive to any component of angular velocity which appears along c^2 . In this connection, let ω_c^3 be the input inertial angular velocity measured about c^3 , while ω_c^2 denotes some positive inertial angular velocity measured about c^2 . Clearly, if α has a positive value, the expression corresponding to Equation 7-9 (for which there was no ω_c^2) is

$$\tau_G^1 = (\omega_c^3 \cos \alpha - \omega_c^2 \sin \alpha) h_w \quad 7-74$$

which indicates that the output torque available to initiate the stabilization mechanism is reduced. Expression 7-74 indicates the genesis of particular *cross-effects* inherent in the conventional platform stabilization mechanism (Figs. 7-1 and 7-7) which have not yet been reviewed. While the stabilization errors which attend these cross-detections are usually suppressed under proper design, high gain, etc., there may occur environmental dynamic conditions which render the suppression inadequate. Environmental conditions which involve cyclic disturbances or residual vibrations transmitted to the gimbals can, under certain time phasing, cause the platform to drift away from the intended behavior. Such drift, or stabilization degradation, is here called *dynamic drift*. The following description of dynamic drift provides some evaluation of a real effect, and should also enhance an appreciation of the nature of cross-effects.

Examination of the specific mechanism of dynamic drift reveals that the nominal unprocessed relative orientations of the spin axes of the three s.d.f. gyros are factors in drift generation. Platform-gimbal system configuration is again that indicated by Fig. 7-1, which is redepicted in Figs. 7-10 and 7-11 for two cases of gyro spin axes nominal orientation. Fig. 7-10 indicates an unrotated attitude of the gimbals and depicts the spin axes arrangement employed in the following description. It is understood that the three gyros of Fig. 7-10 could detect the same angular velocity inputs about *case fixed* input axes with their respective spin axes rotated

to any directions in the planes normal to the respective case input axes. This is illustrated by Fig. 7-11, which depicts a configuration wherein the attitude gyros have had their nominal spin axis directions rotated 90° with respect to the directions employed in Fig. 7-10. On the other hand, *detection of angular velocity components (projections) which are normal to the case fixed input axes, possible only when the spin axes are precessed,*

