HANDBOOK OF ANALOG COMPUTATION



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HANDBOOK OF
ANALOG COMPUTATION

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HANDBOOK OF ANALOG COMPUTATION

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PREFACE

These notes on Analog Simulation have been developed from the experience gained by the Education and Training Department of EAI in presenting intensive short courses in analog computer operation, programming, and applications for nearly a decade.

The objective of these courses has been to provide scientists and engineers with a working knowledge of the analog computer and its uses. They have proven to be most effective when lectures and demonstrations are supplemented with laboratory sessions allowing students to put theory into practice. The solution of problems on an analog computer, using effective and efficient programming techniques and check-out procedures, has proven to be invaluable in gaining familiarity both with the machine and its potential as an engineering tool.

Many of the procedures and techniques described in the notes have been used and found to be effective in EAI Computation Centers throughout the world.

A course in analog computation utilizing these notes could readily meet the requirements of an accredited, one semester, 3-credit-hour university course. A course in differential equations as a prerequisite is desirable.

The wide range of application for the analog computer permits the introduction of actual applications appropriate to courses in all scientific disciplines. The EAI Applications Reference Library is a source of a large number of such studies describing applications in such areas as electronics, chemical processing, aerospace engineering, and life sciences.

These notes represent the combined efforts of a large number of people within the EAI organization. Contribution to the notes and the editing were made by A. I. Katz, O. Serlin, H. Davidson, and J. J. Kennedy, as well as many others. Many sections in the notes were derived from material generated by the various Departments in the Research and Computation Division of EAI.

The editors, in particular, would like to acknowledge the efforts of the secretarial staff, Helen Lynch, Bette Davis, and Ginny Gafgen in helping organize the typing and production of these notes on a time schedule that was agreed upon by all as being impossible.

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CHAPTER I

THE ANALOG COMPUTER AND ITS ROLE IN ENGINEERING ANALYSIS

A. Introduction

The role of the electronic, general purpose analog computer in modern-day industry best can be explained by considering the concept of engineering design. When a design is required, one or more engineers or scientists propose a system which they feel will satisfy the design criteria. Design proposals, however, involve approximations and estimates and there may not be concrete agreement as to which design is best. Therefore, some form of evaluation of the proposed system is desirable.

In evaluating proposed systems or designs, one can, in general, select either of two paths: an experimental program, or an analytical evaluation of the system. The experimental approach is usually characterized by a minimum of of analysis, the construction of a prototype of the system, and considerable "trial-and-error" experimental work. The objectives are to evaluate the experimental data and suggest appropriate modifications which will result eventually in an optimum or nearly optimum design. The cost and time required for this experimental approach are normally much greater than those incurred in an analytical evaluation. In the analytical approach, the task is to derive a set of equations (a mathematical model) whose solution will describe the behavior of the system in terms of its geometry, time, and parameters. These solutions then can be used to obtain operating conditions and parameters which will result in optimum system performance.

Since the derivation of mathematical models frequently requires approximations, and the results obtained are often based on limited input data, prototype experimentation usually is required. However, pilot plants designed on the basis of extensive analytical investigations frequently are near optimum and require little or no modification. The only experimental results required are those which validate the mathematical model. Once the model is validated, additional experimentation can be performed analytically, which results in a considerable cost reduction compared to the experimental approach.

Not all proposed designs, unfortunately, lend themselves to a choice of evaluation programs. If one cannot obtain a mathematical model, there is no recourse except an experimental program. On the other hand, if the cost of a prototype is prohibitive (e.g. a nuclear reactor), its design is restricted to analysis. The major considerations in selecting the proper evaluation path are a compromise between cost, time, and objectives.

B. Mathematical Models

A system is best described analytically in terms of the causal relationship between its component parts, such as one would find on a detailed block diagram of the system.

The analyst then can derive equations for each subsystem, and the set of equations is the mathematical model for the entire system. The individual equations are derived from basic mathematics and physical laws such as the conservation of energy, matter, etc. and, at times, from empirical and semi-empirical equations such as fluid film resistance in heat transfer. The mathematical model can be a collection of integral or algebraic equations, although differential equations are most frequently obtained.

Typical examples of equations encountered in practical applications are:

1) Algebraic and Transcendental Equations, e.g. the effect of temperature on physical properties of materials. Thus

   k = thermal conductivity of a metal = $k_o + aT$

   C_p = specific heat of a gas = $a + bT + cT^2$

   μ = viscosity of a fluid = $\mu_o e^{-A T}$

2) Ordinary Differential Equations, e.g. the kinetics of a chemical reaction

$$A \xrightarrow{k_1} 2B \xrightarrow{k_2} X$$

whose mathematical model is

$$\frac{dA}{dt} = - k_1 A$$

and

$$\frac{dB}{dt} = 2k_1 A - k_2 B^2$$

3) Partial Differential Equations, e.g. a concentric pipe, heat exchanger (Figure I-1)

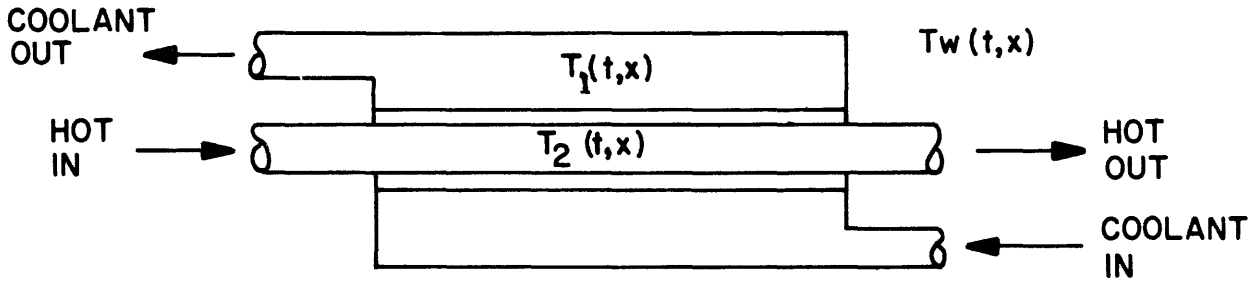


Figure I-1. Simple Heat Exchanger

whose mathematical model is

$$\frac{\partial T_1}{\partial t} + V_1 \frac{\partial T_1}{\partial x} + a_1(T_1 - T_w) = 0$$

$$\frac{\partial T_w}{\partial t} = b_2(T_2 - T_w) - b_1(T_w - T_1)$$

$$\frac{\partial T_2}{\partial t} + V_2 \frac{\partial T_2}{\partial x} + a_2 (T_2 - T_w) = 0$$

where

$T_w(t,x)$ = wall temperature

$T_1(t,x)$ = coolant temperature

$T_2(t,x)$ = primary (hot) fluid temperature

Two types of models, linear or nonlinear, are possible and are a measure of the complexity of the system. Simple linear models are "nicer" since they lend themselves to rapid analytical solutions. Unfortunately, because of the interaction of physical laws, the need for semi-empirical or empirical equations to describe this interaction, and the nature of most physical systems themselves, the majority of the mathematical models encountered in practice are nonlinear. This is unfortunate because little is known about the analytical solutions to nonlinear equations, and those solutions that are obtained are usually difficult to interpret and evaluate. If a system is nonlinear, its behavior is a function of its initial conditions, which makes its analysis even more essential if optimum performance is desired.

C. Solving Mathematical Models

Solutions of mathematical models can be obtained analytically by classical methods, numerical methods, or by electronic computation.

Classical solutions of simple models are possible if the model is composed of ordinary linear and/or partial differential equations and certain classes of non-linear differential equations. Frequently, this technique can be applied to limiting cases of complex models if approximations are acceptable. Analytical solutions for nonlinear models are rare, and, hence, variable substitutions are made to linearize the model as required. Depending upon the model and the results required of a study, phase-plane techniques may be applicable. Unfortunately, as was previously mentioned, linear systems seldom arise in practice and classical solutions are usually reserved for limiting cases and linearized approximations.

Numerical solutions involve the transformation of a mathematical model into a set of algebraic equations by replacing all derivatives in the model with appropriate algebraic, finite difference approximations. The resultant set of algebraic equations is then solved simultaneously to affect a solution. This technique not only is time consuming but may suffer from accuracy, stability and convergence problems.

To illustrate the classical and numerical solutions for a differential equation, consider the problem of a solvent tank (Figure I-2) which can be filled by two feed streams (Q_1 and Q_2) in 4 and 5 hours respectively. Two drain pipes, D_1 and D_2,

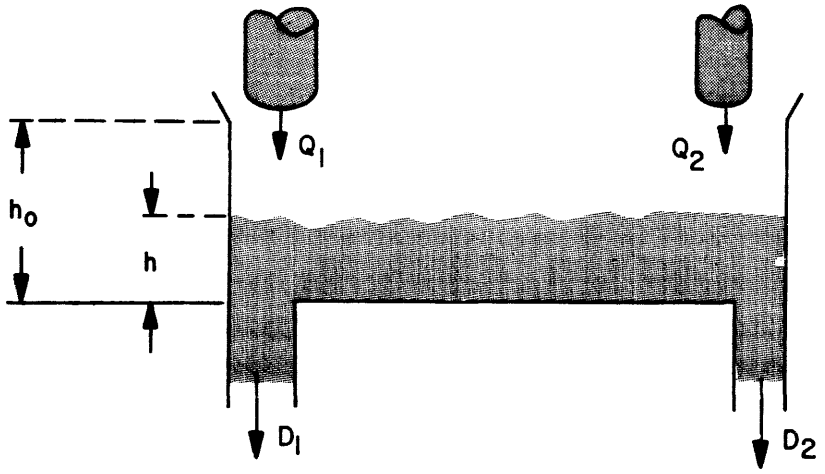


Figure I-2. Solvent Tank System

can empty the tank in 3and 6 hours respectively. If the tank is half full and all feed and effluent streams are used will the tank fill, empty, or reach steady-state? How long will it take?

The mathematical model for the tank is the nonlinear differential equation

$$\frac{dy}{dt} = 0.45 - y^{\frac{1}{2}} \tag{1}$$

where

$$y = h/h_o \tag{2}$$

The analytical solution of this equation, which can easily be obtained by consulting a table of integrals, is,

$$\int_{y_o=\frac{1}{2}}^{y} \frac{dy}{0.45-y^{\frac{1}{2}}} = \int_{\sqrt{y_o}=\sqrt{\frac{1}{2}}}^{\sqrt{y}} \frac{2\mu d\mu}{0.45-\mu} = t \qquad (3)$$

or,

$$0.9 \ln \left| \frac{0.45 - \sqrt{\frac{1}{2}}}{0.45 - \sqrt{y}} \right| + 2\left(\sqrt{\frac{1}{2}} - \sqrt{4y}\right) = t \qquad (4)$$

The obvious difficulty in applying this equation is that y, the level in the tank, does not appear as an explicit function of time, t. Even though we have an analytical solution, considerable effort is still required to produce a useful relation between y and t, say, in the form of a graph. The eventual height of the solvent in the tank will be the steady-state solution, y_s, of equation (1) (obtainedby letting dy/dt equal zero)

$$y_s = (0.45)^2 = 0.203 \qquad (5)$$

The time required to reach this height in theory is infinite; therefore, a practical value of the steady state time must be obtained graphically from a plot of y versus t.

Since the time required to attain the equilibrium height also can be obtained from a numerical solution of equation (1), let us now consider this method of solution.

Integrating equation (1), one obtains

$$y = 0.45\ t - \int_o^t y^{\frac{1}{2}}\, dt + y_o \qquad (6)$$

where the initial value of y, y_o is 0.500. Recalling that integration is the area under a curve, Figure 1-3, equation (6) can be rewritten in terms of finite or discrete intervals of time:

$$y_n = y_o + 0.45\ n\Delta\ t\ - \sum_{n=1}^{n=\infty} y_{n-1}^{\frac{1}{2}}\ \Delta t \qquad (7)$$

where

$$t = n\ \Delta\ t \qquad (8)$$

The accuracy of the solution obtained from this equation depends on the magnitude of the time interval (accuracy increases as Δ t decreases).

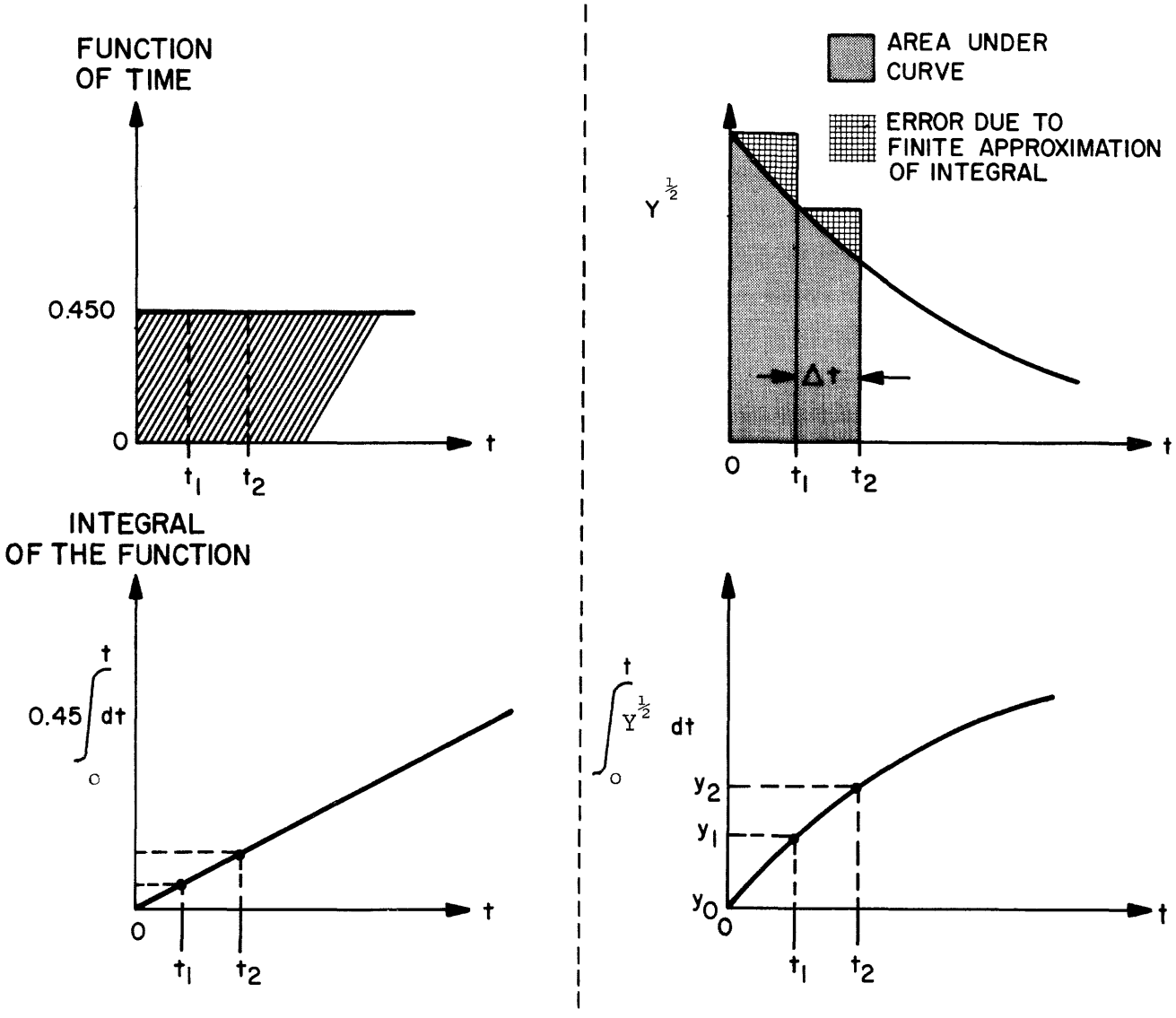


Figure I-3 Illustrations of Numerical Integration

Equation (7) is solved in the following manner, after Δt has been selected, at say 0.5 hours.

1) Compute $y_{n-1}^{\frac{1}{2}}$ (y_o is known and is used as a starting point)

2) Compute $y_{n-1}^{\frac{1}{2}} \quad \Delta t$

3) Compute $\sum_{n=1} \left(y_{n-1}^{\frac{1}{2}} \right) \Delta t$

4) Compute $0.45\ n\Delta t$

5) Compute y_n

6) Let $n = n + 1$ and return to step 1.

Results obtained from both the numerical and analytical solutions are shown in Figure I-4.

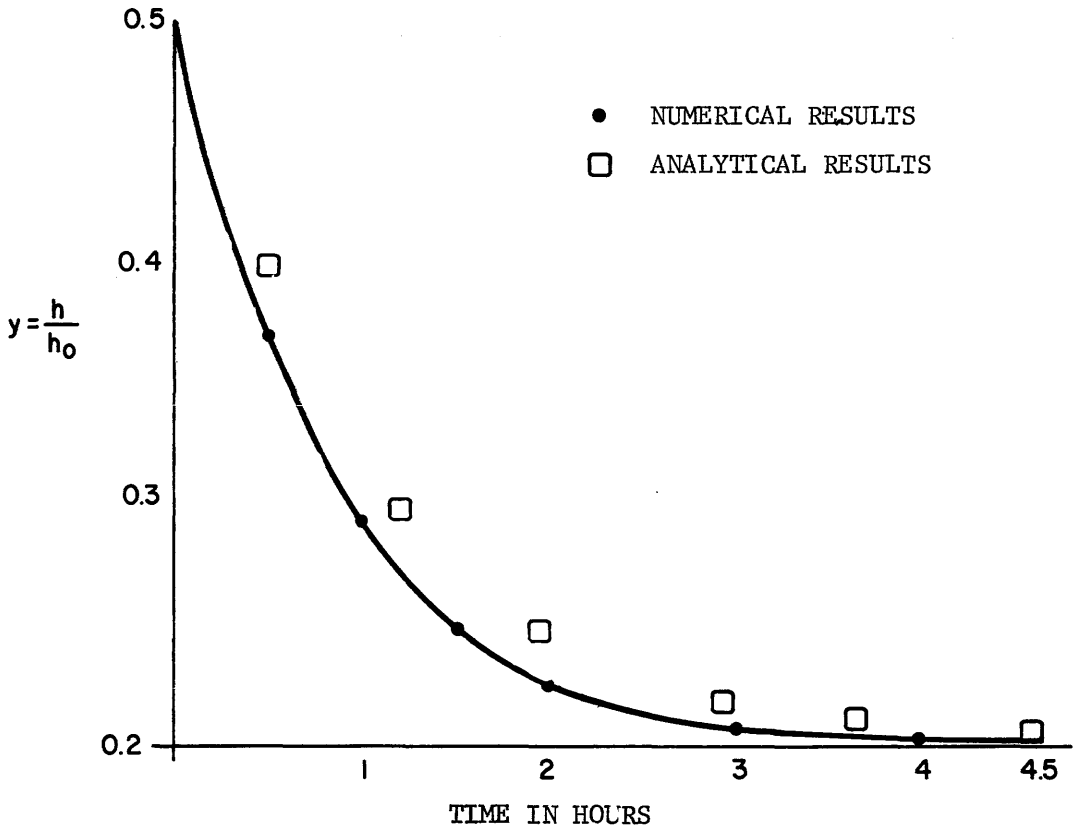


Figure I-4 Numerical Solution of Solvent Tank Problem

From the curve shown in Figure I-4, it is apparent that the tank (Figure I-2) will fill and reach equilibrium in approximately 4.5 hours. It should be noted that an increase in the accuracy of the numerical solution would have required additional computations and, hence, increased computation time. The same procedure would have been followed for a smaller increment of time, Δt. The error of the numerical solution is indicated by comparison to analytical results obtained from equation (4). Computer solutions are best understood after an explanation of the type and methods of computer operation is presented. However, it is convenient to point out at this time that the numerical solution illustrated above is typical of digital computer solutions and the itemized instructions are typical of a digital computer flow chart.

If one considers a flow diagram for the solution of equation (6), which is shown in Figure I-5, insight to the analog computer solution can be obtained. It will be shown later that the analog computer is composed of components which perform the mathematical operations described in Figure I-5.

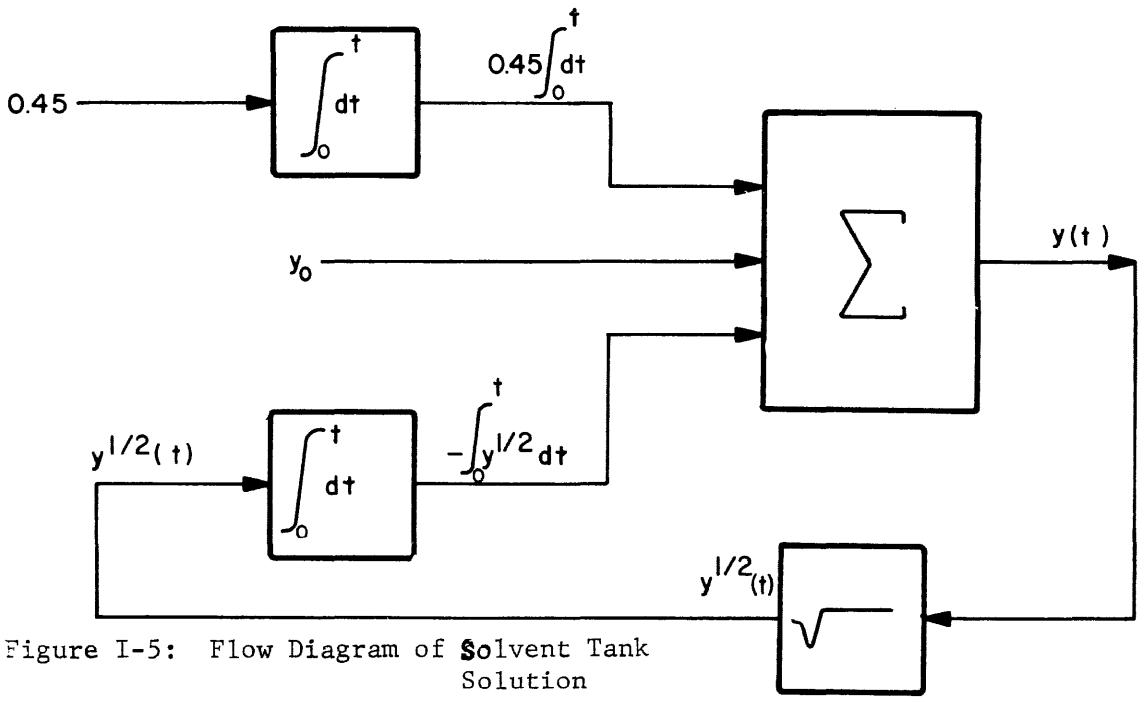


Figure I-5: Flow Diagram of Solvent Tank Solution

At this point, the justification for using computers can be considered.

In our modern society, machinery of various sorts has relieved human muscle from a great deal of routine and repetitive operation. In doing so, it has multiplied the effectiveness of that human muscle both in industry and in the home.

The computer has performed a similar service for the mind essentially by mechanizing routine mental processes, leaving the mind free to examine new problem areas. Studies of the behavior of entire complex systems can be performed with great speed and, consequently, our actual knowledge of complex systems has increased greatly. Equally important, our capacity for control and prediction, and for insight into these complex systems also has been extended. The influence of the computer on our common life, therefore, lies in its contribution, in the broadest sense, to science and technology.

Investigations in science and engineering can be carried out on a scale unheard of only one or two decades ago. Scientific principles and models can be verified against experimental facts at small cost, without hazard and with

considerable flexibility. Thus, new areas of scientific knowledge have been established and will continue to grow as a result of research and development performed on computers.

D. Computer History and Characteristics

A computer is a device that is able to receive information (equations, instructions, data, etc.) and process it in a predetermined manner to obtain useable results.

For example, a human being may be a computer. He can take information in through his senses, use principles stored in his memory to process or perform operations on this information in many ways, and produce an answer, perhaps in the form of an action.

Similarly, a machine may be able to accept information of a suitable form, receive instructions on how to operate on this information, perform the required operations, and give the answers. Machines may take many forms varying from simple beads on a frame to the incredibly complex, expensive and highly sophisticated modern machines.

1. Early Computers---The history of computing devices may well extend to the very beginning of civilization. For our purposes, they can be divided into two categories (see Figure I-6):

- Mathematical instruments, the more complex of which are known as analog computers. These are exemplified in simple form by the slide rule.

- Calculating machines, more often known as digital computers. These can be represented simply by the desk calculator.

Early forms of digital computations could be considered to exist when man first started to use his fingers or pebbles for counting.

The earliest known record of analog computation is its use in surveying and map making for the purpose of taxation (Babylonia, 3800 BC). The earliest digital machine is probably the Abacus. In its early form, it consisted of a clay board with grooves in which pebbles were placed. It later appeared in the form of a wire frame with beads. It is still used extensively in Asia and the East for remarkably rapid calculations.

The development of computational aids can be traced from these early instruments through the invention of logarithms, slide rules, linkages, analytical engines, and desk calculators to the large-scale general-purpose machines of the present day.

The first large-scale general-purpose digital computer was completed at Harvard in 1944. This machine, the Harvard Mark I Calculator, was built jointly by IBM and Harvard, and used electromechanical relays. The Moore School of Engineering also completed its all-electronic digital computer for the Aberdeen

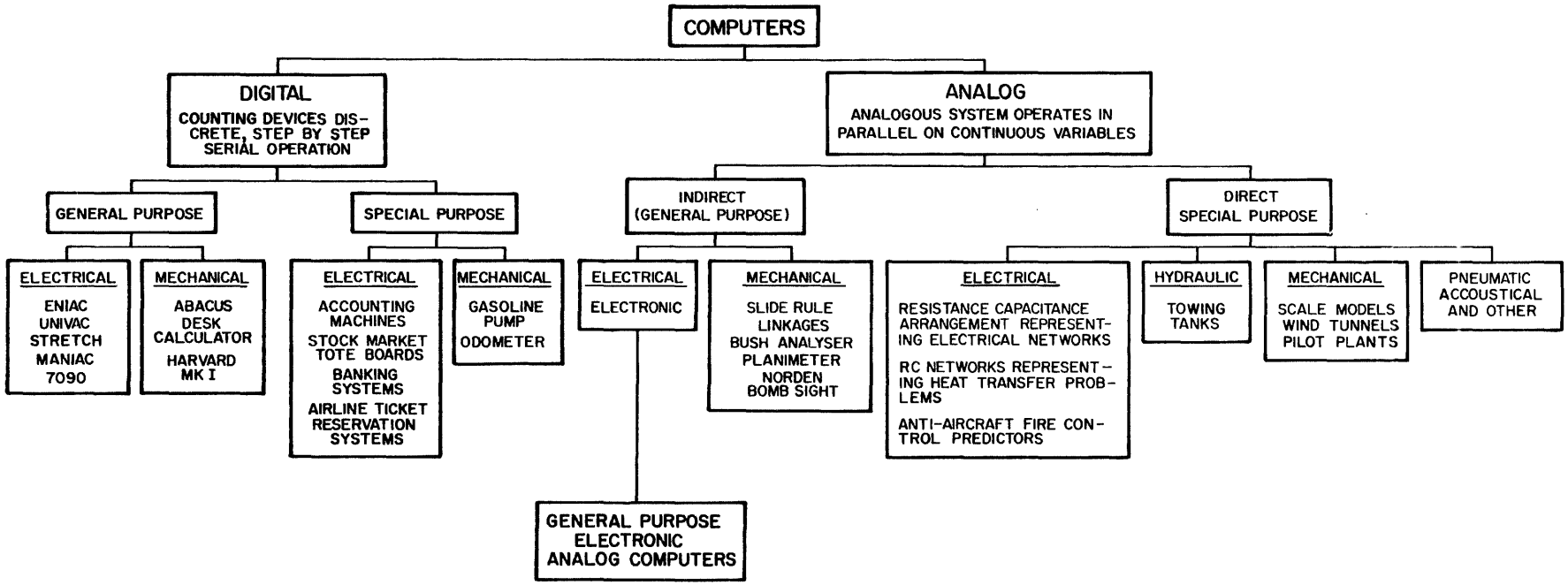


Figure I-6 Computer Devices

Proving Grounds in 1944. This machine, the ENIAC, which contained 18000 vacuum tubes, now has many direct descendents.

Mechanical integrating devices of the late 19th century were improved on during World War I, when Hannibal Ford increased the torque output of the ball-and-disc integrator and used it to make a naval gun fire computer. This was followed by more experimentation in the 1920's.

At M.I.T., Dr. Vannevar Bush completed the first large-scale mechanical differential analyzer in 1931. This machine is now installed at Wayne University in Detroit where it is still being used effectively. At the present time, there are several large scale mechanical machines in operation. Simultaneous equation solvers and harmonic analyzers of many types also appeared in the 1930's.

Special computers, in the form of network analyzers for the simulation of power networks, appeared around 1925. The network analyzer is a passive element analog. A scale model of the particular network to be studied is made with resistors, capacitors, etc. The early network analyzers could be used to investigate only steady state problems; that is, voltage drops along lines, possible current flow in lines, etc. The most recent network analyzers can be used to investigate transient conditions during faults on networks or switching on networks. These may be considered to be true general-purpose computers.

2. Analog and Digital Computers---In digital computers, numbers are operated upon directly. The basic operation in these machines is counting. This enables the machine to perform the four fundamental operations of arithmetic, addition, subtraction, multiplication and division. The basic operation of any digital computer is similar to that of the abacus where numbers are represented by beads and the counting of these beads is the basis of addition and subtraction. In digital computers, all mathematical calculations depend ultimately on counting, whether it be beads, gear teeth, or electrical pulses.

In analog machines, numbers are represented by physical quantities whose magnitude is determined by the magnitude of the number. Mathematical operations are represented by physical events; that is, the machines do not count, but perform continuous manipulations equivalent to the mathematical operation required. The result of these manipulations is another physical quantity, whose magnitude and behavior represents the solution to the problem.

Probably the most useful example of the analog computer is the slide rule. Here, to multiply one number by another, the discrete numbers are converted to logarithms, the logarithms are converted to linear distances on sticks which, when placed end to end (i.e. added together--the continuous operation in this case), give another length representing the product of the numbers.

There exists on the analog a complete analogy between the physical quantities, events, and the mathematical numbers and manipulations. If many events are taking place at the same time in the physical world, they will also take place at the same time, or in parallel, on the machine. An analog device will, consequently, arrive at a result in a shorter time than a digital machine which must perform all its operations serially.

The precision of a digital machine is theoretically boundless. To increase by ten the precision of a decimal counting device, it is a matter simply of accomodating one more place (decimal) throughout the equipment. However, to achieve the same end on the analog, e.g. the slide rule, the length of the slide rule would have to increase by a factor of ten. This is not always practicable.

Analog devices, are characterized by continuous operations performed in parallel, as opposed to digital machines which are discrete, serial devices. The analog solutions are obtained in a continuous manner since all parts of these devices operate simultaneously.

3. General Purpose Analog Computers---We have said that in analog devices numbers are represented by physical quantities. Theoretically, any physical quantity may be used as long as it can be made to obey those laws necessary to represent the mathematical relationships involved in the original problem. Purely electrical relationships, which have the mechanical advantage of no moving parts and, a high speed of operation, have been found most suitable for analog devices.

The introduction of the operational amplifier made possible the newest class of general purpose analog computers using voltages as the 'physical quantity'.

Lovell of Bell Telephone Laboratories is generally credited with the introduction of the operational amplifier during the Second World War. These amplifiers can be divided into two groups, those which operate on a-c voltages and those which operate on d-c voltages. The a-c amplifiers exhibit certain difficulties and do not lend themselves to any direct form of integration. Therefore, only d-c amplifiers are considered in these notes since they are most common in commercially available general purpose analog computers.

E. Industrial Uses of the Analog Computer

As a result of the tremendous competition in industry following World War II, more economical designs and more thorough evaluations were needed. The concept of a fully automated plant, or system, operating at an economic optimum, demanded from the engineer a more extensive knowledge of each element, and its behavior. The engineers, in turn, demanded a more complete analysis of mechanisms and transport properties from the basic research scientists.

It is important, at this point, to state the range of the computer's usefulness, and to delineate those areas where it is not suited.

1. Initial Research and Development---The initial work in development usually takes place in the laboratory where bench-scale studies, thermodynamic calculations of feasibility and other preliminary calculations are made. Because this stage of the work is so intimately concerned with mechanisms, most of which are dynamic in nature, the use of the computer is particularly advantageous. Consider, for example, the determination of chemical reaction velocity constants. A series of isothermal batch reactions may be run and data collected on the compositions of the various components as functions of time. A kinetic model then is assumed, i.e., the orders of the various reactions are estimated, and programmed on the computer.

The problem is one of matching the results of the computer with the data from the test runs. Different reaction velocity constants can be tried or different models assumed until a good match is obtained. In this way, a reliable model of the isothermal chemical kinetics is quickly obtained.

The laboratory work then may be extended to include temperature changes and, possibly, other types of reactors. The computer is used in each step to simulate the mechanisms, check the model and the assumptions, and obtain system parameters for design purposes.

2. Intermediate Development---The use of the analog computer in the prototype stage of development represents a powerful tool for improving the overall efficiency of the development procedure. By combining the philosophy of model building with the philosophy of simulation, a complete study of a component or system can be obtained. The conditions of optimum operation can be determined and quickly evaluated over a wider range of variables than is often possible with the hardware or plant itself.

Consider, for example, a development program in which a pilot plant is simulated with an analog computer. In order to achieve a meaningful simulation, certain basic facts must be known, and these are found from preliminary pilot plant or bench-scale data. Certain heat-transfer coefficients or diffusion constants might, for example, be determined from specific tests in the pilot unit. The simulation then is checked against normal operating data obtained from the plant on the computer where "runs" can be made in a more economical fashion.

Three areas of study thus are defined. In the initial phase of investigation it can be seen that the knowledge gained is, perhaps, not immediately useful as design data. The second phase is equivalent to normal operation, except that the computer is added and the model obtained. Finally, the parallel operation of computer and pilot plant, or prototype, results in a greater amount of information at a substantial decrease in cost and time, since the simulated plant runs faster than the actual plant and does not require any raw materials.

3. Final Development---After pilot plant work is completed, final design calculations are undertaken. Most design calculations are based upon a steady-state type of operation and, hence, are primarily algebraic equations. In many such cases--for example, multicomponent distillation calculations, heat exchanger sizes and capacities, vessel specifications, structural rigidity, etc.--complete digital computer programs have been worked out. In such cases, even though the analog is capable of solution, it is obvious that the digital computer should be used if available.

One of the areas in which the analog computer is particularly applicable is the choice of process instrumentation and control. The large amount of work that has taken place in control engineering recently, in fact, is an excellent illustration of the fact that, while design may be steady-state, the operation of a process is always dynamic. With the analog computer model of a process, the interrelation of its various unit operations can be examined easily, suitable control systems can be tested, and the proper settings on controllers determined. In most cases, the control system itself is simulated; in others, the controllers to be used in the plant are connected directly to the computer. The use of the analog computer in such applications is expanding rapidly, and is one of the primary means the process engineer has available to improve process efficiency.

The last step in the development of a process is start-up--often a difficult and expensive task. If a computer model has been determined, the proper values of the flow rates and other variables can be tested under different start-up conditions, and the optimum ones selected. In many cases, an appropriate start-up procedure for a plant can be determined long before the unit is ready for operation.

4. Post Development Work---After a plant is running satisfactorily, the computer model can be adjusted to match the particular idiosyncrasies of the unit.

Further experimentation is then possible with the computer. As with the pilot plant, the real plant can be tested for different optimum conditions. The economics of the operation can be investigated under changing values of the products. The range of operation can be extended to determine some of the safety precautions to be observed in the plant. Finally, the computer model can be tested for use of the equipment with different materials, reactions, etc., in the event that a changeover ever became necessary.

It is interesting to note that these suggested areas of application are not aimed at replacing with the analog computer important procedures in the standard process development program. Rather, they supplement the ways and means by which decisions can be made. Thus, in the pilot plant simulation, it was necessary to retain the pilot plant as a check on the simulation, but the simulation could be extrapolated outside the range of the actual plant capabilities.

The general program involving the analog computer in the process development scheme is characterized by the high rate of information exchange between experiment and simulation. Such a program shows a great improvement over the usual procedures because at no time does it become necessary for the development program to become "boxed in" by previous studies. The computer provides

an economical means for the complete evaluation of the investigation, since these studies are brought into focus, gaps in data are filled and predictions of major importance are obtainable.

5. Inappropriate Areas for the Analog Computer---As mentioned previously, steady-state algebraic equations, can be and have been solved on the analog computer. As a general rule, however, large scale algebraic problems are better solved with a digital computer.

In general, problems involving high accuracy are not suitable for the analog computer. Some perturbation schemes have been developed for handling problems up to five and six places but only certain problems can be solved in this way. Under ordinary circumstances, the computer is accurate to about 0.1% for small simulations, and, depending on the type of problem, may range from 0.5% to 1.0% or more for very large simulations. Most engineering data, however. are not that accurate. For example, heat transfer coefficients, and reaction velocity constants and modules of elasticity, are usually in the 5% to 20% range of accuracy. Consequently, the computation accuracy is usually not a problem.

Specific Illustrations of Analog Computer Applications

The above discussion has been of a general nature. However, a selected bibliography of computer applications , categorized by specific industrial areas, is presented in Appendix D.

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CHAPTER II

THE GENERAL PURPOSE ANALOG COMPUTER

A. Introduction

Analog computers have been constructed in a number of forms which, by definition, appeal to the similarity between the laws of nature. For example, consider the analogy between mechanical, electrical and thermal systems:

$$F = \frac{W}{g}\frac{dv}{dt} = \quad \text{Force Acting on a Mass}$$

$$i = C\frac{de}{dt} = \quad \text{Current Flow Through a Capacitor}$$

$$Q = WC\frac{dT}{dt} = \quad \text{Heat Flow into a Solid}$$

The similarity in mathematical form among these expressions, with the addition of suitable scale factors, allows the heat flow into a solid to be investigated using an electrical circuit or mechanical system.

This similarity permits the translation of a problem in a given physical system...a problem for which computations would be difficult, a system for which test models would be expensive and inflexible..into another physical system where relatively cheap models with easily varied parameters can be constructed. The physical forms that have been used for models include mechanical, hydraulic, electrostatic, etc. By far the most useful and versatile, however, is the electrical system.

When one uses an electrical system, it is usual for voltages to represent the physical variables. Variation of these voltages with time, under the influence of forcing functions, corresponds directly, through scaling factors, to the variation with time of the original problem variables under the action of the original problem forces. Models using electrical elements can take a number of forms. However, for our present purposes, we shall restrict our attention to just one of these forms, namely, the "Electronic Differential Analyzer". It is this kind of physical model that normally is implied when talking of the general-purpose analog computer (GPAC).

The general-purpose analog computer is an assembly of electronic and electromechanical components which, individually using d-c voltages as variables, can perform specific mathematical operations. The equations most suitable for solution on such a computer are ordinary differential equations containing one independent variable which is represented by time in the computer. Partial

differential equations also can be solved by the analog computer using somewhat more advanced techniques than those required for the solution of ordinary differential equations.

Before discussing programming of the analog computer for the solution of physical problems, it is essential, for effective use of the computer, that a familiarity be gained with:

1) the principle of operation, and the capability of individual computer components

2) the theory of operation and control of the computer itself

and

3) the available computational accessories and readout devices.

The presentation of this material is the purpose of this chapter.

B. Differences between Analog Computers

From Figures II-1, II-2, and II-3, it is obvious that analog computers differ in physical appearance. The basic differences, however, lie much deeper. Computers differ also in:

1) Capacity...the number of computing components

2) Capability...the quality of computing components and the operations they perform

3) Reference Voltage Level...the operating voltage range of the computer:

   $\pm$ 10 or $\pm$ 100 volts is typical

4) Convenience Factors...operator control, the accessibility of equipment, and others, some of which are less meaningful.

Since the basic principles of operation of all d-c general purpose analog computers are similar, the material presented in this chapter is applicable to any GPAC computing system.

Modern analog computers are equipped with removable patch panels which contain the input, output and control terminations of the various analog components in a computer system. The input and output terminations of the components are connected in a particular configuration which is defined by the problem being solved. The control termination connections depend upon the mathematical operation required of a particular component, and the manufacturers method of implementing the operating principles of that component . The term "patching" refers to the inter-connection of patch panel termination.

Specific patching information for a particular computing system can be obtained from the manufacturer's reference handbook for the computer.

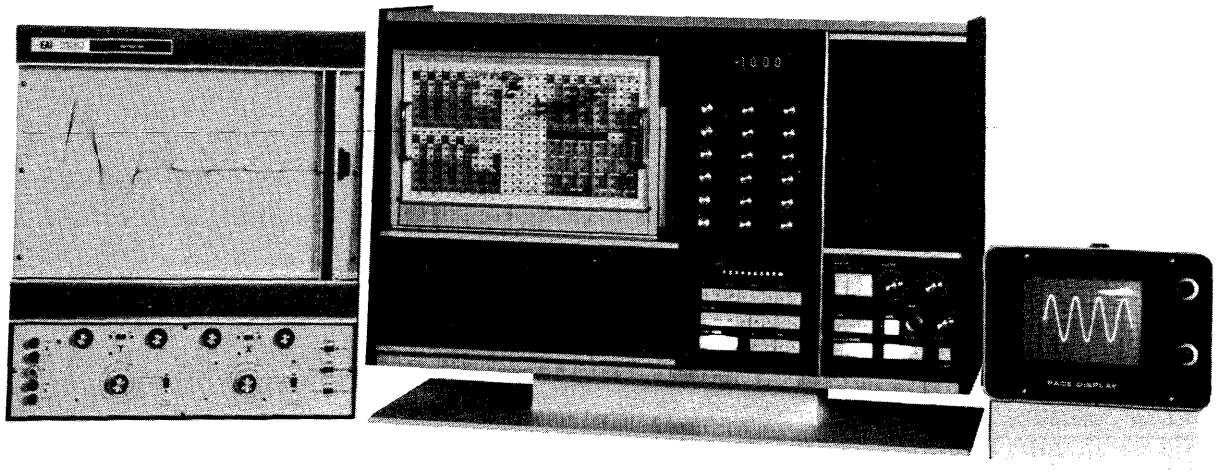


Figure II-1: Typical MiniAC Computing System

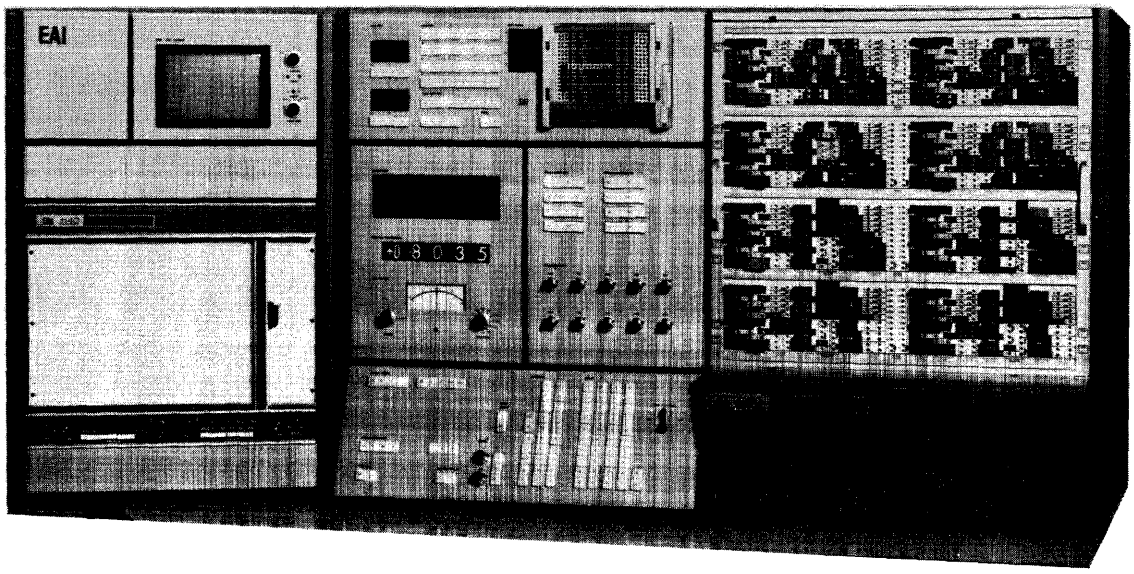


Figure II-2: 580 Analog/Hybrid Computing System

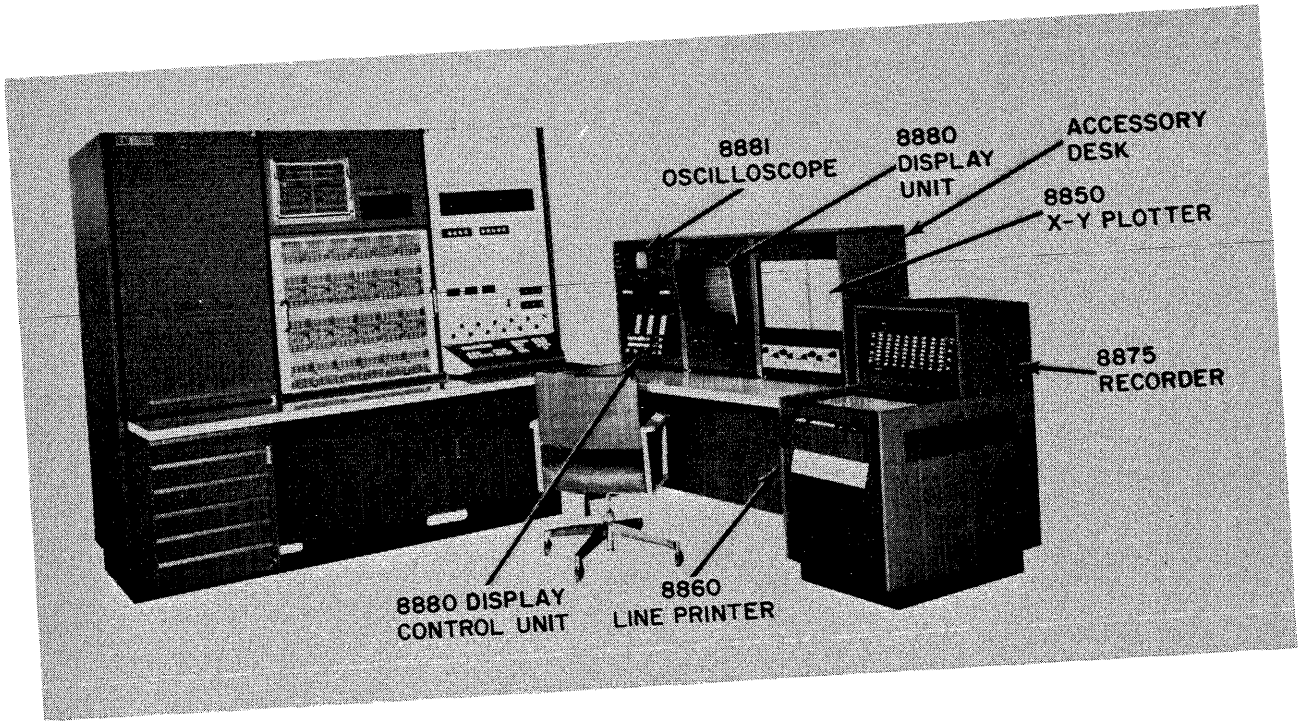


Figure II-3: Typical 7800 Scientific Computing System

Recommended programming symbols for the various components described in this chapter are summarized on Pages 30-31 of this volume. These symbols may require slight modification to indicate specific component interconnections for a particular computing system.

C. Classification of Analog Components

Analog computer components, each of which performs a specific mathematical operation, are classified either as linear or nonlinear components. The linear components perform the mathematical operations of

1) multiplication by a constant

2) inversion

3) algebraic summation

4) continuous integration.

These operations are sufficient to solve linear differential equations with constant coefficients.

The mathematical operations performed by nonlinear components are

1) multiplication and division of variables

2) the generation of arbitrary functions

3) the mechanization of constraints and elementary logic operations.

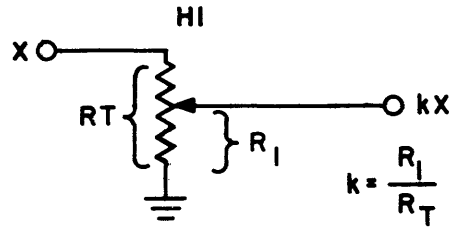
These components, together with the linear components, permit the analog computer to simulate the complex nonlinear systems which occur in practice.

1. LINEAR COMPONENTS

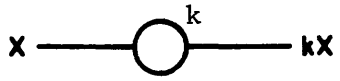
a. Attenuators---Multiplication of a d-c voltage by a positive constant which is less than unity is accomplished by a potentiometer or pot, also called an attenuator. This device, shown in Figure II-4, is simply a fixed resistor with a movable wiper arm. Carbon or wire wound resistances which, for greater accuracy, have multiturn wiper arms are used in most computers.

Two types of pots, "grounded" and "ungrounded", are used in modern analog computers. These names are derived from the termination at the bottom, or "LO" end, of the pot, as shown in the figure. The total resistance of a pot is of the order of 2,000 to 30,000 ohms and depends on the design of the computer.

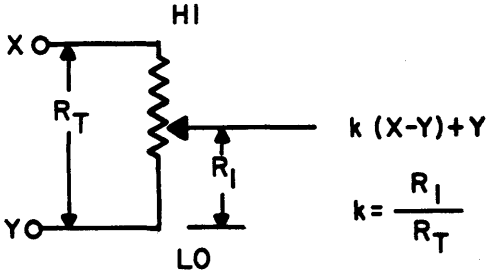
Grounded potentiometers are used in conjunction with a reference voltage (a constant voltage source equal to the upper limit of the computer's operating voltage range) to obtain a fixed voltage less than reference voltage, or to multiply a problem variable by a constant less than unity. The input to the potentiometer is applied



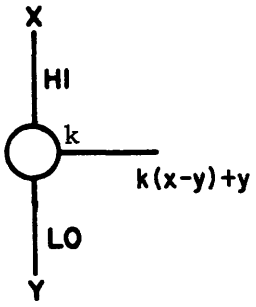
GROUNDED POT SCHEMATIC



GROUNDED POT
PROGRAMMER SYMBOL



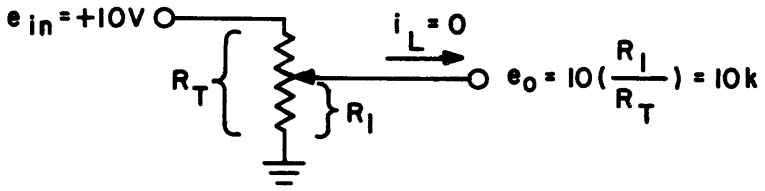
UNGROUNDED POT SCHEMATIC



UNGROUNDED POT
PROGRAMMER SYMBOL

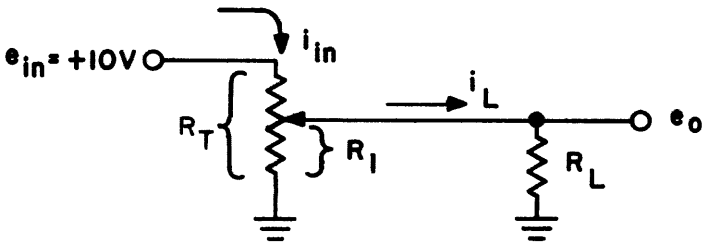
Figure II-4: Potentiometer Schematic Diagram and Programming Symbols

Unloaded



If R_T = 5KΩ and R_1 = 4KΩ, then k = 0.800
and e_o = 8.00 volts.

Loaded



If R_T = 5KΩ, R_1 = 4KΩ, and R_L = 10KΩ, then

$$k = \frac{(R_1/R_T)}{1 + (R_1/R_L)\ (1 - R_1/R_T)}$$

and e_o = +7.41 volts. Thus, $\frac{e_o}{e_{in}} \neq \frac{R_1}{R_T}$.

Figure II-5: Potentiometer Loading

at its top or "HI" end, and the resultant output is obtained through the wiper arm.

Figure II-4 shows programmer symbols for both grounded and ungrounded pots.

The ungrounded pot has special applications in addition to the attenuation of two variables, indicated in Figure II-4, which will be discussed in later chapters.

Normally, an analog computer will contain one and one-half as many pots as it has amplifiers, and 80% of these will usually be grounded.

i. Loading and Setting of Attenuators---the potentiometers shown in Figure II-4 are "unloaded" which means that no current is being drawn through the wiper arm (i.e. they are feeding an infinite resistance-open circuit). Therefore, the mechanical ratio, R_1/R_T, which can be set by a calibrated dial, is equal to its electrical ratio, e_o/e_{in}. However, this is not the case when the infinite load is replaced by a finite load as shown in Figure II-5.

In practice, the wiper arm of a pot will be "looking into" a load ranging from 10^3 to 10^6 ohms since a potentiometer generally feeds resistor inputs to operational amplifiers. The effect of a 10K* resistive load on a 5K pot set at 4/5 is shown in Figure II-5.

In order to eliminate the effects of loading, potentiometers are set by monitoring the wiper voltage while the pot is "feeding" its normal load. In this way, it is possible to set potentiometers to three or four places depending upon the precision of the monitoring device.

In most computers, each potentiometer has switching associated with it similar to that shown in Figure II-6, below:

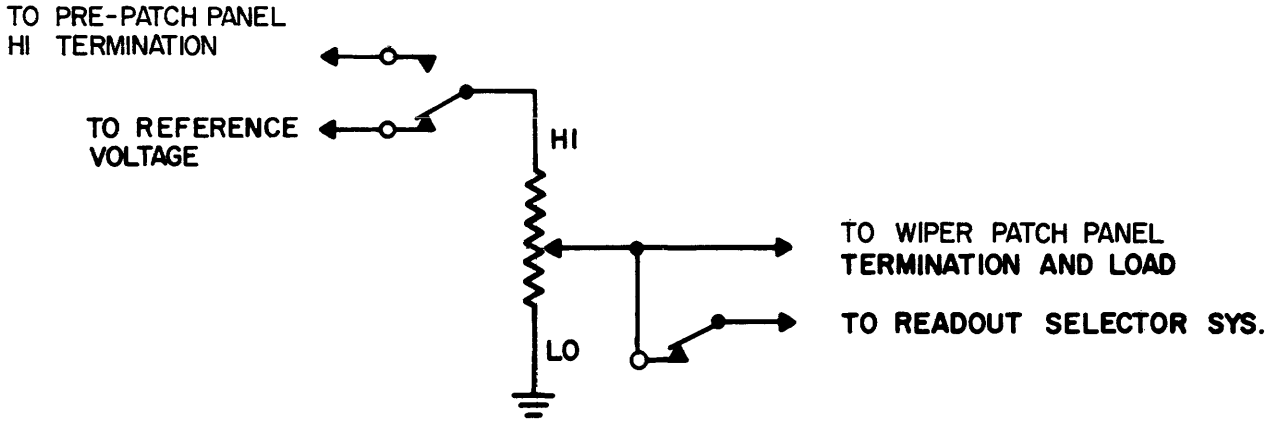


Figure II-6

*In these Notes, the following notation is used:

$K = 10^3$, $M = 10^6$, $m = 10^{-3}$ and $\mu = 10^{-6}$

When the switch is thrown, the patched input to the pot is replaced by a reference voltage, and the loaded wiper arm is connected to a monitoring device via a readout selector system. The readout device can be either a high impedance, digital voltmeter (DVM) or a null meter.

The more sophisticated analog computer systems have digitally-set pots. Here, the potentiometer is selected through a pushbutton system, and then is set by a servo device also controlled by push buttons.

Operational Amplifiers---The operational amplifier is the basic unit in the analog computer. It can be used in a "summing mode" to perform any or all of the three linear operations: inversion, summation, and multiplication by a constant. It also can be used in an "integrating mode" to integrate a voltage or the sum of a number of voltages with respect to time.

Analog computer programs for investigating the behavior of physical systems require some operational amplifiers to be used as integrators, while others are used as "summers," "inverters," "high gain amplifiers," or in conjunction with special networks to perform nonlinear operations. Therefore, it is not necessary for all of the amplifiers to perform as integrators. In modern analog computers, a typical amplifier breakdown would be:

1) combination amplifiers capable of performing integration, summation or inversion...30%

2) summing amplifiers capable of performing summation and inversion...45%

3) inverting amplifiers, capable of performing inversion only...25%

i. Inversion and Multiplication by a Constant...to understand the principle of the operational amplifier, consider the circuit*

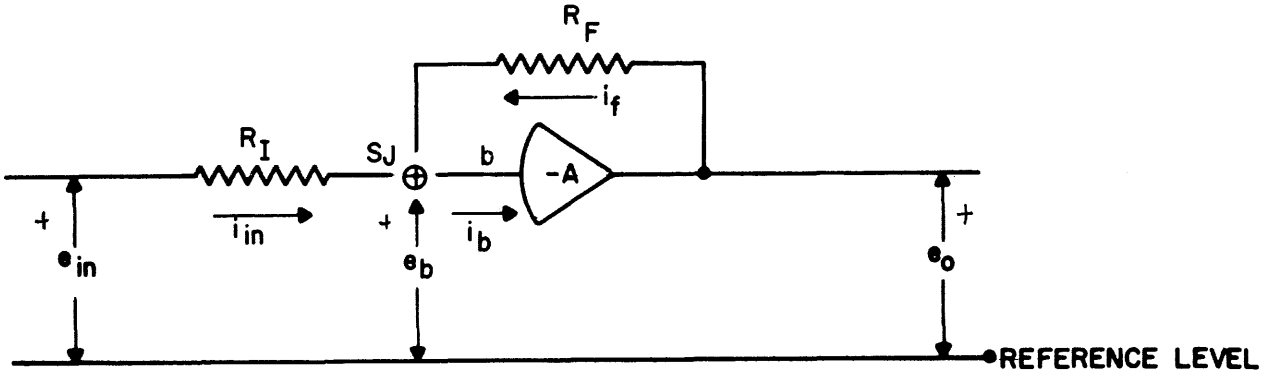


Figure II-7. Simple Amplifier Circuit

*In this circuit, the input, e_{in}, summing junction, e_b, and output voltages, e_o, are referred to a reference level, such as ground. However, in the interest of simplicity, future circuits will omit the reference level terminal and consider it to be grounded. The gain of the amplifier, -A, will also be omitted in future circuit diagrams.

where a high gain d-c amplifier (gain = -A) has a feedback resistor, R_F, and an input resistor, R_I (Figure II-7). The d-c amplifier is designed so that

1) the amplifier output, e_o, is related to the summing junction voltage e_b, by the gain of the amplifier (i.e.,

$$e_o = -A\, e_b$$

within the reference voltage range of the computer),

2) the amplifier draws negligible current, $i_b \simeq 10^{-9}$ amps, and

3) the gain of the amplifier is extremely high, usually on the order of 10^8 at d-c.

Using Kirchhoff's laws, the nodal current equation at the summing junction, SJ, is

$$i_b = i_{in} + i_f$$

or, from Ohm's law,

$$i_b = \frac{e_{in} - e_b}{R_I} + \frac{e_o - e_b}{R_F}$$

Since $i_b \simeq 0$, it can be neglected. Replacing e_b by $\frac{e_o}{-A}$ we obtain:

$$\frac{e_{in}}{R_I} + \frac{e_o}{AR_I} = -\frac{e_o}{AR_F} - \frac{e_o}{R_F}$$

$$e_o = \frac{-\frac{R_F}{R_I}\, e_{in}}{1 + \frac{1}{A}\left(\frac{R_F}{R_I} + 1\right)}$$

Since the ratio of R_F to R_I usually is less than thirty, and A is much greater than 1,

$$e_o = -\frac{R_F}{R_I}\, e_{in} \qquad (1)$$

From this equation we can see a most important characteristic of the operational amplifer: <u>The input-output relationship is solely dependent on the ratio of the feedback to the input impedances (resistances)</u>.

Using this equation as a basis for discussion, some of the various uses of the operational amplifier can be illustrated.

When both resistors are of equal magnitude, R, the amplifier output voltage has the same amplitude as the input voltage but is of the opposite polarity. Thus, the mathematical operation of inversion is performed:

$$e_o = - \frac{R_F}{R_I} e_{in} = \frac{-R}{R} e_{in} = - e_{in}$$

If the resistors are not of equal magnitude, the result is multiplication of the input by a constant. For example, if R_F were 1M and R_I were 100K,

$$e_o = - \frac{R_F}{R_I} e_{in} = - \frac{1M}{0.1M} e_{in} = - 10 e_{in}$$

or. if the resistance ratio is inverted,

$$e_o = - \frac{e_{in}}{10}$$

ii. Summation---the addition of two parallel input resistors to the previous circuit, yields

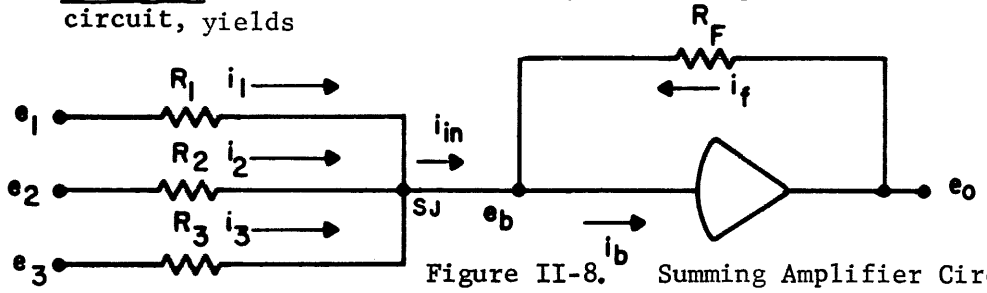


Figure II-8. Summing Amplifier Circuit

And, the SJ node equation,

$$i_1 + i_2 + i_3 + i_f - i_b = 0$$

Using Ohm's law, this equation becomes

$$\frac{e_1 - e_b}{R_1} + \frac{e_2 - e_b}{R_2} + \frac{e_3 - e_b}{R_3} + \frac{e_o - e_b}{R_f} - i_b = 0.$$

Since e_b and $i_b \simeq 0$, we have

$$e_o = - \left[\frac{R_F}{R_1} e_1 + \frac{R_F}{R_2} e_2 + \frac{R_F}{R_3} e_3 \right]. \qquad (2)$$

If the number of input resistors is increased to, say, N, the generalized summer equation becomes

$$e_o = - \left[\frac{R_F}{R_1} e_1 + \frac{R_F}{R_2} e_2 + \text{-----} + \frac{R_F}{R_N} e_N \right] \qquad (3)$$

iii. Integration---when the feedback resistor used in previous circuits is replaced by a capacitor, the amplifier circuit for a single input becomes (Figure II-9)

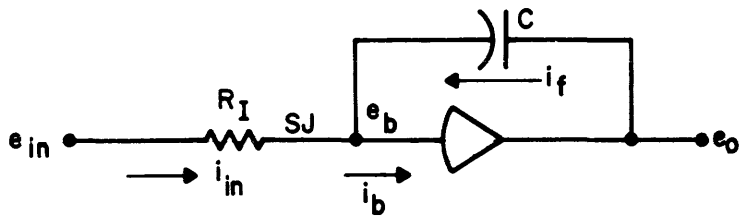


Figure II-9. Simple Integrator Circuit

The relations among capacitance, voltage drop, and current for a capacitor with no initial charge is:

$$e = \frac{1}{C} \int_o^t i \; dt$$

Thus, the voltage drop, $e_o - e_b$, across the feedback capacitor can be expressed as

$$e_o - e_b = \frac{1}{C} \int_o^t i_f \; dt$$

which can be differentiated to obtain an equation for i_f

$$i_f = C \frac{d}{dt} (e_o - e_b)$$

A current summation at SJ ($e_b \simeq 0$, $i_b \simeq 0$) is now

$$\frac{e_{in}}{R_I} + C \frac{de_o}{dt} = 0$$

whose solution is

$$e_o = - \frac{1}{R_I C} \int_o^t e_{in} \; dt. \qquad (4)$$

We now have a device which can perform the operation of integration (with respect to time) on an input voltage.

For multiple resistor inputs, the integrator output is described by the equation:

$$e_o = - \int_o^t \left[\frac{e_1}{R_1 C} + \frac{e_2}{R_2 C} + \text{-----} + \frac{e_N}{R_N C} \right] dt \qquad (5)$$

It should be noted that the amplifier output voltage in this instance is the integral of the algebraic sum of the input voltages.

iv. **Generalized** Amplifier Equations---if one defines the impedance, Z, of: **a passive element as**

$$Z = \frac{E}{I}$$

where E is the voltage drop across the element, and I is the current passing through it, the input-output expression for the generalized circuit (Figure II-10)is:

$$e_o = - \frac{Z_F}{Z_I} e_{in}.$$

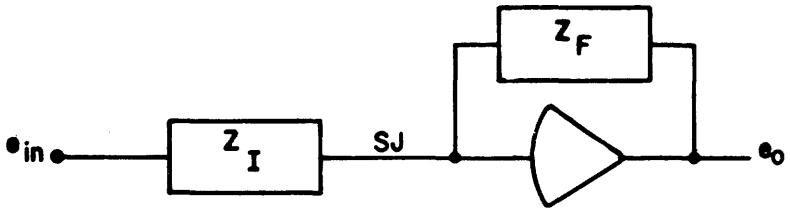


Figure II-10: Generalized Amplifier Circuit

For a multiple input amplifier circuit (Figure II-11),

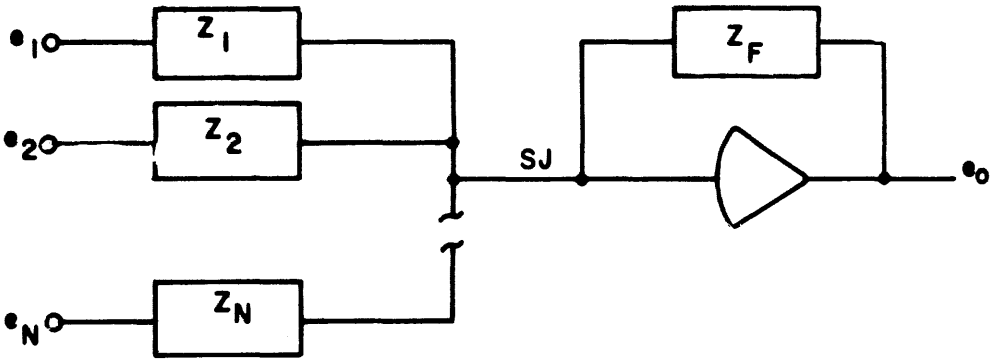


Figure II-11: Generalized Multiple Input Amplifier Circuit.

*For simplicity further reference to $e_i(t)$, $Z_i(t)$, etc. will be considered as functions of time and will be noted simply as e_i, Z_i etc. unless indicated otherwise.

the input-output relationship is

$$e_o = -\sum_{n=1}^{n=N} \frac{Z_F}{Z_n} e_n = -\left[\frac{Z_F}{Z_1} e_1 + \frac{Z_F}{Z_2} e_2 + \ldots + \frac{Z_F}{Z_N} e_N\right] \quad (6)$$

The impedance of a resistor is equal to its resistance in ohms

$$Z_R = R. \quad (7)$$

The impedance of a capacitor is time dependent. Recalling that the voltage drop across a capacitor is

$$e = \frac{1}{C} \int_o^t i \, dt$$

and defining the operators

$$p \equiv \frac{d}{dt} \text{ and } \frac{1}{p} \equiv \int_o^t dt, \text{ the}$$

relation between voltage and current for a capacitor is

$$e = \frac{i}{pC}$$

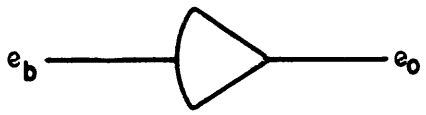
Since impedance is defined as the ratio of voltage drop to current, the capacitor impedance is

$$Z_c = \frac{1}{pC} \quad (8)$$

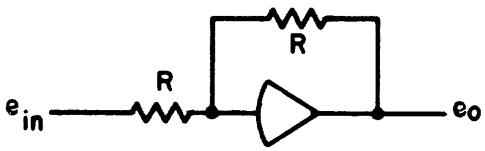
v. Programming Symbols---before illustrating the programming symbols for the circuits just presented, it is important that one realizes how amplifiers and their associated passive elements are packaged in modern day computers. Each amplifier has associated with it an input network (resistors) and a feedback capacitor and/or resistor.

The input resistors are not equal in magnitude. Normally, one finds the input network containing from four to six resistors of two different magnitudes. For example, a six resistor input network may have three 0.1M and three 1M resistors.

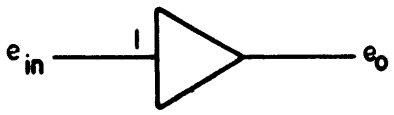
The symbol used for a high gain d-c amplifier is simply



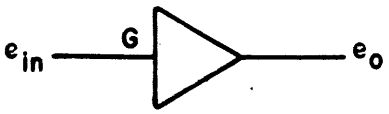
An inverter



whose overall gain is unity because it has identical input and feedback resistors, is denoted by the programming symbol

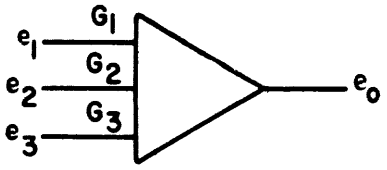


If the passive elements were not identical the symbol would be



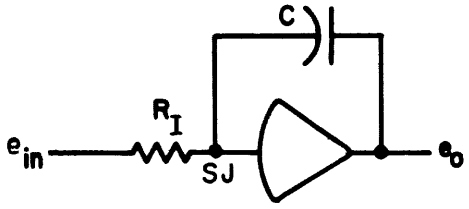
where G is the resistance ratio $\frac{R_f}{R_I}$.

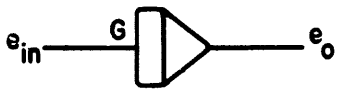
In the case of summing amplifiers which can have multiple inputs, the programming symbol is



where $G_1 = \frac{R_F}{R_1}$, $G_2 = \frac{R_F}{R_2}$, and $G_3 = \frac{R_F}{R_3}$.

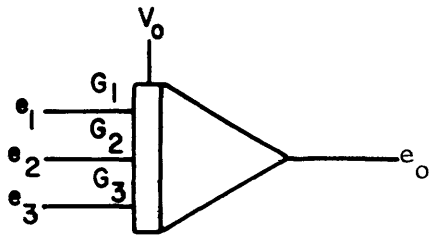
The symbol for an integrator, where $G = 1/R_I C$,





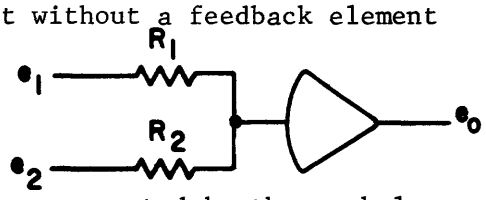
differs from that of a summer by a small rectangle which is adjacent to the base of the triangle. For multiple inputs, the symbol for an

integrator becomes:

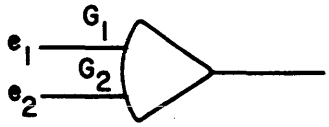


where $G_1 = \frac{1}{R_1C}$, $G_2 = \frac{1}{R_2C}$, and $G_3 = \frac{1}{R_3C}$. The V_o input to the top of the integrator represents the initial value of e_o, or initial charge on the feedback capacitor, which will be discussed in the next section of this chapter.

Finally, one may have occasion to use a high gain amplifier with an input network but without a feedback element



This is commonly represented by the symbol



where G_1 and G_2 are inversely proportional to the size of the input resistors.

To coordinate the packaging of amplifiers and passive elements with the symbols just presented, it must be realized that the input terminations of the input networks usually are not labeled with the magnitude of the input resistors. They are labeled, rather, with gain factors which are based on standard feedback resistors and capacitors selected by the computer manufacturer for a specific computer system.

For example, consider the input or patch panel terminations for a 580 computer shown in Figure II-12. Each input labeled 10 is a 10K resistor, and each input labeled 1 is a 100K resistor. Therefore, the <u>standard</u> feedback resistor for this system must be 100K.

It follows, then, that if this notation is to be used throughout this computing system, the standard integrating capacitor must be 10 μf (1/RC = 1, 10 for 100K and 10K input resistors respectively). Patching details are a function of the specific computer.

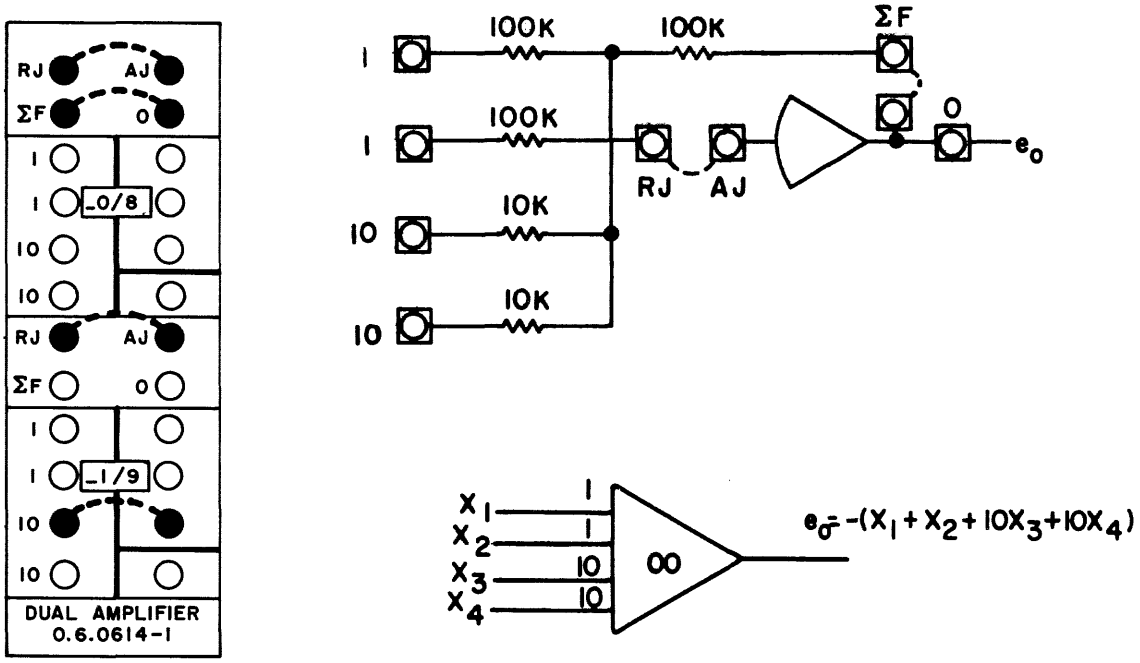


Figure II-12: Summer Amplifier Patching

vi. Drift

D-C amplifiers show a tendency to drift.‡ That is, the output does not necessarily remain steady and, in particular, does not necessarily remain zero for zero input. This is due to changing characteristics within the amplifier and, particularly, to changes in the first stage of the amplifier. It is a most undesirable feature and can lead to serious errors. To compensate for drift which might occur (i.e., to stabilize the d-c amplifier,) an a-c amplifier is used as shown in Figure II-13.

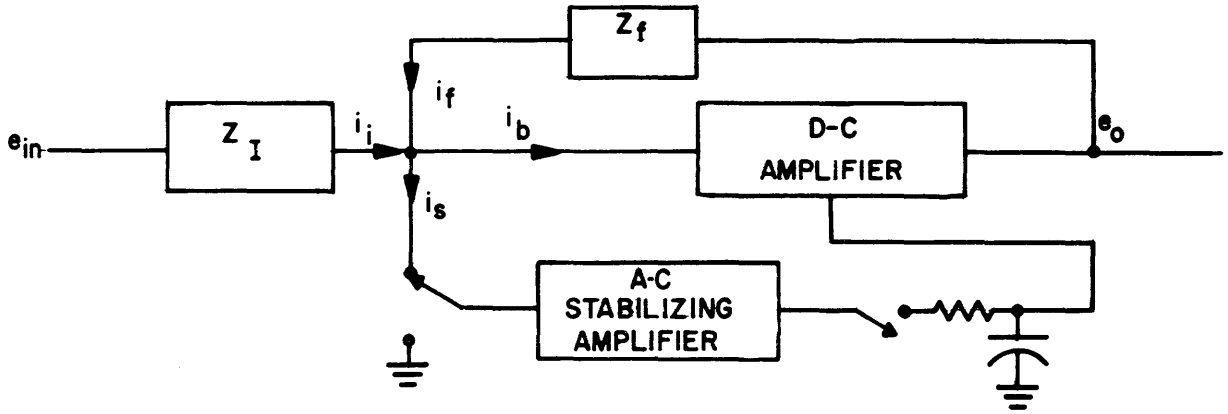


Figure II-13.

The effect of the a-c stabilizer is to increase the overall d-c gain of the amplifier and further attenuate the drift of the operational amplifier by a factor equal to the gain of the a-c amplifier.

vii. Overload Indication -- If the amplifier is overloaded (required output current or voltage is greater than design capabilities, $e_b \neq 0$, etc.) the overload signal connected to the output of the a-c amplifier is excited. This overload indicator ensures that the computer is not used when errors exist due to operating the amplifiers beyond their capacities.

---

‡ For a complete analysis of stabilization of the d-c amplifier, see "Introduction to Analogue Computers" by C.A.A. Wass, pages 76-89.

c. Computer Mode Controls -- The operator controls the mode of operation of the analog computer from push buttons or switches located on the control panel of the computer. The modes of computer operation are classified as follows:

computational modes

Pot Set (PS)
Reset (RS or IC)
Hold (HD)
Operate (OP)

check modes

Static Check (ST)
Rate Test (RT)

slave modes

Slave (SL)
Tape (TP)

special

Repetitive Operation (RO)

A mode control panel which incorporates some of these modes is shown in Figure II-14.

The Operate, Hold and Reset modes are the basic modes required for operation of an analog computer. The additional modes mentioned above are for operator convenience in large computing systems.

i. Integrator Control Switches -analog computer control is accomplished by switches in the integrator circuitry since the integrator is the only dynamic element in the computer. All other components are static in the sense that their output is directly related to their input at all times. A simplified schematic diagram (Figure II-15) of an integrator includes input, feedback and initial condition (I. C.) networks, as well as Operate/Hold and Reset (I. C.) switches.

The IC switch is used to charge the integrator capacitor through the I. C. network to introduce an initial condition on the output of the integrator. The operate-hold (OP-HD) switch, in effect, starts and stops integration since problem variables can not be summed and integrated unless the input network is connected to the amplifier. Note that this switch separates the inputs e_i to the amplifier from the input terminal, b, of the d-c amplifier.

Referring to Figure II-16, consider the modes of operation and switch positions.

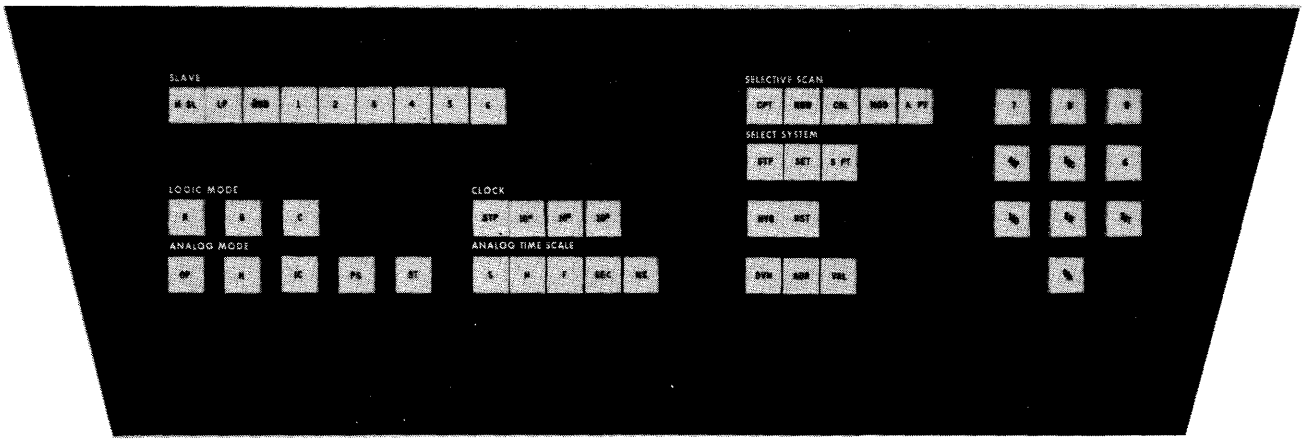


Figure II-14: Mode Control and Signal Selection Panel (EAI 7800 Computer)

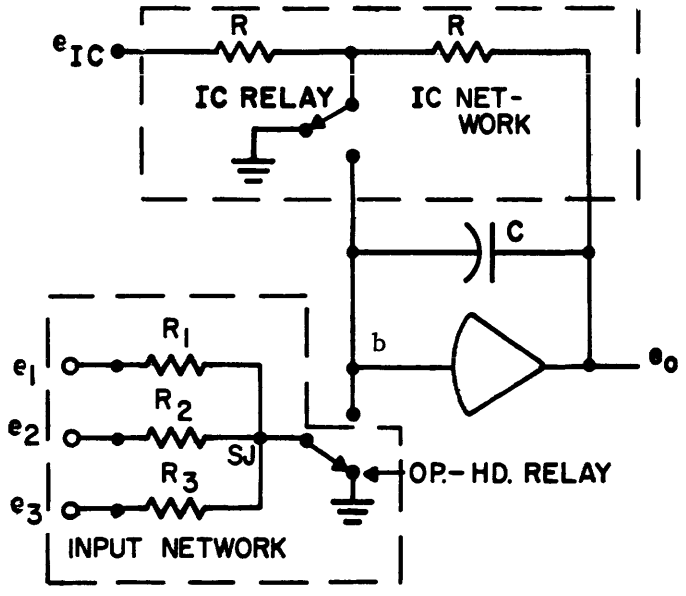


Figure II-15: Simplified Integrator Schematic

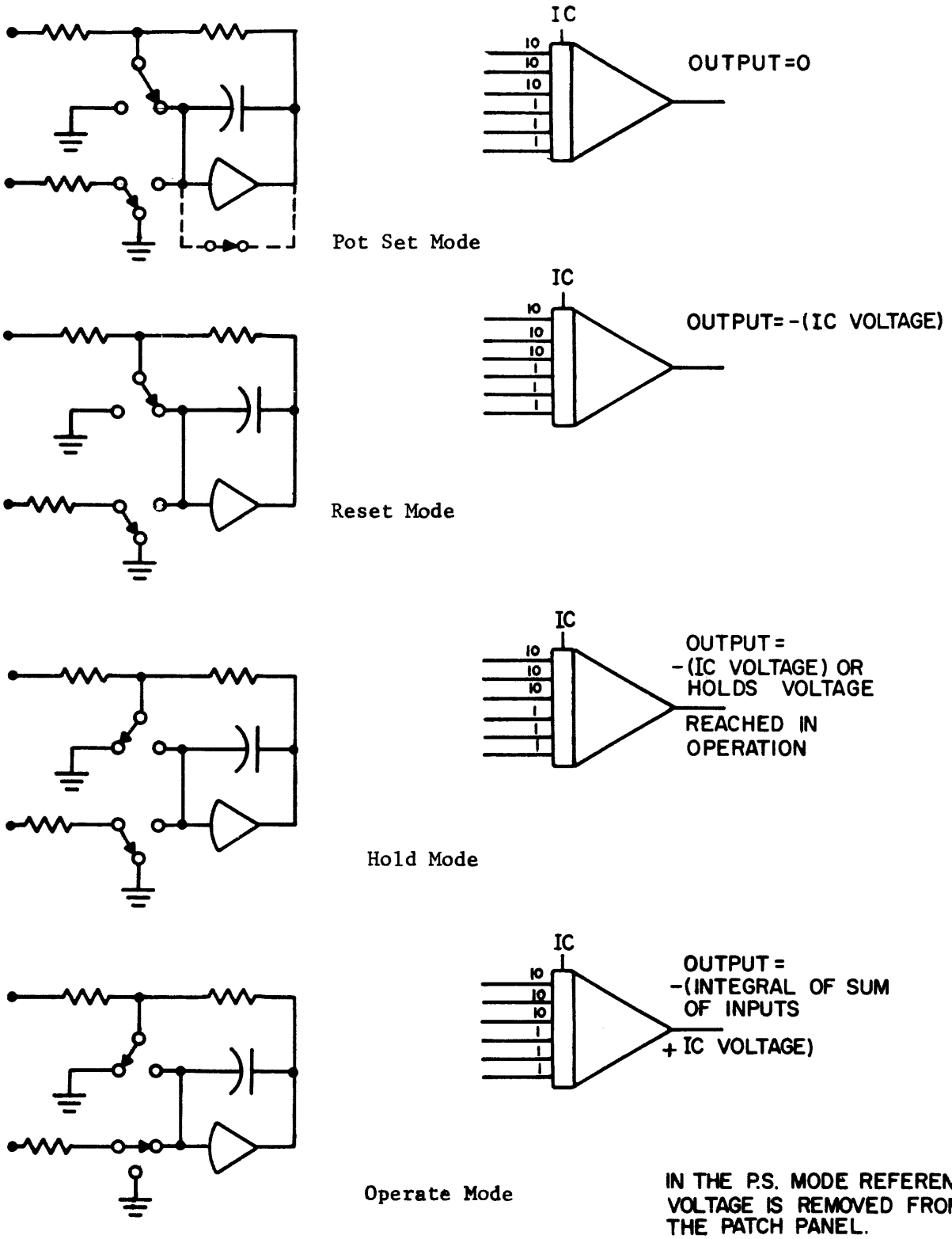


Figure II-16: Integrator Mode Control

ii. Pot Set Mode -- to solve a problem on the analog computer the operator first must introduce the system parameters by setting attenuators. In most computers this is done in the pot set mode; however, in a few computers this function is performed in the reset mode. In the pot set mode, either the reference voltage terminals on the patch panel are de-energized, or the amplifiers are set to zero gain.

In pot set, the IC switch is connected to the input of the d-c amplifier, and the summing junction of the input network is grounded through the OP-HD relay. This allows the attenuators to be set "looking into" their respective loads.

iii. Reset Mode -- the purpose of the reset mode is to introduce initial conditions on the integrator outputs. This can be done only if the IC voltage, e_{IC}, is connected to the amplifier; therefore the IC relay must change position. The OP-HD switch remains in the same position since problem solution is not required at this time.

In the IC mode, the integrator circuit becomes (Figure II-17)

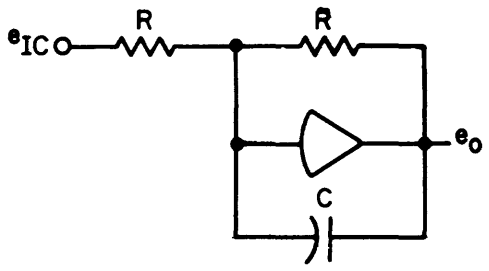


Figure II-17:

The input network is not shown as it is not physically connected to the amplifier in this mode.

The initial condition circuit is solving the equation:

$$-\frac{de_o}{dt} = (e_{IC} + e_o)\, G$$

where e_{IC} is a constant voltage and $G = \frac{1}{RC}$. The solution of this equation is

$$e_o = -e_{IC}\,(1 - e^{-t/RC}) \qquad (9)$$

indicating that the output voltage equals the negative of the initial condition voltage after 10 RC time constants ($t = 10RC$, $e^{-10} \simeq 0$). This time constant is usually

0.1 seconds, (G = 10); therefore, $-e_{IC}$ appears at the amplifier output in less than one second.

iv. Operate Mode -- in the operate mode, the OP-HD switch closes and the integrator operates in the normal manner. In this mode, the IC switch must return to the ground position to remove the IC input, and the IC voltage does not affect the integrator output in the operate mode.

v. Hold Mode -- in practice, it may be necessary to stop a problem solution, obtain intermediate results, and then complete the solution. Therefore, a capability of stopping the solution of a problem without destroying or losing e_o is desirable. This is accomplished by disconnecting the input network. Since the integrator input is zero, the amplifier output in hold is stored on the feedback capacitor, and will be constant.

The capacitors used in analog computers should be of very high quality to minimize "leakage" effects on the amplifier output voltage.

vi. Static Test Mode -- the static test mode is similar to the reset mode with one exception; special "reference voltage" terminations on the patch panel are energized in this mode only. Their purpose is to provide initial condition voltages for checking purposes for integrators, whose initial-condition voltages are zero.

After a problem is mechanized on the computer, a static check (to be discussed in a later chapter) is performed to insure that the patching, etc. are correct.

A zero voltage is not a valid check, however, and, therefore, all integrator outputs must have initial condition voltages. If, as in the case of many analog computers, no static test mode exists, the static check can be performed in the reset mode. Here, one must connect integrator initial conditions to reference voltage sources physically and disconnect them after the check has been completed.

check amplifier

In performing a static check, it is necessary that the net sum of the input voltages to all integrators be available. (i.e. the derivatives of variables). Consider an integrator in the reset mode (Figure II-18):

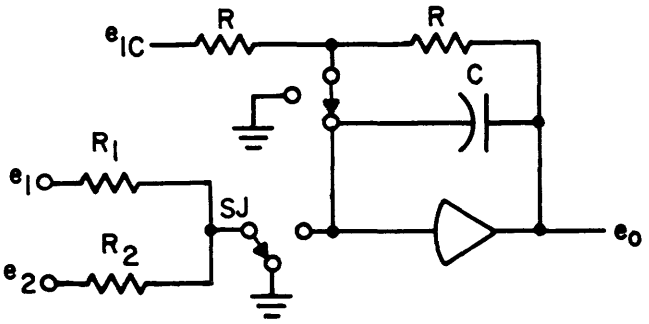


Figure II-18: Integrator Circuit in the Reset Mode

The following operations must be performed to read the check point (derivative) on an integrator:

1) the integrator summing junction ground (through the OP-HD switch) must be removed;

2) the summing junction must be connected to the input (b) of the check amplifier which has its own feedback resistor;

and

3) the output of the check amplifier must be connected to a voltmeter.

The implementation of these operations is either manual, which places the entire burden on the operator, or automatic, which is implemented by relays through a pushbutton selector system.

A choice of check amplifier feedback resistors is usually available to the operator by switching, so that he has a choice of feedback resistors. In the circuit shown in Figure II-19, the feedback resistor, R, is equal in magnitude to a gain one integrator input resistor.

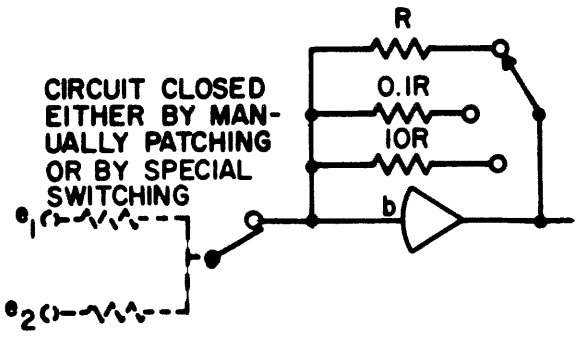


Figure II-19:

The necessity for being able to select the proper feedback resistor is best understood by recalling that only the amplifier outputs must be kept within the reference voltage range of the computer. The net sum of several integrator input voltages could be much greater than the maximum permissable amplifier output voltage. Since the check amplifier output is subject to the same limitations as the other amplifiers in the computer, some check amplifier output voltages (known as check points) will require either amplification or attenuation.

vii. Rate Test Mode -- the purpose of the rate test (RT) mode, which is found only in the more sophisticated and costly analog computers, is to check the integration rate of all the integrators in the computer. For maximum solution accuracy, all integrators must integrate at the same rate.

In the rate test mode, all integrators are fed from a common voltage source, through a grounded potentiometer called a rate test pot, to a gain "one" input on all integrators. The initial condition voltage of the integrators is zero (Figure II-20).

There are no open switches between the voltage source and the integrator inputs (including the operate-hold switch) to eliminate switch closing - time as a factor in determining integration rates.

In performing a rate test, one normally observes the output of one integrator on a DVM and integrates alternately in a positive and then a negative direction (using the three-position switch and the rate test pot) before removing rate test pot input (center position of the switch). All integrator output voltages are then monitored to determine whether or not the capacitors require adjustment.

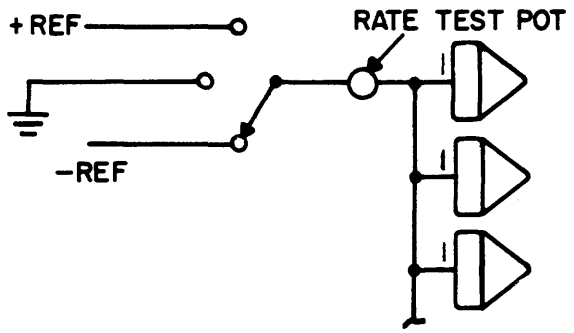


Figure II-20:

viii. Slave Mode -- the slave mode is used when more than one computer console is required in a simulation. If this is the case, all consoles must change modes simultaneously.

One console is designated as the "master" console. Its mode control will govern the remaining consoles which are known as "slaved" consoles.

To slave computers to a master console, one must perform two operations:

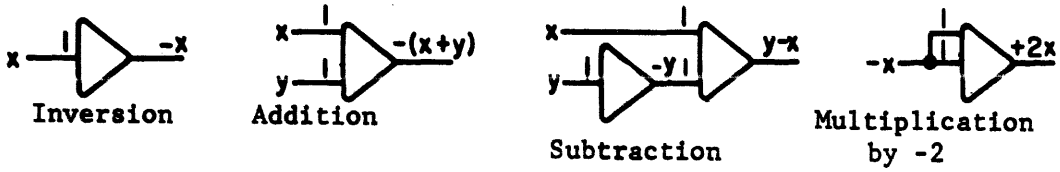
1) put the slaved computers in the slave mode, and

2) using a multiposition switch, select the master console on the slaved computers.

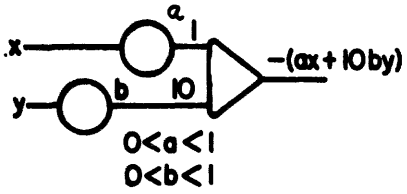
The latter function presumes a large, multiconsole installation. In the case of smaller installations, say, two desk-top, transistorized computers, the multiposition switch is replaced by an interconsole cable which must be connected by the operator. The slaved computer still must be put into the slave mode.

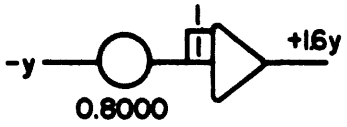
ix. Tape Mode -- the tape mode is used on large analog computers to permit computer operation to be controlled from paper or magnetic tape. This feature is very desirable in a multishift computer laboratory for quickly setting potentiometers and diode function generators (using servo-set pots).

Details regarding tape input are beyond the scope of these notes. However, additional information may be obtained from the reference handbook for a specific model computer.



Arbitrary gains



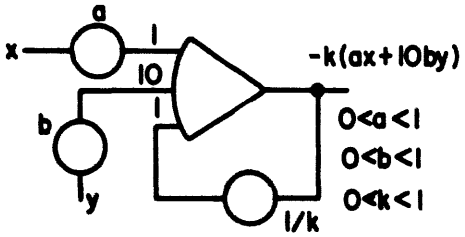


For high gain amplifiers:

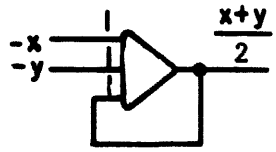
Σ input voltages x gains = 0

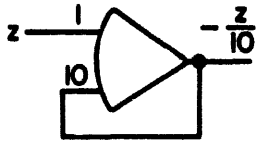
$\therefore$ $ax + 10\,by + \frac{e_o}{k} = 0$

$\therefore$ $e_o = -k\,(ax + 10\,by)$

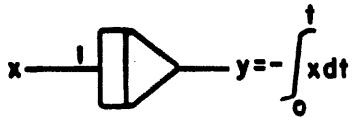


Gains of 1/2 and 1/10 without potentiometers





Integration



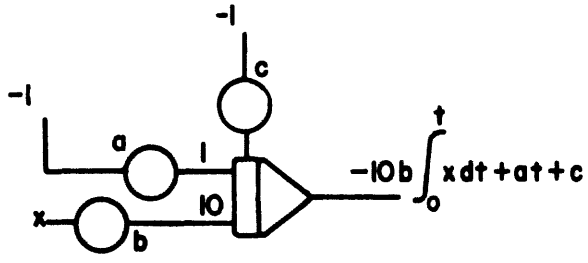


Figure II-21: Linear Component Circuits

d. Linear Component Summary -- To summarize linear components, several simple circuits are presented in Figure II-21. It should be recalled that

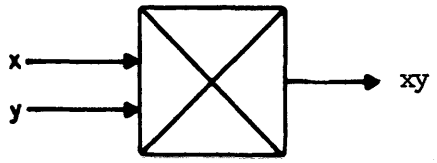
1) voltages may or may not appear at the outputs of amplifiers in the pot set mode (depends on the computer model);

2) voltages undergo a sign inversion when passing through an amplifier;

3) the gains of input resistors are their relative magnitudes, when no feedback resistor is present.

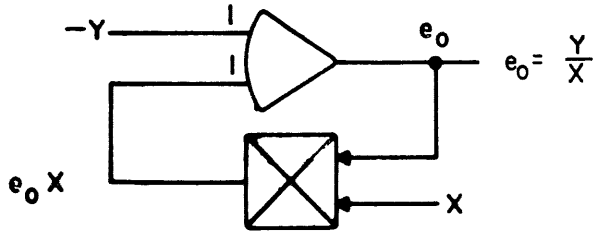
2. NONLINEAR COMPONENTS

a. Multipliers ---To gain an appreciation for the computational capability realized by being able to multiply two time varying voltages, consider a "black box" multiplier.



This multiplier accepts two inputs, x and y and produces their product.

DIVISION



SQUARE ROOTING

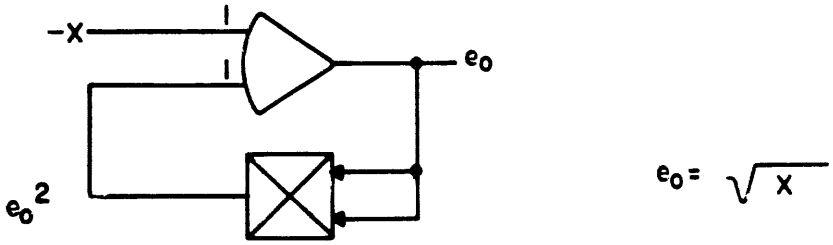


Figure II-22: Mathematical Operations Performed Using Multipliers

If the multiplier is put into the feedback loop of a high gain amplifier, the inverse of multiplication, i.e. division, of two voltages can be accomplished (Figure II-23).

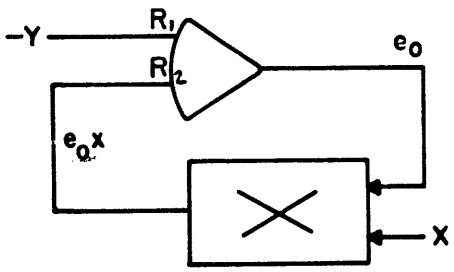


Figure II-23.

Since the sum of the currents feeding the SJ of the amplifier through the two input resistors must be zero

$$-\frac{Y}{R_1} + \frac{e_o X}{R_2} = 0$$

or

$$e_o = + \left(\frac{R_2}{R_1}\right) \frac{Y}{X} = + \frac{Y}{X} \qquad \text{if } R_1 = R_2.$$

One also can obtain the square root of a voltage using a multiplier in the feedback loop, as indicated in Figure II- 22

Two principles of computer component operation which have been presented here are worthy of comment. They are:

1) inverse operations are performed by placing the nonlinear component in the feedback loop of a high gain amplifier,

and

2) implicit algebraic operations, where the sum of the input currents is zero, are performed when a high gain amplifier is used for inverse operations.

The remainder of this section will be devoted to explaining the operation of the two types of multipliers commonly encountered in computing systems. They are:

1) servo or electromechanical multipliers

2) electronic or quarter square (QSM) multipliers.

i. Servo or Electromechanical Multipliers -- the operation of the servo multiplier can be explained best by considering Figure II-24 which is a schematic diagram of a typical servomultiplier. The servo amplifier unit, an a-c amplifier, containing an electromechanical vibrator or "chopper", accepts two inputs of identical polarity, x and e_f, and, using the "chopper", converts the difference between these voltages into an a-c error voltage. The error voltage is amplified and fed to the windings of a two phase motor which is connected to a potentiometer shaft by a gear train.

If the error voltage is positive, the motor will rotate the shaft in a positive direction; for a negative error signal in a negative direction. An audio and/or visual overload alarm is activated by the amplified error signal fed to the motor windings when the error signal is excessive.

To clarify the positive and negative directions, consider the shaft driven by the motor and gear train. On this shaft are several wiper arms which are aligned mechanically. They are not connected electrically. Each of these wipers rotates inside a cylinderical wire wound resistor. One of these resistors (often called "cups",) is referred to as the "follow-up" cup, and the remaining resistors are known as "multiplying" cups. The positive direction refers to the plus (+) end of the cups, and the negative direction refers to the minus (-) end of the cups. Note that the cups each have a centertap.

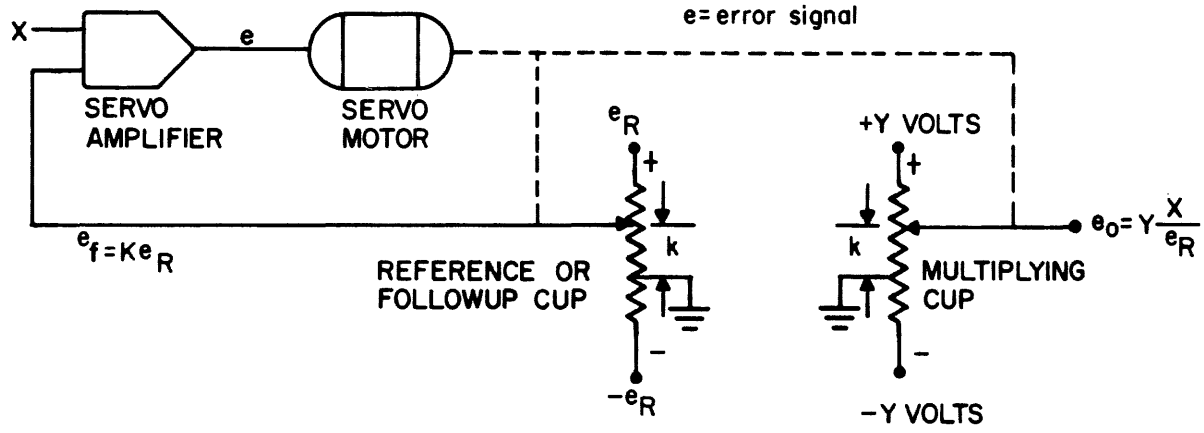


Figure II-24: Schematic Diagram of a Servo-Mechanical Multiplier

To illustrate how a servo multiplier operates, assume that the input voltage, x, shown in Figure II-24, is constant. When e = 0,

$$e_f = x.$$

Note that $e_f = k\, e_R$,

where k is the fraction of the "follow-up" cup resistance between the centertap and $+e_R$ termination. Solving for k yields

$$k = \frac{e_f}{e_R}$$

If the wipers on the shaft are aligned, the output of the multiplying cup is

$$e_o = ky$$

and thus,

$$e_o = \frac{xy}{e_R}$$

One may draw an analogy between "multiplying" cups and attenuators whose input-output relationship is also $e_o = ky$. In this instance, the a-c amplifier, motor, and "follow-up" cup are used to vary k as a function of x.

The "follow-up" cup is used to make a nulling voltage available to the a-c amplifier. Therefore, the voltage range of the input variable determines the "follow-up" cup voltage range. This in turn governs the multiplying cup connections made by the operator.

There are three possible input variable ranges:

$-e_R \leq x \leq 0$; x always negative,

$-e_R \leq x \leq +e_R$; x both positive and negative, and

$0 \leq x \leq +e_R$; x always positive.

If the input variable can be both positive and negative, a grounded center tap and both positive and negative reference voltage would be provided as indicated in Figure II-24. This is called a "normal" connection (N).

Care must be exercised not to reverse $+ e_R$ and $- e_R$. If this is done, the servo would drive in a positive direction for a positive input only to find an increasing negative voltage. It then would be impossible to obtain a null voltage.

If the input variable is always positive, the voltage gradient across the "follow up" cup can be reduced to increase resolution by grounding the bottom (or minus end) of the cup and removing the centertap ground. This is known as a positive connection (+). When the positive connection is used, the multiplying cups must be connected in a similar manner; the low end of the cups must be grounded and the centertap ground must be removed.

A negative connection(-), which is used for input variables that are always negative, is similar to the positive connection in that all center tap grounds are removed. In this instance, the tops (+) of all cups are grounded and the bottom (-) of the "follow-up" cup is connected to a negative reference voltage source.

The follow-up cup connections described on the previous page are summarized in Table II-1. It should be noted that the location of the ground connections on the multiplying cups is of considerable inportance if shaft displacement, k, is to be identical on all cups.

TABLE II-I FOLLOW-UP CUP CONNECTIONS

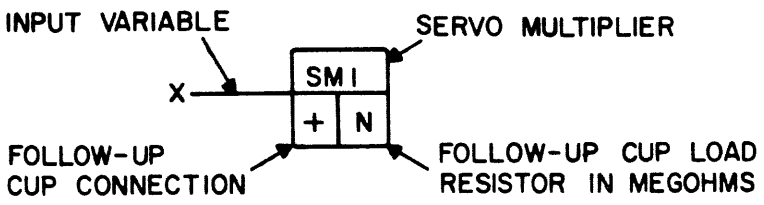
Input Variable Range	FOLLOW-UP CUP TERMINATIONS			Required Ground Connections on Multiplying Cup:
	Top (+)	Center Tap	Bottom (-)	
$-e_R \leq x \leq +e_R$	$+ e_R$	GND	$-e_R$	Center Tap
$0 \leq x \leq +e_R$	$+ e_R$	OPEN	GND	Bottom (-)
$-e_R \leq x \leq 0$	GND	OPEN	$-e_R$	Top (+)

loading

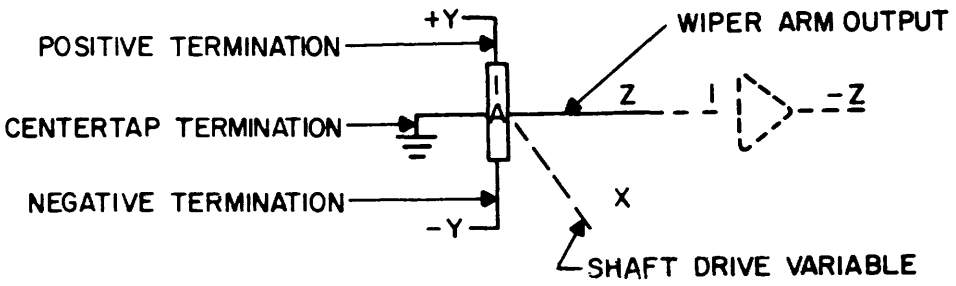
Multiplying cups, like attenuators, are "loaded" when fed to a load such as an amplifier input (recall that the multiplier derivation did not account for current drawn through the wiper arm). This is further complicated by the fact that there are several cups including the "follow-up" cup on the same shaft, which must be "loaded" identically if the multiplier is to function correctly (k identical for all cups). Loading follow-up and multiplying cups equally is accomplished by patching passive resistors which correspond to multiplying cup loads from the follow-up cup wiper arm to ground. This requirement virtually eliminates the possibility of feeding potentiometers from multiplier cups. Also, attenuator outputs can not be fed directly to a multiplying cup because of variable loading.

programming symbols

The servo drive portion of the servo multiplier, including the amplifier, motor, gear train, and follow-up cup, is denoted by a subdivided, rectangular box.



The individual multiplying cups, which do not have to be located adjacent to multiplier on a computer diagram, are denoted by a rectangle.



The upper portion of the drive symbol contains the number of the servo multiplier being used (i.e., SM1 - servo multiplier number 1). The nomenclature in the multiplying cup symbol denotes the multiplier number and the specific multiplying cup (i.e. 1A).

reversing output polarity

To reverse the polarity of the output product, Z, the positive (+) and negative (-) multiplying cup terminations shown above should be reversed. In this instance, replacing +y with -y results in a negative output for a positive x input.

Polarity reversals always should be made by manipulating multiplying cup terminations only. The result of reversing patching terminations for "normally" connected multiplying cups is straightforward. For positive and negative multiplying cup connections, the following rule applies:

$$\text{Polarity of Wiper Arm Voltage} = \begin{cases} \text{(Polarity of the input signal)} \\ \text{times} \\ \text{(Polarity of the cup patching termination)} \\ \text{times} \\ \text{(Polarity of the cup input voltage)} \end{cases}$$

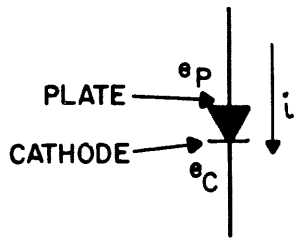
ii. Quarter Square Multipliers -- the operation of the quarter square multiplier (QSM) is based on the relationship

$$xy = \frac{(x + y)^2 - (x - y)^2}{4}$$

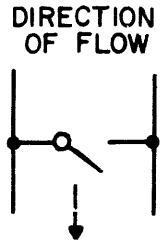
which requires the capability of obtaining the square of a voltage. To appreciate how the squaring operation is performed in the QSM, a completely electronic multiplier, one must become familar first with the principle of diodes and fixed diode function generators which are themselves GPAC components used to generate analytic functions.

Diodes

The diodes used in modern analog computer circuitry are of solid state rather than vacuum tube construction, and have forward resistance of the order of one hundred ohms. They consist of an anode (or plate) and a cathode, and are represented by the symbol shown below.



DIODE



CHECK VALVE

If the plate voltage, e_p, is greater than the cathode voltage, e_c, the diode will "conduct", and a current, i, will pass through. If e_c is greater than e_p, the diode represents an infinite resistance and no current will pass through it. When the diode is conducting, its resistance is a nonlinear function of current and voltage.

For example, if e_c were negative and e_p were at ground potential, the diode would "conduct". If e_c were positive, the diode would not conduct.

Diode operation is analogous to the operation of a check valve.

For analog computer component purposes, diodes are used as switches which switch currents rather than voltages. The state of the switch and the magnitude of the current depend on the voltage drop across the diode.

fixed function generator operation

The method of generating $(x \pm y)^2$ for QSM operation is identical to the operating principles of the fixed diode function generators which are used in the GPAC to generate such analytic functions as x^2 and log x. Therefore, these topics will be discussed simultaneously. The function to be generated will be referred to as f(x), where x is the independent variable or function generator input. In the case of the QSM, the inputs are $(x + y)$ and $(x - y)$.

The desired function can be approximated by a number of straight-line segments as shown in Figure II-25. The point of intersection of two adjacent segments is known as a breakpoint, and the number of segments governs the accuracy of the approximation.

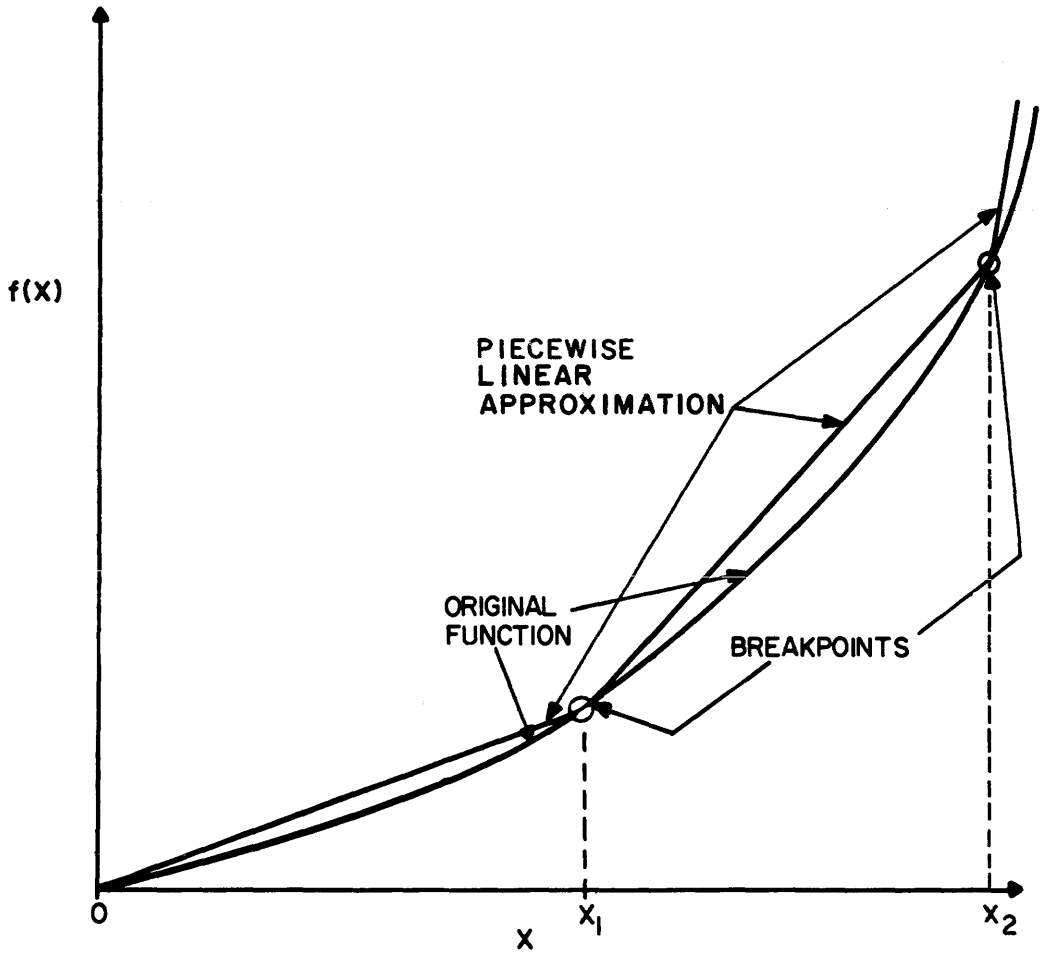


Figure II-25: Approximation by Straight-Line Segments

Consider a simple inverter whose input resistor has been replaced by resistor-diode networks such as shown in Figure II-26. In this circuit, the input voltage, x, is positive and the bias voltage source, $-e_R$, is negative. The total number of two resistor-diode circuits (or straight-line segments) are fabricated on a printed circuit board which is known as a "card".

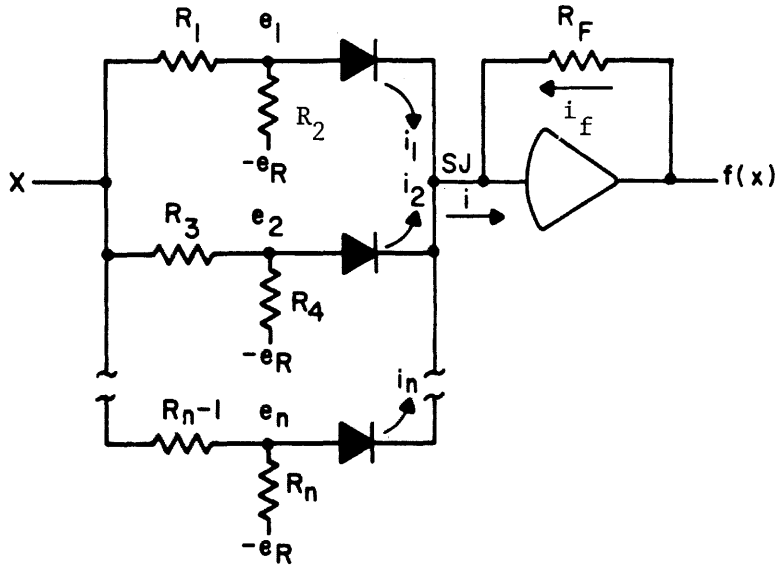
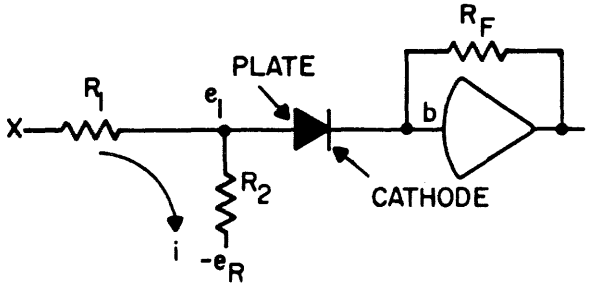


Figure II-26: Simplified Schematic Diagram of Fixed Diode Function Generator.

Let us confine our attention to the first resistor-diode circuit. If x were zero, a negative voltage gradient would exist between $-e_R$ and x.



Therefore, the voltage at e_1, the intersection of R_1 and R_2, would be negative. Since the base, b, of the amplifier is at zero potential, the diode will not conduct because its plate voltage, e_1, is negative with respect to the cathode, and it acts as an open switch.

As x becomes more positive, e_1 becomes less negative until it reaches zero volts. With a further increase, the plate potential of the diode is greater than its cathode potential and it conducts, allowing a current flow to the grid of the amplifier. A voltage appears at the amplifier output which is proportional to -x.

