S2. A 0.178-kg particle moves along the x axis from $x = 12.8$ m to $x = 23.7$ m under the influence of a force

$$F = \frac{375}{x^3 + 3.75x}$$

where F is in newtons and x is in meters. Use numerical integration and set up a spreadsheet to determine the total work done by this force during this displacement. Your calculations should have an accuracy of at least 2%.

ANSWERS TO CONCEPTUAL PROBLEMS

1. Since there is no motion taking place, the rope experiences no displacement. Thus, no work is done on it. For the same reason, no work is being done on the pullers or the ground. Work is only being done within the bodies of the pullers. For example, the heart of each puller is applying forces on the blood to move it through the body.
2. Less force will be necessary with a longer ramp, but the force must act over a longer distance to do the same amount of work. Suppose the refrigerator is rolled up the ramp at constant speed. The normal force does no work because it acts at 90° to the motion. The work by gravity is just the weight of the refrigerator times the vertical height through which it is displaced times (cos 180°), or $W_g = -mgh$. Therefore, the movers must do work mgh on the refrigerator, however long the ramp.
3. If we ignore the effects of rolling friction on the tires of the car, the same amount of work would be done in driving up the switchback and driving straight up the mountain, since the weight of the car is moved upward against gravity by the same vertical distance in each case. If we include friction, there is more work done in driving the switchback, since the distance over which the friction force acts is much longer. So why do we use the switchback? The answer lies in the force required, not the work. The force from the engine required to follow a gentle rise is much smaller than that required to drive straight up the hill. Roadways running straight uphill would require redesigning engines so as to be able to apply much larger forces. This is similar to the ease with which heavy objects can be rolled up ramps into moving trucks, compared to lifting the object straight up from the ground.
4. If the spring is cut in half and stretched by the same distance, each adjacent pair of coils must now be twice as far apart as in the original stretched spring. Thus, the force applied by the spring is twice as great for any given extension. This leads to an increase in the spring constant by a factor of 2. Thus, if the cut spring is stretched by the same amount, it must require twice as much work.
5. We focus on the book as the system. You indeed did some work in applying an upward force to lift the book. But there is another force on the book—gravity. The work done by gravity is negative, since the force of gravity is opposite to the displacement. Assuming that you lifted the book slowly, the upward force that you applied is approximately the same in magnitude as the weight of the book. Thus, the positive work done by you and the negative work done by gravity cancel. Thus, there is no net work performed, and no net change in the kinetic energy—the work-kinetic energy theorem is satisfied. If we consider the book and the Earth to be the system, then there is net work done by you on this system. This work appears in the system as potential energy, which will be investigated in Chapter 7.
6. When walking downstairs, a person has to stop the motion of the entire body at each step. This requires work to be done by the muscles. The work done by the muscles is negative, since the forces applied are opposite to the displacements representing the movement of the body. Negative work represents a transfer of energy out of the muscles (from the store of energy from the food eaten by the walker). When walking horizontally, the body can be kept in motion with an almost constant horizontal speed—only the feet and legs start and stop. Thus, more energy is transferred from the body in bringing the entire body to a stop in the trip downstairs than in walking.
7. Since the time periods are the same for both cars, we need only to compare the work done. Since the sports car is moving twice as fast as the older car at the end of the time interval, it has four times the kinetic energy. Thus, according to the work-kinetic energy theorem, four times as much work was done, and the engine must have expended four times the power.

7

Potential Energy and Conservation of Energy

In Chapter 6 we introduced the concept of kinetic energy, which is associated with the motion of an object. In this chapter we introduce another form of mechanical energy, called *potential energy*, that is associated with the position or configuration of a system. The potential energy of a system can be thought of as stored energy that can be converted to kinetic energy or other forms of energy.

The potential energy concept can be used only with a special class of forces called *conservative forces*. When only internal conservative forces, such as gravitational or spring forces, act within a system, the kinetic energy gained (or lost) by the system as its members change their relative positions is compensated by an equal loss (or gain) in potential energy.



◀ **Twin Falls on the Island of Kauai, Hawaii. The gravitational potential energy of the water-Earth system when the water is at the top of the falls is converted to kinetic energy of the water at the bottom. In many locations, this mechanical energy is used to produce electrical energy.** *(Bruce Byers, FPG)*

CHAPTER OUTLINE

7.1 Potential Energy
7.2 Conservative and Nonconservative Forces
7.3 Conservative Forces and Potential Energy
7.4 Conservation of Mechanical Energy
7.5 Work Done by Nonconservative Forces
7.6 Conservation of Energy in General
7.7 Gravitational Potential Energy Revisited
7.8 Energy Diagrams and Stability of Equilibrium (Optional)

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7.1 • POTENTIAL ENERGY

In Chapter 6 we saw that an object with kinetic energy can do work on another object, as illustrated by the moving hammer driving a nail into a wall. Now we shall see that an object can also do work because of the energy resulting from its *position* in space.

As an object falls in a gravitational field, the field exerts a force on the object, does work on it, and thereby changes its kinetic energy. Consider a brick dropped from rest directly above a nail in a board that is lying horizontally on the ground. When the brick is released, it falls toward the ground, gaining speed and therefore gaining kinetic energy. The potential energy of the brick–Earth system is converted into kinetic energy as the brick falls. When the brick reaches the ground, it does work on the nail, driving it into the board. The potential energy of a system consisting of any object and the Earth is called **gravitational potential energy.**

Gravitational potential energy

Let us now derive an expression for the gravitational potential energy associated with an object at a given location in space. To do this, consider a block of mass m at an initial height y_i above the ground, as in Figure 7.1. With air resistance neglected as the block falls, the only force that does work on it is the gravitational force, $m\mathbf{g}$. The work done by the gravitational force as the block undergoes a downward displacement $\mathbf{d}$ is given by the product of the downward force $m\mathbf{g}$ and the displacement, or

$$W_g = (m\mathbf{g}) \cdot \mathbf{d} = (-mg\mathbf{j}) \cdot [(y_f - y_i)\mathbf{j}] = mgy_i - mgy_f$$

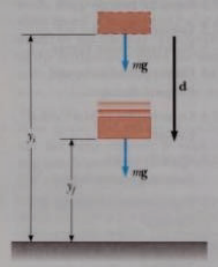


Figure 7.1 The work done by the gravitational force as the block falls from y_i to y_f is equal to $mgy_i - mgy_f$.

We can represent the quantity mgy to be the gravitational potential energy, U_g:

$$U_g = mgy \qquad [7.1]$$

In this representation, the gravitational potential energy associated with an object at any point in space is the product of the object's weight and its vertical coordinate. The origin of the coordinate system could be located at the surface of the Earth or at any other convenient point.

If we substitute U for the mgy terms in the expression for W_g, we have

$$W_g = U_i - U_f \qquad [7.2]$$

From this result, we see that the work done on any object by the force of gravity —that is, the energy transferred to the object from the gravitational field—is equal to the initial value of the potential energy minus the final value of potential energy.

The units of gravitational potential energy are the same as those of work. That is, potential energy may be expressed in joules, ergs, or foot-pounds. Potential energy, like work and kinetic energy, is a scalar quantity.

Note that the gravitational potential energy depends only on the vertical height of the object above the surface of the Earth. From this result, we see that the same amount of work is done on an object if it falls vertically to the Earth as if it starts at the same point and slides down a frictionless incline to the Earth. Also note that Equation 7.1 is valid only for objects near the surface of the Earth, where $\mathbf{g}$ is approximately constant.

In working problems involving gravitational potential energy, it is always necessary to choose a location at which to set the gravitational potential energy equal to zero. The choice of zero level is completely arbitrary, because the important quantity is the *difference* in potential energy, and this difference is independent of the choice of zero level.

It is often convenient to choose the surface of the Earth as the reference position for zero potential energy, but this is not essential. Often, the statement of the problem suggests a convenient level to use.

Thinking Physics 1

Elevators that are lifted by a cable have a counterweight, as shown in Figure 7.2. The elevator and the counterweight are connected by a cable that passes over a drive wheel. The counterweight moves in the direction opposite to that of the elevator. What is the purpose of the counterweight?

Reasoning Without the counterweight, the elevator motor would need to lift the weight of the elevator and the people riding it. This would represent a large increase in gravitational potential energy, which means a large expenditure of energy from the motor. With the counterweight, as the elevator moves upward, the counterweight moves down. Thus, the center of mass of the combined system of the elevator and counterweight moves only over a short distance near the middle region of the vertical shaft. This results in a much smaller variation of potential energy of the system than without the counterweight. In the particular case of the elevator and occupants having exactly the same mass as the counterweight, the center of mass is midway between them and never moves as the elevator changes height. In this case, once the elevator is in motion, the motor is no longer required—the system will continue to move as described by Newton's first law (as long as we ignore friction). The motor is needed again to stop the elevator at the desired floor. Thus, the presence of the counterweight reduces the energy requirements on the motor—instead of having to pull the elevator through its entire motion, it is needed only to start and stop the elevator, to overcome friction, and to account for any difference in mass between the counterweight and the elevator, associated with the number of passengers on the elevator.

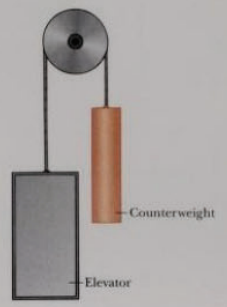


Figure 7.2 (Thinking Physics 1)

EXERCISE 1 What is the gravitational potential energy, relative to the ground, of a 0.15-kg baseball at the top of a 100-m-tall building? Answer 147 J

7.2 • CONSERVATIVE AND NONCONSERVATIVE FORCES

Forces found in nature can be divided into two categories: conservative and nonconservative. We shall describe the properties of conservative and nonconservative forces separately in this section.

Conservative Forces

A force is conservative if the work it does on an object moving between any two points is independent of the path taken by the object. The work done by a conservative force depends only on the initial and final coordinates of the object. A conservative force can also be defined in a second way. **A force is conservative if the work it does on an object moving through any closed path is zero.**

Definition of a conservative force

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The force of gravity is conservative. As we learned in the preceding section, the work done by the gravitational force on an object moving between any two points near the Earth's surface is

$$W_g = mgy_i - mgy_f$$

From this, we see that W_g depends only on the initial and final coordinates of the object and hence is independent of path. Furthermore, W_g is zero when the object moves over any closed path (where $y_i = y_f$).

We can associate a potential energy function with any conservative force. In the preceding section, the potential energy function associated with the gravitational force was found to be

$$U_g = mgy$$

Potential energy functions can be defined only for conservative forces. In general, the work, W_c, done on an object by a conservative force is given by the initial value of the potential energy associated with the object minus the final value:

$$W_c = U_i - U_f \qquad [7.3]$$

Another example of a conservative force is the force of a spring on an object attached to the spring, where the spring force is given by $F_s = -kx$. As we learned in Chapter 6 (Eq. 6.15), the work done by the spring force is

$$W_s = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2$$

where the initial and final coordinates of the object are measured from its equilibrium position, $x = 0$. Again we see that W_s depends only on the initial and final coordinates of the object and is zero for any closed path. Hence, the spring force is conservative. The **elastic potential energy** function associated with the spring force is defined by

Potential energy stored in a spring

$$U_s \equiv \tfrac{1}{2}kx^2 \qquad [7.4]$$

where x is measured from the uncompressed position of the spring. The elastic potential energy can be thought of as the energy stored in the deformed spring (one that is either compressed or stretched from its equilibrium position). To visualize this, consider Figure 7.3a, which shows an undeformed spring on a frictionless, horizontal surface. When the block is pushed against the spring (Fig. 7.3b), compressing the spring a distance x, the elastic potential energy stored in the spring is $kx^2/2$. When the block is released, the spring snaps back to its original length and the stored elastic potential energy is transformed into kinetic energy of the block (Fig. 7.3c). The elastic potential energy stored in the spring is zero whenever the spring is undeformed ($x = 0$). Energy is stored in the spring only when the spring is either stretched or compressed. Furthermore, the elastic potential energy is a maximum when the spring has reached its maximum compression or extension (that is, when $|x|$ is a maximum). Finally, because the elastic potential energy is proportional to x^2, we see that U_s is always positive in a deformed spring. If the spring and object are taken together as the system, then no work is done as the spring changes length because the forces are internal.

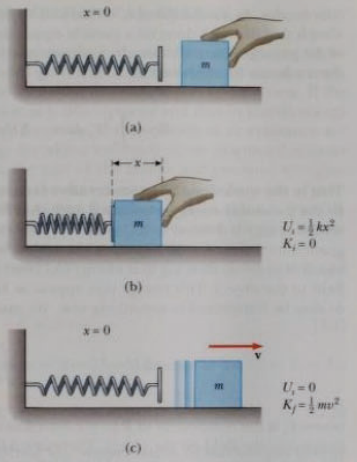


Figure 7.3 A block of mass m on a frictionless horizontal surface is pushed against a spring and then released. If x is the compression in the spring as in (b), the elastic potential energy stored in the spring is $\frac{1}{2}kx^2$. This energy is transferred to the block in the form of kinetic energy as in (c).

Nonconservative Forces

A force is called **nonconservative** if it leads to dissipation of mechanical energy. For example, if you move an object on a horizontal surface, returning to the same location and same state of motion, but you found that it was necessary to do a net amount of work on the object, then something must have dissipated that energy transferred to the object. That dissipative force is recognized as friction between the surface and the object. Friction is a dissipative, or nonconservative, force. By contrast, if the object is lifted, work is required, but that energy is recovered when the object is lowered. The gravitational force is a nondissipative or conservative force.

Definition of a nonconservative force

Suppose you displace a book between two points on a table. If the book is displaced in a straight line along the blue path between points A and B in Figure 7.4, the loss in mechanical energy due to the friction force f is simply $-fd$, where d is the distance between the two points. However, if the book is moved along any other path between the two points, the loss in mechanical energy due to friction is greater (in absolute magnitude) than $-fd$. For example, the loss in mechanical energy due to friction along the red semicircular path in Figure 7.4 is equal to $-f(\pi d/2)$, where d is the diameter of the circle.

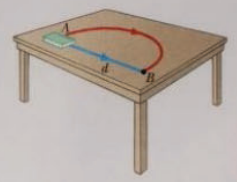


Figure 7.4 The decrease in mechanical energy due to the force of friction depends on the path taken as the book is moved from A to B, and hence friction is a nonconservative force. The decrease in mechanical energy is greater along the red path compared to the blue path.

7.3 · CONSERVATIVE FORCES AND POTENTIAL ENERGY

In the preceding section we found that the work done on a particle by a conservative force does not depend on the path taken by the particle and is independent of the particle's velocity. The work done is a function only of the particle's initial and final

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coordinates. As a consequence, we are able to define a **potential energy function** U such that the work done on a particle equals the decrease in the potential energy of the particle. The work done by a conservative force $\mathbf{F}$ as the particle moves along the x axis can be expressed as[1]

Work done by a conservative force

$$W_c = \int_{x_i}^{x_f} F_x \, dx = -\Delta U = U_i - U_f \qquad [7.5]$$

That is, **the work done by a conservative force equals the negative of the change in the potential energy associated with that force,** where the change in the potential energy is defined as $\Delta U = U_f - U_i$. For example, the work, W_c, done by the gravitational field on an object, as the object is lowered in the field, is $U_i - U_f$, which is positive, showing that energy has been transferred from the gravitational field to the object. This energy may appear as kinetic energy of the falling object or may be transferred to something else. We can also express Equation 7.5 as

Change in potential energy

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x \, dx \qquad [7.6]$$

where F_x is the component of $\mathbf{F}$ in the direction of the displacement: $\mathbf{F}$ is the force exerted by the field on the object. Therefore ΔU is negative when F_x and dx are in the same direction, as when an object is lowered in a gravitational field or a spring pushes an object toward equilibrium.

The term *potential energy* implies that an object has the potential, or capability, of either gaining kinetic energy or doing work when released from some point under the influence of gravity. It is often convenient to establish some particular location, x_i, to be a reference point and measure all potential energy differences with respect to that point. We can then define the potential energy function as

$$U_f = -\int_{x_i}^{x_f} F_x \, dx + U_i \qquad [7.7]$$

Furthermore, as we discussed earlier, the value of U_i is often taken to be zero at some arbitrary reference point. It really doesn't matter what value we assign to U_i, because any value only shifts U_f by a constant, and it is only the *change* in potential energy that is physically meaningful. If the conservative force is known as a function of position, we can use Equation 7.7 to calculate the change in potential energy of a body as it moves from x_i to x_f.

The amount of mechanical energy dissipated by a nonconservative force depends on the path as an object moves from one position to another and can also depend on the object's speed or on other quantities. Because the work done by a nonconservative force is not simply a function of the initial and final coordinates, we conclude that there is no potential energy function associated with a nonconservative force.

[1]For a general displacement, the work done in two or three dimensions also equals $U_i - U_f$, where $U = U(x, y, z)$. We write this formally as $W = \int_i^f \mathbf{F} \cdot d\mathbf{s} = U_i - U_f$.

7.4 · CONSERVATION OF MECHANICAL ENERGY

An object held at some height h above the floor has no kinetic energy, but, as we learned earlier, there is an associated gravitational potential energy equal to mgh, relative to the floor if the gravitational field is included as part of the system. If the object is dropped, it falls to the floor; as it falls, its speed and thus its kinetic energy increase while the potential energy decreases. If factors such as air resistance are ignored, whatever potential energy the object loses as it moves downward appears as kinetic energy. In other words, the sum of the kinetic and potential energies, called the *mechanical energy* E, remains constant in time. This is an example of principle of **conservation of mechanical energy.** For the case of an object in free-fall, this principle tells us that any increase (or decrease) in potential energy is accompanied by an equal decrease (or increase) in kinetic energy.

Because the total mechanical energy E is defined as the sum of the kinetic and potential energies, we can write

$$E \equiv K + U \qquad [7.8]$$

Total mechanical energy

Therefore, we can apply conservation of mechanical energy in the form $E_i = E_f$, or

$$K_i + U_i = K_f + U_f \qquad [7.9]$$

Conservation of mechanical energy

We can state this in a more formal way: **conservation of mechanical energy requires that the total mechanical energy of a system remains constant in any isolated system of objects which interact only through conservative forces.** It is important to note that Equation 7.9 is valid *provided* no energy is added to or removed from the system. Furthermore, there must be no nonconservative forces within the system.

A formal statement of conservation of energy

Because mechanical energy E remains constant with time, $dE/dt = 0$. Taking the derivative of Equation 7.8 with respect to time,

$$\frac{dE}{dt} = 0 = \frac{dK}{dt} + \frac{dU}{dt} \qquad [7.10]$$

Because $K = \frac{1}{2}mv^2$, then

$$\frac{dK}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = mv\frac{dv}{dt} = mva = F_x v$$

Applying the chain rule (see Appendix B.6) to dU/dt, we have

$$\frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} = \left(\frac{dU}{dx}\right)v$$

Substituting these expressions for dK/dt and dU/dt into Equation 7.10 gives

$$F_x v + \left(\frac{dU}{dx}\right)v = 0$$

$$F_x = -\frac{dU}{dx} \qquad [7.11]$$

Relation between a conservative force and potential energy

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That is, **the conservative internal force acting between parts of a system equals the negative derivative of the potential energy associated with that system.**

We can easily check this relationship for the two instances already discussed. In the case of the deformed spring, $U_s = \frac{1}{2}kx^2$, and therefore

$$F_s = -\frac{dU_s}{dx} = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$

which corresponds to the restoring force exerted by the spring. In the case of an object located a distance y above some reference point, the gravitational potential energy function is given by $U_g = mgy$, and it follows from Equation 7.11 that $F_g = -mg$.

We now see that U is an important function; from it can be derived the conservative force acting in any system. Furthermore, Equation 7.11 should clarify the fact that adding a constant to the potential energy is unimportant, because the location of the reference point is arbitrary.

Equation 7.11 can also be written in the form $dU = -F\,dx$, which, when integrated between the initial and final position values, gives

$$U_f - U_i = -\int_{x_i}^{x_f} F\,dx \qquad [7.12]$$

This result, which is identical to Equation 7.6, tells us that if the conservative force F acting on an object within a system is known as a function of x, we can calculate the *difference* in the potential energy associated with the object between the initial and final positions.

If more than one conservative force acts on the object, then a potential energy function is associated with *each* force. In such a case, we can apply the law of conservation of energy for the system as

Conservation of mechanical energy

$$K_i + \sum U_i = K_f + \sum U_f \qquad [7.13]$$

where the number of terms in the sums equals the number of conservative forces present. For example, if a mass connected to a spring oscillates vertically, two conservative forces act on it: the spring force and the force of gravity. (We will discuss this situation later in a worked example.)

If the force of gravity is the *only* force acting on a body, then the total mechanical energy of the body is constant. Therefore, the law of conservation of energy for a freely falling body can be written

Conservation of mechanical energy for a freely falling body

$$\tfrac{1}{2}mv_i^2 + mgy_i = \tfrac{1}{2}mv_f^2 + mgy_f \qquad [7.14]$$

Thinking Physics 2

You have graduated from college and are designing roller coasters for a living. You design a roller coaster in which a car is pulled to the top of a hill of height h and then, starting from a momentary rest, rolls freely down the hill and upward toward the peak of the next hill, which is at height $1.1h$. Will you have a long career in this business?

Reasoning Your career will probably not be long, because this roller coaster will not work! At the top of the first hill, the roller coaster train has no kinetic energy, and gravitational potential energy associated with a height h. If it were to reach the top of the next hill, it would have higher potential energy, that associated with height $1.1h$.

This would violate the principle of conservation of mechanical energy. If this coaster were actually built, the car would move upward on the second hill to a height h (ignoring the effects of friction), stop short of the peak, and then start rolling backward, becoming trapped between the two peaks.

CONCEPTUAL PROBLEM 1

Discuss the energy transformations that occur during the pole vault event pictured in the multiflash photograph shown in Figure 7.5. Ignore rotational motion.



Figure 7.5 (Conceptual Problem 1) Multiflash photograph of a pole vault event. How many forms of energy can you identify in this picture? *(© Harold E. Edgerton, Courtesy of Palm Press, Inc.)*

CONCEPTUAL PROBLEM 2

Three identical balls are thrown from the top of a building, all with the same initial speed. The first ball is thrown horizontally, the second at some angle above the horizontal, and the third at some angle below the horizontal as in Figure 7.6. Neglecting air resistance, describe their motions and compare the speeds of the balls as they reach the ground.

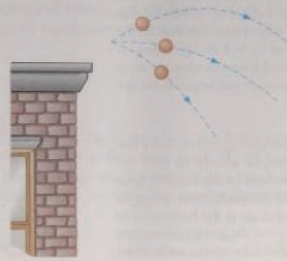


Figure 7.6 (Conceptual Problem 2) Three identical balls are thrown with the same initial speed from the top of a building.

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Example 7.1 Ball in Free-Fall

A ball of mass m is dropped from a height h above the ground, as in Figure 7.7. (a) Neglecting air resistance, determine the speed of the ball when it is at a height y above the ground.

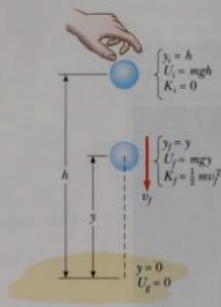


Figure 7.7 (Example 7.1) A ball is dropped from a height h above the ground. Initially, the total energy is gravitational potential energy, equal to mgh relative to the ground. At the elevation y, the total energy is the sum of kinetic and potential energies.

Reasoning Because the ball is in free-fall, the only force acting on it is the gravitational force. Therefore, we can use the principle of conservation of mechanical energy. Initially, the ball has potential energy and no kinetic energy. As it falls, its total energy (the sum of kinetic and potential energies) remains constant and equal to its initial potential energy.

Solution When the ball is released from rest at a height h above the ground, its kinetic energy is $K_i = 0$ and its potential energy is $U_i = mgh$, where the y coordinate is measured from ground level. When the ball is at a distance y above the ground, its kinetic energy is $K_f = \frac{1}{2}mv_f^2$ and its potential energy relative to the ground is $U_f = mgy$. Applying Equation 7.9, we get

$$K_i + U_i = K_f + U_f$$
$$0 + mgh = \tfrac{1}{2}mv_f^2 + mgy$$
$$v_f^2 = 2g(h - y)$$
$$v_f = \sqrt{2g(h - y)}$$

(b) Determine the speed of the ball at y if it is given an initial speed v_i at the initial altitude h.

Solution In this case, the initial energy includes kinetic energy equal to $\frac{1}{2}mv_i^2$, and Equation 7.14 gives

$$\tfrac{1}{2}mv_i^2 + mgh = \tfrac{1}{2}mv_f^2 + mgy$$
$$v_f^2 = v_i^2 + 2g(h - y)$$
$$v_f = \sqrt{v_i^2 + 2g(h - y)}$$

This result is consistent with the expression from kinematics, $v_y^2 = v_{y0}^2 - 2g(y - y_0)$, where $y_0 = h$. Furthermore, this result is valid even if the initial velocity is at an angle to the horizontal (the projectile situation).

Example 7.2 One Way to Lift an Object

Two blocks are connected by a massless cord that passes over a frictionless pulley and a frictionless peg, as in Figure 7.8. One end of the cord is attached to a mass $m_1 = 3.00$ kg that is a distance $R = 1.20$ m from the peg. The other end of the cord is connected to a block of mass $m_2 = 6.00$ kg resting on a table. From what angle θ (measured from the vertical) must the 3.00-kg mass be released in order to just begin to lift the 6.00-kg block off the table?

Reasoning It is necessary to use several concepts to solve this problem. First, we use conservation of energy to find the speed of the 3.00-kg mass at the bottom of the circular path as a function of θ and the radius of the path. Next, we apply Newton's second law to the 3.00-kg mass at the bottom of its path to find the tension as a function of the given parameters. Finally, we note that the 6.00-kg block lifts off the table when the upward force exerted on it by the cord exceeds the force of gravity acting on the block. This procedure enables us to find the required angle.

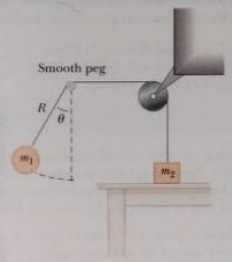


Figure 7.8 (Example 7.2)

Solution Applying conservation of energy to the 3.00-kg mass gives

$$K_i + U_i = K_f + U_f$$
$$(1) \quad 0 + m_1gy_i = \tfrac{1}{2}m_1v^2 + 0$$

where v is the speed of the 3.00-kg mass at the bottom of its path. (Note that $K_i = 0$, because the 3.00-kg mass starts from rest and $U_f = 0$, because the bottom of the circle is the zero level of potential energy.) From the geometry in Figure 7.8, we see that $y_i = R - R\cos\theta = R(1 - \cos\theta)$. Using this relation in (1) gives

$$(2) \quad v^2 = 2gR(1 - \cos\theta)$$

Now we apply Newton's second law to the 3.00-kg mass when it is at the bottom of the circular path:

$$T - m_1g = m_1\frac{v^2}{R}$$

$$(3) \quad T = m_1g + m_1\frac{v^2}{R}$$

This same force is transmitted to the 6.00-kg block, and if it is to be just lifted off the table, the normal force on it becomes zero, and we require that $T = m_2g$. Using this condition, together with (2) and (3), gives

$$m_2g = m_1g + m_1\frac{2gR(1 - \cos\theta)}{R}$$

Solving for θ, and substituting in the given parameters, we get

$$\cos\theta = \frac{3m_1 - m_2}{2m_1} = \frac{3(3.00\text{ kg}) - 6.00\text{ kg}}{2(3.00\text{ kg})} = \frac{1}{2}$$
$$\theta = 60.0°$$

EXERCISE 2 If the initial angle is $\theta = 40.0°$, find the speed of the 3.00-kg mass and the tension in the cord when the 3.00-kg mass is at the bottom of its circular path.
Answer 2.35 m/s; 43.2 N

EXERCISE 3 A 4.0-kg particle moves along the x axis under the influence of a single conservative force. If the work done on the particle is 80.0 J as it moves from the point $x = 2.0$ m to $x = 5.0$ m, find (a) the change in its kinetic energy, (b) the change in its potential energy, and (c) its speed at $x = 5.0$ m if it starts at rest at $x = 2.0$ m. Answer (a) 80.0 J (b) −80.0 J (c) 6.32 m/s

EXERCISE 4 A rocket is launched at an angle of 53° to the horizontal from an altitude h with a speed v_0. Use energy methods to find its speed when its altitude is $h/2$.
Answer $v = \sqrt{v_0^2 + gh}$

7.5 · WORK DONE BY NONCONSERVATIVE FORCES

As we have seen, if the forces acting on a system are conservative, the mechanical energy of the system remains constant. However, if some of the forces acting on the system are not conservative, the mechanical energy of the system does not remain constant. Let us examine two types of nonconservative forces: an applied force and the force of kinetic friction.

Work Done by an Applied Force

When you lift a book through some distance by applying a force to it, the force you apply does work W_{app} on the book, and the force of gravity does work W_g on the book. If we treat the book as a particle, then the net work done on the book is related to the change in its kinetic energy through the work-kinetic energy theorem given by Equation 6.19:

$$W_{app} + W_g = \Delta K \qquad [7.15]$$

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Because the force of gravity is conservative, we can use Equation 7.2 to express the work done by the force of gravity in terms of the change in gravitational potential energy or $W_g = -\Delta U$. Substituting this into Equation 7.15 gives

$$W_{app} = \Delta K + \Delta U \quad [7.16]$$

Note that the right side of this equation is the change in the mechanical energy of the book–Earth system. This result says that your applied force transfers energy to the system in the form of kinetic energy of the book and to the gravitational potential energy of the book–Earth system. Thus, we conclude that if an object is part of a system, then **an applied force can transfer energy into or out of the system.**

Situations Involving Kinetic Friction



Figure 7.9 A book that is given an initial velocity $\mathbf{v}_0$ on a horizontal surface comes to rest due to the force of kinetic friction after traveling a distance d.

Kinetic friction is an example of a nonconservative force. If a book is given some initial velocity on a horizontal surface that is not frictionless, as in Figure 7.9, the force of kinetic friction acting on the book opposes its motion and negative work is done by the friction force. The book slows down and eventually stops after undergoing a displacement **d**. The force of friction reduces the kinetic energy of the book by transferring energy to thermal (internal) energy of the book and part of the horizontal surface. Note that only part of the book's kinetic energy is transferred to thermal energy in the book. The rest is transferred as thermal energy from the book to the surface.

As the book in Figure 7.9 moves through a distance d, the only force that does work is the force of kinetic friction. In this situation, we can apply Equation 7.16, where $\Delta U = 0$ and note that the force of kinetic friction is opposite the displacement. Thus, we see that the amount by which the force of friction decreases the kinetic energy is $-f_k d$, or

$$\Delta K = -f_k d \quad [7.17]$$

If the book moves on an *incline* that is not frictionless, then a change in gravitational potential energy also occurs, and $-f_k d$ is the amount by which the force of kinetic friction reduces the mechanical energy $E = K + U$ of the system. In such cases,

$$\Delta K + \Delta U = -f_k d \quad [7.18]$$

CONCEPTUAL PROBLEM 3

A rock is thrown vertically in the air. Ignoring air friction, it takes the same time to go up as it does to come down. Is this still true in the presence of air friction?

CONCEPTUAL PROBLEM 4

A driver brings an automobile to a stop. If the brakes lock, so that the car skids, where is the energy, and in what form is it, after the car stops? Answer the same question for the case in which the brakes do not lock, but the wheels continue to turn.

PROBLEM-SOLVING STRATEGIES • Conservation of Energy

Many problems in physics can be solved using the principle of conservation of energy. The following procedure should be used when you apply this principle:

1. Define your system, which may consist of more than one object and may or may not include fields, springs, or other sources of potential energy. Choose instants to call the initial and final points.
2. Select a reference position for the zero point of potential energy (both gravitational and spring), and use this throughout your analysis. If there is more than one conservative force, write an expression for the potential energy associated with each force.
3. Determine whether any nonconservative forces are present. Remember that if friction or air resistance is present, mechanical energy *is not constant.*
4. If mechanical energy is *constant,* you can write the total initial energy, E_i, at some point as the sum of the kinetic and potential energy at that point. Then write an expression for the total final energy, $E_f = K_f + U_f$, at the final point that is of interest. Because mechanical energy is *constant,* you can equate the two total energies and solve for the quantity that is unknown.
5. If nonconservative forces are present (and thus mechanical energy is not *constant*), first write expressions for the total initial and total final energies. In this case, the difference between the total final mechanical energy and the total initial mechanical energy equals the energy transferred to or from the system by the nonconservative forces. It is convenient to think of a general work-energy theorem as a combination of Equations 7.16 and 7.18, written as

$$K_i + U_i + W_{app} - f_k d = K_f + U_f$$

Example 7.3 Crate Sliding Down a Ramp

A 3.00-kg crate slides down a ramp at a loading dock. The ramp is 1.00 m in length, and inclined at an angle of 30.0°, as shown in Figure 7.10. The crate starts from rest at the top, experiences a constant frictional force of magnitude 5.00 N, and continues to move a short distance on the flat floor. Use energy methods to determine the speed of the crate when it reaches the bottom of the ramp.

