With concave mirrors, note what happens as the object is moved closer to the mirror. The real, inverted image in Figure 26.11a moves to the left as the object approaches the focal point. When the object is at the focal point, the image is infinitely far to the left. However, when the object lies between the focal point and the mirror surface, as in Figure 26.11b, the image is virtual and upright.

For the convex mirror shown in Figure 26.11c, the image of a real object is always virtual and upright. As the object distance increases, the virtual image gets smaller and approaches the focal point as p approaches infinity. You should construct other diagrams to verify how the image position varies with object position.

Thinking Physics 3

For a concave mirror, a virtual image can be anywhere behind the mirror. For a convex mirror, however, there is a maximum distance at which the image can exist behind the mirror. Why?

Reasoning Let us consider the concave mirror first, and imagine two different light rays leaving a tiny object and striking the mirror. If the object is inside the focal point, but infinitesimally close to it, the light rays reflecting from the mirror will be parallel to the mirror axis. They can be interpreted as forming a virtual image infinitely far away behind the mirror. As the object is brought closer to the mirror, the reflected rays will diverge through larger and larger angles, resulting in their extensions converging closer and closer to the back of the mirror. When the object is brought right up to the mirror, the image is just behind the mirror. Conceptually, we can argue that when the object is much closer to the mirror than the focal length, it looks like a flat mirror, so the image is just as far behind the mirror as the object is in front of it. Thus, the image can be anywhere from infinitely far away to right at the surface of the mirror.

For the convex mirror, an object at infinity produces a virtual image at the focal point. As the object is brought closer, the reflected rays diverge more sharply and the image moves closer to the mirror. Thus, the virtual image is restricted to the region between the mirror and the focal point.

Example 26.2 The Image for a Concave Mirror

Assume that a certain concave spherical mirror has a focal length of 10.0 cm. Find the location of the image for object distances of (a) 25.0 cm, (b) 10.0 cm, and (c) 5.00 cm. Describe the image in each case.

Solution (a) For an object distance of 25.0 cm, we find the image distance using the mirror equation:

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

$$\frac{1}{25.0\text{ cm}}+\frac{1}{q}=\frac{1}{10.0\text{ cm}}$$

$$q = 16.7\text{ cm}$$

The magnification is given by Equation 26.2:

$$M=-\frac{q}{p}=-\frac{16.7\text{cm}}{25.0\text{ cm}}=-0.668$$

The value for M less than unity tells us that the image is smaller than the object. The negative sign for M tells us that the image is inverted. Finally, because q is positive, the image is located on the front side of the mirror and is real. This situation is pictured in Figure 26.11a.

(b) When the object distance is 10.0 cm, the object is located at the focal point. Substituting the values $p = 10.0$ cm and $f = 10.0$ cm into the mirror equation, we find

$$\frac{1}{10.0\text{ cm}}+\frac{1}{q}=\frac{1}{10.0\text{ cm}}$$

$$q = \infty$$

Thus, we see that rays of light originating from an object located at the focal point of a mirror are reflected so that the image is formed at an infinite distance from the mirror—that is, the rays travel parallel to one another after reflection.

(c) When the object is at the position $p = 5.00$ cm, it lies between the focal point and the mirror surface. In this case, the mirror equation gives

$$\frac{1}{5.00\text{ cm}}+\frac{1}{q}=\frac{1}{10.0\text{ cm}}$$

$$q = -10.0\text{ cm}$$

That is, the image is virtual because it is located behind the mirror. The magnification is

$$M=-\frac{q}{p}=-\left(\frac{-10.0\text{ cm}}{5.00\text{ cm}}\right)=2.00$$

From this, we see that the image is magnified by a factor of 2, and the positive sign for M indicates that the image is upright (Fig. 26.11b).

Note the characteristics of the images formed by a concave spherical mirror. When the focal point lies between the object and mirror surface, the image is inverted and real; with the object at the focal point, the image is formed at infinity; with the object between the focal point and mirror surface, the image is upright and virtual.

EXERCISE 2 If the object distance is 20.0 cm, find the image distance and the magnification of the mirror.
Answer $q = 20.0$ cm, $M = -1.00$

Example 26.3 The Image for a Convex Mirror

An object 3.00 cm high is placed 20.0 cm from a convex mirror having a focal length of 8.00 cm. Find (a) the position of the final image and (b) the magnification.

Solution (a) Because the mirror is convex, its focal length is negative. To find the image position, we use the mirror equation:

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}=-\frac{1}{8.00\text{ cm}}$$

$$\frac{1}{q}=-\frac{1}{8.00\text{ cm}}-\frac{1}{20.0\text{ cm}}$$

$$q = -5.71\text{ cm}$$

The negative value of q indicates that the image is virtual, or behind the mirror, as in Figure 26.11c.

(b) The magnification is

$$M=-\frac{q}{p}=-\left(\frac{-5.71\text{ cm}}{20.0\text{ cm}}\right)=0.286$$

The image is about 30% of the size of the object and upright because M is positive.

EXERCISE 3 Find the height of the image.
Answer 0.857 cm

26.3 · IMAGES FORMED BY REFRACTION

In this section we shall describe how images are formed by the refraction of rays at a spherical surface of a transparent material. Consider two transparent media with indices of refraction n_1 and n_2, where the boundary between the two media is a spherical surface with radius of curvature R (Fig. 26.12). We shall assume that the object at point O is in the medium with index of refraction n_1. Furthermore, let us consider only paraxial rays leaving O. As we shall see, all such rays are refracted at the spherical surface and focus at a single point, I, the image point.

Let us proceed by considering the geometric construction in Figure 26.13, which shows a single ray leaving point O and passing through point I. Snell's law applied to this refracted ray gives

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Because the angles θ_1 and θ_2 are assumed to be small, we can use the approximation $\sin\theta \approx \theta$ (angles in radians). Therefore, Snell's law becomes

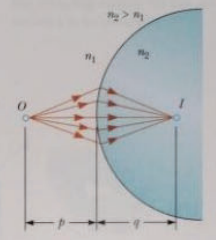


Figure 26.12 An image formed by refraction at a spherical surface. Rays making small angles with the optic axis diverge from a point object at O and pass through the image point, I.

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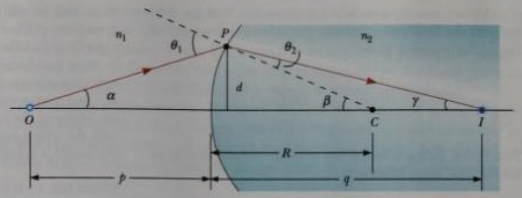


Figure 26.13 Geometry used to derive Equation 26.8.

$$n_1\theta_1 = n_2\theta_2$$

Now we make use of the fact that an exterior angle of any triangle equals the sum of the two opposite interior angles. Applying this to the triangles *OPC* and *PIC* in Figure 26.13 gives

$$\theta_1 = \alpha + \beta$$

$$\beta = \theta_2 + \gamma$$

If we combine the last three equations and eliminate θ_1 and θ_2, we find

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta \qquad [26.7]$$

In the small angle approximation, $\tan\theta \approx \theta$, and so we can write the approximate relations

$$\alpha = \frac{d}{p} \qquad \beta = \frac{d}{R} \qquad \gamma = \frac{d}{q}$$

where d is the distance shown in Figure 26.13. We substitute these into Equation 26.7 and divide through by d to give

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \qquad [26.8]$$

For a fixed object distance of p, the image distance, q, is independent of the angle that the ray makes with the axis. This tells us that all paraxial rays focus at the same point, I. The magnification of a refracting surface is

$$M = \frac{h'}{h} = -\frac{n_1 q}{n_2 p} \qquad [26.9]$$

As we did with mirrors, we must use a sign convention if we are to apply Equation 26.8 to a variety of circumstances. First note that real images are formed on the side of the surface that is *opposite* the side from which the light comes. This is in contrast to mirrors, where real images are formed on the side where the light is. Therefore, **the sign conventions for spherical refracting surfaces are similar to the conventions for mirrors, recognizing the change in sides of the surface for real and virtual images.** For example, in Figure 26.13, p, q, and R are all positive.

TABLE 26.2 Sign Convention for Refracting Surfaces

p is + if the object is in front of the surface (real object).
p is − if the object is in back of the surface (virtual object).
q is + if the image is in back of the surface (real image).
q is − if the image is in front of the surface (virtual image).
R is + if the center of curvature is in back of the surface.
R is − if the center of curvature is in front of the surface.

Note: p = object distance; q = image distance; R = radius of curvature of the surface.

The sign conventions for spherical refracting surfaces are summarized in Table 26.2. The same conventions will be used for thin lenses, discussed in the next section. As with mirrors, we assume that the front of the refracting surface is the side from which the light approaches the surface.

Flat Refracting Surfaces

If the refracting surface is flat, then R approaches infinity and Equation 26.8 reduces to

$$\frac{n_1}{p} = -\frac{n_2}{q}$$

or

$$q = -\frac{n_2}{n_1}p \qquad [26.10]$$

From Equation 26.10 we see that the sign of q is opposite that of p. Thus, **the image formed by a flat refracting surface is on the same side of the surface as the object.** This is illustrated in Figure 26.14 for the situation in which n_1 is greater than n_2, where a virtual image is formed between the object and the surface. Note that the refracted ray bends *away* from the normal in this case, because $n_1 > n_2$.

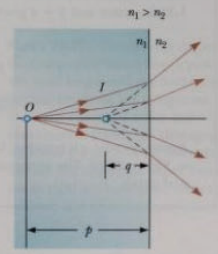


Figure 26.14 The image formed by a flat refracting surface is virtual; that is, it forms to the left of the refracting surface. All rays are assumed to be paraxial.

Example 26.4 Gaze into the Crystal Ball

A coin 2.00 cm in diameter is embedded in a solid glass ball of radius 30.0 cm (Fig. 26.15). The index of refraction of the ball is $n_1 = 1.5$, and the coin is 20.0 cm from the surface. Find the position of the image.

Solution Because $n_1 > n_2$, where $n_2 = 1.00$ is the index of refraction for air, the rays originating from the object are refracted away from the normal at the surface and diverge outward. Hence, the image is formed in the glass and is virtual. Applying Equation 26.8, we get

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

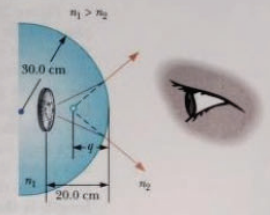


Figure 26.15 (Example 26.4) A coin embedded in a glass ball forms a virtual image between the coin and the glass surface. All rays are assumed to be paraxial.

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$$\frac{1.50}{20.0\text{ cm}} + \frac{1}{q} = \frac{1.00 - 1.50}{-30.0\text{ cm}}$$

$$q = -17.1\text{ cm}$$

The negative sign indicates that the image is in the same medium as the object (the side of incident light), in agreement with our ray diagram. Being in the same medium as the object, the image must be virtual.

Example 26.5 **The One That Got Away**

A small fish is swimming at a depth d below the surface of a pond (Fig. 26.16). What is the apparent depth of the fish as viewed from directly overhead?

Solution In this example, the refracting surface is flat, and so R is infinite. Hence, we can use Equation 26.10 to determine the location of the image. Using the facts that $n_1 = 1.33$ for water and $p = d$ gives

$$q = -\frac{n_2}{n_1}p = -\frac{1}{1.33}d$$

$$= -0.750d$$

Again, because q is negative, the image is virtual, as indicated in Figure 26.16. The apparent depth is three fourths the actual depth.

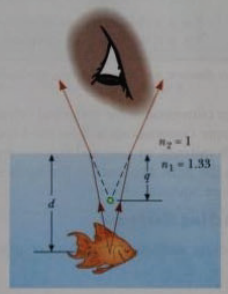


Figure 26.16 (Example 26.5) The apparent depth, q, of the fish is less than the true depth, d. All rays are assumed to be paraxial.

26.4 · THIN LENSES

A typical **thin lens** consists of a piece of glass or plastic, ground so that its two surfaces are either segments of spheres or planes. Lenses are commonly used in optical instruments, such as cameras, telescopes, and microscopes, to form images by refraction. The equation that relates object distances and image distances for a lens is virtually identical to the mirror equation derived earlier, and the method used to derive it is also similar.

Figure 26.17 shows some representative shapes of lenses. Note that we have placed these lenses in two groups. Those in Figure 26.17a are thicker at the center than at the rim, and those in Figure 26.17b are thinner at the center than at the rim. The lenses in the first group are examples of **converging lenses,** and those in the second group are called **diverging lenses.** The reason for these names will become apparent shortly.