If x were negative, e_1 would always be negative and the diode would never conduct. However, if the bias voltage ($-e_R$) and the diode were reversed, a negative input voltage would enable the diode to conduct.

Referring to Figure II-26, a function can be approximated by using a circuit containing a number of diode/resistor networks. The form of the function is governed by the selection of the bias resistors, R_1 to R_n, and the sum of the currents is

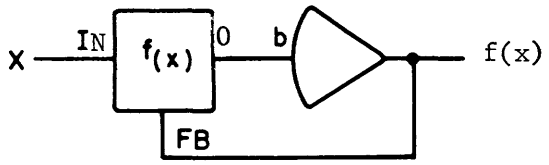
$$i_f = i_1 + i_2 + i_3 + \text{-------} + i_n = \frac{-f(x)}{R_F}$$

(The nonlinearity of the diodes results in rounded rather than sharp breakpoints.)

Note that a potentiometer should never feed directly into a diode function generator since its load would be continually changing.

Common types of fixed DFG's are: x^2, x^4, log x, 1/2 log x (= log $\sqrt{x}$), and sin x/cos x. The sinusoid generator differs somewhat from the others in that it uses diode switching in the feedback path while the others use diode switching in the input path.

The conventional programming symbol for a fixed function generator is



where FB, IN and 0 denote patching notation characteristic of a specific computer system. Some function generation devices of this type require two external amplifiers; this, again, is dependent on the particular computer used.

QSM operation

A QSM, simplified diagram is shown in Figure II-27, uses four x^2 diode function generator circuits; however, only two of these circuits operate at any given time. The multiplier requires two plus and two minus circuits to provide four quadrant multiplication; therefore, card operation depends on the polarity of the input voltage. Referring to Figure II-27, and recalling that the circuits have a fixed bias voltage, the currents are subject to the following constraints:

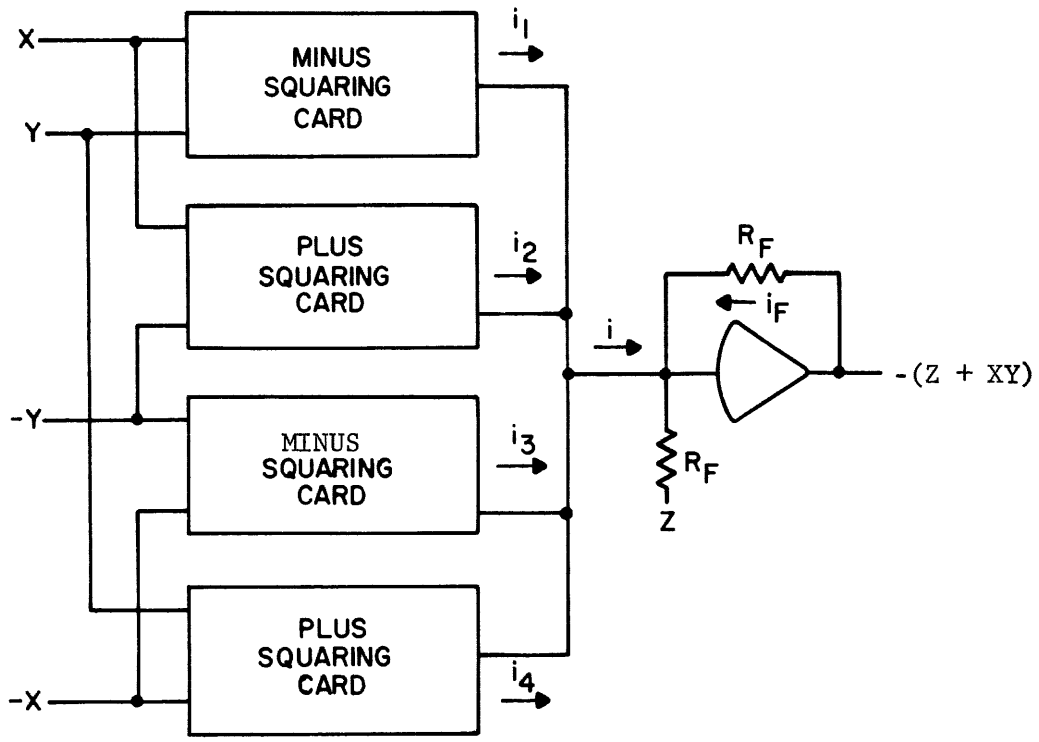


Figure II-27: Simplified Schematic Diagram of a Quarter Square Multiplier.

$$i_1 = \frac{(x + y)^2}{4 R_F e_R} \quad \text{. . . . . if } x + y > 0$$

$$i_1 = 0 \quad \text{. . . . . if } x + y < 0$$

$$i_2 = \frac{-(x - y)^2}{4 R_F e_R} \quad \text{. . . . . if } x - y < 0$$

$$i_2 = 0 \quad \text{. . . . . if } x - y > 0$$

$$i_3 = \frac{(x + y)^2}{4 R_F e_R} \quad \text{. . . . . if } (x + y) < 0$$

$$i_3 = 0 \quad \text{. . . . . if } x + y > 0$$

$$i_4 = \frac{-(x - y)^2}{4 R_F e_R} \quad \text{. . . . . if } (x - y) > 0$$

$$i_4 = 0 \quad \text{. . . . . if } (x - y) < 0$$

$$i = i_1 + i_2 + i_3 + i_4 = \frac{xy}{e_R R_F}$$

Since the output amplifier of the multiplier has a feedback resistor, another input can be summed to the output product by selecting the proper input resistor. This input can be fed by a potentiometer because the load is fixed.

Note that the current sum from the cards is

$$i = \frac{xy}{e_R R_F}$$

for normal operation. Replacing the normal feedback resistor with a resistance $R < R_F$ attenuates the xy product.

$$i + i_f = o$$

$$\frac{e_o}{R} + \frac{xy}{R_F e_R} = o$$

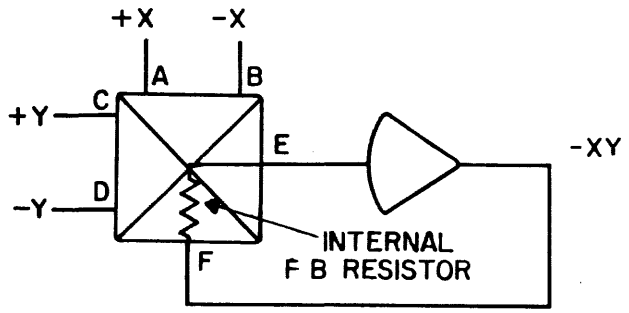
$$e_o = -\left(\frac{R}{R_F}\right) \frac{xy}{e_R}$$

The product may be amplified by introducing a potentiometer, α , between the amplifier output and the normal feedback resistor, R_F, so that

$$e_o = \frac{-xy}{\alpha e_R}$$

where α is less than one.

The programming symbol for a QSM is



where A through F represent patching terminations appropriate to a specific computer. Note that the amplifier feedback resistor is not indicated; in most computers it is included with the four squaring cards. In this instance, the feedback path is a resistor connecting the E and F terminations.

In most modern analog computers, the multiplier is simply the squaring cards and feedback resistor, which means that anywhere from one to three additional amplifiers are required per multiplication. One amplifier is required to sum currents and output the product ; the number of additional amplifiers depends upon the availability of +x, -x, +y, and -y (all four inputs are required). Many analog computers have multipliers with <u>two</u> internally packaged amplifiers, which requires only three inputs. Computer manuals, reference handbooks, etc., should be consulted for further information and programming symbol and notation.

The QSM output amplifier, like all amplifiers in a computer system, has its own visual and/or audio overload alarm.

iii. <u>Choice of Multiplier</u>--there is often a need to decide which type of multiplier to use and the specific circuit which will best meet the problem requirements. Some of the advantages and disadvantages of the different types are:

<u>Advantages</u>	<u>Disadvantages</u>
Servo - Mechanical - Single Turn Potentiometers	
a) is simple and robust	1) has velocity and accelleration limitations due to dynamic response of mechanical components

b) does not require additional amplifiers	2) granularity of windings limits resolution
c) can give, per unit, products of a number of variables with one other	NOTE: requires care in programming to minimize the effect of servo dynamics in circuit loops.
d) with tapped potentiometers and a padding unit, can be used as a function generator	

Servo - Mechanical - Ten Turn Potentiometers

e) resolution is improved by factor of ten compared with single turn device	3) gains resolution at the expense of response which is decreased by a factor between 5-10

Quarter Square Multiplier

a) has higher frequency response than servo type (essentially flat to several Kcps)	1) may be limited by phase error when working beyond 1 Kcps
b) can give up to three separate multiplications per unit, depending on model.	2) requires external operational amplifiers
	3) may require care in programming to minimize effect of static error and noise for low voltage inputs

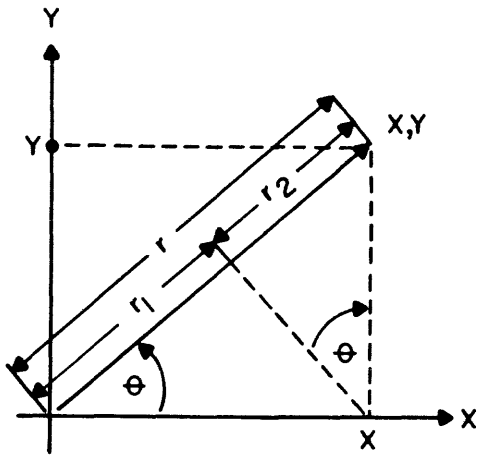
A consideration of the above factors, should serve only as a guide as to which type of multiplier would be best suited to a specific task.

b. Resolvers--The primary function of resolvers in a computing system is to generate the sine and cosine of a variable, convert variables from rectangular to polar coordinates (or polar to rectangular), and rotate axes in a two dimensional system. The mathematical relationships required for coordinate transformation and axes rotation are shown in Figure II-28.

As in the case of multipliers, two types of resolvers are available: servo or electromechanical, and electronic. However, resolvers differ from multipliers in the form of their input which can be either an angular position or the derivative of the angle (rate resolver).

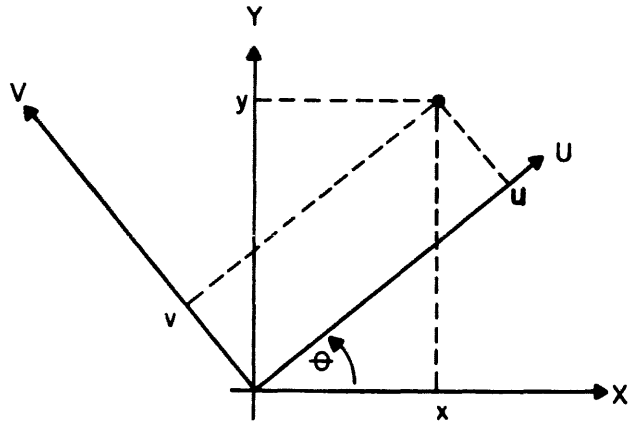
i. Servo-Resolver--A servo-resolver is an electro-mechanical system that operates in a manner similar to that of the servo multiplier. The linear multiplying potentiometers of the multiplier are replaced by potentiometers that are "shaped" to produce a sinusoidal variation in output voltage as the wiper sweeps across the potentiometer.

A simplified diagram of a typical servo resolver (in the PR mode) is shown in Figure II-29 (note the similarity to the servo multiplier). The shaft of the servo motor is positioned as a function of the input, θ, by the feedback system. The linear follow-up cup provides the necessary feedback to null the input to the servo amplifier. Also attached to the motor shaft are wipers that travel along the shaped sine-cosine potentiometers.



x, y to r, θ	r, θ to x, y
$r = r_1 + r_2 = x \cos \theta + y \sin \theta$	$x = r \cos \theta$
$y \cos \theta = x \sin \theta$	$y = r \sin \theta$

Coordinate Transformation



y, x to u , v

$u = x \cos \theta + y \sin \theta$

$v = y \cos \theta - x \sin \theta$

Axis Rotation

Figure II-28: The Mathematical Relationships Required for Resolution

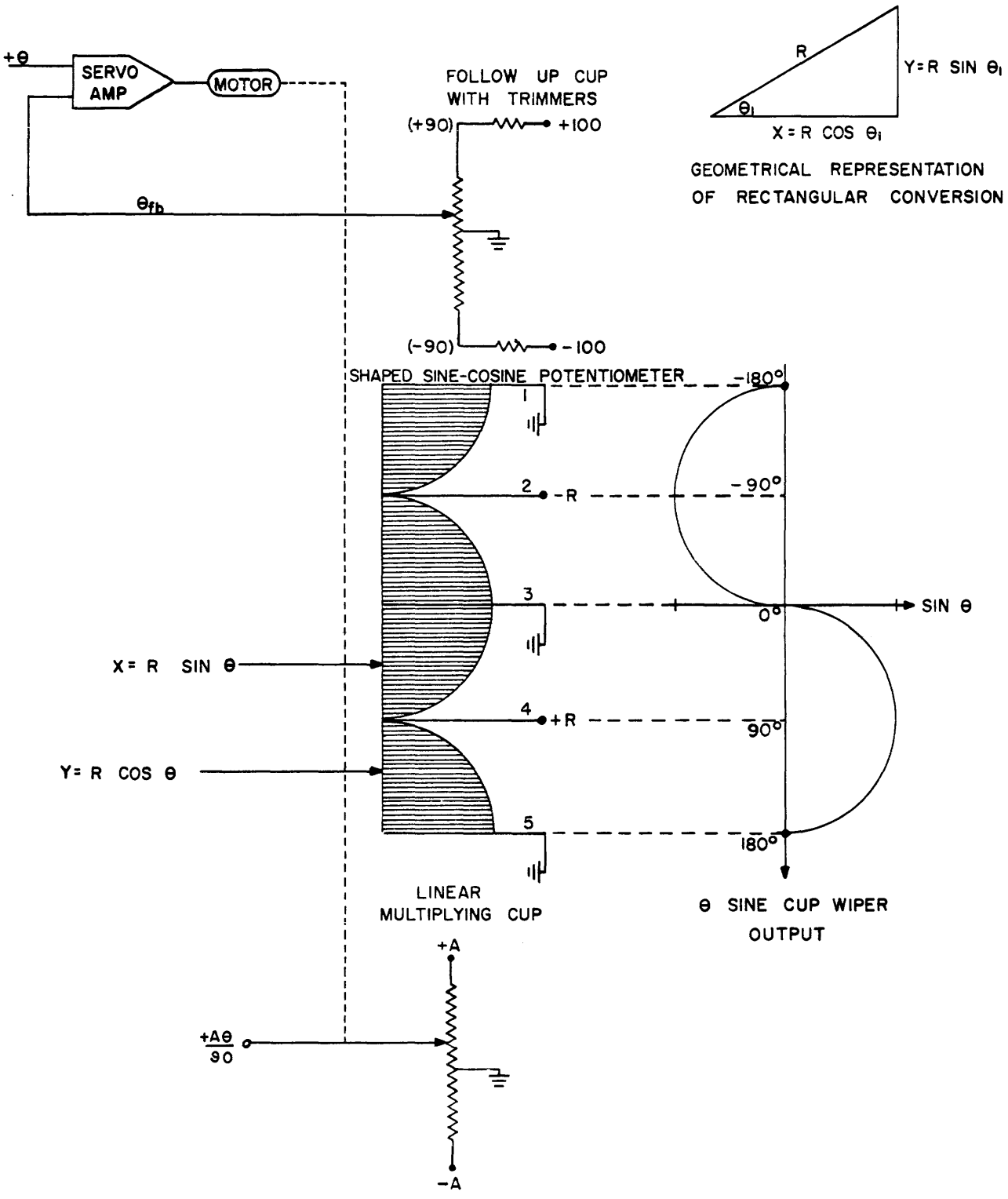


Figure II-29: Simplified Diagram of Servo Resolver in Polar-Rectangular (PR) Mode Inputs $\pm$ R & θ; Outputs x & y

follow-up cup, sine-cosine cup

The sine-cosine cup is a circular, wire-wound resistance element with four taps spaced 90 degrees apart. Therefore, points "1" and "5" in Figure II-29 represent the same electrical tap. The wipers are spaced 90 degrees apart; therefore, the outputs are R cos θ and R Sin θ. If the polarity of the inputs to points "2" and "4" are reversed, the signs of the resolver outputs are inverted.

Since the resolver outputs are obtained from potentiometers, they are subject to loading errors. These errors cannot be compensated for by loading the follow-up cup and sine-cosine cup with equal resistance, because one pot is linear while the other is non-linear. Instead, the sine-cosine cup is wound so that it has the correct output when working into a specific load. The follow-up cup does not require an external load in the resolver mode.

The follow-up cup is equipped with trim pots so that the maximum available feedback voltage is $\pm$90 volts. Thus, the input voltage representing θ should not be allowed to exceed $\pm$90 volts. When the servo system is equipped with mechanical stops, these can be positioned to limit the wiper of the sine-cosine cup as the input voltage ranges from -90 to +90 volts. In this case, the input scale factor for θ is one volt per two degrees, allowing θ to vary between $\pm 180^{o}$. If the system gearing and mechanical stops are arranged so that the wiper of the sine-cosine cup makes ten revolutions as the input voltage ranges from -90 to +90 volts, the input scale factor for θ is one volt per twenty degrees, allowing θ to vary between $\pm 1800^{o}$.

A servo resolver in the Rectangular (Polar-Rectangular, PR) mode is simply a position servo equipped with a non-linear multiplying cup. Hence, linear potentiometers can be added so that multiplication can be performed with the unit servo multiplier. Figure II-29 shows a linear cup (potentiometer) whose output is + $A\theta/100$. Note that since θ is limited to $\pm$90 V, the multiplying cup product is limited to

$$\frac{A\theta}{90}$$

The servo resolver chassis normally has a neon lamp overload indicator that will light whenever the servo tracking error exceeds the allowable limit. Also located on the front of the servo resolver chassis is a dial which indicates the angular position of the resolver wiper.

The PR mode can be used to generate $e_R \sin \theta$ by replacing +R and -R with reference voltage sources.

A simplified diagram of a servo resolver in the Polar (Rectangular-Polar, RP) mode is shown in Figure II-30. Note the polarity of the x and y inputs to the sine-cosine cups, the use of an external operational amplifier, and the fact that a voltage proportional to θ is generated at the follow-up cup wiper. The vector magnitude, R, is formed at the output of the operational amplifier. The equation x Sin θ = y Cos θ provides the null relationship necessary to drive the servo wiper to position θ.

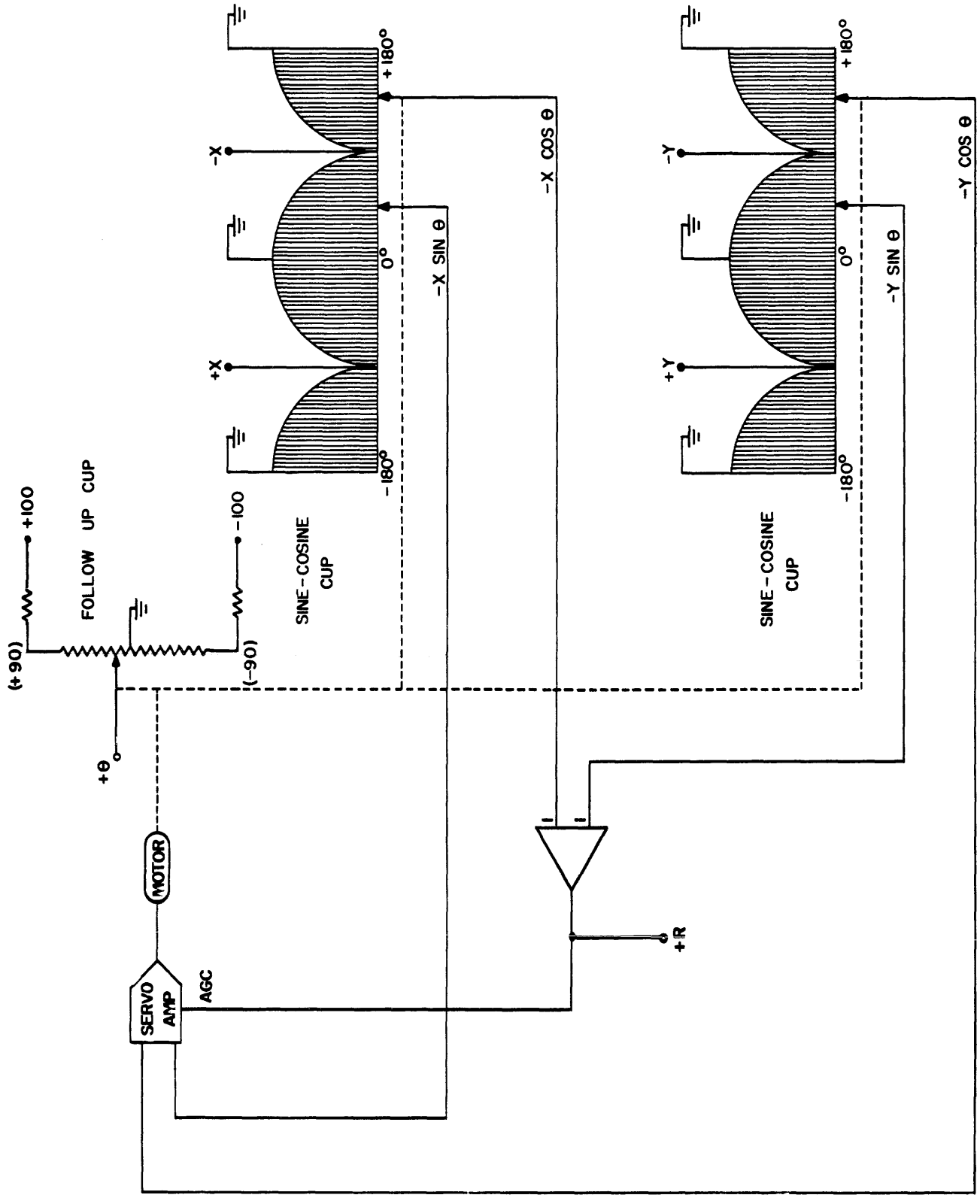


Figure II-30: Simplified Diagram of Servo Resolver in RP Mode (Inputs $\pm$x & y, Outputs R & θ)

With inputs of $\pm$ x and $\pm$ y, the sine-cosine cups will produce -x sin θ', -x cos θ', -y sin θ', and -y cos θ'. The servo will not null unless $\theta' = \theta$. Therefore, the servo error voltage, e, is

$$e = x \sin \theta' - y \cos \theta'$$
$$e = R \cos \theta \sin \theta - R \sin \theta \cos \theta'$$
$$e = R \sin (\theta - \theta')$$

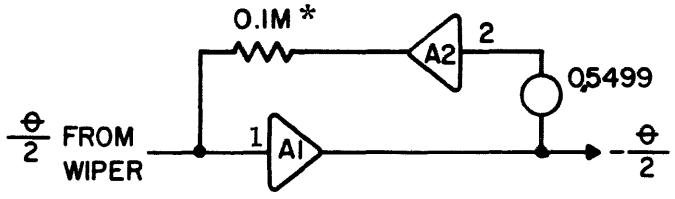
or for small values of $(\theta - \theta')$

$$e = R (\theta - \theta')$$

For accurate computation, the servo must always be close to a null. Therefore, sin $(\theta - \theta')$ can be approximated by $(\theta - \theta')$. The servo error voltage, thus, is proportional to R whereas, ideally, it should be independent of R and simply proportional to $(\theta - \theta')$. Hence, for small values of R the servo will be sluggish, while for large values of R the servo will tend to oscillate. To overcome this undesirable effect, the gain of the servo amplifier is made inversely proportional to R. An automatic gain control (AGC) circuit, which receives R as an input, will increase the servo amplifier gain when R is amall, and decrease the gain when R is large.

The mechanical position of the follow-up cup is a function of θ, and the voltage on its wiper is equal to $\theta/2$ (for a single turn resolver) provided the wiper is not loaded. An unloading circuit must be used to provide θ for computational purposes. All of the sine-cosine cup wipers in Figure II-30 are feeding either one specified load or an approximately infinite inpedence (shown or implied). If any of the wiper voltages are to be used for computational purposes, they must be taken to an unloading circuit.

An unloading circuit that can be used to obtain $\theta/2$ (for a single turn resolver) in RP resolution without loading the follow-up cup is shown in Figure II-31.



*if gain one resistor is 1.0 megohm.

Figure II-31

The unloading circuit (which is essentially a positive feedback system) functions as follows. Let the wiper voltage be y. We want to make the current flowing in the 0.1M resistor just equal to the current flowing in

the 1M input to A1 so that no current is drawn from the wiper. Assume a total gain of K for A2 and the pot, so its output is +Ky. By Kirchoff's current law, the,

$$\frac{Ky - y}{10^5} = \frac{y}{10^6}$$

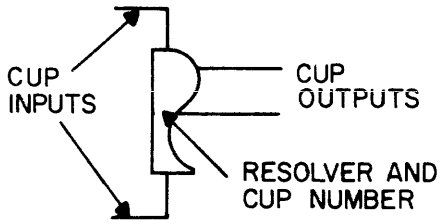
The solution of this equation yields K = 1.1, calling for a pot setting of 0.5500 and a gain of 2. Since at this gain the circuit is just on the verge of instability, a slightly lower gain of the order of 1.09 should be used.

Coordinate transformation is accomplished easily by replacing the $\pm R$ inputs shown in Figure II-29 by $\pm x$. Another resolver cup on the same unit is fed by the $\pm y$ to provide all possible x, y, Sin θ, and Cos θ products. These variables then may be conbimed to form u and v coordinates using the algebraic relationships shown in Figure II-28

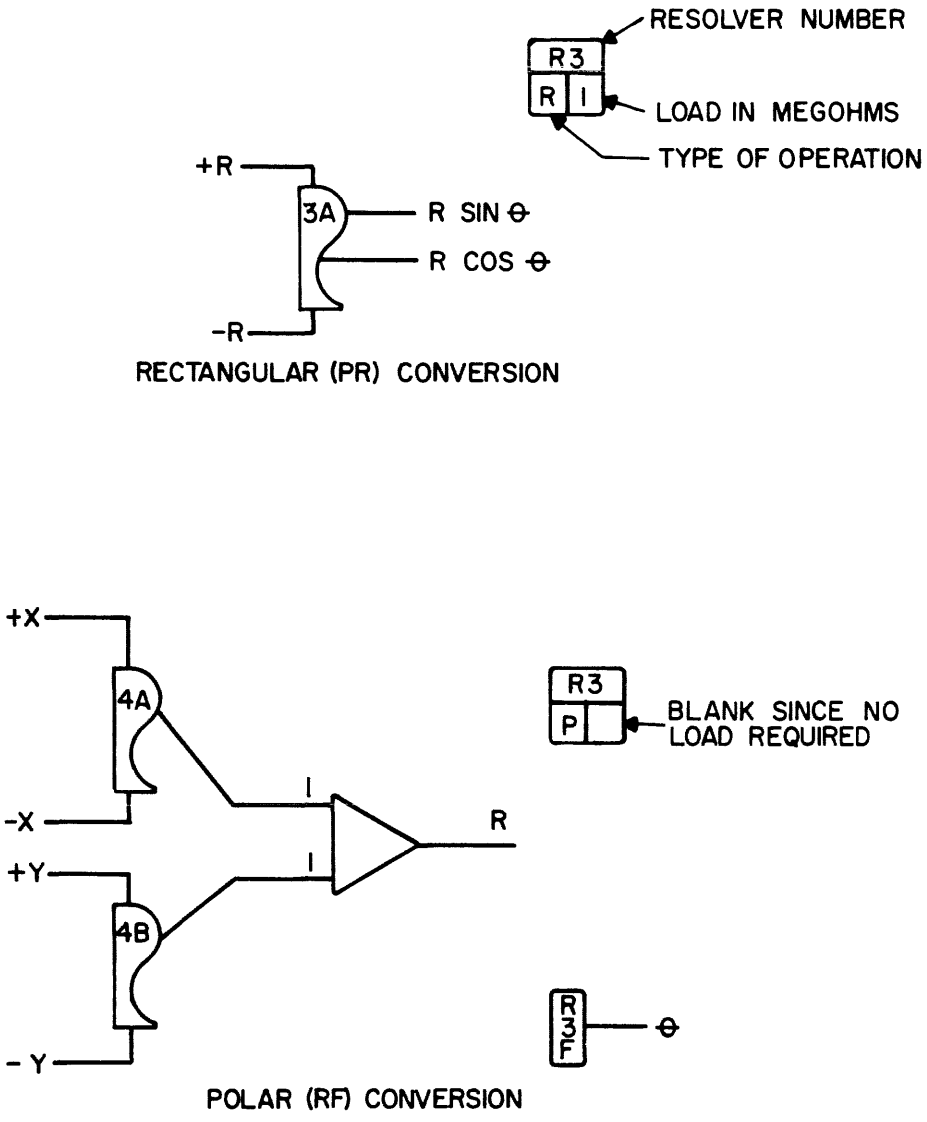
A servo rate resolver differs from a position resolver in that it accepts a rate input, $d\theta/dt$ rather than a position input. It is useful in applications where $\dot{\theta}$ is available and it is not desirable to produce θ by integration. because the system performs many revolutions and θ would have to be scaled at a small scale factor.

programming symbols

Programming symbols for the three operations described above are shown in Figure II-32. These symbols differ from linear servomultiplier symbols only in so far as the nonlinear, sinusoidally-wound cup is concerned, which is represented by the symbol



ii. Electronic Resolvers -- Servo resolvers, like all servo computing components, can not function efficiently if driven by high frequency inputs which often occur in practice. The obvious solution to the frequency response problem is the construction of an all-electronic resolver using sin θ and cos θ fixed diode function generators, operational amplifiers, and quarter square multipliers, to solve the resolver equations presented in Figure II-15.



Additional sinusoidal cup and summing amplifiers required for axis rotation

Figure II-32: Servo Resolver Programming Symbols

The drawbacks to mechanizing servo resolver circuits using components available on a computer patch panel are twofold:

1) The non-linear components required to mechanize the resolution circuits detracts from the problem solving capacity of the computer;

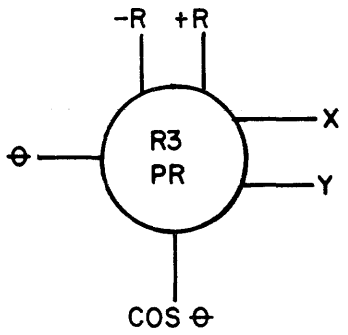
and

2) The sin θ and cos θ function generators have limited ranges ($\pm$ 90° or $\pm$ 180° is typical).

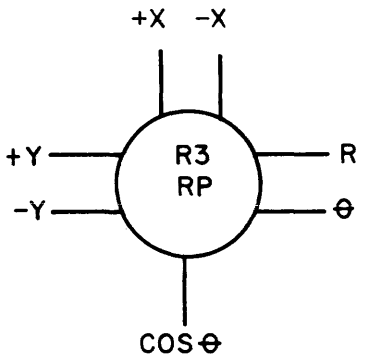
The electronic resolver solves both these problems and manufacture's literature should be consulted for more detailed information.

A detailed explanation of the electronic resolver is beyond the scope of this text and will not be presented.

The electronic resolver programming symbol in the polar-to-rectangular mode is



and, for the rectangular-to-polar mode



iii. Comparison of Resolvers -- as in the case of multipliers, the type of resolver required to perform a specific computation task must be defined by the programmer and he must, therefore, be familiar with their properties. Servo resolvers are desirable because of their accuracy, and also because several resolutions can be carried on the same signal at once, as well as multiplication. However, servo resolvers are not suited for high speed operation. The choice of rate resolvers usually depends on whether continuous or limited rotation is desired. Electronic resolvers suffer from increasing percentage error for decreasing input levels, but they are capable of operating at high speeds and generally offer continuous rotation without difficulty.

c. Function Generators---The generation of arbitrary as well as analytic functions by means of function generators normally is required to obtain variable coefficients or parameters of a system, or to generate an input forcing function.

The topic of the fixed DFG has been discussed already in detail. However, there are other function generation components whose theory of operation is worthy of presentation. They are the Potentiometer Padding Unit (pot padder), variable diode function generator (VDFG), curve follower and bivariant function generator. The latter two devices require a familiarity with the x-y plotter which is one type of analog computer readout device.

i. Potentiometer Padding Unit -- the Potentiometer Padding unit, (pot padder) provides a means whereby a number of predetermined voltages can be set up and applied to taps on a servo multiplier potentiometer. In normal use, a servo multiplying cup is a nearly-linear device, the output from which is proportional to the position of the wiper, as shown in Figure II-33. However, if we apply a number of known voltages to the potentiometer at fixed points along its length, then the voltage at wiper will no longer be proportional only to its position but will be dependent on the voltages applied.

A typical pot padder permits the application of 17 voltages at 17 fixed and equally-spaced points along the potentiometer. As the wiper moves past these points, it will pick up those voltages which have been applied and which correspond exactly to points on the desired function. Between these points the voltage picked up will be approximately correct, being a straight line approximation ("linear interpolation") of values between two points close together on the curve.

A typical pot padder circuit, together with its programming symbol, is shown in Figure II-34. Note that the pot padder is a separate unit which converts a standard servo multiplier to a function generator. The pot padder contains slope polarity and control switches, and potentiometers which force voltages on tapped cups using complex resistor networks.

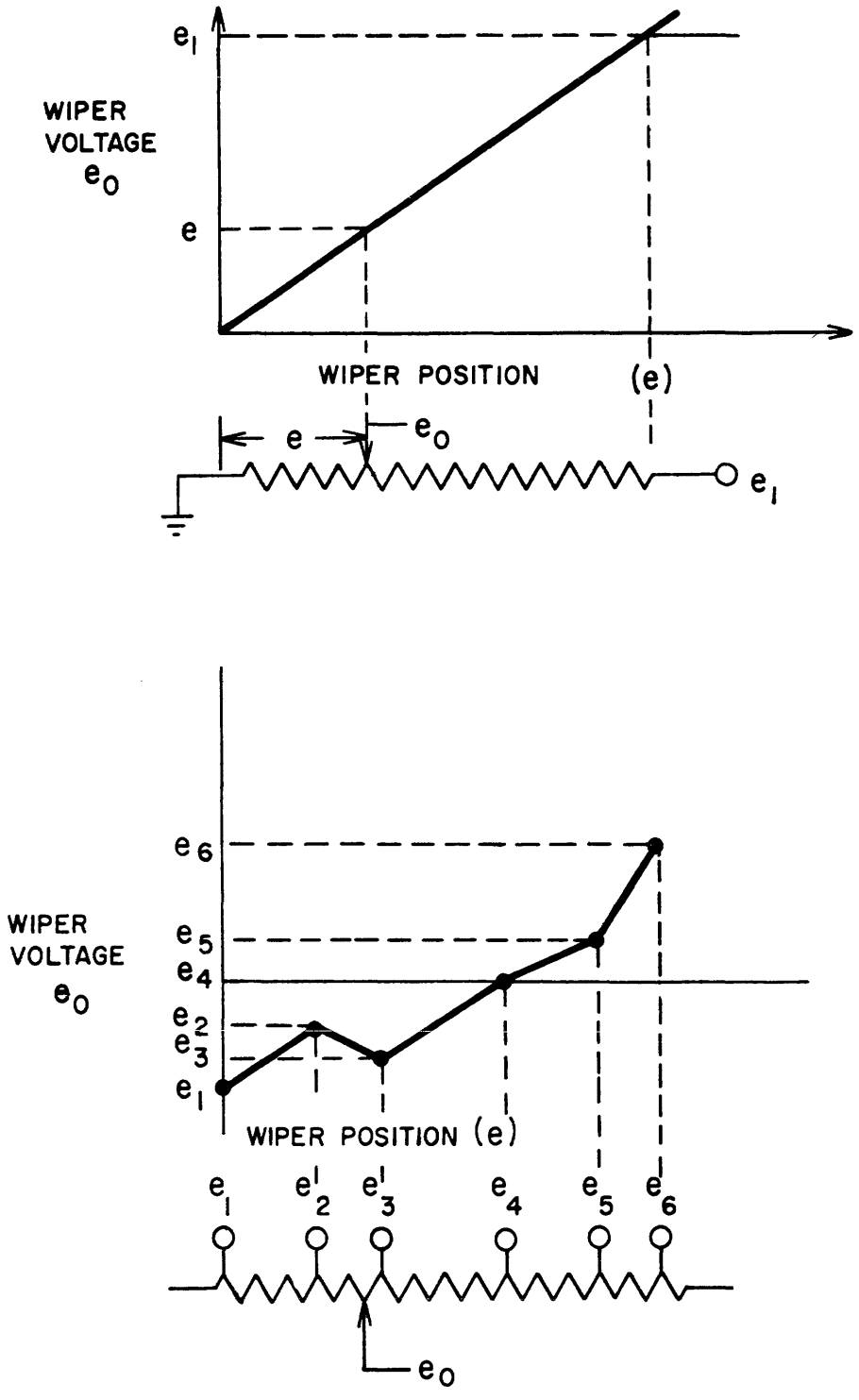


Figure II-33: Function Generation With Potentiometers

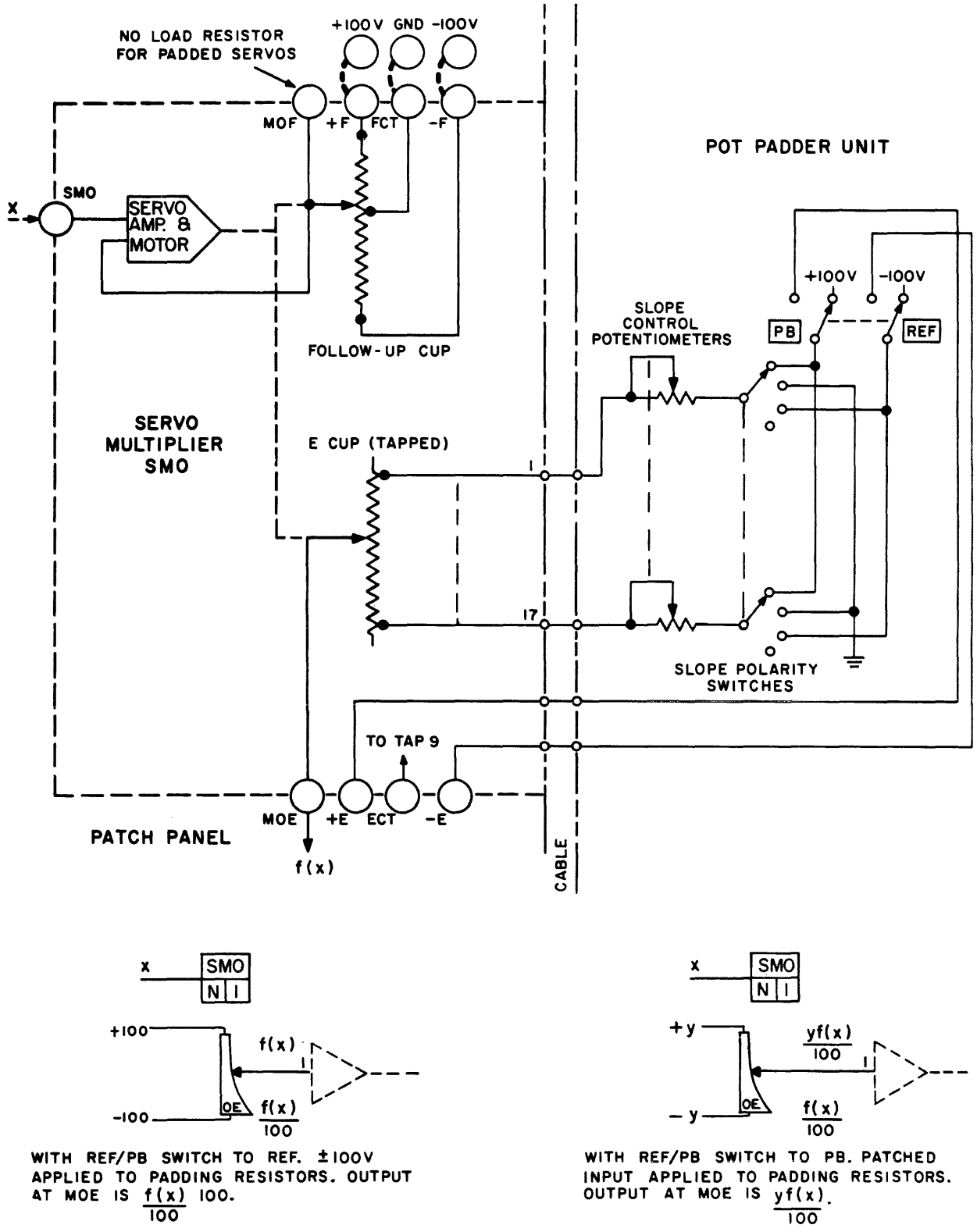


Figure II-34: Function Generator Circuit and Symbols

Although the pot padder can feed a wide selection of input resistors, its follow-up cup is not loaded. Loading is accounted for in the procedure used to set-up the unit.

One advantage of this unit is that it can be used to form the product of two variables, one of which is generated in the pot padder itself, as shown in Figure II-34.

The fact that the taps are equally spaced is sometimes disadvantageous since functions often have regions where the slope is very steep, and pot padders are designed with a voltage limitation between adjacent taps. This restriction is required to protect the multiplier cup windings.

ii. Variable Diode Function Generators -- it is evident that any reasonably-behaved, single-valued function can be approximated by straight line segments if the breakpoints of the diodes and the values of the input resistors are properly chosen. While, theoretically, it is possible to achieve any degree of accuracy desired by increasing the number of segments (crowding the breakpoints), in practice, the non-ideal switching characteristics of the diodes and other considerations limit the number of segments used. Also, all fixed DFG's suffer from constant level errors due to the diode switches and these errors, of course, are more noticeable at low input levels. It should be noted also that while a straight line approximation may give a good fit to the function itself, the slope of the DFG segments do not necessarily fit the slope of the function equally well. Consequently, differentiation of a DFG output should not be attempted.

If, in a DFG, the input resistors are made adjustable, then it is possible to select the slope of each segment generated. If, in addition, the breakpoint of each diode is allowed to vary (by adjusting the bias voltage), then the operator essentially can adapt a single DFG to any arbitrary, well-behaved, single-valued function by selecting b p (break points) and slopes appropriately. A DFG provided with b p and slope controls is a VDFG*, Figure II-35, illustrates the basic VDFG arrangement.

In Figure II-35 (a), Z_1, Z_2, Z_3 represent biased diode networks which control the currents i_1, i_2, i_3. The diode within each network will conduct only when the input voltage, x, exceeds a certain value determined by the setting of a BREAKPOINT potentiometer. Thus, if the input voltage, x, is less than the breakpoint setting of Z_1, the current, i_1, is zero. If x is greater than the breakpoint voltage, x_{b_1}, i_1 is controlled by the setting of the SLOPE potentiometer. By setting the breakpoint point and slope potentiometers of Z_1 and Z_2 appropriately, one obtains the output curve shown in Figure II-35 (b).

* VDFG, variable diode function generator

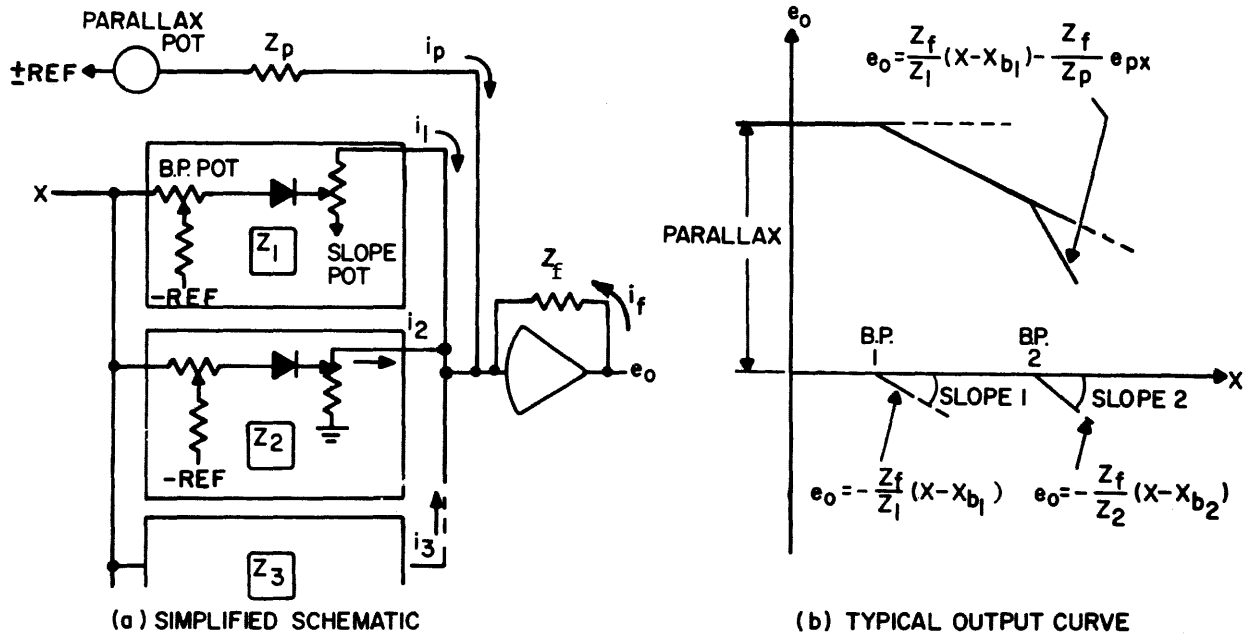


Figure II-35: Simplified Schematic Diagram of VDFG with Typical Output Curve

The parallax potentiometer allows a constant voltage (e_{px}) to be set into the amplifier as a bias. This shifts the whole curve along the Y axis, as shown.

d. Curve Follower -- The addition of a curve follower unit converts the convential X-Y plotter into a function generator by replacing the pen with an R - F pick up device, as shown in Figure II-36.

The curve to be generated is drawn in special conducting ink or wire and a high frequency signal is applied to its ends. The X arm is driven by the independent variable and the pick-up device stays close to the curve by correcting the high frequency signal error. The voltage representing the position of the pick-up along the Y axis for a given X is the output of the generator. Figure II-37 contains curve-follower programming symbols, and a simplified schematic diagram.

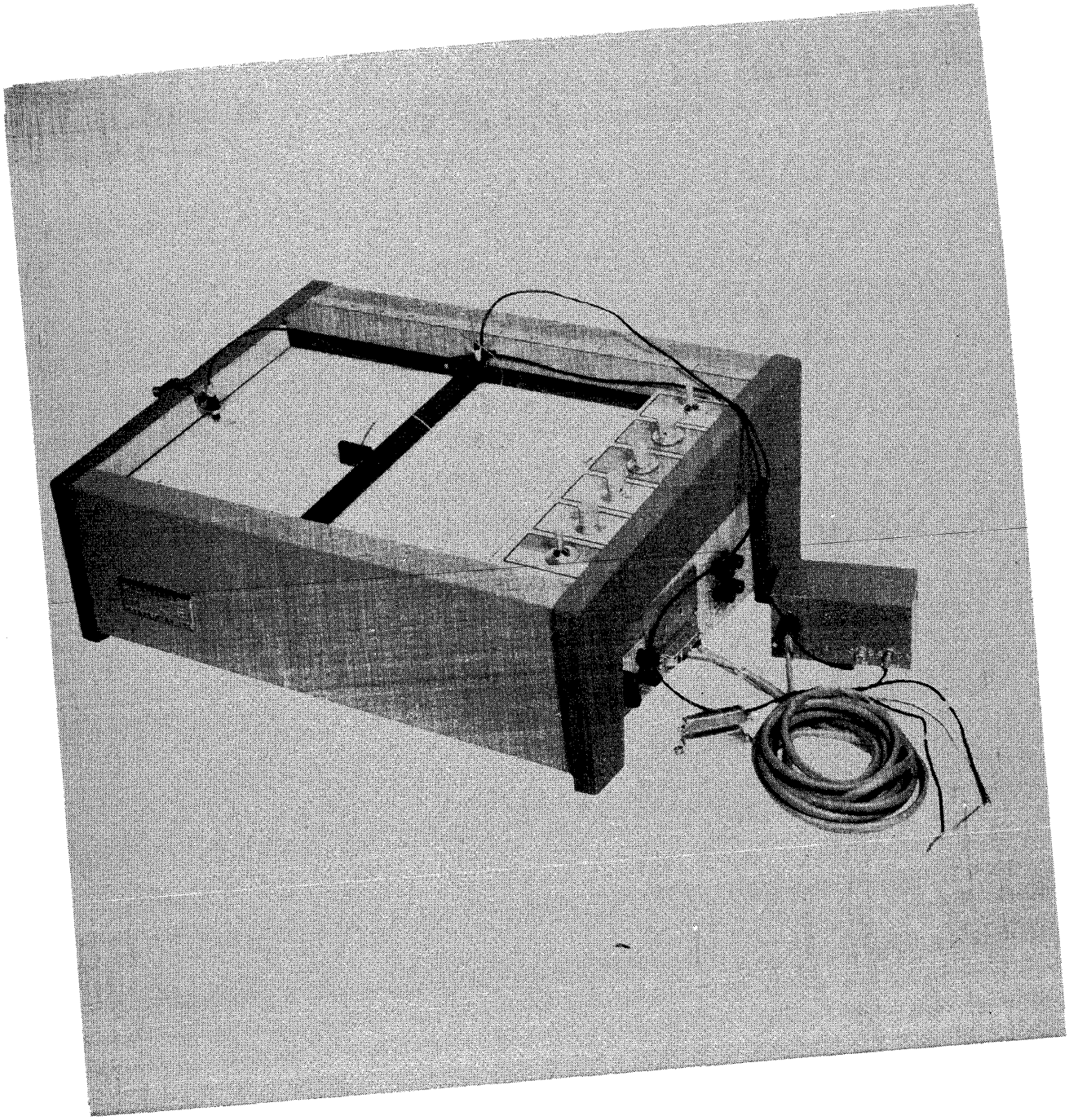
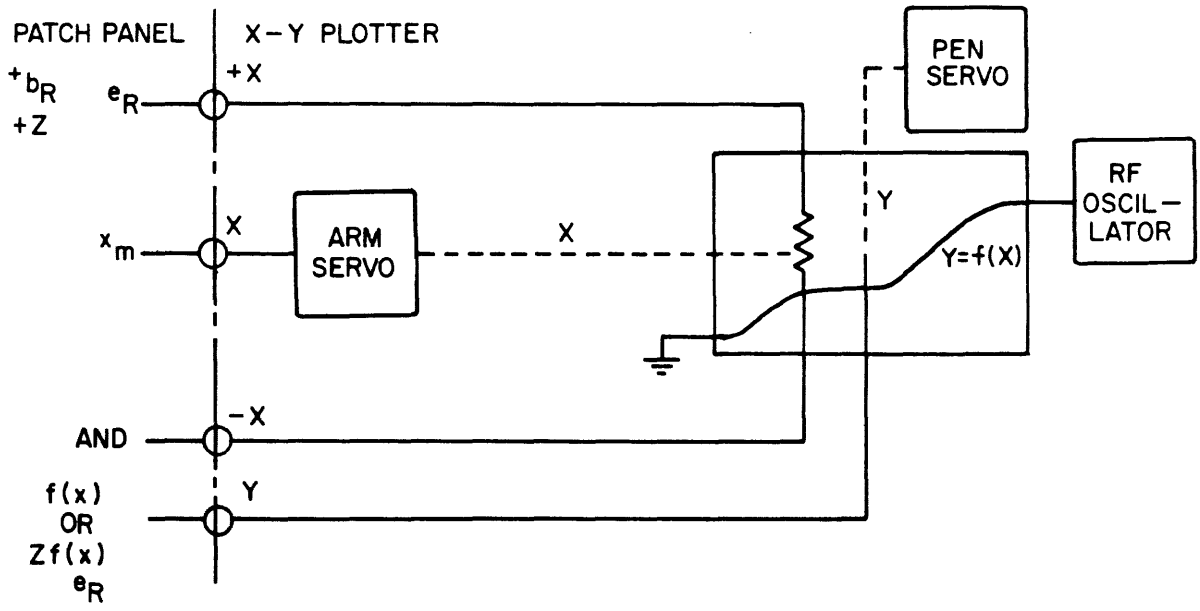
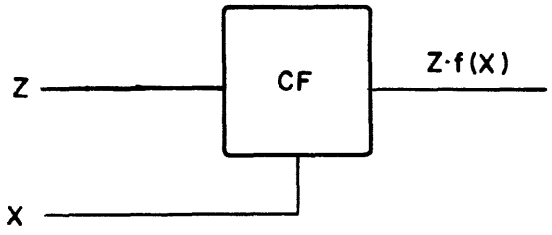


Figure II-36: A Typical Curve Follower



(a) Simplified Schematic of Curve Follower



(b) Computer Symbols for Curve Follower to obtain the product $z \cdot f(x)/e_R$.

Figure II-37: Simplified Schematic Diagram and Programming Symbols for Curve Follower

e. Bivariant Function Generators -- Generation of arbitrary functions of two (or more) variables, such as those arising in studies involving aerodynamic functions, presents a formidable problem in that the analog computer works with only one independent variable, time. However, a simple 2-variable function generator, commercially available, operates as follows using resistive paper and an X-Y plotter. Lines of constant Z, where Z = f(x,y), are drawn on the paper with low-resistance, conducting paint (silver paint, for example), and energized by d-c potentials corresponding to the value of Z. The voltage gradient between these lines is linear. The X and Y inputs position the pick-up device above the point (X,Y) where the voltage is approximately Z. This voltage is picked up and presented as the output of the function generator.

Other techniques for 2-variable function generators, using standard pot padders or DFG's have been developed. However, these are neither simple nor economical in terms of equipment.

f. Selection of Function Generators -- Effective function generation is essential in the efficient utilization of an analog computer, and Chapter V is devoted to presenting material on this topic. The chapter also contains a tabulation of the relative merits of function generation devices, which should govern the selection of the proper component to generate a specific function.

3. COMPUTATIONAL ACCESORIES

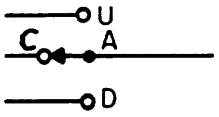
In addition to linear and non-linear equipment, several other components are usually available in the modern GPAC. Typical of these are

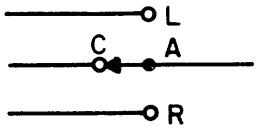
1) Passive Elements, specifically resistors, capacitors, and diodes

2) Switching and Limiting Devices, specifically function switches, comparators, and limiters

a. Passive Elements -- Resistors, diodes, and capacitors, besides those incorporated in the amplifier networks, can be used effectively in transfer function simulation where it is desired to minimize the number of integrators used. Thus, a single amplifier can perform first, as well as second and even third, degree transfer functions. (See Chapter VIII). For convenience, passive elements are available with patch cord terminations.

b. Function Switches -- These are used to make programmed changes in the computer set-up so that several slightly different problems can be investigated without repatching. The symbols for some typical function switches are





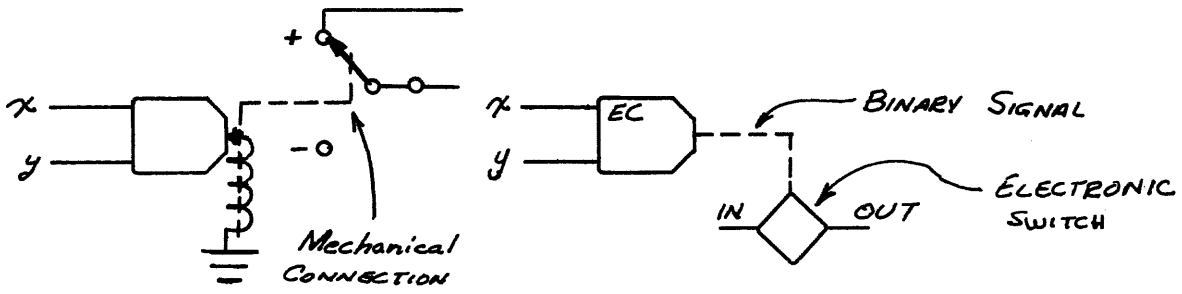
L = left
R = right
C = center
U = up
D = down

3 position SPDT horizontal switch

3 position, SPDT vertical switch

C. Comparators - Comparators are either of the relay or electronic type. The former consists of an amplifier which drives a relay having at least one set of contacts; the latter consists of an amplifier and associated circuitry which produce a binary (two-valued) signal. The binary signal can be used to control an electronic switch (usually SPST). Relay comparators have poor dynamic response - they require milliseconds of time to switch and often suffer from contact bounce at high speeds. Electronic comparator-controlled switches are extremely fast - microsecond response times, no moving parts.

Both types of comparators operate from a pair of input signals, one of which is usually a constant value. When $x + y > 0$, the relay arm moves to close the circuit between the arm and the contact labeled "+". (The electronic comparator produces a signal which would cause the electronic switch to conduct.) When $x + y < 0$, the closed path is arm-to-minus contact (or the binary signal turns off the electronic switch).



Relay Comparator

Electronic Comparator

4. PRE-PATCH PANELS

Analog computers, designed for operator convenience and rapid problem "turn-around", are equipped with removable pre-patch panels. This permits patch boards to be stored pending future investigation, and allows patching and problem preparation for analog solution while the computer itself is in use on another problem. Typical patch panels are shown in Figures II-38 and II-39.

Modern GPAC patch panels, which on first impression appear to be complex (and colorful), are in reality quite simple. They are modular in format; therefore, one need only become familiar with the patching terminations of few computer components (i.e. amplifiers, multipliers, etc.) to understand patching for the complete computing system. The components contained in one module are duplicated in remaining modules, the only exception being the number designation of each component.

Patch panels (Figure II-39) indicate the gains of input terminations, summing junctions, relay terminations, etc., of individual components.

The purpose of the color code is to guide the operator in patching. For example, red terminations are usually component outputs and green terminations are inputs. Small groupings of patching terminations which are not part of the basic module pertain to trunk lines (console interconnections), readout devices, and miscellaneous components such as Noise Generators.

Interconnections between patch panel terminations are made using patch cords and bottle plugs. Patch cords, which come in various lengths, are color coded to facilitate storing and patching.

Bottle plugs, which are encapsulated patch cords, are used to interconnect adjacent terminations, or to perform frequently-occuring interconnections. For example, bottle plugs are used to interconnect all relay coils, relay-buses, input networks, and feedback paths associated with an amplifier to convert it to an integrator. The use of bottle plugs reduces patch panel clutter, and relieves the operator of making repetitive, commonly-occuring patching connections.

Specific patching connections required to interconnect analog computer components are a function of the particular computer used. Therefore, one must consult reference material associated with a computer prior to problem patching.

a. Electronic Digital Voltmeter (EDVM) -- The EDVM is a precision device that displays (in Figure II-40) component address and analog value. The EDVM combines speed and accuracy with high input impedance, usually greater than 100 megohms. The voltage measured is shown in illuminated arabic numerals.

   The high conversion speed -- an average of 200 conversions per second -- permits almost instantaneous readings of slowly varying voltages. This speed, plus the feature of push-button readout by which any monitored voltage is displayed on the 5-digit-plus-sign display panel at the operator's option, reduces pot-setting time by 50%.

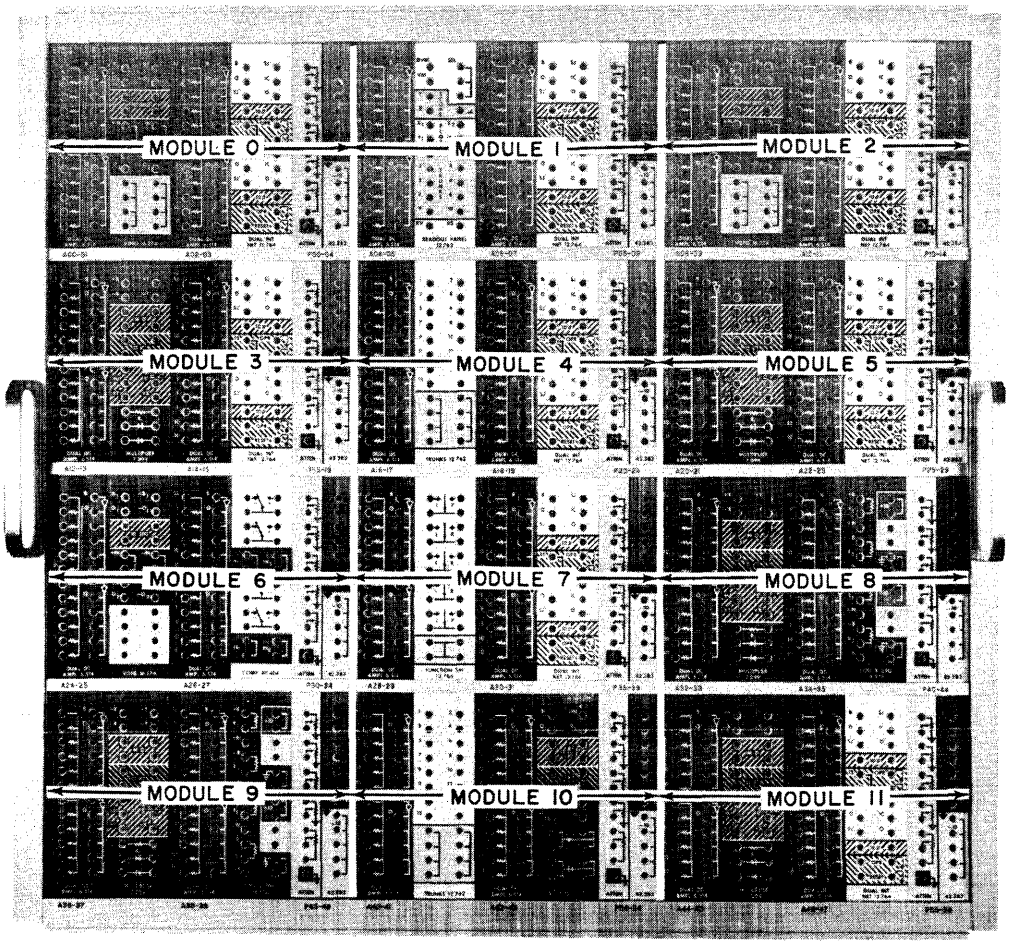


Figure II-38: Typical Patch Panel.

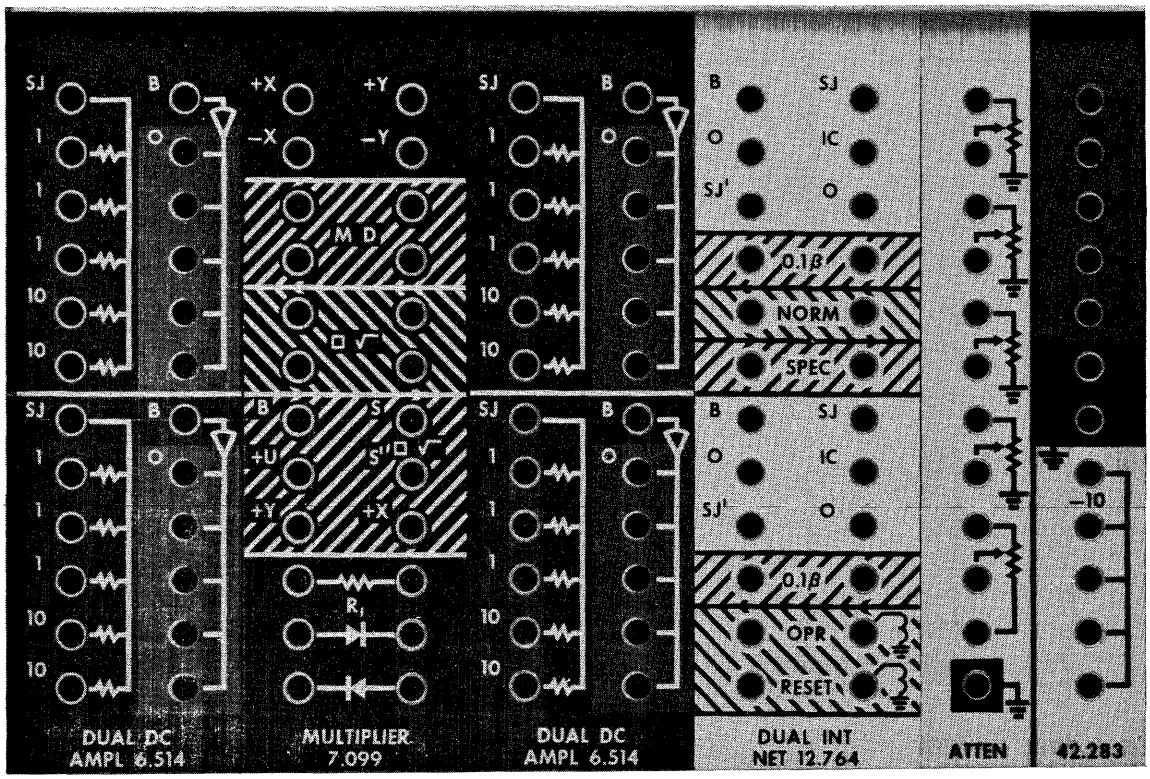
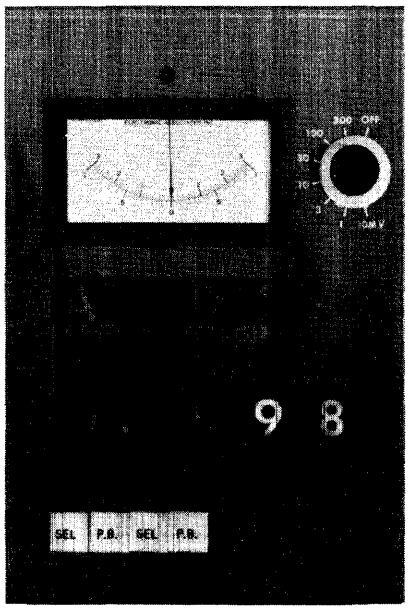


Figure II-39. Typical Patch Panel Module.



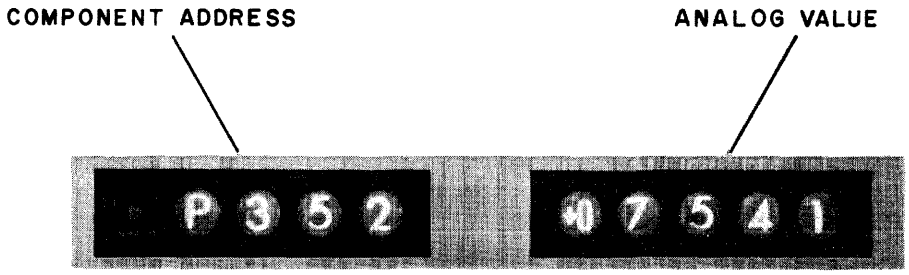


Figure II-40: Typical Digital Voltmeter Displays

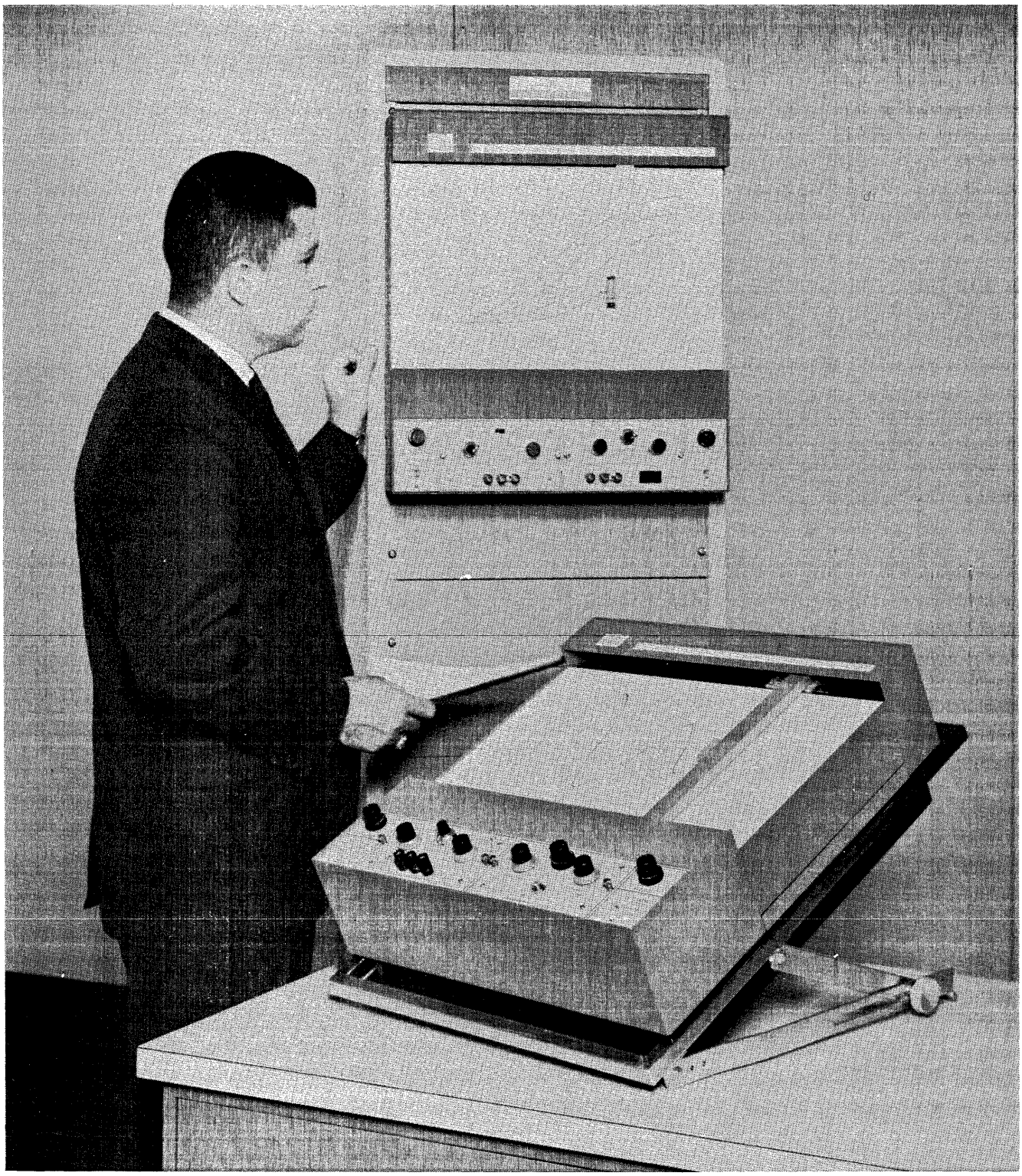


Figure II-41: 1110 X Y Plotter

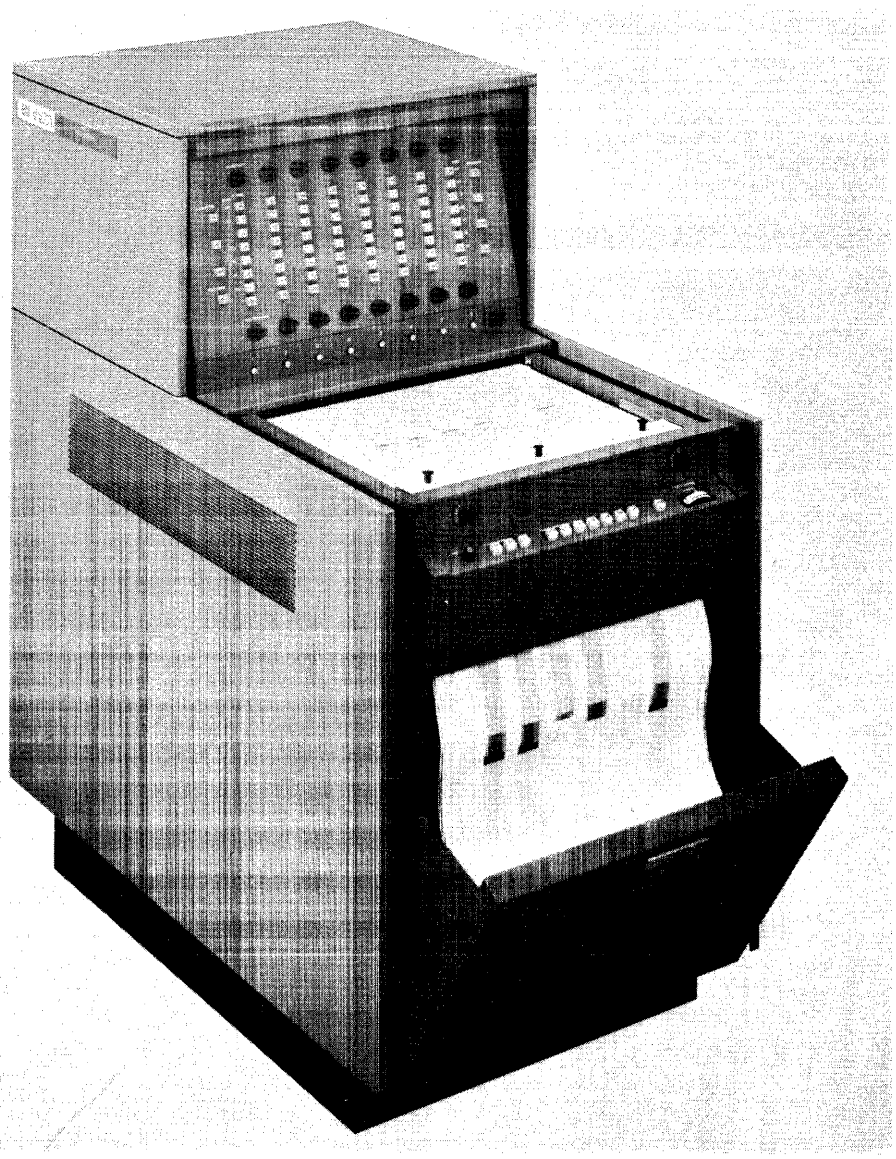


Figure II-42: **Typical Strip Chart Recorder**

b. X-Y Plotter -- The X-Y plotter is shown in Figure II-41. It is used to record two variables simultaneously in the form of a rectangular coordinate graph. It employs two servo systems similar in principle to that of the servo multiplier. The arm is positioned by a drive tape driven by an electric motor. This tape is connected to the wiper of a potentiometer energized by reference voltage. The wiper voltage is, therefore, proportional to the arm position. This voltage is compared with the "arm input" voltage from the computer by means of a different amplifier which produces an error voltage proportional to the difference between the two signals. This error voltage controls the servo-motor, moving the arm in the direction of decreasing error until a null is obtained. A similar servo-mechanism positions the pen on the arm in accordance with a second input voltage. The result is a rectangular plot of one voltage versus another.

The plot is made in ink on paper, the paper being held in position by a vacuum system. The variplotter table is made of plastic. In this surface, there are many shallow grooves. At intervals in in the grooves, holes are drilled through the table and these are connected to a vacuum pump. The subatmospheric pressure in the grooves holds the paper in place.

c. The Multi-Channel Recorder--The multi-channel recorder shown in Figure II-42 employs a moving strip of paper, drawn at constant speed past a set of pens. Most of the pens are deflected proportionally to input voltages, and the result is a set of graphs of voltages as functions of time. Additional pens, called "event" pens, one at each edge of the paper, produce timing marks at one-second intervals.

d. Oscilloscope for Repetitive Operation Displays -- oscilloscopes are used for computer readout during high speed repetitive operation which will be discussed in later chapters. In repetitive operation, solution speeds of 50 solutions per second are typical. This speed, of course, precludes the use of the X-Y plotters or multichannel recorders as readout devices.

The oscilloscopes used in computing systems (Figure II-43) have from four to eight curves displayed simultaneously, and the capability of cross-plotting.

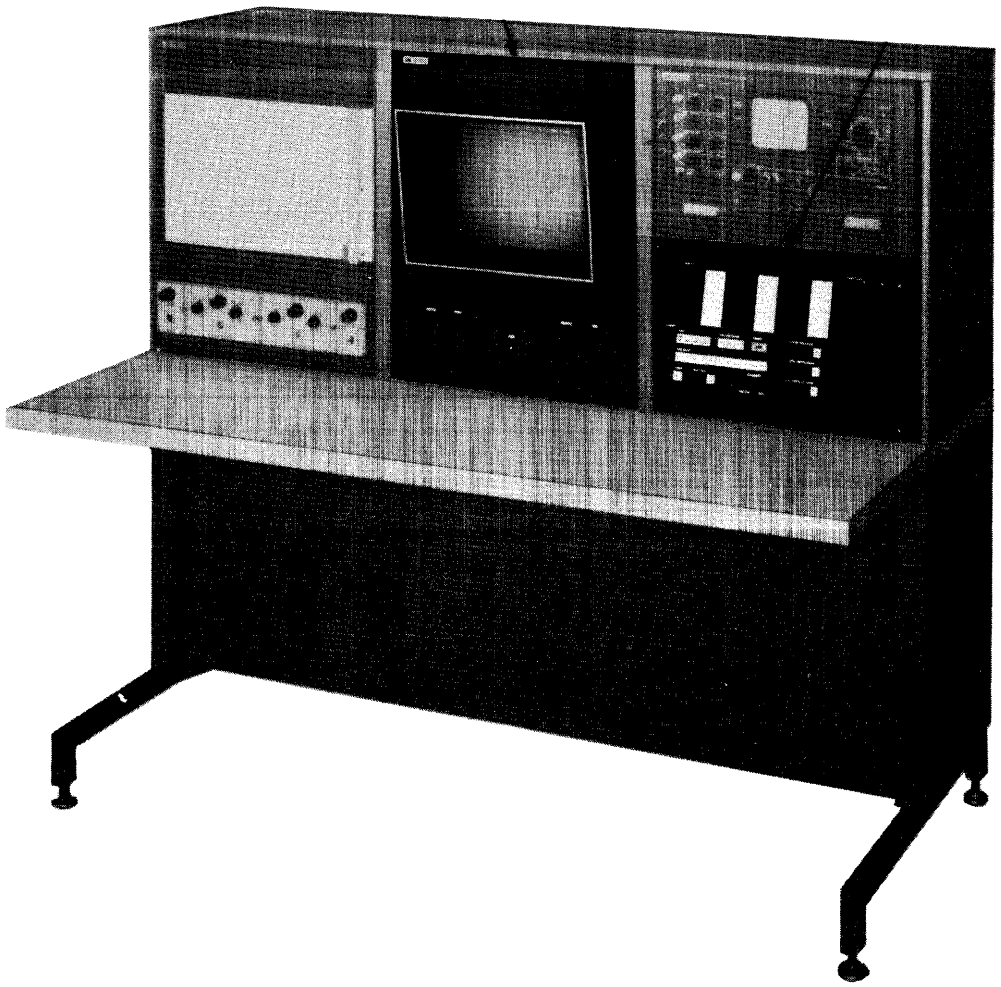


Figure II-43: 8880 Display System

CHAPTER III
ANALOG COMPUTER PROGRAMMING AND CHECKING PROCEDURES

This chapter covers the basic techniques of programming, scaling, and checking out a problem on a general purpose analog computer. It is divided into five sections:

A. Programming
B. Scaling
C. Equipment Assignment
D. Documentation
E. Static Check

Since the most common mathematical models solved on analog computers are sets of ordinary differential equations, this chapter will concentrate on differential equation solution. Some information and references on the solution of algebraic equations are given in Chapter IV, and partial differential equations are covered in Chapter XI.

A. PROGRAMMING

1. Example: A First Order Equation

Let us start with a simple first order equation:

$$\frac{dx}{dt} = - kx \text{ (k constant)} \qquad (1)$$

This equation may describe a number of different physical phenomena, such as radioactive decay, dilution in a stirred tank, the discharge of a capacitor, a first order chemical reaction, and many others.

The two variables in this equation are x and dx/dt. How can we represent the relation between them with analog components? Clearly, the component that accomplishes this is the integrator:

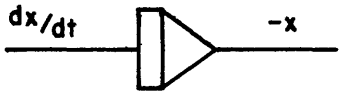


Figure 1

If we feed dx/dt into an integrator, the output will be -x. But where can we obtain dx/dt? From Equation 1, we see that this is simply -kx. Since -x is available at the integrator output, we can use a pot to multiply it by k:



Figure 2

This produces -kx at the output of the pot. Since this is exactly what is needed at the integrator input, we can provide the desired input by connecting the pot output to the integrator input:

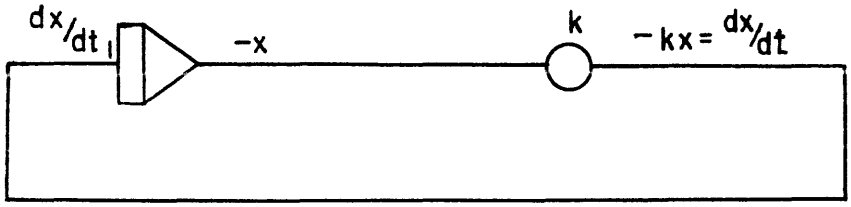


Figure 3

In order to specify the solution completely, we need to know the initial value of x. (Equation 1 actually has infinitely many solutions — one for each initial condition.) If this initial value is given, we may implement it on the computer by means of the IC terminal on the integrator. The appropriate IC voltage will be achieved by means of reference voltage and a pot:

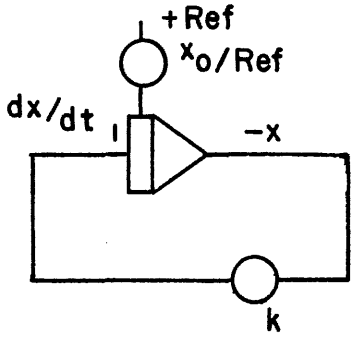


Figure 4

Note that the setting on the IC pot is x_0/Reference Voltage. The pot is connected to +Reference so that the initial voltage out of the integrator will be negative.

Note that this circuit produces the output -x. If it is desired to generate +x instead, this can be done by integrating -dx/dt:

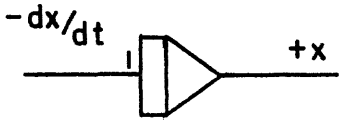


Figure 5

Since -dx/dt = +kx, we can close the loop to obtain Figure 6:

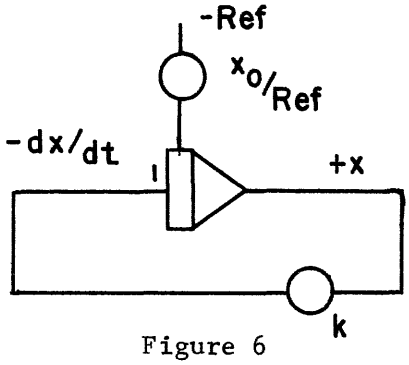


Figure 6

Note that the only difference between Figure 4 and Figure 6 is that Figure 6 uses minus Reference on the IC pot to obtain a positive initial condition on the integrator.

The programming technique used in solving the above problem is simply stated as follows. We start by assuming that dx/dt (or -dx/dt) is available, and then integrate it to produce the variable x at the integrator output. We then use this integrator output to generate the derivative that we started with. The reasoning appears circular: If we have -dx/dt available, then we can integrate it to obtain x; if we have x, we can multiply it by k to obtain -dx/dt. Does this simple approach really lead to a valid solution of the given equation?

Perhaps the best way to see that it does, is to consider what happens if we put an initial condition on the integrator and put the computer into the OPERATE mode. For concreteness, assume x_o = 5 volts and k = 0.2. Referring to Figure 6, we see that the initial condition of +5 volts at the integrator output produces +1 volt at the integrator input. Therefore, the integrator output will decrease at the rate of 1 volt per second initially.

Of course, the rate of change will not remain constant at 1 volt per second. As the integrator output decreases, its rate of change will also decrease. In fact, the rate of change of the integrator output x at any given time will be proportional to the value of x at that time. If we translate this last statement into an equation, we obtain:

$$dx/dt = - kx$$

which is Equation 1, the equation we started out to solve.

The circuit in Figure 6 is mathematically similar to a radioactive isotope, an R-C circuit, a stirred tank, or any other system that satisfies the same equation. Starting with a 5 volt initial condition on the integrator and watching it integrate down to zero is analogous to starting with 5 grams of an unstable isotope and watching it decay, or starting with 5 pounds of salt in a well-stirred tank and flushing it out by forcing a stream of fresh water through it at a steady rate. The tank, the isotope, and the computer circuit all satisfy the same equation, and therefore exhibit similar behavior. Each of these systems is said to be a model or analog of the others (hence the name "analog computer"). The computer circuit is also said to simulate the other systems.

2. Example: A Second Order System

As a second example, consider a slightly more complex equation:

$$a \ddot{x} + b \dot{x} + c x = 0 \qquad (2)$$

where the dot over a variable represents differentiation with respect to time, e.g. $\dot{x} = dx/dt$, and $\ddot{x} = d^2x/dt^2$. A spring-mass system with damping, and an R-L-C circuit provide examples of physical systems described by this equation.

a. Obtaining the Circuit

This equation may be solved on the computer in essentially the same manner

as Equation 1. In solving Equation 1, we started with an integrator, and assumed that the desired input was available. In this case, since the equation is second order, we must integrate $\ddot{x}$ to produce $-\dot{x}$, and integrate $-\dot{x}$ to provide $+x$:



Figure 7

Now how do we provide the necessary input $\ddot{x}$? To do this, we must solve Equation 2 for $\ddot{x}$:

$$\ddot{x} = -\frac{1}{a}(b\dot{x} + cx) = -\frac{b}{a}\dot{x} - \frac{c}{a}x \qquad (3)$$

This equation says that we can obtain $\ddot{x}$ from $\dot{x}$ and x by means of a summing amplifier and two pots:

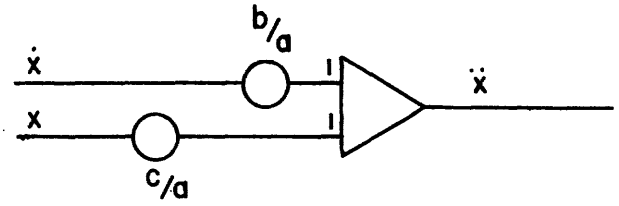


Figure 8

Observe the effect of the sign inversion in the amplifier. The sum of the inputs is $-\ddot{x}$ (from Equation 3) so that the output is $+\ddot{x}$.

To close the loop, we must combine Figure 7 and Figure 8.

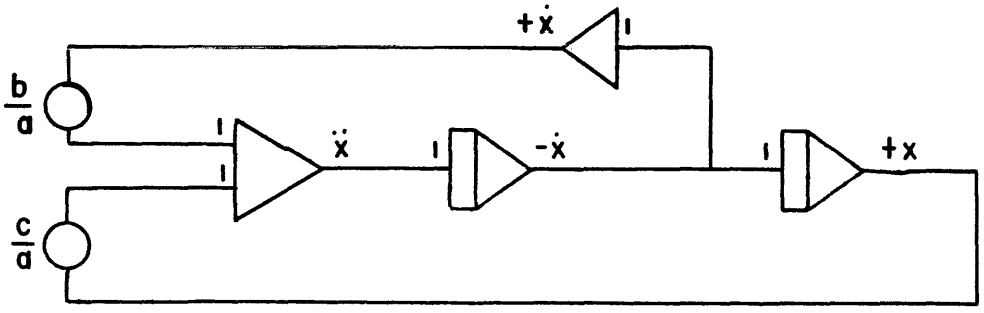


Figure 9

Note that an inverter is required, since $+\dot{x}$ is called for, but $-\dot{x}$ is available. Since the system contains two integrators, two initial conditions must be specified (the initial values of x and $\dot{x}$). Once these are given, they may be

established at the integrator outputs by means of pots and reference voltage.

b. Justification of the Circuit

The circuit of Figure 9 was obtained by the same simple technique that was used to obtain Figure 4 and Figure 6. To increase our understanding of the circuit (and our confidence in the method) we may analyze this circuit by considering the effect of a 5 volt IC on the integrator producing x. For simplicity, set b=0 and c/a=1, so that the inverter and the pots are eliminated (Figure 10). Note that in this case, the summer producing $\ddot{x}$ becomes simply an inverter.

Figure 10a shows the state of the circuit in the IC (or RESET) mode. Amplifier #3 has its 5 volt initial condition, and amplifier #1, which is acting as an inverter, produces -5 volts. With no IC patched into amplifier #2, its output is zero.

Figure 10b shows a plot of the amplifier outputs after a short period in the OPERATE mode. The reader should be able to verify that all amplifier outputs will change in the direction indicated. For example, amplifier #2 receives a negative input, which causes it to integrate upward (remember the sign inversion in an amplifier!). As the output of amplifier #2 becomes positive, amplifier #3 begins to integrate downward. Since amplifier #1 is simply an inverter, its output follows the output of amplifier #3.

As amplifier #3 continues to integrate downward, it eventually will cross zero (Figure 10c). At that moment, amplifier #1 also crosses zero. The reader should be able to convince himself from Figure 10c that the output of amplifier #1 is about to cross zero and become positive. Note that as the output of amplifier #3 becomes positive, amplifier #2 will start integrating downward. Therefore, at the instant when amplifiers 3 and 1 are crossing zero, amplifier #2 is passing through a maximum.

Continuing this reasoning, it becomes clear that the outputs of all amplifiers will oscillate. Figure 10d shows the oscillations for one complete cycle.

The oscillations observed in this system are to be expected in view of the fact (proved in any textbook on differential equations) that the solution of Equation 2 is a sinewave (in case b=0). In case $b > 0$, the inverter and pot in Figure 9 must be added, which will cause the oscillations to damp out.

The fact that the solution to Equation 2 oscillates is worth remembering. An analog programmer soon learns to recognize a loop with two integrators in it as an oscillator. Such loops frequently appear as part of a larger system. (For example, the automobile suspension system analyzed below may be thought of as two coupled oscillators.)

c. Alternative Circuits

The response of the system can now be investigated for a large number of values of the parameters a, b, and c. To change the value of c, all we have to do is change the pot setting labeled "c/a". Similarly, changing the pot setting labeled "b/a" will change the value of b, and changing both pots in the same proportion will change the value of a.

It would be more convenient to change a if this parameter were on a single pot.

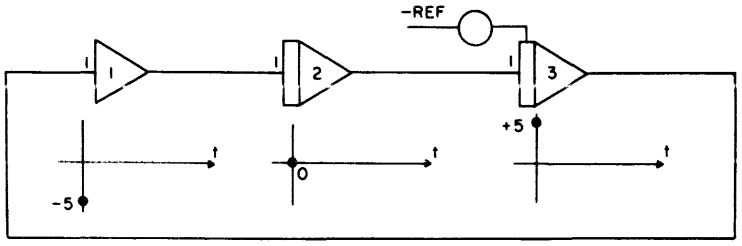


Figure 10a. The spring-mass circuit of Figure 4 shown in the initial condition, or RESET mode (t = 0). For simplicity, b = 0, and c/a = 1.

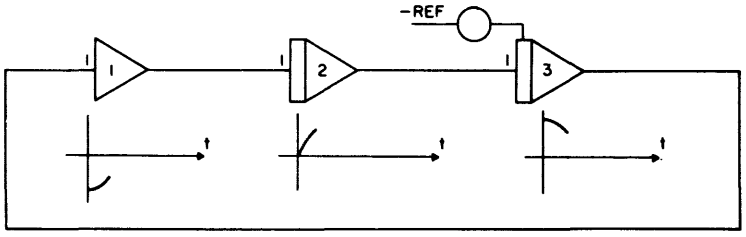


Figure 10b. Plot of output voltages versus time for the first few seconds of operation. Amplifier #1 is inverting Amplifier #3. Amplifier #2 is receiving a negative input voltage and integrating upward. Amplifier #3 is receiving a positive voltage and integrating down.

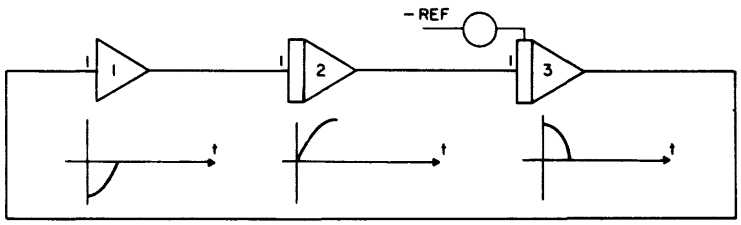


Figure 10c. Plot of output voltages up to the time Amplifier #3 crosses zero. Note that Amplifier #2 will soon be integrating downward, since Amplifier #1 is about to change sign.

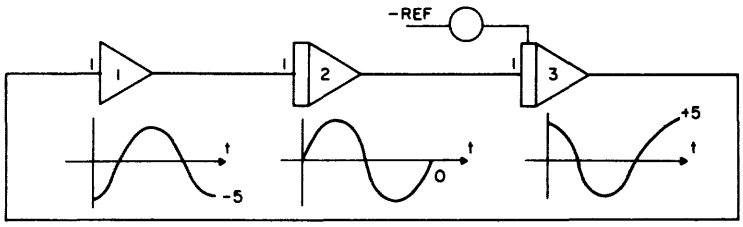


Figure 10d. Plot of output voltages for one complete cycle. Since all output voltages have returned to their initial conditions, the cycle will repeat itself.

This goal may be achieved with pot feedback, as in Figure 11.

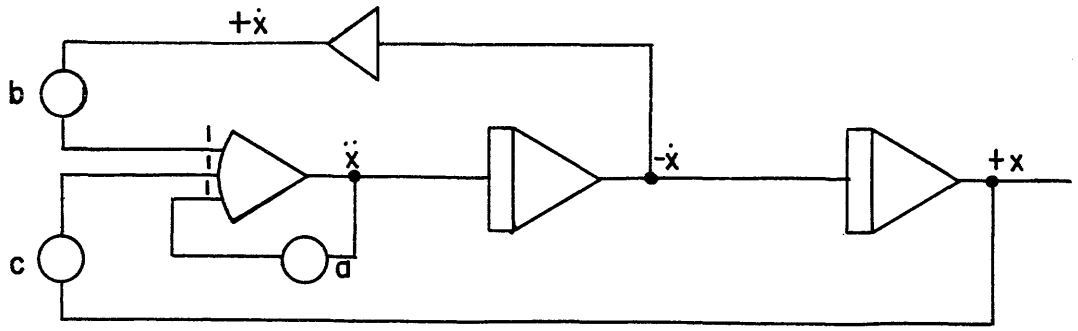


Figure 11

This circuit isolates each system parameter on a single pot, making parameter changes easier.

Another modification, useful when economy of amplifiers is important, is to make use of the fact that an integrator can add as well as integrate. If the two terms $(b/a)\dot{x}$ and $(c/a)\,x$ are fed directly into an integrator, the output is not $\ddot{x}$, but $\dot{x}$. This leads to the circuit in Figure 12.

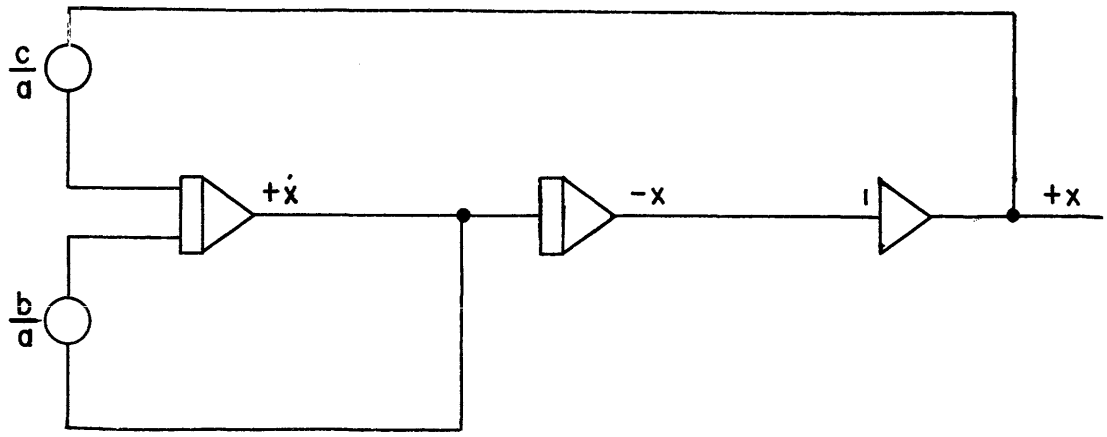


Figure 12

This circuit uses one less amplifier than the circuits of Figure 9 and Figure 11, but it has the disadvantages that $\ddot{x}$ is not available for recording, and that the parameters a, b, and c are not isolated on single pots.

It should be emphasized that all three of the above circuits are legitimate programs for solving Equation 2, and all will produce the same result. They differ in convenience, flexibility and economy of equipment.

3. Summary of Steps

1) Obtain a description of the system to be studied in terms of ordinary differential equations.

2) Solve each equation for the highest derivative that occurs in it.

3) Integrate these derivatives to obtain the lower-order derivatives and the variables themselves.

4) Feed these lower-order terms into the appropriate components, as called for by the equations, to generate the highest derivatives and "close the loop".

5) Provide initial conditions from pots and reference voltages, as required.

Step 4 can be carried out in more than one way since there are generally many different but equally correct circuits for any given problem. (See Figures 9, 11, and 12.)

4. Coupled Systems

The above examples required the solution of a single equation in each problem. However, problems involving systems of coupled equations may be solved in a similar manner.

For example, consider the system in Figure 13, which is a simplified model of one wheel of an automobile suspension system. The spring-action of the tire provides us with one equation, and the action of the auto spring and the shock absorber provide another. A force-balance yields the equations:

$$M_1 \ddot{x}_1 + D(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2) = 0 \qquad (4)$$

$$M_2 \ddot{x}_2 + D(\dot{x}_2 - \dot{x}_1) + K_1(x_2 - x_1) + K_2(x_2 - x_3) = 0 \qquad (5)$$

In these equations, x_1 represents the displacement of the chassis, x_2, the tire displacement, and x_3, the road function.

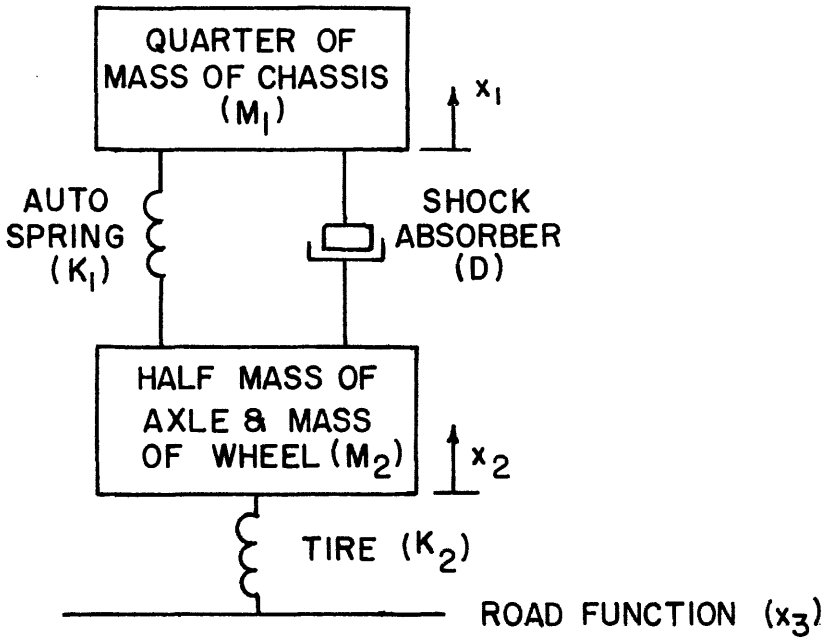


Figure 13 Simplified Model of an Auto Suspension System

For simplicity, we assume that x_3 is a step function, (for example, the car might be riding up over a curb). Other, more complicated, functions are possible, of course, to simulate a "wash board" road, etc. However, a step input keeps the program simple — we may represent x_3 by a pot connected to reference voltage. The quantities K_1, K_2, M_1, M_2, D, and x_3 are therefore all constant for a given computer run.

a. Programming

Following the rules outlined above, we start by solving each equation for the highest derivative:

$$\ddot{x}_1 = -\frac{D}{M_1}(\dot{x}_1 - \dot{x}_2) - \frac{K_1}{M_1}(x_1 - x_2) \tag{6}$$

$$\ddot{x}_2 = -\frac{D}{M_2}(\dot{x}_2 - \dot{x}_1) - \frac{K_1}{M_2}(x_2 - x_1) - \frac{K_2}{M_2}(x_2 - x_3) \tag{7}$$

The highest derivatives $\ddot{x}_1$ and $\ddot{x}_2$ may be generated by means of summers, or we may make use of the multiple inputs available on integrators to add and integrate at the same time:

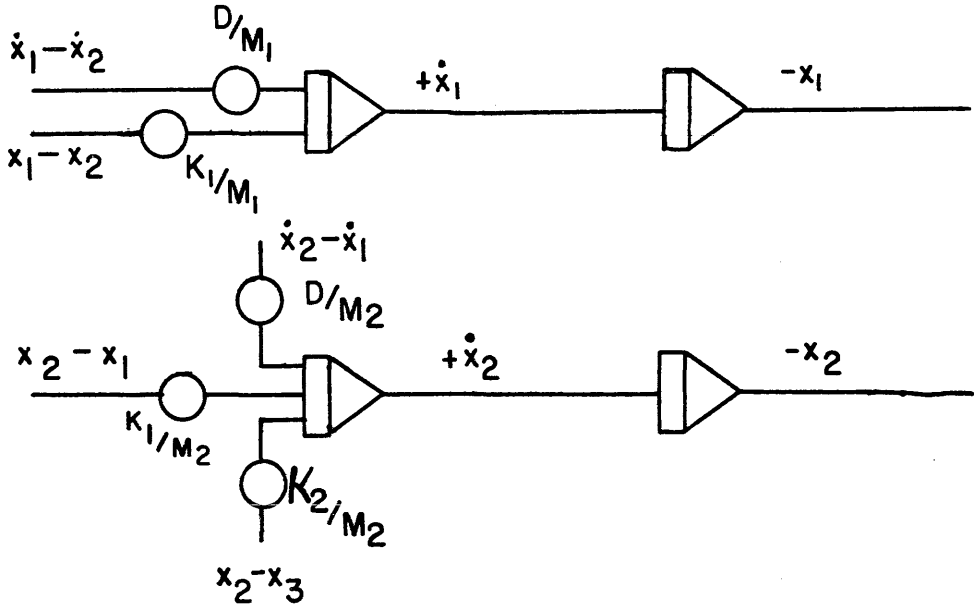


Figure 14

Note the effect of the sign inversion in each integrator: the sum of all inputs to the first integrator is $-\ddot{x}_1$, and the output is $+\dot{x}_1$.

The next step is to generate the terms needed at the integrator inputs. This will illustrate the coupling between the two systems. Note that each term required by the $\ddot{x}_1$ integrator is also required by the $\ddot{x}_2$ integrator, but with opposite sign. (Incidentally, this fact has a physical interpretation. By

Newton's Third Law, the spring and the shock absorber exert equal and opposite forces on M_1 and M_2). These terms may be generated with summing amplifiers:

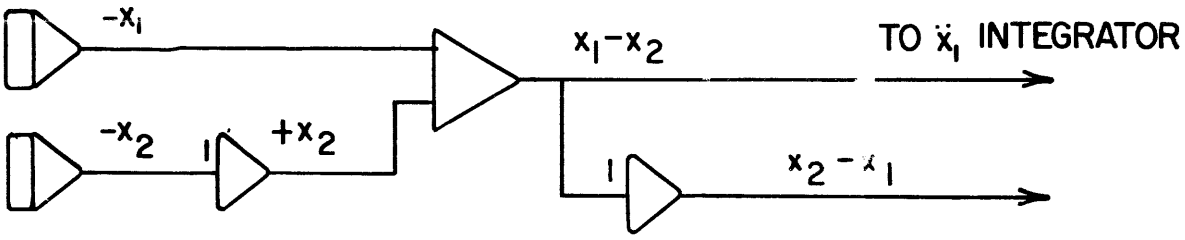


Figure 15

The damping term $(\dot{x}_1 - \dot{x}_2)$ can be generated similarly.

The $x_2 - x_3$ input required by the $\ddot{x}_2$ integrator may be generated by a summing amplifier:

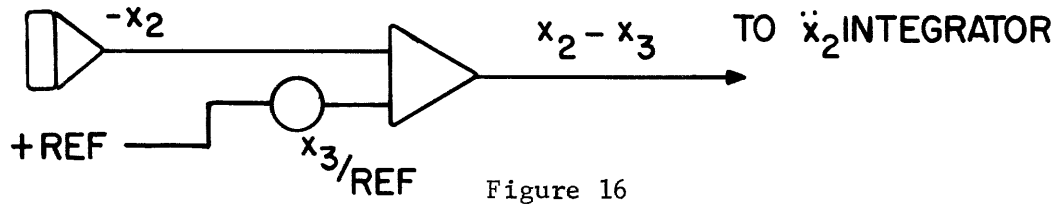


Figure 16

Combining Figures 14, 15, and 16, we arrive at a complete circuit:

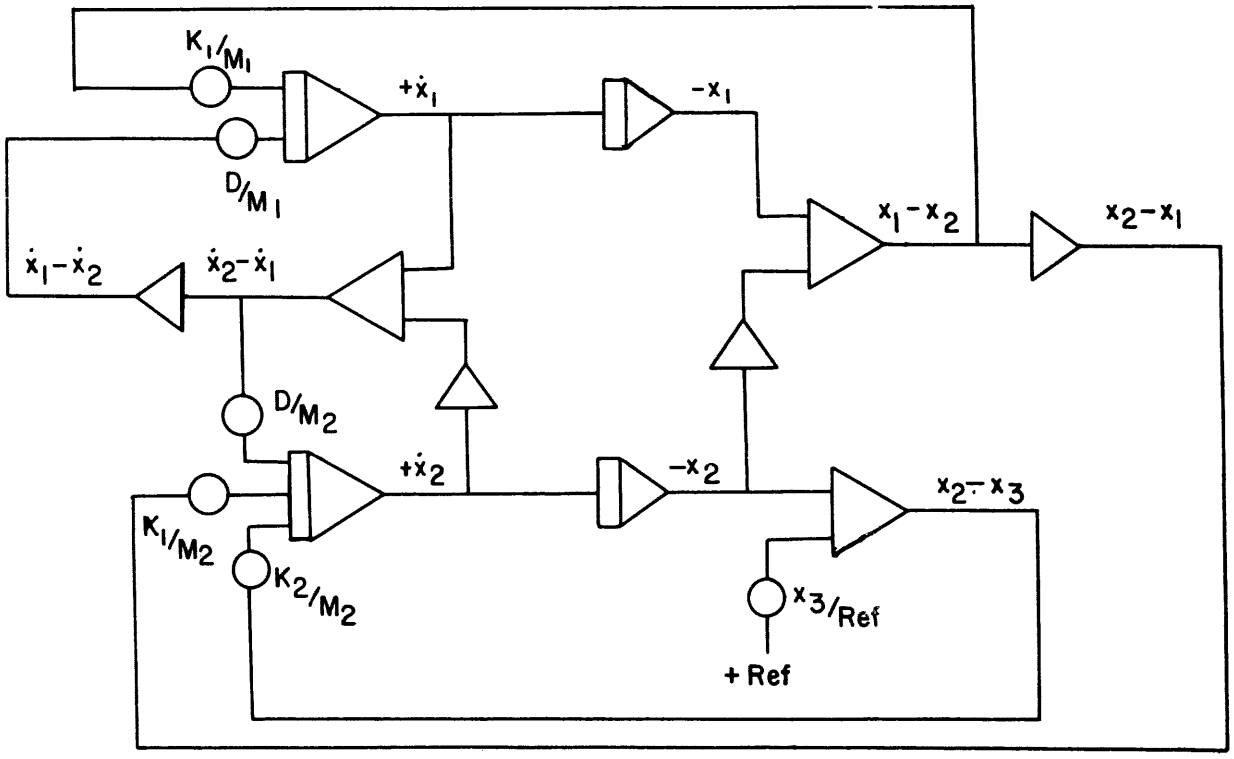


Figure 17: Complete Circuit for Auto Suspension Problem

b. Amplifier Economy

Note that four inverters are required to generate the coupling terms x_2-x_1, x_1-x_2, etc. Since both the $\ddot{x}_2$ integrator require the same inputs, but with different signs, we may save several of these inverters by generating $-\dot{x}_2$ instead of $+\dot{x}_2$:

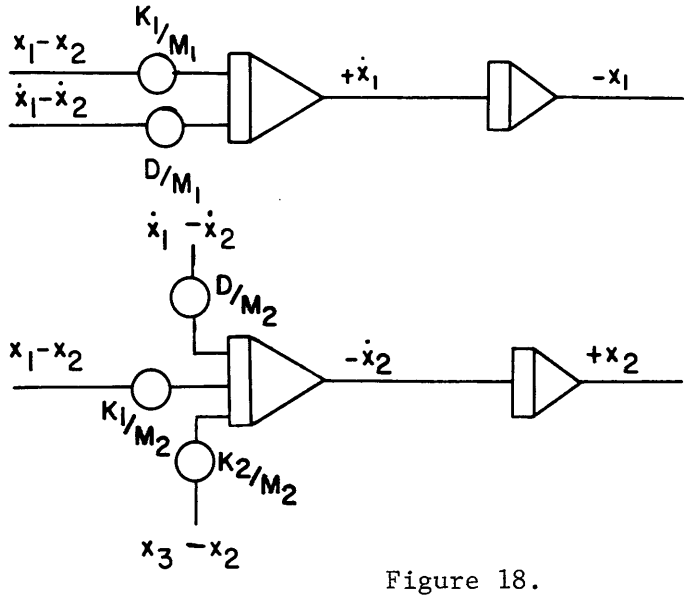


Figure 18.

Comparing Figure 18 with Figure 14, we see that in Figure 18, the variables to be subtracted appear at the integrator outputs with opposite signs, so that we do not need inverters in order to subtract. The complete program, based on Figure 18 as a starting point, is given in Figure 19. This circuit uses only eight amplifiers, as compared to eleven for Figure 17. In general, when the equations call for subtraction, inverters may be saved by generating the terms with opposite sign.

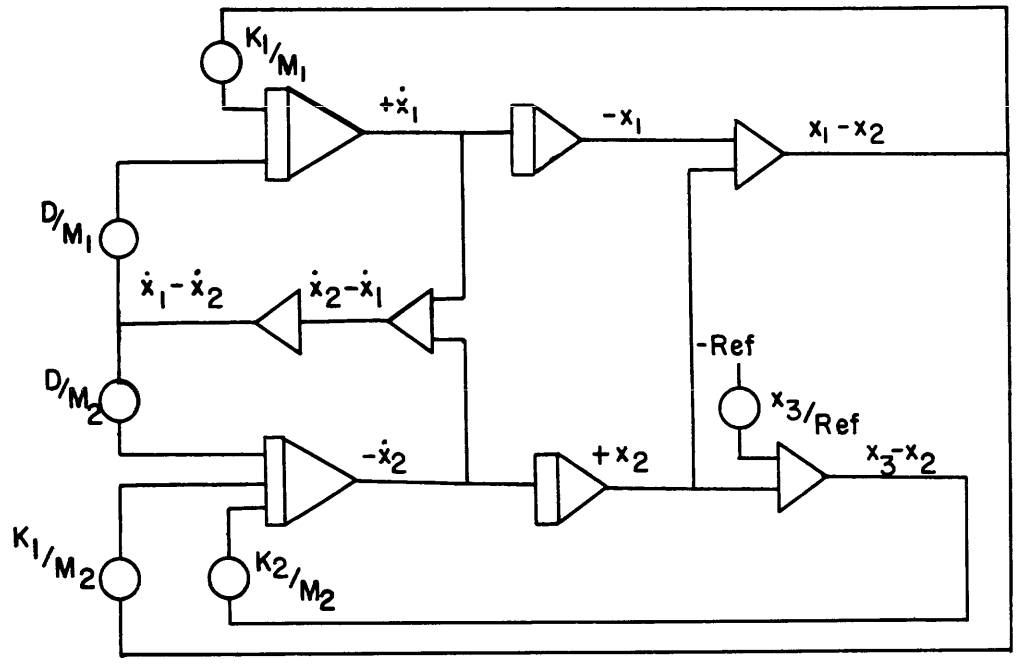


Figure 19: Modified Circuit for Auto Suspension Problems Using Fewer Amplifiers

Note that the three upper amplifiers in Figure 19 form an oscillator, and the three lower amplifiers form another one. Thus the entire circuit may be regarded as two oscillators with coupling terms added. This suggests that the response of the circuit should consist of two superimposed oscillations with different frequencies. This turns out to be the case, which can be verified either by running the program on the computer or by solving the equations analytically. The basic point to keep in mind is that we can actually identify parts of the circuit with parts of the original physical system — the three upper amplifiers represent the chassis, the three lower amplifiers represent the axle, and the coupling components in between represent the spring and shock absorber.

5. Exercises

1. Examine Figure 19 and verify that all integrators have the appropriate inputs, with the appropriate signs.

2. Draw a circuit diagram for the auto syspension problem requiring only six amplifiers. (Hint: remove all parentheses in Equation 6 and Equation 7 so that the $\dot{x}_1$ integrator has four inputs and the $\ddot{x}_2$ integrator has six inputs.)

3. Compare your circuit diagram from exercise 2 with the one in Figure 19. Consider

   a) The number of amplifiers used

   b) The number of pots used

   c) The number of pots that have to be adjusted to change a problem parameter, such as D (the shock absorber damping coefficient)

B. SCALING

1. Need for Scaling

Up to now, no mention has been made of the range of the problem variables or the magnitude of the parameters. Amplifier outputs have been simply labeled x, $\dot{x}$, x_1-x_2, etc., and pot-settings have been labeled K_2/M_2, D/M_1, etc. It should be clear that, except in very special cases, such a program cannot be run "as is". For example, if D/M turns out to be 150, it will be impossible to set a pot to this value. If a variable x has a maximum value of 1000 feet, then an amplifier whose output is x would have to put out 1000 volts, which is also impossible. Hence the amplifier outputs cannot be numerically equal to the problem variables they represent, but merely proportional to them. We must multiply the problem variables by appropriate constants, called scale factors, to assure that the amplifier outputs and pot-settings will not be too large.

It is also true, although somewhat less obvious, that amplifier outputs and pot-settings should not be too small. For example, a pot-setting of 0.5000 can be set, on a high-accuracy computer, to within 0.0001, which is 1 part in 5000. A pot-setting of 0.0010 can be set, on the same computer, to 1 part in 10, and a setting of 0.000005 cannot be obtained at all! Similarly, a variable with a maximum value of 0.002 foot cannot be accurately represented by an amplifier output whose maximum value is 0.002 volt, since this is of the same order of magnitude as the noise and computing error in many computing components.

The process of choosing scale factors is a common one in technical applications, even outside the field of analog computation. For example, suppose we are given the following data to be put into graphical form:

HP vs SPEED FOR INTERNAL-COMBUSTION ENGINE

Speed (RPM)	HP
250	20
500	50
1000	82
1500	95
2000	75
2500	40

Before we can plot the data, we must choose a scale factor — the number of inches that will correspond to one RPM or one HP. This is easily done by observing the maximum value of each variable and comparing it with the size of the graph paper. The object is to produce a graph large enough to use most of the graph paper, yet not so large that it runs off scale.

Figure 20 illustrates the correct graph, together with examples of poor graphs that result from an inappropriate choice of scale factors. Curve #1 is analogous to a poorly scaled amplifier, which overloads during the computation, while Curve #3 is analogous to an amplifier whose output never gets larger than 10% of reference voltage — with a corresponding loss of accuracy

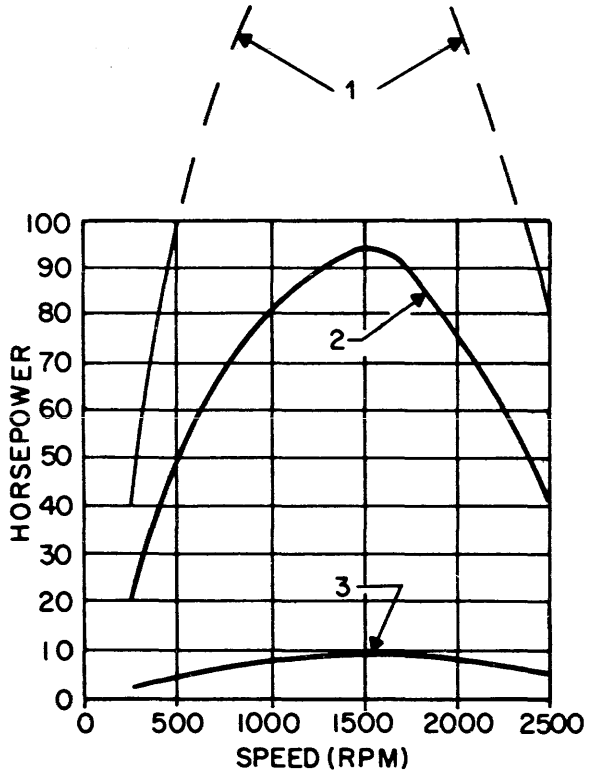


Figure 20 Graph of Speed vs Horsepower, Internal Combustion Engine

2. Choosing Scale Factors

As pointed out above, the output of each amplifier should be proportional to the problem variable that it represents. The constant relating the two is called the scale factor, and is defined formally by the relation:

$$\text{Amplifier Output} = K \cdot (\text{Problem Variable}) \tag{8}$$

or, solving for the scale factor K:

$$K = \frac{\text{Amplifier Output}}{\text{Problem Variable}} \tag{9}$$

To determine an appropriate value for K, we need an estimate of the maximum value of the problem variable. Assume that this estimate is available. When the problem variable reaches its maximum value, then the amplifier output will be at its maximum value, which should be the reference voltage of the computer:

$$K = \frac{\text{Reference Voltage}}{\text{Maximum Value of Problem Variable}} \tag{10}$$

If we substitute this value of K into Equation 8, we obtain Equation 10:

$$\text{Amplifier Output} = \frac{(\text{Reference Voltage})\ (\text{Problem Variable})}{\text{Maximum Value of Problem Variable}} \tag{11}$$

From this equation, it is obvious that when the problem variable is at its maximum, the amplifier output will be reference voltage, as it should be.

