Angular speed has units of rad/s (or s^{-1} because radians are not dimensional). Let us adopt the convention that the fixed axis of rotation for the disk is the z axis, as in Figure 10.1. We shall take ω to be positive when θ is increasing (counterclockwise motion) and negative when θ is decreasing (clockwise motion).

If the instantaneous angular speed of a particle changes from ω_1 to ω_2 in the time interval Δt, the particle has an angular acceleration. The **average angular acceleration,** $\bar{\alpha}$ (alpha), of a rotating particle is defined as the ratio of the change in the angular speed to the time interval, Δt:

Average angular acceleration

$$\bar{\alpha} \equiv \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad [10.4]$$

By analogy with linear acceleration, the **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

Instantaneous angular acceleration

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad [10.5]$$

Angular acceleration has units of rad/s^2 or s^{-2}.

Let us now generalize our argument from the circular disk to any rigid body. A rigid body is any object the elements of which remain fixed with respect to one another. **For rotation about a fixed axis, every particle on a rigid body, such as a circular disk, has the same angular velocity and the same angular acceleration.** That is, the vector quantities $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ characterize the rotational motion of the *entire* rigid body. Using these quantities, we can greatly simplify the analysis of rigid-body rotation.

The angular displacement ($\boldsymbol{\theta}$), angular velocity ($\boldsymbol{\omega}$), and angular acceleration ($\boldsymbol{\alpha}$) are analogous to linear displacement ($\mathbf{x}$), linear velocity ($\mathbf{v}$), and linear acceleration ($\mathbf{a}$), respectively, for the corresponding motion discussed in Chapter 2. The variables θ, ω, and α differ dimensionally from the linear, variables, x, v, and a, only by a length factor.

We have indicated how the signs for $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are determined, but we have not specified any direction in space associated with these vector quantities.[1] For rotation about a fixed axis, the only direction in space that uniquely specifies the rotational motion is the direction along the axis of rotation. However, we must also decide on the sense of these quantities—that is, whether they point into or out of the plane of Figure 10.1.

The direction of $\boldsymbol{\omega}$ is along the axis of rotation, which is the z axis in Figure 10.1. By convention, we take the direction of $\boldsymbol{\omega}$ to be *out of* the plane of the diagram when the rotation is counterclockwise and *into* the plane of the diagram when the rotation is clockwise. To further illustrate this convention, it is convenient to use the **right-hand rule** illustrated by Figure 10.3a. The four fingers of the right hand are wrapped in the direction of the rotation. The extended right thumb points in the direction of $\boldsymbol{\omega}$. Figure 10.3b illustrates that $\boldsymbol{\omega}$ is also in the direction of advance of a similarly rotating right-handed screw.

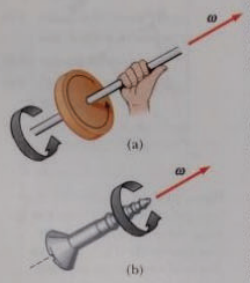


Figure 10.3 (a) The right-hand rule for determining the direction of the angular velocity vector. (b) The direction of $\boldsymbol{\omega}$ is in the direction of advance of a right-handed screw.

[1]Although we do not verify it here, the instantaneous angular velocity and instantaneous angular acceleration are vector quantities, but the corresponding average values are not. This is because angular displacement is not a vector quantity for finite rotations.

The sense of $\boldsymbol{\alpha}$ follows from its definition as $d\boldsymbol{\omega}/dt$. It is the same as $\boldsymbol{\omega}$ if the angular speed (the magnitude of $\boldsymbol{\omega}$) is increasing in time and antiparallel to $\boldsymbol{\omega}$ if the angular speed is decreasing in time.

10.2 • ROTATIONAL KINEMATICS

In our study of linear motion, we found that the simplest accelerated motion to analyze is motion under constant linear acceleration (Chapter 2). Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration. Therefore, we shall next develop kinematic relations for rotational motion under constant angular acceleration.

If we write Equation 10.5 in the form $d\omega = \alpha\, dt$ and let $\omega = \omega_0$ at $t_0 = 0$, we can integrate this expression directly:

$$\omega = \omega_0 + \alpha t \qquad (\alpha = \text{constant}) \qquad [10.6]$$

Likewise, substituting Equation 10.6 into Equation 10.3 and integrating once more (with $\theta = \theta_0$ at $t_0 = 0$), we get

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad [10.7]$$

• *Rotational kinematic equations (α = constant)*

If we eliminate t from Equations 10.6 and 10.7, we get

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad [10.8]$$

If we eliminate α, we obtain

$$\theta = \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \qquad [10.9]$$

Notice that these kinematic expressions for rotational motion under constant angular acceleration are of the *same form* as those for linear motion under constant linear acceleration, with the substitutions $x \to \theta$, $v \to \omega$, and $a \to \alpha$ (Table 10.1). Furthermore, the expressions are valid for both rigid-body rotation about a *fixed* axis and particle motion about a *fixed* axis.

TABLE 10.1 A Comparison of Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis with α = Constant	Linear Motion with a = Constant
Variables: θ and ω	Variables: x and v
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t$	$x = x_0 + \frac{1}{2}(v_0 + v)t$
$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	$v^2 = v_0^2 + 2a(x - x_0)$

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Example 10.1 **Rotating Wheel**

A wheel rotates with a constant angular acceleration of 3.50 rad/s². If the angular speed of the wheel is 2.00 rad/s at $t_0 = 0$, (a) what angle does the wheel rotate through in 2.00 s?

Solution

$$\theta - \theta_0 = \omega_0 t + \tfrac{1}{2}\alpha t^2 = \left(2.00\ \frac{\text{rad}}{\text{s}}\right)(2.00\ \text{s}) + \tfrac{1}{2}\left(3.50\ \frac{\text{rad}}{\text{s}^2}\right)(2.00\ \text{s})^2$$

$$= 11.0\ \text{rad} = 630^\circ = 1.75\ \text{rev}$$

(b) What is the angular speed at $t = 2.00$ s?

Solution

$$\omega = \omega_0 + \alpha t = 2.00\ \text{rad/s} + \left(3.50\ \frac{\text{rad}}{\text{s}^2}\right)(2.00\ \text{s})$$

$$= 9.00\ \text{rad/s}$$

We could also obtain this result using Equation 10.8 and the results of part (a). Try it!

EXERCISE 1 Find the angle that the wheel rotates through between $t = 2.00$ s and $t = 3.00$ s. Answer 10.8 rad

EXERCISE 2 A wheel starts from rest and rotates with constant angular acceleration to an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time. Answer (a) 4.00 rad/s² (b) 18.0 rad

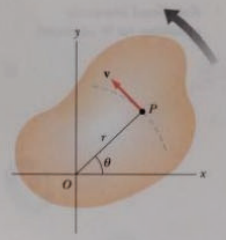


Figure 10.4 As a rigid object rotates about the fixed axis through O, the point P has a linear velocity $\mathbf{v}$ that is always tangent to the circular path of radius r.

10.3 • RELATIONS BETWEEN ANGULAR AND LINEAR QUANTITIES

In this section we shall derive some useful relations between the angular speed and angular acceleration of a rotating particle and its linear speed and linear acceleration. Keep in mind that, when a rigid body rotates about a fixed axis, *every* particle of the body moves in a circle whose center is the axis of rotation.

Consider a particle rotating in a circle of radius r about the z axis, as in Figure 10.4. Because the particle moves along a circular path, its linear velocity vector $\mathbf{v}$ is always tangent to the path; hence, we often call this quantity *tangential velocity*. The magnitude of the tangential velocity of the particle is, by definition, ds/dt, where s is the distance traveled by the particle along the circular path. Recalling from Equation 10.1 that $s = r\theta$, and noting that r is a constant, we get

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Relationship between linear and angular speed

$$v = r\omega \qquad [10.10]$$

That is, the magnitude of the tangential velocity of the particle equals the distance of the particle from the axis of rotation multiplied by the particle's angular speed.

We can relate the angular acceleration of the particle to its tangential acceleration, a_t—which is the component of its acceleration tangent to the path of motion—by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a_t = r\alpha \qquad [10.11]$$

Relationship between tangential and angular acceleration

That is, the tangential component of the linear acceleration of a particle undergoing circular motion equals the distance of the particle from the axis of rotation multiplied by the angular acceleration.

In Chapter 3 we found that a particle rotating in a circular path undergoes a centripetal, or radial, acceleration of magnitude v^2/r directed toward the center of rotation (Fig. 10.5). Because $v = r\omega$, we can express the centripetal acceleration of the particle as

$$a_r = \frac{v^2}{r} = r\omega^2 \qquad [10.12]$$

The *total linear acceleration* of the particle is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$. Therefore, the magnitude of the total linear acceleration of the particle is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \qquad [10.13]$$

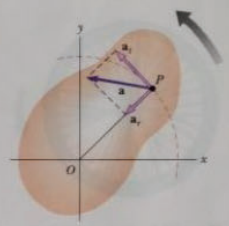


Figure 10.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration, $\mathbf{a}_t$, and a centripetal component of linear acceleration, $\mathbf{a}_r$. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

Thinking Physics 1

A phonograph record (LP—for *long-playing*) is rotated at a constant *angular* speed. A compact disk (CD) is rotated so that the surface sweeps past the laser at a constant *tangential* speed. Consider two circular "grooves" of information on an LP—one near the outer edge and one near the inner edge. Suppose the outer groove "contains" 1.8 seconds of music. Does the inner groove also contain 1.8 seconds of music? How about the CD—do the inner and outer grooves contain the same time interval of music?

Reasoning On the LP, the inner and outer grooves must both rotate once in the same time period. Thus, each groove, regardless of where it is on the record, contains the same time interval of information. Of course, on the inner grooves, this same information must be compressed into a smaller circumference. On a CD, the constant tangential speed requires that there be no such compression—the digital pits representing the information are spaced uniformly everywhere on the surface. Thus, there is more information in an outer groove, due to its larger circumference, and, as a result, a longer time interval of music than the inner groove.

Thinking Physics 2

The launch area for the European Space Agency is not in Europe—it is in French Guiana, near the Equator in South America. Why?

Reasoning Placing a satellite in Earth orbit requires providing a large tangential speed to the satellite. This is the task of the rocket propulsion system. Anything that reduces the requirements on the rocket system is a welcome contribution. The surface of the Earth is already traveling toward the east at a high speed, due to the rotation of the Earth. Thus, if rockets are launched toward the east, the rotation of the Earth provides some initial tangential speed, reducing somewhat the requirements on the rocket. If rockets were launched from Europe, which is at a relatively large angle latitude, the contribution of the Earth's rotation is relatively small, because the distance between Europe and the rotation axis of the Earth is relatively small. The ideal place for launch-

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ing is at the Equator, which is as far as one can be from the rotation axis of the Earth and still be on the surface of the Earth. This results in the largest possible tangential speed due to the Earth's rotation. The European Space Agency exploits this advantage by launching from French Guiana, which is only a few degrees north of the Equator.

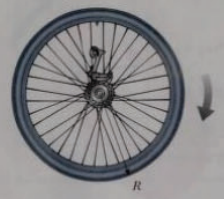


Figure 10.6 (Conceptual Problem 2) A wheel of radius R.

CONCEPTUAL PROBLEM 1

When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground. Why does this happen?

CONCEPTUAL PROBLEM 2

When a wheel of radius R rotates about a fixed axis as in Figure 10.6, do all points on the wheel have the same angular speed? Do they all have the same linear speed? If the angular speed is constant and equal to ω, describe the linear speeds and linear accelerations of the points located at $r = 0$, $r = R/2$, and $r = R$, where the points are measured from the center of the wheel.

EXERCISE 3 A racing car travels on a circular track of radius 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration. Answer (a) 0.180 rad/s (b) 8.10 m/s^2 toward the center of the track

EXERCISE 4 An automobile accelerates from zero to 30 m/s in 6.0 s. The wheels have diameters of 0.40 m. What is the angular acceleration of each wheel?
Answer 25 rad/s^2

10.4 • ROTATIONAL KINETIC ENERGY

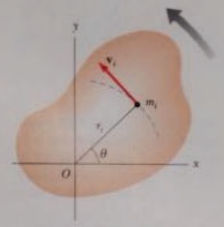


Figure 10.7 A rigid object rotating about the z axis with angular speed ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_iv_i^2$. The total energy of the object is $\frac{1}{2}I\omega^2$.

Let us consider a rigid body as a collection of particles, and let us assume that the body rotates about the fixed z axis with an angular speed of ω (Fig. 10.7). Each particle of the body has some kinetic energy, determined by its mass and speed. If the mass of the ith particle is m_i and its speed is v_i, the kinetic energy of this particle is

$$K_i = \tfrac{1}{2}m_iv_i^2$$

To proceed further, we must recall that, although every particle in the rigid body has the same angular speed ω, the individual linear speeds depend on the distance r_i from the axis of rotation, according to the expression $v_i = r_i\omega$ (Eq. 10.10). The *total* kinetic energy of the rotating rigid body is the sum of the kinetic energies of the individual particles:

$$K_R = \sum K_i = \sum \tfrac{1}{2}m_iv_i^2 = \tfrac{1}{2}\sum m_ir_i^2\omega^2$$

$$K_R = \tfrac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

where we have factored ω^2 from the sum because it is common to every particle. The quantity in parentheses is called the **moment of inertia,** I:

$$I = \sum m_ir_i^2 \qquad [10.14]$$

• *Moment of inertia for a system of particles*

Therefore, we can express the kinetic energy of the rotating rigid body as

$$K_R = \tfrac{1}{2}I\omega^2 \qquad [10.15]$$

• *Kinetic energy of a rotating rigid body*

From the definition of moment of inertia, we see that it has dimensions of ML^2 (kg·m^2 in SI units). The moment of inertia is a measure of a system's resistance to change in its angular velocity. It plays the role of mass in *all* rotational equations. However, the moment of inertia depends on the location and orientation of the axis, and is not a single property of an object the way its mass is. Although we shall commonly refer to the quantity $\frac{1}{2}I\omega^2$ as the **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy, because it was derived from a sum over individual kinetic energies of the particles contained in the rigid body. However, the form of the kinetic energy given by Equation 10.15 is a convenient one for dealing with rotational motion, provided we know how to calculate I. It is important to recognize the analogy between kinetic energy associated with linear motion, $\frac{1}{2}mv^2$, and rotational kinetic energy, $\frac{1}{2}I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively.

Thinking Physics 3

The temperature of a gas is related to the average *translational* kinetic energy of the molecules of the gas. If we add energy to a gas, the average translational kinetic energy increases and, therefore, the temperature increases. Consider adding the same amount of energy to a monatomic gas, such as helium, and a diatomic gas, such as oxygen (O_2). The temperature of the monatomic gas will rise by more than that of the diatomic gas. Why?

Reasoning In a monatomic gas, the only type of kinetic energy possible is translational kinetic energy. Thus, all of the energy added to the gas appears as translational kinetic energy, with a corresponding large rise in temperature. In the diatomic gas, some of the energy added to the gas can appear as rotational kinetic energy of the molecules. Thus, only part of the energy added appears as translational kinetic energy, and the temperature rises by a lesser amount than for the monatomic gas.

Example 10.2 The Oxygen Molecule

Consider the diatomic molecule oxygen, O_2, which is rotating in the xy plane about the z axis passing through its center, perpendicular to its length. The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature, the average separation between the two oxygen atoms is $d = 1.21 \times 10^{-10}$ m (the atoms are treated as point masses). (a) Calculate the moment of inertia of the molecule about the z axis.

Solution Because the distance of each atom from the z axis is $d/2$, the moment of inertia about the z axis is

$$I = \sum m_ir_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2}$$

$$= \left(\frac{2.66 \times 10^{-26}}{2}\text{ kg}\right)(1.21 \times 10^{-10}\text{ m})^2$$

$$= 1.95 \times 10^{-46}\text{ kg·m}^2$$

(b) If the angular speed of the molecule about the z axis is 4.60×10^{12} rad/s, what is its rotational kinetic energy?

Solution

$$K_R = \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}(1.95 \times 10^{-46}\text{ kg·m}^2)\left(4.60 \times 10^{12}\,\frac{\text{rad}}{\text{s}}\right)^2$$

$$= 2.06 \times 10^{-22}\text{ J}$$

This is approximately equal to the average kinetic energy associated with the linear motion of the molecule at room temperature, which is about 6.2×10^{-21} J.

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Example 10.3 Four Rotating Masses

Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane (Fig. 10.8). (a) If the rotation of the system occurs about the y axis with an angular speed ω, find the moment of inertia about the y axis and the rotational kinetic energy about this axis.

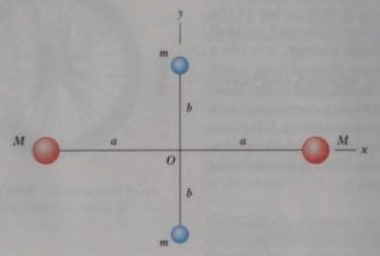


Figure 10.8 (Example 10.3) All four masses are at a fixed separation as shown. The moment of inertia depends on the axis about which it is evaluated.

Solution First, note that the two masses m that lie on the y axis do not contribute to I_y (that is, $r_i = 0$ for these masses about this axis). Applying Equation 10.14, we get

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

Therefore, the rotational kinetic energy about the y axis is

$$K_R = \tfrac{1}{2} I_y \omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2$$

The fact that the masses m do not enter into this result makes sense, because they have no motion about the chosen axis of rotation; hence, they have no kinetic energy.

(b) Suppose the system rotates in the xy plane about an axis through O (the z axis). Calculate the moment of inertia about the z axis and the rotational energy about this axis.

Solution Because r_i in Equation 10.14 is the *perpendicular* distance to the axis of rotation, we get

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2mb^2$$

$$K_R = \tfrac{1}{2} I_z \omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = (Ma^2 + mb^2)\omega^2$$

Comparing the results for (a) and (b), we conclude that the moment of inertia and, therefore, the rotational energy associated with a given angular speed depend on the axis of rotation. In (b), we expect the result to include all masses and distances because all four masses are in motion for rotation in the xy plane. Furthermore, the fact that the rotational energy in (a) is smaller than in (b) indicates that it would take less effort (work) to set the system into rotation about the y axis than about the z axis.

10.5 · TORQUE AND THE VECTOR PRODUCT

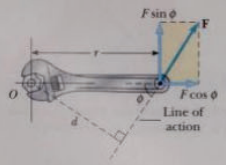


Figure 10.9 The force **F** has a greater rotating tendency about O as F increases and as the moment arm, d, increases. It is the component $F \sin \phi$ that tends to rotate the system about O.

When a force is exerted on a rigid body pivoted about some axis, the body tends to rotate about that axis. The tendency of a force to rotate a body about some axis is measured by a vector quantity called **torque** ($\boldsymbol{\tau}$). Consider the wrench pivoted about the axis through O in Figure 10.9. The applied force **F** generally can act at an angle of ϕ to the horizontal. We define the magnitude of the torque, τ (Greek letter tau), resulting from the force **F** with the expression

$$\tau \equiv rF \sin \phi = Fd \qquad [10.16]$$

It is very important to recognize that **torque is defined only when a reference axis is specified.** The quantity $d = r \sin \phi$, called the **moment arm** (or *lever arm*) of the force **F**, represents the perpendicular distance from the rotation axis to the line of action of **F**. Note that the only component of **F** that tends to cause a rotation is $F \sin \phi$, the component perpendicular to r. The horizontal component, $F \cos \phi$, passes through O and has no tendency to produce a rotation.

If two or more forces are acting on a rigid body, as in Figure 10.10, then each has a tendency to produce a rotation about the pivot at O. For example, $\mathbf{F}_2$ tends to rotate the body clockwise, and $\mathbf{F}_1$ tends to rotate the body counterclockwise. We shall use the convention that the sign of the torque resulting from a force is positive

if its turning tendency is counterclockwise and negative if its turning tendency is clockwise. For example, in Figure 10.10, the torque resulting from $\mathbf{F}_1$, which has a moment arm of d_1, is *positive* and equal to $+F_1 d_1$; the torque from $\mathbf{F}_2$ is *negative* and equal to $-F_2 d_2$. Hence, the *net* torque acting on the rigid body about O is

$$\tau_{net} = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

From the definition of torque, we see that the rotating tendency increases as F increases and as d increases. For example, it is easier to close a door if we push at the doorknob rather than at a point close to the hinge. **Torque should not be confused with force.** Torque has units of force times length—N·m in SI units.

Now consider a force **F** acting on a particle located at the vector position **r** (Fig. 10.11). The origin O is assumed to be in an inertial frame, so Newton's second law is valid. The *magnitude* of the torque due to this force relative to the origin is, by definition, equal to $rF \sin \phi$, where ϕ is the angle between **r** and **F**. The axis about which **F** would tend to produce rotation is perpendicular to the plane formed by **r** and **F**. If the force lies in the xy plane, as in Figure 10.11, then the torque is represented by a vector parallel to the z axis. The force in Figure 10.11 creates a torque that tends to rotate the body counterclockwise looking down the z axis, and so the sense of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is in the positive z direction. If we reversed the direction of **F** in Figure 10.11, $\boldsymbol{\tau}$ would then be in the negative z direction. The torque involves two vectors, **r** and **F**, and is in fact defined to be equal to the **vector product,** or **cross product,** of **r** and **F**:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad [10.17]$$

We now give a formal definition of the vector product. Given any two vectors **A** and **B**, the vector product $\mathbf{A} \times \mathbf{B}$ is defined as a third vector, **C**, the *magnitude* of which is $AB \sin \theta$, where θ is the angle included between **A** and **B**:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \qquad [10.18]$$

$$C \equiv |\mathbf{C}| = |AB \sin \theta| \qquad [10.19]$$

Note that the quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 10.12. The *direction* of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the advance of a right-handed screw when the screw is turned from **A** to **B** through the angle θ. A more convenient rule to use for the direction of $\mathbf{A} \times \mathbf{B}$ is the right-hand rule illustrated in Figure 10.12. The four fingers of the right hand are pointed along **A** and then

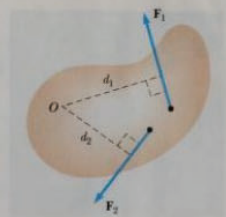


Figure 10.10 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate the object clockwise.

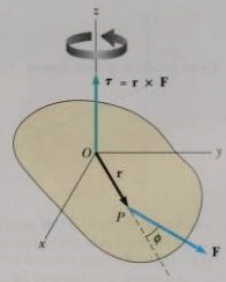


Figure 10.11 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force **F**.

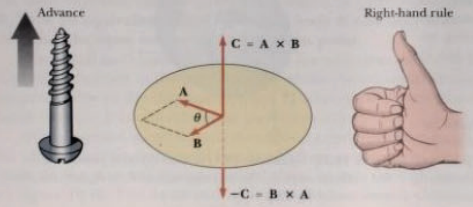


Figure 10.12 The vector product $\mathbf{A} \times \mathbf{B}$ is a third vector **C** having a magnitude $AB \sin \theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the right-hand rule.

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The torque exerted on the nail by the hammer increases as the length of the handle (lever arm) increases. *(© Richard Megna, 1991, Fundamental Photographs)*

"wrapped" into **B** through the angle θ. The direction of the erect right thumb is the direction of $\mathbf{A} \times \mathbf{B}$. Because of the notation, $\mathbf{A} \times \mathbf{B}$ is often read "**A** cross **B**"; hence the term *cross product*.

Some properties of the vector product follow from its definition:

- Unlike the case of the scalar product, the order in which the two vectors are multiplied in a cross product is important; that is

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A}) \qquad [10.20]$$

  Therefore, if you change the order of the cross product, you must change the sign. One could easily verify this relation with the right-hand rule (Fig. 10.12).
- If **A** is parallel to **B** ($\theta = 0°$ or $180°$), then $\mathbf{A} \times \mathbf{B} = 0$; therefore, it follows that $\mathbf{A} \times \mathbf{A} = 0$.
- If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.

From Equations 10.18 and 10.19 and the definition of unit vectors, one finds that the cross products of the rectangular unit vectors **i**, **j**, and **k** obey the following expressions:

Cross products of unit vectors •

$$\begin{aligned} \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\ \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j} \end{aligned} \qquad [10.21]$$

Signs are interchangeable. For example, $\mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$.

CONCEPTUAL PROBLEM 3

Both torque and work are products of force and distance. How are they different? Do they have the same units?

Example 10.4 The Net Torque on a Cylinder

A one-piece cylinder is shaped as in Figure 10.13, with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. A rope wrapped around the drum, of radius R_1, exerts a force $\mathbf{F}_1$ to the right on the cylinder. A rope wrapped around the core, of radius R_2, exerts a force $\mathbf{F}_2$ downward on the cylinder. (a) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in Fig. 10.13)?

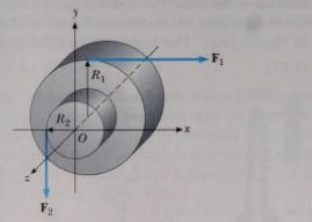


Figure 10.13 (Example 10.4) A solid cylinder pivoted about the z axis through O. The moment arm of $\mathbf{F}_1$ is R_1, and the moment arm of $\mathbf{F}_2$ is R_2.

Solution The torque due to $\mathbf{F}_1$ is $-R_1F_1$ and is negative because it tends to produce a clockwise rotation. The torque due to $\mathbf{F}_2$ is $+R_2F_2$ and is positive because it tends to produce a counterclockwise rotation. Therefore, the net torque about the rotation axis is

$$\tau_{net} = \tau_1 + \tau_2 = R_2F_2 - R_1F_1$$

(b) Suppose $F_1 = 5.0$ N, $R_1 = 1.0$ m, $F_2 = 6.0$ N, and $R_2 = 0.50$ m. What is the net torque about the rotation axis and which way does the cylinder rotate?

Solution

$$\tau_{net} = (6.0\text{ N})(0.50\text{ m}) - (5.0\text{ N})(1.0\text{ m}) = -2.0\text{ N}\cdot\text{m}$$

Because the net torque is negative, the cylinder rotates clockwise.

Example 10.5 The Cross Product

Two vectors lying in the xy plane are given by the equations $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{A} \times \mathbf{B}$, and verify explicitly that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 10.21 for the cross product of unit vectors gives

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (2\mathbf{i} + 3\mathbf{j}) \times (-\mathbf{i} + 2\mathbf{j}) \\ &= 2\mathbf{i} \times 2\mathbf{j} + 3\mathbf{j} \times (-\mathbf{i}) = 4\mathbf{k} + 3\mathbf{k} = 7\mathbf{k} \end{aligned}$$

(We have omitted the terms with $\mathbf{i} \times \mathbf{i}$ and $\mathbf{j} \times \mathbf{j}$, because they are zero.)

$$\begin{aligned} \mathbf{B} \times \mathbf{A} &= (-\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j}) \\ &= -\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times 2\mathbf{i} = -3\mathbf{k} - 4\mathbf{k} = -7\mathbf{k} \end{aligned}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

EXERCISE 5 Use the results to this example and Equation 10.19 to find the angle between **A** and **B**. Answer 60.3°

EXERCISE 6 A particle is located at the vector position $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})$ m, and the force acting on it is $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0,6) m? Answer (a) $(-7\mathbf{k})$ N·m (b) $(11\mathbf{k})$ N·m

EXERCISE 7 If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$, what is the angle between **A** and **B**? Answer 45.0°

10.6 • EQUILIBRIUM OF A RIGID OBJECT

Consider a single force **F** acting on a rigid object as in Figure 10.14. The effect of the force depends on its point of application, P. If **r** is the position vector of this point relative to O, the torque associated with the force **F** about O is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

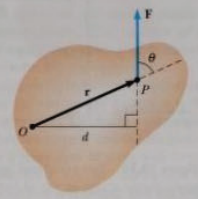


Figure 10.14 A single force **F** acts on a rigid object at the point P. The moment arm of **F** relative to O is the perpendicular distance d from O to the line of action of **F**.

As you can see from Figure 10.14, the tendency of **F** to make the object rotate about an axis through O depends on the moment arm d as well as on the magnitude of **F**. By definition, the magnitude of τ is Fd.

Now suppose either of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, act on a rigid object. The two forces will have the same effect on the object only if they have the same magnitude, the same direction, and the same line of action. In other words, **two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have an equivalent effect on a rigid body if and only if $F_1 = F_2$ and if the two have the same torque about any given point**.

Two equal and opposite forces that are *not* equivalent are shown in Figure 10.15. The force directed to the right tends to rotate the object clockwise about an axis perpendicular to the diagram through O, whereas the force directed to the left tends to rotate it counterclockwise about that axis.

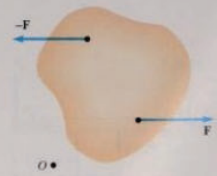


Figure 10.15 The two forces acting on the object are equal in magnitude and opposite in direction, yet the object is not in equilibrium.

When pivoted about an axis through its center of mass, an object undergoes an angular acceleration about this axis if there is a nonzero torque acting. As an example, suppose an object is pivoted about an axis through its center of mass as in Figure 10.16. Two equal and opposite forces act in the directions shown, such

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that their lines of action do not pass through the center of mass. A pair of forces acting in this manner form what is called a **couple.** (The two forces shown in Figure 10.15 also form a couple.) Because each force produces the same torque, Fd, the net torque has a magnitude $2Fd$. Clearly, the object rotates clockwise and undergoes an angular acceleration about the axis. This is a nonequilibrium situation as far as the rotational motion is concerned.

In general, an object is in rotational equilibrium only if its angular acceleration $\alpha = 0$. Because torque causes angular acceleration, **a necessary condition of equilibrium is that the net torque about any origin must be zero.** We now have **two necessary conditions for equilibrium of an object,** which can be stated as follows:

Two conditions for equilibrium of an object

The resultant external force must equal zero. $\sum \mathbf{F} = 0$ **[10.22]**

The resultant external torque must be zero about *any* origin. $\sum \tau = 0$ **[10.23]**

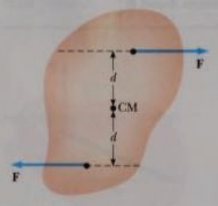


Figure 10.16 Two equal and opposite forces acting on the object form a couple. In this case, the object rotates clockwise. The magnitude of the net torque about the center of mass is $2Fd$.

The first condition is a statement of translational equilibrium; it tells us that the linear acceleration of the center of mass of the object must be zero when viewed from an inertial reference frame. The second condition is a statement of rotational equilibrium and tells us that the angular acceleration about any axis must be zero. In the special case of **static equilibrium,** the object is at rest so that it has no linear or angular speed (that is, $v_{CM} = 0$ and $\omega = 0$). Note that if the resultant external force is zero, and the resultant external torque is zero about some origin, then the resultant external torque is zero about every other origin.

The two vector expressions given by Equations 10.22 and 10.23 are equivalent, in general, to six scalar equations, three from the first condition of equilibrium and three from the second (corresponding to x, y, and z components). Hence, in a complex system involving several forces acting in various directions, you would be faced with solving a set of equations with many unknowns. Here, we restrict our discussion to situations in which all the forces lie in the xy plane. (When a set of forces lie in the same plane, they are said to be *coplanar.*) With this restriction, we shall have to deal with only three scalar equations. Two of these come from balancing the forces in the x and y directions. The third comes from the torque equation—namely, that the net torque about *any* point in the xy plane must be zero. Hence, the two conditions of equilibrium provide the equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau_z = 0 \qquad \textbf{[10.24]}$$

where the axis of the torque equation is arbitrary, as we show later. In working static equilibrium problems, it is important to recognize external forces acting on the object. Failure to do so will result in an incorrect analysis. The following procedure is recommended when analyzing an object in equilibrium under the action of several external forces:

PROBLEM-SOLVING STRATEGY • Objects in Equilibrium

1. Make a sketch of the object under consideration.
2. Draw a free-body diagram and label all external forces acting on the object. Try to guess the correct direction for each force. If you select a direction that leads to a negative sign in your solution for a force, do not be alarmed; this merely means that the direction of the force is the opposite of what you guessed.
3. Resolve all forces into rectangular components, choosing a convenient coordinate system. Then apply the first condition for equilibrium. Remember to keep track of the signs of the various force components.
4. Choose a convenient axis for calculating the net torque on the object. Remember that the choice of the origin for the torque equation is arbitrary; therefore, choose an origin that will simplify your calculation as much as possible. Usually the most convenient origin for calculating torques is the one at which most of the forces act.
5. The first and second conditions of equilibrium give a set of linear equations with several unknowns. All that is left is to solve the simultaneous equations for the unknowns in terms of the known quantities.

Example 10.6 Standing on a Horizontal Beam

A uniform horizontal beam of length 8.00 m and weight 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0° with the horizontal (Fig. 10.17a). If a 600-N person stands 2.00 m from the wall, find the tension in the cable and the force exerted by the wall on the beam.

Solution First we must identify all the external forces acting on the beam. These are the gravitational force, the force **T** exerted by the cable, the force **R** exerted by the wall at the pivot (the direction of this force is unknown), and the force exerted by the person on the beam. These are all indicated in the free-body diagram for the beam (Fig. 10.17b). If we resolve **T** and **R** into horizontal and vertical components and apply the first condition for equilibrium, we get

(1) $\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$

(2) $\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600\ \text{N} - 200\ \text{N} = 0$

Because R, T, and θ are all unknown, we cannot obtain a solution from these expressions alone. (The number of simultaneous equations must equal the number of unknowns in order for us to be able to solve for the unknowns.)