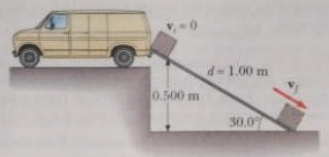


Figure 7.10 (Example 7.3) A crate slides down a ramp under the influence of gravity. The potential energy of the system decreases, and the kinetic energy of the crate increases.

Solution Because $v_i = 0$, the initial kinetic energy is zero. If the y coordinate is measured from the bottom of the ramp, then $y_i = 0.500$ m. Therefore, the total mechanical energy of the crate–Earth system at the top is all gravitational potential energy:

$$U_i = mgy_i = (3.00 \text{ kg})\left(9.80 \frac{\text{m}}{\text{s}^2}\right)(0.5000 \text{ m}) = 14.7 \text{ J}$$

When the crate reaches the bottom, the gravitational potential energy of the system is zero, because the elevation of

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the crate is $y_f = 0$. Therefore, the total mechanical energy at the bottom is all kinetic energy,

$$K_f = \tfrac{1}{2}mv_f^2$$

However, we cannot say that $U_i = K_f$ in this case, because there is an external nonconservative force that reduces the mechanical energy of the system: the force of kinetic friction. In this case, $\Delta E = -f_k d$ where d is the displacement along the ramp. (Remember that the forces normal to the ramp do no work on the crate because they are perpendicular to the displacement.) With $f_k = 5.00$ N and $d = 1.00$ m, we have

$$\Delta E = -f_k d = (-5.00\ \text{N})(1.00\ \text{m}) = -5.00\ \text{J}$$

This says that the decrease in mechanical energy is due to the presence of the force of kinetic friction that opposes the motion. Because $\Delta E = \frac{1}{2}mv_f^2 - mgy_i$ in this situation, Equation 7.18 gives

$$-f_k d = \tfrac{1}{2}mv_f^2 - mgy_i$$

$$\tfrac{1}{2}mv_f^2 = 14.7\ \text{J} - 5.00\ \text{J} = 9.70\ \text{J}$$

$$v_f^2 = \frac{19.4\ \text{J}}{3.00\ \text{kg}} = 6.47\ \text{m}^2/\text{s}^2$$

$$v_f = 2.54\ \text{m/s}$$

EXERCISE 5 Use Newton's second law to find the acceleration of the crate along the ramp and the equations of kinematics to determine the final speed of the crate. Answer 3.23 m/s²; 2.54 m/s

EXERCISE 6 If the ramp is assumed to be frictionless, find the final speed of the crate and its acceleration along the ramp. Answer 3.13 m/s; 4.90 m/s²

Example 7.4 Motion on a Curved Track

A child of mass m takes a ride on an irregularly curved slide of height $h = 6.00$ m, as in Figure 7.11. The child starts from rest at the top. (a) Determine the speed of the child at the bottom, assuming no friction is present.

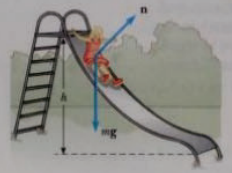


Figure 7.11 (Example 7.4) If the slide is frictionless, the speed of the child at the bottom depends only on the height of the slide.

Reasoning The normal force, **n**, does no work on the child, because this force is always perpendicular to each element of the displacement. Furthermore, because there is no friction, mechanical energy is constant—that is, $K + U =$ constant.

Solution If we measure the y coordinate from the bottom of the slide, then $y_i = h$, $y_f = 0$, and we get

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \tfrac{1}{2}mv_f^2 + 0$$

$$v_f = \sqrt{2gh}$$

Note that the result is the same as it would be if the child fell vertically through a distance h! In this example, $h = 6.00$ m, giving

$$v_f = \sqrt{2gh} = \sqrt{2\left(9.80\ \frac{\text{m}}{\text{s}^2}\right)(6.00\ \text{m})} = 10.8\ \text{m/s}$$

(b) If a frictional force acts on the child, how much mechanical energy is dissipated by this force? Assume that $v_f = 8.00$ m/s and $m = 20.0$ kg.

Solution In this case, $\Delta E \neq 0$ and mechanical energy is *not* constant. We can use Equation 7.18 to find the loss of mechanical energy due to friction, assuming the final speed at the bottom is known:

$$\Delta E = E_f - E_i = \tfrac{1}{2}mv_f^2 - mgh$$

$$\Delta E = \tfrac{1}{2}(20.0\ \text{kg})(8.00\ \text{m/s})^2 - (20.0\ \text{kg})\left(9.80\ \frac{\text{m}}{\text{s}^2}\right)(6.00\ \text{m})$$

$$= -536\ \text{J}$$

Again, ΔE is negative because friction reduces the mechanical energy of the system. Note, however, that because the slide is curved, the normal force changes in magnitude and direction during the motion. Therefore, the frictional force, which is proportional to n, also changes during the motion. Do you think it would be possible to determine μ from these data?

Example 7.5 Mass–Spring Collision

A mass of 0.80 kg is given an initial velocity $v_i = 1.2$ m/s to the right and collides with a light spring of force constant $k = 50$ N/m, as in Figure 7.12. (a) If the surface is frictionless, calculate the initial maximum compression of the spring after the collision.

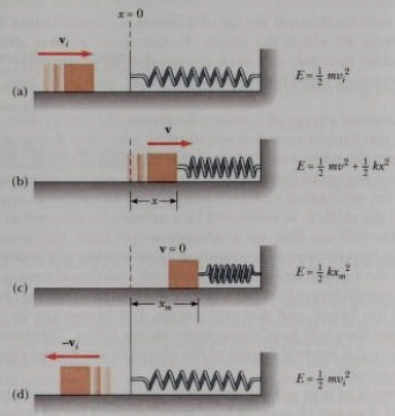


Figure 7.12 (Example 7.12) A block sliding on a smooth, horizontal surface collides with a light spring. (a) Initially the mechanical energy is all kinetic energy. (b) The mechanical energy is the sum of the kinetic energy of the block and the elastic potential energy in the spring. (c) The energy is entirely potential energy. (d) The energy is converted back to the kinetic energy of the block. The total energy remains constant throughout the motion.

Reasoning Before the collision, the mass has kinetic energy and the spring is uncompressed, so the energy stored in the spring is zero. Thus, the total energy of the system (mass plus spring) before the collision is $\frac{1}{2}mv_i^2$. After the collision, and when the spring is fully compressed, the mass is momentarily at rest and has zero kinetic energy, and the energy stored in the spring has its maximum value, $\frac{1}{2}kx_f^2$. The total mechanical energy of the system is constant because no nonconservative forces act on the system.

Solution Because mechanical energy is conserved, the kinetic energy of the mass before the collision must equal the maximum energy stored in the spring when it is fully compressed, or

$$\tfrac{1}{2}mv_i^2 = \tfrac{1}{2}kx_f^2$$

$$x_f = \sqrt{\frac{m}{k}}\,v_i = \sqrt{\frac{0.80\ \text{kg}}{50\ \text{N/m}}}\,(1.2\ \text{m/s}) = 0.15\ \text{m}$$

(b) If a constant force of kinetic friction acts between block and surface with $\mu_k = 0.50$ and if the speed of the block just as it collides with the spring is $v_i = 1.2$ m/s, what is the maximum compression in the spring?

Solution In this case, mechanical energy is *not* conserved because of friction. The magnitude of the frictional force is

$$f_k = \mu_k n = \mu_k mg = 0.50(0.80\ \text{kg})\left(9.80\ \frac{\text{m}}{\text{s}^2}\right) = 3.9\ \text{N}$$

Therefore, the decrease in kinetic energy due to friction as the block is displaced from $x_i = 0$ to $x_f = x$ is

$$\Delta K = -f_k x = (-3.92x)\ \text{J}$$

Substituting this into Equation 7.16 gives

$$\Delta K = (0 + \tfrac{1}{2}kx^2) - (\tfrac{1}{2}mv_i^2 + 0)$$

$$-3.92x = \tfrac{50}{2}x^2 - \tfrac{1}{2}(0.80)(1.2)^2$$

$$25x^2 + 3.92x - 0.576 = 0$$

Solving the quadratic equation for x gives $x = 0.092$ m and $x = -0.25$ m. The physically meaningful root is $x = 0.092$ m = 9.2 cm. The negative root is meaningless because the block must be to the right of the origin when it comes to rest. Note that 9.2 cm is less than the distance obtained in the frictionless case (a). This result is what we expect because friction retards the motion of the system.

EXERCISE 7 A 3.0-kg block starts at a height $h = 60$ cm on a plane that has an inclination angle of 30°. On reaching the bottom, the block slides along a horizontal surface. If the coefficient of friction on both surfaces is $\mu_k = 0.20$, how far does the block slide on the horizontal surface before coming to rest? Answer 1.96 m

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EXERCISE 8 A child starts from rest at the top of a slide of height 4.0 m. (a) What is her speed at the bottom if the incline is frictionless? (b) If she reaches the bottom with a speed of 6.0 m/s, what percentage of her total energy at the top of the slide was lost as a result of friction? Answer (a) 8.85 m/s (b) 54.1%

7.6 · CONSERVATION OF ENERGY IN GENERAL

We have seen that the total mechanical energy of a system is constant when only conservative internal forces act within the system. Furthermore, we were able to associate a potential energy function with each conservative force. However, mechanical energy is not constant when nonconservative forces, such as friction, are present.

Whenever the mechanical energy of a system decreases, the energy does not disappear. Instead, the mechanical energy is transformed into other forms of energy. In the study of thermodynamics we shall find that mechanical energy can be transformed into internal energy of the system. For example, when a block slides over a surface, part of the mechanical energy is transformed into internal energy stored in the block and the surface, as evidenced by a measurable increase in the block's temperature. We shall see that, on a submicroscopic scale, this internal energy is associated with the vibration of atoms about their equilibrium positions. The energy acquired by the atoms is unevenly and randomly distributed among the atoms. This disorderly internal form of energy is often called **thermal energy.** Such internal atomic motion has kinetic and potential energy, and so one can say that frictional forces arise fundamentally from conservative atomic forces. Therefore, if we include this increase in the internal energy of the system in our energy expression, the total energy is conserved.

This is just one example of how you can analyze an isolated system and always find that its total energy does not change, as long as you account for all forms of energy. That is, **energy can never be created or destroyed. Energy may be transformed from one form to another, but the total energy of an isolated system is always constant.** From a universal point of view, we can say that **the total energy of the Universe is constant:** If one part of the Universe gains energy in some form, another part must lose an equal amount of energy. No violation of this principle has been found.

Total energy is always conserved.

Other examples of energy transformations include the energy carried by sound waves resulting from the collision of two objects, the energy radiated by an accelerating charge in the form of electromagnetic waves (a radio antenna), and the elaborate sequence of energy conversions in a thermonuclear reaction.

In subsequent chapters we shall see that the energy concept, and especially transformations of energy, unite the branches of physics. The subjects of mechanics, thermodynamics, and electromagnetism cannot really be separated. In practical terms, all mechanical and electronic devices rely on energy transformations.

Thinking Physics 3

An automobile with kinetic energy and potential energy carried within the gasoline strikes a tree and comes to rest. The total mechanical energy in the system (the car) is now less than before. How did the energy leave the system, and in what form is energy left in the system?

Reasoning There are a number of ways that energy transferred out of the system. We shall describe the main mechanisms. The automobile did some *work* on the tree during the collision, causing the tree to become warmer and deform. There was a large crash during the collision, representing transfer of energy by *sound*. If the gasoline tank leaks after the collision, the automobile is losing energy by means of *mass transfer*. After the car has come to rest, there is no more kinetic energy. There is some *potential energy* left in any gas remaining within the tank. There is also more *internal energy* in the automobile, because its temperature is likely to be higher after the collision than before.

CONCEPTUAL PROBLEM 5

A toaster is turned on. Discuss the forms of energy and energy transfer occurring in the coils of the toaster.

CONCEPTUAL PROBLEM 6

Soft steel can be made red hot by continued hammering on it. This doesn't work as well for hard steel. Why is there this difference?

7.7 · GRAVITATIONAL POTENTIAL ENERGY REVISITED

Earlier in this chapter we introduced the concept of gravitational potential energy, that is, the energy associated with the position of a particle. We emphasized the fact that the gravitational potential energy function, $U = mgy$, is valid only when a particle is near the Earth's surface. Because the gravitational force between two particles varies as $1/r^2$, it follows that the correct potential energy function of the system depends on the amount of separation between the particles.

Consider a particle of mass m moving between two points P and Q above the Earth's surface, as in Figure 7.13. The gravitational force acting on m is

$$\mathbf{F}_g = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} \qquad [7.19]$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the Earth to the particle and the negative sign indicates that the force is attractive. This expression shows that the gravitational force depends only on the polar coordinate r. Furthermore, the gravitational force is conservative. Because the change in potential energy associated with a given displacement of the particle is defined as the negative of the work done by the conservative gravitational force during that displacement, Equation 7.12 gives

$$U_f - U_i = -\int_{r_i}^{r_f} F(r)\,dr = GM_e m\int_{r_i}^{r_f}\frac{dr}{r^2} = GM_e m\left[-\frac{1}{r}\right]_{r_i}^{r_f}$$

or

$$U_f - U_i = -GM_e m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \qquad [7.20]$$

As always, the choice of a reference point for the potential energy is completely arbitrary. It is customary to locate the reference point where the force is zero. Taking $U_i = 0$ at $r_i = \infty$, we obtain the important result

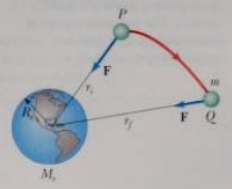


Figure 7.13 As a particle of mass m moves from P to Q above the Earth's surface, the potential energy changes according to Equation 7.20.

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Gravitational potential energy $r > R_e$

$$U(r) = -\frac{GM_e m}{r} \qquad [7.21]$$

This equation applies to the Earth–particle system separated by a distance r, provided that $r > R_e$. The result is not valid for particles moving inside the Earth, where $r < R_e$. Because of our choice of U_i, the function $U(r)$ is always negative (Fig. 7.14).

Although Equation 7.21 was derived for the particle–Earth system, it can be applied to *any* two particles. That is, the gravitational potential energy associated with *any pair* of particles of masses m_1 and m_2 separated by a distance r is

$$U_g = -\frac{Gm_1 m_2}{r} \qquad [7.22]$$

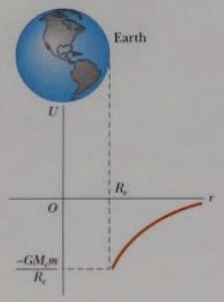


Figure 7.14 Graph of the gravitational potential energy, U_g, versus r for a particle above the Earth's surface. The potential energy of the system goes to zero as r approaches ∞.

This expression also applies to larger objects *if they are spherically symmetric*, as first shown by Newton using integral calculus. Equation 7.22 shows that the gravitational potential energy for any pair of particles varies as $1/r$ (whereas the force between them varies as $1/r^2$). Furthermore, the potential energy is *negative*, because the force is attractive, and we have taken the potential energy as zero when the particle separation is infinity. Because the force between the particles is attractive, we know that an external agent must do positive work to increase the separation between the two particles. The work done by the external agent produces an increase in the potential energy as the two particles are separated. That is, U_g becomes less negative as r increases. (Note that part of the work done can also produce a change in kinetic energy of the system. That is, if the work done in separating the particles exceeds the increase in potential energy, the excess energy is accounted for by the increase in kinetic energy of the system.) When the two particles are separated by a distance r, an external agent would have to supply an energy *at least* equal to $+Gm_1 m_2/r$ in order to separate the particles by an infinite distance.

It is convenient to think of the absolute value of the gravitational potential energy defined this way as the **binding energy** of the system. If the external agent supplies an energy *greater than* the binding energy, $Gm_1 m_2/r$, the additional energy of the system is in the form of kinetic energy when the particles are at an infinite separation.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all *pairs* of particles.[2] Each pair contributes a term of the form given by Equation 7.22. For example, if the system contains three particles, as in Figure 7.15, we find that

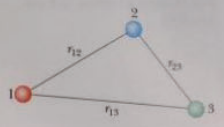


Figure 7.15 Diagram of three interacting particles.

$$U_{\text{total}} = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right) \qquad [7.23]$$

The absolute value of U_{total} represents the work needed to separate the particles by an infinite distance. If the system consists of four particles, there are six terms in the sum, corresponding to the six distinct pairs of interaction forces. The total mechanical energy of this system of four particles includes ten terms: four kinetic energy terms (one for each particle) and six potential energy terms.

[2]The fact that one can add potential energy terms for all pairs of particles stems from the experimental fact that gravitational forces obey the superposition principle. That is, if $\Sigma\mathbf{F} = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{23} + \ldots$, then there exists a potential energy term for each interaction $\mathbf{F}_{ij}$.

Thinking Physics 4

Why is the Sun hot?

Reasoning The Sun was formed when a cloud of gas and dust coalesced, due to gravitational attraction, into a massive astronomical object. Before this occurred, the particles were widely scattered, representing a large amount of gravitational potential energy. As the particles came together to form the Sun, the gravitational potential energy decreased. According to the principle of conservation of energy, this potential energy must be transformed to another form. The form to which it transformed was internal energy, representing an increase in temperature. If enough particles come together, the temperature can rise to a point at which nuclear fusion occurs, and the object becomes a star. If there are not enough particles, the temperature rises, but not to a point at which fusion occurs. The object is a planet if it is in orbit around a star. Jupiter is an example of a large planet that might have been a star if more particles were available to be collected.

Example 7.6 The Change in Potential Energy

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Let us show that the general expression for the change in gravitational potential energy given by Equation 7.20 reduces to the familiar relationship $\Delta U_g = mg\,\Delta y$.

Solution We can express Equation 7.20 in the form

$$\Delta U_g = -GM_e m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GM_e m\left(\frac{r_f - r_i}{r_i r_f}\right)$$

If both the initial and final positions of the particle are close to the Earth's surface, then $r_f - r_i = \Delta y$ and $r_i r_f \approx R_e^2$. (Recall that r is measured from the center of the Earth.) Therefore, the *change* in potential energy becomes

$$\Delta U_g \approx \frac{GM_e m}{R_e^2}\Delta y = mg\,\Delta y$$

where we have used the fact that $g = GM_e/R_e^2$. Keep in mind that the reference point is arbitrary, because it is the *change* in potential energy that is meaningful.

EXERCISE 9 A satellite of the Earth has a mass of 100 kg and its altitude is 2.0×10^6 m. (a) What is the gravitational potential energy of the satellite–Earth system? (b) What is the magnitude of the force on the satellite? Answer (a) -4.8×10^9 J (b) 570 N

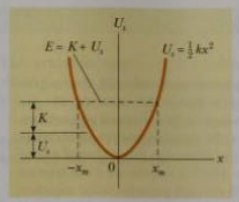
7.8 • ENERGY DIAGRAMS AND STABILITY OF EQUILIBRIUM OPTIONAL

The motion of a system can often be understood qualitatively through an analysis of the system's potential energy curve. Consider the potential energy function for the mass–spring system, given by $U_s = \frac{1}{2}kx^2$. This function is plotted versus x in Figure 7.16a. The spring force is related to U through Equation 7.11:

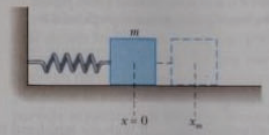
$$F_s = -\frac{dU_s}{dx} = -kx$$

That is, the force is equal to the negative of the *slope* of the U-versus-x curve. When the mass is placed at rest at the equilibrium position ($x = 0$), where $F = 0$, it will remain there unless some external force acts on it. If the spring is stretched from equilibrium, x is positive and the slope dU/dx is positive; therefore, F_s is negative and the mass accelerates back toward $x = 0$. If the spring is compressed, x is negative

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(a)



(b)

Figure 7.16 (a) The potential energy as a function of x for the mass–spring system shown in (b). The mass oscillates between the turning points, which have the coordinates $x = \pm x_m$. Note that the restoring force of the spring always acts toward $x = 0$, the position of stable equilibrium.

Stable equilibrium

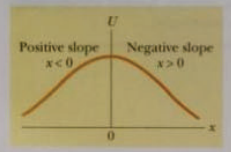


Figure 7.17 A plot of U versus x for a particle that has a position of unstable equilibrium, located at $x = 0$. For any finite displacement of the particle, the force on the particle is directed away from $x = 0$.

Unstable equilibrium

and the slope is negative; therefore, F_x is positive and again the mass accelerates toward $x = 0$.

From this analysis we conclude that the $x = 0$ position is one of **stable equilibrium.** That is, any movement away from this position results in a force that is directed back toward $x = 0$. In general, **positions of stable equilibrium correspond to those points for which $U(x)$ has a relative minimum value.**

From Figure 7.16 we see that if the mass is given an initial displacement x_m and released from rest, its total energy initially is the potential energy stored in the spring, given by $\frac{1}{2}kx_m^2$. As motion commences, the system acquires kinetic energy and loses an equal amount of potential energy. Because the total energy must remain constant, the mass oscillates between the two points $x = \pm x_m$, called the *turning points*. In fact, because no energy loss (no friction) takes place, the mass oscillates between $-x_m$ and $+x_m$ forever. (We shall discuss these oscillations further in Chapter 12). From an energy viewpoint, the energy of the system cannot exceed $\frac{1}{2}kx_m^2$; therefore, the mass must stop at these points and, because of the spring force, accelerate toward $x = 0$.

Another simple mechanical system that has a position of stable equilibrium is a ball rolling around in the bottom of a spherical bowl. If the ball is displaced from its lowest position, it always tends to return to that position when released.

Now consider an example in which the U-versus-x curve is as shown in Figure 7.17. In this case, $F_x = 0$ at $x = 0$, and so the particle is in equilibrium at this point. However, this is a position of **unstable equilibrium** for the following reason. Suppose the particle is displaced to the *right* ($x > 0$). Because the slope is negative for $x > 0$, $F_x = -dU/dx$ is positive and the particle accelerates away from $x = 0$. Now suppose the particle is displaced to the left ($x < 0$). In this case the force is *negative*, because the slope is positive for $x < 0$, and the particle again accelerates away from the equilibrium position. The $x = 0$ position in this situation is called a position of *unstable equilibrium* because, for any displacement from this point, the force pushes the particle farther away from equilibrium. In fact, the force pushes the particle toward a position of lower potential energy. A ball placed on the top of an inverted spherical bowl is obviously in a position of unstable equilibrium. If the ball is displaced slightly from the top and released, it will surely roll off the bowl. In general, **positions of unstable equilibrium correspond to those points for which $U(x)$ has a relative maximum value.**[3]

Finally, a situation may arise where U is constant over some region, and hence $F = 0$. This is called a position of **neutral equilibrium.** Small displacements from this position produce neither restoring nor disrupting forces. A ball lying on a flat horizontal surface is an example of an object in neutral equilibrium.

Neutral equilibrium

SUMMARY

If a particle of mass m is elevated a distance y near the Earth's surface, the **gravitational potential energy** of the particle–Earth system can be represented as

$$U_g = mgy \qquad [7.1]$$

The **elastic potential energy** stored in a spring of force constant k is

$$U_s = \tfrac{1}{2}kx^2 \qquad [7.4]$$

A force is **conservative** if the work it does on a particle is independent of the path the particle takes between two given points. Alternatively, a force is conservative if the work it does is zero when the particle moves through an arbitrary closed path and returns to its initial position. A force that does not meet these criteria is said to be **nonconservative.**

A **potential energy** function U can be associated only with a conservative force. If a conservative force **F** acts on a particle that moves along the x axis from x_i to x_f, *the change in the potential energy equals the negative of the work done by that force:*

$$U_f - U_i = -\int_{x_i}^{x_f} F_x\,dx \qquad [7.6]$$

The **total mechanical energy of a system** is defined as the sum of the kinetic energy and potential energy:

$$E \equiv K + U \qquad [7.8]$$

The principle of **conservation of mechanical energy** states that if no external forces do work on the system, and there are no nonconservative forces, the total mechanical energy is constant:

$$K_i + U_i = K_f + U_f \qquad [7.9]$$

If some of the forces acting on a system are not conservative, the mechanical energy of the system does not remain constant. If an object is part of a system, the work done by an applied force is equal to the change in mechanical energy of the system:

$$W_{app} = \Delta K + \Delta U \qquad [7.16]$$

An applied force transfers energy to or from the system in the form of changes in kinetic energy and gravitational potential energy.

The gravitational force is conservative, and therefore a potential energy function can be

[3] You can test mathematically whether an extreme of U is stable or unstable by examining the sign of d^2U/dx^2. A positive sign gives stable equilibrium and a negative sign gives unstable equilibrium.

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defined. The **gravitational potential energy** associated with two particles separated by a distance r is

$$U_g = -\frac{Gm_1m_2}{r} \qquad [7.22]$$

where U_g is taken to be zero at $r = \infty$. The total gravitational potential energy for a system of particles is the sum of energies for all pairs of particles, with each pair represented by a term of the form given by Equation 7.22.

CONCEPTUAL QUESTIONS

1. One person drops a ball from the top of a building, while another person at the bottom observes its motion. Will these two people agree on the value of the gravitational potential energy? On the change in potential energy? On its kinetic energy?
2. Discuss the production and dissipation of mechanical energy in (a) lifting a weight, (b) holding the weight up, and (c) lowering the weight slowly. Include the muscles in your discussion.
3. A skier prepares to take off down a snow-covered hill. Is it correct to say that gravity provides the energy for the trip down the hill?
4. You walk leisurely up a flight of stairs. You then return to the bottom and run up the same set of stairs as fast as you can. Have you performed a different amount of work in these two cases?
5. Many mountain roads are built so that they spiral around the mountain rather than go straight up the slope. Discuss this design from the viewpoint of energy and power.
6. In an earthquake, a large amount of energy is "released," and spreads outward, potentially causing severe damage. In what form does this energy exist before the earthquake, and by what energy transfer mechanism does it travel from the focus?
7. You ride a bicycle. In what sense is your bicycle solar-powered?
8. Can the gravitational potential energy of a system ever have a negative value? Explain.
9. A bowling ball is suspended from the ceiling of a lecture hall by a strong cord. The bowling ball is drawn away from its equilibrium position and released from rest at the tip of the demonstrator's nose. If the demonstrator remains stationary, explain why she will not be struck by the ball on its return swing. Would the demonstrator be safe if she pushed the ball as she released it?
10. A pile driver is a device used to drive objects into the Earth by repeatedly dropping a heavy weight on them. By how much does the energy of a pile driver increase when the weight it drops is doubled? (Assume the weight is dropped from the same height each time.)
11. Our body muscles exert forces when we lift, push, run, jump, and so forth. Are these forces conservative?
12. When nonconservative forces act on a system, does the total mechanical energy remain constant?
13. A block is connected to a spring that is suspended from the ceiling. If the block is set in motion and air resistance is neglected, describe the energy transformations that occur within the system consisting of the block and spring.
14. What would the curve of U versus x look like if a particle were in a region of neutral equilibrium?
15. A ball rolls on a horizontal surface. Is the ball in stable, unstable, or neutral equilibrium?
16. Discuss the energy transformations that occur during the operation of an automobile.
17. A ball is thrown straight up into the air. At what position is its kinetic energy a maximum? At what position is the gravitational potential energy a maximum?

PROBLEMS

Section 7.1 Potential Energy

Section 7.2 Conservative and Nonconservative Forces

1. A 1000-kg roller coaster is initially at the top of a rise, at point A. It then moves 135 ft, at an angle of 40.0° below the horizontal, to a lower point, B. (a) Choose point B to be the zero level for gravitational potential energy, and find the potential energy at points A and B and the difference in potential energy, $U_A - U_B$, between these points. (b) Repeat part (a), setting the zero reference level at point A.
2. A 2.00-kg ball is attached to the bottom end of a 1.00-m–long string hanging from the ceiling of a room. The height of the room is 3.00 m. What is the gravitational potential energy relative to (a) the ceiling, (b) the floor, and (c) a point at the same elevation as the ball?
3. A 40.0-N child is in a swing that is attached to ropes 2.00 m long. Find the gravitational potential energy of the system relative to the child's lowest position when (a) the ropes are horizontal, (b) the ropes make a 30.0° angle with the vertical, and (c) the child is at the bottom of the circular arc.
4. A 4.00-kg particle moves from the origin to the position having coordinates $x = 5.00$ m and $y = 5.00$ m under the influence of gravity acting in the negative y direction (Fig. P7.4). Using Equation 6.3, calculate the work done by gravity in going from O to C along (a) OAC, (b) OBC, (c) OC. Your results should all be identical. Why?

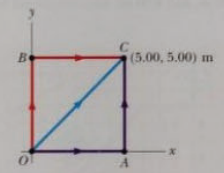


Figure P7.4

Section 7.3 Conservative Forces and Potential Energy

Section 7.4 Conservation of Mechanical Energy

5. A single conservative force $F_x = (2.0x + 4.0)$ N acts on a 5.00-kg particle, where x is in meters. As the particle moves along the x axis from $x = 1.00$ m to $x = 5.00$ m, calculate (a) the work done by this force, (b) the change in the potential energy of the system, and (c) the kinetic energy of the particle at $x = 5.00$ m if its speed at $x = 1.00$ m is 3.00 m/s.
6. A single constant force $\mathbf{F} = (3.0\mathbf{i} + 5.0\mathbf{j})$ N acts on a 4.00-kg particle. (a) Calculate the work done by this force if the particle moves from the origin to the point having the vector position $\mathbf{r} = (2.0\mathbf{i} - 3.0\mathbf{j})$ m. Does this result depend on the path? Explain. (b) What is the speed of the particle at $\mathbf{r}$ if its speed at the origin is 4.00 m/s? (c) What is the change in potential energy?
7. At time t_i, the kinetic energy of a particle is 30.0 J and the potential energy is 10.0 J. At some later time t_f, its kinetic energy is 18.0 J. (a) If only conservative forces act on the particle, what are the potential energy and the total energy at time t_f? (b) If the potential energy at time t_f is 5.00 J, are there any nonconservative forces acting on the particle? Explain.
8. A particle of mass 0.500 kg is shot from P as shown in Figure P7.8 with an initial velocity $\mathbf{v}_0$ having a horizontal component of 30.0 m/s. The particle rises to a maximum height of 20.0 m above P. Using conservation of energy, determine (a) the vertical component of $\mathbf{v}_0$, (b) the work done by the gravitational force on the particle during its motion from P to B, and (c) the horizontal and the vertical components of the velocity vector when the particle reaches B.

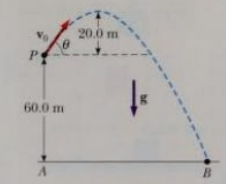


Figure P7.8

9. A bead slides without friction around a loop-the-loop (Fig. P7.9). If the bead is released from a height $h = 3.50R$, what is its speed at point A? How large is the normal force on it if its mass is 5.00 g?

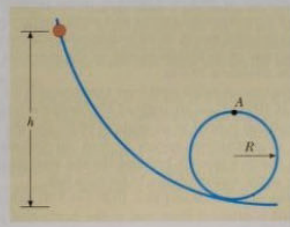


Figure P7.9

10. A simple 2.00-m–long pendulum is released from rest when the support string is at an angle of 25.0° from the vertical. What is the speed of the suspended mass at the bottom of the swing?
11. Two masses are connected by a light string passing over a light frictionless pulley as shown in Figure P7.11. The 5.00-kg mass is released from rest at a height of 4.00 m above the ground. Using the law of conservation of energy, (a) determine the speed of the 3.00-kg mass just as the 5.00-kg mass hits the ground. (b) Find the maximum height to which the 3.00-kg mass rises above the ground.
12. Two masses are connected by a light string passing over a light frictionless pulley as in Figure P7.11. The mass m_1 is released from rest at height h. Using the law of conservation of energy, (a) determine the speed of m_2 just as m_1 hits the ground. (b) Find the maximum height to which m_2 rises.