As an example, suppose a problem variable x has a maximum value of 1000 feet. On a ten-volt computer, the scale factor would be, from Equation 10:

$$K = \frac{10 \text{ volts}}{1000 \text{ feet}} = 0.01 \text{ volts per foot}$$

and the amplifier output would be 0.01x or x/100. If a hundred-volt computer were used, the scale factor would be

$$K = \frac{100 \text{ volts}}{1000 \text{ feet}} = 0.1 \text{ volts per foot}$$

and the amplifier output would be 0.1x or x/10.

As the above example illustrates, the scale factors depend not only on the behavior of the problem variables, but on the reference voltage of the machine. It is possible to avoid this dependence by an appropriate choice of units for voltage measurement. The previous example assumed that the voltages were measured in volts, but, of course, we may use other units, such as millivolts, microvolts, or kilovolts, if these units turn out to be more convenient.

For the purpose of analog scaling, it turns out that the most convenient unit of measurement is the reference voltage of the computer. This means that on a 10 volt computer, the unit would be the dekavolt (10 volts), and on a 100 volt computer, it would be the hectavolt (100 volts). While these units are not as common as the more familiar millivolts, microvolts, and kilovolts, the programmer need not actually concern himself with remembering the correct prefixes. It is common practice to refer to the reference voltage of the computer as one unit, and ignore the terms "dekavolt" and "hectavolt" altogether. Sometimes the terms "analog unit", "machine unit", or "normalized unit" are used, but this chapter will use the shorter term.

The reason for measuring amplifier outputs in terms of the computer reference

voltage is that all amplifier outputs range between -1 and +1 unit on any machine, regardless of the reference level. In the above example, the scale factor would be

$$K = \frac{\text{Reference}}{\text{Maximum of Problem Variable}} = \frac{1 \text{ unit}}{1000 \text{ feet}} = 10^{-3} \frac{\text{units}}{\text{feet}}$$

and the amplifier output would be

$$\frac{(\text{Reference})\ (\text{Problem Variable})}{\text{Maximum of Problem Variable}} = \frac{x}{1000}$$

Note that, since the reference voltage is always one unit, the computer variable is simply the ratio of the problem variable to its maximum value, and clearly will be less than one unit in magnitude.

3. Scaled Equations

When maximum values have been estimated and appropriate scale factors selected, the next step is to make the computer circuit consistent with the original problem equations. The original problem equations are stated in terms of problem variables, such as x, $\dot{x}$, T, P, etc., while the actual amplifier outputs are computer variables, such as $x/10$, $5\dot{x}$, $T/1000$, $P/5$, etc. We must translate the original problem equations into computer oriented terms, i.e. we must express each scaled amplifier output in terms of its scaled inputs.

An expression giving a scaled amplifier output in terms of the scaled inputs is called a scaled equation. A scaled equation can be written down for every amplifier in a problem, and should have the following properties:

a) It should be consistent with the original problem equations.

b) It should be machine-oriented; that is, the individual terms and factors in the equation should correspond to various parts of the machine. To aid in identifying the various parts of the equation, the following convention is widely used:

Scaled variables (either amplifier outputs or reference voltage) are enclosed in square brackets [ ]

Pot-settings are enclosed in parentheses ( )

Amplifier gains are not bracketed.

c) The pot-settings and gains in the scaled equation should be "reasonable", i.e. neither too big nor too small. Exactly what is "too big" or "too small" is to some extent a matter of taste, but as a general rule of thumb, most pot-settings should be greater than 0.1, and most gains less than 10. Occasional smaller pot-settings or larger gains can be tolerated, but any pot-setting less than about 0.01 or gain larger than 20 should be looked upon with suspicion.

Actually, it turns out that in most cases, if we are consistent in satisfying conditions a) and b), then condition c) will "take care of itself". In a

few cases, some additional manipulations are required to satisfy condition c), but these are generally quite easy, as we shall see below.

The three criteria for a properly scaled equation suggest immediately the steps for obtaining it:

a) Write down the expression for the amplifier output in terms of its inputs <u>in unscaled form</u>.

b) Eliminate each problem variable by substituting an equivalent expression in terms of the corresponding computer variable, taking care to maintain the balance in the equation. For example, if the unscaled equation contains a variable x, and the corresponding scaled variable is $[x/50]$, then we may replace x by $50[x/50]$ without unbalancing the equation. Note the square brackets around the computer variable.

c) Solve this equation for the <u>scaled output</u> in terms of the <u>scaled inputs</u>. If pots are required, show their settings in parentheses (remember a pot-setting must be less than unity) and leave amplifier gains unbracketed.

d) Examine the pot-settings and gains in the scaled equation, and, if necessary, manipulate the equation further to make the pot-settings and gains "reasonable".

The above steps provide a general guide to the procedure. The details will depend upon the type of amplifier being scaled (e.g. summer, inverter, high-gain amplifier, etc.). The detailed procedure is outlined below for the more common types of amplifier:

a. <u>Summers</u>

The general form for a scaled equation for a summer is

$$\left[\text{Output}\right] = -\sum\left[\text{Input}\right]\left(\text{Pot-Setting}\right) \text{ Gain}$$

To see how such a scaled equation is obtained, suppose we have the following <u>unscaled</u> diagram.

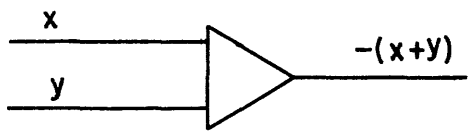


Figure 21

and assume we are given the following maximum values.

PROBLEM VARIABLE	MAXIMUM VALUE	COMPUTER VARIABLE
x	5 feet	$[x/5]$
y	3 feet	$[y/3]$
x+y	?	

If maximum values are given for x and y, then the maximum value of the sum (x+y) may be easily estimated. Clearly, the sum can never be greater than 8 feet. Is this the true maximum value? Probably not, since this value can only be achieved if x and y reach their maximum values in the same direction at the same time. If x is near zero when y is at its maximum, or if x is negative when y is positive, then the 8 foot level will never be reached. However, if nothing is known about x and y except their maximum values, it makes good sense to be conservative and assume the "worst case". Accordingly, we estimate the maximum value of the sum as 8 feet, and the scaled output will be $\left[\frac{x+y}{8}\right]$.

Once these maximum values are determined, we may follow the above outline of steps to obtain the scaled equation:

The relation between the output and the inputs in unscaled form is simply

$$-(x+y) = -(x+y) \tag{12}$$

that is, the output is minus the sum of the inputs.

Eliminating problem variables in favor of computer variables, we obtain

$$8\left[-\left(\frac{x+y}{8}\right)\right] = -\left\{5\left[\frac{x}{5}\right] + 3\left[\frac{y}{3}\right]\right\} \tag{13}$$

This equation is equivalent to Equation 12, since we have substituted equivalent expressions. Note that the minus sign on the left is inside the brackets, since the brackets, by convention are placed around amplifier outputs, and the amplifier output is $\left[-\left(\frac{x+y}{8}\right)\right]$.

To solve for the amplifier output in terms of the inputs, we divide both sides of the equation by 8:

$$\left[-\left(\frac{x+y}{8}\right)\right] = -\left\{\left(\frac{5}{8}\right)\left[\frac{x}{5}\right] + \left(\frac{3}{8}\right)\left[\frac{y}{3}\right]\right\} \tag{14}$$

Note that the factor of 8 has been grouped with the factors of 5 and 3 on the right-hand side, to produce pot-settings of (5/8) and (3/8). Since these numbers are less than unity, we may use gains of 1 on the amplifier, and the final scaled equation is

$$\left[-\left(\frac{x+y}{8}\right)\right] = -\left\{1\left(\frac{5}{8}\right)\left[\frac{x}{5}\right] + 1\left(\frac{3}{8}\right)\left[\frac{y}{3}\right]\right\} \tag{15}$$

where amplifier outputs, pot-settings, and gains are explicitly displayed. The scaled diagram corresponding to this equation is given in Figure 22.

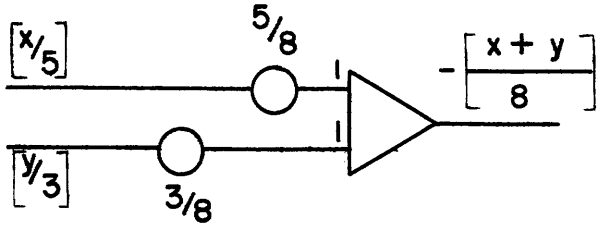


Figure 22

The pot-settings (5/8) and (3/8) are certainly "reasonable", i.e. neither too large nor too small. Is this because we were "lucky" in this case, or is this something that can be reasonably expected to happen when a summer is properly scaled? Going back over the scaling process, we observe that the factors of 5, 3, and 8 that arose in the scaled equation are simply the estimated maximum values for x, y, and (x+y) respectively. The fact that the pot-settings turned out to be reasonable is a direct consequence of the way in which the maximum values were estimated.

To see that this is true in general, consider an arbitrary summer with m inputs (Figure 23).

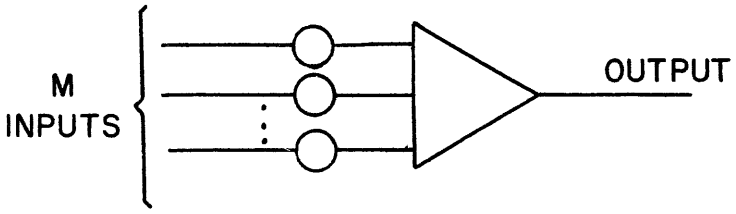


Figure 23

Once estimates are known for the maximum of the output may be estimated by the "worst case" method, i.e. by assuming that the inputs all reach their maximum values in the same direction at the same time. Suppose the output is scaled on this basis, and the scaled equations written accordingly. If the "worst case" actually occurs when the problem is run, then all scaled inputs will reach their maximum values in the same direction at the same time, i.e. all inputs will have values of +1 unit. At this time, the scaled output will have its maximum value (1 unit). In order for this to happen, the sum of all pot-settings in Figure 23 must be unity. This rules out the possibility of pot-settings greater than unity, and also rules out the case where the pot-settings are all very small. If we choose the "worst case" scale factor for the output, or any scale factor reasonably close to it, then we should not expect very large or very small gains in the final scaled equation.

b. Pot-Feedback Amplifiers

Consider the equation

$$z = \frac{1}{a}(bx + cy) \qquad (16)$$

This equation can be mechanized with a summing amplifier and scaled in the same way as the above example. However, a high-gain amplifier with pot-feedback offers better parameter isolation, as shown (unscaled) in Figure 24.

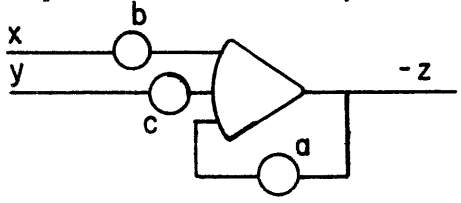


Figure 24

Suppose we are given the following data:

$$|x| \leq 500 \qquad a = 75$$
$$|y| \leq 4 \qquad b = 2$$
$$c = 180$$

To scale the circuit, we first estimate the maximum value of z, using the "worst case" assumption:

$$|z| \leq \frac{1}{75} \left| (2)\ (500) + (180)\ (4) \right| = \frac{1}{75}\ (1000 + 720)$$
$$|z| \leq 22.8$$

Since it is desirable to have a round number for a scale factor, we round this off to 25. In general, it is a good idea to round the estimated maximum up, rather than down. If we scaled z for a maximum value of 20, then the amplifier producing $\left[z/20\right]$ might overload. Hence we have the following scaled variables:

PROBLEM VARIABLE	ESTIMATED MAXIMUM	SCALED VARIABLE
x	500	$\left[x/500\right]$
y	4	$\left[y/4\right]$
z	25	$\left[z/25\right]$

Next, we must take equation 16 and convert it to a scaled equation containing the computer variables. Substituting equivalent expressions into Equation 16, we obtain

$$25\ \left[z/25\right] = \frac{1}{a} \left(b \cdot 500\ \left[x/500\right] + c \cdot 4\ \left[y/4\right] \right) \qquad (17)$$

The desired amplifier output is $-\left[z/25\right]$ (note the minus sign — the unscaled diagram in Figure 24 shows that the output will be proportional to minus z, because of the sign inversion in the amplifier). Solving for $-\left[z/25\right]$, we divide both sides of the equation by -25

$$\left[-z/25\right] = -\frac{1}{(25a)} \left\{ (500b)\ \left[x/500\right] + (4c)\ \left[y/4\right] \right\} \qquad (18)$$

Equation 18 is almost in the right form for the output of a pot-feedback amplifier; namely,

$$\left[\text{Output}\right] = -\ \frac{\sum \left[\text{Input}\right] (\text{Pot-setting})\ \text{Gain}}{(\text{Pot-setting})\ \text{Gain}} \qquad (19)$$

in which the factors in the denominator are the feedback pot-setting and the gain into which it is patched. The remaining step is to separate each term in parentheses into the product of a pot-setting and an amplifier gain. Looking at the numerical values of these expressions, we note that they are all quite large:

$$500b = 1{,}000$$
$$4c = 720$$
$$25a = 1{,}875$$

If we separate these terms into pot-settings (which must be less than unity) and amplifier gains, we discover we need a gain of almost 2000. We must find some way of eliminating these large gains without disturbing either the balance of the equation or the scaling of x, y, and z.

Equations 18 and 19 involve ratios, with pot-settings in both the numerator and the denominator. Hence we may multiply or divide both numerator and denominator by an arbitrary constant without changing the value of the ratio. We may choose this ratio so that the largest pot-setting in the equation is less than unity, thus assuring reasonable settings. In the present case, a good choice is to divide the numerator and denominator in Equation 18 by 2000:

$$\left[-z/25\right] = -\frac{(b/4)\left[x/500\right] + (c/500)\left[y/4\right]}{(a/80)} \qquad (20)$$

This equation is equivalent to Equation 16, which may be checked by cancelling the common factors. However, it contains only scaled variables and reasonable pot-settings. The numerical values of the pot-settings are

$$\begin{aligned} b/4 &= 0.5000 \\ c/500 &= 0.3600 \\ a/80 &= 0.9375 \end{aligned}$$

and thus we may use gains of 1 in all cases. The final scaled diagram is given in Figure 25:

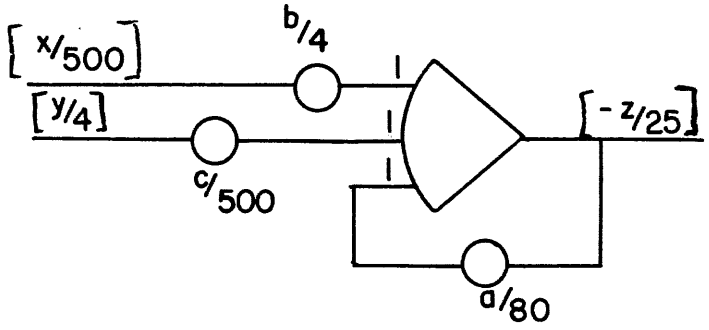


Figure 25

c. Inverters

The scaling of an inverter is very simple, since the maximum magnitude of -x is the same as for +x. Hence the inverter input and output should have the same scale factor — if the input is $[x/25]$, the output will be $[-x/25]$ Note that no pots are necessary. It is possible to write a scaled equation for an inverter:

$$\left[-x/25\right] = -\left[x/25\right] \qquad (21)$$

which says that the output of the inverter is minus the input. However, the component is so simple that the scaled equation is generally omitted.

d. Multipliers

Multipliers, like inverters, are easy to scale; properly scaled inputs almost always produce properly scaled outputs. This is because properly scaled inputs will have values between -1 unit and +1 unit; when both inputs are at their maximum (1 unit) the output is also 1 unit, and this is the maximum possible output.

As an example, suppose we want to generate a voltage proportional to xy, where $|x| \leq 50$ feet and $|y| \leq 10$ feet, so that the corresponding scaled variables are $[x/50]$ and $[y/10]$. How do we estimate the maximum value of the product xy? Clearly, the product can never exceed 500 ft^2, and will only get this large if x and y "peak" simultaneously. (Compare this situation with the situation for the summer). If we know nothing about x and y except their maximum values, then the safest procedure is to use the "worst case" estimate, as we did for the summer. In this case, the estimate is 500 ft^2, so that we should generate the scaled output $[xy/500]$. If we multiply the scaled inputs $[x/50]$ and $[y/10]$, the product is $[xy/500]$, which is exactly what we want! For multipliers, the correct scale factor (or more precisely, the "worst case" scale factor) is built in. The circuit diagram is given in Figure 26.

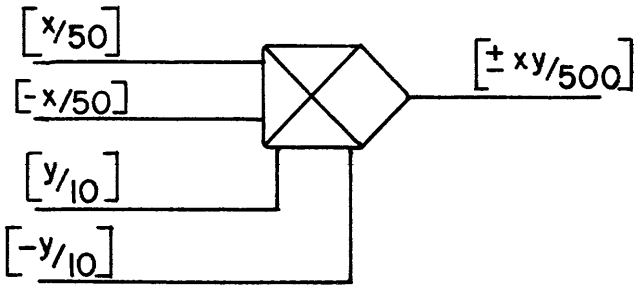


Figure 26

The scaled equation can be written down easily:

$$\left[\pm \frac{xy}{500}\right] = \left[\frac{x}{50}\right] \cdot \left[\frac{y}{10}\right] \qquad (22)$$

The $\pm$ sign, of course, is determined by the input patching.

Note that pots at the inputs to the multiplier are neither necessary nor desirable. Since the "worst case" scale factor is built in, the output $[xy/500]$ can never be larger than 1 unit, and the maximum value will be less, if the inputs do not peak simultaneously. Hence it would be poor practice to put pots at the input, which would only attenuate the output further.

Incidentally, there are sound electrical reasons as well as mathematical reasons for not using pots at the input to a multiplier — the use of pots at the input will result in serious loading errors on most multipliers. Although the electronic construction of the components is beyond the scope of this chapter (see Chapter II), it is interesting to note that both electrical considerations and scaling considerations lead to the same conclusion: multipliers should receive their inputs directly from amplifiers, not from pots.

We have seen that the output of a multiplier automatically has the "worst case" scale factor. However, if some other scale factor is desired, it can usually be obtained. Suppose, in the above example, we know on physical or mathematical grounds, that the maximum value of the product is 200 ft^2, instead of the "worst case" estimate of 500, because x and y do not peak simultaneously. Then the desired output is [xy/200], rather than [xy/500]. Can we obtain this scale factor without changing the scaling of the inputs [x/50] and [y/10]? In most computers, the answer is yes. The output scale factor can be modified by changing the feedback resistor, or by using pot feedback. Most multipliers provide some method of disconnecting the feedback resistor, so that an additional resistor may be patched in series with it, or a feedback pot may be used.

In the present case, the output may be increased by the desired factor(5/2) by connecting a pot in the feedback path and setting the pot for (2/5). The circuit diagram is given in Figure 27:

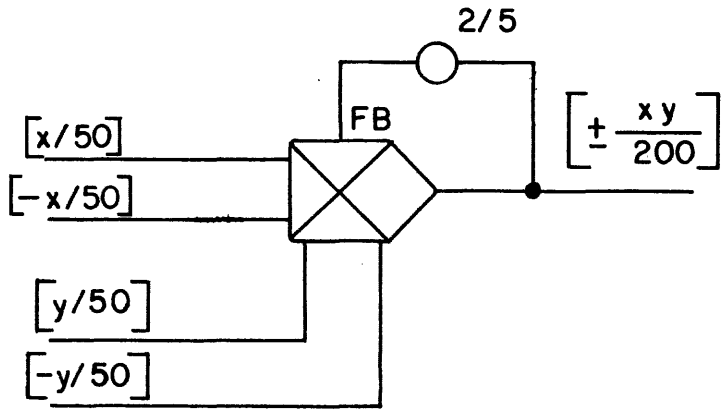


Figure 27

This patching does not violate the above-mentioned rule that multiplier inputs should not be patched to pots, since this is a feedback connection, not an input connection, and the load on the pot is simply the built-in feedback resistor — not the multiplier itself. (See Chapter II for details).

e. Division Circuits

Suppose we want to generate x/y, where the maximum values are given as $|x| \leq 50$ lbs, and $|y| \leq 10$ feet. How can the maximum value of x/y be estimated?

Dividing the maximum value of x by the maximum value of y produces an "estimate" of 5 lbs/foot. This is clearly not a good estimate of the maximum value, since the ratio can get much larger than this, especially when y is small. However, we can draw one conclusion: if the maximum values of x and y are correct, then the maximum value of x/y must be at least 5 lbs/foot. For x must reach its maximum value (50 lbs) at some time during the solution. At that moment, the value of the denominator y cannot be larger than its maximum value of 10 feet, so that the ratio must be $\geq$ 5 lbs/foot at that time. Hence the maximum value is at least 5 lbs/foot, and may get much larger.

Suppose we know, on physical grounds, that the actual maximum is 10 lbs/foot. Then the scaled output should be [x/10y]. How can we obtain this output? The desired relation between input and output is, in unscaled form,

Substituting equivalent expressions in terms of scaled variables, we obtain

$$10\left[\frac{x}{10y}\right] = \frac{50\,[x/50]}{10\,[y/10]} \tag{24}$$

And, solving for the desired output:

$$\left[\frac{x}{10y}\right] = \frac{50[x/50]}{100[y/10]} = (1/2)\,\frac{[x/50]}{[y/10]} \tag{25}$$

From this scaled equation, it follows that we can obtain the desired output by multiplying the numerator by a pot-setting of (1/2), which leads to the diagram of Figure 28:

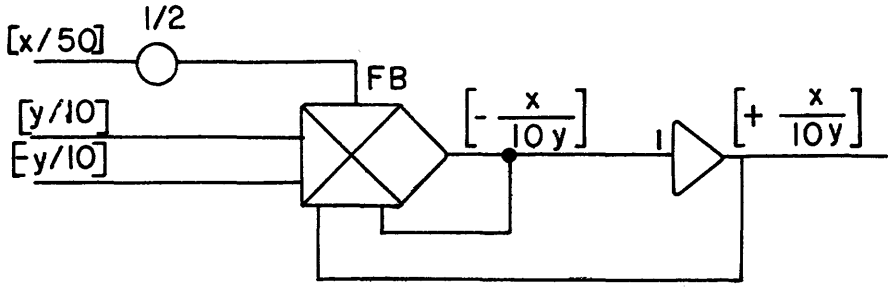


Figure 28

This circuit does not violate the rule against feeding a pot into a multiplier, as the numerator input actually goes into a resistor, rather than into the multiplier itself. In most multipliers, the resistor used is the same one that is used as the feedback resistor for multiplication (See Chapter II for circuit details).

Note that in the multiplication circuit, the output is either correctly scaled or too small; we can use a pot in the feedback to increase it if necessary. In the division circuit, the output is either correctly scaled or too large; we may use a pot in the input to decrease it if necessary. Since multiplication and division are inverse operations, this result is to be expected.

f. Integrators

Suppose we have the unscaled equation

$$\frac{dz}{dt} = x + y \tag{26}$$

which leads to the following unscaled diagram:

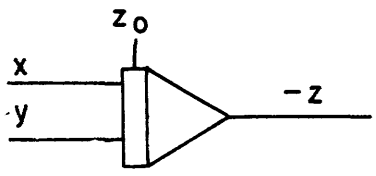


Figure 29

Assume that $|x| \leq 10$; $|y| \leq 5$ and $|z| \leq 2$. Then the scaled computer variables will be $[x/10]$, $[y/5]$ and $[z/2]$. Substituting into Equation 26, we obtain

$$2\frac{d}{dt}[z/2] = 10[x/10] + 5[y/5] \qquad (26a)$$

Note carefully the bracketing on the left-hand side — we have $\frac{d}{dt}[z/2]$, and not $[\frac{dz}{dt}/2]$. The square brackets serve to set off amplifier outputs, and there is no amplifier in the circuit whose output is proportional to $\frac{dz}{dt}$. The expression $\frac{d}{dt}[z/2]$ can be interpreted as the rate of change of the amplifier output $[z/2]$. Note that no estimate was given for the maximum value of $\frac{dz}{dt}$, and none is needed, since $\frac{dz}{dt}$ does not appear as an amplifier output.

Dividing both sides by -2, we get

$$\frac{d}{dt}[-z/2] = -\left\{(5)[x/10] + \left(\frac{5}{2}\right)[y/5]\right\} \qquad (27)$$

Solving for the scaled output $[-z/2]$:

$$[-z/2] = -z_o/2 - \int_o^t \left\{(5)[x/10] + (5/2)[y/5]\right\} dt \qquad (27a)$$

The co-efficients on the right-hand side are greater than unity, so that we must use amplifier gains of 10. Separating these factors into gains and pot-settings, we have

$$[-z/2] = -z_o/2 - \int_o^t \left\{10(1/2)[x/10] + 10(1/4)[y/5]\right\} dt \qquad (28)$$

This equation is almost in the right form. All that remains is to put the IC into the appropriate form — that is, to write it as the product of a pot-setting and reference voltage. Whenever reference voltage appears in a scaled equation, it is customarily written within square brackets, like an amplifier output. Hence the IC term should be written as $-(z_o/2)[+1]$, which shows that it is the product of reference voltage (which is 1 unit) and a pot-setting of $(z_o/2)$. Thus the final scaled equation is

$$[-z/2] = -(z_o/2)[+1] - \int_o^t \left\{10(1/2)[x/10] + 10(1/4)[y/5]\right\} dt \qquad (29$$

and the corresponding scaled diagram is given in Figure 30.

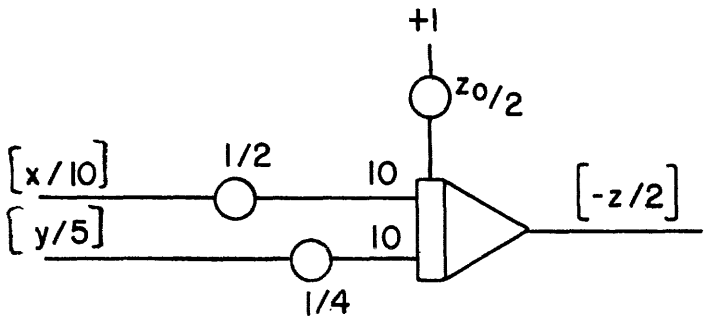


Figure 30

The scaled equation can be made to look simpler if we write it in derivative form instead of integral form:

$$\frac{d}{dt}\left[-z/2\right] = -\left\{10(1/2)\left[x/10\right] + 10(1/4)\left[y/5\right]\right\} \qquad (30)$$

In this form, it says that the output of the integrator changes at a rate equal to minus the sum of the inputs. The scaled diagram can be drawn as easily from Equation 30 as from Equation 29. Note, however, that this form of the scaled equation does not include the IC, which must therefore be scaled separately. However, this is easy enough to do. From Figure 30, we observe that the IC pot-setting has the same scale factor as the integrator output itself. A little reflection will show that this will be the case for any integrator output, since reference voltage is regarded as 1 unit. The amplifier output is the ratio of the variable to its maximum value, the IC pot-setting is the ratio of the initial value to its maximum value, and both are ≤ 1 in magnitude. Incidentally, this last remark shows that the IC pot-setting is always less than unity, if the output has been scaled consistently.

Having verified that the IC pot-setting will turn out reasonable, what can we say about the other inputs? In this case, the required co-efficients were obtained with gains of 10 and pot-settings of 1/4 and 1/2. Is this a special case, or will integrator gains always turn out reasonable in a practical problem? The answer is that we can, in general, assure reasonable gains on amplifiers, but the method is somewhat different for an integrator. This is because the gain on an integrator does not merely determine the magnitude of the output, but determines how fast the output changes. For this reason, the magnitude of integrator gains is related to time scaling, that is, to the process of speeding up or slowing down the problem as a whole. Therefore, the question of the magnitude of the integrator gains will be deferred until the section on time-scaling.

4. Outline of Scaling Procedure

The previous section indicated the procedure for scaling the most commonly used individual components. This section provides a sequence of steps for "putting the pieces together" to scale an entire problem.

a. Prepare an unscaled circuit diagram
b. Prepare a Scaling Table, listing every variable that appears at the output of an amplifier
c. Estimate the maximum value of each of the variables in the scaling table.
d. Divide the variable by its maximum value to obtain the scaled computer variable, which will be ≤ 1 in magnitude. The scaling table should contain 3 columns — one for the problem variable, one for its maximum value, and one for the scaled computer variable.
e. Prepare a scaled equation for each amplifier in the problem, following the procedure given in the previous section.
f. Prepare a scaled diagram from the scaled equations, showing the scaled outputs and the necessary pot-settings and gains.

5. A Scaling Example

To illustrate the scaling process, consider the equation

$$a\ddot{x} + b\dot{x} + cx = 0 \qquad (31)$$

which was programmed, but not scaled, earlier in this chapter. This time, let us assume specific parameter values and initial conditions:

$$\begin{aligned} a &= 25 \\ b &= 50 \\ c &= 100 \\ x(0) &= 5 \\ \dot{x}(0) &= 0 \end{aligned}$$

Note that no specific physical interpretation is given -- the equation might represent the oscillation of a spring-mass system with a 5-inch initial displacement, or an R-L-C circuit with an initial 5-volt charge on the capacitor. The emphasis is on the programming and scaling techniques, rather than on the physics of any particular system.

Solving the equation for the highest derivative yields the equation:

$$\ddot{x} = -\frac{b}{a}\dot{x} - \frac{c}{a}x \qquad (32)$$

and straightforward programming leads to the diagram of Figure 9. For convenience, Figure 9 is repeated here as Figure 31:

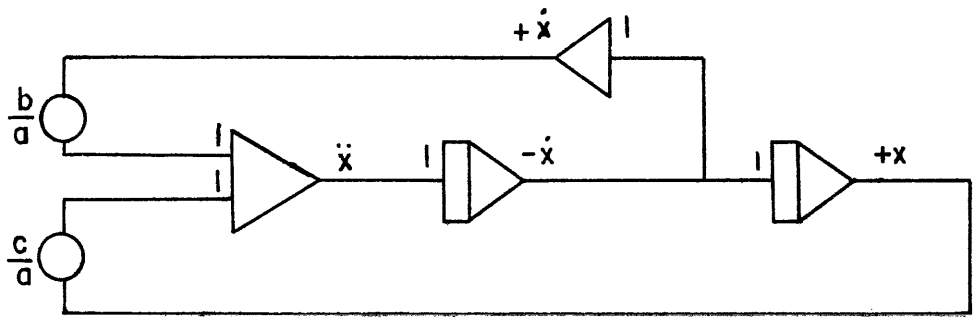


Figure 31

Having prepared the unscaled diagram, the next step is to prepare the scaling table:

Problem Variable	Estimated Maximum	Computer Variable
x	5	$[x/5]$
$\dot{x}$	10	$[\dot{x}/10]$
$-\dot{x}$	10	$[-\dot{x}/10]$
$\ddot{x}$	40	$[\ddot{x}/40]$

Note that there are four entries in the scaling table -- one for each amplifier in the diagram. (In practice, most programmers would not bother to list $\dot{x}$ and $-\dot{x}$ separately, but, since they both appear as amplifier outputs, they are listed here for completeness. Certain "short cuts" will be discussed later).

Although this chapter is not concerned with how the estimates for maximum values are obtained, it should be noted that once maximum values for x and $\dot{x}$ are known, the maximum value for $\ddot{x}$ may be estimated from equation 32, since $\ddot{x}$ appears at the output of a summer.

The scaled equations may be easily written down, following the procedures of the previous section:

The Summer

The unscaled equation for the summer output is Equation 32:

$$\ddot{x} = -\frac{b}{a}\dot{x} - \frac{c}{a}x$$

and substitution of computer variables into this equation gives Equation 33:

$$40\,[\ddot{x}/40] = -\{10\,\frac{b}{a}\,[\dot{x}/10] + 5\,\frac{c}{a}\,[x/5]\} \tag{33}$$

Solving for $[\ddot{x}/40]$, we obtain

$$[\ddot{x}/40] = -\{(\frac{b}{4a})[\dot{x}/10] + (\frac{c}{8a})[x/5]\} \tag{34}$$

Examination of the factors in parenthesis indicates that they are less than unity, so that no gains of 10 are required.

The Integrators

The unscaled equation for the integrator producing x is:

$$\frac{dx}{dt} = \dot{x} \tag{35}$$

substituting computer variables:

$$5\,\frac{d}{dt}\,[x/5] = -10\,[-\dot{x}/10] \tag{36}$$

Solving for $\frac{d}{dt}\,[x/5]$:

$$\frac{d}{dt}\,[x/5] = -2\,[-\dot{x}/10]$$

Separating the factor on the right into the product of a pot-setting and a gain, we obtain:

$$\frac{d}{dt}\,[x/5] = -\,10\,(0.2)\,[-\dot{x}/10] \tag{37}$$

The IC on this integrator must be $x_o/5$, this may be obtained from reference voltage and a pot set to $(x_o/5)$. We may write a scaled equation for the I.C.

$$[x_o/5] = -\,(x_o/5)[-1] \tag{38}$$

This equation shows that we need minus reference on the I.C. pot to produce a positive I.C. As pointed out in the previous section, the I.C. pot-setting has the same scale factor as the amplifier output.

The integrator producing $-\dot{x}$ has the unscaled equation:

$$\frac{d\dot{x}}{dt} = \dddot{x} \tag{39}$$

After appropriate manipulation of signs and scale factors, this becomes:

$$\frac{d[-\dot{x}/10]}{dt} = -\ 10(0.4)[\ddot{x}/40] \tag{40}$$

The Inverter

Since the input and output of the inverter have the same scale factor, the scaled equation is:

$$[\dot{x}/10] = -1\ [-\dot{x}/10] \tag{41}$$

This equation gives the output of the inverter in terms of its input $[-\dot{x}/10$ In practice, the scaled equation would probably not be written, but it is given here for completeness.

The scaled circuit diagram may now be drawn from the scaled equations:

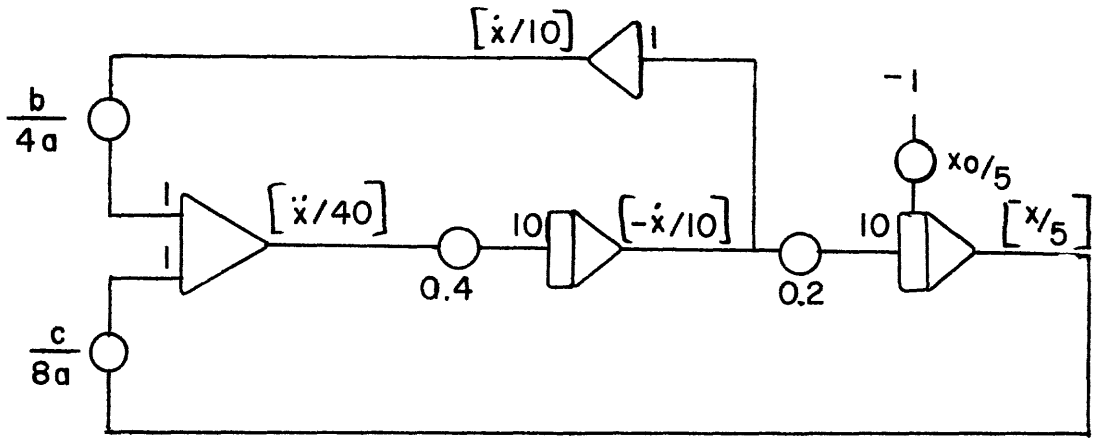


Figure 32

6. Alternative Circuits

Earlier in this chapter, the second-order equation $a\dddot{x} + b\ddot{x} + cx = 0$ was programmed in three different ways -- with a summer, with a pot-feedback amplifier, and with a multiple-input integrator. The previous section illustrated the scaling of the circuit with the summer. This section will illustrate the scaling of the alternative circuits, covering only those parts of the circuit that are different from the circuit of Figure 32.

The Pot-Feedback Circuit

The unscaled diagram for the pot-feedback circuit is given in Figure 11. A scaling table for this circuit looks exactly like the scaling table for the circuit with a summer -- the same variables appear at amplifier outputs, and, of course, the maximum values and scale factors are the same.

The only difference is in the scaled equation for the high-gain amplifier itself. The unscaled equation is:

$$\ddot{x} = -\frac{1}{a}(b\dot{x} + cx) \tag{42}$$

and substituting scaled variables, we obtain:

$$40[\ddot{x}/40] = -\frac{1}{a}(10b\,[\dot{x}/10] + 5c\,[x/5]) \tag{43}$$

This equation must be solved for $[\ddot{x}/40]$ and written in the appropriate form for a pot-feedback amplifier, that is, with a pot-setting proportional to a in the denominator. Solving for $[\ddot{x}/40]$, we obtain

$$[\ddot{x}/40] = -\frac{10b\,[\dot{x}/10] + 5c\,[x/5]}{40a} \tag{44}$$

The numerical values of the coefficients are:

$$10b = 500;\ 5c = 500;\ 40a = 1000$$

To convert these coefficients into pot-settings, we divide both numerator and denominator by 1000, giving the final scaled equation:

$$[\ddot{x}/40] = -\frac{(b/100)[\dot{x}/10] + (c/200)[x/5]}{(a/25)} \tag{45}$$

and the scaled diagram:

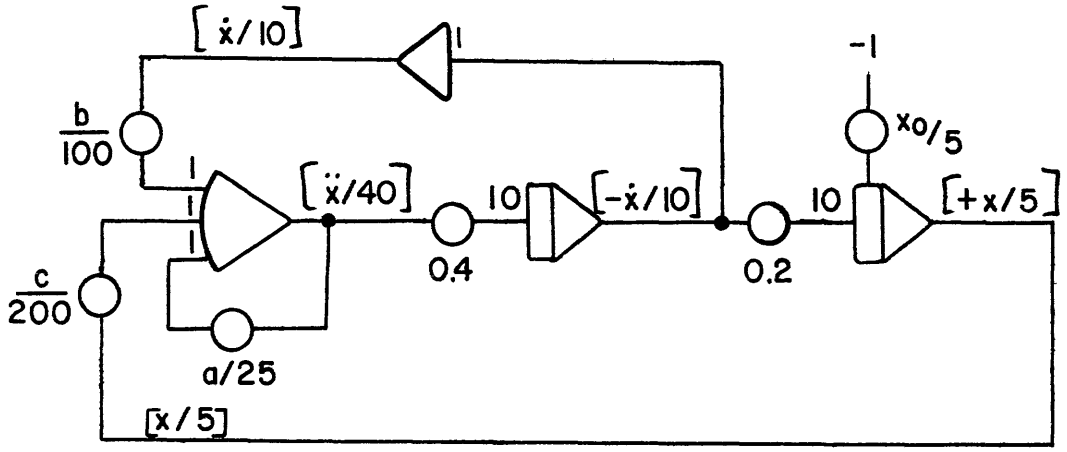


Figure 33

The Multiple-Input Integrator

The circuit diagram given in Figure 12 shows another method of programming the problem, with a net saving of one amplifier. The scaling table for this circuit will therefore contain fewer entries -- there is no need to estimate a maximum value for $\ddot{x}$, since it does not appear as an amplifier output. The scale factors for x and $\dot{x}$ are, of course, the same as before. Taking the unscaled equation:

$$\ddot{x} = -\frac{1}{a}(b\dot{x} + cx) \tag{46}$$

we substitute computer variables to obtain:

$$10\frac{d}{dt}[\dot{x}/10] = \frac{-1}{a}(10b\,[\dot{x}/10] + 5c\,[x/5]) \tag{47}$$

Note the form of the left-hand side. The equation must be put in the right form for an integrator, so that the left-hand side must be the rate of change of the output. Solving for this rate of change, we obtain:

$$\frac{d}{dt}[\dot{x}/10] = -\{(\frac{b}{a})\,[\dot{x}/10] + (\frac{c}{2a})\,[x/5]\} \tag{48}$$

Since the factors in parenthesis are greater than unity, we must use gains of 10, which gives the scaled equation:

$$\frac{d}{dt}[\dot{x}/10] = -\{10\,(\frac{b}{10a})[\dot{x}/10] + 10(\frac{c}{20a})[x/5]\} \tag{49}$$

which leads to the circuit diagram in Figure 34.

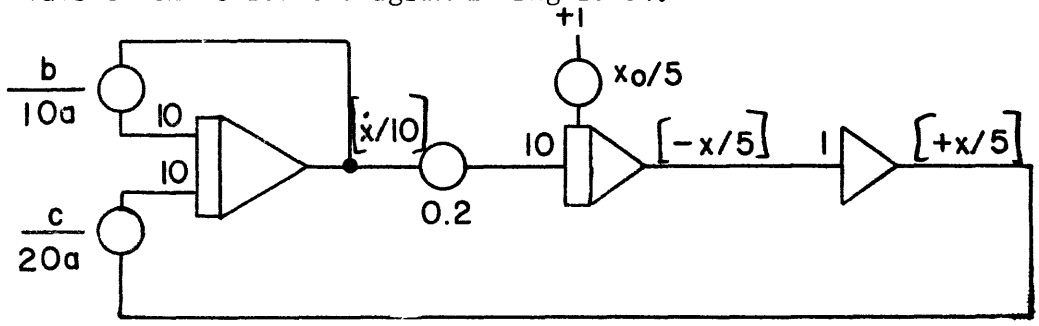


Figure 34

Note that the second integrator produces [-x/5] rather than [+x/5]; hence + Reference is used as input to the I.C. pot.

7. Time Scaling

The scaling process described in the previous sections is called magnitude scaling or voltage scaling; its purpose is to assure that all amplifier outputs cover the appropriate range (from 0 to 1 unit in magnitude). It is equally important to assure that the rate of change of the computer variables is consistent with the dynamic properties of the computer, and that the solution takes place in a reasonable amount of time.

Many physical processes take hours or days to complete, and many take only a few milliseconds or microseconds. To assure that the computer simulation takes place in a reasonable length of time, it is necessary to have some method of speeding up or slowing down the solution; that is, a method of scaling the independent variable (time) as well as the dependent variables (the amplifier outputs).

It is customary to distinguish between time in the original problem and time on the computer by using the letters t and τ respectively. Thus:

t = "problem time" = the independent variable in the original problem statement.

τ = "machine time" or "computer time" = the independent variable on the computer.

In time scaling, as in magnitude scaling, the computer variable is made proportional to the corresponding problem variable:

$$\tau = \beta t \tag{50}$$

where β is the time-scale factor, Solving for β:

$$\beta = \frac{\tau}{t} = \frac{\text{computer time}}{\text{problem time}} \tag{51}$$

Thus β has the units of computer time (usually seconds) divided by problem time. If problem time is also measured in seconds, then β is dimensionless. In this case, the magnitude of β indicates the factor by which the problem is speeded up or slowed down. If $\beta > 1$, then the solution on the computer is slower than the original process; if $\beta < 1$, the computer solution is faster than the original process. For example, if $\beta = 5$, then, from equation 50, we see that $\tau = 5t$. When t = 1 second, τ = 5 seconds, i.e. an event that takes place in one second in the original problem requires 5 seconds on the computer, the solution is slowed down by a factor of 5.

How does one instrument a time-scale change on the computer? Examination of the standard analog-computer components indicates that only one of them has anything to do with time -- the integrator. For all other components, the relation between the inputs and outputs can be described without mentioning time; therefore all components other than integrators are unaffected by time-scaling.

If we examine the input/output relation for an integrator, we find that it integrates with respect to machine time τ, not problem time t.

Hence, if we want an output proportional to a variable x, the input should be proportional to $dx/d\tau$, rather than dx/dt. Since the problem is stated in terms of dx/dt, we must find a way of converting one type of derivative into another.

From equation 50, it follows that:

$$\frac{dx}{d\tau} = \frac{1}{\beta}\frac{dx}{dt} \tag{52}$$

From this equation, we see that if a voltage proportional to dx/dt is available as an input to an integrator, we need only multiply the integrator gain (or the pot-setting preceding it) by $1/\beta$ to convert from dx/dt to $dx/d\tau$.

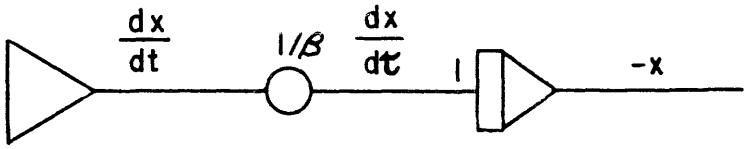


Figure 35

Thus we arrive at the general rule: <u>to speed up or slow down the computer solution by a factor β, every integrator input must be divided by β</u>.

This factor $1/\beta$ is in addition to the voltage scale factors and other coefficients required by the problem. For example, suppose maximum values have been estimated for $\dot{x}$ (= dx/dt) and x. If $\dot{x}$ is scaled as, say, $[\dot{x}/50]$ and x is scaled as [x/2], then we may write a scaled equation starting from equation 52 as follows:

$$\frac{dx}{d\tau} = \frac{1}{\beta}\frac{dx}{dt} = \frac{1}{\beta}\dot{x}$$

Substituting the appropriate <u>scaled</u> variables,

$$2 \cdot \frac{d}{d\tau}[x/2] = \frac{50}{\beta}[\dot{x}/50] \qquad (53)$$

Solving for $\frac{d}{d\tau}[x/2]$ which is the rate of change of the integrator output we obtain

$$\frac{d}{d\tau}[x/2] = \frac{25}{\beta}[\dot{x}/50] \qquad (54)$$

If the signs of x and $\dot{x}$ are as given in Figure 35, we must multiply by -1 to obtain:

$$\frac{d}{d\tau}[-x/2] = -\left(\frac{25}{\beta}\right)[\dot{x}/50] \qquad (55)$$

which is in the correct form for an integrator output. The factor $25/\beta$ in parenthesis is the product of the pot-setting and the integrator gain. If it is greater than one, we may use an appropriate input gain on the integrator. For example, if $\beta = 100$, then the pot-setting $25/\beta$ is less than one, and a gain of one may be used, leading to the final scaled equation:

$$\frac{d}{d\tau}[-x/2] = -1\left(\frac{25}{\beta}\right)[\dot{x}/50] \qquad (56)$$

If $\beta = 10$, then $25/\beta$ is greater than one but less than 10. We may re-write the scaled equation as follows:

$$\frac{d}{d\tau}[-x/2] = -10\left(\frac{25}{10\beta}\right)[\dot{x}/50] \qquad (57)$$

which shows that a gain of 10 is necessary (see Figure 36)

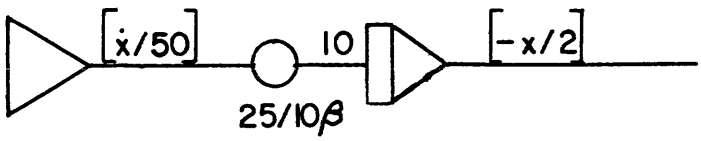


Figure 36

Note that the pot-setting is labeled on the circuit diagram as 25/10β (or it may be reduced to 5/2β). Merely writing the numerical value (.2500) on the circuit would not reveal the time-scale factor explicitly.

So far, the question of choosing the time-scale factor has not been discussed. Although a number of methods are available, the approach that has proved easiest in practice is to choose the time-scale factor to make the pot-settings and gains at integrator inputs reasonable. In section 3, we saw that proper magnitude scaling would produce reasonable gains on most components, although occasionally an auxilliary adjustment of the scaled equation is necessary (e.g. the pot-feedback amplifier). The question of reasonable gains on integrators was left open. Time scaling provides the key to obtaining reasonable integrator gains.

Since the magnitude of an integrator input determines the rate of change of its output, large integrator gains indicate a fast solution, and small integrator gains indicate a slow solution. Hence, dividing all integrator inputs by the same factor to make the gains reasonable will also make the solution speed reasonable. Note that the same time-scale factor must be used for every integrator, since the independent variable, time, is common to all integrators in the problem. The recommended time-scaling procedure is therefore as follows:

1) Program and scale the problem without regard to time-scaling.

2) For each integrator input (except I.C. inputs) determine the necessary coefficient (i.e. the product of pot-setting and gain) from the scaled equation. Estimate its numerical value.

3) Examine the numerical values of all such coefficients, and determine a time-scale factor that will make them all "reasonable". A good general rule is that the majority of these coefficients should be between 0.1 and 5.0.

4) Re-write the scaled equations for the integrators to show the factor 1/β in the pot-setting, and separate each coefficient into the product of a pot-setting and a gain, as required. (See Equation 57 as an example).

8. Scaling the Automobile Suspension System.

The Automobile Suspension System, which was programmed on Page 91 without scaling, provides an excellent example of the procedures for both voltage and time scaling. The equations are:

$$M_1\ddot{x}_1 + D\,(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2) = 0 \qquad (58)$$

$$M_2\ddot{x}_2 + D\,(\dot{x}_2 - \dot{x}_1) + K_1\,(x_2 - x_1) + K_2\,(x_2 - x_3) = 0 \qquad (59)$$

Programming these equations led to the circuit of Figure 19. For more detailed programming and scaling, we need specific numerical values for the parameters. Typical values are:

M_1 = 25 slugs; K_1 = 1000 lb/foot
M_2 = 2 slugs; K_2 = 5000 lb/foot
x_3 = 5 inches = 5/12 foot
$20 \leq D \leq 200$ lbs/foot per second.

Under these conditions, the maximum values of x_1 and x_2 will be approximately one foot, and the velocities $\dot{x}_1$ and $\dot{x}_2$ will have maximum values of approximately 5 feet/sec and 50 feet/sec respectively.

The scaling table for Figure 19 looks like this:

Problem Variable	Estimated Maximum	Computer Variable
x_1	1 foot	x_1
$\dot{x}_1$	5 feet/sec	$\dot{x}_1/5$
x_2	1 foot	x_2
$\dot{x}_2$	50 feet/sec	$\dot{x}_2/50$
$x_1 - x_2$	2 feet	$(x_1 - x_2)/2$
$\dot{x}_1 - \dot{x}_2$	50 feet/sec	$(\dot{x}_1 - \dot{x}_2)/50$
$x_2 - x_3$	2 feet	$(x_2 - x_3)/2$

Note that there is one entry in the scaling table for each amplifier in Figure 19, except that the inverter has been omitted.

The maximum values of x_1, $\dot{x}_1$, x_2, and $\dot{x}_2$ may be regarded as given data. The maximum values of the differences $x_1 - x_2$ and $\dot{x}_1 - \dot{x}_2$ were obtained as "worst case" estimates, based on the maximum values of x_1, x_2, $\dot{x}_1$, and $\dot{x}_2$.

The scaled equations for the summing amplifiers are straightforward:

$$\left[\frac{\dot{x}_2 - \dot{x}_1}{50}\right] = -\left\{(1/10)\,[\dot{x}_1/5] + (1)\,[-\dot{x}_2/50]\right\} \qquad (60)$$

$$\left[\frac{x_3 - x_2}{2}\right] = -\left\{(1/2)\,[+x_2] + (x_3/2)\,[-1]\right\} \qquad (61)$$

$$\left[\frac{x_1 - x_2}{2}\right] = -\left\{(1/2)\,[-x_1] + (1/2)\,[+x_2]\right\} \qquad (62)$$

The reader should be able, by now, to derive these equations by starting with the unscaled equations (e.g. $x_1 - x_2 = x_1 - x_2$) and making appropriate substitutions. Note that the [-1] in brackets in Equation 61 represents negative reference voltage, and indicates the source of this constant term, namely reference voltage connected to a pot.

The scaled equations for the integrators, without time scaling, are given below:

$$\frac{d}{dt}\left[\dot{x}_1/5\right] = -\left\{(2K_1/5M_1)\left[\frac{x_1 - x_2}{2}\right] + (10D/M_1)\left[\frac{\dot{x}_1 - \dot{x}_2}{50}\right]\right\} \tag{63}$$

$$\frac{d}{dt}\left[-\dot{x}_2/50\right] = -\left\{(D/M_2)\left[\frac{\dot{x}_1 - \dot{x}_2}{50}\right] + (K_1/25M_2)\left[\frac{x_1 - x_2}{2}\right] + (K_2/25M_2)\left[\frac{x_3 - x_2}{2}\right]\right\} \tag{64}$$

$$\frac{d}{dt}\left[-x_1\right] = -(5)\left[\dot{x}_1/5\right] \tag{65}$$

$$\frac{d}{dt}\left[+x_2\right] = -(50)\left[-\dot{x}_2/50\right] \tag{66}$$

Examining the coefficients in parenthesis, we see that they are all large:

$(2K_1/5M_1) = 16$
$(10D/M_1) = 40$
$(D/M_2) = 50$
$(K_1/25M_2) = 20$
$(K_2/25M_2) = 100$

Including the gains of 5 and 50 in equations 65 and 66, we have a total of seven integrator inputs, all of which are greater than one. The largest gain is 100; the smallest gain is 5. A value of $\beta = 10$ will make most of these gains "reasonable", i.e. between 0.1 and 5.0. Dividing the integrator inputs by β, and separating each coefficient into a pot-setting and a gain, we arrive at the final scaled equations:

$$\frac{d}{d\tau}\left[\dot{x}_1/5\right] = -\left\{10\ (2K_1/50M_1\beta)\left[\frac{x_1 - x_2}{2}\right] + 10(D/M_1\beta)\left[\frac{\dot{x}_1 - \dot{x}_2}{50}\right]\right\} \tag{67}$$

$$\frac{d}{d\tau}\left[-\dot{x}_2/50\right] = -\left\{10\ (D/10M_2\beta)\left[\frac{\dot{x}_1 - \dot{x}_2}{50}\right] + 10(K_1/250M_2\beta)\left[\frac{x_1 - x_2}{2}\right] + 10(K_2/250M_2\beta)\left[\frac{x_3 - x_2}{2}\right]\right\} \tag{68}$$

$$\frac{d}{d\tau}\,[-x_1] = -(5/\beta)\,[\dot{x}_1/5] \tag{69}$$

$$\frac{d}{d\tau}\,[+x_2] = -\,10\,\left(\frac{5}{\beta}\right)\,[-\dot{x}_2/50] \tag{70}$$

These equations are now in proper form for computer solution. All pot-settings are less than one, and only gains of 1 and 10 are used. The scaled circuit diagram is given in Figure 37.

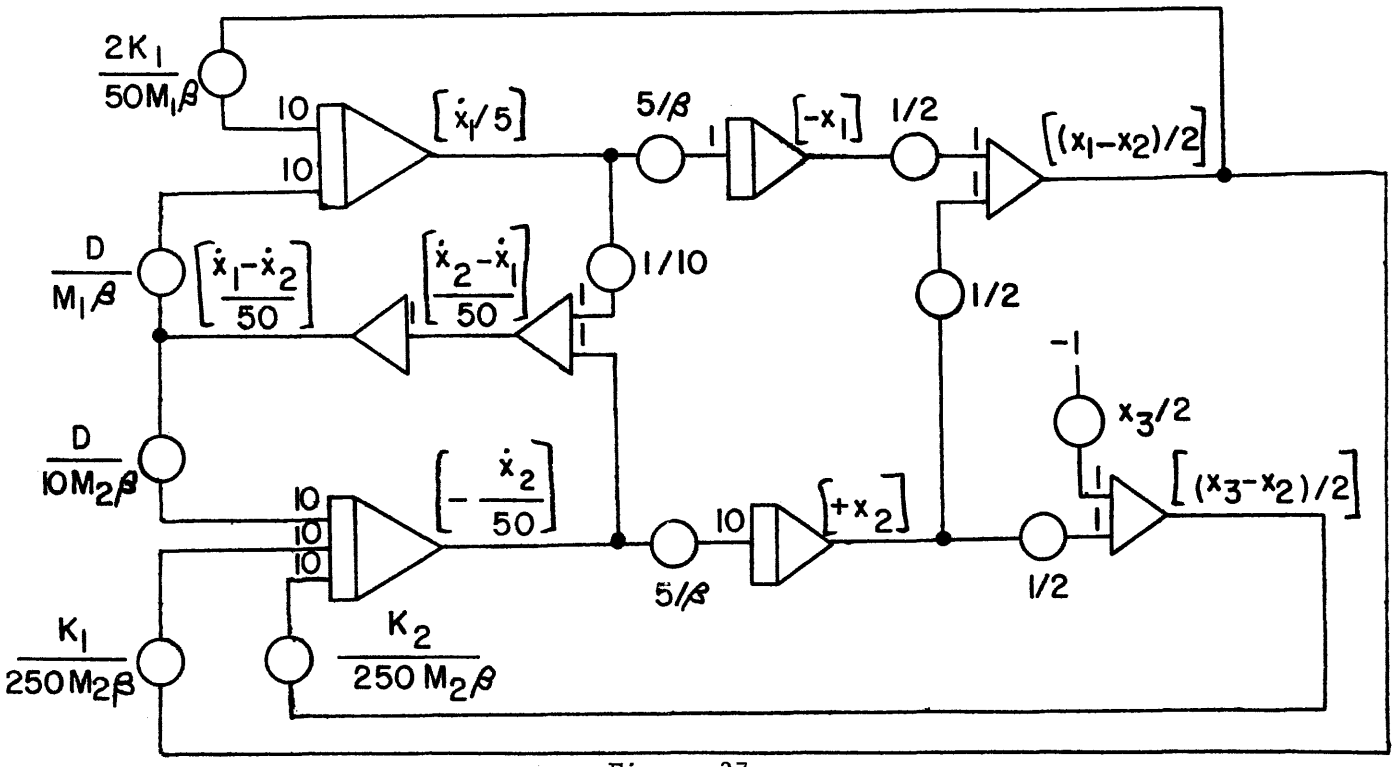


Figure 37

9. Short Cuts

The method of scaling illustrated above has the virtue of being systematic and indicating the logical equivalence between the scaled equations (which are the equations actually solved on the computer) and the original problem equations. However, it can become quite tedious in large problems, and, in practice, it is possible to scale many amplifier outputs without writing scaled equations. For example, we have already seen that an inverter virtually "scales itself", and a scaled equation is not necessary. It is also possible to scale many other outputs merely by inspection of the circuit diagram. For example, consider the portion of a circuit diagram

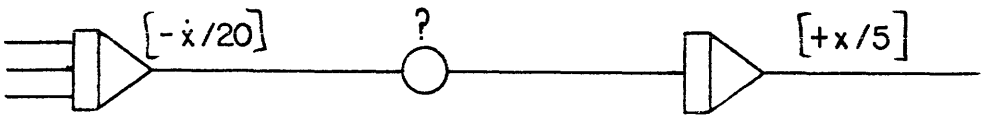


Figure 38

The problem is to calculate the value of the pot-setting necessary to reconcile the scale factor of x with the scale factor for $\dot{x}$. A "common-sense" rule is that the pot-setting must contain the ratio of the two scale factors. This rule might be stated informally as follows "cancel the incoming scale factor and introduce the desired output scale factor". In the diagram of Figure 38, it is fairly obvious that we need to multiply the input by 20 (to cancel the incoming factor of 1/20) and divide by 5 (to introduce the desired scale factor of 1/5 at the output). Hence the product of the pot-setting and the gain should be $20/5\beta = 4/\beta$, including the time-scale factor. Thus the diagram may be scaled directly, giving Figure 39a or 39b depending on the numerical value of β.

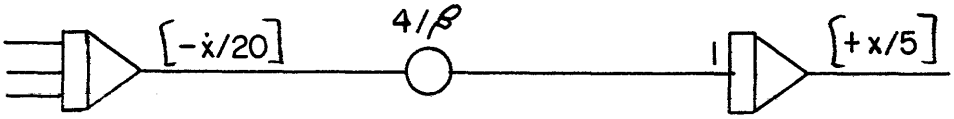


Figure 39a.

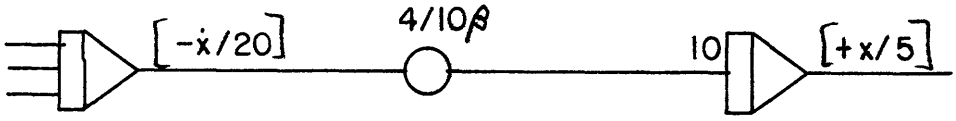


Figure 39b.

Note that the product of the pot-setting and gain is the same in Figure 39a and 39b. These are exactly the scaled diagrams that would be obtained by writing the scaled equations:

$$\frac{d}{d\tau}[x/5] = -(4/\beta)[-\dot{x}/20] \qquad (71\text{-a})$$

and

$$\frac{d}{d\tau}[x/5] = -10\,(4/10\beta)\,[-\dot{x}/20] \qquad (71\text{-b})$$

The "ratio of scale factors" method may also be applied to amplifiers with multiple inputs. Consider, for example, the integrator producing $\dot{x}_1/5$ in Figure 37. The unscaled diagram for this integrator is given in Figure 40:

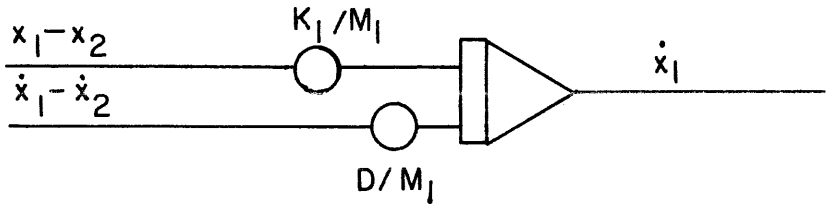


Figure 40

Replacing these unscaled variables by the corresponding scaled variables, we obtain Figure 41.

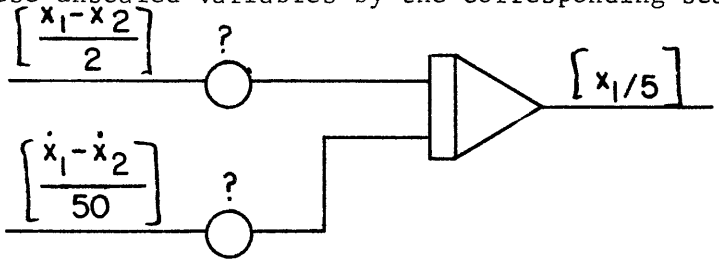


Figure 41

The first pot must contain, besides the problem parameters K_1 and M_1, the ratio of the two scale factors involved. We must multiply the pot-setting by 2 to cancel the input scale factor and divide by 5 to introduce the output scale factor. Hence, including time-scaling, the pot-setting should be

$$(K_1/M_1)(2/5)(1/\beta) = 2K_1/5M_1\beta$$

and the second pot-setting can be similarly calculated:

$$(D/M_1)(50/5)(1/\beta) = (10D/M_1\beta)$$

Finally, numerical calculation of these coefficients shows that with $\beta = 10$, both coefficients are between 1 and 10. Therefore, they should be divided by 10 and gains of 10 used on the integrator, thus keeping the product of pot-settings and gains unchanged. This leads to Figure 42.

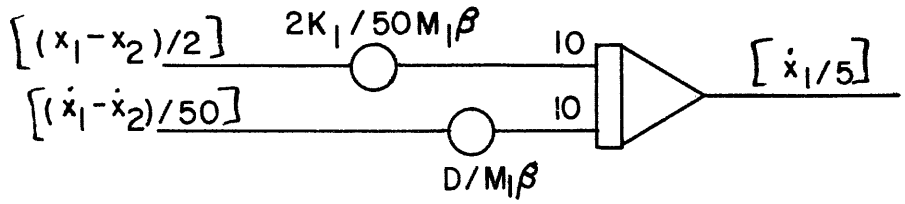


Figure 42

These are the same pot-settings obtained by writing scaled equations (compare Figure 42 with Figure 37 and Equation 67). In fact, the manipulations carried out in the "ratio of scale factors" approach are equivalent to those involved in writing a scaled equation, but they are performed on the diagram, rather than on the equations.

Whether the programmer writes a scaled equation or not for a given amplifier is a matter of personal preference. He should, however, understand scaled equations, and whatever method of scaling he uses, he should at least be able to justify his procedure by writing a scaled equation if called upon to do so.

Note, incidentally, if Figure 37 is scaled by the "ratio of scale factors" method, it is possible to write down an equation such as equation 67 by inspecting the scaled inputs and outputs of various amplifiers. The scaled equation obtained in this manner may then be reduced by cancellation, and the result will be either one of the original problem equations or an identity like $x + y = x + y$. Note that the process of reducing by cancellation is simply the opposite of deriving the scaled equation from the original equation. This process can be used as a check against arithmetic errors and sign errors in the original scaling process.

In practice, both the scaled equation method and the "ratio of scale factors" method are in wide use by experienced programmers. Some programmers prefer to write a scaled equation for every amplifier in the circuit (usually omitting the inverters), while others prefer to not to write scaled equations at all, or to write them only as a check.

10. Alternative Approach to Scaling

In section 2, the convention was established that reference voltage would be used as the unit of measurement. As a consequence, all amplifier outputs, and indeed all signals on the computer, are ≤ 1 in magnitude. This convention is not universally adopted, and it is worth while considering several alternative approaches, since many of the problems in the existing literature are scaled with different notation. In this section, two alternative approaches will be considered.

Alternative #1. This alternative consists of measuring amplifier outputs in volts, regardless of the computer reference level. The scaled equations will assume different forms, depending upon the reference voltage. The scaled variables will be larger, but the pot-settings will be numerically the same as those obtained by the procedure recommended in this chapter. (This last fact should not be surprising -- since a pot-setting is the ratio of output voltage to input voltage, and this ratio is independent of the units of measurement.)

To illustrate the differences between the recommended approach and the first alternative, consider a 100-volt computer. From Equation 11, section 2,

$$\text{Amplifier Output} = \frac{(\text{Reference})(\text{Problem Variable})}{\text{Maximum Value of Problem Variable}}$$

If reference is regarded as 100-volts rather than 1 unit, then all amplifier outputs will be numerically 100 times larger than with the method recommended in this chapter. Of course they will not be physically larger -- one man will refer to an amplifier output as 80.00 volts and the other will refer to it as 0.8000 units, but the amplifier output is physically unchanged.

As an example of scaling in volts, consider the second-order equation $a\ddot{x} + b\dot{x} + cx = 0$, which was scaled in Section 5. If computer variables are measured in volts rather than units, then the scaling table becomes:

PROBLEM VARIABLE	ESTIMATED MAXIMUM	COMPUTER VARIABLE (IN VOLTS)
x	5 feet	$[20\ x]$ (not $x/5$)
$\dot{x}$	10 feet/sec.	$[10\ \dot{x}]$ (not $\dot{x}/10$)
$\ddot{x}$	40 feet/sec.^{2}	$[2.5\ \ddot{x}]$(not $\ddot{x}/40$)

Note that the computer variables are numerically 100 times larger than before, and that they reach a maximum of 100 volts when the corresponding problem variables are at their maximum values. Scaled equations may be derived in the usual manner, and are given below:

$$[2.5\ \ddot{x}] = -\left\{\left(\frac{b}{4a}\right)[10\ \dot{x}] + \left(\frac{c}{8a}\right)[20\ x]\right\} \tag{72}$$

$$\frac{d}{dt}[20\ x] = -\ 10\ (0.2)\ [-10\ \dot{x}] \tag{73}$$

$$[20\ x_o] = -\ (x_o/5)\ [-100] \tag{74}$$

$$\frac{d}{dt}[-10\ \dot{x}] = -\ 10\ (0.4)\ [2.5\ \ddot{x}] \tag{75}$$

Comparing these scaled equations with equations 34, 37, 38, and 40 respectively, we see that the pot-settings and gains are the same, but the amplifier outputs are numerically 100 times larger. The corresponding circuit diagram is given in Figure 43.

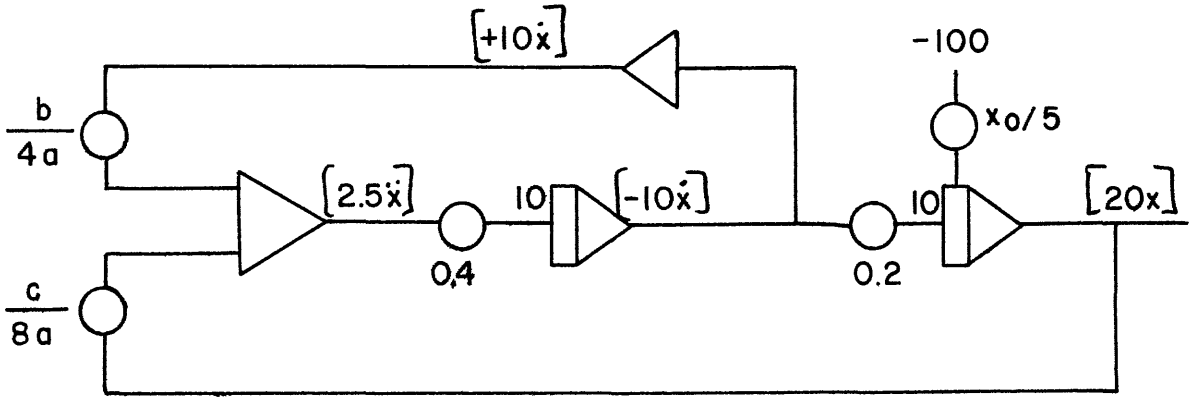


Figure 43

This is the same as Figure 32, except that the amplifier outputs and the reference voltage input to the I.C. pot are numerically 100 times larger. (On a 10-volt computer, they would be 10 times larger). Both circuits will, of course, yield the same results when translated back into problem variables. For example, if the output of the second integrator at a given instant is 0.8000 units or 80.00 volts, then, reasoning in units, we obtain:

$$[x/5] = 0.8 \therefore x = 5 \times 0.8 = 4 \text{ feet.}$$

and reasoning in volts, we obtain:

$$[\,20x\,] = 80 \therefore x = \frac{80}{20} = 4 \text{ feet.}$$

Scaled equations for nonlinear elements contain an additional complexity when computer variables are measured in volts. For example, the output of a multiplier is not simply the product of the inputs, but rather the product of the inputs divided by reference voltage. Thus, on a 100-volt machine, the output of a multiplier with inputs x and y would be xy/100. This factor of 100 is necessary because the unit is designed so that when both inputs are 100 volts, the output will also be 100 volts.

The factor of 100 complicates the scaling of a multiplier. For example, suppose it is desired to multiply two variables x and y with maximum values of 50 and 10 respectively. The scaled voltages would be [2 x] and [10 y], and the output (which has a "worst case" maximum of 50 x 10 = 500) would be scaled as [xy/5]. The scaled equation would then be

$$[xy/5] = \frac{[2\ x][10\ y]}{100} \tag{76}$$

Compare this to the more straightforward equation 22; in terms of units, one simply multiplies [x/50] by [y/10] to obtain [xy/500].

In summary, scaling in terms of volts will lead to different (and sometimes more complicated) scaled equations. All amplifier outputs will be numerically larger, but pot-settings will be the same. Of course, the final solution, stated in terms of problem variables, will be the same.

The principal advantage of scaling in volts, rather than in units, is that the programmer is using a familiar unit of measurement that is in common use throughout the electronics industry. Interfacing the computer with external devices, such as transducers, oscillators, meters, and other devices is simplified, since these devices are generally calibrated in volts. The programmer who scales in units has to go through a mental translation from units to volts every time he connects or adjusts an external device. Note that this applies only to external meters, recorders, etc. Readout devices such as scopes, plotters, meters, etc. which are built-in or are intended for use only with a given computer should be calibrated in units. Most DVM's are designed this way, so that when connected to reference voltage, they display + 1.0000, i.e. one unit. The trend in modern, large-scale computers is toward built-in plotters, scopes, etc. If these are regarded as an integral part of the machine,

they can be calibrated in terms of the computer reference voltage, and labeled in units, rather than volts.

Scaling in terms of units has several advantages, of which the most significant are given below:

1. Scaling is machine-independent. The same scaled equations and scaled diagrams result for any machine. This is the reason originally given in section 2 for using units, and is in line with the general trend (in both digital and analog computation) toward programs that are less oriented toward a given machine.

2. As mentioned above, most DVM's are calibrated in units. The programmer who thinks in terms of volts must distinguish between an amplifier reading of 39.82 volts and a pot-setting of .3982, both of which are displayed as +.3982 on a 4-digit DVM. In terms of units, the decimal point is correctly placed in all cases.

3. Scaling of nonlinear devices is simpler. For example, the scaling of products, quotients, squares, and square roots is free from extraneous factors of 10 or 100, which have nothing to do with the problem variables.

4. Since reference is 1 unit, the scaling of pots connected to reference voltage is somewhat simpler. For example, an I.C. pot always has the same scale factor as the integrator output itself, which is not true if volts are used instead of units.

Alternative #2.

This is really a minor variant on the recommended procedure, rather than a genuine alternative. In this approach, a computer variable is regarded not as a voltage, but as a dimensionless variable (a "pure" number) between - 1.0000 and + 1.0000. A variable with a maximum value of 5 feet would be represented on the computer by the "dimensionless" or "normalized" variable [x/5].

With this approach, the scaled equations are scaled computer diagrams will be the same as those produced by measuring voltages in units instead of volts. In other words, the difference between this "alternative" and the recommended procedure is purely psychological. Nevertheless, the difference can be important, since it goes to the heart of a person's basic attitude toward an analog computer.

If one thinks of the analog as an electrical model, or simulator of a physical system, then it is desirable to think of computer variables as voltages. It turns out that scaling is simplified if we use somewhat unconventional units (dekavolts, hectavolts), but the voltages lose none of their physical or electrical significance.

If, on the other hand, one regards the machine as a computer, receiving numerical information (in the form of pot-settings, for example), and producing numerical or graphical output, then this "alternative" approach becomes applicable. Note that this is very similar to the way one thinks of a "number" in a digital computer -- the digital programmer generally regards a piece of data either as a "pure number" or as a bit-pattern, but he rarely, if ever, concerns himself with the physical "meaning" of these bits -- i.e. in terms of whether or not certain transistors are conducting,

or whether a given core is magnetized.

Both approaches are legitimate, and since it makes no practical difference in scaling, the programmer may adopt either option.

C. EQUIPMENT ASSIGNMENT

Before the scaled problem can be put on the computer, it is necessary to assign to each component (amplifier, pot, multiplier, etc.) a number, or address, consistent with the numbering system for the particular computer used. For example, one computer may have 100 amplifiers numbered 00-99, while another may have 20 amplifiers, numbered 1-20. Information on the number of components in a computer, their locations, and addresses, is found in the Reference Handbook for that computer.

Component addresses should be written on the scaled diagram, inside the amplifier or pot symbol to aid in patching. In addition, it is helpful to list the components in assignment sheets, which provide the operator with the necessary information for pot-setting, checkout, etc. in tabulated form. The use of these assignment sheets is described more fully below.

D. DOCUMENTATION

An important part of any analog simulation is recordkeeping. A reasonable compromise must be reached between the extremes of inadequate documentation on one hand, and excessive paperwork on the other. To provide the programmer and operator with sufficient information to carry through the simulation efficiently, and to establish confidence in the validity of the final results, it is recommended that the following information be combined in order, in a single place:

1. The original, unscaled problem equations, together with the values (or expected ranges of values) of all parameters.
2. The unscaled circuit diagram.
3. The scaling table.
4. The scaled circuit diagram, and scaled equations.
5. The Potentiometer Assignment Sheet (Pot Sheet). This should contain the following information for each pot:
   a. Its number (address).
   b. Its setting in terms of problem parameters (e.g. $2K_1/50M_1\beta$).
   c. Its numerical value (e.g. 0.1600)

   Note that the numerical values themselves do not appear on the circuit diagram. The setting in terms of problem parameters is what is needed on the diagram, so that the programmer or operator can see the relation between parameters and pot-settings at a glance. Pots that do not contain problem parameters, but only scale factors, may be indicated on the diagram as in Figure 27 (e.g. 1/2).
6. The Amplifier Sheet. This should contain the following information for each amplifier:
   a. Its address.
   b. The type of amplifier (e.g. summer, integrator, high-gain, inverter)
   c. The output variable, including sign and scale factor (e.g. $-\ddot{x}/50$)
   d. Check calculations (see next section).

With this information available in a single place, the patching, checkout, and operation of the computer, and the interpretation of results can be carried out much more efficiently and quickly.

E. STATIC CHECK

1. Need for Static Check

In the process of programming, scaling and patching a problem, numerous opportunities for error arise. Since even one error in a program can invalidate the entire result, it is absolutely necessary to detect and correct these errors before actual computation starts.

On moderately large problems (over 100 amplifiers), it is very rare that the entire program will be completed without error. Even on fairly small problem (50 amplifiers or less), an experienced programmer still may make a mistake or two. Component malfunctions occasionally occur, even with well-maintained equipment, and, again, one faulty resistor can invalidate the entire result.

Occasional programming errors and component malfunctions will cause no harm if a systematic checkout procedure is adopted that will assure their detection. Only if this is done can a programmer have confidence in the correctness of the results. This section describes in detail the most commonly accepted checkout procedure: the static check (sometimes called static test).

This procedure will be illustrated by applying it to the Automobile Suspension problem, which has been programmed and scaled above.

A complete program for the Auto Suspension System, including pot and amplifier sheets, is given in Figures 44 and 45. A number of programming errors have been deliberately included:

1. The term $\dot{x}_2 - \dot{x}_1$ has the wrong sign. An inverter has been omitted. (Compare Figure 44 with the correct program in Figure 37).

2. Amplifier 03 (the integrator producing $-x_1$) has the wrong input gain.

3. The numerical value of the setting on 004 has been miscalculated. (If $x_3 = 5/12$ foot, then $x_3/2 = .2083$).

In addition, let us assume the following sources of error in the machine setup:

4. The patching connection from the output of P05 to the gain of 1 on A01 has been omitted.

5. The input resistor on A07 is defective.

These errors and malfunctions are typical of the difficulties encountered in programming and running an analog problem. A systematic check must locate all such errors, since a single error (or component malfunction) can make the entire simulation invalid.

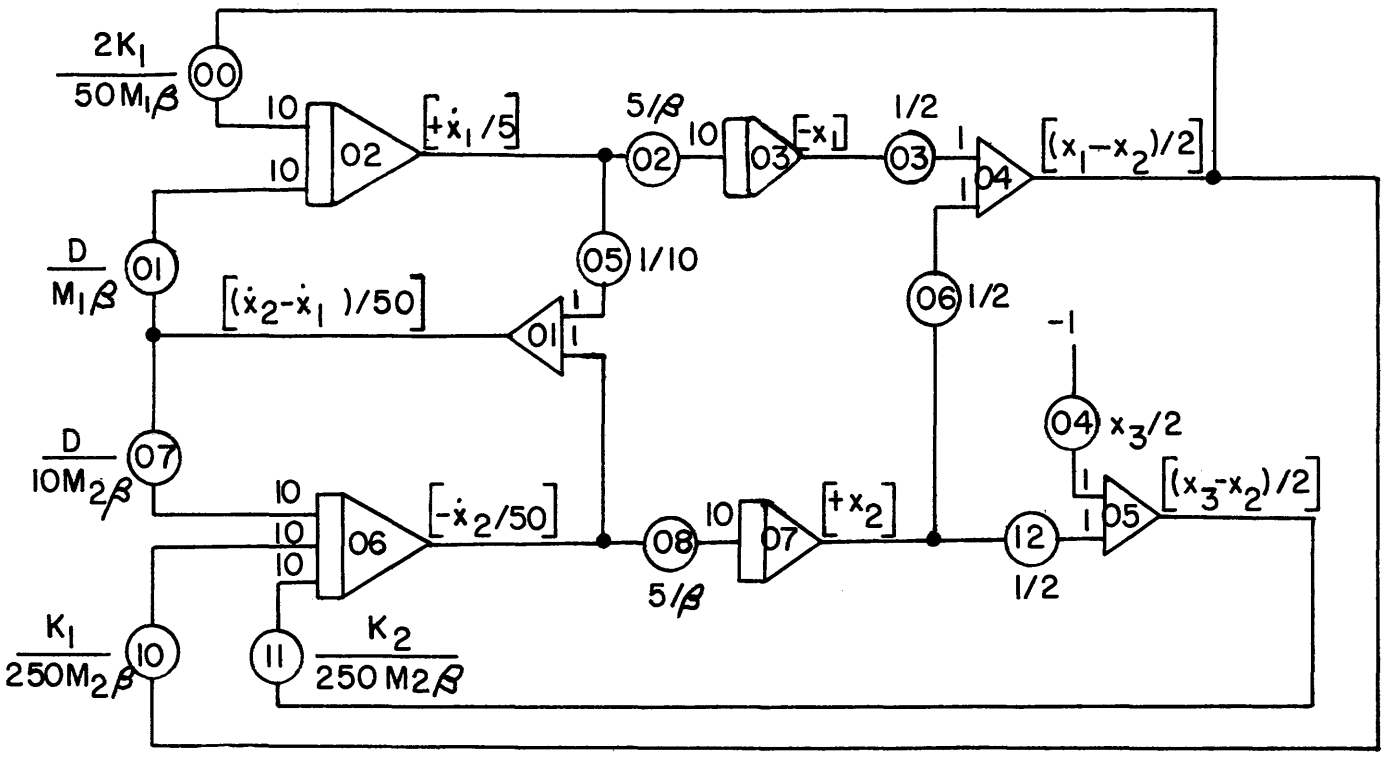


Figure 44: Program for Auto Suspension Problem, Including Programming Errors

ELECTRONIC ASSOCIATES INC

TR - 48 AMPLIFIER ASSIGNMENT SHEET

A00 – A23

DATE 3/10/65 PROBLEM Auto Suspension

AMP NO	FB	OUTPUT VARIABLE	STATIC CHECK				NOTES
			CALCULATED		MEASURED		
			DERIV	OUTPUT	DERIV	OUTPUT	
00							
01	Σ	$(\dot{x}_2-\dot{x}_1)/50$					
02	∫	$\dot{x}_1/5$					
03	∫	$-x_1$					
04	Σ	$(x_1-x_2)/2$					
05	Σ	$(x_3-x_2)/2$					
06	∫	$-x_2/50$					
07	∫	$+x_2$					
08							
09							

ELECTRONIC ASSOCIATES INC

TR-48 POTENTIOMETER ASSIGNMENT SHEET

P00 - P29

DATE 3/10/65 PROBLEM Auto Suspension

POT NO	PARAMETER DESCRIPTION	SETTING STATIC CHECK	STATIC CHECK OUTPUT VOLTAGE	SETTING RUN NUMBER 1	NOTES	POT NO
00	$2K_1/50m_1\beta$	.1600				00
01	$D/m_1\beta$	.8000				01
02	$5/\beta$	.5000				02
03	$1/2$	.5000				03
04	$x_3/2$	.1083				04
05	$1/10$	.1000				05
06	$1/2$	.5000				06
07	$D/10m_2\beta$	1.000				07
08	$5/\beta$	.5000				08
09						09
10	$K_1/250m_2\beta$	.2000				10
11	$K_2/250m_2\beta$	1.000				11
12	$1/2$	.5000				12
13						13
14						14
15						15

Figure 45. Pot Sheet and Amplifier Sheet for the Auto Suspension Problem

The standard check procedure, known as the static check, or static test, is divided into two parts:

> A program check, to determine whether the program (i.e. the circuit diagram, pot sheets, and amplifier sheets) actually represents the original equations. The program check should detect the first three errors listed above.
>
> A circuit check, to determine whether the actual computer setup corresponds to the program. The circuit check should detect the remaining sources of error (i.e. mis-patching, faulty components, etc.).

In practice, the two checks are closely related; the calculations made in the program check serve as the basis for the measurements made in the circuit check. In both checks, the computer is put in the RESET or INITIAL CONDITION mode, establishing test signals at the outputs of integrators, and calculating (or measuring) the outputs produced at other points in the circuit.

2. Program Check

The program check starts by assuming arbitrary values for all integrator outputs. These values should, of course, be within the range of values for which the problem was scaled, since otherwise, they would correspond to overloads at the integrator outputs. In addition, they should be chosen to avoid producing either overloads or extremely small outputs any where in the circuit.

In the auto suspension problem, there are four variables appearing at integrator outputs: x_1, $\dot{x}_1$, x_2, and $\dot{x}_2$. Let us start by assuming values for these four variables:

$$x_1 = + 0.8 \text{ feet}$$
$$\dot{x}_1 = + 3 \text{ feet/sec.}$$
$$x_2 = + 0.5 \text{ feet}$$
$$\dot{x}_2 = + 35 \text{ feet/sec.}$$

These are original problem variables, and the units are the original problem units (e.g. feet and feet/sec.). Each of these problem variables corresponds to a computer variable (an amplifier output) measured in analog units:

$$A02 = + \dot{x}_1/5 = + 0.6 \text{ unit}$$
$$A03 = - x_1 = - 0.8 \text{ unit}$$
$$A06 = - \dot{x}_2/50 = - 0.7 \text{ unit}$$
$$A07 = + x_2 = + 0.5 \text{ unit}$$

These assumed values form the starting point for two sets of calculations; one set will be made from the program (i.e. circuit diagram and pot-sheet) and the other from the original, unscaled equations. The results of the two sets of calculations will then be compared; if they disagree, it is an indication of a programming error.

a. Calculations Based on the Program. The calculations based on the program start with the assumed values of the integrator outputs (in analog units) listed above. The only information needed for this series of calculations is a copy of the circuit diagram and the pot-sheet.

Knowing the integrator outputs and the numerical setting on every pot, it is possible to calculate the output of every component in the circuit. For example, the output of A02 is + 0.6 unit. This output is connected to P05. From the pot-sheet, we see that the setting on P05 is 0.1000. Multiplying the pot input by the numerical pot-setting, we obtain the pot output (+ 0.06 unit). This is one of the inputs to A01, and the other input is - 0.7 unit from A06. Therefore the output of A01 will be -(- 0.7 + 0.06) = + 0.64 unit. Similar calculations can be made for every component on the diagram. For convenience, it is recommended that the calculated output be written on the circuit diagram as soon as it is calculated (See Figure 46).

In addition to the outputs of the amplifiers and pots, the initial derivative of every integrator output should also be calculated, since this is the only way to check the programming and patching of the integrators. The derivative of an integrator output is, of course, simply the sum of the inputs. Thus to calculate the initial derivative of integrator 02 (abbreviated D02):

$$D02 = + \left\{10\ (0.024) + 10\ (0.512)\right\} = + 5.36 \text{ units/sec.}$$

Of course, the integrator output will not really be changing, if the system remains in the I.C. mode. The physical meaning of D02 is that if the computer were put into the OPERATE mode at this point, the output would initially decrease at the rate of 5.36 units/sec. In practice, there is a simpler and more accurate way to measure this derivative, which will be discussed below when the circuit check is covered.

Note that all calculations so far have been based on the program, i.e. on the circuit diagram and the numerical settings on the pot sheet. The person making these calculations need not know anything about the original problem variables or the original equations.

b. Calculations Based on the Original Problem. Once these calculations on the circuit diagram are completed, they are set aside and another set of calculations is made from the original equations, in terms of the original problem variables. The same quantities are calculated (outputs of amplifiers and derivatives of integrators) but from the original data, without looking at the pot-sheet or amplifier sheet.

The starting point for these calculations consists of the original problem parameters:

$M_1 = 25$ slugs; $K_1 = 1000$ lb/foot.

$M_2 = 2$ slugs; $K_2 = 5000$ lb/foot.

$x_3 = 5/12$ foot; $D = 200$ lb/foot per sec.

and the assumed values of the original problem variables:

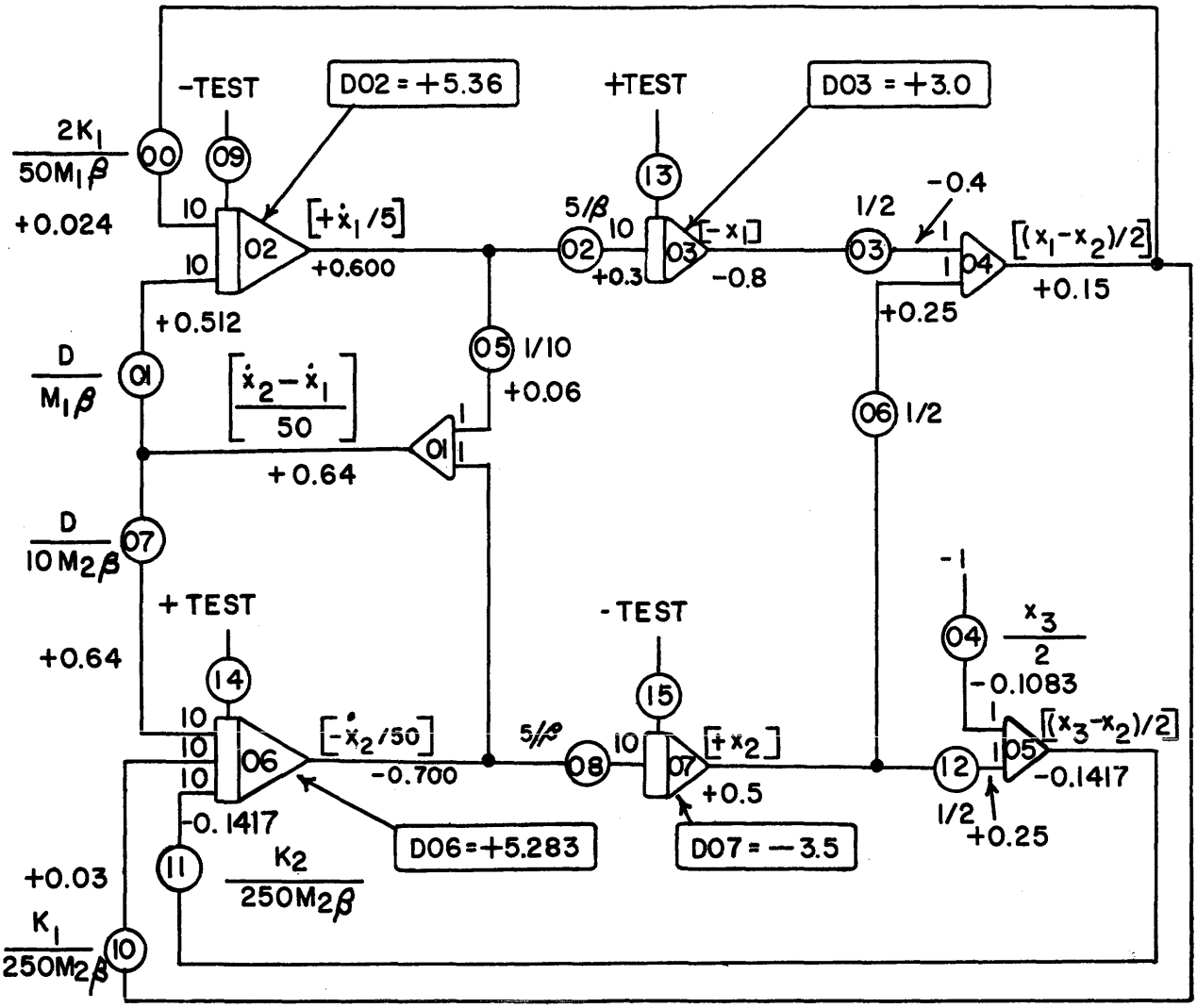


Figure 46: Circuit Diagram for Auto Suspension System With Static Check Values Written In

$x_1 = +\ 0.8$ feet

$\dot{x}_1 = +\ 3$ feet/sec.

$x_2 = +\ 0.5$ feet

$\dot{x}_2 = +\ 35$ feet/sec.

To calculate amplifier outputs from this set of data, we need to know the expression for each amplifier output in terms of problem variables. These expressions are listed in the amplifier sheet (see Figure 45). For example, the amplifier sheet tells us that the output of A01 is $(\dot{x}_2 - \dot{x}_1)/50$. Substituting the assumed values of $\dot{x}_2$ and $\dot{x}_1$ into this equation gives:

$$A01 = (\dot{x}_2 - \dot{x}_1)/50 = (35 - 3)/50 = 32/50 = 0.64 \text{ units.}$$

Similar calcuations can be made for the other amplifiers:

$A02 = \dot{x}_1/5 = +\ 0.6$ units

$A03 = -\ x_1 = -\ 0.8$ units

$A04 = (x_1 - x_2)/2 = +\ 0.15$ units

$A05 = (x_3 - x_2)/2 = -\ 0.0417$ units

$A06 = -\ \dot{x}_2/50 = -\ 0.7$ units

$A07 = +\ x_2 = +\ 0.5$ units

Note that these calculations can be made without looking at the circuit diagram or the numerical pot-settings listed on the pot sheet. The calculations are made by substituting the assumed values of the problem variables into the algebraic expressions listed on the amplifier sheet. It is recommended that these calculated values be filled in on the amplifier sheet in the space provided (see Figure 45).

We have now calculated all amplifier outputs and listed them on the amplifier sheet. To complete the check, we must calculate the derivatives of the integrator outputs as well. For example, the output of A02 is $+\ \dot{x}_1/5$. Its derivative (in analog units per second) is therefore:

$$-\frac{d}{d\tau}\left[\dot{x}_1/5\right] = -\ddot{x}_1/5\beta.$$

Note that in order to translate the problem derivative into a machine derivative, we must divide by the time-scale factor β. The result is the derivative of the amplifier output in analog units per second of machine time τ. An expression for the derivative of any integrator output may be obtained by differentiating the output and dividing by β:

$A03 = +\ x_1 \quad \therefore\ D03 = +\ \dot{x}_1/\beta$

$A07 = -\ x_2 \quad \therefore\ D07 = -\ \dot{x}_2/\beta$

$A02 = -\dot{x}1/5. \quad \therefore\ D02 = -\ \ddot{x}_1/5\beta$

$A06 = +\dot{x}_{2/50} \quad \therefore\ D06 = +\ \ddot{x}_2/50\beta$

We can now calculate these derivatives by substituting the known values into these expressions. For example, we started out by assuming values for $\dot{x}_1$ and $\dot{x}_2$. Substituting these values, we obtain:

$$D03 = + \dot{x}_1/\beta = + \frac{3}{10} = + 0.3 \text{ units/sec.}$$

$$D07 = - \dot{x}_2/\beta = - \frac{35}{10} = - 3.5 \text{ units/sec.}$$

The expressions for D02 and D06 require knowledge of the highest derivatives $\dddot{x}_1$ and $\ddot{x}_2$. To calculate these, we must use the original unscaled equations:

$$\dddot{x}_1 = - \frac{K_1}{M_1} (x_1 - x_2) - \frac{D}{M_1} (\dot{x}_1 - \dot{x}_2)$$

$$= - \frac{1000}{25} (0.8 - 0.5) - \frac{200}{25} (3 - 35)$$

$$= - 12 + 256$$

$$\boxed{\dddot{x}_1 = + 244 \text{ feet/sec.}^2}$$

$$\ddot{x}_2 = - \frac{K_1}{M_2} (x_2 - x_1) - \frac{K_2}{M_2} (x_2 - x_3) - \frac{D}{M_2} (\dot{x}_2 - \dot{x}_1)$$

$$= - \frac{1000}{2} (- 0.3) - \frac{5000}{2} (\frac{1}{2} - \frac{5}{12}) - \frac{200}{2} (32)$$

$$= + 150 - 208.3 - 3200$$

$$\boxed{\ddot{x}_2 = - 3258.3 \text{ feet/sec.}^2}$$

Thus the calculated derivatives become:

$$D02 = - \dddot{x}_1/5\beta = - \frac{244}{50} = - 4.88 \text{ units/sec.}$$

$$D06 = + \ddot{x}_2/50\beta = + 6.5166 \text{ units/sec.}$$

The calculated derivatives, in units/sec. should be listed in the appropriate places on the amplifier sheet, along with the amplifier outputs.

C. DETECTION OF ERRORS

We have now calculated every amplifier output, and the derivative of every integrator in two ways, one based on the circuit diagram and one based on the original equations. Comparing the two should detect any errors in programming.

When the calculated values of an amplifier output or derivative disagree, the error can be located by working backward through the circuit. If the outputs of a component do not check, we examine the inputs. If the inputs check, then the error lies in the programming or scaling of that particular component. If the inputs do not check, we apply the same process, working backward until the error is isolated. For example, the derivative D02 was calculated on the diagram as + 5.36 units/sec., while the calculation from the original equations yields - 4.88 units/sec. This amplifier receives inputs from A01 and A04, and the calculated values for these outputs agree. The trouble must therefore lie in the programming of the $\ddot{x}_1$ equation. If we compare the scaled equation for $d/d\tau[\dot{x}_1/5]$ with the original equation for $\ddot{x}_1$, it becomes clear that P01 should have the input $(\dot{x}_1 - \dot{x}_2)/5$; so that an inverter is necessary. Adding this inverter to the circuit, we find that the output of P01 becomes - 0.512 units, and the calculated value of D02 becomes - 4.88 units/sec., as it should be.

Comparison of the two calculated values for D03 will indicate a factor-of-ten difference, and inspection of the diagram indicates that a gain of 1 is needed, rather than 10.

The calculated values of D06 also show disagreement. Checking the three inputs to this amplifier, we see that the calculated output of A05 does not check. Working backwards, we check the two inputs to this amplifier. One comes from A07, which does check, (the calculated output is + 0.5 units in both cases) and the other comes from reference through P04. Thus the source of disagreement has been narrowed down to this pot. Using the given value of $x_3 = 5/12$ feet, we find that the pot-setting should be 0.2083, not 0.1083, and the arithmetic error is detected. After this error is corrected on the pot-sheet, and the sign error on the $\dot{x}_2 - \dot{x}_1$ term is also corrected, the derivative D06; as calculated on the diagram, becomes:

$$D06 = + \left\{10\ (-0.64) + 10(0.03) - 10(0.0417)\right\} = - 6.517 \text{ units/sec.}$$

which agrees with the value calculated from the original equations. The program check is now complete. Note that the entire program check can be carried out off-line (i.e. without using the computer at all).

3. Circuit Check

When the program check is complete, the programmer has a list of amplifier outputs and integrator derivatives that can be used for the circuit check. This check is made on the computer by establishing the appropriate initial conditions, measuring all amplifier outputs, and comparing the measured values to the previously calculated values. Any disagreement larger than the effect of component tolerances indicates a patching error or a component malfunction.

The isolation of the source of error follows the same procedure used in the program check. If the output of a component is wrong and the inputs are correct, the error is isolated in that component. The trouble is quickly narrowed down to a patching error, a blown fuse, or a faulty component. When the faulty component is replaced or the patching error corrected, the calculated and measured values should agree.

Although the measurement of amplifier outputs is straightforward, the measurement of integrator derivatives may require a temporary patching change. Since the derivative of an integrator output is minus the sum of the inputs, it may be read by momentarily disconnecting the network of input resistors from the integrator, and connecting them to the summing junction of a spare summer. This summer, which is referred to as the check amplifier, will then read the derivative directly. Large machines generally have a built-in check amplifier, so that no re-patching is necessary --the operator merely depresses a series of buttons marked D02, and the derivative of amplifier 02 can then be read on the DVM.

The test signals must, of course, be established at the integrator outputs by means of the initial condition input. If these test values are different from the actual I.C.'s used in the problem, they must be removed and the actual I.C.'s established before the problem is run. In many cases, the actual problem I.C. is zero, so that the only change required is to remove the I.C. leads on the integrators when the static check is complete.

Some computers have an output on the patch panel known as test reference to avoid the need for this patching change. The test reference terminals are energized with $\pm$ reference voltage only in the static test mode. In all other respects, the static test mode is equivalent to the I.C. mode.

4. Summary of Steps

1. Assume arbitrary values for all variables appearing at integrator outputs. For each assumed value of a problem variable (e.g. $\dot{x}_2 = +35$ feet/sec.) there is a corresponding value of the scaled computer variable (e.g., $-\dot{x}_2/50 = -0.7$ units).

2. Using the circuit diagram, the numerical values on the pot-sheet, and the arbitrary I.C.'s from step 1, calculate on the diagram the output of every amplifier and the derivative of every integrator output.

3. Using the algebraic expressions for the amplifier outputs, as given in the amplifier sheet, the arbitrary I.C.'s for the problem variables from step 1, and the original equations, calculate every amplifier output and integrator derivative a second time, and enter these calculated values on the amplifier sheet.

4. Compare the calculated values on the circuit diagram (from step 2) with these on the amplifier sheet (from step 3). Any disagreement in calculated values indicates a programming error. Isolate the sources of error by working backwards through the circuit. After the programming errors are detected and corrected, the calculated values should agree.

5. Patch the problem and put it on the computer. Establish the appropriate static test I.C.'s on the integrators, measure the outputs and derivatives, and compare them with the calculated values. If discrepancies occur, a patching error or component malfunction is indicated. Such sources of error can be located by working backward through the circuit, as in step 4.

6. After all patching errors are corrected and faulty components replaced, the measured and calculated values should agree (allowing for normal component tolerances). The static check is now complete. Before running the problem, restore all I.C.'s and parameters to the proper values for the first run. (This task is made somewhat easier if the computer has a static test mode).

CHAPTER IV.

ANALYSIS OF LINEAR AND NON-LINEAR SYSTEMS

A. Introduction

The most common type of problem generally solved on analog computers is one involving sets of ordinary differential equations. This chapter surveys differential equations in general, with major emphasis on the most easily solved cases (linear, constant co-efficient, unforced equations). The objective is not to develop facility in analytical solution of differential equations since, for most of the interesting and useful equations that arise in practice, such methods are either impractical or impossible. Rather, the objective is to draw attention to the mathematical and physical concepts (time constants, natural frequencies, resonance) that arise from a study of linear systems, and to emphasize the relation between the form of an equation and its solution.

Nonlinear equations are discussed, and two examples are given, chiefly to point out the differences in behavior as compared with linear equations.

The chapter concludes with some general remarks that are often useful in estimating maximum values for scaling purposes.

B. Linear, Constant-Coefficient Equations

A differential equation is linear if it has the form

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_o x = f(t) \qquad (1)$$

where the co-efficients, a_i $(0 \leq i \leq n)$, are either constants, or known functions of the independent variable, t. The equation is called a constant co-efficient equation if all the co-efficients are constant; otherwise, it is a variable coefficient equation.

It is customary to refer to the function, f(t), on the right-hand side as a forcing function or input, and to the unknown, x(t), (the solution of the equation), as the response or output. An examination of the most common applications, as cited below, will make the reason for this terminology clear.

1. Physical Examples

In a subsequent section of this chapter, the following first-and second-order linear differential equations with constant coefficients will be solved:

$$a\dot{x} + bx = f(t) \qquad (2)$$

and

$$a\ddot{x} + b\dot{x} + cx = f(t) \qquad (3)$$

Both equations apply to a variety of systems. For example, equation (2) may represent any of the following:

a) A thermal system - suppose a small mass of hot material is quenched suddenly in a large bath of fluid whose temperature is f(t), which need not be constant. Then, the rate of heat loss of the quenched mass is approximately proportional to the difference between its temperature and that of the fluid. A heat balance equation yields

$$k \frac{dx}{dt} = f(t) - x(t) \tag{4}$$

where x(t) is the time-varying temperature of the quenched substance and the constant, k, depends on the thermal conductivities and specific heats of the materials. This is equation (2) with a = k and b = 1. Clearly, f(t) is, physically speaking, an input since we may pick the quenching temperature arbitrarily. On the other hand, the solution x(t) is an output or result due to the input, f(t).

b) The concentration, C(t), of a chemical compound in a reactor--this may be described (if the reaction is a sufficiently simple one) by the equation

$$\frac{dC}{dt} = -kC + f(t) \tag{5}$$

which says that the concentration, C(t), decreases at a rate proportional to the remaining concentration (as C reacts to form another substance), but that new amounts of the material are being fed into the reactor at a rate f(t) which may represent the rate of formation of C by some other reaction(s), or the rate at which the component is piped in from outside. In any case, this is equation (2) with a = 1 and b = k.

c) The radioactive decay of radium into lead and other products.

d) The concentration, C, in a stirred tank-- in this case, the component is not necessarily reacting but is simply being pumped in and out. A material balance yields

$$V \cdot \frac{d\, C_{out}}{dt} = q(C_{in} - C_{out}) \tag{6}$$

where C_{IN} is the inlet concentration and C_{OUT} is the outlet concentration (which is the same as the average tank concentration, if the tank is well-stirred). This, again, is equation (2) where we may take a = V, b = q, and $f(t) = qC_{IN}$ or, more conveniently, a = V/q, b = 1 and $f(t) = C_{IN}$. The interpretation of the variables as input and output is clear.

e) The output voltage of the R-C filter (as shown in Figure IV-1)-- here, the equation is

$$RC \cdot \frac{dV_{out}}{dt} = V_{in} - V_{out} \tag{7}$$

which is equation (2) again.

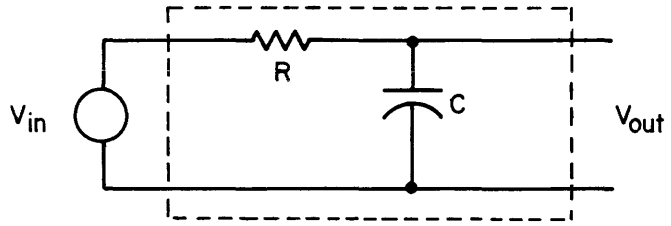


Figure IV-1. An R-C filter, showing input and output voltages.

f) Population growth--here, x is the population of an area, a = 1, b = the relative annual excess of births over deaths, and f(t) represents input via immigration (or emigration when f(t) < 0.)

g) Compound interest in a savings account-- here a = 1, b = the interest rate, f(t) represents the deposits or withdrawals, and x(t) is the amount of money in the account at time, t.

Typical examples of equation (3) are the following:

h) An R-L-C Series Circuit

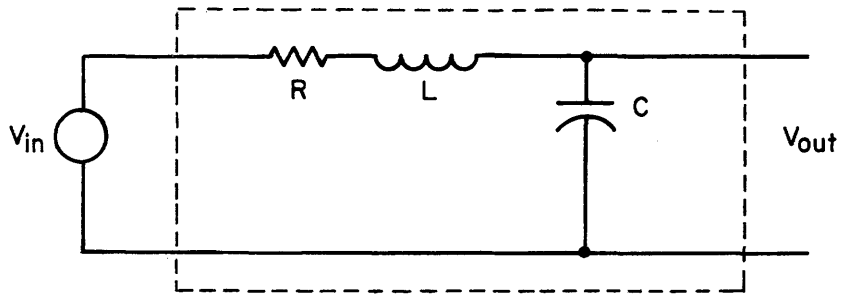


Figure IV-2

The output voltage is described by the equation

$$LC\frac{d^2V_{out}}{dt^2} + RC\frac{d\,V_{out}}{dt} + V_{out} = V_{in} \qquad (8)$$

which follows from Kirchhoff's Voltage Law.

i) A spring-mass-dashpot system

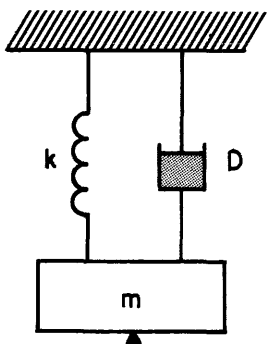


Figure IV-3.

Summing the forces on the mass and applying Newton's second law:

$$m\ddot{x} = -kx - D\dot{x} + f(t) \tag{9}$$

where x is the displacement of the mass, $\dot{x}$ and $\ddot{x}$ are, of course, its velocity and acceleration, k, m, and D are the spring stiffness, mass and damping coefficient, and f(t) is the externally applied force. This can be re-written in standard form:

$$m\frac{d^2x}{dt^2} + D\frac{dx}{dt} + kx = f(t) \tag{10}$$

and is, thus, an example of equation (3).

All these examples indicate the reason for referring to f(t) as the input or forcing function, and x(t) as the output or response. If f(t) = 0, the system is called unforced (sometimes the term homogeneous is used). The unforced system represents the decay of a chemical substance that is not being replenished, the discharging of a capacitor, the accumulation of capital in an account in which no withdrawals or deposits are being made, or the transient oscillations of a spring-mass system.

2. General Properties of Linear Systems

In solving these equations analytically, two general principles which apply to any linear differential equation of any order are used. These are the principles of superposition and the principle of proportionality. These principles take slightly different forms in the forced and unforced cases:

unforced case (Equation 1, with f(t) = 0).

Superposition Principle: If $x_1(t)$ and $x_2(t)$ are solutions of equation (1) with f(t) = 0, then so is $x_1(t) + x_2(t)$. In other words, the sum of two solutions is itself a solution.

Proportionality Principle: If x(t) is a solution of equation 1 with f(t) = 0, then so is kx(t), where k is any constant.

forced case (Equation 1, with $f(t) \neq 0$, but with zero initial conditions).

Superposition: If the input $f_1(t)$ produces the output $x_1(t)$, and the input $f_2(t)$ produces $x_2(t)$, then the input $f_1(t)+f_2(t)$ will produce the output $x_1(t) + x_2(t)$.

Proportionality: If the input f(t) produces the output x(t), then kf(t) will produce kx(t).

The proofs, in every case, follow by differentiation and substitution. These two properties are characteristic of linear systems; in fact, it can be proved that any first-or second-order differential equation that has these properties must be linear; that is, it must have the form of equations (1) or (2).

Both these principles can be given physical interpretation . The principle of superposition means that it makes sense to talk about cause and effect in a linear system. If a linear system is subject to many inputs and initial conditions, then its output can be broken down into parts (effects) corresponding to the various causes. When several causes operate simultaneously, the total effect is the sum of the individual effects.

The principle of proportionality means that if the magnitude of the forcing function to a linear system is doubled the magnitude of the response will also be doubled.

3. Analytical Solution

a. Unforced Systems ($f(t) = 0$) -- the key to tackling the unforced case lies in the principles of superposition and linearity.

Together, they imply that if solutions $x_1(t)$, $x_2(t)$, ... $x_n(t)$ can be found, then the sum

$$\sum_{i=1}^{n} A_i x_i(t) \tag{11}$$

is also a solution for any constants, A_1, $A_2 \ldots A_n$. If n initial conditions must be matched, there should be n constants, A_1, $A_2, \ldots A_n$, to meet these conditions. The problem is, therefore, one of finding enough different solutions so that, by combining them, a solution to match an arbitrary set of IC's can be found.

i. First-order equation, $a\frac{dx}{dt} + bx = 0$.

This equation is so simple that it can be solved easily by separation of variables.

$$\frac{dx}{dt} = -\frac{b}{a} x \tag{12}$$

Separating variables:

$$\frac{dx}{x} = -\frac{b}{a} dt \tag{13}$$

Integrating:

$$\log x = -(b/a)t + c \tag{14}$$

$$x = e^{c} \cdot e^{-(b/a)t} = Ke^{-(b/a)t} \tag{15}$$

This equation contains one arbitrary constant, K, derived from the constant, c, which we obtained as a constant of integration. This constant is just enough to permit matching of one arbitrary initial condition. Given x_o, K can be determined by setting $t = 0$; since $e^0 = 1$:

$$x_o = K e^{o} = K \tag{16}$$

Therefore

$$x(t) = x_o e^{-(b/a)t} \tag{17}$$

The graph of the solution is a decaying exponential (Figure IV-4) if b and a have the same sign, and an increasing exponential (Figure IV-5) if they have different signs. If b = 0, x is a constant.

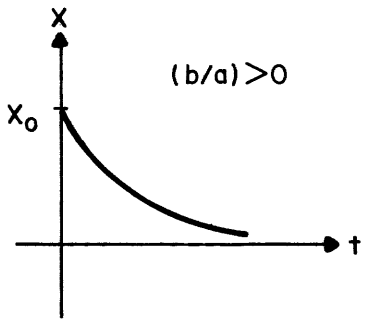


Figure IV-4
Decaying Exponential

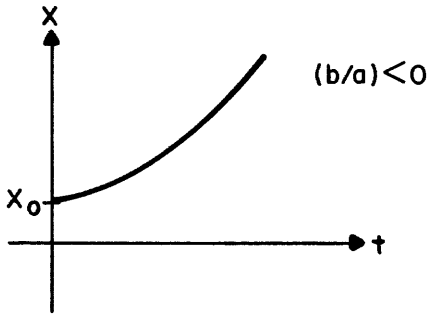


Figure IV-5
Increasing Exponential

Both the decaying and the increasing exponential occur in practice. Stirred tanks, capacitors and radioactive decay follow the decaying exponential curve, while compound interest, population growth and supercritical nuclear reactors follow the expanding exponential.

The behavior of this unforced system is described completely by specifying the initial condition and the time constant (a/b) of exponential growth or decay. The reader should be able to verify, directly from equation 2, that a/b must, in fact, have the dimensions of time. This time constant, or its reciprocal b/a which has the units of $(\text{time})^{-1}$, is recognized as an important concept in many fields of study and has been given a bewildering variety of names which tend to obscure the basic similarity of the systems involved. While the term time constant seems to have won the day in electronics and in applied mathematics, a chemical engineer is likely to refer to a rate constant if he is thinking about a reaction and a holdup time if he is thinking about mixing in a tank. A nuclear physicist talks about the half-life of radium (which is proportional to the time constant), while a banker refers to a 5%/year interest rate instead of a 20-year time-constant. Again, a demographer, instead of referring to a 50-year time constant for population growth, speaks of a birth rate (more precisely, birth rate minus death rate) of 20-per-year-per-thousand population.

ii. Second-order equation, $a\ddot{x} + b\dot{x} + cx = 0$

This equation does not appear easy to solve by separation of variables. However, the solution to the first-order system turned out to be an exponential

so maybe the second-order system will behave similarly. Let us try $x = e^{kt}$ and see whether it works. Clearly, the derivatives will be given by

$$\dot{x} = k\, e^{kt} \tag{18}$$

$$\ddot{x} = k^2 e^{kt} \tag{19}$$

Substituting into the original equation, we obtain

$$ak^2 e^{kt} + b\, k\, e^{kt} + c\, e^{kt} = 0 \tag{20}$$

Factoring out the exponential gives us

$$(ak^2 + b\, k + c)\, e^{kt} = 0 \tag{21}$$

and canceling, we obtain

$$ak^2 + b\, k + c = 0 \tag{22}$$

This is the so-called characteristic equation corresponding to the original differential equation. The assumed solution, e^{kt}, will work if (and only if) k is a root of this equation. Since this is a quadratic equation, the two roots are given by the quadratic formula:

$$k_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} \qquad k_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

Thus, $x_1(t) = e^{k_1 t}$ is a solution and so is $x_2(t) = e^{k_2 t}$. By using the superposition and proportionality principles, we conclude that $x(t) = A_1 x_1(t) + A_2 x_2(t)$ is also a solution for any values of the constants, A_1 and A_2. This gives enough constants to match an arbitrary set of initial conditions (displacement and velocity), and provides the general solution.

The graph of the solution takes different forms depending on whether (b^2-4ac) is greater than, less than, or equal to zero.

Case I. $b^2 > 4ac$. There are two real and distinct roots. The solution is the sum of two real exponentials. If both roots are negative, the system is stable and the graph looks like Figure IV-6a or Figure IV-7a, depending on the initial conditions. If either or both roots are positive, the graph looks like Figure IV-6b or IV-7b.

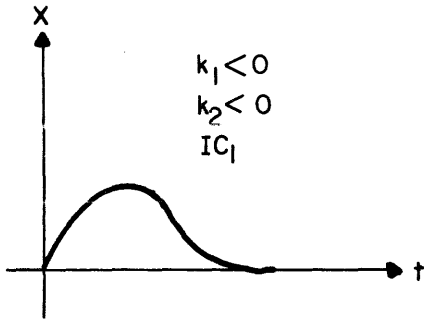


Figure IV-6a

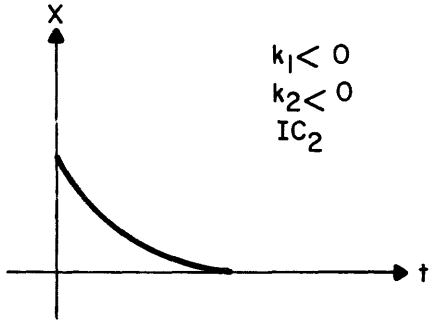


Figure IV-7a

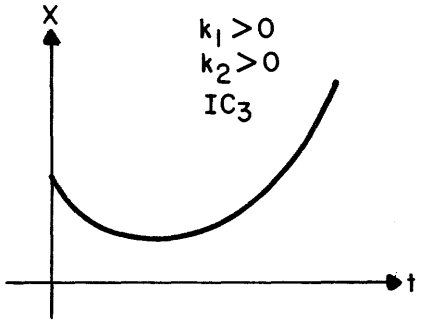


Figure IV-6b

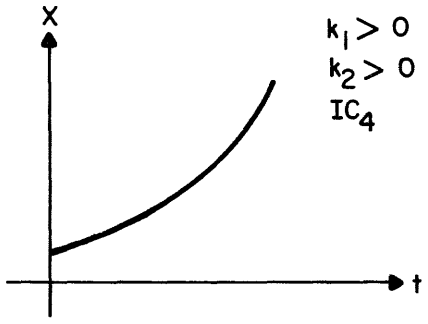


Figure IV-7b

Case I. $b^2 > 4ac$ Exponential Solutions

Case II. $b^2 < 4ac$. In this case, the roots are complex conjugates and they can be rewritten for convenience as $\sigma + j\omega$ and $\sigma - j\omega$, where $\sigma = -b/2a$ and

$$\omega = \frac{1}{2a}\sqrt{|b^2 - 4ac|} \tag{23}$$

The general solution is still

$$x(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t} = A_1 e^{(\sigma + j\omega)t} + A_2 e^{(\sigma - j\omega)t} \tag{24}$$

but now, since the solution is written in terms of complex numbers, it requires some interpretation; in physical situations, x(t) is generally a real variable.