As we did with mirrors, it is convenient to define a point called the **focal point** for a lens. For example, in Figure 26.18a a group of rays parallel to the principal axis passes through the focal point, F, after being converged by the lens. The distance from the focal point to the lens is again called the **focal length, f. The focal length is the image distance that corresponds to an infinite object distance.** Recall that we are considering the lens to be very thin. As a result, it makes no difference whether we take the focal length to be the distance from the focal point to the surface of the lens or the distance from the focal point to the center of the lens, because the difference in these two lengths is negligible. A thin lens has one focal length and *two* focal points, as illustrated in Figure 26.18b, corresponding to parallel light rays traveling from the left or right.

Rays parallel to the axis diverge after passing through a lens of the shape shown in Figure 26.18b. In this case, the focal point is defined as the point at which the diverged rays appear to originate, labeled F in the figure. Figures 26.18a and 26.18b indicate why the names *converging* and *diverging* are applied to these lenses.

Consider a ray of light passing through the center of a lens, shown as ray 1 in Figure 26.19. If we apply Snell's law at both surfaces, we find that this ray is deflected from its original direction of travel by a distance of δ, shown in Figure 26.20. Here, in order to avoid the complications arising from this deflection, we shall make what is called the **thin-lens approximation: The thickness of the lens is assumed to be negligible.** As a result, δ becomes vanishingly small and we see that the ray passes through the lens undeflected. Ray 2 in Figure 26.17 is parallel to the principal axis of the lens (the horizontal axis passing through O), and as a result it passes through the focal point, F, after refraction. The point at which these two rays intersect is the image point.

We first note that the tangent of the angle α can be found by using the shaded triangles in Figure 26.19:

$$\tan\alpha = \frac{h}{p} \quad \text{or} \quad \tan\alpha = -\frac{h'}{q}$$

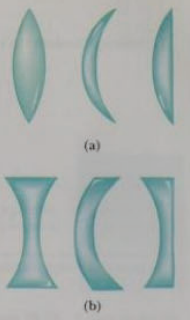


Figure 26.17 Various lens shapes: (a) Converging lenses have a positive focal length and are thickest at the middle. From left to right are biconvex, convex-concave, and plano-convex lenses. (b) Diverging lenses have a negative focal length and are thickest at the edges. From left to right are biconcave, convex-concave, and plano-concave lenses.

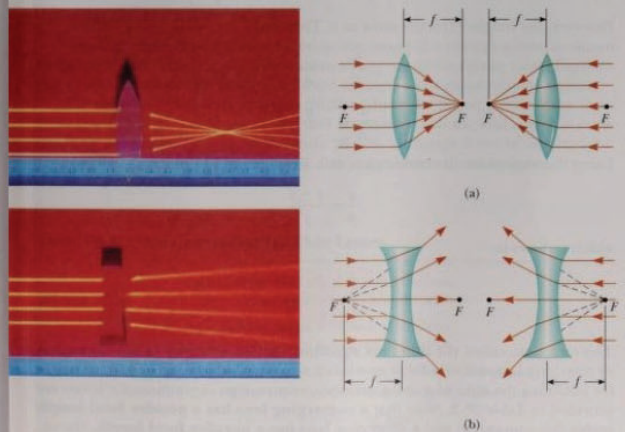


Figure 26.18 *(left)* Photographs of the effect of converging and diverging lenses on parallel rays. *(Henry Leap and Jim Lehman)* *(right)* The object and image focal points of (a) the biconvex lens and (b) the biconcave lens.

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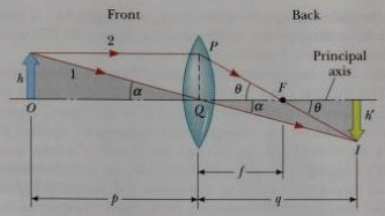


Figure 26.19 A geometric construction for developing the thin-lens equation.

from which

$$M = \frac{h'}{h} = -\frac{q}{p} \qquad [26.11]$$

Thus, the equation for magnification by a lens is the same as the equation for magnification by a mirror (Eq. 26.2). We also note from Figure 26.19 that the tangent of θ is

$$\tan\theta = \frac{PQ}{f} \quad \text{or} \quad \tan\theta = -\frac{h'}{q-f}$$

However, the height PQ is the same as h. Therefore,

$$\frac{h}{f} = -\frac{h'}{q-f}$$

$$\frac{h'}{h} = -\frac{q-f}{f}$$

Using this expression in combination with Equation 26.11 gives

$$\frac{q}{p} = \frac{q-f}{f}$$

which reduces to

Thin-lens equation •

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \qquad [26.12]$$

This equation, called the **thin-lens equation,** can be used with either converging or diverging lenses if we adhere to a set of sign conventions. Figure 26.21 is useful for obtaining the signs of p and q, and the complete sign conventions for lenses are provided in Table 26.3. Note that **a converging lens has a positive focal length** under this convention and **a diverging lens has a negative focal length.** Hence, the names *positive* and *negative* are often given to these lenses.

The focal length for a lens in air is related to the curvatures of its surfaces and to the index of refraction, n, of the lens material by

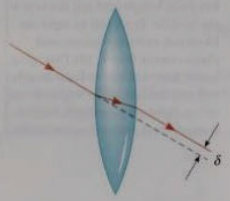


Figure 26.20 A ray passing through the center of the lens is deviated from its path by a distance of δ. In the thin-lens approximation, this deviation is ignored.

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad [26.13]$$

Lens makers' equation •

where R_1 is the radius of curvature of the front surface and R_2 is the radius of curvature of the back surface. (As with mirrors, we arbitrarily call the side from which the light approaches the *front* of the lens.) Equation 26.13 enables us to calculate the focal length from the known properties of the lens. It is called the **lens makers' equation.**

Ray Diagrams for Thin Lenses

Ray diagrams are very convenient for locating the image of a thin lens or system of lenses. They should also help clarify the sign conventions we have already discussed. Figure 26.22 illustrates this method for three single-lens situations. To locate the image of a converging lens (Figs. 26.22a and b), the following three rays are drawn from the top of the object:

- The first ray is drawn parallel to the principal axis. After being refracted by the lens, this ray passes through (or appears to come from) one of the focal points.
- The second ray is drawn through the center of the lens. This ray continues in a straight line.
- The third ray is drawn through the focal point, F, and emerges from the lens parallel to the principal axis.

A similar construction is used to locate the image of a diverging lens, as shown in Figure 26.22c. The point of intersection of *any two* of the rays in these diagrams can be used to locate the image. The third ray serves as a check of construction.

For the converging lens in Figure 26.22a, where the object is *outside* the front focal point ($p > f$), the image is real and inverted. When the real object is *inside* the front focal point ($p < f$), as in Figure 26.22b, the image is virtual and upright. For the diverging lens of Figure 26.22c, the image is virtual and upright.

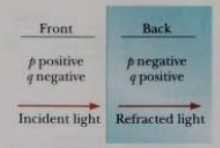


Figure 26.21 A diagram for obtaining the signs of p and q for a thin lens or a refracting surface. Although the interface between the two media is shown flat here, the diagram is also valid for convex and concave surfaces.

TABLE 26.3 Sign Convention for Thin Lenses

p is + if the object is in front of the lens.
p is − if the object is in back of the lens.
q is + if the image is in back of the lens.
q is − if the image is in front of the lens.
R_1 and R_2 are + if the center of curvature is in back of the lens.
R_1 and R_2 are − if the center of curvature is in front of the lens.
f is + for a converging lens.
f is − for a diverging lens.

Note: p = object distance; q = image distance; R_1 = radius of curvature of front surface; R_2 = radius of curvature of back surface; f = focal length.

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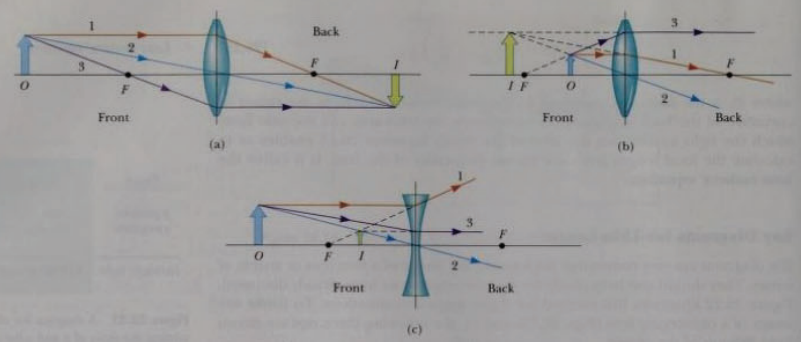


Figure 26.22 Ray diagrams for locating the image formed by a thin lens. (a) The object is located to the left of the focal point of a converging lens. (b) The object is located between the focal point and a converging lens. (c) The object is located to the left of the focal point of a diverging lens.

PROBLEM-SOLVING STRATEGY • Lenses and Mirrors

Success or failure in working lens and mirror problems is largely determined by whether or not you make sign errors when substituting into the equations, and the only way to ensure that you don't make such errors is to become adept at using the sign conventions. The best way to do this is to work a multitude of problems on your own. Watching your instructor or reading the example problems is no substitute for practice.

Thinking Physics 4

Diving masks often have a lens built into the glass for divers who do not have perfect vision. This allows the individual to dive without the necessity of glasses, because the lenses in the faceplate perform the necessary refraction to provide clear vision. Normal glasses have lenses that are curved on both the front and rear surfaces. The lenses in a diving mask faceplate often only have curved surfaces on the *inside* of the glass. Why is this design desirable?

Reasoning The main reason for curving only the inner surface of the lenses in the diving mask faceplate is so that the diver can see clearly while underwater *and* in the air. If there were curved surfaces on both the front and the back of the diving lens, there would be two refractions. The lens could be designed so that these two refractions would give clear vision while the diver is in air. When the diver goes underwater, however, the refraction between the water and the glass at the first interface is now different, because the index of refraction of water is different from that of air. Thus, the vision will not be clear underwater.

By making the outer surface of the lens flat, there is no refraction at the outer surface *in either air or water*—all of the refraction occurs at the inner glass–air surface. Thus, there is the same refractive correction in water and in air, and the diver can see clearly in both environments.

Thinking Physics 5

Consider a glass plano-convex lens—flat on one side and convex on the other. You project three parallel laser beams through it, as shown in Figure 26.23a, and measure the focal length, f—that is, the distance from the lens to the point at which the three beams cross. Now you hold the flat side of the lens against the glass of an aquarium filled with water, as in Figure 26.23b. When you shine the laser beams through the lens from the outside of the aquarium, will the beams cross at a point closer to the lens than before, farther away, or at the same distance? What if you direct the laser beams through the lens from the other side of the aquarium, as in Figure 26.23c? What happens in this case?

Reasoning The laser beam going through the center of the lens is unaffected by the lens in all three cases. We need to look at what happens to the outer beams. In Figure 26.23a, these two laser beams will refract toward the normal as they pass from the air into the glass of the lens. Thus, they will be deviated *toward* the central beam. As they reach the other side, and pass from the glass back into the air, they will deviate away from the normal, which causes additional deviation toward the central beam. As a result, all three beams cross at the focal point. When the flat side of the lens is held against the glass side of the aquarium, as in Figure 26.23b, the outgoing laser beams pass through the glass of the aquarium without additional refraction, but then enter *water*. The change in index of refraction in going from glass to water is much smaller than that for going from glass to air. Thus, the outgoing refraction is smaller than in the previous case, and the deviation toward the central beam is less. As a result, the crossing point for the three laser beams is farther from the lens. When the laser beams are directed through the water and then through the lens, as in Figure 26.23c, we must consider two factors. First, the focal length of a lens is independent of which way the light passes through it. Second, in the situation shown, the laser beams experience no refraction on entering the flat side of the lens, because they strike the glass at normal incidence. Thus, all of the refraction occurs at the interface of the air with the curved side of the lens. This is exactly the same situation as if the laser beams had struck the flat side of the lens while it was in air. Thus, there is no effect of the water in the aquarium. The laser beams cross at the same distance from the lens as they did in the first diagram. This third situation is very similar to the curvature of a lens on the inside of a diving mask, discussed in Thinking Physics 4.

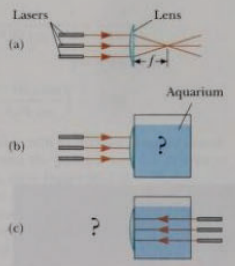


Figure 26.23 (Thinking Physics 5)

CONCEPTUAL PROBLEM 3

You are taking a picture of yourself with a camera that uses an ultrasonic range finder to measure the distance to the subject. When you take a picture of yourself in a mirror with this camera, your image is out of focus. Why?

CONCEPTUAL PROBLEM 4

A virtual image is often described as one through which light rays do not actually travel, as they do for a real image. Can a virtual image be photographed?