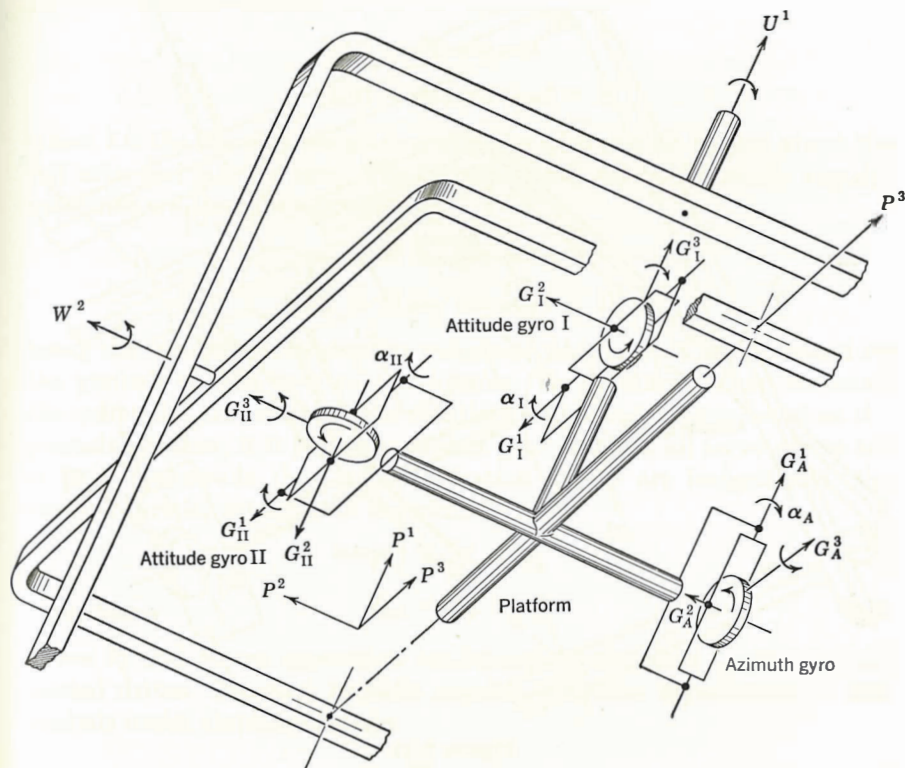


Figure 7-10

definitely involves the relative orientations of the precession planes. It is the detection of such *cross* angular velocity components, possible only when gyro spin axes are precessed (even ever so slightly), which caused dynamic drift under vibration, and which makes the relative orientation of nominal (or null) directions of spin axes a factor in dynamic drift generation. Preliminary appreciation of these statements may be enhanced by a comparison of Figs. 7-10 and 7-11.

As it is only vibratory angular dynamics which is of current concern, a discussion of platform accelerometers, which lead to slow rate torquing signals involved in platform leveling under terrestrial curvilinear motion,

is omitted. This involves the tacit assumption that the accelerometers are not involved in the drift mechanism. If this assumption is objectionable, one may then rather stipulate that the system is operating to achieve *inertial* angular stabilization, and thus that there are neither accelerometers or gyro torquers involved in the stabilization mechanism. It is

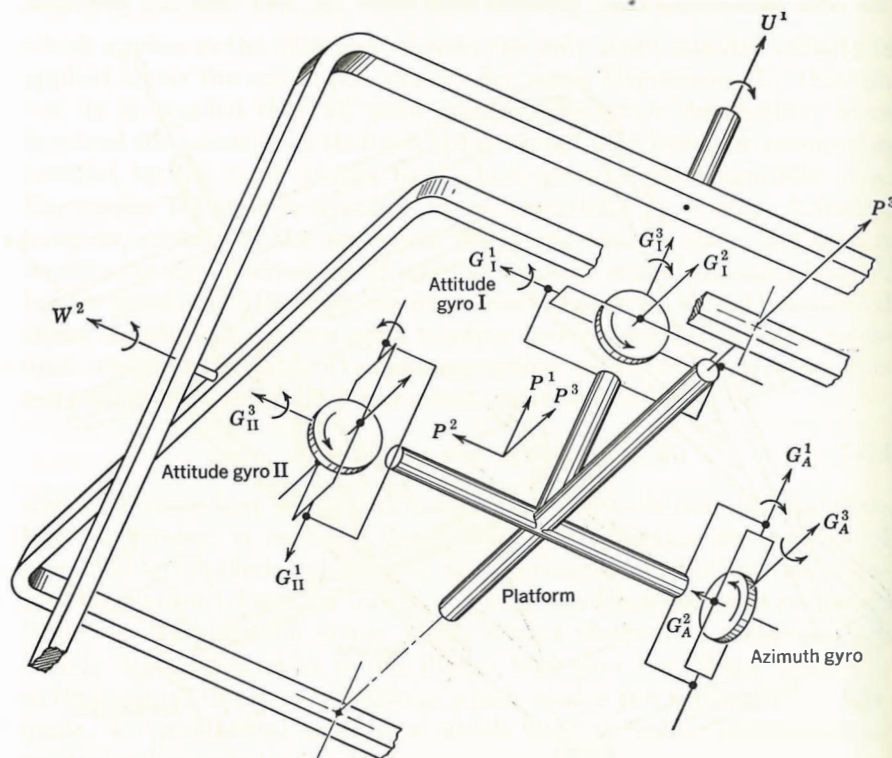


Figure 7-11

concluded that the dynamic drift of a platform operating to maintain inertial directional reference must be of essentially the same nature as the dynamic drift of a platform operating to stabilize with respect to a slowly rotating reference. For simplicity, it is assumed that the initial azimuth displacement of the platform is zero, and that the gimbals are unrotated from their coincident attitudes of Figs. 7-1, 7-10, and 7-11. Recall that the gyros are here oriented as depicted by Fig. 7-10. It is now stipulated that the only inertial motions externally applied to the system is a rotational oscillation, $\theta_b^1(t)$, applied to the vehicle about U^1 , and a rotational oscillation, $\theta_w^2(t)$, applied about the oscillating pitch axis, W^2 . The three gyros simultaneously detect the residual angular velocities which appear about the instantaneous directions of their actual input axes