Now let us invoke the condition for rotational equilibrium. A convenient axis to choose for our torque equation is the one that passes through the pivot at O. The feature that makes this point so convenient is that the force **R** and the horizontal component of **T** both have a lever arm of zero, and hence zero torque, about this pivot. Recalling our convention for the sign of the torque about an axis and noting that the lever

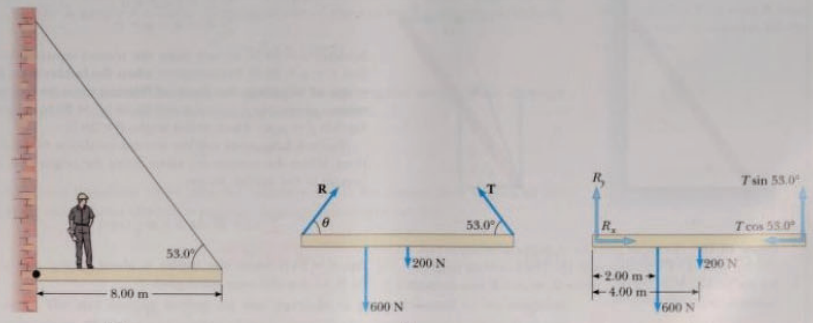


Figure 10.17 (Example 10.6) A uniform beam supported by a cable. (b) and (c) The free-body diagram for the beam.

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arms of the 600-N, 200-N, and T sin 53° forces are 2.00 m, 4.00 m, and 8.00 m, respectively, we get

$$\Sigma\tau_O = (T \sin 53.0°)(8.00 \text{ m}) - (600 \text{ N})(2.00 \text{ m}) - (200 \text{ N})(4.00 \text{ m}) = 0$$

$$T = 313 \text{ N}$$

Thus the torque equation with this axis gives us one of the unknowns directly! This value is substituted into (1) and (2) to give

$$R \cos\theta = 188 \text{ N}$$
$$R \sin\theta = 550 \text{ N}$$

We divide these two equations and recall the trigonometric identity $\sin\theta/\cos\theta = \tan\theta$ to get

$$\tan\theta = \frac{550 \text{ N}}{188 \text{ N}} = 2.93$$
$$\theta = 71.1°$$

Finally,

$$R = \frac{188 \text{ N}}{\cos\theta} = \frac{188 \text{ N}}{\cos 71.1°} = 581 \text{ N}$$

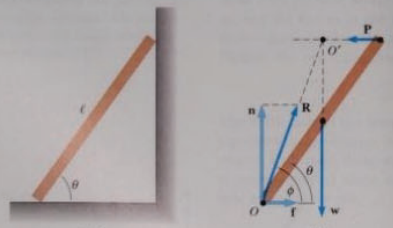
If we had selected some other axis for the torque equation, the solution would have been the same. For example, if we had chosen to have the axis pass through the center of gravity of the beam, the torque equation would involve both T and R. However, this equation, coupled with (1) and (2), could still be solved for the unknowns. Try it!

When many forces are involved in a problem of this nature, it is convenient to set up a table of forces, lever arms, and torques. For instance, in the example just given, we would construct the following table. Setting the sum of the terms in the last column equal to zero represents the condition of rotational equilibrium.

Force Component	Lever Arm Relative to O (m)	Torque About O (N·m)
T sin 53.0°	8.00	(8.00) T sin 53°
T cos 53.0°	0	0
200 N	4.00	−(4.00)(200)
600 N	2.00	−(2.00)(600)
R sin θ	0	0
R cos θ	0	0

Example 10.7 The Leaning Ladder

A uniform ladder of length ℓ and weight $w = 50$ N rests against a smooth, vertical wall (Fig. 10.18a). If the coefficient of static friction between ladder and ground is $\mu_s = 0.40$, find the minimum angle θ_{min} such that the ladder does not slip.



(a) (b)

Figure 10.18 (Example 10.7) (a) A uniform ladder at rest, leaning against a frictionless wall. (b) The free-body diagram for the ladder. Note that the forces **R**, **w**, and **P** pass through a common point O'.

Solution The free-body diagram showing all the external forces acting on the ladder is illustrated in Figure 10.18b. The reaction **R** exerted by the ground on the ladder is the vector sum of a normal force, **n**, and the force of friction, **f**. The reaction force **P** exerted by the wall on the ladder is horizontal, because the wall is smooth. From the first condition of equilibrium applied to the ladder, we have

$$\sum F_x = f - P = 0$$
$$\sum F_y = n - w = 0$$

Because $w = 50$ N, we see from the second equation here that $n = w = 50$ N. Furthermore, **when the ladder is on the verge of slipping, the force of friction must be a maximum,** given by $f_{max} = \mu_s n = 0.40(50 \text{ N}) = 20$ N. (Recall Eq. 5.9: $f_s \leq \mu_s n$.) Thus, at this angle, $P = 20$ N.

To find θ, we must use the second condition of equilibrium. When the torques are taken about the origin O at the bottom of the ladder, we get

$$\sum \tau_O = P\ell \sin\theta - w\frac{\ell}{2}\cos\theta = 0$$

But $P = 20$ N when the ladder is about to slip and $w = 50$ N, so that this expression gives

$$\tan\theta_{min} = \frac{w}{2P} = \frac{50 \text{ N}}{40 \text{ N}} = 1.25$$
$$\theta_{min} = 51°$$

It is interesting to note that the result does not depend on ℓ or w. The answer depends only on μ_s.

An alternative approach to analyzing this problem is to consider the intersection O' of the lines of action of forces **w** and **P**. Because the torque about any origin must be zero, the torque about O' must be zero. This requires that the line of action of **R** (the resultant of **n** and **f**) pass through O'. That is, because three forces act on this stationary object, the forces must be concurrent. With this condition, one could then obtain the angle ϕ that **R** makes with the horizontal (where ϕ is greater than θ), assuming the length of the ladder is known.

EXERCISE 8 For the angles labeled in Figure 10.18b, show that $\tan\phi = 2\tan\theta$.

EXERCISE 9 Two children weighing 500 N and 350 N are on a uniform board weighing 40.0 N supported at its center (a see-saw). If the 500-N child is 1.50 m from the center, determine (a) the upward force exerted on the board by the support and (b) where the 350-N child must sit to balance the system. Answer (a) 890 N (b) 2.14 m from the center

10.7 · RELATION BETWEEN TORQUE AND ANGULAR ACCELERATION

In this section we shall show that the angular acceleration of a particle rotating about a fixed axis is proportional to the net torque acting about that axis. The ideas embodied in this situation can easily be extended to the case of a rigid body rotating about a fixed axis.

Consider a particle of mass m rotating in a circle of radius r under the influence of both a tangential force, $\mathbf{F}_t$, and a radial force, $\mathbf{F}_r$, as in Figure 10.19. (The force $\mathbf{F}_r$, whose nature is not yet specified, *must* be present to keep the particle moving in its circular path.) According to Newton's second law, $\mathbf{F}_t = m\mathbf{a}_t$ and $\mathbf{F}_r = m\mathbf{a}_r$, where $\mathbf{a}_t$ is the tangential acceleration of the particle and $\mathbf{a}_r$ is its centripetal acceleration. The torque due to the force $\mathbf{F}_r$ about the origin is zero because $\mathbf{F}_r$ is antiparallel to $\mathbf{r}$; hence, $\mathbf{r} \times \mathbf{F}_r = 0$. The torque about the origin due to the tangential force is $\mathbf{r} \times \mathbf{F}_t$, but because $\mathbf{r}$ is perpendicular to $\mathbf{F}_t$, the magnitude of the torque is simply $F_t r$. Thus, the magnitude of the net torque on the particle is

$$\sum \tau = F_t r = (ma_t) r$$

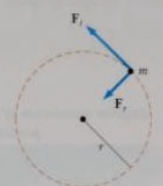


Figure 10.19 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ must also be present to maintain the circular motion.

Because the tangential acceleration is related to the angular acceleration through Equation 10.11, $a_t = r\alpha$, the torque can be expressed as

$$\sum \tau = (mr\alpha) r = (mr^2)\alpha$$

Recall from Equation 10.14 that the quantity mr^2 is the moment of inertia of the rotating mass about the z axis passing through the origin, so that

$$\sum \tau = I\alpha \qquad [10.25]$$

• *Relationship between net torque and angular acceleration*

That is, **the net torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. It is important to note that $\Sigma\tau = I\alpha$ is the rotational analogue of Newton's second law of motion, $\Sigma F = ma$.

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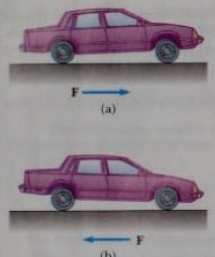


Figure TP10.4
(Thinking Physics 4)

Thinking Physics 4

When an automobile driver steps on the accelerator, the nose of the car moves upward. When the driver brakes, the nose moves downward. Why do these effects occur?

Reasoning When the driver steps on the accelerator, there is an increased force on the tires from the roadway. This force is parallel to the roadway and directed toward the front of the automobile, as suggested in Figure TP10.4a. This force provides a torque about the center of mass that tends to cause the car to rotate in the counterclockwise direction in the diagram. The result of this rotation is a "nosing up" of the car. When the driver steps on the brake, there is an increased force on the tires from the roadway, directed toward the rear of the automobile, as suggested in Figure TP10.4b. This force results in a torque that causes a clockwise rotation and the subsequent "nosing down" of the automobile.

10.8 · ANGULAR MOMENTUM

Again let us consider a particle of mass m, situated at the vector position $\mathbf{r}$ and moving with a momentum $\mathbf{p}$, as shown in Figure 10.20. The **instantaneous angular momentum, L,** of the particle relative to the origin O is defined by the cross product of its instantaneous vector position and the instantaneous linear momentum, $\mathbf{p}$:

Angular momentum of a particle

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [10.26]$$

The SI units of angular momentum are $\text{kg} \cdot \text{m}^2/\text{s}$. It is important to note that both the magnitude and the direction of $\mathbf{L}$ depend on the choice of origin. The direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$, and the sense of $\mathbf{L}$ is governed by the right-hand rule. For example, in Figure 10.20, $\mathbf{r}$ and $\mathbf{p}$ are assumed to be in the xy plane, and $\mathbf{L}$ points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of $\mathbf{L}$ is

$$L = mvr \sin \phi \qquad [10.27]$$

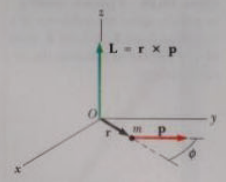


Figure 10.20 The angular momentum $\mathbf{L}$ of a particle of mass m and momentum $\mathbf{p}$ located at the position $\mathbf{r}$ is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of $\mathbf{L}$ depends on the origin and is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$. It follows that $\mathbf{L}$ is zero when $\mathbf{r}$ is parallel to $\mathbf{p}$ ($\phi = 0°$ or $180°$). In other words, when the particle moves along a line that passes through the origin, it has zero angular momentum with respect to the origin. This is equivalent to stating that it has no tendency to rotate about the origin. However, if $\mathbf{r}$ is perpendicular to $\mathbf{p}$ ($\phi = 90°$), L is a maximum and equal to mvr. In this case, the particle has maximum tendency to rotate about the origin. In fact, at that instant the particle moves exactly as though it were on the rim of a wheel rotating about the origin in a plane defined by $\mathbf{r}$ and $\mathbf{p}$. A particle has nonzero angular momentum about some point if the position vector of the particle measured from that point rotates about the point.

For linear motion, we found that the resultant force on a particle equals the time rate of change of the particle's linear momentum (Eq. 8.3). We shall now show that Newton's second law implies that the resultant torque acting on a particle equals the time rate of change of the particle's angular momentum. Let us start by writing the torque on the particle in the form

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad [10.28]$$

where we have used the fact that $\mathbf{F} = d\mathbf{p}/dt$. Now let us differentiate Equation 10.26 with respect to time, using the product rule.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

It is important to adhere to the order of terms, because $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

The last term on the right in the preceding equation is zero because $\mathbf{v} = d\mathbf{r}/dt$ is parallel to $\mathbf{p}$. Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad [10.29]$$

Comparing Equations 10.28 and 10.29, we see that

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \qquad [10.30]$$

Torque equals time rate of change of angular momentum

This result, $\boldsymbol{\tau} = d\mathbf{L}/dt$, is the rotational analog of Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. Equation 10.30 says that the **torque** acting on a particle is equal to the time rate of change of the particle's angular momentum. It is important to note that Equation 10.30 is valid only if the origins of $\boldsymbol{\tau}$ and $\mathbf{L}$ are the *same*. Equation 10.30 is also valid when several forces are acting on the particle, in which case $\boldsymbol{\tau}$ is the *net* torque on the particle. **Furthermore, the expression is valid for any origin fixed in an inertial frame.** Of course, the same origin must be used in calculating all torques as well as the angular momentum.

A System of Particles

The total angular momentum, $\mathbf{L}$, of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum \mathbf{L}_i$$

where the vector sum is over all of the n particles in the system.

Because the individual momenta of the particles may change in time, the total angular momentum may also vary in time. In fact, from Equations 10.28 and 10.29, we find that the time rate of change of the total angular momentum equals the vector sum of *all* torques, including those associated with internal forces between particles and those associated with external forces. However, the net torque associated with internal forces is zero. To understand this, recall that Newton's third law tells us that the internal forces occur in equal and opposite pairs that lie along the line of separation of each pair of particles. Therefore, the torque due to each action–reaction force pair is zero. By summation, we see that *the net internal torque vanishes*. Finally, we conclude that the total angular momentum can vary with time *only* if there is a net *external* torque on the system, so that we have

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt} \sum \mathbf{L}_i = \frac{d\mathbf{L}}{dt} \qquad [10.31]$$



This is an aerial view of Hurricane Fran. The spiral-shaped, nonrigid mass of air undergoes rotation and has angular momentum. *(Courtesy of NASA)*

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That is, the time rate of change of the total angular momentum of the system about some origin in an inertial frame equals the net external torque acting on the system about that origin. Note that Equation 10.31 is the rotational analog of $\mathbf{F}_{ext} = d\mathbf{p}/dt$ (Eq. 8.35) for a system of particles.

Thinking Physics 5

Can a particle moving in a straight line have nonzero angular momentum?

Reasoning A particle has angular momentum about an origin when moving in a straight line as long as its line of motion does not pass through the origin. Its angular momentum is zero only if the line of motion passes through the origin, in which case $\mathbf{r}$ is parallel to $\mathbf{v}$, and $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = 0$.

Example 10.8 Circular Motion

A particle moves in the xy plane in a circular path of radius r, as in Figure 10.21. (a) Find the magnitude and direction of its angular momentum relative to O when its velocity is $\mathbf{v}$.

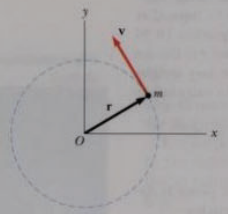


Figure 10.21 (Example 10.8) A particle moving in a circle of radius r has an angular momentum equal in magnitude to mvr relative to the center. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (not shown) points *out* of the diagram in this case.

Solution Because $\mathbf{r}$ is perpendicular to $\mathbf{v}$, $\phi = 90°$ and the magnitude of $\mathbf{L}$ is simply

$$L = mvr \sin 90° = mvr \qquad \text{(for } \mathbf{r} \text{ perpendicular to } \mathbf{v}\text{)}$$

The direction of $\mathbf{L}$ is perpendicular to the plane of the circle, and its sense depends on the direction of $\mathbf{v}$. If the sense of the rotation is counterclockwise, as in Figure 10.21, then by the right-hand rule, the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is *out* of the paper. Hence, we can write the vector expression for the angular momentum as $\mathbf{L} = (mvr)\mathbf{k}$. If the particle were to move clockwise, $\mathbf{L}$ would point into the paper.

(b) Find an alternative expression for L in terms of the angular velocity, ω.

Solution Because $v = r\omega$ for a particle rotating in a circle, we can express L as

$$L = mvr = m(r\omega)r = mr^2\omega = I\omega$$

where I is the moment of inertia of the particle about the z axis through O. In this case the angular momentum is in the *same* direction as the angular velocity vector, $\boldsymbol{\omega}$ (see Section 10.1), and so we can write $\mathbf{L} = I\boldsymbol{\omega} = I\omega\mathbf{k}$.

Example 10.9 Two Connected Masses

Two masses, m_1 and m_2, are connected by a light cord that passes over a pulley of radius R and moment of inertia I about its axle, as in Figure 10.22. The mass m_2 slides on a frictionless, horizontal surface. Determine the acceleration of the two masses using the concepts of angular momentum and torque.

Reasoning and Solution First, calculate the angular momentum of the system, which consists of the two masses plus the pulley. Then calculate the torque about an axis along the axle of the pulley through O. At the instant m_1 and m_2 have a speed v, the angular momentum of m_1 is m_1vR, and that of m_2 is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Therefore, the total angular momentum of the system is

$$(1) \quad L = m_1vR + m_2vR + I\frac{v}{R}$$

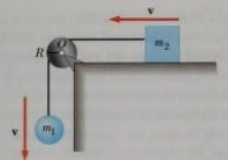


Figure 10.22 (Example 10.9)

Now let us evaluate the total external torque on the system about the axle. Because they have zero moment arms, the forces exerted by the axle on the pulley and by the weight of the pulley do not contribute to the torque. Furthermore, the normal force acting on m_2 is balanced by the force of gravity, $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The external force $m_1\mathbf{g}$ produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about O; that is, $\tau_{ext} = m_1gR$. Using this result, together with (1) and Equation 10.30 gives

$$\tau_{ext} = \frac{dL}{dt}$$

$$m_1gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right]$$

$$(2) \quad m_1gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to get

$$a = \frac{m_1g}{(m_1 + m_2) + I/R^2}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration. Only the external torques contribute to the change in angular momentum.

EXERCISE 10 A car of mass 1500 kg moves on a circular racetrack of radius 50 m with a speed of 40 m/s. What is the magnitude of its angular momentum relative to the center of the track? Answer 3.0×10^6 kg·m²/s

10.9 · CONSERVATION OF ANGULAR MOMENTUM

In Chapter 8 we found that the total linear momentum of a system of particles remains constant when the net external force acting on the system is zero. In rotational motion, we have an analogous conservation law that states that **the total angular momentum of a system remains constant if the net external torque acting on the system is zero.**

Because the net torque acting on the system equals the time rate of change of the system's angular momentum, we see that if

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = 0 \qquad [10.32]$$

then

$$\mathbf{L} = \text{constant} \qquad [10.33]$$

For a system of particles, we can write this conservation law as $\Sigma\mathbf{L}_i = \text{constant}$, where $\mathbf{L}_i$ is the angular momentum of the ith particle, given by $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$. Hence, we can express the conservation of angular momentum for a system of particles as

$$\sum \mathbf{L}_i = \sum \mathbf{r}_i \times \mathbf{p}_i = \text{constant} \qquad [10.34]$$

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Figure 10.23 Nancy Kerrigan. In this spin, her angular momentum increases when she pulls her arms in close to her body, demonstrating that angular momentum is conserved. *(Reuters/Corbis-Bettmann)*

From Equation 10.34 we derive a third conservation law to add to our list of conserved quantities. We can now state that **the total energy, linear momentum, and angular momentum of an isolated system all remain constant.**

If the system of "particles" under consideration is a rigid body rotating about a fixed axis, then the angular momentum of the body about the axis has a magnitude given by $L = I\omega$, where I is the moment of inertia about the axis. In this case, if the net external torque on the body is zero, we can express the conservation of angular momentum as $I\omega$ = constant.

Although we do not prove it here, an important theorem concerns the angular momentum of a system of particles relative to the center of mass of the system. It can be stated as follows:

> The net torque acting on a system of particles about the center of mass equals the time rate of change of angular momentum, regardless of the motion of the center of mass.

This theorem applies even if the center of mass is accelerating, provided that both $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

Many examples can be used to demonstrate conservation of angular momentum; some of them should be familiar to you. You may have observed a figure skater



Figure 10.24 The Crab Nebula, in the constellation Taurus. This nebula is the remnant of a supernova explosion, which was seen on Earth in the year A.D. 1054. It is located some 6300 lightyears away and is approximately 6 lightyears in diameter, still expanding outward. *(David Malin, Anglo-Australian Observatory)*

spinning (Fig. 10.23). The angular speed of the skater increases as she pulls her hands and feet close to the trunk of her body. Neglecting friction between skater and ice, we see that there are no external torques on the skater. The moment of inertia of her body decreases as her hands and feet are brought in. The resulting change in angular speed is accounted for as follows. Because angular momentum must be conserved, the product $I\omega$ remains constant, and a decrease in I causes a corresponding increase in ω.

An interesting astrophysical example of conservation of angular momentum occurs when, at the end of its lifetime, a massive star uses up all its fuel and collapses under the influence of gravitational forces, causing a gigantic outburst of energy called a supernova explosion. The best-studied example of a remnant of a supernova explosion is the Crab Nebula, a chaotic, expanding mass of gas (Fig. 10.24). Part of the star's mass is released into space, where it eventually condenses into new stars and planets. Most of what is left behind makes up a **neutron star,** an extremely dense sphere of matter with a diameter of about 10 km in comparison with the 10^6-km diameter of the original star. As the rotational inertia of the system decreases, the star's rotational speed increases. More than 300 rapidly rotating neutron stars have been identified, with periods of rotation ranging from 1.6 ms to 4 s. The neutron star is a most dramatic system—an object with a mass greater than the Sun, rotating about its axis a few times each second!

EXERCISE 11 A particle of mass m moves in the xy plane with a velocity $\mathbf{v}$ along a straight line, as in Figure E10.11. What are the magnitude and direction of its angular momentum (a) with respect to the origin O and (b) with respect to the origin O'? Answer (a) mvd into the diagram (b) zero

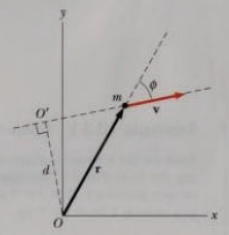


Figure E10.11 (Exercise 11)

Example 10.10 A Revolving Ball on a Horizontal, Frictionless Surface

A ball of mass m on a horizontal, frictionless table is connected to a string that passes through a small hole in the table. The ball is set into circular motion of radius R, at which time its speed is v_0 (Fig. 10.25). If the string is pulled from the bottom so that the radius of the circular path is decreased to r, what is the final speed v of the ball?

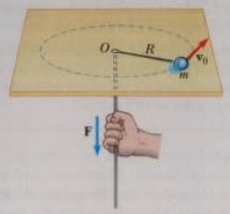


Figure 10.25 (Example 10.10)

Solution Let us take the torque about the center of rotation, O. Note that the force of gravity acting on the ball is balanced by the upward normal force, and so these forces cancel. The force, $\mathbf{F}$, of the string on the ball acts toward the center of rotation, and the vector position $\mathbf{r}$ is directed away from O. Thus, we see that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$. Because $\boldsymbol{\tau} = d\mathbf{L}/dt = 0$, $\mathbf{L}$ is a constant of the motion. That is, $mv_0R = mvr$, or

$$v = \frac{v_0 R}{r}$$

From this result, we see that as r decreases, the speed v increases.

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OPTIONAL

10.10 · QUANTIZATION OF ANGULAR MOMENTUM

We have seen that the concept of angular momentum is very useful for describing the motions of macroscopic systems. The concept is also valid and equally useful on a submicroscopic scale, and it has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it was found that the angular momentum of a system is a *fundamental* quantity. The word "fundamental" in this context implies that angular momentum is an inherent property of atoms, molecules, and their constituents.

In order to explain the results of a variety of experiments on atomic and molecular systems, we must assign discrete values to—in modern terminology, **quantize**—the angular momentum. These discrete values are multiples of a fundamental unit of angular momentum that equals $h/2\pi$, where h is called **Planck's constant.**

$$\text{Fundamental unit of angular momentum} = \frac{h}{2\pi} = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Example 10.11 Estimating the Rotational Frequency of the Oxygen Molecule

Estimate the rotational frequency of the oxygen molecule using the facts that the "average" separation between the two oxygen atoms is 1.21×10^{-10} m and that the mass of an oxygen atom is 2.66×10^{-26} kg.

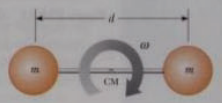


Figure 10.26 (Example 10.11) The rigid-rotor model of the diatomic molecule. The rotation occurs about the center of mass in the plane of the diagram.

Solution Consider the O_2 molecule as a rigid rotor—that is, two oxygen atoms separated by a fixed distance d and rotating about their center of mass (Fig. 10.26). To find the rotational frequency, first we must evaluate the angular momentum of the molecule about an axis through its center of mass, perpendicular to the plane of rotation.

$$L = L_1 + L_2 = mvr + mvr = 2mvr$$

In this example, $r = d/2 = 0.605 \times 10^{-10}$ m. Also, it is useful to use the relation $v = r\omega$ (Eq. 10.10), giving

$$\begin{aligned} L &= 2mvr = 2mr^2\omega \\ &= 2(2.66 \times 10^{-26}\ \text{kg})(0.605 \times 10^{-10}\ \text{m})^2\omega \\ &= (1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2)\omega \end{aligned}$$

We can now estimate the lowest rotational frequency of the molecule by equating the rotational angular momentum with the fundamental unit of angular momentum, $h/2\pi$.

$$\begin{aligned} L &= (1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2)\omega = \frac{h}{2\pi} \\ &= 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s} \end{aligned}$$

Therefore,

$$\omega = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} = 5.41 \times 10^{11}\ \text{rad/s}$$

This result is in good agreement with measured rotational frequencies. The rotational frequencies are much lower than the vibrational frequencies of the molecule, which are typically of the order of 10^{13} rad/s.

Classical concepts and mechanical models can be useful in describing some features of atomic and molecular systems. However, as shown by the preceding example, a wide variety of submicroscopic phenomena can be explained only if one assumes discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962), in his theory of the hydrogen atom, suggested the radical idea of angular momentum quantization. Strictly classical models were unsuccessful in describing many properties of the hydrogen atom, such as the fact that the atom emits radiation at discrete frequencies. Bohr found that the electron could occupy only those orbits about the proton for which the orbital angular momentum was equal to $nh/2\pi$, where n is an integer, or quantum number. From this simple model, one can estimate the rotational frequencies of the electron in the allowed orbits.

Although Bohr's theory provided some insight concerning the behavior of matter at the atomic level, it is basically incorrect. Subsequent developments in quantum mechanics from 1924 to 1930 provided models and interpretations that are still accepted.

Later developments in atomic physics indicated that an electron possesses another kind of angular momentum, called **spin,** which is an inherent property of the electron itself, unrelated to its motion. The spin angular momentum is also restricted to discrete values. Later we shall return to this important property and discuss its great impact on modern physical science.

10.11 · ROTATION OF RIGID BODIES

OPTIONAL

In this section we shall investigate the rotational motion of rigid bodies, using as a basis much of what we have learned concerning the circular motion of particles. First we shall discuss the dynamics of a rigid body rotating about a fixed axis. Next we shall show how to determine the kinetic energy of a rotating rigid body and how its change is related to the work done by external forces. Finally we shall use the principle of conservation of energy to describe an object that rolls on a surface without slipping.

Rotational Dynamics

In Section 10.7 we learned that the magnitude of the net torque acting on a particle is given by Equation 10.25:

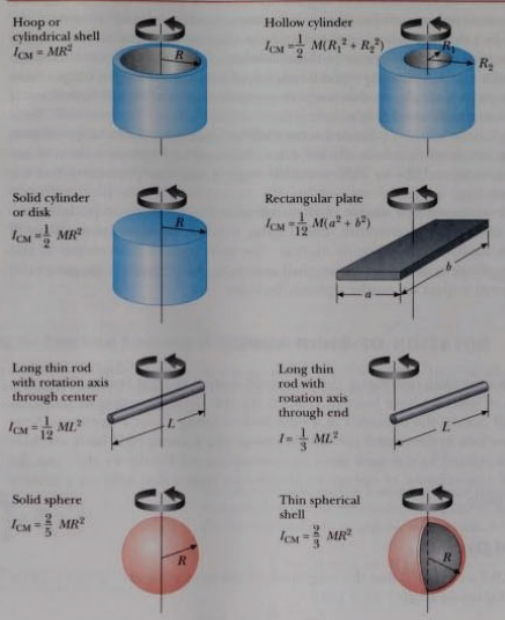
$$\tau_{\text{net}} = I\alpha$$

where I is the moment of inertia and α is the angular acceleration. It is important to note that this result also applies to a rigid body of arbitrary shape rotating about a fixed axis, because the body can be regarded as an infinite number of mass elements of infinitesimal size, each rotating in a circle about the rotation axis. The important and strikingly simple result given by $\tau_{\text{net}} = I\alpha$ is in complete agreement with experimental observations. Its simplicity is a result of how the motion is described.

Although the points on a rigid body rotating about a fixed axis may not experience the same force, linear acceleration, or linear velocity, every point on the body has the same angular acceleration and angular velocity at any instant. Therefore, at any instant the rigid body as a whole is characterized by specific values for angular acceleration, net torque, and angular velocity.

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TABLE 10.2 Moments of Inertia of Some Rigid Objects



The specific form of I depends on the axis of rotation and on the size and shape of the body. The moments of inertia of rigid bodies can be evaluated by methods of integral calculus. Table 10.2 gives the moments of inertia for a number of bodies about specific axes. In all cases, note that I is proportional to the mass of the object and to the square of a geometric factor.

Example 10.12 Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as in Figure 10.27. A light cord wrapped around the wheel supports an object of mass m. Calculate the linear acceleration of the object, the angular acceleration of the wheel, and the tension in the cord.

Reasoning and Solution The torque acting on the wheel about its axis of rotation is $\tau = TR$, where T is the force exerted by the cord on the rim of the wheel. (The weight of the wheel and the normal force exerted by the axle on the wheel pass through the axis of rotation and produce no torque.) Because $\tau = I\alpha$, we get

$$\tau = I\alpha = TR$$

$$(1) \quad \alpha = \frac{TR}{I}$$

Now let us apply Newton's second law to the motion of the suspended object, making use of the free-body diagram in Figure 10.27, taking the upward direction to be positive:

$$\sum F_y = T - mg = -ma$$

$$(2) \quad a = \frac{mg - T}{m}$$

The linear acceleration of the suspended object is equal to the tangential acceleration of a point on the rim of the wheel. Therefore, the angular acceleration of the wheel and this linear acceleration are related by $a = R\alpha$. Using this fact together with (1) and (2), gives

$$(3) \quad a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$(4) \quad T = \frac{mg}{1 + mR^2/I}$$

Substituting (4) into (3), and solving for a and α gives

$$a = \frac{g}{1 + I/mR^2}$$

$$\alpha = \frac{a}{R} = \frac{g}{R + I/mR}$$

EXERCISE 12 The wheel in Figure 10.27 is a solid disk of $M = 2.00$ kg, $R = 30.0$ cm, and $I = 0.0900$ kg·m^2. The suspended object has a mass of $m = 0.500$ kg. Find the tension in the cord and the angular acceleration of the wheel.
Answer 3.27 N; 10.9 rad/s^2

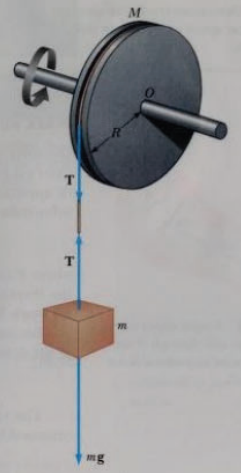


Figure 10.27 (Example 10.12) The cord attached to an object of mass m is wrapped around the wheel, and the tension in the cord produces a torque about the axle through O.

Example 10.13 Rotating Rod Revisited

A uniform rod of length L and mass M is free to rotate on a frictionless pin through one end (Fig. 10.28). The rod is released from rest in the horizontal position. (a) What is the angular speed of the rod at its lowest position?

Reasoning and Solution The question can be answered by considering the mechanical energy of the system. When the rod is horizontal, it has no rotational energy. The potential energy relative to the lowest position of its center of mass (O') is $MgL/2$. When it reaches its lowest position, the energy is entirely rotational energy, $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the pivot. Because $I = \frac{1}{3}ML^2$ (Table 10.2) and because mechanical energy is constant, we have

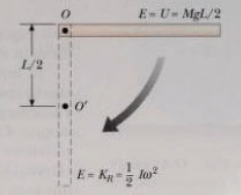


Figure 10.28 (Example 10.13) A uniform rigid rod pivoted at O rotates in a vertical plane under the action of the force of gravity.

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$$\tfrac{1}{2}MgL = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(\tfrac{1}{3}ML^2)\,\omega^2$$

$$\omega = \sqrt{\frac{3g}{L}}$$

(b) Determine the linear speed of the center of mass and the linear speed of the lowest point on the rod in the vertical position.

Solution

$$v_{CM} = r\omega = \frac{L}{2}\,\omega = \tfrac{1}{2}\sqrt{3gL}$$

The lowest point on the rod has a linear speed equal to $2v_{CM} = \sqrt{3gL}$.

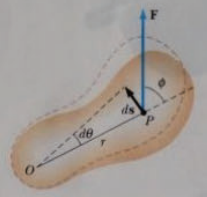


Figure 10.29 A rigid object rotates about an axis through O under the action of an external force **F** applied at P.

Work and Energy in Rotational Motion

Consider a rigid body pivoted at the point O in Figure 10.29. Suppose a single external force, **F**, is applied at the point P and $d\mathbf{s}$ is the displacement of the point of application of the force. The work done by **F** as the body rotates through an infinitesimal distance $ds = r\,d\theta$ in a time dt is

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)\,r\,d\theta$$

where $F\sin\phi$ is the tangential component of **F** or the component of the force along the displacement. Note from Figure 10.29 that **the radial component of F does no work because it is perpendicular to the displacement.**

Because the magnitude of the torque due to **F** about the origin was defined as $rF\sin\phi$, we can write the work done for the infinitesimal rotation in the form

$$dW = \tau d\theta \qquad [10.35]$$

The rate at which work is being done by **F** for rotation about the fixed axis is obtained by formally dividing the left and right sides of Equation 10.35 by dt:

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} \qquad [10.36]$$

But the quantity dW/dt is, by definition, the instantaneous power, P, delivered by the force. Furthermore, because $d\theta/dt = \omega$, Equation 10.36 reduces to

Power delivered to a rigid body

$$P = \frac{dW}{dt} = \tau\omega \qquad [10.37]$$

This expression is analogous to $P = Fv$ in the case of linear motion, and the expression $dW = \tau d\theta$ is analogous to $dW = F_x dx$.