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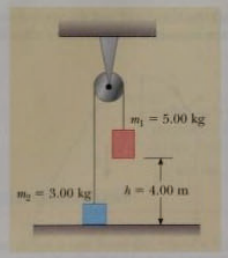


Figure P7.11

13. Dave Johnson, the bronze medalist at the 1992 Olympic decathlon in Barcelona, leaves the ground at the high jump with a vertical velocity of 6.00 m/s. How far does his center of gravity move up as he makes the jump?
14. A 0.400-kg ball is thrown into the air and reaches a maximum altitude of 20.0 m. Taking its initial position as the point of zero potential energy and using energy methods, find (a) its initial speed, (b) its total mechanical energy, and (c) the ratio of its kinetic energy to the potential energy when its altitude is 10.0 m.
15. A 20.0-kg cannon ball is fired from a cannon with the muzzle speed of 1000 m/s at an angle of 37.0° with the horizontal. A second ball is fired at an angle of 90.0°. Use the conservation of mechanical energy to find (a) the maximum height reached by each ball and (b) the total mechanical energy at the maximum height for each ball.
16. A 2.00-kg ball is attached to a 10-lb (44.5 N) fishing line. The ball is released from rest at the horizontal position ($\theta = 90.0°$). At what angle, θ, (measured from the vertical) will the fishing line break?
17. A child starts from rest and slides down the frictionless slide shown in Figure P7.17. In terms of R and H, at what height h will he lose contact with the section of radius R?



Figure P7.17

Section 7.5 Work Done by Nonconservative Forces

18. A 70.0-kg diver steps off a 10.0-m tower and drops straight down into the water. If he comes to rest 5.00 m beneath the surface of the water, determine the average resistance force exerted on the diver by the water.
19. A force F_x, shown as a function of distance in Figure P7.19, acts on a 5.00-kg mass. If the particle starts from rest at $x = 0$ m, determine the speed of the particle at $x =$ 2.00 m, 4.00 m, and 6.00 m.

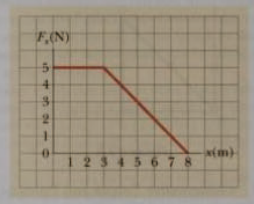


Figure P7.19

20. A softball pitcher swings in her hand a ball of mass 0.250 kg around a vertical circular path of radius 60.0 cm before releasing it. The pitcher maintains a component of force on the ball of constant magnitude 30.0 N in the direction of motion around the complete path. The speed of the ball at the top of the circle is 15.0 m/s. If the ball is released at the bottom of the circle, what is its speed on release?
21. The coefficient of friction between the 3.00-kg mass and surface in Figure P7.21 is 0.400. The system starts from rest. What is the speed of the 5.00-kg mass when it has fallen 1.50 m?

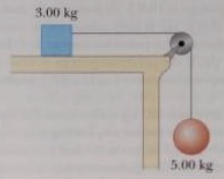


Figure P7.21

22. A 2000-kg car starts from rest at the top of a 5.00–m-long driveway that is sloped at an angle of 20.0° with the horizontal. If an average friction force of 4000 N impedes the motion of the car, find the speed of the car at the bottom of the driveway.
23. A 5.00-kg block is set into motion up an inclined plane with an initial speed of 8.00 m/s (Fig. P7.23). The block comes to rest after traveling 3.00 m along the plane, which is inclined at an angle of 30.0° to the horizontal. Determine (a) the change in the block's kinetic energy, (b) the change in potential energy, and (c) the frictional force exerted on it (assumed to be constant). (d) What is the coefficient of kinetic friction?

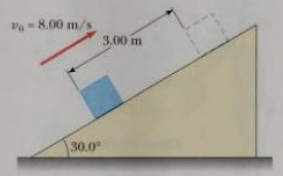


Figure P7.23

24. A parachutist of mass 50.0 kg jumps out of an airplane at a height of 1000 m and lands on the ground with a speed of 5.00 m/s. How much mechanical energy was lost to air friction during this jump?
25. A 80.0-kg skydiver jumps out of an airplane at an altitude of 1000 m and opens the parachute at an altitude of 200.0 m. (a) Assuming that the total retarding force on the diver is constant at 50.0 N with the parachute closed and constant at 3600 N with the parachute open, what is the speed of the diver when he lands on the ground? (b) Do you think the skydiver will get hurt? Explain. (c) At what height should the parachute be opened so that the final speed of the skydiver when he hits the ground is 5.00 m/s? (d) How realistic is the assumption that the total retarding force is constant? Explain.
26. A toy gun uses a spring to project a 5.30-g soft rubber ball. The spring is originally compressed by 5.00 cm and has stiffness constant 8.00 N/m. When it is fired, the ball moves 15.0 cm through the barrel of the gun, and there is a constant frictional force of 0.0320 N between barrel and ball. (a) With what speed does the projectile leave the barrel of the gun? (b) Just where does the ball have maximum speed? (c) What is this maximum speed?
27. A block slides down a curved frictionless track and then up an inclined plane, as in Figure P7.27. The coefficient of kinetic friction between block and incline is μ_k. Use energy methods to show that the maximum height reached by the block is

$$y_{max} = \frac{h}{1 + \mu_k \cot \theta}$$

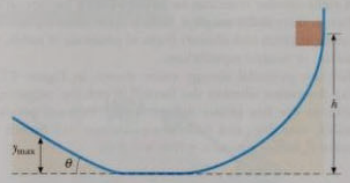


Figure P7.27

28. A 1.50-kg mass is held 1.20 m above a relaxed massless spring with a spring constant of 320 N/m. The mass is dropped onto the spring. (a) How far does it compress the spring? (b) How far does it compress the spring if the same experiment is performed on the moon where $g =$ 1.63 m/s²? (c) Repeat part (a), but this time assume a constant air-resistance force of 0.700 N acts on the mass during its motion.
29. A skier starts from rest at the top of a hill that is inclined at an angle of 10.5° to the horizontal. The hill is 200 m long, and the coefficient of friction between the snow and the skis is 0.0750. At the bottom of the hill the snow is level and the coefficient of friction is unchanged. How far does the skier move along the horizontal portion of the snow before coming to rest?

Section 7.7 Gravitational Potential Energy Revisited

(Assume $U = 0$ at $r = \infty$.)

30. How much energy is required to move a 1000-kg mass from the Earth's surface to an altitude twice the Earth's radius?
31. After our Sun exhausts its nuclear fuel, its ultimate fate is possibly to collapse to a *white-dwarf* state, in which it has approximately the mass of the Sun but the radius of the Earth. Calculate (a) the average density of the white dwarf, (b) the free-fall acceleration at its surface, and (c) the gravitational potential energy of a 1.00-kg object at its surface. (Take $U_g = 0$ at infinity.)
32. At the Earth's surface a projectile is launched straight up at a speed of 10.0 km/s. To what height will it rise? Ignore air resistance.
33. A system consists of three particles, each of mass 5.00 g, located at the corners of an equilateral triangle with sides of 30.0 cm. (a) Calculate the potential energy of the system.

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(b) If the particles are released simultaneously, where will they collide?

Section 7.8 Energy Diagrams and Stability of Equilibrium (Optional)

34. A right circular cone can be balanced on a horizontal surface in three different ways. Sketch these three equilibrium configurations and identify them as positions of stable, unstable, or neutral equilibrium.
35. For the potential energy curve shown in Figure P7.35, (a) determine whether the force F_x is positive, negative, or zero at the five points indicated. (b) Indicate points of stable, unstable, and neutral equilibrium. (c) Sketch the curve F_x versus x from $x = 0$ to $x = 8$ m.

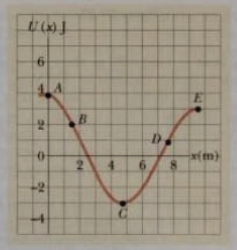


Figure P7.35

36. A hollow pipe has one or two weights attached to its inner surface as shown in Figure P7.36. Characterize each configuration as being stable, unstable, or neutral equilibrium and explain each of your choices.

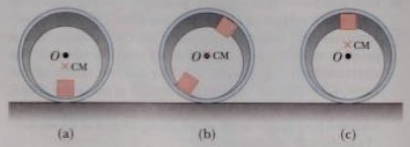


Figure P7.36

37. The potential energy of a two-particle system separated by a distance r is given by $U(r) = A/r$, where A is a constant. Find the radial force $\mathbf{F}_r$.

Additional Problems

38. A 200-g particle is released from rest at point A along the horizontal diameter on the inside of a frictionless, hemispherical bowl of radius $R = 30.0$ cm (Fig. P7.38). Calculate (a) the gravitational potential energy at point A relative to point B, (b) the kinetic energy of the particle at point B, (c) its speed at point B, and (d) its kinetic energy and the potential energy at point C.

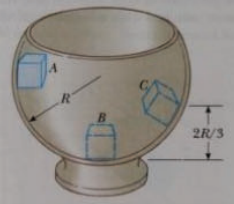


Figure P7.38

39. The particle described in Problem 38 (Fig. P7.38) is released from rest at A, and the surface of the bowl is rough. The speed of the particle at B is 1.50 m/s. (a) What is its kinetic energy at B? (b) How much mechanical energy is lost due to friction as the particle moves from A to B? (c) Is it possible to determine μ from these results in any simple manner? Explain.
40. Make an order-of-magnitude estimate of your power output. Let the work you do be climbing stairs. In your solution state the physical quantities you take as data and the values you measure or estimate for them. Do you consider your peak power or your sustainable power?
41. A 10.0-kg block is released from point A in Figure P7.41. The track is frictionless except for the portion BC, of length 6.00 m. The block travels down the track, hits a spring of force constant $k = 2250$ N/m, and compresses it 0.300 m from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between surface BC and block.
42. A child's pogo stick (Fig. P7.42) stores energy in a spring ($k = 2.50 \times 10^4$ N/m). At position A ($x_1 = -0.100$ m), the spring compression is a maximum and the child is momentarily at rest. At position B ($x = 0$), the spring is relaxed and the child is moving upward. At position C, the child is again momentarily at rest at the top of the jump. Assuming that the combined mass of child and pogo stick is 25.0 kg, (a) calculate the total energy of the system if both potential energies are zero at $x = 0$, (b) determine x_2, (c) calculate the speed of the child at $x = 0$, (d) determine the value of x for which the kinetic energy of the system is a maximum, and (e) obtain the child's maximum upward speed.

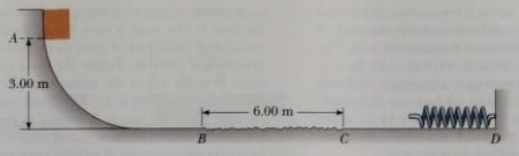


Figure P7.41

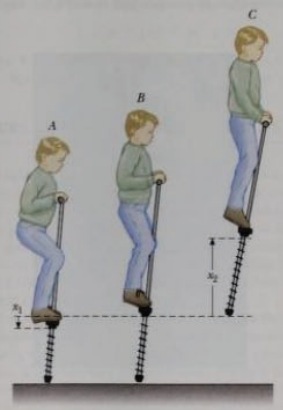


Figure P7.42

43. A 2.00-kg block situated on a rough incline is connected to a spring of negligible mass having a spring constant of 100 N/m (Fig. P7.43). The block is released from rest when the spring is unstretched, and the pulley is frictionless. The block moves 20.0 cm down the incline before coming to rest. Find the coefficient of kinetic friction between block and incline.
44. Suppose the incline is frictionless for the system described in Problem 43 (Fig. P7.43). The block is released from rest with the spring initially unstretched. (a) How far does it move down the incline before coming to rest? (b) What is its acceleration at its lowest point? Is the acceleration constant? (c) Describe the energy transformations that occur during the descent.

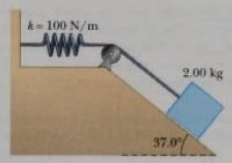


Figure P7.43

45. A 20.0-kg block is connected to a 30.0-kg block by a string that passes over a frictionless pulley. The 30.0-kg block is connected to a spring that has negligible mass and a force constant of 250 N/m, as in Figure P7.45. The spring is unstretched when the system is as shown in the figure, and the incline is frictionless. The 20.0-kg block is pulled 20.0 cm down the incline (so that the 30.0-kg block is 40.0 cm above the floor) and released from rest. Find the speed of each block when the 30.0-kg block is again 20.0 cm above the floor (that is, when the spring is unstretched).

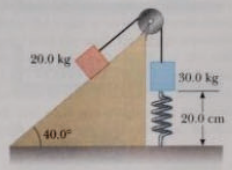


Figure P7.45

46. A potential energy function for a system is given by $U(x) = -x^3 + 2x^2 + 3x$. (a) Determine the force F_x as a function of x. (b) For what values of x is the force equal to zero? (c) Plot $U(x)$ versus x and F_x versus x, and indicate points of stable and unstable equilibrium.
47. A block of mass 0.500 kg is pushed against a horizontal spring of negligible mass, compressing the spring a distance of Δx (Fig. P7.47). The spring constant is 450 N/m. When

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released, the block travels along a frictionless, horizontal surface to point *B*, the bottom of a vertical circular track of radius $R = 1.00$ m, and continues to move up the track. The speed of the block at the bottom of the track is $v_B = 12.0$ m/s, and the block experiences an average frictional force of magnitude 7.00 N while sliding up the track. (a) What is Δx? (b) What is the speed of the block at the top of the track? (c) Does the block reach the top of the track, or does it fall off before reaching the top?

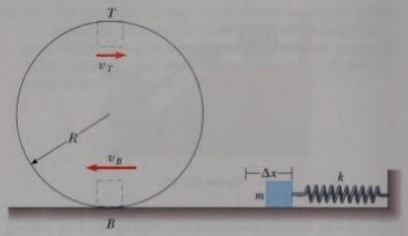


Figure P7.47

48. A 1.00-kg mass slides to the right on a surface having a coefficient of kinetic friction $\mu_k = 0.250$ (Fig. P7.48). The mass has a speed of $v_i = 3.00$ m/s when contact is made with a spring that has a spring constant $k = 50.0$ N/m. The mass comes to rest after the spring has been compressed a distance *d*. The mass is then forced toward the left by the spring and continues to move in that direction beyond the unstretched position. Finally the mass comes to rest a distance *D* to the left of the unstretched spring. Find (a) the compressed distance *d*, (b) the speed *v* at the unstretched position when the system is moving to the left, and (c) the distance *D* where the mass comes to rest.

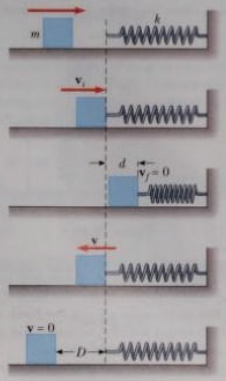


Figure P7.48

49. In the dangerous sport of bungee jumping, a student jumps from a balloon with a specially designed elastic cord attached to his ankles, as shown in the photograph. The unstretched length of the cord is 25.0 m, the student weighs 700 N, and the balloon is 36.0 m above the surface of a river below. Calculate the required force constant of the cord if the student is to stop safely 4.00 m above the river.



Bungee jumping (Problem 49). *(Gamma)*

50. A uniform chain of length 8.00 m initially lies stretched out on a horizontal table. (a) If the coefficient of static friction between chain and table is 0.600, show that the chain will begin to slide off the table if at least 3.00 m of it hangs over the edge of the table. (b) Determine the speed of the chain as all of it leaves the table, given that the coefficient of kinetic friction between chain and table is 0.400.

51. An object of mass *m* is suspended from the top of a cart by a string of length *L* as in Figure P7.51a. The cart and object are initially moving to the right at constant speed v_0. The cart comes to rest after colliding and sticking to a bumper as in Figure P7.51b, and the suspended object swings through an angle θ. (a) Show that $v_0 = \sqrt{2gL(1 - \cos\theta)}$. (b) If $L = 1.20$ m and $\theta = 35.0°$, find the initial speed of the cart. (*Hint:* The force exerted by the string on the object does no work on the object.)

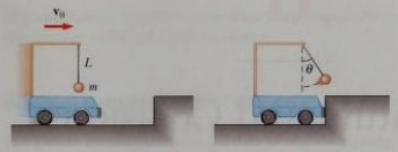


Figure P7.51

Spreadsheet Problems

S1. The potential energy function of a particle is

$$U(x) = \tfrac{1}{3}kx^2 + bx^3 + c$$

where $k = 300$ N/m, $b = -12.0$ N/m^2, and $c = -1000$ J. Use Spreadsheet 7.1 to plot the function from $x = -5.00$ m to $x = +15.0$ m. Describe the general motion of a particle under the influence of this potential energy function. How does the motion change when the initial energy of the object is increased?

S2. The potential energy function associated with the force between two atoms is modeled by the Lennard-Jones potential

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$

In this model, there are two adjustable parameters, σ and ϵ, that are determined from experiments (σ is a range parameter and ϵ is the well depth). Modify Spreadsheet 7.1 to plot $U(x)$ versus x for $\sigma = 0.263$ nm and $\epsilon = 1.51 \times 10^{-22}$ J. Numerically integrate $U(x)$ to find the force F_x.

S3. Using Spreadsheet 7.1, numerically differentiate the potential energy function given in Problem S1. Find the equilibrium points. Are they stable or unstable?

ANSWERS TO CONCEPTUAL PROBLEMS

1. As an athlete runs, chemical energy in the body is converted mostly into thermal energy but also into kinetic energy. The kinetic energy of the athlete just before ascending becomes elastic potential energy stored in the bent pole at the bottom, then gravitational potential energy in the elevated body of the vaulter, and then kinetic energy as he falls. When the vaulter lands on the pad, this energy turns into additional thermal energy, mostly in the pad.
2. The first and third balls speed up after they are thrown, while the second ball first slows down and then speeds up after reaching its peak. The paths of all three are portions of parabolas. The three take different times to reach the ground. However, all have the same impact speed because all start with the same kinetic energy and undergo the same change in gravitational potential energy. In other words, $E_{\text{total}} = \frac{1}{2}mv^2 + mgy$ is the same for all three balls.
3. Air friction is a nonconservative force, so it will result in a transformation of mechanical energy to internal energy. On the way up, the rock slows down from its projection speed of v to zero. On the way down, it speeds up to v_f. Because of the air friction, the mechanical energy of the system as the rock lands must be less than that when it was projected. The potential energy of the rock-Earth system is the same at the initial and final points, since the two points are at the same height. Thus, the final kinetic energy must be less than the initial kinetic energy. In turn, the final speed, v_f, must be less than the initial speed, v. Finally, since the trip on the way down went from zero velocity to a speed lower than v, the average velocity on the way down must be less than that on the way up, and it must have taken longer for the rock to come down than to go up.
4. If the brakes lock, then the car skids over the roadway. The friction force between the roadway and the tires brings the car to a stop. In this case, the energy is transformed to internal energy in the tires and is also laid out as internal energy along the skid path of the tires. If the brakes do not lock, then the major friction force is between the brake pads and the discs (or the brake shoes and the drums), so that the energy appears as internal energy within the brake structure of the car.
5. The coils change their *internal energy*, since the temperature of the coils rises. The energy transfer mechanism for energy coming into the coils is *electrical transmission*, through the wire plugged into the wall. Energy is transferring out of the coils by *electromagnetic radiation*, since the coils are hot and glowing. There is also some transfer of thermal energy from the hot surfaces of the coils into the air. After a short warm-up period, the coils will stabilize in temperature, and the internal energy will no longer change. In this situation, the energy input and output will be balanced.
6. The rise in temperature of the soft steel is an example of transferring energy into a system by work and having it appear as an increase in the internal energy in the system. This works well for the soft steel because it is *soft*. This softness results in a deformation of the steel under the blow of the hammer. Thus, there is a displacement of the point of application of the force from the hammer, and work is done. With the hard steel, there is much less deformation. Thus, there is less displacement of the point of application of the force, and less work done. The soft steel is thus more efficient at absorbing energy from the hammer by means of work, and its temperature rises more rapidly.

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8

Momentum and Collisions

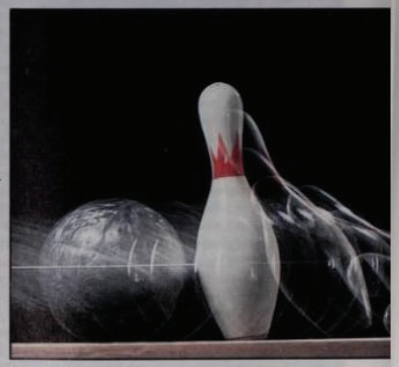
CHAPTER OUTLINE

8.1 Linear Momentum and Its Conservation
8.2 Impulse and Momentum
8.3 Collisions
8.4 Elastic and Inelastic Collisions in One Dimension
8.5 Two-Dimensional Collisions
8.6 The Center of Mass
8.7 Motion of a System of Particles
8.8 Rocket Propulsion (Optional)

Consider what happens when a golf ball is struck by a club. The ball is given a very large initial velocity as a result of the collision; as a consequence, it is able to travel more than 100 m through the air. The ball experiences a large change in velocity and a correspondingly large acceleration. Furthermore, because the ball experiences this acceleration over a short time interval, the average force on it during the collision is very large. As Newton's third law predicts, the club experiences a reaction force that is equal to and opposite the force on the ball. This reaction force produces a change in the velocity of the club. Because the club is much more massive than the ball, however, the change in velocity of the club is much less than the change in velocity of the ball.

One of the main objectives of this chapter is to enable you to understand and analyze such events. As a first step, we shall introduce the concept of *momentum*, a term that is used in describing objects in motion. For example, a massive football player is often said to have a great deal of momentum as he runs down the field. A much less massive player, such as a halfback, can have equal or greater momentum if his speed is greater than that of the more massive player. This follows from the fact that momentum is defined as the product of mass and velocity.

The concept of momentum leads us to a second conservation law, that



As a result of the collision between the bowling ball and pin, part of the ball's momentum is transferred to the pin. As a result, the pin acquires momentum and kinetic energy, and the ball loses momentum and kinetic energy. However, the total momentum of the system (ball and pin) remains constant. *(Ben Rose/The Image Bank)*

of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects.

8.1 · LINEAR MOMENTUM AND ITS CONSERVATION

The **linear momentum** of a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:[1]

$$\mathbf{p} = m\mathbf{v} \qquad [8.1]$$

• *Definition of linear momentum of a particle*

Because momentum equals the product of a scalar, m, and a vector, $\mathbf{v}$, it is a vector quantity. Its direction is along $\mathbf{v}$, and it has dimensions of ML/T. In the SI system, momentum has the units kg · m/s.

If a particle is moving in an arbitrary direction in three-dimensional space, $\mathbf{p}$ will have three components, and Equation 8.1 is equivalent to the component equations

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \qquad [8.2]$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$ *quantity of motion*, perhaps a more graphic description than *momentum*, which comes from the Latin word for movement.

By using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle. In Chapter 4 we learned that Newton's second law can be written as $\Sigma\mathbf{F} = m\mathbf{a}$. However, this form applies only when the mass of the system remains constant. In situations in which the mass is changing with time, one must use an alternative statement of Newton's second law: **The time rate of change of momentum of a particle is equal to the resultant force acting on the particle.**

$$\Sigma\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad [8.3]$$

• *Newton's second law for a particle*

From Equation 8.3 we see that if the resultant force is zero, the time derivative of the momentum is zero, and therefore the momentum of any object must be constant. In other words, the linear momentum of an object is *constant* when $\Sigma\mathbf{F} = 0$. Of course, if the particle is *isolated* (that is, if it does not interact with its environment), then by necessity $\Sigma\mathbf{F} = 0$ and $\mathbf{p}$ remains unchanged.

In a certain sense, conservation of linear momentum is just another way of stating Newton's first law. If an object is in motion, its linear momentum and, as a consequence, its velocity do not change unless an external force acts on the system. A good example is a rocket with its engines disengaged, which coasts through space far from any gravitational sources.

[1] This expression is nonrelativistic and is valid only when $v \ll c$, where c is the speed of light.

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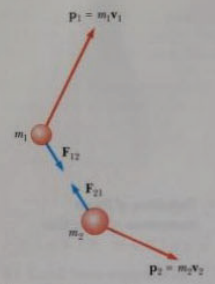


Figure 8.1 At some instant, the momentum of m_1 is $\mathbf{p}_1 = m_1\mathbf{v}_1$ and the momentum of m_2 is $\mathbf{p}_2 = m_2\mathbf{v}_2$. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Conservation of Linear Momentum for a Two-Particle System

To see how to apply the principle of *conservation of linear momentum*, consider a system of two particles that can interact with each other but are isolated from their surroundings (Fig. 8.1). That is, the particles may exert a force on each other, but no *external* forces are present. It is important to note the impact of Newton's third law on this analysis. Recall from Chapter 4 that Newton's third law states that the forces on these two particles are equal in magnitude and opposite in direction; that is, *forces always occur in pairs*. Thus, if an *internal* force (say a gravitational force) acts on particle 1, then there must be a second *internal* force, equal in magnitude but opposite in direction, that acts on particle 2.

Suppose that at some instant, the momentum of particle 1 is $\mathbf{p}_1$ and the momentum of particle 2 is $\mathbf{p}_2$. Applying Newton's second law to each particle, we can write

$$\mathbf{F}_{12} = \frac{d\mathbf{p}_1}{dt} \quad \text{and} \quad \mathbf{F}_{21} = \frac{d\mathbf{p}_2}{dt}$$

where $\mathbf{F}_{12}$ is the force exerted on particle 1 by particle 2 and $\mathbf{F}_{21}$ is the force exerted on particle 2 by particle 1. (These forces could be gravitational forces, or they could have some other origin. The source of the forces isn't important for the present discussion.) Newton's third law tells us that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal in magnitude and opposite in direction. That is, they form an action–reaction pair, and $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can also express this condition as

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0 \quad \text{or as} \quad \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

Because the time derivative of the total momentum, $\mathbf{p}_{tot} = \mathbf{p}_1 + \mathbf{p}_2$, is *zero*, we conclude that the *total* momentum, $\mathbf{p}_{tot}$, must remain constant:

$$\mathbf{p}_{tot} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \qquad [8.4]$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \qquad [8.5]$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ are final values of the momentum during a time period, dt, over which the reaction pair interacts. Equation 8.5 in component form says that the total momenta in the x, y, and z directions are all *independently conserved;* that is,

$$p_{ix} = p_{fx} \qquad p_{iy} = p_{fy} \qquad p_{iz} = p_{fz} \qquad [8.6]$$

This result is known as the law of **conservation of linear momentum.** It is considered to be one of the most important laws of mechanics. We can state it as follows:

Conservation of momentum

> Whenever two isolated particles interact with each other, their total momentum remains constant.

That is, **the total momentum of an isolated system at all times equals its initial total momentum.**

We can also describe the law of conservation of momentum in another way. Because we require that the system be isolated, no external forces are present, and the total momentum of the system remains constant. Therefore, momentum conservation is an alternative and more general statement of Newton's third law.

Notice that we have made no statement concerning the nature of the forces acting on the system. The only requirement was that the forces must be *internal* to the system. Thus, momentum is constant for a two-particle system *regardless* of the nature of the internal forces. One can use a similar and equivalent argument to show that the law of conservation of momentum also applies to a system of many particles.

Thinking Physics 1

A baseball is projected into the air at an upward angle to the ground. As it moves through its trajectory, its velocity and, therefore, its momentum constantly changes. Is this a violation of conservation of momentum?

Reasoning The principle of conservation of momentum states that the momentum of a particle or a system of particles is conserved *in the absence of external forces.* A ball projected through the air is subject to the external force of gravity, so we would not expect its momentum to be conserved. If we consider this in a little more detail, the gravitational force is in the vertical direction, so it is actually only the vertical component of the momentum that changes due to this force. In the horizontal direction, there is no force (ignoring air friction), so the horizontal component of the momentum is conserved. In Chapter 3, we used the same idea in stating that the horizontal component of the *velocity* of a projectile remains constant.

If we consider the baseball and the Earth as a system of particles, then the gravitational force is an internal force to this system. The momentum of the ball–Earth system remains unchanged. The downward impulse due to the thrower's feet gives the Earth an initial motion. As the ball rises and falls, the Earth initially sinks but then rises (although imperceptibly) due to the upward gravitational force from the ball, so that the total momentum of the system remains unchanged.

Example 8.1 The Recoiling Pitching Machine

A baseball player uses a pitching machine to help him improve his batting average. He places the 50-kg machine on a frozen pond as in Figure 8.2. The machine fires a 0.15-kg baseball horizontally with a velocity of $36\hat{\mathbf{i}}$ m/s. What is the recoil velocity of the machine?

Reasoning We take the system to consist of the baseball and the pitching machine. Because of the force of gravity and the normal force, the system is not really isolated. However, both of these forces are directed perpendicularly to the motion of the system. Therefore, momentum is constant in the x direction because there are no external forces in this direction (assuming the surface is frictionless).



Figure 8.2 (Example 8.1) When the baseball is fired horizontally to the right, the pitching machine recoils to the left. The total momentum of the system before and after firing is zero.

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Solution The total momentum of the system before firing is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$). Therefore, the total momentum after firing must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

With $m_1 = 0.15$ kg, $\mathbf{v}_{1i} = 36\mathbf{i}$ m/s, and $m_2 = 50$ kg, solving for $\mathbf{v}_{2f}$, we find the recoil velocity of the pitching machine to be

$$\mathbf{v}_{2f} = -\frac{m_1}{m_2}\mathbf{v}_{1f} = -\left(\frac{0.15\text{ kg}}{50\text{ kg}}\right)(36\mathbf{i}\text{ m/s}) = -0.11\mathbf{i}\text{ m/s}$$

The negative sign for $\mathbf{v}_{2f}$ indicates that the pitching machine is moving to the left after firing, in the direction opposite the direction of motion of the baseball. In the words of Newton's third law, for every force (to the left) on the pitching machine, there is an equal but opposite force (to the right) on the ball. Because the pitching machine is much more massive than the ball, the acceleration and consequent speed of the pitching machine are much smaller than the acceleration and speed of the ball.

Example 8.2 Decay of the Kaon at Rest

A meson is a nuclear particle that is more massive than an electron but less massive than a proton or neutron. One type of meson, called the neutral kaon (K^0), decays into a pair of charged pions (π^+ and π^-) that are oppositely charged but equal in mass, as in Figure 8.3. A pion is a particle associated with the strong nuclear force that binds the protons and neutrons together in the nucleus. Assuming the kaon is initially at rest, prove that after the decay, the two pions must have momenta that are equal in magnitude and opposite in direction.

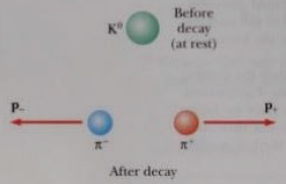


Figure 8.3 (Example 8.2) A kaon at rest decays spontaneously into a pair of oppositely charged pions. The pions move apart with momenta of equal magnitudes but opposite directions.

Solution The decay of the kaon, represented in Figure 8.3, can be written

$$K^0 \longrightarrow \pi^+ + \pi^-$$

If we let $\mathbf{p}_+$ be the momentum of the positive pion and $\mathbf{p}_-$ be the momentum of the negative pion after the decay, then the final momentum of the system can be written

$$\mathbf{p}_f = \mathbf{p}_+ + \mathbf{p}_-$$

Because the kaon is at rest before the decay, we know that $\mathbf{p}_i = 0$. Furthermore, because momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}_+ + \mathbf{p}_- = 0$, or

$$\mathbf{p}_+ = -\mathbf{p}_-$$

Thus, we see that the two momentum vectors of the pions are equal in magnitude and opposite in direction. Furthermore, because the pions have equal mass, they will have equal and opposite velocities.

EXERCISE 1 The momentum of a 1250-kg car is equal to the momentum of a 5000-kg truck traveling at a speed of 10 m/s. What is the speed of the car? Answer 40.0 m/s

EXERCISE 2 A 1500-kg car moving with a speed of 15 m/s collides with a utility pole and is brought to rest in 0.3 s. Find the average force exerted on the car during the collision. Answer 7.5×10^4 N

EXERCISE 3 A 60-kg boy and a 40-kg girl, both wearing skates, face each other at rest. The girl pushes the boy, sending him eastward with a speed of 4 m/s. Describe the subsequent motion of the girl. (Neglect friction.) Answer She moves westward at 6 m/s.