(the axes normal to the planes of the frames supporting the wheels in Fig. 7-10), and thus generate signals proportional to the precessions α_I , α_{II} , and α_A . As previously described, these pickoff signals are employed to control the gimbal motors and azimuth motor (through appropriate resolutions) such that signal suppression to null is intended to stabilize the platform axes, P^1 , P^2 , P^3 , with respect to inertial directions. Impressed oscillations are here stipulated to be

$$\theta_U^1(t) = M_1 \sin \omega_U t$$

$$\theta_W^2(t) = M_2 \sin (\omega_W t - \delta)$$

where δ is the phase of the pitch oscillation with respect to that about the roll axis, and may be zero. The instantaneous external periodic angular velocities are then respectively

$$\omega_U^1 = M_1 \omega_U \cos \omega_U t$$

$$\omega_W^2 = M_2 \omega_W \cos (\omega_W t - \delta)$$

being the instantaneous angular velocities which *would* be impressed on the gimbals *if there were no stabilization*. Under stabilization, however, the quantities above are *not* the rotations and velocities impressed on the gimbals. Rather, if it be assumed that gear ratios in all servo drives are of such magnitude that self-stabilization effects are insignificant, the residual inertial rotations of the gimbals are

$$\theta_{RU}^1 = \theta_U^1 - \theta_S^1 \quad 7-75$$

$$\theta_{RW}^2 = \theta_W^2 - \theta_S^2 \quad 7-76$$

where θ_S^1 and θ_S^2 are opposition (stabilization) rotations applied by the gimbal drives. Residual periodic angular velocities experienced by the gimbals under stabilization are

$$\omega_{RU}^1 = \frac{d\theta_{RU}^1}{dt} = \omega_U^1 - \omega_S^1 \quad 7-77$$

$$\omega_{RW}^2 = \frac{d\theta_{RW}^2}{dt} = \omega_W^2 - \omega_S^2 \quad 7-78$$

Now, after system activation, the angular velocity ω_{RU}^1 has induced a periodic precession, α_I , of the spin axis of attitude gyro I. This precession generates the periodic signal employed to drive the roll gimbal in opposition to the external periodic disturbance θ_U^1 , in the attempt to achieve stabilization of the platform about U^1 . However, as soon as the precession α_I has any nonzero value, gyro I detects a component of the periodic residual angular velocity ω_{RW}^2 , measured about the pitch axis, W^2 . Reference to Fig. 7-10 indicates that the total inertial angular

velocity impressed on gyro I is

$$\omega_I = \omega_{RU}^1 \cos \alpha_I - \omega_{RW}^2 \sin \alpha_I = (\omega_U^1 - \omega_S^1) \cos \alpha_I - (\omega_W^2 - \omega_S^2) \sin \alpha_I \quad 7-79$$

while a similar argument applied to gyro II indicates that

$$\omega_{II} = \omega_{RU}^1 \sin \alpha_{II} + \omega_{RW}^2 \cos \alpha_{II} = (\omega_U^1 - \omega_S^1) \sin \alpha_{II} + (\omega_W^2 - \omega_S^2) \cos \alpha_{II} \quad 7-80$$

Further inspection of Fig. 7-10 indicates that the residual angular velocity impressed on the azimuth gyro is

$$\omega_A = \omega_{RU}^1 \cos \left(\frac{\pi}{2} - \theta_{RU}^1 \right) - \omega_{RW}^2 \sin \alpha_A = (\omega_U^1 - \omega_S^1) \sin \theta_{RU}^1 - (\omega_W^2 - \omega_S^2) \sin \alpha_A$$