Complex exponentials actually are just a shorthand notation for sinewaves, and the general solution can be written in the form:

$$x(t) = A_1 e^{(\sigma + j\omega)t} + A_2 e^{(\sigma - j\omega)t} = Ae^{\sigma t}\sin(\omega t + \phi) \tag{25}$$

where the amplitude, A, and the phase angle, ϕ, are complicated functions of A_1 and A_2. The actual proof of this--the transition from complex exponentials to real damped sinewaves--takes up about 1½ pages of algebraic computation and is omitted. In the usual case, a, b and c are all positive and the solution is a damped sinewave (note that $\sigma = -b/2a$, so that $e^{\sigma t} = e^{-(b/2a)t}$). Figures IV-8a and IV-8b show the usual response ($\sigma < 0$), and Figures IV-9a and IV-9b show what

happens if $\sigma > 0$. This last case (instability) does not occur in a simple spring-mass-dashpot system or an R-L-C filter but, as every control engineer knows, it can happen in a poorly designed servo system.

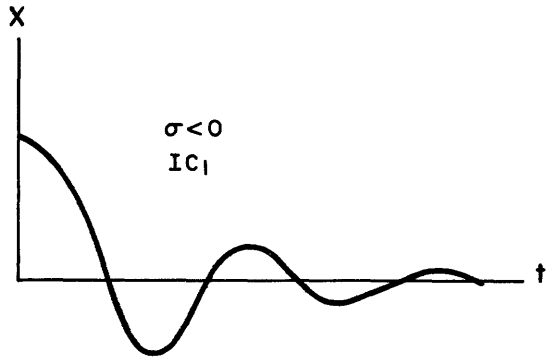


Figure IV-8a

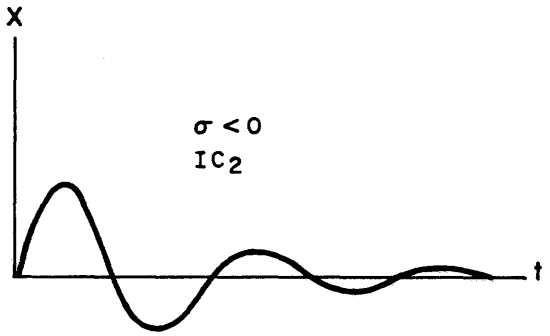


Figure IV-8b

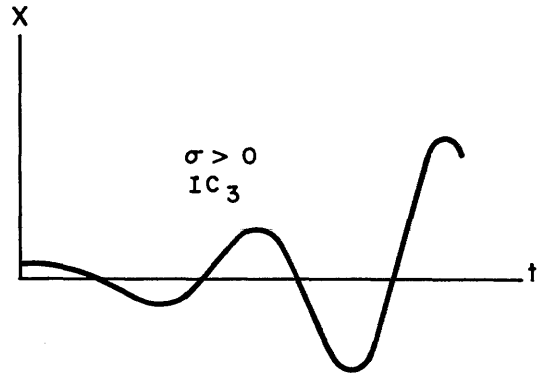


Figure IV-9a

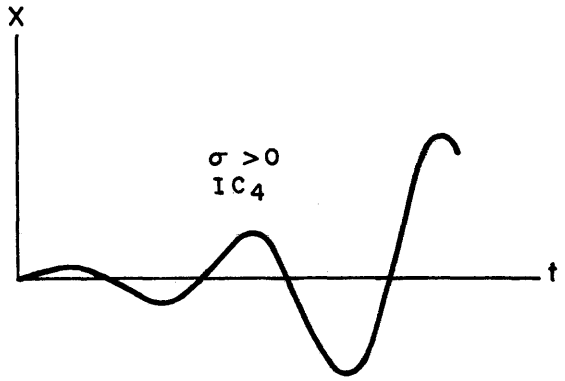


Figure IV-9b

Case II. $b^2 < 4ac$, Solutions

Case III $b^2 = 4ac$. In this case, there is only one root and it is real ($k = -b/2a$). Clearly $x = e^{kt}$ is a solution but, unless two different solutions can be found, only one of the two arbitrary initial conditions can be matched.

It turns out that, in this case, $x = te^{kt}$ is a solution also which can be verified by differentiation and substitution. This saves the day, since $x = A_1e^{kt}+A_2te^{kt}$ is a solution, and the constants A_1 and A_2 can be picked to match two given IC's. The solution can be written in factored form as $x(t) = (A_1+A_2t)e^{kt}$ and, thus, represents a linear term times an exponential. The solution does not oscillate.

The behavior of the solutions depends upon the magnitude of the damping co-efficient, b. If b is large, the system will not oscillate; if b is small, it will. If b is just large enough to prevent oscillations, the system is said to be critically damped; if b is larger than this value, the system is overdamped; if b is smaller than this value, the system is underdamped. The various cases are summarized in Table IV-1.

Physical Interpretation

From Table IV-1, we see that a necessary and sufficient condition for the system to oscillate is $b^2 < 4ac$. For the R-L-C filter (Figure IV-2), this condition reduces to $R^2 < 4L/C$. For the spring-mass-dashpot system (Figure IV-3), the condition

Case	Formula for Solution	Arbitrary Constants (depending on IC's)	System Constants (depending on a, b, and c)	Oscillation?
$b^2 > 4ac$ Overdamped	$x(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t}$	A_1, A_2	k_1, k_2	No
$b^2 = 4ac$ Critically Damped	$x(t)=(A_1+A_2 t)e^{kt}$	A_1, A_2	k	No
$b^2<4ac$ Underdamped	$x(t)=A\ e^{\sigma t}\sin(\omega t+\phi)$	A, ϕ	σ, ω	Yes
$b = 0$ Undamped	$x(t)=A \sin(\omega t + \phi)$	A, ϕ	ω	Yes

Table IV-1 Summary of Solution Behavior with Various Damping Coefficients

is $D^2 < 4km$. These findings confirm the experimental results that a husky shock absorber or a large resistor in a mechanical or electrical system will damp out oscillations. For example, the shock absorbers on a car may be tested by jumping on the bumper and then off again. If the bumper returns to rest with no overshoot or one overshoot, then $D^2 \geq 4km$ and the shock absorbers are all right. If the bumper oscillates several times in settling down, then $D^2 < 4km$ and it is time for new shock absorbers.

The underdamped case is by far the most common and deserves a little extra attention. The roots of the characteristic equation are the complex conjugate pair $\sigma + j\omega$ and $\sigma - j\omega$, where $\sigma = -\frac{b}{2a}$ and $\omega = \frac{1}{2a}\sqrt{|b^2 - 4ac|}$. Since the solution is $x = A\, e^{\sigma t} \sin(\omega t + \phi)$, there will be a decaying exponential (Figure IV-8a or Figure IV-8b) only if $\sigma < 0$. In this case, the system is said to be stable. In general, a linear system is stable if (and only if) all the roots of its characteristic equation have negative real parts.

Looking at the formula for σ in terms of a, b, and c, it is found that $\sigma = -b/2a$ and, hence, if a and b are both positive (or, for that matter, both negative), the system will be stable. In the examples cited above (the spring-mass-dashpot system and the R-L-C filter), the coefficients a, b and c represent physical parameters such as mass, capacitance, etc. which are always positive, and, hence, these systems are always stable.

Figure IV-10 gives a plot of the roots $\sigma + j\omega$. Both the real and imaginary parts of these roots have the units of $(\text{time})^{-1}$. The real part, σ, is the time constant of the exponential factor). The imaginary part, ω, is the radian frequency of the sinusoidal oscillation.

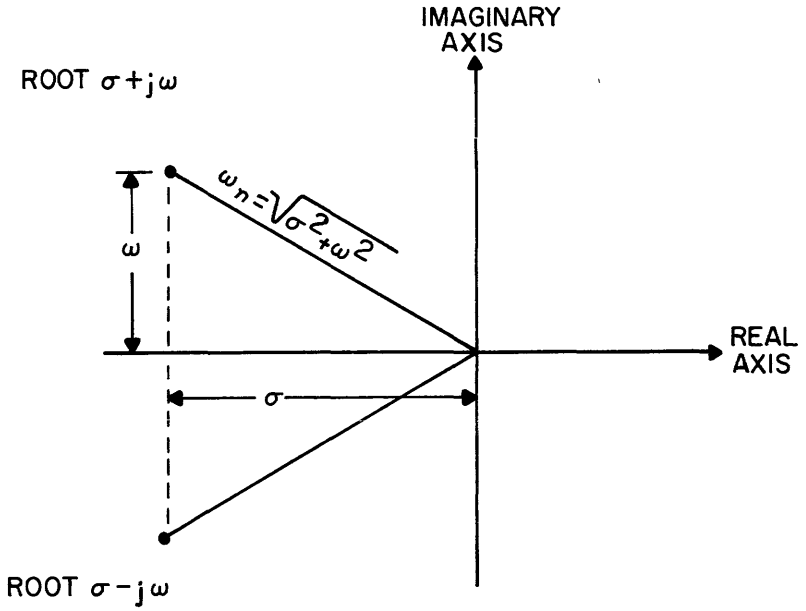


Figure IV-10. Complex Plot of the Roots of the Characteristic Equation.

It is interesting to calculate the length of the complex vectors representing the roots in Figure IV-10 The length is given by

$$\sqrt{\sigma^2 + \omega^2} = \sqrt{\left(\frac{b}{2a}\right)^2 - \left(\frac{1}{2a}\sqrt{b^2-4ac}\right)^2} = \sqrt{\frac{c}{a}} \qquad (26)$$

Note that this is independent of the damping co-efficient, b. The effect of changing b is to rotate these vectors without changing their length. If b=0, then the vectors are vertical (pure imaginary numbers) and their length is the radian frequency of the undamped oscillation. This is called the undamped natural frequency, ω_n, of the system and can be calculated very easily

$$\omega_n = \sqrt{\frac{c}{a}} \qquad (27)$$

As the damping coefficient, b, is increased, the complex vectors rotate downward, maintaining the same length, ω_n. The vertical component, ω, is the actual frequency of the damped oscillation, and the horizontal component is the reciprocal of the exponential time-constant.

The original equation can be divided through by a to give

$$\ddot{x} + \frac{b}{a}\dot{x} + \frac{c}{a}x = 0 \qquad (28)$$

and then re-written in terms of the undamped natural frequency

$$\omega_n = \sqrt{\frac{c}{a}} \qquad (29)$$

and the dimensionless damping factor

$$\xi = \frac{\text{actual damping}}{\text{critical damping}} = \frac{b}{2\sqrt{ac}} = -\frac{\sigma}{\omega_n} \qquad (30)$$

The re-written equation becomes

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (31)$$

which is a common way of writing an unforced second-order system since it displays the parameters (ξ and ω) which describe the actual response.

iii. Higher Order Systems

An nth-order, linear, unforced equation can be attacked by the same method used for the second-order system. The general, linear, unforced, nth-order equation is

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\frac{dx}{dt} + a_o x = 0 \qquad (32)$$

and a solution of the form $x = e^{kt}$ is found to satisfy the equation only if

$$a_n k^n + a_{n-1}k^{n-1} + \ldots + a_1 k + a_o = 0 \qquad (33)$$

This is the characteristic equation corresponding to the original differential equation. Since it is an nth-order polynomial, it will generally have n distinct roots, although multiple roots occasionally will occur. If the roots are all distinct, then there are exactly n of them: $k_1, k_2, \ldots k_n$.

Since $e^{k_1 t}$, $e^{k_2 t}$, etc. are all solutions. The following is also a solution:

$$x(t) = \sum_{i=1}^{n} A_i e^{k_i t} + \sum_{i=1}^{n} A_i e^{\sigma_i t} \sin(\omega_i t + \phi_i) \qquad (34)$$

where the k_i are the real roots of the characteristic equation, and the σ_i and ω_i are, respectively, the real and imaginary parts of the complex roots. Each of the terms in this expression is called a normal mode of the system. Each normal mode is either an exponential (growing or decaying) or a damped (or undamped) sinewave.

A normal mode represents the typical behavior of a first-order or second-order system, and, thus, a higher order system can be thought of as a combination of first-and-second-order systems.

While a first-order system is characterized by a time-constant and a second-order system by a natural frequency and damping coefficient, a higher-order system has several time-constants, damping co-efficients and natural frequencies.

We can "excite" any normal mode independently of the others by appropriate choice of initial conditions. That is, we may choose the IC's so that all but one of the A_i's are zero. The result will be a"pure" normal mode. However, in most actual physical situations, all the system's modes will be excited. To visualize the types of responses that result, one should think a little about what happens when several exponentials and damped sinewaves are added. Figure IV-11 shows a typical combination of an exponential and a damped sinewave such as might be expected from a third-order system. Figure IV-12a shows the sum of two sinewaves whose amplitudes and frequencies differ considerably. The graph presents a clear picture of "one damped sinewave riding on top of another". Figure IV - 12b illustrates the sum of two sinewaves of slightly different amplitude and frequency. If the frequencies are only slightly different, the two sinewaves will drift out of phase, then into phase, then back out of phase again. When they are out of phase, the sum will be near zero since the amplitudes are approximately the same. As they drift into phase, the sum becomes larger eventually reaching a maximum when the two are in phase, and then declining again. The resultant swells in amplitude are known as beats and are familiar to anyone who has listened to two musical instruments that are slightly out of tune with each other playing simultaneously.

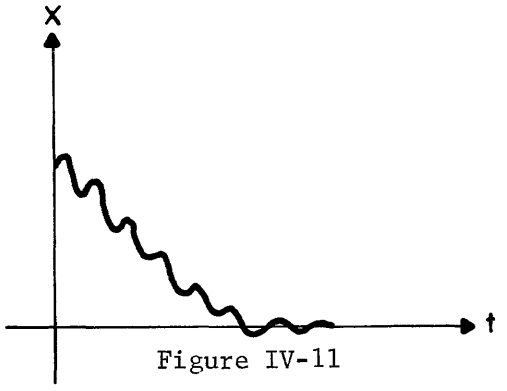


Figure IV-11

Combination of Exponential and damped sinewaves

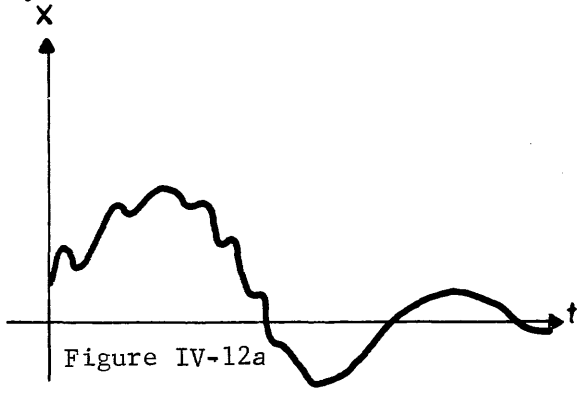


Figure IV-12a

Sum of two completely different sinewaves

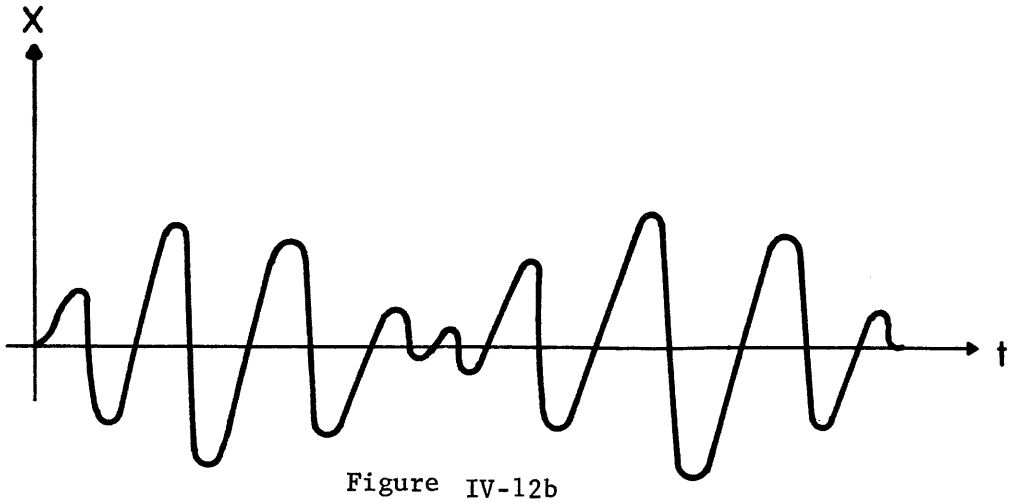


Figure IV-12b
Sum of two slightly different sinusoids

b. Forced Systems ($f(t) \neq 0$)... This section discusses the equation with a forcing function

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = f(t) \qquad (35)$$

i. Relation between Forced and Unforced Systems

One of the first things to be learned about solving such forced-system equations is that it is a very good idea to be on familiar terms with the corresponding unforced equation, for the two are very closely related.

For example, suppose the unforced equation has already been solved completely, with all the time-constants and natural frequencies characterizing the system found and, hence, all solutions obtained. Then, if just one solution for the forced equation can be found, then all solutions for a given forcing function can be found.

To see this, suppose there are two functions, $x_1(t)$ and $x_2(t)$, which satisfy the same linear, forced equation with the same forcing function, $f(t)$. This means

$$a_n \frac{d^n x_1}{dt^n} + a_{n-1} \frac{d^{n-1} x_1}{dt^{n-1}} + \ldots + a_1 \frac{dx_1}{dt} + a_o x_1 = f(t) \qquad (36)$$

and

$$a_n \frac{d^n x_2}{dt^n} + a_{n-1} \frac{d^{n-1} x_2}{dt^{n-1}} + \ldots + a_1 \frac{dx_2}{dt} + a_o x_2 = f(t) \qquad (37)$$

Subtracting, we discover that

$$a_n \frac{d^n (x_1 - x_2)}{dt^n} + a_{n-1} \frac{d^{n-1} (x_1 - x_2)}{dt^{n-1}} + \ldots + a_1 \frac{d(x_1 - x_2)}{dt} + a_o (x_1 - x_2) = 0 \qquad (38)$$

In other words, the difference between two solutions of the forced equation satisfies the unforced equation.

This general principle means that the general solution of the forced system can be obtained by finding just one solution (usually called a particular integral) and adding to it the general solution of the unforced system (usually called the complementary function) which we already know how to obtain.

The problem of finding the general solution in the forced case, therefore, boils down to the problem of finding any solution. Trial-and-error is often very successful here, and there are general methods available to use when trial-and-error fails.

As a starter, consider several cases where f(t) has a particularly simple form.

ii. Solution by Step Input

Suppose f(t) is a step input of height, k. Then we have to find a solution to the equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = k \qquad (39)$$

One idea immediately suggests itself: a constant value for x. It is not hard to see that if $x = k/a_o$, then the equation is satisfied. This is a trivial solution and, in a given physical situation, it almost certainly will not be the solution of interest, but no matter! All solutions can be obtained by adding to the constant, k/a_o, the general solution of the unforced equation. The result will contain enough arbitrary constants to match initial conditions for x and its first n-1 derivatives, and thus find the required solution.

Typical step responses for the first-and second-order systems are given in Figures IV-13 and IV-14. Note that overshoot and oscillations occur only in the underdamped second-order case. The overshoot, if the system is stable, is never more than 100%, and even

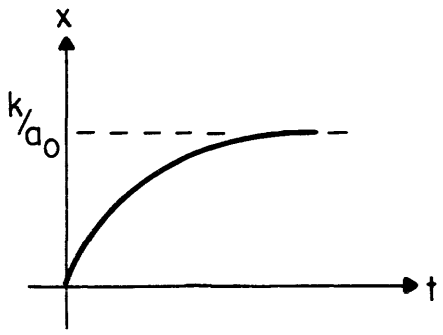


Figure IV-13: Solution to the first-order equation $a\ddot{x} + bx = k$

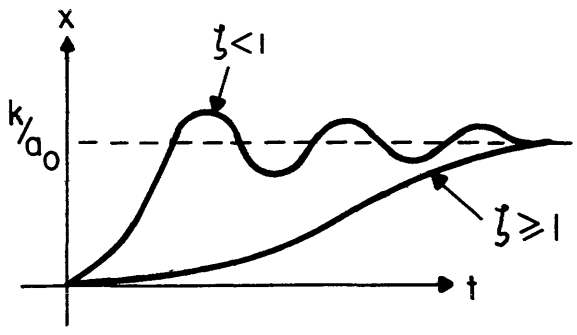


Figure IV-14: Solution to the second-order equation $a\ddot{x} + b\dot{x} + cx = k$

this value is only approached in very lightly damped second-order systems. (This fact is often useful in estimating maximum values for scaling purposes.)

iii. Sinewave Input

Suppose the input is a sinewave, $f(t) = \sin \omega t$. Then, both experience and intuition (mathematical or physical) will lead most people to expect a sinusoidal output.

Usually, this turns out to be the case. One may try for a sinusoidal solution by assuming $x(t) = A \sin (\omega t + \phi)$ and then differentiating and substituting to find the amplitude, A, and the phase angle, ϕ, of the output relative to the input. With luck, two equations in the two unknowns will result which can be solved for A and ϕ. The algebra gets a bit complicated but it can be simplified greatly by regarding the input as the imaginary part of the complex exponential, $ej\omega t = \cos \omega t + j \sin \omega t$. For a detailed description of the process, see Ref. 1, Chapter 6. For the purposes of this volume, it will suffice to observe that a sinusoidal input into a linear, constant co-efficient system generally produces a sinewave output, and that the amplitude and phase of the output relative to the input can be calculated algebraically.

There is one case in which the above method for finding a particular integral does not produce a pure sinewave. This is the case where the unforced system has a normal mode consisting of a pure (i.e. undamped) sinewave with the same frequency as the input. In this case, the solution contains terms like $(A+Bt)(\sin \omega t)$ indicating oscillations whose amplitude increases without bound. In case the normal mode is slightly damped, we have oscillations with a finite, but very large amplitude. In either case, the system is said to be in resonance.

Physically, resonance means that if a system is driven by an input whose frequency is close to a natural frequency of the system (that is, a frequency at which the system would tend to oscillate even if unforced), the result will be an output of very large amplitude. For example, in pushing a child in a swing, large amplitude oscillations are necessary to satisfy the child. To achieve these with minimum effort, the pusher soon learns to time his pushes so that each push reinforces the motion of the swing rather than opposing it. This means that the pusher must match the frequency of his pushes to the natural frequency of the swing.

Resonance is also important in structural design. If an airplane happens to have lightly-damped normal modes whose frequency is close to the rpm of the engines, the resulting vibrations can build up in amplitude until they tear the plane apart. It is the responsibility of the designer to see that this resonance phenomenon does not happen. The same sort of thing can occur, say, when a detachment of infantry march in cadence across a bridge. Often, the leader will tell his men to break step and just walk across to prevent this sort of catastrophe.

Resonance is also important in tuning a radio. To get the maximum output for a given input, it is desirable to adjust the tuner section of the radio so that its natural frequency is as close as possible to the frequency at which the station is actually broadcasting.

iv. Arbitrary Input

Suppose f(t) is an arbitrary function given by some complicated analytical expression or by a graph or table of values. To find one solution (a

particular integral), in this case, there exist a number of techniques, some of which are listed below. For more details, see Ref. 1.

1) The convolution integral — This technique gives the solution in terms of an integral which may or may not be easy to evaluate. It has the advantage of working with tabular or graphical inputs, in which case the integral must be evaluated by some numerical method (Simpson's Rule, counting squares, etc.).

2) Harmonic analysis-- If the input is periodic but not sinusoidal, it may be broken down into an infinite sum of sinewaves. Using the principle of superposition, each sinewave may be handled separately and then added to obtain the resultant output.

3) Integrating Factors-- This method works nicely with first-order systems, including those with variable co-efficients.

4) Undetermined coefficients-- It has already been shown how this works in tackling the unforced equation by assuming an exponential solution, e^{kt}, and solving for k. Sometimes it works very well for the forced case also.

5) Variation of parameters -- This method is similar to the method of undetermined coefficients. It has the advantage of working in the case of time-varying coefficients, and the disadvantage of requiring the solution of the corresponding unforced case first.

6) Laplace Transforms -- This method is very handy for functions that have Laplace transforms that can be easily managed. See Ref. 1, Chapter 9 for details.

7) Infinite series

8) Guessing-- This is the easiest method to apply, if it works. If it does, more complicated techniques are not necessary.

C. Linear Systems with Time-Varying Coefficients

Suppose the co-efficients, a_n, in Equation 1 are not constants but known functions of time. Then, the exponentials and sinewaves encountered in the constant-coefficient case will not always work. Other methods, such as infinite series, integral transforms and integrating factors must be used to find particular solutions. However, it is worth pointing out that the principles of superposition and proportionality still work. If enough different solutions of the unforced equation can be found, they can be multiplied by arbitrary constants and added to obtain the general solution. If the unforced system can be solved completely and just one solution of the forced equation can be found, then the general solution for that forcing function can be obtained by adding the complementary function(general solution of the unforced system) to the particular integral (solution of the forced system), just as was done in the constant-coefficient case. A more detailed discussion of time-varying systems is beyond the scope of this volume.

D. Non-linear Equations

Most of this section is concerned with illustrations showing that the most desirable properties of linear equations (superposition, proportionality, etc.) are not applicable in solving the nonlinear ones.

As an illustration of what can happen with nonlinear systems, consider the pendulum equation

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad \theta(o) = 0;\ \dot{\theta}(o) \neq 0 \tag{40}$$

This equation describes the angular displacement, θ, of the ballistic pendulum in Figure IV-15 which is suddenly given an initial velocity (by striking it with a hammer or a bullet). In case θ is very small (less than about $15^o = \frac{1}{4}$ radian), it is usual to replace sin θ by θ, giving the linear equation

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\theta = 0 \tag{41}$$

which can be solved by the methods outlined above, yielding a damped sinewave solution. In case θ is not restricted to small values, the original nonlinear equation must be solved. Although the analytical techniques described above will not work, an analog computer solution is relatively easy. The results given below were obtained in this manner.

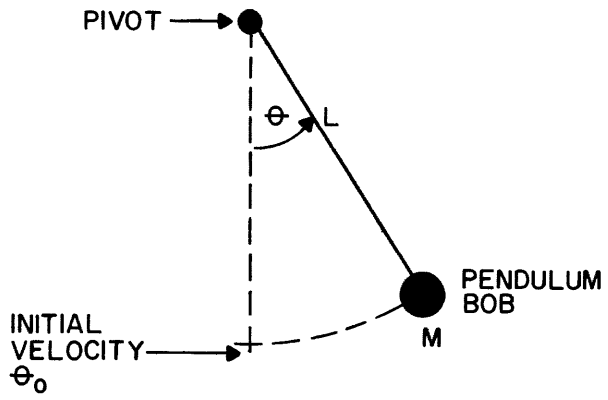


Figure IV-15: A Ballistic Pendulum.

Summation of the torques about the pivot-point gives the non-linear equation.

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{42}$$

Compare the solution of the nonlinear equation with the solution of the small-angle linear approximation. Figure IV-16 gives the solution of the linear equation for three different values of the initial velocity, and Figure IV-17 gives the solution of the nonlinear equation for the same three values.

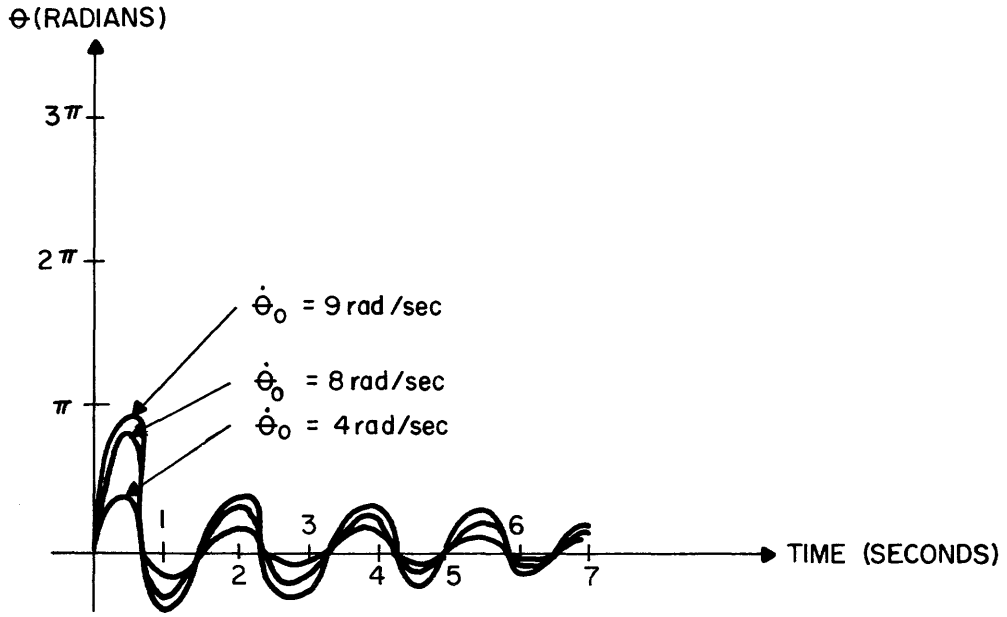


Figure IV-16: Solution of the Linear Equation

$$\ddot{\theta} + (c/m)\dot{\theta} + (g/L)\theta = 0$$ for Different Initial Velocities

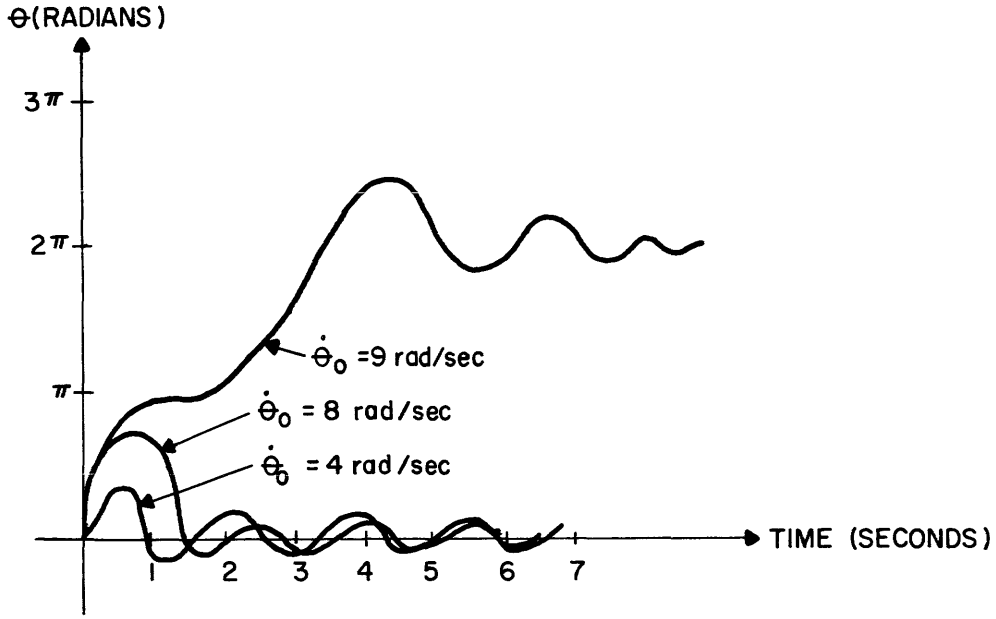


Figure IV-17: Solution of the Non-linear Equation

$$\ddot{\theta} + (c/m)\dot{\theta} + (g/L)\sin\theta = 0$$

for Different Initial Velocities

Inspection of the graphs illustrates very clearly the proportionality principle. In the case of the linear circuit, changing the initial velocity simply changes the entire graph in proportion. The curve corresponding to $\dot{\theta}_o$ = 8 rad/sec is just twice as high as the curve corresponding to $\dot{\theta}_o$ = 4 rad/sec. The curves in Figure IV-16 all cross zero at the same time since they differ only by a constant factor.

For Figure IV-17, corresponding to the nonlinear equation, the results are quite different. If $\dot{\theta}_o$ = 4 rad/sec, then the system produces a small-angle oscillation, and the maximum value of θ is about 1 radian. Since, in this region, sin θ is fairly closely approximated by θ, the solution should look very much like the solution to the linear system. An IC twice as large produces an oscillation <u>about</u> twice as large, but even here the differences are noticeable. The curve with the larger IC is noticeably "flatter" near its first peak as θ approaches π radians. Note, also, that the zero-crossings do not coincide as they would if the second curve were simply twice the first.

With $\dot{\theta}_o$ = 9 radians per second, the differences in the nature of the solution become even more striking. Physically, it is obvious what has happened. The pendulum has been hit so hard that it has gone over the top and will not return. The system settles down to a new value of $\theta = 2\pi$ (i.e. 360°).

The critical value of the initial velocity--the velocity that is just sufficient to put it over the top--is about 8.50 radians per second. Figure IV-18 shows the results of an IC of 8.49 and an IC of 8.51 radians/sec. It is clear that the solution of the nonlinear equation is much more sensitive to small changes in the initial condition. Since the solution of a linear differential equation depends <u>linearly</u> on the IC's, this sort of thing can not happen in the linear case.

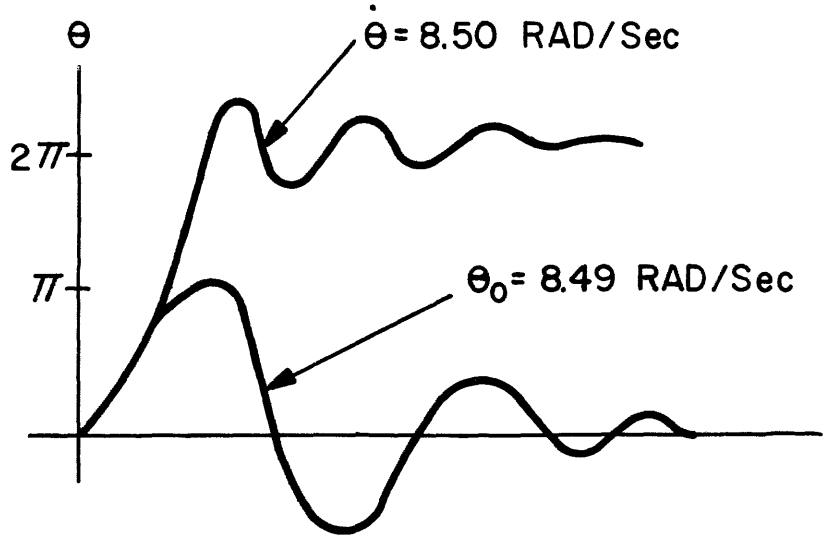


Figure IV-18.
Solution of Non-Linear Equation (40) for Critical Initial Velocities

Clearly, the small-angle approximation is not valid for large initial conditions. As another example of a nonlinear system which does not obey the laws that make linear systems so tractable, compare the two equations:

$$\frac{dC}{dt} = - kC \text{ (linear)} \tag{43}$$

$$\frac{dC}{dt} = - kC^2 \text{ (nonlinear)} \tag{44}$$

Either of these equations may describe the depletion of a component, C, in a reactor. Which equation applies in a given circumstance depends (among other things) upon the particular substance reacting.

Both equations are simple enough that they may be solved analytically (by separating variables). Despite this fact, the nonlinear equation is not nearly so "nice" to handle analytically since the principles of superposition and proportionality do not apply.

Compare the steps taken to solve both equations. Although the steps are very similar, the results are quite different.

Linear Case		Nonlinear Case
$\frac{dC}{dt} = - kC$	Original Equation	$\frac{dC}{dt} = - kC^2$
$\frac{dC}{C} = - kdt$	Separating Variables	$\frac{dC}{C^2} = -kdt$
$\log C = -kt+K$	Integrating	$-\frac{1}{C} = - kt + K$
$C = K'e^{-kt}$, where $K' = e^k$	Solving for C	$C = \frac{1}{kt+K'}$, where $K'=-K$

The result is an exponential in one case, and an algebraic solution in the other. More significantly, the dependence upon the initial conditions is quite different in the two cases. Since the initial condition determines the constant, K', in each case, the fact that $C = C_o$ when $t = 0$ can be used to obtain

$$C = C_o e^{-kt} \quad \text{in the linear case, and} \tag{45}$$

$$C = \frac{C_o}{1 + C_o kt} \quad \text{in the nonlinear case.} \tag{46}$$

Note that doubling the initial concentration simply doubles the entire solution in the linear case. In other words, if the system is started with twice as much of the component initially, twice as much will be obtained at any later time--the entire behavior of the system is simply scaled up be a factor of two. Examination of the above formulas shows that this is not true for the nonlinear case. This result is quite important to someone who is trying to extrapolate the results of a bench-scale or pilot-plant study to predict the overall behavior of a full-size commercial reactor.

As another example, suppose component C is fed into the reactor at a rate, f(t). The following forced equations are then obtained:

$$\frac{dc}{dt} = - kC + f(t) \qquad \text{(linear case)} \tag{47}$$

and

$$\frac{dc}{dt} = - kC^2 + f(t) \qquad \text{(nonlinear case)} \tag{48}$$

In the linear case, the solution to the unforced system tells us a good deal about the forced system as well, while in the nonlinear case this is not so. Suppose, for example, a constant input rate, K, is assumed. Then a particular integral (which represents the steady-state behavior) may be found in both the linear and nonlinear cases by setting $\frac{dC}{dt} = 0$. The results are

$$C = \frac{K}{k} \qquad \text{for the linear case} \tag{49}$$

$$C = \sqrt{\frac{K}{k}} \qquad \text{for the nonlinear case} \tag{50}$$

The steady-state concentration depends linearly on the input in the linear case but not in the nonlinear case. Furthermore, the "complementary function" (the general solution of the unforced equation) may be added to this particular integral and gives in the linear case

$$C = \frac{K}{k} + K'e^{-kt} \tag{51}$$

which is the general solution. Attempting this for the nonlinear equation, the result is

$$C = \sqrt{\frac{K}{k}} + \frac{1}{kt + k'} \tag{52}$$

which does <u>not</u> satisfy the forced equation. Although the nonlinear <u>unforced</u> system has been solved completely, this does no good at all when the same system is tackled in the presence of a forcing function. It must be started from scratch.

It has been shown that most of the "nice" properties of linear systems go out the window when we consider nonlinear systems. Since most physical systems are, at least, somewhat nonlinear and many are highly nonlinear, a great many of the equations that an engineer or scientist would write down are difficult or impossible to solve analytically. (In fact, even the solution of a linear constant-coefficient equation with a simple forcing function can be quite tedious. The amount of algebraic manipulation required to find the roots of the characteristic polynomial, to fit the initial conditions, etc. for a fourth-order system is enough to make most men turn to a computer for relief, even though this is, theoretically, a "solvable" case.)

Many of the above difficulties are eliminated when a computer is brought in to do the work. However, the computer-user must expect occasional extreme sensitivity of the solution to small changes in parameters, initial conditions or inputs, such as the radical change in the solution caused by a very small change in initial velocity in the pendulum problem (Figure IV-18). Such extremely "sensitive" solutions are sometimes taken as an indication of poor programming and scaling, or faulty computer components. Occasionally, this is the case. However, if the programming is carefully done, and both the program and the operation of the computer components are thoroughly checked, such sensitivity should be taken as evidence of a "critical" area in the original problem.

E. Estimating Maximum Values for Scaling

The general principles given in this chapter provide a very useful 'rule of thumb' for estimating maximum values. As pointed out in Chapter III, one of the problems encountered is that of estimating maximum values for higher-order derivatives of a variable, assuming that the maximum value of the variable itself is known. This is made very easy by examining the time-constants and natural frequencies of the system. It is not even necessary to go through all the algebra required to actually solve the equation. Furthermore, the method can be applied to nonlinear systems as well as linear ones, since many non-linear systems can be linearized to obtain "ball park" estimates of maximum values.

Begin by considering the two types of functions which are characteristic solutions of linear systems, namely exponentials and damped sinewaves. The relationship between the maximum value of the variable, and the maximum values of its derivatives is easy to calculate for these functions:

For the exponential function

$$x = A\, e^{kt} \tag{53}$$

$$\dot{x} = k\, A\, e^{kt} \tag{54}$$

$$\ddot{x} = k^2 A\, e^{kt} \tag{55}$$

etc.

If the system is stable ($k < 0$)

$$x\ (MAX) = A \tag{56}$$

$$\dot{x}\ (MAX) = k\, A \tag{57}$$

$$\ddot{x}\ (MAX) = k^2 A \tag{58}$$

etc.

In other words, the maximum values of the variable and its derivatives form a geometric progression; each estimated maximum value for a derivative can be obtained by multiplying the maximum value of the lower-order derivative by k, the root of the characteristic equation.

For the damped sinewave

$$x = Ae^{\sigma t}\sin(\omega t + \phi) \tag{59}$$

$$\dot{x} = A\left[e^{\sigma t} \cdot \omega \cos(\omega t + \phi) + \sigma e^{\sigma t}\sin(\omega t + \phi)\right] \tag{60}$$

$$\dot{x} = Ae^{\sigma t}\left[\omega \cos(\omega t+\phi) + \sigma\sin(\omega t+ \phi)\right] \tag{61}$$

Since the sum of a sinewave and a cosine wave of the same frequency is another sinewave, the term in parentheses can be rewritten as $\sqrt{\sigma^2 + \omega^2}\sin(\omega t+\phi')$, where ϕ' is a phase angle depending on σ and ω. Hence

$$\dot{x} = A\sqrt{\sigma^2 + \omega^2}\left(e^{\sigma t}\right)\sin(\omega t + \phi') \tag{62}$$

Since the maximum of the exponential factor, $e^{\sigma t}$, is unity (assuming, as is usually the case, that the system is stable so that $\sigma < 0$), and that the maximum of the $\sin(\omega t + \phi')$ term is also unity, a reasonable estimate for the maximum is $x(MAX) = A$. The actual maximum will be slightly less than this if the sine term and the exponential terms do not assume their maximum values simultaneously. Whether this happens or not depends on the phase angle, ϕ', which in turn depends upon the initial conditions. However, the estimate $x(MAX) = A$ is seldom very far from the true maximum, and is generally close enough for scaling purposes.

Continuing this reasoning,

$$\dot{x}(MAX) = \sqrt{\sigma^2 + \omega^2}\,(A) = \omega_n A \tag{63}$$

Thus, for damped sinewaves, the estimated maximum values also are in geometric progression. The maximum value for each higher-order derivative then can be found by multiplying the estimate for the preceding derivative by the natural frequency of the system which is the magnitude or absolute value of the complex conjugate roots of the characteristic equation.

Note that it is easy to calculate the undamped natural frequency without actually solving the characteristic equation; the undamped natural frequency for the equation $a\ddot{x} + b\dot{x} + cx = 0$ is simply $\sqrt{c/a}$.

The general nth-order system has several time constants and natural frequencies, and the relationship between the variable and its derivatives will depend upon which normal modes are excited. This, in turn, depends upon the initial conditions. Actually, calculating the natural frequencies and/or time constants and determining which normal modes are excited is a very tedious job for a system of higher than second-order. In fact, it amounts to solving the equation analytically. However, since only approximations are of interest here, perhaps some easily-calculated "average" natural frequency will suffice.

It turns out that, for the general nth-order polynomial equation

$$a_n k^n + a_{n-1} k^{n-1} + \ldots + a_1 k + a_o = 0 \tag{64}$$

a good "average" natural frequency to use for scaling purposes is

$$\bar{\omega} = \sqrt[n]{|a_o/a_n|} \tag{65}$$

This is a direct generalization of the formula for the second-order case

$$\omega_n = \sqrt{c/a} \tag{66}$$

This formula works quite well when the roots of the characteristic equation do not differ too widely in magnitude. (In fact, it is a basic theorem in the theory of polynomials that the product of all the roots is precisely a_o/a_n, so that $\bar{\omega}$, as defined above, is actually the geometric mean of the absolute values of all the roots.)

This seems like a good "average" to use in estimating the maximum magnitudes of the various derivatives.

Hence, to estimate the maximum values of $\dot{x}$, $\ddot{x}$, etc., where x is a solution of the equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = 0 \tag{67}$$

the general rule is, first calculate the "average frequency", $\bar{\omega} = \sqrt[n]{a_o/a_n}$, and then estimate the maximum values of the higher derivatives by using the rule

$$\dot{x}(MAX) \simeq \bar{\omega}\, x(MAX) \tag{68}$$

$$\ddot{x}(MAX) \simeq \bar{\omega}^2\, x(MAX) \tag{69}$$

etc.

This method is quite easy to use and results in approximately equal distribution of gains around the main computing loop.

For nonlinear systems, a rough idea of the "time constants" and "frequencies" involved often may be obtained by a very crude linearization. For example, a spring-mass system with a nonlinear spring might be described by an equation such as

$$m\ddot{x} + D\dot{x} + k_1 x + k_2 x^3 = 0 \tag{70}$$

Assume x(MAX) is known on physical grounds. Then, if we write the spring term as $(k_1 + k_2 x^2)x$, we may linearize the spring by substituting $(k_1 + k_2 x(MAX)^2)$ for the term in parentheses. This amounts to replacing the cubic spring term with a straight line, as in Figure IV-19.

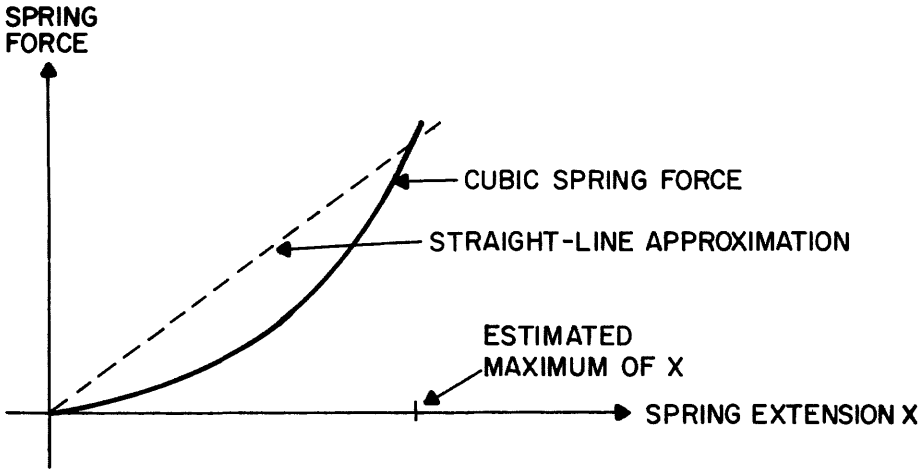


Figure IV-19

Spring-mass System Approximation

It is then possible to calculate a crude "natural frequency,"

$$\omega = \sqrt{\frac{k_1 + k_2\, x(MAX)^2}{m}} \tag{71}$$

and use this to estimate the maximum velocity and, if needed, the acceleration. Though the method is crude, it is fairly easy to use and gives reasonable results in most practical cases.

F. Algebraic Equations

Certain difficulties arise in the solution of algebraic equations on an analog computer which prevent one from programming them in a straightforward manner. The nature of these difficulties is described briefly in the following pages.

Consider a simple set of two simultaneous algebraic equations:

$$a_{11}x_1 + a_{12}x_2 = b_1 \tag{72}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \tag{73}$$

One could attempt to solve these equations using a similar approach as for differential equations, i.e., solve for the unknowns x_1 and x_2,

$$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}\, x_2 \tag{74}$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 \tag{75}$$

and develop a computer circuit in a straightforward manner:

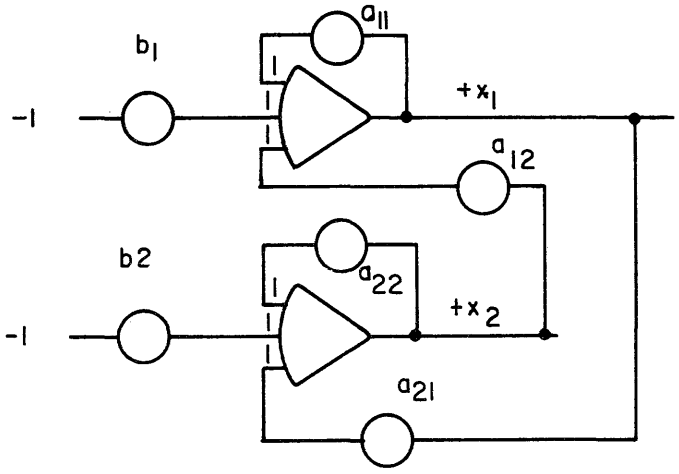


Figure IV-20: Direct Circuit for Solution of Two Simultaneous Linear Algebraic Equations

The resulting circuit is a two amplifier loop, and the gain around this loop is $K = a_{12}a_{21}/a_{11}a_{22}$.

This is a positive feedback loop and it can be shown that if $K \geq 1$ the system will be divergently unstable.

It would appear that if one of the coefficients were negative, adding a third amplifier to the loop a condition of negative feedback would result and there would not be a stability problem. However, one may find that in this case, instead of a divergence one obtains a violent oscillation. This would occur if the frequency dependent characteristic of the amplifiers contributes sufficient undesired phase shift such that the required condition of negative feedback becomes one of positive feedback; if the loop gain is greater than one at that frequency, the system will break into oscillation.

If one considers each amplifier to have a first-order transfer function $-K/(1+Tp)$, it can be shown (See Ref. 2, pp. 195-6) that the maximum permissable loop gain, K, for an odd number of amplifiers is

$$K = \left(\cos \frac{\pi}{n}\right)^{-n} \tag{76}$$

where n = number of amplifiers in the loop.

Thus, for a three amplifier loop, the maximum loop gain is given by

$$K = \left(\cos \frac{\pi}{3}\right)^{-3} = \left(\frac{1}{2}\right)^{-3} = 8 \tag{77}$$

In practice the figure is somewhat lower than this since the transfer function of an amplifier is more complicated than a first-order lag.

A better approach for solving algebraic equations is to transform the set of algebraic equations into a set of differential equations which has, as its steady state solution, the solution of the original algebraic equations.

Considering the above example, the set of differential equations would be

$$\frac{dx_1}{dt} + a_{11}x_1 = b_1 - a_{12}x_2 \tag{78}$$

$$\frac{dx_2}{dt} + a_{22}x_2 = b_2 = a_{21}x_1 \tag{79}$$

These can be programmed readily on the analog computer and will give the desired solution provided a_{11} and a_{22} are positive coefficients.

To guarantee a stable solution for larger sets of equations, requires special techniques which will adjust the equations so that they will always reach steady state.

It is beyond the aim of this course to describe these techniques in detail; if one meets with sets of algebraic equations in programming, one should obtain further information from the available literature.

G. References

1) "Differential Equations," R.P. Agnew, McGraw-Hill Book Company, 1960.

2) "Analog Computer in Engineering Design", A.E. Rogers & T. W. Connolly, McGraw-Hill Book Company, 1960.

3) "Analog Computation", S. Fifer, McGraw-Hill Book Company, 1961.

CHAPTER V

TECHNIQUES IN FUNCTION GENERATION

A. Introduction

The necessity of introducing implicit and explicit (or analytic) functions into a simulation is characteristic of the mechanization of a great many practical problems. A variety of methods may be used to introduce these functions, such as function generators (diode function generators, curve followers, pot padder units, resolvers, etc.); diode and relay circuits; and the solution of equations using amplifiers, potentiometers and multipliers.

The choice of method, therefore, requires careful consideration of several factors:

1) the available equipment
2) the nature of the function
3) the accuracy required
4) the need for flexibility

The mechanization of implicitly defined functions is restricted to function generators, with the exception of selected physical phenomena and mathematical constraints. The simulation of dead zones, absolute values, hysteresis, etc. is readily accomplished by diode and relay circuits.

If a function is analytic, it is advantageous frequently to mechanize the algebraic, differential or integral equation relating it to its independent variables.

If one or more parameters of an analytic function are of interest, it is simpler to vary potentiometer settings than to set up a new function on a function generator per computer run. Accuracy is another consideration, since function generators can create errors when they approximate a function by straight line segments.

In this chapter, the methods of analytic function generation involving power series, differential equations and rational fractions and powers will be considered. In addition, selected diode and relay circuits will be illustrated, and methods of minimizing function generator errors will be discussed. Finally, a portion of this chapter will be devoted to miscellaneous practical applications such as continuous resolution, high speed coordinate transformations, etc.

B. Analytic Function Generation

1. Power Series

In practice, power series generally arise from numerical analyses of experimental data, or as accurate approximations for analytic functions whose argument is in the vicinity of zero or varies over a very small range. Power series applications may be subdivided into two categories: functions of computer time, t, and functions of variables other than computer time, x. They will be considered in that order.

A general representation of a power series in time is the Maclaurin Series

$$y = a_o + a_1 t + \frac{a_2 t^2}{2!} + \ldots + \frac{a_n t^2}{n!} + \ldots \tag{1}$$

whose coefficients are actually initial conditions. For example

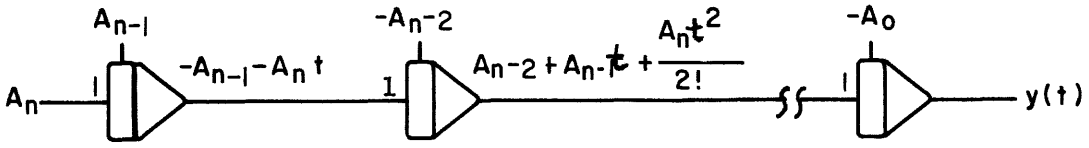
$$y(0) = a_o \tag{2}$$

$$\dot{y}(0) = a_1 \tag{3}$$

and in general

$$\left(\frac{d^n y}{dt^n}\right)_{t=0} = a_n \tag{4}$$

Therefore, a power series in time easily can be generated by cascading integrators whose initial conditions are the coefficients of a series approximation.



This mechanization is "open loop" since no feedback is required.

A practical illustration of this technique is the approximation:

$$y = \tan(t) = t + \frac{t^3}{3} \; ; \; y(0) = 0 \tag{5}$$

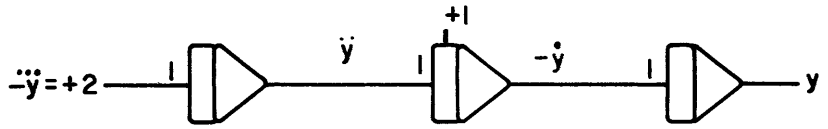
whose derivatives are

$$\dot{y} = 1 + t^2 \; ; \; \dot{y}(0) = 1 \tag{6}$$

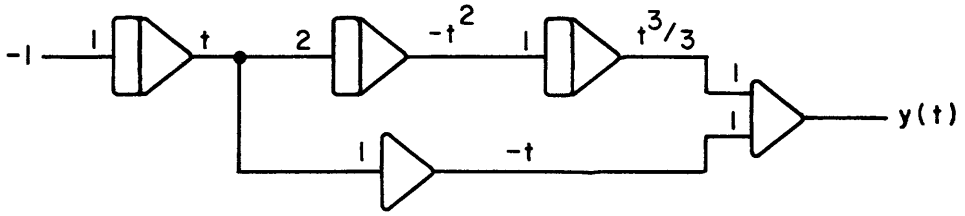
$$\dddot{y} = -2t \; ; \; \ddot{y}(0) = 0 \tag{7}$$

and

$$\ddot{\ddot{y}} = -2 \tag{8}$$



An alternative approach is to generate the required powers of time and then sum them to obtain the desired function.

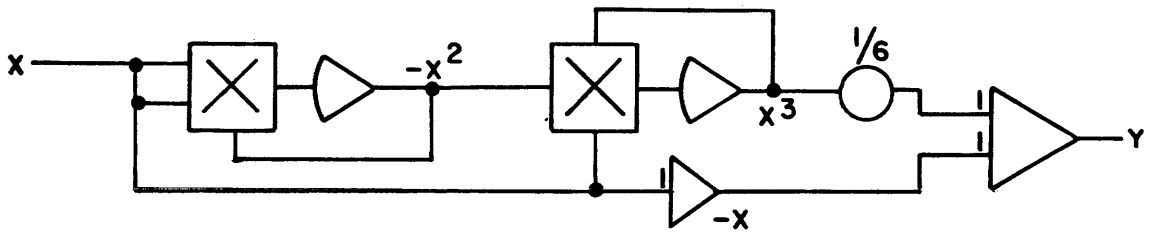


This method obviously requires more components for the generation of a single function but it proves to be the most efficient method of simulating several power series simultaneously.

If the independent variable of the series is not represented by machine time, the generation of its powers will require nonlinear equipment as in the case of the approximation,

$$y = \sin x \simeq x - \frac{x^3}{3!} \tag{9}$$

whose maximum error is 1/1000 for $x < 1.04$. Using a $(1/4)^2$ multiplier, which is good for generating ascending powers, a circuit to generate sin x is

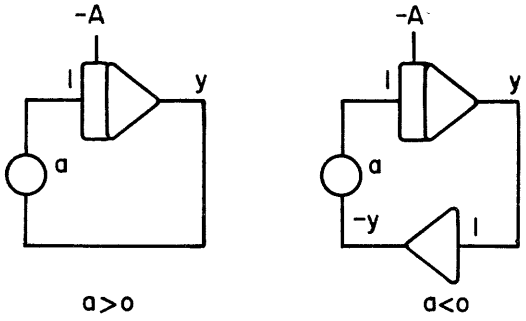


If limited-range, function generation of this type is desired about a fixed point, x_o, a perturbation variable substitution of the form,

$$Z = x - x_o \tag{10}$$

is required.

EXPONENTIAL

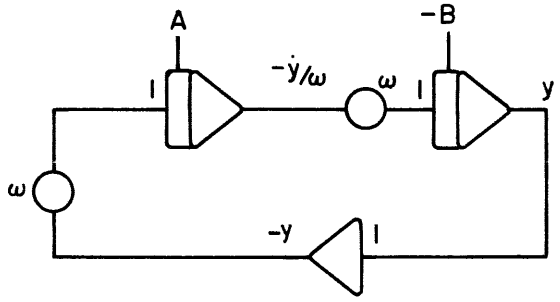


$$y = Ae^{-at}$$

$$\frac{dy}{dt} = -aAe^{-at}$$

$$\boxed{\frac{dy}{dt} + ay = 0 \qquad y(0) = A}$$

SINUSOIDAL



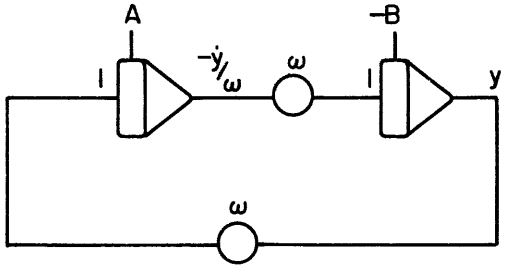
$$y = A\sin(\omega t) + B\cos(\omega t)$$

$$\dot{y} = \omega\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

$$\ddot{y} = -\omega^2\left[A\sin(\omega t) + B\cos(\omega t)\right]$$

$$\boxed{\ddot{y} + \omega^2 y = 0 \qquad y(0) = B;\ \dot{y}(0) = \omega A}$$

HYPERBOLIC



$$y = A\sinh(\omega t) + B\cosh(\omega t)$$

$$\dot{y} = \omega\left[A\cosh(\omega t) + B\sinh(\omega t)\right]$$

$$\ddot{y} = \omega^2\left[A\sinh(\omega t) + B\cosh(\omega t)\right]$$

$$\boxed{\ddot{y} - \omega^2 y = 0 \qquad y(0) = B;\ \dot{y}(0) = \omega A}$$

Figure V-1 Selected Function Generation Circuits

In this case, sin x becomes

$$\sin x = \sin (Z + x_o) = \sin Z \cos x_o + \cos Z \sin x_o \tag{11}$$

or

$$\sin x = a \sin Z + b \cos Z \tag{12}$$

where a and b represent $\cos x_o$ and $\sin x_o$, respectively. An approximation (1) of comparable accuracy for $\cos Z$ ($1/100^o$ for $|z| < \pm 0.7$) is

$$\cos Z = 1 - \frac{Z^2}{2} \tag{13}$$

Therefore, a resultant expression for sin x about point x_o (from (9) and (13)) is,

$$\sin x = b + aZ - \frac{bZ^2}{2} - \frac{aZ^3}{6} \tag{14}$$

which can be mechanized by replacing x with Z in the previous circuit. The high accuracy of this mechanization can be realized only if the input, Z, does not exceed $\pm$ 0.7. Additional illustrations of series approximations are available in the literature (1).

In general, one can obtain high accuracy approximations to functions by using relatively few terms of a power series. This is especially true in the vicinity of zero. Therefore, this method should be given careful consideration when high accuracy is desired over a limited range.

2. Differential Equations

Another method of generating an analytic function is to obtain a differential equation whose solution is the desired function. This equation can then be mechanized on the computer, with the proper initial conditions, to obtain the desired function.

Typical illustrations include exponential and sinusoidal functions.

$$y = A\, \epsilon^{-at} \tag{15}$$

and

$$y = A \sin \omega t + B \cos \omega t \tag{16}$$

whose differential equations, which were discussed in Chapter IV, are

$$\dot{y} + ay = 0\ ,\ y(0) = A \tag{17}$$

and

$$\ddot{y} + \omega^2 y = 0\ ,\ y(0) = B,\ \dot{y}(0) = \omega A \tag{18}$$

respectively. The mechanization of these equations and the equation generating the hyperbolic functions, Sinh ωt and Cosh ωt, is shown in Figure V-1.

If a positive exponential or hyperbolic function is of interest, the fact that its mechanization will produce overloads eventually should not affect its utilization. One should scale these quantities such that overloads occur after results have been obtained from the simulation.

A more complex illustration is the generation of the probability distribution integral

$$y = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt \tag{19}$$

whose derivatives are

$$\dot{y} = \frac{2}{\sqrt{\pi}} e^{-t^2} \tag{20}$$

and

$$\ddot{y} = -2t \left(\frac{2 e^{-t^2}}{\sqrt{\pi}} \right) \tag{21}$$

Substituting equation (20) into equation (21), a convenient differential equation is obtained

$$\ddot{y} = -2t \dot{y} \tag{22}$$

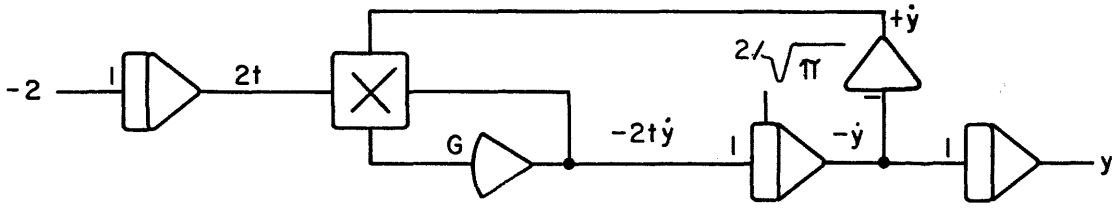
whose initial conditions are

$$y(0) = 0 \tag{23}$$

and

$$\dot{y}(0) = \frac{2}{\sqrt{\pi}} \tag{24}$$

Therefore, the integral function may be obtained from the circuit



The mechanization and analysis of functions involving variables other than machine time are treated in a similar manner. For example, consider the generation of the exponential

$$y = A e^{ax} \tag{25}$$

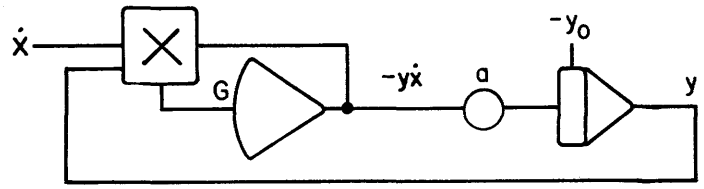
whose first derivative is

$$\dot{y} = a A e^{ax} \dot{x} \tag{26}$$

or

$$\dot{y} = a\, y\, \dot{x} \tag{27}$$

The solution to this equation is obtained from the circuit



where the initial value of the function, y_o, depends on the initial value of the independent variable, x_o,

$$y_o = A\, e^{ax_o} \tag{28}$$

Note that this circuit requires a derivative input, $\dot{x}$, which is typical of many function generation applications.

3. Fractional Powers, Reciprocals, and Logarithmic Functions

The procedure for generating fractional powers, reciprocals and logarithmic functions is similar to that described in the previous section. An attempt is made to obtain a differential equation whose solution is the desired function.

If the desired function is

$$y = A\,(t + a)^{\alpha} \tag{29}$$

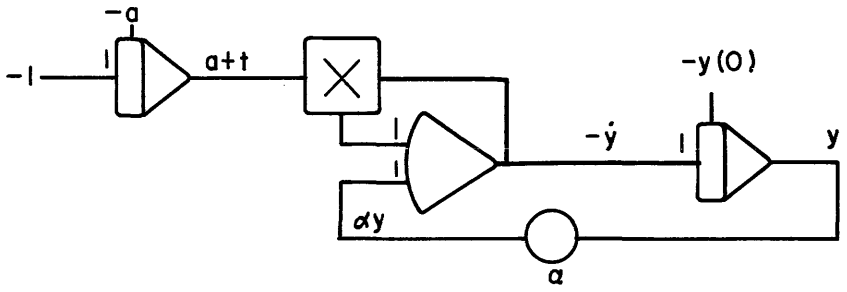
where α is a fraction, it can be differentiated to obtain

$$\dot{y} = \alpha\, A(t + a)^{\alpha - 1} \tag{30}$$

or

$$\dot{y} = \frac{\alpha y}{t + a} \tag{31}$$

The circuit for generating this function is



(32)

Circuits for the generation of logarithms and reciprocals of the form

$$y = \ln A(t + a) \tag{33}$$

and

$$y = \frac{A}{t + a} \tag{34}$$

are similar to the fractional power circuit. Their differential equations are:

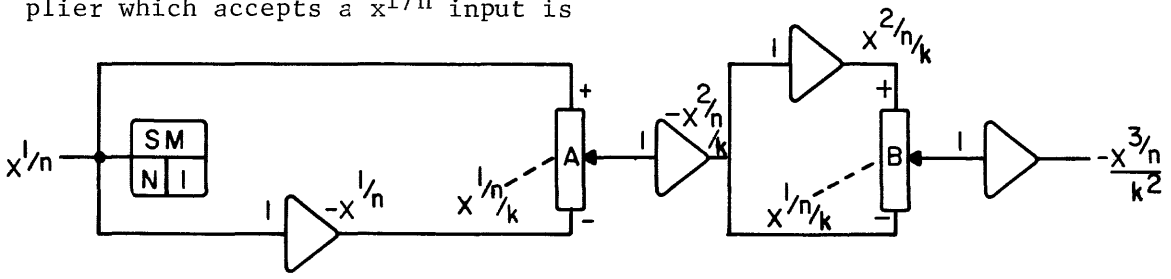
$$\dot{y} = \frac{A}{t + a} \tag{35}$$

and,

$$\dot{y} = -\frac{A}{(t + a)^2} = -\frac{y^2}{A} \tag{36}$$

respectively, whose mechanization requires one integrator and one multiplier.

An alternate circuit for generating fractional powers requiring a servo multiplier which accepts a $x^{1/n}$ input is



However, this method is not very efficient if the functional power is a parametric variable.

4. Comments on Analytic Function Generation

It is impractical to consider additional examples due to space limitations in this volume and the complete coverage of this topic in the literature (1). The material presented illustrates the simplicity of analytic function generation and its basic principle. Convert the desired function into a differential equation whose solution is the function of interest and mechanize it on the computer.

C. Resolution and Coordinate Transformation

If a coordinate transformation involving high frequency variables or repetitive operation is required, servo rate and position resolvers cannot be used. However, solid state sine-cos fixed diode function generators with high frequency response can be combined with amplifiers and multipliers, as shown in Figures V-2, V-3 and V-4, to perform:

1. Polar-to-rectangular coordinate transformations

2. Rectangular-to-polar coordinate transformations

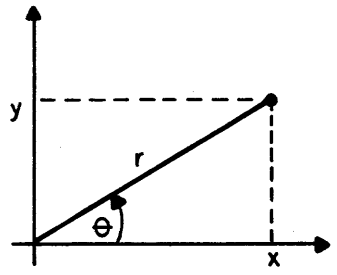
These circuits have limited resolution because the range of the function generators will, as a rule, be limited to $\pm$ 90° or $\pm$ 180°.

The multipliers shown in these three figures are quarter square multipliers with internally-packaged feedback resistors. In Figure V-3, the angle θ is generated in a high speed error loop from an error equation obtained from the equality

$$x \sin \theta - y \cos \theta = 0 \tag{37}$$

$y = r \sin \theta$

$x = r \cos \theta$



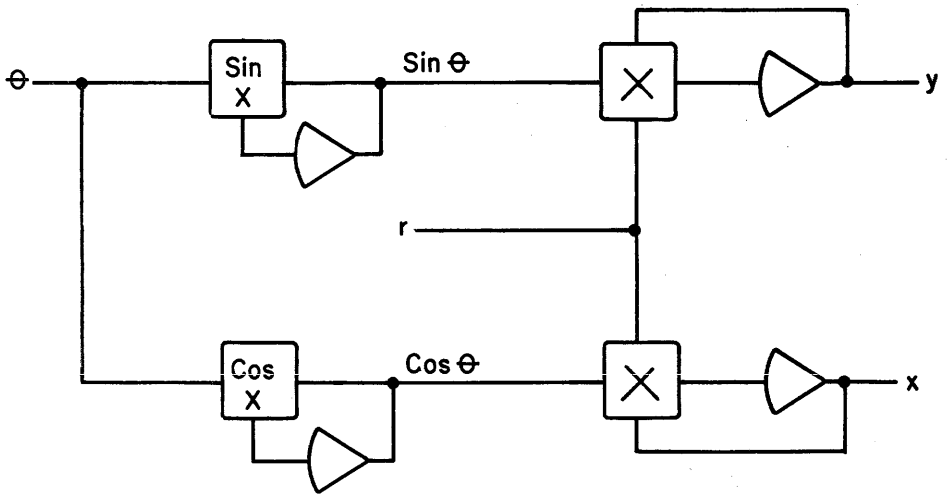
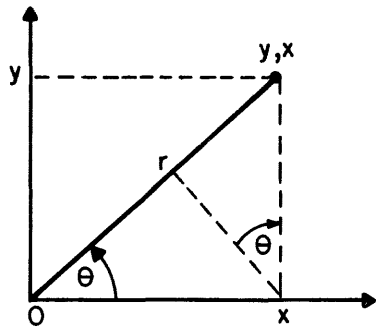


Figure V-2 Polar-To-Rectangular Coordinate Transformation

$x \sin\theta - y \cos\theta = 0$

$x \cos\theta + y \sin\theta = r$



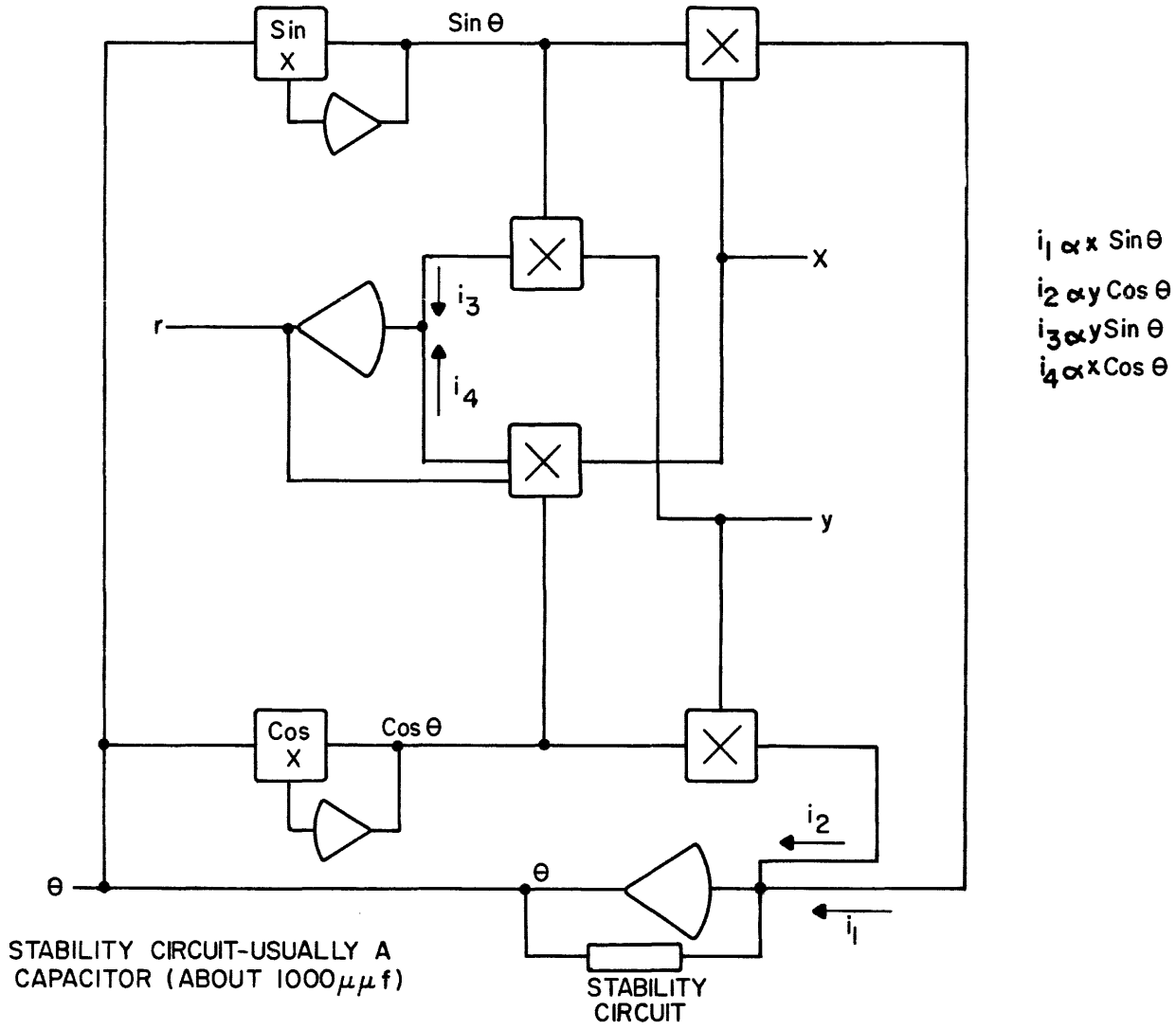
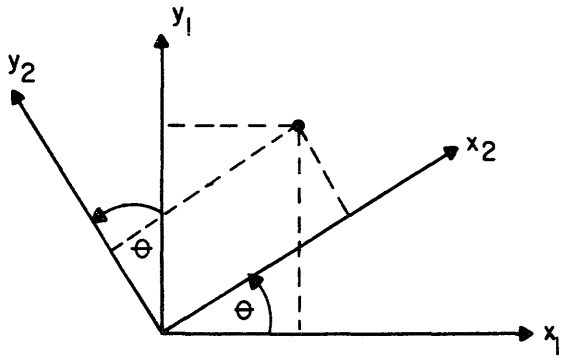


Figure V-3 Rectangular-To-Polar Coordinate Transformation

$$x_2 = x_1 \cos\theta + y_1 \sin\theta$$
$$y_2 = y_1 \cos\theta - x_1 \sin\theta$$



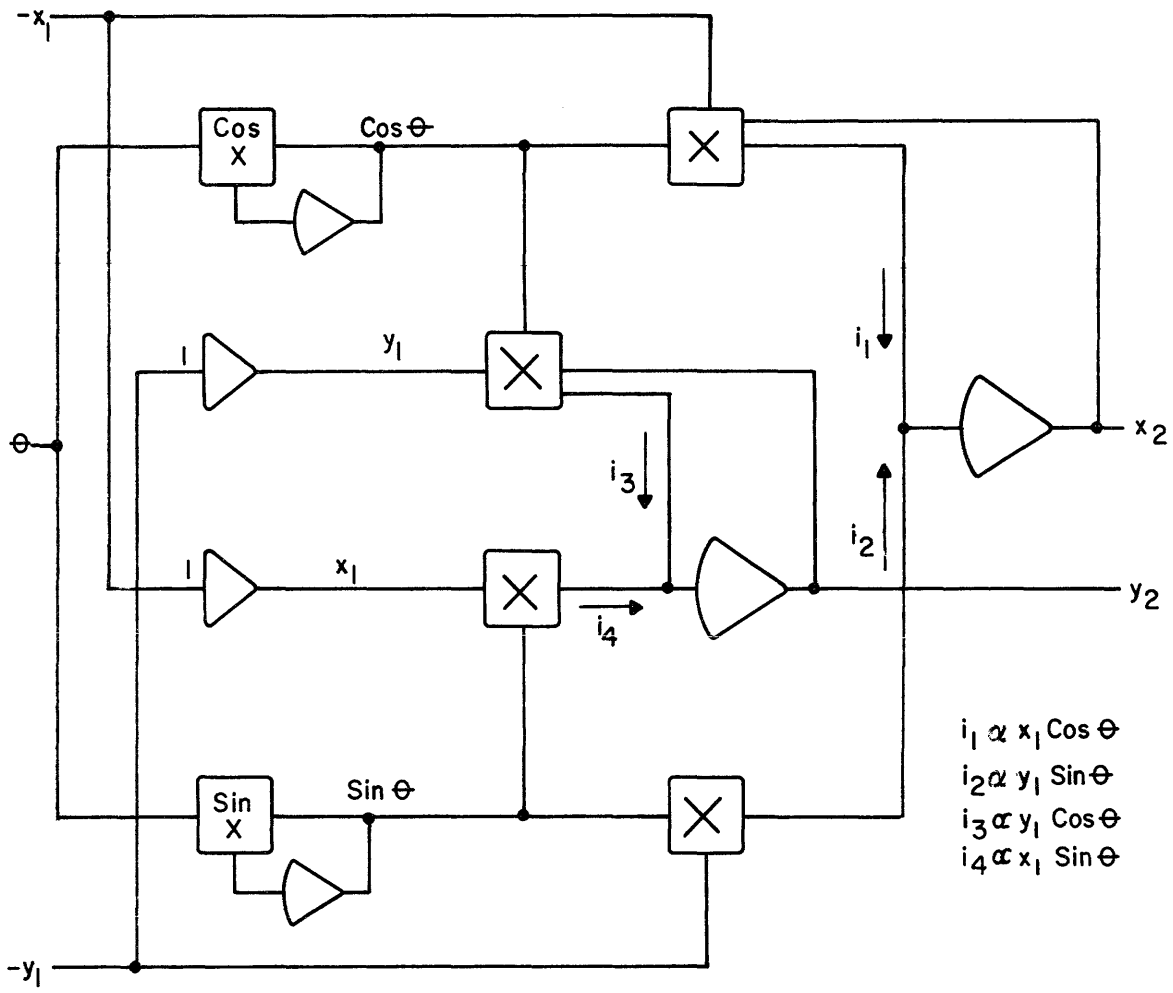


Figure V-4 Axis Rotation

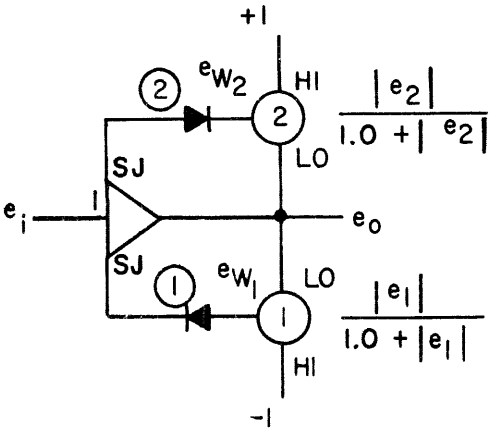
The size of the capacitor in the stability circuit must be determined by trial and error. The smallest possible capacitor should be used to keep the loop gain as high as possible

D. Representation of Discontinuities (2)

Frequently, nonlinear effects in physical systems can be described graphically by continuous straight line relationships which will be termed discontinuities to separate them from normally-defined non-linearities. The discontinuities occuring in physical systems are many, but those most commonly encountered include simple limiting, coulomb (dry) friction, dead zone, gear backlash and hysteresis. All can be represented reasonably well on the computer by standard circuits containing diodes and/or relay comparators which were discussed in Chapter II. A few circuits will be presented in this section for familiarity purposes. However, additional diode and relay circuits are presented in Appendix C in tabular form.

1. Simple Limiting

Probably the discontinuity that occurs most frequently, causing a physical system to become nonlinear, is simple limiting. This limiting effect can be simulated by the diode circuit (known as a feedback limiter)



whose behavior is shown in Figure V-5. Between the limits, e_1 and e_2, neither diode can conduct if potentiometers 1 and 2 have the proper settings. If, for example, the output voltage exceeds its upper limit ($e_o > e_1$), diode 1 conducts. This places a low impedance path in parallel with the normal feedback resistor of the amplifier, which greatly reduces its gain (by a factor of approximately 100). Since the gain is not reduced to zero, a "soft limit" is obtained, as indicated in Figure V-5. The other diode cannot conduct when e_o is positive because its plate-to-cathode voltage difference is always negative. It behaves in a manner similar to diode 1 when the output voltage is negative.

The potentiometer settings for a specified limit can be estimated by considering potentiometer 1 and its input-output relationship

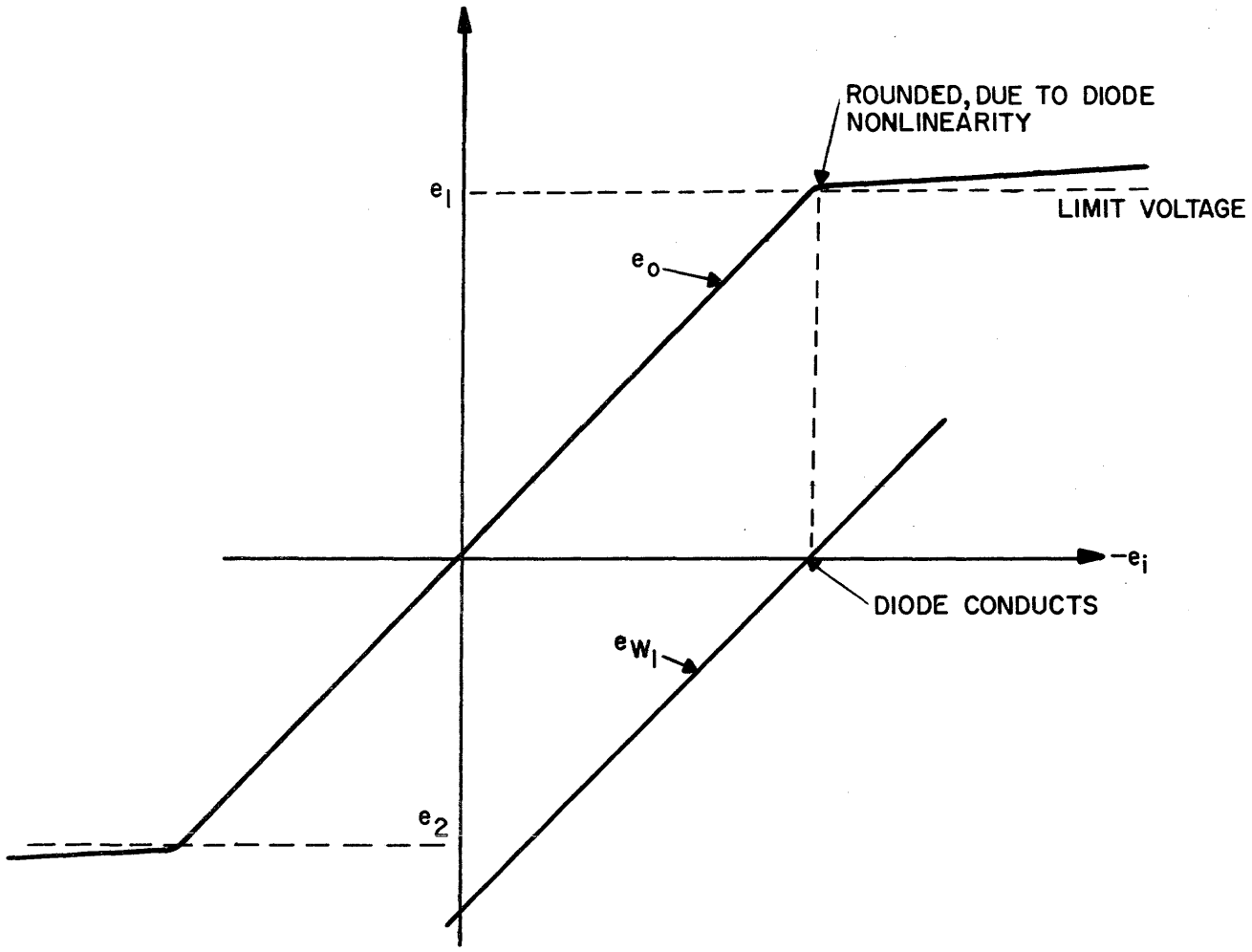


Figure V-5 Feedback Limiter Behavior

$$e_{w_1} = e_o - k\,(1 \quad + e_o) \tag{38}$$

When the diode conducts, e_{w_1} is zero and e_o equals the limit voltage, e_1. Therefore, the potentiometer setting, k, from equation (38) is

$$k = \frac{e_1}{1 \quad + e_1} \tag{39}$$

In general, the potentiometer setting for either limit is

$$k = \frac{|e_1|}{1 \quad + |e_1|} \tag{40}$$

These settings are approximate due to the impedance of the diodes; therefore, accurate limits must be set visually.

If hard limiting is required, the relay circuit or bridge limiter, shown in Appendix C, can be used.

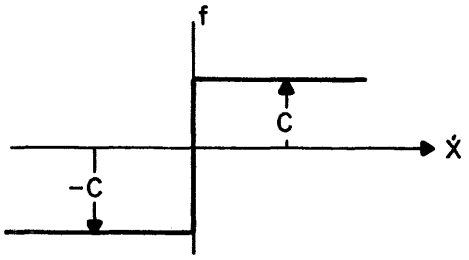
2. Coulomb (Dry) Friction

Dry friction, which is present in many physical systems, is ideally represented as a force,

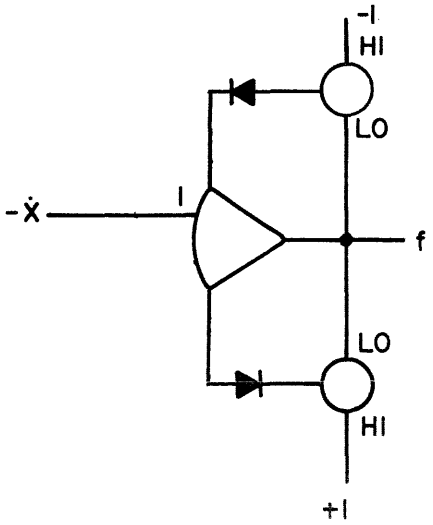
$$-\dot{x} < 0, \qquad f = c$$

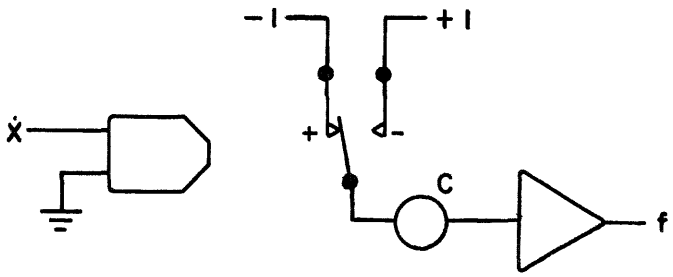
and if

$$-\dot{x} > 0, \qquad f = -c$$



This can be generated using either a relay or a diode "bang-bang" circuit

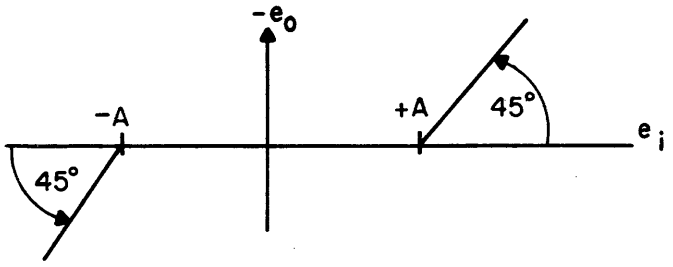




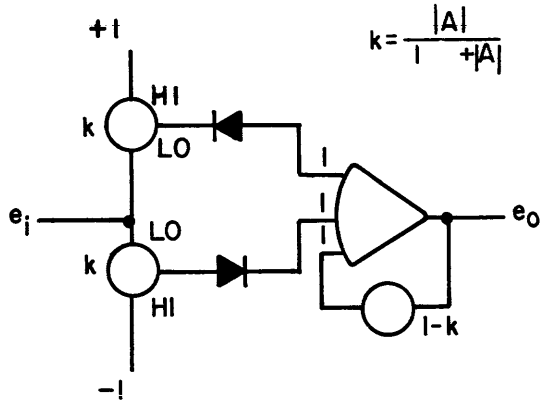
The diode circuit, which has no feedback path other than the limiter, is a "soft" dry friction simulator. More sophisticated friction circuits are shown in Appendix C.

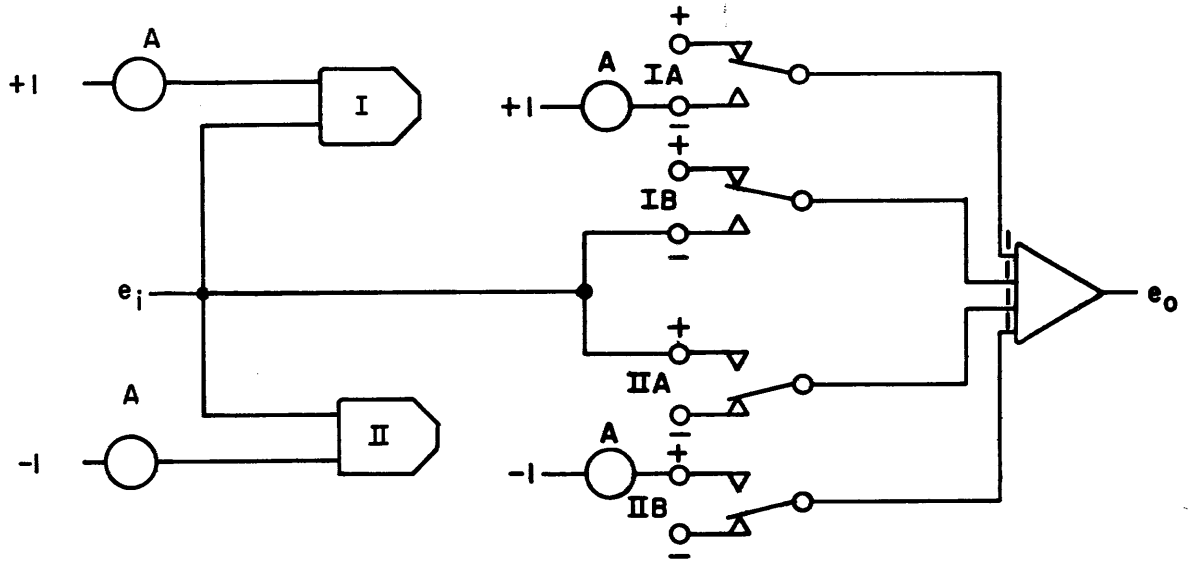
3. Dead Zone

In control systems, one is frequently required to simulate a dead zone



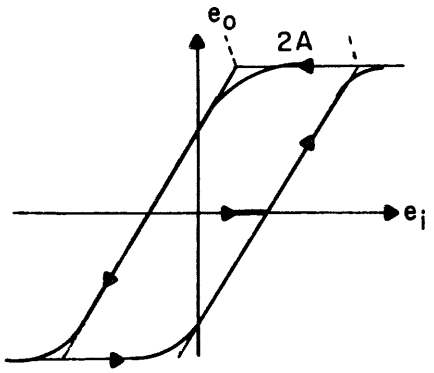
which can be mechanized on the computer by either of the following circuits:



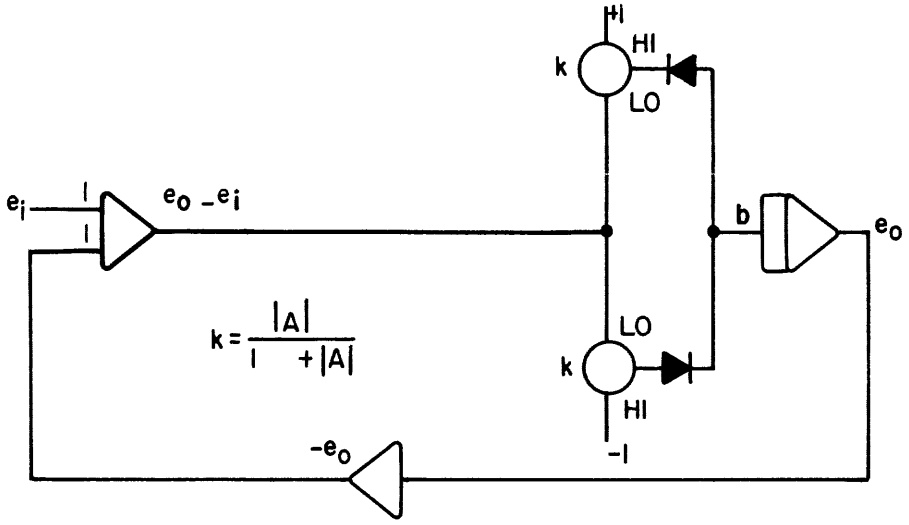


4. Backlash and Hysteresis

The dead zone circuit leads quite naturally to one representing backlash or any ideal form of hysteresis loop



The circuit which will produce this effect (using a diode dead zone) is



As e_i increases from zero, the dead zone does not permit any input to the integrator until e_i - e_o input exceeds the A limit. Then the integrator operates at a very high rate (input is to the grid of the integrator) and tracks the e_i - e_o input with no appreciable lag. When e_i decreases, the integrator output will not change until e_o - e_i becomes less than -A. There is no fixed upper limit to the loop. Only its width is fixed by the deadband which can be varied by replacing the potentiometers by multipliers.

E. Function Generator Techniques

The guiding principle to use in choosing the computer formulation of the function is to minimize the effect of any errors which may arise in the function generator itself. For example, if the computer-scaled diagram shows that the output of a particular function generator is small in comparison with other outputs at the same point, then obviously any errors in

the function generator would be second-order effects, and one could probably go right ahead and use the simplest and most straightforward method available for generating the function.

On the other hand, if errors of the order of a fraction of a volt in the output of a function generator can produce changes at other output points of the order of volts, then an investigation of the best method for setting-in the function is in order.

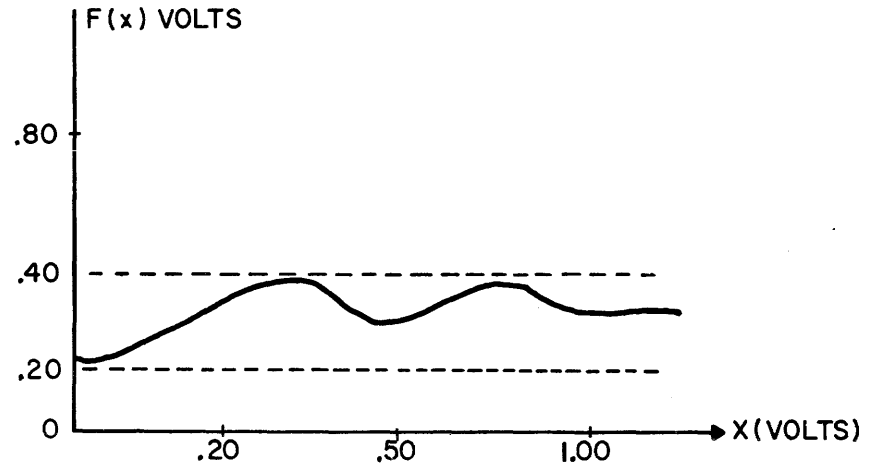
The following set of ground rules always should be considered before setting-in functions on a function generator:

Rule 1 Functions placed on function-generating equipment should be corrections only to analytic approximations to the desired function so that a high computing accuracy can be preserved.

Rule 2 The function f(x) set in the computer should go full scale or as close to full scale as practicable. In other words, the output of the function generator should cover as much of the range between the reference voltage levels as possible.

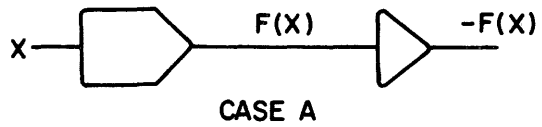
Rule 3 Every attempt should be made to have the independent variable which drives the function generator traverse a full range.

If the above rules are followed, then one will be able to utilize the maximum accuracy available from the function generation equipment. The following example, considered over the range 0 -1 unit for simplicity, will help to illustrate Rules 2 and 3. Rule 1 will be discussed later.

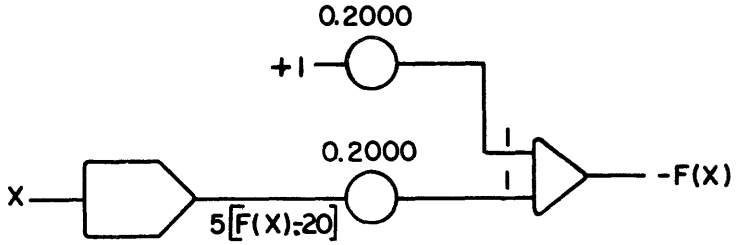


The peak value of F(x) is less than .40 and, hence, it easily can be amplified by a factor of 2. On the other hand, the minimum value is about .22 volts, so that if .20 units were subtracted from the function, our new function (F(x) -.20) would range between 0 and .20 units. This new function can then be amplified by a factor of 5, yielding the function $\boxed{5\ (F(x) - .20)}$ which would range between 0 and 1 unit. The difference in the computer set-up of the function is indicated below.

In Case B, the effect of an error in the output of the function generator has been reduced by a factor of 5.

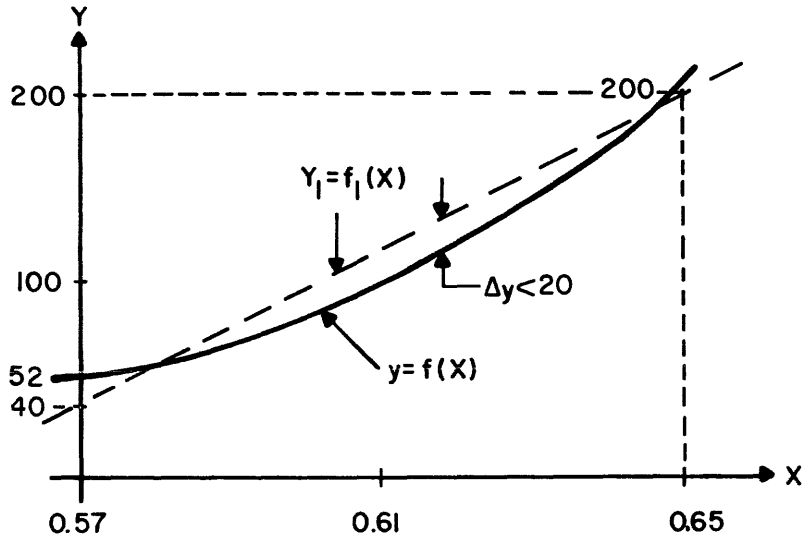


CASE A



CASE B

As another example, consider the curve



where the independent variable, x, ranges between 0.57 and 0.65. It would be more appropriate to use x' = x - 0.61 as the driving variable where x' would vary from +0.04 to -0.04. We can scale x' so that the voltage representing it at the input to the function generator ranges between -1 and +1 unit, the scaled input to the function generator would be $[25x']$. (We might also assume, for this example, that it was appropriate to scale x in another part of the problem as $[x]$).

The function, y_1, is a straight line ($y_1 = mx + b$) approximation of the curve, y, so that the correction term, Δy, is small at any point along the line. The equation for this line is

$$y_1 = 120 + 80\,[25x'] \tag{41}$$

The function, y, would be scaled suitably as $\left[\frac{y}{500}\right]$ since its maximum value is greater than 200. Since Δy has a maximum of less than 20, it may be scaled as $[\Delta y/20]$ on the function generator. For convenience in programming, we shall scale y_1 as $\left[\frac{y_1}{500}\right]$

Now, since $y_1 = y + \Delta y$, the scaled variable will be related by:

$$\left[\frac{y}{500}\right] = \left[\frac{y_1}{500}\right] + \frac{1}{25}\left[\frac{\Delta y}{20}\right] \tag{42}$$

From this, we see immediately that any error produced in generating $\left[\frac{\Delta y}{20}\right]$ is reduced by a factor of 25.

F. Comparison of Function Generators

Generally, there are three types of function generators available: pot padders, diode function generators and curve followers. Each of these has an area of greatest utility, although in many instances the choice of function generator depends on the availability of equipment.

The characteristics of each generator (see the Table V-I) determine the area where maximum usefulness, efficiency and accuracy can be obtained.

On the basis both of the information presented in Table V-I and of experience, one can set down a relative rating of the various function generators in regard to specific factor--all other things being considered equal. The comparisons in Table V-II are shown only as a guide to the novice in analog computing and should by no means be considered as absolute.

TABLE V-I Characteristics of Function Generators*

Characteristic	Pot Padder	Curve Follower	D.F.G.
Time required for set-up of function	15-30 min.	15-30 min.	15-30 min.
Time required to change function	15-30 min.	1 min.	15-30 min.
Driver Input Impedance (ohms)	Very high (Approaches ∞)	Very high (Approaches ∞)	5 K to 25K depending on value of Input Function
Output Impedance (ohms)	0 to 15K (Depends on Driver)	0 to 6K (Depends on Driver)	≈ 0
Maximum Number of of Points	17	Comtinuous line	21
Spacing Between Points	Fixed	None	Variable
Maximum Voltage or Slope Limits	25 volts between points	Maximum slope determined by speed of driver	1,4; 2.5, 6 Volts/Volt/diode (Depends on diode location and scale)
Isolation Amplifier Required (Output)	Yes	Yes	No
Multiplication of Function by Variable Possible	Yes	Yes	No
Input Impedance to Variable (ohms)	3K to 15K (Depends on Driver position)	6K*	-----
Maximum Frequency of Variable	No inherent limit	No inherent limit	-----

* Typical values

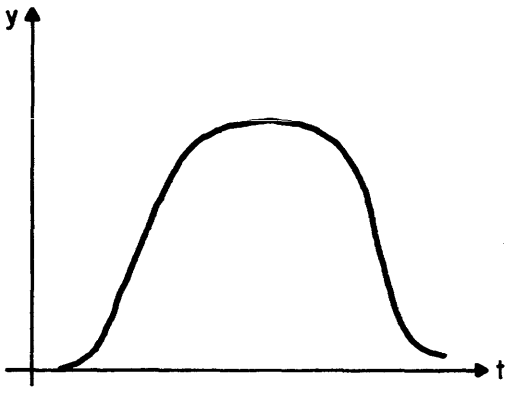
TABLE V-II Suitability of Function Generators

Area of Utility	Pot Padder	Curve Follower	DFG
Monotonic Functions	II	I	III
Sharply Changing Slopes	III	II	I
Ease of Changing Function	II	I	III
Accuracy Near Zero	II	III	I
Long Term Stability	I	II	III

G. Practical Applications - Function Generation

1. Impulse Functions

An illustration of how analytic functions can be combined to generate a complex function can be made with the impulse function



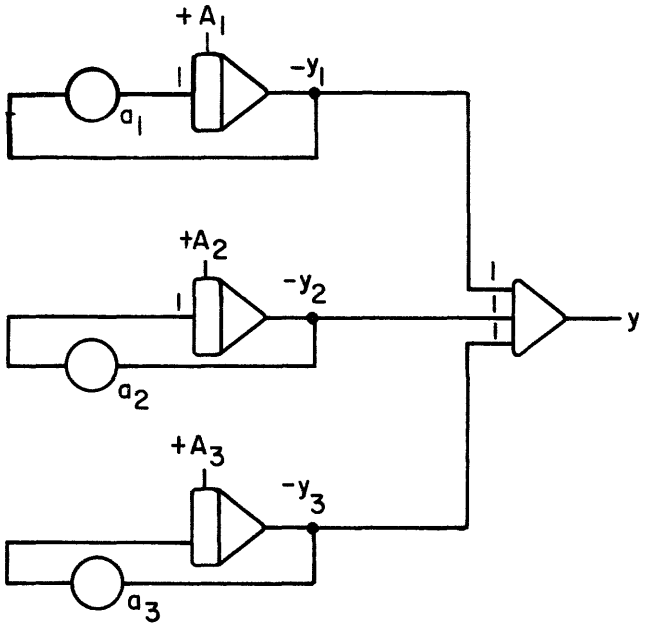
which can be generated from the linear combination

$$y(t) = y_1 + y_2 + y_3 \tag{43}$$

of three exponentials

$$y_i = A_i \, \epsilon^{-a_i t} \qquad i = 1, 2, 3 \tag{44}$$

The correct time constants and initial conditions for the circuit



depend on the form of the impulse function. This circuit is more desirable than a polynomial fit of the impulse function because it can decay to a hard zero which is difficult to obtain from a power series approximation.

2. Pressure Waves

The pressure disturbance due to a gun blast is characterized graphically by the curve



and analytically by the expression

$$P = P_o \, \varepsilon^{-t/t_1} \, (1 - t/t_1) \tag{45}$$

Note that the above relationship satisfies the major points on the curve

$$P(0) = P_o \tag{46}$$

$$P(t_1) = 0 \tag{47}$$

and

$$\lim_{t \to \infty} P(t) = 0 \quad \text{(from the negative side)} \tag{48}$$

Differentiation of the pressure equation yields

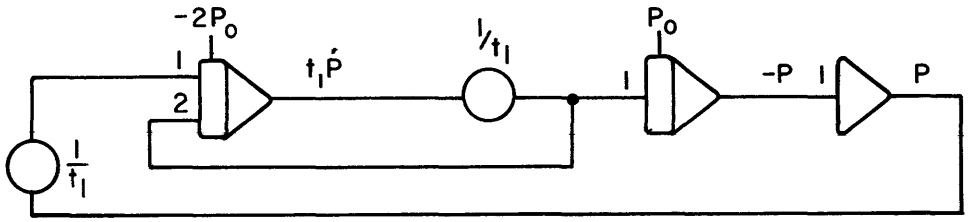
$$\dot{P} = -\frac{P_o}{t_1}(2 - t/t_1)\, \epsilon^{-t/t_1} \tag{49}$$

$$\ddot{P} = \left(P_o/t_1^{\,2}\right)(3-t/t_1)\left(\epsilon^{-t/t_1}\right) \tag{50}$$

or

$$\ddot{P} + \left(2/t_1\right)\dot{P} + \left(1/t_1^{\,2}\right)(P) = 0 \tag{51}$$

The circuit for the generation of this function is simply



The initial conditions

$$P(0) = P_o \tag{52}$$

and

$$\dot{P}(0) = \frac{-2P_o}{t_1} \tag{53}$$

were obtained from equations (49) and (50).

3. Continuous Resolution

If continuous-resolution coordinate transformation is desired, the circuits previously shown can not be used because of the limited resolution of the sinusoidal function generators. Therefore, one must use the solution of differential equations for coordinate transformation rather than the familiar algebraic trigonometric relationships.

For example, consider rectangular-to-polar coordinate transformation where

$$x = r \cos \theta \tag{54}$$

and

$$y = r \sin \theta \tag{55}$$

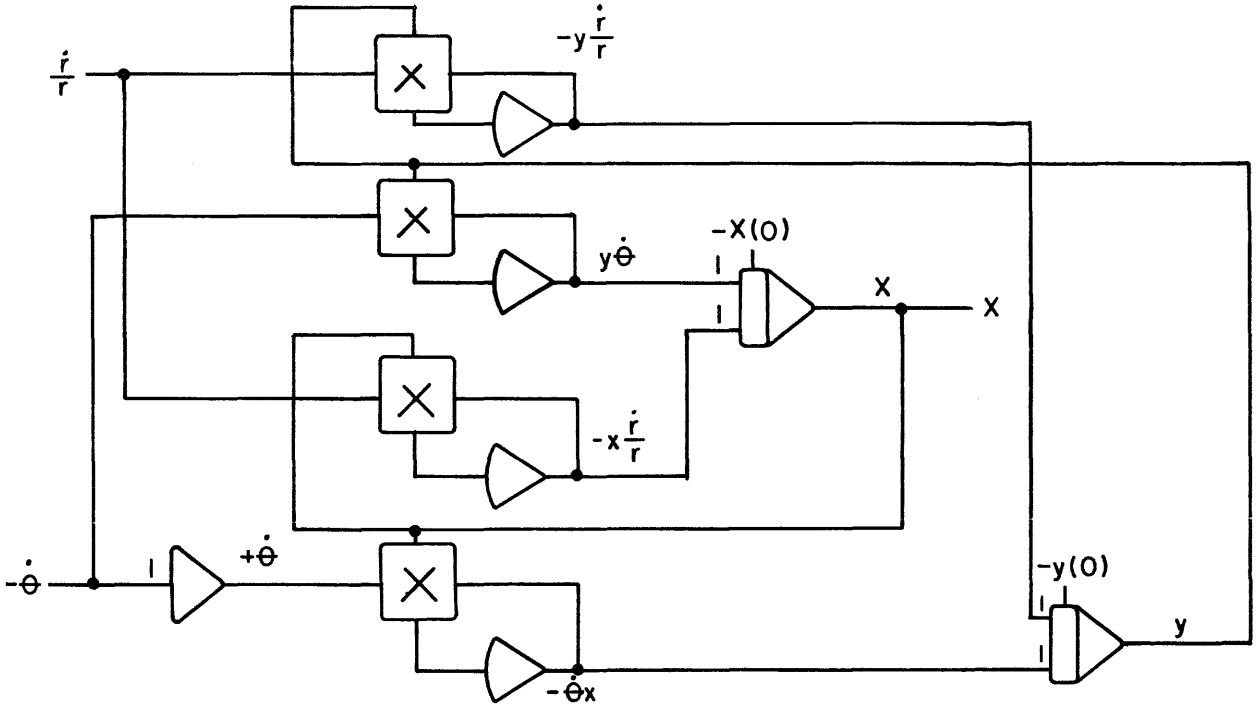
Differential equations obtained from these relationships are

$$\dot{y} = y \left(\frac{\dot{r}}{r} \right) + x \dot{\theta} \tag{56}$$

and

$$\dot{x} = x \left(\frac{\dot{r}}{r} \right) - y \dot{\theta} \tag{57}$$

which can be solved simultaneously for x and y if $\dot{r}/r$ and $\dot{\theta}$ are available.



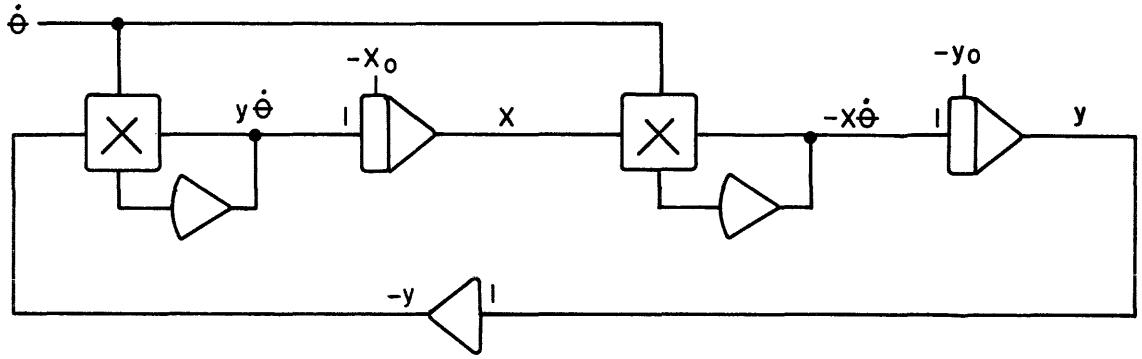
A simplified version of this circuit can be used to generate

$$y = \sin \theta \tag{58}$$

and

$$x = \cos \theta \tag{59}$$

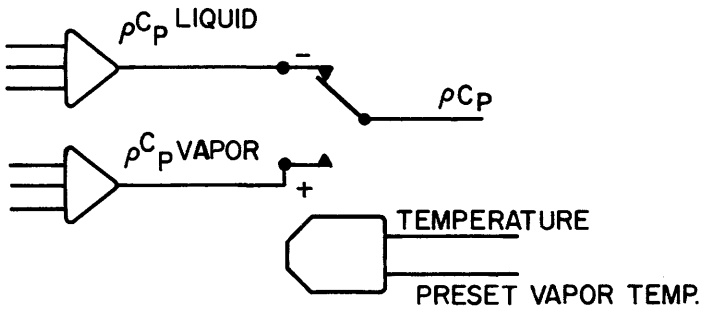
continuously. If $\dot{r}$ is zero, the previous circuit becomes



4. Practical Applications

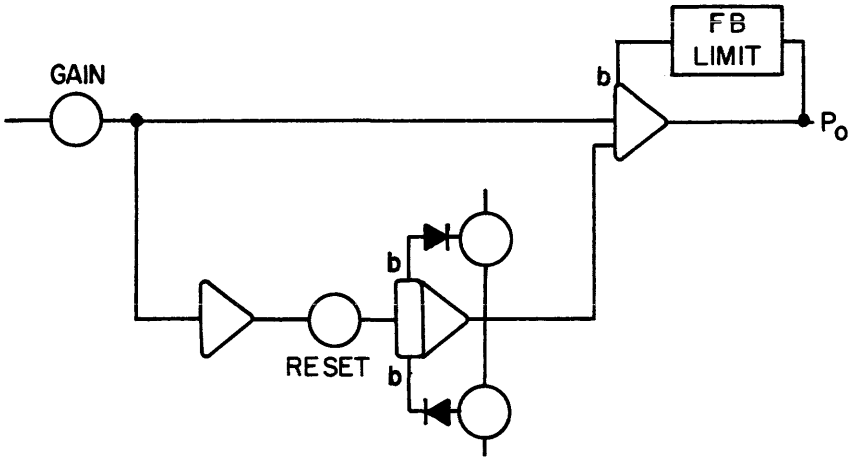
With diode or relay circuitry, many nonlinear phenomena which are difficult to deal with analytically can be simulated very simply on an analog computer as shown in the following examples.

1. In heat transfer problems, when it is necessary to calculate $\rho C_p = a + bT + CT^2$, and the coefficients a, b, and c depend upon whether the product is in liquid or vapor form, a simple **switching** circuit may be acceptable



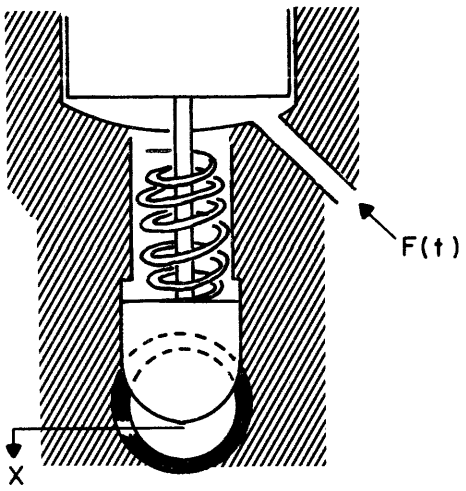
Here, the liquid and vapor heat terms are generated continuously in the computer and the one selected is dependent upon the temperature.

2. The output of a controller may be limited to, say, 15 psi. To include this effect on the computer, simple limit circuits can be connected around the output amplifier and, depending on the characteristics of the controller, around the integrator.



3. The motion of the flow control valve is given by

$$F(t) = M\ddot{x} + C\dot{x} + kx \qquad (60)$$



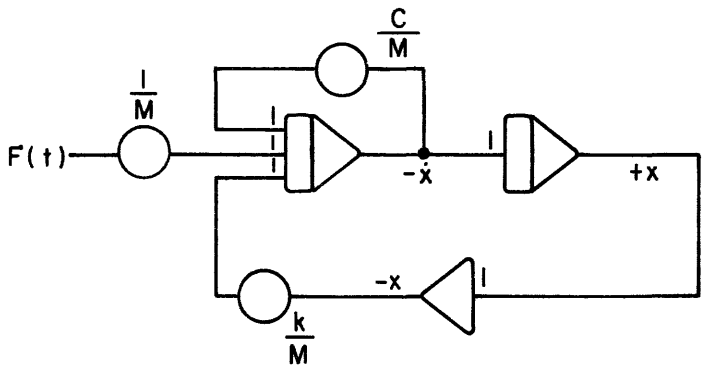
where M = mass of valve, valve stem and piston

C = damping

k = valve spring constant

F(t) = lifting function

If the motion, x, is mechanically limited such that the flow cannot exceed a certain rate, it might appear that a simple limit on the output quantity, x, is sufficient.