In linear motion, we found the energy concept, and in particular the work-kinetic energy theorem, to be extremely useful in describing the motion of a system. The energy concept can be equally useful in simplifying the analysis of rotational motion. From what we learned of linear motion, we expect that for rotation of a symmetric object (such as a symmetric wheel) about a fixed axis, the work done by external forces will equal the change in the rotational kinetic energy. To show that this is in fact the case, let us begin with $\tau = I\alpha$. Using the chain rule from the calculus, we can express the torque as

$$\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\,\omega$$

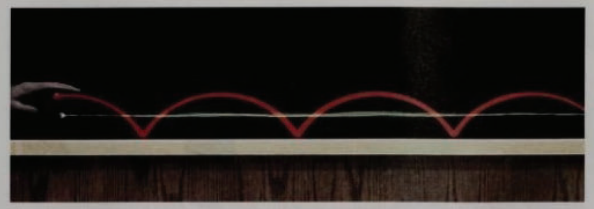


Figure 10.30 Light sources attached at the center and rim of a rolling cylinder illustrate the different paths these points take. The center moves in a straight line, as indicated by the green line, and a point on the rim moves in the path of a cycloid, as indicated by the red curve. *(Henry Leap and Jim Lehman)*

Rearranging this expression and noting that $\tau d\theta = dW$, we get

$$\tau d\theta = dW = I\omega\,d\omega$$

Integrating this expression, we get the total work done:

$$W = \int_{\theta_0}^{\theta}\tau\,d\theta = \int_{\omega_0}^{\omega} I\omega\,d\omega = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_0^2 \qquad [10.38]$$

Work-kinetic energy relationship for rotational motion

Rolling Motion of a Rigid Body

Suppose a cylinder is rolling on a straight path as in Figure 10.30. The center of mass moves in a straight line, and a point on the rim moves in a more complex path corresponding to the shape of a cycloid. Let us further assume that the cylinder of radius R is uniform and rolls on a surface with friction. As the cylinder rotates through an angle of θ, its center of mass moves a distance of $s = r\theta$. Therefore, the speed and acceleration of the center of mass for **pure rolling motion** are

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad [10.39]$$

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \qquad [10.40]$$

The linear velocities of various points on the rolling cylinder are illustrated in Figure 10.31. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point. At any instant, the point P is at rest relative to the surface, because sliding does not occur.

A general point on the cylinder, such as Q, has both horizontal and vertical components of velocity. However, the points P and P' and the point at the center of mass are unique and of special interest. Relative to the surface on which the cylinder is moving, the center of mass moves with a speed of $v_{CM} = R\omega$, whereas the contact point P has zero velocity. The point P' has a speed equal to $2v_{CM} = 2R\omega$, because all points on the cylinder have the same angular speed.

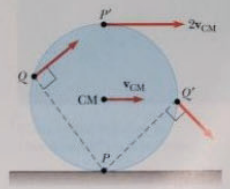


Figure 10.31 All points on a rolling body move in a direction perpendicular to an axis through the contact point P. The center of the body moves with a velocity $\mathbf{v}_{CM}$, and the point P' moves with a velocity $2\mathbf{v}_{CM}$.

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We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2} I_p \omega^2 \qquad [10.41]$$

where I_p is the moment of inertia about the axis through P.

A useful theorem called the **parallel axis theorem** enables us to calculate the moment of inertia, I_p, through any axis parallel to the axis through the center of mass of a body. This theorem states that

$$I_p = I_{CM} + MR^2 \qquad [10.42]$$

where R is the distance from the center-of-mass axis to the parallel axis, and M is the total mass of the body. Substitution of this expression into Equation 10.41 gives

$$K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} MR^2 \omega^2$$

Total kinetic energy of a rolling body

$$K = \tfrac{1}{2} I_{CM} \omega^2 + \tfrac{1}{2} M v_{CM}^2 \qquad [10.43]$$

where we have used the fact that $v_{CM} = R\omega$.

We can think of Equation 10.43 as follows: The first term on the right, $\frac{1}{2} I_{CM} \omega^2$, represents the rotational kinetic energy about the center of mass, and the term $\frac{1}{2} M v_{CM}^2$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that **the total kinetic energy of an object undergoing rolling motion is the sum of a rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

We can use energy methods to treat a class of problems concerning the rolling motion of a rigid body down a rough incline. We shall assume that the rigid body in Figure 10.32 does not slip and is released from rest at the top of the incline. Note that rolling motion is possible only if a frictional force is present between the object and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy takes place because the contact point is at rest relative to the surface at any instant. However, if the rigid body were to slide, mechanical energy would decrease as motion progressed.

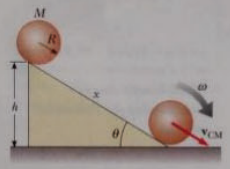


Figure 10.32 A round object rolling down an incline. Mechanical energy remains constant if no slipping occurs and there is no rolling friction.

Thinking Physics 6

A common classroom demonstration is to spin on a rotating stool with heavy weights in your outstretched hands and bring the weights close to your body. The result is an increase in the angular velocity, in agreement with the principle of conservation of angular momentum (see Fig. 10.33). Let us imagine we perform such a demonstration, and moving the weights inward results in halving the moment of inertia, and, therefore, doubling the angular velocity. If we consider the rotational kinetic energy, we see that the energy is *doubled* in this situation. Thus, angular momentum is conserved, but kinetic energy is not. Where does this extra energy come from?

Reasoning As you spin with the weights in your hands, there is a tension force in your arms that is causing the weights to move in a circular path. As you bring the weights in, this force from your arms is doing work on the weights, because it is moving them through a radial displacement. As the angular velocity increases in response to the weights moving inward, you must apply more and more force to keep the weights moving in the smaller radius circular path. Thus, the work that you do in bringing the weights in increases as they are brought in. It is this work that you do with your arms

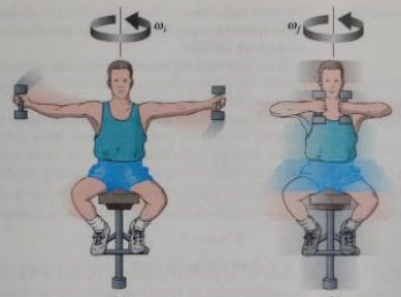


Figure 10.33 (Thinking Physics 6) (a) The student is given an initial angular speed while holding two masses as shown. (b) When the masses are pulled in close to the body, the angular speed of the system increases.

that appears as the increase in rotational kinetic energy of the system. You can perform this work due to the energy stored in your body from previous meals, so you are transforming potential energy stored in your body into rotational kinetic energy.

CONCEPTUAL PROBLEM 4

Suppose a pencil is balanced on a perfectly frictionless table. If it falls over, what is the path followed by the center of mass of the pencil?

CONCEPTUAL PROBLEM 5

In some motorcycle races, the riders drive over small hills, and the motorcycle becomes airborne for a short time. If the motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?

CONCEPTUAL PROBLEM 6

Suppose you are designing a car for a coasting race—the cars in this race have no engines, they simply coast down a hill. Do you want large wheels or small wheels? Do you want solid, disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?



(Conceptual Problem 5)
(Tom Raymond/Tony Stone Images)

Example 10.14 Sphere Rolling Down an Incline

If the object in Figure 10.32 is a solid sphere, calculate the speed of its center of mass at the bottom and determine the magnitude of the linear acceleration of the center of mass.

Solution For a uniform solid sphere, $I_{CM} = \frac{2}{5} MR^2$, and therefore, conservation of mechanical energy gives

$$v_{CM} = \left(\frac{2gh}{1 + \frac{\frac{2}{5} MR^2}{MR^2}} \right)^{1/2} = \left(\tfrac{10}{7} gh \right)^{1/2}$$

The vertical displacement is related to the displacement x along the incline through the relationship $h = x \sin \theta$.

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Hence, after squaring both sides, we can express the equation above as

$$v_{CM}^2 = \frac{10}{7} g x \sin\theta$$

Comparing this with the familiar expression from kinematics, $v_{CM}^2 = 2a_{CM}x$, we see that the acceleration of the center of mass is

$$a_{CM} = \frac{5}{7} g \sin\theta$$

These results are quite interesting in that both the speed and the acceleration of the center of mass are *independent* of the mass and radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline.**

If we repeated the calculations for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific body. In all cases, the acceleration of the center of mass is *less* than $g \sin\theta$, the value it would have if the plane were frictionless and no rolling occurred.

EXERCISE 13 The center of mass of a pitched baseball (radius = 3.8 cm) moves at 38 m/s. The ball spins about an axis through its center of mass with an angular speed of 125 rad/s. Calculate the ratio of the rotational energy to the translational kinetic energy. Treat the ball as a uniform sphere. Answer 1/160

EXERCISE 14 An automobile tire, considered as a solid disk, has a radius of 25 cm and a mass of 6.0 kg. Find its rotational kinetic energy when it is rotating about an axis through its center at an angular speed of 2.0 rev/s. Answer 15 J

SUMMARY

The **instantaneous angular speed** of a particle rotating in a circle or of a rigid body rotating about a fixed axis is

$$\omega = \frac{d\theta}{dt} \qquad [10.3]$$

where ω is in rad/s or in s^{-1}.

The **instantaneous angular acceleration** of a rotating body is

$$\alpha = \frac{d\omega}{dt} \qquad [10.5]$$

and has units of rad/s^2 or s^{-2}.

When a rigid body rotates about a fixed axis, every part of the body has the same angular velocity and the same angular acceleration. However, different parts of the body, in general, have different linear velocities and different linear accelerations.

If a particle (or body) undergoes rotational motion about a fixed axis under constant angular acceleration, α, one can apply equations of kinematics by analogy with kinematic equations for linear motion with constant linear acceleration:

$$\omega = \omega_0 + \alpha t \qquad [10.6]$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad [10.7]$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad [10.8]$$

$$\theta = \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \qquad [10.9]$$

When a particle rotates about a fixed axis, the angular speed and angular acceleration are related to the linear speed and tangential linear acceleration through the relationships

$$v = r\omega \qquad [10.10]$$

$$a_t = r\alpha \qquad [10.11]$$

The **moment of inertia of a system of particles** is

$$I = \sum m_i r_i^2 \qquad [10.14]$$

If a rigid body rotates about a fixed axis with angular speed ω, its **kinetic energy** can be written

$$K = \tfrac{1}{2} I\omega^2 \qquad [10.15]$$

where I is the moment of inertia about the axis of rotation.

The **torque,** τ, due to a force **F** about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad [10.17]$$

Given two vectors, **A** and **B**, their **cross product, A × B,** is a vector **C** having the magnitude

$$C = |AB \sin\theta| \qquad [10.19]$$

where θ is the angle between **A** and **B**. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and its sense is determined by the right-hand rule. Some properties of the cross product include the facts that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{A} = 0$.

The net torque acting on a particle is proportional to the angular acceleration of the particle, and the proportionality constant is the moment of inertia, I:

$$\sum \tau = I\alpha \qquad [10.22]$$

The **angular momentum, L,** of a particle of linear momentum $\mathbf{p} = m\mathbf{v}$ is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad [10.23]$$

where **r** is the vector position of the particle relative to an origin in an inertial frame. If ϕ is the angle between **r** and **p**, the magnitude of **L** is

$$L = mvr \sin\phi = I\omega \qquad [10.24]$$

The **net torque** acting on a particle is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \qquad [10.27]$$

The law of conservation of angular momentum states that the total angular momentum of a system remains constant if the net external torque acting on the system is zero:

$$\text{When } \sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = 0, \text{ then} \qquad [10.29]$$

$$\mathbf{L} = \text{constant} \qquad [10.30]$$

Angular momentum is an inherent property of atoms, molecules, and their constituents. Such systems have discrete (quantized) values of angular momenta that are integer multiples of $h/2\pi$, where h is Planck's constant.

The **total kinetic energy** of a rigid body, such as a cylinder, that is rolling on a rough surface without slipping equals the rotational kinetic energy about the body's center of mass $\frac{1}{2} I_{CM}\omega^2$ plus the translational kinetic energy of the center of mass $\frac{1}{2} M v_{CM}^2$:

$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M v_{CM}^2 \qquad [10.43]$$

In this expression, v_{CM} is the speed of the center of mass and $v_{CM} = R\omega$ for pure rolling

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CONCEPTUAL QUESTIONS

1. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled—as the tape is pulled off it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular velocity of the reel change with time? If the tape mechanism is suddenly turned on so that the tape is quickly pulled with a large force, is the tape more likely to break when pulled from a nearly full reel or a nearly empty reel?
2. If a car's wheels are replaced with wheels of a larger diameter, will the reading of the speedometer change? Explain.
3. What is the magnitude of the angular velocity, ω, of the second hand of a clock? What is the direction of ω as you view a clock hanging vertically? What is the magnitude of the angular acceleration, α, of the second hand?
4. If you see an object rotating, is there necessarily a net torque acting on it?
5. In order for a helicopter to be stable as it flies, it must have two propellers. Why?
6. The moment of inertia of an object depends on the choice of rotation axis, as suggested by the parallel axis theorem. Argue that an axis passing through the center of mass of an object must be the axis with the smallest moment of inertia.
7. Suppose you remove two eggs from the refrigerator, one hard-boiled and the other uncooked. You wish to determine which is the hard-boiled egg without breaking the eggs. This can be done by spinning the two eggs on the floor and comparing the rotational motions. Which egg spins faster? Which rotates more uniformly? Explain.
8. A ladder rests inclined against a wall. Would you feel safer climbing up the ladder if you were told that the floor is frictionless but the wall is rough, or that the wall is frictionless but the floor is rough? Justify your answer.
9. Often when a high diver wants to flip in midair, she will draw her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?
10. If the net force acting on a system is zero, then is it necessarily true that the net torque on it is also zero?
11. Why do tightrope walkers carry a long pole to help balance themselves?
12. Two solid spheres are rolled down a hill—a large, massive sphere and a small sphere with low mass. Which one reaches the bottom of the hill first? Next, we roll a large, low density sphere, and a small high density sphere, and both spheres have the same mass. Which one wins in this case?
13. Consider an object in the shape of a hoop lying in the xy plane with all of its mass concentrated on its rim. In two separate experiments, the hoop is rotated by an external agent from rest to an angular speed ω. In one experiment, the rotation occurs about the z axis through the center of the hoop. In the other experiment, the rotation occurs about an axis parallel to z passing through a point P on the rim of the hoop. Which rotation requires more work?
14. Vector $\mathbf{A}$ is in the negative y direction and vector $\mathbf{B}$ is in the negative x direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ and (b) $\mathbf{B} \times \mathbf{A}$?
15. If global warming occurs over the next century, it is likely that the polar ice caps of the Earth will melt and the water will be distributed closer to the Equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

PROBLEMS

Section 10.2 Rotational Kinematics

1. What is the angular speed in radians per second of (a) the Earth in its orbit about the Sun and (b) the Moon in its orbit about the Earth?
2. An airliner arrives at the terminal and the engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take the rotor to come to rest?
3. An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn during the time found in (a)?
4. A centrifuge in a medical laboratory rotates at an angular speed of 3600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.
5. A dentist's drill starts from rest. After 3.20 s of constant angular acceleration, the drill rotates at a rate of 2.51×10^4 rev/min. (a) Find the drill's angular acceleration. (b) Determine the angle (in radians) through which the drill rotates during this period.
6. The angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$ rad. Determine the angular position, angular speed, and angular acceleration of the door at $t = 0$ and $t = 3.00$ s.
7. The tub of a washer goes into its spin cycle, starting from rest and reaching an angular speed of 5.00 rev/s in 8.00 s. At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The tub slows to rest in 12.0 s. Through how many revolutions does the tub turn? Assume constant angular acceleration while the machine is starting and stopping.
8. A rotating wheel requires 3.00 s to rotate 37.0 revolutions. Its angular velocity at the end of the 3.00 s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?

Section 10.3 Relations Between Angular and Linear Quantities

9. A discus thrower accelerates a discus from rest to a speed of 25.0 m/s by whirling it through 1.25 rev. Assume the discus moves on the arc of a circle 1.00 m in radius. (a) Calculate the final angular speed of the discus. (b) Determine the magnitude of the angular acceleration of the discus, assuming it to be constant. (c) Calculate the acceleration time.
10. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming no slipping. (b) What is the final rotational speed of a tire in revolutions per second?
11. A disk 8.00 cm in radius rotates at a constant rate of 1200 rev/min about its central axis. Determine (a) its angular speed, (b) the linear speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.
12. A 6.00-kg block is released from A on a frictionless track shown in Figure P10.12. Determine the radial and tangential components of acceleration for the block at P.

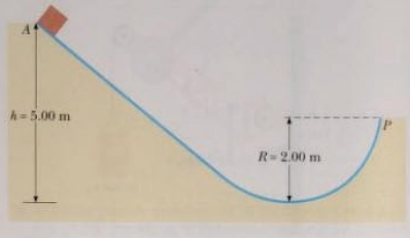


Figure P10.12

13. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s^2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

Section 10.4 Rotational Kinetic Energy

14. Three particles are connected by rigid rods of negligible mass lying along the y axis (Fig. P10.14). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the linear speed of each particle and the total energy evaluated from $\Sigma\frac{1}{2}m_iv_i^2$.

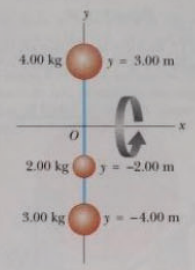


Figure P10.14

15. The four particles in Figure P10.15 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy plane about the z axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the z axis and (b) the rotational energy of the system.

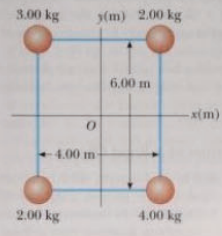


Figure P10.15

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Section 10.5 Torque and the Vector Product

16. The fishing pole in Figure P10.16 makes an angle of 20.0° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the fisher's hand?

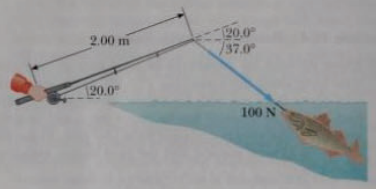


Figure P10.16

17. In Figure P10.17, find the net torque on the wheel about the axle through O if $a = 10.0$ cm and $b = 25.0$ cm.

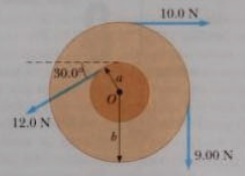


Figure P10.17

18. Given $\mathbf{M} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{N} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.
19. Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between $\mathbf{A}$ and $\mathbf{B}$.
20. A force of $\mathbf{F} = 2.00\mathbf{i} + 3.00\mathbf{j}$ N is applied to an object that is pivoted about a fixed axis aligned along the z coordinate axis. If the force is applied at the point whose vector position is $\mathbf{r} = (4.00\mathbf{i} + 5.00\mathbf{j} + 0\mathbf{k})$ m, find (a) the magnitude of the net torque about the z axis and (b) the direction of the torque vector, $\boldsymbol{\tau}$.
21. A student claims that she has found a vector $\mathbf{A}$ such that $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Do you believe this claim? Explain.

Section 10.6 Equilibrium of a Rigid Object

22. A ladder of weight 400 N and length 10.0 m is placed against a smooth vertical wall. A person weighing 800 N stands on the ladder 2.00 m from the bottom as measured along the ladder. The foot of the ladder is 8.00 m from the bottom of the wall. Draw a free-body diagram of the ladder. Calculate the force exerted by the wall and the normal force exerted by the floor on the ladder.
23. A 1500 kg automobile has a wheel base (the distance between the axles) of 3.00 m. The center of mass of the automobile is on the center line at a point 1.20 m behind the front axle. Find the force exerted by the ground on each wheel.
24. A uniform plank of length 6.00 m and mass 30.0 kg rests horizontally on a scaffold, with 1.50 m of the plank hanging over one end of the scaffold. Draw a free-body diagram of the plank. How far can a painter of mass 70.0 kg walk on the overhanging part of the plank before it tips?
25. A 1200-N uniform boom is supported by a cable as in Figure P10.25. The boom is pivoted at the bottom, and a 2000-N object hangs from its top. Find the tension in the cable and the components of the reaction force on the boom by the floor.

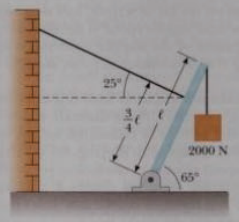


Figure P10.25

26. A crane of mass 3000 kg supports a load of 10 000 kg as in Figure P10.26. The crane is pivoted with a smooth pin at A and rests against a smooth support at B. Find the reaction forces at A and B.

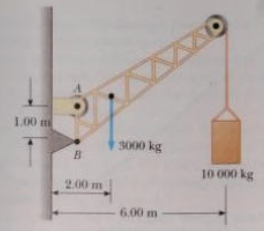


Figure P10.26

Section 10.7 Relation Between Torque and Angular Acceleration

27. A model airplane the mass of which is 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque that the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.
28. The combination of an applied force and a frictional force produces a constant total torque of 36.0 N·m on a wheel rotating about a fixed axis. The applied force acts for 6.00 s, during which time the angular speed of the wheel increases from 0 to 10.0 rad/s. The applied force is then removed, and the wheel comes to rest in 60.0 s. Find (a) the moment of inertia of the wheel, (b) the magnitude of the frictional torque, and (c) the total number of revolutions of the wheel.

Section 10.8 Angular Momentum

29. A light rigid rod 1.00 m in length rotates in the xy plane about a pivot through the rod's center. Two particles of masses of 4.00 kg and 3.00 kg are connected to its ends (Fig. P10.29). Determine the angular momentum of the system about the origin at the instant the speed of each particle is 5.00 m/s.

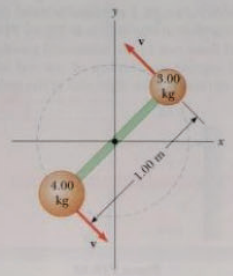


Figure P10.29

30. A 1.50-kg particle moves in the xy plane with a velocity of $\mathbf{v} = (4.20\mathbf{i} - 3.60\mathbf{j})$ m/s. Determine the angular momentum of the particle when its position vector is given by $\mathbf{r} = (1.50\mathbf{i} + 2.20\mathbf{j})$ m.
31. The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\,\mathbf{i} + 5.00t\mathbf{j})$ m. Determine the angular momentum of the particle as a function of time.
32. An airplane of mass 12 000 kg flies level to the ground at an altitude of 10.0 km with a constant speed of 175 m/s relative to the Earth. (a) What is the magnitude of the airplane's angular momentum relative to a ground observer directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line?

Section 10.9 Conservation of Angular Momentum

Section 10.10 Quantization of Angular Momentum (Optional)

33. A ball having mass m is fastened at the end of a flagpole that is connected to the side of a tall building at point P shown in Figure P10.33. The length of the flagpole is ℓ and it makes an angle θ with the horizontal. If the ball become loose and starts to fall, determine the angular momentum (as a function of time) of the ball about point P. Neglect air resistance.

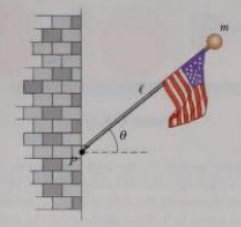


Figure P10.33

34. A student sits on a rotating stool holding two weights, each of mass 3.00 kg. When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is 3.00 kg·m² and is assumed to be constant. The student pulls the weights horizontally to 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the student before and after the weights are pulled in.
35. A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of 500 kg·m² and a radius of

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2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

36. The ball in Figure 10.25 has a mass of 0.120 kg. The distance of the ball from the center of rotation is originally 40.0 cm, and the ball is moving with a speed of 80.0 cm/s. The string is pulled downward 15.0 cm through the hole in the frictionless table. Determine the work done on the ball. (*Hint:* Consider the change of kinetic energy.)

37. A puck of mass 80.0 g and radius 4.00 cm slides along an air table at a speed of 1.50 m/s as shown in Figure P10.37a. It makes a glancing collision with a second puck of radius 6.00 cm and mass 120 g (initially at rest) such that their rims just touch. The pucks stick together and spin after the collision (Fig. P10.37b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular velocity about the center of mass?



Figure P10.37

38. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular frequency of the electron's motion.

Section 10.11 Rotation of Rigid Bodies (Optional)

39. Attention! About face! Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn around an axis through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

40. A horizontal 800-N merry-go-round is a solid disk of radius 1.50 m, started from rest by a constant horizontal force of 50.0 N applied tangentially at its edge. Find the kinetic energy of the disk after 3.00 s.

41. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational energy about its center of mass, and (c) its total energy.

42. Consider two masses connected by a string that passes over a pulley having a moment of inertia of I about its axis of rotation, as in Figure P10.42. The string does not slip on the pulley, and the system is released from rest. Use the principle of conservation of energy to find the linear speeds of the masses after m_2 descends through a distance h and the angular speed of the pulley at this time.

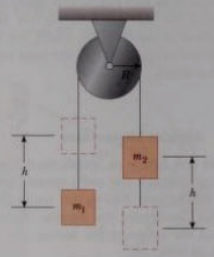


Figure P10.42

43. A uniform rod of length L and mass M is free to rotate about a frictionless pivot at one end, as in Figure P10.43. The rod is released from rest in the horizontal position. What are the *initial* angular acceleration of the rod and the *initial* linear acceleration of the right end of the rod?

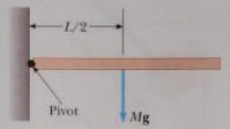


Figure P10.43

44. A bicycle wheel has a diameter of 64.0 cm and a mass of 1.80 kg. The bicycle is placed on a stationary stand on rollers and a resistive force of 120 N is applied to the rim of the tire. (Assume that the wheel is a hoop with all the mass concentrated on the outside radius.) What force must be applied by a chain passing over a 9.00-cm diameter sprocket in order to give the wheel an acceleration of 4.50 rad/s^2? What if you shifted to a 5.60-cm diameter sprocket?

45. (a) A uniform solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.45). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) Repeat part (a) using a uniform hoop.

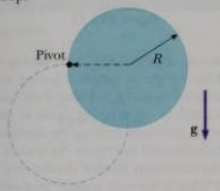


Figure P10.45

46. A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a disk having radius $R = 0.25$ m, mass $M = 10.0$ kg. In addition, they are allowed to move on a fixed block-wedge of angle $\theta = 30.0°$ as in Figure P10.46. The coefficient of kinetic friction is 0.360 for both blocks. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks, and (b) the tensions in the string on both sides of the pulley.

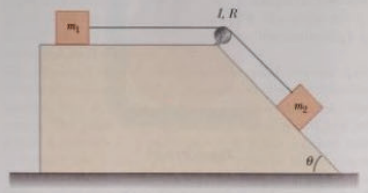


Figure P10.46

47. A can of condensed mushroom soup has mass 215 g, height 10.8 cm, and outer diameter 6.38 cm. It is placed at rest on the top of a 3.00-m long incline, which is at 25.0° to the horizontal. Assuming energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline.

Additional Problems

48. An electric motor can accelerate a Ferris wheel of moment of inertia $I = 20\,000$ kg·m^2 from rest to 10.0 rev/min in 12.0 s. When the motor is turned off, friction causes the wheel to slow down from 10.0 to 8.00 rev/min in 10.0 s. Determine (a) the torque generated by the motor to bring the wheel to 10.0 rev/min and (b) the power needed to maintain this rotational speed.

49. Two blocks, as shown in Figure P10.49, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving up with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

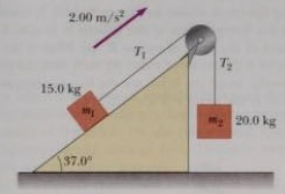


Figure P10.49

50. As a result of friction, the angular speed of a wheel changes with time according to

$$\frac{d\theta}{dt} = \omega_0 e^{-\sigma t}$$

where ω_0 and σ are constants. The angular speed changes from 3.50 rad/s at $t = 0$ to 2.00 rad/s at $t = 9.30$ s. Use this information to determine σ and ω_0. Then determine (a) the magnitude of the angular acceleration at $t = 3.00$ s, (b) the number of revolutions the wheel makes in the first 2.50 s, and (c) the number of revolutions it makes before coming to rest.

51. The pulley shown in Figure P10.51 has radius R and moment of inertia I. One end of the mass m is connected to a spring of force constant k, and the other end is fastened to a cord wrapped around the pulley. The pulley axle and the incline are frictionless. If the pulley is wound counterclock-

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wise in order to stretch the spring a distance d from its unstretched position and then released from rest, find (a) the angular speed of the pulley when the spring is again unstretched and (b) a numerical value for the angular speed at this point if $I = 1.00\ \text{kg} \cdot \text{m}^2$, $R = 0.300\ \text{m}$, $k = 50.0\ \text{N/m}$, $m = 0.500\ \text{kg}$, $d = 0.200\ \text{m}$, and $\theta = 37.0°$.

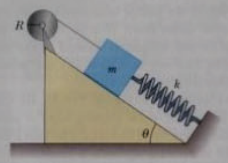


Figure P10.51

52. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P10.52. It starts from rest with the lowest point of the sphere at height h above the bottom of the loop of radius R. (a) What is the minimum value of h (in terms of r and R) such that the sphere completes the loop? (b) What are the force components on the sphere at the point P if $h = 3R$?

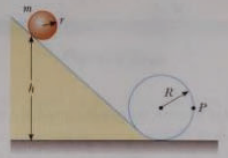


Figure P10.52

53. Two astronauts (Fig. P10.53), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. (a) Calculate the magnitude of the angular momentum of the system by treating the astronauts as particles and (b) the rotational energy of the system. By pulling on the rope, the astronauts shorten the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are their new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronauts in shortening the rope?

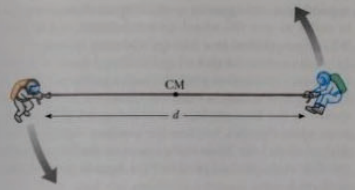


Figure P10.53

54. Two astronauts (Fig. P10.53), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Calculate (a) the magnitude of the angular momentum of the system by treating the astronauts as particles and (b) the rotational energy of the system. By pulling on the rope, the astronauts shorten the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are their new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronauts in shortening the rope?

55. Figure P10.55 shows a vertical force applied tangentially to a uniform cylinder of weight w. The coefficient of static friction between the cylinder and all surfaces is 0.500. Find, in terms of w, the maximum force **F** that can be applied without causing the cylinder to rotate. (*Hint:* When the cylinder is on the verge of slipping, both friction forces are at their maximum values. Why?)

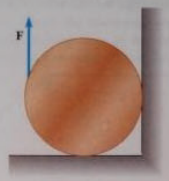


Figure P10.55

56. A common physics demonstration (Fig. P10.56) consists of a ball resting at the end of a board of length ℓ that is elevated at angle θ with the horizontal. A light cup is attached to the board at r_c so that it will catch the ball when the support stick is suddenly removed. (a) Show that the ball will lag behind the falling board when θ is less than 35.3°, and (b) the ball will fall into the cup when the board is supported at this limiting angle and the cup is placed at

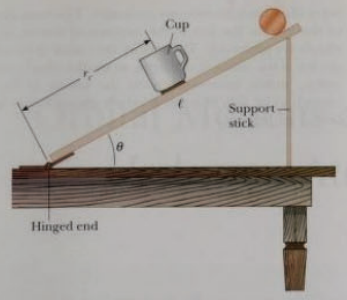


Figure P10.56

$$r_c = \frac{2\ell}{3\cos\theta}$$

(c) If a ball is at the end of a 1.00-m stick at this critical angle, show that the cup must be 18.4 cm from the moving end.

Spreadsheet Problem

S1. A disk having a moment of inertia of 100 kg·m² is free to rotate about a fixed axis through its center as in Figure 10.1. A tangential force whose magnitude ranges from $F = 0$ to $F = 50.0$ N can be applied at any distance ranging from $R = 0$ to $R = 3.00$ m measured from the axis of rotation. Using Spreadsheet 10.1, find values of F and R that will cause the disk to complete two revolutions in 10.0 s. Are there unique values for F and R?

ANSWERS TO CONCEPTUAL PROBLEMS

1. As the smokestack rotates around its base, each higher portion of the smokestack falls with an increasing tangential acceleration. The tangential acceleration of a given point on the smokestack is proportional to the distance of that portion from the base. As the acceleration increases, eventually higher portions of the smokestack will experience an acceleration higher than that which could result from gravity alone. This can only happen if these portions are being pulled downward by a force in addition to the gravitational force. The force that does this is the shear force from lower portions of the smokestack. Eventually, the shear force to provide this acceleration is larger than the smokestack can withstand, and it breaks.
2. Yes, all points on the wheel have the same angular speed. This is why we use angular quantities to describe rotational motion. Not all points on the wheel have the same linear speed. The point at $r = 0$ has zero linear speed and zero linear acceleration; a point at $r = R/2$ has a linear speed $v = R\omega/2$ and a linear acceleration equal to the centripetal acceleration $v^2/(R/2) = R\omega^2/2$ (the tangential acceleration is zero at all points because ω is constant). A point on the rim at $r = R$ has a linear speed $v = R\omega$ and a linear acceleration $R\omega^2$.
3. There are two major differences between torque and work. The primary difference is that the displacement in the expression for work is directed *along* the force, while the important distance in the torque expression is *perpendicular* to the force. The second difference involves whether there is motion or not—in the case of work, there is work done only if the force succeeds in causing a displacement of the point of application of the force. On the other hand, a force applied at a perpendicular distance from a rotation axis results in a torque whether there is motion or not.

   As far as units are concerned, the mathematical expressions for both work and torque result in the product of newtons and meters, but this product is called a joule in the case of work and remains as a newton-meter in the case of torque.
4. On a frictionless table, the only forces on the pencil are the gravitational force, at the center of mass, and the normal force from the table, at the tip. These forces result in a torque on the pencil, causing it to rotate. Both of these forces, however, are vertical—there are no horizontal forces on the pencil. As a result, the center of mass of the pencil must fall straight downward. As the pencil falls, the tip slides to the side, so that the center of mass of the pencil follows a straight line downward to the table.
5. As the motorcycle tire leaves the ground, the friction between the tire and the ground suddenly disappears. If the motorcycle driver keeps the throttle open while leaving the ground, the drive tire will increase its angular velocity, since it no longer experiences the friction force from the ground. The airborne motorcycle is now an isolated system, and its angular momentum must be conserved. The increase in angular momentum of the tire (directed to the left of the motorcycle) must be compensated by an increase in angular momentum of the entire motorcycle (to the right). This ro-

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tation results in the nose of the motorcycle rising and the tail dropping.