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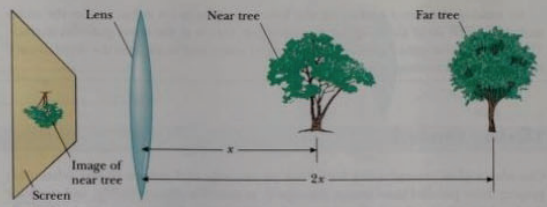
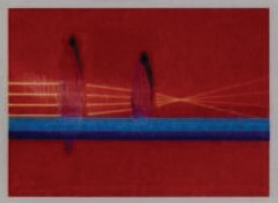


Figure 26.24 (Conceptual Problem 5)

CONCEPTUAL PROBLEM 5

Suppose you want to use a converging lens to project the image of two trees onto a screen. One tree is at a distance x from the lens, the other is at $2x$, as in Figure 26.24. You adjust the screen so that the near tree is in focus. If you now want to move the screen so that the far tree is in focus, do you move the screen toward, or away from, the lens?



Light from a distant object brought into focus by two converging lenses. *(Henry Leap and Jim Lehman)*

CONCEPTUAL PROBLEM 6

Can a converging lens be made to diverge light if placed in a liquid? How about a converging mirror?

CONCEPTUAL PROBLEM 7

A plastic sandwich bag filled with water can act as a crude converging lens in air. If the bag is filled with air and placed under water, is it a lens? If so, is it converging or diverging?

CONCEPTUAL PROBLEM 8

Why does the focal length of a mirror not depend on the mirror material when the focal length of a lens *does* depend on the lens material?

Example 26.6 The Lens Makers' Equation

The biconvex lens of Figure 26.25 has an index of refraction of 1.50. The curvature of the front surface is 10 cm, and that of the back surface is 15 cm. Find the focal length of the lens.

Solution From the sign conventions in Table 26.3 we find that $R_1 = +10$ cm and $R_2 = -15$ cm. Thus, using the lens makers' equation, we have

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5-1)\left(\frac{1}{10\text{ cm}} - \frac{1}{-15\text{ cm}}\right)$$

$$f = 12\text{ cm}$$

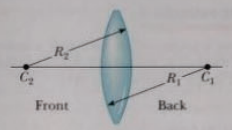


Figure 26.25 (Example 26.6) This lens has two curved surfaces with radii of curvature R_1 and R_2.

Example 26.7 An Image Formed by a Converging Lens

A converging lens of focal length 10.0 cm forms an image of an object placed (a) 30.0 cm, (b) 10.0 cm, and (c) 5.00 cm from the lens. Find the image distance and describe the image in each case.

Solution (a) The thin lens equation, Equation 26.12, can be used to find the image distance:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{30.0\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = 15.0\text{ cm}$$

The positive sign tells us that the image is real. The magnification is

$$M = -\frac{q}{p} = -\frac{15.0\text{ cm}}{30.0\text{ cm}} = -0.500$$

Thus, the image is reduced in size by one half, and the negative sign for M tells us that the image is inverted. The situation is like that pictured in Figure 26.22a.

(b) No calculation is necessary for this case because we know that when the object is placed at the focal point, the image is formed at infinity. This is readily verified by substituting $p = 10.0$ cm into the lens equation.

(c) We now move inside the focal point, to an object distance of 5.00 cm. In this case, the lens equation gives

$$\frac{1}{5.00\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = -10.0\text{ cm}$$

$$M = -\frac{q}{p} = -\left(\frac{-10.0\text{ cm}}{5.00\text{ cm}}\right) = 2.00$$

The negative image distance tells us that the image is virtual. The image is enlarged, and the positive sign for M tells us that the image is upright, as in Figure 26.22b.

There are two general cases for a converging lens. When the object distance is greater than the focal length ($p > f$), the image is real and inverted. When the object is between the focal point and lens ($p < f$), the image is virtual, upright, and enlarged.

Combination of Thin Lenses

If two thin lenses are used to form an image, the system can be treated in the following manner. First, the image of the first lens is calculated as though the second lens were not present. The light then approaches the second lens *as if* it had come from the image formed by the first lens. Hence, the image of the first lens is treated as the object of the second lens. The image of the second lens is the final image of the system. If the image of the first lens lies on the back side of the second lens, then the image is treated as a virtual object for the second lens (that is, p is negative). The same procedure can be extended to a system of three or more lenses. The overall magnification of a system of thin lenses equals the *product* of the magnifications of the separate lenses.

Example 26.8 Where Is the Final Image?

Two thin converging lenses of focal lengths 10.0 cm and 20.0 cm are separated by 20.0 cm, as in Figure 26.26. An object is placed 15.0 cm to the left of the first lens. Find the position of the final image and the magnification of the system.

Solution First, we find the image position for the first lens and neglect the second lens:

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{15.0\text{ cm}} + \frac{1}{q_1} = \frac{1}{10.0\text{ cm}}$$

$$q_1 = 30.0\text{ cm}$$

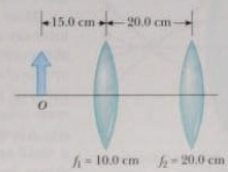


Figure 26.26 (Example 26.8) A combination of two converging lenses.

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where q_1 is measured from the first lens.

Because q_1 is greater than the separation between the two lenses, the image of the first lens lies 10.0 cm to the right of the second lens. We take this as the object distance for the second lens. That is, we apply the thin lens equation to the second lens with $p_2 = -10.0$ cm, where distances are now measured from the second lens, whose focal length is 20.0 cm:

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$$

$$\frac{1}{-10.0\text{ cm}} + \frac{1}{q_2} = \frac{1}{20.0\text{ cm}}$$

$$q_2 = 6.67\text{ cm}$$

That is, the final image lies 6.67 cm to the right of the second lens.

The magnification of each lens separately is given by

$$M_1 = \frac{-q_1}{p_1} = -\frac{30.0\text{ cm}}{15.0\text{ cm}} = -2.00$$

$$M_2 = \frac{-q_2}{p_2} = -\frac{6.67\text{ cm}}{-10.0\text{ cm}} = 0.667$$

The total magnification M of the two lenses is the product $M_1M_2 = (-2.00)(0.667) = -1.33$. Hence, the final image is real, inverted, and enlarged.

OPTIONAL

26.5 · LENS ABERRATIONS

One of the basic problems of lenses and lens systems is the imperfect quality of the images. The simple theory of mirrors and lenses assumes that rays make small angles with the principal axis and that all rays reaching the lens or mirror from a point source are focused at a single point, producing a sharp image. This is not always true in the real world. Where the approximations used in this theory do not hold, imperfect images are formed.

If one wishes to precisely analyze image formation, it is necessary to trace each ray, using Snell's law, at each refracting surface. This procedure shows that there is no single point image; instead, the image is *blurred*. The departures of real (imperfect) images from the ideal predicted by the simple theory are called **aberrations.** Two types of aberrations will now be described.

Spherical Aberrations

Spherical aberrations result from the fact that the focal points of light rays far from the principal axis of a spherical lens (or mirror) are different from the focal points of rays of the same wavelength passing near the axis. Figure 26.27 illustrates spherical aberration for parallel rays passing through a converging lens. Rays near the middle of the lens are imaged farther from the lens than rays at the edges. Hence, there is no single focal length for a lens.

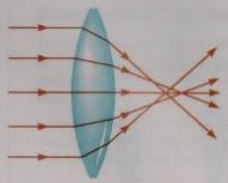


Figure 26.27 Spherical aberration caused by a converging lens. Does a diverging lens cause spherical aberration?

Most cameras are equipped with an adjustable aperture to control the light intensity and reduce spherical aberration when possible. (An aperture is an opening that controls the amount of light transmitted through the lens.) Sharper images are produced as the aperture size is reduced, because only the central portion of the lens is exposed to the incident light when the aperture is very small. At the same time, however, less light is imaged. To compensate for this loss in light intensity, a longer exposure time is used. An example of the good results obtained with a small aperture is the sharp image produced by a "pinhole" camera, with an aperture size of approximately 1 mm.

In the case of mirrors used for very distant objects, one can eliminate, or at least minimize, spherical aberration by employing a parabolic surface rather than a spherical surface. Parabolic surfaces are not used in many applications, however, because they are very expensive to make with high-quality optics. Parallel light rays incident on such a surface focus at a common point. Parabolic reflecting surfaces are used in many astronomical telescopes in order to enhance the image quality. They are also used in searchlights, where a nearly parallel light beam is produced from a small lamp placed at the focus of the reflecting surface.

Chromatic Aberrations

The fact that different wavelengths of light refracted by a lens focus at different points gives rise to *chromatic aberrations.* In Chapter 25 we described how the index of refraction of a material varies with wavelength. When white light passes through a lens, one finds, for example, that violet light rays are refracted more than red light rays (Fig. 26.28). From this we see that the focal length is larger for red light than for violet light. Other wavelengths (not shown in Fig. 26.28) would have intermediate focal points. The chromatic aberration for a diverging lens is opposite that for a converging lens. Chromatic aberration can be greatly reduced by using a combination of a converging and diverging lens made from two different types of glass.

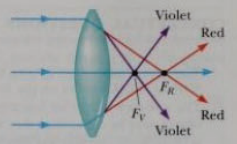


Figure 26.28 Chromatic aberration caused by a converging lens. Rays of different wavelengths focus at different points.

SUMMARY

The **magnification,** M, of a mirror or lens is defined as the ratio of the image height, h', to the object height, h:

$$M = \frac{h'}{h} = -\frac{q}{p} \qquad [26.2, 26.11]$$

In the paraxial ray approximation, the object distance, p, and image distance, q, for a spherical mirror of radius R are related by the **mirror equation,**

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} = \frac{1}{f} \qquad [26.4, 26.6]$$

where $f = R/2$ is the **focal length** of the mirror.

An image can be formed by refraction from a spherical surface of radius R. The object and image distances for refraction from such a surface are related by

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \qquad [26.8]$$

where the light is incident in the medium of index of refraction n_1 and is refracted in the medium the index of refraction of which is n_2.

For a thin lens, and in the paraxial ray approximation, the object and image distances are related by the **thin lens equation:**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \qquad [26.12]$$

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The **focal length,** f, of a thin lens in air is related to the curvature of its surfaces and to the index of refraction, n, of the lens material by

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad [26.13]$$

Converging lenses have positive focal lengths, and **diverging lenses** have negative focal lengths.

CONCEPTUAL QUESTIONS

1. Why do some emergency vehicles have the symbol ƎƆИAꓶUЯAMA written on the front?
2. The side-view mirror on late-model cars warns the user that objects may be closer than they appear. What kind of mirror is being used and why was this type selected?
3. Consider a concave spherical mirror with a real object. Is the image always inverted? Is the image always real? Give conditions for your answers.
4. Why does a clear stream always appear to be shallower than it actually is?
5. A person spearfishing from a boat sees a fish that is 3 m from the boat at a depth of 1 m. In order to spear the fish, should the person aim at, above, or below the image of the fish?
6. Explain why a fish in a spherical goldfish bowl appears larger than it really is.
7. If a cylinder of solid glass or clear plastic is placed above the words LEAD OXIDE and viewed from the side, as shown in Figure Q26.7, the LEAD appears inverted but the OXIDE does not. Explain.



Figure Q26.7 *(Courtesy of Henry Leap and Jim Lehman)*

8. A mirage is formed when the air gets gradually cooler as the height above the ground increases. What might happen if the air grows gradually warmer as the height is increased? This often happens over bodies of water or snow-covered ground: the effect is called *looming*.
9. Lenses used in eyeglasses, whether converging or diverging, are always designed such that the middle of the lens curves away from the eye, like the center lenses of Figure 26.17a and 26.17b. Why?
10. Describe lenses that can be used to start a fire.
11. Explain this statement: "The focal point of a lens is the location of the image of a point object at infinity." Discuss the notion of infinity in real terms as it applies to object distances. Based on this statement, can you think of a quick and dirty method for determining the focal length of a positive lens?
12. Consider the image formed by a thin converging lens. Under what conditions is the image (a) inverted, (b) upright, (c) real, (d) virtual, (e) larger than the object, and (f) smaller than the object?
13. Discuss the proper position of a slide relative to the lens in a slide projector. What type of lens must the slide projector have?
14. Discuss why each of the following statements is true or false. (a) A virtual image can be a virtual object. (b) A virtual image can be a real object. (c) A real image can be a virtual object. (d) A real image can be a real object.
15. Discuss the type of aberration involved in each of the following situations. (a) The edges of the image appear reddish. (b) A clear focus of the image's central portion cannot be obtained. (c) A clear focus of the image's outer portion cannot be obtained. (d) The central portion of the image is enlarged relative to the outer portions.
16. In a Jules Verne novel, a piece of ice is shaped to form a magnifying lens to focus sunlight to start a fire. Is this possible?
17. A solar furnace can be constructed by using a concave mirror to reflect and focus sunlight into a furnace enclosure. What factors in the design of the reflecting mirror would guarantee very high temperatures?
18. A pinhole camera can be constructed by punching a small hole in one side of a cardboard box. If the opposite side is cut out and replaced with tissue paper, the image of a distant object is formed by light passing through the pinhole onto the tissue paper (the screen). No lens is involved here. In effect, the pinhole replaces the lens of the camera. Explain why an image is formed, and describe the nature of the image.
19. If you want to examine the fine detail of an object with a magnifying lens of focal length 15 cm, where should the object be placed in order to observe a magnified image of the object?
20. Large telescopes are usually reflecting rather than refracting. List some reasons for this choice.
21. A classic science fiction novel, *The Invisible Man*, tells of a person who becomes invisible by changing the index of refraction of his body to that of air. This story has been criticized by students who know how the eye works; they claim the invisible man himself would be unable to see. On the basis of your knowledge of the eye, could he see or not?