However, this is not the case. It must be remembered that for the displacement to be limited to a constant value, the velocity must be zero. A simple limit on the quantity, x, does not accomplish this. Methods of limiting displacement are shown in the literature (2).

5. Dynamic Pressure

In aircraft and aerospace applications, one is usually confronted with generating dynamic pressure, Q. Thus

$$Q = k \rho V^2 \tag{61}$$

where

ρ = air density, varies exponentially with altitude, $<0.01 \frac{\text{lbs (mass)}}{\text{ft}^3}$

V = vehicle velocity, of the order of 2×10^4 ft/sec for a space vehicle

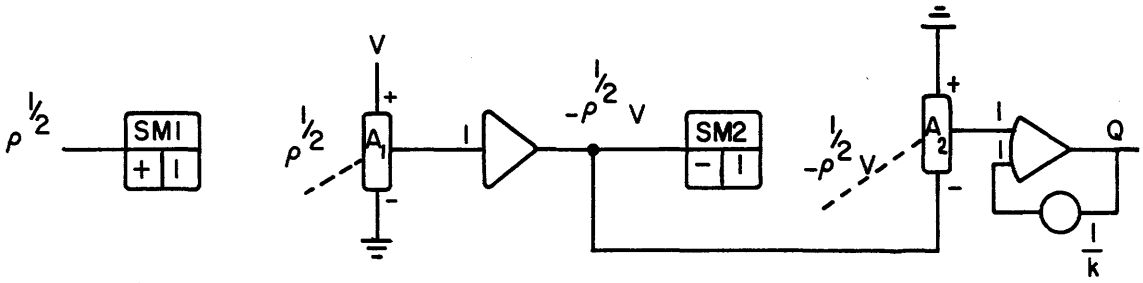
and

k = a constant

The problem in this illustration is accuracy since both ρ and V vary over a wide range and straight-forward programming will lead to the undesirable product of a very large and a very small voltage.

The accuracy and voltage range of the density computation is increased by mechanizing the simulation to obtain $\sqrt{\rho}$ which then forms the $\sqrt{\rho} \cdot V$ product and eventually the dynamic pressure.

The air density is a low frequency variable; therefore, servo multipliers are applicable if they are driven by ρ and $\rho^{\frac{1}{2}} V$.



6. Euler-Angle Axis Rotation and Matrix Inversion

One method of determining the relationships between earth coordinates of a point and its wind coordinates is the Euler-Angle technique. This technique, which is described in detail in the literature (2), rotates the axes of the earth until they are aligned with the wind or body axis. The rotation is done in a definite sequence, accounting for the changes in coordinate values caused by the rotation.

A typical example is the conversion of X, Y, Z to x, y, z coordinates by three rotations in the following order, where X_1, Y_1, Z_1 are interim coordinates

1) Rotate about the Z axis by an angle Ψ

$$X_1 = X \cos \Psi + Y \sin \Psi \quad (62)$$

$$Y_1 = -X \sin \Psi + Y \cos \Psi \quad (63)$$

$$Z = Z \quad (64)$$

2) Now rotate about the Y_1 axis by an angle θ

$$x = -Z \sin \theta + X_1 \cos \theta \quad (65)$$

$$Y_1 = Y_1 \quad (66)$$

$$Z_2 = Z \cos \theta + X_1 \sin \theta \quad (67)$$

and

3) Finally rotate about the x-axis by an angle ϕ

$$x = x \quad (68)$$

$$y = Y_1 \cos \phi + Z_2 \sin \phi \quad (69)$$

$$z = -Y_1 \sin \phi + Z_2 \cos \phi \quad (70)$$

The mechanization of this transformation requires approximately 15-amplifiers, 3-servo resolvers and all angular and original coordinage inputs. Due to the complexity of the circuitry, it will not be illustrated; however, the circuit diagram is available in the literature (2). Even more significant than the actual mechanization is the fact that the analog computer is capable of performing matrix inversion. Te above transformation in matrix notation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

which must be inverted in a definite order.

1. References

1) Fifer, S.; Analogue Computation; McGraw-Hill Book Col, Inc. New York; 1961.

2) Rogers, A. E. and Connolly, T. N.; Analog Computation in Engineering Design, McGraw Hill Book Co; Inc.; New York, 1960.

CHAPTER VI

TRANSFER FUNCTION SIMULATION

A. Introduction

The purpose of this chapter is to present the methods commonly used to simulate linear transfer functions which frequently arise in the formulation of analog computer models. Prior to discussing simulation methods, a brief introduction to transfer function theory is necessary for background purposes.

A transfer function is, in reality, a linear, ordinary differential equation which describes a particular phenomena or system in shorthand notation. It is usually expressed as the ratio of the response (or output) of a system to its forcing function (or input). It is a function of the parameters of the system, and either the classical operator, D, or the Laplace transform variable, s. The system parameters are expressed in terms of gains, time constants, etc., and the classical operator and Laplace transform are defined as:

$$D \equiv \frac{d}{dt} \text{ or } \frac{1}{D} = \int_o^t dt \tag{1}$$

and

$$y(s) = \int_o^t \epsilon^{-st}\, y(t)dt \tag{2}$$

where s is a complex variable.

Transfer functions obtained from differential equations using the classical operator are not necessarily equal to those obtained via Laplace transforms because the classical operator does not account for non-zero initial conditions. For example, the transfer function for the first order equation

$$\frac{dy}{dt} + y = x \tag{3}$$

in terms of the classical operator is

$$\frac{y}{x} = \frac{1}{D+1} \tag{4}$$

as compared to its Laplace transform counterpart

$$\frac{y}{x + y(o)} = \frac{1}{s+1} \tag{5}$$

which was obtained using the table of Laplace transform pairs in Appendix A. If the system described by equation (3) was initially at rest, (y(o)=o), the two transfer functions would be identical.

In many instances, variables can be redefined to obtain zero initial condition. In the case of the first order equation, if one defines

$$z = x - x(o) \tag{6}$$

and

$$w = y - y(o) \tag{7}$$

the transfer function in Laplace transform notation becomes

$$\frac{w}{z} = \frac{1}{s+1} \tag{8}$$

since the steady-state solution of equation 3 is y equals x.

In the remainder of this chapter, it will be assumed that all initial conditions are zero and all transfer functions will be written in Laplace transform notation.

B. Properties of Transfer Functions

Two properties of transfer functions which account, to a large extent, for their high utilization in practical applications are

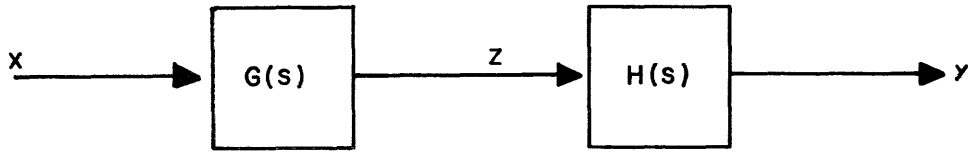
1) their ability to combine with each other to form an overall input-output transfer function for a system,

and

2) their ability to predict the response of a system to a sinusoidal input.

1. Combining Transfer Functions

If one considers the block diagram of a system where each block represents a transfer function, an overall transfer function can be obtained using block diagram algebra. For example, consider two transfer functions in series



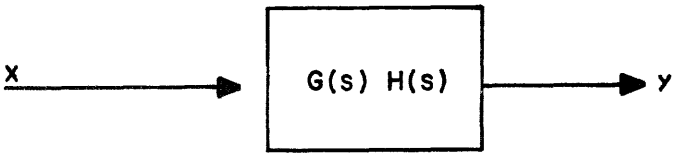
where $\frac{z}{x} = G(s)$ (9)

and $\frac{y}{z} = H(s).$ (10)

Since $\frac{y}{z}\,\frac{z}{x} = \frac{y}{x}$ (11)

the over-all transfer function becomes $\frac{y}{x} = G(s)\ H(s)$ (12)

and the block diagram reduces to



Several of the most frequently used block diagram identities are shown in Figure VI-1

The form of complex over-all transfer functions in generalized notation is

$$\frac{y}{x} = \frac{b_m S^m + b_{m-1} S^{m-1} + - - - - - - + b_1 S + b_o}{A_n S^n + A_{n-1} S^{n-1} + - - - - - - + A_1 S + A_o} \tag{13}$$

which can be represented in terms of first and second order transfer functions

$$\frac{y}{x} = \frac{S^n \left[\pi_{i=1}^{n_1} (1+T_i s)\right]\left[\pi_{k=1}^{n_2} \left\{1 + \frac{2\xi_k}{\omega_k} S + \frac{S^2}{\omega_k 2}\right\}\right]}{\left[\pi_{j=1}^{n_3} (1+T_j S)\right]\left[\pi_{m=1}^{n_4} \left\{1 + \frac{2\xi_m S}{\omega_m} + \frac{S^2}{\omega_m^2}\right\}\right]} \tag{14}$$

2. Frequency Response

The frequency response of a system to a sinusoidal input can be obtained directly from its transfer function by letting

$$s = j\omega \tag{15}$$

where j is $\sqrt{-1}$ and ω is the frequency of the sinusoid. The resultant expression is a complex number which can be represented by a gain and a phase angle.

For example, consider the frequency response of the first order system described by equation 3

$$\frac{y}{x} = \frac{1}{1 + j\omega} \left(\frac{1 - j\omega}{1 - j\omega}\right) = \left(\frac{1}{1 + \omega 2}\right) + j \left(\frac{-\omega}{1 + \omega^2}\right) \tag{16}$$

which is a complex number made up of real (Re) and imaginary (Im) parts.

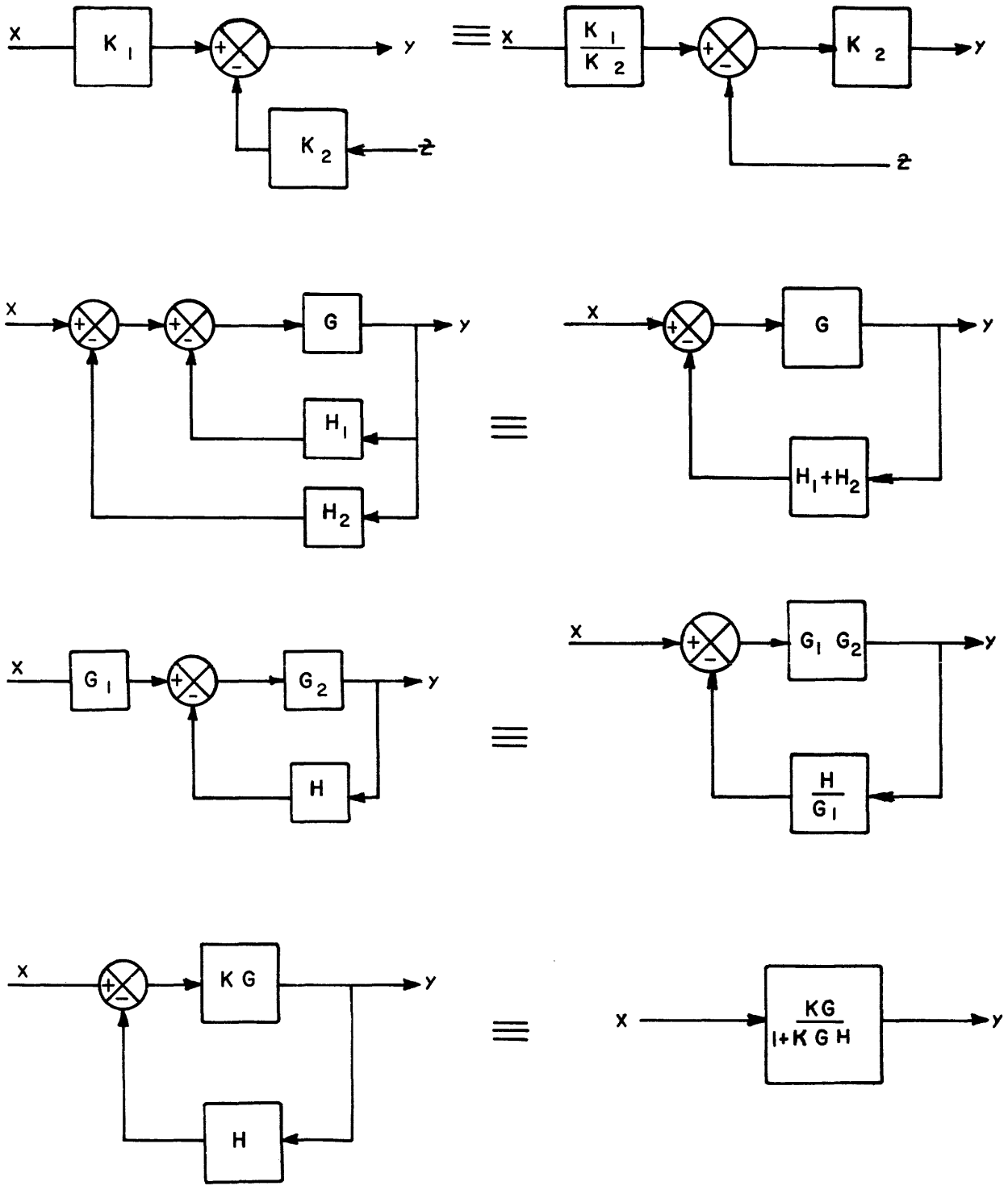
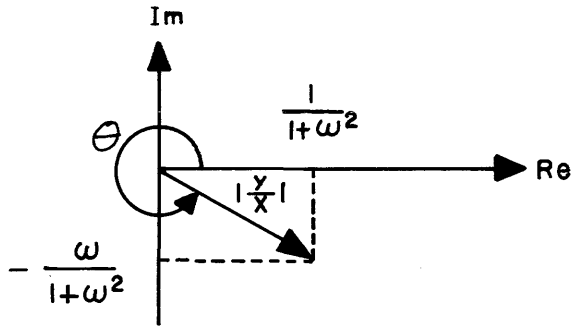


Figure VI-1: Selected Block Diagram Identities

In the complex plane,



any imaginary number can be represented by a vector whose magnitude and phase angle are

$$\left|\frac{y}{x}\right| = \sqrt{(Re)^2 + (Im)^2} \tag{17}$$

and

$$\theta = Tan^{-1}\left(\frac{Im}{Re}\right) \tag{18}$$

therefore,

$$\frac{y}{x} = \left|\frac{y}{x}\right| \epsilon^{j\theta} = \left|\frac{y}{x}\right| \left[Cos\theta + j\ Sin\theta\right] \tag{19}$$

where

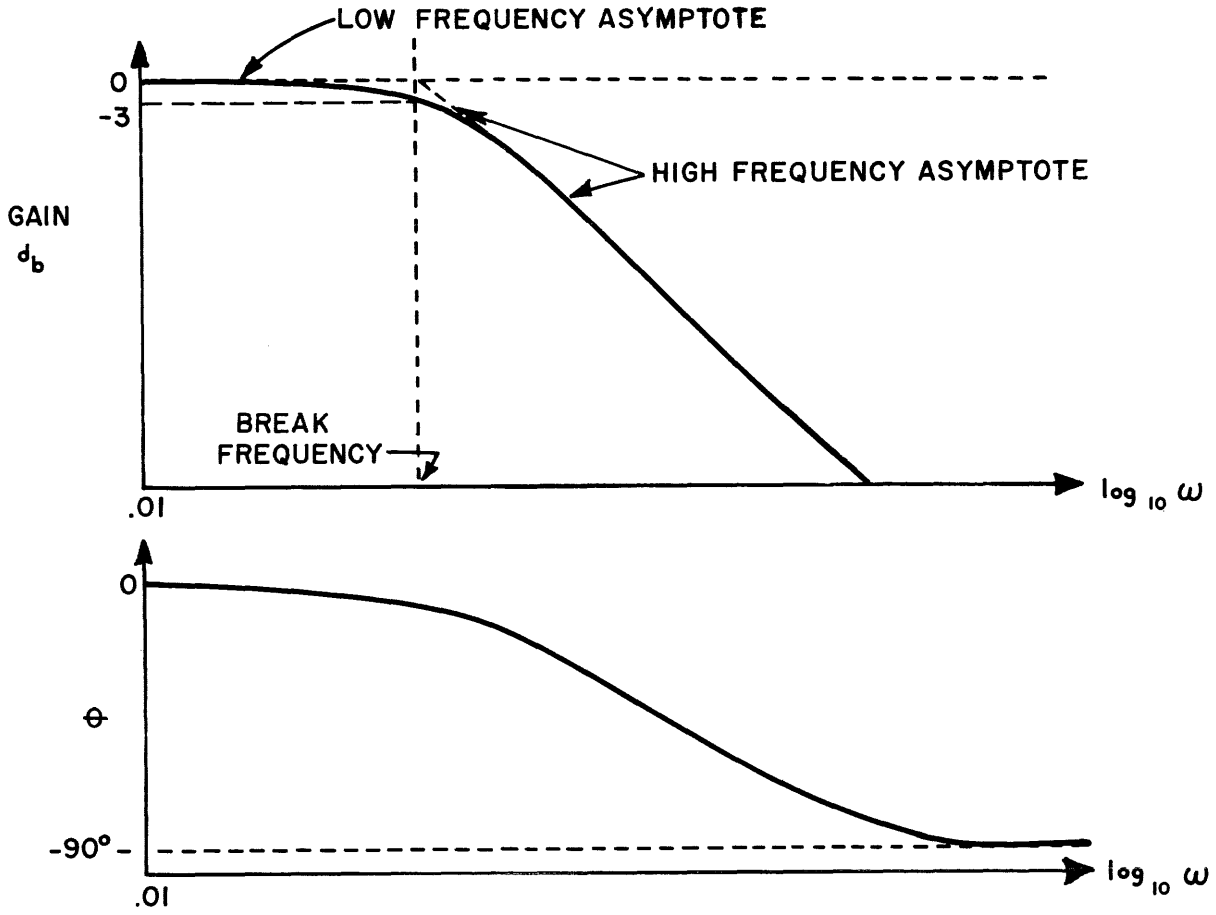
$$\left|\frac{y}{x}\right| = \left(\frac{1}{1+\omega^2}\right)^{1/2} \tag{20}$$

$$\theta = -Tan^{-1}(\omega) \tag{21}$$

The magnitude of the vector in decibel form (X in decibels = 20 log X) is

$$20 \log \left|\frac{y}{x}\right| = -10 \log \left(1 + \omega^2\right) \tag{22}$$

which is defined as the gain of the system. Plots of the gain and phase angle versus frequency yield curves which are characteristic of this transfer function. These curves, which are referred to as Bode plots, are of the form,



In practice, high and low frequency asymptotes, which are shown above, are used to approximate the gain frequency curve. At low frequencies, the gain is approximately zero

$$20 \log \left| \frac{y}{x} \right| = -10 \log \left(1 + \omega^2\right) \approx -10 \log 1 = 0 \qquad (23)$$

while at high frequencies it is proportional to frequency

$$20 \log \left| \frac{y}{x} \right| = -10 \log \omega^2 \left(1 + \frac{1}{\omega^2}\right) \approx -20 \log \omega \qquad (24)$$

The "break frequency", which is the point of intersection of the asymptotes, is defined as the frequency where the asymptote and the ideal curve differ by 3 db. Additional Bode plot theory, manipulations, and applications are available in the literature (1),(3).

The practical aspect of frequency response theory permits the determination of a mathematical model in transfer function form from experimental data. (These models are linear approximations of non-linear systems.) This technique frequently represents the only practical method of obtaining a mathematical model for an existing system, and is one method of introducing transfer functions into a simulation.

C. Transfer Function Simulation

Transfer functions can be simulated by three different methods:

1) Passive elements

2) Amplifiers and passive elements

3) Computing components -- potentiometers and amplifiers -- which will now be presented in detail. This presentation will be restricted to first and second order transfer functions for the following reasons:

   a) Few mathematical models for physical systems exceed second order.

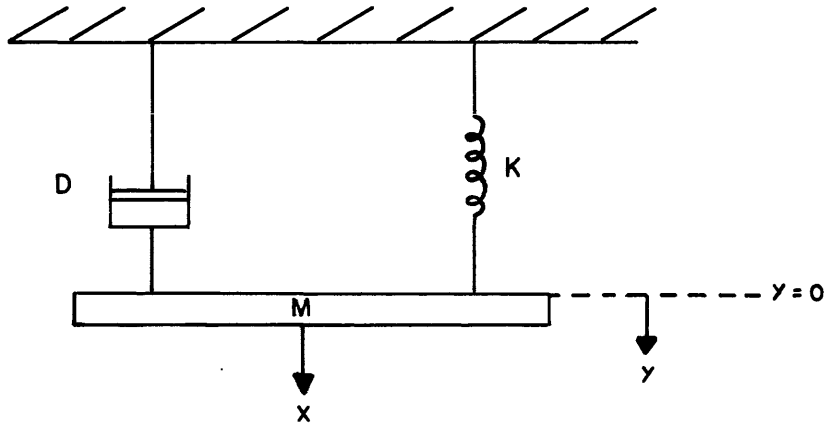
   b) Higher order transfer functions can be reduced to lower order transfer functions.

   c) Lower order transfer functions are simpler to scale and mechanize on the computer.

   d) Complex, overall transfer functions combine the individual parts of a system which nullifies the simulation concept of the analog computer.

1. Passive Element Simulation

Prior to the development of modern-day, electronic analog computers, linear equations were solved by direct analogy to electrical equations for passive element circuits. The analogies between electrical, mechanical and thermal parameters and variables, which is presented in Table VI-I are typical of the direct analogy approach. To illustrate the application of this technique, consider the mechanical system,

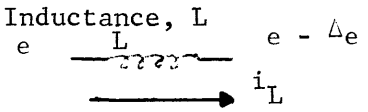
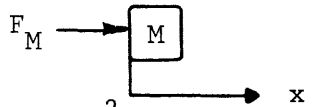
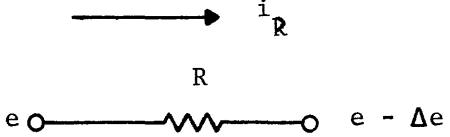
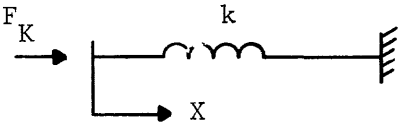
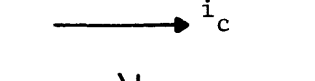
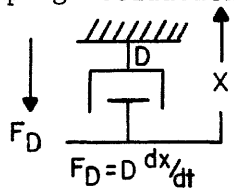


whose differential equation and transfer function are

$$M\frac{d^2y}{dt^2} + D\frac{dy}{dt} + Ky = x \tag{25}$$

$$\frac{y}{x} = \frac{1}{MS^2 + DS + K} \tag{26}$$

TABLE VI-I: Comparison of Electrical, Mechanical, and Thermal Variables and Parameters

Electrical	Mechanical	Thermal
Inductance, L  $\Delta e = L \frac{di_L}{dt}$	Mass, M  $F_M = M \frac{d^2x}{dt^2}$	Not Applicable
Resistance, R  $\Delta e = i_R R$	Compliance, K  $F_K = kx$	Heat Transfer Resistance $\frac{L}{ka}$ or $\frac{1}{ha} = \frac{\Delta T}{q}$ L = Thickness k = Thermal Conductivity A = Area h = Fluid Film Coefficient
Capacitance, C  $\Delta e = 1/C \int_0^t i_c dt$ or $i_c = C \frac{d\Delta e}{dt}$	Damping Coefficient, D  $F_D = D \frac{dx}{dt}$	Unit Capacity $C_p \rho V = q / \frac{dT}{dt}$ C_p = Specific Heat ρ = Density V = Volume
Voltage - e	Force - F	Temperature - T
Current - i	Velocity - $\frac{dx}{dt}$	Rate of Heat Flow = q = $\frac{dQ}{dt}$
Charge - q $\frac{dq}{dt} = i$	Displacement - x	Heat - Q

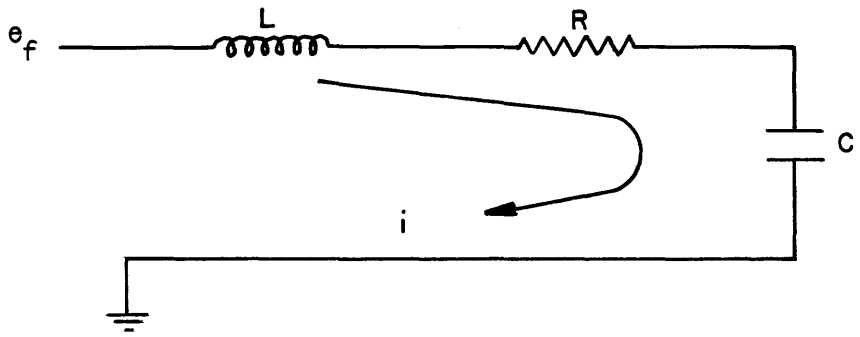
Since the sum of the forces acting on the system are

$$F_M + F_D + F_K = x \tag{27}$$

and voltage is analogous to force

$$e_m + e_D + e_k = e_f \tag{28}$$

a series R-L-C circuit is indicated.



From Table VI-I, the voltage across each passive element can be expressed in terms of charge, q_y, to obtain

$$e_f = \frac{q_y}{C} + R\frac{dq_y}{dt} + L\frac{d^2q_y}{dt^2} \tag{29}$$

whose transfer function is

$$\frac{q_y}{e_f} = \frac{1}{LS^2 + RS + \frac{1}{C}} \tag{30}$$

indicating that L, R, and 1/C must equal M, D, and K, respectively. This eqality may require modification since scaled voltage equations must be used, which introduce scale factors into coefficients of the original equations.

In determining the transfer function for an electrical circuit, it should be recalled that the impedance of a resistor and a capacitor are

$$Z_R = R \tag{31}$$

$$Z_c = \frac{1}{CS} \tag{32}$$

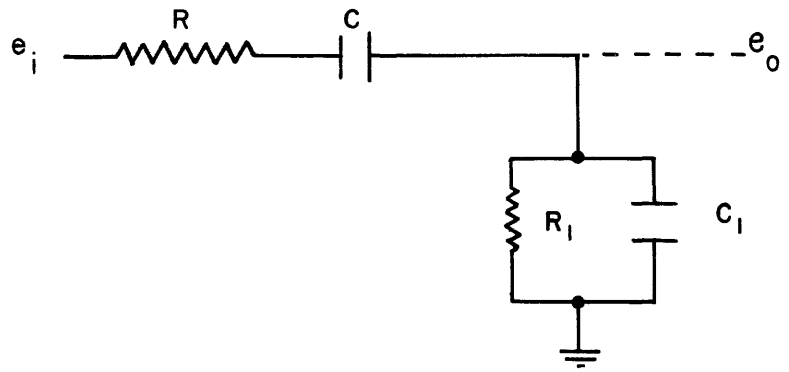
In addition, impedances in series may be represented by a single impedance, Z_s

$$Z_s = Z_1 + Z_2 + Z_3 \tag{33}$$

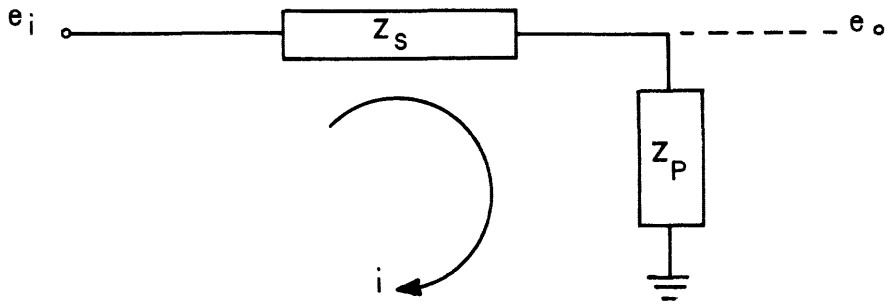
and the series equivalent of parallel impedances, Z_p, is

$$\frac{1}{Z_p} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \tag{34}$$

To illustrate the application of these relationships consider the circuit,



The circuit reduces to,



where

$$Z_s = R + \frac{1}{CS} = \frac{RCS + 1}{CS} \tag{35}$$

and

$$Z_p = \frac{R_1}{1 + R_1 C_1 S} \tag{36}$$

The transfer function is

$$\frac{e_o}{e_i} = \frac{Z_p}{Z_s + Z_p} = \frac{\left(R_1C\right) S}{\left(R_1C_1RC\right) S^2 + \left(RC + R_1C_1 + R_1C\right) S + 1} \qquad (37)$$

Since extensive tables of passive circuits and their corresponding transfer functions are available in the literature (1),(2), additional illustrations will not be presented.

The advantage of using passive elements to simulate linear systems is that no amplifiers or other computing components are required. However, there are several disadvantages which must be considered before this technique can be applied effectively:

1) The derivation of all transfer functions does not account for loading effects which can only be eliminated by adding compensating passive elements or unloading circuits.

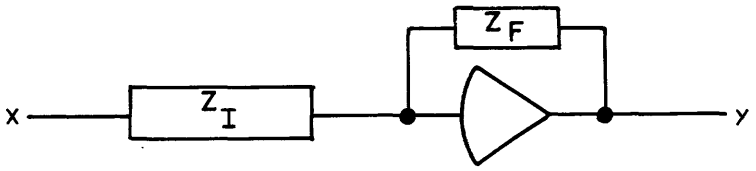
2) The cost of precision passive elements is sometimes prohibitive.

3) The lack of flexibility, ie, a complete redesign is required to change the time scale factor, frequently is a major handicap.

In practice, only long term, high-computer-utilization simulations can justify the time and effort required to mechanize passive element circuitry with its lack of flexibility.

2. Passive Element, Amplifier Simulation

The loading problem associated with passive element simulations is eliminated if operational amplifiers are used in conjunction with the passive circuits. The generalized amplifier-impedance circuit which was presented in Chapter II,



has the transfer function,

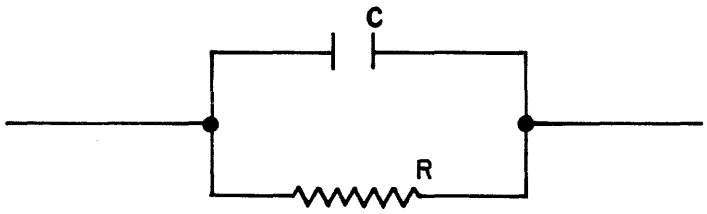
$$\frac{y}{x} = -\frac{Z_F}{Z_I} \qquad (38)$$

where the input, Z_I, and feedback, Z_F, impedances can be complex R-C circuits. These impedances are known as "transfer impedances" which are tabulated in the literature (1),(2),(3) and in Appendix B.

These tables can be used to associate a computer circuit quickly with a particular transfer function. For example, consider the lead-lag (or lag-lead) transfer function

$$\frac{y}{x} = -K \left[\frac{T_1 S + 1}{T_2 S + 1}\right] \tag{39}$$

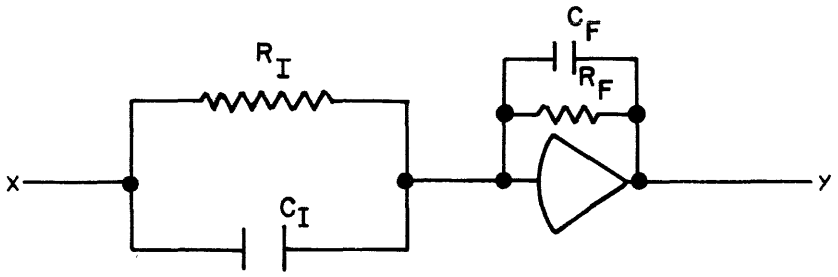
Since the electrical impedance of a parallel R-C network



is

$$Z = \frac{R}{RCS + 1} \tag{40}$$

the lead-lag transfer function can be simulated using R-C networks in the amplifier feedback and input



The resultant transfer function for this circuit is

$$\frac{y}{x} = -\left(\frac{R_F}{R_I}\right)\left(\frac{R_I C_I S + 1}{R_F C_F S + 1}\right) \tag{41}$$

Therefore, the correct circuit is obtained if

$$K = R_F / R_I \tag{42}$$

$$T_1 = R_I \cdot C_I \tag{43}$$

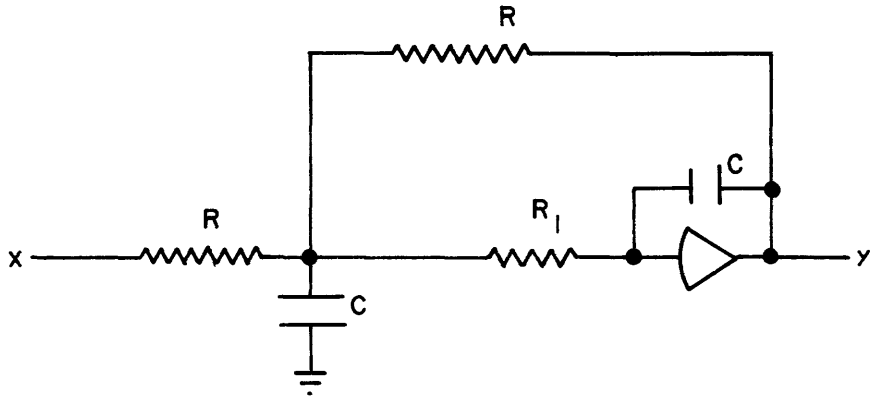
and

$$T_2 = R_F C_F \tag{44}$$

If it is necessary to derive a complex transfer function, the transfer function for the five-impedance amplifier circuit (shown in Figure VI-2)

$$\frac{y}{x} = \frac{-Z_4 / Z_1}{1 + \frac{Z_4 Z_3}{Z_5}\left[\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4}\right]} \tag{45}$$

can be used. For example, to determine the transfer function for the circuit



one need only evaluate the individual impedances, make a comparison with Figure VI-2.

$$Z_1 = Z_4 = R \tag{46}$$

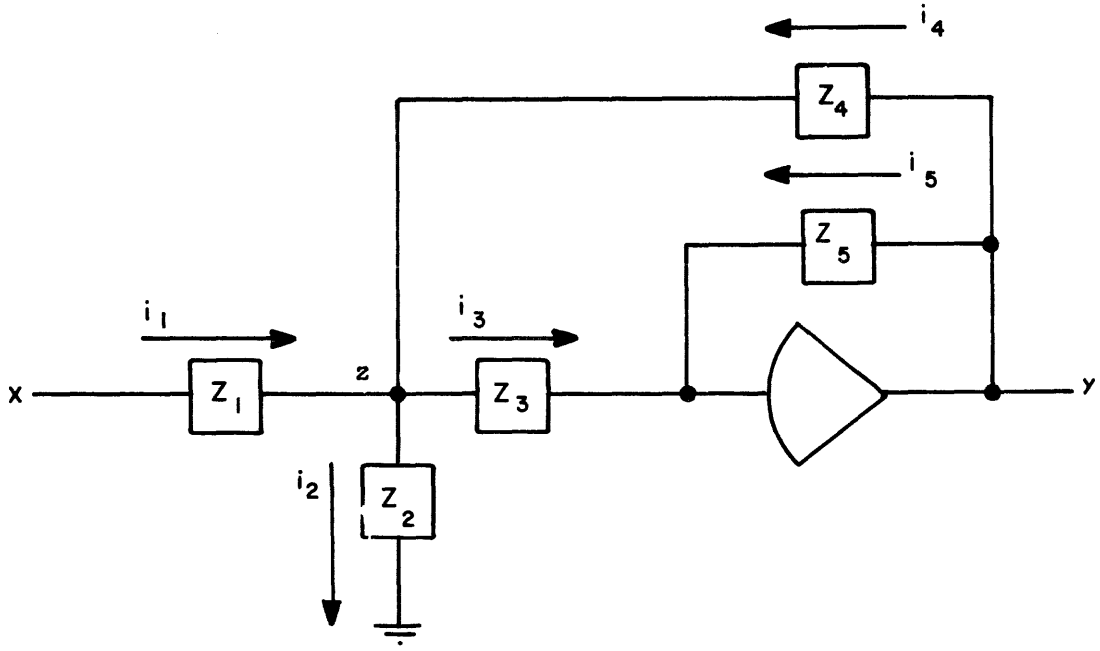
$$Z_3 = R_1 \tag{47}$$

$$Z_5 = Z_2 = \frac{1}{CS} \tag{48}$$

and substitute into equation 45. In this case, the resultant equation is the second-order transfer function,

(49)

$$\frac{y}{x} = \frac{-1}{1 + (2R_1 + R)\,CS + RR_1(CS)^2}$$



$$i_1 + i_4 = i_2 + i_3 \qquad i_4 = \frac{y - z}{Z_4}$$

$$i_3 + i_5 = 0 \qquad i_5 = \frac{y}{Z_5}$$

Figure VI-2 Five-Impedance, Amplifier Circuit

which can be compared to the familiar equation,

$$\frac{y}{x} = \frac{1}{1 + \frac{2\xi}{\omega_n} S + \frac{S^2}{\omega_n^2}} \tag{50}$$

to determine the parameter-passive element relationships,

$$\omega_n = \frac{1}{C}\sqrt{\frac{1}{RR_1}} \tag{51}$$

and

$$\xi^2 = \frac{\left(2R_1 + R\right)^2}{4RR_1} \tag{52}$$

This simulation method, with the exception of the loading problem, has the same drawbacks as the passive element method. It can be justified only on a long-term usage basis. One typical example is the simulation of U^{235} kinetics in a nuclear reactor*, which we described by seven first order equations.

*This simulation is described in detail in EAI Application Study 13.4.2a: Simulation of the Primary Loop of a Nuclear Power Plant with a Small General Purpose Analog Computer. Order by Bulletin No. ALAC 6300-1ab.

These equations can be simulated using the one amplifier circuit shown in Figure VI-3.

3. Potentiometer and Amplifier Simulation

When transfer functions are simulated using standard analog computer components only, several factors must be considered. They are:

1) several circuits can simulate the same transfer function,

2) transfer functions, which are, in reality, differential equations, must be scaled, programmed and static-checked, and

3) in a parameter study, transfer function parameters should be isolated on individual potentiometers.

This section will present the established techniques for the simulation of first and second order transfer functions, and comment on the use of tabulated transfer function circuits.

To illustrate the various analytical techniques for manipulating transfer functions into the proper form for computer solution, and their corresponding computer circuits, consider the transfer function

$$\frac{y}{x} = \frac{K\left(T_2S + 1\right)}{\left(T_1S + 1\right)\left(T_3S + 1\right)} \tag{53}$$

Equation 53 can be expressed as

$$\frac{y}{x} = \left(\frac{K}{T_1S + 1}\right)\left(\frac{T_2}{T_3} + \frac{1 - T_{2/T_3}}{T_3S + 1}\right) \tag{54}$$

expanded using partial fractions to obtain

$$\frac{y}{x} = \frac{K\left(\frac{T_1 - T_2}{T_1 - T_3}\right)}{T_1S + 1} + \frac{K\left(1 - \frac{T_1 - T_2}{T_1 - T_3}\right)}{T_3S + 1} \tag{55}$$

or solved for powers of S

$$T_1T_3S^2y + (T_1 + T_3)\,Sy + y = Kx + KT_2\,Sx \tag{56}$$

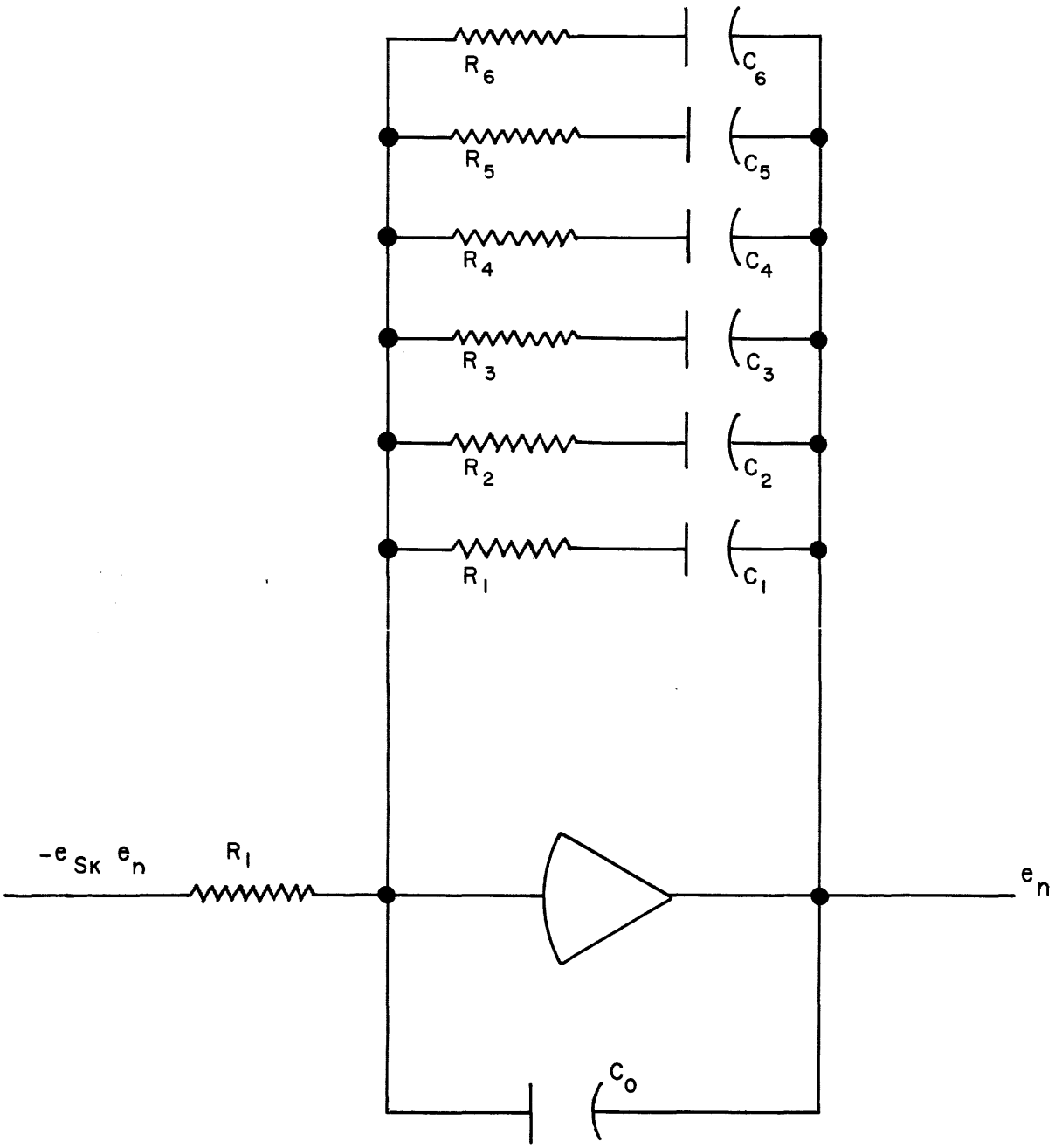


Figure VI-3 Reactor Kinetics Network

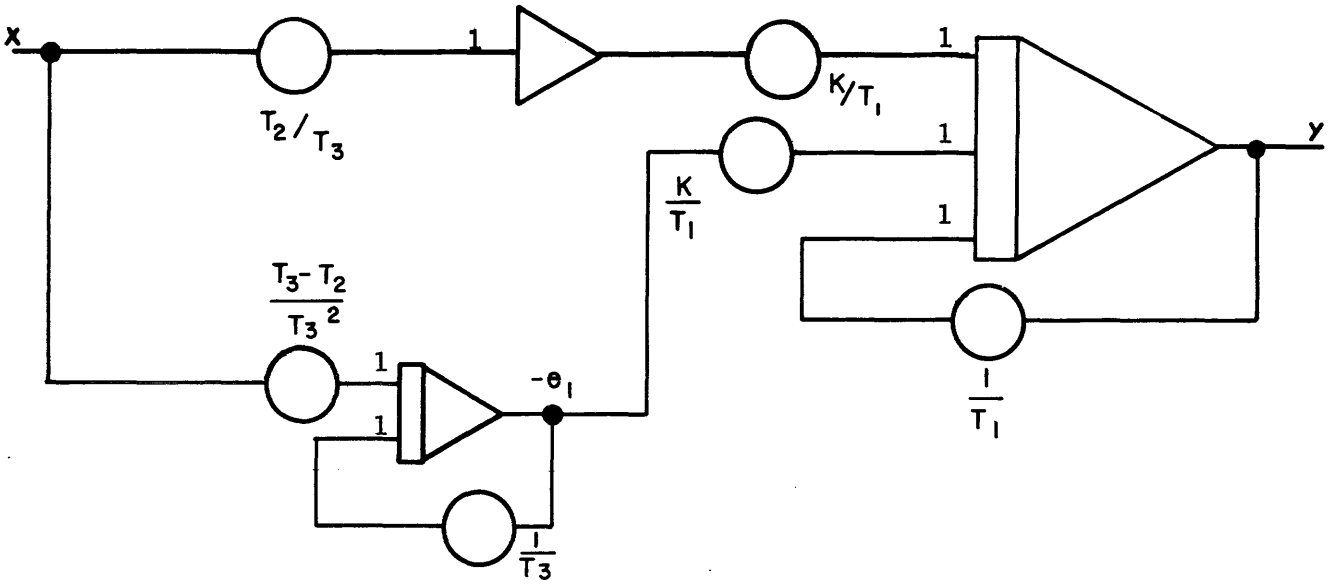
which yields

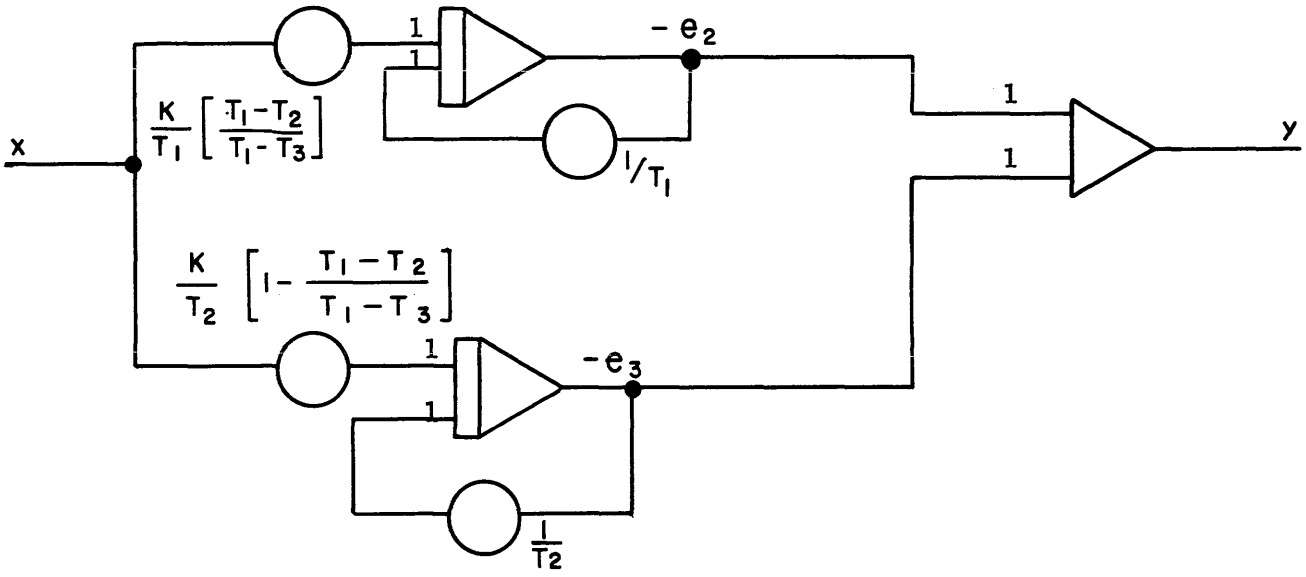
$$y = \frac{1}{T_1 T_3 S} \left\{ \frac{(Kx - y)}{S} + KT_2 x - (T_1 + T_3)\, y \right\} \qquad (57)$$

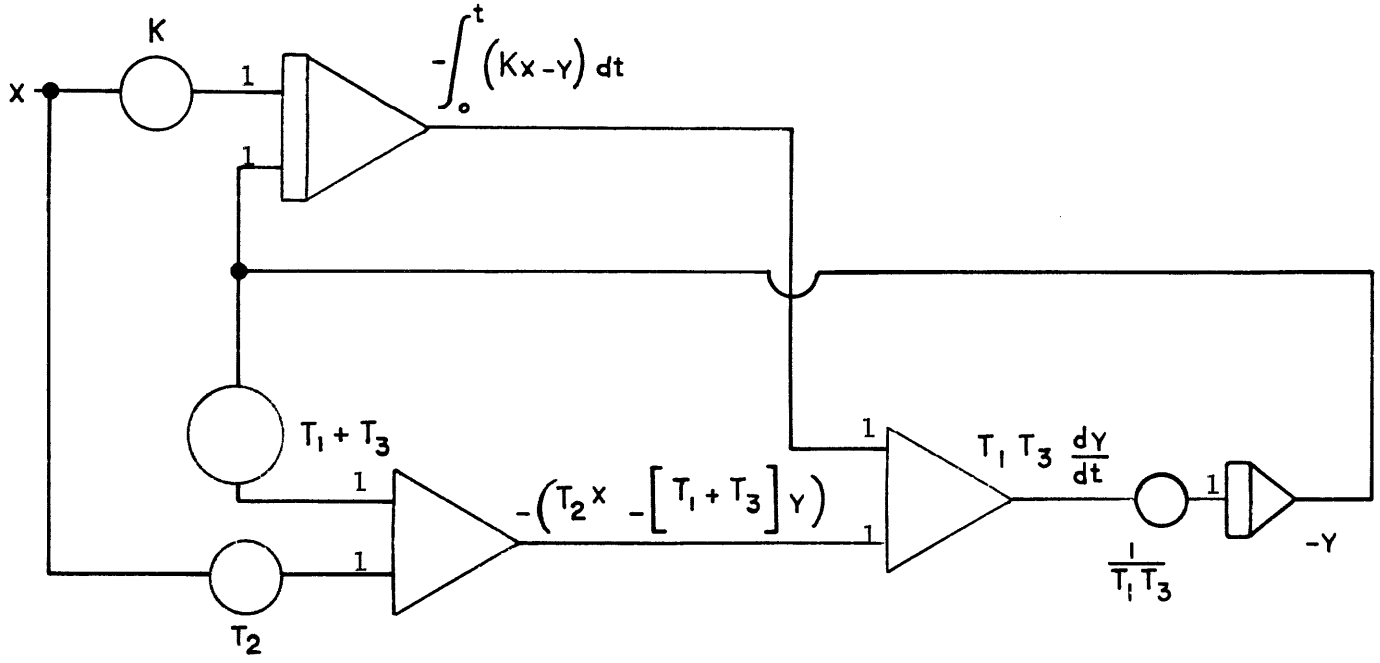
or

$$y = \frac{1}{T_1 T_3} \int_o^t \left\{ \int_o^t (Kx - y)\, dt + KT_2 x - \left(T_1 + T_3\right) y \right\} dt \qquad (58)$$

The computer diagrams associated with equation 54, 55, and 58 are







where

$$\frac{e_1}{x} = \frac{1 - T_2/T_3}{T_3 S + 1} \tag{59}$$

$$\frac{e_2}{x} = \frac{K\left[\frac{T_1 - T_2}{T_1 - T_3}\right]}{T_1 S + 1} \tag{60}$$

and

$$\frac{e_3}{x} = \frac{K\left[\frac{T_2 - T_3}{T_1 - T_3}\right]}{T_2 S + 1} \tag{61}$$

There is little difference among the three transfer function equations and their respective circuits. In the first two circuits, the maximum integrator outputs can be obtained from their steady-state equations

$$e_{1_{MAX}} = \left(1 - T_2/T_3\right) x_{MAX} \tag{62}$$

$$e_{2_{MAX}} = K \left[\frac{T_1 - T_2}{T_1 - T_3}\right] x_{MAX} \tag{63}$$

$$e_{3_{MAX}} = K \left[\frac{T_2 - T_3}{T_1 - T_3}\right] x_{MAX} \tag{64}$$

which facilitates scaling. The third circuit has no particular assets or liabilities. Since all amplifier outputs are defined, only the parameter ranges and the maximum values of x and y are required for programming. The maximum value of y usually can be estimated from the transfer function, using the steady-state relationship (S = 0), if the maximum value of x is known

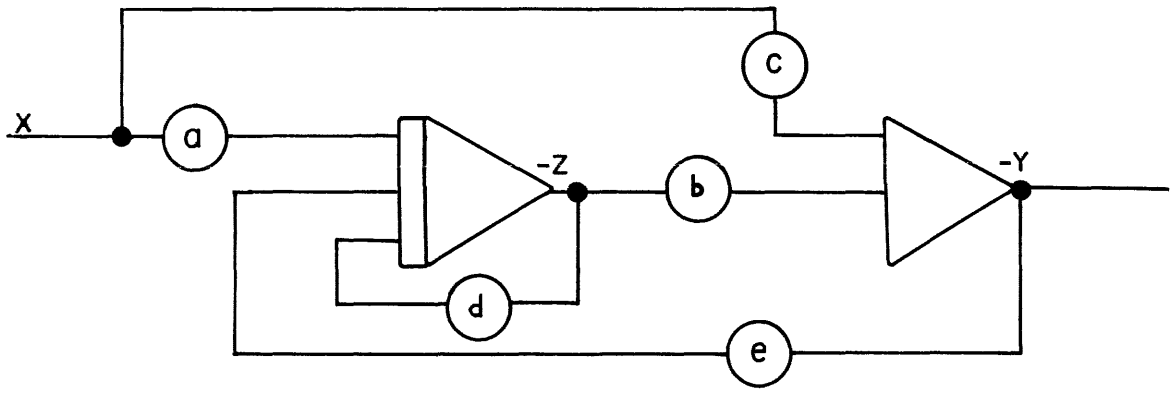
$$y_{MAX} = Kx_{MAX} \tag{65}$$

On the assumption that all amplifier outputs can be defined and scaled from a knowledge of the physical system, from steady-state considerations or by trial and error, the selection of the proper transfer function becomes obvious. One either selects the most efficient (fewest amplifiers) circuit, which tends to combine and distribute system parameters, or the most practical circuit for a parameter study, which isolates all parameters on individual potentiometers.

D. Efficiently Programmed Transfer Functions

The most efficient programming of first and second order transfer functions is based on the circuits and equations shown in Figures VI-4 and VI-5. Tabulations of transfer function circuits, which are derived using these efficient circuit relationships, are available in the literature (1) (2) (3), and are illustrated in Appendix B. Tables of transfer function circuits frequently contain errors; therefore, one should use these circuits with caution.

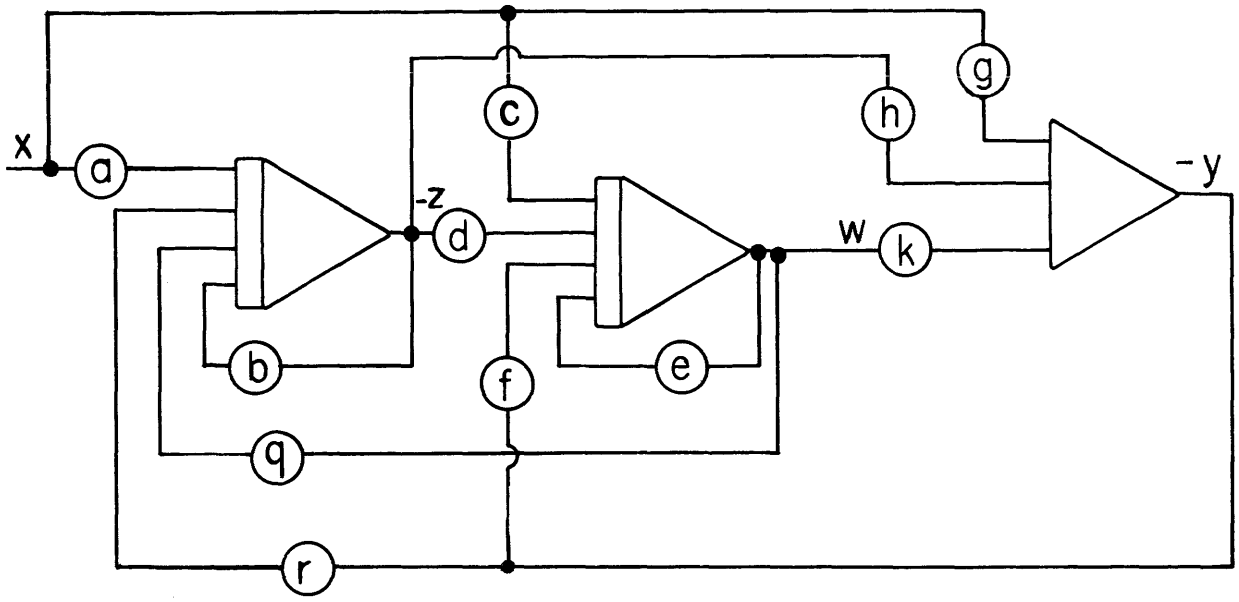
These circuits are normally accompanied by a rough Bode plot which indicates the break frequencies of the transfer function time constants, and the potentiometer-parameter interrelationships. The Bode plot provides the approximate response of the transfer function of interest in terms of asymptotes, and the relative magnitudes of its time constants.



$$\frac{y}{x} = \frac{cs + cd - ab}{s + d - eb}$$

$$\frac{z}{x} = \frac{a - ec}{s + d - eb}$$

Figure VI-4 Generalized First Order Transfer Function Circuit



$$\frac{y}{x} = \frac{gs^2 + (bg + eg - ah - ck)\ s + (beg + adk + chq - dgq - aeh - bck)}{s^2 + (b + e - hr - fk)\ s + (be + fhq + dkr - dq - bfk - ehr)}$$

$$\frac{z}{x} = \frac{(a - rg)\ s + (rck + ae + gqf - cq - rge - kfa)}{s^2 + (b+e - hr - fk)s + (be + fhq + dkr-dq-bfk-ehr)}$$

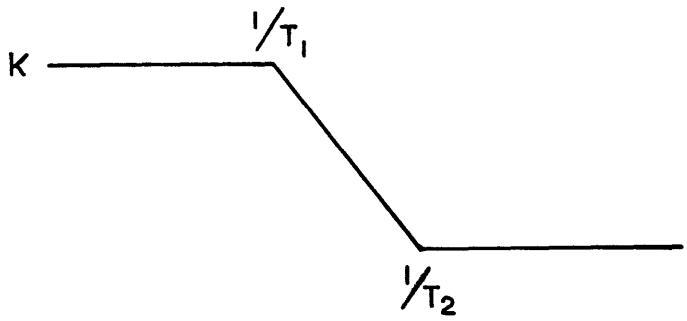
$$\frac{w}{x} = \frac{(gf - c)\ s + (gfb + rch + ad - afh - cb - rdg)}{s^2 + (b + e - hr - fk)\ s + (be + fhq + dkr - dq - ehr - bfk)}$$

Figure VI-5 Generalized Second Order Transfer Function Circuit

To illustrate the use of the generalized equations and the tabulated circuits, consider the transfer function

$$\frac{Y}{X} = K\left[\frac{T_2S + 1}{T_1S + 1}\right] = \left(\frac{K\,T_2}{T_1}\right)\left(\frac{S + 1/T_2}{S + 1/T_1}\right) \tag{66}$$

and its Bode plot



which indicates that $T_1 > T_2$. A comparison of this transfer function to the generalized equation yields

$$a = c = KT_2/T_1 \tag{67}$$

$$d = b + 1/T_2 \tag{68}$$

and

$$e = 1 + \frac{(T_1 - T_2)}{bT_1T_2} \tag{69}$$

where b is the degree of freedom which must be specified.

The transfer function for Z is

$$\frac{Z}{X} = \frac{KT_2/T_1\left[(T_2 - T_1)/T_1\,T_2b\right]}{S + \frac{1}{T_1}} = -\frac{\frac{K}{b}\left[1 - T_2/T_1\right]}{T_1\,S + 1} \tag{70}$$

Therefore, the maximum value of z can be estimated from

$$Z_{MAX} = \frac{K}{b}\left[1 - \frac{T_2}{T_1}\right] X_{MAX} \qquad (71)$$

The degree of freedom, b, which controls the magnitude of the output of the integrator must be selected to insure that **Z** is properly scaled.

Second order transfer functions are treated in a similar manner. However, in the interest of brevity, they will not be discussed at this time.

E. Isolated-Parameter Transfer Function Circuits

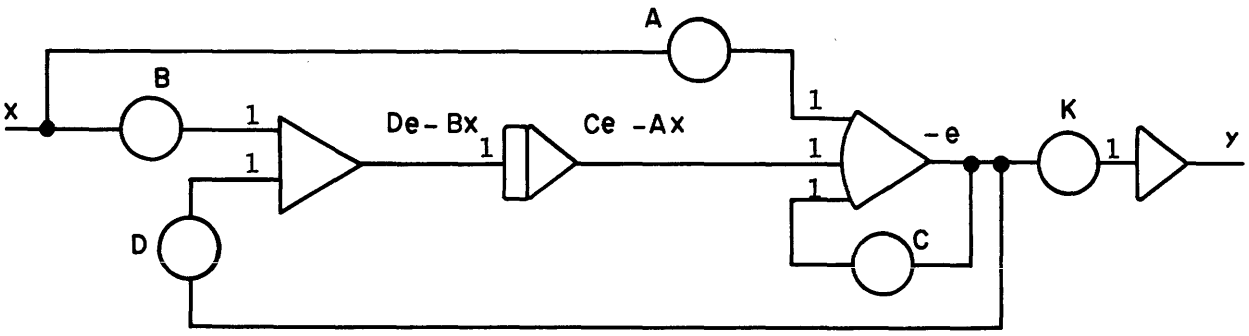
To obtain isolated parameter circuits, one need only consider the generalized transfer functions

$$\frac{y}{x} = K\left(\frac{AS + B}{CS + D}\right) \qquad (72)$$

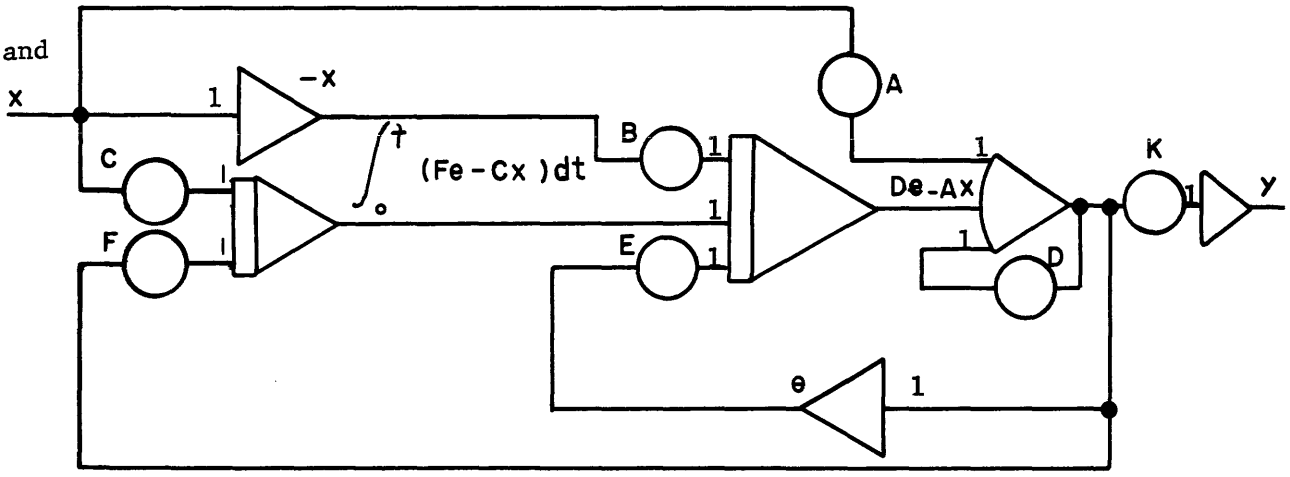
and

$$\frac{y}{x} = K\left[\frac{AS^2 + BS + C}{DS^2 + ES + F}\right] \qquad (73)$$

whose circuit diagrams are



and



where

$$y = kc \tag{74}$$

These generalized circuits must be applied intelligently if any of the coefficients are equal or zero, which leads to circuit simplifications. For example, if

$$\frac{y}{x} = \frac{k}{S^2 + aS + b} \tag{75}$$

then (by comparing equations 73 and 75)

$$A = B = 0 \tag{76}$$

$$C = k \tag{77}$$

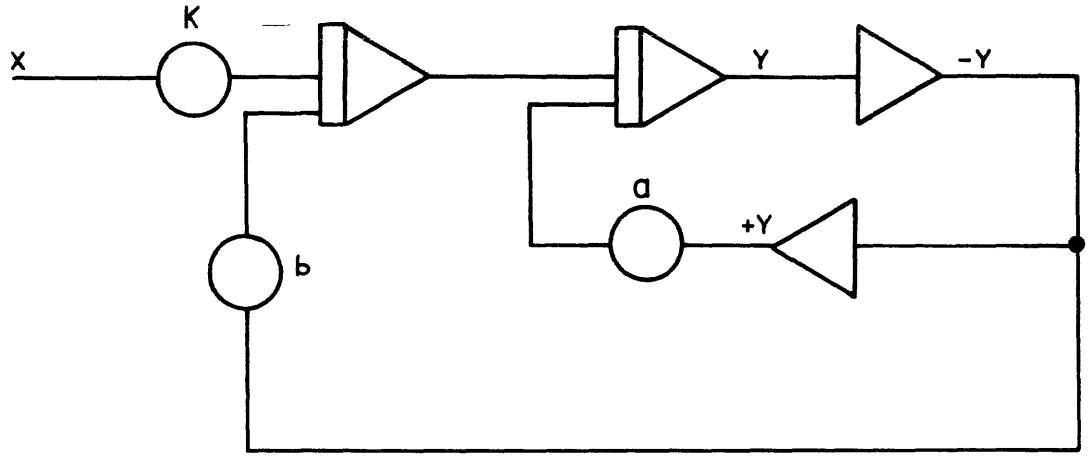
$$D = 1 \tag{78}$$

$$E = a \tag{79}$$

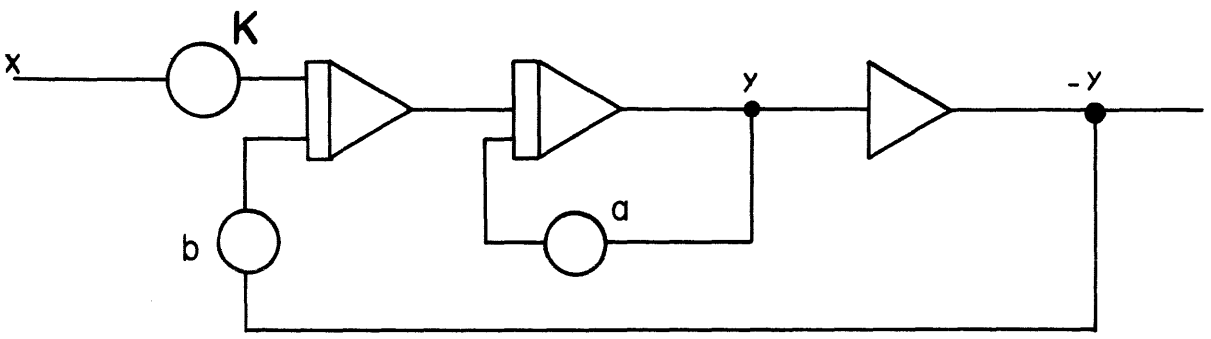
and

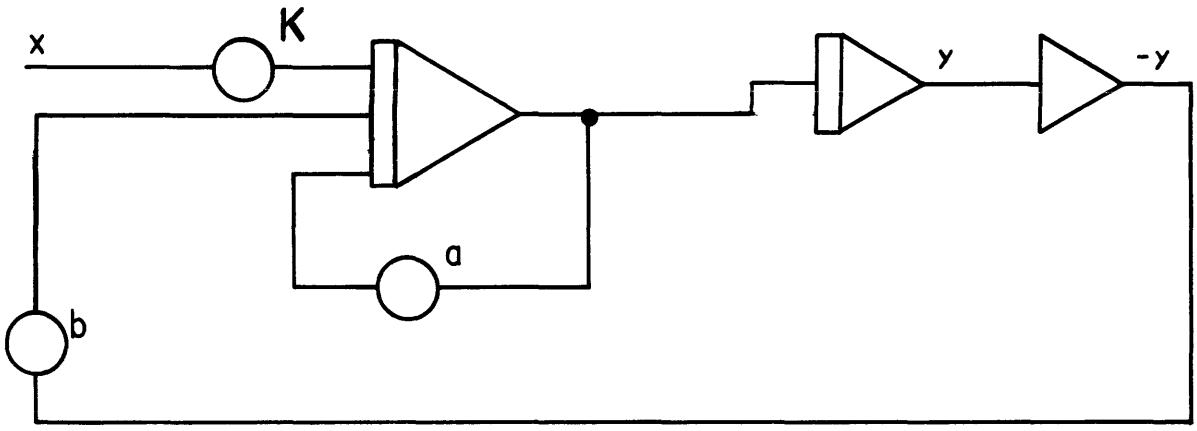
$$F = b. \tag{80}$$

The resultant circuit is



which can be simplified to





F. Scaling and Static Check Considerations

Transfer functions are scaled and static checked in the same manner as ordinary differential equations. This is readily illustrated by example. Consider the transfer function for a 2-mode pneumatic controller

$$\frac{\Delta P}{\epsilon} = K_c \left(\frac{T_R S + 1}{T_R S + \frac{1}{K_r}} \right) \tag{81}$$

where

ΔP = control output

T_R = reset time

K_r = reset gain

and

K_c = controller gain

In practice, controllers are simulated on computers to determine the optimum controller settings, K_c and T_R. The output pressure difference, ΔP, normally has fixed limits. Assume that a preliminary problem analysis indicates indicates

$$0.5 \leq K_c \leq 1.0$$

$$-7.5 \leq \Delta P \leq 7.5$$

$$K_r = 200$$

$$1 < T_R < 10$$

and

$$-0.2 \leq \epsilon \leq 0.2$$

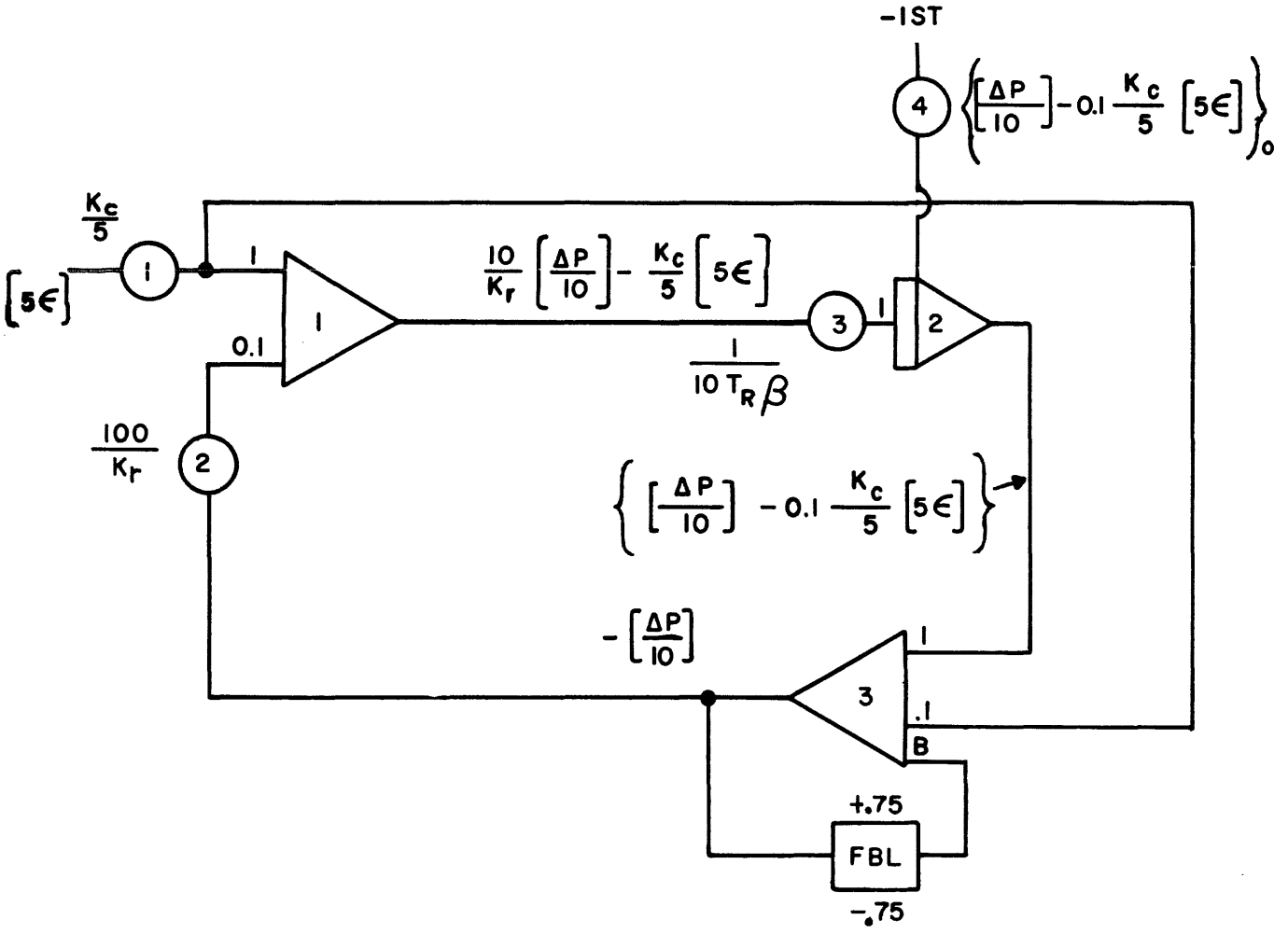
The scaled equation is

$$\left[\frac{\Delta P}{10}/5\epsilon\right] = \left(\frac{K_c}{50}\right)\left[\frac{S + 1/T_R}{S + 1/K_r T_R}\right] \tag{82}$$

or

$$\frac{d}{dt}\left[\left[\frac{\Delta P}{10}\right] - 0.1\ \frac{K_c}{5}\left[5\epsilon\right]\right] = \left(\frac{1}{10T_R}\right)\left[\left(\frac{K_c}{5}\right)\left[5\epsilon\right] - \left(\frac{10}{K_r}\right)\left[\frac{\Delta P}{10}\right]\right] \tag{83}$$

The computer diagram for equation 83 is



which can be static checked by assuming ϵ is 1/10, β and T_R are one, K_R is 200,

K_C is 3 and the integrator initial condition, $\Delta P - K_C\epsilon$, is 2. Therefore the potentiometer settings are

Pot #	Function	Setting
1	$K_C/5$	0.6000
2	$100/K_r$	0.5000
3	$1/10\ T_R\beta$	0.1000
4	$\frac{\Delta P - K_c\epsilon}{10}$	0.2000

Since the integrator output is

$$\left(\left[\frac{\Delta P}{10}\right] - 0.1\ \frac{Kc}{5}\ \left[5\epsilon\right]\right) = 0.2\ MU \qquad (84)$$

the output of amplifier 1 is

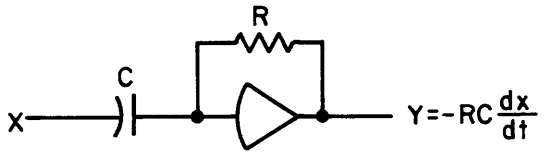
$$\left(\frac{10}{Kr}\right)\left[\frac{\Delta P}{10}\right] - \frac{Kc}{5}\left[5\epsilon\right] = \frac{0.23}{0.20} - 0.30 = 0.28\ MU \qquad (85)$$

and the check point reading of amplifier 2 is

In computing the static check for this and other transfer functions, the input and integrator outputs must be specified in order to compute the static response.

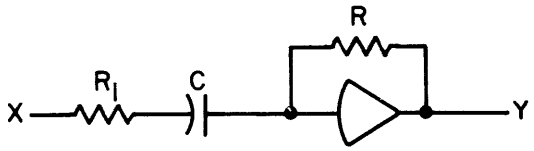
G. Differentiation Circuits

An ideal differentiation circuit is



However, it is seldom used in practice because it amplifies low level noise and hum which are present in all analog computers. If differentiation is required, several approximate differentiation circuits are available which do not have this problem.

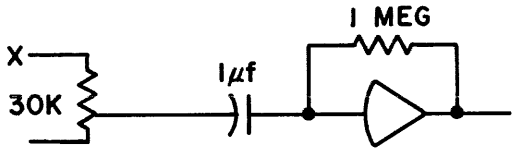
If a resistor, R_1 is placed in the input of the ideal differentiation circuit



its transfer function becomes

$$\frac{y}{x} = - \frac{RCS}{R_1CS + 1} \qquad (86)$$

which approaches a true differential circuit if R_1 is small. For effective operation, the time constant, R_1C, should be five to ten times smaller than the smallest time constant in the input. In practice, an ungrounded potentiometer can be used as R_1

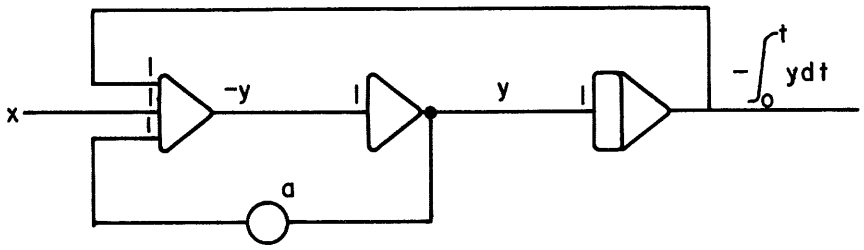


to determine the effect of its magnitude on the output by trial and error.

Another approximate differentiation circuit is obtained from the equation

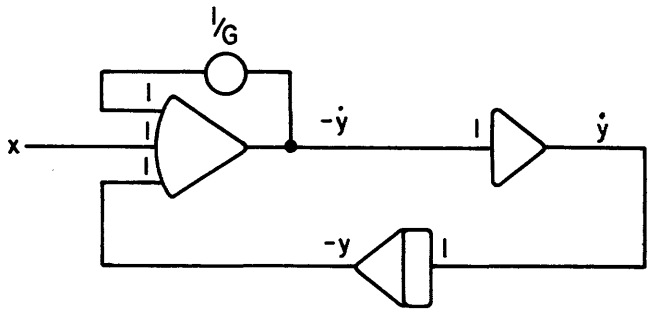
$$(1\text{-}a)\ \frac{dy}{dt} + y = \frac{dx}{dt} \qquad (87)$$

whose circuit is



As "a" approaches unity, y approaches $\frac{dx}{dt}$. The magnitude of "a" depends on the frequency of x. If "a" is too large, a high frequency input will cause oscillations in the circuit.

A final example is the circuit



If G is large and x is not a high frequency variable, than y and x will be equal and the integrator input must be $\dot{y}$ or $\dot{x}$. If G is too large, the circuit will break into oscillations; therefore, it must be determined by trial and error.

Fortunately, differentiation is not required in most simulations. It should be avoided if possible.

H. Conclusion

In conclusion, the analog computer simulation of transfer functions offers flexibility, accuracy and the elimination of frequency response problems which are adverse effects encountered using passive elements. In spite of the cost differential, computing components still represent the most practical means of simulating transfer functions, especially for short term simulations.

1. REFERENCES

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2. Jackson, A. S., Analog Computation; McGraw-Hill Book Co., Inc., New York, 1960.
3. Smith, G. N. and Wood, R. C., Principles of Analog Computations; McGraw-Hill Book Co., Inc., New York, 1959.
4. King, R. E., "A Dual Amplifier Differentiator", IEEE Transactions on Electronic Computer, Pages 120-121, February 1966.

CHAPTER VII

TRANSPORT DELAY SIMULATION

A. Introduction

This chapter describes typical analog computer applications requiring transport delay, and outlines the most common techniques for achieving it. Although it is written for readers with no previous experience with delay circuits, the circuits given in sections C and I will be of interest to readers familiar with the literature on delay approximations.

Many of the published delay circuits are either incorrect, wasteful of equipment or poorly scaled. The circuits given in this chapter have been carefully checked for accuracy, and care has been taken to optimize them from the standpoint of scaling and equipment economy.

B. Definition of Delay; Need for Delay

In many studies, it is necessary to represent on the computer a phenomenon called transport lag, dead time or delay. This need arises in many problems in wave propagation, and in studying or controlling systems containing a delay device.

Mathematically, a delay box is a device that accepts a time-varying input and produces an output which is equal to the input, but displaced in time. This property can be expressed by the equation

$$y(t) = x(t-T) \qquad \text{Eq. 1}$$

where x (t) is the time-varying input, y (t) the output and T the length of the delay. Figure VII-1 shows a block diagram of the relation between the input and the output. Figure VII-2 shows a typical plot of input and output versus time.

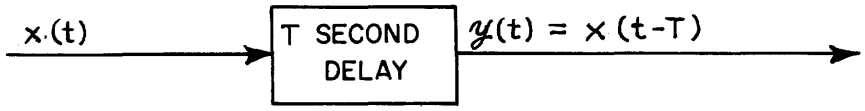


Figure VII-1

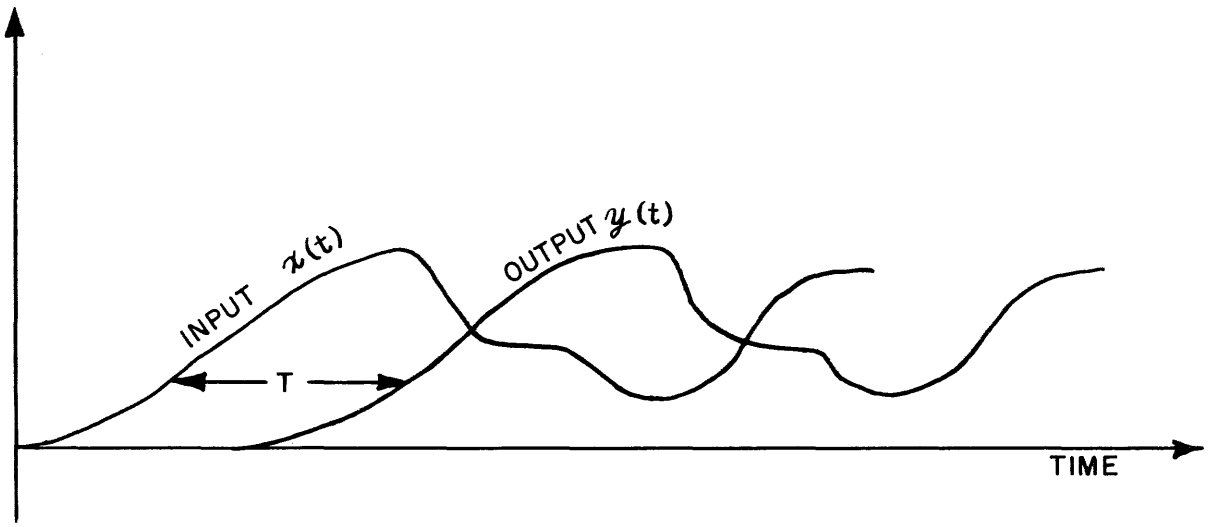


Figure VII-2

In most applications, the delay time, T, is constant but occasionally it is necessary to regard it as a function of time. Most of the circuits described below are applied most easily to fixed delay times. Variable delay times are discussed in Section I.

The need for a delay box in an analog computer arises whenever the system to be simulated contains a delay. Typical examples are:

1) Industrial processes: chemical reactors, heat exchangers, and similar systems in which fluid flows through pipes. If fluid flows with a steady velocity, V, through a pipe of length, L, and if plug-flow conditions are assumed (no mixing within the pipe), then the material that leaves the pipe at time, t, is the same material that entered it at time t-T, where T = L/V. Any variable property of the fluid, such as temperature or the concentration of some dissolved substance, will be subject to delay.

2) Wave propagation studies: radar, sonar, mechanical shock waves, etc

3) Steel-strip rolling mill control, where the thickness of the strip must be regulated. Although the thickness is determined by the position of the rollers, the thickness is not measured until some time after the strip leaves the rollers.

4) Biological studies, in which a drug is injected into the bloodstream and its effect "downstream" is measured at some later time. Here, unlike example 1), plug-flow assumptions are usually unrealistic and the "pure delay" may have to be combined with a "mixing effect", generally a first-order lag. Of course, this mixing may be included in industrial simulations also.

5) Statistical data reduction, where it is necessary to evaluate integrals such as

$$\int f(t) \cdot g(t-T)\ dt \qquad \text{(Cross-correlation)} \qquad (1)$$

$$\int f(t) \cdot f(t-T)\ dt \qquad \text{(Auto-correlation)} \qquad (2)$$

6) Design studies of suspension systems for multi-axled vehicles. In an automobile suspension system, for example, any input disturbance (say a bump in the road) that strikes the front wheels will strike the rear wheels T seconds later. Here, as in example 1), we can determine T by the equation $T = L/V$, where L is the wheel-base and V the car's velocity.

In every one of the above cases, and in many other applications, there is need for an electrical delay box, one that will accept a time-varying voltage as an input, store it, and produce a time-varying output that represents the delayed input.

C. Methods of Achieving Delay on a Computer

1. Tape Transport

Probably the most straightforward approach to the problem is to record the signal on magnetic tape and play it back into the system by means of a suitably-placed playback head. Figure VII-3 presents the basic idea. Note that the tape moves in a continuous loop. The delay time is given by $T = L/V$, where L is the length of tape between the recording and playback heads, and V is the tape speed.

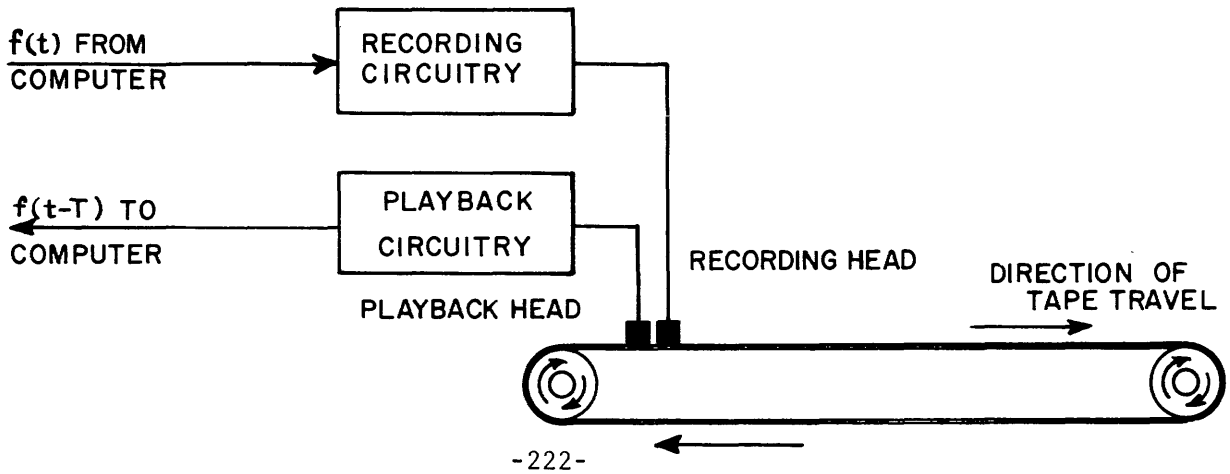


Figure VII-3

Note that the playback head is placed immediately before the recording head so that new values of f(t) are recorded on a section of tape immediately after the old values are read out.

2. Analog Point Storage--the Capacitor Wheel

Another technique is to sample the input voltage at discrete points in time, store the sampled values on charged capacitors and read these discrete values out later. This technique is illustrated in Figure VII-4 which shows a so-called "capacitor wheel." A number of capacitors are mounted on a wheel which is allowed to rotate. The voltage, f(t), is fed into one capacitor after another and a discrete value of f(t) is stored on it as the contact is broken. Note that each capacitor is read out to produce f(t-T) just before it is re-charged, to produce the new value of f(t). The output is a "staircase" approximation to f(t-T), as in Figure VII-5. In actual practice, the capacitors are generally stationary and relays are used to switch them in and out of the circuits instead of brushes, slip-rings, etc.

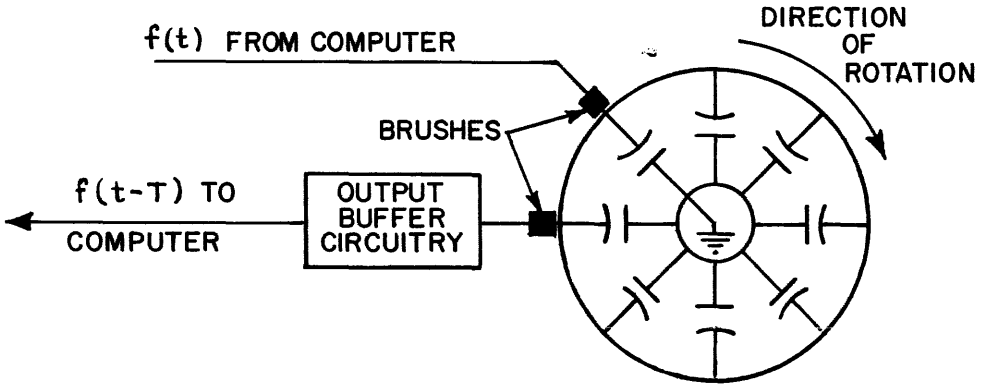


Figure VII-4 A "Capacitor Wheel"

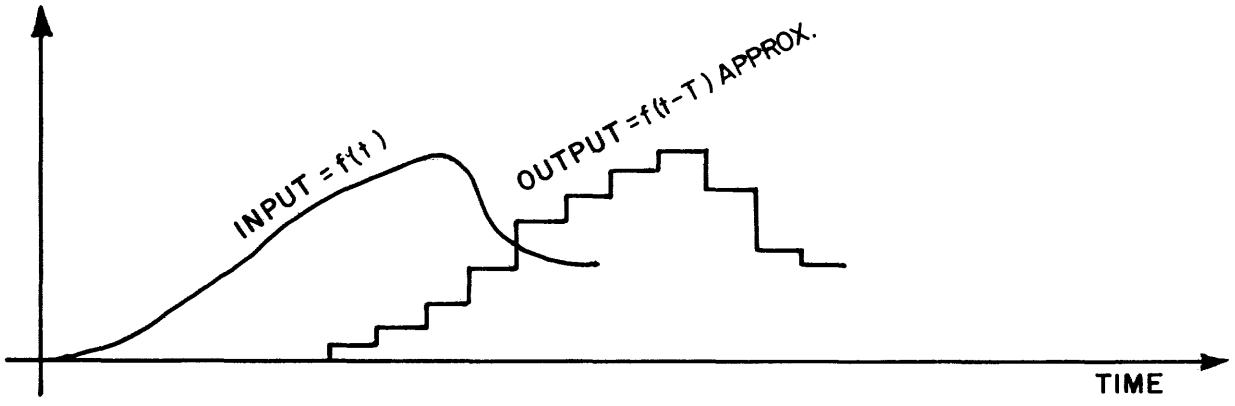


Figure VII-5 "Staircase" Output Produced by a Capacitor Wheel

3. Digital Point-Storage-- Hybrid Delay Program

An approach similar to the capacitor wheel, but using digital storage instead of capacitors, is blocked out in Figure VII-6. The analog voltage is sampled, converted into a digital number and stored in digital form. Core storage, tape, drums, delay lines or any other form of digital storage may be used At a later time, the number is brought out of storage and converted to an analog voltage, producing the output f(t-T). The output is a staircase function, as in Figure VII-5.

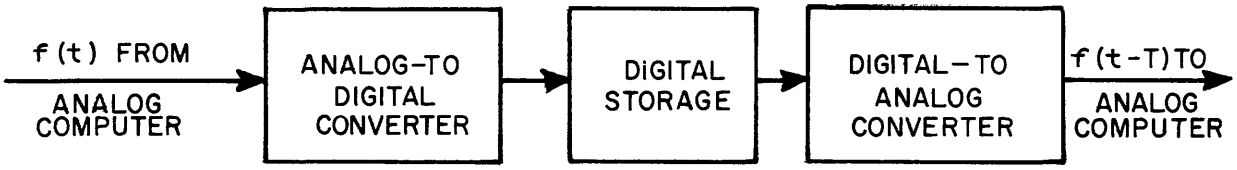


Figure VII-6

Both this hybrid method, and the capacitor wheel mentioned above, can be modified to produce linear interpolation between stored values rather than the "staircase" output of Figure VII-5. The same curve with linear interpolation is shown in Figure VII-7.

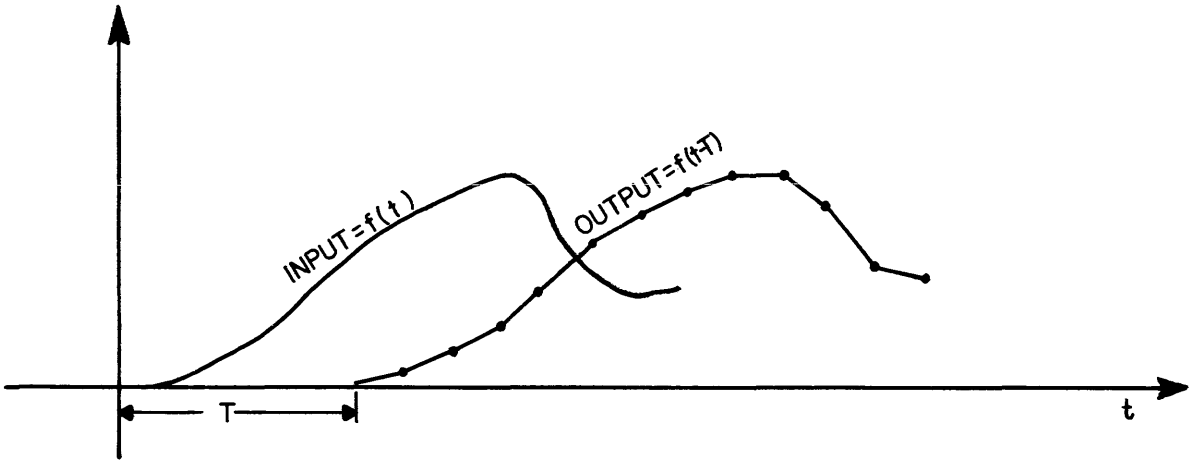


Figure VII-7 The Curve of Figure VII-5 Reproduced with Linear Interpolation

4. Curve-Follower Techniques

A delay box can be constructed from a strip-chart recorder by having an oscillograph pen produce a graph of the input, x (t). "Downstream" of the writing pen, a sensing unit tracks the variable with a delay of T seconds where T = L/V. (Here, L is the length of the strip-chart paper between the recording pen and the sensing unit, and V is the paper speed.)

To enable the sensing unit to track the curve; the pen must produce the graph in conducting ink. The curve is energized by a high-frequency signal, and the sensing unit positions itself above the curve by sensing its magnetic field. Alternatively, a photo-sensitive unit may be used and the graph may be drawn in ordinary ink (opaque, but non-conducting). In either case, the sensing unit must be connected to a suitable transducer (such as a linear potentiometer) to produce an output voltage proportional to its position.

This method is clumsy to set up, is limited in frequency response and requires a considerable amount of expensive and "touchy" special-purpose equipment, but it represents one approach to the problem that has been tried. A modification of this procedure involves having an operator position the transducer manually, thus eliminating the need for an electrical or photo-sensitive sensor.

5. Relay Circuits for Inputs of Known Form

If the signal to be delayed is an input signal (independent of the signals generated on the rest of the computer), and if it has a simple known form (a step, a ramp or a sinewave), then relay comparators (or solid-state switches, if available) provide the easiest and most accurate solution.

An example would be a step response study of the automobile suspension system described above. If it is desired to have the same disturbance (a step) strike the rear wheels T seconds after striking the front, a comparator can be used to introduce the desired step after T seconds of computer operation.

If ramps or sinewaves are the inputs, these can be generated by integrators in the usual manner. The comparator can then be used to put these integrators into the OPERATE mode after T seconds of computer operation.

6. Padé and Padé-Type Approximations

A technique which involves only standard analog computer components (integrators, pots, summers, and inverters) is the so-called Padé approximation. In this approach, the delay box is represented by a linear transfer function

$$\frac{Y\ (s)}{X\ (s)} = e^{-Ts} \qquad (3)$$

This transfer function can be derived easily by means of Laplace transforms, or by making use of the fact that the amplitude response for a sinusoidal input must be 1, and the phase shift for all frequencies must be ωt

No electrical network has this transfer function exactly, but it can be approximated. Since the most general transfer function that can be represented on the analog computer is a rational fraction, it is desirable to approximate the exponential function as a rational fraction, that is, as a ratio of two polynomials. The higher the degree of the polynomials, the better the approximation and the more amplifiers will be required to achieve the transfer function.

Four of the most commonly used transfer-function approximations to the exponential function are given below (the subscript refers to the order of the transfer function):

<u>first-order Padé approximation</u>

$$P_1(s) = \frac{1 - Ts/2}{1 + Ts/2} \tag{4}$$

<u>second-order Padé approximation</u>

$$P_2(s) = \frac{1 - (1/2)Ts + (1/12)\,T^2s^2}{1 + (1/2)Ts + (1/12)\,T^2s^2} \tag{5}$$

<u>fourth-order Padé approximation</u>

$$P_4(s) = \frac{1 - (1/2)Ts + (3/28)\,T^2s^2 - (1/84)\,T^3s^3 + (1/1680)\,T^4s^4}{1 + (1/2)Ts + (3/28)\,T^2s^2 + (1/84)\,T^3s^3 + (1/1680)\,T^4s^4} \tag{6}$$

<u>fourth-order Stubbs-Single approximation</u>

$$S_4(s) = \frac{1 - (1/2)Ts + (15/134)\,T^2s^2 - (13.55/1072)\,T^3s^3 + (1/1072)\,T^4s^4}{1 + (1/2)Ts + (15/134)\,T^2s^2 + (13.55/1072)\,T^3s^3 + (1/1072)\,T^4s^4} \tag{7}$$

Figures VII-8 through VII-11 are recommended circuits for obtaining these transfer functions.

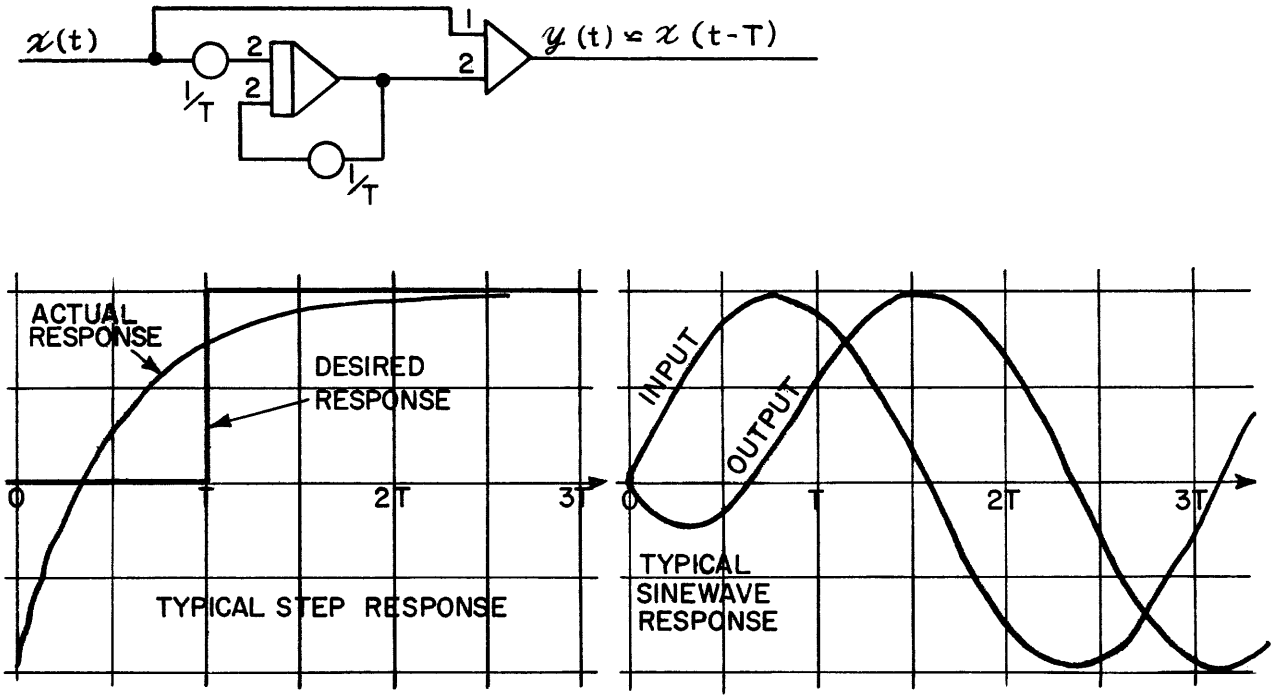


Figure VII-8 Circuit for the First-Order Padé Approximation, with Typical Responses

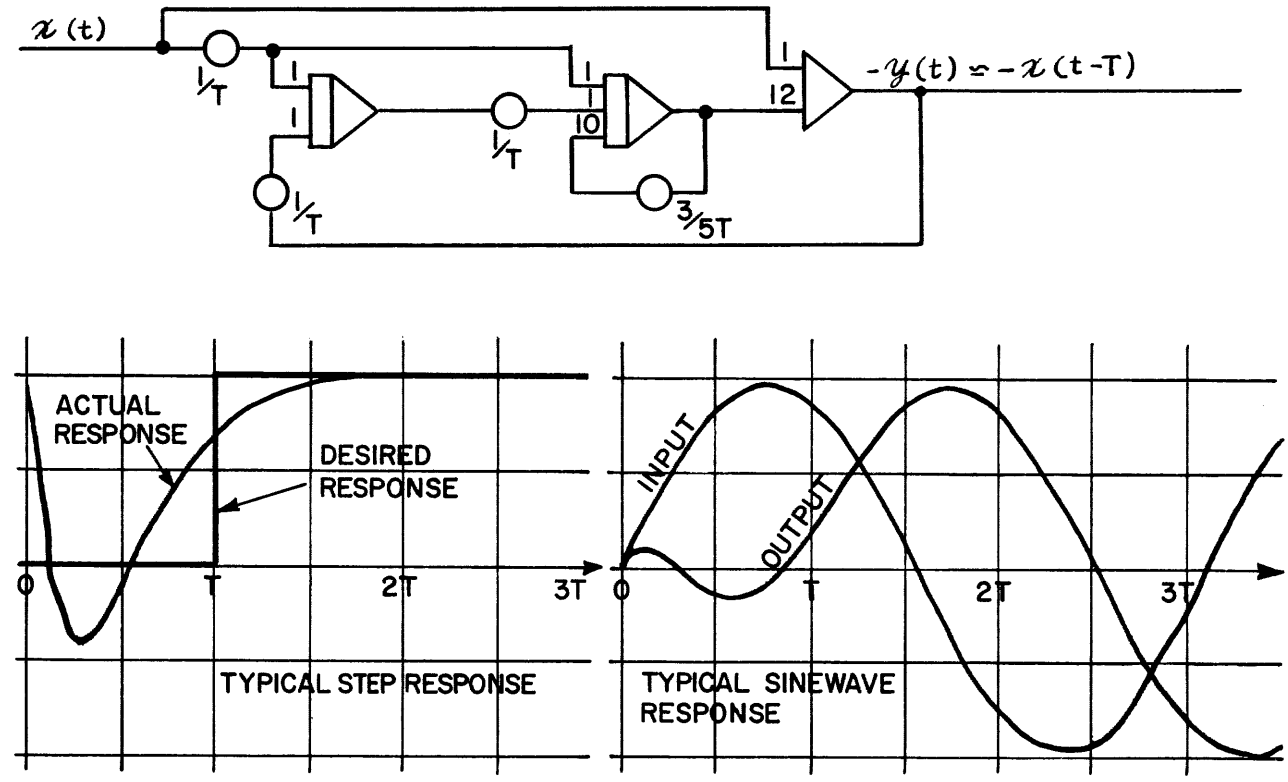


Figure VII-9 Circuit for the Second-Order Padé Approximation, with Typical Responses

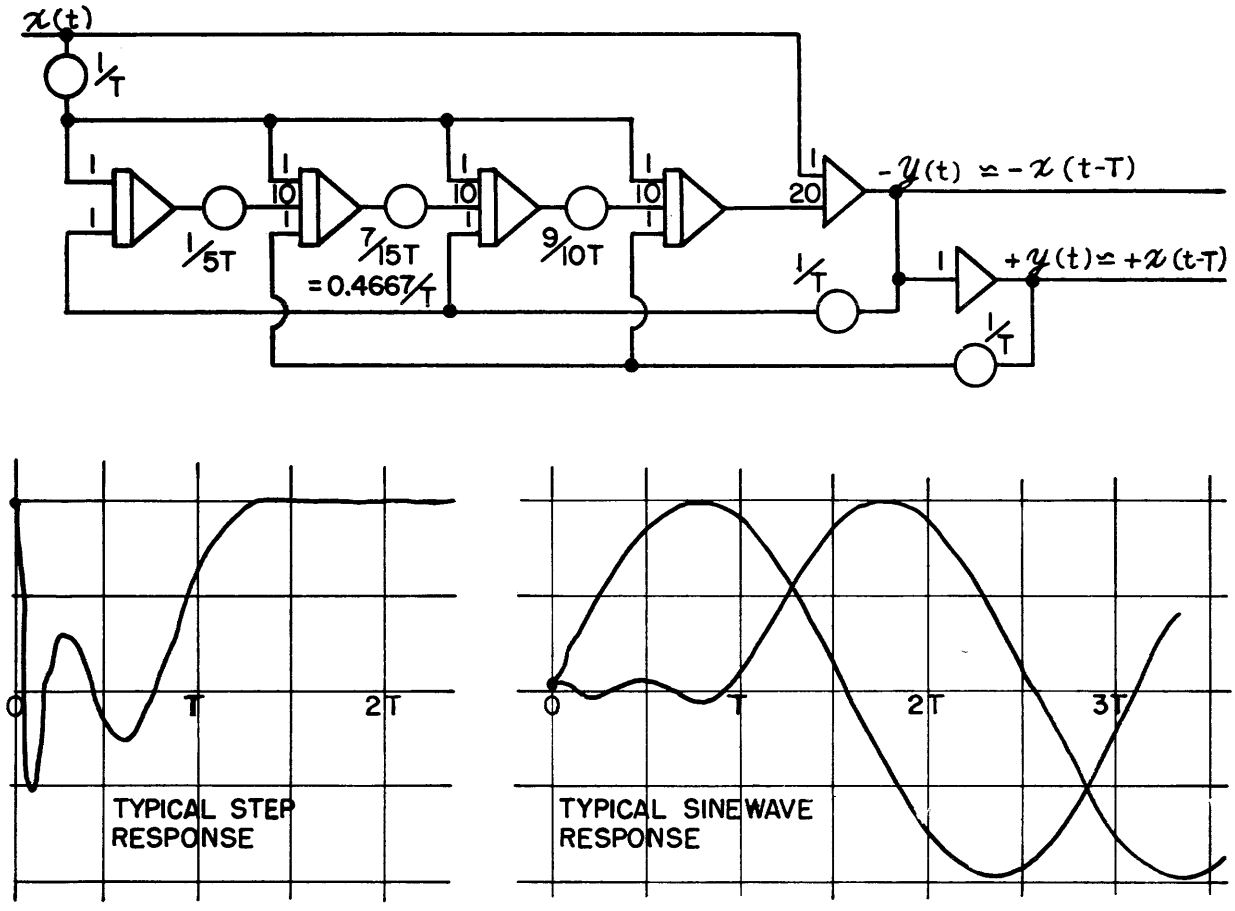


Figure VII-10 Circuit for the Fourth-Order Padé Approximation, with Typical Responses

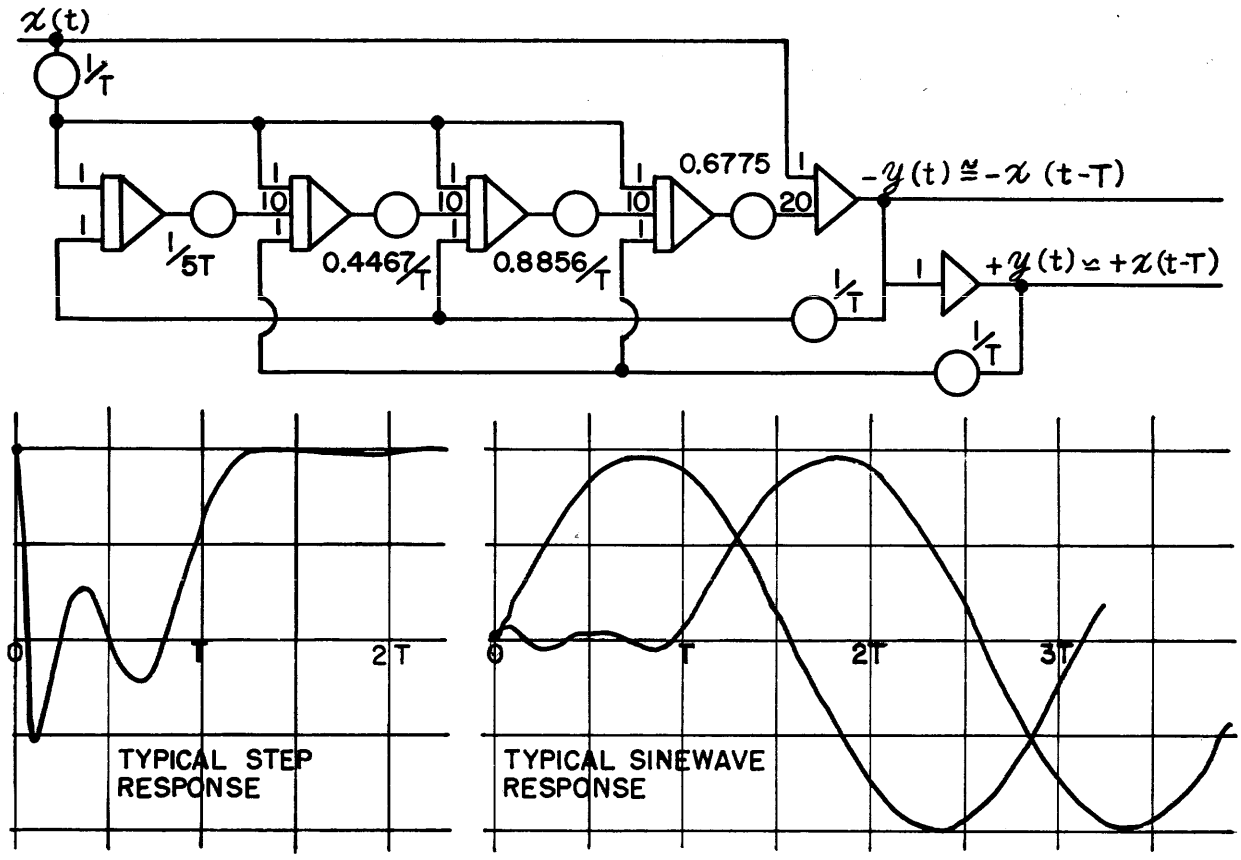


Figure VII-11 Circuit for the Fourth-Order Stubbs-Single Approximation, with Typical Responses

These fractions are all good approximations to the exponential function for small values of Ts (low frequencies and/or short time delays). Of course, the higher-order approximations are better. In each case, dividing the numerator by the denominator will produce a power-series in Ts that closely approximates the power-series for the exponential function:

$$e^{-Ts} = 1 - Ts + (Ts)^2/\,2! - (Ts)^3/\,3! \qquad (8)$$

For the first-order Pade approximation, the first term to differ from the corresponding term in the exponential series is the fourth term (the term involving $(Ts)^3$.).

For the second-order Padé approximation, the first term to deviate is the sixth. For the fourth-order Padé approximation, the first deviation is in the tenth term. Higher order Padé approximations are seldom used since the number of amplifiers involved becomes prohibitive.

The coefficients of the Padé approximation are defined and calculated so that the greatest possible number of terms in the quotient will agree with the corresponding terms in the exponential series. This gives the best possible fit to the exponential function for small values of Ts (low frequencies or short term delays). The coefficients of the Stubbs-Single approximation (Ref. 1) are close to the Padé coefficients, but they have been slightly modified. The result is an approximation that is not quite as good as the Padé for small values of Ts, but is slightly better for larger values (see the section on accuracy, below).

The fourth-order Padé and Stubbs-Single approximations are very close, as inspection of the co-efficients indicates. The step and sinewave responses are very similar. Note, however, that the true Padé circuit uses one less pot.

7. Scaling

If the input, x(t), to the delay box is a properly scaled voltage, then the output, y(t) = x(t-T), is properly scaled also. It is, therefore, not necessary to worry about the output scale factor if the input is properly scaled.

As for the outputs of the integrators, they must be scaled to achieve reasonable pot-settings and integrator gains, and to prevent amplifier overloads. The recommended circuits accomplish this. For time delays greater than 1 second, all pot-settings are ≤ 1.0. For time-delays less than 1 second, the 1/T pot-settings become greater than 1, and additional gain is required on the integrators. The easiest way to do this is to change the feedback capacitors. A ten to one decrease in feedback capacitors on all integrators will extend the range down to 0.1 second.

Each of the recommended circuits has been tested on a computer with full-scale step inputs, sinewave inputs of various frequencies and a sufficiently wide variety of other inputs to assure that the integrators will not overload in a practical problem with a properly scaled input.

The reader should be warned that there are a number of poorly scaled Padé circuits in the literature. Figure VII-12 gives an example of such a circuit. This circuit is obtained from the circuit of Figure VII-10 by

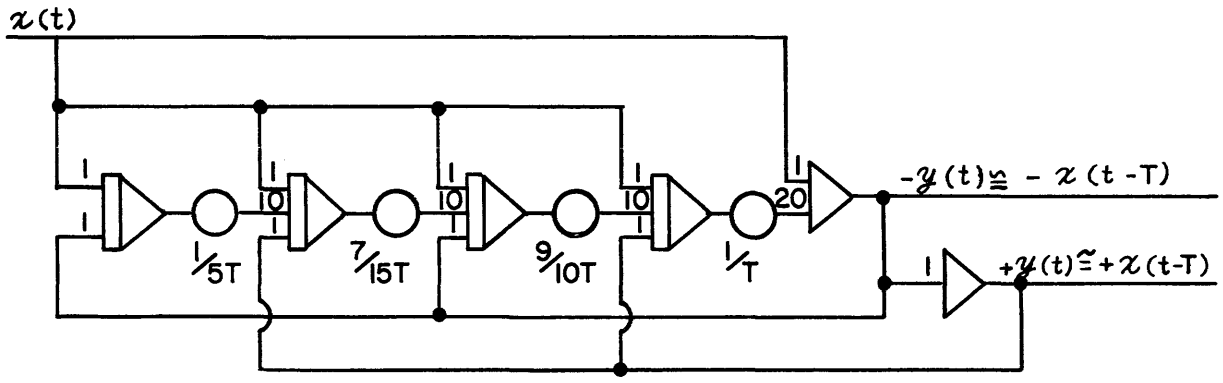


Figure VII-12 Fourth-Order Pade Circuit similar to many found in the literature. Although it uses two fewer pots than the recommended circuit, it is very poorly scaled and overloads can occur even with properly scaled inputs.

"pulling the factor 1/T through the integrators". In other words, each integrator output is re-scaled by a factor of T. The circuit uses two fewer pots and is simpler to patch (no pot feeds more than one amplifier). However, it is almost impossible to avoid overloads with properly scaled inputs. For example, it is not hard to show that, if T = 3 seconds, the steady-state output of the first integrator is 3 times the steady-state value of the input. The steady-state output of the first integrator is directly proportional to the length of the time delay. This circuit is not recommended.

With the recommended circuit, changing the delay time consists of changing all pot-settings in front of integrators by the same factor. Since this amounts to changing the time-scaling of the circuit, it does not interfere with magnitude scaling. The poorly-scaled circuit does not have this property since some integrators do not contain the factor, 1/T, and one of the summer inputs does. Hence, a change in the delay time changes the amplitude scaling.

D. Accuracy

All the methods listed above are most accurate when the frequency of the signal to be delayed is low and the delay time is short. The accuracy deteriorates for high frequencies and long delay times. The error is a function of the product of the input frequency and the delay time.

This is most obvious for the Padé approximation since this technique is based on approximating the exponential function by a rational fraction, and the magnitude of the error depends upon the size of the product, Ts. If we set $s = j\omega$, it becomes apparent that it is the product of the time delay and the radian frequency of the signal to be delayed that is the significant factor in determining the error. We may increase either one, provided we decrease the other proportionately, and the product, and therefore the error, will remain the same.