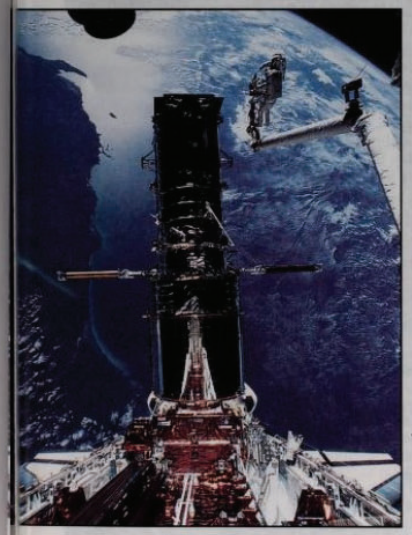
6. In general, you want the rotational kinetic energy of the system to be as small a fraction as possible of the total energy—you want translation, not rotation! You want the wheels to have as little moment of inertia as possible, so that they represent the lowest resistance to changes in rotational motion. Disk-like wheels would have lower moment of inertia than hoop-like wheels, so disks are preferable. The lower the mass of the wheels, the less is the moment of inertia, so light wheels are preferable. The smaller the radius of the wheels, the less is the moment of inertia, so smaller wheels are preferable, within limits—you want the wheels to be large enough to be able to travel relatively smoothly over irregularities in the road.

11

Orbital Motions and the Hydrogen Atom

We began our study of mechanics with translational motion and the forces that cause it. If we know the forces acting on a system and its initial conditions, we can predict the future of the system. However, describing motion in this manner is often tedious and time consuming. Fortunately, we can often follow the simpler approach of using conservation principles. We easily solved many interesting problems involving motion by recognizing that certain fundamental quantities—such as energy and momentum—were conserved.

In this chapter we return to Newton's universal law of gravitation—one of the fundamental force laws in nature—and show how it, together with Newton's laws of motion, enables us to understand a variety of familiar orbital motions, such as the motions of planets and Earth satellites. Newton's theory of gravitation evolved out of his studies of the motions of the



◀ **Astronauts F. Story Musgrave and Jeffrey A. Hoffman complete the final space walks needed to repair the Hubble Space Telescope in December 1993.** *(Courtesy NASA)*

CHAPTER OUTLINE

11.1 Newton's Universal Law of Gravity Revisited
11.2 Kepler's Laws
11.3 The Universal Law of Gravity and the Motions of Planets
11.4 Energy Considerations in Planetary and Satellite Motion
11.5 Atomic Spectra and the Bohr Theory of Hydrogen

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Moon and planets, for which he used the groundwork provided by Copernicus, Brahe, Kepler, and other astronomers. In examining this evolution, we shall once again make use of conservation of energy and angular momentum.

We conclude this chapter with a discussion of Niels Bohr's famous model of the hydrogen atom, which represents an interesting mixture of classical physics (Newton's laws of motion and the electromagnetic force) and quantum physics (quantization of angular momentum). Although Bohr's theory contains ideas that are contrary to classical physics, his model successfully predicts the observed spectral lines of hydrogen.

11.1 • NEWTON'S UNIVERSAL LAW OF GRAVITY REVISITED

Prior to 1686, many data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces that caused those motions was not yet attainable. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from the first law, that a net force had to be acting on the Moon. If not, the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force between Moon and Earth was an attractive force. He also concluded that there could be nothing special about the Earth–Moon system or the Sun and its planets that would cause gravitational forces to act on them alone. He wrote,

> I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centers about which they revolve; and thereby compared the force requisite to keep the Moon in her orb with force of gravity at the surface of the Earth; and found them answer pretty nearly.

As you should recall from Chapter 5, every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. If two particles have masses m_1 and m_2 and are separated by a distance r, the magnitude of the gravitational force between them is

Universal law of gravity •

$$F_g = G\frac{m_1 m_2}{r^2} \qquad [11.1]$$

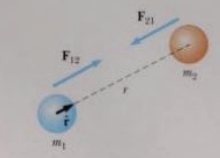


Figure 11.1 The gravitational force between two particles is attractive. The unit vector $\hat{\mathbf{r}}$ is directed from m_1 to m_2. Note that $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

where G is the *gravitational constant* the value of which in SI units is

$$G = 6.672 \times 10^{-11}\,\frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}} \qquad [11.2]$$

The force law given by Equation 11.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles. We can express this attractive force in vector form by defining a unit vector, $\hat{\mathbf{r}}$, directed from m_1 to m_2, as shown in Figure 11.1. The force on m_1 due to m_2 is

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \qquad [11.3]$$

Likewise, by Newton's third law, the force on m_2 due to m_1, designated $\mathbf{F}_{21}$, is equal in magnitude to $\mathbf{F}_{12}$ and in the opposite direction. That is, these forces form an action–reaction pair, and $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

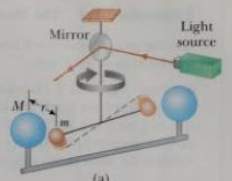
The gravitational force exerted by a finite-sized, spherically symmetric mass distribution on a particle outside the sphere is the same as if the entire mass of the sphere were concentrated at its center. For example, the force on a particle of mass m at the Earth's surface has the magnitude

$$F_g = G\frac{M_E m}{R_E^2}$$

where M_E is the Earth's mass and R_E is the Earth's radius. This force is directed toward the center of the Earth.

Measurement of the Gravitational Constant

The gravitational constant, G, was first measured in an important experiment by Sir Henry Cavendish in 1798. The apparatus he used consisted of two small spheres, each of mass m, fixed to the ends of a light horizontal rod suspended by a thin wire, as in Figure 11.2a. Two large spheres, each of mass M, are then placed near the smaller spheres. The attractive force between the smaller and larger spheres causes the rod to rotate and twist the wire. If the system is oriented as shown in Figure 11.2a, the rod rotates clockwise when viewed from the top. The angle through which it rotates is measured by the deflection of a light beam that is reflected from a mirror attached to the wire. The experiment is carefully repeated with different masses at various separations. In addition to providing a value for G, the results show that the force is attractive, proportional to the product mM, and inversely proportional to the square of the distance r.



(a)



(b)

Figure 11.2 (a) Schematic diagram of the Cavendish apparatus for measuring G. The smaller spheres of mass m are attracted to the large spheres of mass M, and the rod rotates through a small angle. A light beam reflected from a mirror on the rotating apparatus measures the angle of rotation. The dashed line represents the original position of the rod. (b) Photograph of a student Cavendish apparatus. *(Courtesy of PASCO Scientific)*

Thinking Physics 1

A student drops a ball and considers why the ball falls to the ground as opposed to the situation in which the ball stays stationary and the Earth moves up to meet it. The student comes up with the following explanation: "The Earth is much more massive than the ball, so the Earth pulls much harder on the ball than the ball pulls on the Earth. Thus, the ball falls while the Earth remains stationary." What do you think about this explanation?

Reasoning According to Newton's universal law of gravity, the force between the ball and the Earth depends on the product of their masses, so both forces—that of the ball on the Earth and that of the Earth on the ball—are equal in magnitude. This follows also, of course, from Newton's third law. The ball has large motion compared to the Earth because, according to Newton's second law, the force of equal magnitude gives a much greater acceleration to the small mass of the ball.

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Example 11.1 The Mass of the Earth

Use the gravitational force law to find an approximate value for the mass of the Earth.

Solution Figure 11.3, obviously not to scale, shows a baseball falling toward the Earth at a location where the acceleration due to gravity is g. We know from Chapter 5 that the magnitude of the gravitational force exerted on the baseball by the Earth is the same as the weight of the ball, $w = m_b g$. That is,

$$m_b g = G\frac{M_E m_b}{R_E^2}$$

We can divide each side of this equation by m_b and solve for the mass of the Earth, M_E:

$$M_E = \frac{gR_E^2}{G}$$

The falling baseball is close enough to the Earth so that the distance of separation between the center of the ball and the center of the Earth can be taken as the radius of the Earth, 6.38×10^6 m. Thus, the mass of the Earth is

$$M_E = \frac{(9.80\ \text{m/s}^2)(6.38 \times 10^6\ \text{m})^2}{6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2} = 5.98 \times 10^{24}\ \text{kg}$$



Figure 11.3 (Example 11.2) A baseball falling toward the Earth (not drawn to scale).

Example 11.2 Gravity and Altitude

Derive an expression that shows, for any point above the Earth's surface, how the acceleration due to gravity varies with distance from the center of the Earth.

Solution The falling baseball of Example 11.1 can be used here also. Now, however, assume that the ball is situated some arbitrary distance, $r > R_E$, from the Earth's center. The first equation in Example 11.1, with r replacing R_E and m_b removed from both sides, becomes

$$g = G\frac{M_E}{r^2}$$

This indicates that the acceleration due to gravity at a point above the Earth's surface decreases as the inverse square of the distance between the point and the center of the Earth. Our assumption of Chapter 2 that objects fall with a constant acceleration is obviously incorrect in light of this example. For short falls, however, the change in g is so small that neglecting the variation does not introduce a significant effect in our results.

Because the weight of an object is mg, a change in the value of g produces a change in weight. For example, if you weigh 800 N at the surface of the Earth, you will weigh only 200 N at a height above the Earth equal to the radius of the Earth. Also, if the distance of an object from the Earth becomes infinitely large, the weight approaches zero. Values of g at various altitudes are given in Table 11.1.

TABLE 11.1 Free-Fall Acceleration, g, at Various Altitudes Above the Earth's Surface

Altitude h (km)	g(m/s^2)
1000	7.33
2000	5.68
3000	4.53
4000	3.70
5000	3.08
6000	2.60
7000	2.23
8000	1.93
9000	1.69
10 000	1.49
50 000	0.13
∞	0

EXERCISE 1 If an object weighs 270 N at the Earth's surface, what will it weigh at an altitude equal to twice the radius of the Earth? Answer 30 N

EXERCISE 2 Determine the magnitude of the acceleration due to gravity at an altitude of 500 km. By what percentage is the weight of a body reduced at this altitude? Answer 8.43 m/s^2; 14%

11.2 · KEPLER'S LAWS

The movements of the planets, stars, and other celestial bodies have been observed by people for thousands of years. Early in history, scientists regarded the Earth as the center of the Universe. This so-called geocentric model was elaborated and formalized by the Greek astronomer Claudius Ptolemy in the second century A.D. and was accepted for the next 1400 years. In 1543 the Polish astronomer Nicolaus Copernicus (1473–1543) suggested that the Earth and the other planets revolve in circular orbits about the Sun (the heliocentric hypothesis).

The Danish astronomer Tycho Brahe (1546–1601) made accurate astronomical measurements over a period of 20 years and provided the basis for the currently accepted model of the Solar System. It is interesting to note that these precise observations, made on the planets and 777 stars, were carried out with nothing more elaborate than a large sextant and compass; the telescope had not yet been invented.

The German astronomer Johannes Kepler, who was Brahe's assistant, acquired Brahe's astronomical data and spent about 16 years trying to deduce a mathematical model for the motions of the planets. After many laborious calculations, he found that Brahe's precise data on the revolution of Mars about the Sun provided the answer. Kepler's analysis first showed that the concept of circular orbits about the Sun had to be abandoned. He eventually discovered that the orbit of Mars could be accurately described by an ellipse with the Sun at one focus. He then generalized this analysis to include the motions of all planets. The complete analysis is summarized in three statements, known as **Kepler's laws:**



Johannes Kepler (1571–1630)
A German astronomer, Kepler is best known for developing the laws of planetary motion based on the careful observations of Tycho Brahe. After spending several years trying to work out a "regular-solid theory" of the planets, he concluded that the Copernican view of circular planetary orbits had to be abandoned for the view that the planetary orbits are ellipses with the Sun always at one of the foci. *(Art Resource)*

1. Every planet moves in an elliptical orbit with the Sun at one of the focal points.
2. The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

· *Kepler's laws*

Newton demonstrated that these laws were consequences of a simple force that exists between any two masses. Newton's universal law of gravity, together with his laws of motion, provides the basis for a full mathematical solution to the motion of planets and satellites. More important, Newton's universal law of gravity correctly describes the gravitational attractive force between *any* two masses.

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11.3 • THE UNIVERSAL LAW OF GRAVITY AND THE MOTIONS OF PLANETS

In formulating his universal law of gravity, Newton built on an observation that suggests that the gravitational force between two bodies is proportional to the inverse square of the separation. Let us compare the acceleration of the Moon in its orbit with the acceleration of an object falling near the Earth's surface, such as an apple (Fig. 11.4). Assume that the two accelerations have the same cause—namely, the gravitational attraction of the Earth. From the inverse-square law, Newton found that the acceleration of the Moon toward the Earth should be proportional to $1/r_M^2$, where r_M is the separation between centers of the Earth and Moon. Furthermore, the acceleration of the apple toward the Earth should be proportional to $1/R_E^2$, where R_E is the radius of the Earth. When the values $r_M = 3.84 \times 10^8$ m and $R_E = 6.37 \times 10^6$ m are used, the ratio of the Moon's acceleration, a_M, to the apple's acceleration, g, is predicted to be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6 \text{ m}}{3.84 \times 10^8 \text{ m}}\right)^2 = 2.75 \times 10^{-4}$$

Therefore,

$$a_M = (2.75 \times 10^{-4})(9.80 \text{ m/s}^2) = 2.70 \times 10^{-3} \text{ m/s}^2$$

The centripetal acceleration of the Moon can also be calculated kinematically from a knowledge of the Moon's orbital period, T, where $T = 27.32$ days $= 2.36 \times 10^6$ s and its mean distance from the Earth, r_M. In the time T, the Moon travels a distance $2\pi r_M$, which equals the circumference of its orbit. Therefore, its orbital speed is $2\pi r_M/T$, and its centripetal acceleration is

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2(3.84 \times 10^8 \text{ m})}{(2.36 \times 10^6 \text{ s})^2} = 2.72 \times 10^{-3} \text{ m/s}^2$$

This agreement provides strong evidence that the inverse-square law of force is correct.

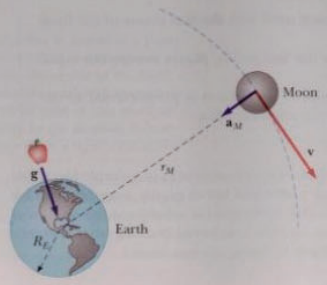


Figure 11.4 As it revolves about the Earth, the Moon experiences a centripetal acceleration $\mathbf{a}_M$ directed toward the Earth. An object near the Earth's surface, such as the apple shown here, experiences an acceleration $\mathbf{g}$. (Dimensions are not to scale.)

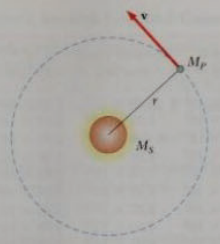


Figure 11.5 A planet of mass M_P moving in a circular orbit about the Sun. The orbits of all planets except Mars, Mercury, and Pluto are nearly circular.

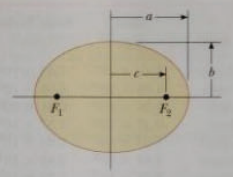


Figure 11.6 Plot of an ellipse. The semimajor axis has a length a, and the semiminor axis has a length b. The focal points are located at a distance c from the center, where $a^2 = b^2 + c^2$, and the eccentricity is defined as $e = c/a$.

Kepler's Third Law

Kepler's third law can be predicted from the inverse-square law for circular orbits.[1] Consider a planet of mass M_P which is assumed to be moving about the Sun (mass M_S) in a circular orbit, as in Figure 11.5. Because the gravitational force on the planet provides the centripetal acceleration as it moves in a circle, we can use Equation 5.3 and write

$$\frac{GM_SM_P}{r^2} = \frac{M_Pv^2}{r}$$

But the orbital speed of the planet is simply $2\pi r/T$, where T is the period; therefore, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3 \qquad [11.4] \quad \bullet \textit{ Kepler's third law}$$

where K_S is a constant given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19} \text{ s}^2/\text{m}^3$$

Equation 11.4 is Kepler's third law. It is also valid for elliptical orbits if we replace r with the length of the semimajor axis, a (Fig. 11.6). Note that the constant of proportionality, K_S, is independent of the mass of the planet. Equation 11.4 is therefore valid for *any* planet. If we were to consider the orbit of a satellite about the Earth, such as the Moon, then the constant would have a different value, with

[1]The orbits of all planets except Mars, Mercury, and Pluto are very close to being circular. For example, the ratio of the semiminor axis to the semimajor axis for the Earth is $b/a = 0.999\,86$. The closeness of the ratio b/a to 1 indicates that the difference between perihelion and aphelion is small as expected.

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TABLE 11.2 Useful Planetary Data

Body	Mass (kg)	Mean Radius (m)	Period (s)	Distance from Sun (m)	$\frac{T^2}{r^3}\left(\frac{s^2}{m^3}\right)$
Mercury	3.18×10^{23}	2.43×10^{6}	7.60×10^{6}	5.79×10^{10}	2.97×10^{-19}
Venus	4.88×10^{24}	6.06×10^{6}	1.94×10^{7}	1.08×10^{11}	2.99×10^{-19}
Earth	5.98×10^{24}	6.37×10^{6}	3.156×10^{7}	1.496×10^{11}	2.97×10^{-19}
Mars	6.42×10^{23}	3.37×10^{6}	5.94×10^{7}	2.28×10^{11}	2.98×10^{-19}
Jupiter	1.90×10^{27}	6.99×10^{7}	3.74×10^{8}	7.78×10^{11}	2.97×10^{-19}
Saturn	5.68×10^{26}	5.85×10^{7}	9.35×10^{8}	1.43×10^{12}	2.99×10^{-19}
Uranus	8.68×10^{25}	2.33×10^{7}	2.64×10^{9}	2.87×10^{12}	2.95×10^{-19}
Neptune	1.03×10^{26}	2.21×10^{7}	5.22×10^{9}	4.50×10^{12}	2.99×10^{-19}
Pluto	$\approx 1.4 \times 10^{22}$	$\approx 1.5 \times 10^{6}$	7.82×10^{9}	5.91×10^{12}	2.96×10^{-19}
Moon	7.36×10^{22}	1.74×10^{6}	—	—	—
Sun	1.991×10^{30}	6.96×10^{8}	—	—	—

the Sun's mass replaced by the Earth's mass. In this case, the proportionality constant would equal $4\pi^2/GM_E$.

Table 11.2 is a collection of useful planetary data. The last column verifies that the ratio T^2/r^3 is constant.

Example 11.3 An Earth Satellite

A satellite of mass m moves in a circular orbit about the Earth with a constant speed of v, at a height of $h = 1000$ km above the Earth's surface, as in Figure 11.7. (For clarity, this figure is not drawn to scale.) Find the orbital speed of the satellite. The radius of the Earth is 6.37×10^6 m, and its mass is 5.98×10^{24} kg.

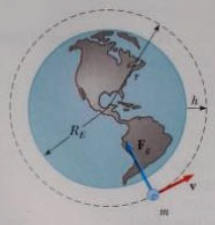


Figure 11.7 (Example 11.3) A satellite of mass m moving around the Earth in a circular orbit of radius r and with constant speed v. The only force acting on the satellite is the gravitational force $\mathbf{F}_g$. (Not drawn to scale.)

Solution The only external force on the satellite is the gravitational attraction exerted by the Earth. This force is directed toward the center of the satellite's circular path and is the force causing the centripetal acceleration of the satellite. Because the magnitude of the force of gravity is GM_Em/r^2, we find that

$$F_g = G\frac{M_E m}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM_E}{r}$$

In this expression, the distance r is the Earth's radius plus the height of the satellite; that is, $r = R_E + h = 7.37 \times 10^6$ m, so that

$$v^2 = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{7.37 \times 10^6\ \text{m}}$$

$$= 5.41 \times 10^7\ \text{m}^2/\text{s}^2$$

Therefore,

$$v = 7.36 \times 10^3\ \text{m/s} \approx 16\,400\ \text{mi/h}$$

Note that *v is independent of the mass of the satellite!*

EXERCISE 3 Calculate the period of revolution, T, of the satellite. Answer 105 min

Kepler's Second Law and Conservation of Angular Momentum

Consider a planet of mass M_P moving about the Sun in an elliptical orbit (Fig. 11.8). The gravitational force acting on the planet is a central force, always along the radius vector, directed toward the Sun. The torque on the planet due to this central force is clearly zero, because $\mathbf{F}$ is parallel to $\mathbf{r}$. That is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F(r)\hat{\mathbf{r}} = 0$$

But recall that the torque equals the time rate of change of angular momentum; that is, $\boldsymbol{\tau} = d\mathbf{L}/dt$. Therefore,

because $\boldsymbol{\tau} = 0$, the angular momentum, $\mathbf{L}$, of the planet is a constant of the motion:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = \text{constant}$$

Because $\mathbf{L}$ is a constant of the motion, the planet's motion at any instant is restricted to the plane formed by $\mathbf{r}$ and $\mathbf{v}$.

We can relate this result to the following geometric consideration. In a time of dt, the radius vector $\mathbf{r}$ in Figure 11.9b sweeps out the area dA, which equals one half the area $|\mathbf{r} \times d\mathbf{r}|$ of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r}$. Because the displacement of the planet in the time dt is given by $d\mathbf{r} = \mathbf{v}\,dt$, we get

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \tfrac{1}{2}|\mathbf{r} \times \mathbf{v}\,dt| = \frac{L}{2m}\,dt$$

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant} \qquad [11.5]$$

• *Kepler's second law*

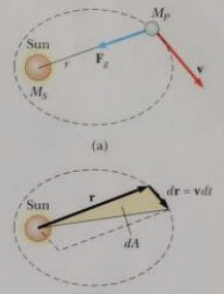
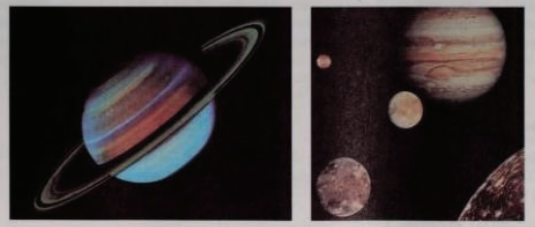


Figure 11.8 (a) The gravitational force acting on a planet acts toward the Sun, along the radius vector. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time dt is equal to one half the area of the parallelogram formed by the vectors $\mathbf{r}$ and $d\mathbf{r} = \mathbf{v}\,dt$.



(Left) This image of Saturn, taken by the *Voyager I* spacecraft on October 18, 1980, was color-enhanced to increase the visibility of large, bright features in Saturn's North Temperate Belt. The distinct color difference between the North Equatorial Belt and other belts may be due to a thicker haze layer covering the northern belt. *(Right)* Jupiter and its four planet-size moons were photographed by *Voyager I* and assembled into this collage. The moons are not to scale but are in their relative positions. Nine other smaller moons circle Jupiter. Not visible is Jupiter's faint ring of particles, seen for the first time by *Voyager I*. *(NASA photos)*

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where L and m are both constants of the motion. Thus, we conclude that

> the radius vector from the Sun to any planet sweeps out equal areas in equal times.

It is important to recognize that this result, which is Kepler's second law, is a consequence of the fact that the force of gravity is a central force, which in turn implies that angular momentum of the planet is constant. Therefore, the law applies to *any* situation that involves a central force, whether inverse square or not.

The inverse-square nature of the force of gravity is not revealed by Kepler's second law. Although we do not prove it here, Kepler's first law (as well as his third law) is a direct consequence of the fact that the gravitational force varies as $1/r^2$. That is, under an inverse-square force law, the orbits of the planets can be shown to be ellipses with the Sun at one focus.

Thinking Physics 2

The Earth is closer to the Sun when it is winter in the Northern Hemisphere than when it is summer. July and January both have 31 days. In which month, if either, does the Earth move through a longer distance in its orbit?

Reasoning The Earth is in a slightly elliptical orbit around the Sun. In order to conserve angular momentum, the Earth must move more rapidly when it is close to the Sun and more slowly when it is farther away. Thus, since it is closer to the Sun in January, it is moving faster, and will cover more distance in its orbit than it will in July.

CONCEPTUAL PROBLEM 1

Kepler's first law indicates that planets travel in elliptical orbits. This is intimately related to the fact that the gravitational force varies as the inverse square of the separation between the Sun and the planet. Suppose the gravitational force varied as the inverse *cube*, rather than the inverse square. This opens up some new types of orbits. What about the second and third laws? Will they change if the force depends on the inverse cube of the separation?

CONCEPTUAL PROBLEM 2

A satellite in orbit is not truly traveling through a vacuum—it is moving through very thin air. Does the resulting air friction cause the satellite to slow down?

CONCEPTUAL PROBLEM 3

How would you explain the fact that Saturn and Jupiter have periods much greater than one year?

EXERCISE 4 At its aphelion the planet Mercury is 6.99×10^{10} m from the Sun, and at its perihelion it is 4.60×10^{10} m from the Sun. If its orbital speed is 3.88×10^4 m/s at the aphelion, what is its orbital speed at the perihelion? Answer 5.90×10^4 m/s

11.4 · ENERGY CONSIDERATIONS IN PLANETARY AND SATELLITE MOTION

Consider a body of mass m moving with a speed of v in the vicinity of a massive body of mass $M \gg m$. The system might be a planet moving around the Sun or a satellite in orbit around the Earth. If we assume that M is at rest in an inertial reference frame,[2] then the total mechanical energy, E, of the two-body system when the bodies are separated by a distance r is the sum of the kinetic energy of the mass m and the potential energy of the system:

$$E = K + U$$

Recall from Chapter 7, Equation 7.22, that the gravitational potential energy, U_g, associated with *any pair* of particles of masses m_1 and m_2 separated by a distance r is given by

$$U_g = -\frac{Gm_1m_2}{r}$$

Therefore, in our case,

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad [11.6]$$

Furthermore, the total mechanical energy is constant if we assume that the system is isolated. Therefore, as the mass m moves from P to Q in Figure 11.9, the total energy remains constant and Equation 11.6 gives

$$E = \tfrac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \tfrac{1}{2}mv_f^2 - \frac{GMm}{r_f} \qquad [11.7]$$

Equation 11.6 shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system, such as the Earth and Sun, E is necessarily *less than zero* if we use the arbitrary convention $U \to 0$ as $r \to \infty$. We can easily establish that $E < 0$ for the system consisting of a mass m moving in a circular orbit about a body of mass M. Newton's second law applied to the body of mass m gives

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

Multiplying both sides by r and dividing by 2,

$$\tfrac{1}{2}mv^2 = \frac{GMm}{2r} \qquad [11.8]$$

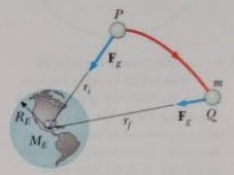


Figure 11.9 As a particle of mass m moves from P to Q above the Earth's surface, the total mechanical energy of the system remains constant.

[2] To see that this is reasonable, consider an object of mass m falling toward the Earth. Because the center of mass of the object–Earth system is stationary, it follows that $mv = M_E v_E$. Thus, the Earth acquires a kinetic energy equal to

$$\tfrac{1}{2}M_E v_E^2 = \tfrac{1}{2}\frac{m^2}{M_E}v^2 = \frac{m}{M_E}K$$

where K is the kinetic energy of the object. Because $M_E \gg m$, the kinetic energy of the Earth is negligible.

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Substituting this into Equation 11.6, we obtain

$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

Total mechanical energy for circular orbits

$$E = -\frac{GMm}{2r} \qquad [11.9]$$

This clearly shows that **the total mechanical energy must be negative in the case of circular orbits.** Furthermore, **the kinetic energy is positive and equal to one half the magnitude of the potential energy** (when the potential energy is chosen to be zero at infinity). The absolute value of E is also equal to the binding energy of the system.

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 11.9, with r replaced by the semimajor axis length, a. The total energy, the total angular momentum, and the total linear momentum of a planet–Sun system are constants of the motion.



" . . . the greater the velocity . . . with which (a stone) is projected, the farther it goes before it falls to the Earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at last, exceeding the limits of the Earth, it should pass into space without touching."—Newton, *System of the World.*

Thinking Physics 3

Icebound, by Dean Koontz (Ballantine Books, New York, 1995), is a story of a group of scientists trapped on a floating iceberg near the North Pole. One of the devices that the scientists have with them on their task is a transmitter with which they can fix their position with "the aid of a *geosynchronous polar satellite.*" Can a satellite in a *polar* orbit be *geosynchronous?*

Reasoning A geosynchronous satellite is one that stays over one location on the Earth's surface at all times. Thus, an antenna receiving signals from the satellite, such as a television dish, can stay pointed in a fixed direction toward the sky. The satellite must be in an orbit with the correct radius such that its orbital period is the same as that of the Earth's rotation. This would result in the satellite appearing to have no east–west motion relative to the observer at the chosen location. Another requirement is that a geosynchronous satellite *must be in orbit over the Equator.* Otherwise it would appear to undergo a north–south oscillation during one orbit. Thus, it would be impossible to have a geosynchronous satellite in a *polar* orbit. Even if such a satellite were at the proper distance from the Earth, it would be moving rapidly in the north–south direction, resulting in the necessity of accurate tracking equipment. What's more, it would be below the horizon for long periods of time, making it useless for determining one's position.

Thinking Physics 4

A satellite is in a circular orbit in space. Is it necessary that there be a massive object at the center of the orbit?

Reasoning This is the usual situation—satellites are normally in orbit around the Earth, a massive object. Imagine, however, tying a rock somewhere along a length of string and twirling the rock in a circle as shown in Figure 11.10. In this case, there is no entity at the center of the circular path of the rock, nor along its radius, that is exerting a force on the rock. The net force holding the rock in the circular orbit is a

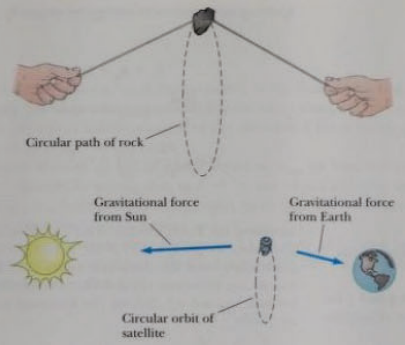


Figure 11.10 (Thinking Physics 4)

combination of the radial components of two forces—one from the string on each side of the rock.

The question posed was not about rocks on strings; it was about satellites. Is there a gravitational counterpart to this situation? There is. Imagine moving from the Earth to the Sun along the line between them. At some point along that line, the gravitational force of attraction to the Earth will balance that toward the Sun. This is similar to the balance along the string of the tension on either side of the rock—the rock is in equilibrium between two balanced forces. Now imagine pulling the satellite perpendicularly off the Earth–Sun line a small amount—as long as the distance is relatively small, the components of the forces from the Earth and Sun along the Sun–Earth centerline will cancel. The components of the two forces perpendicular to the centerline will add, to provide a net force pulling the satellite back toward the centerline. Now, suppose we set the satellite into circular motion with just the right velocity. The satellite will be in orbit around a center point with no mass located there.

The International Sun–Earth Explorer 3 satellite (ISEE-3) was launched in 1978 and placed in just such an orbit in order to study the solar wind. The advantage over a Sun-centered orbit is that the satellite always resides on the side of the Sun toward the Earth, so that it is constantly in communication with Earth. In the absence of a specific mission to Comet Halley, the ISEE-3 satellite was removed from this orbit in 1982 and placed in orbit around the Moon, in anticipation of a flyby of Comet Halley. On the way to this flyby, it passed through the tail of Comet Giacobini-Zinner and successfully gathered important data on both comets.

Example 11.4 A Satellite in an Elliptical Orbit

A satellite moves in an elliptical orbit about the Earth, as in Figure 11.11. The minimum and maximum distances from the surface of the Earth are 400 km and 3000 km. Find the speeds of the satellite at apogee and perigee.

Solution Because the mass of the satellite is negligible compared with the Earth's mass, we take the center of mass of the Earth to be at rest. Gravity is a *central* force, and so the angular momentum of the satellite about the Earth's center of

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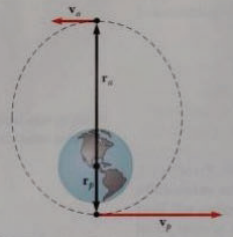


Figure 11.11 (Example 11.4) A satellite in an elliptical orbit about the Earth.

mass remains constant in time. With subscripts a and p for the apogee and perigee positions, conservation of angular momentum gives $L_p = L_a$, or

$$mv_p r_p = mv_a r_a$$
$$v_p r_p = v_a r_a \qquad [1]$$

Applying conservation of energy, we obtain $E_p = E_a$, or

$$U_p + K_p = U_a + K_a$$
$$-G\frac{M_E m}{r_p} + \tfrac{1}{2}mv_p^2 = -G\frac{M_E m}{r_a} + \tfrac{1}{2}mv_a^2$$
$$2GM_E\left(\frac{1}{r_a} - \frac{1}{r_p}\right) = (v_a^2 - v_p^2) \qquad [2]$$

Assuming the Earth's radius is 6.37×10^6 m and using the given data, we get $r_a = 9.37 \times 10^6$ m and $r_p = 6.77 \times 10^6$ m. Because we know the numerical values of G, M_E, r_p, and r_a, we can use Equations (1) and (2) to determine the two unknowns, (v_p and v_a). Solving the equations simultaneously, we obtain

$$v_p = 8.27 \text{ km/s} \qquad v_a = 5.98 \text{ km/s}$$

Escape Speed

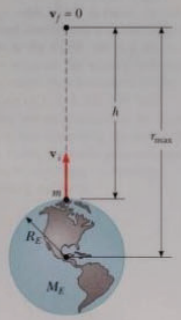


Figure 11.12 An object of mass m projected upward from the Earth's surface with an initial velocity $\mathbf{v}_i$ reaches a maximum altitude h.