PROBLEMS

Section 26.1 Images Formed by Flat Mirrors

1. Does your bathroom mirror show you older or younger than your actual age? Compute an order-of-magnitude estimate for the age difference, based on data that you specify.
2. In a physics laboratory experiment, a torque is applied to a small-diameter wire suspended vertically under tensile stress. The small angle through which the wire turns as a consequence of the net torque is measured by attaching a small mirror to the wire and reflecting a beam of light off the mirror and onto a circular scale. Such an arrangement is known as an optical lever and is shown from a top view in Figure P26.2. Show that when the mirror turns through an angle θ, the reflected beam turns through angle 2θ.

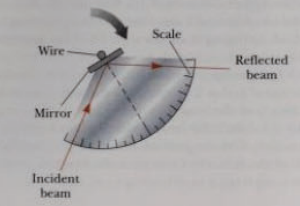


Figure P26.2

3. Determine the minimum height of a vertical flat mirror in which a person 5′10″ in height can see his or her full image. (A ray diagram would be helpful.)
4. Two flat mirrors have their reflecting surfaces facing one another, with the edge of one mirror in contact with an edge of the other, so that the angle between the mirrors is α. When an object is placed between the mirrors, a number of images are formed. In general, if the angle α is such that $n\alpha = 360°$, where n is an integer, the number of images formed is $n - 1$. Graphically, find all the image positions for the case $n = 6$ when a point object is between the mirrors (but not on the angle bisector).
5. A person walks into a room with two flat mirrors on opposite walls, which produce multiple images. When the person is 5.00 ft from the mirror on the left wall and 10.0 ft from the mirror on the right wall, find the distance from the person to the first three images seen in the mirror on the left.

Section 26.2 Images Formed by Spherical Mirrors

6. A concave spherical mirror has a radius of curvature of 20.0 cm. Find the location of the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted. Find the magnification in each case.
7. A convex mirror monitors the aisles in a store, as in Figure P26.7. The mirror has a radius of curvature of 0.550 m. Locate and describe the image of a customer 10.0 m from the mirror. Determine the magnification.



Figure P26.7 Convex mirrors, often used for security in department stores, provide wide-angle viewing. *(© Paul Silverman 1990, Fundamental Photographs)*

8. A large church has a niche in one wall. On the floor plan it appears as a semicircular indentation of radius 2.50 m. A worshiper stands on the center line of the niche, 2.00 m out from its deepest point, and whispers a prayer. Where is the sound concentrated after reflection from the back wall of the niche?

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9. A spherical convex mirror has a radius of curvature of 40.0 cm. Determine the position of the virtual image and magnification for object distances of (a) 30.0 cm and (b) 60.0 cm. (c) Are the images upright or inverted?
10. An object 2.00 cm in height is placed 3.00 cm in front of a concave mirror. If the image is 5.00 cm in height and virtual, what is the focal length of the mirror?
11. A spherical mirror is to be used to form, on a screen located 5.00 m from the object, an image five times the size of the object. (a) Describe the type of mirror required. (b) Where should the mirror be positioned relative to the object?
12. (a) A concave mirror forms an inverted image four times larger than the object. Find the focal length of the mirror if the distance between object and image is 0.600 m. (b) A convex mirror forms a virtual image half the size of the object. If the distance between image and object is 20.0 cm, determine the radius of curvature of the mirror.
13. A concave mirror has a focal length of 40.0 cm. Determine the object position for which the resulting image is upright and four times the size of the object.

Section 26.3 Images Formed by Refraction

14. A simple model of the human eye ignores its lens entirely. Most of what the eye does to light happens at the transparent cornea. Assume that this outer surface has a radius of curvature 6.00 mm, and assume that the eyeball contains just one fluid of index of refraction 1.40. Prove that a very distant object will be imaged on the retina, 21.0 mm behind the cornea. Describe the image.
15. A cubical block of ice 50.0 cm on a side is placed on a level floor over a speck of dust. Find the location of the image of the speck if the index of refraction of ice is 1.309.
16. A colored marble is dropped into a large tank filled with benzene ($n = 1.50$). (a) What is the depth of the tank if the apparent depth of the marble when viewed from directly above the tank is 35.0 cm? (b) If the marble has a diameter of 1.50 cm, what is its apparent diameter when viewed from directly above the tank?
17. A glass sphere ($n = 1.50$) of radius 15.0 cm has a tiny air bubble located 5.00 cm from the center. The sphere is viewed along a direction parallel to the radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?
18. A flint glass plate ($n = 1.66$) rests on the bottom of an aquarium tank. The plate is 8.00 cm thick (vertical dimension) and covered with water ($n = 1.33$) to a depth of 12.0 cm. Calculate the apparent thickness of the plate as viewed from above the water. (Assume nearly normal incidence.)
19. A glass hemisphere is used as a paperweight with its flat face resting on a stack of papers. The radius of the circular cross-section is 4.00 cm, and the index of refraction of the glass is 1.55. The center of the hemisphere is directly over a letter "O" that is 2.50 mm in diameter. What is the diameter of the image of the letter as seen looking along a vertical radius?
20. A transparent sphere of unknown composition is observed to form an image of the Sun on the surface of the sphere opposite the Sun. What is the refractive index of the sphere material?

Section 26.4 Thin Lenses

21. The left face of a biconvex lens has a radius of curvature of 12.0 cm, and the right face has a radius of curvature of 18.0 cm. The index of refraction of the glass is 1.44. (a) Calculate the focal length of the lens. (b) Calculate the focal length if the radii of curvature of the two faces are interchanged.
22. A contact lens is made of plastic with an index of refraction of 1.50. The lens has an outer radius of curvature of +2.00 cm and an inner radius of curvature of +2.50 cm. What is the focal length of the lens?
23. A thin lens has a focal length of 25.0 cm. Locate the image when the object is placed (a) 26.0 cm and (b) 24.0 cm in front of the lens. Describe the image in each case.
24. A converging lens has a focal length of 20.0 cm. Locate the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted. Find the magnification in each case.
25. The nickel's image in Figure P26.25 has twice the diameter of the nickel and is 2.84 cm from the lens. Determine the focal length of the lens.
26. An object located 32.0 cm in front of a lens forms an image on a screen 8.00 cm behind the lens. (a) Find the focal length of the lens. (b) Determine the magnification. (c) Is the lens converging or diverging?



Figure P26.25

27. A magnifying glass is a converging lens of focal length 15.0 cm. At what distance from a postage stamp should you hold this lens to get a magnification of +2.00?
28. A microscope slide is placed in front of a converging lens with a focal length of 2.44 cm. The lens forms an image of the slide 12.9 cm from the slide. How far is the lens from the slide if the image is (a) real? (b) virtual?
29. A person looks at a gem with a jeweler's loupe—a converging lens that has a focal length of 12.5 cm. The loupe forms a virtual image 30.0 cm from the lens. (a) Determine the magnification. Is the image upright or inverted? (b) Construct a ray diagram for this arrangement.
30. An object is placed 30.0 cm from a lens, and a virtual image is formed 15.0 cm from the lens. (a) What is the focal length of the lens? (b) Is the lens converging or diverging?
31. Suppose an object has thickness dp so that it extends from object distance p to $p + dp$. Prove that the thickness dq of its image is given by $(-q^2/p^2)\,dp$, so that the longitudinal magnification $dq/dp = -M^2$, where M is the lateral magnification.
32. An object is placed 50.0 cm from a screen. Where should a converging lens with a 10.0-cm focal length be placed in order to form an image on the screen? Find the magnification(s).
33. The projection lens in a certain slide projector is a single thin lens. A slide 24.0 mm high is to be projected so that its image fills a screen 1.80 m high. The slide-to-screen distance is 3.00 m. (a) Determine the focal length of the projection lens. (b) How far from the slide should the lens of the projector be placed in order to form the image on the screen?
34. A runner passes a spectator at 10.0 m/s. The spectator photographs the runner using a camera having a focal length of 5.00 cm and an exposure time of 20.0 ms. For a good photograph, the maximum image blurring allowed on the film is 0.500 mm. Find the minimum distance between runner and camera in order to obtain a good photograph.

Additional Problems

35. The lens and mirror in Figure P26.35 have focal lengths of +80.0 cm and −50.0 cm, respectively. An object is placed 1.00 m to the left of the lens as drawn. Locate the final image, formed by light that has gone through the lens twice. State whether the image is upright or inverted, and determine the overall magnification.

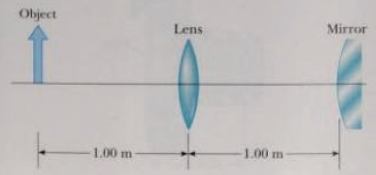


Figure P26.35

36. The object in Figure P26.36 is midway between the lens and the mirror. The mirror's radius of curvature is 20.0 cm, and the lens has a focal length of −16.7 cm. Considering only the light that leaves the object and travels first toward the mirror, locate the final image formed by this system. Is this image real or virtual? Is it upright or inverted? What is the overall magnification?

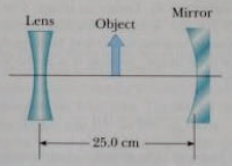


Figure P26.36

37. The distance between an object and its upright image is 20.0 cm. If the magnification is 0.500, what is the focal length of the lens being used to form the image?
38. The distance between an object and its upright image is d. If the magnification is M, what is the focal length of the lens being used to form the image?
39. An object placed 10.0 cm from a concave spherical mirror produces a real image 8.00 cm from the mirror. If the object is moved to a new position 20.0 cm from the mirror, what is the position of the image? Is the latter image real or virtual?
40. Two rays traveling parallel to the principal axis strike a plano-convex lens having a refractive index of 1.60 (Fig. P26.40). If the convex face is spherical, a ray near the edge

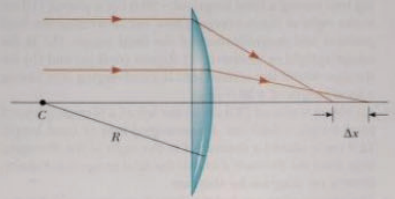


Figure P26.40

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does not strike the focal point (spherical aberration). If this face has a radius of curvature of 20.0 cm and the two rays are $h_1 = 0.500$ cm and $h_2 = 12.0$ cm from the principal axis, find the difference in the positions where each strikes the principal axis.

41. A real object is located at the zero end of a meter stick. A large concave mirror at the 100-cm end of the meter stick forms an image of the object at the 70.0-cm position. A small convex mirror placed at the 20.0-cm position forms a final image at the 10.0-cm point. What is the radius of curvature of the convex mirror?

42. Derive the lens-makers' equation as follows: Consider an object in vacuum at $p_1 = \infty$ from a first refracting surface of radius of curvature R_1. Locate its image. Use this image as the object for the second refracting surface, which has nearly the same location as the first, because the lens is thin. Locate the final image, proving it is at the image distance q_2 given by

$$\frac{1}{q_2} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

43. A parallel beam of light enters a glass hemisphere perpendicular to the flat face, as shown in Figure P26.43. The radius is $R = 6.00$ cm, and the index of refraction is $n = 1.560$. Determine the point at which the beam is focused. (Assume paraxial rays.)

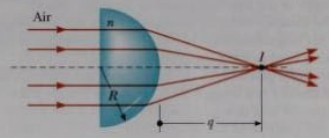


Figure P26.43

44. An object 2.00 cm high is placed 40.0 cm to the left of a converging lens having a focal length of 30.0 cm. A diverging lens having a focal length of −20.0 cm is placed 110 cm to the right of the converging lens. (a) Determine the final position and magnification of the final image. (b) Is the image upright or inverted? (c) Repeat parts (a) and (b) for the case where the second lens is a converging lens having a focal length of +20.0 cm.