It is somewhat less obvious but equally true that the other methods are subject to essentially the same limitation. For instance, tape transport mechanisms are limited in their frequency response. To improve the frequency response, we can speed up the tape, which results in a proportional decrease in delay time.

As for the point storage techniques, both the capacitor wheel and the hybrid point storage program are subject to a similar limitation. The limitation here lies in the fact that these are sampled data systems and, therefore, limited in their frequency response by the sampling rate. To increase fidelity at high frequencies, we must sample more frequently; this means speeding up the rotation of the capacitor wheel or speeding up the sampling rate of the hybrid program. If the amount of storage is left unchanged (that is, the number of capacitors on the capacitor wheel or the number of words in digital memory), then the delay time is proportionately shortened.

In each case, we can increase both frequency response and delay time only by increasing the total amount of storage available in the system. For the Padé approximation, this means increasing the order of the approximation since the number of integrators in the system is a measure of the amount of information that can be stored. For the tape transport system, this means increasing the amount of tape between the recording and playback heads. For the curve follower technique, it means, of course, increasing the spacing between the writing pen and the pickup. For the point-storage techniques, it means increasing the number of capacitors or the number of words of available memory.

It should now be apparent that we can evaluate any delay device in terms of the total amount of storage of which it is capable. For the Padé or Padé-type approximations, this storage is measured in radians (the product of the frequency in radians per second and the delay time in seconds).

Typical upper limits for ωT are 0.6 radians for the first order Padé approximation, 2 radians for the second order, and 6 radians for the fourth order. The Stubbs-Single approximation is slightly worse than the 4th order Padé for $\omega T <$ about 6 radians, and slightly better for ωT between 6 and 7.5 radians. At about 7.5 radians, both approximations are poor.

With point storage techniques, the amount of storage is usually measured in words or bits. A direct translation between the number of radians available in a Padé approximation, and the number of bits or words available in a point storage approach is not possible since the number of bits required to represent a cycle of a sinewave is dependent upon the interpolation scheme used, and also upon the precision required. However, this method is capable of a much greater amount of storage than any Padé or Padé-type approximation that uses a reasonable number of amplifiers.

Note that the Padé and Padé-type approximations all show poor step response. This is not a serious drawback, since it is not very often that an input as "violent" as a step actually will be fed into a delay box in a practical problem. In most applications, step inputs occur, if at all, only at the very beginning of a problem. In this case, most of the initial "ripple" can be removed by the method of the next section.

E. Initial Conditions

Equation 1 does not completely specify the output, $y(t)$, for all $t > 0$. If $x(t)$ is defined for $t \geq 0$ only, then it should be possible to "load" the delay box with an arbitrary time - history for $0 \leq t \leq T$.

This is possible with any of the above methods. In most cases, it is fairly obvious what has to be done: record an arbitrary function on the tape between the heads at the start of computation, charge each capacitor in the capacitor wheel to an appropriate voltage, etc. For the Pade approximation, the desired initial time-history can be approximated by establishing appropriate initial conditions on the integrators.

However, in most cases, it is not worthwhile; most of the time it is desired merely to have the output equal to zero for $0 \leq t \leq T$. This is usually easier to accomplish than loading an arbitrary time-history, and systems like the capacitor wheel are generally provided with a method of clearing

the storage, but none for charging each individual capacitor. For the Pade approximation, putting no IC on the integrators will assure that the output remains zero for $0 \leq t \leq T$ except for a slight initial "ripple" dependent upon the behavior of the input function.

If desired, a circuit such as shown in Figure VII-13 can be used to provide an arbitrary time-history for $0 \leq t \leq T$

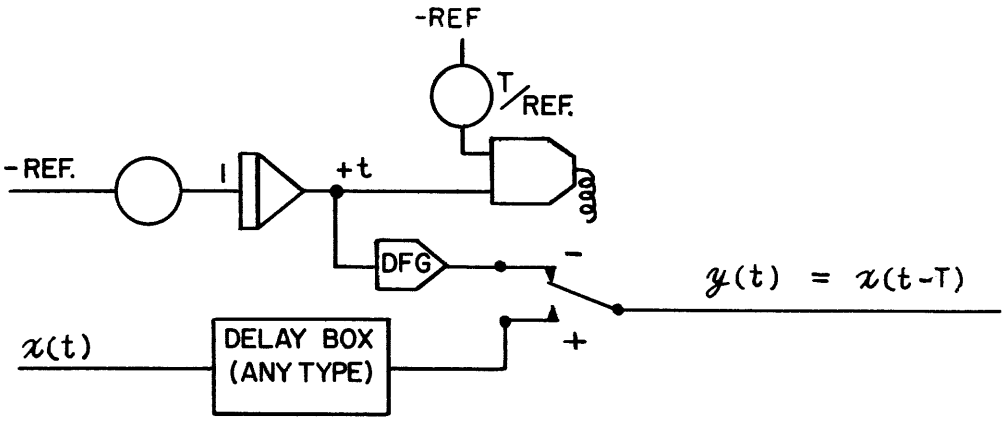


Figure VII-13

Of course, any appropriate function generation technique can be used instead of the DFG to generate the desired time-history. If the DFG is omitted and the "-" contact of the relay grounded, then this method assures $y(t) = 0$ for $0 \leq t \leq T$. This is one way to take most of the initial "ripple" out of the Padé approximation. The relay comparator may, of course, be replaced by an electronic switch if one is available.

F. Time Scaling

The modifications to be made, if the problem is not run in real time, are fairly simple. A change of time scale obviously means a proportional change in delay time. For example, if the system has 100 second time delay and the problem is run 50 times faster than real time, the delay time on the computer will be 2 seconds.

The Pade approximation is by far the simplest to implement in case the problem is to be run both in repetitive and non-repetitive operation. The necessary shortening of the delay time for repetitive operation is taken care of automatically if ordinary analog computing integrators are used, since the feedback capacitors on these integrators will change along with the feedback capacitors on every other integrator in the problem.

G. Choice Among Various Methods

Only a few of the methods mentioned above are suitable for general purpose use. The tape transport method, while it is very simple in concept, puts very stringent requirements on the behavior of the recording and playback circuitry. Tape transport mechanisms of a required quality are fairly

clumsy and expensive at the present state of the art.

The capacitor wheel offers a considerable saving in amplifiers in comparison with the Pade approximation of equivalent quality. However, it requires logic and timing signals, switches and buffer circuitry which are not easily adapted to other applications. Therefore, such devices are practical only in installations where problems involving transport delay are run fairly frequently. A dual channel capacitor memory, capable of delaying two independent electrical signals and using 16 capacitors per delayed signal, is available from Electronic Associates as an accessory for its TR/20 computer. All the necessary timing and switching circuitry is included. The sampling rate is adjustable from 10 samples per second to 1 sample per 10 seconds, offering delay times from 1.6 to 160 seconds.

Point storage techniques involving digital storage are quite accurate and economical for large problems. Programs have been written for hybrid computers utilizing linear interpolation between stored values and providing several hundred radians of storage. Although this technique is not practical for the occasional small problem, it is the only possible approach for large problems involving hundreds of radians of storage and many delayed signals.

Curve followers are generally awkward to set up, limited in frequency response and unreliable. In addition, the use of a curve follower as a delay unit ties up the mechanism of a recorder or plotter which could otherwise be used to record signals of more interest.

The relay circuits described in section C are, of course, the best approach for delaying inputs of known form. However, their area of applicability is very limited.

For the man who has only an occasional problem involving a moderate amount of transport delay and who does not want to tie up money in a special-purpose unit, the Pade approximation represents the best approach. Since it involves using only general purpose analog components which are part of the computer anyway, it should be the first method to be considered in any application. Only if sufficient components for a Padé approximation are not available, or if the problem requires too much storage for a Pade approximation of reasonable size (fourth order or less) should serious consideration be given to the other methods.

H. Practical Programming Steps

1. Determine the approximate natural frequency of the signal to be delayed. This can be determined by examining a linearized model of the differential equations to be solved, or by inspection of the computer circuit generating the signal. A good rule is to look at the integrator gains. If the problem has been properly voltage-and time-scaled, the total input coefficient on every integrator should be approximately the same as that on any other integrator. (By input coefficient is meant the product of the pot-setting and the amplifier gain.) The average of the integrator input coefficients in the main computing loop is the approximate natural frequency of the solution in radians per second of computer time.

2. Multiply the radian frequency by the delay time. Note that this resulting figure, in radians, is independent of time scaling. It is, of course, essential to be consistent. If the radian frequency has been calculated in radians per second of computer time, the delay time should also be expressed in seconds of computer time. If the radian frequency of the original system is used, then the original problem delay time should be used. In either case, the result should be the same.

3. If the total amount of storage required is less than about 6 or 7 radians, then it is practical to use a Padé approximation of fourth order or less. See section D for accuracy estimations of the various Padé circuits. If more than 6 radians of storage are required, it may be necessary to use a delay device of greater storage capability such as one of the point storage devices outlined above. If such point storage devices are not available, it is, of course, possible to use a higher order Padé approximation. However, the total number of amplifiers required for higher-order approximation is quite large. Section I gives instructions on programming a higher order Padé transfer function. It should be observed that an eighth order Padé approximation is considerably better than two fourth-order Padé approximations in cascade, and uses fewer amplifiers.

4. Check the adequacy of whatever delay device you are using on the computer. The delay device should be checked <u>while it is connected</u> as part of the circuit so that the input is the <u>actual input signal</u> from the problem itself, rather than an artificial input such as a step, ramp or sinewave. If both the input and the output are plotted versus time (this will probably take two successive computer runs), the result should be a graph such as that of figure VII-2. If the delay device is inadequate in its total storage capability, this will produce distortion in the output, and the extent of such distortion can easily be checked visually by examining the graph.

I. Variable Delay Times

In many of the problems mentioned above, the delay time can be variable. That is, T may be a function of time. For instance, in industrial processes where fluid is flowing through pipes, the flow velocity, V, may be increasing or decreasing with time as additional pumps are turned on and off. In wave propogation studies, the so-called Doppler effect is due to the fact that the time required for a sound wave, light wave or shock wave to get from one body to another is variable in case the bodies are moving with significant relative velocity. In automobile suspension studies, if a vehicle is accelerating or decelerating, then the delay time between a bump hitting the front wheels and the same bump hitting the back wheels is inversely proportional to the car's velocity.

Variable delay times can be produced, in principle, by any of the above delay devices. In tape transport delay devices, the changing delay time may be produced by changing the tape speed. Similarly, changing the speed

of rotation of the capacitor wheel or changing the sampling rate in a hybrid storage circuit will change the delay time appropriately. In all of these cases, information is flowing in and out of the storage medium at a certain rate. Changing the speed of the storage medium changes the rate at which information flows in and out. There is strong analogy between changing the rate at which information flows through the delay box, and changing the flow velocity at which the fluid flows through a pipe. It is, therefore, very clear what to do to obtain a variable delay time. How to do it is a different matter. It may prove impractical to design a tape transport mechanism that will increase or decrease the speed of tape motion on a continuous basis in response to an analog signal representing the flow velocity. On the other hand, with hybrid point storage, it requires only a slight bit extra of timing and logical circuitry to change the rate of the sampling pulses in response to an analog signal proportional to the variable flow velocity.

With the Padé approximation, the delay time is represented by a pot-setting at the input to every integrator. Replacing these pots by multipliers enables a variable delay time to be achieved. Although, in principle, any type of multiplier could be used, the presence of the same variable factor, namely 1/T, in several different places indicates a servo multiplier will be the logical choice. Since the delay time is inversely proportional to the flow velocity, we may take the voltage proportional to the flow velocity, which is coming from some other analog circuit, and feed it directly into the drive on the servo multiplier. The coefficient on each of the servo multiplier cups then will be directly proportional to the flow velocity, and therefore inversely proportional to the time. Note that the Padé circuits given in section C all involve pot-settings which are inversely proportional to the delay time but are not equal. Since the cups on the servo multiplier must have the same setting, it is necessary to modify these circuits by using unequal amplifier gains on the various integrators. This creates a loading problem, since all cups on a servo multiplier should be loaded the same way for accurate multiplication.

However, loading error will result merely in a small distortion of the coefficients of the transfer function. Since the Padé transfer function is only an approximation to the desired exponential, it is not worthwhile to worry about slight errors in the Padé coefficients (the low-order terms are important because they determine steady-state response, but the higher-order terms are often approximated as in the Stubbs-Single approximation.

Figure VII-14 gives a recommended circuit for achieving variable delay times, together with instructions for loading compensation. If these instructions are followed, there will be no loading error on the A, B and E cups. This assures that the first two terms in the numerator and denominator of the transfer function will be correct. It means, too, that the steady-state gain will be unity and the low-frequency delay will be correct, which is important to maintain the desired steady-state relationships (See reference 6).

In Figure VII-14, every cup except the C and D cups "sees" a "gain 4" load -- four gain-one resistors in parallel. The follow-up should be loaded accordingly.

Note the gain on the third integrator, fed by the C - cup. Comparison with Figure VII-10 indicates that this should be 70/15 = 14/3 = 4.667. Instead, it is 5. The gain is about 5% higher than it should be for a true Pade, but the loading error will partially compensate for this as this cup is more heavily loaded than the follow-up cup. Similarly, the input gain on the last integrator should be 9, not 10. The gain is 11% higher than it should be, but loading error will partially compensate for the too-high gain. The result is a transfer function with co-efficients only slightly different from those of the true Padé approximation.

For the purist, Figure VII-15 gives a circuit for an exact Pade approximation with no loading error. Every cup on the servo "sees" a "gain 2" load. Note, however, that the circuit uses 3 more amplifiers and two additional pots.

The "gain 2 inverters" in Figure VII-15 will not overload even if the preceding servo cups are set to a coefficient of unity, since the preceding integrators are scaled at less than 50% of reference.

Note that each fourth-order circuit uses five multiplying cups. If fewer than five multiplying cups per servo are available, then it is necessary to use two servos or a lower-order approximation.

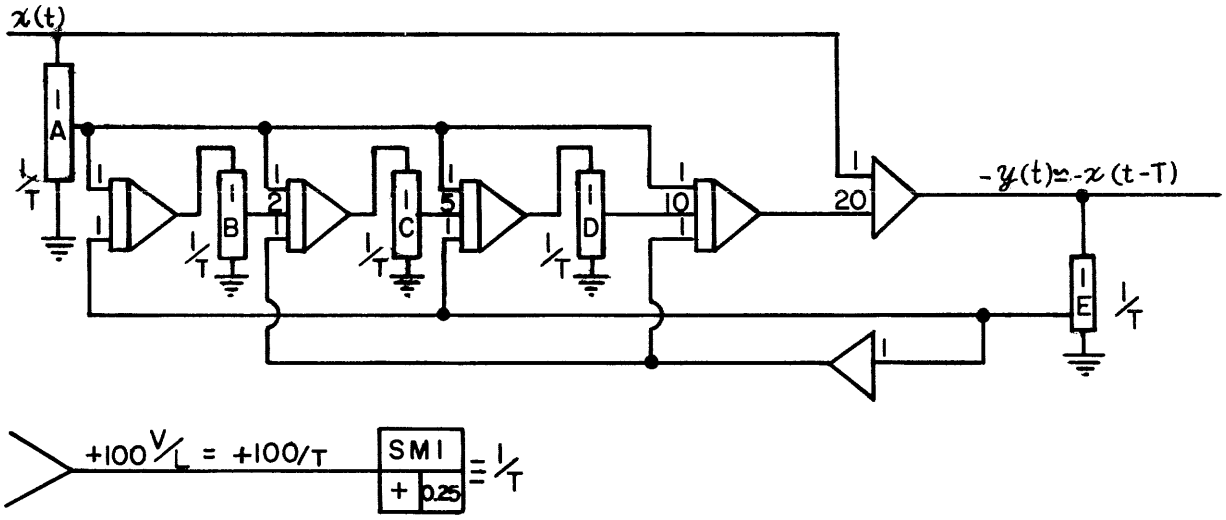


Figure VII-14. Recommended Circuit for Variable Delay Time

Loading Correction should be as follows:

Load the "B" cup with two additional "Gain 1" resistors.
Load the "E" cup with one additional "Gain 1" resistor.
Load the Follow-up cup with four "Gain 1" resistors.

All cups except the C and D cups will be correctly loaded. Loading errors on the C and D cups will be small and will not affect the response greatly. If "gain 5" input resistors are not available on a particular computer, two "gain 10" resistors in series may be used.

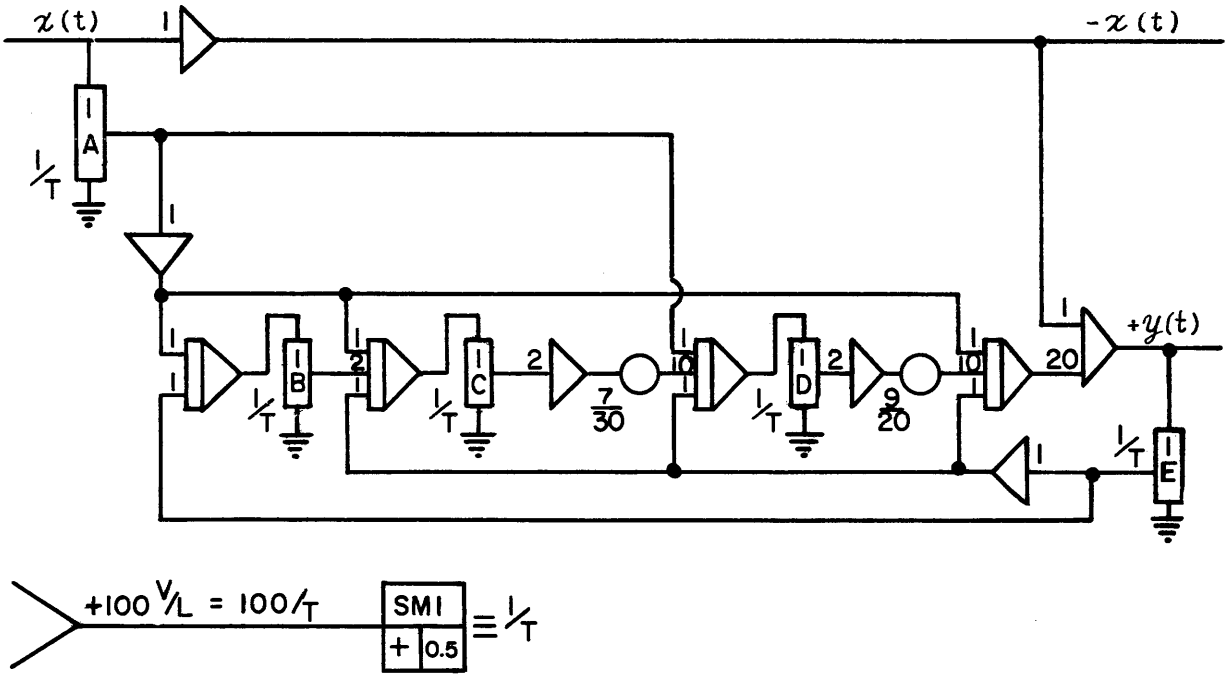


Figure VII-15. Circuit for obtaining a variable delay time without loading error. The follow-up cup should be loaded with <u>two</u> gain-one resistors.

The circuits in Figures VII-14 and VII-15 include the 1/T factor <u>before</u> the integrators. If the delay time, T, is fixed, it makes no difference, theoretically, whether the 1/T factor appears before the integrator or after it, and only scaling convenience makes it desirable to put the 1/T factors in front. If the input is scaled low enough so that no overloads occur, the outputs will be the same regardless of the location of the pots. <u>For time-varying delays, this is not true</u>.

Consider the circuit of Figures VII-16 and VII-17.

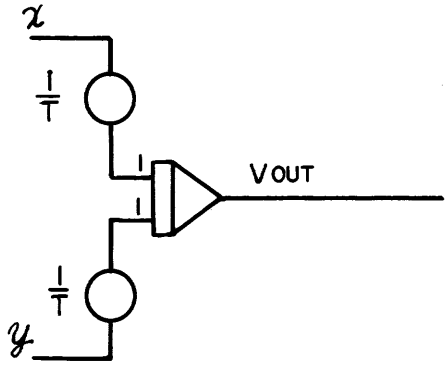


Figure VII-16

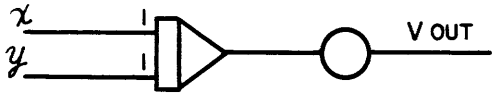


Figure VII-17

In the first case

$$V_{out} = -\int \frac{1}{T(t)} \left[x(t) + y(t) \right] \, dt \qquad (9)$$

In the second case

$$V_{out} = -\frac{1}{T(t)} \int \left[x(t) + y(t) \right] \, dt \qquad (10)$$

If T is constant, it may be brought out from under the integral sign, and the two circuits are, therefore, equivalent. If T is a function of t, the two are clearly not equivalent. It is, therefore, very important where the time-dependent coefficients are placed.

In order to decide which of the above arrangements is correct, consider the typical case of fluid flow through a pipe. If the fluid is assumed incompressible, then the flow velocity is the same throughout the pipe, and if the pumping rate is varied, the flow velocity will be a continuous function of time. The variable to be delayed is the temperature of the input stream, or the concentration of dissolved solids it contains, etc., and will also be a function of time.

If the 1/T factor, (V/L), is put before each integrator, then a change in flow velocity speeds up the integration rate of every integrator in the system proportionately. This changes the rate at which the signal flows throughout the entire circuit. This isanalogous to changing the flow velocity throughout the pipe.

If a circuit analogous to VII-12 is used, with the flow velocity factors <u>after</u> the integrators, some integrator inputs will change as the flow velocity changes and some will not. Some integrators will have <u>one</u> input that contains the flow velocity factor, 1/T, and two that do not. Instead of simply increasing its integration rate in response to an increase in flow velocity, the integrator will actually be adding and integrating these inputs with different coefficients. The result is a distortion of the input signal -- a distortion depending on the rate of increase or decrease in flow velocity. This distortion is <u>in addition to any distortion caused by the Pade approximation itself</u>. If the pots proportional to flow velocity are put <u>before</u> the integrators, then there will be no such additional distortion. In this case, if the Padé approximation is adequate for the <u>given input signal</u> and the <u>longest delay time</u> (slowest velocity) is involved, the circuit will be adequate for varying flow velocity.

J. Deriving Circuit Diagrams for the Padé Approximation

A Padé approximation for any order may be derived by writing

$$P(m, n, x) = \frac{a_o + a_1 x + \text{-----} + a_m x^m}{b_o + b_1 x + \text{-----} + b_n x^n} \qquad (11)$$

where m is the degree of the polynomial in the numerator, and n the degree of the polynomial in the denominator. We must have $m \leq n$ for a stable transfer function.

Dividing the numerator by the denominator gives a power series in x that should match as many terms of the power series for the exponential function as possible. The coefficients a_i and b_i are determined by this criterion.

Since there are n + 1 coefficients in the denominator and m+1 in the numerator, there are m + n+2 coefficients to be determined. One of these may be fixed arbitrarily (for instance, we may divide the numerator and denominator by b_n, obtaining a new transfer function in which $b_n = 1$).

The remaining m + n + 1 coefficients can be chosen so that the first m + n + 1 terms in the quotient agree with the corresponding terms in the series for e^{-x}. This means solving m + n + 1 equations for the same number of unknowns. The solution is

$$a_i = (-1)^i \frac{m(m-1) \ \ldots \ (m-i+1)}{p(p-1) \ \ldots \ (p-i+1) \ i!} \tag{12}$$

$$b_i = \frac{n(n-1) \ \ldots \ (n-i+1)}{p(p-1) \ \ldots \ (p-i+1) \ i!} \tag{13}$$

where p = m + n. The proof is long and is omitted.

Note that the numerator coefficients alternate in sign while those in the denominator are all positive. If m = n, we have

$$|a_i| = |b_i|, \tag{14}$$

and the circuit is an all-pass filter.

This formula allows Padé circuits of any desired order to be programmed, if sufficient amplifiers are available. A single Pade circuit gives better response than two smaller circuits in cascade (for example, an eighth-order Padé circuit is better than two cascaded fourth-order circuits and uses fewer amplifiers as well, if standard analog components are used).

The circuits for these transfer functions can be programmed using the general schemes on pages 14 and 15 of Reference 3. The fourth-order circuits in this chapter were programmed in this way. The first-order Padé approximation is easily recognized as a lead-lag and can be programmed as in figure 2a of Reference 3. The second-order circuit given in Figure VII-9 of this paper is adapted from a circuit in Reference 2. It uses one less amplifier than the circuit that follows from standard techniques.

K. References

1) Rogers and Connolly Analog Computation in Engineering Design McGraw Hill, 1960, Appendix 4

2) King and Rideout "Improved Transport Delay Circuits for Analog Computer Use" 3rd International Analogue Computation Meetings, Page 560

3) Hannauer, George "A Discussion and Listing of Commonly-Used Transfer Functions for Analog Programming" E & T Memo #21, Electronic Associates, Inc., Box 582, Princeton, N. J.

4) Vichnevetsky "A Particular Aspect of Dead-Time Simulation on the Analog Computer" ECC Report #37, Electronic Associates, Inc.

CHAPTER VIII

REPETITIVE OPERATION

A. Introduction

In many design studies, a large number of combinations of parameters must be investigated to optimize a system. Curve-fitting problems and boundary-value problems involve many trial-and-error runs, and only the final runs--the runs that actually match the desired curve or satisfy the boundary values--are of permanent interest. Such problems require trial-and-error adjustment of parameters and initial conditions, with the operator examining the results and re-adjusting parameters until the desired boundary value or optimum value is met. This procedure can be time-consuming if a large number of parameters are involved. A method of speeding up the computer and providing virtually instantaneous display of the solution is a great benefit in such cases.

High-speed repetitive operation provides one answer to such needs. By greatly decreasing problem solution time and cycling the computer repetitively between the OPERATE and RESET modes, a continuous trace of the results can be displayed on an oscilloscope, where the effects of any parameter changes can be observed immediately.

B. Rep-Op Circuitry

To obtain a very rapid display of problem variables for purposes of parameter exploration, curve-fitting, etc, the speed of problem solution must be greatly increased so that the solution is obtained in a matter of milliseconds, rather than seconds.

To alter the time-scale uniformly in a simulation, all integrator gains must be increased in the same proportion. This is accomplished easily by decreasing the magnitude of the feedback capacitor on all integrators and is done at the push of a button by adding a relay for capacitor selection to each integrator, as shown in Figure VIII-1.

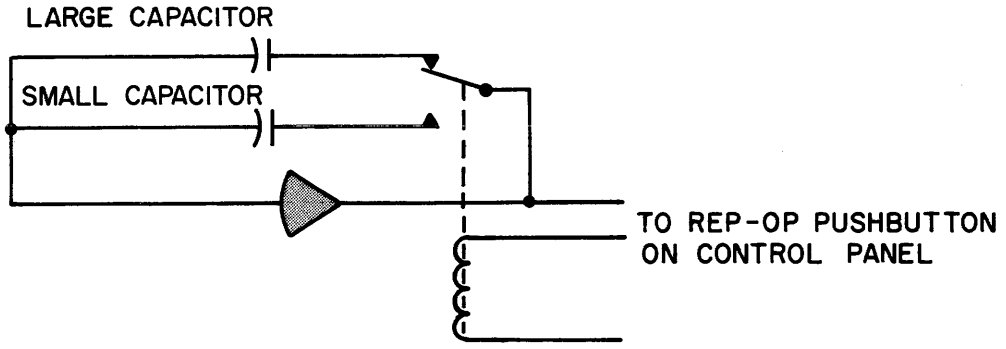


Figure VIII-1. Simple capacitor selection circuit.

The large capacitor in Figure VIII-1 is the standard feedback capacitor (normally 1 or 10 microfarads) while the other is of much smaller size (generally 100, 500, or 1,000 times smaller). Thus, switching in the smaller capacitor speeds up the problem by a factor of 100, 500, or 1,000 since the time constant, RC, of the integrator network is decreased. The specific factor depends, of course, on the design of the machine.

Figure VIII-2 shows a modification of the capacitor selection circuitry in common use. The smaller capacitor is in the amplifier feedback path all the time, and the larger capacitor is switched in parallel as required. The total feedback capacitance with both capacitors in the circuit is the sum of the two values. For example, if a 1000 to 1 speed up were desired, we might choose a value of 0.001 μf for the smaller capacitor and 0.999 μf for the larger one. When both are switched into the circuit, the total feedback capacitance is 1.000 μf. When only the smaller capacitor is in the circuit, the value is 0.001 μf. This arrangement provides feedback around the amplifier at all times, even during switching.

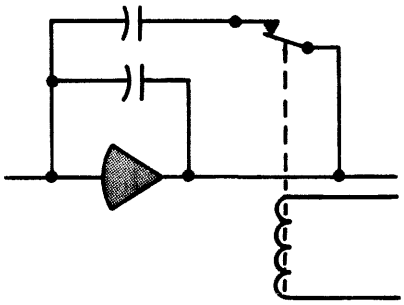


Figure VIII-2: Modified Circuitry for Capacitor Selection

With the solution time decreased by a factor of 1000, a solution that formerly took 20 to 50 seconds now will require 20 to 50 milliseconds. Such a solution is clearly too fast for servo-driven plotters and recorders, but can be observed easily on a cathode-ray oscilloscope. To provide a continuous display of the solution which will change as parameters and initial conditions are changed, we must generate the solution repetitively, that is, by cycling the computer at high speed between the OPERATE and RESET modes.

This is done by using a high-speed relay (a "chopper"), as shown in Figure VIII-3. This high-speed relay is driven by an oscillator which cycles the computer between the two modes. The reader should verify from Figure VIII-3 that when the relay is de-energized (as shown in the figure), the integrator is in the RESET mode and when it is energized, the integrator is in the OPERATE mode. By cycling all integrators back and forth between the two positions, the entire problem is repeatedly solved, reset and solved again, at a rate determined by the programmer. (Some computers use an electronic chopper in place of the mechanical chopper. The circuitry is slightly different but the principle is the same.)

The oscillator that drives the mode control circuitry normally has two synchronized outputs: a square wave to drive the choppers and a sawtooth wave to drive the horizontal axis on the oscilloscope (Figure VIII-4).

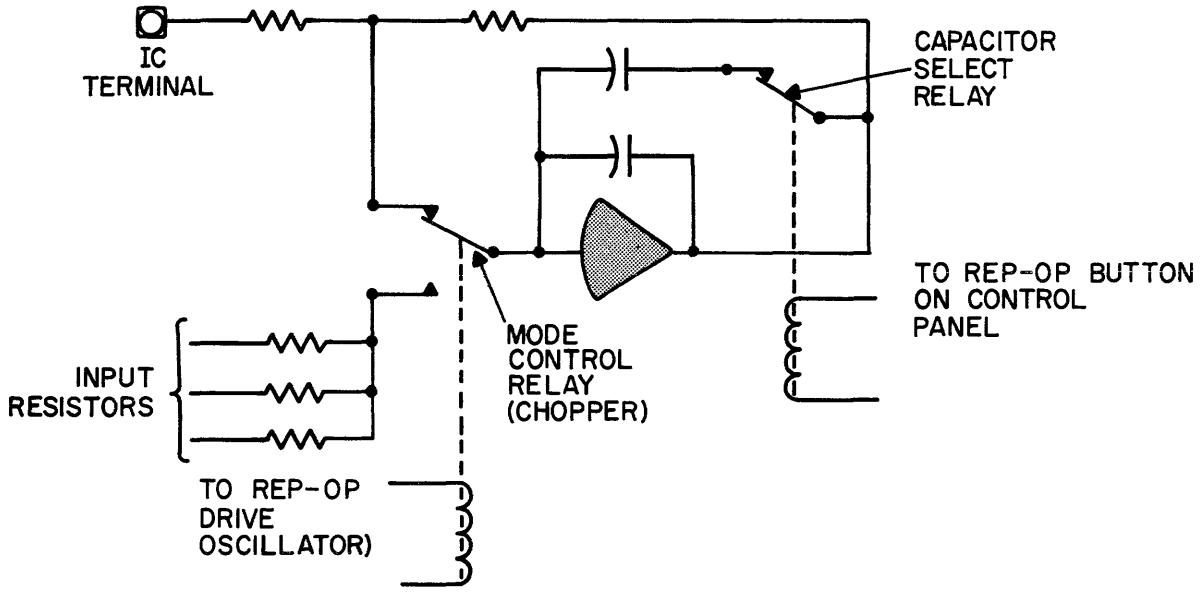


Figure VIII-3: Repetitive Operation Circuitry

The length of the OPERATE cycle is continuously adjustable by the operator, while the length of the RESET cycle is held fixed (controlled by the RC time constant of the I.C. network). The RESET cycle must be long enough to allow all integrators to return to their initial values. Thus, in a computer with an RC of one millisecond, the RESET cycle is under 10 milliseconds and the OPERATE cycle is adjustable between 10 and 200 milliseconds.

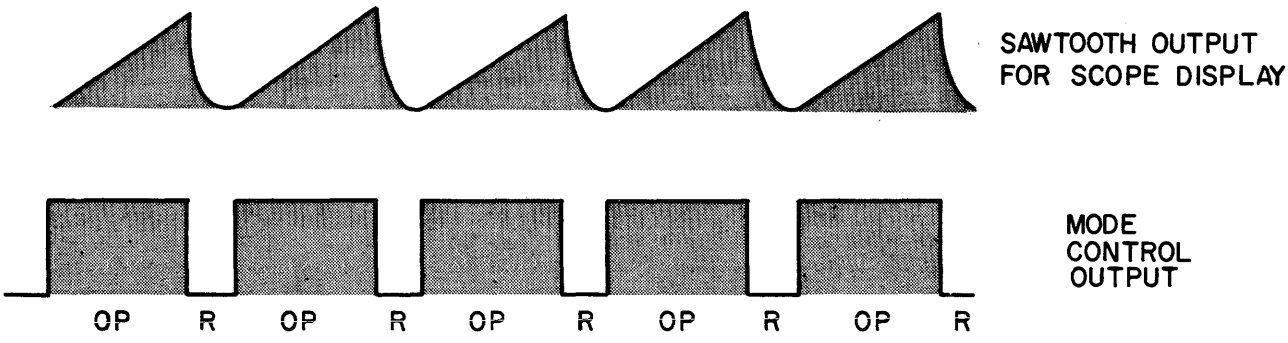


Figure VIII-4: Rep-Op Oscillator Outputs

It should be emphasized that all this circuitry is built into the computer. Generally speaking, the operator has only two controls to concern himself with: a pushbutton to put the computer into repetitive operation, and an adjustment for the OPERATE time. (This adjustment usually takes the form of a switch with several discrete settings, such as 20, 50, 100 and 200 milliseconds, and a vernier dial to bridge the gap between these values.) When the button is pushed, a graph of the desired variable versus time (or of one variable versus another) appears immediately on the face of the scope. If the problem conditions are changed by throwing a function switch or adjusting a potentiometer, the curve will appear to adjust itself instantaneously to the new conditions since a new curve is being generated every 20 to 200 milliseconds. The result is that the curve appears to shift continuously in response to a gradual parameter change.

C. Programming Considerations.

Programming a problem for repetitive operation differs very little from programming for ordinary "slow" or "real time" operation. In fact, the same problem can be run both ways, with the operator changing from "slow" operation to repetitive operation and back again at the touch of a button. The time-scale change brought about by changing capacitors is a purely mechanical change and is done automatically when the computer is put into repetitive operation.

The programmer should choose his time-scale factor for the problem in the usual manner, by examining integrator gains and dividing them all by the proper factor, so that all pot-settings are reasonable and the solution takes a "reasonable" amount of time, say 20 to 100 seconds, without repetitive operation. The automatic speedup for rep-op in no way affects this part of the programming.

There are, however, two things that the programmer should keep in mind:

1) If a problem is to run in repetitive operation, mechanical computing components, such as servo multipliers, servo resolvers, and relay comparators can not be used since they are too slow.

The program should use only electronic computing components such as quarter-square multipliers, electronic resolvers and solid-state switches which can function either in repetitive operation or in "slow" time. (Exception: certain problems can be run in repetitive operation with relay comparators. If the comparator throws only once or twice per OPERATE cycle and the operate cycle lasts 100 to 200 milliseconds, then a 1-2 millisecond relay may be fast enough. However, this is a "borderline" case and the relay may or may not be fast enough, depending on the problem. Electronic comparators, with switching times in the microsecond range, are preferable if available. Servomultipliers, incidentally, do have their uses in rep-op, as indicated below.

2) A parameter that is to be varied in repetitive operation should, whenever possible, appear on only one pot as it is difficult to change two or more pot-settings simultaneously in rep-op. For example, the circuit in Figure III-9, in which the mass m appears on a single pot, is to be preferred to the circuit in Figure III-4, in which two pots must be re-set to change the mass.

In some cases, where it appears that the parameter cannot be isolated on one pot, ganged potentiometers may be used (potentiometers mounted on a common shaft, so that all turn together). If the computer is not equipped with ganged pots, servo multiplier cups (which, of course, are a form of ganged pot) may be substituted.

The servo may be driven directly from a hand-set pot patched to reference voltage. As the hand-set pot is adjusted, all servo cups will turn in response to vary the single parameter that appears on several pots.

D. Readout

As mentioned above, the oscilloscope is the main form of readout for repetitive operation. Although other oscilloscopes may be used, best results are achieved with a unit that is built into the computer so that it is always available. Access to the built-in scope is by patching the appropriate amplifier directly into a scope-input terminal on the patchpanel; all necessary cabling is internal.

Oscilloscopes suitable for use with general-purpose analog computers have the following features:

1) Multi-Channel Display. Four-channel and eight-channel units are available, allowing several variables to be observed simultaneously. Depending upon the design of the unit, the plots may be superimposed, as in Figure VIII-5, or "split" as in Figure VIII-6.

2) Crossplot Feature. Since an oscilloscope is, basically, a device for plotting one voltage against another, it is quite easy to remove the saw tooth sweep from the horizontal drive terminal of the built-in scope and patch in an arbitrary computer voltage. Thus, the scope may be used in the same manner as an X-Y plotter to display plots of one variable versus another (Phase-plane plots, etc.)

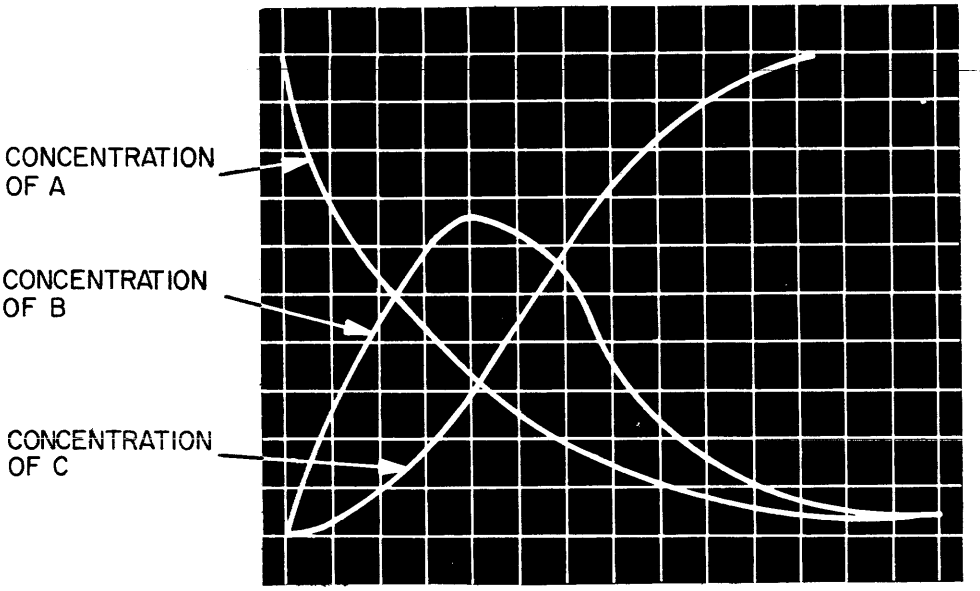


Figure VIII-5: Superimposed Display of the variables in the chemical reaction A→ B→ C as displayed on a multi-channel scope.

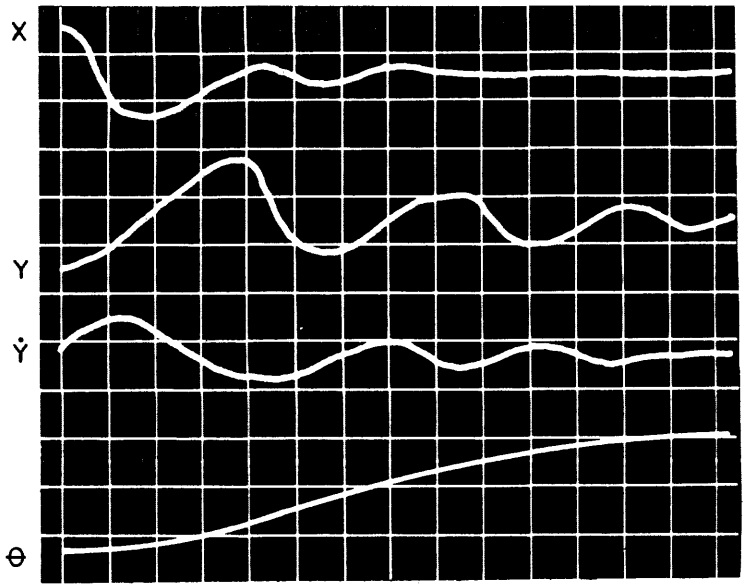


Figure VIII-6: Simultaneous "Parallel track" display of four variables in a mechanical problem, using a multi-channel scope.

3) High Persistence. A repetition rate likely to be used in repetitive operation is 50 solutions per second (10 milliseconds OPERATE time followed by 10 milliseconds RESET time). This repetition rate is much slower than the repetition rate for which most laboratory-type oscilloscopes are designed. The relatively slow repetition rate makes a high-persistence display desirable and practical. A typical unit can easily be read at a distance of 10 feet or more in a brightly lighted room; there is no need for dimming lights, or using masks or shields.

4) Accuracy of Display. Display accuracy is typically 1% or better.

5) Reset Blanking. Most units have built-in circuitry to dim or turn off the electron gun while the computer is resetting, eliminating the "flyback trace" that would otherwise be displayed.

For curve fitting purposes, the empirical curve to be matched may be sketched on the face of the scope with a grease pencil, or a few individual data points may be individually marked on, using the grid lines as a guide.

In addition to oscilloscope readout, x-y plotters and voltmeters may be used in conjunction with track-store units to read out certain repetitively-generated signals.

E. Operating Procedure

The recommended operating procedure is quite straightforward:

1) Program in the normal fashion, as described above.

2) After patching and static check are complete, make a run in "slow" time with typical parameter values and record significant variables on an x-y plotter.

3) Put the computer into rep-op and adjust the compute time as desired. Compare the rep-op solution with the solution produced on the plotter. This checks the operation of the rep-op relays and allows the operator to verify that the rep-op solution matches the slow speed solution.

4) Perform the wide-range parameter exploration, curve-fitting, optimization, etc. by varying potentiometers and observing results on the scope.

5) Return to "slow" time for permanent recording of specific solutions on the x-y plotter.

CHAPTER IX

ANALOG MEMORY

A. Introduction

Memory, or storage -- the ability of a computer to "remember" the results of past computations -- is a concept more frequently associated with a digital, rather than an analog computer. However, the inclusion of a moderate amount of storage capability in an analog computer can greatly increase its problem-solving capacity.

At the outset, we should distinguish between point storage (the storage of a single number, or a few numbers) and curve storage (the storage of an entire function). The two are closely related -- curve storage is usually accomplished in practice by storing a large number of points and interpolating between them. However, the amount of storage required and the complex switching circuitry necessary for interpolation make curve storage an order of magnitude more difficult than point storage. This chapter will be limited to point storage.

B. Typical Storage Applications

Typical classes of applications for storage techniques include the following:

1. Automatic Iteration (trial-and-error)

Here, a problem limitation must be met such as a boundary-value, or end-value, or a maximum or minimum condition. If the computer is to perform the trial-and-error computations automatically, it is necessary to provide the computer with some method of "remembering" the results of previous computations.

2. Parameter Sweep

As another example of a storage application, consider the effect of an adjustable parameter α on an analog computer solution. For example, one may want to study the steady-state error of a position servo as a function of controller gain or some other control parameter. On the other hand one may be interested in the miss distance of a missile as a function of initial launching error. In either case, one is interested in a single number, X, (steady-state error, miss distance, etc.) which is computed during an OPERATE cycle using a particular value of some parameter, α, (controller gain, launching error, etc.). A value of α is chosen at the beginning of the OPERATE cycle, and the corresponding value of X is computed. Usually X will be a final value, i.e. the output of some component at the end of the OPERATE cycle. It is desired to obtain a plot of X versus α.

Obtaining such a plot manually would be quite time consuming. The operator would have to choose several values of α, change pots manually between runs, and tabulate the result, X, after each run. He could then make a point-plot of X versus α and connect the points by a smooth curve. The procedure outlined below replaces this

tedious work with a single computer run in which the desired graph is produced automatically on an X-Y plotter. The necessary circuit is described in section C.

3. Multi-Speed Computation

In the parameter-sweep examples, the high-speed circuit behaves very much as a function generator, accepting one or more slowly-varying inputs while producing a stair-step output which approximates a continuous function of the inputs. These function-generation problems are characterized by the fact that an entire computer run is necessary to produce one value of the function. If an entire curve, or family of curves, is desired, many runs must be made, and this is very time-consuming unless the runs are made at high speed.

In the circuit of Figure IX-7, the high-speed "function generator" is used merely for plotting purposes. The input variable is simply swept slowly through its range to provide a graph of the output X versus the input α. In the case of a function of several variables, one variable may be swept through its range while the others are held constant, and the result will be a family of curves (one curve for each slow-speed sweep).

There is no reason however, why the use of this function generation technique must be limited to simple plotting. It may also be used when the function is needed as part of a dynamic simulation in which the inputs to the high-speed rep-op circuit (the low-speed variables) are being generated by solving a low-speed system of differential equations. In this case, the rep-op circuit plays the same role as any function generator in a dynamic simulation: it accepts varying inputs and produces one or more of the output variables which are a necessary part of the dynamic loop.

Since an entire differential equation must be solved to obtain a single function value, the high-speed circuit is capable of solving problems that can not be solved easily by any other function generation technique. On the other hand, it may be possible to solve the high-speed equations analytically and, thus, replace the high-speed circuit with conventional function - generators to implement the analytical solution. In this case, the high-speed circuit may be considered simple as a substitute for a conventional function generator. Examples of both cases are given below in section D.

C. Storage Circuits

Since an analog signal is a voltage, it may be stored by using it to charge a capacitor. Readout of the stored information must take place without drawing appreciable current from the capacitor since drawing current would partially discharge the capacitor, and result in an erroneous reading. This requirement for "nondestructive readout" implies that the use of an amplifier will be necessary.

1. Simple Track/Store Unit

A simple track-store unit is shown in Figure IX-1. It may be readily assembled using an amplifier, its resistor network and a relay comparator. Alternatively, an integrator IC network may be used. Many computers contain packaged track/store units with a capacitor, an amplifier and a built-in electronic switch.

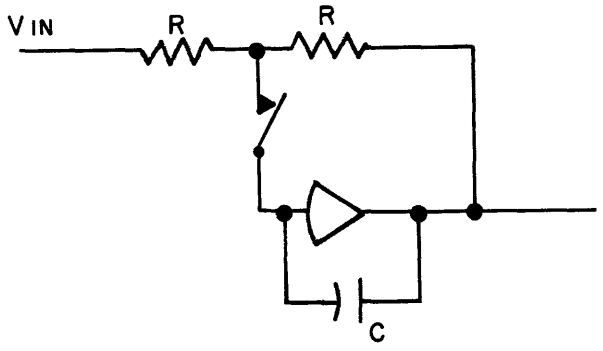


Figure IX-1: Simple Track-Store Circuit

When the relay is closed, the amplifier is essentially a gain-one inverter with a short lag due to the capacitor. The transfer function is $\frac{-1}{1 + RCS}$. The time constant can be kept short by choosing low values for the resistors and capacitor. When the relay is opened, the circuit behaves like an integrator in the HOLD mode: it stores the last value it had when the relay opened.

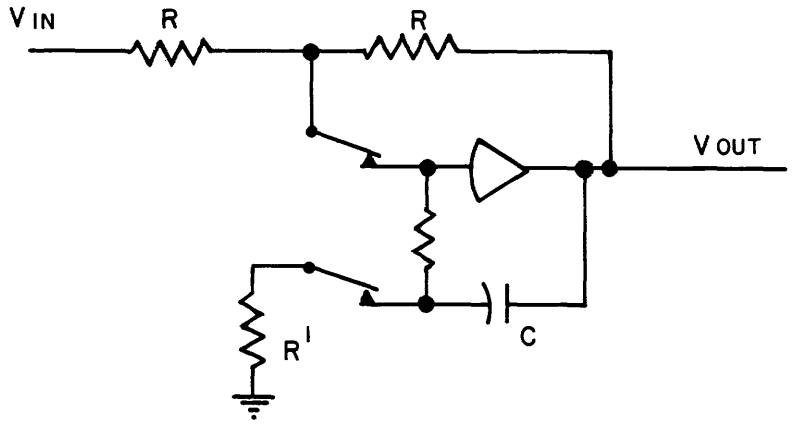


Figure IX-2: Improved Track-Store Circuit

2. Improved Track-Store Unit

Figure IX-2 shows a modified version of the track-store circuit. Grounding the capacitor by means of another set of contacts on the same relay shortens the tracking time constant which is now R'C, instead of RC. It is practical to make R' much smaller than R without loading the circuit excessively. Of course, C should be small in both circuits. If several input resistors are used, the unit can track and store the sum of several voltages without the need for an additional summer.

Modern track-store units use the switching scheme of Figure IX-2, except that the mechanical relays are replaced with solid-state switches. Both the RC time constant and the switching time are in the microsecond range. Many computers use this type of circuit for the initial condition networks of integrators as well, since it is inherently faster than the circuit in Figure IX-1.

A programmer's symbol for a track-store unit is shown in Figure IX-3. The logic input signal controls the switching between the track and store modes. The unit tracks when the logic input level is high and stores when it is low.

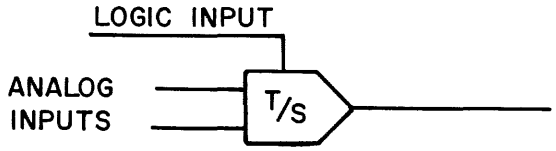


Figure IX-3. Programmer's Symbol for Track/Store Unit.

3. "Bucket Brigade"

Certain combinations of track-store circuits occur frequently. Figure IX-4 shows a circuit for storing the final values of a computer variable during high-speed repetitive or iterative operation. It is assumed that the computer is being cycled automatically between the RESET and OPERATE modes. The logic input signal driving the track-store units is generally supplied by the same square-wave source which provides the mode-control signals, but is may also be supplied by an electronic comparator.

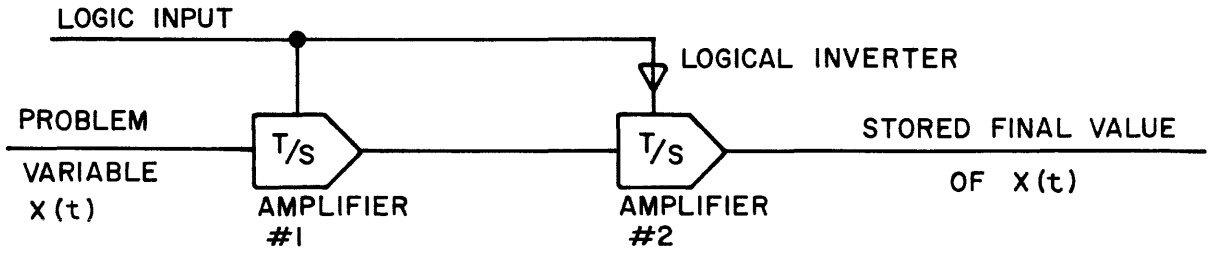


Figure IX-4. "Bucket Brigade" Circuit

Table IX-1 indicates the sequence of events. Amplifier #1 tracks the problem variable while it is being generated in the OPERATE mode. Then, when the computer is reset, the logic input level changes from 1 to 0, and amplifier #1 stores this value and passes it along to amplifier #2. Because of the way in which information is "passed along" this is sometimes referred to as a "bucket brigade" circuit. When the computer is switched back to the OPERATE mode, amplifier #1 starts tracking the variable again while amplifier #2 stores the final value from the preceding run. If the rep-op circuitry has provided for a HOLD mode to follow the OPERATE mode, amplifier #1 can be eliminated, and amplifier #2 can pick up its new value while the problem is in the HOLD mode.

Table IX-1: Sequence of events in "bucket brigade" circuits.

Logic Input	Computer Mode	Behavior of Amplifier 1	Behavior of Amplifier 2
0	RST	stores (output =0)	tracks (output = 0)
1	OPR	tracks $-X_1$	stores (output is still 0)
0	RST	stores $-X_1$ (final value)	tracks X_1 (final value)
1	OPR	tracks $-X_2$	stores X_1 (final value)
0	RST	stores $-X_2$ (final value)	tracks X_2 (final value)
1	OPR	tracks $-X_3$	stores X_2 (final value)

The subscripts refer to the successive runs (thus X_n is generated during the nth run).

4. Analog Accumulator

By adding a feedback connection to a bucket brigade circuit, an analog accumulator is obtained (Figure IX-5). The output, x, changes by an amount, Δx, each time the logic input changes from 0 to 1 and back again. The value, Δx, need not be constant.

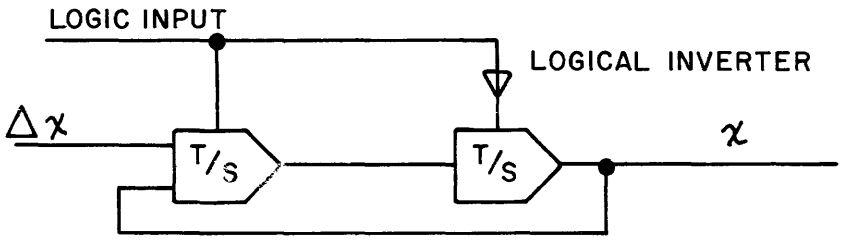


Figure IX-5: Accumulator Circuit.

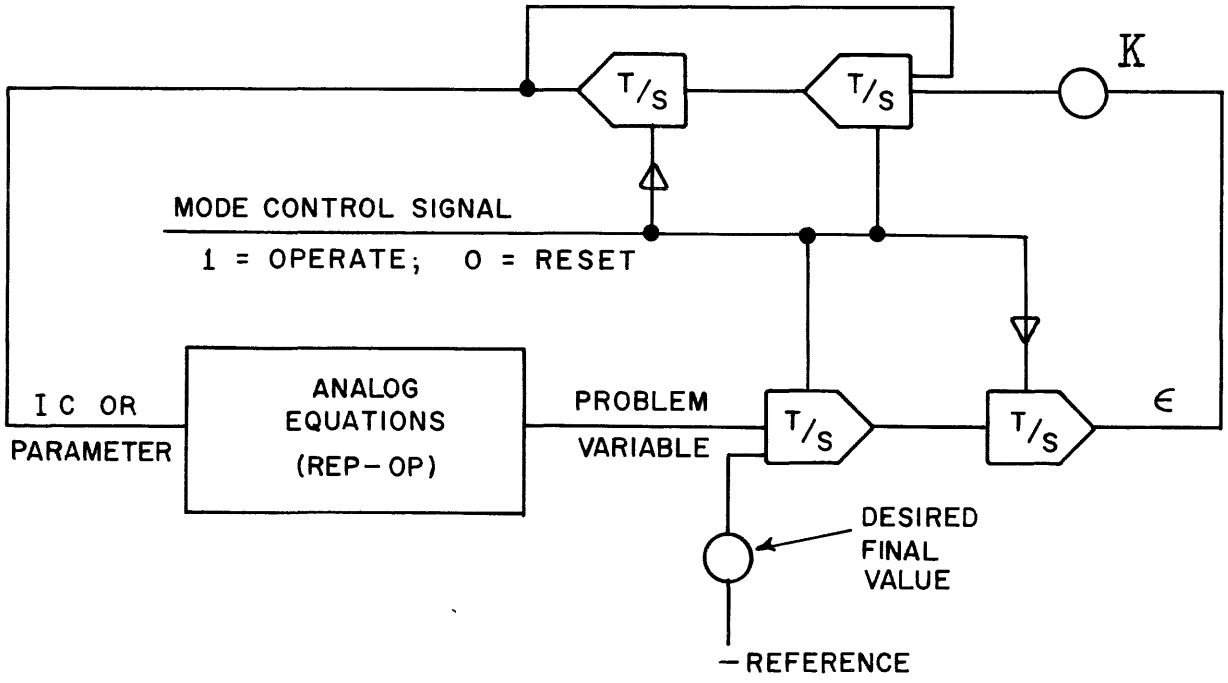


Figure IX-6. Iterative Solution Circuit

). Examples

1. Iterative Solution to Boundary-Value Problems

Figure IX-6 shows a general scheme for matching a boundary value by automatic iteration. The problem variable is compared with the desired final value, and the resulting error, ϵ , is stored at the end of the run by a "bucket brigade". This error is applied to an accumulator in the RESET mode to provide a correction proportional to the error. Thus, large corrections are made for large errors and small corrections for small errors. If due attention is paid to the sign, so that the correction is made in the right direction, the iteration will usually converge. If excessive overshoots occur, the gain, K, should be reduced.

2. Parameter Sweep Circuit

A general scheme for parameter-sweep plotting is shown in Figure IX-7. The system of differential equations is run in rep-op, and the parameter or initial condition is fed into the system via a slow-speed (non-repetitive) integrator. The integtator sweeps the parameter slowly through its range of values and, if the sweep rate is low enough, the value of α may be considered to be approximately constant throughout the run. The final value of X, which is to be plotted versus α, is tracked and stored by a "bucket brigade" circuit resulting in a new final value after

every run. The output plot is a "staircase" approximation to the desired smooth curve (Figure IX-8). However, if we vary the parameter over the desired range in 50 seconds and the rep-op rate is 20 solutions per second, then 1000 solutions will be generated at this time. The plot will then consist of 1000 tiny steps, so small as to be barely noticeable.

A trivial special case occurs when the parameter, α, is simply the elapsed time for the computer run. The result will then simply be a plot of X versus time, which is exactly what would be obtained by direct plotting without rep-op or track-store units.

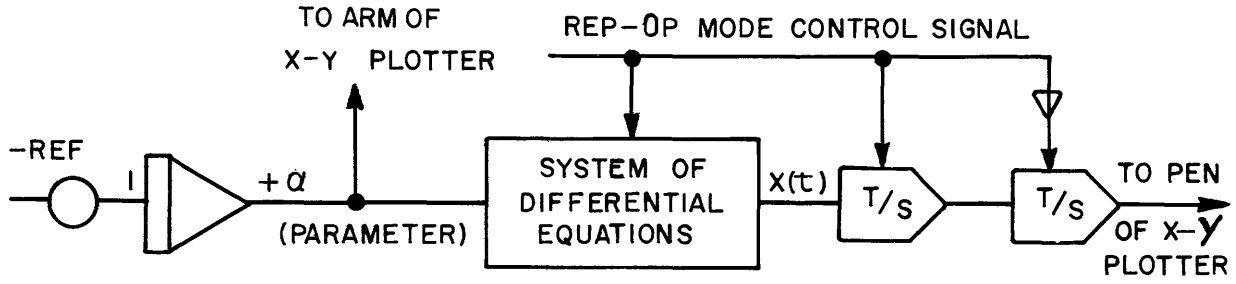


Figure IX-7: Parameter Sweep Unit

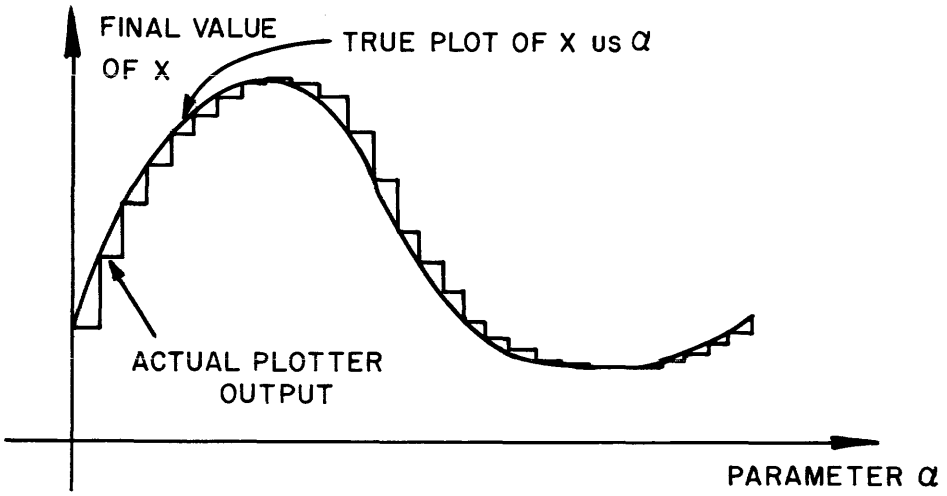


Figure IX-8: Typical Plot produced by Parameter Sweep Circuit
(Step size exaggerated for clarity)

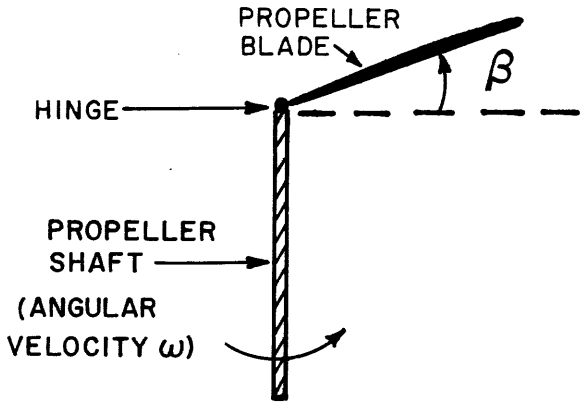


Figure IX-9: Basic Geometry of Hinged Propeller.

3. Applications of Multi-Speed Operation

a. A Helicopter Design Problem -- In one type of helicopter design, the propeller is connected by a hinge to the vertical driveshaft (Figure IX-9). When the shaft is not rotating, the blade hangs vertically. As the shaft rotates, centrifugal force makes the blades stand outward, so that at full speed they will be in a horizontal position. The determination of the angle, β, which the blade makes with the horizontal is crucial to the study of the stability and control of the craft. This angle will vary with time in a manner determined by the centrifugal force and the aerodynamic torque on the blade. (Note that $\beta \leq 0$) if the aerodynamic torque is zero.)
The basic dynamic equation is

$$I_o \ddot{\beta} = - K \beta + T \qquad (1)$$

where I_o is the moment of inertia of the blade about hinge and T is the total aerodynamic torque on the blade. The need for two-speed computation comes from the fact that the total torque on the blade must be determined by integration.

If the distance along the blade, measured from the hinge, is called x, then the torque on the short length between x and x +dx is a function of x, and also of $\beta, \dot{\beta}$, and the centrifugal forcing function (sin ωt)

$$dT = f(x, \beta, \dot{\beta}, \sin \omega t)\ dx \qquad (2)$$

The total torque is obtained by combining the torques on each individual section:

$$T = \int_0^L f(x, \beta, \dot{\beta}, \sin \omega t)\ dx = F(\beta, \dot{\beta}, \sin \omega t) \qquad (3)$$

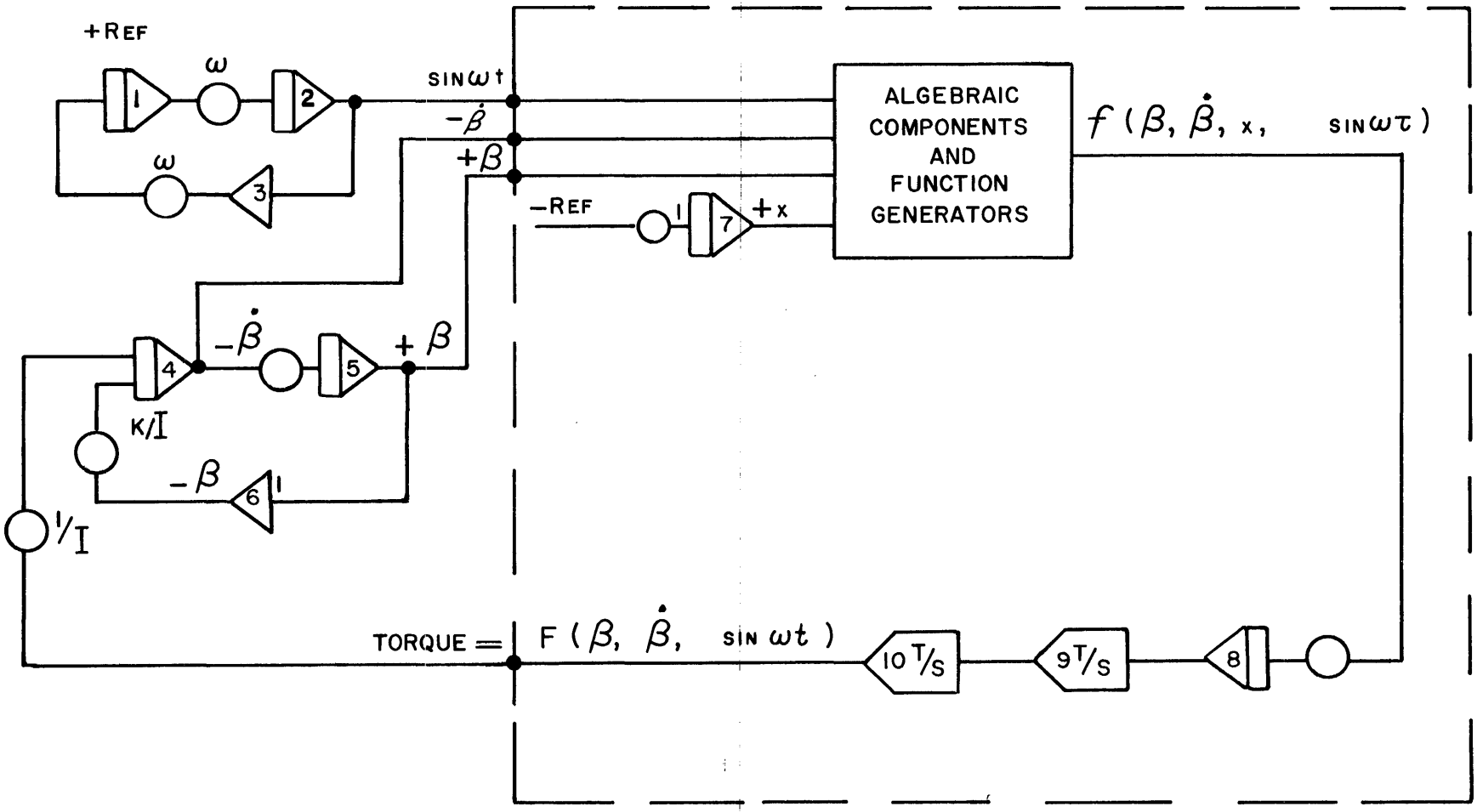


Figure IX-10: Two Speed Circuit for Solving Helicopter Blade Problem

Since the integrand is a complicated function of all four variables, and involves empirically measured aerodynamic co-efficients, it is impossible to perform this integration analytically. It is, of course, a simple matter to perform the integration on the computer. The computer time is made proportional to the distance x along the rod, and the length of the OPERATE cycle is proportional to the length of the propeller blade. A single OPERATE cycle gives the torque for <u>one</u> set of values for β, $\dot{\beta}$, and sin ωt. What is needed is a <u>continuous</u> function of these variables as they change in response to the main dynamic equation for β. Here is where two-speed operation comes in.

The circuit is shown in Figure IX-10. All integrators within the dotted lines (and both track-store units) are running in high-speed repetitive operation; the main problem is run in "slow" time.

The high-speed rep-op circuit receives β, $-\dot{\beta}$ and sin ωt as input variables from the main program, and generates the integrand (the torque-per-unit-length) by means of multipliers, DFG's, etc. This is integrated by a rep-op integrator, 8, over a very short time interval (e.g. 20 milliseconds), with the rep-op time scale chosen so that this corresponds to integration over the interval $0 \leq x \leq L$. At the end of the rep-op cycle, the output voltage of integrator 8 is proportional to the total torque. Amplifier 9 stores this value during the RESET cycle, and passes it on to amplifier 10. At the beginning of the next OPERATE cycle, the rep-op circuit repeats the process with slightly different values of the input variables from the main program. If the rep-op rate is fast enough, the values of these variables will not change very much from one rep-op run to the next, and the input variables may be considered essentially as constant for any given run.

From the point of view of the main program, the rep-op circuit appears as a <u>function generator</u> or <u>subroutine</u>, generating a complicated function of its three input variables.

b. <u>A Multi-speed resolver</u> -- As another example of multi-speed operation, consider the problem of polar-to-rectangular conversion. A polar-to-rectangular resolver accepts input voltages proportional to R and θ and produces outputs proportional to X and Y, where

$$X = R \cos\theta \tag{4}$$

$$Y = R \sin\theta \tag{5}$$

These equations may be mechanized with DFG's and quarter-square multipliers, as shown in Figure IX-11. This is the circuit used in conventional electronic resolvers. It used 9 amplifiers, two multipliers and two DFG's. The range of the input angle, θ, is generally limited to $\pm 180^{\circ}$, although some models are designed for a range of $\pm 360^{\circ}$.

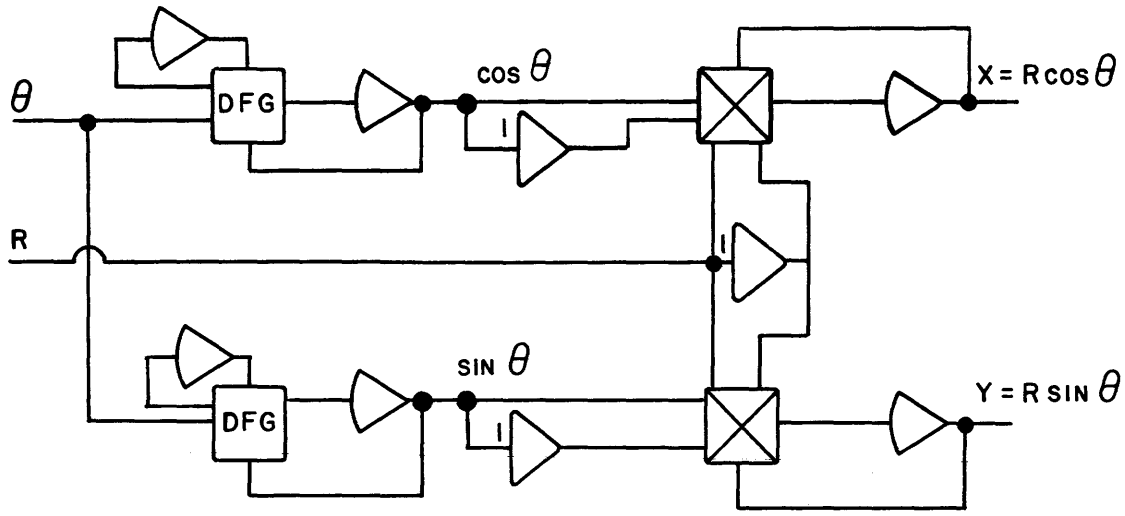


Figure IX-11: Conventional Circuitry for Polar-to-Rectangular Conversion.

The same outputs can be produced by a two-speed circuit with less equipment and without limiting θ to $\pm 180^{o}$. The circuit is based on the solution (in rep-op) of the equations

$$\dot{U} = \omega V \qquad (6)$$

where ω is a conveniently chosen constant

$$\dot{V} = -\omega U \qquad (7)$$

with the IC's $U(0) = R$; $V(0) = 0$. The equations may be solved analytically to obtain

$$U = R \cos \omega t \qquad (8)$$

$$V = R \sin \omega t \qquad (9)$$

where t is time in the rep-op system. If the variables U and V are sampled by comparator-controlled track-store units at time $t = \theta/\omega$ (i.e., when $\omega t = \theta$), the values will be $R \cos \theta$ and $R \sin \theta$. The circuit is given in Figure IX-12. All integrators in the circuit are operating in high-speed rep-op. The variable, θ, (an input into the system from the slow-speed part of the problem) is compared with the high-speed ramp, ωt. When $\theta = \omega t$, an electronic comparator generates a logic signal which causes the outputs of the high-speed oscillator to be held in the track-store units. The usual arrangement of cascaded track-store units (bucket brigade) provides a stair-step output. Note the fact that no multipliers

are needed to produce the products R cos θ and R sin θ. The variable R enters into the high-speed system as an IC on the oscillator.

The multi-speed circuit uses only eight amplifiers (counting those in the track/store units), one comparator and four track-store units. The four track-store units can be reduced to two if the rep-op integrators have a HOLD mode capability. The value of θ is not restricted to $\pm 180^{o}$. As it stands, the circuit works only forpositive values of θ. However, it may be modified easily by adding an IC to the high-speed ramp integrator so that its output is $\omega t - \phi$, where ϕ is determined by the maximum expected negative value of θ. For simplicity, it is easiest to make ϕ a multiple of 360°.

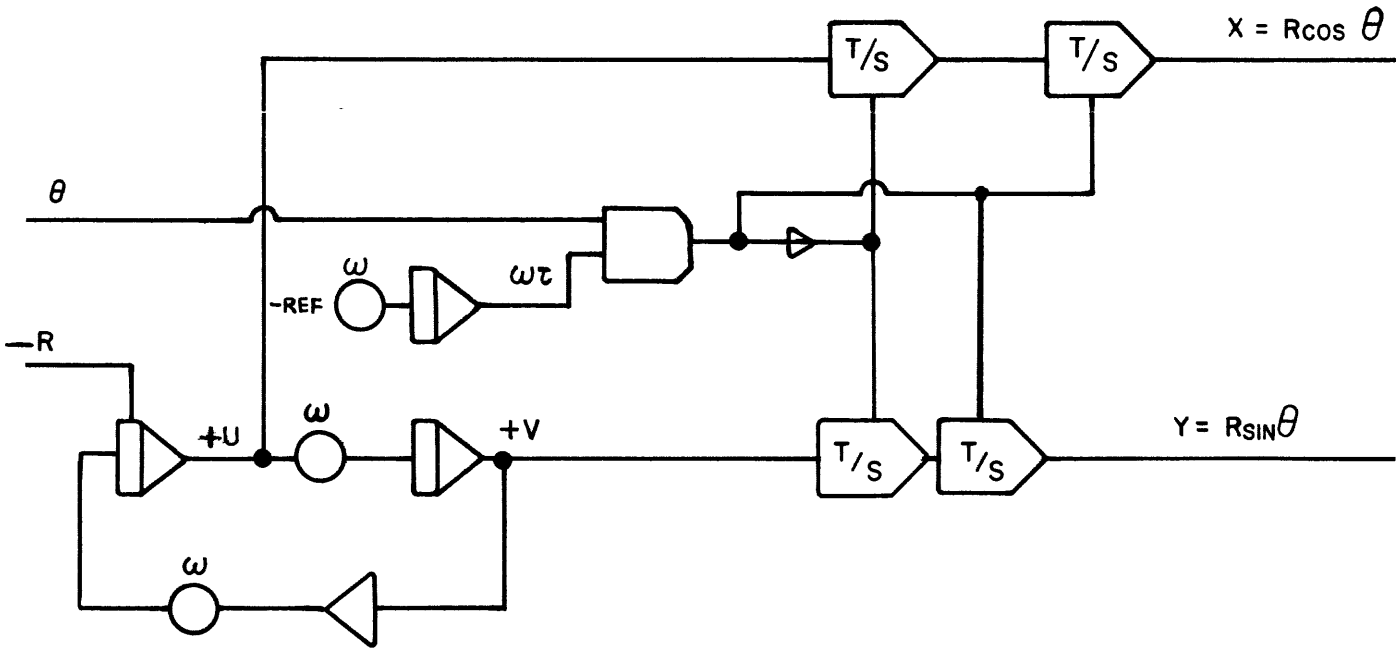


Figure IX-12: Two Speed Circuit for Polar-to-Rectangular Conversion. Inputs R and θ amd Outputs X and Y are slow-speed variables. All others are rep-op variables.

CHAPTER X

PROBLEM PREPARATION PROCEDURE

A. Introduction

The ability to solve physical problems on an analog computer is gained only with practice. However, less individual ability is required for an applications engineer to achieve competence in the art of solving problems on analog computers when he has a definite procedure to follow rather than when he has the added task of deciding separately the steps he should take. This does not imply that any single set of rules can hold for all problems, nor that when two engineers follow a given procedure on a problem they will arrive at the same set-up. It does imply, however, that their chances of obtaining the same solutions are improved and that the time taken to produce a solution is reduced.

The procedure that follows is recommended as a guide to the systematic preparation of problems for solution to insure that all information important to a problem will be at hand immediately through all phases of the problem. The standardization involved is especially valuable when the work of a problem is to be shared among several members of the staff, or when the problem must be transferred from one engineer to another.

The procedure should be regarded rather as a general guide to logical problem organization than as an inflexible set of rules to be followed in every case. On the basis of extensive experience gained at the EAI Computation Centers, it is suggested that where part of this procedure does apply to a problem, then it should be used intelligently. Time spent in preparation is usually more than balanced by the resultant saving in time on the computer.

B. Statement of the Problem in the Original Form

When a physical problem is under consideration, some effort should be given to making a precise statement of the problem. This statement should assemble in one place, all the information which describes the problem in its original form. It should be so organized that a complete understanding of the problem can be grasped easily by anyone reading it. Apart from the description, it also should contain all the basic data required for the solution of the problem, and the general or specific objectives to be attained. As the most concise form, an attempt should be made to include a description of the physical system in terms of mathematical equations, relating those dependent variables whose values determine the state of the system. These equations should be as complete as possible. Approximations may be made at a later point in the study but, once omitted at this early stage, an influence on the system is unlikely to be reconsidered. Care must be exercised so that redundancy is not included in the description e.g., using too many variables or writing equations with too many degrees of freedom (too high an order) can lead to trouble in computation. The available data which determines the pecularities of the particular system of interest must be reviewed to discover its character. The source and form

of the data is important in determining its accuracy and reliability. Much can depend on the confidence one places in this input to the study at hand. Having reviewed the form of the problem, pinned down the objectives and determined the method of solution, one then can decide the form of computation necessary. Possibly a slide rule or a desk calculator is sufficient for the purpose. On the other hand the method of problem solving obviously may be suited to an analog computer. If the problem is arithmetical by nature (an inventory or accounting problem) or requires accuracy greater than 1/1000, the use of a digital computer is suggested. With a little experience and common sense, there is usually little difficulty in deciding which computer should be used.

From this point onward, we will assume that the problem is to be solved using an analog computer.

C. Preliminary Problem Investigation

With all the background material available, the next step is to treat the problem as an analog computer task. After changing the variables or modifying the mathematical model to facilitate programming (such as, transforming a partial differential equation into a set of ordinary differential equations by using finite difference approximations for the derivatives with respect to all but one of the independent variables), the original equations should be written as:

1) First and second order differential equations which are solved for the highest derivative, and/or transfer function.
2) Open loop algebraic equations.
3) Implicit equations
4) Closed loop algebraic equations, if unavoidable
5) Logical conditions (relay comparator arrangements).

At this time, a mathematical block or flow diagram of the problem should be drawn. The purpose of this preliminary computer diagram is to show all the major computing components, and to determine if all necessary inputs and data are available to solve the problem. This diagram should contain all integrators, multipliers, resolvers, function generators, comparators and their interconnections It should ignore scale factors, signs, gains and element number designation.

From the preliminary diagram, an estimate of the equipment required can be obtained for comparison with the available computing equipment. If insufficient equipment is available, simplifications of the original equations may be necessary. At this stage, answers should be provided for all questions which could be raised as to the specific methods to be used. Do any difficult problem areas exist, such as closed algebraic loops, etc., and how will they be handled? Are approximations and simplifications required to stay within the size of the installation, and if so, what should they be? What magnitude of error is to be expected in the solution and is this acceptable? What kind of recordings are to be taken? How much equipment will be required? What scheduling should be done? At this point in the problem investigation, it is advisable to begin planning the future use of the computer. If an "open

shop" operation of the computing facility is the practice within an organization, it is well to tell the computation laboratory manager when you expect to be ready to use the computer. Do not underestimate the work still to be done in the preparation. Allow plenty of time, but do prepare the laboratory for your requirements with respect to the expected size of computer necessary, the output devices you wish to use and when you will be ready to use the equipment.

D. Analog Computer Equations

With the equations describing the problem adapted to a form suitable for computer solution, they now must be written so that no signal voltage on the computer, representing a physical variable or quantity in the problem, can exceed the limitations of the computer equipment. This phase of the preparation should contain all the manipulations required to arrive at the "scaled" set of equations which actually will be solved on the computer.

In writing the original differential equations, it is recommended that the following rules be observed:

1) All constant coefficients or parameters that will be varied at some stage in the study should be indicated.
2) All constants should be defined as positive, so that their signs become explicit.
3) If a constant will change sign during a study, it should be mechanized with appropriate switching circuitry.

In addition, a complete list of coefficients with their units and ranges should be made up as well as a list of all initial conditions.

1. Scale Factors

In selecting scale factors, one should:

a) Identify, on the flow diagram, which variables and expressions require scaling, and list them. This list should include all variables appearing as amplifier outputs, along with the sign of the variables.
b) Evaluate the ranges and maximum values of all listed quantities.

Scaling is generally based on:

a) Knowledge of the physical system under investigation. This alone is sufficient to determine scaling in most practical problems.
b) Consideration of the original equations. (Steady-state relations, time constants and characteristic frequencies help to evaluate the ranges of variables for the initial conditions, (and inputs).
e) Trial and error on the computer. It is often possible to run a simplified version of the computer program to determine a scale factor.

When a variable has either a positive or negative range, variable substitutions should be made to increase the accuracy of the computation. (e.g. $600 \leq T \leq 700$ could be replaced by a new variable $T_1 = T - 650.$)

The computer variables should be tabulated and should include:

a) the physical variables and their units
b) the estimated maximum values or ranges
c) the scale factors (1/maximum value)
d) the computer variables (scale factor x physical variable) in machine units

2. Scaled Voltage Equations

These scale factors, together with those factors inherent in the computer components performing certain mathematical operations (e.g. multiplication), should be written into the original equations. To obtain scaled equations, it is suggested that the notation used in the original equations be maintained and that the equations be manipulated as illustrated in Chapter III.

Potentiometer settings and ranges, and amplifier input gains now can be computed from the tabulated parametric data.

Thought should be given also to "time scaling" in order that the dynamic response of the computer elements will not be exceeded. However, it is not recommended that this type of scaling be included in the equations. Simply change the gain of all integrators by a factor , $1/\beta$, in order to achieve a change in time-scale of β. The output of any external equipment being used in conjunction with the computer also must be adjusted. In the case when passive element networks are used, the values of the time constants also must be multiplied by β. In fact, all time constants must be changed by the same factor, β.

For many problems, large gains or very small attenuator settings often will be required. If these are made to occur at the inputs of integrators, inclusion of the factor, $1/\beta$, at each and every input of the integrators often will remove most difficulties. This will also result in frequencies for the computer model which are suitable for computer operation (0.1 - 10 rad/sec), ensuring problem solution on the computer in a time convenient for interpretation (10-60 seconds).

3. Time Scaling Considerations

If all integrator input potentiometer settings are small or input gains are high, the selection of the proper time scale factor is obvious. If the time-constant-ratio (highest to lowest) of the equations is larger than 10^3, however, a difficulty exists since no matter what time scale factor, β, is selected, some equations will be "too slow" and others will be "too fast".

If a problem is made up of equations that can be divided into fast and slow groups, with a large gap in time constants in between, it can be split as follows:

a) Fast problem -- Consider all the outputs of the slow integrators as constants (equal to the initial conditions of these variables). You are left with a problem in the fast equations only and its solution tells you in what manner the fast variables change from their initial conditions into the steady-state solution for given constant values of the slow variables (the "initial jump" of the full problem).

This problem is much smaller than the original problem, and the range of its time constants is reasonable.

b) Slow problem -- Assume that the fast equations are always in equilibrium. This implies that all fast-variable-derivatives are zero and, therefore, the equations become algebraic, For example, if

$\frac{dx}{dt} = \frac{y - x}{T}$ and T is small, we can substitute x = y,

eliminating a variable and an equation.

The slow problem solution closely approximates the full problem solution from the point where the latter starts with the "steady state" initial conditions of the fast variables reached after the "initial jump".

c) Stability -- If either the fast or slow problem alone is unstable, the full problem is unstable. If we have kept a few fast differential equations but not all of them, the "half-slow" problem may be unstable while the full problem is stable. If slow and fast problems are both stable, the full problem is generally stable for sure if the gap in time constants is large enough.

If the problem cannot be split, we will have to program very large and very small machine time constants. An optimum mechanization can be obtained if the standard feedback capacitors of the integrators are replaced by capacitors whose magnitudes result in accurate integrator input potentiometer settings.

This phase completes any mathematical manipulations required to obtain solutions.

E. Analog Computer Circuit Diagram

A computer circuit diagram can be produced directly from the scaled equations. This will follow the lines of the data flow diagram but will also include all the detailed information with regard to components. The standard symbols for computer diagrams should be used, and voltages at all points in the diagram should be indicated. The diagram should be drawn on such a material that it can be reproduced easily, and the master copy should be retained in the problem file for reference.

One should select a definite address for each element of the computer diagram in a way that minimizes patch-panel clutter. If interchangeable units are used, make a list of which units should be inserted in which slots and issue this as an instruction to technicians. These instructions should also request routine maintenance testing of any unit that is going to be used, which has not been checked for a long period of time. Also, construction of any special elements that may be needed should be requested.

If several machines are used, allocate the elements to minimize trunk connections so that physically-significant partial systems can be set-up on each patch-panel. If several patch-panels are used, as well as several sheets for the diagram, one should attempt, as far as possible, to have all elements on a given sheet allocated to the same patch-panel.

Select, for each pot, the input gain of the amplifier(s) to which it is connected, e.g., 1 or 10, or parallel combinations like 20, 2, etc.

This allocation is made amplifier by amplifier. If it is found that the number of inputs on some amplifiers is not sufficient, proceed as follows:

1) Precision resistors from passive element groups can be patched to the summing junction for extra inputs.

2) A summing amplifier, not used otherwise, can have its input network paralleled with that of another summing or combination amplifier.

The amplifier which has lost its input network still can be used as an inverter by patching a precision resistor directly to its grid. It also can be used as a two-input high gain amplifier by using the feed-back resistor as an input resistor. The grid must then be grounded by a patchcord during pot setting since there is no pot set relay. It is often practical to compromise in gain allocation by using a gain of 10 where an extra gain of 1 is needed, unless the drop in relative accuracy of the pot-setting (which becomes 10 times smaller) cannot be tolerated. As to which setting should be modified, it should be either the highest (among coefficients < 1) for adequate accuracy, or the lowest as the term represented by this setting is known to be quasi-negligible anyway. In general, one should apply common sense in constructing a circuit diagram, and attempt to utilize the equipment efficiently, anticipating mechanization problems.

Typical illustrations include:

1) Programming a potentiometer in the numerator of a division circuit which may overload
2) Distribution of loop gains where possible
3) The isolation of parametric variables.

With the computer circuit diagram complete, the following lists should be prepared, indicating the purpose of each component and the signal voltages or settings involved:

1) Potentiometer sheets
2) Amplifier sheets
3) Multiplier sheets
4) Resolver sheets
5) Function generator sheets
6) Recorder sheets
7) Trunks and external connections
8) Special circuit description

Examples of these sheets are shown in Figures X-1, X-2, X-3, and X-4.

Potentiometer assignments sheets normally contain:

1) the address of the individual potentiometers
2) the parameter description or what parameters are represented by the potentiometer

and

3) precalculated settings for both the static check and operating runs.

Amplifier assignment sheets contain the address and feedback of the individual amplifiers, in addition to:

1) a description of the variables or functions appearing at the amplifier outputs.
2) precalculated static check amplifier output voltages

and

3) precalculated integrator check point or derivative voltages.

If the feedback resistor of the check amplifier used to validate (3) is not explicit, it should be defined on the amplifier assignment sheet to prevent confusion. Since diode function generators are slope limited, provision is made on their set-up sheets for computing the slope of each segment and the change in slopesbetween segments. Frequently, the various segments of variable breakpoint DFG's will have different maximum slopes and tabulations of this type will aid in matching function slopes to their corresponding DFG segments.

Now that the final diagram is available, the boards reserved for the problem can be patched. The diagram must show all information necessary for patching, otherwise it cannot be considered complete.

Using a copy of the diagram, mark connections in red pencil or ink as they are patched, and proceed in a systematic way from one corner of the diagram to the other to minimize the chances of missing a connection.

In the end, all connections should stand out in red. However, it is not easy to check on the basis of connections alone, since a short line from a pot to an amplifier easily can be overlooked. The best check is to go over all elements on the diagram and see if their inputs have been marked. This is necessary and sufficient.

In the same way, the patching itself can be double-checked. After the "inputs" check, the outputs of each element should be counted to make sure that no extra connections have been made.

The larger the distance between two patching terminations, the greater the length that should be left in the connection. This leads to the following pattern



which makes checking and modification easier later on.

After patching has been completed, the underside of the patch panel should be checked to determine if all leads have been completely inserted into their patching terminations.

F. Static Check Preparations

The static check, which is illustrated in Chapter III, is a procedure ensuring that the problem set on the machine corresponds to the original equations. It shows up errors that may have occured in:

1) derivation of scaled equations from original equations.
2) computation of pot settings
3) drawing of the computer diagram.
4) patching.
5) actual setting of potentiometers.
6) operation of machine elements. (blown fuses, wrong servo gain settings, defective amplifiers, etc...).

Most problems are set-up in order to investigate solutions for different combinations of parameters, in which lies one of the main advantages of analog computation. For the purpose of checking, one set of parameters is selected. If sets with different degrees of complexity occur, (with one simple one implying that several coefficients are equal to zero), the set selected as basic for the check should be representative of the most complex case where all coefficients are different from zero. The choice of an arbitrary set of static check variables is guided by the following considerations:

1) When compatible with the other requirements that follow, the check values should be chosen equal to the initial conditions for the first or basic run.
2) The check values should give reasonable voltages after scaling, say from 10 to 90% of reference.
3) Intermediate quantities which had to be scaled (the result of a division, for instance) must be within the scaling range.

4) No check values should be zero since parts of the circuit thus would not be checked at all.
5) The voltages corresponding to the scaled values of the problem variables should all be different in magnitude to detect patching errors.
6) Constant values for all external inputs (signal generators, etc.) should be chosen.
7) In case of a dead-band simulation, the static check values should be chosen to fall outside of this dead-band.

If no complex high-gain equations or other algebraic loops are present, the computation of check points is straightforward and should be carried out by means of a desk computer. Generally, four significant figures should be retained. This numerical work should be written clearly so that the values found for intermediate expressions and for the different terms in sums can be identified easily. Results should be entered on the amplifier assignment sheets.

1. Computer Preparations -- Make sure that the proper units are inserted where required; also, connect pot-padder cables and calibrate recorders and plotters.

Set-up all DFG's and pot-padders, using blank patch-panels. Remember that the output of a tapped pot should not be loaded during set-up since the load is already simulated by the load selector switch circuitry.

After setting, record functions on a plotter, taking the input from a time base integrator. Note that during checking, tapped cups are loaded (by an amplifier input) as in the problem. Check the recorder curves and file them for reference.

2. Operational Procedure -- The operational procedure to be followed, once the computer is checked out and solutions are available, should be detailed as far as possible before one moves to the computer. There should be a plan of campaign so that the data gathering period on the computer is as efficient as possible. This plan should include a run schedule with changes in potentiometer settings listed, the type of recording required, the expected results (if any) and any instructions needed in the operating period.

This completes the preparation stage of the problem, and the patch panels now can be prepared for insertion into the computer. With this achieved, work may begin using the computer.

3. Introduction of Patch-Panels -- The patch panels prepared for the problem now are inserted in all patch-bays, the machines are slaved and the PS (pot set) mode is selected.

The problem log is now started. This is an extremely important part of the work . It should record, with the appropriate time, all events that may affect the results. For example, starting, ending and transition points in the program; changes to patching and diagrams; malfunctions; etc.

This record should be designed to organize and describe the status of a problem when it is removed from the computer, or is interrupted for more than a short time (e.g. overnight). Ideally, the problem could continue from the last run, with a list of potentiometer settings as read-out at the end of the period, a list of switch positions and special conditions, strip chart recordings of all important amplifier outputs, special functions, etc.

If there are any overloads in the PS mode, they can appear in:

1) high-gain amplifier circuits (this does not matter at this stage)
2) circuits involving simple inverters without pot-set relays (if this is a consequence of (1), it does not matter)
3) faulty amplifiers (this should be investigated, and if the cause of the trouble is not in the patching, a grounded output terminal, for instance, the amplifier should be replaced)

4. Potentiometer Setting -- The pots are set according to the pot sheets. Recall that:

1) for pots in cascade, the innermost pot must be set first for correct loading
2) pots loaded by servo cups should be set for a 70% attenuation setting of the servo
3) pots loaded via function switches must be set with the load switched on
4) pots loaded via comparator relay contact must be set with the load switched on
5) pots patched to the open end of the feedback resistor of high-gain amplifiers require special handling: the grid of the amplifier must be temporarily grounded during the setting since the pot set relay does not affect the FB resistor

and

6) pots loaded by diode circuits are set by considering the operation and ultimate output of the circuit.

5. Check of Special Circuits -- Special circuits, such as diode and relay circuits, can be checked at this stage by patching their input terminals to appropriate signal generators or to pots set by hand and then observing and recording the output signals. It is necessary to switch the machines to the IC (RESET) mode or to the OPERATE mode to accomplish this. The overloads that occur during the test in components other than the circuits being checked are immaterial. After correction of errors, if any, and completion of the check, restore the original patching as required for the problem.

G. Static Check Debugging

When patching-modifications necessary for the static check have been made, as in the case of breaking an algebraic loop, record all modifications in the problem log. This will result generally in a number of amplifier and servo overloads.

It is best to start looking at points following the parts of the circuit containing the overloaded elements to find out why they are overloading. In this way, a number of gross errors will be discovered and eliminated. It may be necessary to introduce small feedback capacitors on critical high-gain amplifiers and algebraic loops. This will become apparent immediately. When there are no further overloads evident, the static check can be carried out in the usual manner. If everything checks, the static debugging is terminated. The output of all integrators should be measured to check (against the tabulation) that the static check values actually have been imposed. Then, the check-points or derivatives are measured and compared to the tabulation. More often than not, a number of gross disagreements between measured and calculated derivatives will be found and the real static check work then begins.

The general idea is to start by checking end results, and to work backwards in the signal flow until the error is detected. This reduces the debugging time because:

1) If end results check, the inputs need not be investigated (though this is recommended if a detailed study of small errors is intended).
2) If one works forward in the direction of signal flow, all accessible points will have to be measured and compared with precalculated values even if no errors are present.

When an integrator has a wrong derivative, the inputs of this integrator are measured separately (in general, they will be pot wipers), noted down and multiplied by appropriate input gains as given in the circuit diagram.

Using the static check calculation,the individual terms corresponding to these inputs are identified and scaled. A term by term comparison then can be carried out between the statically computed values and the measured values. This will lead to detection of the cause of the discrepancy and error.

If one or more of the derivative inputs does not check:

1) The pot is not connected to the right signal; check the patching.
2) If the patching is correct, either the pot setting calculation, the actual pot setting or the static check calculation is wrong. Find out which is the cause by referring to the scaled equations and pot sheets.
3) The pot is defective (input shorted, blown fuse, etc.).

In any case, take corrective action and make the proper modifications in the computer diagram, pot sheets, amplifier sheets or static check calculations.

If the signal that should be coupled to the pot does not check, do not consider the pot until this signal has been made to check. In order to find the origin of the error, work gradually, opposite to the direction of signal flow, always comparing with static check calculations and keeping in mind that the latter often may contain errors and must not be taken for granted.

When an error has been found and corrected, go back to the derivative term under investigation and see if it checks. If it does not, another error is located somewhere between the integrator input terminal and the point where corrective action was taken. Work opposite to the direction of signal flow, again starting from the input terminal of the integrator until this error has been found and corrected, and so on.

Finally, the voltage supplying the pot will be correct and, if the wiper voltage still does not check, we must return to the previous case.

If all individual input terms check but not the derivative, start by checking the addition in the static check calculation. If this is correct, then;

1) One or more pot output terminals are patched to the wrong amplifiers or are not patched to anything.
2) A pot is patched to the right amplifier but to a wrong input terminal (gain).
3) The pots are patched correctly but a network input resistor is defective.
4) A faulty patchcord is used in the input connection.

Check the patching to find out which is the case.

If the patching is correct the trouble will be caused by a defective resistor unless the derivative (summing junction current) measuring system is not functioning properly. To check this, patch the summing junction (SJ) to the grid of another amplifier with resistive feedback, and measure the output of this amplifier. If the output is not correct and is the same as the value measured previously, an input resistor is defective. Such defects in equipment resistors and measuring circuits are very rare, and have been considered here only in order to cover all possibilities.

1. Implicit Algebra -- If high gain amplifiers and/or algebraic loops are used (and turn out to be stable or capable of being stabilized), the calculation of the static check may prove difficult. The machine then is, in fact, doing in the static state something which may be quite lengthy to do by hand computation such as: solving a system of 3 or more simultaneous linear equations, solving a system of 2 simultaneous equations of the third degree in 2 variables, etc.

Apart from the difficulty in doing the check calculation, difficulty will be experienced in carrying out the debugging. In the static check condition, integrators are static. The main computing loops thus are open, and it is convenient to work backwards against the direction of the signal flow to identify the sources of error. When algebraic loops are present, all the variables in them (and they may be numerous) react on each other. If a single error exists, all these variables will be incorrect and it is difficult to identify the exact cause of the error.

If the loops are simple and each one involves only 2 or 3 variables, a little logic and some calculations based on the actual, false outcome of the loop will detect the error.

If the loops are more complex, one may find serious difficulty in detecting an error. In such a case, an alternative approach is preferable, as described below.

For high-gain equations, convert the high-gain amplifier to an integrator for the static check. This means that the algebraic relation will not be satisfied in the static check mode, the idea being to have a large error and to check the measured error signal at the input of the temporary integrator against the computed value of this error. This procedure has the advantage of breaking the loops so that the debugging can proceed in the same systematic way as for derivatives.

If a high gain or summing amplifier should have a value imposed on its output during the static check, but it cannot be replaced by an integrator, it should be transformed into an inverter fed by reference voltage through a potentiometer. Its normal inputs are fed to a check amplifier which effectively reduces the original amplifier to an integrator circuit in the IC mode.

After these checks, it is vitally important that the amplifiers are turned back into high-gain amplifiers. It is worthwhile to identify patching changes temporarily with labels or pieces of colored tape.

H. Completeness of the Static Check

The static check is the most complete check that can be done quickly and without major patching changes. The latter point is very important. Any check which necessitates important patching changes to restore the problem conditions after the check conditions introduces the very serious danger of an error in these patching changes, an error that cannot be detected.

The limitations of the static check are:

1) The operation of integrators is not checked (value of capacitor, drift). This requires a rate test which will show whether integrator bottle plugs have been forgotten or incorrectly inserted.
2) The hold relay operation is not checked. However, it establishes a contact between summing junction and amplifier grid in the OPERATE mode which can be checked only by actually putting the computer into the Operate mode. This is done best with a different patch-panel, as a regular routine test. If the contact does not close, the integrator output will stay in its initial condition a fact that is easy to detect on recordings.

3) The static check is made effectively at only one point which is conclusive for linear elements, electronic multipliers and untapped servos. It is not entirely conclusive for servo cups operating in the "normal" mode with the center tap grounded. When a servo is used with a grounded center-tap (normal connection), and one end is not receiving its correct input voltage (due, for instance, to the omission of a patchcord), the unit will work properly if the wiper stands between the center tap and the other end, assuming it to be correctly patched. The error can, therefore, only be detected if the static check values happen to be chosen so that the wiper stays in the "wrong" half of the cup. This possibility should always be kept in mind and if there is any doubt, it is recommended that the sign of an appropriate initial condition be changed temporarily so that the wiper moves to the other section. The output value can be measured and compared to a computed value. It is not necessary to repeat the whole static check with this new initial condition. To prevent any errors due to oversight or omission, the sign of the initial condition should be restored to the original value immediately after the test. Similar remarks apply to the sine-cosine cups of resolvers.

I. Dynamic Check

Dynamic debugging is action taken between actual computer runs in order to correct or improve the set-up. It deals mainly with scaling accuracy, noise and stability. As far as scale factors are concerned, the overload alarm will warn about excessive voltages (or rates of change of voltage on servos), which may necessitate a change in scale factors.

It is just as important, particularly in non-linear problems, to make sure that all voltages are sufficiently large, especially multiplier inputs and divisors (for division). With servo-multipliers, this can be checked by watching the dials.

Non-linear algebraic loops may become unstable for certain input values reached during runs, while being stable for the static check values. Stabilizing action (small feedback capacitors or changes in set-up) must be taken.

Inaccuracies, such as non-repeatability or non-conservation of energy, matter, etc., can be reduced by:

1) better scaling
2) more accurate non-linear computing elements
3) better arrangement of non-linear circuits
4) suppression of redundancy

Non-repeatability may be due to the fact that one has an unstable solution or one on the borderline of instability. In such cases, an error of 10^{-4} in a pot setting may result in a 100% solution error as a consequence. Such a phenomenon is not considered as an error since the same dispersion (and often a larger one)

would be experienced if the real system were tested under the same conditions.

A dynamic check, which is part of the normal check-out routine, is easily implemented to determine if the dynamics of any of the computing components are being exceeded. This check consists of comparing the results of a typical run to those obtained from the same run with an increased time scale factor. If no dynamic problems exist, the results will be identical. If they are not identical, the time scale factors should be increased until agreement is obtained between two sets of results. Computer results should be checked against:

1) analytical solutions
2) experimental data
3) digital computer results whenever possible.

J. Production

Now that the problem is operational, results may be obtained. During this period it is suggested that:

1) All actions and observations be recorded in the problem log.
2) All results be identified by run number and properly labeled.
3) If the problem is left for any period of time (several hours, over night, etc.), the last run prior to shut-down should be repeated for check purposes.
4) At the end of the computer solution period, a brief review of important items should be written immediately, giving a brief history of the operation, the degree of success, and indicating any future plans.

K. Comments

It will be realized from these notes that the solution of problems by computer methods requires an organized approach. This is particularly true for problems requiring large numbers of computing units with many opportunities for errors. Human error is most usually the cause of breakdowns in problem solution on a computer. Rarely does the fault lie in the machine.

If by organization and correct recording of steps taken, these errors can be eliminated, reduced or even quickly corrected, then the extra time required to organize one's methods will result in an over-all saving of time and certainly a reduction in the number of frustrating doubts that can arise in a poorly prepared computer solution.

EAI 8800 SCIENTIFIC COMPUTING SYSTEM

PROBLEM _____ PROJECT NO. _____

AMPLIFIER ASSIGNMENT SHEET

PROGRAMMER _____ CONSOLE _____ DATE _____

AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A000					A004		
A001					M004		
A002					A104		
A003					M104		
A200					A204		
A201					M204		
A202					A304		
A203					M304		
A400					A404		
A401					M404		
A402					A504		
A403					M504		

AMP MODE	VARIABLE OUTPUT	S J INVERTER
A510		A511 A512 A513
A520		A521 A522 A523

AMP MODE	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A600				A604		
A601				M604		
A602				A704		
A603				M704		
A800				R800	R900	
				R801	R901	
A801				R802	R902	
				R803	R903	
A802				R804	M800	
				R805	M801	
A803				R806	M802	
				R807	M803	

AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A010					A014		
A011					M014		
A012					A114		
A013					M114		
A210					A214		
A211					M214		
A212					A314		
A213					M314		
A410					A414		
A411					M414		
A412					A514		
A413					M514		

AMP MODE	VARIABLE OUTPUT	S J INVERTER
A610		
A611		
A612		
A613		

AMP MODE	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A610				A614		
A611				M614		
A612				A714		
A613				M714		
A810				R810	R910	
				R811	R911	
A811				R812	R912	
				R813	R913	
A812				R814	M810	
				R815	M811	
A813				R816	M812	
				R817	M813	

AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A020					A024		
A021					M024		
A022					A124		
A023					M124		
A220					A224		
A221					M224		
A222					A324		
A223					M324		
A420					A424		
A421					M424		
A422					A524		
A423					M524		

AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A620					A624		
A621					M624		
A622					A724		
A623					M724		
A820					R820	R920	
					R821	R921	
A821					R822	R922	
					R823	R923	
A822					R824	M820	
					R825	M821	
A823					R826	M822	
					R827	M823	

AMP MODE	T/S CAP	VARIABLE	OUTPUT	RESOLVER MODE	FC CRF	PRI CPRI	PR2 CPR2	RP CRP	RA CRA

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EAI 8800 SCIENTIFIC COMPUTING SYSTEM

PROBLEM _____ PROJECT NO. _____

AMPLIFIER ASSIGNMENT SHEET

(COLUMNS 3, 4 AND 5)

PROGRAMMER _____ CONSOLE _____ DATE _____

AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT	AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT	AMP MODE	CAP SEL	VARIABLE	DERIV	OUTPUT	INV/MULT MODE	VARIABLE	OUTPUT
A030					A034			A040					A044			A050					A054		
A031					M034			A041					M044			A051					M054		
A032					AI34			A042					AI44			A052					AI54		
A033					MI34			A043					MI44			A053					MI54		
A230					A234			A240					A244			A250					A254		
A231					M234			A241					M244			A251					M254		
A232					A334			A242					A344			A252					A354		
A233					M334			A243					M344			A253					M354		
A430					A434			A440					A444			A450					A454		
A431					M434			A441					M444			A451					M454		
A432					A534			A442					A544			A452					A554		
A433					M534			A443					M544			A453					M554		
A530		A531		A532		A533		A540		A541		A542		A543		A550		A551		A552		A553	
A630					A634			A640					A644			A650					A654		
A631					M634			A641					M644			A651					M654		
A632					A734			A642					A744			A652					A754		
A633					M734			A643					M744			A653					M754		
A830					R830		R930	A840					R840		R940	A850					R850		R950
					R831		R931						R841		R941						R851		R951
A831					R832		R932	A841					R842		R942	A851					R852		R952
					R833		R933						R843		R943						R853		R953
A832					R834		M830	A842					R844		M840	A852					R854		M850
					R835		M831						R845		M841						R855		M851
A833					R836		M832	A843					R846		M842	A853					R856		M852
					R837		M833						R847		M843						R857		M853

AMP MODE	T/S CAP	VARIABLE	OUTPUT	RESOLVER MODE	FC CRF	PRI CPRI	PR2 CPR2	RP CRP	RA CRA

AMP MODE	T/S CAP	VARIABLE	OUTPUT	RESOLVER MODE	FC CRF	PRI CPRI	PR2 CPR2	RP CRP	RA CRA

AMP MODE	T/S CAP	VARIABLE	OUTPUT	RESOLVER MODE	FC CRF	PRI CPRI	PR2 CPR2	RP CRP	RA CRA

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PROBLEM _____ PROJECT NO. _____

PROGRAMMER _____ CONSOLE _____ DATE _____

POTENTIOMETER ASSIGNMENT SHEET

POT	PARAMETER	VALUE	COEFFICIENT	POT	PARAMETER	VALUE	COEFFICIENT	POT	PARAMETER	VALUE	COEFFICIENT	POT	PARAMETER	VALUE	COEFFICIENT	POT	PARAMETER	VALUE	COEFFICIENT	POT	PARAMETER	VALUE	COEFFICIENT
000				010				020				030				040				050			
001				011				021				031				041				051			
002				012				022				032				042				052			
003				013				023				033				043				053			
100				110				120				130				140				150			
101				111				121				131				141				151			
102				112				122				132				142				152			
103				113				123				133				143				153			
200				210				220				230				240				250			
201				211				221				231				241				251			
202				212				222				232				242				252			
203				213				223				233				243				253			
300				310				320				330				340				350			
301				311				321				331				341				351			
302				312				322				332				342				352			
303				313				323				333				343				353			
400				410				420				430				440				450			
401				411				421				431				441				451			
402				412				422				432				442				452			
403				413				423				433				443				453			
500				510				520				530				540				550			
501				511				521				531				541				551			
502				512				522				532				542				552			
503				513				523				533				543				553			
600				610				620				630				640				650			
601				611				621				631				641				651			
602				612				622				632				642				652			
603				613				623				633				643				653			
700				710				720				730				740				750			
701				711				721				731				741				751			
702				712				722				732				742				752			
703				713				723				733				743				753			
800				810				820				830				840				850			
801				811				821				831				841				851			
802				812				822				832				842				852			
803				813				823				833				843				853			
900				910				920				930				940				950			
901				911				921				931				941				951			
902				912				922				932				942				952			
903				913				923				933				943				953			

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EAI 8800 SCIENTIFIC COMPUTING SYSTEM

PROBLEM _____ PROJECT NO. _____

PROGRAMMER _____

NON-LINEAR SET-UP SHEET

CONSOLE _____ DATE _____

L 000	VARIABLE	+	−	L 010	VARIABLE	+	−	L 020	VARIABLE	+	−	L 030	VARIABLE	+	−	L 040	VARIABLE	+	−	L 050	VARIABLE	+	−
C 000	INPUT	INPUT	OUTPUT	C 010	INPUT	INPUT	OUTPUT	C 020	INPUT	INPUT	OUTPUT	C 030	INPUT	INPUT	OUTPUT	C 040	INPUT	INPUT	OUTPUT	C 050	INPUT	INPUT	OUTPUT
DA 002	INPUT	OUTPUT	CONTROL	DA 012	INPUT	OUTPUT	CONTROL	DA 022	INPUT	OUTPUT	CONTROL	DA 032	INPUT	OUTPUT	CONTROL	DA 042	INPUT	OUTPUT	CONTROL	DA 052	INPUT	OUTPUT	CONTROL
DA 003				DA 013				DA 023				DA 032				DA 043				DA 053			
FS 001	L C R A			FS 011	L C R A			FS 021	L C R A			FS 031	L C R A			FS 041	L C R A			FS 051	L C R A		
FR 004				FR 014				FR 024				FR 034				FR 044				FR 054			
F 204	TYPE	VARIABLE	VALUE	F 214	TYPE	VARIABLE	VALUE	F 224	TYPE	VARIABLE	VALUE	F 234	TYPE	VARIABLE	VALUE	F 244	TYPE	VARIABLE	VALUE	F 254	TYPE	VARIABLE	VALUE
F 304				F 314				F 324				F 334				F 344				F 354			
L 200	VARIABLE	+	−	L 210	VARIABLE	+	−	L 220	VARIABLE	+	−	L 230	VARIABLE	+	−	L 240	VARIABLE	+	−	L 250	VARIABLE	+	−
C 200				C 210				C 220				C 230				C 240				C 250			
FR 204				FR 214				FR 224				FR 234				FR 244				FR 254			
												F 434	TYPE	VARIABLE	VALUE	F 444	TYPE	VARIABLE	VALUE	F 454	TYPE	VARIABLE	VALUE
												F 534				F 544				F 554			
L 400	VARIABLE	+	−	L 410	VARIABLE	+	−	L 420	VARIABLE	+	−	L 430	VARIABLE	+	−	L 440	VARIABLE	+	−	L 450	VARIABLE	+	−
C 400	INPUT	INPUT	OUTPUT	C 410	INPUT	INPUT	OUTPUT	C 420	INPUT	INPUT	OUTPUT	C 430	INPUT	INPUT	OUTPUT	C 440	INPUT	INPUT	OUTPUT	C 450	INPUT	INPUT	OUTPUT
DA 402	INPUT	OUTPUT	CONTROL	DA 412	INPUT	OUTPUT	CONTROL	DA 422	INPUT	OUTPUT	CONTROL												
DA 403				DA 413				DA 423															
FS 401	L C R A			FS 411	L C R A			FS 421	L C R A														
FR 404				FR 414				FR 424				FR 434				FR 444				FR 454			
F 604	TYPE	VARIABLE	VALUE	F 614	TYPE	VARIABLE	VALUE	F 624	TYPE	VARIABLE	VALUE	F 634	TYPE	VARIABLE	VALUE	F 644	TYPE	VARIABLE	VALUE	F 654	TYPE	VARIABLE	VALUE
F 704				F 714				F 724				F 734				F 744				F 754			
L 600	VARIABLE	+	−	L 610	VARIABLE	+	−	L 620	VARIABLE	+	−	L 630	VARIABLE	+	−	L 640	VARIABLE	+	−	L 650	VARIABLE	+	−
C 600				C 610				C 620				C 630				C 640				C 650			
FR 604				FR 614				FR 624				FR 634				FR 644				FR 654			
L 800	VARIABLE	+	−	L 810	VARIABLE	+	−	L 820	VARIABLE	+	−	L 830	VARIABLE	+	−	L 840	VARIABLE	+	−	L 850	VARIABLE	+	−
C 800	INPUT	INPUT	OUTPUT	C 810	INPUT	INPUT	OUTPUT	C 820	INPUT	INPUT	OUTPUT	C 830	INPUT	INPUT	OUTPUT	C 840	INPUT	INPUT	OUTPUT	C 850	INPUT	INPUT	OUTPUT
DA 802	INPUT	OUTPUT	CONTROL	DA 812	INPUT	OUTPUT	CONTROL	DA 822	INPUT	OUTPUT	CONTROL	DA 832	INPUT	OUTPUT	CONTROL	DA 842	INPUT	OUTPUT	CONTROL	DA 852	INPUT	OUTPUT	CONTROL
DA 803				DA 813				DA 823				DA 833				DA 843				DA 853			
FS 801	L C R A			FS 811	L C R A			FS 821	L C R A			FS 831	L C R A			FS 841	L C R A			FS 851	L C R A		

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CHAPTER XI

ANALOG COMPUTER SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

A. Introduction

In formulating mathematical descriptions of physical systems, one must be aware of the fact that all physical phenomena actually are described by partial differential equations, and that ordinary differential equations are only approximations. Therefore, one is confronted with the task of mechanizing one or more partial differential equations on the analog computer. The purpose of this chapter is to indicate how the analog computer solutions of these distributed parameter systems are obtained. Although there are several methods of simulating distributed parameter systems, we will restrict ourselves to "eigenvalue" and lumped parameter model methods since they are most frequently used in practice.

A tabulation of selected, linear partial differential equations and the physical phenomena they describe are contained in Table XI-I. The ∇^2 notation, which is common practice in the literature (1) (2), is a linear combination of space derivatives.

TABLE XI - I SELECTED LINEAR PARTIAL DIFFERENTIAL EQUATIONS

NAME	EQUATION	PHYSICAL PHENOMENA
Laplace's Equation	$\nabla^2 y = 0$	Steady-State Diffusion
Diffusion Equation	$k\nabla^2 y = \frac{\partial y}{\partial t}$	Transient Diffusion
Wave Equation	$k^2\nabla^2 y = \frac{\partial^2 y}{\partial t^2}$	Wave Motion
Poisson's Equation	$\nabla^2 y = e$	Diffusion From A Source
Transport Equation	$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} = 0$	Delay Due to Fluid Motion
Transverse Motion Equation	$\frac{1}{\rho A}\frac{\partial^2}{\partial x^2}\left[E\,I\,\frac{\partial^2 y}{\partial x^2}\right] + \frac{\partial^2 y}{\partial t} = 0$	Transverse Vibrations In A Rod

B. Eigenvalue Method

The eigenvalue method is related directly to the analytical "separation of variables" technique which is used to solve linear partial differential equations. To illustrate this technique, consider heat transfer in an insulated bar as shown in Figure XI-I, and described by the one-dimensional diffusion

equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C} \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where

T = temperature
k = thermal conductivity
ρ = density
C = specific heat
t = time
x = length variable $(0 \leq x \leq L)$

If it is assumed that the solution is the product of a function of length and a function of time

$$T(x,t) = X(x)\theta(t) \tag{2}$$

Equation (1) can be reduced to two ordinary differential equations

$$\frac{d^2X}{dx^2} + a^2X = 0 \tag{3}$$

and

$$\frac{d\theta}{dt} = -a^2\theta \tag{4}$$

where a and X(x) are known as the eigenvalue and eigenfunction respectively.