8.2 • IMPULSE AND MOMENTUM

As we have seen, the momentum of a particle changes if a net force acts on the particle. Let us assume that a force $\mathbf{F}$ acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F}\,dt \qquad [8.7]$$

We can integrate this expression to find the change in the momentum of a particle as it changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f. Integrating Equation 8.7 gives

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f}\mathbf{F}\,dt \qquad [8.8]$$

• *Impulse of a force*

The integral of a force over the time for which it acts is called the **impulse** of the force. The impulse of the force $\mathbf{F}$ for the time interval $\Delta t = t_f - t_i$ is a vector defined by

$$\mathbf{I} \equiv \int_{t_i}^{t_f}\mathbf{F}\,dt = \Delta\mathbf{p} \qquad [8.9]$$

• *Impulse-momentum theorem*

That is, the **impulse** of the force $\mathbf{F}$ equals the change in the momentum of any object.[2] This statement, known as the **impulse-momentum theorem**, is equivalent to Newton's second law. From this definition we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 8.4. In this figure it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum, ML_x/T. Note that impulse is *not* a property of the particle itself; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, it is implied that momentum is transferred from an external agent to that particle.

Because the force can generally vary in time as in Figure 8.4a, it is convenient to define a time-averaged force $\bar{\mathbf{F}}$, given by

$$\bar{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f}\mathbf{F}\,dt \qquad [8.10]$$

where $\Delta t = t_f - t_i$. Therefore, we can express Equation 8.9 as

$$\mathbf{I} = \Delta\mathbf{p} = \bar{\mathbf{F}}\,\Delta t \qquad [8.11]$$

This average force, described in Figure 8.4b, can be thought of as the constant force that would give the same impulse to the particle in the time interval Δt as the actual time-varying force gives over this same interval.

[2] In general, if several forces act on a particle, the total impulse injected into the particle by the forces acting on it constitutes the change in momentum of the particle.

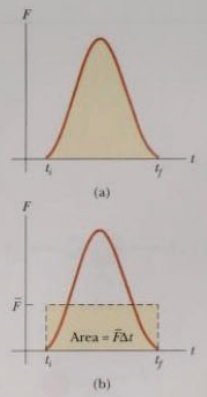


Figure 8.4 (a) A force acting on a particle may vary in time. The impulse is the area under the force-versus-time curve. (b) The average force (horizontal dashed line) gives the same impulse to the particle in the time Δt as the time-varying force described in (a).

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In principle, if $\mathbf{F}$ is known as a function of time, the impulse can be calculated from Equation 8.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 8.11 becomes

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{F}\,\Delta t \qquad [8.12]$$

In many physical situations, we shall use what is called the **impulse approximation: We assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions, where the duration of the collision is very short. When this approximation is made, we refer to the force as an *impulsive force*. For example, when a baseball is struck with a bat, the duration of the collision is about 0.01 s, and the average force the bat exerts on the ball in this time is typically several thousand newtons. This is much greater than the force of gravity, so the impulse approximation is justified. It is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, respectively. Therefore, in the impulse approximation very little motion of the particle takes place during the collision.

Thinking Physics 2

A race car travels rapidly around a circular race track at constant speed. For a given portion of the track, what is the direction of the impulse vector? Once the car returns to the starting point, there is no net change in momentum. Thus, according to the impulse-momentum theorem, there must be no impulse. Yet a nonzero force acted over the time interval for one rotation around the track. How can the impulse be zero?

Reasoning For the movement around a short portion of the track, the direction of the impulse vector is the same as that of the change in momentum. The direction of the vector representing the change in momentum is the same as the direction of the vector representing the change in velocity. From our discussion of circular motion, we know that the direction of the vector representing the change in velocity is that of the acceleration vector, which is toward the center of the circle, because this is uniform circular motion. Thus, the impulse vector is directed toward the center of the circle.

If the car makes one rotation around the track, there is indeed no net change in momentum, so there must be zero net impulse. Keeping in mind that the force on the car causing the circular motion is the friction between the tires and the roadway, and that this force is always directed toward the center of the circle, we can argue the zero value of the total impulse. For every location of the car on the track, there is another point of its motion diametrically opposed across the circle, at which its force vector will be directed in the opposite direction. Thus, as we add up the vector impulses $\mathbf{F}\,dt$ around the circle, they will cancel in pairs for a net impulse of zero.

CONCEPTUAL PROBLEM 1

In boxing matches of the nineteenth century, bare fists were used. In modern boxing, fighters wear padded gloves. How does this better protect the brain of the boxer from injury? Boxers often "roll with the punch." How does this maneuver protect their health?

Example 8.3 How Good Are the Bumpers?

In a particular crash test, an automobile of mass 1500 kg collides with a wall, as in Figure 8.5. The initial and final velocities of the automobile are $\mathbf{v}_i = -15.0\mathbf{i}$ m/s and $\mathbf{v}_f = 2.6\mathbf{i}$ m/s. If the collision lasts for 0.150 s, find the impulse due to the collision and the average force exerted on the automobile.

Solution The initial and final momenta of the automobile are

$$\mathbf{p}_i = m\mathbf{v}_i = (1500\text{ kg})(-15.0\mathbf{i}\text{ m/s}) = -2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

$$\mathbf{p}_f = m\mathbf{v}_f = (1500\text{ kg})(2.6\mathbf{i}\text{ m/s}) = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

Hence, the impulse is

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$$
$$= 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s})$$
$$\mathbf{I} = 2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

The average force exerted on the automobile is

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = 1.76 \times 10^5\mathbf{i}\text{ N}$$

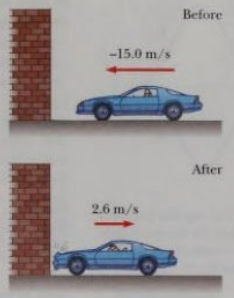


Figure 8.5 (Example 8.3)

EXERCISE 4 A child bounces a superball on the sidewalk. The linear impulse delivered by the sidewalk to the ball is 2.00 N·s during the 1/800 s of contact. What is the magnitude of the average force exerted on the ball by the sidewalk? Answer 1.60 kN upward

EXERCISE 5 A 0.15-kg baseball is thrown with a speed of 40 m/s. It is hit straight back at the pitcher with a speed of 50 m/s. (a) What is the impulse delivered to the baseball? (b) Find the average force exerted by the bat on the ball if the two are in contact for 2.0×10^{-3} s. Answer (a) 13.5 kg·m/s (b) 6.75 kN

8.3 · COLLISIONS

In this section we use the law of conservation of momentum to describe what happens when two objects collide. **The force due to the collision is assumed to be much larger than any external forces present.**

A collision may be the result of physical contact between two objects, as described in Figure 8.6a. This is a common observation when two macroscopic objects, such as two billiard balls or a baseball and a bat, collide. The notion of what we mean by *collision* must be generalized because "contact" on a submicroscopic scale is ill-defined and hence meaningless. More accurately, contact forces between two bodies arise from the electrostatic interaction of the electrons in the surface atoms of the bodies.

To understand this distinction between macroscopic and microscopic collisions, consider the collision of a proton with an alpha particle (the nucleus of the helium atom), such as occurs in Figure 8.6b. Because the two particles are positively charged, they repel each other.

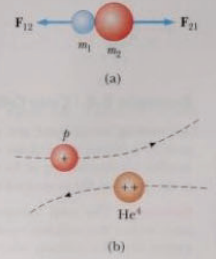


Figure 8.6 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

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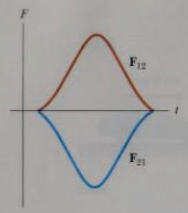


Figure 8.7 The impulse force as a function of time for the two colliding particles described in Figure 8.6a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

When two particles of masses m_1 and m_2 collide, the collision forces may vary in time in a complicated way (Fig. 8.7). If $\mathbf{F}_{12}$ is the force on m_1 due to m_2, then the change in momentum of m_1 due to the collision is given by Equation 8.8:

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

Likewise, if $\mathbf{F}_{21}$ is the force exerted on m_2 by m_1, the change in momentum of m_2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Newton's third law states that the force on m_1 by m_2 is equal to and opposite the force exerted on m_2 by m_1; that is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. (This is described graphically in Fig. 8.7.) Hence, we conclude that

$$\Delta \mathbf{p}_1 = \Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect if no external forces are acting on the system (Section 8.2), or if the external forces are assumed to act for too short a time to make a significant change in momentum. However, the result is also valid if we consider the motion just before and just after the collision. Because the forces due to the collision are internal to the system, they do not change the total momentum of the system. Therefore, for any type of collision, the total momentum of a system just before the collision equals the total momentum of a system just after the collision.

CONCEPTUAL PROBLEM 2

You are watching a movie about a superhero and notice that the superhero hovers in the air and throws a piano at some bad guys while remaining stationary in the air. What's wrong with this scenario?

Example 8.4 Carry Collision Insurance

An 1800-kg car stopped at a traffic light is struck from the rear by a 900-kg car and the two become entangled. If the smaller car was moving at 20 m/s before the collision, what is the speed of the entangled mass after the collision?

Reasoning The total momentum of the system (the two cars) before the collision equals the total momentum of the system after the collision because momentum is conserved.

Solution The magnitude of the total momentum of the system before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_i = (900\ \text{kg})(20\ \text{m/s}) = 1.80 \times 10^4\ \text{kg·m/s}$$

After the collision, the mass that moves is the sum of the masses of the cars. The magnitude of the momentum of the combination is

$$p_f = (m_1 + m_2)v_f = (2700\ \text{kg})v_f$$

Equating the momentum before to the momentum after and solving for v_f, the speed of the combined mass, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4\ \text{kg·m/s}}{2700\ \text{kg}} = 6.67\ \text{m/s}$$

8.4 • ELASTIC AND INELASTIC COLLISIONS IN ONE DIMENSION

As we have seen, momentum is conserved in any type of collision. Kinetic energy, however, is generally *not* constant in a collision because some of it is converted to thermal energy, to internal elastic potential energy when the objects are deformed, and to rotational kinetic energy.

• *Inelastic collision*

We define an inelastic collision as one in which total kinetic energy is not constant (even though momentum is constant). The collision of a rubber ball with a hard surface is inelastic, because some of the kinetic energy of the ball is lost when it is deformed while in contact with the surface. When two objects collide and stick together after a collision, some kinetic energy is lost, and the collision is called **perfectly inelastic.** For example, if two vehicles collide and become entangled, as in Example 8.4, they move with some common velocity after the perfectly inelastic collision. If a meteorite collides with the Earth, it becomes buried and the collision is perfectly inelastic.

• *Elastic collision*

An elastic collision is defined as one in which total kinetic energy is constant (as well as momentum). Real collisions in the macroscopic world, such as those between billiard balls, are only approximately elastic because some deformation and loss of kinetic energy takes place. Billiard-ball collisions and the collisions of air molecules with the walls of a container at ordinary temperatures are highly elastic. Truly elastic collisions do occur, however, between atomic and subatomic particles. Elastic and perfectly inelastic collisions are *limiting* cases; most collisions fall in a category between them.

In the remainder of this section we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum is conserved in all cases, but kinetic energy is constant only in elastic collisions.**

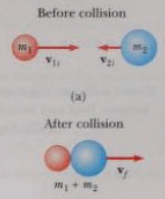


Figure 8.8 A perfectly inelastic head-on collision between two particles.

Perfectly Inelastic Collisions

Consider two objects of masses m_1 and m_2 moving with initial velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ along a straight line, as in Figure 8.8. If the two objects collide head-on, stick together, and move with some common velocity $\mathbf{v}_f$ after the collision, the collision is perfectly inelastic. Because the total momentum before the collision equals the total momentum of the composite system after the collision, we have

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \qquad [8.13]$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \qquad [8.14]$$

Elastic Collisions

Now consider two particles that undergo an elastic head-on collision (Fig. 8.9). In this case, both momentum and kinetic energy are constant; therefore, we can write

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} \qquad [8.15]$$

$$\tfrac{1}{2}m_1v_{1i}^2 + \tfrac{1}{2}m_2v_{2i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \qquad [8.16]$$

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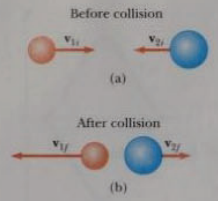


Figure 8.9 An elastic head-on collision between two particles.

where v is positive if a particle moves to the right and negative if it moves to the left.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 8.15 and 8.16 can be solved simultaneously to find them. An alternative approach, employing a little mathematical manipulation of Equation 8.16, often simplifies this process. To see this, let us cancel the factor of $\frac{1}{2}$ in Equation 8.16 and rewrite it as

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

Here we have moved the terms containing m_1 to one side of the equation and those containing m_2 to the other. Next, let us factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \qquad [8.17]$$

We now separate the terms containing m_1 and m_2 in the equation for the conservation of momentum (Eq. 8.15) to get

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \qquad [8.18]$$

To obtain our final result, we divide Equation 8.17 by Equation 8.18 and get

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$
$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \qquad [8.19]$$

This equation, in combination with the condition for conservation of momentum, can be used to solve problems dealing with elastic collisions. According to Equation 8.19, the relative speed of the two objects before the collision, $v_{1i} - v_{2i}$, equals the negative of their relative speed after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and the initial velocities of both particles are known. Equations 8.15 and 8.16 can be solved for the final speeds in terms of the initial speeds, because there are two equations and two unknowns.

Elastic collision: relations between final and initial velocities

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \qquad [8.20]$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \qquad [8.21]$$

It is important to remember that the appropriate signs for v_{1i} and v_{2i} must be included in Equations 8.20 and 8.21. For example, if m_2 is moving to the left initially, as in Figure 8.9, then v_{2i} is negative.

Let us consider some special cases. If $m_1 = m_2$, then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange speeds if they have equal masses. This is what one observes in head-on billiard ball collisions.

If m_2 is initially at rest, $v_{2i} = 0$, and Equations 8.20 and 8.21 become

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \qquad [8.22]$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \qquad [8.23]$$

If m_1 is very large compared with m_2, we see from Equations 8.22 and 8.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one initially at rest, the heavy particle continues its motion unaltered after the collision, and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, with a light atom, such as hydrogen.

If m_2 is much larger than m_1, and m_2 is initially at rest, then we find from Equations 8.22 and 8.23 that $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$. That is, when a very light particle collides head-on with a very heavy particle initially at rest, the velocity of the light particle is reversed, and the heavy particle remains approximately at rest. For example, imagine what happens when a marble hits a stationary bowling ball.



The photograph of a car crash test (an inelastic collision) illustrates that much of the car's initial kinetic energy is transformed into the energy it took to damage the vehicle. Why do safety belts and air bags help prevent serious injury in such collisions? *(Courtesy of General Motors)*

CONCEPTUAL PROBLEM 3

A great deal of expense is involved in repairing automobiles after collisions. It seems that much of this money could be saved if cars were built from hard rubber, so that they made approximately elastic collisions and simply bounced off each other without denting. Why are cars not made of hard rubber?

CONCEPTUAL PROBLEM 4

In perfectly inelastic collisions between two objects, under what conditions is all of the original kinetic energy transformed to forms other than kinetic?

Example 8.5 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when a $^{235}_{92}U$ atom splits in a process called fission. These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by being passed through a solid or liquid material called a *moderator*. The slowing-down process involves elastic collisions. Let us show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in "heavy water," D_2O) or carbon (in graphite).

Reasoning Because momentum and energy are constant, Equations 8.22 and 8.23 can be applied to the head-on collision of a neutron with the moderator nucleus.

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that the neutron of mass m_n and initial speed v_{ni} collides head-on with it. The initial kinetic energy of the neutron is

$$K_{ni} = \tfrac{1}{2}m_n v_{ni}^2$$

After the collision, the neutron has a kinetic energy $\frac{1}{2}m_n v_{nf}^2$, where v_{nf} is given by Equation 8.22.

$$K_{nf} = \tfrac{1}{2}m_n v_{nf}^2 = \frac{m_n}{2}\left(\frac{m_n - m_m}{m_n + m_m}\right)^2 v_{ni}^2$$

Therefore, the fraction of the total kinetic energy possessed by the neutron after the collision is

$$(1) \quad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m}\right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_n is close to m_m and is zero when $m_n = m_m$.

We can calculate the kinetic energy of the moderator nucleus after the collision using Equation 8.23:

$$K_{mf} = \tfrac{1}{2}m_m v_{mf}^2 = \frac{2m_n^2 m_m}{(m_n + m_m)^2}v_{ni}^2$$

Hence, the fraction of the total kinetic energy transferred to the moderator nucleus is

$$(2) \quad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4m_n m_m}{(m_n + m_m)^2}$$

Because the total energy is constant, (2) can also be obtained from (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O ($m_m = 2m_n$), $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely to occur.

How do the results differ with graphite as the moderator?

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Example 8.6 A Two-Body Collision with Spring

A block of mass $m_1 = 1.60$ kg, initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track, collides with a spring attached to a second block of mass $m_2 = 2.10$ kg, moving to the left with a speed of 2.50 m/s, as in Figure 8.10a. The spring has a spring constant of 600 N/m. (a) At the instant when m_1 is moving to the right with a speed of 3.00 m/s, as in Figure 8.10b, determine the speed of m_2.

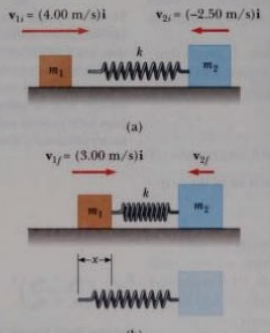


Figure 8.10 (Example 8.6)

Solution First, note that the initial velocity of m_2 is -2.50 m/s because its direction is to the left. Because the total momentum is conserved, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(1.60\ \text{kg})(4.00\ \text{m/s}) + (2.10\ \text{kg})(-2.50\ \text{m/s}) = (1.60\ \text{kg})(3.00\ \text{m/s}) + (2.10\ \text{kg})v_{2f}$$

$$v_{2f} = -1.74\ \text{m/s}$$

The negative value for v_{2f} means that m_2 is still moving to the left at the instant we are considering.

(b) Determine the distance the spring is compressed at that instant.

Solution To determine the compression in the spring, x, shown in Figure 8.10b, we use conservation of energy, because there are no friction or other nonconservative forces acting on the system. Thus, we have

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 + \tfrac{1}{2}kx^2$$

Substituting the given values and the result to part (a) into this expression gives

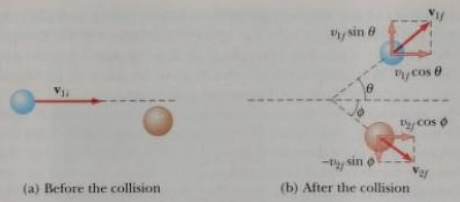
$$x = 0.173\ \text{m}$$

EXERCISE 6 Find the velocity of m_1 and the compression in the spring at the instant that m_2 is at rest.
Answer 0.719 m/s to the right; 0.251 m

EXERCISE 7 A 2.5-kg mass, moving initially with a speed of 10 m/s, makes a perfectly inelastic head-on collision with a 5.0-kg mass that is initially at rest. (a) Find the final velocity of the composite particle. (b) How much kinetic energy is lost in the collision?
Answer (a) 3.33 m/s (b) 83.4 J

8.5 • TWO-DIMENSIONAL COLLISIONS

In Section 8.1 we showed that the total momentum of a system of particles is constant when the system is isolated (that is, when no external forces act on the system). For a general collision of two particles in three-dimensional space, the conservation-of-momentum law implies that the total momentum in each direction is constant. However, an important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for the conservation of momentum:



(a) Before the collision (b) After the collision

Figure 8.11 An elastic glancing collision between two particles.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Consider a two-dimensional problem in which a particle of mass m_1 collides elastically with a particle of mass m_2 that is initially at rest, as in Figure 8.11. After the collision, m_1 moves at an angle θ with respect to the horizontal, and m_2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form, and noting that the total y component of momentum is zero, we get

• *Conservation of momentum*

$$x \text{ component:} \quad m_1 v_{1i} + 0 = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \qquad [8.24]$$

$$y \text{ component:} \quad 0 + 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \qquad [8.25]$$

We now have two independent equations. So long as only two of the preceding quantities are unknown, we can solve the problem completely.

Because the collision is elastic, we can write a third equation for kinetic energy, in the form

• *Conservation of energy*

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 \qquad [8.26]$$

If we know the initial velocity, v_{1i}, and the masses, we are left with four unknowns. Because we have only three equations, one of the four remaining quantities (v_{1f}, v_{2f}, θ, or ϕ) must be given to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* constant, and Equation 8.26 does *not* apply.

PROBLEM-SOLVING STRATEGY • Collisions

The following procedure is recommended when dealing with problems involving collisions between two objects.

1. Set up a coordinate system and define your velocities with respect to that system. It is convenient to have the x axis coincide with one of the initial velocities.
2. In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information. Write expressions for the x and y

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components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors. For example, if an object is moving in the negative x direction, its x component of velocity must be taken to be negative. It is essential that you pay careful attention to signs.

3. Write expressions for the *total* momentum in the x direction *before* and *after* the collision, and equate the two. Repeat this procedure for the total momentum in the y direction. These steps follow from the fact that, because the momentum is conserved in any collision, the total momentum in any direction must be constant. It is important to emphasize that it is the momentum of the *system* (the two colliding objects) that is constant, not the momenta of the individual objects.
4. If the collision is inelastic, kinetic energy is *not* constant, and additional information is probably required. If the collision is perfectly inelastic, the final velocities of the two objects are equal. Proceed to solve the momentum equations for the unknown quantities.
5. If the collision is elastic, kinetic energy is constant, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision. This provides an additional relationship between the velocities.

Example 8.7 Proton–Proton Collision

A proton collides in an elastic fashion with another proton that is initially at rest. The incoming proton has an initial speed of 3.5×10^5 m/s and makes a glancing collision with the second proton, as in Figure 8.11. (At close separations, the protons exert a repulsive electrostatic force on each other.) After the collision, one proton moves off at an angle of 37° to the original direction of motion, and the second deflects at an angle of ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

Solution Both momentum and kinetic energy are constant in this glancing elastic collision. Because $m_1 = m_2$, $\theta = 37°$, and we are given $v_{1i} = 3.5 \times 10^5$ m/s, Equations 8.24, 8.25, and 8.26 become

$$v_{1f} \cos 37° + v_{2f} \cos \phi = 3.5 \times 10^5$$

$$v_{1f} \sin 37° - v_{2f} \sin \phi = 0$$

$$v_{1f}^2 + v_{2f}^2 = (3.5 \times 10^5)^2$$

Simultaneous solution of these three equations with three unknowns gives

$$v_{1f} = 2.80 \times 10^5 \text{ m/s} \qquad v_{2f} = 2.11 \times 10^5 \text{ m/s}$$

$$\phi = 53.0°$$

It is interesting to note that $\theta + \phi = 90°$. This result is *not* accidental. **Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are *always* at right angles to each other.**

Example 8.8 Collision at an Intersection

A 1500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 8.12. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive x direction and north to be along the positive y direction, as in Figure 8.12. Before the collision, the only object having momentum in the x direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the x direction is

$$\sum p_{xi} = (1500 \text{ kg})(25.0 \text{ m/s}) = 3.75 \times 10^4 \text{ kg·m/s}$$

Now let us assume that the wreckage moves at an angle θ and speed v after the collision. The magnitude of the total momentum in the x direction after the collision is

$$\sum p_{xf} = (4000 \text{ kg}) v \cos \theta$$

Because the total momentum in the x direction is constant, we can equate these two equations to get

$$(1) \quad (3.75 \times 10^4) \text{ kg·m/s} = (4000 \text{ kg}) v \cos \theta$$

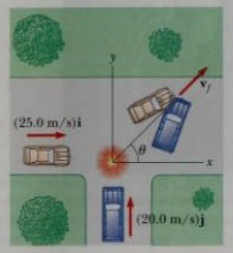


Figure 8.12 (Example 8.8) Top view of an eastbound car colliding with a northbound van.

In a similar way, the total initial momentum of the system in the y direction is that of the van, the magnitude of which is equal to (2500 kg)(20.0 m/s). Applying conservation of momentum to the y direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2500 \text{ kg})(20.0 \text{ m/s}) = (4000 \text{ kg}) v \sin \theta$$

$$(2) \quad 5.00 \times 10^4 \text{ kg·m/s} = (4000 \text{ kg}) v \sin \theta$$

If we divide (2) by (1), we get

$$\tan \theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = 53.1°$$

When this angle is substituted into (2), the value of v is

$$v = \frac{5.00 \times 10^4 \text{ kg·m/s}}{(4000 \text{ kg}) \sin 53°} = 15.6 \text{ m/s}$$

EXERCISE 8 A 3.00-kg mass with an initial velocity of 5.00**i** m/s collides with and sticks to a 2.00-kg mass with an initial velocity of −3.00**j** m/s. Find the final velocity of the composite mass. Answer (3.00**i** − 1.20**j**) m/s

EXERCISE 9 A bird and a bee are approaching each other at right angles. The bird has a mass of 0.125 kg and a speed of 0.600 m/s, and the bee has a mass of 5.00×10^{-3} kg and a speed of 15.0 m/s. If the bird catches the bee, what is the new speed of the bird? Answer 0.816 m/s

8.6 · THE CENTER OF MASS

In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a system of particles or any object. We shall see that the center of mass accelerates as if all of the system's mass were concentrated at that point and all external forces acted there. Furthermore, if the resultant external force on the system is $\mathbf{F}$ and the total mass of the system is M, the center of mass moves with an acceleration given by $\mathbf{a} = \mathbf{F}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This result was implicitly assumed in earlier chapters, because nearly all examples referred to the motion of extended objects.

Consider a system consisting of a pair of particles connected by a light, rigid rod (Fig. 8.13). The center of mass is located somewhere on the rod and is closer to the larger mass. If a single force is applied at some point on the rod that is closer to the smaller mass than to the larger one, the system rotates clockwise (Fig. 8.13a). If the force is applied at a point on the rod closer to the larger mass, the system rotates counterclockwise (Fig. 8.13b). If the force is applied at the center of mass,

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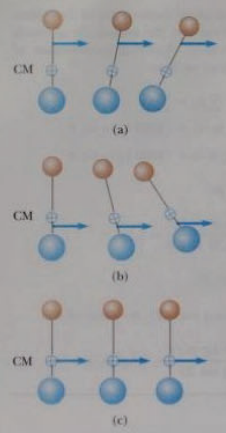


Figure 8.13 Two unequal masses are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the smaller mass and the center of mass. (b) The system rotates counterclockwise when a force is applied between the larger mass and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.



Figure 8.14 The center of mass of two particles on the x axis is located at x_{CM}, a point between the particles, closer to the larger mass.



This multiflash photograph shows that as the diver executes a back flip, his center of mass follows a parabolic path, the same path that a particle would follow. *(© the Harold E. Edgerton 1992 Trust. Courtesy of Palm Press, Inc.)*

the system moves in the direction of **F** without rotating (Fig. 8.13c). Thus, the center of mass can be easily located.

The position of the center of mass of a system can be described as being the *average position* of the system's mass. For example, the center of mass of the pair of particles described in Figure 8.14 is located on the x axis, somewhere between the particles. The x coordinate of the center of mass in this case is

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad [8.27]$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend the center of mass concept to a system of many particles in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum m_i x_i}{\sum m_i} \qquad [8.28]$$

where x_i is the x coordinate of the ith particle and $\sum m_i$ is the *total mass* of the system. For convenience, we shall express the total mass as $M = \sum m_i$, where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{CM} \equiv \frac{\sum m_i y_i}{M} \quad \text{and} \quad z_{CM} \equiv \frac{\sum m_i z_i}{M} \qquad [8.29]$$

The center of mass can also be located by its position vector, $\mathbf{r}_{CM}$. The rectangular coordinates of this vector are x_{CM}, y_{CM}, and z_{CM}, defined in Equations 8.28 and 8.29. Therefore,

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k} = \frac{\sum m_i x_i \mathbf{i} + \sum m_i y_i \mathbf{j} + \sum m_i z_i \mathbf{k}}{M}$$

$$\mathbf{r}_{CM} \equiv \frac{\sum m_i \mathbf{r}_i}{M} \qquad [8.30]$$

where $\mathbf{r}_i$ is the position vector of the ith particle, defined by

$$\mathbf{r}_i \equiv x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

The center of mass of a homogeneous, symmetric body must lie on an axis of symmetry. For example, the center of mass of a homogeneous rod must lie midway between the ends of the rod. The center of mass of a homogeneous sphere or a homogeneous cube must lie at the geometric center of the object. One can experimentally determine the center of mass of an irregularly shaped object, such as a wrench, by suspending the wrench from two different points (Fig. 8.15). The wrench is first hung from point A, and a vertical line, AB, is drawn (which can be established with a plumb bob) when the wrench is in equilibrium. The wrench is then hung from point C, and a second vertical line, CD, is drawn. The center of mass coincides with the intersection of these two lines. In fact, if the wrench is hung freely from any point, the vertical line through that point will pass through the center of mass.

Because a rigid body is a continuous distribution of mass, each portion is acted on by the force of gravity. The net effect of all of these forces is equivalent to the effect of a single force, $M\mathbf{g}$, acting through a special point called the **center of gravity.** If $\mathbf{g}$ is constant over the mass distribution, then the center of gravity coincides with the center of mass. If a rigid body is pivoted at its center of gravity, it will be balanced in any orientation.

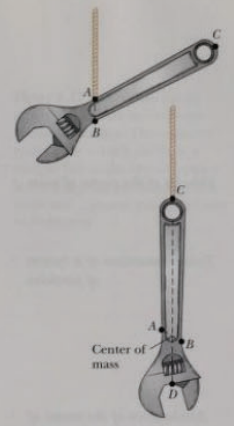


Figure 8.15 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely from two different pivots, A and C. The intersection of the two vertical lines AB and CD locates the center of mass.

Example 8.9 The Center of Mass of Three Particles

A system consists of three particles located at the corners of a right triangle as in Figure 8.16. Find the center of mass of the system.

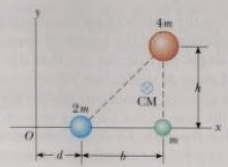


Figure 8.16 (Example 8.9) Locating the center of mass for a system of three particles.

Solution Using the basic defining equations for the coordinates of the center of mass, and noting that $z_{CM} = 0$, we get

$$x_{CM} = \frac{\sum m_i x_i}{M} = \frac{2md + m(d+b) + 4m(d+b)}{7m} = d + \tfrac{5}{7}b$$

$$y_{CM} = \frac{\sum m_i y_i}{M} = \frac{2m(0) + m(0) + 4mh}{7m} = \tfrac{4}{7}h$$

Therefore, we can express the position vector to the center of mass measured from the origin as

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k} = \left(d + \tfrac{5}{7}b\right)\mathbf{i} + \tfrac{4}{7}h\mathbf{j}$$

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8.7 • MOTION OF A SYSTEM OF PARTICLES

We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector of the center of mass $\mathbf{r}_{CM}$, given by Equation 8.30. Assuming that M remains constant—that is, no particles enter or leave the system—we get the following expression for the **velocity of the center of mass:**

Velocity of the center of mass •

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M}\sum m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad [8.31]$$

where $\mathbf{v}_i$ is the velocity of the ith particle. Rearranging Equation 8.31 gives

Total momentum of a system of particles •

$$M\mathbf{v}_{CM} = \sum m_i \mathbf{v}_i = \sum \mathbf{p}_i = \mathbf{p}_{tot} \qquad [8.32]$$

This result tells us that **the total momentum of the system equals the total mass multiplied by the velocity of the center of mass**—in other words, the total momentum of a single particle of mass M moving with a velocity $\mathbf{v}_{CM}$.

If we now differentiate Equation 8.32 with respect to time, we get the **acceleration of the center of mass:**

Acceleration of the center of mass for a system of particles •

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}_{CM}}{dt} = \frac{1}{M}\sum m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum m_i \mathbf{a}_i \qquad [8.33]$$

Rearranging this expression and using Newton's second law, we get

$$M\mathbf{a}_{CM} = \sum m_i \mathbf{a}_i = \sum \mathbf{F}_i \qquad [8.34]$$

where $\mathbf{F}_i$ is the force on particle i.