As it must be assumed that the system does indeed achieve some measure of adequate stabilization, the periodic precession angles α_I , α_{II} , and α_A must be of small amplitude, say, one degree or less. Consequently

$$\omega_I \cong (\omega_U^1 - \omega_S^1) - (\omega_W^2 - \omega_S^2) \alpha_I \quad (\text{gyro I}) \quad 7-81$$

$$\omega_{II} \cong (\omega_U^1 - \omega_S^1) \alpha_{II} + (\omega_W^2 - \omega_S^2) \quad (\text{gyro II}) \quad 7-82$$

$$\omega_A \cong (\omega_U^1 - \omega_S^1) \theta_{RU}^1 - (\omega_W^2 - \omega_S^2) \alpha_A$$

where it has also been recognized that $\theta_{RU}^1 = \theta_U^1 - \theta_S^1$ must be small.

It is now stipulated that the spin angular momenta of the three gyros are of the same value, h_w , and that their output axis moments of inertia, J , and damping coefficients, f , are also respectively equal. Equations of motion about the gyro output axes are thus essentially and respectively

$$J \frac{d^2 \alpha_I}{dt^2} + f \frac{d \alpha_I}{dt} = h_w \omega_I = h_w (\omega_U^1 - \omega_S^1) - h_w (\omega_W^2 - \omega_S^2) \alpha_I \quad 7-83$$

$$J \frac{d^2 \alpha_{II}}{dt^2} + f \frac{d \alpha_{II}}{dt} = h_w \omega_{II} = h_w (\omega_U^1 - \omega_S^1) \alpha_{II} + h_w (\omega_W^2 - \omega_S^2) \quad 7-84$$

$$J \frac{d^2 \alpha_A}{dt^2} + f \frac{d \alpha_A}{dt} = h_w \omega_A = h_w (\omega_U^1 - \omega_S^1) \theta_{RU}^1 - h_w (\omega_W^2 - \omega_S^2) \alpha_A$$

These expressions could be employed with the drive system equations to write the closed-loop stabilization equations, as demonstrated in Section 7.3. However, this approach may be circumvented by recognition of the fact that a *cardinal feature* of the stabilization mechanism is the *torque balance* which must be effected about the respective gyro output axes. Thus, considering a system with undamped gyros (that is, in behalf of simplicity $f = 0$), and recognizing that under stabilization the $J \frac{d^2 \alpha}{dt^2}$

terms are small, stabilization of the platform then essentially dictates that

$$h_w(\omega_U^1 - \omega_S^1) - h_w(\omega_W^2 - \omega_S^2)\alpha_I \cong 0$$

$$h_w(\omega_U^1 - \omega_S^1)\alpha_{II} + h_w(\omega_W^2 - \omega_S^2) \cong 0$$

which may be written

$$h_w\omega_{RU}^1 \cong h_w\omega_{RW}^2\alpha_I \quad 7-85$$

$$h_w\omega_{RU}^1\alpha_{II} \cong -h_w\omega_{RW}^2 \quad 7-86$$

where ω_{RU}^1 and ω_{RW}^2 are the residual oscillatory velocities which the stabilization system does not quite null. The precessions, α_I and α_{II} , are also, of course, oscillatory and of small amplitudes, as previously indicated. It is logical to assume that residual oscillations, oscillation velocities, and precessions are of the same frequency as the environmental (impressed) oscillations, $\theta_U^1(t)$, and $\theta_W^2(t)$. Interpretation of dynamic drift from the torque balance implied by Expressions 7-85 and 7-86 then becomes a matter of average values and phase relations.