Suppose an object of mass m is projected vertically from the Earth's surface with an initial speed v_i, as in Figure 11.12. We can use energy considerations to find the minimum value of the initial speed such that the object will escape the Earth's gravitational field. Equation 11.6 gives the total energy of the object at any point when its speed and distance from the center of the Earth are known. At the surface of the Earth, where $v_i = v$, $r_i = R_E$. When the object reaches its maximum altitude, $v_f = 0$ and $r_f = r_{max}$. Because the total energy of the system is constant, substitution of these conditions into Equation 11.7 gives

$$\tfrac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{max}}$$

Solving for v_i^2 gives

$$v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right) \qquad [11.10]$$

Therefore, if the initial speed is known, this expression can be used to calculate the maximum altitude, h, because we know that $h = r_{max} - R_E$.

We are now in a position to calculate the minimum speed the object must have at the Earth's surface in order to escape from the influence of the Earth's gravitational field. This corresponds to the situation in which the object will continue to move away forever, with the speed asymptotically approaching *zero*. Setting $r_{max} = \infty$ in Equation 11.10 and taking $v_i = v_{esc}$ (the escape speed), we get

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} \qquad [11.11]$$

• *Escape speed*

Note that this expression for v_{esc} is independent of the mass of the object projected from the Earth. For example, a spacecraft has the same escape speed as a molecule. Furthermore, the result is independent of the *direction* of the velocity, provided the object is not thrown into the ground.

If the object is given an initial speed equal to v_{esc}, its *total* energy is equal to zero. This can be seen by noting that as $r \to \infty$, the object's kinetic energy and its potential energy are both zero. If v_i is greater than v_{esc}, the *total* energy is greater than zero and the object is still moving as $r \to \infty$.

Finally, you should note that Equations 11.10 and 11.11 can be applied to objects projected from *any* planet. That is, in general, the escape speed from any planet of mass M and radius R is

$$v_{esc} = \sqrt{\frac{2GM}{R}} \qquad [11.12]$$

A list of escape speeds for the planets, the Moon, and the Sun is given in Table 11.3. Note that the values vary from 1.1 km/s for Pluto to about 618 km/s for the Sun. These results, together with some ideas from the kinetic theory of gases (Chapter 16), explain why some planets have atmospheres and others do not. As we shall see later, a gas molecule has an average kinetic energy that depends on its temperature. Lighter atoms, such as hydrogen and helium, have higher average speeds than do the heavier species. When the average speed of the lighter atoms is not much less than the escape speed, a significant fraction of the molecules have a chance to escape from the planet. This mechanism also explains why the Earth does not retain hydrogen and helium molecules in its atmosphere but does retain much heavier molecules, such as oxygen and nitrogen. By contrast, Jupiter has a very large escape speed (60 km/s), which enables it to retain hydrogen, the primary component of its atmosphere.

TABLE 11.3 Escape Speeds from the Surfaces of the Planets, the Moon, and the Sun

Planet	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Mars	5.0
Jupiter	60.0
Saturn	36.0
Uranus	22.0
Neptune	24.0
Pluto	1.1
Moon	2.3
Sun	618.0

Black Holes

In Chapter 10 we briefly described a rare event called a supernova—the catastrophic explosion of a very massive star. The material that remains in the central core of such an object continues to collapse, but the core's ultimate fate depends on its mass. If the core has a mass less than 1.4 times the mass of our Sun, it gradually cools down and ends its life as a white dwarf star. However, if the core's mass is greater than this, the star may collapse further due to gravitational forces. What remains is a neutron star, a region compressed to a radius of about 10 km. (On Earth, a teaspoon of this material would weigh about 5 billion tons!)

An even more unusual star death may occur when the core has a mass greater than about three solar masses. No known forces in nature are strong enough to prevent the collapse of such a star. The collapse may continue until the star becomes a small object in space, commonly referred to as a **black hole.** In effect, black holes are remains of stars that have collapsed under their own weight. Once an object such as a spaceship comes near a black hole, it experiences an extremely strong gravitational force and is trapped forever (Fig. 11.13).

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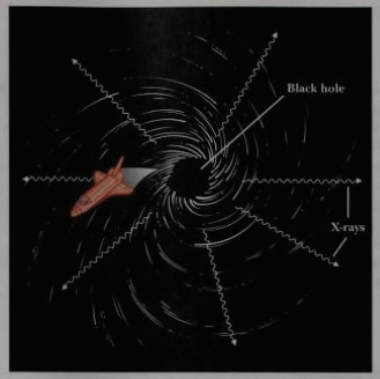


Figure 11.13 The gravitational field in the vicinity of a black hole is so strong that nothing can escape. Any object moving close to a black hole can emit x-rays.

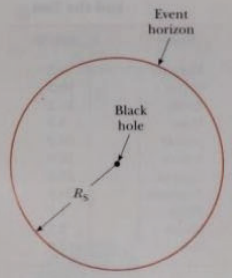


Figure 11.14 A black hole. The distance R_S equals the Schwarzschild radius. Any event occurring within the boundary of radius R_S, called the event horizon, is invisible to an outside observer.

The escape speed from any spherical body can be calculated from Equation 11.12. If the escape speed exceeds the speed of light, c, radiation within the body (such as visible light) cannot escape and the body appears to be black; hence the origin of the terminology "black hole." The critical radius, R_S, for which this occurs is called the **Schwarzschild radius** (Fig. 11.14). Taking $v_{esc} = c$ in Equation 11.12 and solving for R_S, we obtain $R_S = 2GM/c^2$. For example, the value for R_S for a black hole with a mass equal to that of the Sun is calculated to be 3.0 km (about 2 mi); a black hole the mass of which equals that of the Earth has a radius of about 9 mm (about the size of a dime).

Although light from a black hole cannot escape, light from events taking place near the black hole should be visible. A companion star captured by the strong gravitational field of a black hole should emit x-rays (Fig. 11.13). Based on this reasoning, several candidates for black holes have been detected, the most famous being Cygnus X-1, the first x-ray source detected in the constellation Cygnus. There is also evidence that supermassive black holes exist at the centers of galaxies.

EXERCISE 5 The escape speed from the surface of the Earth is 11.2 km/s. Estimate the escape speed for a spacecraft from the surface of the Moon. The Moon has a mass $\frac{1}{81}$ that of Earth and a radius $\frac{1}{4}$ that of Earth. Answer 2.49 km/s

EXERCISE 6 (a) Calculate the minimum energy required to send a 3000-kg spacecraft from the Earth to a distant point in space where Earth's gravity is negligible. (b) If the journey is to take three weeks, what *average* power will the engines have to supply?
Answer (a) 1.88×10^{11} J (b) 103 kW

EXERCISE 7 A satellite moves in an elliptical orbit about the Earth such that, at perigee and apogee positions, the distances from the Earth's center are, respectively, D and $4D$. Find the ratio of the speeds at the two positions, v_p/v_a. Answer 4

11.5 · ATOMIC SPECTRA AND THE BOHR THEORY OF HYDROGEN

As you may have already learned in chemistry, the hydrogen atom is the simplest known atomic system and an especially important one to understand. Much of what is learned about the hydrogen atom (which consists of one proton and one electron) can be extended to other single-electron ions such as He^+ and Li^{2+}. Furthermore, a thorough understanding of the physics underlying the hydrogen atom can then be used to describe more complex atoms and the periodic table of the elements.

Suppose an evacuated glass tube is filled with hydrogen (or some other gas). If a voltage applied between metal electrodes in the tube is great enough to produce an electric current in the gas, the tube emits light the color of which is characteristic of the gas (this is how a neon sign works). When the emitted light is analyzed with a device called a spectroscope, a series of discrete lines is observed, each line corresponding to a different wavelength, or color, of light. Such a series of spectral lines is commonly referred to as an **emission spectrum.** The wavelengths contained in a given line spectrum are characteristic of the element emitting the light (Fig. 11.15). Because no two elements emit the same line spectrum, this phenomenon represents a marvelous and reliable technique for identifying elements in a substance.

As you shall learn in more detail in Chapter 13, a mechanical wave is a disturbance that transports energy as it moves through a system without transporting matter. A common form of periodic wave is the sinusoidal wave, the shape of which

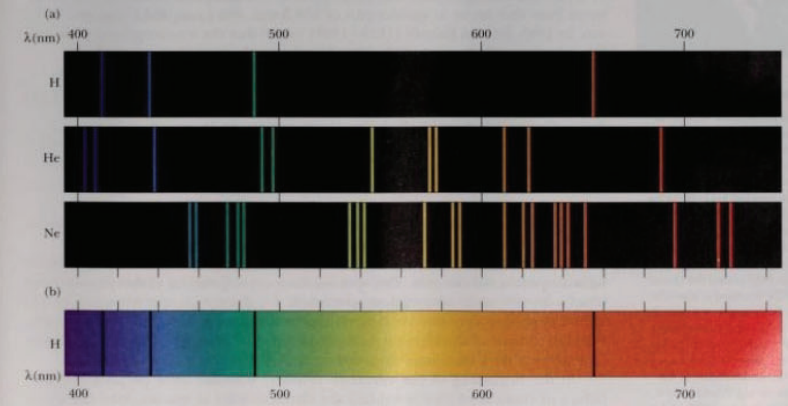


Figure 11.15 Visible spectra. (a) Line spectra produced by emission in the visible range for the elements hydrogen, helium, and neon. (b) The absorption spectrum for hydrogen. The dark absorption lines occur at the same wavelengths as the emission line for hydrogen shown in (a).

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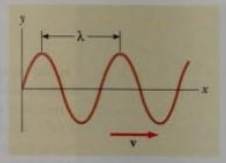


Figure 11.16 A sinusoidal wave traveling to the right. Any point on the wave moves a distance of one wavelength, λ, in a time equal to the period of the wave.

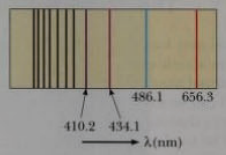
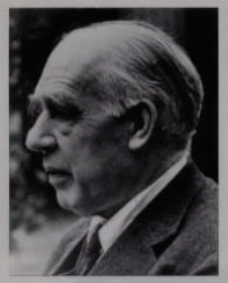


Figure 11.17 A series of spectral lines for atomic hydrogen. The prominent lines labeled are part of the Balmer series.



Niels Bohr (1885–1962)

A Danish physicist, Bohr was an active participant in the early development of quantum mechanics and provided much of its philosophical framework. During the 1920s and 1930s, Bohr headed the Institute for Advanced Studies in Copenhagen. This institute was a magnet for many of the world's best physicists and provided a forum for the exchange of ideas. When Bohr visited the United States in 1939 to attend a scientific conference, he brought news that the fission of uranium had been discovered by Hahn and Strassman in Berlin. The results were the foundations of the atomic bomb developed in the United States during World War II. Bohr was awarded the 1922 Nobel prize for his investigation of the structure of atoms and of the radiation emanating from them.

is depicted in Figure 11.16. The distance between two consecutive crests of the wave is called the **wavelength,** λ. As the wave travels to the right with speed v, any point on the wave travels a distance of one wavelength in a time interval of one period, T (the time for one cycle), so the wave speed is $v = \lambda/T$. The inverse of the period, $1/T$, is called the **frequency,** f, of the wave; it represents the number of cycles per second. Thus, the speed of the wave is often written as $v = \lambda f$. In this section, because we shall deal with electromagnetic waves—which travel at the speed of light, c—the appropriate relation is

$$c = \lambda f \qquad [11.13]$$

The emission spectrum of hydrogen shown in Figure 11.17 includes four prominent lines that occur at wavelengths of 656.3 nm, 486.1 nm, 434.1 nm, and 410.2 nm. In 1885, Johann Balmer (1825–1898) found that the wavelengths of these and less prominent lines can be described by this simple empirical equation:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \qquad \text{Balmer series} \qquad [11.14]$$

where n may have integral values of 3, 4, 5, . . . , and R_H is a constant, now called the **Rydberg constant.** If the wavelength is in meters, R_H has the value

$$R_H = 1.0973732 \times 10^7\ \text{m}^{-1}$$

The first line in the Balmer series, at 656.3 nm, corresponds to $n = 3$ in Equation 11.14; the line of 486.1 nm corresponds to $n = 4$; and so on.

In addition to emitting light at specific wavelengths, an element can also absorb light at specific wavelengths. The spectral lines corresponding to this process form what is known as an **absorption spectrum.** An absorption spectrum can be obtained by passing a continuous radiation spectrum (one containing all wavelengths) through a vapor of the element being analyzed. The absorption spectrum consists of a series of dark lines superimposed on the otherwise continuous spectrum.

At the beginning of the twentieth century, scientists were perplexed by the failure of classical physics to explain the characteristics of spectra. Why did atoms of a given element emit only certain wavelengths of radiation, so that the emission spectrum displayed discrete lines? Furthermore, why did the atoms absorb only those wavelengths that they emitted? In 1913 Bohr provided an explanation of

atomic spectra that included some features of the currently accepted theory. Using the simplest atom, hydrogen, Bohr described a model of what he thought must be the atom's structure. His model of the hydrogen atom contains some classical features as well as some revolutionary postulates that could not be justified within the framework of classical physics.[3] The basic assumptions of the Bohr theory as it applies to the hydrogen atom are as follows:

- *Assumptions of the Bohr theory*

1. The electron moves in circular orbits about the proton under the influence of the Coulomb force of attraction, as in Figure 11.18.
2. Only certain electron orbits are stable. These are orbits in which the hydrogen atom does not emit energy in the form of radiation. Hence, the total energy of the atom remains constant, and classical mechanics can be used to describe the electron's motion.
3. Radiation is emitted by the hydrogen atom when the electron "jumps" from a more energetic initial state to a lower state. The jump cannot be visualized or treated classically. In particular, the frequency, f, of the radiation emitted in the jump is related to the change in the atom's energy and **is independent of the frequency of the electron's orbital motion.** The frequency of the emitted radiation is found from

$$E_i - E_f = hf \qquad [11.15]$$

where E_i is the energy of the initial state, E_f is the energy of the final state, h is Planck's constant (see Section 10.9), and $E_i > E_f$.

4. The size of the allowed electron orbits is determined by a condition imposed on the electron's orbital angular momentum: the allowed orbits are those for which the electron's orbital angular momentum about the nucleus is an integral multiple of $\hbar = h/2\pi$.

$$m_e v r = n\hbar \qquad n = 1, 2, 3, \ldots \qquad [11.16]$$

Using these four assumptions, we can calculate the allowed energy levels and emission wavelengths of the hydrogen atom. The electrical potential energy of the system shown in Figure 11.18 is given by $U_e = -k_e e^2/r$, where k_e is the Coulomb constant, e is the charge on the electron, and r is the electron–proton separation. Thus, the total energy of the atom, which contains both kinetic and potential energy terms, is

$$E = K + U_e = \tfrac{1}{2}m_e v^2 - k_e\frac{e^2}{r} \qquad [11.17]$$

Applying Newton's second law to this system, we see that the Coulomb attractive force on the electron, $k_e e^2/r^2$ (Eq. 5.15), must equal the mass times the centripetal acceleration ($a = v^2/r$) of the electron:

$$\frac{k_e e^2}{r^2} = \frac{m_e v^2}{r}$$

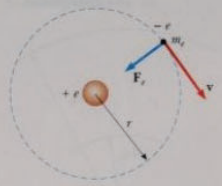


Figure 11.18 Diagram representing Bohr's model of the hydrogen atom in which the orbiting electron is allowed to be only in specific orbits of discrete radii.

[3]The Bohr model can be applied successfully to such hydrogen-like ions as singly ionized helium and doubly ionized lithium, but the theory does not properly describe the visible spectra of more complex atoms and ions.

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From this expression, we find the kinetic energy to be

$$K = \tfrac{1}{2} m_e v^2 = \frac{k_e e^2}{2r} \qquad [11.18]$$

Substituting this value of K into Equation 11.17, we find that the total energy of the atom is

Total energy of the hydrogen atom

$$E = -\frac{k_e e^2}{2r} \qquad [11.19]$$

Note that the total energy is negative, indicating a bound electron–proton system. This means that energy in the amount of $k_e e^2/2r$ must be added to the atom just to remove the electron and make the total energy zero. An expression for r, the radius of the allowed orbits, can be obtained by eliminating v by substitution between Equations 11.16 and 11.18:

Radii of Bohr orbits in hydrogen

$$r_n = \frac{n^2 \hbar^2}{m_e k_e e^2} \qquad n = 1, 2, 3, \ldots \qquad [11.20]$$

This result shows that the radii have discrete values, or are *quantized.*

The orbit for which $n = 1$ has the smallest radius; it is called the **Bohr radius,** a_0, and has the value

The Bohr radius

$$a_0 = \frac{\hbar^2}{m_e k_e e^2} = 0.0529 \text{ nm} \qquad [11.21]$$

The first three Bohr orbits are shown to scale in Figure 11.19.

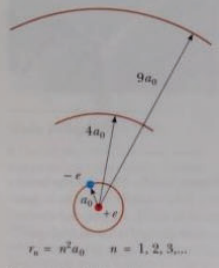


Figure 11.19 The first three Bohr orbits for hydrogen.

The quantization of the orbit radii immediately leads to energy quantization. This can be seen by substituting $r_n = n^2 a_0$ into Equation 11.19. The allowed energy levels are found to be

$$E_n = -\frac{k_e e^2}{2a_0}\left(\frac{1}{n^2}\right) \qquad n = 1, 2, 3, \ldots \qquad [11.22]$$

Insertion of numerical values into Equation 11.22 gives

$$E_n = -\frac{13.6}{n^2} \text{ eV} \qquad n = 1, 2, 3, \ldots \qquad [11.23]$$

(Recall from Section 9.7 that 1 eV = 1.6×10^{-19} J.) The lowest stationary state corresponding to $n = 1$, called the **ground state,** has an energy of $E_1 = -13.6$ eV. The next state, the **first excited state,** has $n = 2$ and an energy of $E_2 = E_1/2^2 = -3.4$ eV. Figure 11.20 is an energy-level diagram showing the energies of these discrete energy states and the corresponding quantum numbers. The uppermost level, corresponding to $n = \infty$ (or $r = \infty$) and $E = 0$, represents the state for which the electron is removed from the atom. The minimum energy required to ionize the atom (that is, to completely remove an electron in the ground state from the proton's influence) is called the **ionization energy.** As can be seen from Figure 11.20, the ionization energy for hydrogen, based on Bohr's calculation, is 13.6 eV. This constituted another major achievement for the Bohr theory, because the ionization energy for hydrogen had already been measured to be precisely 13.6 eV.

Figure 11.20 also shows other spectral series (the Lyman series and the Paschen series) that were found after Balmer's discovery. These spectra obey other empirical formulas, which are reconciled with the Bohr model.

Equation 11.22, together with Bohr's third postulate, can be used to calculate the frequency of the radiation that is emitted when the electron jumps from an outer orbit to an inner orbit:

$$f = \frac{E_i - E_f}{h} = \frac{k_e e^2}{2a_0 h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad [11.24]$$

Because the quantity being measured is wavelength, it is convenient to convert frequency to wavelength, using $c = f\lambda$, to get

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{k_e e^2}{2a_0 hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad [11.25]$$

The remarkable fact is that the *theoretical* expression, Equation 11.25, is identical to a generalized form of the empirical relations discovered by Balmer and others (see Eq. 11.14),

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad [11.26]$$

provided that the combination of constants $k_e e^2/2a_0 hc$ is equal to the experimentally determined Rydberg constant. After Bohr demonstrated the agreement of these two quantities to a precision of about 1%, it was soon recognized as the crowning achievement of his new theory of quantum mechanics. Furthermore, Bohr showed that all of the spectral series for hydrogen have a natural interpretation in his theory. Figure 11.20 shows these spectral series as transitions between energy levels.

Bohr immediately extended his model for hydrogen to other elements in which all but one electron had been removed. Ionized elements such as He^+, Li^{2+}, and Be^{3+} were suspected to exist in hot stellar atmospheres, where frequent atomic collisions occur with enough energy to completely remove one or more atomic electrons. Bohr showed that many mysterious lines observed in the Sun and several stars could not be due to hydrogen but were correctly predicted by his theory if attributed to singly ionized helium.

Although the Bohr model had some success in predicting the spectra of single valence electron atoms, it has some serious drawbacks. For example, it cannot account for the visible spectra of more complex atoms, and it is unable to predict many subtle spectral details of hydrogen and other simple atoms.

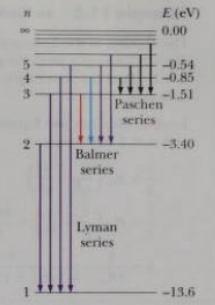


Figure 11.20 An energy level diagram for hydrogen. The discrete allowed energies are plotted on the vertical axis. Nothing is plotted on the horizontal axis, but the horizontal extent of the diagram is made large enough to show allowed transitions. Quantum numbers are given on the left.

CONCEPTUAL PROBLEM 4

Under normal experimental conditions, hydrogen atoms show more lines in emission than in absorption. Why?

CONCEPTUAL PROBLEM 5

Suppose the energy of the electron in a hydrogen atom is $-E$. What is the kinetic energy? What is the potential energy?

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Example 11.5 **An Electronic Transition in Hydrogen**

The electron in the hydrogen atom makes a transition from the $n = 2$ energy state to the ground state (corresponding to $n = 1$). Find the wavelength and frequency of the emitted radiation.

Solution We can use Equation 11.26 directly to obtain λ, with $n_i = 2$ and $n_f = 1$:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$\frac{1}{\lambda} = R_H\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3R_H}{4}$$

$$\lambda = \frac{4}{3R_H} = \frac{4}{3(1.097 \times 10^{-7}\ \text{m}^{-1})}$$

$$= 1.215 \times 10^{-7}\ \text{m} = 121.5\ \text{nm} \quad \text{(ultraviolet)}$$

Because $c = f\lambda$, the frequency of the radiation is

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^8\ \text{m/s}}{1.215 \times 10^{-7}\ \text{m}} = 2.47 \times 10^{15}\ \text{s}^{-1}$$

EXERCISE 8 What is the wavelength of the radiation emitted by hydrogen when the electron makes a transition from the $n = 3$ state to the $n = 1$ state?

Answer $\dfrac{9}{8R_H} = 102.6\ \text{nm}$

Bohr's Correspondence Principle

In our study of relativity in Chapter 9, we found that newtonian mechanics cannot be used to describe phenomena that occur at speeds approaching the speed of light. Newtonian mechanics is a special case of relativistic mechanics and is usable only when v is much less than c. In a similar way, **quantum mechanics is in agreement with classical physics where the energy differences between quantized levels vanish.** This principle, first set forth by Bohr, is called the **correspondence principle.**

For example, consider an electron orbiting the hydrogen atom with $n > 10\,000$. For such large values of n, the energy differences between adjacent levels approach zero and the levels are nearly continuous. As a consequence, the classical model is reasonably accurate in describing the system for large values of n. According to the classical picture, the frequency of the light emitted by the atom is equal to the frequency of revolution of the electron in its orbit about the nucleus. Calculations show that for $n > 10\,000$, this frequency differs from that predicted by quantum mechanics by less than 0.015%.

SUMMARY

Newton's universal law of gravity states that the gravitational force of attraction between any two particles of masses m_1 and m_2 separated by a distance r has the magnitude

$$F_g = G\frac{m_1 m_2}{r^2} \qquad [11.1]$$

where G is the gravitational constant, $6.672 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.

Kepler's laws of planetary motion state the following:

1. Every planet moves in an elliptical orbit with the Sun at one of the focal points.
2. The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis for the elliptical orbit.

Kepler's second law is a consequence of the fact that the force of gravity is a *central force.* This implies that the angular momentum of the planet–Sun system is a constant of the motion.

Kepler's first and third laws are a consequence of the inverse-square nature of the universal law of gravity. Newton's second law, together with the force law given by Equation 11.1, verifies that the period, T, and radius, r, of the orbit of a planet about the Sun are related by

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3 \qquad [11.4]$$

where M_s is the mass of the Sun. Most planets have nearly circular orbits about the Sun. For elliptical orbits, Equation 11.4 is valid if r is replaced by the semimajor axis, a.

If an isolated system consists of a particle of mass m moving with a speed of v in the vicinity of a massive body of mass M, the *total energy* of the system is constant and is

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad [11.6]$$

If m moves in a circular orbit of radius r about M, where $M \gg m$, the total energy of the system is

$$E = -\frac{GMm}{2r} \qquad [11.9]$$

The total energy is negative for any bound system—that is, one in which the orbit is closed, such as a circular or an elliptical orbit.

The minimum speed an object must have to escape the gravitational field of a uniform sphere of mass M and radius R is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \qquad [11.12]$$

The Bohr model of the atom is successful in describing the spectra of atomic hydrogen and hydrogen-like ions. One of the basic assumptions of the model is that the electron can exist only in discrete orbits such that the angular momentum mvr is an integral multiple of $h/2\pi = \hbar$. Assuming circular orbits and a simple coulombic attraction between the electron and proton, the energies of the quantum states for hydrogen are calculated to be

$$E_n = -\frac{k_e e^2}{2a_0}\left(\frac{1}{n^2}\right) \qquad [11.22]$$

where k_e is the Coulomb constant, e is electronic charge, $n = 1, 2, 3, \cdots$ is a positive integer called the **quantum number** of the state, and $a_0 = 0.0529$ nm is the **Bohr radius.**

If the electron in the hydrogen atom makes a transition from an orbit with a quantum number of n_i to one with a quantum number of n_f, where $n_f < n_i$, the frequency of the radiation emitted by the atom is

$$f = \frac{k_e e^2}{2a_0 h}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad [11.24]$$

Using $E = hf = hc/\lambda$, one can calculate the wavelengths of the radiation for various transitions in which there is a change in quantum number, $n_i \rightarrow n_f$. The calculated wavelengths are in excellent agreement with observed line spectra.

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CONCEPTUAL QUESTIONS

1. If the gravitational force on an object is directly proportional to its mass, why don't large masses fall with greater acceleration than small ones?
2. The gravitational force that the Sun exerts on the Moon is about twice as great as the gravitational force that the Earth exerts on the Moon. Why doesn't the Sun pull the Moon away from the Earth during a total eclipse of the Sun?
3. Explain why it takes more fuel for a spacecraft to travel from the Earth to the Moon than for the return trip. Estimate the difference.
4. Explain why there is no work done on a planet as it moves in a circular orbit around the Sun, even though a gravitational force is acting on the planet. What is the *net* work done on a planet during each revolution as it moves around the Sun in an elliptical orbit?
5. At what position in its elliptical orbit is the speed of a planet a maximum? At what position is the speed a minimum?
6. Why don't we put a communications satellite in orbit around the 45th parallel? Wouldn't this be more useful in the United States than one in orbit around the Equator?
7. In his 1798 experiment, Cavendish was said to have "weighed the Earth." Explain this statement.
8. If a hole could be dug to the center of the Earth, do you think that the force on a mass m would still obey Equation 11.1 there? What do you think the force on m would be at the center of the Earth?
9. The *Voyager* spacecraft was accelerated toward escape speed from the Sun by the gravitational force exerted on the spacecraft by Jupiter. How is this possible?
10. The *Apollo 13* spaceship developed trouble in the oxygen system about halfway to the Moon. Why did the mission continue on around the Moon and then return home rather than immediately turn back to Earth?
11. Discuss the similarities and differences between the classical description of planetary motion and the Bohr model of the hydrogen atom.
12. The Bohr theory of the hydrogen atom is based on several assumptions. Discuss those assumptions and their significance. Do any of them contradict classical physics?
13. Explain the significance behind the fact that the total energy of the atom in the Bohr model is negative.
14. Suppose that the electron in the hydrogen atom obeyed classical mechanics rather than quantum mechanics. Why should such a "hypothetical" atom emit a continuous spectrum rather than the observed line spectrum?

PROBLEMS

Section 11.1 Newton's Universal Law of Gravity Revisited

1. Which exerts a greater force of gravitational attraction on objects on the Earth: the Moon or the Sun? Calculate these forces on a 1.00-kg mass.
2. Two ocean liners, each with a mass of 40 000 metric tons, are moving on parallel courses, 100 m apart. What is the magnitude of the acceleration of one of the liners toward the other due to the mutual gravitational attraction? (Treat the ships as spheres.)
3. The gravitational field on the surface of the Moon is about one sixth that on the surface of the Earth. If the radius of the Moon is about one quarter that of the Earth, find the ratio of the average mass density of the Moon to the average mass density of the Earth.
4. A student proposes to measure the gravitational constant G by suspending two spherical masses from the ceiling of a tall cathedral and measuring the deflection of the cables from the vertical. Draw a free-body diagram for one of the masses. If two 100.0 kg masses are suspended at the end of 45.00 m long cables and the cables are attached to the ceiling 1.000 m apart, what is the separation of the masses?
5. On the way to the Moon the Apollo astronauts reach a point at which the Moon's gravitational pull becomes equal to that of Earth. Determine the distance of this point from the center of the Earth.

Section 11.2 Kepler's Laws

Section 11.3 The Universal Law of Gravity and the Motions of Planets

6. A satellite is in a circular orbit just above the surface of the Moon. (The radius of the Moon is 1738 km.) (a) What is the acceleration of the satellite? (b) What is the speed of the satellite? (c) What is the period of the satellite orbit?
7. The *Explorer VIII* satellite, placed into orbit November 3, 1960, to investigate the ionosphere, had the following orbit parameters: perigee 459 km and apogee 2289 km (both distances above the Earth's surface); period, 112.7 min. Find the ratio v_p/v_a.
8. Halley's Comet approaches the Sun to within 0.570 A.U., and its orbital period is 75.6 years. (A.U. is the abbreviation for astronomical unit, where 1 A.U. = 1.50×10^{11} m is the mean Earth–Sun distance.) How far from the Sun will Halley's Comet travel before it starts its return journey? (See Fig. P11.8.)

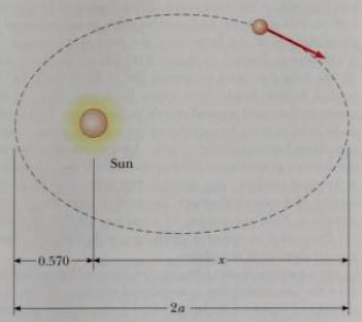


Figure P11.8

9. Io, a satellite of Jupiter, has an orbital period of 1.77 days and an orbital radius of 4.22×10^5 km. From these data, determine the mass of Jupiter.
10. Two planets X and Y travel counterclockwise in circular orbits about a star, as in Figure P11.10. The radii of their orbits is in the ratio 3:1. At some time, they are aligned as in Figure P11.10a, making a straight line with the star. Five years later, Planet X has rotated through 90° as in Figure P11.10b. Where is planet Y at this time?

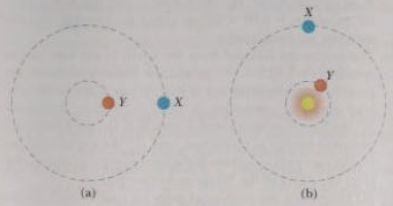


Figure P11.10

11. A synchronous satellite, which always remains above the same point on a planet's equator, is put in orbit around Jupiter to study the famous red spot. Jupiter rotates once every 9.84 h. Use the data of Table 11.2 to find the altitude of the satellite.

Section 11.4 Energy Considerations in Planetary and Satellite Motion

12. A satellite of mass 200 kg is placed in Earth orbit at a height of 200 km above the surface. (a) Assuming a circular orbit, how long does the satellite take to complete one orbit? (b) What is the satellite's speed? (c) What is the minimum energy necessary to place this satellite in orbit (assuming no air friction)?
13. A spaceship is fired from the Earth's surface with an initial speed 2.00×10^4 m/s. What will its speed be when it is very far from the Earth? (Neglect friction.)
14. A satellite of the Earth has a mass of 100 kg and is at an altitude of 2.00×10^6 m. (a) What is the potential energy of the satellite–Earth system? (b) What is the magnitude of the gravitational force exerted by the Earth on the satellite?
15. A 1000-kg satellite orbits the Earth at an altitude of 100 km. It is desired to increase the altitude of the circular orbit to 200 km. How much energy must be added to the system to effect this change in altitude?
16. A satellite of mass m orbits the Earth at an altitude h_1. It is desired to increase the altitude of the circular orbit to h_2. How much energy must be added to the system to effect this change in altitude?
17. A satellite moves in a circular orbit just above the surface of a planet. Show that the orbital speed v and escape speed of the satellite are related by the expression $v_{esc} = \sqrt{2}v$.
18. The planet Uranus has a mass about 14.0 times the Earth's mass, and its radius is equal to about 3.70 Earth radii. (a) By setting up ratios with the corresponding Earth values, find the acceleration due to gravity at the cloud tops of Uranus. (b) Ignoring the rotation of the planet, find the minimum escape speed from Uranus.
19. Determine the escape velocity for a rocket on the far side of Ganymede, the largest of Jupiter's moons. Ganymede's radius is 2.64×10^6 m, and its mass is 1.495×10^{23} kg. The mass of Jupiter is 1.90×10^{27} kg, and the distance between Jupiter and Ganymede is 1.071×10^9 m. Be sure to include the gravitational effect due to Jupiter, but you may ignore the motion of Jupiter and Ganymede as they revolve about their center of mass.
20. How much work is done by the Moon's gravitational field as a 1000 kg meteor comes in from outer space and impacts on the Moon's surface?
21. In Robert Heinlein's *The Moon Is a Harsh Mistress*, the colonial inhabitants of the Moon threaten to launch rocks down onto the Earth if they are not given independence (or at least representation). Assume that a rail gun can launch a rock at twice the speed that would be required to escape from an isolated stationary Moon. Calculate the speed of this rock as it enters the Earth's atmosphere.
22. A spacecraft in the shape of a long cylinder has a length of 100 m and its mass with occupants is 1000 kg. It has strayed in too close to a 1.0-m radius black hole having a mass 100 times that of the Sun (Fig. P11.22). If the nose of the space-

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craft points toward the center of the black hole, and if distance between the nose of the spaceship and the black hole's center is 10.0 km, (a) determine the average acceleration of the spaceship. (b) What is the difference in the acceleration experienced by the occupants in the nose of the ship and those in the rear of the ship farthest from the black hole?