The forces on any particle in the system may include both external and internal forces. However, by Newton's third law, the force exerted by particle 1 on particle 2, for example, is equal to and opposite the force exerted by particle 2 on particle 1. When we sum over all internal forces in Equation 8.34, they cancel in pairs and the net force on the system is due *only* to external forces. Thus, we can write Equation 8.34 in the form

Newton's second law for a system of particles •

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \frac{d\mathbf{p}_{tot}}{dt} \qquad [8.35]$$

That is, the resultant external force on the system of particles equals the total mass of the system multiplied by the acceleration of the center of mass. If we compare this to Newton's second law for a single particle, we see that the center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. In the absence of external forces, the center of mass moves with uniform velocity, as in the case of the rotating wrench in Figure 8.17. If the resultant force acts along a line through the center of mass of an extended body such as the wrench, the body is accelerated without rotation, and its kinetic energy is associated entirely with its linear motion. If the resultant force does not act through the center of mass, the body will be rotationally accelerated, and the body will acquire a rotational kinetic energy in addition to the kinetic energy of its linear motion. The linear acceleration of the center of mass is the same in either case, as given by Equation 8.35.



Figure 8.17 Multiflash photograph of a wrench moving on a horizontal surface. The center of mass of the wrench moves in a straight line as the wrench rotates about this point as marked with a white dot. *(Richard Megna, Fundamental Photographs)*

Finally, we see that if the resultant external force is zero, then from Equation 8.35 it follows that

$$\frac{d\mathbf{p}_{tot}}{dt} = M\mathbf{a}_{CM} = 0$$

so that

$$\mathbf{p}_{tot} = M\mathbf{v}_{CM} = \text{constant} \qquad (\text{when } \sum \mathbf{F}_{ext} = 0) \qquad [8.36]$$

That is, the total linear momentum of a system of particles is constant if there are no external forces acting on the system. It follows that, for an *isolated* system of particles, both the total momentum and the velocity of the center of mass are constant in time. The law of conservation of momentum that was derived in Section 8.1 for a two-particle system is thus generalized to a many-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted on by an external force. For example, consider a system, initially at rest, made up of a swimmer and a raft. When the swimmer dives off the raft, the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the horizontal momentum of the diver is equal in magnitude to the momentum of the raft, but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly decays into two fragments of masses M_1 and M_2, with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Because the total momentum of the system before the decay is zero, the total momentum of the system after the decay must also be zero. Therefore, we see that $M_1\mathbf{v}_1 + M_2\mathbf{v}_2 = 0$. If the velocity of one of the fragments after the decay is known, the recoil velocity of the other fragment can be calculated.

Thinking Physics 3

A boy stands at one end of a floating canoe that is stationary relative to the shore (Fig. 8.18). He then walks to the opposite end of the canoe, away from the shore. Does the canoe move?

Reasoning Yes, the canoe moves toward the shore. Neglecting friction between the canoe and water, there are no horizontal forces acting on the system consisting of the boy and canoe. Therefore, the center of mass of the system remains fixed relative to the shore (or any stationary point). As the boy moves away from the shore, the canoe

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Figure 8.18 (Thinking Physics 3)

must move toward the shore such that the center of mass of the system remains constant. An alternative explanation is that the momentum of the system remains constant if friction is neglected. As the boy acquires a momentum away from the shore, the canoe must acquire an equal momentum toward the shore such that the total momentum of the system is always zero.

CONCEPTUAL PROBLEM 5

Suppose you tranquilize a polar bear on a glacier as part of a research effort. How might you be able to estimate the weight of the polar bear using a measuring tape, a rope, and a knowledge of your own weight?

CONCEPTUAL PROBLEM 6

A projectile is fired into the air and suddenly explodes into several fragments (Fig. 8.19). What can be said about the motion of the center of mass of the fragments after the explosion?

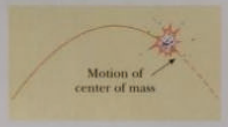


Figure 8.19 (Conceptual Problem 6)

EXERCISE 10 A 5.0-kg particle moves along the x axis with a velocity of 3.0 m/s. A 2.0-kg particle moves along the x axis with a velocity of -2.5 m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system. Answer (a) 1.4 m/s to the right (b) 10 kg·m/s to the right

EXERCISE 11 A 2.0-kg particle has a velocity $(2.0\mathbf{i} - 3.0\mathbf{j})$ m/s, and a 3.0-kg particle has a velocity $(1.0\mathbf{i} + 6.0\mathbf{j})$ m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system. Answer (a) $(1.4\mathbf{i} + 2.4\mathbf{j})$ m/s (b) $(7.0\mathbf{i} + 12\mathbf{j})$ kg·m/s

OPTIONAL

8.8 · ROCKET PROPULSION

When ordinary vehicles, such as automobiles and locomotives, are propelled, the driving force for the motion is one of friction. In the case of the automobile, the driving force is the force exerted by the road on the car. A locomotive "pushes" against the tracks; hence, the driving force is the reaction force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to "push" against. Therefore, the source of the propulsion of a rocket must be different. **The operation of a rocket depends on the law of conservation of momentum as applied to a system of particles, where the system is the rocket plus its ejected exhaust gases.**



Figure 8.20 Lift-off of the space shuttle *Columbia*. Enormous thrust is generated by the shuttle's liquid-fueled engines, aided by the two solid-fuel boosters. Many physical principles from mechanics, thermodynamics, and electricity and magnetism are involved in such a launch. *(Courtesy of NASA)*

The propulsion of a rocket (Fig. 8.20) can be understood by first considering the mechanical system consisting of a machine gun mounted on a cart on wheels. As the machine gun is fired, each bullet receives a momentum $m\mathbf{v}$ in some direction, where $\mathbf{v}$ is the velocity of the bullet relative to the gun. For each bullet that is fired, the gun and cart must receive a compensating momentum in the opposite direction (neglecting the momentum imparted to the escaping gas in the barrel of the gun). That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun. If there are n bullets fired each second, then the average force exerted on the gun is $\mathbf{F}_{av} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space (a vacuum), **its momentum changes when some of its mass is released in the form of ejected gases. Because the ejected gases acquire some momentum, the rocket receives a compensating momentum in the opposite direction.** Therefore, **the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases.** In free space, the center of mass of the system moves uniformly, independent of the propulsion process. It is interesting to note that the rocket and machine gun represent cases of the *inverse* of an inelastic collision—that is, momentum is conserved, but the kinetic energy of the system is *increased* (at the expense of potential energy carried in the fuel).

Suppose that at some time t, the momentum of the rocket plus the fuel is $(M + \Delta m)v$ (Fig. 8.21a). At some short time later, Δt, the rocket ejects fuel of mass Δm and the rocket's speed therefore increases to $v + \Delta v$ (Fig. 8.21b). If the fuel is ejected with a velocity v_e *relative to the rocket*, then the speed of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial momentum of the system with the total final momentum, we get

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

Simplifying this expression gives

$$M\,\Delta v = \Delta m(v_e)$$

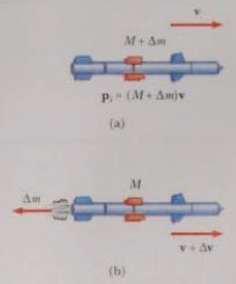


Figure 8.21 Rocket propulsion. (a) The initial mass of the rocket is $M + \Delta m$ at a time t, and its speed is v. (b) At a time $t + \Delta t$, the rocket's mass has reduced to M, and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv.

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We can also arrive at this result by considering the system in the center-of-mass frame of reference—that is, a frame the velocity of which equals the center-of-mass velocity. In this frame, the total momentum is zero; therefore, if the rocket gains a momentum $M\,\Delta v$ by ejecting some fuel, the exhaust gases obtain a momentum $\Delta m(v_e)$ in the *opposite* direction, and so $M\,\Delta v - \Delta m(v_e) = 0$. If we now take the limit as Δt goes to zero, the $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Furthermore, the increase in the exhaust mass, dm, corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is given a negative sign because it represents a decrease in mass. Using this fact, we get

$$M\,dv = -v_e\,dM \qquad [8.37]$$

Integrating this equation and taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f we get

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

Basic expression for rocket propulsion

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \qquad [8.38]$$

This is the basic expression of rocket propulsion. First, it tells us that the increase in speed is proportional to the exhaust speed, v_e. Therefore, the exhaust velocity should be very high. Second, the increase in speed is proportional to the logarithm of the ratio M_i/M_f. Therefore, this ratio should be as large as possible, which means that the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on the rocket by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 8.37:

$$\text{Thrust} = M\frac{dv}{dt} = \left|v_e\frac{dM}{dt}\right| \qquad [8.39]$$

Here we see that the thrust increases as the exhaust speed increases and as the rate of change of mass (burn rate) increases.

Thinking Physics 4

When Robert Goddard proposed the possibility of rocket-propelled vehicles, the *New York Times* agreed that such vehicles would be useful and successful within the Earth's atmosphere ("Topics of the Times," *New York Times*, Jan. 13, 1920, p. 12). But the *Times* balked at the idea of using such a rocket in the vacuum of space, noting that, ". . . its flight would be neither accelerated nor maintained by the explosion of the charges it then might have left. To claim that it would be is to deny a fundamental law of dynamics . . . That Professor Goddard, with his 'chair' in Clark College and the countenancing of the Smithsonian Institution, does not know the relation of action to reaction, and of the need to have something better than a vacuum against which to react—to say that would be absurd."

What did the writer of these passages overlook?

Reasoning The writer was making a common mistake in believing that a rocket works by expelling gases that *push* on something, propelling the rocket forward. With this belief, it is impossible to see how a rocket fired in empty space *would* work.

Gases do not need to push on anything—it is the act of expelling the gases itself that pushes the rocket forward. This can be argued from Newton's third law—the rocket pushes the gases backward, resulting in the gases pushing the rocket forward. It can also be argued from conservation of momentum—as the gases gain momentum in one direction, the rocket must gain momentum in the opposite direction to conserve the original momentum of the rocket–gas system.

The *New York Times* did publish a retraction 49 years later ("A Correction," *New York Times*, July 17, 1969, p. 43), at the same time that the Apollo astronauts were on their way to the Moon. It appeared on a page with two other articles, one titled, "Fundamentals of Space Travel," and the other, "Spacecraft, Like Squid, Maneuver by 'Squirts.'" The following passages were part of these articles: ". . . an editorial feature of the *New York Times* dismissed the notion that a rocket could function in a vacuum and commented on the ideas of Robert H. Goddard. . . . Further investigation and experimentation have confirmed the findings of Isaac Newton in the 17th century, and it is now definitely established that a rocket can function in a vacuum as well as in an atmosphere. The *Times* regrets the error."

Example 8.10 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to Earth. Its engines are turned on, and exhaust is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to Earth once its mass is reduced to one half its mass before ignition?

Solution Applying Equation 8.38, we get

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right)$$

$$= 3.0 \times 10^3 + 5.0 \times 10^3 \ln\left(\frac{M_i}{0.5M_i}\right)$$

$$= 6.5 \times 10^3 \text{ m/s}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution

$$\text{Thrust} = \left|v_e\frac{dM}{dt}\right| = \left(5.0 \times 10^3 \frac{\text{m}}{\text{s}}\right)\left(50 \frac{\text{kg}}{\text{s}}\right)$$

$$= 2.5 \times 10^5 \text{ N}$$

EXERCISE 12 A rocket engine consumes 80 kg of fuel per second. If the exhaust speed is 2.5×10^3 m/s, calculate the thrust on the rocket. Answer 200 kN

SUMMARY

The **linear momentum** of any object of mass m moving with a velocity $\mathbf{v}$ is

$$\mathbf{p} = m\mathbf{v} \qquad [8.1]$$

Conservation of momentum applied to two interacting objects states that if the two objects form an isolated system, their total momentum is constant regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum:

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \qquad [8.5]$$

The **impulse** of a force $\mathbf{F}$ on any object is equal to the change in the momentum of the object and is given by

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\,dt = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \qquad [8.9]$$

This is known as the **impulse-momentum theorem.**

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Impulsive forces are forces that are very strong compared with other forces on the system. They usually act for a short time, as in the case of collisions.

When two bodies collide, the total momentum of the system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which kinetic energy is not constant, but momentum is. A perfectly inelastic collision is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which both momentum and kinetic energy are constant.

In a two- or three-dimensional collision, the components of momentum in each of the directions are conserved independently.

The **vector position of the center of mass of a system of particles** is defined as

$$\mathbf{r}_{CM} \equiv \frac{\sum m_i \mathbf{r}_i}{M} \qquad [8.30]$$

where $M = \sum m_i$ is the total mass of the system and $\mathbf{r}_i$ is the vector position of the ith particle.

The **velocity of the center of mass for a system of particles** is

$$\mathbf{v}_{CM} = \frac{\sum m_i \mathbf{v}_i}{M} \qquad [8.31]$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass—that is, $\mathbf{p}_{tot} = M\mathbf{v}_{CM}$.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} = \frac{d\mathbf{p}_{tot}}{dt} \qquad [8.35]$$

where $\mathbf{a}_{CM}$ is the acceleration of the center of mass and the sum is over all external forces. Therefore, the center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. It follows from Equation 8.35 that the total momentum of the system is constant if there are no external forces acting on it.

CONCEPTUAL QUESTIONS

1. Does a large force always produce a larger impulse on a body than a smaller force does? Explain.
2. If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.
3. Explain how linear momentum is conserved when a ball bounces from a floor.
4. Consider a perfectly inelastic collision between a car and a large truck. Which vehicle loses more kinetic energy as a result of the collision?
5. Your physical education teacher, who knows something about physics, throws you a tennis ball at a certain velocity, and you catch it. You are now given the following choice: The teacher can throw you a medicine ball, which is much more massive than the tennis ball, with the same velocity as the tennis ball, the same momentum, or the same kinetic energy. Which choice would you make in order to make the easiest catch, and why?
6. Can the center of mass of a body lie outside the body? If so, give examples.
7. Is the center of mass of a mountain higher or lower than the center of gravity?
8. In golf, novice players are often advised to be sure to "follow through" with their swing. Why does this make the ball travel a longer distance? What about a chip shot? There is very little follow-through here—why?

9. A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why isn't it as dangerous to be hit by the gun as by the bullet?
10. A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Could you calculate the force exerted on him due to the collision? Explain.
11. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
12. An airbag is inflated when a collision occurs, which protects the passenger (the dummy, in this case) from serious injury (Fig. Q8.12). Why does the airbag soften the blow? Discuss the physics involved in this dramatic photograph.
13. A large bedsheet is held vertically by two students. A third student, who happens to be the star pitcher on the baseball team, throws a raw egg at the sheet. Explain why the egg does not break when it hits the sheet, regardless of its initial speed. (If you try this, make sure the pitcher hits the sheet near its center, and do not allow the egg to fall on the floor after being caught in the sheet.)



Figure Q8.12 *(Courtesy of Saab)*

PROBLEMS

Section 8.1 Linear Momentum and Its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\mathbf{i} - 4.00\mathbf{j})$ m/s. Find its x and y components of momentum and the magnitude of its total momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. A 40.0-kg child standing on a frozen pond throws a 0.500-kg stone to the east with a speed of 5.00 m/s. Neglecting friction between child and ice, find the recoil velocity of the child.
4. A pitcher claims she can throw a baseball with as much momentum as a 3.00-g bullet moving with a speed of 1500 m/s. A baseball has a mass of 0.145 kg. What must be its speed if the pitcher's claim is valid?
5. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them, and the blocks are pushed together with the spring between them (Fig. P8.5). A cord holding them together is burned, after which the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the speed of the block of mass M? (b) Find the original elastic potential energy in the spring if $M = 0.350$ kg.

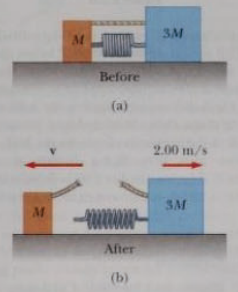


Figure P8.5

Section 8.2 Impulse and Momentum

6. A car is stopped for a traffic signal. When the light turns green, the car accelerates, increasing its speed from 0 to 5.20 m/s in 0.832 s. What linear impulse and average force does a 70.0-kg passenger in the car experience?
7. An estimated force–time curve for a baseball struck by a bat is shown in Figure P8.7. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

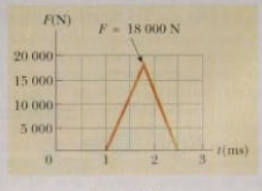


Figure P8.7

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8. A tennis player receives a shot with the ball (0.0600 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racket? (b) What work does the racket do on the ball?

9. A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P8.9). If the ball is in contact with the wall for 0.200 s, what is the average force exerted on the ball by the wall?

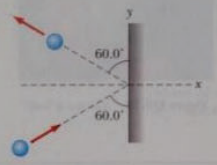


Figure P8.9

10. In a slow-pitch softball game, a 0.200-kg softball crossed the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The ball was hit at 40.0 m/s, 30.0° above the horizontal. (a) Determine the impulse applied to the ball. (b) If the force on the ball increased linearly for 4.00 ms, held constant for 20.0 ms, then decreased to 0 linearly in another 4.00 ms, find the maximum force on the ball.

11. A machine gun fires 35.0 g bullets at a speed of 750.0 m/s. If the gun can fire 200 bullets/min, what is the average force the shooter must exert to keep the gun from moving?

Section 8.3 Collisions

Section 8.4 Elastic and Inelastic Collisions in One Dimension

12. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s. Find the speed of the golf ball just after impact.

13. A 10.0-g bullet is stopped in a block of wood ($m = 5.00$ kg). The speed of the bullet-plus-wood combination immediately after the collision is 0.600 m/s. What was the original speed of the bullet?

14. A 75.0-kg ice skater, moving at 10.0 m/s, crashes into a stationary skater of equal mass. After the collision, the two skaters move as a unit at 5.00 m/s. The average force a skater can experience without breaking a bone is 4500 N. If the impact time is 0.100 s, does a bone break?

15. As shown in Figure P8.15, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

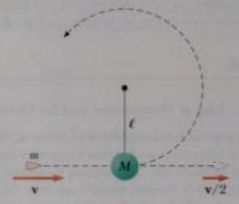


Figure P8.15

16. Gayle runs at a speed of 4.00 m/s and dives on a sled, initially at rest on the top of a frictionless snow-covered hill. After she has descended a vertical distance of 5.00 m, her brother, who is initially at rest, hops on her back and together they continue down the hill. What is their speed at the bottom of the hill if the total vertical drop is 15.0 m? Gayle's mass is 50.0 kg, the sled has a mass of 5.00 kg, and her brother has a mass of 30.0 kg.

17. A 1200-kg car traveling initially with a speed of 25.0 m/s in an easterly direction crashes into the rear end of a 9000-kg truck moving in the same direction at 20.0 m/s (Fig. P8.17). The velocity of the car right after the collision is 18.0 m/s to the east. (a) What is the velocity of the truck right after the collision? (b) How much mechanical energy is lost in the collision? Account for this loss in energy.



Figure P8.17

18. A railroad car of mass 2.50×10^4 kg, moving with a speed of 4.00 m/s, collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much energy is lost in the collision?

19. A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is about 12.0 times the mass of the neutron.)

20. A 7.00-kg bowling ball collides head-on with a 2.00-kg bowling pin. The pin flies forward with a speed of 3.00 m/s. If the ball continues forward with a speed of 1.80 m/s, what was the initial speed of the ball? Ignore rotation of the ball.

21. A 12.0-g bullet is fired into a 100-g wooden block initially at rest on a horizontal surface. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between block and surface is 0.650, what was the speed of the bullet immediately before impact?

22. A bullet of mass m_1 is fired into a wooden block of mass m_2 initially at rest on a horizontal surface. After impact, the block slides a distance d before coming to rest. If the coefficient of friction between block and surface is μ, what was the speed of the bullet immediately before impact?

23. Consider a frictionless track ABC as shown in Figure P8.23. A block of mass $m_1 = 5.00$ kg is released from A. It makes a head-on elastic collision with a block of mass $m_2 = 10.0$ kg at B, initially at rest. Calculate the maximum height to which m_1 rises after the collision.

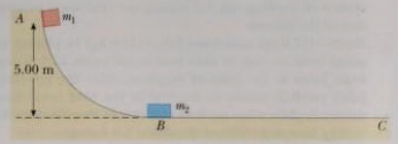


Figure P8.23

24. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, penetrates the block to a depth of 8.00 cm. This block of wood is placed on a frictionless horizontal surface, and a 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

Section 8.5 Two-Dimensional Collisions

25. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the energy lost as a result of the collision. Account for the missing energy.

26. The mass of the blue puck in Figure P8.26 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with equal and opposite momenta, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

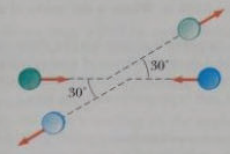


Figure P8.26

27. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves at 4.33 m/s, at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision, find the struck ball's velocity after the collision.

28. A proton, moving with a velocity of $v_0\mathbf{i}$, collides elastically with another proton that is initially at rest. If the two protons have the same speed after the collision, find (a) the speed of each proton after the collision in terms of v_0 and (b) the direction of the velocity vectors after the collision.

29. A 0.300-kg puck, initially at rest on a horizontal, frictionless surface, is struck by a 0.200-kg puck moving initially along the x axis with a speed of 2.00 m/s. After the collision, the 0.200-kg puck has a speed of 1.00 m/s at an angle of $\theta = 53.0°$ to the positive x axis (Fig. 8.11). (a) Determine the velocity of the 0.300-kg puck after the collision. (b) Find the fraction of kinetic energy lost in the collision.

30. During the battle of Gettysburg, the gunfire was so intense that several bullets collided in mid-air and fused together. Assume a 5.00-g Union musket ball was moving to the right at a speed of 250 m/s, 20.0° above the horizontal, and that a 3.00-g Confederate ball was moving to the left at a speed of 280 m/s, 15.0° above the horizontal. Immediately after they fuse together, what is their velocity?

31. An unstable nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a velocity of 6.00×10^6 m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of $4.00 \times$

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10^6 m/s. Find (a) the velocity of the third particle and (b) the total energy given off in the process.

32. At an intersection, a 1500-kg car, traveling east with a speed of 20.0 m/s, collides with a 2500-kg van traveling south at a speed of 15.0 m/s. The vehicles undergo a perfectly inelastic collision, and the wreckage slides 6.00 m before coming to rest. Find the magnitude and direction of the constant force that decelerated them.

Section 8.6 The Center of Mass

33. Four objects are situated along the *y* axis as follows: a 2.00-kg object is at +3.00 m, a 3.00-kg object is at +2.50 m, a 2.50-kg object is at the origin, and a 4.00-kg object is at −0.500 m. Where is the center of mass of these objects?

34. A uniform carpenter's square has the shape of an L, as in Figure P8.34. Locate the center of mass relative to an origin at the lower left corner. (*Hint:* Note that the mass of each rectangular part is proportional to its area.)

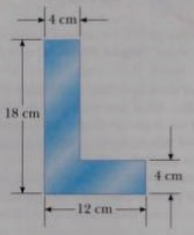


Figure P8.34

35. A uniform piece of sheet steel is shaped as in Figure P8.35. Compute the *x* and *y* coordinates of the center of mass of the piece.

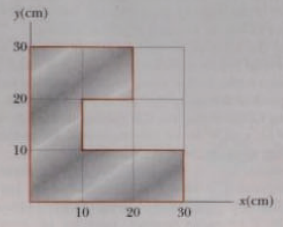


Figure P8.35

36. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^8 m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

37. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P8.37). The angle between the two bonds is 106°. If the bonds are 0.100 nm long, where is the center of mass of the molecule?

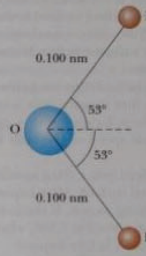


Figure P8.37

Section 8.7 Motion of a System of Particles

38. Consider a system of two particles in the *xy* plane: m_1 = 2.00 kg is at $\mathbf{r}_1 = (1.00\mathbf{i} + 2.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} + 0.500\mathbf{j})$ m/s; m_2 = 3.00 kg is at $\mathbf{r}_2 = (-4.00\mathbf{i} - 3.00\mathbf{j})$ m and has velocity $(3.00\,\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

39. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat in still water, 2.70 m away from Juliet in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

40. A ball of mass 0.200 kg has velocity $1.50\mathbf{i}$ m/s; a ball of mass 0.300 kg has velocity $-0.400\mathbf{i}$ m/s. Then they meet in a head-on elastic collision. (a) Find their velocities thereafter. (b) Find the velocity of their center of mass before the collision and afterward.

Section 8.8 Rocket Propulsion (Optional)

41. The first stage of a Saturn V space vehicle consumes fuel at the rate of 1.50×10^4 kg/s, with an exhaust speed of 2.60×10^3 m/s. (a) Calculate the thrust produced by these engines. (b) Find the initial acceleration of the vehicle on the launch pad if its initial mass is 3.00×10^6 kg. [*Hint:* You must include the force of gravity to solve part (b).]

42. A large rocket with an exhaust speed of v_e = 3000 m/s develops a thrust of 24.0 million newtons. (a) How much mass is being blasted out of the rocket exhaust per second? (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with v_e = 3.00 km/s and if 90.0% of its initial mass is fuel?

43. A rocket for use in deep space is to have the capability of accelerating a payload (including the rocket frame and engine) of 3.00 metric tons from rest to a final speed of 10 000 m/s with an engine and fuel designed to produce an exhaust speed of 2000 m/s. (a) How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5000 m/s, what amount of fuel and oxidizer would be required for the same task?

Additional Problems

44. A golf ball (m = 46.0 g) is struck a blow that makes an angle of 45.0° with the horizontal. The drive lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

45. An 8.00-g bullet is fired into a 2.50-kg block initially at rest at the edge of a frictionless table of height 1.00 m (Fig. P8.45). The bullet remains in the block, and after impact the block lands 2.00 m from the bottom of the table. Determine the initial speed of the bullet.

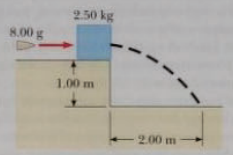


Figure P8.45

46. A spacecraft is stationary in deep space when its rocket engine is ignited for a 100-s "burn." Hot gases are ejected at a constant rate of 150 kg/s, with a speed of 3000 m/s relative to the spacecraft. The initial mass of the spacecraft (plus fuel and oxidizer) is 25 000 kg. Determine the thrust of the rocket engine and the initial acceleration in units of *g*. What is the final speed of the spacecraft?

47. A small block of mass m_1 = 0.500 kg is released from rest at the top of a curved-shaped frictionless wedge of mass m_2 = 3.00 kg, which sits on a frictionless horizontal surface as in Figure P8.47a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P8.47b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height, *h*, of the wedge?

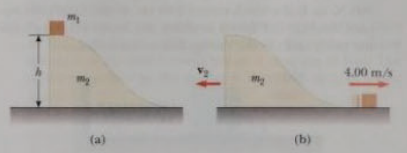


Figure P8.47

48. Tarzan, whose mass is 80.0 kg, swings from a 3.00-m vine that is horizontal when he starts. At the bottom of his arc, he picks up 60.0-kg Jane in a perfectly inelastic collision. What is the height of the highest tree limb they can reach on their upward swing?

49. A 80.0-kg astronaut is working on the engines of her ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds herself 30.0 m behind the ship. Without a thruster, the only way to return to the ship is to throw her 0.500-kg wrench directly away from the ship. If she throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

50. A 40.0-kg child stands at one end of a 70.0-kg boat that is 4.00 m in length (Fig. P8.50). The boat is initially 3.00 m from the pier. The child notices a turtle on a rock at the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction between boat and water, (a) describe the subsequent motion of the system (child plus boat). (b) Where is the child *relative to the pier* when he reaches the far end of the boat? (c) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat.)



Figure P8.50

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51. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as in Figure P8.51. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the energy lost in the collision.

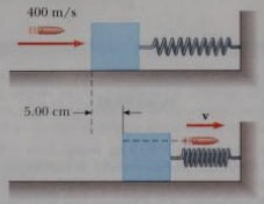


Figure P8.51

52. A student performs a ballistic pendulum experiment using the apparatus shown in Figure P8.52a. A ball of mass 68.8 g is fired into a pendulum of mass 263 g hanging from a low friction pivot. The ball is caught inside the pendulum, which then swings up through a height 8.68 cm. (a) Determine the initial speed v_{1i} of the ball. (b) The second part of her experiment is to obtain v_{1i} by firing the same ball horizontally (with the pendulum removed from the path), by measuring its horizontal displacement, x, and vertical displacement, y (Fig. P8.52b). Show that the initial speed of the ball is related to x and y through the relation

$$v_{1i} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1i} based on her measured values of $x = 257$ cm and $y = 85.3$ cm? What factors might account for the difference in this value compared to that obtained in part (a)?

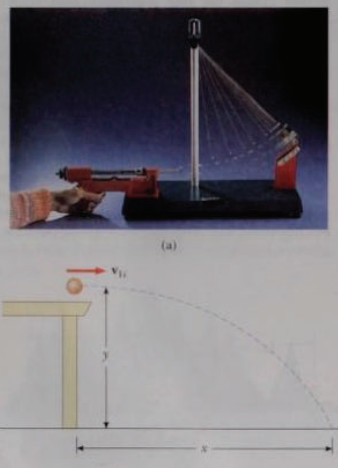


Figure P8.52 (a) and (b) *(Photo courtesy of CENCO)*

53. Two particles of masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_0. Mass m is traveling to the left, and mass $3m$ is traveling to the right. They undergo a head-on elastic collision. Find the final speeds of the particles.

54. Two particles of masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_0. Mass m is traveling to the left, and mass $3m$ is traveling to the right. They undergo an elastic glancing collision such that mass m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two masses. (b) What is the angle θ at which the mass $3m$ is scattered?

55. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, you give the Earth a maximum recoil speed of what order of magnitude? Visualize the Earth as a perfectly rigid object. In your solution state the physical quantities you take as data and the values you measure or estimate for them.

56. There are (one can say) three co-equal theories of motion: Newton's second law, holding that the total force on an object causes its acceleration; the work-kinetic energy theorem, holding that the total work on an object causes its change in kinetic energy; and the impulse-momentum theorem, holding that the total impulse on an object causes its change in momentum. In this problem you compare the predictions of the three theories in one particular case. A 3.00-kg object has velocity $7.00\hat{\mathbf{j}}$ m/s. Then a total force $12.0\hat{\mathbf{i}}$ N acts on the object for 5.00 s. (a) Calculate its final velocity from the impulse-momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_0)/t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma\mathbf{F}/m$. (d) Find the object's vector displacement from $\mathbf{s} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$. (e) Find the work done on the object from $W = \mathbf{F}\cdot\mathbf{s}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f\cdot\mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_0^2 + W$.

Spreadsheet Problems

S1. A rocket has an initial mass of 20 000 kg, of which 20% is the payload. The rocket burns fuel at a rate of 200 kg/s and exhausts gas at a relative speed of 2.00 km/s. Its acceleration dv/dt is determined from the equation of motion

$$M\frac{dv}{dt} = v_e\left|\frac{dM}{dt}\right| + F_{\text{ext}}$$

Assume that there are no external forces and that the rocket's initial velocity is zero. When $F_{\text{ext}} = 0$, the rocket's velocity is $v(t) = v_e \ln(M_i/M)$, where M_i is the rocket's initial mass. Spreadsheet 8.1 calculates the acceleration and velocity as a function of time. (a) Using the given parameters, find the maximum acceleration and velocity. (b) At what time is the velocity equal to half its maximum value? Why isn't this time half of the burn time?

S2. Modify Spreadsheet 8.1 to calculate the distance traveled by the rocket in Problem S1. Add a new column to the spreadsheet to find the new position x_{i+1}. Estimate the new position of the rocket by

$$x_{i+1} = x_i + \tfrac{1}{2}(v_{i+1} + v_i)\Delta t$$

where x_i is the old position, v_i is the old velocity, and v_{i+1} is the new velocity.

S3. If the rocket in the previous problem burns its fuel twice as fast (400 kg/s) while maintaining the same exhaust speed, how far does it travel before it runs out of fuel? The velocity reached by the rocket is the same in both cases; that is

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

Is there an advantage in burning the fuel faster? Is there a disadvantage? (*Hint:* It may be useful to plot the acceleration, velocities, and position for the same time intervals in both cases. In which case does the rocket travel farther? In which case does the rocket have to withstand the largest acceleration? By how much?)