45. An object is placed 12.0 cm to the left of a diverging lens of focal length −6.00 cm. A converging lens of focal length 12.0 cm is placed a distance d to the right of the diverging lens. Find the distance d so that the final image is at infinity. Draw a ray diagram for this case.

46. The cornea of an eye has a radius of curvature of 0.800 cm. (a) What is the focal length of the reflecting surface of the eye? (b) If a $20 gold piece 3.40 cm in diameter is held 25.0 cm from the cornea, what are the size and location of the reflected image?

47. The disk of the Sun subtends an angle of 0.533° at the Earth. What are the position and diameter of the solar image formed by a concave spherical mirror of radius 3.00 m?

48. A floating coin illusion consists of two parabolic mirrors, each having a focal length 7.50 cm, facing each other so that their centers are 7.50 cm apart (Fig. P26.48). If a few coins are placed on the lower mirror, an image of the coins is formed at the small opening at the center of the top mirror. Show that the final image is formed at that location and describe its characteristics. (*Note:* A very startling effect is to shine a flashlight beam on these images. Even at a glancing angle, the incoming light beam is seemingly reflected off the images! Do you understand why?)

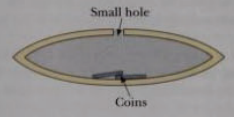


Figure P26.48

49. In a darkened room, a burning candle is placed 1.50 m from a white wall. A lens is placed between candle and wall at a location that causes a larger, inverted image to form on the wall. When the lens is moved 90.0 cm toward the wall, another image of the candle is formed. Find (a) the two object distances that produce the images stated above and (b) the focal length of the lens. (c) Characterize the second image.

50. The fixed lens L_1 in Figure P26.50 has a focal length of 15.0 cm, whereas lens L_2 has a focal length of 13.3 cm. The distance d of lens L_2 from the film plane can be varied from 5.00 cm to 10.0 cm. Determine the range of distances for which objects can be focused on the film.

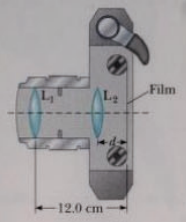


Figure P26.50

Spreadsheet Problem

S1. The equation for refraction by a spherical surface separating two media (Fig. 26.13) having indices of refraction n_1 and n_2 is

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

This equation was derived using small-angle approximations. Spreadsheet 26.1 tests the range of validity of these approximations. The spreadsheet calculates the image distance for a series of incident angles α without making small-angle approximations. Input parameters are the indices of refraction, the radius of curvature of the surface, and the object distance. (a) Take $n_1 = 1.00$, $n_2 = 1.52$, $R = 10.0$ cm, and $p = 30.0$ cm. If the maximum acceptable error in the image distance is 3.00%, for what maximum angle α can the refraction equation be used? (b) Repeat part (a) using $p = 50.0$ cm. (c) Choose other values for the input parameters and repeat (a). (d) Is the error linearly proportional to α?

ANSWERS TO CONCEPTUAL PROBLEMS

1. You will not be able to focus your eyes on both the picture and your image at the same time. To focus on the picture, you must adjust your eyes so that an object several feet away (the picture) is in focus. Thus, you are focusing *on the mirror surface.* But your image in the mirror is behind the mirror, as far behind the mirror as you are in front of it. Thus, you must focus your eyes beyond the mirror, twice as far away as the picture, to bring the image into focus.
2. The actor will see *the movie camera.* Light from the actor's face must reflect from the mirror in the direction of the camera lens, so that his face is recorded on film. If we reverse that light path, light from the camera reflects from the mirror into the eyes of the actor.
3. The ultrasonic range finder sends out a sound wave and measures the time for the echo to return. Using this information, the camera calculates the distance to the subject and sets the camera lens. When facing a mirror, the ultrasonic signal reflects from the mirror surface and the camera adjusts its focus so that the mirror surface is at the correct focusing distance from the camera. But your image in the mirror is twice this distance from the camera, so it is blurry.
4. Light rays diverge from the position of a virtual image just as they do from an actual object. Thus, a virtual image can be as easily photographed as any object can. Of course, the camera would have to be placed near the axis of the lens or mirror in order to intercept the light rays.
5. We consider the two trees to be two separate objects. The far tree is an object that is farther from the lens than the near tree. Thus, the image of the far tree will be closer to the lens than the image of the near tree. The screen must be moved closer to the lens to put the far tree in focus.
6. If a converging lens is placed in a liquid the index of refraction of which is larger than that of the lens material, the relative size of the angles at the interfaces will be reversed, and the lens will diverge light. A mirror depends only on reflection, which is independent of the surrounding material, so a converging mirror will be converging in any liquid.
7. The bag of air will act as a crude lens underwater. The direction of refraction at the bag surface as light goes from water to air will be the opposite of the directions for the light entering from air into the water-filled bag, so the air lens will diverge light underwater.
8. The focal length for a mirror is determined by the law of reflection from the mirror surface. The law of reflection is independent of the material of which the mirror is made and of the surrounding medium. Thus, the focal length depends only on the radius of curvature and not the material. The focal length of a lens depends on the law of refraction, and refraction is highly dependent on the indices of refraction of the lens material and surrounding medium. Thus, the focal length of a lens depends on the lens material.

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Wave Optics

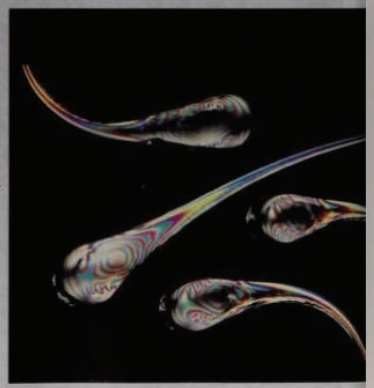
CHAPTER OUTLINE

27.1 Conditions for Interference
27.2 Young's Double-Slit Experiment
27.3 Change of Phase Due to Reflection
27.4 Interference in Thin Films
27.5 Diffraction
27.6 Resolution of Single-Slit and Circular Apertures
27.7 The Diffraction Grating
27.8 Diffraction of X-Rays by Crystals (Optional)

In the preceding chapter, we used the concept of light rays to examine what happens when light passes through a lens or reflects from a mirror. This chapter is concerned with the subject of **wave optics,** which addresses the related optical phenomena of interference and diffraction. These phenomena cannot be adequately explained with ray (geometric) optics, but we shall discuss how the wave nature of light leads to satisfying descriptions of such events.

27.1 • CONDITIONS FOR INTERFERENCE

In our discussion of wave interference of mechanical waves in Chapter 14, we found that two waves could add together either constructively or destructively. In constructive interference, the amplitude of the resultant wave is greater than that of either individual wave, whereas in destructive interference, the resultant amplitude is less than that of either individual wave. Electromagnetic waves also undergo interference. All interference associated with electromagnetic waves arises fundamentally as a result of combining the electromagnetic fields that constitute the individual waves.



These glass objects, called Prince Rupert drops, are made by dropping molten glass into water. The photograph was made by placing the objects between two crossed polarizers. The patterns observed represent the strain distribution in the glass. Studies of such patterns led to the development of tempered glass. *(James L. Amos, Peter Arnold, Inc.)*

Interference effects in visible electromagnetic waves are not easy to observe because of their short wavelengths (from about 4×10^{-7} m to 7×10^{-7} m). In order to observe sustained interference in light waves, three conditions must be met:

- The sources must be **coherent**—that is, they must maintain a constant phase with respect to each other.
- The sources must be of identical wavelength.
- The superposition principle must apply.

• *Conditions for interference*

Let us examine the characteristics of coherent sources. Two sources (producing two traveling waves) are needed to create interference. However, in order to produce a stable interference pattern, **the individual waves must maintain a constant phase with one another.** When this situation prevails, the sources are said to be **coherent.** As an example, the sound waves emitted by two side-by-side loudspeakers driven by a single amplifier can produce interference because the two speakers respond to the amplifier in the same way at the same time.

Now, if two light sources are placed side by side, no interference effects are observed because the light waves from one source are emitted independently of the other source; hence, the emissions from the two sources do not maintain a constant phase relationship with each other over the time of observation. An ordinary light source undergoes random changes about once every 10^{-8} s. Therefore, the conditions for constructive interference, destructive interference, or some intermediate state last for times on the order of 10^{-8} s. The result is that no interference effects are observed, because the eye cannot follow such short-term changes. Such light sources are said to be **noncoherent.**

A common method for producing two coherent light sources is to use one single wavelength source to illuminate a screen containing two small slits. The light emerging from the two slits is coherent because a single source produces the original light beam and the two slits serve only to separate the original beam into two parts (which is exactly what was done to the sound signal just discussed). A random change in the light emitted by the source will occur in the two separate beams at the same time, and interference effects can still be observed.

27.2 • YOUNG'S DOUBLE-SLIT EXPERIMENT

Interference in light waves from two slits was first demonstrated by Thomas Young in 1801. A schematic diagram of the apparatus used in this experiment is shown in Figure 27.1a. (Young used pinholes in his original experiments, rather than slits.) Light is incident on a screen in which there is a narrow slit, S_0. The light waves emerging from this slit arrive at a second screen that contains two narrow, parallel slits, S_1 and S_2. These two slits serve as a pair of coherent light sources because waves emerging from them originate from the same source, S_0, and therefore maintain a constant phase relationship. The light from the two slits produces a visible pattern on screen C; the pattern consists of a series of bright and dark parallel bands called **fringes** (Fig. 27.1b). When the light from S_1 and S_2 arrives at a point on the screen so that constructive interference occurs at that location, a bright fringe appears. When the light from the two slits combines destructively at any location, a dark fringe results.

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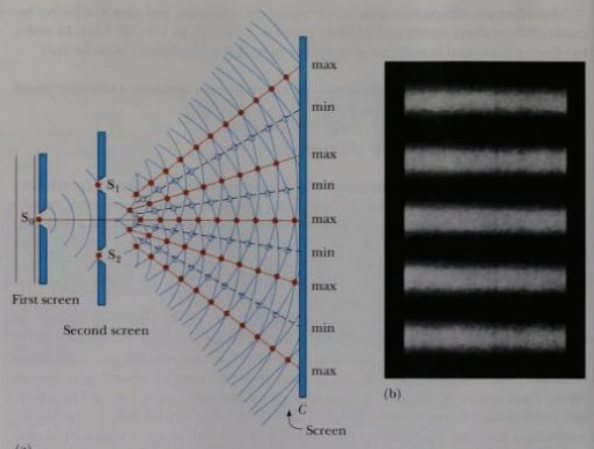


Figure 27.1 (a) Schematic diagram of Young's double-slit experiment. The narrow slits act as wave sources. Slits S_1 and S_2 behave as coherent sources that produce an interference pattern on screen C. (Note that this drawing is not to scale.) (b) The fringe pattern formed on screen C could look like this.

Figure 27.2 is a schematic diagram of some of the ways the two waves in Young's experiment can combine at screen C. In Figure 27.2a, the two waves, which leave the two slits in phase, strike the screen at the central point, P. Because these waves travel equal distances, they arrive in phase at P, and as a result constructive interference occurs at this location and a bright fringe is observed. In Figure 27.2b, the

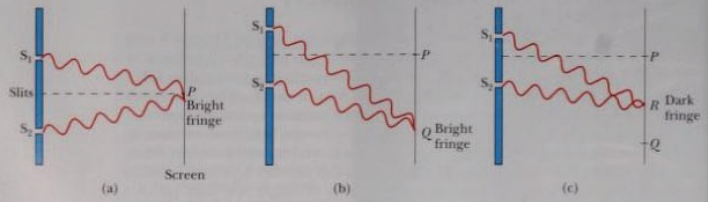


Figure 27.2 (a) Constructive interference occurs at P when the waves combine. (b) Constructive interference also occurs at Q. (c) Destructive interference occurs at R because the wave from the upper slit falls half a wavelength behind the wave from the lower slit. (Note that these figures are not drawn to scale.)

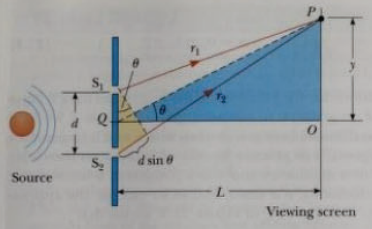


Figure 27.3 Geometric construction for describing Young's double-slit experiment. The path difference between the two rays is $r_2 - r_1 = d \sin \theta$. (Note that this figure is not drawn to scale.)

two light waves again start in phase, but the upper wave has to travel one wavelength farther to reach point Q on the screen. Because the upper wave falls behind the lower one by exactly one wavelength, they still arrive in phase at Q, and so a second bright fringe appears at this location. Now consider point R, midway between P and Q in Figure 27.2c. At this location, the upper wave has fallen half a wavelength behind the lower wave. This means that the trough from the lower wave overlaps the crest from the upper wave, giving rise to destructive interference at R. For this reason, one observes a dark fringe at this location.