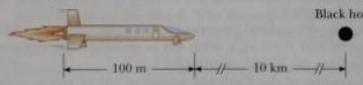


Figure P11.22

Section 11.5 Atomic Spectra and the Bohr Theory of Hydrogen

23. For a hydrogen atom in its ground state, use the Bohr model to compute (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the electrical potential energy of the atom.
24. Radiation is emitted from a hydrogen atom, which undergoes a transition from the $n = 3$ state to the $n = 2$ state. Calculate (a) the energy, (b) the wavelength, and (c) the frequency of the emitted light.
25. A hydrogen atom is in its first excited state ($n = 2$). Using the Bohr theory of the atom, calculate (a) the radius of the orbit, (b) the linear momentum of the electron, (c) the angular momentum of the electron, and its (d) kinetic energy, (e) potential energy, and (f) total energy.
26. How much energy is required to ionize hydrogen (a) when it is in the ground state? (b) when it is in the state for which $n = 3$?
27. Show that the speed of the electron in the nth Bohr orbit in hydrogen is given by

$$v_n = \frac{k_e e^2}{n\hbar}$$

28. (a) Calculate the angular momentum of the Moon due to its orbital motion about the Earth. In your calculation, use 3.84×10^8 m as the average Earth–Moon distance and 2.36×10^6 s as the period of the Moon in its orbit. (b) Determine the corresponding quantum number. (c) By what fraction would the Earth–Moon distance have to be increased to increase the quantum number by 1?

Additional Problems

29. *Voyagers 1* and *2* surveyed the surface of Jupiter's moon Io and photographed active volcanoes spewing liquid sulfur to heights of 70 km above the surface of this moon. Estimate the speed with which the liquid sulfur left the volcano. Io's mass is 8.9×10^{22} kg and its radius is 1820 km.
30. You are an astronaut, and you observe a small planet to be spherical. After landing on the planet, you set off, walk straight ahead, and find yourself returning to your spacecraft from the opposite side after completing a lap of 25.0 km. You hold a hammer and a falcon feather at a height of 1.40 m, release them, and observe them to fall together to the surface in 29.2 s. Determine the mass of the planet.
31. A cylindrical habitat in space 6.00 km in diameter and 30 km long has been proposed (by G. K. O'Neill, 1974). Such a habitat would have cities, land, and lakes on the inside surface and air and clouds in the center. This would all be held in place by rotation of the cylinder about its long axis. How fast would the cylinder have to rotate to imitate the Earth's gravitational field at the walls of the cylinder?
32. A "treetop satellite" is a satellite that orbits just above the surface of a spherical object, assumed to offer no air resistance. (a) Find the period of a treetop satellite of Earth. (b) Prove that its speed is given by $v = R\sqrt{4\pi G\rho/3}$, where ρ is the density of the planet.
33. In introductory physics laboratories, a typical Cavendish balance for measuring the gravitational constant G uses lead spheres of masses 1.50 kg and 15.0 g the centers of which are separated by about 4.50 cm. Calculate the gravitational force between these spheres, treating each as a point mass located at the center of the sphere.
34. Show that the escape speed from the surface of a planet of uniform density is directly proportional to the radius of the planet.
35. Two hypothetical planets of masses m_1 and m_2 and radii r_1 and r_2, respectively, are at rest when they are an infinite distance apart. Because of their gravitational attraction, they head toward each other on a collision course. (a) When their center-to-center separation is d, find the speed of each planet and their *relative* velocity. (b) Find the kinetic energy of each planet *just* before they collide if $m_1 = 2.00 \times 10^{24}$ kg, $m_2 = 8.00 \times 10^{24}$ kg, $r_1 = 3.00 \times 10^6$ m, and $r_2 = 5.00 \times 10^6$ m. (*Hint:* Both energy and momentum are conserved.)
36. The maximum distance from the Earth to the Sun (at our aphelion) is 1.521×10^{11} m and the distance of closest approach (at perihelion) is 1.471×10^{11} m. If the Earth's orbital speed at perihelion is 3.027×10^4 m/s, determine (a) the Earth's orbital speed at aphelion, (b) the kinetic and potential energy at perihelion, and (c) the kinetic and potential energy at aphelion. Is the total energy constant? (Neglect the effect of the Moon and other planets.)
37. (a) Determine the amount of work (in joules) that must be done on a 100-kg payload to elevate it to a height of 1000 km above the Earth's surface. (b) Determine the amount of additional work that is required to put the payload into circular orbit at this elevation.
38. After a supernova explosion, a star may undergo a gravitational collapse to an extremely dense state known as a neutron star, in which all the electrons and protons are squeezed together to form neutrons. A neutron star having a mass about equal to that of the Sun would have a radius of about 10.0 km. Find (a) the free-fall acceleration at its surface, (b) the weight of a 70.0-kg person at its surface, and (c) the energy required to remove a neutron of mass 1.67×10^{-27} kg from its surface to infinity.
39. When it orbited the Moon, the *Apollo 11* spacecraft's mass was 9.979×10^3 kg, its period was 119 min, and its mean distance from the Moon's center was 1.849×10^6 m. Assuming its orbit to be circular and the Moon to be a uniform sphere, find (a) the mass of the Moon, (b) the orbital speed of the spacecraft, and (c) the minimum energy required for the craft to leave the orbit and escape the Moon's gravitational field.
40. Three point objects having masses m, $2m$, and $3m$, are fixed at the corners of a square of side length a such that the lighter object is at the upper left-hand corner, the heavier object is at the lower left-hand corner, and the remaining object is at the upper right-hand corner. Determine the magnitude and direction of the resulting gravitational acceleration at the center of the square.
41. X-ray pulses from Cygnus X-1, a celestial x-ray source, have been recorded during high-altitude rocket flights. The signals can be interpreted as originating when a blob of ionized matter orbits a black hole with a period of 5.00 ms. If the blob were in a circular orbit about a black hole the mass of which is $20.0M_{Sun}$, what is the orbital radius?
42. Studies of the relationship of the Sun to its galaxy—the Milky Way—have revealed that the Sun is located near the outer edge of the galactic disc, about 30 000 lightyears from the center. Furthermore, it has been found that the Sun has an orbital speed of approximately 250 km/s around the galactic center. (a) What is the period of the Sun's galactic motion? (b) What is the order of magnitude of the mass of the Milky Way galaxy? Supposing that the galaxy is made mostly of stars of which the Sun is typical, estimate the number of stars in the Milky Way.
43. Four possible transitions for a hydrogen atom are as follows:

(A) $n_i = 2$; $n_f = 5$ (B) $n_i = 5$; $n_f = 3$
(C) $n_i = 7$; $n_f = 4$ (D) $n_i = 4$; $n_f = 7$

(a) Which transition emits the shortest wavelength photon? (b) In which transition does the atom gain the most energy? (c) In which transition(s) does the atom lose energy?
44. *Vanguard I*, launched March 3, 1958, is the oldest human-made satellite still in orbit. Its initial orbit had an apogee of 3970 km and a perigee of 650 km. Its maximum speed was 8.23 km/s and it had a mass of 1.60 kg. (a) Determine the period of the orbit. (Use the semimajor axis.) (b) Determine the speeds at apogee and perigee. (c) Find the total energy of the satellite.
45. Two stars of masses M and m, separated by a distance d, revolve in circular orbits about their center of mass (Fig. P11.45). Show that each star has a period given by

$$T^2 = \frac{4\pi^2}{G(M+m)}d^3$$

(*Hint:* Apply Newton's second law to each star, and note that the center-of-mass condition requires that $Mr_2 = mr_1$, where $r_1 + r_2 = d$.)

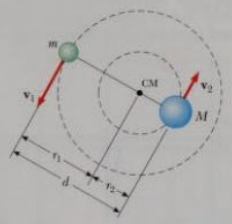


Figure P11.45

46. In 1978, astronomers at the U.S. Naval Observatory discovered that Pluto has a moon, called Charon, that eclipses the planet every 6.39 days. Given that the center-to-center separation between Pluto and Charon is 19 700 km, find the total mass ($M + m$) of the two bodies. (*Hint:* See Problem 45.)
47. Two hydrogen atoms collide head-on and end up with zero kinetic energy. Each then emits a 121.6-nm photon ($n = 2$ to $n = 1$ transition). At what speed were the atoms moving before the collision?
48. A rocket is given an initial speed of $v_0 = 2\sqrt{Rg}$ vertically upward at the surface of the Earth, which has radius R and surface free-fall acceleration g. The rocket motors are then cut off, and thereafter the rocket coasts under the action of gravitational forces only. (Ignore atmospheric friction and the Earth's rotation.) Derive an expression for the subsequent speed, v, as a function of the distance, r, from the center of the Earth in terms of g, R, and r.

Spreadsheet Problems

S1. Four point masses are fixed as shown in Figure PS11.1. The gravitational potential energy for a test particle of mass m moving along the x axis in the gravitational field of the fixed masses can be written as

$$U(x) = -\frac{GM_1 m}{r_1} - \frac{GM_2 m}{r_2} - \frac{GM_3 m}{r_3} - \frac{GM_4 m}{r_4}$$

where $r_2 = r_4 = [b^2 + (x + a)^2]^{1/2}$ and $r_2 = r_3 = [b^2 + (x - a)^2]^{1/2}$. Spreadsheet 11.1 calculates $U(x)$ and the force $F(x) = -dU(x)/dx$ exerted on the test particle. The quantities M_1, M_2, M_3, M_4, a, and b are input parameters.

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The mass of the test particle is 1.00 kg. (This is the case of two particles on the y axis at y = +1.00 m and −1.00 m.) (a) Plot $U(x)$ and $F(x)$ versus x for $x = -4.00$ m to $+4.00$ m. (b) If the test particle has an energy of -7.50×10^{-11} J, describe the particle's motion. (c) How does the force on the particle vary as the particle moves along the x axis?

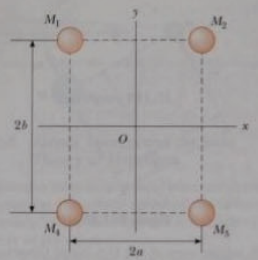


Figure PS11.1

S2. In Spreadsheet 11.1, let $a = 1.00$ m, $b = 0.50$ m, and $M_1 = M_2 = M_3 = M_4 = 1.00$ kg. The mass of the particle is 1.00 kg. (a) Plot $U(x)$ and $F(x)$ versus x. (b) If the test particle has an energy of -2.00×10^{-10} J, what motion is possible?

S3. If a projectile is fired straight up from the surface of the Earth with an initial speed v_0 that is less than the escape speed $v_{esc} = \sqrt{2gR_E} = 11.2$ km/s, the projectile will rise to a maximum height h above the Earth's surface and return. Neglecting air resistance and using conservation of energy, show that

$$h = \frac{R_E v_0^2}{2gR_E - v_0}$$

where R_E is the Earth's radius. Write a spreadsheet to calculate h for $v_0 = 1, 2, 3, \ldots,$ 11 km/s. Plot h/R_E versus v_0. What happens as v_0 approaches the escape speed v_{esc}?

S4. The acceleration of an object moving in the gravitational field of the Earth is

$$\mathbf{a} = -GM_E \frac{\mathbf{r}}{r^3}$$

where $\mathbf{r}$ is the position vector directed from the center of the Earth to the object. Choosing the origin at the center of the Earth and assuming the object is moving in the xy plane, the acceleration has rectangular components

$$a_x = -\frac{GM_E x}{(x^2 + y^2)^{3/2}} \qquad a_y = -\frac{GM_E y}{(x^2 + y^2)^{3/2}}$$

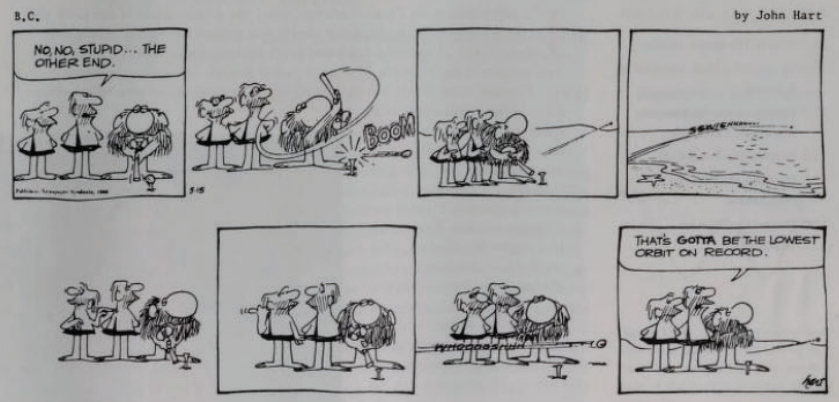
Write a spreadsheet (or computer program) to find the position of the object as a function of time. (Use the techniques developed in Spreadsheet 11.1 as a model for your own program. Assume that the initial position of the object is at $x = 0$ and $y = 2R_E$, where R_E is the radius of the Earth, and give the object an initial velocity of 5 km/s in the x direction. The time increment should be made as small as practical. Try 5 s. Plot the x and y coordinates of the object as functions of time. Does the object hit the Earth? Vary the initial velocity until a circular orbit is found.

S5. Modify your spreadsheet (or program) from Problem S4 to calculate the kinetic, potential, and total energies as functions of time. How do they vary with time for the various cases?

S6. Modify your spreadsheet (or program) from Problem S4 to calculate the angular momentum of the object. In rectangular coordinates, $L = m(xv_y - yv_x)$. How does the angular momentum vary with time? Kepler's second law of planetary motion implies that L remains constant. Is your calculated L constant?

ANSWERS TO CONCEPTUAL PROBLEMS

1. Kepler's second law is a consequence of conservation of angular momentum. Angular momentum is conserved as long as there is no torque, which is true for any central force, regardless of the particular dependence on separation. Thus, Kepler's second law would be unchanged. The particular mathematical form of Kepler's third law depends on the inverse square character of the force. If the force were an inverse cube, there would still be a proportional relationship between a power of the period and a power of the semimajor axis, but the powers would be different than in the existing law.
2. Air friction causes a decrease in the mechanical energy of the satellite–Earth system. This reduces the major axis of the orbit, bringing the satellite closer to the surface of the Earth. A satellite in a smaller orbit, however, must travel faster. Thus, the effect of the air friction is to speed up the satellite!
3. Kepler's third law (Eq. 11.4), which applies to all the planets, tells us that the period of a planet is proportional to $r^{3/2}$. Because Saturn and Jupiter are farther from the Sun than Earth, they have longer periods. The Sun's gravitational field (whose magnitude is GM_S/r^2) is much weaker at a distant Jovian planet. Thus, an outer planet experiences much smaller centripetal acceleration than Earth, and a correspondingly longer period.
4. In an emission spectrum, a gas is initially excited to higher quantum states. The emission spectrum is provided by subsequent downward transitions which can occur to all lower states. If the transition is not to the lowest state, the atom can make a subsequent transition from the intermediate state to the lowest state. For example, a transition can be made from $n = 10$ to $n = 5$, followed by a transition from $n = 5$ to $n = 1$. Thus, there is a rich mixture of spectral lines in the spectrum, corresponding to this large number of possible transitions. An absorption spectrum is normally performed with the gas at room temperature, at which almost all of the atoms are in the ground state. Thus, when radiation is absorbed, upward transitions involving *only the ground state* are possible. Thus, there are fewer transitions than for the emission spectrum and fewer lines.
5. In the discussion of the hydrogen atom, it can be seen that the kinetic energy is equal to half of the magnitude of the potential energy. Thus, we must have $K = E$ and $U_e = -2E$.



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12

Oscillatory Motion

CHAPTER OUTLINE

12.1 Simple Harmonic Motion
12.2 Motion of a Mass Attached to a Spring
12.3 Energy of the Simple Harmonic Oscillator
12.4 Motion of a Pendulum
12.5 Damped Oscillations (Optional)
12.6 Forced Oscillations (Optional)

If a force acting on a body varies in time, the acceleration of the body also changes with time. If this force always acts toward the equilibrium position of the body, a repetitive back-and-forth motion about that position results. The motion is an example of what is called *periodic* or *oscillatory* motion.

You are most likely familiar with several examples of periodic motion, such as the oscillations of a mass on a spring, the motion of a pendulum, and the vibrations of a stringed musical instrument. Numerous systems exhibit oscillatory motion. For example, the molecules in a solid oscillate about their equilibrium positions; electromagnetic waves, such as light waves, radar, and radio waves, are characterized by oscillating electric and magnetic field vectors; in alternating-current circuits, voltage, current, and electrical charge vary periodically with time.



Time exposure of a pendulum, which consists of a small sphere suspended by a light string. The time it takes the pendulum to undergo one complete oscillation is called the period of its motion, and the maximum displacement of the pendulum from the vertical position is called the *amplitude*. As we shall see in this chapter, the period of its motion for small amplitudes depends only on the length of the pendulum and the value of the free-fall acceleration. *(James Stevenson/SPL/Photo Researchers)*

Much of the material in this chapter deals with **simple harmonic motion.** This special kind of motion occurs when a restoring force acts on a body. In simple harmonic motion, an object oscillates between two spatial positions for an indefinite period of time. In real mechanical systems, retarding (frictional) forces are always present. Such forces reduce the mechanical energy of the system with time, and the oscillations are said to be *damped*. If an external driving force is applied, we call the motion a *forced oscillation*.

12.1 • SIMPLE HARMONIC MOTION

A particle moving along the x axis is said to exhibit **simple harmonic motion** when x, its displacement from equilibrium, varies in time according to the relationship

$$x = A\cos(\omega t + \phi) \qquad [12.1]$$

• ***Displacement versus time for simple harmonic motion***

where A, ω, and ϕ are constants of the motion. In order to give physical significance to these constants, it is convenient to plot x as a function of t, as in Figure 12.1. First, we note that A, called the **amplitude** of the motion, is simply **the maximum displacement of the particle in either the positive or negative x direction.** The constant ω is called the **angular frequency** (defined in Eq. 12.4). The constant angle ϕ is called the **phase constant** (or phase angle) and, along with the amplitude A, is determined uniquely by the initial displacement and velocity of the particle. The constants ϕ and A tell us what the displacement and velocity were at time $t = 0$. The quantity $(\omega t + \phi)$ is called the **phase** of the motion and is useful for comparing the motions of two particles. Note that the function x is periodic and repeats itself each time ωt increases by 2π radians.

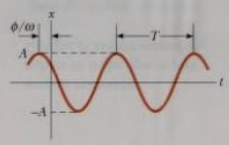


Figure 12.1 Displacement versus time for a particle undergoing simple harmonic motion. The amplitude of the motion is A and the period is T.

The **period,** T, of the motion is the time required for the particle to go through one full cycle of its motion. That is, the value of x at time t equals the value of x at time $t + T$. We can show that the period is given by $T = 2\pi/\omega$ by using the fact that the phase increases by 2π radians in a time of T:

$$\omega t + \phi + 2\pi = \omega(t + T) + \phi$$

Simplifying this expression, we see that $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\omega} \qquad [12.2]$$

• ***Period***

The inverse of the period is called the **frequency** of the motion, f. The frequency represents the **number of oscillations the particle makes per unit time:**

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad [12.3]$$

• ***Frequency***

The units of f are cycles per second, or hertz (Hz). Rearranging Equation 12.3 gives

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad [12.4]$$

• ***Angular frequency***

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The angular frequency ω has the units radians per second. We shall discuss the geometric significance of ω in Section 12.4.

We can obtain the velocity of a particle undergoing simple harmonic motion by differentiating Equation 12.1 with respect to time:

Velocity in simple harmonic motion

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \qquad [12.5]$$

The acceleration of the particle is dv/dt:

Acceleration in simple harmonic motion

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \qquad [12.6]$$

Because $x = A\cos(\omega t + \phi)$, we can express Equation 12.6 in the form

$$a = -\omega^2 x \qquad [12.7]$$

From Equation 12.5 we see that, because the sine and cosine functions oscillate between ± 1, the extreme values of v are $\pm \omega A$. Likewise, Equation 12.6 tells us that the extreme values of the acceleration are $\pm \omega^2 A$. Therefore, the *maximum* values of the velocity and acceleration are

Maximum values of velocity and acceleration in simple harmonic motion

$$v_{max} = \omega A \qquad [12.8]$$

$$a_{max} = \omega^2 A \qquad [12.9]$$

Figure 12.2a plots displacement versus time for an arbitrary value of the phase constant. The velocity and acceleration-versus-time curves are illustrated in Figures 12.2b and 12.2c. They show that the phase of the velocity differs from the phase of the displacement by $\pi/2$ rad, or 90°. That is, when x is a maximum or a minimum,

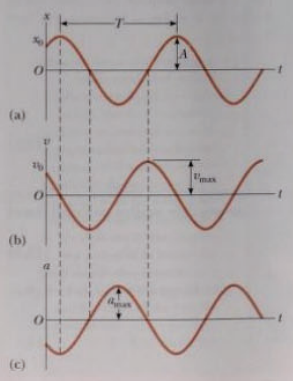


Figure 12.2 Graphical representation of simple harmonic motion: (a) displacement versus time, (b) velocity versus time, and (c) acceleration versus time. Note that at any specified time the velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement.

the velocity is zero. Likewise, when x is zero, the speed is a maximum. Furthermore, note that the phase of the acceleration differs from the phase of the displacement by π radians, or 180°. That is, when x is a maximum, a is a maximum in the opposite direction.

As we stated earlier, $x = A\cos(\omega t + \phi)$ is a general expression for the displacement of the particle from equilibrium, where the phase constant ϕ and the amplitude A must be chosen to meet the initial conditions of the motion. Suppose that the initial position, x_0, and initial velocity, v_0, of a single oscillator of known angular frequency are given; that is, at $t = 0$, $x = x_0$ and $v = v_0$. Under these conditions, Equations 12.1 and 12.5 give

$$x_0 = A\cos\phi \quad \text{and} \quad v_0 = -\omega A \sin\phi \qquad [12.10]$$

Dividing these two equations eliminates A, giving $\frac{v_0}{x_0} = -\omega\tan\phi$, or

$$\tan\phi = -\frac{v_0}{\omega x_0} \qquad [12.11]$$

Furthermore, if we take the sum $x_0^2 + (v_0/\omega)^2 = A^2\cos^2\phi + A^2\sin^2\phi$ (where we have used Eq. 12.10) and solve for A, we find that

The phase angle ϕ and amplitude A can be obtained from the initial conditions.

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2} \qquad [12.12]$$

Thus, we see that ϕ and A are known if x_0, ω, and v_0 are specified. We shall treat a few specific cases in the next section.

We conclude this section by pointing out the following important properties of a particle moving in simple harmonic motion:

Properties of simple harmonic motion

- The displacement, velocity, and acceleration all vary sinusoidally with time but are not in phase, as shown in Figure 12.2.
- The acceleration is proportional to the displacement, but in the opposite direction.
- The frequency and the period of motion are independent of the amplitude.

Example 12.1 An Oscillating Body

A body oscillates with simple harmonic motion along the x axis. Its displacement varies with time according to the equation

$$x = (4.00\text{ m})\cos\left(\pi t + \frac{\pi}{4}\right)$$

where t is in seconds and the angles in the parentheses are in radians.

(a) Determine the amplitude, frequency, and period of the motion.

Solution By comparing this equation with the general equation for simple harmonic motion, $x = A\cos(\omega t + \phi)$, we see that $A = 4.00$ m and $\omega = \pi$ rad/s; therefore we find $f = \omega/2\pi = \pi/2\pi = 0.500\text{ s}^{-1}$ and $T = 1/f = 2.00$ s.

(b) Calculate the velocity and acceleration of the body at any time t.

Solution

$$v = \frac{dx}{dt} = -4.00\sin\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi\text{ m/s})\sin\left(\pi t + \frac{\pi}{4}\right)$$

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$$a = \frac{dv}{dt} = -4.00\pi \cos\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$= -(4.00\pi^2\ \text{m/s}^2)\cos\left(\pi t + \frac{\pi}{4}\right)$$

EXERCISE 1 Using the results to part (b), determine the position, velocity, and acceleration of the body at $t = 1.00$ s.
Answer $x = -2.83$ m; $v = 8.89$ m/s; $a = 27.9$ m/s^2

EXERCISE 2 Determine the maximum speed and the maximum acceleration of the body. Answer From the general expressions for v and a found in part (b), we see that the maximum values of the sine and cosine functions are unity. Therefore, v varies between $\pm 4.00\pi$ m/s, and a varies between $\pm 4.00\pi^2$ m/s^2. Thus, $v_{max} = 4.00\pi$ m/s and $a_{max} = 4.00\pi^2$ m/s^2.

EXERCISE 3 A particle moving with simple harmonic motion travels a total distance of 20.0 cm in each cycle of its motion, and its maximum acceleration is 50.0 m/s^2. Find (a) the angular frequency of the motion and (b) the maximum speed of the particle.
Answer (a) 31.6 rad/s (b) 1.58 m/s

12.2 · MOTION OF A MASS ATTACHED TO A SPRING

Consider a physical system consisting of a mass, m, attached to the end of a spring where the mass is free to move on a horizontal track (Fig. 12.3). We know from experience that such a system oscillates back and forth if disturbed from the equilibrium position $x = 0$, where the spring is unstretched. If the surface is frictionless, the mass can exhibit simple harmonic motion. One experimental arrangement that clearly demonstrates that such a system exhibits simple harmonic motion is illustrated in Figure 12.4. A mass oscillating vertically on a spring has a marking pen attached to it. While the mass is in motion, a sheet of paper is moved horizontally as shown, and the marking pen traces out a sinusoidal pattern. We can understand this qualitatively by first recalling that when the mass is displaced a small distance, x, from equilibrium, the spring exerts a force on it, given by **Hooke's law,**

$$F_s = -kx \qquad [12.13]$$

where k is the force constant of the spring. We call this a **linear restoring force** because it is linearly proportional to the displacement and always directed toward the equilibrium position *opposite* the displacement. That is, when the mass is displaced to the right in Figure 12.3, x is positive and the restoring force is to the left. When the mass is displaced to the left of $x = 0$, then x is negative and the restoring force is to the right.

If we apply Newton's second law to the motion of the mass in the x direction, we get

$$F_s = -kx = ma$$

$$a = -\frac{k}{m}x \qquad [12.14]$$

That is, just as we learned in Section 12.1, **the acceleration is proportional to the displacement of the mass from equilibrium and is in the opposite direction.** If the mass is displaced a maximum distance, $x = A$, at some initial time and released from rest, its *initial* acceleration is $-kA/m$ (that is, the acceleration has its extreme negative value). When the mass passes through the equilibrium position, $x = 0$ and its acceleration is zero. At this instant, its speed is a maximum. It then continues to travel to the left of equilibrium and finally reaches $x = -A$, at which time its acceleration is kA/m (maximum positive) and its speed is again zero. Thus, we see that the mass oscillates between the turning points $x = \pm A$. In one full cycle of its motion it travels a distance of $4A$.

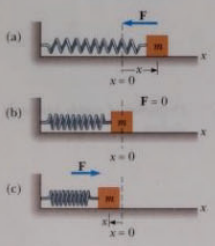


Figure 12.3 A mass attached to a spring on a frictionless track moves in simple harmonic motion. (a) When the mass is displaced to the right of equilibrium, the displacement is positive and the acceleration is negative. (b) At the equilibrium position, $x = 0$, the acceleration is zero but the speed is a maximum. (c) When the displacement is negative, the acceleration is positive.

Let us now describe this motion in a quantitative fashion. Recall that, by definition, $a = dv/dt = d^2x/dt^2$, and so we can express Equation 12.14 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad [12.15]$$

If we denote the ratio k/m with the symbol ω^2,

$$\omega^2 = \frac{k}{m} \qquad [12.16]$$

then Equation 12.15 can be written in the form

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad [12.17]$$

What we now require is a solution to Equation 12.17—that is, a function $x(t)$ that satisfies this second-order differential equation. Because Equations 12.17 and 12.7 are equivalent, we see that the solution must be that of simple harmonic motion:

$$x(t) = A\cos(\omega t + \phi)$$

To see this explicitly, note that if $x = A\cos(\omega t + \phi)$, then

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 A\cos(\omega t + \phi)$$

Comparing the expressions for x and d^2x/dt^2, we see that $d^2x/dt^2 = -\omega^2 x$ and Equation 12.17 is satisfied.

The following general statement can be made based on the foregoing discussion:

> Whenever the force acting on a particle is linearly proportional to the displacement and in the opposite direction, the particle exhibits simple harmonic motion.

We shall give additional physical examples in subsequent sections.

Because the period of simple harmonic motion is $T = 2\pi/\omega$ and the frequency is the inverse of the period, we can express the period and frequency of the motion for this mass–spring system as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad [12.18]$$

• *Period of a mass–spring system*

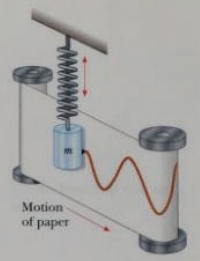


Figure 12.4 An experimental apparatus for demonstrating simple harmonic motion. A pen attached to the oscillating mass traces out a sine wave on the moving chart paper.

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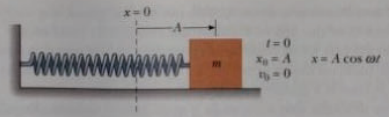


Figure 12.5 A mass–spring system that starts from rest at $x_0 = A$. In this case, $\phi = 0$, and so $x = A \cos \omega t$.

Frequency of a mass–spring system

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega}{2\pi} \qquad [12.19]$$

That is, the period and frequency depend *only* on the mass and on the force constant of the spring. As we might expect, the frequency is larger for a stiffer spring (larger value of k) and decreases with increasing mass.

Special Case I In order to better understand the physical significance of our solution of the equation of motion, let us consider the following special case. Suppose we pull the mass from equilibrium by a distance of A and release it from rest in this stretched position, as in Figure 12.5. We must then require that our solution for $x(t)$ obey the initial conditions that at $t = 0$, $x_0 = A$ and $v_0 = 0$. These conditions are met if we choose $\phi = 0$, giving $x = A \cos \omega t$ as our solution. To check this solution, we note that it satisfies the condition that $x_0 = A$ at $t = 0$, because $\cos 0 = 1$. Thus, we see that A and ϕ contain the information on initial conditions.

Now let us investigate the behavior of the velocity and acceleration in this special case. Because $x = A \cos \omega t$, we have

$$v = \frac{dx}{dt} = -\omega A \sin \omega t \qquad \text{and} \qquad a = \frac{dv}{dt} = -\omega^2 A \cos \omega t$$

From the preceding velocity expression we see that at $t = 0$, $v_0 = 0$, as we require. The expression for the acceleration tells us that at $t = 0$, $a = -\omega^2 A$. Physically this makes sense, because the force on the mass is to the left when the displacement is positive. In fact, at this position $F_x = -kA$ (to the left), and the initial acceleration is $-kA/m$.

We could also use a more formal approach to show that $x = A \cos \omega t$ is the correct solution by using the relation $\tan \phi = -v_0/\omega x_0$ (Eq. 12.10). Because $v_0 = 0$ at $t = 0$, $\tan \phi = 0$ and so $\phi = 0$.

The displacement, velocity, and acceleration versus time are plotted in Figure 12.6 for this special case. Note that the acceleration reaches extreme values of $\pm \omega^2 A$ when the displacement has extreme values of $\mp A$. Furthermore, the velocity has extreme values of $\pm \omega A$, which both occur at $x = 0$. Hence, the quantitative solution agrees with our qualitative description of this system.

Special Case II Now suppose that the mass is given an initial velocity of $\mathbf{v}_0$ to the *right* at the unstretched position of the spring, so that at $t = 0$, $x_0 = 0$ and $v = v_0$ (Fig. 12.7). Our particular solution must now satisfy these initial conditions.

Applying Equation 12.11, $\tan \phi = -v_0/\omega x_0$, and the initial condition that $x_0 = 0$ at $t = 0$ gives $\tan \phi = -\infty$ or $\phi = -\pi/2$. Hence, the solution is given by $x = A \cos(\omega t - \pi/2)$, which can be written $x = A \sin \omega t$. Furthermore, from Equation 12.11 we see that $A = v_0/\omega$; therefore, we can express our solution as

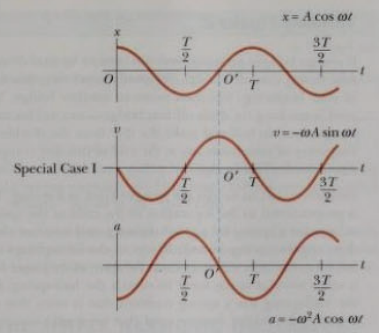


Figure 12.6 Displacement, velocity, and acceleration versus time for a particle undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x = A$, and $v = 0$.

$$x = \frac{v_0}{\omega} \sin \omega t$$

The velocity and acceleration in this case are

$$v = \frac{dx}{dt} = v_0 \cos \omega t \qquad \text{and} \qquad a = \frac{dv}{dt} = -\omega v_0 \sin \omega t$$

This is consistent with the fact that the mass always has its maximum speed at $x = 0$, and the force and acceleration are zero at that position. The graphs of these functions versus time in Figure 12.6 correspond to the origin at O'. What would be the solution for x if the mass were initially moving to the left in Figure 12.7?

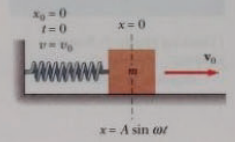


Figure 12.7 The mass–spring system starts its motion at the equilibrium position, $x = 0$ at $t = 0$. If its initial velocity is v_0 to the right, its x coordinate varies as $x = \frac{v_0}{\omega} \sin \omega t$.