Expression 7-85 indicates an approximate torque balance about the precessional axis of gyro I, which, under the unrotated azimuth orientation of Fig. 7-10, is that gyro which acts as the roll sensor. *If* there were *no* drift of the roll gimbal, the following relation would remain valid

$$(h_w\omega_{RU}^1)_{\text{Mean}} \cong (h_w\omega_{RW}^2\alpha_I)_{\text{Mean}} \quad 7-87$$

Comparison of the sides of this expression reveals a measure of the dynamic drift of the platform about the roll axis which attends the residual vibrations. Suppose now that stabilization performance about the roll axis is such that the residual oscillation of the pitch axis, $\theta_{RU}^1(t)$, is

$$\theta_{RU}^1(t) = a \sin \omega_U t \quad 7-88$$

being the oscillation of amplitude a which is not quite nulled by the roll stabilization system. The roll gyro, being here gyro I (as depicted by Fig. 7-10, responds essentially to ω_{RU}^1 , being

$$\omega_{RU}^1 = \frac{d\theta_{RU}^1}{dt} = a\omega_U \cos \omega_U t \quad 7-89$$

It is thus assumed that

$$\frac{d^2\alpha_I}{dt^2} = \frac{h_w}{J} \omega_{RU}^1 = \frac{ah_w\omega_U}{J} \cos \omega_U t \quad 7-90$$

Integration with respect to time then gives

$$\frac{d\alpha_I}{dt} = \frac{ah_w}{J} \int \cos \omega_U t (\omega_U dt) = \frac{ah_w}{J} \sin \omega_U t$$

and

$$\alpha_I = \frac{ah_w}{J\omega_U} \int \sin \omega_U t (\omega_U dt) = -\frac{ah_w}{J\omega_U} \cos \omega_U t \quad 7-91$$

where it has been assumed that the initial value of α_I and that of its first derivative are both zero. Expression 7-91 is employed presently. Consider now the *mean value* of the torque $h_w \omega_{RU}^1$, being the left side of Expression 7-87. This is evaluated by employment of Equation 7-89. Thus

$$(h_w \omega_{RU}^1)_{\text{Mean}} = \frac{\omega_U}{2\pi} \int_0^{2\pi/\omega_U} h_w \omega_{RU}^1 dt = \frac{ah_w \omega_U}{2\pi} \sin \omega_U t \Big|_0^{2\pi/\omega_U} = 0$$

That is, the left side of Expression 7-87 is zero. Evaluation of the average value of the torque $h_w \omega_{RW}^2 \alpha_I$, being the right side of Expression 7-87, involves the residual oscillation about the pitch axis. This is assumed to be

$$\theta_{RW}^2(t) = b \sin (\omega_W t - \phi) \quad 7-92$$

where b is the amplitude, and ϕ is the time phase of θ_{RW}^2 with respect to θ_{RU}^1 . That is

$$\phi = (\theta_{RU}^1 - \theta_{RW}^2)_{\text{Phase}}$$

even though the oscillations occur in planes nominally at right angles to each other. Oscillatory residual angular velocity about the pitch axis is

$$\begin{aligned} \omega_{RW}^2 &= \frac{d\theta_{RW}^2}{dt} = b\omega_W \cos (\omega_W t - \phi) \\ \omega_{RW}^2 &= b\omega_W \cos \phi \cos \omega_W t + b\omega_W \sin \phi \sin \omega_W t \end{aligned} \quad 7-93$$

Now, as the residual oscillations about the roll and pitch gimbals could certainly be induced by the same external vibration (from the vehicle engine, for example), it is logical to assume that they are of the same frequency. If so, $\omega_W = \omega_U$, from which it follows that

$$h_w \omega_{RW}^2 \alpha_I = -\frac{ab(h_w)^2}{J} \cos \phi \cos^2 \omega_U t - \frac{ab(h_w)^2}{J} \sin \phi \sin \omega_U t \cos \omega_U t \quad 7-94$$

which, by virtue of the *cosine squared* term, is seen to involve a *rectification* of torque. That is, there is torque which remains undulatory but which is *unidirectional*. The average or mean value of $h_w \omega_{RW}^2 \alpha_I$ is

$$(h_w \omega_{RW}^2 \alpha_I)_{\text{Mean}} = \frac{\omega_U}{2\pi} \int_0^{2\pi/\omega_U} h_w \omega_{RW}^2 \alpha_I dt$$

which involves the integrals

$$\int_0^{2\pi/\omega_U} \cos^2 \omega_U t dt = \frac{\pi}{\omega_U} \quad \int_0^{2\pi/\omega_U} \sin \omega_U t \cos \omega_U t dt = 0$$