We can obtain a quantitative description of Young's experiment with the help of Figure 27.3. Consider point P on the viewing screen; the screen is located a perpendicular distance of L from the screen containing slits S_1 and S_2, which are separated by a distance of d, and r_1 and r_2 are the distances the waves travel from slit to screen. Let us assume that the source is monochromatic. Under these conditions, the waves emerging from S_1 and S_2 have the same frequency and amplitude and are in phase. The light intensity on the screen at P is the resultant of the light coming from both slits. Note that a wave from the lower slit travels farther than a wave from the upper slit by an amount equal to $d \sin \theta$. This distance is called the **path difference, δ** (lowercase Greek delta), where

$$\delta = r_2 - r_1 = d \sin \theta \qquad [27.1]$$

• *Path difference*

This equation assumes that the two waves are parallel to each other, which is approximately true because L is much greater than d. As noted earlier, the value of this path difference determines whether or not the two waves are in phase or out of phase when they arrive at P. If the path difference is either zero or some integral multiple of the wavelength, the two waves are in phase at P and constructive interference results. Therefore, the condition for bright fringes, or **constructive interference,** at P is

$$\delta = d \sin \theta = m\lambda \qquad (m = 0, \pm 1, \pm 2, \ldots) \qquad [27.2]$$

• *Conditions for constructive interference*

The number m is called the **order number.** The central bright fringe at $\theta = 0$ ($m = 0$) is called the **zeroth-order maximum.** The first maximum on either side, when $m = \pm 1$, is called the **first-order maximum,** and so forth.

In a similar way, when the path difference is an odd multiple of $\lambda/2$, the two waves arriving at P are 180° out of phase and give rise to destructive interference. Therefore, the condition for dark fringes, or **destructive interference,** at P is

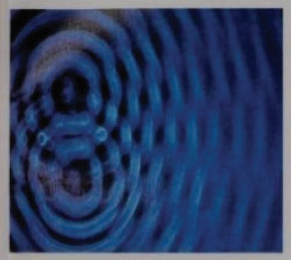
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Conditions for destructive interference

$$\delta = d \sin\theta = \left(m + \frac{1}{2}\right)\lambda \qquad (m = 0, \pm 1, \pm 2, \ldots) \qquad [27.3]$$

It is useful to obtain expressions for the positions of the bright and dark fringes measured vertically from O to P. In addition to our assumption that $L \gg d$, we shall assume that $d \gg \lambda$—that is, the distance between the two slits is much larger than the wavelength. This situation prevails in practice because L is often on the order of 1 m whereas d is a fraction of a millimeter and λ is a fraction of a micrometer for visible light. Under these conditions, θ is small, and so we can use the approximation $\sin\theta \approx \tan\theta$. From the triangle OPQ in Figure 27.3, we see that

$$\sin\theta \approx \tan\theta = \frac{y}{L} \qquad [27.4]$$



An interference pattern involving water waves is produced by two vibrating sources at the water's surface. The pattern is analogous to that observed in Young's double-slit experiment. Note the regions of constructive and destructive interference. *(Richard Megna, Fundamental Photographs)*

Using this result and making the substitution $\sin\theta = m\lambda/d$ from Equation 27.2, we see that the positions of the bright fringes measured from O are given by

$$y_{\text{bright}} = \frac{\lambda L}{d} m \qquad [27.5]$$

Likewise, using Equations 27.3 and 27.4, we find that the dark fringes are located at

$$y_{\text{dark}} = \frac{\lambda L}{d}\left(m + \tfrac{1}{2}\right) \qquad [27.6]$$

As we shall demonstrate in Example 27.1, Young's double-slit experiment provides a method for measuring the wavelength of light. In fact, Young used this technique to make the first measurement of the wavelength of light. In addition, the experiment gave the wave model of light a great deal of credibility. Today we still use the phenomenon of interference to describe many observations of wavelike behavior.

Thinking Physics 1

Consider a double-slit experiment in which a laser beam is passed through a pair of very closely spaced slits, and a clear interference pattern is displayed on a distant screen. Now, suppose you place smoke particles between the double slit and the screen. With the presence of the smoke particles, will you see the effects of the interference in the space between the slits and the screen, or will you only see the effects on the screen?

Reasoning You will see the effects in the area filled with smoke. There will be bright lines directed toward the bright areas on the screen and dark lines directed toward the dark areas on the screen. In Figure 27.3, the geometrical construction is important for developing the mathematical description of interference. It is subject to misinterpretation, however, because it might suggest that the interference does not occur until the light rays from the two slits strike the screen at the same position. The more fruitful diagram for this situation is Figure 27.1, in which it is clear that there are *paths* of destructive and constructive interference all the way from the slits to the screen. These paths will be made visible by the smoke.

Thinking Physics 2

If your stereo speakers are connected "out of phase"—that is, with one speaker connected correctly and the other with its wires reversed, the bass in the music tends to be weak. Why does this happen, and why is it a problem for the bass and not the treble notes?

Reasoning This is an acoustic analog to double-slit interference. The two speakers act as sources of waves, just like the two slits in a Young's double-slit experiment. If the speakers are connected correctly, and the same sound signal is fed to each speaker, both speakers move inward and outward at the same time in response to the signal. Thus, the sound waves are in phase as they leave the speakers. If you are sitting at a point in front of the speakers and midway between them, you will be located at the zero-order maximum—the interference is constructive and the sound will be loud. If one speaker is wired backward, then one speaker will be moving outward while the other is moving inward. The sound leaves the two speakers half a wavelength out of phase. Thus, if you are sitting at the same place, you will be at an interference minimum. This is a particular problem for the bass due to the long wavelength of low frequency notes. This results in a very large region of destructive interference in front of the speakers, on the order of the size of the room. The much shorter wavelengths of the high-frequency notes result in closely spaced maxima and minima. The spacing can be on the order of the size of the head and smaller. Thus, if one ear is at a minimum, the other might be at a maximum. What's more, small movements of the head will result in a shift from the position of a minimum to that of a maximum.

Thinking Physics 3

Suppose you are watching television by means of an antenna rather than a cable system. If an airplane flies near your location, you may notice wavering ghost images in the television picture. What might cause this?

Reasoning Your television antenna will receive two signals—the direct signal from the transmitting antenna and a reflected signal from the surface of the airplane. As the airplane changes position, there are some times when these two signals are in phase and other times when they are out of phase. As a result, there is a variation in the intensity of the combined signal received at your antenna. This variation is evidenced by the wavering ghost images of the picture.

Example 27.1 Measuring the Wavelength of a Light Source

A viewing screen is separated from a double-slit source by 1.2 m. The distance between the two slits is 0.030 mm. The second-order bright fringe ($m = 2$) is 4.5 cm from the center line. (a) Determine the wavelength of the light.

Solution We can use Equation 27.5, with $y_2 = 4.5 \times 10^{-2}$ m, $m = 2$, $L = 1.2$ m, and $d = 3.0 \times 10^{-5}$ m:

$$\lambda = \frac{dy_2}{mL} = \frac{(3.0 \times 10^{-5}\ \text{m})(4.5 \times 10^{-2}\ \text{m})}{2 \times 1.2\ \text{m}}$$

$$= 5.6 \times 10^{-7}\ \text{m} = 560\ \text{nm}$$

(b) Calculate the distance between adjacent bright fringes.

Solution From Equation 27.5 and the results to (a), we get

$$y_{m+1} - y_m = \frac{\lambda L(m+1)}{d} - \frac{\lambda L m}{d}$$

$$= \frac{\lambda L}{d} = \frac{(5.6 \times 10^{-7}\ \text{m})(1.2\ \text{m})}{3.0 \times 10^{-5}\ \text{m}}$$

$$= 2.2 \times 10^{-2}\ \text{m} = 2.2\ \text{cm}$$

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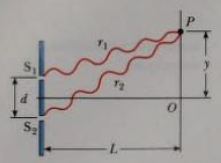


Figure 27.4 Construction for analyzing the double-slit interference pattern. A bright region, or intensity maximum, is observed at *O*.

Intensity Distribution of the Double-Slit Interference Pattern

We shall now calculate the distribution of light intensity (the energy delivered by the wave per unit area per unit time) associated with the double-slit interference pattern. Again, suppose that the two slits represent coherent sources of sinusoidal waves. Hence, the two waves have the same angular frequency, ω, and a constant phase difference, ϕ. The total electric field at the point P on the screen in Figure 27.4 is the *vector superposition* of the two waves. Assuming the two waves have the same amplitude, E_0, we can write the electric field at P due to each wave separately as

$$E_1 = E_0 \sin \omega t \quad \text{and} \quad E_2 = E_0 \sin(\omega t + \phi) \qquad [27.7]$$

Although the waves have equal phase at the slits, **their phase difference, ϕ, at P depends on the path difference, $\delta = r_2 - r_1 = d \sin \theta$.** Because a path difference of λ corresponds to a phase difference of 2π rad (constructive interference), whereas a path difference of $\lambda/2$ corresponds to a phase difference of π rad (destructive interference), we obtain the ratio

$$\frac{\delta}{\phi} = \frac{\lambda}{2\pi}$$

Phase difference •

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d \sin \theta \qquad [27.8]$$

This equation tells us precisely how the phase difference ϕ depends on the angle θ.

Using the superposition principle and Equation 27.7, we can obtain the resultant electric field at the point P:

$$E_P = E_1 + E_2 = E_0[\sin \omega t + \sin(\omega t + \phi)] \qquad [27.9]$$

To simplify this expression, we use the trigonometric identity

$$\sin A + \sin B = 2 \sin\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right)$$

Taking $A = \omega t + \phi$ and $B = \omega t$, we can write Equation 27.9 in the form

$$E_P = 2E_0 \cos\left(\frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \qquad [27.10]$$

Hence, the electric field at P has the same frequency ω as the original two waves, but its amplitude is multiplied by the factor $2 \cos(\phi/2)$. To check the consistency of this result, note that if $\phi = 0, 2\pi, 4\pi, \ldots$, the amplitude at P is $2E_0$, corresponding to the condition for constructive interference. Referring to Equation 27.8, we find that our result is consistent with Equation 27.2. Likewise, if $\phi = \pi, 3\pi, 5\pi, \ldots$, the amplitude at P is zero, which is consistent with Equation 27.3 for destructive interference.

Finally, to obtain an expression for the light intensity at P, recall that **the intensity of a wave is proportional to the square of the resultant electric field at that point** (Chapter 24, Section 24.6). Using Equation 27.10, we can therefore express the intensity at P as

$$I \propto E_P^2 = 4E_0^2 \cos^2(\phi/2) \sin^2\left(\omega t + \frac{\phi}{2}\right)$$

Because most light-detecting instruments measure the time-averaged light intensity, and the time-averaged value of $\sin^2(\omega t + \phi/2)$ over one cycle is $1/2$, we can write the average intensity at P as

$$I_{av} = I_0 \cos^2(\phi/2) \qquad [27.11]$$

where I_0 is the *maximum* possible time-averaged light intensity. [Note that $I_0 \propto (E_0 + E_0)^2 = (2E_0)^2 = 4E_0^2$.] Substituting Equation 27.8 into Equation 27.11, we find that

$$I_{av} = I_0 \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right) \qquad [27.12]$$

* *Average light intensity for the double-slit interference pattern*

Alternatively, because $\sin \theta \approx y/L$ for small values of θ, we can write Equation 27.12 in the form

$$I_{av} = I_0 \cos^2\left(\frac{\pi d}{\lambda L} y\right) \qquad [27.13]$$

Constructive interference, which produces intensity maxima, occurs when the quantity $\pi yd/\lambda L$ is an integral multiple of π, corresponding to $y = (\lambda L/d)m$. This is consistent with Equation 27.5. Intensity distribution versus θ is plotted in Figure 27.5. Note that the interference pattern consists of equally spaced fringes of equal intensity. However, the result is valid only if the slit-to-screen distance, L, is large relative to the slit separation, and only for small values of θ.