Thinking Physics 1

We know that the period of oscillation of a mass hung on a spring is proportional to the square root of the mass. Thus, if we perform an experiment in which we hang a range of masses on the end of a spring and measure the period of oscillation, a graph of the square of the period against the mass will result in a straight line, as suggested in Figure TP12.1. But we find that the line does not go through the origin. Why not?

Reasoning The reason that the line does not go through the origin is that the spring itself has mass. Thus, the resistance to changes in motion of the system is a combination of the mass hung on the end of the spring and the mass of the oscillating coils of the spring. The entire mass of the spring is not oscillating, however. The bottom-most coil is oscillating over the same amplitude as the mass, and the top-most coil is not oscillating at all. For a cylindrical spring, energy arguments can be used to show that the effective additional mass representing the oscillations of the spring is one third of the mass of the spring. The square of the period is proportional to the total oscillating mass, but the graph in the diagram shows the square of the period against only the mass hung on the spring. A graph of period squared against total mass (mass hung on the spring plus the effective oscillating mass of the spring) would pass through the origin.

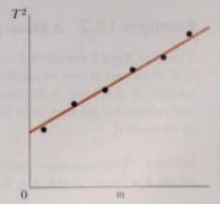


Figure TP12.1 (Thinking Physics 1)

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Thinking Physics 2



(Thinking Physics 2) Bungee jumping. *(Gamma)*

If you go bungee jumping, you will bounce up and down at the end of the cord after your daring dive off a bridge. Suppose you perform this dive and measure the frequency of your bouncing. You then move to another bridge. You discover that the bungee cord is too long for dives off this bridge—you will hit the ground. Thus, you fold the bungee cord in half and make the dive from the doubled bungee cord. How does the frequency of your bouncing at the end of this dive compare to the frequency after the dive from the first bridge?

Reasoning Let us model the bungee cord as a spring. The force exerted by a spring is proportional to the separation of the coils as the spring is extended. Imagine that we extend a spring by a given distance and measure the distance between coils. We then cut the spring in half. If one of the half-springs is now extended by the same distance, the coils will be twice as far apart as they were for the complete spring. Thus, it takes twice as much force to stretch the half-spring, from which we conclude that the half-spring has a spring constant that is twice that of the complete spring. Now consider the folded bungee cord that we model as two half-springs in parallel. Each half has a spring constant that is twice the original spring constant of the bungee cord. In addition, a given mass hanging on the folded bungee cord will experience two forces—one from each half-spring. As a result, the required force for a given extension will be four times as much as for the original bungee cord. The effective spring constant of the folded bungee cord is, therefore, four times as large as the original spring constant. Because the frequency of oscillation is proportional to the square root of the spring constant, your bouncing frequency on the folded cord will be twice that of the original cord.

CONCEPTUAL PROBLEM 1

Is a bouncing ball an example of simple harmonic motion? Is the daily movement of a student from home to school and back simple harmonic motion?

CONCEPTUAL PROBLEM 2

A mass is hung on a spring and the frequency of oscillation of the system, f, is measured. The mass, a second identical mass, and the spring are carried in the Space Shuttle to space. The two masses are attached to the ends of the spring and the system is taken out into space on a space walk. The spring is extended and the system is released to oscillate while floating in space. What is the frequency of oscillation for this system, in terms of f?

Example 12.2 A Mass–Spring System

A mass of 200 g is connected to a light spring of force constant 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. If the mass is displaced 5.00 cm from equilibrium and released from rest, as in Figure 12.5, (a) find the period of its motion.

Solution This situation corresponds to Special Case I, in which $x = A\cos\omega t$ and $A = 5.00 \times 10^{-2}$ m. Therefore,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00\ \text{N/m}}{200 \times 10^{-3}\ \text{kg}}} = 5.00\ \text{rad/s}$$

and

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00} = 1.26\ \text{s}$$

(b) Determine the maximum speed and maximum acceleration of the mass.

Solution

$$v_{\max} = \omega A = (5.00\ \text{rad/s})(5.00 \times 10^{-2}\ \text{m}) = 0.250\ \text{m/s}$$

$$a_{\max} = \omega^2 A = (5.00\ \text{rad/s})^2(5.00 \times 10^{-2}\ \text{m}) = 1.25\ \text{m/s}^2$$

(c) Express the displacement, speed, and acceleration as functions of time.

Solution The expression $x = A\cos\omega t$ is our solution for Special Case I, and so we can use the results from (a), (b), and (c) to get

$$x = A\cos\omega t = (0.0500\ \text{m})\cos 5.00t$$

$$v = -\omega A\sin\omega t = -(0.250\ \text{m/s})\sin 5.00t$$

$$a = -\omega^2 A\cos\omega t = -(1.25\ \text{m/s}^2)\cos 5.00t$$

EXERCISE 4 (a) A 400-g mass is suspended from a spring hanging vertically, and the spring is found to stretch 8.0 cm. Find the spring constant. (b) How much will the spring stretch if the suspended mass is 575 g? Answer (a) 49 N/m (b) 12 cm

EXERCISE 5 A 3.0-kg mass is attached to a spring and pulled out horizontally to a maximum displacement from equilibrium of 0.50 m. What spring constant must the spring have if the mass is to achieve an acceleration equal to the free-fall acceleration? Answer 59 N/m

12.3 · ENERGY OF THE SIMPLE HARMONIC OSCILLATOR

Let us examine the mechanical energy of the mass–spring described in Figure 12.5. Because the surface is frictionless, we expect that the total mechanical energy is constant. We can use Equation 12.5 to express the kinetic energy as

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi) \qquad [12.20]$$

• ***Kinetic energy of a simple harmonic oscillator***

The elastic potential energy stored in the spring for any elongation x is $\frac{1}{2}kx^2$. Using Equation 12.1, we get

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\cos^2(\omega t + \phi) \qquad [12.21]$$

• ***Potential energy of a simple harmonic oscillator***

We see that K and U are always positive quantities. Because $\omega^2 = k/m$, we can express the *total energy* of the simple harmonic oscillator as

$$E = K + U = \tfrac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

But $\sin^2\theta + \cos^2\theta = 1$; therefore, this equation reduces to

$$E = \tfrac{1}{2}kA^2 \qquad [12.22]$$

• ***Total energy of a simple harmonic oscillator***

That is, the energy of a simple harmonic oscillator is a constant of the motion and proportional to the square of the amplitude. In fact, the total mechanical energy is just equal to the maximum potential energy stored in the spring when $x = \pm A$. At these points, $v = 0$ and there is no kinetic energy. At the equilibrium position, $x = 0$ and $U = 0$, so that the total energy is all in the form of kinetic energy. That is, at $x = 0$, $E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m\omega^2 A^2$.

Plots of the kinetic and potential energies versus time are shown in Figure 12.8a, where $\phi = 0$. In this situation, both K and U are always positive, and their sum at all times is a constant equal to $\frac{1}{2}kA^2$, the total energy of the system. The variations of K and U with displacement are plotted in Figure 12.8b. Energy is continuously being transferred between potential energy stored in the spring and the kinetic energy of the mass. Figure 12.9 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the mass–spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure. We suggest that you study it carefully.

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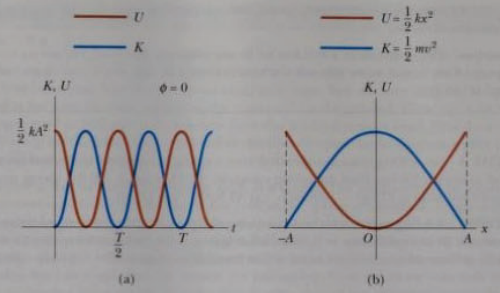


Figure 12.8 (a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with $\phi = 0$. (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In either plot, note that $K + U =$ constant.

Finally, we can use energy to obtain the velocity for an arbitrary displacement, x, expressing the total energy at some arbitrary position as

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

Velocity as a function of position for a simple harmonic oscillator

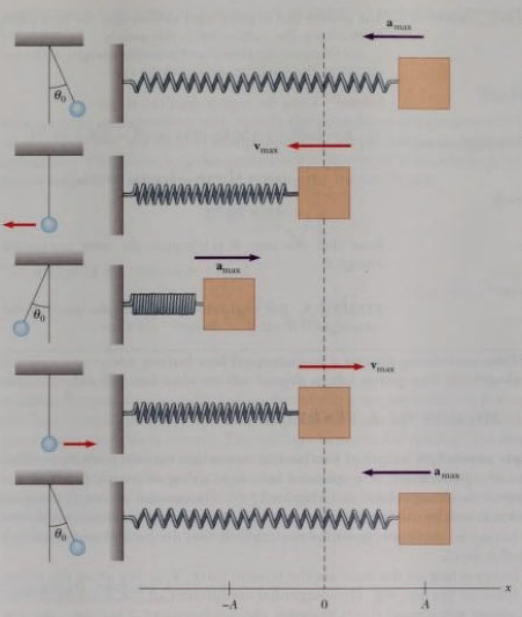
$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2} \qquad [12.23]$$

Again, this expression substantiates the fact that the speed is a maximum at $x = 0$ and zero at the turning points, $x = \pm A$.

Thinking Physics 3

A mass oscillating on the end of a horizontal spring slides back and forth over a frictionless surface. During one oscillation, you set an identical mass with Velcro (a sticky fabric) attached at the maximum displacement point. Just as the oscillating mass reaches its largest displacement and is momentarily at rest, it adheres to the new mass by means of the Velcro, and the two masses continue the oscillation together. Does the period of the oscillation change? Does the amplitude of oscillation change? Does the energy of the oscillation change?

Reasoning The period of oscillation *does* change, because the period depends on the mass that is oscillating. The amplitude does not change. Because the new mass was added while the original mass was at rest, the combined masses are at rest at this point, also, defining the amplitude as the same as in the original oscillation. The energy does not change, either. At the maximum displacement point, the energy is all potential energy stored in the spring, which depends only on the spring constant and the amplitude, not the mass. The increased mass will pass through the equilibrium point with less velocity than in the original oscillation, but with the same kinetic energy. Another approach is to think about how energy could be transferred into the oscillating system—no work was done (nor was there any other form of energy transfer), so the energy in the system cannot change.



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
T/4	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
T/2	-A	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
3T/4	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Figure 12.9 Simple harmonic motion for a mass–spring system and its analogy to the motion of a simple pendulum. The parameters in the table at the right refer to the mass–spring system, assuming that at $t = 0$, $x = A$ so that $x = A\cos\omega t$ (Special Case I).

CONCEPTUAL PROBLEM 3

A mass–spring system undergoes simple harmonic motion with an amplitude A. Does the total energy change if the mass is doubled but the amplitude is not changed? Are the kinetic and potential energies at a given point in its motion affected by the change in mass? Explain.

Example 12.3 Oscillations on a Horizontal Surface

A 0.50-kg mass connected to a light spring of force constant 20.0 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum speed of the mass if the amplitude of the motion is 3.0 cm.

Solution Using Equation 12.22, we get

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}\left(20.0\,\frac{\text{N}}{\text{m}}\right)(3.0 \times 10^{-2}\ \text{m})^2 = 9.0 \times 10^{-3}\ \text{J}$$

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When the mass is at $x = 0$, $U = 0$ and $E = \frac{1}{2}mv_{max}^2$; therefore

$$\tfrac{1}{2}mv_{max}^2 = 9.0 \times 10^{-3}\ \text{J}$$

$$v_{max} = \sqrt{\frac{18.0 \times 10^{-3}\ \text{J}}{0.50\ \text{kg}}} = 0.19\ \text{m/s}$$

(b) What is the velocity of the mass when the displacement is equal to 2.0 cm?

Solution We can apply Equation 12.23 directly:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$= \pm\sqrt{\frac{20.0}{0.50}(3.0^2 - 2.0^2) \times 10^{-4}}$$

$$= \pm 0.14\ \text{m/s}$$

The positive and negative signs indicate that the mass could be moving to the right or left at this instant.

(c) Compute the kinetic and potential energies of the system when the displacement equals 2.0 cm.

Solution Using the result to part (b), we get

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.50\ \text{kg})(0.14\ \text{m/s})^2 = 5.0 \times 10^{-3}\ \text{J}$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}\left(20.0\ \frac{\text{N}}{\text{m}}\right)(2.0 \times 10^{-2}\ \text{m})^2$$

$$= 4.0 \times 10^{-3}\ \text{J}$$

Note that the sum $K + U$ equals the total mechanical energy, E.

EXERCISE 6 For what values of x does the speed of the mass equal 0.10 m/s? Answer ± 2.6 cm

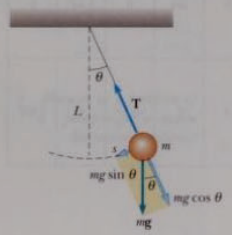


Figure 12.10 When θ is small, the simple pendulum oscillates in simple harmonic motion about the equilibrium position ($\theta = 0$). The restoring force is $mg \sin \theta$, the component of the force of gravity tangent to the circle.

12.4 • MOTION OF A PENDULUM

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a point mass, m, suspended by a light string of length L, where the upper end of the string is fixed as in Figure 12.10. The motion occurs in a vertical plane and is driven by the force of gravity. We shall show that the motion is that of a simple harmonic oscillator, provided the angle, θ, that the pendulum makes with the vertical is small.

The forces acting on the mass are the tension force, $\mathbf{T}$, acting along the string, and the force of gravity, $m\mathbf{g}$. The tangential component of the force of gravity, $mg \sin \theta$, always acts toward $\theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force, and we can use Newton's second law to write the equation of motion in the tangential direction as

$$F_t = -mg \sin \theta = m\frac{d^2s}{dt^2}$$

where s is the displacement measured along the arc in Figure 12.10 and the minus sign indicates that F_t acts toward the equilibrium position. Because $s = L\theta$ and L is constant, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin \theta$$

The right side is proportional to $\sin \theta$ rather than to θ; hence, we conclude that the motion is not simple harmonic motion, because it is not of the form of Equation 12.17. However, if we assume that θ is *small*, we can use the approximation $\sin \theta \approx \theta$, where θ is measured in radians,[1] and the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \qquad [12.24]$$

• ***Equation of motion for the simple pendulum (small θ)***

Now we have an expression with exactly the same form as Equation 12.17, and so we conclude that the motion is approximately simple harmonic motion for small amplitudes. Therefore, θ can be written as $\theta = \theta_0 \cos(\omega t + \phi)$, where θ_0 is the **maximum angular displacement** and the angular frequency ω is

$$\omega = \sqrt{\frac{g}{L}} \qquad [12.25]$$

• ***Angular frequency of motion for a simple pendulum***

The period of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad [12.26]$$

• ***Period of motion for a simple pendulum***

In other words, **the period and frequency of a simple pendulum oscillating at small angles depend only on the length of the string and the free-fall acceleration.** Because the period is *independent* of the mass, we conclude that *all* simple pendula of equal length at the same location oscillate with equal periods. Experiment shows that this is correct. The analogy between the motion of a simple pendulum and the mass–spring system is illustrated in Figure 12.9.

The simple pendulum can be used as a timekeeper. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important, because variations in local values of $\mathbf{g}$ can provide information on the locations of oil and other valuable underground resources.

CONCEPTUAL PROBLEM 4

A simple pendulum is suspended from the ceiling of a stationary elevator, and the period is determined. Describe the changes, if any, in the period when the elevator (a) accelerates upward, (b) accelerates downward, and (c) moves with constant velocity.

CONCEPTUAL PROBLEM 5

Imagine that a pendulum is hanging from the ceiling of a car. As the car coasts freely down a hill, is the equilibrium position of the pendulum vertical? Does the period of oscillation change from that in a stationary car?

CONCEPTUAL PROBLEM 6

A grandfather clock depends on the period of a pendulum to keep correct time. Suppose a grandfather clock is calibrated correctly and then the temperature of the room in which it resides increases. Does the grandfather clock run slow, fast, or correctly? *Hint:* A metal expands when its temperature is raised.

[1] See Appendix A for more details on the small-angle approximation.

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Example 12.4 A Measure of Height

A man enters a tall tower, needing to know its height. He notes that a long pendulum extends from the ceiling almost to the floor and that its period is 12.0 s. How tall is the tower?

Solution If we use $T = 2\pi\sqrt{L/g}$ and solve for L, we get

$$L = \frac{gT^2}{4\pi^2} = \frac{(9.80\ \text{m/s}^2)(12.0\ \text{s})^2}{4\pi^2} = 35.7\ \text{m}$$

EXERCISE 7 If the pendulum described in this example is taken to the moon, where the free-fall acceleration is 1.67 m/s², what is the period there? Answer 29.1 s

OPTIONAL

The Physical Pendulum

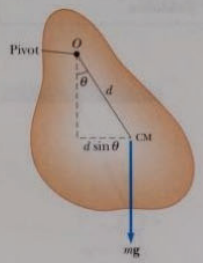


Figure 12.11 The physical pendulum consists of a rigid body pivoted at the point O, which is not at the center of mass. At equilibrium, the weight vector passes through O, corresponding to $\theta = 0$. The restoring torque about O when the system is displaced through an angle θ is $mgd \sin\theta$.

If a hanging object oscillates about a fixed axis that does not pass through its center of mass, and the object cannot be accurately approximated as a point mass, then it must be treated as a physical, or compound, pendulum. Consider a rigid body pivoted at a point O that is a distance of d from the center of mass (Fig. 12.11). The torque about O is provided by the force of gravity, and its magnitude is $mgd \sin\theta$. Using the fact that $\tau = I\alpha$, where I is the moment of inertia about the axis through O, we get

$$-mgd \sin\theta = I\frac{d^2\theta}{dt^2}$$

The minus sign on the left indicates that the torque about O tends to decrease θ. That is, the force of gravity produces a restoring torque.

If we again assume that θ is small, then the approximation $\sin\theta \approx \theta$ is valid and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta \qquad [12.27]$$

Note that the equation has the same form as Equation 12.17, and so the motion is approximately simple harmonic motion for small amplitudes. That is, the solution of Equation 12.27 is $\theta = \theta_0 \cos(\omega t + \phi)$, where θ_0 is the maximum angular displacement and

$$\omega = \sqrt{\frac{mgd}{I}}$$

The period is

Period of motion for a physical pendulum

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \qquad [12.28]$$

One can use this result to measure the moment of inertia of a planar rigid body. If the location of the center of mass and, hence, the distance d are known, the moment of inertia can be obtained through a measurement of the period. Finally, note that Equation 12.28 reduces to the period of a simple pendulum (Eq. 12.26) when $I = md^2$—that is, when all the mass is concentrated at the center of mass.

CONCEPTUAL PROBLEM 7

Two students are watching the swinging of a Foucault pendulum with a large bob in a museum. One student says, "I'm going to sneak past the fence and stick some chewing gum on the top of the pendulum bob, to change its frequency of oscillation." The other student says, "That won't change the frequency—the frequency of a pendulum is independent of mass." Which student is correct?

Example 12.5 A Swinging Rod

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane (Fig. 12.12). Find the period of oscillation if the amplitude of the motion is small.

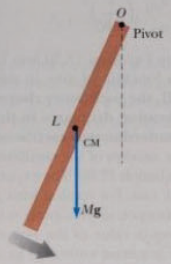


Figure 12.12 (Example 12.5) A rigid rod oscillating about a pivot through one end is a physical pendulum with $d = L/2$ and from Table 10.2, $I_0 = \frac{1}{3}ML^2$.

Solution The moment of inertia of a uniform rod about an axis through one end is $\frac{1}{3}ML^2$. The distance d from the pivot to the center of mass is $L/2$. Substituting these quantities into Equation 12.28 gives

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Comment In one of the Moon landings, an astronaut walking on the Moon's surface had a belt hanging from his space suit, and the belt oscillated as a compound pendulum. A scientist on Earth observed this motion on TV and from it was able to estimate the free-fall acceleration on the Moon. How do you suppose this calculation was done?

EXERCISE 8 Calculate the period of a meter stick pivoted about one end and oscillating in a vertical plane as in Figure 12.12. Answer 1.64 s

EXERCISE 9 A simple pendulum has a length of 3.00 m. Determine the change in its period if it is taken from a point where $g = 9.80\ \text{m/s}^2$ to an elevation where the free-fall acceleration decreases to 9.79 m/s². Answer 1.78×10^{-3} s

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OPTIONAL

12.5 • DAMPED OSCILLATIONS

The oscillatory motions we have considered so far have occurred in the context of an ideal system—that is, one that oscillates indefinitely under the action of a linear restoring force. In realistic systems, dissipative forces, such as friction, are present and retard the motion of the system. As a consequence, the mechanical energy of the system diminishes in time, and the motion is said to be *damped*.

One common type of drag force, which we discussed in Chapter 5, is proportional to the velocity and acts in the direction opposite the motion. This type of drag is often observed when an object is oscillating in air, for instance. Because the drag force can be expressed as $\mathbf{R} = -b\mathbf{v}$, where b is a constant, and the restoring force exerted on the system is $-kx$, we can write Newton's second law as

$$\sum F_x = -kx - bv = ma_x \qquad [12.29]$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

The solution of this equation requires mathematics that may not yet be familiar to you, and so it will simply be stated without proof. When the parameters of the system are such that $b < \sqrt{4mk}$, the solution to Equation 12.29 is

$$x = Ae^{-(b/2m)t}\cos(\omega t + \phi) \qquad [12.30]$$

where the angular frequency of motion is

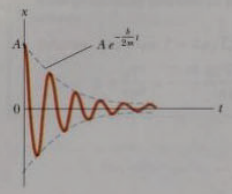
$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad [12.31]$$

This result can be verified by substituting Equation 12.30 into Equation 12.29. Figure 12.13a shows the displacement as a function of time in this case. We see that **when the drag force is relatively small, the oscillatory character of the motion is preserved but the amplitude of vibration decreases in time** and the motion ultimately ceases. This is known as an **underdamped oscillator.** The dashed blue lines in Figure 12.13a, which outline the *envelope* of the oscillatory curve, represent the exponential factor that appears in Equation 12.30. The exponential factor shows that *the amplitude decays exponentially with time.* For motion with a given spring constant and particle mass, the oscillations dampen more rapidly as the maximum value of the drag force approaches the maximum value of the restoring force. One example of a damped harmonic oscillator is a mass immersed in a fluid, as in Figure 12.13b.

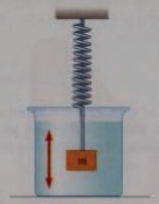
It is convenient to express the angular frequency of vibration of a damped system (Eq. 12.31) in the form

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency of oscillation in the absence of a drag force (the undamped oscillator). In other words, when $b = 0$, the drag force is zero and the system oscillates with its natural frequency, ω_0. As the magnitude of the drag force increases, the oscillations dampen more rapidly. When b reaches a critical value, b_c, so that $b_c/2m = \omega_0$, the system does not oscillate and is said to be



(a)



(b)

Figure 12.13 (a) Graph of the displacement versus time for a damped oscillator. Note the decrease in amplitude with time. (b) One example of a damped oscillator is a mass on a spring submersed in a liquid.

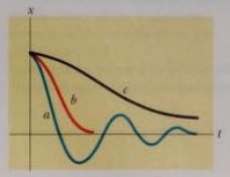


Figure 12.14 Plots of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

critically damped. In this case it returns to equilibrium in an exponential manner with time, as in Figure 12.14.

If the medium is highly viscous and the parameters meet the condition that $b/2m > \omega_0$, the system is **overdamped.** Again, the displaced system does not oscillate but simply returns to its equilibrium position. As the damping increases, the time it takes the displacement to reach equilibrium also increases, as indicated in Figure 12.14. In any case, when a drag force is present, the energy of the oscillator eventually falls to zero. The lost mechanical energy dissipates into thermal energy in the resistive medium.

12.6 • FORCED OSCILLATIONS

OPTIONAL

We have seen that the energy of a damped oscillator decreases in time as a result of the drag force. It is possible to compensate for this energy loss by applying an external force that does positive work on the system. At any instant, energy can be put into the system by an applied force that acts in the direction of motion of the oscillator. For example, a child on a swing can be kept in motion by appropriately timed "pushes." The amplitude of motion remains constant if the energy input per cycle of motion exactly equals the energy lost as a result of air drag.

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F = F_0\cos\omega t$, where ω is the angular frequency of the force and F_0 is a constant. Adding this driving force to the left side of Equation 12.29 gives

$$F_0\cos\omega t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \qquad [12.32]$$

Again, the solution of this equation is rather lengthy and will not be presented. However, after a sufficiently long period of time, when the energy input per cycle equals the energy lost per cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. At this time, Equation 12.32 has the solution

$$x = A\cos(\omega t + \phi) \qquad [12.33]$$

where

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}} \qquad [12.34]$$

and where $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator ($b = 0$). From a physical point of view, one can argue that in a steady state the oscillator must have the same angular frequency as the driving force, and so the solution given by Equation 12.33 is expected.

Equation 12.34 shows that the amplitude of the forced oscillator is constant, because it is being driven by an external force. That is, the external agent provides the energy necessary to overcome the losses due to the drag force. Note that the mass oscillates at the angular frequency of the driving force, ω. For small damping, the amplitude becomes large when the frequency of the driving force is near the



This is a computer-generated simulation of the chaotic pattern of a driven pendulum. The nonlinearity in the motion is provided by the gravitational torque, which is proportional to the sine of the angle. An important research problem is to predict the conditions under which different physical systems exhibit chaotic behavior. *(© Yoav Levy/Phototake)*

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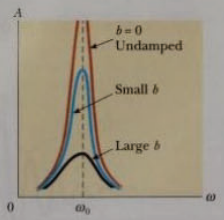


Figure 12.15 Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. When the frequency of the driving force equals the natural frequency, ω_0, resonance occurs. Note that the shape of the resonance curve depends on the size of the damping coefficient, b.

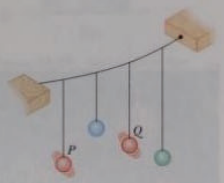


Figure 12.16 If pendulum P is set into oscillation, pendulum Q will eventually oscillate with the greatest amplitude because its length is equal to that of P and so they have the same natural frequency of vibration. The pendula are oscillating in a direction perpendicular to the plane formed by the stationary strings.

natural frequency of oscillation or when $\omega \approx \omega_0$. The dramatic increase in amplitude near the natural frequency is called **resonance,** and the angular frequency, ω_0, is called the **resonance frequency** of the system.

The reason for large-amplitude oscillations at the resonance frequency is that energy is being transferred to the system under the most favorable conditions. This can be better understood by taking the first time derivative of x, which gives an expression for the velocity of the oscillator. In doing so, one finds that v is proportional to $\sin(\omega t + \phi)$. When the applied force is in phase with $\mathbf{v}$, the rate at which work is done on the oscillator by the force $\mathbf{F}$ (in other words, the power) equals Fv. Because the quantity Fv is always positive when $\mathbf{F}$ and $\mathbf{v}$ are in phase, we conclude that **at resonance the applied force is in phase with the velocity, and the power transferred to the oscillator is a maximum.**

Figure 12.15 is a graph of amplitude as a function of frequency for the forced oscillator, with and without a resistive force. Note that the amplitude increases with decreasing damping ($b \rightarrow 0$) and that the resonance curve broadens as the damping increases. Under steady-state conditions, and at any driving frequency, the energy transferred into the system equals the energy lost because of the damping force; hence, the average total energy of the oscillator remains constant. In the absence of a damping force ($b = 0$), we see from Equation 12.34 that the steady-state amplitude approaches infinity as $\omega \rightarrow \omega_0$. In other words, if there are no losses in the system and we continue to drive an initially motionless oscillator with a sinusoidal force that is in phase with the velocity, the amplitude of motion will build up without limit (Fig. 12.15). This does not occur in practice because some damping is always present. That is, at resonance the amplitude is large but finite for small damping.

One experiment that demonstrates a resonance phenomenon is illustrated in Figure 12.16. Several pendula of different lengths are suspended from a stretched string. If one of them, such as P, is set swinging in the plane perpendicular to the plane of the string, the others will begin to oscillate, because they are coupled by the stretched string. Of those that are forced into oscillation by this coupling, pendulum Q, which is the same length as P (hence, the two pendula have the same natural frequency), will oscillate with the greatest amplitude.

Resonance appears in other areas of physics. For example, certain electrical circuits have natural (resonant) frequencies. A structure such as a bridge has natural frequencies and can be set into resonance by an appropriate driving force. A striking example of such a structural resonance occurred in 1940, when the Tacoma Narrows Bridge in Washington was destroyed by resonant vibrations (Fig. 12.17). The winds were not particularly strong on that occasion, but the bridge still collapsed because vortices (turbulences) generated by the wind blowing through the bridge occurred at a frequency that matched the natural frequency of the bridge.

Many other examples of resonant vibrations can be cited. A resonant vibration you may have experienced is the "singing" of telephone wires in the wind. Machines often break if one vibrating part is at resonance with some other moving part.

Thinking Physics 4

In an earthquake region, should damping in a building be large or small?

Reasoning The damping in a building should be large, so that the resonance curve is broad and flat. The building should respond to a wide range of earthquake frequencies, with a relatively small response. If the damping were small, so that the resonance curve were tall and skinny, the building would reject many earthquake frequencies but would have a disastrously large response to a narrow range of frequencies.



(a) (b)

Figure 12.17 (a) High winds set up vibrations in the bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge's collapse. *(UPI/Bettmann Newsphotos)*

CONCEPTUAL PROBLEM 8

You stand on the end of a diving board and bounce to set it into oscillation. You find a maximum response, in terms of the amplitude of oscillation of the end of the board, when you bounce at frequency f. You now move to the middle of the board and repeat the experiment. Is the resonance frequency for forced oscillations at this point higher than f, lower than f, or the same as f? Why?

CONCEPTUAL PROBLEM 9

Some parachutes have holes in them to allow air to move smoothly through the chute. Without the holes, the air gathered under the chute as the parachutist falls is sometimes released from under the edges of the chute alternately and periodically from one side and then the other. Why might this periodic release of air cause a problem?

SUMMARY

The position of a simple harmonic oscillator varies periodically in time according to the relation

$$x = A\cos(\omega t + \phi) \qquad [12.1]$$

where A is the amplitude of the motion, ω is the angular frequency, and ϕ is the phase constant. The values of A and ϕ depend on the initial position and velocity of the oscillator.

The time for one complete oscillation is called the **period** of the motion, defined by

$$T = \frac{2\pi}{\omega} \qquad [12.2]$$

The inverse of the period is the **frequency** of the motion, which equals the number of oscillations per second.

The **velocity** and **acceleration** of a simple harmonic oscillator are

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad [12.5]$$

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$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \qquad [12.6]$$

Thus, the maximum velocity is ωA, and the maximum acceleration is $\omega^2 A$. The velocity is zero when the oscillator is at its turning points, $x = \pm A$, and the speed is a maximum at the equilibrium position, $x = 0$. The magnitude of the acceleration is a maximum at the turning points and is zero at the equilibrium position.

A mass–spring system moving on a frictionless track exhibits simple harmonic motion with the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad [12.18]$$

where k is the force constant of the spring and m is the mass attached to the spring.

The kinetic energy and potential energy of a simple harmonic oscillator vary with time and are

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \qquad [12.20]$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \cos^2(\omega t + \phi) \qquad [12.21]$$

The **total energy** of a simple harmonic oscillator is a constant of the motion and is

$$E = \tfrac{1}{2}kA^2 \qquad [12.22]$$

The potential energy of a simple harmonic oscillator is a maximum when the particle is at its turning points (maximum displacement from equilibrium) and is zero at the equilibrium position. The kinetic energy is zero at the turning points and is a maximum at the equilibrium position.

A **simple pendulum** of length L exhibits simple harmonic motion for small angular displacements from the vertical, with a period of

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad [12.26]$$

That is, the period is independent of the suspended mass.

A **physical pendulum** exhibits simple harmonic motion about a pivot that does not go through the center of mass. The period of this motion is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \qquad [12.28]$$

where I is the moment of inertia about an axis through the pivot and d is the distance from the pivot to the center of mass.

Damped oscillations occur in a system in which a drag force opposes the linear restoring force. If such a system is set in motion and then left to itself, its mechanical energy decreases in time because of the presence of the nonconservative drag force. It is possible to compensate for this loss in energy by driving the system with an external periodic force that is in phase with the motion of the system. When the frequency of the driving force matches the natural frequency of the *undamped* oscillator that starts its motion from rest, energy is continuously transferred to the oscillator and its amplitude increases without limit.

CONCEPTUAL QUESTIONS

1. Does the acceleration of a simple harmonic oscillator remain constant during its motion? Is the acceleration ever zero? Explain.
2. Explain why the kinetic and potential energies of a mass–spring system can never be negative.
3. What is the total distance traveled by a body executing simple harmonic motion in a time equal to its period if its amplitude is A?
4. If a mass–spring system is hung vertically and set into oscillation, why does the motion eventually stop?
5. If a pendulum clock keeps perfect time at the base of a mountain, will it also keep perfect time when moved to the top of the mountain? Explain.
6. If a grandfather clock were running slow, how could we adjust the length of the pendulum to correct the time?
7. A pendulum bob is made from a ball filled with water. What happens to the frequency of vibration of this pendulum if the ball has a hole that allows water to slowly leak out?
8. A platoon of soldiers marches in step along a road. Why are they ordered to break step when crossing a bridge?
9. If the length of a simple pendulum is quadrupled, what happens to (a) its frequency and (b) its period?
10. The amplitude of a system moving in simple harmonic motion is doubled. Determine the change in (a) the total energy, (b) the maximum speed, (c) the maximum acceleration, and (d) the period.
11. A simple harmonic oscillator has a total energy of E. (a) Determine the kinetic and potential energies when the displacement equals one-half the amplitude. (b) For what value of the displacement does the kinetic energy equal the potential energy?
12. If the coordinate of a particle varies as $x = -A\cos\omega t$, what is the phase constant in Equation 12.1? At what position does the particle begin its motion?
13. Does the displacement of an oscillating particle between $t = 0$ and a later time t necessarily equal the position of the particle at time t? Explain.
14. Determine whether or not the following quantities can be in the same direction for a simple harmonic oscillator: (a) displacement and velocity, (b) velocity and acceleration, (c) displacement and acceleration.
15. Can the amplitude A and phase constant ϕ be determined for an oscillator if only the position is specified at $t = 0$? Explain.
16. A simple pendulum undergoes simple harmonic motion when θ is small. Is the motion periodic when θ is large? How does the period of motion change as θ increases?
17. Will damped oscillations occur for any values of b and k? Explain.
18. Is it possible to have damped oscillations when a system is at resonance? Explain.