ANSWERS TO CONCEPTUAL PROBLEMS

1. The brain is immersed in a cushioning fluid inside the skull. If the head is struck suddenly by a bare fist, there is a rapid acceleration of the skull. The brain matches this acceleration only because of the large impulsive force of the skull on the brain. This large and sudden force can severely injure the brain. Padded gloves extend the time over which the force is applied to the head. Thus, for a given impulse, the gloves provide a longer time interval than the bare fist, decreasing the average force. Because the average force is decreased, the acceleration of the skull is decreased, reducing (but not eliminating) the chance for brain injury. The same argument can be made for "rolling with the punch." If the head is held steady while being struck, the time interval over which the force is applied is relatively short and the average force is large. If the head is allowed to move in the same direction of the punch, the time interval is lengthened, and the average force reduced.
2. The scenario described violates the principle of conservation of momentum. If the superhero, initially at rest, throws the piano in one direction, he must recoil in the other direction, in order to maintain the zero momentum of the system.
3. Making cars out of hard rubber would save on automobile repair bills, but the results for the drivers and passengers would be disastrous. Imagine two identical cars traveling toward each other at identical speeds, and making a head-on collision. In the approximation that the collision is elastic, the cars bounce off each other and leave the collision with their velocities reversed in direction. Thus, the inhabitants of the cars would suffer a large change in momentum in a short period of time. As a result, the force on the inhabitants would become very large, causing significant bodily injury. For real cars in the above situation, the cars would come to rest during the collision, as the front sections of the cars crumpled. Thus, the change in momentum is only half what it was in the elastic case, and the change occurs over the longer time period during which the cars crumpled. The resulting smaller force on the inhabitants will result in fewer injuries.
4. If all of the kinetic energy is transformed into other forms, then there is no motion of either of the objects after the collision. If nothing is moving, there is no momentum. Thus, if the final momentum of the system is zero, then the initial momentum of the system must have been zero. The required condition, then, is that the two objects are approaching each other so as to make a head-on collision, with equal, but oppositely directed, momenta.
5. Tie one end of the rope around the bear. Lay out the tape measure on the ice between the bear's original position and yours, as you hold the opposite end of the rope. Take off your spiked shoes and pull on the rope. Both you and the bear will slide over the ice until you meet. From the tape observe how far you have slid, x_y, and how far the bear has slid, x_b. The point where you meet the bear is the constant location of the center of mass of the system (bear plus you),

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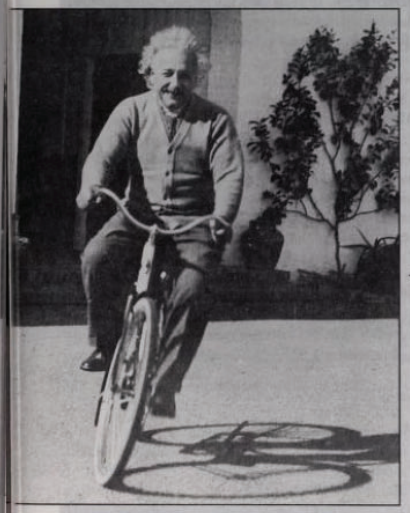
so you can determine the mass of the bear from $m_b x_b = m_y x_y$. (Unfortunately, if the bear now wakes up you cannot get back to your spiked shoes.)

6. Neglecting air resistance, the only external force on the projectile is the force of gravity. Thus, the projectile follows a parabolic path. If the projectile did not explode, it would continue to move along the parabolic path indicated by the broken line in Figure 8.19. Because the forces due to the explosion are internal, they do not affect the motion of the center of mass. Thus, after the explosion, and until the first fragment hits the ground, the center of mass of the fragments follows the same parabolic path the projectile would have followed if there had been no explosion.

9

Relativity

Most of our everyday experiences and observations have to do with objects that move at speeds much less than that of light. Newtonian mechanics and early ideas on space and time were formulated to describe the motion of such objects. This formalism is successful in describing a wide range of phenomena that occur at low speeds. It fails, however, when applied to particles the speeds of which approach that of light. The predictions of Newtonian theory can be tested experimentally at high speeds by accelerating electrons or other charged particles. For example, it is possible to accelerate an electron to a speed of $0.99c$ (where c is the speed of light) by using particle accelerators. According to Newtonian mechanics, if the energy transferred to an electron is increased by a factor of 4, the electron speed should jump to $1.98c$. However, experiments show that the speed of the electron—as well as the speeds of all other particles in the Universe—always remains less than the speed of light, regardless of the particle energy. Because it places no upper limit on speed, New-



◀ **Albert Einstein (1879–1955), one of the greatest physicists of all times, is best known for developing the theory of relativity. He is shown here in a playful mood riding a bicycle. The photograph was taken in 1933 in Santa Barbara, California.** *(From the California Institute of Technology archives)*

"Imagination is more important than knowledge."
Albert Einstein

CHAPTER OUTLINE

9.1 The Principle of Newtonian Relativity
9.2 The Michelson–Morley Experiment
9.3 Einstein's Principle of Relativity
9.4 Consequences of Special Relativity
9.5 The Lorentz Transformation Equations
9.6 Relativistic Momentum and the Relativistic Form of Newton's Laws
9.7 Relativistic Energy
9.8 Mass as a Measure of Energy
9.9 General Relativity (Optional)

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tonian mechanics is contrary to modern experimental results and is clearly a limited theory.

In 1905, at the age of 26, Einstein published his special theory of relativity. Einstein wrote,

> The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions. . . . [1]

Although Einstein made many important contributions to science, special relativity alone represents one of the greatest intellectual achievements of the twentieth century. With special relativity, experimental observations can be correctly predicted over the range of speeds from $v = 0$ to speeds approaching the speed of light. Newtonian mechanics, which was accepted for more than 200 years, is in fact a special case of special relativity. This chapter gives an introduction to special relativity, with emphasis on some of its consequences.

Special relativity covers phenomena such as the slowing down of clocks and the contraction of lengths in moving reference frames as measured by a laboratory observer. We also discuss the relativistic forms of momentum and energy, and some consequences of the famous mass–energy formula, $E = mc^2$.

In addition to its well known and essential role in theoretical physics, special relativity has practical applications, including the design of accelerators and other devices that use high-speed particles. These devices will not work if designed according to nonrelativistic principles. We shall have occasion to use special relativity in some subsequent chapters of this textbook, most often presenting only the outcome of relativistic effects.

9.1 • THE PRINCIPLE OF NEWTONIAN RELATIVITY

To describe a physical event, it is necessary to establish a frame of reference. You should recall from Chapter 5 that Newton's laws are valid in all inertial frames of reference. Because an inertial frame is defined as one in which Newton's first law is valid, it can be said that **an inertial system is one in which a free body experiences no acceleration.** Furthermore, any system moving with constant velocity with respect to an inertial system must also be an inertial system.

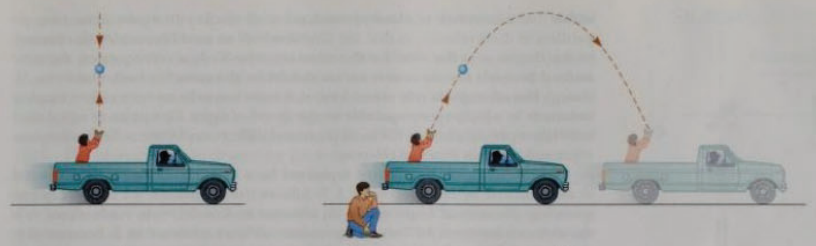
Inertial frame of reference •

There is no preferred frame. This means that the results of an experiment performed in a vehicle moving with uniform velocity will be identical to the results of the same experiment performed in a stationary vehicle. The formal statement of this result is called the principle of **Newtonian relativity:**

> The laws of mechanics must be the same in all inertial frames of reference.

Let us consider an observation that illustrates the equivalence of the laws of mechanics in different inertial frames. Consider a pickup truck moving with a constant velocity, as in Figure 9.1a. If a passenger in the truck throws a ball straight up in the air, the passenger observes that the ball moves in a vertical path. The motion

[1] A. Einstein and L. Infeld, *The Evolution of Physics*, New York, Simon and Schuster, 1961.



(a) (b)

Figure 9.1 (a) The observer in the truck sees the ball move in a vertical path when thrown upward. (b) The Earth observer views the path of the ball to be a parabola.

of the ball appears to be precisely the same as if the ball were thrown by a person at rest on Earth. The law of gravity and the equations of kinematics are obeyed whether the truck is at rest or in uniform motion. Now consider the same experiment viewed by an observer at rest on Earth. This stationary observer sees the path of the ball as a parabola, as in Figure 9.1b. Furthermore, according to this observer, the ball has a horizontal component of velocity equal to the velocity of the truck. Although the two observers see different velocities and rates of energy change, they see the same forces and agree on the validity of Newton's laws as well as on principles like conservation of energy and conservation of momentum. All differences between the two views stem from the relative motion of one frame with respect to the other. That is, the notion of absolute motion through space plays no role in what is seen.

Suppose that some physical phenomenon that we call an *event* occurs in an inertial system. The event's location and time of occurrence can be specified by the coordinates (x, y, z, t). We would like to be able to transform these coordinates from one inertial system to another moving with uniform relative velocity. This is accomplished by using what is called a *Galilean transformation*, which owes its origin to Galileo.

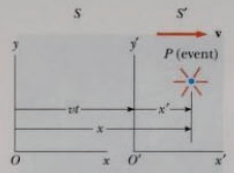


Figure 9.2 An event occurs at a point P. The event is observed by two observers in inertial frames S and S', where S' moves with a velocity $\mathbf{v}$ relative to S.

Consider two inertial systems S and S' (Fig. 9.2). The system S' moves with a constant velocity $\mathbf{v}$ along the xx' axes, where $\mathbf{v}$ is measured relative to S. We assume that an event occurs at the point P and that the origins of S and S' coincide at $t = 0$. An observer in S describes the event with space–time coordinates (x, y, z, t), while an observer in S' uses (x', y', z', t') to describe the same event. As we can see from Figure 9.2, these coordinates are related by the equations

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \qquad [9.1]$$

• *Galilean transformation of coordinates*

These equations constitute what is known as a **Galilean transformation of coordinates.** Note that time is assumed to be the same in both inertial systems. That is,

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within the framework of classical mechanics, all clocks run at the same rate, regardless of their velocity, so that the time at which an event occurs for an observer in S is the same as the time for the same event in S'. As a consequence, the time interval between two successive events should be the same for both observers. Although this assumption may seem obvious, it turns out to be incorrect when treating situations in which v is comparable to the speed of light. This point of equal time intervals represents one of the most profound differences between Newtonian concepts and Einstein's special relativity.

Now suppose two events are separated by a distance dx and a time interval dt as measured by an observer in S. It follows from Equation 9.1 that the corresponding distance dx' measured by an observer in S' is $dx' = dx - v dt$, where dx is the distance between the two events measured by an observer in S. Because $dt = dt'$, we find that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

or

Galilean addition law for velocities •

$$u_x' = u_x - v \qquad [9.2]$$

where u_x and u_x' are the instantaneous velocities of the object relative to S and S', respectively. This result, which is called the **Galilean addition law for velocities** (or Galilean velocity transformation), is used in everyday observations and is consistent with our intuitive notion of time and space. As we will soon see, however, it leads to serious contradictions when applied to electromagnetic waves.



Albert A. Michelson (1852–1931)

A German American physicist, Michelson spent much of his life making accurate measurements of the speed of light. In 1907 he was the first American to be awarded the Nobel prize, which he received for his work in optics. His most famous experiment, conducted with Edward Morley in 1887, implied that it was impossible to measure the absolute velocity of the Earth with respect to the ether. *(AIP Emilio Segrè Visual Archives, Michelson Collection)*

9.2 • THE MICHELSON–MORLEY EXPERIMENT

Many experiments similar to the one described in the preceding section show us that the laws of mechanics are the same in all inertial frames of reference. When similar inquiries are made into the laws of other branches of physics, the results are contradictory. In particular, the laws of electricity and magnetism are found to depend on the frame of reference used. It might be argued that it is these laws that are wrong, but that is difficult to accept because the laws are in total agreement with all known experimental results. The Michelson–Morley experiment was an attempt to explain this dilemma.

The experiment stemmed from a misconception early physicists had concerning the manner in which light propagates. The properties of mechanical waves, such as water and sound waves, were well known, and all of these waves require a medium to support the disturbances. In the nineteenth century, physicists thought that electromagnetic waves also required a medium through which to propagate. They proposed that such a medium existed, and they named it the **luminiferous ether.** This ether was assumed to be present everywhere, even in a vacuum, and to have the unusual property of being a massless but rigid medium (a strange concept indeed!). In addition, it was found that the troublesome laws of electricity and magnetism took on their simplest forms in a frame of reference *at rest* with respect to the luminiferous ether; this frame was called the **absolute frame.** In any reference frame moving with respect to it, the laws of electricity and magnetism would remain valid but would have to be modified.

As a result of the importance attached to this absolute frame, experimental proof of its existence became of considerable interest in physics. However, all attempts to detect the presence of the ether (and hence the absolute frame) proved futile. The most famous experiment designed to show the presence of the ether was performed in 1887 by A. A. Michelson (1852–1931) and E. W. Morley (1838–1923). The objective was to determine the velocity of the Earth with respect to the ether, and the experimental tool used was a device called the interferometer, shown in Figure 9.3.

Suppose one arm of an interferometer (M_0M_2 in Fig. 9.3) were aligned along the direction of the motion of the Earth through space. The Earth moving through the ether would be equivalent to the ether flowing past the Earth in the opposite direction. This "ether wind" blowing in the direction opposite the Earth's motion should cause the speed of light, as measured in the Earth's frame of reference, to be $c - v$ as the light approaches mirror M_2 in Figure 9.3, and $c + v$ after reflection, where c is the speed of light in the ether frame and v is the speed of the Earth through space and hence the speed of the ether wind. The incident and reflected beams of light would recombine, and a pattern would form as the two beams interfere with each other.

During the experiment, the interference pattern was observed while the interferometer was rotated through an angle of 90°. The idea was that this rotation would change the speed of the ether wind along the direction of the arms of the interferometer, and as a consequence the interference pattern would shift slightly but measurably. Measurements failed to show any change in the pattern. The Michelson–Morley experiment was repeated by other researchers under varying conditions and at different locations, but the results were always the same: *no interference-pattern shift of the magnitude required was ever observed.*[2]

The negative result of the Michelson–Morley experiment not only contradicted the ether hypothesis; it also meant that it was impossible to measure the absolute (orbital) velocity of the Earth with respect to the ether frame. From a theoretical viewpoint, this meant that it was impossible to find the absolute frame. However, as we shall see in the next section, Einstein offered a postulate that places a different interpretation on the negative result. In later years, when more was known about the nature of light, the idea of an ether that permeates all space was relegated to the ash heap of worn-out concepts. **Light is now understood to be an electromagnetic wave that requires no medium for its propagation.** As a result, an ether in which light could travel became an unnecessary construct.

Modern versions of the Michelson–Morley experiment have compared the frequencies of resonant laser cavities of identical length, oriented at right angles to each other. Most recently, Doppler shift experiments using gamma rays emitted by a radioactive sample of ^{57}Fe have placed an upper limit of about 5 cm/s on ether wind velocity. These results have shown quite conclusively that the motion of the Earth has no effect on the speed of light!

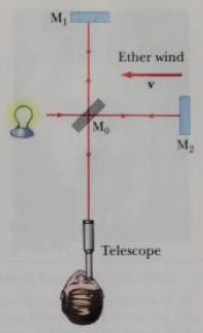


Figure 9.3 According to the ether wind theory, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

[2]From an Earth-observer's point of view, changes in the Earth's speed and direction of motion in the course of a year are viewed as ether wind shifts. Even if the speed of the Earth with respect to the ether were zero at some time, 6 months later the speed of the Earth would be 60 km/s with respect to the ether and a clear fringe shift should be found. None has ever been observed, however.

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9.3 • EINSTEIN'S PRINCIPLE OF RELATIVITY

In the previous section we noted the impossibility of measuring the speed of the ether with respect to the Earth and the failure of the Galilean addition law for velocities in the case of light. Albert Einstein proposed a theory that boldly removed these difficulties and at the same time completely altered our notion of space and time.[3] Einstein based special relativity on two postulates:

The postulates of special relativity •

1. **The Principle of Relativity:** All the laws of physics are the same in all inertial reference frames.
2. **The Constancy of the Speed of Light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

The first postulate asserts that *all* the laws of physics, those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on, are the same in all reference frames moving with constant velocity relative to each other. This postulate is a sweeping generalization of the principle of Newtonian relativity that only refers to the laws of mechanics. From an experimental point of view, Einstein's principle of relativity means that any kind of experiment (measuring the speed of light, for example), performed in a laboratory at rest must give the same result when performed in a laboratory moving at constant velocity past the first one. Hence no preferred inertial reference frame exists and it is impossible to detect absolute motion.

Note that postulate 2, the principle of the constancy of the speed of light, is required by postulate 1: If the speed of light were not the same in all inertial frames, it would be possible to distinguish between inertial frames and a preferred, absolute frame could be identified in contradiction to postulate 1. Postulate 2 also eliminates the problem of measuring the speed of the ether by denying the existence of the ether and boldly asserting that light always moves with speed c relative to all inertial observers.

Although the Michelson–Morley experiment was performed before Einstein published his work on relativity, it is not clear whether or not Einstein was aware of the details of the experiment. Nonetheless, the null result of the experiment can be readily understood within the framework of Einstein's theory. According to his principle of relativity, the premises of the Michelson–Morley experiment were incorrect. In the process of trying to explain the expected results, we stated that when light traveled against the ether wind its speed was $c - v$, in accordance with the Galilean addition law for velocities. However, if the state of motion of the observer or of the source has no influence on the value found for the speed of light, one will always measure the value to be c. Likewise, the light makes the return trip after reflection from the mirror at speed c, not at speed $c + v$. Thus, the motion of the Earth does not influence the interference pattern observed in the Michelson–Morley experiment, and a null result should be expected.

[3] A. Einstein, "On the Electrodynamics of Moving Bodies," *Ann. Physik* 17:891, 1905. For an English translation of this article and other publications by Einstein, see the book by H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity*, New York, Dover, 1958.

If we accept special relativity, we must conclude that relative motion is unimportant when measuring the speed of light. At the same time, we must alter our common-sense notion of space and time and be prepared for some bizarre consequences.

Thinking Physics 1

Imagine a very powerful lighthouse, with a rotating beacon. Imagine also drawing a horizontal circle around the lighthouse, with the lighthouse at the center. Along the circumference of the circle, the light beam lights up a portion of the circle and the lit portion of the circle moves around the circle at a certain tangential speed. If we now imagine a circle twice as big in radius, the tangential speed of the lit portion is faster, because it must travel a larger circumference in the time of one rotation of the light source. Imagine that we continue to make the circle larger and larger, eventually moving it out into space. The tangential speed of the lit portion will keep increasing. Is it possible that the tangential speed could become larger than the speed of light? Would this violate a principle of special relativity?

Reasoning For a large enough circle, it is possible that the tangential speed of the lit portion of the circle could be larger than the speed of light. This does not violate a principle of special relativity, however, because no matter or information is traveling faster than the speed of light.

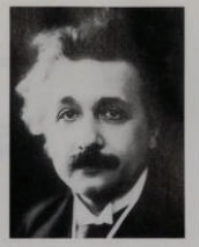
CONCEPTUAL PROBLEM 1

You are in a speedboat on a lake. You see ahead of you a wavefront, caused by the previous passage of another boat, moving away from you. You accelerate, catch up with, and pass the wavefront. Is this scenario possible if you are in a rocket and you detect a wavefront of *light* ahead of you?

9.4 • CONSEQUENCES OF SPECIAL RELATIVITY

Before we discuss the consequences of special relativity, we must first understand how an observer located in an inertial reference frame describes an event. As mentioned earlier, an event is an occurrence describable by three space coordinates and one time coordinate. Different observers in different inertial frames usually describe the same event with different space–time coordinates.

The reference frame used to describe an event consists of a coordinate grid and a set of synchronized clocks located at the grid intersections as shown in Figure 9.4 in two dimensions. The clocks can be synchronized in many ways with the help of light signals. For example, suppose the observer is located at the origin with his master clock and sends out a pulse of light at $t = 0$. The light pulse takes a time r/c to reach a second clock located a distance r from the origin. Hence, the second clock is synchronized with the clock at the origin if the second clock reads a time r/c at the instant the pulse reaches it. This procedure of synchronization assumes that the speed of light has the same value in all directions and in all inertial frames. Furthermore, the procedure concerns an event recorded by an observer in a specific inertial reference frame. An observer in some other inertial frame would assign different space–time coordinates to events being observed by using another coordinate grid and another array of clocks.



Albert Einstein (1879–1955)

One of the greatest physicists of all times, Einstein was born in Ulm, Germany. In 1905, at the age of 26, he published four scientific papers that revolutionized physics. Two of these papers were concerned with what is now considered his most important contribution of all, the special theory of relativity. In 1916, Einstein published his work on the general theory of relativity. The most dramatic prediction of this theory is the degree to which light is deflected by a gravitational field. Measurements made by astronomers on bright stars in the vicinity of the eclipsed Sun in 1919 confirmed Einstein's prediction, and Einstein suddenly became a world celebrity. Einstein was deeply disturbed by the development of quantum mechanics in the 1920s despite his own role as a scientific revolutionary. In particular, he could never accept the probabilistic view of events in nature that is a central feature of quantum theory. The last few decades of his life were devoted to an unsuccessful search for a unified theory that would combine gravitation and electromagnetism into one picture.

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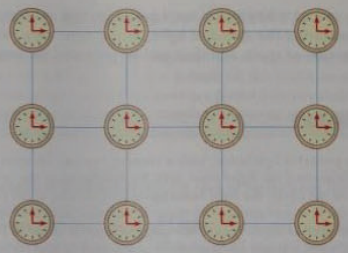


Figure 9.4 In relativity, we use a reference frame consisting of a coordinate grid and a set of synchronized clocks.

Almost everyone who has dabbled even superficially with science is aware of some of the startling predictions that arise because of Einstein's approach to relative motion. As we examine some of the consequences of relativity in the following three sections, we see that they conflict with our basic notions of space and time. We restrict our discussion to the concepts of length, time, and simultaneity, which are quite different in relativistic mechanics than in Newtonian mechanics. For example, **the distance between two points and the time interval between two events depend on the frame of reference in which they are measured.** That is, **there is no such thing as absolute length or absolute time in relativity.** Furthermore, **events at different locations that occur simultaneously in one frame are not simultaneous in another frame moving uniformly past the first.**

Simultaneity and the Relativity of Time

A basic premise of Newtonian mechanics is that a universal time scale exists that is the same for all observers. In fact, Newton wrote, "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external." Thus, Newton and his followers simply took simultaneity for granted. In his development of special relativity, Einstein abandoned this assumption. According to Einstein, **a time interval measurement depends on the reference frame in which the measurement is made.**

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike its ends, as in Figure 9.5a, leaving marks on the boxcar and on the ground. The marks on the boxcar are labeled A' and B', and those on the ground are labeled A and B. An observer at O' moving with the boxcar is midway between A' and B', and a ground observer at O is midway between A and B. The events recorded by the observers are the light signals from the lightning bolts.

The two light signals reach observer O at the same time, as indicated in Figure 9.5b. This observer realizes that the light signals have traveled at the same speed over equal distances, and so rightly concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by the observer on the boxcar at O'. The lightning strikes as A' passes A, O' passes O, and B' passes B. By

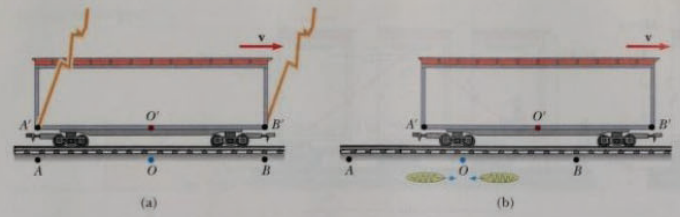


Figure 9.5 Two lightning bolts strike the end of a moving boxcar. (a) The events appear to be simultaneous to the stationary observer at O, who is midway between A and B. (b) The events do not appear to be simultaneous to the observer at O', who claims that the front of the train is struck before the rear.

the time the light has reached observer O, observer O' has moved as indicated in Figure 9.5b. Thus, the light signal from B' has already swept past O', and the light from A' has not yet reached O'. According to Einstein, **observer O' must find that light travels at the same speed as that measured by observer O.** Therefore, the observer O' concludes that the lightning struck the front of the boxcar before it struck the back. This thought experiment clearly demonstrates that the two events, which appear to be simultaneous to observer O, do not appear to be simultaneous to observer O'. In other words,

> two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first. That is, simultaneity is not an absolute concept but one that depends on the state of motion of the observer.

At this point, you might wonder which observer is right concerning the two events. The answer is that *both are correct*, because the principle of relativity states that **there is no preferred inertial frame of reference.** Although the two observers reach different conclusions, both are correct in their own reference frame because the concept of simultaneity is not absolute. This, in fact, is the central point of relativity—any uniformly moving frame of reference can be used to describe events and do physics. However, observers in different inertial frames of reference always measure different time intervals with their clocks and different distances with their meter sticks. Nevertheless, all observers agree on the forms of the laws of physics in their respective frames, because these laws must be the same for all observers in uniform motion. It is the alteration of time and space that allows the laws of physics (including Maxwell's equations) to be the same for all observers in uniform motion.

Time Dilation

The fact that observers in different inertial frames always measure different time intervals between a pair of events can be illustrated by considering a vehicle moving to the right with a speed v, as in Figure 9.6a. A mirror is fixed to the ceiling of the

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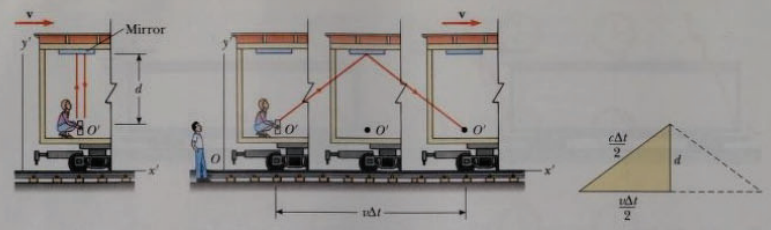


Figure 9.6 (a) A mirror is fixed to a moving vehicle, and a light pulse leaves O' at rest in the vehicle. (b) Relative to a stationary observer on Earth, the mirror and O' move with a speed v. Note that the distance the pulse travels is greater than $2d$ as measured by the stationary observer. (c) The right triangle for calculating the relationship between Δt and $\Delta t'$.

vehicle and observer O' at rest in this system holds a laser a distance d below the mirror. At some instant, the laser emits a pulse of light directed towards the mirror (event 1), and at some later time after reflecting from the mirror, the pulse arrives back at the laser (event 2). Observer O' carries a clock C' that she uses to measure the time interval Δt_p between these two events. Because the light pulse has a speed c, the time it takes the pulse to travel from O' to the mirror and back to O' can be found from the definition of speed:

$$\Delta t_p = \frac{\text{distance traveled}}{\text{speed}} = \frac{2d}{c} \qquad [9.3]$$

This time interval Δt_p measured by O', who is at rest in the moving vehicle, requires only a single clock C' located at the same place in this frame.

Now consider the same pair of events as viewed by observer O in a second frame, as in Figure 9.6b. According to this observer, the mirror and laser are moving to the right with a speed v. The sequence of events appears entirely different as viewed by this observer. By the time the light from the laser reaches the mirror, the mirror has moved a distance $v\,\Delta t/2$, where Δt is the time it takes the light to travel from O' to the mirror and back to O' as measured by observer O. In other words, the second observer concludes that, because of the motion of the vehicle, if the light is to hit the mirror, it must leave the laser at an angle with respect to the vertical direction. Comparing Figures 9.6a and 9.6b, we see that the light must travel farther in the second frame than in the first frame.

According to the second postulate of special relativity, both observers must measure c for the speed of light. Because the light travels farther in the second frame, it follows that the time interval Δt measured by the observer in the second frame is longer than the time interval Δt_p measured by the observer in the first frame. To obtain a relationship between these two time intervals, it is convenient to use the right triangle shown in Figure 9.6c. The Pythagorean theorem gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for Δt gives

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - \frac{v^2}{c^2}}} \qquad [9.4]$$

Because $\Delta t_p = 2d/c$, we can express Equation 9.4 as

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\,\Delta t_p \qquad [9.5]$$

• ***Time dilation***

where $\gamma = (1 - v^2/c^2)^{-1/2}$. This result says that **the time interval Δt measured by an observer moving with respect to the clock is longer than the time interval Δt_p measured by an observer at rest with respect to the clock** because γ is always greater than unity. That is, $\Delta t > \Delta t_p$. This effect is known as **time dilation.**

The time interval Δt_p in Equation 9.5 is called the **proper time.** In general, proper time is defined as **the time interval between two events measured by an observer who sees the events occur at the same point in space.** In our case, observer O' measures the proper time. That is, **proper time is always the time interval measured with a single clock at rest in the frame in which the events take place at the same position.**

• ***Proper time***

Because the time between ticks of a moving clock, $\gamma(2d/c)$, is observed to be longer than the time between ticks of an identical clock at rest, $2d/c$, it is often said, "*A moving clock runs slower than a clock at rest by a factor* γ." This is true for ordinary mechanical clocks as well as for the light clock just described. In fact, we can generalize these results by stating that **all physical processes, including chemical and biological ones, slow down relative to a stationary clock when those processes occur in a moving frame.** For example, the heartbeat of an astronaut moving through space would keep time with a clock inside the spaceship. Both the astronaut's clock and heartbeat are slowed down relative to a stationary clock. The astronaut would not have any sensation of life slowing down in the spaceship. For the astronaut, it is the clock on Earth and the companions at Mission Control that are moving and therefore keep a slow time.

Time dilation is a verifiable phenomenon. For example, muons are unstable elementary particles that have a charge equal to that of the electron and a mass 207 times that of the electron. Muons can be produced by the collision of cosmic radiation with atoms high in the atmosphere. Slow-moving muons have a lifetime of only 2.2 μs in the laboratory. If we take 2.2 μs as the average lifetime of a muon and assume that its speed is close to the speed of light, we find that these particles can travel a distance of only approximately 600 m before they decay (Fig. 9.7a). Hence, they cannot reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons *do* reach the Earth. The phenomenon of time dilation explains this effect. Relative to an observer on Earth, the muons have a lifetime equal to $\gamma\tau$, where $\tau = 2.2$ μs is the lifetime in a frame of reference traveling with the muons. For example, for $v = 0.99c$, $\gamma \approx 7.1$ and $\gamma\tau \approx 16$ μs. Hence, the average distance traveled as measured by an observer on Earth is $\gamma v\tau \approx 4800$ m, as indicated in Figure 9.7b.

In 1976, at the laboratory of the European Council for Nuclear Research in

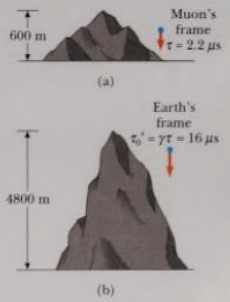


Figure 9.7 (a) Muons traveling with a speed of $0.99c$ travel only approximately 600 m as measured in the muons' reference frame, where their proper lifetime is about 2.2 μs. (b) The muons travel approximately 4800 m as measured by an observer on Earth. Because of time dilation, the muons' lifetime is longer as measured by the Earth observer.

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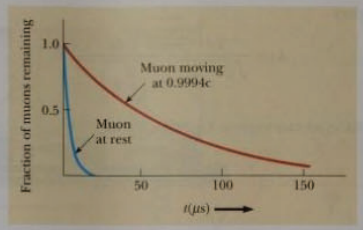


Figure 9.8 Decay curves for muons traveling at a speed of $0.9994c$ and for muons at rest.

Geneva, muons injected into a large storage ring reached speeds of approximately $0.9994c$. Electrons produced by the decaying muons were detected by counters around the ring, enabling scientists to measure the decay rate and hence the muon lifetime. The lifetime of the moving muons was measured to be approximately 30 times as long as that of the stationary muon (Fig. 9.8), in agreement with the prediction of relativity to within 2 parts in 1000.

The results of an experiment reported by Hafele and Keating provided direct evidence of time dilation.[4] The experiment involved the use of very stable cesium-beam atomic clocks. Time intervals measured with four such clocks in jet flight were compared with time intervals measured by reference clocks located at the U.S. Naval Observatory. To compare these results with the theory, many factors had to be considered, including periods of acceleration and deceleration relative to the Earth, variations in direction of travel, and the weaker gravitational field experienced by the flying clocks compared to the Earth-based clock. Their results were in good agreement with the predictions of special relativity and can be explained in terms of the relative motion between the Earth's rotation and the jet aircraft. In their paper, Hafele and Keating report the following: "Relative to the atomic time scale of the U.S. Naval Observatory, the flying clocks lost 59 ± 10 ns during the eastward trip and gained 273 ± 7 ns during the westward trip. . . . These results provide an unambiguous empirical resolution of the famous clock paradox with macroscopic clocks."