It follows that

$$(h_w \omega_{RW}^2 \alpha_1)_{\text{Mean}} = - \frac{ab(h_w)^2}{2J} \cos \phi \quad 7-95$$

which is seen to have the dimensions

$$\frac{\left(\frac{(\text{gm})(\text{cm})^2}{\text{sec}} \right)^2}{(\text{gm})(\text{cm})^2} = \frac{(\text{gm})(\text{cm})^2}{(\text{sec})^2} = (\text{dyne})(\text{cm})$$

as is required. Expression 7-95 represents an excess of unidirectional torque which disturbs the average balance implied by Expression 7-87. This excess unidirectional torque is concluded to reflect that the roll gimbal is *not* quite stabilized about the roll axis, but rather drifts away from the desired stable orientation at the rate

$$\omega_D^1 = \frac{(h_w \omega_{RW}^2 \alpha_1)_{\text{Mean}}}{h_w} = - \frac{abh_w}{2J} \cos \phi, \frac{\text{rad}}{\text{sec}} \quad 7-96$$

By way of example, suppose that

$$\begin{aligned} a = b &= 10^{-3} \text{ rad} & J &= 10^3 (\text{gm})(\text{cm})^2 \\ h_w &= 10^6 \frac{(\text{gm})(\text{cm})^2}{\text{sec}} & \phi &= 0 \end{aligned}$$

These values give

$$-\omega_D^1 = \frac{10^{-3} \cdot 10^{-3} \cdot 10^6}{2 \cdot 10^3} = 5 \cdot 10^{-4} \frac{\text{rad}}{\text{sec}} = 1.8 \frac{\text{rad}}{\text{hr}}$$

which may be attenuated by inducing phase between the pitch and roll residual oscillations.

An expression for the drift rate about the pitch axis may be established by application of the same procedure to Expression 7-86. This drift rate is found to be of the same form as Equation 7-96. It is of interest to note that the dynamic drift rates do not explicitly involve the frequency of the inducing oscillations.

Reference to Fig. 7-11 indicates that the description of dynamic drift should be rederived for that placement of nominal spin axes directions, even though the case input axes retain the same directions. Further examination of Figs. 7-10 and 7-11 may suggest the possibility that dynamic drift may be altered by spin axes orientations intermediate to those indicated by the figures.

7.6 | Certain Other Basic Features of Platform Theory and Operation

Introductory simplicity has dictated that the current rudimentary review of platform stabilization, emanating essentially from the gimbal

configuration of Figs. 7-1, 7-10, and 7-11 and the uniaxial descriptions of Section 7.3, should exclude several important considerations. A qualitative description of some of these omitted properties should precede following developments, in which platform stabilization participates in the larger mechanisms of inertial guidance.

Although the basic intent and philosophy of this book relate specifically to theory, as opposed to explicit engineering detail, some brief recognition of the salient physical features of a typical s.d.f. gyro is certainly consistent with purpose. Individual s.d.f. gyros of the general configuration employed in the uniaxial discussions (Fig. 7-2), and in the angular sensor triads depicted by Figs. 7-10 and 7-11, are available as separate purchasable entities. Such instrument gyros, which have been developed to an impressive level of predictability by employment of certain friction suppressing suspension techniques, are self-sufficient components to the extent that they include a motor-driven rotor (of precisely controlled angular momentum) carried by a gimbal mounted within a sealed case containing the pickoff and torquer. Frictional torque at the gimbal (output axis) bearings is suppressed by a hydrostatic technique whereby the entire precessing assembly (spin motor stator, spin motor rotor, gimbal with pickoff and torquer rotors affixed, and an internal case or *damping sleeve*) is floated within the case. Within this flotation technique, the *temperature-controlled* fluid provides the buoyancy which suppresses the output axis frictional torque, and also introduces the output axis viscous damping, previously identified by the damping coefficient f_a . Under flotation of the precessing assembly, the output-axis bearings become essentially centering agencies which may be jewel pivots or may involve magnetic suspension. Guidance errors due to platform drift attributable to gyro output axis spurious torque, and to other origins, will be described in a subsequent chapter. Such errors are partially suppressed by reduction of this frictional torque, notably achieved by the flotation technique.