PROBLEMS

Section 12.1 Simple Harmonic Motion

1. The displacement of a particle is given by the expression $x = (4.00\text{ m})\cos(3.00\pi t + \pi)$, where x is in meters and t is in seconds. Determine (a) the frequency and period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the displacement of the particle at $t = 0.250$ s.
2. A particle oscillates with simple harmonic motion so that its displacement varies according to the expression $x = (5.00\text{ cm})\cos(2t + \pi/6)$, where x is in centimeters and t is in seconds. At $t = 0$, find (a) the displacement of the particle, (b) its velocity, and (c) its acceleration. (d) Find the period and amplitude of the motion.
3. A piston in an automobile engine is in simple harmonic motion. If its amplitude of oscillation from the centerline is ± 5.00 cm and its mass is 2.00 kg, find the maximum velocity and acceleration of the piston when the auto engine is running at the rate of 3600 rev/min.
4. A 20.0-g particle moves in simple harmonic motion with a frequency of 3.00 oscillations/s and an amplitude of 5.00 cm. (a) Through what total distance does the particle move during one cycle of its motion? (b) What is its maximum speed? Where does this occur? (c) Find the maximum acceleration of the particle. Where in the motion does the maximum acceleration occur?
5. A particle moving along the x axis in simple harmonic motion starts from the origin at $t = 0$ and moves to the right. If the amplitude of its motion is 2.00 cm and the frequency is 1.50 Hz, (a) show that its displacement is given by the expression $x = (2.00\text{ cm})\sin(3.00\pi t)$. Determine (b) the maximum speed and the earliest time ($t > 0$) at which the particle has this speed, (c) the maximum acceleration and the earliest time ($t > 0$) at which the particle has this acceleration to the right, and (d) the total distance traveled between $t = 0$ and $t = 1.00$ s.
6. If the initial position, velocity, and acceleration of an object moving in simple harmonic motion are x_0, v_0, and a_0, and

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if the angular frequency of oscillation is ω, (a) show that the position and velocity of the object for all time can be written as

$$x(t) = x_0 \cos \omega t + \left(\frac{v_0}{\omega}\right) \sin \omega t$$

$$v(t) = -x_0 \omega \sin \omega t + v_0 \cos \omega t$$

(b) If the amplitude of the motion is A, show that

$$v^2 - ax = {v_0}^2 - a_0 x_0 = A^2 \omega^2$$

Section 12.2 Motion of a Mass Attached to a Spring

(*Note:* Neglect the mass of the spring in all of these problems.)

7. An archer pulls her bow string back 0.400 m by exerting a force that increases uniformly from zero to 230 N. (a) What is the equivalent spring constant of the bow? (b) How much work is done in pulling the bow?
8. A 1.00-kg mass attached to a spring of force constant 25.0 N/m oscillates on a horizontal, frictionless track. At $t = 0$, the mass is released from rest at $x = -3.00$ cm. (That is, the spring is compressed by 3.00 cm.) Find (a) the period of its motion, (b) the maximum values of its speed and acceleration, and (c) the displacement, velocity, and acceleration as functions of time.
9. A 0.500-kg mass attached to a spring of force constant 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate (a) the maximum value of its speed and acceleration, (b) the speed and acceleration when the mass is 6.00 cm from the equilibrium position, and (c) the time it takes the mass to move from $x = 0$ to $x = 8.00$ cm.
10. A block of unknown mass is attached to a spring of force constant 6.50 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the mass is halfway between its equilibrium position and the endpoint, its speed is measured to be +30.0 cm/s. Calculate (a) the mass of the block, (b) the period of the motion, and (c) the maximum acceleration of the block.
11. A 1.00-kg mass is attached to a horizontal spring. The spring is initially stretched by 0.100 m and the mass is released from rest there. After 0.500 s, the speed of the mass is zero. What is the maximum speed of the mass?
12. A particle that hangs from a spring oscillates with an angular frequency of 2.00 rad/s. The spring is suspended from the ceiling of an elevator car and hangs motionless (relative to the elevator car) as the car descends at a constant speed of 1.50 m/s. The car then stops suddenly. (a) With what amplitude does the particle oscillate? (b) What is the equation of motion for the particle? (Choose the upward direction to be positive.)

Section 12.3 Energy of the Simple Harmonic Oscillator

(Neglect spring masses.)

13. An automobile having a mass of 1000 kg is driven into a brick wall in a safety test. The bumper behaves like a spring of constant 5.00×10^6 N/m and compresses 3.16 cm as the car is brought to rest. What was the speed of the car before impact, assuming no energy is lost during impact with the wall?
14. An ore car of mass 4000 kg starts from rest and rolls downhill from the mine on tracks. At the end of the tracks, 10.0 m lower in elevation, is a spring with force constant $k = 400\,000$ N/m. How much is the spring compressed in stopping the ore car? Ignore friction.
15. A 200-g mass is attached to a spring and executes simple harmonic motion with a period of 0.250 s. If the total energy of the system is 2.00 J, find (a) the force constant of the spring and (b) the amplitude of the motion.
16. A mass–spring system oscillates with an amplitude of 3.50 cm. If the force constant of the spring is 250 N/m and the mass is 0.500 kg, determine (a) the mechanical energy of the system, (b) the maximum speed of the mass, and (c) the maximum acceleration.
17. A 50.0-g mass connected to a spring of force constant 35.0 N/m oscillates on a horizontal, frictionless surface with an amplitude of 4.00 cm. Find (a) the total energy of the system and (b) the speed of the mass when the displacement is 1.00 cm. When the displacement is 3.00 cm, find (c) the kinetic energy and (d) the potential energy.
18. A 2.00-kg mass is attached to a spring and placed on a horizontal, smooth surface. A horizontal force of 20.0 N is required to hold the mass at rest when it is pulled 0.200 m from its equilibrium position (the origin of the x axis). The mass is now released from rest with an initial displacement of $x_0 = 0.200$ m, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the mass. Where does this maximum speed occur? (d) Find the maximum acceleration of the mass. Where does it occur? (e) Find the total energy of the oscillating system. When the displacement equals one third the maximum value, find (f) the speed and (g) the acceleration.

Section 12.4 Motion of a Pendulum

19. A simple pendulum has a period of 2.50 s. (a) What is its length? (b) What would its period be on the Moon, where $g_M = 1.67$ m/s^2?
20. A "seconds" pendulum is one that moves through its equilibrium position once each second. (The period of the pendulum is 2.000 s.) The length of a seconds pendulum is 0.9927 m at Tokyo and 0.9942 m at Cambridge, England. What is the ratio of the free-fall accelerations at these two locations?
21. A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced through an angle of 15.0° and then released. What are (a) the maximum speed? (b) the maximum angular acceleration? (c) the maximum restoring force?
22. The angular displacement of a pendulum is represented by the equation $\theta = 0.320 \cos \omega t$, where θ is in radians and $\omega = 4.43$ rad/s. Determine the period and length of the pendulum.
23. A particle of mass m slides inside a frictionless hemispherical bowl of radius R. Show that if it starts from rest with a small displacement from equilibrium, the particle moves in simple harmonic motion with an angular frequency equal to that of a simple pendulum of length R. That is, $\omega = \sqrt{g/R}$.
24. A simple pendulum is 5.00 m long. (a) What is the period of simple harmonic motion for this pendulum if it is located in an elevator accelerating upward at 5.00 m/s^2? (b) What is its period if the elevator is accelerating downward at 5.00 m/s^2? (c) What is the period of simple harmonic motion for this pendulum if it is placed in a truck that is accelerating horizontally at 5.00 m/s^2?
25. A very light rod extends rigidly 0.500 m out from one end of a meter stick. The stick is suspended from a pivot at the far end of the rod and set into oscillation. (a) Determine the period of oscillation. (b) By what percentage does this differ from a 1.00-m-long simple pendulum?
26. Consider the physical pendulum of Figure 12.11. (a) If its moment of inertia about an axis passing through its center of mass and parallel to the axis passing through its pivot point is I_{CM}, show that its period is

$$T = 2\pi \sqrt{\frac{I_{CM} + md^2}{mgd}}$$

where d is the distance between the pivot point and center of mass. (b) Show that the period has a minimum value when d satisfies $md^2 = I_{CM}$.

Section 12.5 Damped Oscillations (Optional)

27. A pendulum of length 1.00 m is released from an initial angle of 15.0°. After 1000 s, its amplitude is reduced by friction to 5.50°. What is the value of $b/2m$?
28. Show that Equation 12.30 is a solution of Equation 12.29 provided that $b^2 < 4mk$.
29. Show that the time rate of change of mechanical energy for a damped, undriven oscillator is given by $dE/dt = -bv^2$ and hence is always negative. (*Hint:* Differentiate the expression for the mechanical energy of an oscillator, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, and use Eq. 12.29.)

Section 12.6 Forced Oscillations (Optional)

30. Calculate the resonant frequencies of (a) a 3.00-kg mass attached to a spring of force constant 240 N/m and (b) a simple pendulum 1.50 m in length.
31. A 2.00-kg mass attached to a spring is driven by an external force $F = (3.00\text{ N}) \cos(2\pi t)$. If the force constant of the spring is 20.0 N/m, determine (a) the period and (b) the amplitude of the motion. (*Hint:* Assume there is no damping—that is, $b = 0$, and use Eq. 12.34.)
32. Consider an *undamped*, forced oscillator ($b = 0$), and show that Equation 12.33 is a solution of Equation 12.32, with an amplitude given by Equation 12.34.

Additional Problems

33. A mass m is oscillating freely on a vertical spring (Fig. P12.33). When $m = 0.810$ kg, the period is 0.910 s. An unknown mass on the same spring has a period of 1.16 s. Determine (a) the spring constant k and (b) the unknown mass.

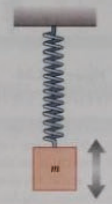


Figure P12.33

34. A mass m is oscillating freely on a vertical spring with a period T (Fig. P12.33). An unknown mass m' on the same spring oscillates with a period T'. Determine (a) the force constant k and (b) the unknown mass m'.
35. A pendulum of length L and mass M has a spring of force constant k connected to it at a distance h below its point of suspension (Fig. P12.35). Find the frequency of vibration of the system for small values of the amplitude (small θ). (Assume the vertical suspension of length L is rigid but neglect its mass.)
36. A horizontal plank of mass m and length L is pivoted at one end, and the opposite end is attached to a spring of force constant k (Fig. P12.36). The moment of inertia of the plank about the pivot is $\frac{1}{3}mL^2$. When the plank is displaced a small angle θ from its horizontal equilibrium position and released, show that it moves with simple harmonic motion with an angular frequency $\omega = \sqrt{3k/m}$.

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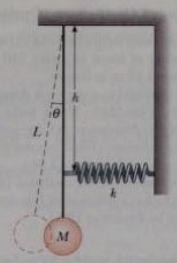


Figure P12.35

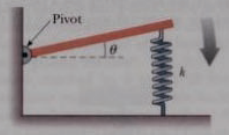


Figure P12.36

37. A large block P executes horizontal simple harmonic motion by sliding across a frictionless surface with a frequency $f = 1.50$ Hz. Block B rests on it, as shown in Figure P12.37, and the coefficient of static friction between the two is $\mu_s = 0.600$. What maximum amplitude of oscillation can the system have if the block is not to slip?

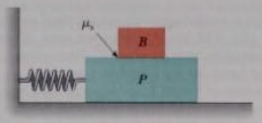


Figure P12.37

38. A 2.00-kg block hangs without vibrating at the end of a spring ($k = 500$ N/m) that is attached to the ceiling of an elevator car. The car is rising with an upward acceleration of $g/3$ when the acceleration suddenly ceases (at $t = 0$). (a) What is the angular frequency of oscillation of the block after the acceleration ceases? (b) By what amount is the spring stretched during the time that the elevator car is accelerating? (c) What are the amplitude of the oscillation and the initial phase angle observed by a rider in the car? Take the upward direction to be positive.

39. A simple pendulum having a length of 2.23 m and a mass of 6.74 kg is given an initial speed of 2.06 m/s at its equilibrium position. Assume it undergoes simple harmonic motion and determine its (a) period, (b) total energy, and (c) maximum angular displacement.

40. People who ride motorcycles and bicycles learn to look out for bumps in the road—and especially for washboarding, a condition of many equally spaced ridges or bumps a certain distance apart. What is so bad about washboarding? A motorcycle has several springs and shock absorbers in its suspension, but you can model it as a single spring supporting a mass. You can estimate the spring constant by thinking about how far the spring compresses when a big biker sits down on the seat. Compute, on the basis of reasonable estimates, the distance between the washboard bumps of which a motorcyclist must be particularly careful when he or she is traveling at highway speed.

41. A mass M is connected to a spring of mass m and oscillates in simple harmonic motion on a horizontal, frictionless track (Fig. P12.41). The force constant of the spring is k and the equilibrium length is ℓ. Find (a) the kinetic energy of the system when the mass has a speed of v and (b) the period of oscillation. [*Hint:* Assume that all portions of the spring oscillate in phase and that the velocity of a segment dx is proportional to the distance from the fixed end; that is, $v_x = (x/\ell)v$. Also, note that the mass of a segment of the spring is $dm = (m/\ell)\,dx$.]

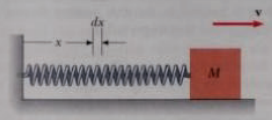


Figure P12.41

42. When a mass M, connected to the end of a spring of mass $m_s = 7.40$ g and force constant k is set into simple harmonic motion, the period of its motion is

$$T = 2\pi\sqrt{\frac{M + (m_s/3)}{k}}$$

A two-part experiment is conducted using various masses suspended vertically from the spring, as in Figure P12.33. (a) Displacements of 17.0, 29.3, 35.3, 41.3, 47.1, and 49.3 cm are measured for M values of 20.0, 40.0, 50.0, 60.0, 70.0, and 80.0 g, respectively. Construct a graph of Mg versus x, and perform a linear least-squares fit to the data. From the slope of your graph, determine a value for k for this spring. (b) The system is now set into simple harmonic motion, and periods are measured with a stopwatch. With $M = 80.0$ g, the total time for 10 oscillations is measured to be 13.41 s. The experiment is repeated with M values of 70.0, 60.0, 50.0, 40.0, and 20.0 g, with corresponding times for 10 oscillations of 12.52, 11.67, 10.67, 9.62, and 7.03 s. Obtain experimental values for T for each of these M values. Plot a graph of T^2 versus M and determine a value for k from the slope of the linear least-squares fit through the data points. Compare this value of k with that obtained in part (a). (c) Obtain a value for m_s from your graph and compare it with the given value of 7.40 g.

43. A mass m is connected to two rubber bands of length L, each under tension F, as in Figure P12.43. The mass is displaced vertically by a small distance, y. Assuming the tension does not change, show that (a) the restoring force is $-(2F/L)y$ and (b) the system exhibits simple harmonic motion with an angular frequency $\omega = \sqrt{2F/mL}$.



Figure P12.43

44. Consider a damped oscillator as illustrated in Figure 12.13. Assume the mass is 375 g, the spring constant is 100 N/m, and $b = 0.100$ kg/s. (a) How long does it take for the amplitude to drop to half its initial value? (b) How long does it take for the mechanical energy to drop to half its initial value? (c) Show that, in general, the fractional rate at which the amplitude decreases in a damped harmonic oscillator is one half the fractional rate at which the mechanical energy decreases.

45. A small, thin disk of radius r and mass m is attached rigidly to the face of a second thin disk of radius R and mass M as shown in Figure P12.45. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Show that the speed of the center of the small disk as it passes through the equilibrium position is

$$v = 2\left[\frac{Rg(1 - \cos\theta)}{(M/m) + (r/R)^2 + 2}\right]^{1/2}$$

(b) Show that the period of the motion is

$$T = 2\pi\left[\frac{(M + 2m)R^2 + mr^2}{2mgR}\right]^{1/2}$$

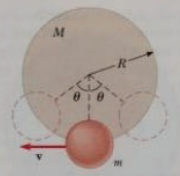


Figure P12.45

46. A mass m is connected to two springs of force constants k_1 and k_2, as in Figures P12.46a and P12.46b. In each case, the mass moves on a frictionless table and is displaced from equilibrium and released. Show that in each case the mass exhibits simple harmonic motion with periods

(a) $T = 2\pi\sqrt{\dfrac{m(k_1 + k_2)}{k_1 k_2}}$

(b) $T = 2\pi\sqrt{\dfrac{m}{k_1 + k_2}}$

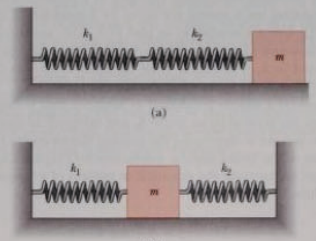


Figure P12.46

47. Imagine that a hole is drilled through the center of the Earth to its other side. It can be shown that an object of mass m at a distance r from the center of the Earth is pulled toward the center of the Earth only by the mass within the sphere of radius r (the reddish region in Fig. P12.47). Write Newton's law of gravitation for an object at the distance r from the center of the Earth, and show that the force on it is of Hooke's law form, $F = -kr$, where the effective force constant is $k = (4/3)\pi\rho Gm$, where ρ is the density of the Earth (assumed uniform) and G is the gravitational constant. Show that a sack of mail dropped into the hole will execute simple harmonic motion. When will it arrive at the other side of the Earth?

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Figure P12.47

Spreadsheet Problems

S1. Use a spreadsheet to plot the position x of an object undergoing simple harmonic motion as a function of time. (a) Use $A = 2.00$ m, $\omega = 5.00$ rad/s, and $\phi = 0$. On the same graph, plot the function for $\phi = \pi/4$, $\pi/2$, and π. Explain how the different phase constants affect the position of the object. (b) Repeat part (a) with $\omega = 5.00$ rad/s. How does the change in ω affect the plots?

S2. The potential and kinetic energies of a particle of mass m oscillating on a spring are given by Equations 12.20 and 12.21. Use a spreadsheet to plot $U(t)$, $K(t)$, and their sum as functions of time on the same graph. The input variables should be k, A, m, and ϕ. What happens to the sum of $U(t)$ and $K(t)$ as the input variables are changed?

S3. Consider a simple pendulum of length L that is displaced from the vertical by an angle θ_0 and released. In this problem, you are to determine the angular displacement for any value of the initial angular displacement. Prepare a spreadsheet to integrate the equation of motion of a simple pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Take the initial condition to be $\theta = \theta_0$ and $d\theta/dt = 0$ at $t = 0$. Choose values for θ_0 and L, and find the angular displacement θ as a function of time. Using the same values of θ_0, compare your results for θ with those obtained from $\theta(t) = \theta_0 \cos \omega_0 t$ where $\omega_0 = \sqrt{g/L}$. Repeat for different values of θ_0. Be sure to include large initial angular displacements. Does the period change when the initial angular displacement is changed? How do the periods for large values of θ_0 compare to those for small values of θ_0? (*Note:* Using the modified Euler's method to solve this differential equation you may find that the amplitude tends to increase with time. The fourth-order Runge-Kutta method discussed in the *Spreadsheet Investigations* Supplement would be a better choice to solve the differential equation. However, if you choose dt small enough, the solution using the modified Euler's method will still be good.)

S4. In the real world friction is always present. Modify your spreadsheet for Problem S3 to include the force of air resistance: $F_{air} = -bv$. In this situation, the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta - \frac{b}{m}\frac{d\theta}{dt}$$

where m is the mass of the pendulum and b is the drag coefficient. Integrate this equation and calculate the sum of the kinetic and potential energies as functions of time using your calculated values of θ and $d\theta/dt$. Because energy is not constant, when in the pendulum's cycle is energy dissipated the fastest?

S5. The differential equation for the position x of a driven, damped harmonic oscillator of mass m is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + m\omega_0^2 x = F_0 \cos \omega t$$

where ω_0 and b are constants, F_0 is the amplitude of the driving force, and ω is the driving angular frequency. For the case of a mass on a spring, $\omega_0 = \sqrt{k/m}$, where k is the force constant. Write a computer program or spreadsheet to integrate this equation of motion. Your input variables are the damping coefficient, b, the mass m, ω (the natural frequency of the system), and F_0. Choose the values for the initial velocity and position and calculate the position x as a function of time. Vary the driving angular frequency ω. Pick some values near ω_0. Does resonance occur? (*Note:* Your program may crash or blow up near resonance. The amplitudes of the motion may have large values when $\omega = \omega_0$. Also, the Euler method usually tends to overestimate the solution. Other methods such as the fourth-order Runge-Kutta method discussed in the *Spreadsheet Investigations* Supplement give a more accurate solution.)

ANSWERS TO CONCEPTUAL PROBLEMS

1. The bouncing ball is not an example of simple harmonic motion. The ball does not follow a sinusoidal function for its position as a function of time. The daily movement of a student is also not simple harmonic motion, since the student stays at a fixed position—school—for a long period of time. If this motion were sinusoidal, the student would move more and more slowly as she approached her desk, and, as soon as she sat down at the desk, she would start to move back toward home again!
2. When the spring with two masses is set into oscillation in space, the coil in the exact center of the spring does not move. Thus, we can imagine clamping the center coil in place without affecting the motion. If we do this, we have two separate oscillating systems, one on each side of the clamp. The half-spring on each side of the clamp has twice the spring constant of the full spring, as shown by the following argument. The force exerted by a spring is proportional to the separation of the coils as the spring is extended. Imagine that we extend a spring by a given distance and measure the distance between coils. We then cut the spring in half. If one of the half-springs is now extended by the same distance, the coils will be twice as far apart as they were for the complete spring. Thus, it takes twice as much force to stretch the half-spring, from which we conclude that the half-spring has a spring constant which is twice that of the complete spring. Thus, our clamped system of masses on two half-springs will vibrate with a frequency that is higher than f by a factor of $\sqrt{2}$.
3. No. Because $E = \frac{1}{2}kA^2$, changing the mass while keeping A constant has no effect on the total energy. The potential energy $U = \frac{1}{2}kx^2$ depends only on position, not mass. Because $K = E - U$, the kinetic energy will also depend on position and not mass.
4. If it accelerates upward, the effective "g" is greater than the free-fall acceleration, so the period decreases. If it accelerates downward, the effective "g" is less than the free-fall acceleration, so the period increases. If it moves with constant velocity, the period does not change. (If the pendulum is in free-fall, it does not oscillate.)
5. Since the car is accelerating downhill, the principle of equivalence tells us that there is an effective gravitational field component uphill. Combining this with the actual gravitational field, the net effective gravitational field is no longer vertical. Thus, the equilibrium position of the pendulum will not be vertical. In addition, the net effective gravitational field has changed in value from the actual field, so the period of the pendulum will change.
6. As the temperature increases, the length of the pendulum will increase, due to thermal expansion. With a longer length, the period of the pendulum will increase. Thus, it will take longer to execute each swing, so that each second according to the clock will take longer than an actual second. Thus, the clock will run *slow*.
7. The first student is correct. While changing the mass of a simple pendulum does not change the frequency, the Foucault pendulum is a *physical* pendulum—the bob has a size. Assuming that the cord is very light, when the gum is placed on top of the bob, the center of mass of the physical pendulum is shifted very slightly upward. This shortens the effective length of the pendulum, which alters the frequency.
8. The diving board is acting as a physical pendulum. With your mass at the end of the board, you and the board have the maximum possible moment of inertia. When you move to the center of the board, the moment of inertia of the system decreases, and the natural frequency of oscillation increases above f.
9. If the period of the alternating release of air matched the swinging period of the parachutist, there could be a resonance response. The swinging of the parachutist could become so energetic that the parachute could become fouled and its braking action could disappear.

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13

Wave Motion

CHAPTER OUTLINE

13.1 Three Wave Characteristics
13.2 Types of Waves
13.3 One-Dimensional Transverse Traveling Waves
13.4 Sinusoidal Traveling Waves
13.5 Superposition and Interference of Waves
13.6 The Speed of Transverse Waves on Strings
13.7 Reflection and Transmission of Waves
13.8 Energy Transmitted by Sinusoidal Waves on Strings
13.9 Sound Waves
13.10 The Doppler Effect

Most of us experienced waves as children, when we dropped pebbles into a pond. The disturbance created by a pebble manifests itself as ripple waves that move outward until they finally reach the shore. If you were to carefully examine the motion of a leaf floating near the point where the pebble entered the water, you would see that the leaf moves up and down and sideways about its original position, but does not undergo any net displacement away from or toward the source of the disturbance. The water wave (the disturbance) moves from one place to another, *yet the water is not carried with it.*

The world is full of other kinds of waves, including sound waves, waves on strings, earthquake waves, radio waves, and x-rays. Most waves can be placed in one of two categories. **Mechanical waves** are waves that disturb and propagate through a medium; sound waves, for which air is the medium, and earthquake waves, for which the Earth's crust is the medium, are two examples of mechanical waves. **Electromagnetic waves** are a special class of waves that do not require a medium in order to propagate; light waves and radio waves are two familiar examples. In this chapter we shall confine our attention to the study of mechanical waves, deferring our study of electromagnetic waves to Chapter 24.

The wave concept is abstract. When we observe a water wave, what we see is a rearrangement of the water's surface. Without the water, there would be no wave. In the case of this or any other mechanical wave, what we interpret as a wave corresponds to the disturbance of a medium. Therefore, we can consider a mechanical wave to be **the motion of a disturbance.** This motion is not to be confused with the motion of the particles that make up the medium. In general, we describe mechanical wave motion by specifying the positions of all particles of the disturbed medium as a function of time.



Large waves such as this travel great distances over the surface of the ocean, yet the water does not flow with the wave. The crests and troughs of the wave often form repetitive patterns. *(Superstock)*

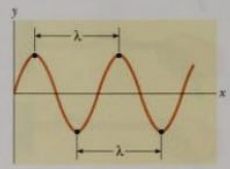
13.1 · THREE WAVE CHARACTERISTICS

All mechanical waves require (1) some source of disturbance, (2) a medium that can be disturbed, and (3) some physical mechanism through which particles of the medium can influence each other. All waves carry energy and momentum, but the amount of energy transmitted through a medium and the mechanism responsible for the energy transport differ from case to case. For instance, the power of ocean waves during a storm is much greater than the power of sound waves generated by a musical instrument.

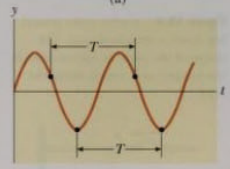
Three physical characteristics are important in describing waves: wavelength, frequency, and wave speed. **One wavelength is the minimum distance between any two points on a wave that behave identically**—for example, adjacent crests or adjacent troughs. Figure 13.1a shows displacement versus position for a sinusoidal wave at a specific time. The symbol λ (Greek lambda) is used to denote wavelength.

Most waves are periodic. **The frequency of such waves is the rate at which the disturbance repeats itself.** The period of the wave is the minimum time it takes the disturbance to repeat itself and is equal to the inverse of the frequency. Figure 13.1b shows displacement versus time for a sinusoidal wave at a fixed position, where the period is the time between identical displacements of a point on the wave.

Waves travel, or *propagate*, with a specific speed, which depends on the properties of the medium being disturbed. For instance, sound waves travel through air at 20°C with a speed of about 343 m/s, whereas the speed of sound in most solids is higher than 343 m/s.



(a)



(b)

Figure 13.1 (a) The wavelength λ of a wave is the distance between adjacent crests or adjacent troughs. (b) The period T of a wave is the time it takes the wave to travel one wavelength.

Thinking Physics 1

Why is it important to be quiet in avalanche country? In the movie *On Her Majesty's Secret Service* (United Artists, 1969), the bad guys try to stop James Bond, who is escaping on skis, by firing a gun and causing an avalanche. Why did this happen?

Reasoning The essence of a wave is the propagation of a *disturbance* through a medium. An impulsive sound, like the gunshot in the James Bond movie, can cause an acoustical disturbance that propagates through the air and can impact a ledge of snow that is just ready to break free to begin an avalanche. Such a disastrous event occurred in 1916 during World War I, when Austrian soldiers in the Alps were smothered by an avalanche caused by cannon fire.

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13.2 · TYPES OF WAVES

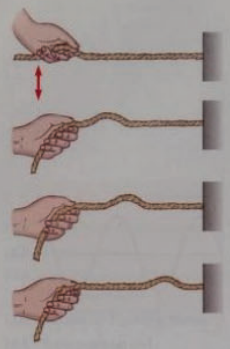


Figure 13.2 A wave pulse traveling down a stretched rope. The shape of the pulse is approximately unchanged as it travels along the rope.

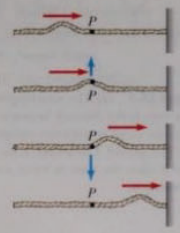


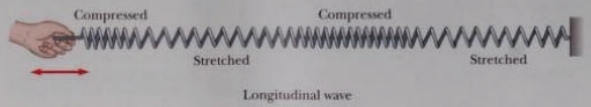
Figure 13.3 A pulse traveling on a stretched rope is a transverse wave. That is, any element *P* on the rope moves (blue arrows) in a direction perpendicular to the wave motion (red arrows).

One way to demonstrate wave motion is to flip the free end of a long rope that is under tension and has its opposite end fixed, as in Figure 13.2. In this manner, a single wave bump (or pulse) is formed and travels (to the right in Fig. 13.2) with a definite speed. This type of disturbance is called a **traveling wave,** and the rope is the medium through which it travels. Figure 13.2 represents four consecutive "snapshots" of the traveling wave. The shape of the wave pulse changes very little as it travels along the rope.[1]

Note that **as the wave pulse travels, each rope segment that is disturbed moves in a direction perpendicular to the wave motion.** Figure 13.3 illustrates this point for a particular segment, labeled *P*. Note that there is no motion of any part of the rope in the direction of the wave. A traveling wave such as this, in which the particles of the disturbed medium move perpendicularly to the wave velocity, is called a **transverse wave.**

In another class of waves, called **longitudinal waves,** the particles of the medium undergo displacements *parallel* to the direction of wave motion. Sound waves in air, for instance, are longitudinal. Their disturbance corresponds to a series of high- and low-pressure regions that may travel through air or through any material medium with a certain speed. A longitudinal pulse can be easily produced in a stretched spring, as in Figure 13.4. The free end is pumped back and forth along the length of the spring. This action produces compressed and stretched regions of the coil that travel along the spring, parallel to the wave motion.[2]

Some waves are neither transverse nor longitudinal but a combination of the two. Surface water waves are a good example. Figure 13.5 shows the motion of water particles at the surface as a wave moves to the right. Each particle moves in a circular path, and hence the disturbance has both transverse and longitudinal components. As the wave passes, water particles at the crests are moving in the direction of the wave, and particles at the troughs move in the opposite direction. Hence, no *net* displacement of water particles takes place. A cork bobbing on a pond surface as a wave passes exhibits this circular motion.



Longitudinal wave

Figure 13.4 A longitudinal pulse along a stretched spring. The displacement of the coils is in the direction of the wave motion. For the starting motion described in the text, the compressed region is followed by a stretched region.

[1]Strictly speaking, the pulse changes its shape and gradually spreads out during the motion. This effect is called *dispersion* and is common to many mechanical waves.

[2]In the case of longitudinal waves in a gas, each compressed area is a region of higher-than-average pressure and density, and each stretched region is a region of lower-than-average pressure and density.

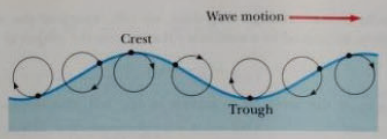


Figure 13.5 Wave motion on the surface of water. The molecules at the water's surface move in nearly circular paths. Each molecule is displaced horizontally and vertically from its equilibrium position, represented by circles.

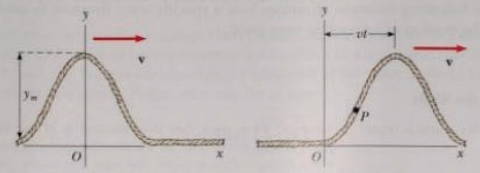
CONCEPTUAL PROBLEM 1

In a long line of people waiting to buy tickets at a movie theater, when the first person leaves, a pulse of motion occurs, as people step forward to fill in the gap. The gap moves through the line of people. What determines the speed of this pulse? Is it transverse or longitudinal? How about "the wave" at a baseball game, where people in the stands stand up and shout as the wave arrives at their location, and this pulse moves around the stadium—what determines the speed of this pulse? Is it transverse or longitudinal?

13.3 · ONE-DIMENSIONAL TRANSVERSE TRAVELING WAVES

So far we have provided only verbal and graphical descriptions of a traveling wave. Let us now develop a mathematical description of a one-dimensional traveling wave. Consider a wave pulse traveling to the right with constant speed, v, along a long, stretched string, as in Figure 13.6. The pulse moves along the x axis (the axis of the string), and the transverse (up-and-down) displacement of the string is measured with the coordinate y.

Figure 13.6a represents the shape and position of the pulse at time $t = 0$. At this time, the shape of the pulse, whatever it may be, can be represented as $y = f(x)$. That is, y is some definite function of x. The *maximum displacement* of the string, y_m, is called the **amplitude** of the wave. Because the speed of the wave pulse is v, the pulse travels to the right a distance of vt in the time t (Fig. 13.6b). If the shape



(a) Pulse at $t = 0$ (b) Pulse at time t

Figure 13.6 A one-dimensional wave pulse traveling to the right with a speed v. (a) At $t = 0$, the shape of the pulse is given by $y = f(x)$. (b) At some later time t, the shape remains unchanged and the vertical displacement of any point P of the medium is given by $y = f(x - vt)$.

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