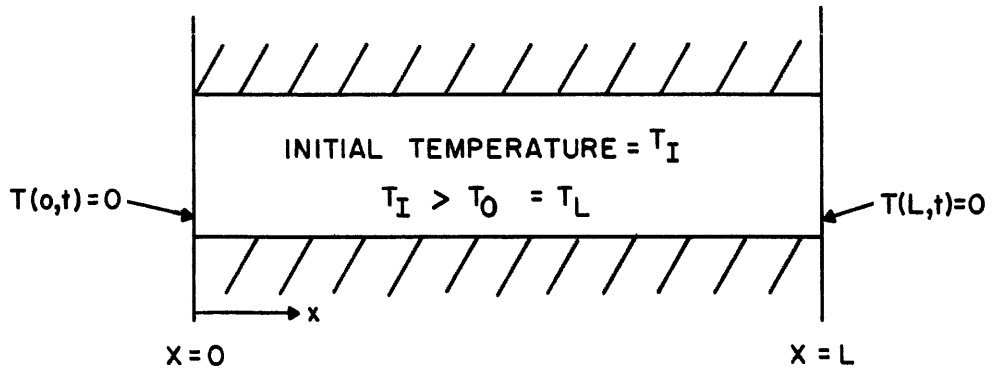


Figure XI-I: Insulated Bar With Fixed End Temperatures

A general solution to equation (1) is obtained by combining the solutions of equations (3) and (4)*. This resultant solution contains constants of integration, and eigenvalues or frequencies which are evaluated by subjecting the solution to the boundary values and initial conditions of the system. The result of this evaluation yields a multivalued eigenvalue

$$La_n = n\pi, \; n = 1, 2, 3, 4 \text{ ---} \tag{5}$$

since there are an infinite number of eigenvalues and eigenfunctions that

* $X(x) = \left(A' \text{ Sin } ax + B' \text{ Cos } ax\right)$, $\theta(t) = \left(C'\epsilon^{-a^2 t}\right)$, therefore, $T(x, t) = (A \text{ Sin } ax + B \text{ Cos } ax)(C'\epsilon^{-a2t})$.

satisfy the zero temperature boundary conditions at the ends of the bar. Therefore, the solution to equation (1) is the sum of all possible solutions

$$T(x, t) = \sum_{n=1}^{n=\infty} A_n X_n \theta_n \qquad (6)$$

where A_n is an amplitude coefficient.

The actual mechanics of obtaining the solution to this problem will be omitted; however, they are readily available in the literature (2) (3). The solution to this problem is:

$$T(x,t) = T_I \sum_{n=1}^{n=\infty} \left(\frac{4}{(2n-1)\pi}\right) \operatorname{Sin}\left[(2n-1)\pi\frac{x}{L}\right]\left[\exp\left(-\left[\frac{(an-1^2)\pi}{L}\right]^2 kt\right)\right] \qquad (7)$$

which is representative of the form of analytical solutions obtained for partial differential equations. Fortunately, these series converge rapidly and only the first few terms are required for reasonable accuracy.

Since the analog computer can integrate continuously with respect to only one independent variable, it can not solve equation (1) in its original form. However, it can be used to solve the eigenfunction equation, equation (3), to obtain the correct eigenfunctions and their corresponding eigenvalues. These results are obtained readily using the circuit shown in Figure XI-2, with high-speed repetitive operation which reduces the time required for the trial and error solution to a few minutes. The objective of the simulation is to determine the eigenvalues which will produce eigenfunctions that are zero at both ends of the bar. The number of eigenfunctions required for a given problem depends on the convergence of its series solution.

The amplitude coefficients, A_n, also can be determined on the analog computer for their associated eigenfunctions (5). Therefore, the first few terms of a series solution can be determined if time is held constant. Typical results are a family of curves, such as temperature versus bar length at fixed values of time.

This method of solution is used generally when the normal modes (eigenfunctions) of a system are required. A typical example is seen in the determination of wing-flutter, a problem familiar to the aircraft industry. The main drawback of this method is that it is restricted to linear systems; therefore, a more general approach to the analog computer solution of partial differential equations is desirable.

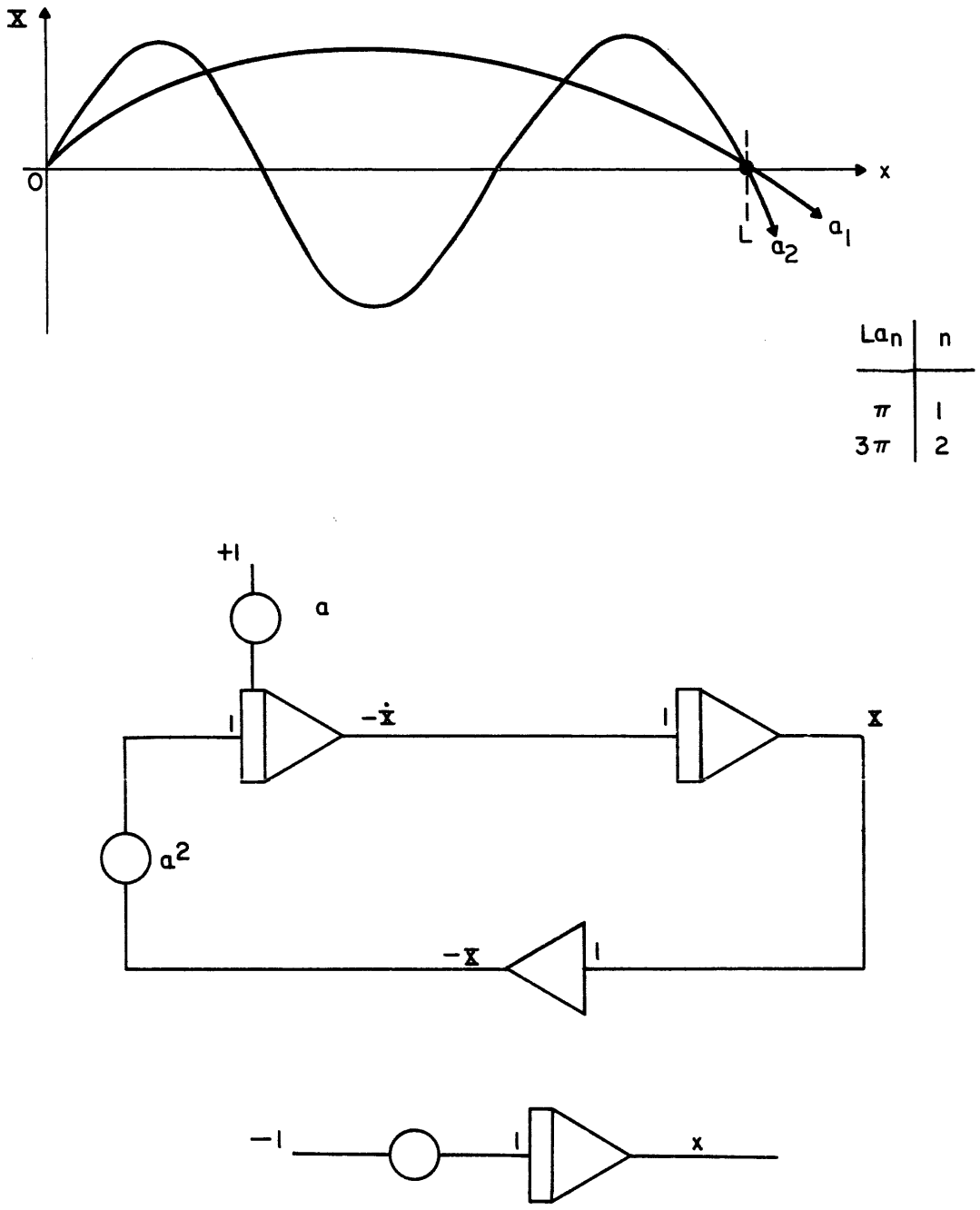
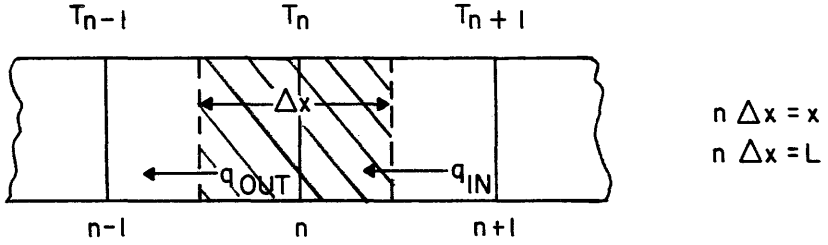


Figure XI-2: Computer Solution by Eigenvalue Method

C. Lumped Parameter Model

To illustrate this method of solution, assume a metal bar is divided into a finite number of equal sections, n as shown below. Restricting our interest to two adjacent sections and to small



values of Δx, the heat transferred into and out of the shaded area can be approximated by the familiar, steady-state heat transfer equation

$$q = \text{Rate of Heat Transfer} = \frac{k\ A\ \Delta T}{\Delta x} \qquad (8)$$

to obtain

$$q_{in} = \frac{kA}{\Delta x}\left[T_{n+1} - T_n\right] \qquad (9)$$

and

$$q_{out} = \frac{kA}{\Delta x}\left[T_n - T_{n-1}\right] \qquad (10)$$

where A is the cross section area of the bar.

From energy balance considerations, the energy gain of the shaded, Δx, section is

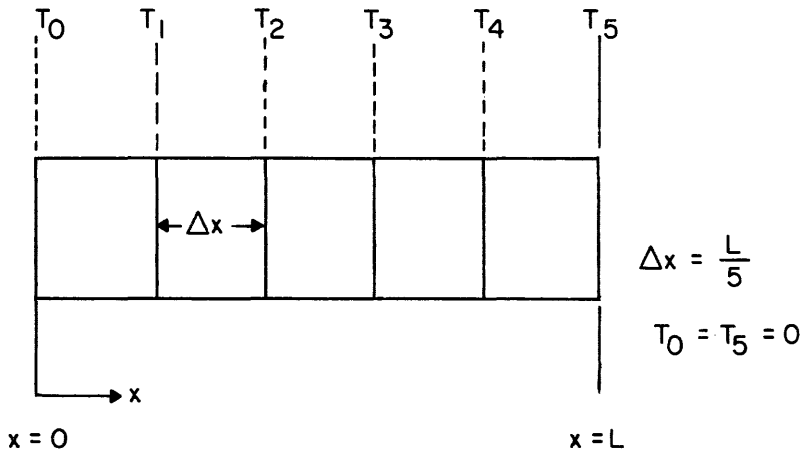
$$(A\ \Delta x\ \rho)\ C\ \frac{dT_n}{dt} = q_{in} - q_{out} = \frac{kA}{\Delta x}\left[T_{n+1} - T_n\right] - \frac{kA}{\Delta x}\left[T_n - T_{n-1}\right] \qquad (11)$$

Similar equations can be obtained for all values of n. Thus, rather than by the original partial differential equations, the system is now represented by a set of ordinary differential equations. This set of equations must be solved simultaneously or in "parallel" to simulate the system. The computer solution of this problem for a bar divided into five equal sections is shown in Figure XI-3. The initial temperature of the bar, T_I, and its zero and end temperatures are mechanized easily on the computer.

There is no need to perform an analysis of the physical system to convert a partial differential equation into a set of ordinary differential equations. The same results could have been obtained directly by finite difference approximations which are obtained from manipulation of the Taylor Series

$$T(x,t) = T(x_o,t) + \left(\frac{\partial T}{\partial x}\right)_{x=x_o} \cdot \frac{(x - x_o)}{1!} + \text{------} + \left(\frac{\partial^n T}{\partial x^n}\right)_{x=x_o} \cdot \frac{(x - x_o)^n}{n!} \qquad (12)$$

System



Circuit Diagram

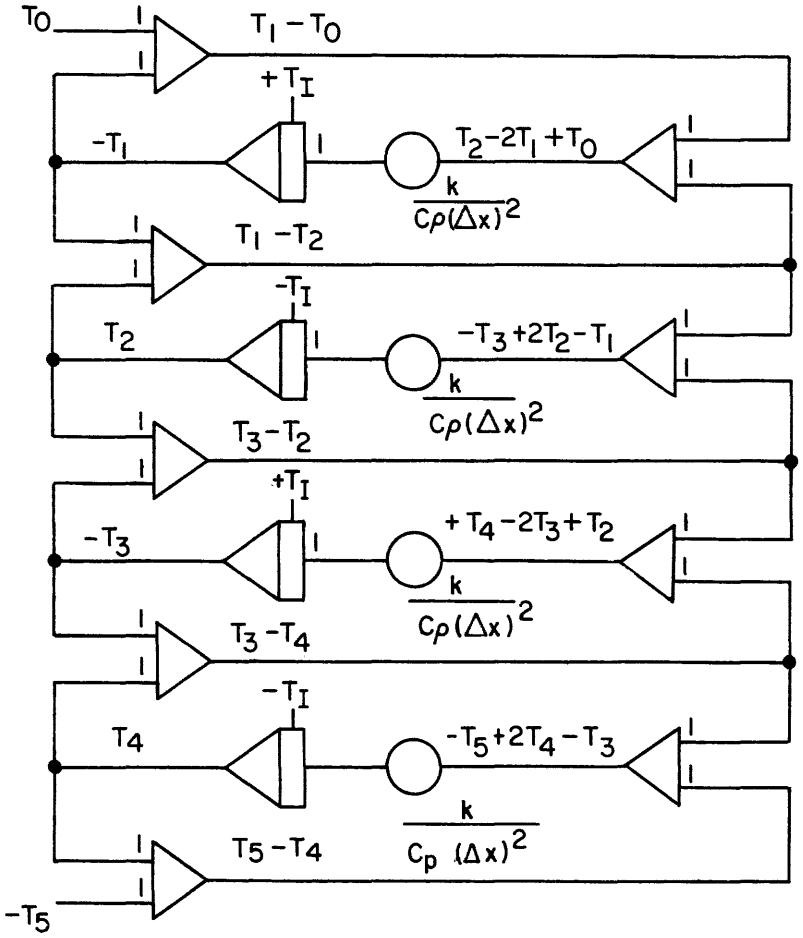


Figure XI-3: Illustration Of Parallel Computer Solution

For example, since

$$T(x + \Delta x,t) = T(x,t) + \left(\frac{\partial T}{\partial x}\right)\Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)\frac{(\Delta x)^2}{2} + \text{----------} \qquad (13)$$

and

$$T(x - \Delta x, t) = T(x,t) - \left(\frac{\partial T}{\partial x}\right)\Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)\frac{(\Delta x)^2}{2} + \text{----------} \qquad (14)$$

their sum represents an approximation for a second derivative

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T(x + \Delta x,t) - 2\ T(x,t) + T(x - \Delta x,t)}{(\Delta x)^2} \qquad (15)$$

which could have been used directly in the previous example. The higher order terms in equation (13) and (14) produce an error in the resultant finite difference approximation, equation (15).

This error, which is difficult to define in convenient form, normally is represented by its initial term which serves as an accuracy guide. For example, consider the elementary finite difference approximations and their initial error terms listed in Table XI-II. The middle difference approximation for a first derivative is more accurate than the backward and forward first derivative approximations. This can be verified by representing the slope of a curve at a given point by two adjacent points compared to one adjacent point.

Fortunately, most partial differential equations can be simulated by a relatively small set of ordinary differential equations. This is illustrated by Table X -III which tabulates the decay constant† error introduced to the first three terms (modes) of the analytical series solution (by the finite difference approximation of the diffusion equation). These errors are small compared to the error usually encountered in parametric data, and equal to or less than the errors obtained from digital solutions of the same equation. The analog computer errors may be greatly improved by using higher order finite difference approximations and other simple techniques, which are discussed in the literature (4) (7).

† Reciprocal Of The Time Constant

Table XI-II, Finite Difference Approximations

Approximations	First Error Term
First Derivatives	
Forward Difference	
$\frac{\partial T}{\partial x} \simeq \frac{T(x + \Delta x,t) - T(x,t)}{\Delta x}$	$- \frac{\Delta x}{2!}\left(\frac{\partial^2 T}{\partial x^2}\right) + \text{----}$
Backward Difference	
$\frac{\partial T}{\partial x} \simeq \frac{T(x,t) - T(x-\Delta x,\ t)}{\Delta x}$	$+ \frac{\Delta x}{2!}\left(\frac{\partial^2 T}{\partial x^2}\right) + \text{----}$
Middle Difference	
$\frac{\partial T}{\partial x} \simeq \frac{T(x + \Delta x,t) - T(x - \Delta x,t)}{2\,\Delta x}$	$- \frac{(\Delta x)^2}{3!}\left(\frac{\partial^3 T}{\partial x^3}\right) + \text{----}$
Second Derivatives	
$\frac{\partial^2 T}{\partial x^2} \simeq \frac{T(x+ \Delta x,t) - 2T(x,t) + T(x- \Delta x,t)}{(\Delta x)^2}$	$- \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 T}{\partial x^4}\right)$

Table XI-III Decay Constant Error Versus Number of Sections (7)

	Percent Error		
Number of Sections	1st Mode	2nd Mode	3rd Mode
5	-0.8	-7.0	-20.0
8	-0.3	-3.0	-8.0
10	-0.2	-2.0	-5.0

Although an increase in the number of equations (or divisions in the physical system) reduces the error of the finite difference approximations, the computer solutions approach their analytical solutions asymptotically. Therefore, the practical number of equations required is relatively low, and additional equations only contribute computer component errors to their analog computer solutions. It is significant also that this method is not restricted by stability and convergence criteria which are commonplace in the digital solution of partial differential equations.

D. Summary

This chapter, which is by no means complete, is intended to be a brief introduction to the methods used to solve partial differential equations on the analog computer. Its purpose is to assure the reader that the modern, electronic analog computer can and has simulated distributed-parameter systems since its inception in the early 1950's. Therefore, additional information on this topic is readily available in the open literature (4) (5) (6) (7).

E. References

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7. Howe, R. M.; Partial Differential Equations and An Introduction To Some Solution Methods; The University of Michigan, 1960.

CHAPTER XII

ACCURACY OF ANALOG COMPUTER SOLUTIONS

A. Introduction

The general techniques of evaluating errors which occur in analog computer solutions are referred to by the broad term "error theory ". Although quite a bit of effort has been applied to the study of error theory, the results have not been encouraging. This is because the problem of determining the errors arising in any analog computer study is usually long and involved depending on the particular study being considered, and is, by its very nature, statistical.

An appreciation of the task of evaluating the total error in an analog computer solution is best obtained by considering first all the possible sources of error as well as by defining exactly what constitutes an error. For our purposes, an error will be defined as the difference between a computed or estimated result, and the actual value. The sources of computer error are indicated in Figure XII-1 which indicates two paths one may take to obtain results from an existing physical system.

For discussion purposes, the analog computer simulation of a physical system may be subdivided into three stages:

1) mathematical model
2) computer programming
3) computer mechanization

As a problem passes through each of these stages, errors arise which effect the overall accuracy of the results obtained. Our objective here is to discuss these stages in turn, and to indicate the sources of errors.

B. Errors in Mathematical Models

When a system is to be analyzed for the express purpose of deriving a mathematical model, simplifying assumptions are made as to its geometry and physical behavior. Typical examples are the assumption of

1) spherical particles to represent irregular solids in a packed bed
2) coplanar forces in a simple pendulum
3) perfect mixing of a fluid
4) infinite and semi-infinite geometry

Although the results obtained from the solutions based on simplified representations of a system are adequate, one must be aware that an error exists. The assumed physical system which is used to describe significant physical phenomena occuring in the actual system does differ from the actual physical system.

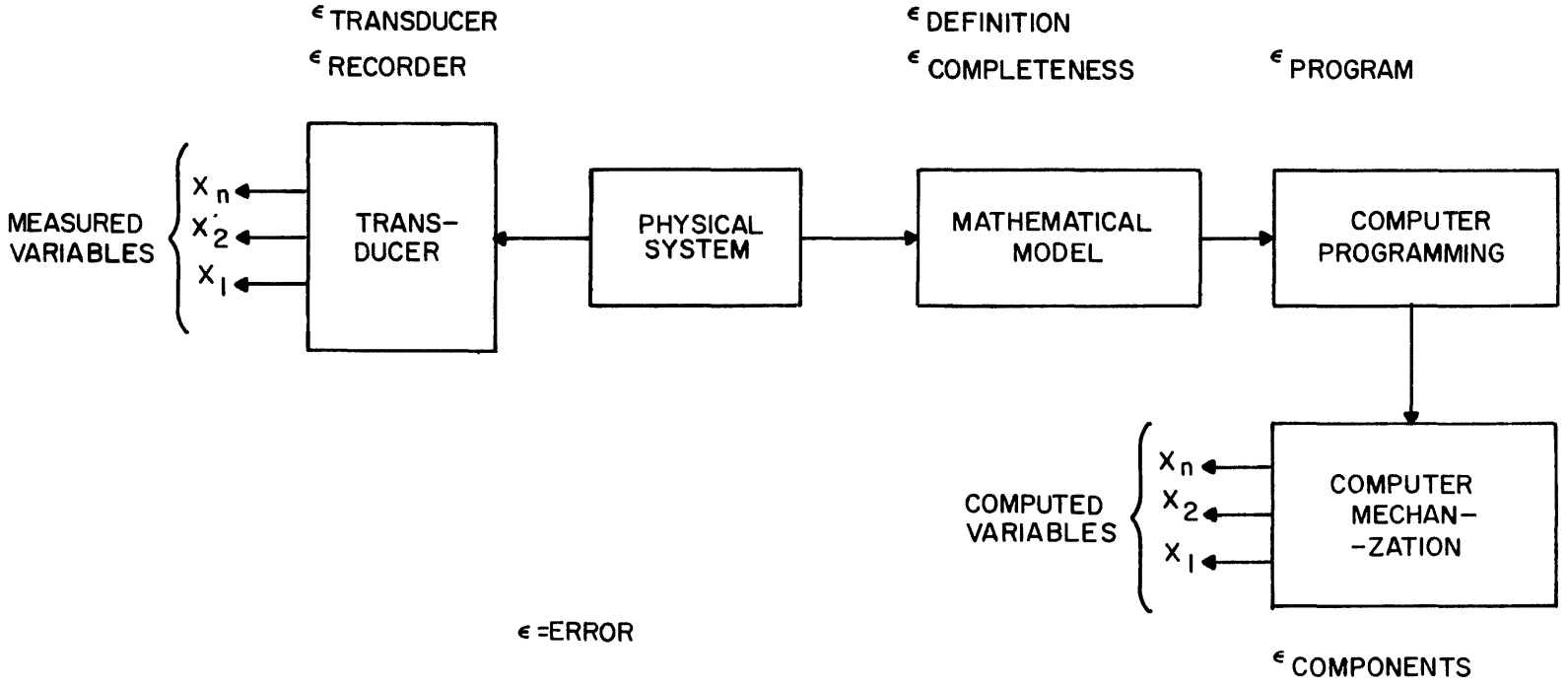


Figure XII-1 Errors in Obtaining Computer Variables

Although a rigorous mathematical model may be derived for the simplified physical system, simplifications will often be made in practice. Examples of this include

1) using steady-state equations for relatively small time-constant equations
2) neglecting minor physical phenomena, such as heat transferred through an insulated wall
3) considering physical properties of substances independent of the state of the system (i.e., temperature and pressure)

Mathematical models containing these simplifications, therefore, have a compound error

1) they are based on a simplified physical model of the system
2) they are reduced in completeness by certain assumptions used to reduce their complexity.

C. Computer Programming and Mechanization Errors

1. Component Errors

Prior to discussing programming and mechanization errors, it should be realized that two types of errors are associated with all analog components:

1) Static Errors -- the deviation of a component's output from its actual value based on constant (zero frequency) inputs
2) Dynamic Errors -- all components have frequency-dependent transfer functions which makes their accuracy a function of their input frequency (this will be discussed later for amplifiers and integrators).

The sum of these errors, which is the total instantaneous dynamic error (TIDE), is, therefore, frequency dependent.

The concept of TIDE is very important in evaluating components for their selection to perform a specific task in a simulation.

2. Mechanization Errors

In programming, several computer circuits or components often are available to perform the same mathematical operation. Since all components have TIDE associated with their operation, one circuit or component will be optimum for a specific task. Therefore, it is desirable that programs be devised in an efficient manner using the guide lines described in Chapter X. Programming errors due to the selection of inaccurate circuits are human errors; they can be corrected.

Typical examples of instances where computer component error can be reduced are

1) the reduction of the number of amplifiers in a mechanization by reprogramming
2) optimize potentiometer settings by rescaling
3) careful consideration of variable frequencies before selecting a multiplier (servo versus quarter square).

D. Data and Readout Errors

Additional errors in analog computer solutions can be attributed to:

1. Parametric Data Inaccuracies

Inaccurate parametric data is frequently the biggest factor in the overall error associated with analog computer results. For example, if parameters are accurate to within 5 or 10% , one would not be concerned with a possible 1% computer error. The same considerations would apply if one were considering a digital computer for a specific application (because of its accuracy) rather than an analog computer.

2. Readout Device Inaccuracies

For non-stationary solutions, the accuracy of recording devices which are frequency-dependent is important and should be considered. For example, the dynamic error of an x-y plotter being driven at 10 inches per second is 0.1% of full scale.

Normally, voltages within the computer are scaled to be as large as possible without overloading the computing components. However, where a quantity changes its magnitude considerably during a solution, one still has a difficulty measuring it for small values. Sometimes it is possible to overcome this difficulty by rewriting the problem. This is particularly so if one can make use of the adjoint technique described in the literature (7),(8). Furthermore, rewriting the problem, possibly making some approximations, may lead to a need for considerably less equipment. This would lead to smaller machine errors by introducing known errors of approximation. Both of these techniques have been used successfully to reduce the order of the machine errors associated with problem variables.

E. Comparison of Analog Solutions and General Comments

In general, one may conclude that definition, completeness and parametric data errors will appear in results regardless of the method of computation used. These errors are invariant with regard to analytical, analog and digital computer solutions. Readout and individual component errors are hardware limitations; they must be recognized and taken into account.

Hardware advancements have been significant in the last decade. For example,

1) quarter square multipliers now have static errors on the order of 0.02% compared to earlier models whose error was 0.5 to 1.0%.
2) linear component errors have been reduced from 0.1% to 0.01%.

Since the error associated with an analog solution is time dependent, its determination by analysis is very difficult and, at times, virtually impossible. Therefore, a comparison of analog, digital and experimental solutions is the most practical method of determining overall solution error.

In this regard, one must realize that experimental data is also subject to errors as shown in Figure XII-1. These errors are attributed to:

1) recording devices
2) transducers
3) sampling geometries and techniques.

Transducer and recording errors are usually on the order of 1.0%.

In comparing analog and digital solutions, it should be realized that the accuracy of a digital solution depends on the digital programmer himself, as well as the equipment. It is his skill, the size and speed of the computer and the time allotted to obtain a solution which determine the accuracy of the final result.

In summary, analog solutions have an inherent resultant error on the order of 0.1 to 5.0%. In most cases, this error is one or two percent.

F. Error Theory Investigations

Having briefly reflected on the overall situation, and indicated the degree of accuracy possible, we will now review some of the work that has been done on error theory during the last few years. It has, for the most part, been limited to an investigation of the errors to be expected in a machine solution of linear ordinary differential equations with constant coefficients.

1. Practical Considerations: Errors in Operational Amplifiers

As pointed out by Bode (1), the transfer characteristic of a summing amplifier can never be expressed as a simple gain. Instead, the simplest transfer function will take the form

$$KG(s) = \frac{K_1}{1 + T_1 s} \qquad (1)$$

where K_1 and T_1 represent the amplifier gain and time constant. The attenuation, phase characteristics and step response of such an amplifier are shown in Figure XII-2.

When used as an integrator, the ideal transfer characteristic would be 1/s. However, this again is physically unrealizable for it requires an infinite gain at zero frequency, and the normal integrating amplifier will have a simple transfer function of the form

$$KG(s) = \frac{K}{s(1 + 1/\mu s)} \qquad (2)$$

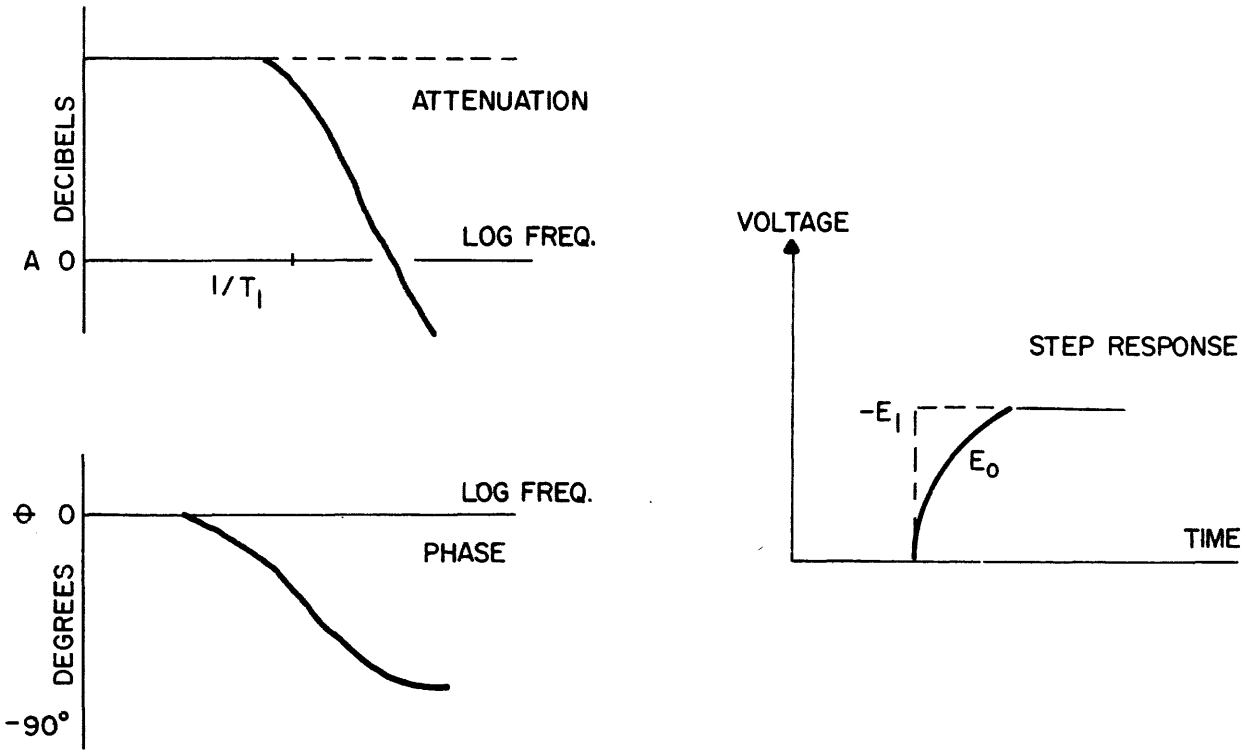


Figure XII-2 Summer Amplifier Characteristics

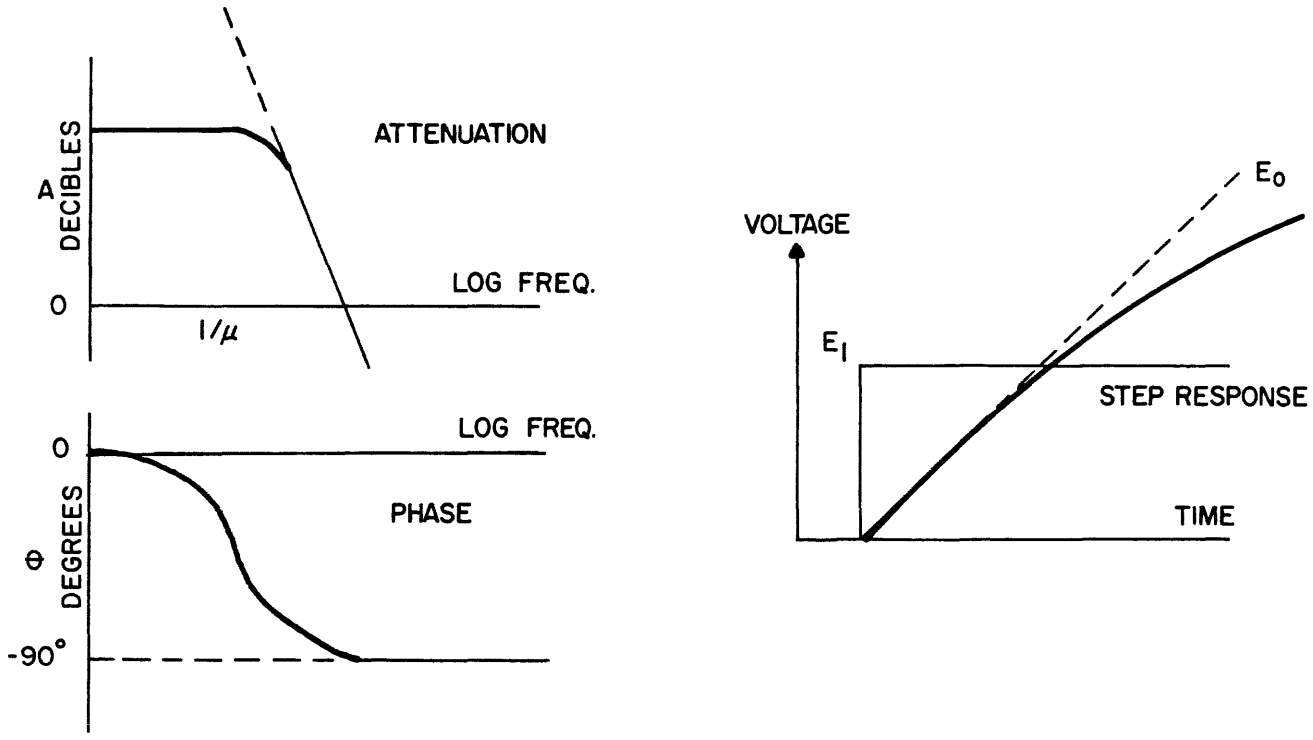


Figure XII-3 Integrator Amplifier Characteristics

where K is the open loop gain of the amplifier. The attenuation, phase characteristics and step response are shown in Figure XII-3. This transfer characteristic implies that the high frequency response of the integrator is perfect. This is not so and, therefore, we should add another term to the above transfer function

$$KG(s) = \frac{K_2}{s(1 + 1/\mu s)(1 + T_2 s)} \qquad (3)$$

where K_2 and T_2 again represent the amplifier gain and time constant.

As shown by Macnee (2), the effect of these two shortcomings in the operational amplifier is to increase the order of any differential equation that is applied to the computer for solution. Provided the bandwidth of the amplifiers is such that the extra roots introduced into the characteristic equation by the computer are far removed from the original roots, and also provided that these extra roots have negative real parts, then the machine solution will be very close to the true solution. The requirement that the real parts are negative is satisfied in the case of summing and integrating amplifiers, but would not be satisfied for any differentiating circuit.

This theoretical consideration implies that the high frequency cut-off of the operational amplifier must be as high as possible in order to accommodate the solutions of differential equations containing high natural frequencies. Where the natural frequencies contained in the solutions of differential equations are too high for the operational amplifiers in use, they have to be reduced by changing the time scale.

If one wishes to estimate the errors contained in a machine solution, an expression developed by Marsocci (3), which gives the change in roots of the characteristic equation explicitly in terms of μ, T_1, T_2, may be of value.

2. The Theoretical Approach: Previous Investigations of Analog Errors

Murray and Miller (4), discuss the problem of error theory applied to analog computers in a more theoretical manner. They define three types of error which could occur in a machine solution, namely:

α errors: these are errors introduced by components of the machine, but which do not arise from an increase in the order of the differential equation solved.

β errors: these are errors which arise in the course of a machine computation as a result of instantaneous disturbances of the solution.

λ errors: these are errors which arise due to an increase in the order of the differential equation.

The approach used by the authors was purely mathematical, and attempted to show how the solution, being analytical in α, β and λ, depended on these errors.

Further theoretical work was done by Winson, which amplified the Murray-Miller theory to a point of practical application to linear systems with constant coefficients. However, the analysis is difficult and does not appear to have any easy practical application.

Howe and Gilbert (5), have obtained practical estimates of the static and dynamic error of a resolver circuit. Their approach to the problem was to add an error term to the individual terms in the resolution equations, and to solve for the errors in the resultant solution. This method, which is applicable for a small problem, becomes prohibitive for large or extremely complex problems.

Meissinger (6), has employed iterative and variational procedures to improve solution accuracy by substituting the analog computer solution back into the original differential equations. These methods, which are too complex to present in detail, have the advantage of using information which is readily available; therefore, results are obtained rapidly compared to numerical check solutions, etc. These techniques also are valuable in locating the sources of computer error. Their only drawback is the amount of computation components required to implement the method, which is a function of problem complexity.

G. Conclusions

One has to conclude that the determination of errors that can occur in an analog computer solution is not an easy task. This is particularly true when the problem being solved demands the use of large numbers of computing components. In consequence, it is normal to use the computer to obtain qualitative results in order to determine the general characteristics of a physical system.

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CHAPTER XIII

EXAMPLES OF EFFICIENT PROGRAMMING

Introduction

The purpose of this chapter is to provide the student with actual examples of the programming procedures described in the previous sections. It is suggested that these examples be used as a basis for review and study by referring back to the appropriate subsections as the problem solution is developed.

Example #1: The Study of an Aircraft Arresting Gear System*

In the design of an aircraft arresting gear system, cable tension specifications are important parameters in determining the maximum landing loads and velocities that a particular system can handle safely. Cable performance may be described in terms of the following specifications:

1. Maximum rated cable tension, that value of cable tension which will cause the cable either to snap or to be permanently deformed, i.e., to exceed its elastic limit.

2. Working limit cable tension, that value which generally is defined as one-half of the maximum rated cable tension. Pragmatically, it is that value of cable tension such that if cable tension is constrained below this value, the cable lifetime will be essentially indefinite.

The cable tension is assumed to be proportional to its elastic stretch under the action of an applied force.

From previous design experience, modern development in equipment, new ideas, etc., a design configuration for the system has been proposed, and this is depicted in Figure XIII-1. The symbols and units used are given in Table XIII-I.

The basic arresting gear system is symmetrical mechanically, and it is assumed that the aircraft arresting hook engages the dead center of the cable about sheaves s_3 and s_3' during landing. One-half of the overall system consists of a piston, p, which produces a drag force proportional to a function of its instantaneous displacement and velocity, linked by means of Cable 2 to a carriage of mass, m_2, which runs on tracks parallel to the runway. The cable tension is proportional to the product of the Cable 2 force constant, and the extension of Cable 2 under the action of an applied force. The carriage is linked to the system by means of Cable 1 which extends from its anchor position at w to fixed pulley, s_1, about carriage pulley, s_2, and fixed pulleys, s_3 and s_3'. The system is originally in static equilibrium with all cables taut.

* Courtesy of All American Engineering Co., Wilmington, Delaware

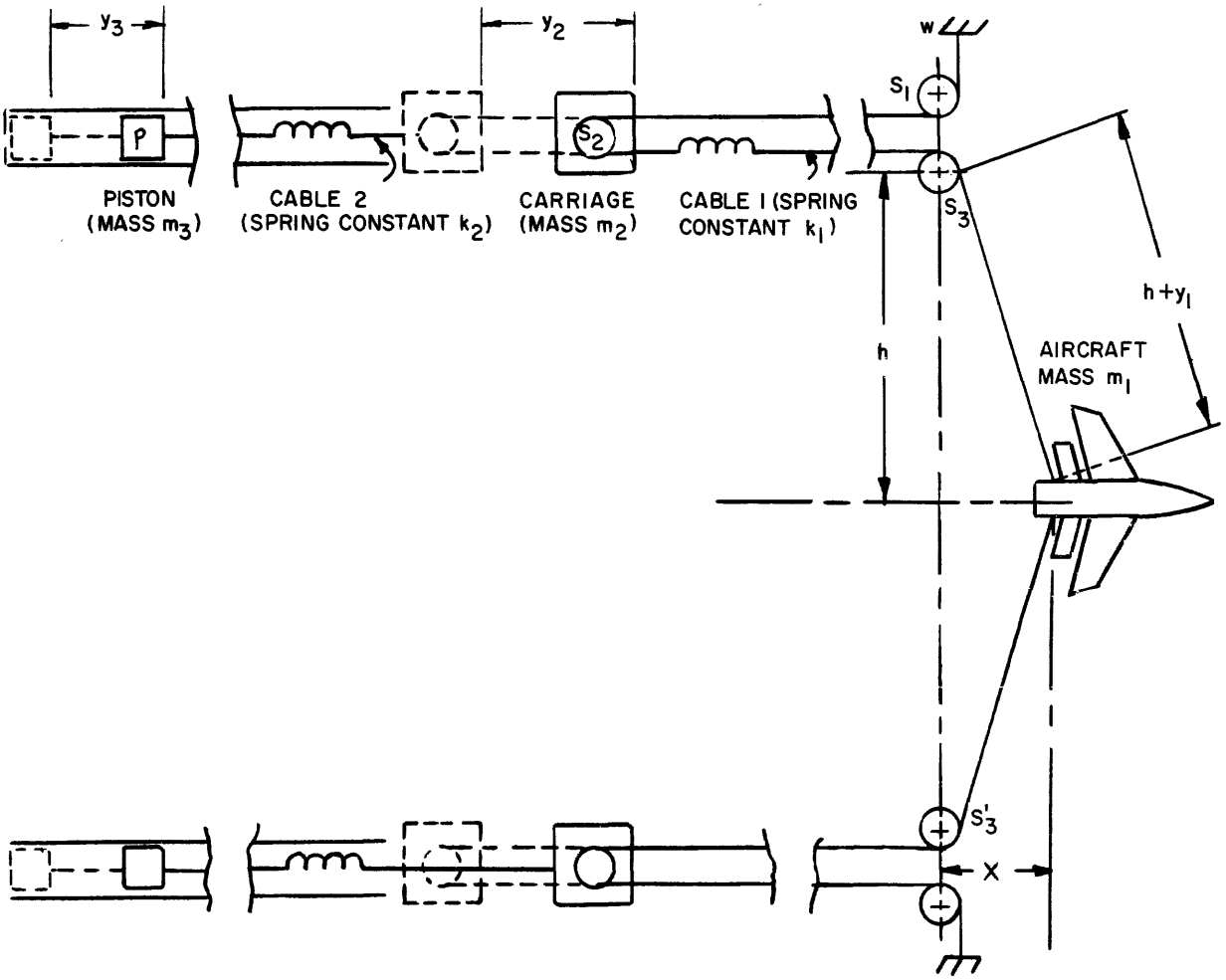


Figure XIII-1 Basic Aircraft Arresting Gear System

The types of aircraft to be considered for this study are classified as follows:

1. Fighter airplanes in the weight range 6-20 tons, with landing speeds of approximately 110-160 mph.

2. Light bomber aircraft in a weight class up to 23 tons, with landing speeds of 140-200 mph.

The objectives of the study are twofold:

1. To determine the system response over a range of aircraft landing speeds and loads.

2. To determine those combinations of aircraft landing speeds and weights within the above range, which would result in cable tension exceeding the working limit cable tension.

Table XIII-I: Symbols and Units

Symbol	Meaning	Units
m_1	mass of aircraft plus effective mass of cable	slugs
m_2	mass of carriage plus effective mass of cable	slugs
m_3	mass of piston plus effective mass of cable	slugs
k_1	Cable No. 1 spring constant	lbs/ft.
k_2	Cable No. 2 spring constant	lbs/ft.
h	One half effective runway width = one half transverse cable	ft.
x	aircraft travel	ft.
$\dot{x}$	aircraft velocity	ft/sec.
$\ddot{x}$	aircraft deceleration	ft/sec^2
y_1	cable payout	ft.
y_2	carriage displacement	ft.
$\dot{y}_2$	carriage velocity	ft/sec.
$\ddot{y}_2$	carriage acceleration	ft/sec^2
y_3	piston displacement	ft.
$\dot{y}_3$	piston velocity	ft/sec.
$\ddot{y}_3$	piston acceleration	ft/sec^2
$f(y_3)$	piston drag coefficient	$lbs/(ft/sec)^2$

1. Mathematical Model

The basic system equations can be derived by considering the forces acting on the system elements. These equations will be summarized at the end of this section.

Piston forces...figure XIII-2 illustrates the forces acting on the piston. It is assumed that the piston arm is weightless and that the piston mass, m_3, includes an "effective weight" of the cable. These assumptions simplify the analysis to that of a single dimension, lumped parameter system.

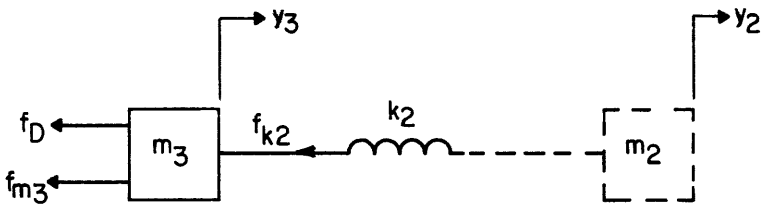


Figure XIII-2: Forces Acting on the Piston

f_D = piston drag force, proportional to the instantaneous displacement and the square of piston velocity

$= f(y_3)\ (\dot{y}_3)^2$

f_{k2} = piston driving force, equal to the tension in Cable 2

$= k_2(y_2-y_3)$

f_{m3} = inertia force

By d'Alembert's principle, the sum of the external forces and the inertial force acting on a particle is zero. From Figure XIII-2,

$$f_{m3} + f_D = f_{k2}$$

or

$$m_3\ddot{y} + f(y_3)\ (\dot{y}_3)^2 = k_2(y_2 - y_3)$$

The piston drag coefficient, $f(y_3)$, is available empirically, and is shown graphically in Figure XIII-3.

Carriage forces...the forces acting on the carriage are shown in Figure XIII-4.

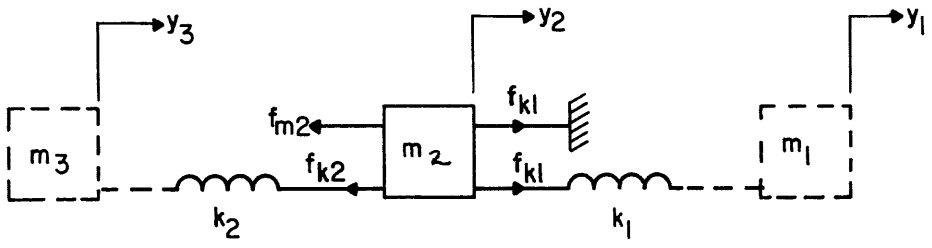


Figure XIII-4: Forces Acting Carriage

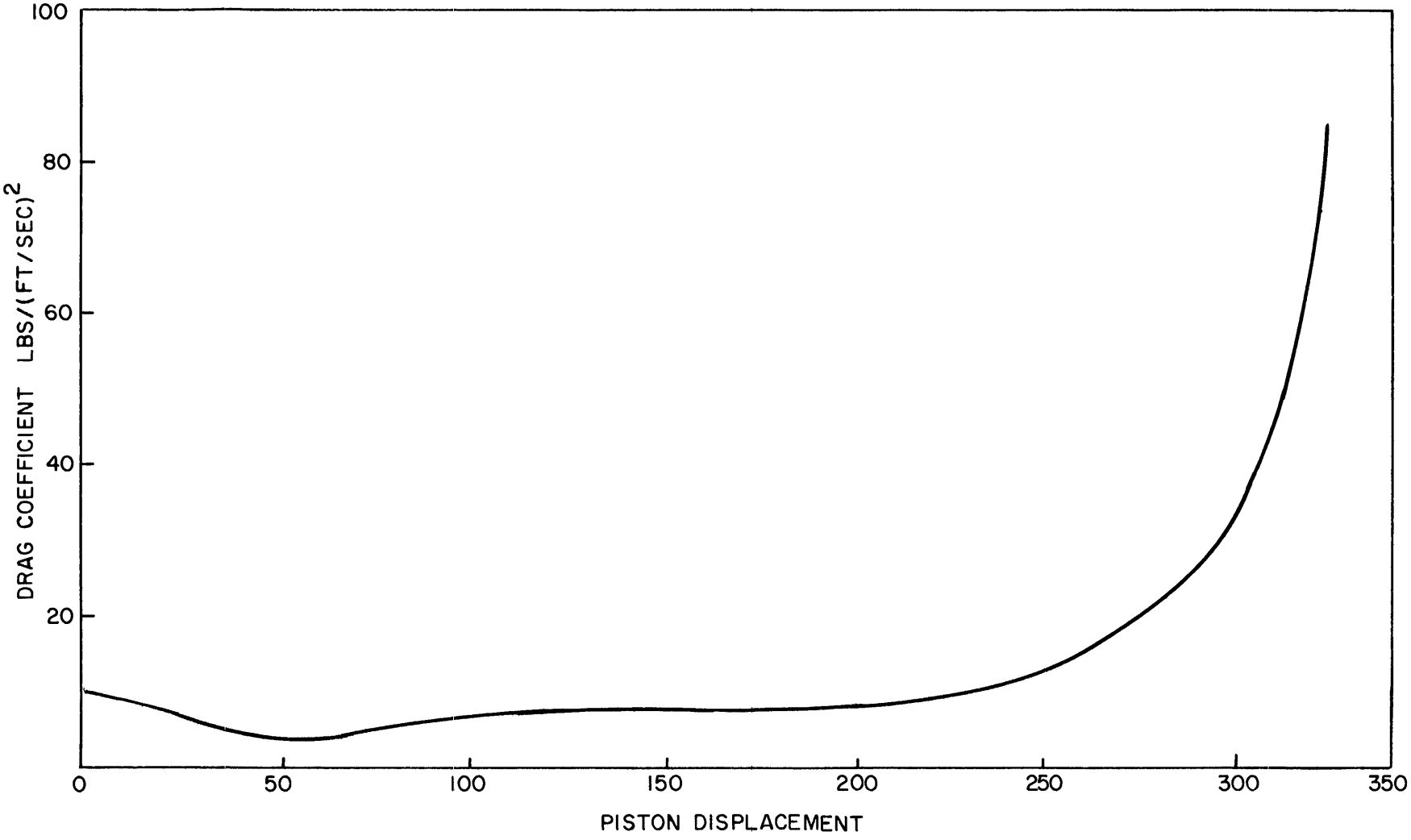


Figure XIII-3: Drag Coefficient $f(y_3)$
vs Piston Displacement Y_3

f_{k1} = tension in Cable 1

$= k_1(y_1 - 2y_2)$

f_{k2} = tension in Cable 2

$= k_2(y_2 - y_3)$

f_{m2} = inertia force

With similar assumptions to those used in the development of the piston equation, the equation of motion of the carriage is written from the force diagram as

$$m_2 \ddot{y}_2 + k_2(y_2 - y_3) = 2k_1 (y_1 - 2y_2)$$

It should be noted at this point that the cable tension cannot be less than zero, i.e., there will be no force transmitted if the cables are slack.

In other words, with the above equations we must include the constraints,

$f_{k1} = k_1(y_1 - 2y_2)$ when $y_1 \geqq 2y_2$

$= 0$ $y_1 < 2y_2$

$f_{k2} = k_2(y_2 - y_3)$ $y_2 \geqq y_3$

$= 0$ $y_2 < y_3$

Forces acting on the aircraft...the aircraft is decelerated by the tension force in Cable 1 acting in a direction opposite to the aircraft's motion, but through an angle θ, as shown in Figure XIII-5.

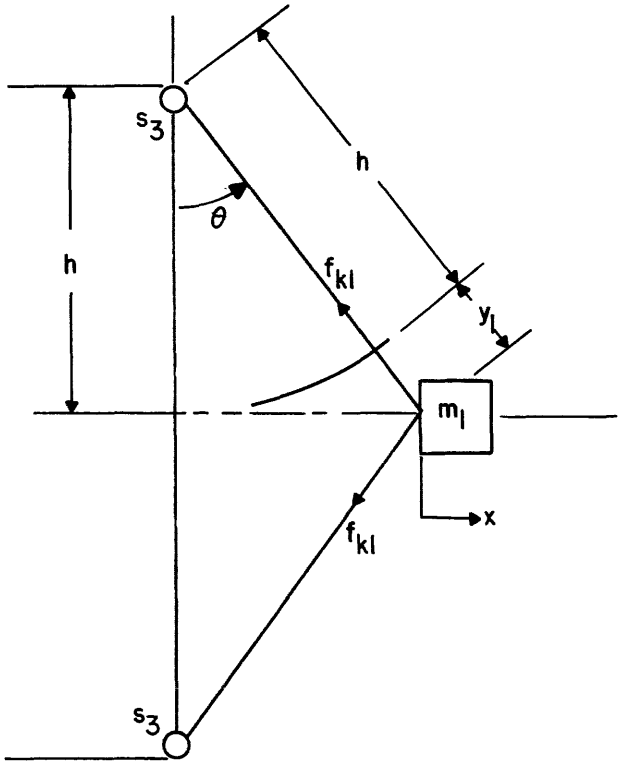


Figure XIII-5: Forces Acting on the Aircraft

From the force diagram, Figure XIII-5, the equation of motion of the aircraft is

$$m_1\ddot{x} + 2k_1 (y_1 - 2y_2) \sin\theta = 0$$

with the constraints

$$\sin\theta = \frac{x}{h + y_1}$$

and

$$y_1 = (x^2 + h^2)^{1/2} - h$$

2. Summary of Equations

Following the usual procedure, the equations to be mechanized on the computer are written for the highest derivative

$$\ddot{y}_3 = \frac{k_2}{m_3}(y_2 - y_3) - \frac{f(y_3)}{m_3} (\dot{y}_3)^2 \qquad (1)$$

$$\ddot{y}_2 = \frac{2k_1}{m_2}(y_1 - 2y_2) - \frac{k_2}{m_2} (y_2 - y_3) \qquad (2)$$

$$\ddot{x} = - \frac{2k_1}{m_1} (y_1 - 2y_2) \sin\theta \qquad (3)$$

$$\text{Sin}\ \theta = \frac{x}{h + y_1} \qquad (4)$$

$$y_1 = (x^2 + h^2)^{1/2} - h \qquad (5)$$

constraints

$$f_{k1} = k_1(y_1 - 2y_2) \qquad y_1 \geq 2y_2 \qquad (6a)$$
$$= 0 \qquad y_1 < 2y_2$$

$$f_{k2} = k_2(y_2 - y_3) \qquad y_2 \geq y_3 \qquad (6b)$$
$$= 0 \qquad y_2 < y_3$$

3. Scaling

Due to the layout of the proposed design, the mechanical limitations of the equipment to be used and from previous knowledge of such systems, the maximum values of some of the variables will be predetermined. Such values and typical values for the parameters are given below.

m_1 = mass of aircraft + cables	= 414 - 1423	slugs
m_2 = mass of carriage + cable	= 45.28	slugs
m_3 = mass of piston + cable	= 29	slugs
h = one-half width of runway	= 125	ft.
y_{3max} = maximum piston drag coeff.	= 90	$lbs/(ft/sec)^2$
$\dot{x}$ = landing speeds of aircraft	= 169 - 295	ft/sec

Due to mechanical restrictions, the size and type of cable which can be used are fixed. Cable parameters can, therefore, be obtained from the manufacturer.

Breaking strength of cable	= 150,000	lbs.
Working limit† cable 2	= 75,000	lbs.
Working limit† cable 1	= 37,500	lbs.
k_1 spring constant cable 1	= 4,550	lbs./ft.
k_2 spring constant cable 2	= 25,300	lbs./ft.

† The working limit, in effect, includes a factor of 2 for repeated cyclic loading. However, in Cable 1, it will be advisable to include an additional factor of 2 for dynamic loading effects. This implies upper limits on forces f_{k1} and f_{k2}. Thus

$$f_{k1} = k_1(y_1 - 2y_2) \leq 37{,}500 \text{ lbs.}$$

$$f_{k2} = k_2(y_2 - y_3) \leq 75{,}000 \text{ lbs.}$$

which will not be programmed since one of the requirements is to be determined by how much these limits are exceeded under certain load conditions.

Referring to the summary of equations, the following variables and derivatives will be represented on the computer and will, therefore, require scaling:

$x, \dot{x}, \ddot{x}$

y_1

$y_2, \dot{y}_2, \ddot{y}_2$

$y_3, \dot{y}_3, \ddot{y}_3$

The acceleration variables, $\ddot{y}_2$, and $\ddot{y}_3$, are not required explicitly in the results. They can be formed at the input to integrators and will not, therefore, require scaling as such. The maximum piston displacement, y_3, is given as 350 ft.

The maximum elongation of both cables based on their breaking strength are

$$y_1 - 2y_2 = \frac{f_{k_1}}{k_1} \simeq 35 \text{ ft.}$$

and

$$y_2 - y_3 = \frac{f_{k_2}}{k_2} \simeq 6 \text{ ft.}$$

Therefore

$$y_1 - 2y_3 = (y_1 - 2y_2) + 2(y_2 - y_3) \simeq 50 \text{ ft.}$$

Since the maximum piston displacement, y_3, is 350 ft., the maximum values of y_1 and y_2 are

$$y_1 = 50 + 2y_3 = 750 \text{ ft.}$$

and

$$y_2 = 6 + y_3 \simeq 360 \text{ ft.}$$

Recalling that

$$x = \sqrt{(y_1 + h)^2 - h^2}$$

and h is 125 ft., the maximum value of x is 860 ft. If the tension in cable 1 is restricted to 25% of its breaking strength by mechanical restrictions as previously stated, the maximum elongation $(y_1 - 2y_2)$ can be reduced from 35 to 20 ft. for scaling purposes.

One method for estimating the maximum deceleration, $\ddot{x}$, would be to consider the Kinetic energy of aircraft (= 1/2 mv^2). Assuming that this energy must be dissipated in the two pistons, then (piston dissipation force) x (displacement) = 1/4 mv^2.

The worst case will be when the complete piston length is required to absorb the energy and from the drag curve, Figure XIII-3, we see that approximately one-half the energy is dissipated in the first 300 ft. and the other half in the remaining 50 ft. This suggests that for the worst case the maximum deceleration will occur in the last 50 ft. of piston travel. Estimating that the force acting on the piston will be twice that required to retard the aircraft, then

$$(f_d)(d) = (2ma)(d) = 1/4\ mv^2$$

where

f_d = dissipation force

d = piston displacement

m = mass of aircraft

v = aircraft velocity = x

a = aircraft acceleration = $\ddot{x}$

$a = \ddot{x} = \frac{1}{8} \frac{v^2}{d}$ can be assumed to be a maximum

when

$v = \dot{x}$ = maximum = 300 ft/sec

d = minimum = 50 ft.

Therefore

$$\ddot{x}_{max} = \frac{1}{8} \frac{(300)^2}{50} = 225 \text{ ft/sec}^2$$

To determine the maximum values of $\dot{y}_2$ and $\dot{y}_3$, assume that

$$\dot{y}_{2max} \approx \dot{y}_{3max} \approx 1/2 \, \dot{y}_{1max}$$

Estimating that $\dot{y}_{1max}$ will be approximately equal to $\dot{x}_{max}$

$$y_{1max} \approx y_{max} \approx 300 \text{ ft/sec.}$$

Then $\dot{y}_2$ and $\dot{y}_3$ have maximum values of 150 ft/sec.

Now that all necessary information is available, the variables may be scaled. Magnitude scaling , and system parameters, are shown in table XIII - 2 and table XIII - 3.

Table XIII-2: Magnitude Scaling Summary

Physical Variables (Units)	Estimated Maximum Values (Units)	Units	Scale Factors MU/Unit	Computer Variable M.U.
y_1	750	ft	1/1000	$[y_1/1000]$
y_2	360	ft	1/500	$[y_2/500]$
$\dot{y}_2$	150	ft/sec	1/200	$[y_2/200]$
y_3	350	ft	1/500	$[y_3/500]$
$\dot{y}_3$	150	ft/sec	1/200	$[y_3/200]$
x	860	ft	1/1000	$[x/1000]$
$\dot{x}$	205	ft/sec	1/500	$[\dot{x}/500]$
$\ddot{x}$	225	ft/sec^2	1/500	$[\ddot{x}/500]$
$f(y_3)$	90	lbs/(ft/sec)^2	1/100	$\left[\frac{f(y_3)}{100}\right]$
$y_2 - y_3$	6	ft	1/10	$\left[\frac{y_2-y_3}{10}\right]$
$y_1 - 2y_2$	20	ft	1/20	$\left[\frac{y_1-2y_2}{20}\right]$
$h + y_1$	(750 + 125)	ft	1/1000	$[(h + y_1)/1000]$

Table XIII-3: Parameter Summary

Constant Parameters	Independent Parameters, m_1 and $\dot{x}_o$
k_1 = 4550 lbs/ft	$414 \leqq m_1 \leqq 1423$ slugs
k_2 = 25300 lbs/ft	$169 \leqq \dot{x}_o \leqq 295$ ft/sec
m_2 = 45.28 slugs	
m_3 = 20 slugs	
h = 125 ft	

The following constraints must be programmed to account for the unidirectional behavior of cable tensions:

$$\text{Cable 2 cable tension} = k_2(y_2 - y_3) \qquad y_2 \geq y_3$$

$$= 0 \qquad y_3 > y_2$$

$$\text{Cable 1 cable tension} = k_1(y_1 - 2y_2) \qquad y_1 \geq 2y_2$$

$$= 0 \qquad 2y_2 > y_1$$

4. Scaled Equations

The scaled equations can now be written by substituting the scaled variables from table XIII-2 into the original equations.

Piston Equation

$$\ddot{y}_3 = \frac{k_2}{m_3}(y_2 - y_3) - \frac{f(y_3)}{m_3}(\dot{y}_3)^2$$

The appropriate scale factor for $\ddot{y}_3$ is the same as that for $\dot{y}_3$. Direct substitution of the scaled variables into the original equation yields;

$$\left[\frac{\ddot{y}_3}{200}\right] = \frac{k_2}{20m_3}\left[\frac{y_2 - y_3}{10}\right] - \frac{20{,}000}{m_3}\left[\frac{f(y_3)}{100}\right]\left[\frac{\dot{y}_3}{200}\right]^2$$

Coefficients of the scaled variables are pot settings. Substitution of the parameters K_2 and m_3 indicate pot settings of much greater than 1 are required. To reduce pot settings to a value < 1 the equation may be rewritten as follows:

$$\left[\frac{\ddot{y}_3}{200}\right] = 100\left(\frac{k_2}{2000m_3}\right)\left[\frac{(y_2 - y_3)}{10}\right] - 1000\left(\frac{20}{m_3}\right)\left[\frac{f(y_3)}{100}\right]\left[\frac{\dot{y}_3}{200}\right]^2$$

Where 100 and 1000 are amplifier gains and termsin curved brackets are pot settings.

The gain difference of the order of 10 is noted. This could imply that the the second term is poorly scaled. Recalling the physics of the problem, initially the piston velocity $\dot{y}_3$ will be a maximum then decrease. The drag coefficient will initially increase from some minimum value. These quantities will not be a maximum together and therefore neither will their product. It will probably be safe to increase the scaling by a factor of five.

With the above modification the final scaled piston equation becomes:

$$\left[\frac{\ddot{y}_3}{200}\right] = 100\left(\frac{k_2}{2000m_3}\right)\left[\frac{(y_2 - y_3)}{10}\right] - 200\left(\frac{20}{m_3}\right)\left[5\left[\frac{f(y_3)}{100}\right]\left[\frac{\dot{y}_3}{200}\right]^2\right]$$

Note that high integrator gains are required. This may indicate that if the magnitude scaling is satisfactory, time scaling is required.

Carriage Equation

$$\ddot{y}_2 = \frac{2k_1}{m_2}(y_1 - 2y_2) - \frac{k_2}{m_2}(y_2 - y_3)$$

Proceeding as before and since $\ddot{y}_2$ may be scaled the same as $\dot{y}_2$ the scaled equation becomes:

$$\left[\frac{\ddot{y}_2}{200}\right] = \frac{k_1}{5m_2}\left[\frac{y_1 - 2y_2}{20}\right] - \frac{k_2}{20m_2}\left[\frac{y_2 - y_3}{10}\right]$$

or

$$\left[\frac{\ddot{y}_2}{200}\right] = 100\left(\frac{k_1}{500m_2}\right)\left[\frac{y_1 - 2y_2}{20}\right] - 100\left(\frac{k_2}{2000m_2}\right)\left[\frac{y_2 - y_3}{10}\right]$$

Aircraft Equation

Equations 4, 5, & 6 of section 2 may be combined to eliminate sin θ: yielding,

$$\ddot{x} = -\frac{2k_1}{m_1}\left(y_1 - 2y_2\right)\left(\frac{x}{h + y_1}\right)$$

The scaled equation becomes:

$$\left[\frac{\ddot{x}}{500}\right] = -\left(\frac{2K_1}{25m_1}\right)\left[\frac{y_1 - 2y_2}{20}\right]\frac{\left[\frac{x}{1000}\right]}{\left[\frac{h + y_1}{1000}\right]}$$

To summarize the material just presented, the scaled equations are:

Piston

$$\left[\frac{\ddot{y}_3}{200}\right] = 100\left(\frac{k_2}{2000m_3}\right)\left[\frac{y_2 - y_3}{10}\right] - 200\left(\frac{20}{m_3}\right)\left[5\left[\frac{f(y_3)}{100}\right]\left[\frac{\dot{y}_3}{200}\right]^2\right]$$

Carriage Equation

$$\left[\frac{\ddot{y}_2}{200}\right] = 100\left(\frac{k_1}{500m_2}\right)\left[\frac{y_1 - 2y_2}{20}\right] - 100\left(\frac{k_2}{2000m_2}\right)\left[\frac{y_2 - y_3}{10}\right]$$

Aircraft Equation

$$\left[\frac{\ddot{x}}{500}\right] = -\left(\frac{2k_1}{25m_1}\right)\left[\frac{y_1 - 2y_2}{20}\right]\frac{\left[\frac{x}{1000}\right]}{\left[\frac{h + y_1}{1000}\right]}$$

Algebraic Equation

$$y_1 + h = \left(x^2 + h^2\right)^{1/2}$$

Recall that the following constraints must be imposed:

Tension in cable 2 $= k_2(y_2 - y_3)$ $\qquad y_2 \geq y_3$

$= 0$ $\qquad y_2 < y_3$

Tension in cable 1 $= k_1(y_1 - 2y_2)$ $\qquad y_1 \geq 2y_2$

$= 0$ $\qquad y_1 < 2y_2$

5. Time Scale

Drawing the circuit diagram directly from the scaled equations would require large gains at the inputs to the integrators. This indicates a need for a time scale change which can be estimated from the maximum natural frequencies of the system. These frequencies, which can be obtained from equation 1-4, are:

$$\omega_{na} = \sqrt{\frac{2k_1 \sin \theta}{m_1}}$$

assuming $\sin \theta = 1$

$$\omega_{na} = \sqrt{\frac{2 \times 4550}{414}} = 4.7 \text{ rad-sec.}$$

$$\omega_{nc} = \sqrt{\frac{4k_1 + k_2}{m_2}} = 31.0 \text{ rad/sec.}$$

$$\omega_{np} = \sqrt{\frac{k_2}{m_3}} = 35.6 \text{ rad/sec.}$$

The natural frequency of the piston is approximately 5.7 cps and, if one considers a frequency range of 0.02 to 3.0 cps as generally suitable for problem solutions on the analog computer, a time scale of $1/\beta = 1/10$ seems to be a reasonable choice. Furthermore, the landing of the aircraft requires approximately 3 to 5 seconds of real time. Therefore, if $\beta = 10$ the solution on the computer would take 30 to 50 seconds which is quite adequate. In order to perform the time scale change in the computer circuit, each input to each and every integrator must be multiplied by $1/\beta$.

6. Generating the Drag Coefficient

In some cases where a function of this nature is required accurately, it may be advantageous to try an analytic approach to the curve or some part of it. For example, in this case, one could try an exponential type function for the last part and use a function generator to give the first part. The advantage would be that the function generator could then be used to plot an error function rather than the function itself. For our purposes, since the experimental method used to determine the drag relationship was far less accurate than any computing technique, it is convenient to use a standard (10 segment) variable diode function generator for the entire curve simulation shown in Figure XIII-3. A scaled graphical representation of the drag coefficient curve with recommended break points is shown in Figure XIII-6.

7. Programming

From the original equations the unscaled computer diagram can be drawn. This step aids the programmer in setting up a neat program. More important, however, it can point out areas where equipment can be often be minimized. The scaled computer diagram is shown in Figure XIII-7. Note that the equations of constraint are mechanized using standard limiter circuits around amplifiers 08 and 09.

Equipment savings can be realized in the square root and squaring circuits since the sign of the input $\left[\frac{x}{1000}\right]$ does not change. Pot 29 in the feedback of amplifier 17 provides a convenient method for rescaling the multiplier output. The output amplifier of the x^2 card serves the dual purpose of a high gain amplifier for the x^2 card and a summing amplifier for h^2.

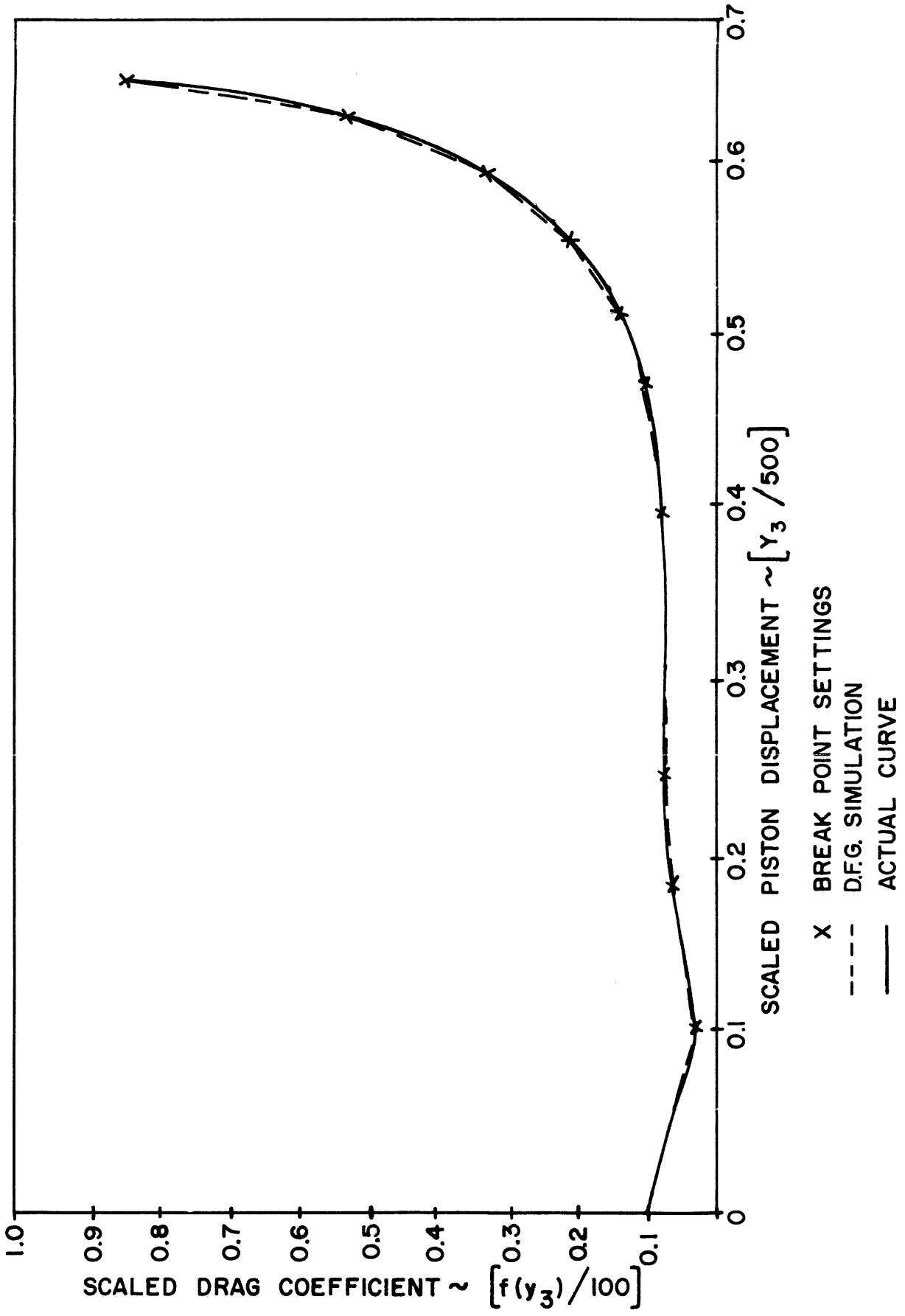


FIGURE XIII - 6

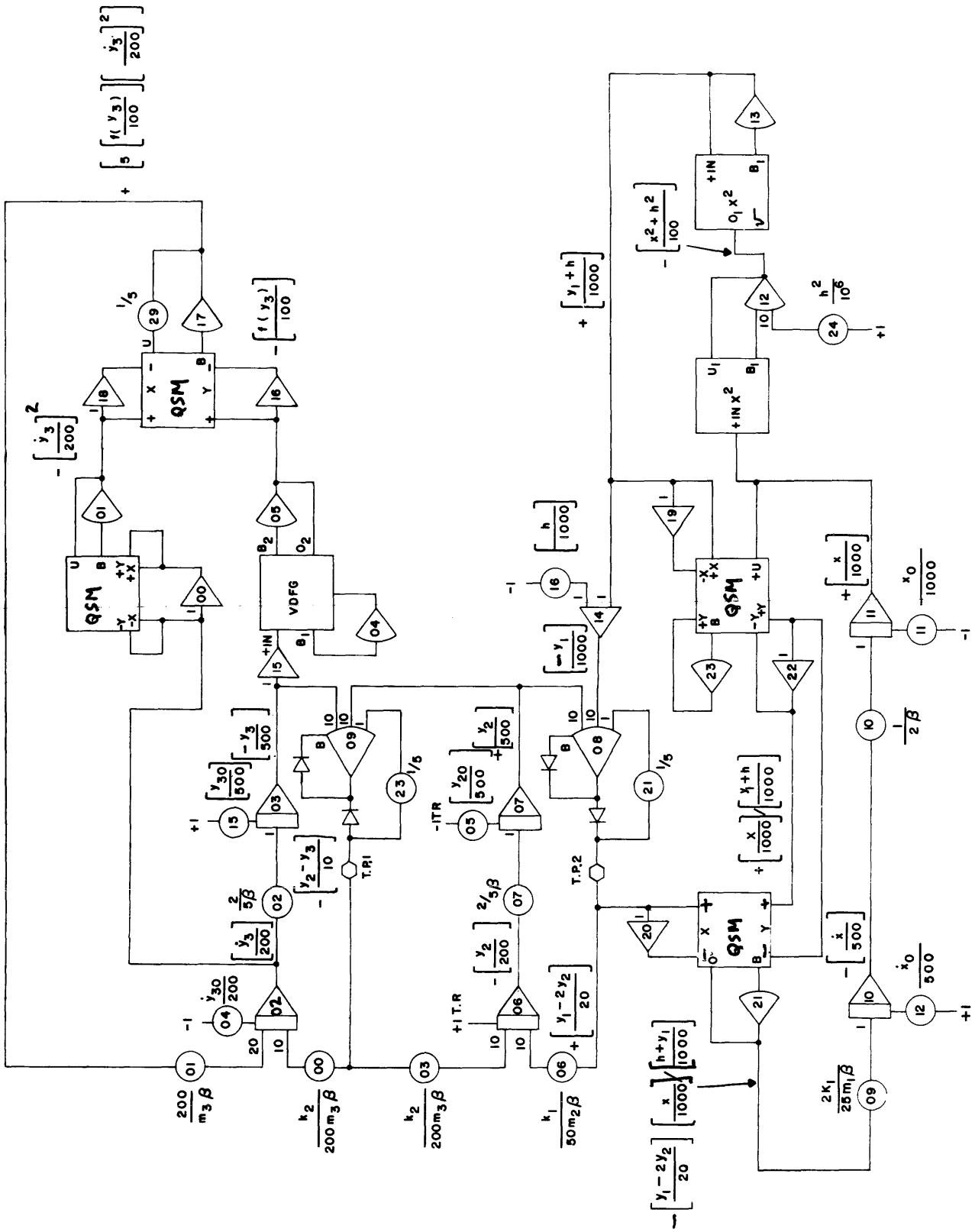


FIGURE XIII - 7 COMPUTER DIAGRAM

ELECTRONIC ASSOCIATES INC.

TR-48 POTENTIOMETER ASSIGNMENT SHEET

P00 – P29

DATE _____ PROBLEM _____

POT NO	PARAMETER DESCRIPTION	SETTING STATIC CHECK	STATIC CHECK OUTPUT	SETTING RUN NUMBER 1	NOTES	POT NO
00	$R_2/200m_3\beta$	.6365				00
01	$200/m_3\beta$	1.000				01
02	$2/5\beta$	.0400				02
03	$R_2/200m_2\beta$	.2793				03
04	$\sqrt{30}/200$	.5000				04
05	$\sqrt{20}/500$	.4200				05
06	$R_1/50m_2\beta$	.2009				06
07	$2/5\beta$	.0400				07
08						08
09	$2k/25m\beta$	.0728				09
10	$1/2\beta$	.0500				10
11	$x_0/1000$	.5510				11
12	$z_0/500$	.500				12
13						13
14						14
15	$\sqrt{30}/500$	.4000				15
16	$h/1000$	.1250				16
17						17
18						18
19						19
20						20
21	$1/5$	.2000			GAIN POT	21
22						22
23	$1/5$	.2000			GAIN POT	23
24	$h^2/10^6$	.0156				24
25						25
26						26
27						27
28						28
29	$1/5$	.2000			GAIN POT	29

M654

ELECTRONIC ASSOCIATES INC.

TR-48 AMPLIFIER ASSIGNMENT SHEET
A00 — A23

DATE _____ PROBLEM _____

AMP NO.	FB	OUTPUT VARIABLE	STATIC CHECK				NOTES
			CALCULATED		MEASURED		
			CHECK PT.	OUTPUT	CHECK PT.	OUTPUT	
00	INV	$+[\dot{y}_3/200]$		-.50			
01	HG	$-[\dot{y}_3/200]^2$		-.25			
02	INT	$+[\dot{y}_3/200]$	+4.45	+.50			
03	INT	$-[y_3/500]$	-.02	-.40			
04	HG	VDFG					
05	HG	$+[f(y_3)/100]$		+.075			
06	INT	$-[\dot{y}_2/200]$	+.784	-1.00			
07	INT	$+[y_2/500]$	+.04	+.42			
08	HG	$+[y_1 - 2y_2/20]$ OUTPUT AT T.P. 2		+1.00			
09	HG	$-[y_2 - y_3/10]$ OUTPUT AT T.P. 1		-1.00			
10	INT	$-[\dot{x}/500]$	-.071	-.50			
11	INT	$+[x/1000]$	-.025	+.551			
12	HG	$-[(x^2+h^2)/10^6]$		-.3236			
13	HG	$+[(y_1+h)/1000]$		+.567			
14	Σ	$-[y_1/1000]$		-.440			
15	INV	$[y_3/500]$		+.40			
16	INV	$-[f(y_3)/100]$		-.075			
17	HG	$+[5[f(y_3)/100][\dot{y}_3/200]^2]$		+.0937			
18	INV	$+[\dot{y}_3/200]^2$		+.25			
19	INV	$-[(y_1+h)/1000]$		-.567			
20	INV	$-[(y_1-2y_2)/20]$		+1.00			
21	HG	$-[(y_1-2y_2)/20][x/1000]/[(h+y_1)/1000]$		-.972			
22	INV	$+[x/1000]/[(y_1+h)/1000]$		+.972			
23	HG	$-[x/1000]/[(y_1+h)/1000]$		-.972			

M653

8. Static Test

The introduction of IC-voltages into the circuit permits the calculation of output and checkpoint voltages to be expected on the amplifiers and integrators. The IC-voltages (which will be chosen for convenience) need not have any physical significance. The outputs and checkpoints will be calculated from the original equations, to ensure that errors have not been made either in the scaled equations or in the mechanization.

Check Calculations

A. $$\ddot{y}_3 = \frac{k_2}{m_3}(y_2 - y_3) - \frac{f(y_3)}{m_3}(\dot{y}_3)^2$$

$k_2 = 25300$ lbs per ft

$m_3 = 20$ slugs

Let $y_3 = 200$ ft

$y_2 = 210$ ft

$\dot{y}_3 = 100$ ft/sec ;

then $f(y_3) = 7.5$ lbs/(ft per sec)^2

and $$\ddot{y}_3 = \frac{25300}{20}(210-200) - \frac{7.5}{20}(100)^2 = \frac{253000 - 75000}{20}$$

$= 8900$ ft/sec.^{2}

and $$\frac{\ddot{y}_3}{200\beta} = +4.45 \text{ MU}$$ at check point C 02

B. $$\ddot{y}_2 = \frac{2k_1}{m_2}(y_1 - 2y_2) - \frac{k_2}{m_2}(y_2 - y_3)$$

$k_1 = 4550$ lbs per ft.

$m_2 = 45.28$ slugs

Let $y_1 = 440$ ft

and $\dot{y}_2 = 200$ ft per sec.;

then $$\ddot{y}_2 = \frac{9100}{45.28}(440-420) - \frac{25300}{45.28}(210-200)$$

$$= \frac{1}{45.28}(182000 - 253000)$$

$= -1567.68$ ft per sec.^{2}

and $$\frac{-\ddot{y}_2}{200\beta} = +0.784\text{MU}$$ at check point C 06

C. $y_1 = (x^2 + h^2)^{1/2} - h$

$h = 125$ ft

$\therefore x^2 = (y_1 + h)^2 - h^2 = 303.500$

and $x = 551$ ft

$$\ddot{x} = - \frac{2k_1}{m_1} (y_1 - 2y_2) \frac{x}{h + y_1}$$

Let $m_1 = 500$ slugs and $\dot{x} = 250$ ft per sec.;

then $\ddot{x} = - \frac{9100}{500} (440-420)\ 0.9754$

$= - 355$ ft per sec.^{2}

and $\frac{\ddot{x}}{500\beta} = -.071MU$ at check point C *10*

D. For the first derivatives chosen, the check points of integrators 03, 07 and 11 should read

C03	-.02M.U.
C07	+.04M.U.
C11	*-.025* M.U.

A typical set of computer results are presented in Figure XIII-8. The system parameters producing these results are extreme in that they represent system demands which lie outside the range of required system performance. It is, therefore, most interesting in terms of system response. For these conditions, the tension in Cable 1 exceeds the working limit but is still below the maximum allowable cable tension specification. It is of interest to note that at the time of complete cable run-out, the aircraft is still in motion. A number of possibilities may arise at this point, i.e., either the system or the aircraft may be damaged or, perhaps, a successful landing without damage to either may result; however, the simulation ends at this point, since further study of the system response at cable run-out would, of course, require an expanded mathematical model and a more complex computer simulation. Nevertheless, with the existent model, adequate information has been obtained to confirm the original design philosophy, and to indicate satisfactory future system operation.

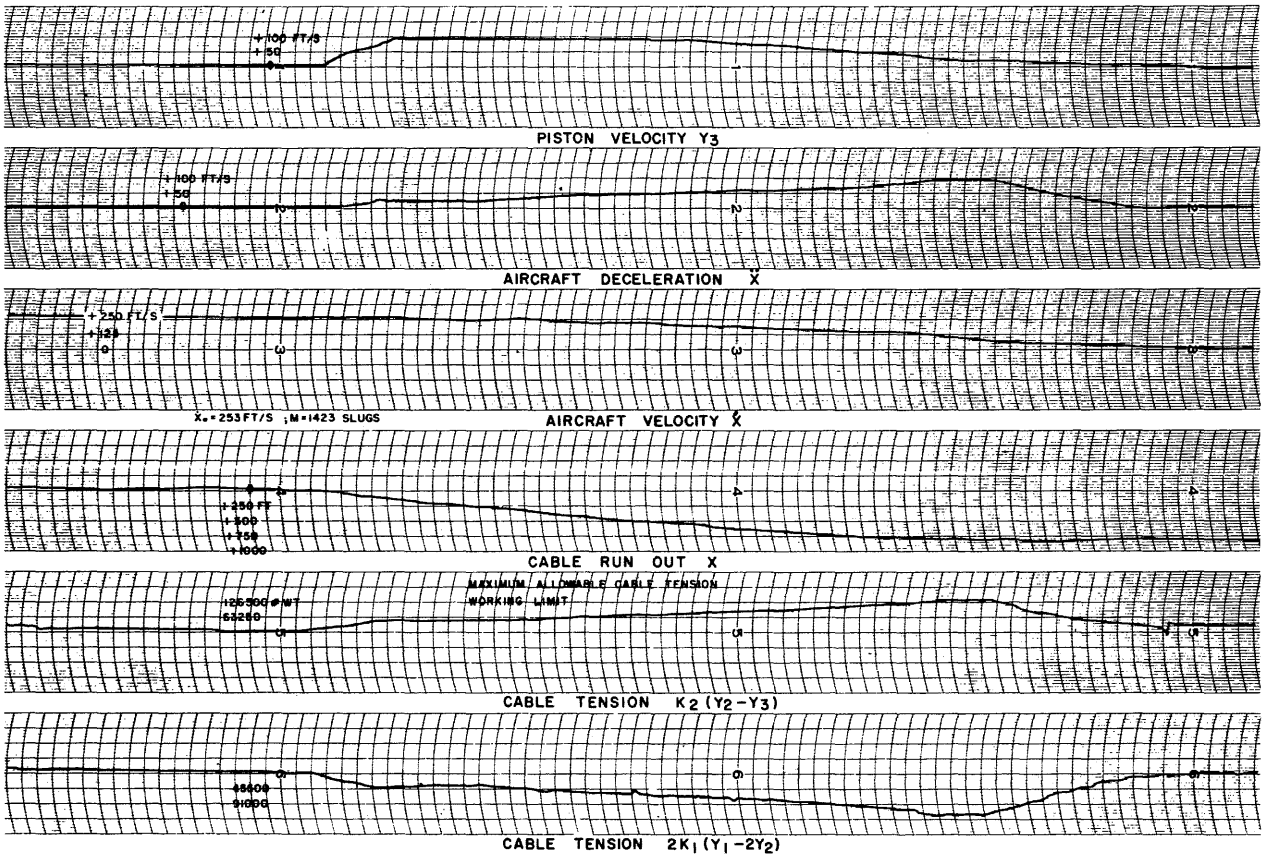


Figure XIII- 8: Typical Results

C. Example #2: Stability of the Molten Zone Refining Process

In the production of transistors, it is necessary to obtain pure, single-crystal germanium or silicon. The principal source of germanium is germanium dioxide, a by-product of zinc mining. It is obtained also from coal ash. The dioxide is reduced in a hydrogen atmosphere, and the resultant metallic germanium is cast into long rods. The purity at this stage is good but still is not sufficient for the manufacture of transistors. Further refining is necessary.

Some manufacturers use the zone refining process which is illustrated in Figure XIII-11, to achieve the high degree of purity required. The rod is placed vertically in a special furnace filled with hydrogen. Starting at the lower end, a small length of the rod is heated to a molten state by using an R-F induction heating coil. As the coil is moved upwards, the molten zone, held together by surface tension, travels along the rod. The impurities in the germanium tend to collect in the molten zone and are removed to the top end. The process can be repeated until the desired purity is achieved. In practice, germanium with impurities of the order of one part in 10^9 is obtained, which is sufficiently pure for use in transistors.

Questions which arise in implementing this refining process include: "Under what conditions should the process be conducted?", "What temperature, what diameter of rod and what length of molten zone will give the most efficient operation with stable conditions?" Obviously, if the molten zone is too long, the surface tension forces will be too small to hold it, and the liquid metal will flow away from the rod. If the molten zone is too small, the process will take too long. The shape of the rod formed below the molten zone must be regular, and similar to the original rod. These questions can be answered by considering the shape of the molten zone under different conditions.

The objectives of this study are to derive and mechanize a mathematical model which describes the surface of revolution of a hanging drop. By considering the resultant curves, together with the necessary constraints of the molten zones (rod diameter, stable shapes, etc.) (1) in the refining process, it will be possible to determine the most efficient and stable operating conditions for the process.

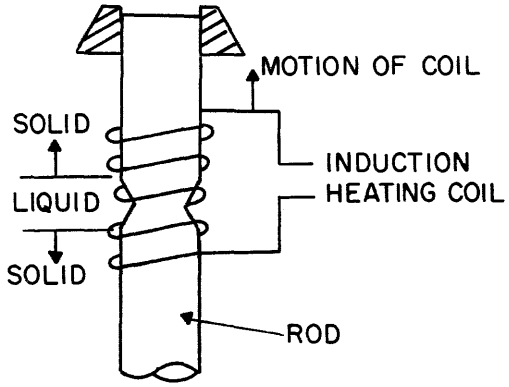


Figure XIII-11: Molten Zone Refining

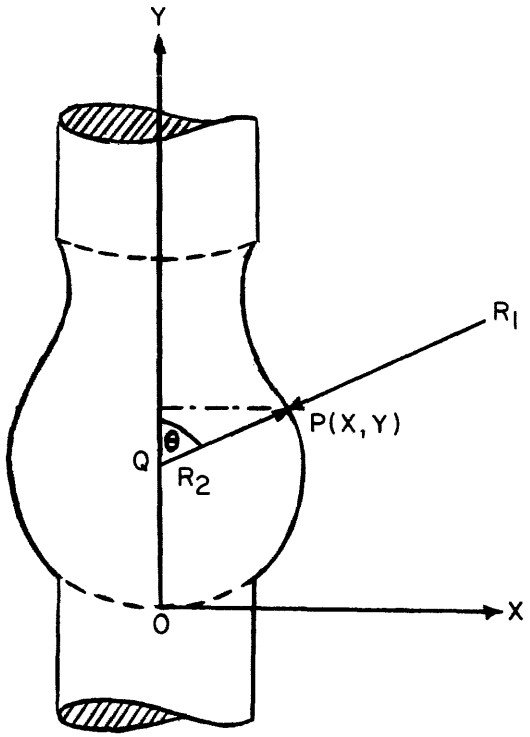


Figure XIII-12: Geometry of Molten Zone

1. Mathematical Model

The general equation for the shapes of liquid surfaces has been developed by many authors (2), and takes the form of the LaPlace equation

$$p = \gamma \quad (1/R_1 + 1/R_2) \tag{1}$$

where

p = pressure difference across the surface membrane at any point

R_1, R_2 = major radii of curvature at a point

and

γ = surface tension of liquid

The coordinates of a point, P, on the surface can be expressed in terms of X and Y, as shown in Figure XIII-12.

The radius of curvature, R_1, is given by the expression

$$R_1 = \frac{\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{2}$$

The radius of curvature, R_2, must be normal to the surface and, as P lies on a surface of revolution, R_2 must be equal to QP where Q lies on the axis OY.

$$R_2 = \left| X \operatorname{cse} \theta \right| = \left| \frac{X \left[1 + \tan^2\theta\right]^{1/2}}{\tan \theta} \right|$$

or

$$R_2 = \left| \frac{X\left[1 + \left\{\frac{dy}{dx}\right\}\right]^2}{\frac{-dy}{dx}} \right|^{1/2} = \frac{X\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\frac{dy}{dx}} \qquad (3)$$

If the pressure at the bottom of a drop is due to the height of liquid, H, then the pressure at the point, P, above the bottom of the drop will be

$$p = \rho g \left[H-Y\right] \qquad (4)$$

Replacing R_1, R_2, and p in the LaPlace equation with their equivalents, we obtain

$$\frac{\frac{d^2y}{dx^2}}{\left[1 + \left\{\frac{d^2y}{dx^2}\right\}^2\right]^{3/2}} + \frac{\frac{dy}{dx}}{x\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]^{1/2}} = \frac{\rho g}{\gamma}\left[H - Y\right] \qquad (5)$$

Introducing the capillary constant, a, having the dimension of length

$$a = \sqrt{\frac{2}{\rho g}}$$

the right hand side of equation 5 becomes

$$\frac{2\left[H-Y\right]}{a^2}$$

By substituting

$$Y = ay, \; X = ax, \; H = ah$$

in equation 5, it can be reduced to dimensionless form and rewritten as

$$\frac{d^2y}{dx^2} = 2\left[h-y\right]\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]^{3/2} - \frac{dy}{dx}\left[1 + \left\{\frac{dy}{dx}\right\}\right]^2 \qquad (6)$$

2. Preliminary Considerations of the Problem

Equation 6 contains two variables, x and y, and a parameter, h, which is a constant for any one condition studied. The general problem is to plot x against y, for values of y in the range from zero to h. To do this, one could regard either x or y as the independent variable and represent it on the computer by time. It is prudent to consider the physical system and how the variables behave with respect to each other before deciding the best approach to mechanizing the equation.

The equation represents the surface of revolution of a hanging liquid drop, and the kind of curves we expect to obtain are similar to that shown in Figure XIII-13.

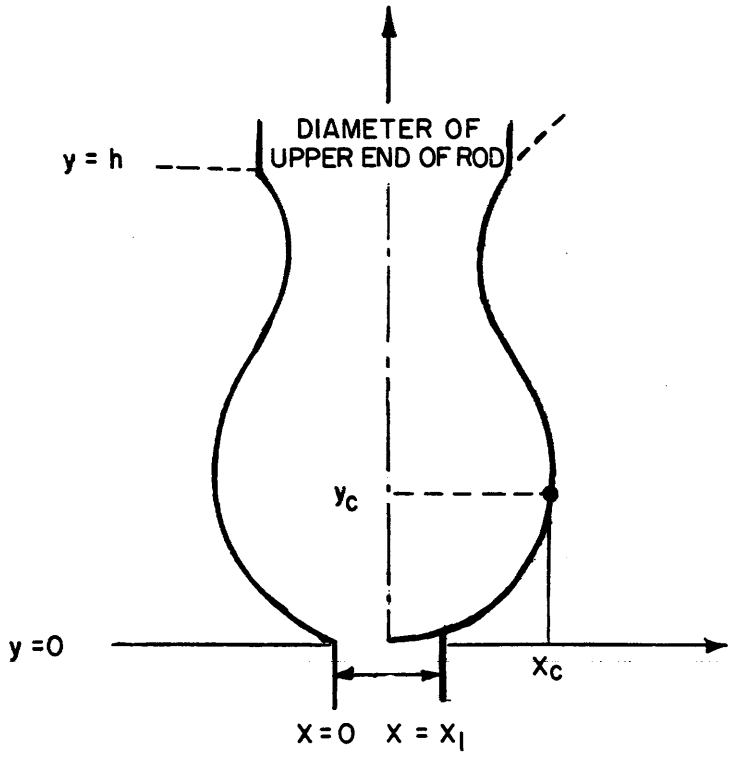


Figure XIII-13: Typical Cross-Section of Molton Zone

For this problem, one is not interested in that portion of the results where $y > h$ or $x < x_1$ (part of the curve shown dotted). However, to obtain a solution, boundary conditions must be determined, and the only point of the curve at which values are known is the origin. Hence, although the region near the origin is of no interest, one must set up the computer solution using the values at the origin as initial conditions.

If x is used as the independent variable, the initial conditions for y(x) are

$$y(0) = 0$$

and

$$\left(\frac{dy}{dx}\right)_{x=0} = 0$$

and a computer circuit appears to be possible. Hoever, in looking at Figure XIII-13, we note that at x_c, dy/dx becomes infinite and the solution could not be completed.

If y is selected as the independent variable, which makes x a function of y, the initial conditions are:

$$x(0) = 0$$

and

$$\left(\frac{dx}{dy}\right)_{y=0} = \infty$$

Therefore, the use of y as the independent variable would be troublesome since it has an infinite initial value.

Based on the above consideration, it would seem necessary to program the problem in two parts (Figure XIII-14). The first part would use x as the independent variable and stop at a reasonable value for dy/dx. At this point, the values of x,y and dy/dx would be recorded to be used as initial conditions for the second part of the solution which uses y as the independent variable. However, further analysis of the problem shows that by transforming equation 6, one can produce a more efficient solution to the problem. The two-part programming description and mechanization can be found in the literature[3].

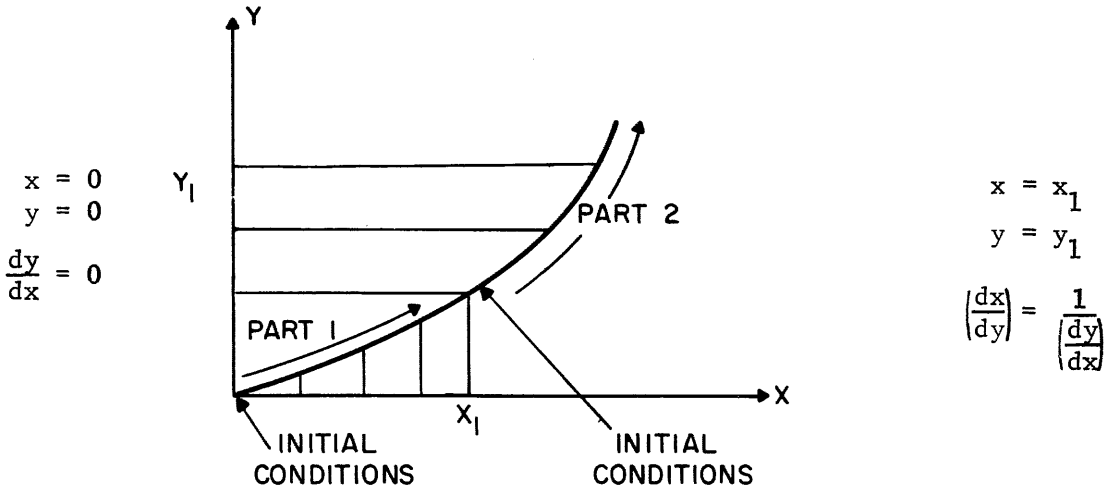


Figure XIII-14: Scheme of Solution of Problem in two parts.

3. Revised Mathematical Model

In this problem, our objective is to plot a family of curves referenced to a set of rectangular coordinates. The device which will produce these curves will be an x-y plotter, where the motion of a carriage will follow one coordinate, and a pen travel normal to the carriage will follow other coordinate. The dynamic response of plotters normally is velocity limited. Since time is not a variable in our original equation (equation 6), we can introduce the constraint

$$\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 = V^2 \tag{7}$$

where V is a suitable constant velocity vector for the pen of the x-y plotter. By fixing an appropriate value for V_1 we can be certain the dynamic response of the x-y plotter will not be exceeded.

The relations between time and space derivatives are†

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$$

and

$$\frac{dy'}{dt} = \left(\frac{dy'}{dx}\right)\left(\frac{dx}{dt}\right) = (y'')(\dot{x})$$

which can be solved for y''

$$y'' = \frac{dy'/dt}{\dot{x}} = \frac{\frac{d}{dt}(y')}{\dot{x}} = \frac{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)}{\dot{x}} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^3}$$

Substituting y' and y'' into equation 6 yields

$$\frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^3} = 2(h-y)\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2} - \frac{\dot{y}}{\dot{x}}\,\frac{\left[1 + (\dot{y}/\dot{x})^2\right]}{x} \tag{8}$$

or

$$\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x} = 2(h-y)\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2} - y\,\frac{\left[(\dot{x})^2 + (\dot{y})^2\right]}{x} \tag{9}$$

which contains both $\dot{x}$ and $\ddot{y}$

Differentiating equation & $(\dot{x}^2 + \dot{y}^2 = V^2)$ with respect to time we obtain

$$\dot{x}\,\ddot{x} + (\dot{y})(\ddot{y}) = 0 \tag{10a}$$

and

$$\ddot{x} = -\frac{\dot{y}}{\dot{x}}\,\ddot{y} \tag{10b}$$

† The dot notation is used for derivatives with respect to time, and the prime notation is used for derivatives with respect to x or y.

or

$$\ddot{y} = -\frac{\dot{x}}{\dot{y}}\ddot{x} \tag{10c}$$

Substituting equations 7 and 10b into equation 9 yields

$$\dot{x}\dot{y} - y\left(\frac{-\dot{y}}{\dot{x}}\right)\ddot{y} = 2\left(h - y\right)v^3 - \frac{\dot{y}}{\dot{x}}v^2$$

or

$$\ddot{y} = \dot{x}\left(2[h-y]V - \frac{\dot{y}}{x}\right) \tag{11}$$

Similarly, an equation for $\ddot{x}$ can be obtained by substituting equations 7 and 10c into equation 9

$$\ddot{x} = -\dot{y}\left(2\left[h-y\right]V - \frac{\dot{y}}{\dot{x}}\right) \tag{12}$$

To define the system completely, the $\dot{y}_o$, $\ddot{y}_o$, $\ddot{x}_o$ in terms of y_o' and y_o'' must be determined.

We know from the geometry of the drop that x_o, y_o, and y_o' are zero and y_o'' is h.

Since

$$y_o' = \frac{\dot{y}_o}{\dot{x}_o} = 0$$

then $\dot{y}_o = 0$ which can be substituted into equation 7 to determine the initial value of $\dot{x}$, $\dot{x}_o = V$

These initial conditions now can be used to determine y_o'' since

$$y_o'' = \frac{1}{\dot{x}_o^3}\left(\ddot{y}_o\,\dot{x}_o - \dot{y}_o\,\ddot{x}_o\right)$$

$$y_o'' = \frac{1}{V^3}\left(y_o V - 0\right)$$

and

$$\ddot{y}_o = V^2 y_o'' = V^2 h$$

then

$$x_o = -\frac{\dot{y}_o}{\dot{x}_o}\ddot{y}_o = \frac{\dot{y}_o}{V}\left(V^2 h\right) = 0$$

Now that the initial conditions are available, equations 11 and 12 can be mechanized readily.

The initial value of $\dot{y}/x$ can be shown by L' Hospital's rule to be equal to Vh. The initial value for this division circuit can be programmed most effectively by using a "steepest descents" division circuit.[3]

4. Scaling and Scaled Voltage Equations

From Figure XIII-3, the maximum value of $\dot{x}$ seems to occur at the origin. Since

$$V^2 = \dot{x}^2 + \dot{y}^2$$

both derivatives can not exceed V, which must be defined. A reasonable value of V, 0.5 units/sec, was selected based on the velocity limitations of the x-y plotter.

The maximum value of x is unity since values of X greater than "a" are of no physical significance (X = ax).

The maximum value of $\dot{y}$ can be obtained directly by specifying h, since $y < h$ has no meaning. From the literature and from experience with the physical system, it was assumed that "h" would not exceed four.

The following scaled table (Table XIII-6) can be used to summarize magnitude scaling.

Table XIII-6: Magnitude Scaling Summary

PHYSICAL VARIABLE (Units)	ESTIMATED MAXIMUM VALUE (Units)	SCALE FACTOR MU/Unit	COMPUTER VARIABLE M.U.
h-y	4	1/4	$\left[\frac{h-y}{4}\right]$
$\dot{x}$	½	2	$[2\,\dot{x}]$
$\dot{y}$	½	2	$[2\,\dot{y}]$
x	1	1	$[x]$
y	4	1/4	$\left[\frac{y}{4}\right]$
$\dot{y}/x$	10	1/10	$\left[\frac{\dot{y}}{10\,x}\right]$

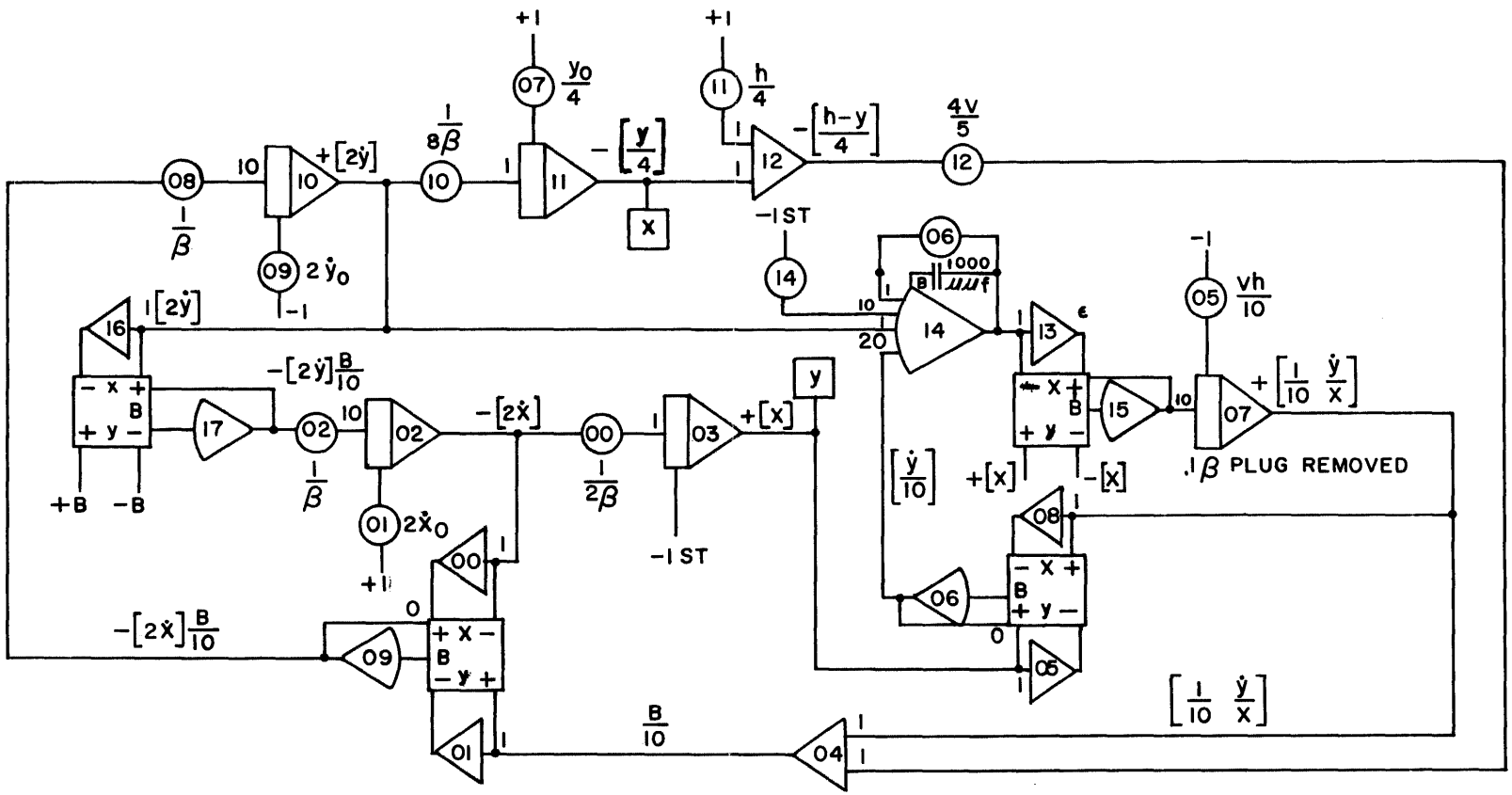
We can now derive the scaled equations

$$[2\,\ddot{y}] = [2\,\dot{x}]\left\{10\left(\frac{8v}{10}\right)\left[\frac{h-y}{4}\right] - 10\left[\frac{\dot{y}}{10x}\right]\right\} \tag{13}$$

and

$$[2\ddot{x}] = -[2\dot{y}]\left\{10\left(\frac{8v}{10}\right)\left[\frac{h-y}{4}\right] - 10\left[\frac{\dot{y}}{10x}\right]\right\} \tag{14}$$

A convenient time scale factor, β, for the simulation, based on potentiometer gain considerations, is ten. The computer diagram for this simulation is shown in Figure XIII-15.



$$\frac{B}{10} = \left\{ \left(\frac{4V}{5}\right) \left[\frac{h-y}{4}\right] - \left[\frac{1}{10}\ \frac{\dot{y}}{X}\right] \right\}$$

FIGURE XIII - 15
COMPUTER DIAGRAM

5. Static Check

The following parameters were selected for the static test run:

$$V = 0.500, \quad h = 3.000$$

The following values were used for initial conditions:

$$\dot{y} = 0.15, \text{ (therefore with } V = 0.5, \quad \dot{x} = 0.476)$$

$$x = 1.0, \quad y = 2$$

Substituting these values in equation 11 yields

$$\ddot{y} = 0.4760\left\{2(0.5)\ (3-2) - \frac{0.1500}{1.0000}\right\}$$

$$= (0.4760)\ (0.8500) = 0.4040$$

which can be multiplied by $200/\beta$ to obtain the check amplifier output ‡ for amplifier 10

$$\left[\frac{2\ \ddot{y}}{\beta}\right] = \frac{(2)(0.404)}{10} = +\ .0808MU$$

Similarly for equation 12

$$x = -\ (0.15)(0.850) = -\ 0.1272$$

and the check amplifier output ‡ for amplifier 02 is

$$\left[\frac{-2\dot{x}}{\beta}\right] = (2)\ \frac{(-.1272)}{10} = +.0254MU$$

The Potentiometer Assignment Sheet is given in Table XIII-7, and the Amplifier Assignment Sheet in Table XIII-8. Calculated values for static test outputs are included in both figures.

6. Results

Curves obtained from method B for values of h = 2.5, 3.0, 3.5, 4.0 are shown in Figure XIII-16. These curves then are used to determine the zone shape for a given value of h, rod radius, and capillary constant. Stable zones were obtained for $h \geq 3.5$.

‡ With the check amplifier function switch in the OUT position.

TR-48 POTENTIOMETER ASSIGNMENT SHEET

P00—P29

DATE _____ PROBLEM _____

POT NO	PARAMETER DESCRIPTION	SETTING STATIC CHECK	STATIC CHECK OUTPUT VOLTAGE	SETTING RUN NUMBER 1 h=2.5	NOTES	POT NO
00	$\frac{1}{2}\beta$	.0500		.0500	β=10	00
01	$2\ \dot{x}_o$	0.952		.0000		01
02	$1/\beta$	.1000		0.100		02
03						03
04						04
05	vh/10	.0150		0.125		05
06	1/G	1.000		.1000		06
07	$yo/_4$	0.500		.0000		07
08	$1/\beta$	0.100		0.100		08
09	$2\dot{y}o$	0.300		.0000		09
10	$1/8\ \beta$	.0125		.0125		10
11	h/4	0.750		0.625		11
12	4 v/5	0.400		0.400		12
13						13
14	Static Test	.0500		.0000		14
15						15
16						16
17						17
18						18
19						19
20						20
21						21
22						22
23						23
24						24
25						25
26						26
27						27
28						28
29						29

TR – 48 AMPLIFIER ASSIGNMENT SHEET
A00 – A23

DATE _____ PROBLEM _____

AMP NO.	FB	OUTPUT VARIABLE	STATIC CHECK				NOTES
			CALCULATED		MEASURED		
			CHECK PT.	OUTPUT	CHECK PT.	OUTPUT	
00	INV	$+ 2\dot{x}$		+0.952			
01	INV	$-\frac{13}{10}$		-.085			
02	INT	$- 2\dot{x}$	-.0254	-0.952			
03	INT	$+ x$	-.0476	+1.00			
04		$\frac{+\beta}{10}$		.085			
05	INV	$- x$		-1.00			
06	HG	$y/10$		-.015			
07	INT	$+\frac{1}{10}\frac{\dot{y}}{\dot{x}}$	-5.0	+.015			Remove .1β plug
08	INV	$-\frac{1}{10}\frac{\dot{y}}{\dot{x}}$		-.015			
09	HG	$-2\dot{x}\frac{\beta}{10}$		-.0808			
10	INT	$+ 2\dot{y}$	-.0808	+0.30			
11	INT	$-y/4$	+.0375	-0.50			
12		$-\frac{h - y}{4}$		-0.250			
13	HG	ϵ		-0.500			
14	HG	ϵ		+0.500			
15	HG	$- x \epsilon$		-0.500			
16	INV	$- 2\dot{y}$		-0.300			
17	HG	$- 2\dot{y}\ \beta/10$		-.0254			
18							
19							
20							
21							
22							
23							

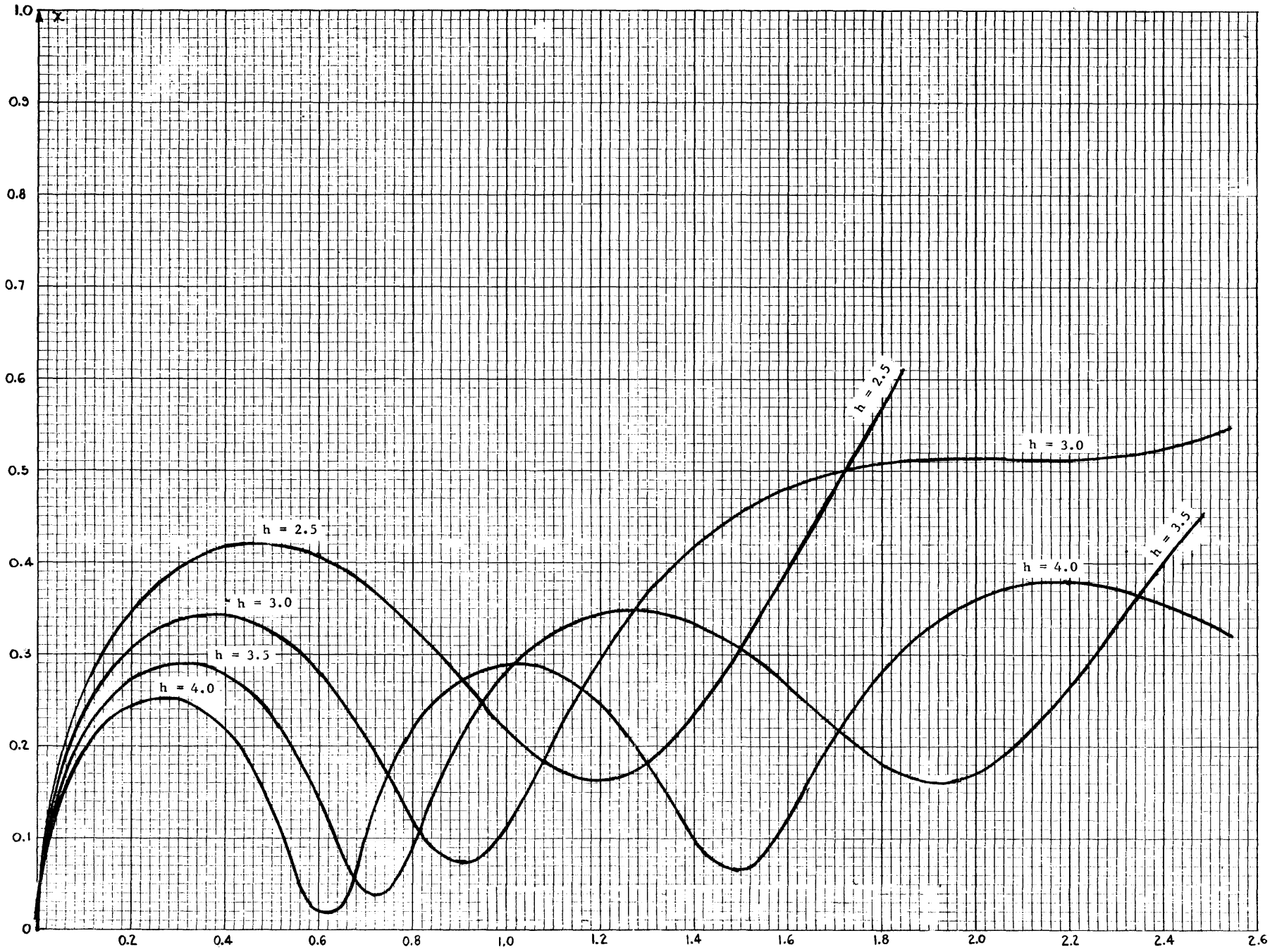


Figure XIII-16: Computer Results

D. Example #3: Transient Behavior of a High Speed Tunnel-Diode Switching Circuit

In the design of high speed switching circuits, such as one might find in a modern digital computer, experimentation with the actual circuit is possible but extremely difficult. The experimental difficulties arise in the measurement of circuit behavior (wave forms, etc.) under high frequency operation. Frequencies, such as those obtained from a digital clock, range up to 10^9 cps. Therefore, analysis and computation are desirable means of evaluating proposed high speed circuits.

This study deals with the analysis and simulation of a proposed tunnel-diode circuit which is shown in Figure XIII-17. For design purposes, a dynamic analysis which takes circuit reactances into account is necessary for at least two reasons:

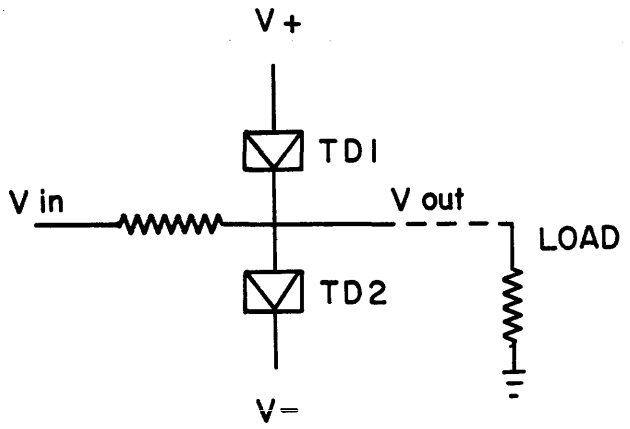


Figure XIII-17: Simplified Schematic for Tunnel-Diode Switching Circuit

1. A knowledge of the switching time is necessary to determine the maximum clock frequency (i.e., the maximum rate of information transfer) at which the system can be operated.