Platform suspension of the two-gimbal configuration (Figs. 7-1, 7-10, 7-11) provides a convenient arrangement for the introduction of the platform stabilization mechanism, and also for real terrestrial leveling systems which are exposed to vehicle rotational perturbations of moderate magnitude only (as, say, in marine applications, or in the case of unaerobatic aircraft). However, as previously implied in Section 7.2, employment of two-gimbal platform suspension in an environment of unrestricted vehicle rotations can induce stabilization singularity. An examination of the implication of this statement reveals a basic restriction inherent in the two-gimbal platform suspension system, and indicates the necessity for a third or so-called *redundant* gimbal in a platform suspension arrange-

ment designed for a terrestrial leveling system which is subjected to unrestricted vehicle rotations.

Consider the relative gimbal orientation of the gimbals of the two-gimbal platform suspension system of Fig. 7-1, when the P^3 axis is slaved along a vertical and the vehicle is in a dive along that vertical. Suppose that this vehicle attitude is achieved by a single positive rotation of $\pi/2$ radians about the pitch axis, and that said rotation departs from the axial coincidence of Fig. 7-1 (that is, before rotation the U^i axes are along the respective P^i axes). Under this diving attitude, the U^1 axis is along the negative P^3 axis, while U^2 and W^2 remain along P^2 until additional rotation occurs. If the stabilization system is one which employs its two-gimbal suspension to achieve a terrestrial level (under less severe attitudes) independent of the azimuth behavior of the platform (that is, a system wherein the normal attitude leveling signals are resolved by that resolver between platform and pitch gimbal, while the azimuth behavior of the platform is of independent implementation), then this vertical vehicle attitude induces a stabilization singularity. Such an attitude generates a singularity in the stabilization mechanism, because roll motion is now detectable only by the azimuth gyro, the output of which has no capacity to drive the leveling servos. Consequent loss of stabilization is sometimes called *gimbal lock*. A two-gimbal terrestrial leveling system is thus inadequate under unlimited vehicle rotation. This inadequacy, and other justifications, such as suppression of dynamic drift, indicate the feasibility of a three-gimbal platform suspension.

A platform suspension system which circumvents the singularity indicated above, and which suppresses the phenomenon of dynamic drift, previously described (Section 7.5), is realized by the inclusion of a third platform gimbal. For the configuration of Fig. 7-1, the third gimbal may be installed by rotating the pitch gimbal $\pi/2$ radians, so that it is now coplanar with the roll gimbal, and installing trunnions, adjacent to the outer-roll trunnions, to support an *inner-roll gimbal*. This inner-roll gimbal carries the upper and lower bearing junctions which support the platform. The platform thus now rotates about P^3 with respect to the inner-roll gimbal, which is nominally vertical under terrestrial level attitude of the platform. Inclusion of the redundant gimbal (which is any one of the three gimbals) necessitates an additional gimbal servo drive system, and thus increases the complexity of the signal distribution in the overall system. Such increased complexity is often justified by the removal of rotational restrictions otherwise applied to the vehicle, and also by an inherent suppression of platform dynamic drift.

As implied by the chapter title, "*Preliminary Theory of Stabilized Platforms*," no inference that the current chapter is a complete or general

discussion of reference platform stabilization should be accepted. Quite apart from many omissions relating to control system detail, such as gain variation under variation of relative gimbal attitudes, etc., the discussion has only faintly suggested the dominant participation of the rectilinear sensors in the mechanism of terrestrial leveling, which is a form of platform stabilization of basic significance in many terrestrial inertial systems. The brief verbal description of the participation of rectilinear sensing (and rectilinear sensor signal processing) in the generation of torquing signals is only a qualitative preliminary to the analytical descriptions of following chapters.