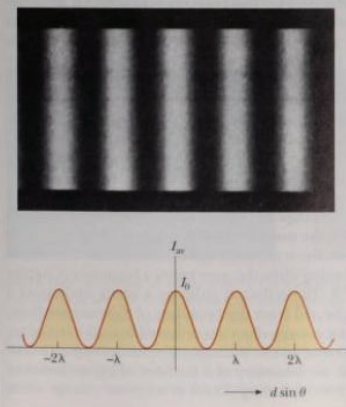


Figure 27.5 Intensity distribution versus $d \sin \theta$ for the double-slit pattern when the screen is far from the two slits ($L \gg d$). *(Photo from M. Cagnet, M. Francon, and J. C. Thierr, Atlas of Optical Phenomena, Berlin, Springer-Verlag, 1962)*

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We have seen that the interference phenomena arising from two coherent sources depend on the relative phase of the waves at a given point. Furthermore, the phase difference at a given point depends on the *path difference* between the two waves. **The resultant intensity at a point is proportional to the square of the resultant amplitude.** That is, the intensity is proportional to $(E_1 + E_2)^2$. It would be *incorrect* to calculate the resultant intensity by adding the intensities of the individual waves. This procedure would give a different quantity, namely $E_1^2 + E_2^2$. Finally, $(E_1 + E_2)^2$ has the same *average* value as $E_1^2 + E_2^2$, when the time average is taken over all values of the phase difference between E_1 and E_2. Hence, the principle of energy conservation is not violated.

CONCEPTUAL PROBLEM 1

Consider a dark fringe in a two-slit interference pattern, at which almost no light energy is arriving. Waves from both slits travel to this point, but the waves cancel. Where does the energy go?

27.3 · CHANGE OF PHASE DUE TO REFLECTION

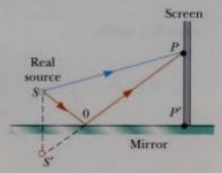


Figure 27.6 Lloyd's mirror. An interference pattern is produced on a screen at P as a result of the combination of the direct ray (blue) and the reflected ray (brown). The reflected ray undergoes a phase change of 180°.

Young's method of producing two coherent light sources involves illuminating a pair of slits with a single source. Another simple arrangement for producing an interference pattern with a single light source is known as *Lloyd's mirror*. A light source is placed at point S close to a mirror, as illustrated in Figure 27.6. Waves can reach the viewing point, P, either by the direct path SP or by the indirect path involving reflection from the mirror. The reflected ray can be treated as a ray originating from a source at S', located behind the mirror. This source S', which is the image of S, can be considered a virtual source.

At points far from the source, one would expect an interference pattern due to waves from S and S', just as is observed for two real coherent sources. An interference pattern is indeed observed. However, the positions of the dark and bright fringes are *reversed* relative to the pattern of two real coherent sources (Young's experiment). This is because the coherent sources at S and S' differ in phase by 180°. This 180° phase change is produced on reflection.

To illustrate this further, consider the point P' at which the mirror meets the screen. This point is equidistant from S and S'. If path difference alone were responsible for the phase difference, one would expect to see a bright fringe at P' (because the path difference is zero for this point), corresponding to the central fringe of the two-slit interference pattern. Instead, one observes a *dark* fringe at P' because of the 180° phase change produced by reflection. In general, **an electromagnetic wave undergoes a phase change of 180° on reflection from a medium of higher index of refraction than the one in which it was traveling.**

It is useful to draw an analogy between reflected light waves and the reflections of a transverse wave on a stretched string when the wave meets a boundary (Chapter 13, Section 13.7), as in Figure 27.7. The reflected pulse on a string undergoes a phase change of 180° when it is reflected from the boundary of a denser medium, such as a heavier string, and no phase change when it is reflected from the boundary of a less dense medium. In a similar way, an electromagnetic wave undergoes a 180° phase change when reflected from the boundary of a medium of higher index of refraction than the one in which it was traveling. There is no phase change when

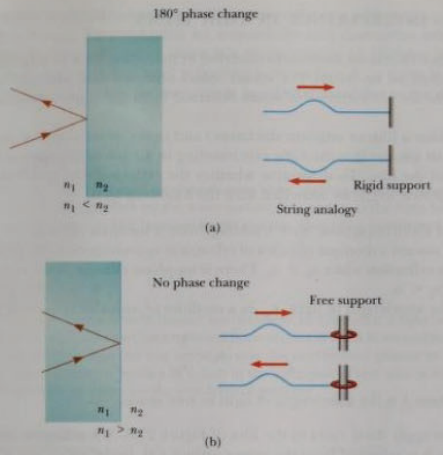
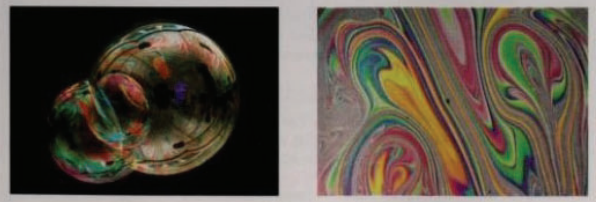


Figure 27.7 (a) A ray traveling in medium 1 reflecting from the surface of medium 2 undergoes a 180° phase change. The right side shows the analogy with a reflected pulse on a string fixed at one end. (b) A ray traveling in medium 1 reflecting from the surface of medium 2 with $n_1 > n_2$ undergoes no phase change. The right side shows the analogy with a reflected pulse on a string the end of which is free.

the wave is reflected from a boundary leading to a medium of lower index of refraction. The part of the wave that crosses the boundary undergoes no phase change.



(left) A layer of bubbles on water, produced by soap film. The colors, produced just before the bubbles burst, are due to interference between light rays reflected from the front and back of the thin film of water making the bubble. The colors depend on the thickness of the film, ranging from black where the film is at its thinnest to red as the film gets thicker. *(Dr. Jeremy Burgess/ Science Photo Library)* *(right)* Thin film interference. A thin film of oil on water displays interference, as shown by the pattern of colors when white light is incident on the film. The film thickness varies, thereby producing the interesting color pattern. *(© Tom Branch 1984, Photo Researchers)*

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27.4 • INTERFERENCE IN THIN FILMS

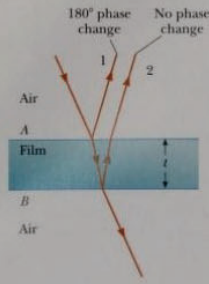


Figure 27.8 Interference in light reflected from a thin film is due to a combination of rays reflected from the upper and lower surfaces of the film.

Interference effects are commonly observed in thin films, such as soap bubbles and thin layers of oil on water. The varied colors observed with ordinary white light result from the interference of waves reflected from the opposite surfaces of the film.

Consider a film of uniform thickness t and index of refraction n, as in Figure 27.8. Let us assume that the light rays traveling in air are nearly normal to the two surfaces of the film. To determine whether the reflected rays interfere constructively or destructively, we must first note the following facts:

- An electromagnetic wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change on reflection when $n_2 > n_1$. There is no phase change in the reflected wave if $n_2 < n_1$.
- The wavelength of light, λ_n, in a medium of index of refraction n is

$$\lambda_n = \frac{\lambda}{n} \qquad [27.14]$$

where λ is the wavelength of light in free space.

Let us apply these rules to the film of Figure 27.8. According to the first rule, ray 1, which is reflected from the upper surface (A), undergoes a phase change of 180° with respect to the incident wave. Ray 2, which is reflected from the lower surface (B), undergoes no phase change with respect to the incident wave. Therefore, ray 1 is 180° out of phase with respect to ray 2, a situation that is equivalent to a path difference of $\lambda_n/2$. However, we must also consider that ray 2 travels an extra distance equal to $2t$ before the waves recombine. For example, if $2t = \lambda_n/2$, rays 1 and 2 will recombine in phase and constructive interference will result. In general, the condition for constructive interference is

$$2t = (m + \tfrac{1}{2})\lambda_n \qquad (m = 0, 1, 2, \ldots) \qquad [27.15]$$

This condition takes into account two factors: (1) the difference in optical path length for the two rays (the term $m\lambda_n$) and (2) the 180° phase change on reflection (the term $\lambda_n/2$). Because $\lambda_n = \lambda/n$, we can write Equation 27.15 in the form

Constructive interference in thin films •

$$2nt = (m + \tfrac{1}{2})\lambda \qquad (m = 0, 1, 2, \ldots) \qquad [27.16]$$

If the extra distance $2t$ traveled by ray 2 corresponds to a multiple of λ_n, the two waves will combine out of phase and destructive interference will result. The general equation for destructive interference is

Destructive interference in thin films •

$$2nt = m\lambda \qquad (m = 0, 1, 2, \ldots) \qquad [27.17]$$

It is important that you realize that two factors influence interference: (1) phase changes and (2) differences in travel distance. The preceding conditions for constructive and destructive interference are valid when the medium above the top surface of the film is the same as the medium below the bottom surface. The surrounding medium may have a refractive index less than or greater than that of the film. In either case, the rays reflected from the two surfaces will be out of phase by 180°. If the film is placed between two different media, one with $n < n_{film}$ and the other with $n > n_{film}$, the conditions for constructive and destructive interference are reversed. In this case, either there is a phase change of 180° for both ray 1 reflecting from surface A and ray 2 reflecting from surface B, or there is no phase change for either ray; hence, the net change in relative phase due to the reflections is zero.

CONCEPTUAL PROBLEM 2

In a laboratory accident, you spill two liquids onto water, neither of which mixes with the water. They both form thin films on the water surface. You notice, as the films become very thin as they spread, that one film becomes bright and the other black in reflected light. Why might this be?

CONCEPTUAL PROBLEM 3

In Chapter 9, the Michelson interferometer was discussed. In this device, a light beam is split into two perpendicular paths and then reflected from mirrors in order to recombine, forming an interference pattern. Suppose you are observing the interference pattern in your laboratory and a joking colleague holds a lit match in the light path of one arm of the interferometer. Will this have an effect on the interference pattern?

CONCEPTUAL PROBLEM 4

In our discussion of thin film interference, we looked at light *reflecting* from a thin film. Consider now the light *transmitting* through a thin film, with air on both sides of the film. Consider one light ray, the direct ray, that transmits through the film without reflecting. Consider a second ray, the reflected ray, that transmits through the first surface, reflects back from the second, reflects again from the first, and then transmits out into the air, parallel to the direct ray. For normal incidence, how thick must the film be, in terms of the wavelength of the light, for the outgoing rays to interfere destructively? Is it the same thickness as for reflected destructive interference?

PROBLEM-SOLVING STRATEGY • Thin Film Interference

The following features should be kept in mind while working thin-film interference problems:

1. Identify the thin film causing the interference.
2. The type of interference that occurs is determined by the phase relationship between the portion of the wave reflected at the upper surface of the film and the portion reflected at the lower surface.
3. Phase differences between the two portions of the wave have two causes: (a) differences in the distances traveled by the two portions and (b) phase changes occurring on reflection. *Both* causes must be considered when you are determining which type of interference occurs.
4. When distance and phase changes on reflection are both taken into account, the interference will be constructive if the phase difference between the two waves is an integral multiple of λ, and destructive if the phase difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, and so forth.

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Example 27.2 Interference in a Soap Film

Calculate the minimum thickness of a soap bubble film ($n = 1.33$) that results in constructive interference in the reflected light if the film is illuminated with light the wavelength in free space of which is 600 nm.

Solution The minimum film thickness for constructive interference in the reflected light corresponds to $m = 0$ in Equation 27.16. This gives $2nt = \lambda/2$, or

$$t = \frac{\lambda}{4n} = \frac{600 \text{ nm}}{4(1.33)} = 113 \text{ nm}$$

EXERCISE 1 What other film thicknesses produce constructive interference? **Answer** 338 nm, 564 nm, 789 nm, and so on

Example 27.3 Nonreflecting Coatings for Solar Cells

Semiconductors such as silicon are used to fabricate solar cells—devices that generate electricity when exposed to sunlight. Solar cells are often coated with a transparent thin film, such as silicon monoxide (SiO, $n = 1.45$), in order to minimize reflective losses from the surface. A silicon solar cell ($n = 3.5$) is coated with a thin film of silicon monoxide for this purpose (Fig. 27.9). Determine the minimum film thickness that produces the least reflection at a wavelength of 550 nm, which is the center of the visible spectrum.

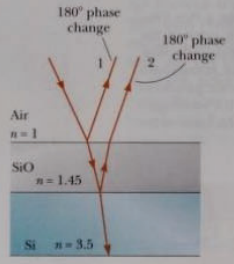


Figure 27.9 (Example 27.3) Reflective losses from a silicon solar cell are minimized by coating it with a thin film of silicon monoxide.

Reasoning The reflected light is a minimum when rays 1 and 2 in Figure 27.9 meet the condition of destructive interference. Note that both rays undergo a 180° phase change on reflection in this case, one from the upper and one from the lower surface. Hence, the net change in phase is zero due to reflection, and the condition for reflection minimum requires a path difference of $\lambda_n/2$; hence, $2t = \lambda/2n$.