Thinking Physics 2

Suppose a student explains time dilation with the following argument: If you start running at $0.99c$ away from a clock at 12:00, you would not see the time change, because the light from the clock representing 12:01 would never reach you. What is the flaw in this argument?

Reasoning The inference in this argument is that the velocity of light relative to the runner is approximately *zero*—"the light . . . would never reach you." This is a Galilean point of view, in which the relative velocity is a simple subtraction of running velocity from the light velocity. From the point of view of special relativity, one of the fundamental postulates is that the speed of light is the same for all observers, *including one running away from the light source at the speed of light.* Thus, the light from 12:01 will move toward the runner at the speed of light.

[4] J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Relativistic Time Gains Observed," *Science*, July 14, 1972, p. 168.

Example 9.1 What Is the Period of the Pendulum?

The period of a pendulum is measured to be 3.0 s in the rest frame of the pendulum's pivot. What is the period when measured by an observer moving at a speed of $0.95c$ relative to the pendulum?

Reasoning Instead of the observer moving at $0.95c$, we can take the equivalent point of view that the observer is at rest and that the pendulum is moving at $0.95c$ past the stationary observer. Hence the pendulum is an example of a moving clock.

Solution The proper time is 3.0 s. Because a moving clock runs slower than a stationary clock by γ, Equation 9.5 gives

$$T = \gamma T_p = \frac{1}{\sqrt{1 - \dfrac{(0.95c)^2}{c^2}}}\, T_p = (3.2)(3.0\text{ s}) = 9.6\text{ s}$$

That is, a moving pendulum slows down or takes longer to complete a period compared to one at rest.

EXERCISE 1 Muons move in circular orbits at a speed of $0.9994c$ in a storage ring of radius 500 m. If a muon at rest decays into other particles after 2.2 μs (proper time), how many trips around the storage ring do we expect the muons to make before they decay?
Answer 6.1 revolutions

Length Contraction

The measured distance between two points also depends on the frame of reference. The **proper length** of an object is defined as **the length of the object measured by someone who is at rest relative to the object.** The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as **length contraction.**

• *Proper length*

Consider a spaceship traveling with a speed v from one star to another. There are two observers, one on Earth and the other in the spaceship. The observer at rest on Earth (and also assumed to be at rest with respect to the two stars) measures the distance between the stars to be L_p, the proper length. According to this observer, the time it takes the spaceship to complete the voyage is $\Delta t = L_p / v$. What does an observer in the moving spaceship measure for the distance between the stars? Because of time dilation, the space traveler measures a smaller time of travel: $\Delta t_p = \Delta t / \gamma$. The space traveler claims to be at rest and sees the destination star moving toward the spaceship with speed v. Because the space traveler reaches the star in the time Δt_p, she or he concludes that the distance, L, between the stars is shorter than L_p. This distance measured by the space traveler is

$$L = v\,\Delta t_p = v\,\frac{\Delta t}{\gamma}$$

Because $L_p = v\,\Delta t$, we see that $L = L_p / \gamma$ or

$$L = L_p \left(1 - \frac{v^2}{c^2}\right)^{1/2} \qquad [9.6]$$

• *Length contraction*

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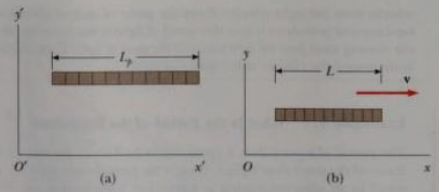


Figure 9.9 (a) A stick as viewed by an observer in a frame attached to the stick (i.e., both have the same velocity). (b) The stick as seen by an observer in a frame in which the stick has a velocity **v** relative to the frame. The length is *shorter* than the proper length, L_p, by a factor $(1 - v^2/c^2)^{1/2}$.

where $(1 - v^2/c^2)^{1/2}$ is a factor less than one. This result may be interpreted as follows:

> A moving observer measures the length L of an object (along the direction of motion) to be shorter than the length L_p measured by an observer at rest with respect to the object (the proper length).

Note that **length contraction takes place only along the direction of motion.** For example, suppose a stick moves past an Earth observer with speed v as in Figure 9.9. The length of the stick as measured by an observer in a frame attached to the stick is the proper length, L_p, as in Figure 9.9a. The length of the stick, L, measured by the Earth observer is shorter than L_p by the factor $(1 - v^2/c^2)^{1/2}$. Furthermore, length contraction is a symmetrical effect: If the stick is at rest on Earth, an observer in the moving frame would also measure its length to be shorter by the same factor $(1 - v^2/c^2)^{1/2}$.

It is important to emphasize that proper length and proper time are measured in different reference frames. As an example of this point, let us return to the decaying muons moving at speeds close to the speed of light. An observer in the muon's reference frame would measure the proper lifetime, whereas an Earth-based observer would measure the proper height of the mountain in Figure 9.7. In the muon's reference frame, there is no time dilation, but the distance of travel is observed to be shorter when measured in this frame. Likewise, in the Earth-based observer's reference frame, there is time dilation, but the distance of travel is measured to be the proper height of the mountain. Thus, when calculations on the muon are performed in both frames, the outcome of the experiment in one frame is the same as the outcome in the other frame!

Example 9.2 A Voyage to Sirius

An astronaut takes a trip to Sirius, located 8 lightyears from Earth. The astronaut measures the time of the one-way journey to be 6 y. If the spaceship moved at a constant speed of $0.8c$, how can the 8-lightyear distance be reconciled with the 6-y duration measured by the astronaut?

Reasoning The 8 lightyears (ly) represents the proper length (distance) from Earth to Sirius measured by an observer seeing both nearly at rest. The astronaut sees Sirius approaching her at $0.8c$ but also sees the distance contracted to

$$\frac{8 \text{ ly}}{\gamma} = (8 \text{ ly})\sqrt{1 - \frac{v^2}{c^2}} = (8 \text{ ly})\sqrt{1 - \frac{(0.8c)^2}{c^2}} = 5 \text{ ly}$$

So the travel time measured on the astronaut's clock is

$$\Delta t = \frac{d}{v} = \frac{5 \text{ ly}}{0.8c} = 6 \text{ y}$$

Example 9.3 The Triangular Spaceship

A spaceship in the form of a triangle flies by an observer with a speed of $0.95c$ along the x-direction. When the ship is at rest relative to the observer (Fig. 9.10a), the distances x and y are measured to be 52 m and 25 m, respectively. What is the shape of the ship as seen by an observer at rest when the ship is in motion along the direction shown in Figure 9.10b?

Solution The observer sees the horizontal length of the ship to be contracted to a length

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} = (52 \text{ m})\sqrt{1 - \frac{(0.95c)^2}{c^2}} = 16 \text{ m}$$

The 25-m vertical height is unchanged because it is perpendicular to the direction of relative motion between observer and spaceship. Figure 9.10b represents the shape of the spaceship as seen by the observer at rest.



Figure 9.10 (Example 9.3) (a) When the spaceship is at rest, its shape is as shown. (b) The spaceship appears to look like this when it moves to the right with a speed v. Note that only its x dimension is contracted in this case.

EXERCISE 2 A spacecraft moves at $0.9c$. If its length is L_0 when measured from inside the spacecraft, what is its length measured by a ground observer? Answer $0.436L_0$

EXERCISE 3 A spaceship is measured to be 120 m long while at rest relative to an observer. If this spaceship now flies by the observer with a speed $0.99c$, what length does the observer measure? Answer 17 m

The Twins Paradox

OPTIONAL

An intriguing consequence of time dilation is the so-called twins paradox. Consider an experiment involving a set of twins named Speedo and Goslo who are, say, 20 years old. The twins carry with them identical clocks that have been synchronized (Fig. 9.11). Speedo, the more adventuresome of the two, sets out on an epic journey



Figure 9.11 (a) As the twins depart, they are the same age. (b) When Speedo returns from his journey to Planet X, he is younger than his twin Goslo who remained on Earth.

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to Planet X, located 10 lightyears from Earth. Furthermore, his spaceship is capable of reaching a speed of $0.500c$ relative to the inertial frame of his twin brother on Earth. After reaching Planet X, Speedo becomes homesick and immediately returns to Earth at the same high speed he had attained on the outbound journey. On his return, Speedo is shocked to discover that many things have changed in his absence. To Speedo, the most significant change to have occurred is that his twin brother Goslo has aged 40 years and is now 60 years old. Speedo, on the other hand, has aged by only 34.6 years.

At this point, it is fair to raise the following question—which twin is the traveler and which twin is really younger as a result of this experiment? From Goslo's frame of reference, he was at rest while his brother traveled at a high speed. From Speedo's perspective, it is he who is at rest while Goslo is on the high-speed space journey. According to Speedo, it is Goslo and the Earth that have raced away on a 17.3-year journey and then headed back for another 17.3 years. This leads to an apparent contradiction. Which twin has developed signs of excess aging?

To resolve this apparent paradox, recall that relative to any chosen inertial frame (like that of the stay-at-home twin), clocks in a moving object's frame move more slowly. However, the trip situation is not symmetrical. Speedo, the space traveler, must experience a series of accelerations during his journey. As a result, his speed is not always uniform and as a consequence Speedo is not in a single inertial frame. He cannot be regarded as always being at rest and Goslo to be in uniform motion because he changes his state of motion at least once. Therefore there is no paradox.

The conclusion that Speedo is in a noninertial frame is inescapable. The time required to accelerate and decelerate Speedo's spaceship may be made very small using large rockets, so that Speedo can claim that he spends most of his time traveling to Planet X at $0.500c$ in an inertial frame. However, Speedo must slow down, reverse his motion, and return to Earth in an altogether different inertial frame. At the very best, Speedo is in two different inertial frames during his journey. Only Goslo, who is in a single inertial frame, can apply the simple time dilation formula to Speedo's trip. Thus, Goslo finds that instead of aging 40 years, Speedo ages only $(1 - v^2/c^2)^{1/2}(40 \text{ years}) = 34.6$ years. However, Speedo spends 17.3 years traveling to Planet X and 17.3 years returning, for a total travel time of 34.6 years, in agreement with our earlier statement.

CONCEPTUAL PROBLEM 2

You are packing for a trip to another star, to which you will be traveling at $0.99c$. Should you buy smaller sizes of your clothing, since you will be skinnier on your trip? Can you sleep in a smaller cabin than usual, since you will be shorter when you lie down?

CONCEPTUAL PROBLEM 3

Suppose astronauts were paid according to the time spent traveling in space. After a long voyage traveling at a speed near that of light, a crew of astronauts return to Earth and open their pay envelopes. What will their reaction be?

CONCEPTUAL PROBLEM 4

You are observing a rocket moving away from you. You notice that it is measured to be shorter than when it was at rest on the ground next to you, and, through the rocket window, you can see a clock. You observe that the passage of time on the clock is measured to be slower than that of the watch on your wrist. What if the rocket turns around and comes toward you? Will it appear to be *longer* and will the rocketbound clock move *faster*?

9.5 · THE LORENTZ TRANSFORMATION EQUATIONS

We have seen that the Galilean transformation is not valid when v approaches the speed of light. In this section, we state the correct transformation equations that apply for all speeds in the range $0 \leq v < c$.

Suppose an event that occurs at some point P is reported by two observers, one at rest in a frame S and another in a frame S' that is moving to the right with speed v as in Figure 9.12. The observer in S reports the event with space–time coordinates (x, y, z, t), and the observer in S' reports the same event using the coordinates (x', y', z', t'). We would like to find a relationship between these coordinates that is valid for all speeds. In Section 9.1, we found that the Galilean transformation of coordinates, given by Equation 9.1, does not agree with experiment at speeds comparable to the speed of light.

The equations that are valid from $v = 0$ to $v = c$ and enable us to transform coordinates from S to S' are given by the **Lorentz transformation equations:**

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Lorentz transformation for $S \rightarrow S'$ [9.7]

These transformation equations, known as the Lorentz transformation, were developed by Hendrik A. Lorentz (1853–1928) in 1890 in connection with electromagnetism. However, it was Einstein who recognized their physical significance and took the bold step of interpreting them within the framework of special relativity.

We see that the value for t' assigned to an event by an observer standing at O' depends both on the time t and on the coordinate x as measured by an observer at O. This is consistent with the notion that an event is characterized by four space–time coordinates (x, y, z, t). In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other. This is unlike the case of the Galilean transformation in which $t = t'$.

If we wish to transform coordinates in the S' frame to coordinates in the S frame, we simply replace v by $-v$ and interchange the primed and unprimed coordinates in Equation 9.7:

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Inverse Lorentz transformation for $S' \rightarrow S$ [9.8]

When $v \ll c$, the Lorentz transformation should reduce to the Galilean transformation. To check this, note that as $v \rightarrow 0$, $v/c^2 \ll 1$ and $v^2/c^2 \ll 1$, so that

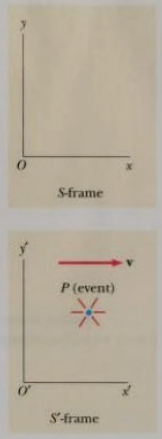


Figure 9.12 Representation of an event that occurs at some point P as observed by an observer at rest in the S frame and another in the S' frame, which is moving to the right with a speed v.

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$\gamma = 1$ and Equation 9.7 reduces in this limit to the Galilean coordinate transformation equations

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

In many situations, we would like to know the difference in coordinates between two events or the time interval between two events as seen by observers at O and O'. This can be accomplished by writing the Lorentz equations in a form suitable for describing pairs of events. From Equations 9.7 and 9.8, we can express the differences between the four variables x, x', t, and t' in the form

$$\Delta x' = \gamma(\Delta x - v\,\Delta t)$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \qquad S \rightarrow S' \qquad [9.9]$$

$$\Delta x = \gamma(\Delta x' + v\,\Delta t')$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \qquad S' \rightarrow S \qquad [9.10]$$

where $\Delta x' = x_2' - x_1'$ and $\Delta t' = t_2' - t_1'$ are the differences measured by the observer at O', and $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ are the differences measured by the observer at O. We have not included the expressions for relating the y and z coordinates because they are unaffected by motion along the x direction.[5]

Example 9.4 Simultaneity and Time Dilation Revisited

Use the Lorentz transformation equations in difference form to show that (a) simultaneity is not an absolute concept and (b) moving clocks run slower than stationary clocks.

Solution (a) Suppose that two events are simultaneous according to a moving observer at O', so that $\Delta t' = 0$. From the expression for Δt given in Equation 9.10, we see that in this case, $\Delta t = \gamma v\,\Delta x'/c^2$. That is, the time interval for the same two events as measured by an observer at O is nonzero, and so they do not appear to be simultaneous in O.

(b) Suppose that an observer at O' finds that two events occur at the same place ($\Delta x' = 0$), but at different times ($\Delta t' \neq 0$). In this situation, the expression for Δt given in Equation 9.10 becomes $\Delta t = \gamma\,\Delta t'$. This is the equation for time dilation found earlier, Equation 9.5, where $\Delta t' = \Delta t$ is the proper time measured by the single clock located at O'.

EXERCISE 4 Use the Lorentz transformation equations in difference form to confirm that $L = L_p/\gamma$.

Lorentz Velocity Transformation

Let us now derive the Lorentz velocity transformation, which is the relativistic counterpart of the Galilean velocity transformation. Once again S is our stationary frame of reference and S' is our frame of reference that moves at a speed v relative to S. Suppose that an object is observed in the S' frame with an instantaneous speed u_x' measured in S' given by

$$u_x' = \frac{dx'}{dt'} \qquad [9.11]$$

[5]Although motion along x does not change y and z coordinates, it does change velocity components along y and z.

Using Equations 9.7, we have

$$dx' = \gamma(dx - v\,dt) \qquad \text{and} \qquad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

Substituting these values into Equation 9.11 gives

$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

But dx/dt is just the velocity component u_x of the object measured in S, and so this expression becomes

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad [9.12]$$

• *Lorentz velocity transformation for $S \rightarrow S'$*

In a similar way, if the object has velocity components along y and z, the components in S' are

$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad \text{and} \qquad u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad [9.13]$$

When u_x and v are both much smaller than c (the nonrelativistic case), the denominator of Equation 9.12 approaches unity and so $u_x' \approx u_x - v$. This corresponds to the Galilean velocity transformations. In the other extreme, when $u_x = c$, Equation 9.12 becomes

$$u_x' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c\left(1 - \frac{v}{c}\right)}{1 - \frac{v}{c}} = c$$

From this result, we see that an object whose speed approaches c relative to an observer in S also has a speed approaching c relative to an observer in S'—independent of the relative motion of S and S'. Note that this conclusion is consistent with Einstein's second postulate—namely, that the speed of light must be c relative to all inertial frames of reference. Furthermore, the speed of an object can never exceed c. That is, the speed of light is the ultimate speed. We return to this point later when we consider the energy of a particle.



The speed of light is the speed limit of the Universe.

To obtain u_x in terms of u_x', we replace v by $-v$ in Equation 9.12 and interchange the roles of u_x and u_x':

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} \qquad [9.14]$$

• *Inverse Lorentz velocity transformation for $S' \rightarrow S$*

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Example 9.5 Relative Velocity of Spaceships

Two spaceships A and B are moving directly toward each other, as in Figure 9.13. An observer on Earth measures the speed of A to be $0.750c$ and the speed of B to be $0.850c$. Find the velocity of B with respect to A.

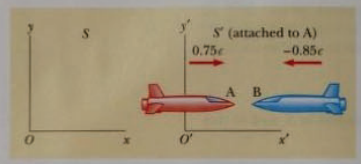


Figure 9.13 (Example 9.5) Two spaceships A and B move in opposite directions. The speed of B relative to A is *less* than c and is obtained by using the relativistic velocity transformation.

Solution This problem can be solved by taking the S' frame as being attached to A, so that $v = 0.750c$ relative to the Earth observer (the S frame). Spaceship B can be considered as an object moving with a velocity $u_x = -0.850c$ relative to the Earth observer. Hence, the velocity of B with respect to A can be obtained using Equation 9.12:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \frac{(-0.850c)(0.750c)}{c^2}} = -0.980c$$

The negative sign indicates that spaceship B is moving in the negative x direction as observed by A. Note that the result is less than c. That is, a body whose speed is less than c in one frame of reference must have a speed less than c in any other frame. (If the Galilean velocity transformation were used in this example, we would find that $u'_x = u_x - v = -0.850c - 0.750c = -1.60c$, which is greater than c. The Galilean transformation does not work in relativistic situations.)

Example 9.6 Relativistic Leaders of the Pack

Two motorcycle pack leaders named David and Emily are racing at relativistic speeds along perpendicular paths, as in Figure 9.14. How fast does Emily recede over David's right shoulder as seen by David?

Solution Figure 9.14 represents the situation as seen by a police officer at rest in frame S, who observes the following:

David: $u_x = 0.75c \qquad u_y = 0$

Emily: $u_x = 0 \qquad u_y = -0.90c$

To get Emily's speed of recession as seen by David, we take S' to move along with David and we calculate u'_x and u'_y for Emily using Equations 9.12 and 9.13:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{0 - 0.75c}{1 - \frac{(0)(0.75c)}{c^2}} = -0.75c$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{\sqrt{1 - \frac{(0.75c)^2}{c^2}}(-0.90c)}{\left(1 - \frac{(0)(0.75c)}{c^2}\right)} = -0.60c$$

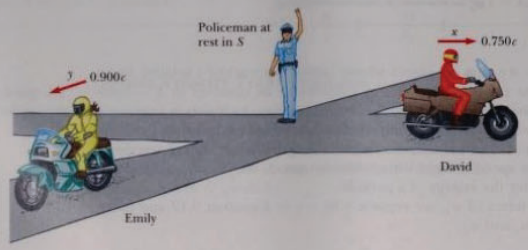


Figure 9.14 (Example 9.6) David moves to the east with a speed $0.750c$ relative to the policeman, while Emily travels south at a speed $0.900c$.

Thus, the speed of Emily as observed by David is

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = \sqrt{(-0.75c)^2 + (-0.60c)^2} = 0.96c$$

Note that this speed is less than c as required by special relativity.

EXERCISE 5 Calculate the classical speed of recession for Emily as observed by David using a Galilean transformation.
Answer $1.2c$

EXERCISE 6 A Klingon spaceship moves away from the Earth at a speed of $0.80c$. The starship *Enterprise* pursues at a speed of $0.90c$ relative to the Earth. Observers on Earth see the *Enterprise* overtaking the Klingon ship at a relative speed of $0.10c$. With what speed is the *Enterprise* overtaking the Klingon ship as seen by the crew of the *Enterprise*?
Answer $0.36c$

9.6 · RELATIVISTIC MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

We have seen that in order to properly describe the motion of particles within the framework of special relativity, the Galilean transformation must be replaced by the Lorentz transformation. Because the laws of physics must remain unchanged under the Lorentz transformation, we must generalize Newton's laws and the definitions of momentum and energy to conform to the Lorentz transform and the principle of relativity. These generalized definitions should reduce to the classical (nonrelativistic) definitions for $v \ll c$.

First, recall that the law of conservation of momentum states that when two bodies collide, the total momentum remains constant, assuming the bodies are isolated. Suppose the collision is described in a reference frame S in which momentum is conserved. If the velocities in a second reference frame S' are calculated using the Lorentz transformation and the classical definition of momentum, $\mathbf{p} = m\mathbf{u}$, it is found that momentum is *not* conserved in the second reference frame. (We use the symbol $\mathbf{u}$ for particle velocity rather than $\mathbf{v}$, which is used for the relative velocity of two reference frames.) However, because the laws of physics are the same in all inertial frames, the momentum must be conserved in all systems. In view of this condition and assuming the Lorentz transformation is correct, we must modify the definition of momentum to satisfy the following conditions:

- $\mathbf{p}$ must be conserved in all collisions
- $\mathbf{p}$ must approach the classical value $m\mathbf{u}$ as $\mathbf{u} \to 0$

The correct relativistic equation for momentum that satisfies these conditions is

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.15]$$

· *Definition of relativistic momentum*

where $\mathbf{u}$ is the velocity of the particle. When u is much less than c, the denominator of Equation 9.15 approaches unity, so that $\mathbf{p}$ approaches $m\mathbf{u}$. Therefore, the rela-

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tivistic equation for $\mathbf{p}$ reduces to the classical expression when u is small compared with c. Because it is simpler, Equation 9.15 is often written as

$$\mathbf{p} = \gamma m\mathbf{u} \qquad [9.16]$$

where $\gamma = (1 - u^2/c^2)^{-1/2}$. Note that γ has the same functional form as the γ in the Lorentz transformation. The transformation is from that of the particle to the frame of the observer moving at speed u relative to the particle.

The relativistic force $\mathbf{F}$ on a particle whose momentum is $\mathbf{p}$ is defined as

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \qquad [9.17]$$

where $\mathbf{p}$ is given by Equation 9.15. This expression is reasonable because it preserves classical mechanics in the limit of low velocities and requires conservation of momentum for an isolated system ($\mathbf{F} = 0$) both relativistically and classically.

It is left as an end-of-chapter problem (Problem 47) to show that the acceleration $\mathbf{a}$ of a particle decreases under the action of a constant force, in which case $a \propto (1 - u^2/c^2)^{3/2}$. From this formula note that as the particle's speed approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed $v \geq c$.

Example 9.7 Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

Solution Using Equation 9.15 with $u = 0.75c$, we have

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$= \frac{(9.11 \times 10^{-31}\ \text{kg})(0.750 \times 3.00 \times 10^{8}\ \text{m/s})}{\sqrt{1 - \frac{(0.750c)^2}{c^2}}}$$

$$= 3.10 \times 10^{-22}\ \text{kg·m/s}$$

The incorrect classical expression gives

$$\text{Momentum} = m_e u = 2.05 \times 10^{-22}\ \text{kg·m/s}$$

Hence, the correct relativistic result is 50% greater than the classical result!

9.7 • RELATIVISTIC ENERGY

We have seen that the definition of momentum and the laws of motion require generalization to make them compatible with the principle of relativity. This implies that the definition of kinetic energy must also be modified.

To derive the relativistic form of the work-kinetic energy theorem, let us start with the definition of the work done on a particle by a force F and use the definition of relativistic force, Equation 9.17:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \qquad [9.18]$$

for force and motion both along the x axis. In order to perform this integration, and find the work done on a particle, or the relativistic kinetic energy as a function of u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m(du/dt)}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 9.18 gives

$$W = \int_0^t \frac{m(du/dt)\,u\,dt}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} = m\int_0^u \frac{u}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}\,du$$

where we have assumed that the particle is accelerated from rest to some final speed u. Evaluating the integral, we find that

$$W = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \qquad [9.19]$$

Recall from Chapter 6 that the work done by a force acting on a particle equals the change in kinetic energy of the particle. Because the initial kinetic energy is zero, we conclude that the work W is equivalent to the relativistic kinetic energy K:

$$K = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2 \qquad [9.20]$$

• *Relativistic kinetic energy*

This equation is routinely confirmed by experiments using high-energy particle accelerators.

At low speeds, where $u/c \ll 1$, Equation 9.20 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \ \ldots$ for $x \ll 1$, where the higher-order powers of x are neglected in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

Substituting this into Equation 9.20 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 = \frac{1}{2}mu^2$$

which agrees with the classical result. Figure 9.15 shows a comparison of the speed–kinetic energy relationships for a particle using the nonrelativistic expression for K (the blue curve) and the relativistic expression for K (the brown curve). The curves are in good agreement at low speeds but deviate at higher speeds. The nonrelativistic expression indicates a violation of physical law because it suggests that sufficient energy can be added to the particle to accelerate it to a speed larger than c. In the relativistic case, the particle speed never exceeds c, regardless of the kinetic energy. When an object's speed is less than one tenth the speed of light, the classical kinetic energy equation differs by less than 1% from the relativistic

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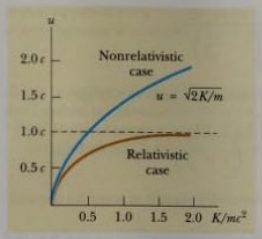


Figure 9.15 A graph comparing relativistic and nonrelativistic kinetic energy. The speeds are plotted versus energy. In the relativistic case, u is always less than c.

equation (which is experimentally verified at all speeds). Thus for practical calculations it is valid to use the classical equation when the object's speed is less than $0.1c$.

The constant term mc^2 in Equation 9.20, which is independent of the speed, is called the **rest energy** of the free particle E_R.

Rest energy

$$E_R = mc^2 \qquad [9.21]$$

The term γmc^2 in Equation 9.20 depends on the particle speed and is the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E—that is, total energy = kinetic energy + rest energy, or

$$E = \gamma mc^2 = K + mc^2 \qquad [9.22]$$

or, when γ is replaced by its equivalent,

Definition of total energy

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.23]$$

This, of course, is Einstein's famous mass–energy equivalence equation. The relation $E = \gamma mc^2 = \gamma E_R$ shows that **mass is one possible manifestation of energy.** Furthermore, this result shows that a small mass corresponds to an enormous amount of energy. This concept is fundamental to much of the field of nuclear physics.

In many situations, the momentum or energy of a particle is measured instead of its speed. It is therefore useful to have an expression relating the total energy E to the relativistic momentum p. This is accomplished by using the expressions $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 35). The result, after some algebra, is

Energy–momentum relationship

$$E^2 = p^2c^2 + (mc^2)^2 \qquad [9.24]$$

When the particle is at rest, $p = 0$, and so $E = E_R = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons

(massless, chargeless particles of light), we set $m = 0$ in Equation 9.24, and we see that

$$E = pc \qquad [9.25]$$

Energy of a photon

This equation is an exact expression relating energy and momentum for photons, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. For this reason, m is often called the *invariant mass*. However, the total energy and momentum of a particle depend on the reference frame in which they are measured, because they both depend on velocity. Because m is a constant, then according to Equation 9.24 the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is invariant (its value remains the same) under a Lorentz transformation. These equations do not yet make provision for potential energy.

When dealing with subatomic particles, it is convenient to express their energy in electron volts (eV), because the particles are usually given this energy in particle accelerators. The conversion factor is

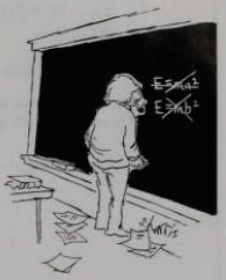
$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

For example, the mass of an electron is 9.11×10^{-31} kg. Hence, the rest energy of the electron is

$$m_e c^2 = (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 8.20 \times 10^{-14} \text{ J}$$

Converting this to eV, we have

$$m_e c^2 = (8.20 \times 10^{-14} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J}) = 0.511 \text{ MeV}$$



Thinking Physics 3

A common principle learned in chemistry is conservation of mass. In practice, if the mass of the reactants is measured before a reaction and the mass of the products is measured afterward, the results will be the same. In light of special relativity, should we stop teaching the principle of conservation of mass in chemistry classes?

Reasoning Consider a reaction that does not require energy input to occur. This type of reaction occurs because the products represent a lower overall rest energy than the reactants, the difference in rest energy carried away as kinetic energy of ejected particles or radiation. Because the rest energy of the reactants is smaller, according to relativity the mass of the reactants should be smaller than that of the products. Thus the law of conservation of mass is violated. The mass changes are so small, however, that, in practice the law of conservation of mass is still useful.

CONCEPTUAL PROBLEM 5

A photon has zero mass. If a photon is reflected from a surface, does it exert a force on the surface?

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Example 9.8 The Energy of a Speedy Proton

The total energy of a proton is three times its rest energy. (a) Find the proton's rest energy in electron volts.

Solution

$$E_R = m_p c^2 = (1.67 \times 10^{-27}\ \text{kg})(3.00 \times 10^8\ \text{m/s})^2$$
$$= (1.50 \times 10^{-10}\ \text{J})(1.00\ \text{eV}/1.60 \times 10^{-19}\ \text{J})$$
$$= 938\ \text{MeV}$$

(b) With what speed is the proton moving?

Solution Because the total energy E is three times the rest energy, $E = \gamma mc^2$ (Eq. 9.23) gives

$$E = 3m_p c^2 = \frac{m_p c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$3 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Solving for u gives

$$\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{9} \quad \text{or} \quad \frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = 2.83 \times 10^8\ \text{m/s}$$

(c) Determine the kinetic energy of the proton in electron volts.

Solution

$$K = E - m_p c^2 = 3m_p c^2 - m_p c^2 = 2m_p c^2$$

Because $m_p c^2 = 938$ MeV, $K = 1876$ MeV

(d) What is the magnitude of the proton's momentum?

Solution We can use Equation 9.24 to calculate the momentum with $E = 3m_p c^2$:

$$E^2 = p^2 c^2 + (m_p c^2)^2 = (3m_p c^2)^2$$

$$p^2 c^2 = 9(m_p c^2)^2 - (m_p c^2)^2 = 8(m_p c^2)^2$$

$$p = \sqrt{8}\frac{m_p c^2}{c} = \sqrt{8}\frac{(938\ \text{MeV})}{c} = 2650\frac{\text{MeV}}{c}$$

The unit of momentum is written MeV/c for convenience.

EXERCISE 7 Find the speed at which the relativistic kinetic energy is two times the nonrelativistic value. Answer $0.786c$

EXERCISE 8 Find the speed of a particle whose total energy is twice its rest energy. Answer $0.866c$

EXERCISE 9 An electron moves with a speed $u = 0.850c$. Find its (a) total energy and (b) kinetic energy in electron volts. Answer (a) 0.970 MeV (b) 0.459 MeV

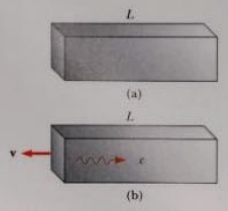


Figure 9.16 (a) A box of length L at rest. (b) When a light pulse is emitted at the left end of the box, the box recoils to the left until the pulse strikes the right end.