2. The circuit may prove unstable and break into oscillations.

Both the stability and the switching time will depend primarily on the inevitable circuit reactances. The determination of the maximum clock frequency and the stability limit of the circuit are the objectives of this study. The characteristic curve of a typical tunnel diode(s) is given in Figure XIII-18. Note that the slope is negative over the greater portion of the operating range (from about 60 to 300 millivolts). The existence of this negative resistance region means that the equilibrium described in the preceding paragraph will be unstable. A very slight disturbance will upset the symmetry described above, causing a very much larger voltage to appear across one diode than across the other. This sensitivity to small input disturbances enables the circuit to function as a high-gain, high-speed switch.

Assume V_+ and V_- are initially zero and that V_+ gradually V_+ gradually increases while V_- decreases, such that $V_+ + V_- = 0$. Then the operating point of each diode will move from the origin in Figure XIII-18 along the characteristic curve toward the peak. The voltage at the junction will remain zero, as before. After the peak has been passed, the voltages and currents for the two diodes should remain identical (by symmetry), and the voltage at the junction should remain zero. However, since the diodes are now in the negative resistance region, this equilibrium will prove extremely unstable.

Suppose, for instance, that the circuit is unbalanced by a small positive voltage at the input terminal. Then tunnel diode #2 will have a slightly greater voltage across it than tunnel diode #1 and, as the bias voltage increases, tunnel diode #2 will pass its peak first. Further increases in the voltage across TD2 will result in a decrease in the current through it, and, hence, a decrease in the current through TD1 as well. Since TD1 is still in its positive resistance region, a decrease in its current will decrease its voltage drop, increasing the voltage drop across TD2 (since the tunnel diodes are acting as a voltage divider, and the sum of their voltage drops must equal $2V_+$). It follows that TD2 will move rapidly through its negative resistance region, at a speed limited only by circuit reactances, and stabilize somewhere in the positive resistance region above 300 millivolts, while TD1 stabilizes in the positive resistance region below 60 millivolts.

Since most of the voltage drop between the two bias sources appears across TD2, it follows that the voltage at the junction (the output voltage) will be positive. A positive input voltage will, therefore, produce a positive output voltage. The output voltage will be of the same order of magnitude as the bias voltage, V_+, while the input voltage need only be large enough to overcome any inherent imbalance in the diode characteristics. The input voltage thus can be made very small, and the gain of the circuit is limited only by the closeness of the match between diode characteristics and between the positive and negative power supplied. If the input and output voltages are equal in magnitude, as in the case with cascaded logic elements of similar design, high gain means that very large resistors can be used, resulting in very large fan-in and fan-out figures.

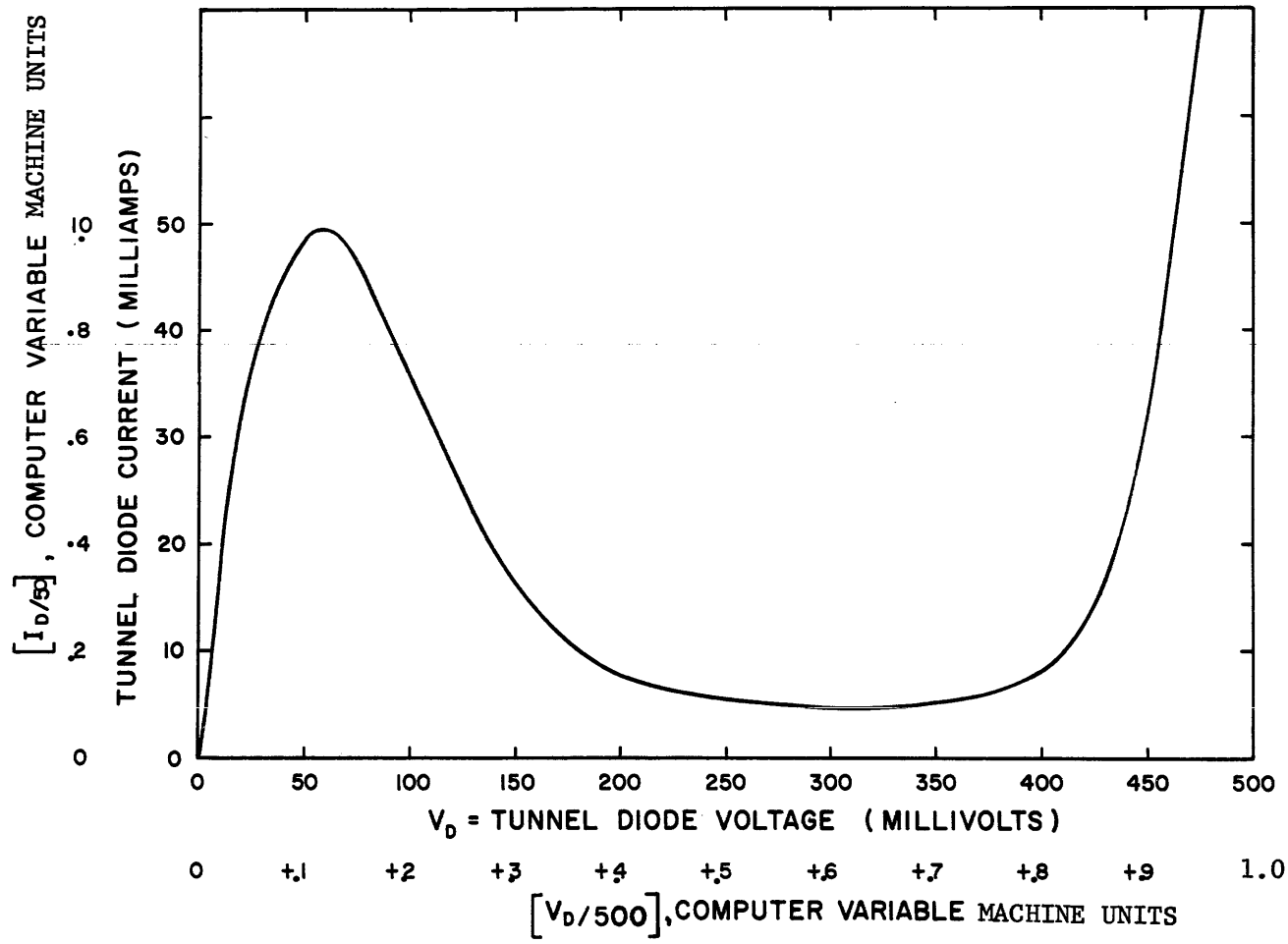


Figure XIII-18: Characteristic Curve of Typical Tunnel Diode

In high speed logic and arithmetic circuits, the voltages V_+ and V_- will be out-of-phase alternating voltages with approximately sinusoidal wave-shapes. D-C levels will be superimposed on these sinusoids and, in most applications, this bias will just equal the amplitude of oscillation, assuring that the bias voltages do not change sign. The alternating bias voltages then will serve as a "clock", supplying the necessary timing signals for high-speed arithmetic and logic. An output can be obtained only during that part of the bias cycle when the bias voltage is large enough to switch one of the diodes into its negative resistance region.

The attractiveness of this circuit for digital applications lies in the fact that the input need only be large enough to unbalance a symmetric circuit. If all componenets were perfectly matched, the slightest input disturbance would unbalance the circuit in the proper direction and the theoretical gain would be infinite. In practice, the gain will be limited by component tolerances. An estimate of the gain(or, equivalently, the fan-in/fan-out capability) of the circuit can be made from a static analysis of the tunnel-diode characteristic and a knowledge of circuit tolerances. Such an analysis has been made by Chow (7) who predicted large gains for closely matched diode.

The equivalent circuit for the system, with circuit reactances and external loading taken into account, is given in Figure XIII-19. The principal reactances involved are lead inductance and case inductance (which may be lumped together), and diode shunt capacitance. Provision is also included for coupling the leads together with mutual inductance which would correspond to dressing the power supply leads close together during construction. A moderate amount of coupling proves to have a beneficial effect on switching time when the circuit is operated at high frequencies.

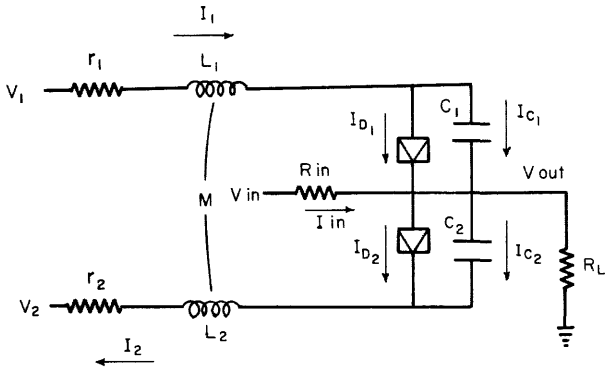


Figure XIII-19: Equivalent Circuit Showing Reactances

1. Mathematical Model

The equations describing the system are given below. These equations are straightforward applications of Kirchoff's laws and the known properties of resistors, capacitors, and inductors. They may be written down immediately upon inspection of Figure XIII-19.

Equations 1 and 2 are Kirchoff voltage equations for the two major loops. Equation 3 is simply the statement that the bias or "clock" voltages are out-of-phase sinusoids with d-c levels superimposed. Equations 4 and 5 state the basic volt-ampere relation of a capacitor, and 6 and 7 are Kirchoff current equations. Equations 8 and 9 state the characteristic relation between voltage and current in the tunnel diodes (see Figure XIII-18). On the computer, these curves will be represented by function generators. Equations 10 and 11 are, of course, statements of Ohm's Law.

System Equations:

$$V_1 = I_1 r_1 + L_1 \dot{I}_1 - M\dot{I}_2 + V_{D_1} + V_{out} \tag{1}$$

$$V_2 = I_2 r_2 + L_2 \dot{I}_2 - M\dot{I}_1 + V_{D_2} - V_{out} \tag{2}$$

$$V_1 = -V_2 = E_{DC} - A \sin \omega t \tag{3}$$

$$\dot{V}_{D_1} = \frac{1}{C_1} I_{C_1} \tag{4}$$

$$\dot{V}_{D_2} = \frac{1}{C_2} I_{C_2} \tag{5}$$

$$I_{C_1} = I_1 - I_{D_1} \tag{6}$$

$$I_{C_2} = I_2 - I_{D_2} \tag{7}$$

$$I_{D_1} = f(V_{D_1}) \tag{8}$$

$$I_{D_2} = f(V_{D_2}) \tag{9}$$

$$V_{out} = (I_1 - I_2 + I_{in})R_L \tag{10}$$

$$I_{in} = \frac{V_{in} - V_{out}}{R_{in}} \tag{11}$$

For the purpose of this simulation, equation 11 was omitted and replaced by the assumption that the input current I_{in} was constant. This simplification was made so that the results could be compared directly with a digital solution which also made this assumption.

For simulation purposes, equation (3) is mechanized readily from its differential equation. It may be differentiated twice to obtain

$$\frac{dv}{dt} = - \omega A \text{ Cos } \omega t$$

and

$$\frac{d^2V}{dt^2} = \omega^2 A \text{ Sin } \omega t$$

where/

$$V = V_1 - E_{DC} = - (V_2 + E_{DC}) = -A \text{ Sin } \omega t$$

Therefore, the mechanization of the oscillator equation

$$\frac{d^2V}{dt^2} + \omega^2 V = 0 \qquad (12)$$

with the initial conditions

$$V(0) = 0 \qquad (13)$$

and

$$\left[\frac{dv}{dt}\right]_{t=0} = - \omega A \qquad (14)$$

can be used to simulate the clock sinusoid.

2. Scaling

Since this switching circuit is part of a much more elaborate system, the system specifications can be used in estimating maximum values. In this case, system specifications call for the clock voltages, V_1 and V_2, not to exceed 500 mv and the current drawn through these inputs, I_1 and I_2, to be less than 50 ma. By consulting Figure XIII-18, it is obvious that the tunnel diode was selected to meet these specifications. Therefore, the maximum values of I_{D_1} and I_{D_2} are also 50 ma. Since the capacitors paralleling the diodes may be called upon to provide the entire current path, I_{C_1} and I_{C_2}, if the diodes do not conduct, their maximum currents are also 50 ma. As seen in the figure, the maximum possible diode voltages, V_{D_1} and V_{D_2}, are 500 mv.

The maximum values of the current derivatives can be estimated directly from equation 1 and 2 after the system parameters have been defined. The parameters are summarized in Table XIII-19 and the estimated maximum value of $\dot{I}_1$ and $\dot{I}_2$ is 2000ma/ns.

Now that all maximum values have been determined, the necessary scale factors can be summarized, as shown in Table XIII-10. This study was carried out on a transistorized, TR-48 analog computer. The scale factors were determined by dividing the maximum values into a one unit reference level.

3. Scaled Equations

From the above information, the following scaled voltage equations can be obtained:

$$\left[\frac{\dot{I}_1}{2000}\right] = \left(\frac{1}{4L_1}\right)\left\{\left[\frac{V_1}{500}\right] - \left(\frac{r_1}{10}\right)\left[\frac{I_1}{50}\right] + (4M)\left[\frac{\dot{I}_2}{2000}\right] - \left[\frac{V_{D_1}}{500}\right] - \left[\frac{V_{out}}{500}\right]\right\} \tag{15}$$

$$\left[\frac{\dot{I}_2}{2000}\right] = \left(\frac{1}{4L_2}\right)\left\{\left[\frac{V_{out}}{500}\right] - \left[\frac{V_{D_2}}{500}\right] - \left[\frac{V_2}{500}\right] + (4M)\left[\frac{I_1}{2000}\right] - \left[\frac{r_2}{10}\right]\left[\frac{I_2}{50}\right]\right\} \tag{16}$$

$$\left[\frac{\dot{V}_{D_1}}{500}\right] = 10\left(\frac{1}{100\ C_1}\right)\left[\frac{I_{C_1}}{50}\right] \tag{17}$$

$$\left[\frac{\dot{V}_{D_2}}{500}\right] = 10\left(\frac{1}{100\ C_2}\right)\left[\frac{I_{C_2}}{50}\right] \tag{18}$$

$$\left[\frac{I_{C_1}}{50}\right] = \left[\frac{I_1}{50}\right] - \left[\frac{I_{D_1}}{50}\right] \tag{19}$$

$$\left[\frac{I_{C_2}}{50}\right] = \left[\frac{I_2}{50}\right] - \left[\frac{I_{D_2}}{50}\right] \tag{20}$$

and

$$\left[\frac{V_{out}}{500}\right] = \left\{\left[\frac{I_1}{50}\right] - \left[\frac{I_2}{50}\right] + \left(\frac{I_{in}}{50}\right)\right\}\left(\frac{1}{10R_L}\right) \tag{21}$$

Table XIII-9: Summary of Parameters

$r_1 = r_2 = 3$ ohms

$R_L = 8$ ohms

$C_1 = C_2 = 10^{-2}$ nanofarads

$L_1 = L_2 = 0.4$ nanohenry

$M \leq \sqrt{L_1 L_2}$ Try M=0 for first run. Probably maximum value around 0.2 nanohenry

$I_{in} = 1.0$ ma.

A = 130 millivolts

Edc = 130 millivolts

f = 150 megacycles/second = 0.15 cycles/nanosecond

$\omega = 2\pi f = 0.9425$ radians/n.s.

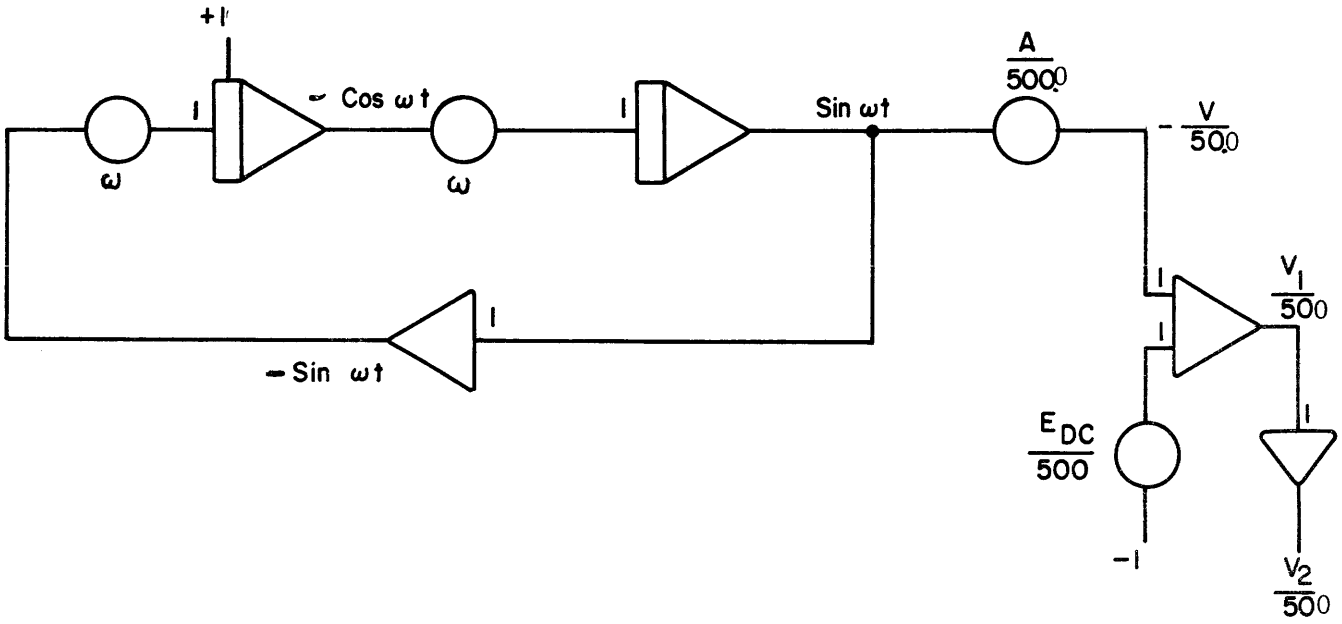
Table XIII-10: Summary of Variables

Physical Variable (Units)	Estimated Max Value (Units)	Units	Scale Factor Machine Units/ Unit	Computer Variable Machine Unit
V_{D1}, V_{D_2}, V_1 V_2	500	mv	1/500	$\left[\frac{V}{500}\right]$
I_{D1}, I_{D_2}	50	ma	1/50	$\left[\frac{I_D}{50}\right]$
I_1, I_2	50	ma	1/50	$\left[\frac{I}{50}\right]$
$\dot{I}_1$, $\dot{I}_2$	2000	ma/ns	1/2000	$\left[\frac{\dot{I}}{1000}\right]$
I_{C1}, I_{C_2}	50	ma	1/50	$\left[\frac{Ic}{50}\right]$

The oscillator equations can be rewritten as

$$\frac{V}{A} = \left[1 \text{ Sin } \omega t\right] = \left(\frac{500}{A}\right)\left[\frac{V}{500}\right]$$

to program (1 Sin ωt) directly.



The scale factors for V, V_1, and V_2 are 1/500 m.u./mv); therefore

$$\left[\frac{V_1}{500}\right] = \left(\frac{E_{DC}}{500}\right) + \left[\frac{V}{500}\right] \tag{22}$$

and

$$\left[\frac{V_2}{500}\right] = -\left[\frac{V_1}{500}\right] \tag{23}$$

4. Time Scale

The simulation must run considerably slower than the physical system; therefore, time scaling is required. A time scale factor

$$\beta = \frac{\text{5 seconds of machine time}}{\text{nanosecond of problem time}}$$

was selected to obtain reasonable potentiometer settings. Since the maximum clock frequency, ω (as shown in Table XIII-9) is approximately one rad/ns, the value of β selected results in a computer frequency ($\overline{\omega}$) of

$$\overline{\omega} = \frac{\omega}{\beta} = \frac{1}{5} = 0.2 \text{ rad/sec of machine time}$$

A frequency of this magnitude is ideal for the analog computer, and insures that the frequency limitations of all components will not be exceeded.

5. Programming

The mechanization of the scaled equations is shown in the computer diagram of Figure XIII-20. Their associated potentiometer and amplifier sheets are contained in Tables XIII-11 and XIII-12.

Two items of interest should be noted:

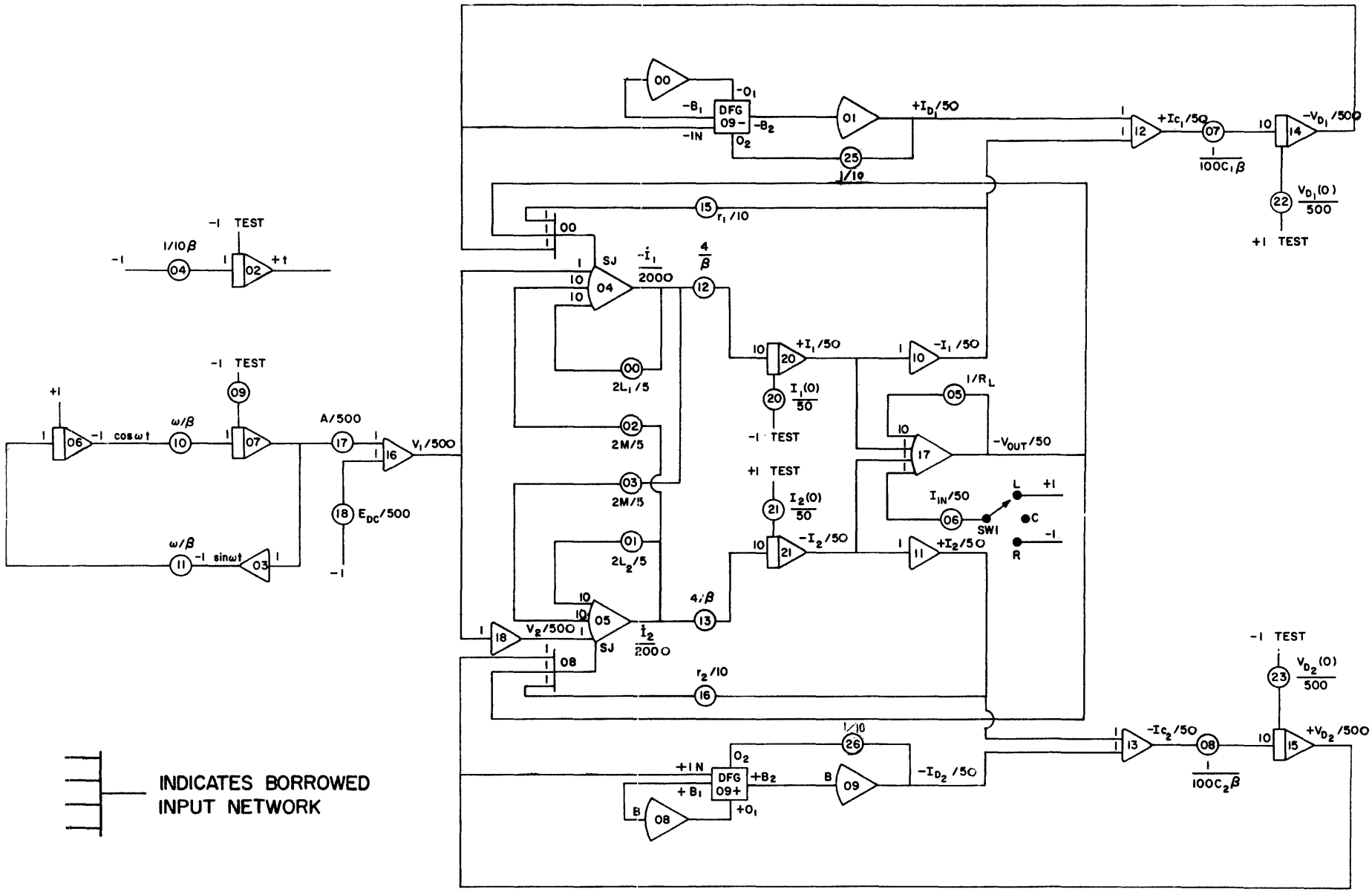


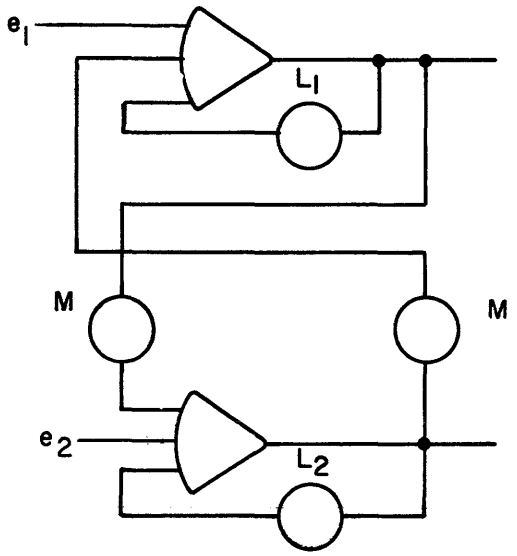
Figure XIII-20: Analog Computer Circuit Diagram

a) The programming of the diode function generators (DFG) which simulate the diode characteristic curves

b) The algebraic loop formed between amplifiers 4 and 5.

6. Algebraic Loop

A simple unscaled circuit representation for this algebraic loop is



whose overall loop gain, K, must be less than one for stability. Since

$$K = \frac{M^2}{L_1 L_2}$$

which is always less than one, the loop is stable (Reference 8). Note that the loop gain of an algebraic loop is independent of scaling.

7. Generating the Characteristic Curve

The tunnel-diode characteristic in Figure XIII-18 presents a problem in function generation no matter which simulation method is used.

On an analog computer, the best approach is the straightforward use of a standard diode function generator (DFG) which uses biased diode networks to approximate a curve by straight-line segments. On the TR-48 variable-breakpoint DFG, ten segments are available. The breakpoints (corners) of the curve may be varied so as to group many short straight-line segments together where the graph is straighter.

Inspection of the curve (Figure XIII-18) shows that the slope at the origin (which must be set by the "center slope" potentiometer in the DFG) is very great. Furthermore, the function curves vary sharply (i.e., has a large second derivative) near the first peak. This indicates that very great changes in slope will be necessary at each breakpoint. Since variable DFG's are limited with respect to the slope change obtainable at a single breakpoint, it is advisable to tabulate the necessary slope changes to see if the slope capability of the DFG is exceeded.

This tabulation appears in Table XIII-13. The function is tabulated at ten breakpoints (which is the number of breakpoints available on the TR-48 DFG) and two endpoints (0 and .95 M. U.). Subtraction and division yield the slopes in column 3, and a second subtraction yields the slope changes in column 4.

The TR-48 DFG is designed to produce a maximum center slope of $\pm$ 2 units per unit and a maximum slope change of $\pm$.1 unit per unit at each segment. Inspection of column 4 indicates that these figures are exceeded by a factor of over 8 to 1 at some points. Therefore, the DFG gain must be increased. This can be done by using a potentiometer in the feedback path of the DFG output amplifier. If the potentiometer is set to 1/10, then the maximum obtainable slopes become 20 units per unit center slope and 10 unit per unit slope change per segment, which is more than adequate to generate this curve.

The fact that this "slope amplification" was necessary indicates that this is a difficult function to generate accurately with straight-line segments. Accordingly, one would expect the DFG to exhibit a much larger error in generating this function than would be the case with functions that curve less sharply. A rough calculation indicates that the maximum error is about .005 units (0.5% of reference). Since the original curve is empirical, and depends upon the particular tunnel diode selected for measurement, this accuracy is sufficient.

8. Static Check

For static check calculations, convenient values are selected for all integrator output signals (in this case I_1, I_2, V_{D_1}, and V_{D_2}), and for the driving functions, V_1, V_2 and I_{in}. These values may be chosen arbitrarily, but they should be small enough to avoid amplifier overloads, and large enough to provide amplifier output signals that can be measured with reasonable accuracy. The values chosen were:

$I_1 = +25$ ma. $\qquad V_{D_1} = +50$ mv.

$I_2 = +25$ ma. $\qquad V_{D_2} = +50$ mv.

$V_1 = +50$ mv. $\qquad V_2 = -V_1 = -50$ mv.

$I_{in} = +1.0$ ma.

Parameter values are the same as those for the first run. The values for all variables may be calculated from equations 1-10. These values are given below.

$I_{D_1} = +49.2$ ma. $\qquad I_{D_2} = +49.2$ ma.

$I_{C_1} = 24.2$ ma. $\qquad I_{C_2} = -24.2$ ma.

$V_{out} = +8$ mv.

Table XIII-11: TR-48 POTENTIOMETER ASSIGNMENT SHEET

P00 – P29

DATE 6/1/63 PROBLEM TUNNEL DIODE

POT NO	PARAMETER DESCRIPTION	SETTING STATIC CHECK	STATIC CHECK OUTPUT	SETTING RUN NUMBER 1	NOTES	POT NO
00	$2L_1/5$	.160	+0.166			00
01	$2L_2/5$	.160	-0.134			01
02	$2M/5$	0.000	0.00			02
03	$2M/5$	0.000	0.00			03
04	$1/10\beta$	.020	-0.20			04
05	$1/R_L$	.125	-0.02			05
06	$I_{IN}/50$	.020	+0.20			06
07	$1/100C_1\beta$	.200	-0.97			07
08	$1/100C_2\beta$	.200	+0.97			08
09		.640	-6.40		STATIC CHECK I.C.	09
10	ω/β	.188	-1.88			10
11	ω/β	.188	-1.20			11
12	$4/\beta$	.800	+0.83			12
13	$4/\beta$	.800	-0.67			13
14						14
15	$r_1/10$	.300	-1.50			15
16	$r_2/10$	.300	+1.50			16
17	$A/500$	.250	+1.60	.260		17
18	$E_{DC}/500$	.260	-2.60			18
19						19
20	$1/50\ I_1(0)$	.500	-5.00			20
21	$1/50\ \lvert I_2(0)\rvert$	.500	+5.00			21
22	$1/500\ V_{D_1}(0)$	.100	+1.00			22
23	$1/500\ \lvert V_{D_2}(0)\rvert$	.100	-1.00			23
24						24
25	1/10	.100	+0.98		INCREASES DFG GAIN	25
26	1/10	.100	-0.98		INCREASES DFG GAIN	26
27						27
28						28
~~29~~	SWITCH #1	LEFT				29

M654

TR-48 AMPLIFIER ASSIGNMENT SHEET

DATE 6/1/63 A00—A23 PROBLEM TUNNEL DIODE

AMP NO.	FB	OUTPUT VARIABLE	STATIC CHECK				NOTES
			CALCULATED		MEASURED		
			DERIV.	OUTPUT	DERIV.	OUTPUT	
00		PART OF DFG					Use input Resistors with 04
01	H	$+ID_1/50$		+.984			
02	∫	+t	+.020	+1.000			
03	-	-1 sin ωt		-.640			
04	H	$-\dot{I}_1/2000$		+.1037			
05	H	$+\dot{I}_2/2000$		-.0837			
06	∫	-1 cos ωt	+.120	-1.000			
07	∫	+1 sin ωt	+.188	+.640			
08		PART OF DFG					Use Input Resistors with 05
09	H	$-ID_2/50$		-.984			
10	-	$-I_1/50$		-.500			
11	-	$+I_2/50$		+.500			
12	Σ	$+Ic_1/50$		-.484			
13	Σ	$-Ic_2/50$		+.484			
14	∫	$-VD_1/500$	+.968	-.100			
15	∫	$+VD_2/500$	-968	+.100			
16	Σ	$\frac{V1}{500} = \frac{-V2}{500}$		+.100			
17	H	-VOUT/500		-.016			
18	-	V2/500		-.100			
19							
20	∫	$+I_1/50$	-.830	+.500			
21	∫	$-I_2/50$	+.670	-.500			
22							
23							

Table XIII-13: Set-Up Table for Variable DFG

INPUT $\left[\frac{V_D}{500}\right]$	OUTPUT $\left[\frac{I_D}{50}\right]$	SLOPE IN $\frac{\text{units}}{\text{unit}}$ $\frac{\Delta\left[\frac{I_D}{50}\right]}{\Delta\left[\frac{V_D}{500}\right]}$	Change in Slope Units/unit
0	0		
.035	0.581	1.66	
.085	0.934	0.71	0.95
0.110	0.993	0.24	0.47
0.145	0.936	0.16	.08
0.285	0.367	0.41	0.25
0.385	0.169	0.20	0.21
0.620	.089	0.23	.03
0.790	0.156	.04	0.17
0.855	0.320	0.25	0.21
0.900	0.675	0.800	0.55
0.950	1.200		

The derivatives may be calculated from equations 11, 12, 4, and 5. These are:

$$\dot{V}_{D_1} = \dot{V}_{D_2} = -2420 \text{ millivolts/nanosecond}$$

$$\dot{I}_1 = -207.50 \text{ milliamps/nanosecond}$$

$$\dot{I}_2 = -167.50 \text{ milliamps/nanosecond}$$

All variables that have been calculated now can be translated into amplifier output units. These units appear on the amplifier assignment sheet (Table XIII-12). They may be checked against values calculated on the dircuit diagram (to check the programming and scaling), and later checked against actual measured units on the computer (to check the patching and the functioning of the components). For integrators, the checkpoints should also be calculated and measured. (The checkpoint of an integrator is minus the sum of its input units. It may be read out by patching the summing junction of the integrator temporarily to the summing junction of a summing amplifier that is not being used in the problem. These calculated values are also tabulated in the amplifier assignment sheet.

The initial condition inputs marked "Test" are for static test purposes only and not for the actual run. They should be removed prior to the first run.

Note that the static test value of the mutual inductance, M, is zero. This value was chosen to break the algebraic loop in the static test mode since, otherwise, it would be very hard to troubleshoot. However, this static test value does not check the algebraic loop itself and a supplementary test should be included with $M \neq 0$. For this supplementary test, we may as well assume that the artificial initial conditions on voltages and currents are zero except for V_1 and V_2 since this part of the circuit has already been checked by the main static check. Equations 1 and 2 then become:

$$V_1 = L_1 \dot{I}_1 - M\dot{I}_2$$

$$-V_2 = L_2 \dot{I}_2 - M\dot{I}_1$$

Solving by determinants:

$$\dot{I}_1 = \frac{V_1 L_2 - MV_2}{L_1 L_2 - M^2} \; ; \qquad \dot{I}_2 = \frac{-L_1 V_2 + MV_1}{L_1 L_2 - M^2}$$

If we let $V_1 = 50$ mv., $V_2 = -50$ mv., $L_1 = L_2 = 0.4$ nanohenry and $M = 0.2$ nanohenry, the $\dot{I}_1 = \dot{I}_2 = +250.0$ milliamps/nanosecond.

The output of amplifier 04 should be $-\dot{I}_1/200 = -.125$ M. U., and the output of amplifier 05 should be +.125 M. U.

9. Results and Conclusions

The switching transients and output waveshpaes for constant input current are shown in Figures XIII-21 and XIII-22. Note, in particular, that at 300 megacycles the output voltage goes through an initial oscillation before switching. The output is high for only a short period of time late in the bias cycle. At 150 megacycles, the output rises much more sharply and gives a good waveshape. Operation at 300 megacycles is possible, but marginal. Further experimentation with the model indicates that for reliable operation with sufficiently large fan in/fan out capability, the circuit should not be operated above about 200 or 250 megacycles.

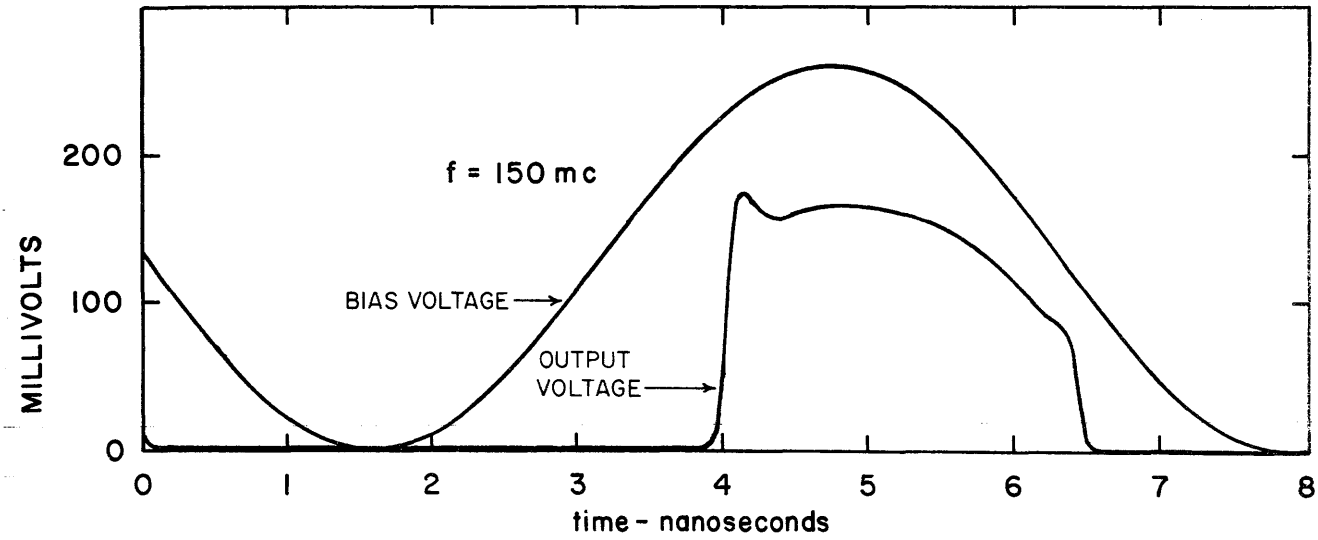


Figure XIII-21: Output Waveshape of Switching Circuit at 150 Megacycle Clock Frequency

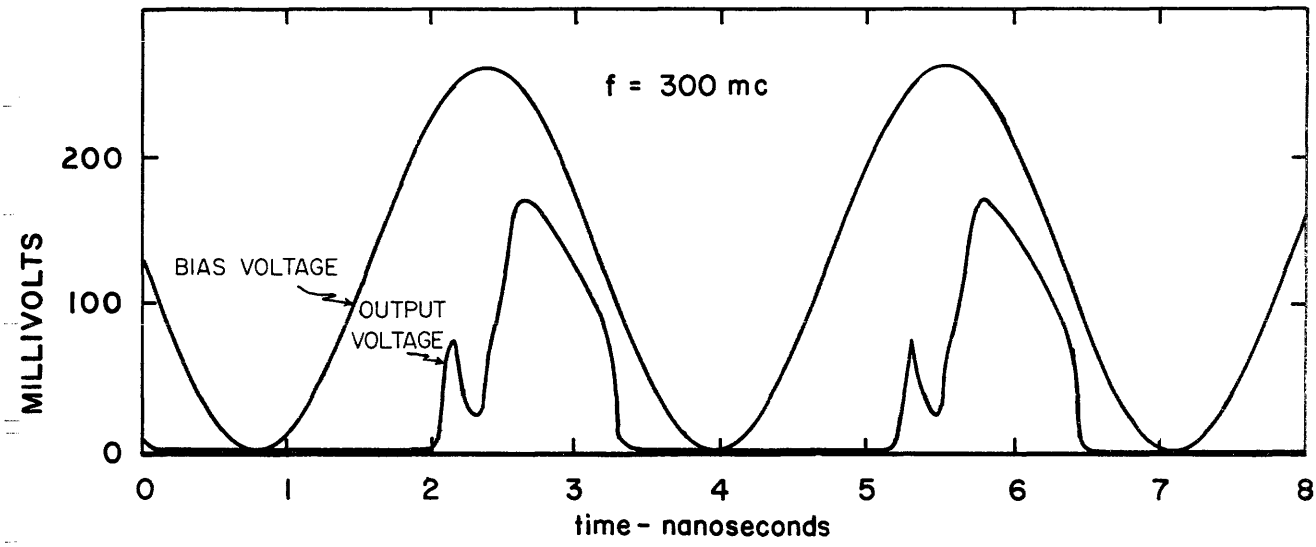


Figure XIII-22: Output Waveshape at 300 Megacycle Clock Frequency

It is significant that a preliminary pencil-and-paper analysis indicated that the system ought to operate at frequencies up to 1000 megacycles. As shown, however, a circuit designed to operate at this frequency would not function properly and would be extremely hard to troubleshoot due to the circuit loading of the measuring devices.

Of course, on-line devices are available for producing graphical readout from a digital computer. Some of Herzog's graphical results were obtained by oscilloscope photography. Although this method eliminates the tedium of point-plotting, it is messy and time-consuming and offers limited resolution. Perhaps the best form of readout, and certainly the most convenient, involves the use of a digital-to-analog converter and an analog X-Y plotter. While this method is acceptable, it requires expensive conversion equipment to translate the data into analog voltages for plotting. Such conversion equipment is unnecessary with the TR-48 computer since the signal is an analog voltage in the first place.

Since only about half of the computing capacity of the TR-48 computer is used, the simulation easily can be expanded to take additional effects into account. Herzog's equations assumed a constant current input and ignored the fact that the load is not purely resistive. These simplifications could easily be removed on an analog simulation by addition of a few more amplifiers and potentiometers. On a digital computer, any additional complexity in the equations would increase the running time.

An attractive alternative, for example, would be to simulate an additional tunnel-diode circuit on the analog computer and feed the output of the first into the second. This would enable the designer to determine how flat the output waveshape of the first circuit would have to be in order to trigger the second successfully.

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CHAPTER XIV

FACTORS IN PLANNING AND OPERATING AN ANALOG LABORATORY

A. Introduction

In establishing an analog facility, a number of important decisions and choices must be made. The purpose of this chapter is to comment on the principal factors that must be considered and to make appropriate suggestions based on the experience of Electronic Associates, Inc., and others.

B. Justification for an Analog Facility

The major justifications for obtaining analog computers are both technical and economic. The technical justification is, in a broad sense, based on solving problems whose solution can be obtained only via electronic computation. The cost justification must answer the question, "Will the computer provide needed information more efficiently, effectively and economically than any other method currently available?"

In formulating a cost justification, one must realize that the efficient use of analog computers will decrease design and construction costs and increase productivity of new and existing processes. These are long range factors, but their consideration reduces the overall cost of an analog facility if one adopts a far-sighted viewpoint.

Additional justification factors are

1) Utilization: Although technical applications exist, are they sufficient in number to justify a computer purchase? Can my problems be solved more economically by renting time at a computation center?

2) Computation Time: When solution speed is an urgent factor, an analog computer may provide information sooner than other methods even though, under circumstances, it might not be the only or least expensive method of solution.

C. Choice of Analog Equipment

The choice of equipment involves not only the size of the computer but the relative numbers of each component. Although the selection must be based on specific requirements, typical applications in related areas are frequently helpful in providing guidelines. For example, in aircraft problems, trigonometric resolvers usually are required, while they seldom are found in a computer in a chemical facility. In chemical engineering problems, however, log function generators are particularly important components. The equipment requirements for a number of small problems

(less than twenty amplifiers) is shown in Table XIV-1 (2). It can be used to obtain insight into component requirements for particular applications and estimates of the ratio of linear to nonlinear equipment requirements.

As problem complexity and size increase, computer facilities, in turn, have grown rapidly in size, speed and flexibility as shown in Figure XIV-1 (3). More specific illustrations of the size of petro-chemical analog facilities are shown in Table XIV-2 (4) and Williams' estimate of an analog facility is contained in Table XIV-3 (5). The latter illustration includes computer personnel, which will be discussed later.

These illustrations also point out the importance of obtaining basic computer equipment which has the flexibility of adding more equipment or making modifications. Equipment should incorporate provisions for both electrical and mechanical expansion.

D. Buildings and Floor Space

The size of a computer installation will, of course, dictate the size of the laboratory required. It is necessary to provide adequate space behind the equipment for service access, and in front of the equipment for the usual recorders and plotters. Additional space should be allotted for chart-examination tables, a station for the convenient patching of panels and storage space for panels. Either in the laboratory or immediately adjacent to it should be a shop area for the servicing of equipment. Nearby should be the offices of staff members.

Of prime importance in the laboratory is adequate electric power; 50KW is typical for vaccum tube computers. The power requirements of transistorized computers are much less, which also reduces air conditioning or cooling requirements. For example, two transistorized PACE TR-48 Computers (96 amplifiers) drain less than 300 watts and require no air conditioning, while a PACE 231-R vaccum tube computer (120 amplifiers) has a 6.5KVA power requirement, necessitating air conditioning.

In addition, power is required for tools, lighting, cooling or ventilation, and for other purposes. Ordinarily, three-phase wiring is used for efficiency of distribution; most computers, themselves, are single phase, however. Power requirements for the apparatus will be specified by the manufacturer. It is also important that the line voltage be reasonably stable and free from spikes or transients. If transients are being caused by some nearby apparatus such as an arc welder or a large motor, it may be necessary to install independent transformers and perhaps voltage regulators.

Ventilation and cooling for the equipment will be specified when needed. Exhausting the heat from the equipment and from the room is extremely important; the means for doing this should be carefully planned.

Floor loading also should be considered in terms of the weight of the equipment and the construction of the building that will house it.

More detailed information about physical facilities is available in the literature (6), and from computer manufacturers. The physical description sheet shown in Figure XIV-2 is typical of manufacturers literature.

Table XIV-1: Equipment required to Solve Example Problems

Problem No.	- Amplifiers - Integrators	Summers or Inverters	- Multipliers - If Servo	If Electronic	Strip Chart	x-y	Function Generators	Potentiometers	Relays	Diodes	Function Switches
1	2	--	--	--	X	--	--	3	--	--	--
2	2	--	--	--	X	--	--	4	--	--	--
3	4	2	--	--	--	X	--	7	4	--	--
4	2-3	1-3	--	--	X	X	--	3-6	--	--	--
5	4	2	--	--	X	X	--	5	--	--	--
6	2	4	--	--	X	--	--	5	--	2	--
7	4	2-4	--	--	--	X	--	4	--	--	--
8	8	4	--	--	X	--	--	10	--	--	--
9	2	2	--	--	X	--	--	6	--	--	--
10	4	3	--	--	--	X	--	--	--	--	--
11	2	1	--	--	X	--	--	4	--	--	--
12	4	2	--	--	--	X	--	6	--	--	2
13	11	15	--	--	X	--	--	--	--	--	2
14	2	3	--	--	X	--	--	--	3	--	--
15	2	2	1	1	X	--	--	2-4	--	--	--
16	4	6	2	4	--	X	--	9	--	--	--
17	5	4	1	2	--	X	--	12	5	--	--
18	2	3	1	2	X	--	--	1	--	2	--
19	2	2	1	1	X	--	--	5	--	--	--
20	4	6	1	2	--	X	--	14	--	--	--
21	4	14	3	5	--	X	1	21	--	--	--
22	3	8	2	2	--	X	--	10	--	4	--
23	7	7	4	8	X	--	--	7-14	--	--	--
24	3	6	4	5	--	X	--	6	--	--	--
25	8	8	2	3	X	--	--	20	--	4	2
26	3	12	1	1	X	--	--	15	--	--	--
27	4	14	2	4	X	--	--	15	--	--	--
28	11	20	1	5	X	--	--	--	--	--	--
29	4	3	1	1	--	X	--	3	--	--	--
30	3	3	1	1	--	X	--	5	--	--	--
31	2	6	1	2	--	X	--	2	--	--	--

Table XIV-1a:

Some Example Problems Suitable for Small-Scale Analog Computation

A. LINEAR SYSTEMS - NO FUNCTION GENERATION OR MULTIPLICATION REQUIRED

1. Falling body subject to gravitational force, no air resistance (11).
2. Falling body subject to a gravitational force and subject to a resistive force proportional to velocity(11)
3. Simulation of the action of a bouncing ball (1, 10).
4. Vibrations in a mass-spring-damper mechanical system (7, 10, 11).
5. Forced vibrations in a mass-spring-damper mechanical system (11).
6. Vibrations in a mass-spring- damper system with elastic stops (6)
7. Oscillations of a tuned torsional pendulum (1, 10).
8. Modes of vibration of a cantilever beam subjected to tip displacement(13)
9. Consecutive and reversible first-order chemical reactions (14).
10. Loading on a bridge caused by a vehicle (8).
11. Mixing of chemical solutions in a series of well-stirred tanks. Example shown is for two tanks (11).
12. Simulation of an isothermal catalytic reactor with a first-order reaction and axial diffusion only. Steady state operation (7).
13. Process dynamics of a shell and tube heat exchanger (16).
14. Dynamics of a high speed adaptive control system (10).

B. NONLINEAR SYSTEMS - MULTIPLICATION AND/OR FUNCTION GENERATION REQUIRED.

15. Falling body subject to a medium whose resistance is proportional to the square of the velocity (11).
16. Trajectory of a bomb or artillery shell (1, 10).
17. Trajectory of a long-range ballistic missile, including firing and ballistic phases of programmed flight (10).
18. Vibrations in a mass-spring-damper system with a nonlinear spring constant and either elastic or inelastic stops(15)
19. Composition in a well-stirred tank with different input and output compositions (11).
20. Kinetics of a set of simultaneous and consecutive second-order reactions (10).
21. Simulation of a tubular reactor with a first-order reaction. Temperature and composition profile of reactor calculated vs. length (7).
22. Simulation of the intake, exhaust and cylinder system of a reciprocating gas compressor (7, 10).
23. Dynamic characteristics of a distillation column. Example shown is for three plates (7).
24. Determination of the shape of the liquid zone in the zone-refining of metals or the shpae of unsupported liquid drops (13).
25. Simulation of the pressure control in a pneumatic system (7).
26. Automatic control of the temperature in a heat transfer process consisting of a well-mixed kettle and jacket (10).
27. Automatic control of the output of a reaction process consisting of a well-stirred reactor and a second order process (10).
28. Automatic control of a shell and tube heat exchanger (16).
29. Solution of Mathieu's equation, $\ddot{y} + (a-2q \cos 2t) = 0$.(17).
30. Solution of Legendre's equation, $\ddot{y}(1-t^2)-2t\,\dot{y} + n\,(n+1)y = 0$. (6).
31. Solution of van der Pol's Equation, $\ddot{x} - \lambda(1-x^2)\,\dot{x} + x = 0$, $\lambda = 0.6$ (13).

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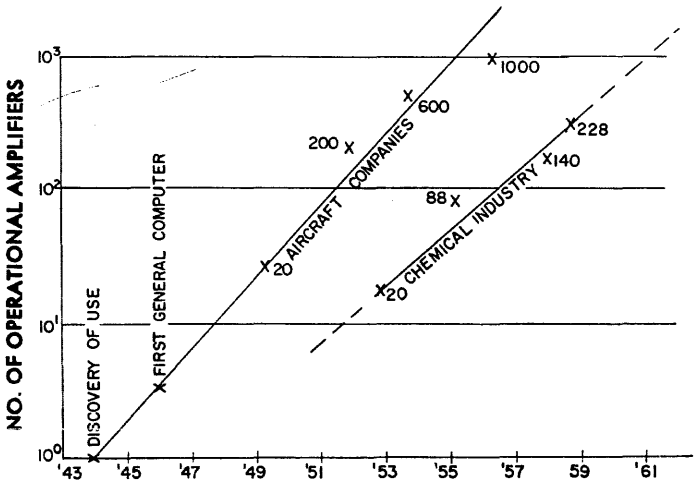


Figure XIV-1: The Typical Expansion of a Large Analog-Computing Facility

Company	*Year Acquired*	*Number of Amplifiers*	*Manufacturer*
Dow Chemical Co.			
Midland Division	1954	20	Beckman (Berkeley)
Midland, Mich.	1961	140 (on order)	Electronic Associates
Texas Division	1956	30	Daystrom (Heath)
Freeport, Tex.	1961	80 (on order)	Philbrick
E. I. du Pont de Nemours & Co.			
Newark, Del.	1950	30	Beckman (Berkeley)
	1955	50	Beckman (Berkeley)
	1958	120	Electronic Associates
	1960	300	Electronic Associates
Experimental Station Wilmington, Del.	1960	70	Computer Systems
Monsanto Chemical Co.	1957	116	Electronic Associates
St. Louis, Mo.	1958	24	Electronic Associates
	1959	88	Electronic Associates
Ohio Oil Co. Denver, Colo.	1957	56	Electronic Associates
Humble Oil & Refining Co.			
Baytown, Tex.	1960	80	Electronic Associates
	1961	80	Electronic Associates
Baton Rouge, La.	1959	80	Electronic Associates
	1960	40	Electronic Associates
Esso Research & Engineering Co.	1959	40	Electronic Associates
Florham Park, N.J.	1959	40	Electronic Associates
	1960	80 (enlarged unit)	Electronic Associates
	~~1960~~	~~80 (enlarged unit)~~	~~Electronic Associates~~
American Oil Co.	1955	—	Electronic Associates
Whiting, Ind.	1957	168 (enlarged unit)	Electronic Associates
Standard Oil Co. (Ohio)	1955	90	Beckman (Berkeley)
Cleveland, Ohio	1957	10	Beckman (Berkeley)
	1961	170 (enlarged unit)	Beckman (Berkeley)
Union Carbide Olefins Co.	1956	30	Electronic Associates
South Charleston, W.Va.	1958	60 (enlarged unit)	Electronic Associates
	1959	60	Electronic Associates
Thiokol Chemical Corp. Brigham City, Utah	1959	168	Electronic Associates
Phillips Petroleum Co.	1959	80	Electronic Associates
Bartlesville, Okla	1960	80	Electronic Associates
Chemstrand Corp. Decatur, Ala.	1960	80	Electronic Associates
Shell Oil Co.			
Shell Chemical Corp. New York, N.Y.	1960	120	Electronic Associates
Shell Development Co.	1956	24	Goodyear
Emeryville, Calif.	1957	24	Goodyear
	1960	10	Donner Scientific
	1960	10	Donner Scientific
Hercules Powder Co. Wilmington, Del.	1960	44	Beckman (Berkeley)
Daystrom, Inc. La Jolla, Calif.	1960	100	Computer Systems

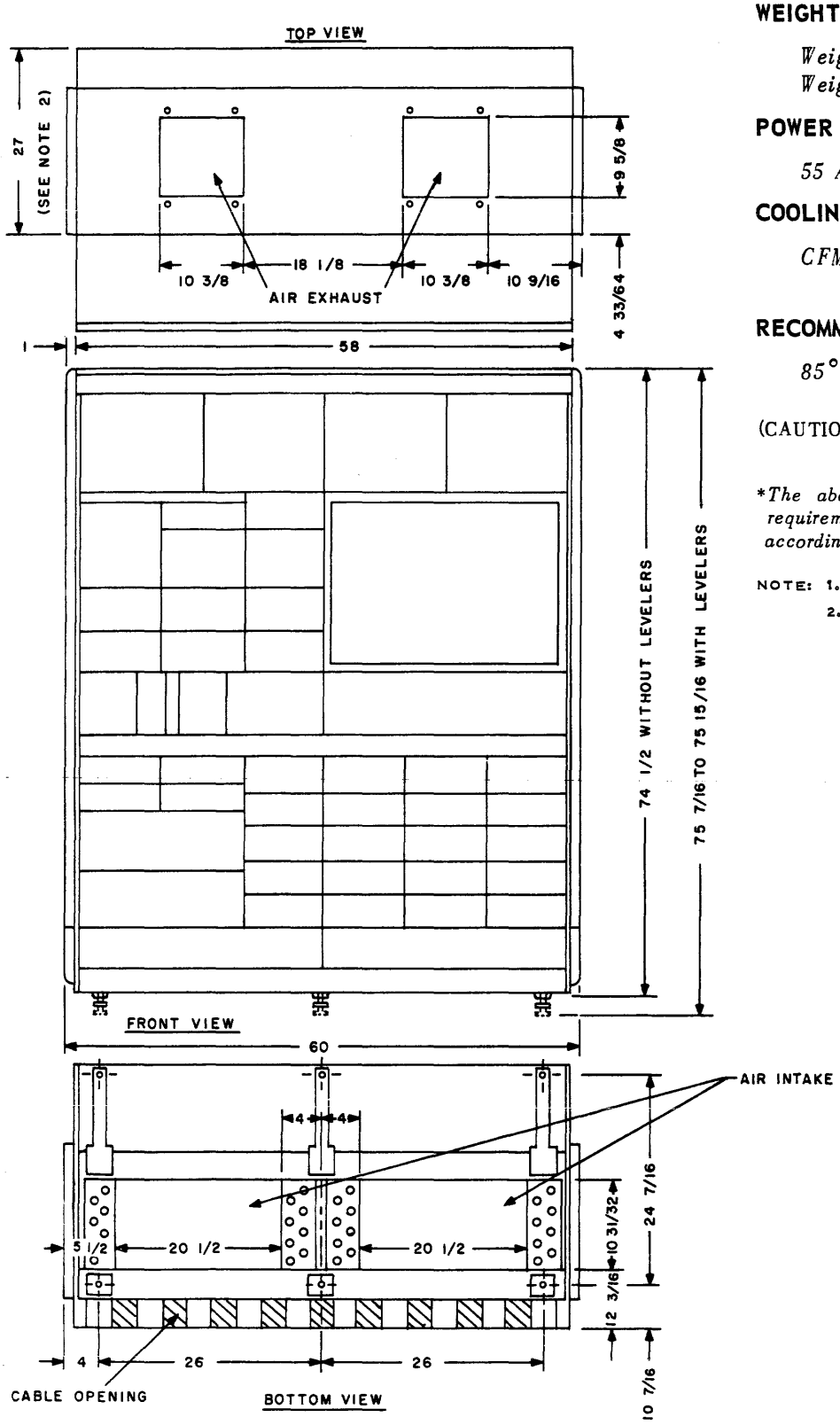
Table XIV-2 - Petro-Chemical Analog Computer Installations*

*Chem. & Eng. News; 34; pg. 118; Feb., 1961

Table XIV-3: Analog Computer Facility in the Chemical Process Industry

Computer Components	Instructional Use Only	Instrumentation Response & Design	Control System Response; Some Process Dynamics	Complete Simulation of Chemical Process Systems
Operational amplifiers	10-20	15-50	40-100	80-200+
Multiplication channels	1-2	2-5	10-30	50-100+
Function generation channels	1-2	2-6	5-10	25-50+
Operational relays	1-2	6	12	25-50+
Function switches	3	6	12	25-50+
Separate control consoles	1	1	1-2	2-3+
Plotting channels (x-y)	1	1	3	3-4+
Plotting channels (x-t)	2-4	6	12	18
Repetitive operation	yes	if desired	if desired	if desired
Computer Personnel				
EE or ME (MS or equivalent)	1	1-2	2-3	3-6+
ChE (MS or PH.D.)	--	1	2-3	6-8+
Computer repairmen	--	--	1-2	2-3+
Shop arrangement	Open	Open	As desired	Closed

(Courtesy of Chemical Engineering, p. 121, February, 1960)



WEIGHT

*Weight, Crated ..... 2100 lbs (typical)**
*Weight, Uncrated ..... 1800 lbs (typical)**

POWER REQUIREMENTS

55 Amps at 117V, 60 cps (approx. 6.5 KVA)*

COOLING AIR

$$CFM\ Req'd = \frac{40{,}000}{100 - (\text{input air temp, } ^\circ F)} *$$

RECOMMENDED INPUT AIR TEMP

85°F Max., 65°F Min.

(CAUTION: Exhaust air temp above 97° indicates a significant cooling deficiency.)

**The above weights and the power and cooling air requirements are average figures which will vary according to the equipment in the rack.*

NOTE: 1. ALL DIMENSIONS REFER TO INCHES.
2. DIMENSION SHOWN DOES NOT INCLUDE HANDLES ON REAR DOORS (APPROX. 1 INCH) OR PROJECTION OF EQUIPMENT ON FRONT PANELS.

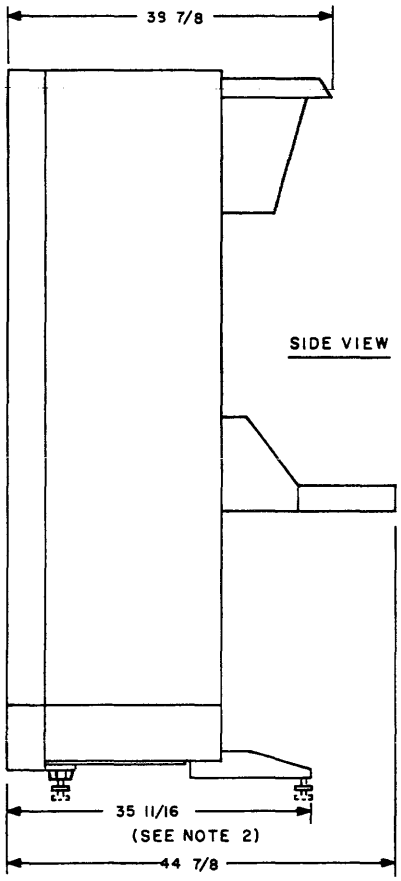


Figure XIV: Typical Physical Description Sheet

Finally, one should consider the operating positions of the computers. If it is likely that many large problems will be solved, the computers should be relatively close together. On the other hand, if several small problems are to be solved simultaneously, then the relative separation of computers is desirable to prevent operating personnel from disturbing each other.

E. Manpower

Staff requirements will depend on the objectives of the facility and on its relation to other groups that may exist in the same organization. In many industrial organizations, the computer group functions as a service facility for many technical groups. It may simply take problems that are already formulated, perform specified operations on them and return the results to the originator. In other organizations, the members of the computer group begin work on a particular problem a considerable time before it reaches the computer, assist in formulating it, bring it to the computer, use the background gained to make the running efficient, assist in analyzing the result and may assist in writing a report. Another type of operation exists in some organizations where the computer is merely a tool available to each and every engineer; the only staff for the computer itself consists of maintenance personnel.

On the assumption that members of the computer group will be active on problems, the following types of personnel are needed:

1. Problem engineers. These should be graduate engineers or the equivalent. They should have a background in their industrial field adequate for understanding the range of problems encountered, and should be thoroughly familiar with the computer.

2. Maintenance technicians. Men with prior service experience on analog computers are often difficult to find. In general, a few years experience in a field such as television or radar, together with a solid foundation of electronic fundamentals, provide a man with an adequate background for computer maintenance. The number of technicians needed depends on many factors and is hard to estimate in advance; in an active facility, a useful approximation would be one technician per computer. If the maintenance group is large and has auxiliary duties such as design or evaluation of computer equipment, it may be headed by an engineer.

3. Secretarial help.

4. Administrative help to handle business matters, library, personnel, supplies and auxiliary services.

F. Computer Maintenance

In modern, general-purpose analog computers, the majority of the computing units are of plug-in construction. This fact is of considerable importance

from the maintenance standpoint for, when a unit fails, it may be replaced very rapidly with a spare unit; the interruption to computer operation need only be very brief. The repair and check-out of repaired equipment is best accomplished using a "test rack", which is a small computer designed for maintenance purposes only. Its use reduces interruption of computer operation; it is used simply to serve as a source of power and control for testing equipment.

Very necessary to this concept, of course, is the establishment of a stock of spare units. The cost of the spares must be balanced against the value of the computational time lost after a failure if an immediate repair cannot be made. In addition, component parts necessary for possible repairs of these and any remaining units should be listed and accumulated insofar as possible. Repair of a malfunction within a unit may be accomplished in the shop after replacement. The repaired unit is tested and then becomes the new spare.

To ensure that each equipment failure is followed up, and that information regarding such failures is available later for evaluation and action, it is important that adequate records be kept. For example, a procedure should be used whereby a detailed sheet follows the progress of the repair, including the substitution and the subsequent repair. Valuable, also, is a set of unit record cards which provides an historical picture of each unit: receipt, initial tests, repairs, routine tests, etc. It is recommended that testing of units on a routine basis be done without removing them from the computer.

Additional maintenance comments are available in the literature (7).

G. Preventive Maintenance

To insure that analog computers are performing correctly at all times, and to minimize time lost due to component failures, a periodic check of individual components is recommended. The method and frequency of preventive maintenance checks required of each component normally can be obtained from its manufacturer.

A typical preventive maintenance schedule for a 400 amplifier facility is shown in Table XIV-5 (8).

Table XIV-5: Preventive Maintenance Schedule

Equipment	Frequency of Testing Wk.	Equipment	Frequency Testing Wk.
Amplifiers	7	Stepping switches	4
Servo multipliers	7	Reference balance	12
Digital Voltmeters	7	Resistors	12
Eight-channel recorders	7	Capacitors	12
Power supplies	7	Noise generator	24
Variplotters	4	Electronic multipliers	24
Summing relays	4	Test instruments	24

(Courtesy of Chemical Engineering, p. 103, April 1963)

As a result of implementing this schedule, a total of 99.6% of the total scheduled computer hours was available over a four-year period. Prewired preventive maintenance patch panels also are effective in reducing maintenance downtime.

H. References

1) Rogers, A. E. and Connelly, T; Analog Computation in Engineering Design; McGraw-Hill Book Co., Inc., New York, 1960.

2) Williams, T. J. and Lanher, V. A.; Chemical Engineering Progress; 56, February, 1960.

3) Williams, T. J.; Instruments and Control Systems, 33; April, 1960.

4) Chemical Engineering News, 39, p. 118, February, 1961.

5) Williams, T. J.; Systems Engineering, p. 121, February, 1960.

6) Horwitz, R. D.; Environmental Prerequisites for Computer Efficiency; Instruments and Automation, 31; July, 1958.

7) Ibid; Maintaining an Analog Computing System; Electronic Industries; December, 1958.

8) Evans, G. E.; Preventive Maintenance of Analog Components Assures Correct Operation of Computation Center; Chemical Engineering, p. 103, April, 1963.

APPENDIX A

LAPLACE TRANSFORMS

The Laplace Transform is a mathematical tool of great use in obtaining analytical solutions for linear systems. Its importance arises from the property that the ordinary differential equations which describe the system's dynamic behavior are transformed into algebraic relations which are much easier to solve. As analog computer users, we are not so much interested in the use of this mathematical technique but in its notation. The definition and table below demonstrate the notation.

Definition: $\mathcal{L}\left[f(t)\right] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

	f(t)	F(s)
1)	A	A/s
2)	t	$1/s^2$
3)	$\frac{t^{n-1}}{(n-1)!}$	$1/s^n$, $n = 1,2, \ldots$
4)	e^{at}	$\frac{1}{s-a}$
5)	$(t^{n-1} e^{at}) / (n-1)!$	$\frac{1}{(s-a)^n}$, $n = 1,2, \ldots$
6)	$\sin at$	$\frac{a}{s^2 + a^2}$
7)	$\cos at$	$\frac{s}{s^2 + a^2}$
8)	$\frac{df}{dt}$	$sF(s) - f(o)$
9)	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(o) - \ldots - \left[\frac{d^{n-1} f}{dt^{n-1}}\right]_{t=o}$
10)	$\int_o^t f(x)\,dx$	$\frac{F(s)}{s}$

	f(t)	F(s)
11)	$\int_0^t f(t-x)\, g(x)\, dx$	$F(s)\, G(s)$
12)	$f(t-b),\ t \geq b$ $0,\ t < b$	$e^{-bs} F(s)$

As an illustration, consider the equation

$$\ddot{y} + 2\dot{y} + y = o,\ y(o) = 1,\ \dot{y}(o) = 0$$

The transform of each term is taken individually as

$$\mathcal{L}\left[\ddot{y}\right] = s^2 Y(s) - sy(o) - \dot{y}(o)$$

$$\mathcal{L}\left[\dot{y}\right] = s\, Y(o) - y(o)$$

$$\mathcal{L}\left[y\right] = Y(s)$$

Thus, we have

$$\left[s^2 Y - s\right] + 2\left[sY - 1\right] + \left[Y\right] = o$$

$$\text{or} \quad Y = \frac{s+2}{s^2 + 2s + 1} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

from which we obtain by numbers 4 and 5

$$y(t) = e^{-t} + te^{-t} = (1 + t)e^{-t}$$

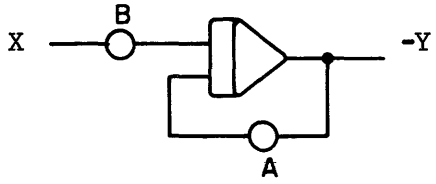
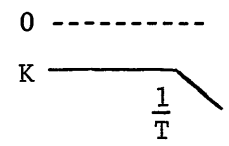
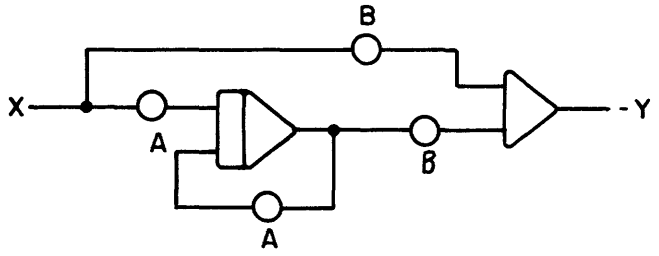
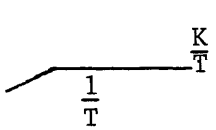
A knowledge of Laplace transform techniques is helpful to the engineer using the analog computer because it permits the rapid analysis of almost all kinds of linear systems. This capability is an asset if simplified models of systems of interest are to be used as checks of computer solutions. Familiarity with transfer functions (see Chapter VI) and their corresponding computer circuits is often helpful in estimating the behavior of a system.

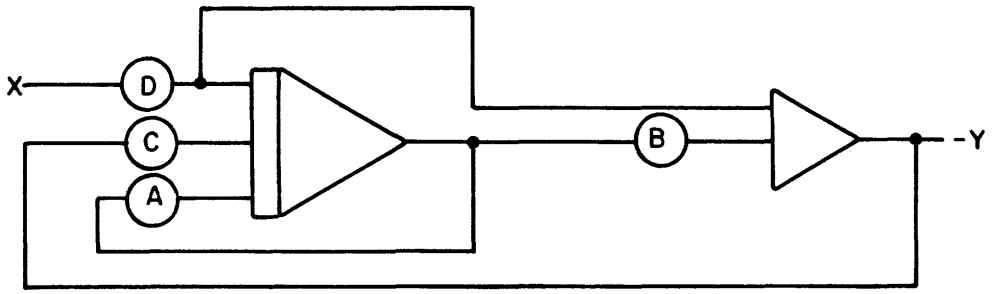
APPENDIX B

TRANSFER FUNCTION CIRCUITS

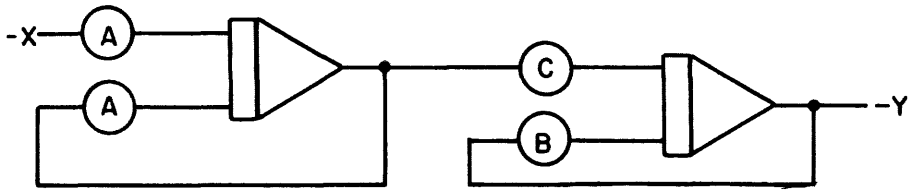
The purpose of this Appendix, which is subdivided into two tables, is to summarize a selected number of passive element and analog computer transfer function circuits. The selection of these circuits was determined by the frequency of their occurence in practice, and by their ability to combine with each other to simulate more complex transfer functions. More extensive tables of this type are readily available in the open literature, and suitable references may be found in Chapter VI.

Table B-I Analog Computer Transfer Function Circuits

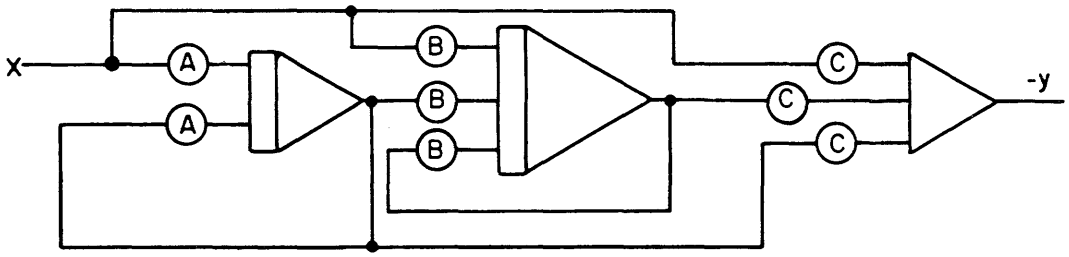
No.	Bode Plot	Transfer Function	Time Constants	Gains
				
1		$\frac{Y}{X} = \frac{K}{1 + Ts}$	$T = \frac{1}{A}$ $K = \frac{B}{A}$	$A = \frac{1}{T}$ $B = \frac{K}{T}$
				
2		$\frac{Y}{X} = \frac{Ks}{1 + Ts}$	$T = \frac{1}{A}$ $K = BT$	$A = \frac{1}{T}$ $B = \frac{K}{T}$



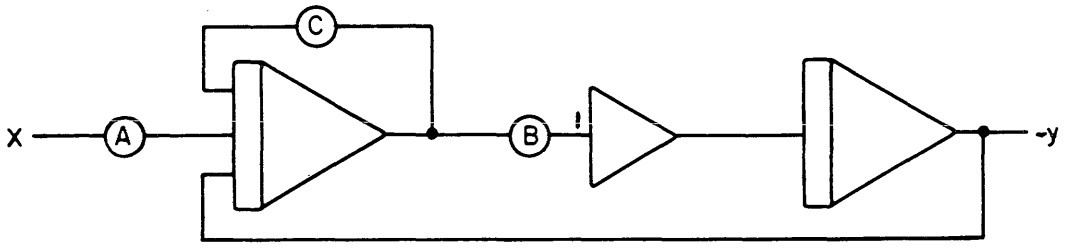
3	0 -------- K ——— $1/T_1$; $1/T_2$	$\frac{Y}{X} = \frac{K(T_2 s + 1)}{T_1 s + 1}$	$T_1 = \frac{1}{A-BC}$ $T_2 = 1/(A-B)$ $K = T_1 D/T_2$	$A = B + 1/T_2$ $C = \left(\frac{T_1 - T_2}{BT_1T_2}\right) + 1$ $D = K\ T_2/T_1$
4	0 -------- K ——— $1/T_2$; $1/T_1$	$\frac{Y}{X} = \frac{K(T_1 s + 1)}{T_2 s + 1}$	$T_1 = \frac{1}{A - B}$ $T_2 = 1/A$ $K = \frac{T_2 D}{T_1}$	$A = 1/T_2$ $B = \frac{T_1 - T_2}{T_1 T_2}$ $C = 0$ $D = \frac{KT_1}{T_2}$



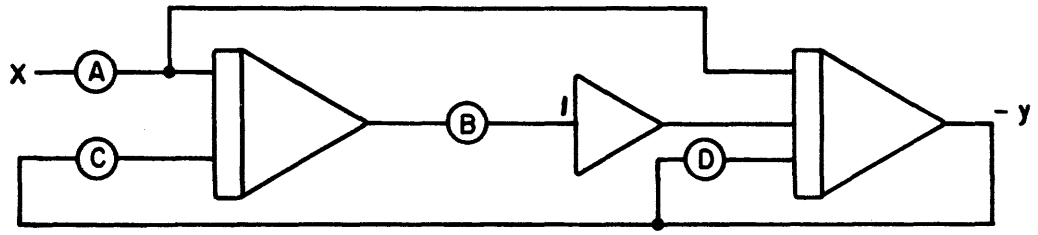
5	0 ---------- $1/T_1$ K ——— $1/T_2$	$\frac{K}{(T_1 s + 1)\ (T_2 s + 1)} = \frac{Y}{X}$	$T_1 = 1/A$ $T_2 = 1/B$	$A = 1/T_1$ $B = 1/T_2$ $C = K/T_2$



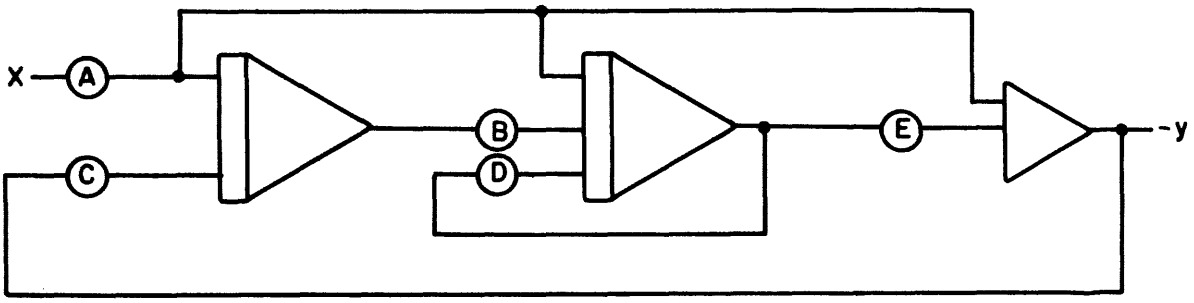
6	0 ------ K/T_1T_2 $\frac{1}{T_1}$ $\frac{1}{T_2}$	$Y/X = \frac{K_s^2}{(T_1 s+1)(T_2 s+1)}$	$T_1 = 1/A$ $T_2 = 1/B$ $K = C/AB$	$A = 1/T_1$ $B = 1/T_2$ $C = K/T_1T_2$



7	$\frac{Y}{X} = \frac{K}{T^2s^2 + 2hTs + 1}$	$K = A$ $T = 1/\sqrt{B}$ $h = \frac{C}{2\sqrt{B}}$	$A = K$ $B = 1/T^2$ $C = \frac{2h}{T}$

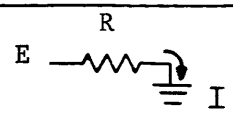
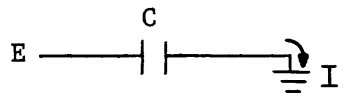
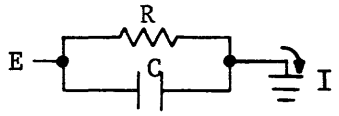
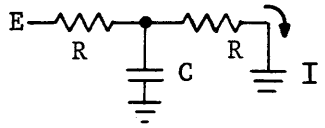
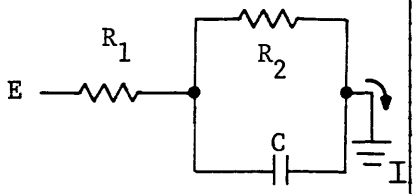
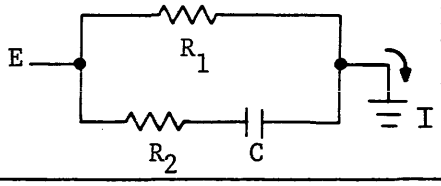
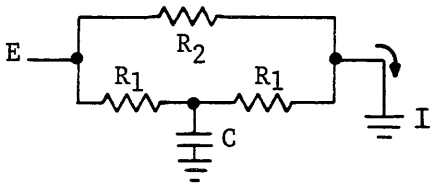
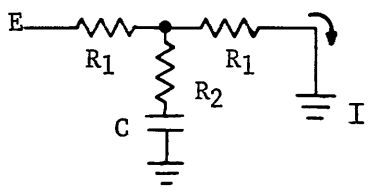


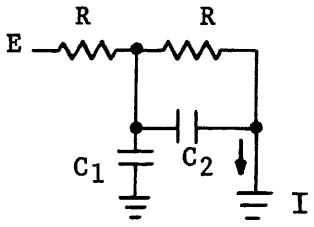
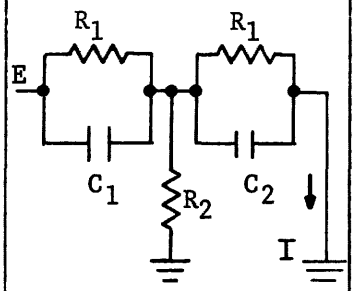
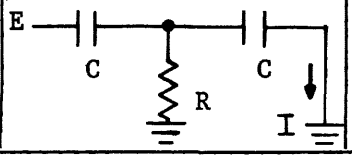
8	$\frac{Y}{X} = \frac{K(sT_1 + 1)}{T_2^2 s^2 + 2hT_2 s + 1}$	$K = A/C$ $T_1 = 1/B$ $T_2 = 1/BC$ $h = DT_2/2$	$A = KT_1/T_2^2$ $B = 1/T_1$ $C = T_1/T_2^2$ $D = 2h/T_2$



9	$\frac{Y}{X} = \frac{K(T_1^2 s^2 + 2h_1 T_1 s + 1)}{T_2^2 s^2 + 2h_2 T_2 s + 1}$	$K = A/C$ $T_1 = 1/\sqrt{BE}$ $h_1 = \frac{D - E}{2\sqrt{BE}}$ $T_2 = \frac{1}{\sqrt{BCE}}$ $h_2 = \frac{D}{2\sqrt{BCE}}$	$A = CK$ $B = \frac{T_2}{2T_1(h_2T_1 - h_1T_2)}$ $C = T_1^2/T_2^2$ $D = 2h_2/T_2$ $E = 2\left(\frac{h_2}{T_2} - \frac{h_1}{T_1}\right)$

Table B-II **Passive Element Transfer Impedance Functions**

No.	Transfer Impedance Function	Network	Parameter Relationships
1	$\frac{E}{I} = K$		$K = R$
2	$\frac{E}{I} \cong K/S$		$K = 1/C$
3	$\frac{E}{I} = \frac{K}{1 + TS}$		$K = R$ $T = RC$
4	$\frac{E}{I} = K\ (1 + TS)$		$K = 2R$ $T = \frac{RC}{2}$
5	$\frac{E}{I} \cong K \frac{(1 + T\theta S)}{(1 + TS)}$ $\theta < 1$		$K = R_1 + R_2$ $T = R_2C$ $\theta = \frac{R_1}{R_1 + R_2}$
6	$\frac{E}{I} = K\left(\frac{1 + T\theta S}{1 + TS}\right)$ $\theta < 1$		$K = R_1$ $T = (R_1 + R_2)\ C$ $\theta = \frac{R_2}{R_1 + R_2}$
7	$\frac{E}{I} = K\left(\frac{1 + TS}{1 + T\theta S}\right)$ $\theta < 1$		$K = \frac{2R_1R_2}{2R_1 + R_2}$ $T = \frac{R_1C}{2}$, $\theta = \frac{2R}{2R_1 + R_2}$
8	$\frac{E}{I} = \frac{1 + TS}{1 + T\theta S}$ $\theta < 1$		$K = 2R_1$ $T = (R_2 + \frac{R_1}{2})\ C$ $\theta = \frac{2R_2}{2R_2 + R_1}$

9	$\frac{E}{I} = K\left(\frac{1 + TS}{1 + \theta TS}\right)$ $\theta < 1$		$K = 2R$ $T = \frac{R}{2}(C_1 + C_2)$ $\theta = \frac{2C_2}{C_1 + C_2}$
10	$\frac{E}{I} = \frac{K(1 + T_2 s)}{(1 + T_1 s)(1 + T_3 s)}$ $T_2 < T_1 < T_3$		$K = 2R_1 + \frac{R_1^2}{R_2}$ $T_1 = R_1 C_1$; $T_3 = R_1 C_2$ $T_2 = \frac{R_1 R_2}{R_1 + 2R_2}(C_1 + C_2)$
11	$\frac{E}{I} = \frac{K}{TS^2}(1 + TS)$		$K = \frac{2}{C}$ $T = 2RC$

APPENDIX C

DIODE AND RELAY CIRCUITS

In simulating physical systems on the analog computer, it is frequently necessary to impose constraints such as absolute values, limits, etc., on one or more computer variables. The purpose of this Appendix is to provide a summary of selected diode and relay circuits which occur most frequently in practice for imposing these constraints. In addition, circuits which describe unusual behavior, such as hysteresis, are also included.

In this collection of circuits, computer reference voltage level is represented as e_R.

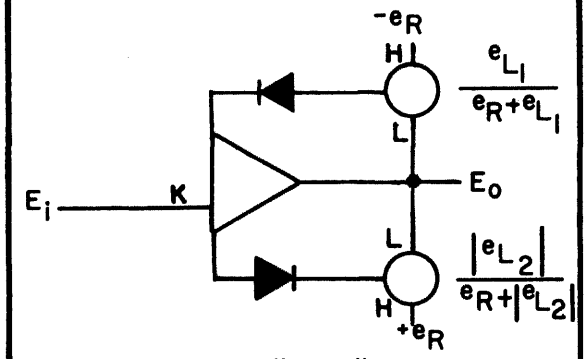
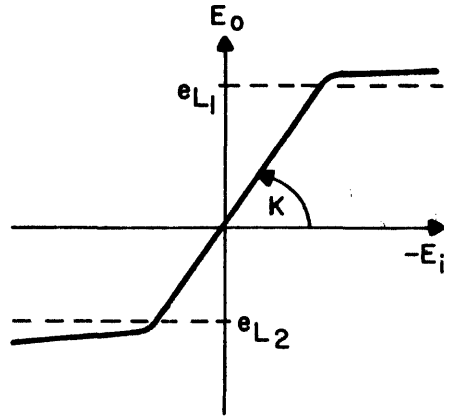
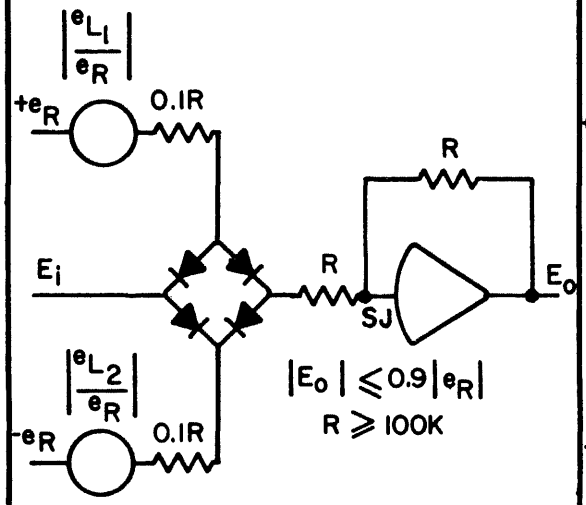
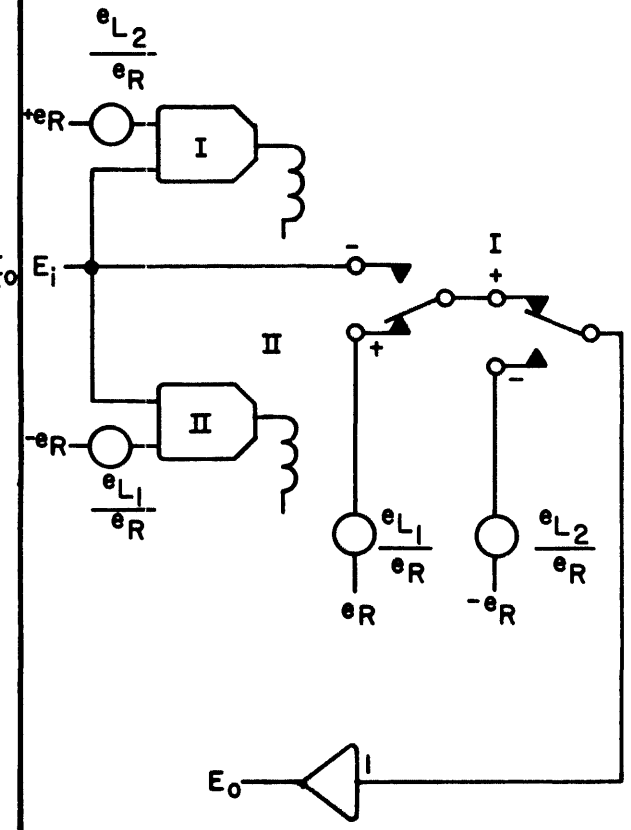
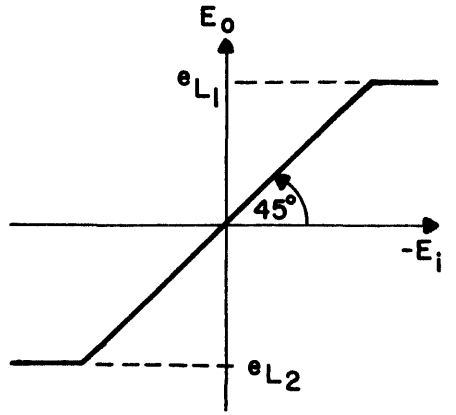
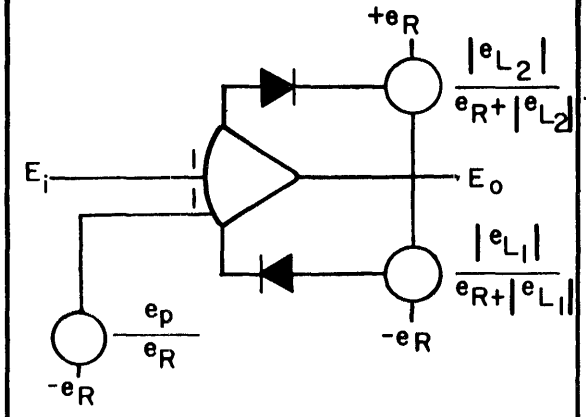
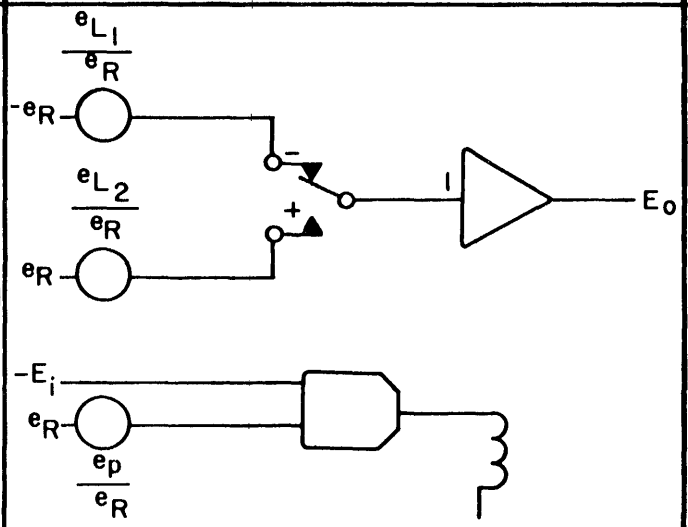
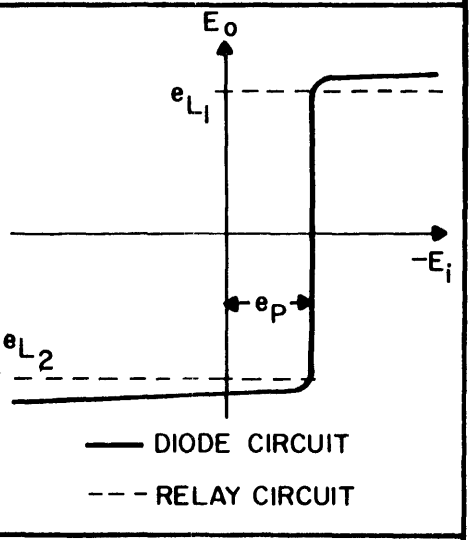
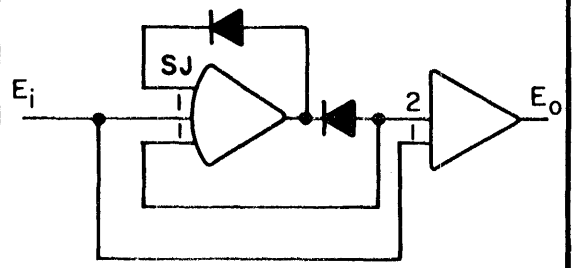
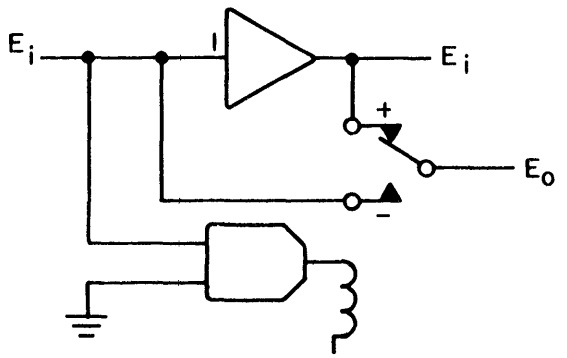
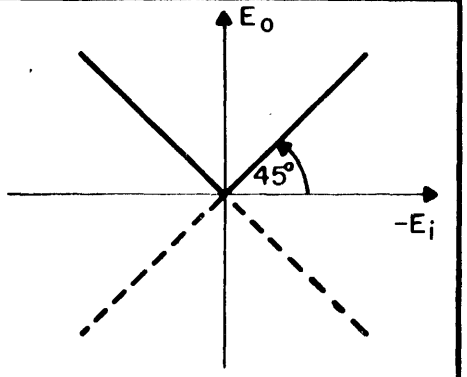
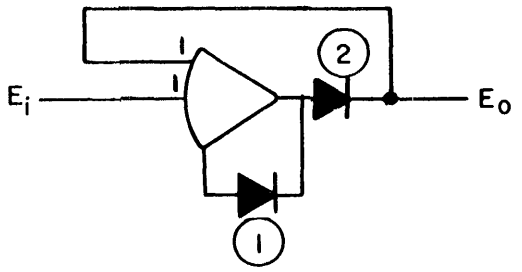
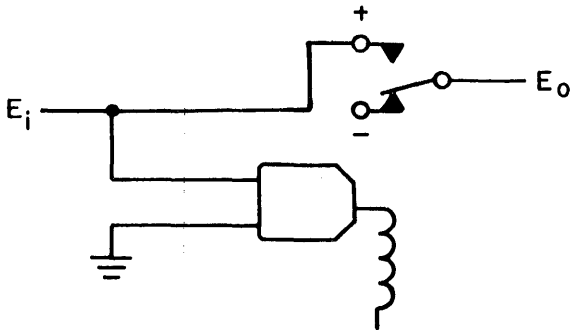
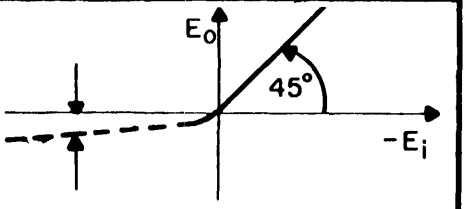
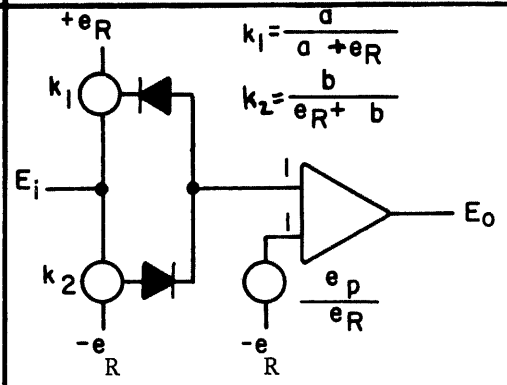
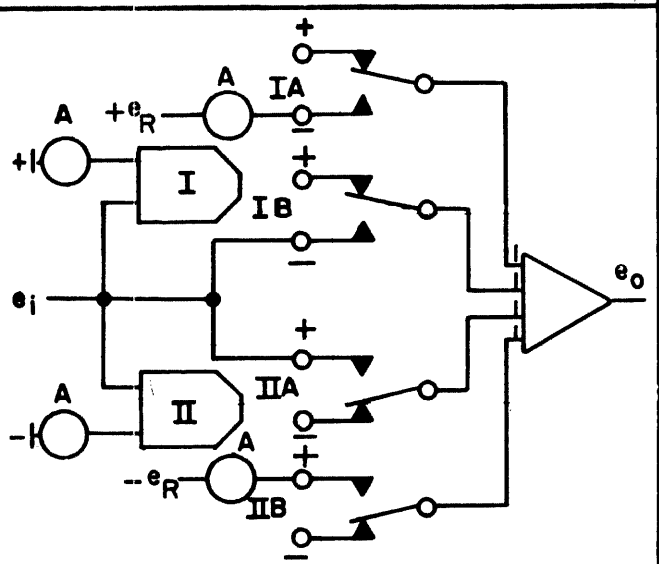
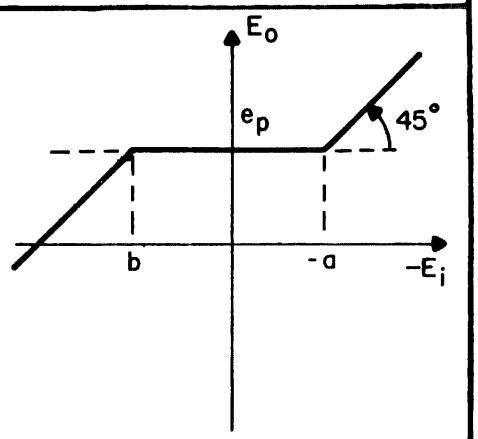
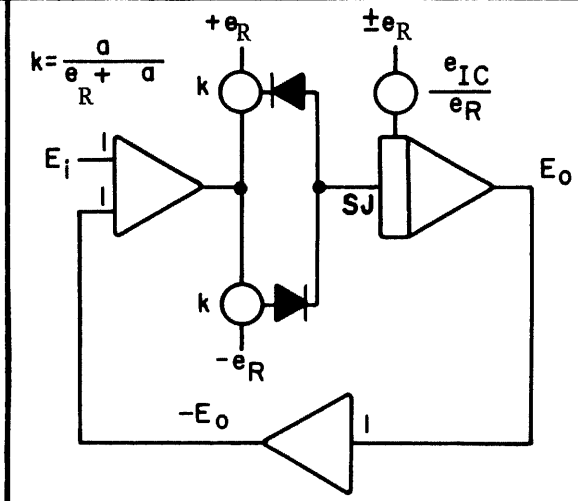
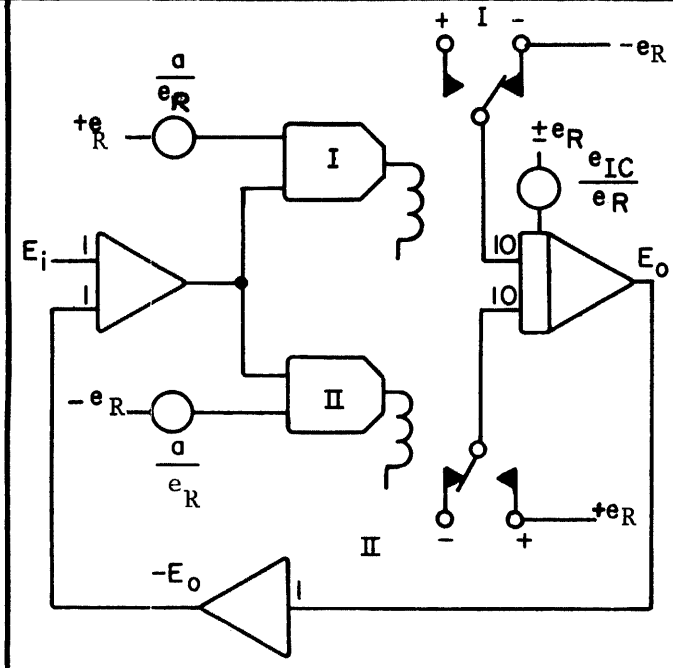
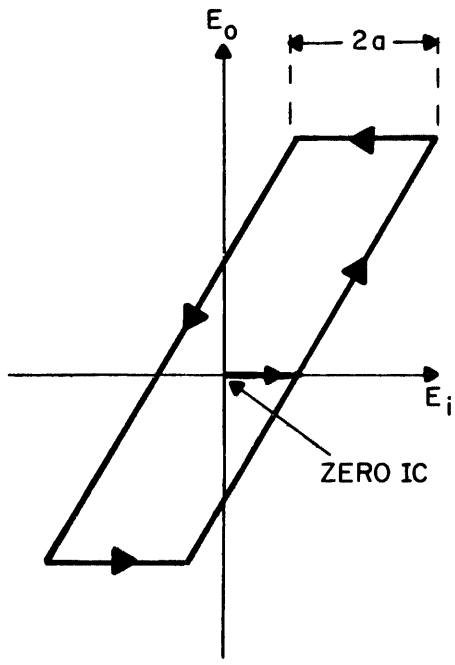
FUNCTION	DIODE CIRCUIT	RELAY CIRCUIT	BEHAVIOR
LIMITER	 FEEDBACK OR "SOFT" LIMITER		 ROUNDING DUE TO DIODE NON-LINEARITIES
	 BRIDGE OR "HARD" LIMITER		

Table C-1: Selected Diode and Relay Circuits

FUNCTION	DIODE CIRCUIT	RELAY CIRCUIT	BEHAVIOR
BANG – BANG			
ABSOLUTE VALUE			
ZERO LIMITING			 DIODE DROP – OCCURS IF DIODE 2 IS OMITTED REVERSE DIODES AND/OR CONTACTS FOR NEGATIVE INPUTS

FUNCTION	DIODE CIRCUIT	RELAY CIRCUIT	BEHAVIOR
DEAD SPACE			
HYSTERESIS OR BACKLASH			

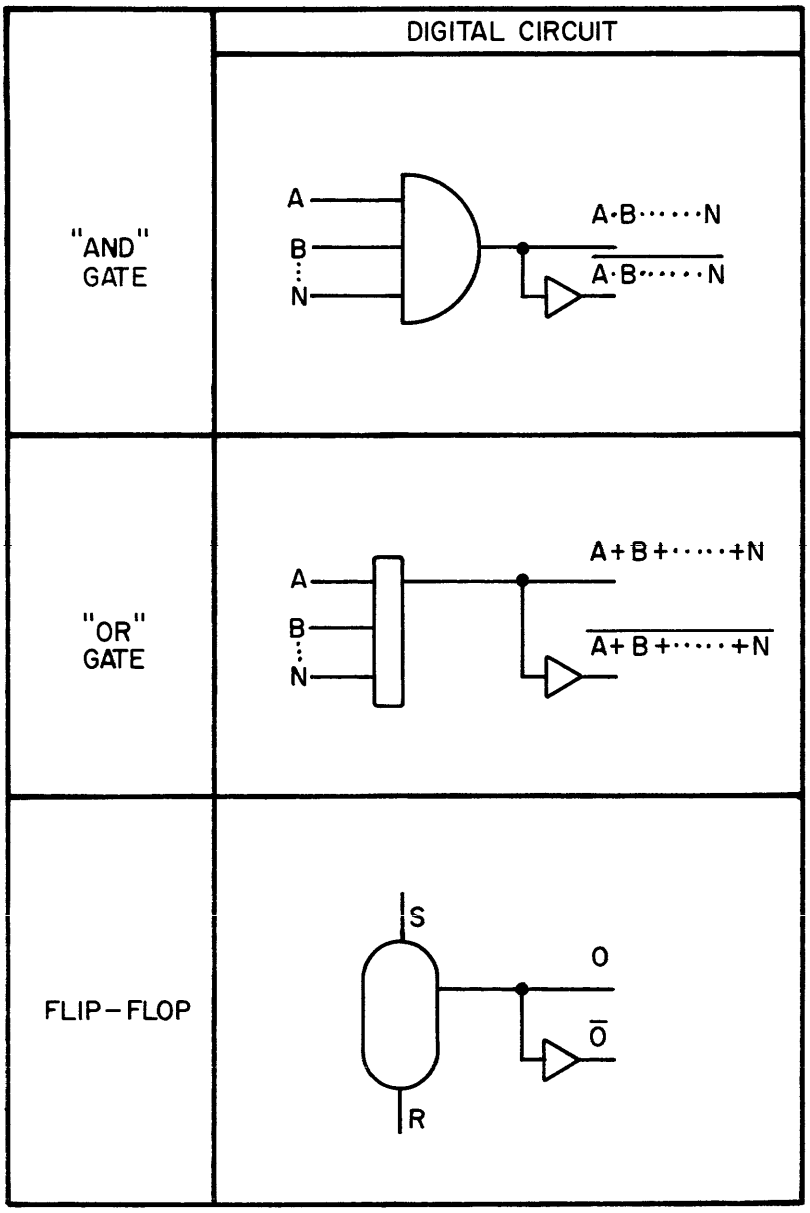


Figure C-1: Analog Equivalent of Digital Logic Components

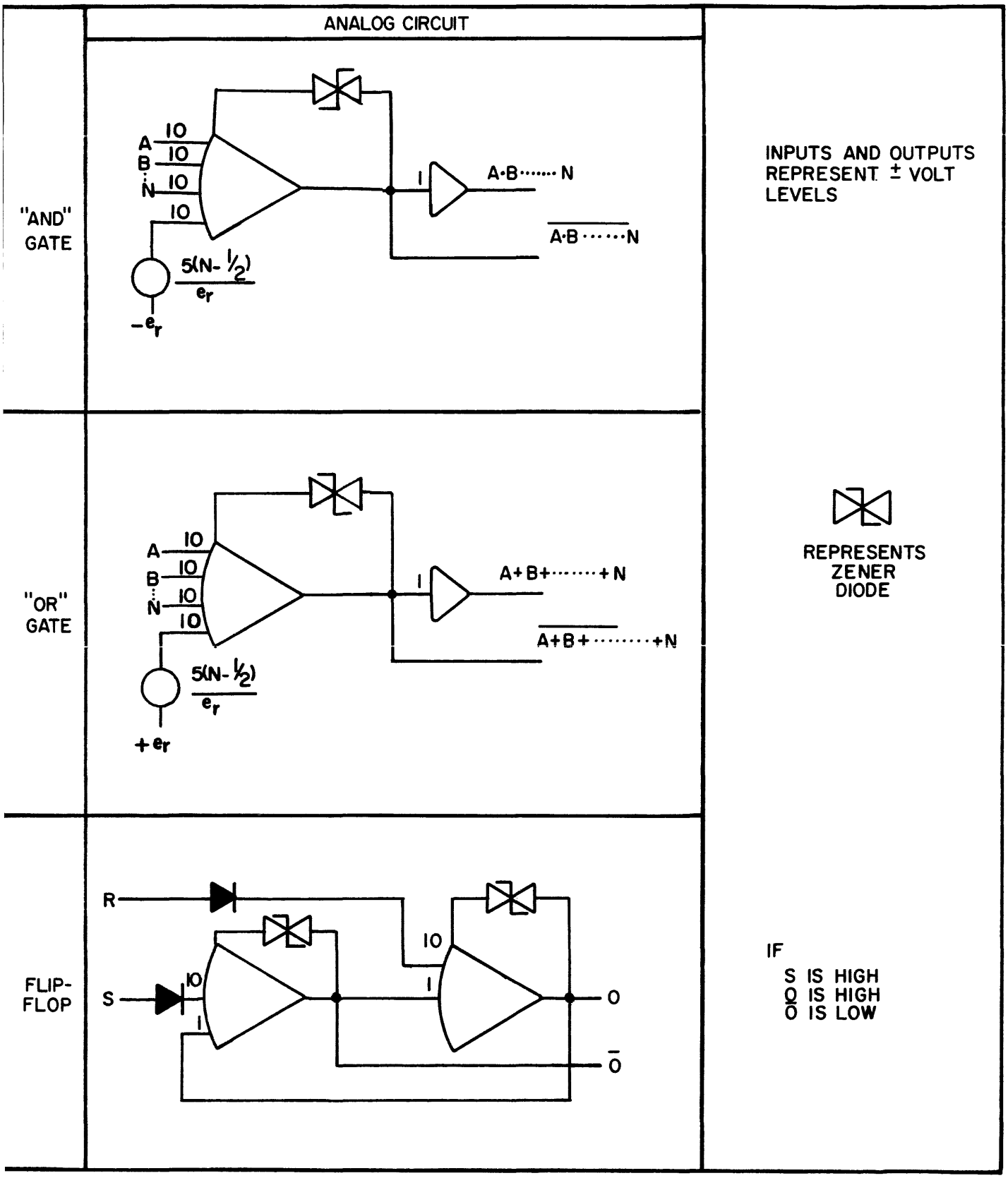
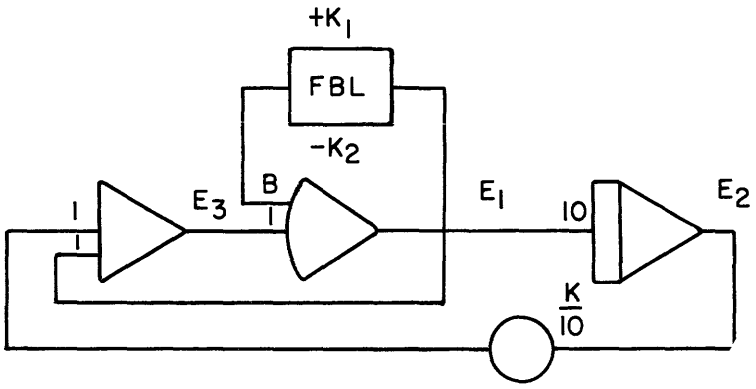


Figure C-1 (Continued)



PERIOD FOR SYMETRICAL WAVE $= \frac{4}{K}$

$$K_1 = \frac{V_1}{V_1 + e_R}$$

$$K_2 = \frac{|V_1|}{|V_1| + e_R}$$

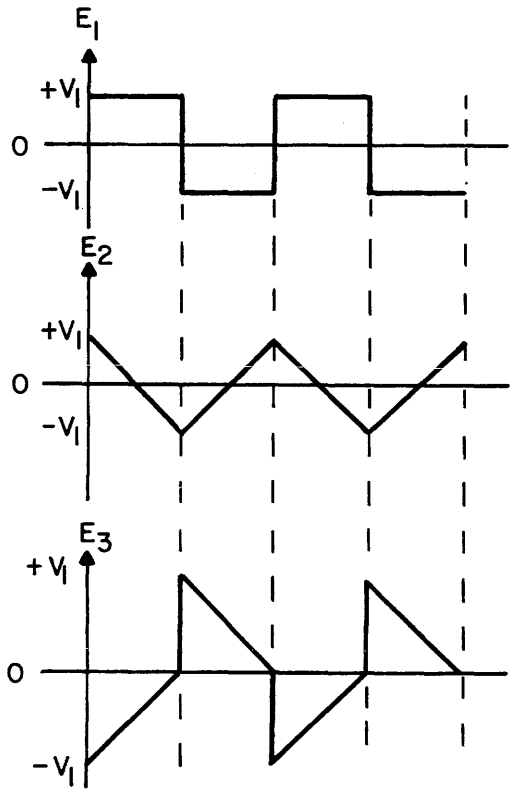


Figure C-2: Free Running Multivibrator Circuit

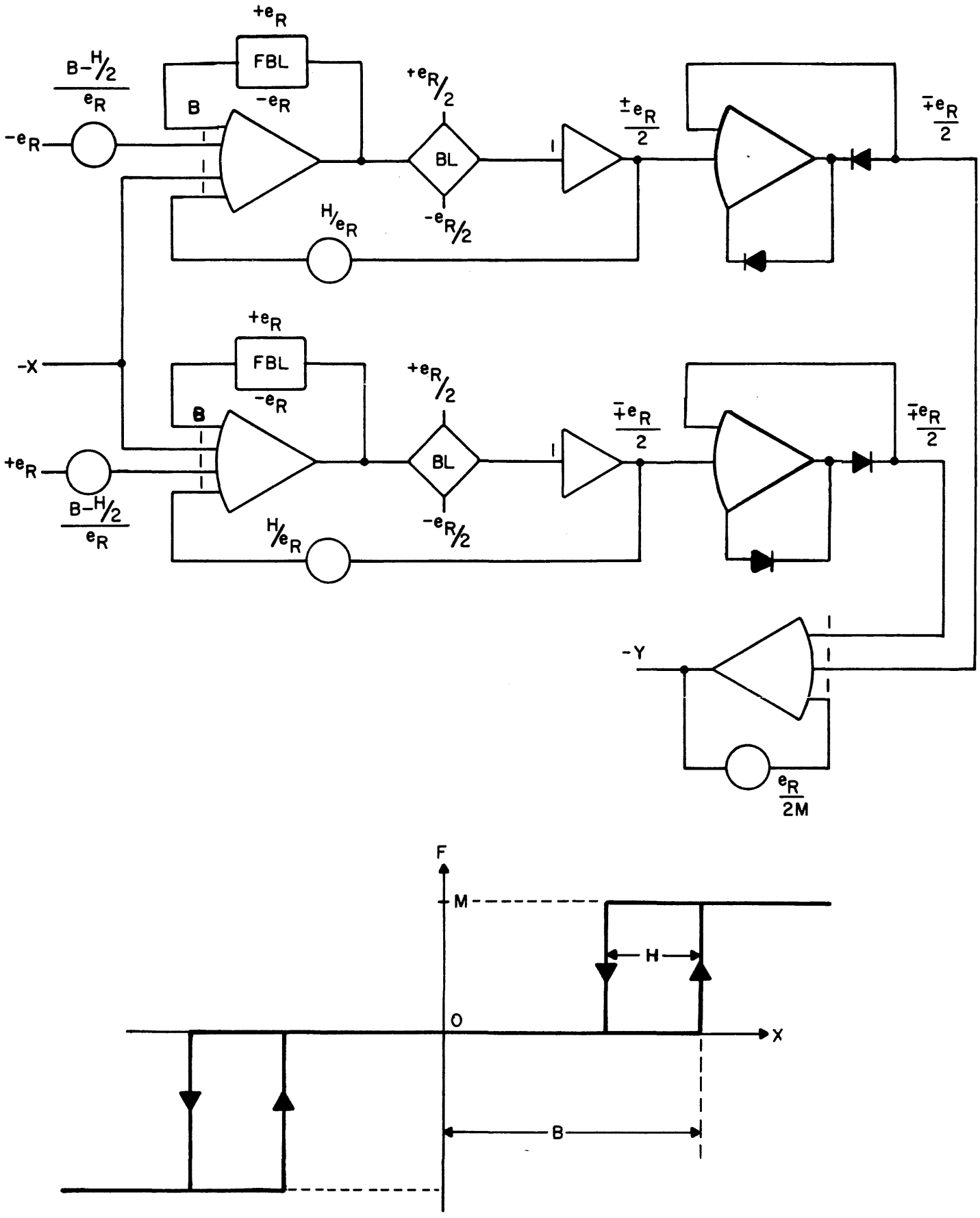


Figure C-3: Hysteresis Dead Band Circuit

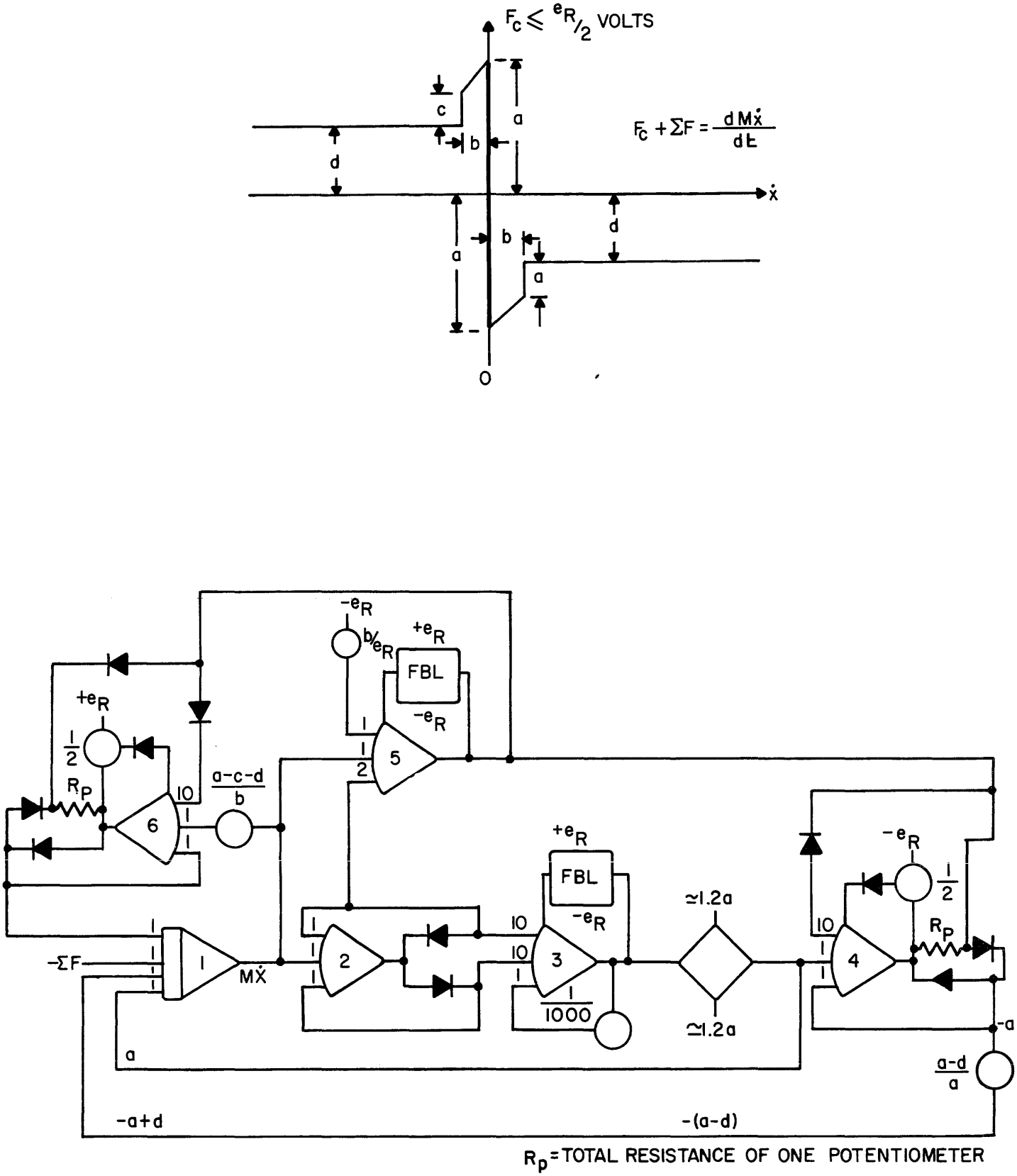


Figure C-4: Coulomb Friction Circuit

APPENDIX D

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