9.8 • MASS AS A MEASURE OF ENERGY

To understand the equivalence of mass and energy, consider the following thought experiment proposed by Einstein. Imagine a box of mass M and length L initially at rest, as in Figure 9.16a. Suppose that a pulse of light is emitted from the left side of the box, as in Figure 9.16b. From Equation 9.25, we know that the light of energy E carries momentum $p = E/c$. Hence, the box must recoil to the left with a speed v to conserve momentum. Assuming the box is very massive, the recoil speed is small compared with the speed of light. Conservation of momentum gives $Mv = E/c$, or

$$v = \frac{E}{Mc}$$

The time it takes the light to move the length of the box is approximately $\Delta t = L/c$ (where again, we assume that $v \ll c$). In this time interval, the box moves a small distance Δx to the left, where

$$\Delta x = v\,\Delta t = \frac{EL}{Mc^2}$$

The light then strikes the right end of the box, transfers its momentum to the box, causing the box to stop. With the box in its new position, it appears as if its center of mass has moved to the left. However, its center of mass cannot move because the box is an isolated system. Einstein resolved this perplexing situation by assuming that in addition to energy and momentum, light energy also has kinetic mass. If m_k is the kinetic mass carried by the pulse of light, and the center of mass of the box is to remain fixed, then

$$m_k L = M\,\Delta x$$

Solving for m_k, and using the previous expression for Δx, we get

$$m_k = \frac{M\,\Delta x}{L} = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2}$$

or

$$E = m_k c^2$$

Thus, Einstein reached the profound conclusion, "If a body gives off the energy E in the form of radiation, its mass diminishes by E/c^2 . . . The mass of a body is a measure of its energy content."

Although we derived the relationship $E = mc^2$ for light energy, the equivalence of mass and energy is universal. Equation 9.22, $E = \gamma mc^2$, which represents the total energy of any particle, suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy through its mass. The clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions in which both the conversion of mass into energy and the conversion of energy into mass takes place. Because of this, we can no longer accept the separate classical laws of conservation of mass and conservation of energy; we must instead speak of a unified law of **conservation of mass–energy.** Simply put, this law requires that **the sum of the mass–energy of a system of particles before interaction must equal the sum of the mass–energy of the system after interaction, where the mass–energy of the i^{th} particle is defined as**

• *Conservation of mass–energy*

$$E_i = \frac{m_i c^2}{\sqrt{1 - \frac{u_i^2}{c^2}}} \qquad [9.26]$$

The release of enormous energy, accompanied by the change in masses of particles after they have lost their excess energy as they are brought to rest, is the basis of atomic and hydrogen bombs. In a conventional nuclear reactor, the uranium nucleus undergoes fission, a reaction that results in several lighter fragments having considerable kinetic energy. In the case of ^{235}U (the parent nucleus), which undergoes fission, the fragments are two lighter nuclei and two neutrons. The total mass of the fragments is less than that of the parent nucleus by an amount Δm. The

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corresponding energy Δmc^2 associated with this mass difference is exactly equal to the total kinetic energy of the fragments. This kinetic energy is then used to produce hot water and steam for the generation of electrical power.

Next, consider the basic fusion reaction in which two deuterium atoms combine to form one helium atom. This reaction is of major importance in current research and development of controlled-fusion reactors. The decrease in mass that results from the creation of one helium atom from two deuterium atoms is $\Delta m = 4.25 \times 10^{-29}$ kg. Hence, the corresponding energy that results from one fusion reaction is $\Delta mc^2 = 3.83 \times 10^{-12}$ J = 23.9 MeV. To appreciate the magnitude of this result, if 1 g of deuterium is converted to helium, the energy released is about 10^{12} J! At the 1997 cost of electrical energy, this would be worth about $65 000.

Example 9.9 Binding Energy of the Deuteron

The mass of the deuteron, which is the nucleus of "heavy hydrogen," is not equal to the sum of the masses of its constituents, which are the proton and neutron. Calculate this mass difference and determine its energy equivalence.

Solution Using atomic mass units (u), we have

$$m_p = \text{mass of proton} = 1.007\,276 \text{ u}$$

$$m_n = \text{mass of neutron} = 1.008\,665 \text{ u}$$

$$m_p + m_n = 2.015\,941 \text{ u}$$

Because the mass of the deuteron is 2.013 553 u (Appendix A), we see that the mass difference Δm is 0.002 388 u. By definition, 1 u = 1.66×10^{-27} kg, and therefore

$$\Delta m = 0.002\,388 \text{ u} = 3.96 \times 10^{-30} \text{ kg}$$

Using $E = \Delta mc^2$, we find that

$$E = \Delta mc^2 = (3.96 \times 10^{-30} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2$$

$$= 3.56 \times 10^{-13} \text{ J} = 2.23 \text{ MeV}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

OPTIONAL

9.9 • GENERAL RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that resists acceleration. To designate these two attributes, we use the subscripts g and i and write

$$\text{Gravitational property} \qquad F_g = m_g g$$

$$\text{Inertial property} \qquad F_i = m_i a$$

The value for the gravitational constant G was chosen to make the magnitudes of m_g and m_i numerically equal. Regardless of how G is chosen, however, the strict proportionality of m_g and m_i has been established experimentally to an extremely high degree: a few parts in 10^{12}. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses and the resistance of a single mass to being accelerated, regardless of what kind of force is producing the acceleration. This question, which puzzled Newton and many other physicists over the years, was answered when Einstein published his theory of gravitation, known as *general relativity*, in 1916. Because it is a mathematically complex theory, we merely offer a hint of its elegance and insight.

In Einstein's view, the remarkable coincidence that m_g and m_i seemed to be exactly proportional was evidence for a very intimate and basic connection between the two concepts. He pointed out that no mechanical experiment (such as dropping a mass) could distinguish between the two situations illustrated in Figures 9.17a and 9.17b. In each case, a mass released by the observer undergoes a downward acceleration of g relative to the floor.

Einstein carried this idea further and proposed that *no* experiment, mechanical or otherwise, could distinguish between the two cases. This extension to include all phenomena (not just mechanical ones) has interesting consequences. For example, suppose that a light pulse is sent horizontally across the box, as in Figure 9.17c. The trajectory of the light pulse bends downward as the box accelerates upward to meet it. Therefore, Einstein proposed that a beam of light should also be bent downward by a gravitational field. (No such bending is predicted in Newton's theory of gravitation.)

The two postulates of Einstein's **general relativity** are as follows:

- All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
- In the vicinity of any given point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the *principle of equivalence*.)

The second postulate implies that gravitational mass and inertial mass are completely equivalent, not just proportional. What were thought to be two different types of mass are actually identical.

One interesting effect predicted by general relativity is that time scales are altered by gravity. A clock in the presence of gravity runs more slowly than one in which gravity is negligible. As a consequence, the frequencies of radiation emitted by atoms in the presence of a strong gravitational field are shifted to lower fre-

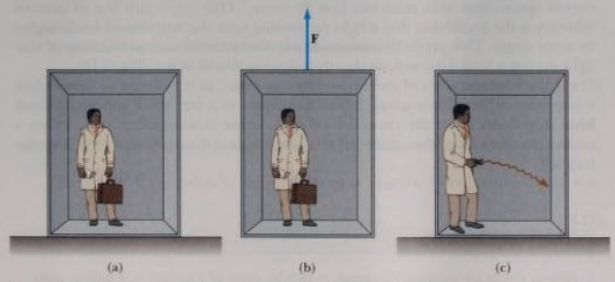


Figure 9.17 (a) The observer is at rest in a uniform gravitational field **g**. (b) The observer is in a region where gravity is negligible, but the frame of reference is accelerated by an external force **F** that produces an acceleration **g**. According to Einstein, the frames of reference in parts (a) and (b) are equivalent in every way. No local experiment could distinguish any difference between the two frames. (c) If parts (a) and (b) are truly equivalent, as Einstein proposed, then a ray of light would bend in a gravitational field.

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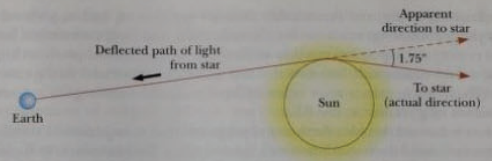


Figure 9.18 Deflection of starlight passing near the Sun. Because of this effect, the Sun and other remote objects can act as a *gravitational lens*. In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75°.

quencies when compared with the same emissions in a weak field. This gravitational shift has been detected in spectral lines emitted by atoms in massive stars. It has also been verified on the Earth by comparing the frequencies of gamma rays (a high energy form of electromagnetic radiation) emitted from nuclei separated vertically by about 20 m.

The second postulate suggests that a gravitational field may be "transformed away" at any point if we choose an appropriate accelerated frame of reference—a freely falling one. Einstein developed an ingenious method of describing the acceleration necessary to make the gravitational field "disappear." He specified a certain quantity, the *curvature of space–time*, that describes the gravitational effect at every point. In fact, the curvature of space–time completely replaces Newton's gravitational theory. According to Einstein, there is no such thing as a gravitational force. Rather, the presence of a mass causes a curvature of space–time in the vicinity of the mass, and this curvature dictates the space–time path that all freely moving objects must follow. As one physicist said, "Mass one tells space–time how to curve; curved space–time tells mass two how to move." One important test of general relativity is the prediction that a light ray passing near the Sun should be deflected by some angle. This prediction was confirmed by astronomers as bending of starlight during a total solar eclipse shortly following World War I (Fig. 9.18).

If the concentration of mass becomes very great, as is believed to occur when a large star exhausts its nuclear fuel and collapses to a very small volume, a **black hole** may form. Here the curvature of space–time is so extreme that, within a certain distance from the center of the black hole, all matter and light become trapped.

Thinking Physics 4

Atomic clocks are extremely accurate; in fact, an error of 1 second in 3 million years is typical. This error can be described as about 1 part in 10^{14}. However, the atomic clock in Boulder, Colorado, is often 15 ns faster than the one in Washington after only 1 day. This is an error of about 1 part in 6×10^{12}, which is about 17 times larger than the previously expressed error. If atomic clocks are so accurate, why does a clock in Boulder not remain in synchronism with one in Washington? (*Hint:* Denver, near Boulder, is known as the mile-high city.)

Reasoning According to the general theory of relativity, the rate of passage of time depends on gravity—time runs more slowly in strong gravitational fields. Washington is at an elevation very close to sea level, whereas Boulder is about 1 mile higher in altitude. This will result in a weaker gravitational field at Boulder than at Washington. As a result, time runs more rapidly in Boulder than in Washington.

SUMMARY

The two basic postulates of special relativity are

- All the laws of physics are the same in all inertial reference frames.
- The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Three consequences of special relativity are

- Events that are simultaneous for one observer are not simultaneous for another observer who is in motion relative to the first.
- Clocks in motion relative to an observer appear to be slowed down by a factor γ. This is known as **time dilation.**
- Lengths of objects in motion appear to be contracted in the direction of motion.

To satisfy the postulates of special relativity, the Galilean transformations must be replaced by the **Lorentz transformations:**

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned} \qquad [9.7]$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

The relativistic form of the **velocity transformation** is

$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad [9.13]$$

where u_x is the speed of an object as measured in the S frame and u'_x is its speed measured in the S' frame.

The relativistic expression for the **momentum** of a particle moving with a velocity **u** is

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \qquad [9.16]$$

The relativistic expression for the **kinetic energy** of a particle is

$$K = \gamma mc^2 - mc^2 \qquad [9.21]$$

where mc^2 is called the **rest energy** of the particle.

The total energy E of a particle is related to the **mass** through the **energy–mass** equivalence expression:

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$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad [9.23]$$

The relativistic momentum is related to the total energy through the equation

$$E^2 = p^2c^2 + (mc^2)^2 \qquad [9.24]$$

CONCEPTUAL QUESTIONS

1. What two speed measurements do two observers in relative motion *always* agree on?
2. The speed of light in water is 2.3×10^8 m/s. Suppose an electron is moving through water at 2.5×10^8 m/s. Does this violate the principles of relativity?
3. Two identical clocks are synchronized. One is put in orbit directed eastward around the Earth and the other remains on Earth. Which clock runs slower? When the moving clock returns to Earth, are the two still synchronized?
4. A train is approaching you as you stand next to the tracks. Just as an observer on the train passes, you both begin to play the same Beethoven symphony on portable compact disk players. (a) According to you, whose player finishes the symphony first? (b) According to the observer on the train, whose player finishes the symphony first? (c) Whose player *really* finishes the symphony first?
5. A spaceship in the shape of a sphere moves past an observer on Earth with a speed $0.5c$. What shape does the observer see as the spaceship moves past?
6. Explain why it is necessary, when defining length, to specify that the positions of the ends of a rod are to be measured simultaneously.
7. When we say that a moving clock runs slower than a stationary one, does this imply that there is something physically unusual about the moving clock?
8. Give a physical argument that shows that it is impossible to accelerate an object of mass m to the speed of light, even with a continuous force acting on it.
9. List some ways our day-to-day lives would change if the speed of light were only 50 m/s.
10. It is said that Einstein, in his teenage years, asked the question, "What would I see in a mirror if I carried it in my hands and ran at the speed of light?" How would you answer this question?
11. Some of the distant star-like objects, called quasars, are receding from us at half the speed of light (or greater). What is the speed of the light we receive from these quasars?
12. How is it possible that photons of light, which have zero rest mass, have momentum?
13. Relativistic quantities must make a smooth transition to their Newtonian counterparts as the speed of a system becomes small compared to the speed of light. Explain.
14. If the speed of a particle is doubled, what effect does it have on its momentum? (Assume that its initial speed is less than $c/2$.)

PROBLEMS

Section 9.1 The Principle of Newtonian Relativity

1. A 2000-kg car moving at 20.0 m/s collides with and sticks to a 1500-kg car at rest at a stop sign. Show that momentum is conserved in a reference frame moving at 10.0 m/s in the direction of the moving car.
2. A ball is thrown at 20.0 m/s inside a boxcar moving along the tracks at 40.0 m/s. What is the speed of the ball relative to the ground if the ball is thrown (a) forward? (b) backward? (c) out the side door?
3. In a laboratory frame of reference, an observer notes that Newton's second law is valid. Show that it is also valid for an observer moving at a constant speed, small compared to the speed of light, relative to the laboratory frame.
4. Show that Newton's second law is *not* valid in a reference frame moving past the laboratory frame of Problem 3 with a constant acceleration.

Section 9.4 Consequences of Special Relativity

5. How fast must a meter stick be moving if its length is observed to shrink to 0.500 m?
6. At what speed does a clock have to move in order to run at a rate that is one half the rate of a clock at rest?
7. An astronaut is traveling in a space vehicle that has a speed of $0.500c$ relative to the Earth. The astronaut measures his pulse rate at 75.0 per minute. Signals generated by the astronaut's pulse are radioed to Earth when the vehicle is moving perpendicularly to a line that connects the vehicle with an Earth observer. What pulse rate does the Earth observer measure? What would be the pulse rate if the speed of the space vehicle were increased to $0.990c$?
8. The proper length of one spaceship is three times that of another. The two spaceships are traveling in the same direction and, while both are passing overhead, an Earth observer measures the two spaceships to have the same length. If the slower spaceship is moving with a speed of $0.350c$, determine the speed of the faster spaceship.
9. An atomic clock moves at 1000 km/h for 1 hour as measured by an identical clock on Earth. How many nanoseconds slow will the moving clock be at the end of the one-hour interval?
10. If astronauts could travel at $v = 0.950c$, we on Earth would say it takes $(4.20/0.950) = 4.42$ years to reach Alpha Centauri, 4.20 lightyears away. The astronauts disagree. (a) How much time passes on the astronauts' clocks? (b) What distance to Alpha Centauri do the astronauts measure?
11. A spaceship of proper length 300 m takes 0.750 μs to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
12. A spaceship of proper length L_p takes time t to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
13. A muon formed high in the Earth's atmosphere travels at speed $v = 0.990c$ for a distance of 4.60 km before it decays into an electron, a neutrino, and an antineutrino ($\mu^- \rightarrow e^- + \nu + \bar{\nu}$). (a) How long does the muon live, as measured in its reference frame? (b) How far does the muon travel, as measured in its frame?

Section 9.5 The Lorentz Transformation Equations

14. For what value of v does $\gamma = 1.01$? Observe that for speeds lower than this value, time dilation and length contraction will be less-than-one-percent effects.
15. A spaceship travels at $0.750c$ relative to Earth. If the spaceship fires a small rocket in the forward direction, how fast (relative to the ship) must it be fired for it to travel at $0.950c$ relative to Earth?
16. A certain quasar recedes from the Earth at $v = 0.870c$. A jet of material ejected from the quasar toward the Earth moves at $0.550c$ relative to the quasar. Find the speed of the ejected material relative to the Earth.
17. Two jets of material from the center of a radio galaxy fly away in opposite directions. Both jets move at $0.750c$ relative to the galaxy. Determine the speed of one jet relative to the other.
18. A friend travels by you at a high speed in a spaceship. He tells you that his ship is 20.0 m long and that the identically constructed ship you are sitting in is 19.0 m long. According to your observations, (a) how long is your ship, (b) how long is your friend's ship, and (c) what is the speed of your friend's ship?
19. Observer A measures the lengths of two rods, one stationary, the other moving with a speed of $0.955c$. She finds that the rods have the same length. A second observer, B, travels along with the moving rod. What is the ratio of the length of A's rod to the length of B's rod, according to observer B?
20. A moving rod is 2.00 m long, and its length is oriented at an angle of 30.0° with respect to the direction of motion. The rod has a speed of $0.995c$. (a) What is the proper length of the rod? (b) What is the orientation angle in the proper frame?

Section 9.6 Relativistic Momentum and the Relativistic Form of Newton's Laws

21. Calculate the momentum of an electron moving with a speed of (a) $0.0100c$, (b) $0.500c$, (c) $0.900c$.
22. The *nonrelativistic* expression for the momentum of a particle, $p = mv$, can be used if $v \ll c$. For what speed does the use of this formula give an error in the momentum of (a) 1.00% and (b) 10.0%?
23. An electron has a momentum that is 90.0% greater than its classical momentum. (a) Find the speed of the electron. (b) How would your result change if the particle were a proton?
24. A golf ball travels with a speed of 90.0 m/s. By what fraction does its relativistic momentum, p, differ from its classical value, mv? That is, find the ratio $(p - mv)/mv$.
25. An unstable particle at rest breaks into two fragments of *unequal* mass. The rest mass of the lighter fragment is 2.50×10^{-28} kg, and that of the heavier fragment is 1.67×10^{-27} kg. If the lighter fragment has a speed of $0.893c$ after the breakup, what is the speed of the heavier fragment?
26. Show that the speed of an object having momentum p and mass m is

$$v = \frac{c}{\sqrt{1 + (mc/p)^2}}$$

Section 9.7 Relativistic Energy

27. A proton moves at $0.950c$. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy (in eV).
28. Show that, for any object moving at less than one tenth the speed of light, the relativistic kinetic energy agrees with the result of the classical equation $K = \frac{1}{2}mv^2$ to within less than 1%. Thus for most purposes, the classical equation is good enough to describe these objects, whose motion we call *nonrelativistic*.
29. Determine the energy required to accelerate an electron from (a) $0.500c$ to $0.900c$ and (b) $0.900c$ to $0.990c$.
30. A spaceship of mass 1.00×10^6 kg is to be accelerated to $0.600c$. (a) How much energy does this require? (b) How

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many kilograms of matter would it take to provide this much energy?

31. Make an order-of-magnitude estimate of the ratio of mass increase to the rest mass of a flag as you run it up a flagpole. In your solution explain what quantities you take as data and the values you estimate or measure for them.
32. When 1.00 g of hydrogen combines with 8.00 g of oxygen, 9.00 g of water is formed. During this chemical reaction, 2.86×10^5 J of energy is released. How much mass do the constituents of this reaction lose? Is the loss of mass likely to be detectable?
33. A cube of steel has a volume of 1.00 cm^3 and a mass of 8.00 g when at rest on the Earth. If this cube is now given a speed $v = 0.900c$, what is its density as measured by a stationary observer? Note that relativistic density is $m/V = E/c^2V$.
34. An electron has a kinetic energy five times greater than its rest energy. Find (a) its total energy and (b) its speed.
35. Show that the energy–momentum relationship $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.
36. An unstable particle with a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of $0.987c$ and $-0.868c$. Find the rest masses of the fragments. (*Hint:* Conserve both mass–energy and momentum.)

Additional Problems

37. The net nuclear reaction inside the Sun is $4p \rightarrow He^4 + \Delta E$. If the rest mass of each proton is 938.2 MeV and the rest mass of the He^4 nucleus is 3727 MeV, calculate the percentage of the starting mass that is released as energy.
38. An electron has a speed of $0.750c$. Find the speed of a proton that has (a) the same kinetic energy as the electron, (b) the same momentum as the electron.
39. The cosmic rays of highest energy are protons, having kinetic energy on the order of 10^{15} MeV. (a) How long would it take a proton of this energy to travel across the Milky Way galaxy, of diameter 10^5 lightyears, as measured in the proton's frame? (b) From the point of view of the proton, how many kilometers across is the galaxy?
40. A spaceship moves away from the Earth at $0.500c$ and fires a shuttle craft in the forward direction at $0.500c$ relative to the ship. The pilot of the shuttle craft launches a probe at forward speed $0.500c$ relative to the shuttle craft. Determine (a) the speed of the shuttle craft relative to the Earth and (b) the speed of the probe relative to the Earth.
41. The average lifetime of a pi meson in its own frame of reference is 2.60×10^{-8} s. If the meson moves with a speed of $0.950c$, what are (a) its mean lifetime as measured by an observer on Earth and (b) the average distance it travels before decaying, as measured by an observer on Earth?
42. An astronaut wishes to visit the Andromeda galaxy (suppose it is 2.00 million lightyears away), making a one-way trip that will take 30.0 years in the spaceship's frame of reference. Assuming that her speed is constant, how fast must she travel relative to the Earth?
43. A physics professor on Earth gives an exam to her students who are on a rocket ship traveling at speed v relative to Earth. The moment the ship passes the professor, she signals the start of the exam. If she wishes her students to have time T_0 (rocket time) to complete the exam, show that she should wait a time (Earth time) of

$$T = T_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

before sending a light signal telling them to stop. (*Hint:* Remember that it takes some time for the second light signal to travel from the professor to the students.)

44. Spaceship I, which contains students taking a physics exam, approaches Earth with a speed of $0.600c$ (relative to Earth), and spaceship II, which contains professors proctoring the exam, moves at $0.280c$ (relative to Earth) directly toward the students. If the professors stop the exam after 50.0 min have passed on their clock, how long does the exam last as measured by (a) the students? (b) an observer on Earth?
45. A supertrain (rest length = 100 m) travels at a speed of $0.950c$ as it passes through a tunnel (rest length = 50.0 m). As seen by a trackside observer, is the train ever completely within the tunnel? If so, with how much space to spare?
46. Energy reaches the upper atmosphere of the Earth from the Sun at the rate of 1.79×10^{17} W. If all of this energy were absorbed by the Earth and not re-emitted, how much would the mass of the Earth increase in 1 year?
47. A charged particle moves along a straight line in a uniform electric field **E** with a speed of v. The electric force on a charge q in an electric field is $q\mathbf{E}$. If the motion and the electric field are both in the x direction, (a) show that the acceleration of the charge q in the x direction is given by

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, how would you proceed to find the speed of the particle and its position after a time t has elapsed?

48. Imagine that the entire Sun collapses to a sphere of radius R_g such that the work required to remove a small mass m from the surface would be equal to its rest energy mc^2. This radius is called the *gravitational radius* for the Sun. Find R_g. (It is believed that the ultimate fate of many stars is to collapse to their gravitational radii or smaller.)

Spreadsheet Problems

S1. Astronomers use the Doppler shift in the Balmer series of the hydrogen spectrum to determine the radial speed of a galaxy. The fractional change in the wavelength of the spectral line is given by

$$Z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1$$

Once this quantity is measured for a particular receding galaxy, the speed of recession can be found by solving for v/c in terms of Z. Spreadsheet 9.1 calculates the speed Z values. (a) What is the speed for galaxies having $Z =$ 0.2, 0.5, 1.0, and 2.0? (b) The largest Z values, $Z \approx 3.8$, have been measured for several quasars (quasi-stellar radio sources). How fast are these quasars moving away from us?

S2. Astronauts in a starship traveling at a speed v relative to a starbase are given instructions from mission control to call back in 1 h as measured by the starship clocks. Spreadsheet 9.2 calculates how long Mission Control has to wait for the call for different starship speeds. How long does Mission Control have to wait if the ship is traveling at $v = 0.1c, 0.4c, 0.6c, 0.8c, 0.9c, 0.995c, 0.9995c$?

S3. Most astronomers believe that the Universe began at some instant with an explosion called the Big Bang and that the observed recession of the galaxies is a direct result of this explosion. If the galaxies recede from each other at a constant rate, then we expect that the galaxies moving fastest are now farthest away from Earth. This result, called Hubble's law after Edwin Hubble, can be written $v = Hr$, where H can be determined from observation, v is the speed of recession of the galaxy, and r is its distance from Earth. Current estimates of H range from 15 to 30 km/s/Mly, where Mly is the distance light travels in one million years, and a conservative estimate is $H = 20$ km/s/Mly. Use Spreadsheet 9.1 to calculate the distance from Earth for each galaxy in Problem S1.

S4. Design a spreadsheet program to calculate and plot the relativistic kinetic energy (Eq. 9.20) and the classic kinetic energy ($\frac{1}{2}mu^2$) of a macroscopic object. Plot the relativistic and classical energies versus speed on the same graph. (a) For an object of mass $m = 3$ kg, at what speed does the classical kinetic energy underestimate the relativistic value by 1%? 5%? 50%? What is the relativistic kinetic energy at these speeds? Repeat part (a) for (b) an electron and (c) a proton.

ANSWERS TO CONCEPTUAL PROBLEMS

1. This scenario is not possible with light. Water waves move through a medium—the surface of the water—and therefore have an absolute velocity relative to the reference frame in which this medium is at rest. Thus, the relative velocity of the waves and your boat is determined by Galilean relativity. This makes it possible for you to move faster than the waves. Light waves, which do not require a medium, are described by the principles of special relativity. As you detect the light wave ahead of you and moving away from you (which would be a pretty good trick—think about it!), its velocity relative to you is c. No matter how fast you accelerate in that direction, trying to catch up to it, its velocity relative to you is still c. Thus, you will not be able to catch up to the light wave.
2. The answers to both of these questions is *no*. Both your clothing and your sleeping cabin are at rest in your reference frame, thus, they will have their proper length. There will be no change in measured lengths of objects within your spacecraft. Another observer, on a spacecraft traveling at a high speed relative to yours, will measure you as thinner (if your body is oriented in a direction perpendicular to your velocity vector relative to the other observer) or will claim that you are able to fit into a shorter sleeping cabin (if your body is oriented in a direction parallel to your velocity vector relative to the other observer).
3. Assuming that their on-duty time was kept on Earth, they will be pleasantly surprised with a large paycheck. Less time will have passed for the astronauts in their frame of reference than for their employer back on Earth.
4. The incoming rocket will not appear to have a longer length and a faster clock. Length contraction and time dilation depend only on the magnitude of the relative velocity, not on the direction.
5. A reflected photon does exert a force on a surface. Although a photon has zero mass, a photon does carry momentum. When it reflects from a surface, there is a change in the momentum, just like the change in momentum of a ball bouncing off the floor. According to the momentum interpretation of Newton's second law, a change in momentum results in a force on the surface. This concept is used in theoretical studies of *space sailing*. These studies propose building nonpowered spacecraft with huge reflective sails oriented perpendicularly to the rays from the Sun. The large number of photons from the Sun reflecting from the surface of the sail will exert a force which, although small, will provide a continuous acceleration. This would allow the spacecraft to travel to other planets without fuel.

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10

Rotational Motion

CHAPTER OUTLINE

10.1 Angular Velocity and Angular Acceleration
10.2 Rotational Kinematics
10.3 Relations Between Angular and Linear Quantities
10.4 Rotational Kinetic Energy
10.5 Torque and the Vector Product
10.6 Equilibrium of a Rigid Object
10.7 Relation Between Torque and Angular Acceleration
10.8 Angular Momentum
10.9 Conservation of Angular Momentum
10.10 Quantization of Angular Momentum (Optional)
10.11 Rotation of Rigid Bodies (Optional)

In this chapter we investigate the dynamics of particles and systems of particles moving along a circular path and rigid objects rotating about a fixed axis. We shall encounter such terms as angular displacement, angular velocity, angular acceleration, torque, and angular momentum and show how these quantities are useful in describing rotational motion. Next we shall define a vector product, which is a convenient mathematical tool for expressing such physical quantities as torque and angular momentum.

One of the central points of this chapter is to develop the concept of the angular momentum of a system of particles. By analogy with the conservation of linear momentum (Chapter 8), we shall find that the angular momentum of any isolated system is always constant. The results we derive here will enable us to understand the rotational motions of a diverse range of objects in our environment, from an electron orbiting a nucleus to clusters of galaxies orbiting a common center. Although the focus of this chapter is the circular motion of particles, the extension of this treatment to rigid bodies is reasonably straightforward and is treated in an optional section.



Derek Swinson, professor of physics at the University of New Mexico, demonstrating the "gyro-ski" technique. The skier initiates a turn by lifting the axle of the rotating bicycle wheel. The direction of the turn depends on whether the left or right hand is used to lift the axle from the horizontal. Ignoring friction and gravity, the angular momentum of the system (the skier and the bicycle wheel) remains constant. *(Courtesy of Derek Swinson)*

10.1 • ANGULAR VELOCITY AND ANGULAR ACCELERATION

We began our study of linear motion by defining the terms *displacement*, *velocity*, and *linear acceleration*. We will take the same basic approach now as we turn to the study of rotational motion. Let us begin by considering a circular disk rotating around a fixed axis that is perpendicular to the disk and goes through the point O (Fig. 10.1). A point P on the disk is at a fixed distance r from the origin and rotates about O in a circle of radius r. In fact, **every point on the disk undergoes circular motion about *O*.**

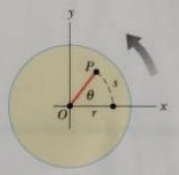


Figure 10.1 Rotation of a disk about a fixed axis through O perpendicular to the plane of the figure. (In other words, the axis of rotation is the z axis.) Note that a particle at P rotates in a circle of radius r centered at O.

It is convenient to represent the position of the point P with its polar coordinates: (r, θ). The origin of the polar coordinates is chosen to coincide with the center of the circle. In this representation, the only coordinate that changes in time is the angle θ; r remains constant. As a point on the disk moves along the circle of radius r from the positive x axis ($\theta = 0$) to the point P, it moves through an arc of length s, which is related to the angular position θ through the relation $s = r\theta$, or

$$\theta = \frac{s}{r} \qquad [10.1]$$

It is important to note the units of θ as expressed by Equation 10.1. The angle θ is the ratio of an arc length and the radius of the circle, and hence is a pure number. However, we commonly refer to the unit of θ as a **radian** (rad). One radian is the angle subtended by an arc length equal to the radius of the arc. Because the circumference of a circle is $2\pi r$, it follows that 360° corresponds to an angle of $2\pi r/r$ rad or 2π rad (one revolution). Hence, 1 rad = $360°/2\pi \approx 57.3°$. To convert an angle in degrees to an angle in radians, we can use the fact that 2π radians = 360°, or π radians = 180°; hence,

$$\theta\,(\text{rad}) = \frac{2\pi}{360°}\,\theta\,(\text{deg}) = \frac{\pi}{180°}\,\theta\,(\text{deg})$$

For example, 60° equals $\pi/3$ rad, and 45° equals $\pi/4$ rad.

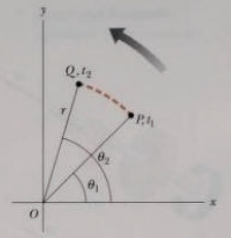


Figure 10.2 A particle on a rotating rigid object moves from P to Q along the arc of a circle. In the time interval $\Delta t = t_2 - t_1$, the radius vector sweeps out an angle $\Delta\theta = \theta_2 - \theta_1$.

In Figure 10.2, as the particle travels from P to Q in a time Δt, the radius vector sweeps out an angle of $\Delta\theta = \theta_2 - \theta_1$, which equals the **angular displacement** during the time interval Δt. We define the **average angular speed,** $\bar{\omega}$ (omega), as the ratio of this angular displacement to the time interval Δt:

$$\bar{\omega} \equiv \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \qquad [10.2]$$

• *Average angular speed*

By analogy with linear speed, the **instantaneous angular speed,** ω, is defined as the limit of the ratio in Equation 10.2 as Δt approaches zero:

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad [10.3]$$

• *Instantaneous angular speed*

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