Solution Because $2t = \lambda/2n$, the required thickness is

$$t = \frac{\lambda}{4n} = \frac{550 \text{ nm}}{4(1.45)} = 94.8 \text{ nm}$$

Typically, such antireflecting coatings reduce the reflective loss from 30% (with no coating) to 10% (with coating), thereby increasing the cell's efficiency, because more light is available to create charge carriers in the cell. In reality, the coating is never perfectly nonreflecting because the required thickness is wavelength-dependent and the incident light covers a wide range of wavelengths.

Glass lenses used in cameras and other optical instruments are usually coated with a transparent thin film, such as magnesium fluoride (MgF_2), to reduce or eliminate unwanted reflection, and thus such coatings enhance the transmission of light through the lenses.

Example 27.4 Interference in a Wedge-Shaped Film

A thin, wedge-shaped film of refractive index n is illuminated with monochromatic light of wavelength λ, as illustrated in Figure 27.10. Describe the interference pattern observed for this case.

Reasoning and Solution The interference pattern is that of a thin film of variable thickness surrounded by air. Hence, the pattern is a series of alternating bright and dark parallel bands. A dark band corresponding to destructive interference appears at point O, the apex, because the upper reflected ray undergoes a 180° phase change and the lower one does not. According to Equation 27.17, other dark bands appear when $2nt = m\lambda$, so that $t_1 = \lambda/2n$, $t_2 = \lambda/n$, $t_3 = 3\lambda/2n$, and so on. In a similar way, bright bands are observed when the thickness satisfies the condition $2nt = (m + \frac{1}{2})\lambda$, corresponding to thicknesses of $\lambda/4n$, $3\lambda/4n$, $5\lambda/4n$, and so on. If white light is used, bands of different colors are observed at different points, corresponding to the different wavelengths of light.

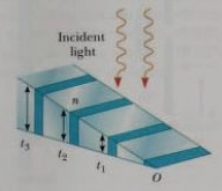


Figure 27.10 (Example 27.4) Interference bands in reflected light can be observed by illuminating a wedge-shaped film with monochromatic light. The dark blue areas correspond to positions of destructive interference.

27.5 • DIFFRACTION

Suppose a light beam is incident on two slits, as in Young's double-slit experiment. If the light truly traveled in straight-line paths after passing through the slits, as in Figure 27.11a, the waves would not overlap and no interference pattern would be seen. Instead, Huygens' principle requires that the waves spread out from the slits, as shown in Figure 27.11b. In other words, the light deviates from a straight-line path and enters the region that would otherwise be shadowed. This divergence of light from its initial line of travel is called **diffraction.**

In general, diffraction occurs when waves pass through small openings, around obstacles, or by sharp edges. For example, when a narrow slit is placed between a distant light source (or a laser beam) and a screen, the light produces a diffraction pattern like that in Figure 27.12. The pattern consists of a broad, intense central band, the **central maximum,** flanked by a series of narrower, less intense secondary

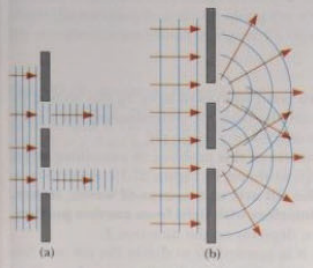


Figure 27.11 (a) If light waves did not spread out after passing through the slits, no interference would occur. (b) The light waves from the two slits overlap as they spread out, filling the expected shadowed regions with light and producing interference fringes.



Figure 27.12 The diffraction pattern that appears on a screen when light passes through a narrow vertical slit. The pattern consists of a broad central band and a series of less intense and narrower side bands.

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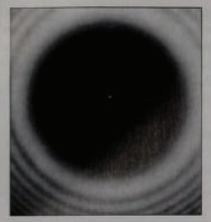


Figure 27.13 Diffraction pattern of a penny, taken with the penny midway between screen and source. *(Courtesy of P. M. Rinard, from Am. J. Phys. 44:70, 1976)*

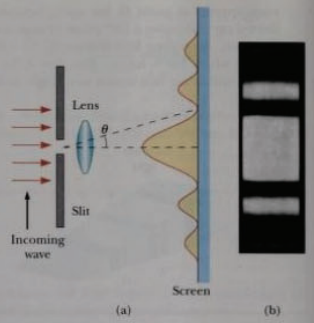


Figure 27.14 (a) Fraunhofer diffraction pattern of a single slit. The pattern consists of a central bright region flanked by much weaker maxima alternating with dark bands. (Note that this is not to scale.) (b) Photograph of a single-slit Fraunhofer diffraction pattern. *(From M. Cagnet, M. Francon, and J. C. Thierr, Atlas of Optical Phenomena, Berlin, Springer-Verlag, 1962, plate 18)*

bands (called **secondary maxima**) and a series of dark bands, or **minima.** This cannot be explained within the framework of geometric optics, which says that light rays traveling in straight lines should cast a sharp rendition of the slit on the screen.

Figure 27.13 shows the diffraction pattern and shadow of a penny. The pattern consists of the shadow, a bright spot at its center, and a series of bright and dark bands of light near the edge of the shadow. The bright spot at the center (called the *Arago bright spot* after its discoverer, Dominique Arago) can be explained through the wave theory of light, which predicts constructive interference at this point. In contrast, from the viewpoint of geometric optics, the center of the pattern would be completely screened by the penny, and so one would never observe a central bright spot.

Fraunhofer diffraction occurs when the rays reaching the observing screen are approximately parallel. This can be achieved experimentally either by placing the observing screen far from the slit or by using a converging lens to focus the parallel rays on the screen, as in Figure 27.14a. A bright fringe is observed along the axis at $\theta = 0$, with alternating dark and bright fringes on each side of the central bright fringe. Figure 27.14b is a photograph of a single-slit Fraunhofer diffraction pattern.

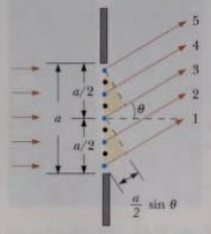


Figure 27.15 Diffraction of light by a narrow slit of width a. Each portion of the slit acts as a point source of waves. The path difference between rays 1 and 3 or between rays 2 and 4 is $(a/2)\sin\theta$. (Note that this is not to scale.)

Single-Slit Diffraction

Until now we have assumed that slits are point sources of light. In this section we shall determine how their finite widths are the basis for understanding the nature of the Fraunhofer diffraction pattern produced by a single slit.

We can deduce some important features of this problem by examining waves coming from various portions of the slit, as shown in Figure 27.15. According to Huygens' principle, **each portion of the slit acts as a source of waves. Hence, light from one portion of the slit can interfere with light from another portion,** and the resultant intensity on the screen depends on the direction θ.

To analyze the diffraction pattern, it is convenient to divide the slit into two halves, as in Figure 27.15. All the waves that originate at the slit are in phase. Consider waves 1 and 3, which originate at the bottom and center of the slit, respectively. To reach the same point on the viewing screen, wave 1 travels farther than wave 3 by an amount equal to the path difference $(a/2)\sin\theta$, where a is the width of the slit. In a similar way, the path difference between waves 3 and 5 is also $(a/2)\sin\theta$. If the path difference is exactly one half of a wavelength (corresponding to a phase difference of 180°), the two waves cancel each other and destructive interference results. This is true, in fact, for any two waves that originate at points separated by half the slit width, because the phase difference between two such points is 180°. Therefore, waves from the upper half of the slit interfere *destructively* with waves from the lower half of the slit when

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

or when

$$\sin\theta = \frac{\lambda}{a}$$

If we divide the slit into four parts rather than two and use similar reasoning, we find that the screen is also dark when

$$\sin\theta = \frac{2\lambda}{a}$$

Likewise, we can divide the slit into six parts and show that darkness occurs on the screen when

$$\sin\theta = \frac{3\lambda}{a}$$

Therefore, the general condition for **destructive interference** is

$$\sin\theta = m\frac{\lambda}{a} \qquad (m = \pm 1, \pm 2, \pm 3, \ldots) \qquad [27.18]$$

• *Condition for destructive interference*

Equation 27.18 gives the values of θ for which the diffraction pattern has zero intensity—that is, a dark fringe is formed. However, Equation 27.18 tells us nothing about the variation in intensity along the screen. The general features of the intensity distribution are shown in Figure 27.16: a broad central bright fringe flanked by

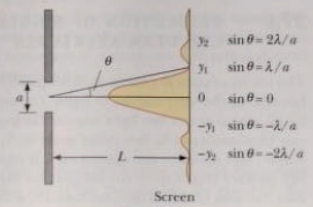


Figure 27.16 Positions of the minima for the Fraunhofer diffraction pattern of a single slit of width a. The pattern is obtained only if $L \gg a$. (Note that this is not to scale.)

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much weaker, alternating bright fringes. The various dark fringes (points of zero intensity) occur at the values of θ that satisfy Equation 27.18. The position of the points of constructive interference lie approximately halfway between the dark fringes. Note that the central bright fringe is twice as wide as the weaker maxima.

Thinking Physics 4

If a classroom door is open even just a small amount, you can hear sounds coming from the hallway. Yet you cannot see what is going on in the hallway. Why is there this difference?

Reasoning The space between the slightly open door and the wall is acting as a single slit for waves. Sound waves have wavelengths larger than the slit width, so sound is effectively diffracted by the opening and spread throughout the room. Light wavelengths are much smaller than the slit width, so there is virtually no diffraction for the light. You must have a direct line of sight view to detect the light waves.

Example 27.5 Where Are the Dark Fringes?

Light of wavelength 580 nm is incident on a slit of width 0.300 mm. The observing screen is 2.00 m from the slit. Find the positions of the first dark fringes and the width of the central bright fringe.

Solution The first dark fringes that flank the central bright fringe correspond to $m = \pm 1$ in Equation 27.18. Hence, we find that

$$\sin\theta = \pm\frac{\lambda}{a} = \pm\frac{5.80\times10^{-7}\text{ m}}{0.300\times10^{-3}\text{ m}} = \pm 1.933\times10^{-3}$$

From the triangle in Figure 27.16, note that $\tan\theta = y_1/L$. Because θ is very small, we can use the approximation $\sin\theta \approx \tan\theta$, so that $\sin\theta \approx y_1/L$. Therefore, the positions of the first minima measured from the central axis are given by

$$y_1 \approx L\sin\theta = \pm L\frac{\lambda}{a} = \pm 3.87\times10^{-3}\text{ m}$$

The positive and negative signs correspond to the dark fringes on either side of the central bright fringe. Hence, the width of the central bright fringe is equal to $2|y_1| = 7.73\times10^{-3}\text{ m} = 7.73\text{ mm}$. Note that this value is much larger than the width of the slit. However, as the width of the slit is increased, the diffraction pattern narrows, corresponding to smaller values of θ. In fact, for large values of a, the various maxima and minima are so closely spaced that the only thing observed is a large central bright area, which resembles the geometric image of the slit. This matter is of great importance in the design of lenses used in telescopes, microscopes, and other optical instruments.

EXERCISE 2 Determine the width of the first-order bright fringe. Answer 3.87 mm

27.6 • RESOLUTION OF SINGLE-SLIT AND CIRCULAR APERTURES

The ability of optical systems to distinguish between closely spaced objects is limited because of the wave nature of light. To understand this difficulty, consider Figure 27.17, which shows two light sources far from a narrow slit of width a. The sources can be considered as two point sources, S_1 and S_2, that are noncoherent. For example, they could be two distant stars. If no diffraction occurred, one would observe two distinct bright spots (or images) on the screen at the right in the figure. However, because of diffraction, each source is imaged as a bright central region flanked

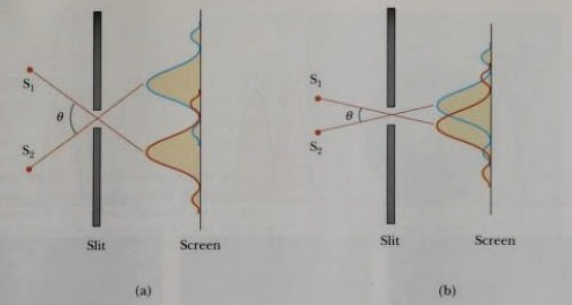


Figure 27.17 Two point sources far from a small aperture each produce a diffraction pattern. (a) The angle subtended by the sources at the aperture is large enough so that the diffraction patterns are distinguishable. (b) The angle subtended by the sources is so small that their diffraction patterns overlap and the images are not well resolved. (Note that the angles are greatly exaggerated.)

by weaker bright and dark bands. What is observed on the screen is the sum of two diffraction patterns, one from S_1 and the other from S_2.

If the two sources are far enough apart to ensure that their central maxima do not overlap, as in Figure 27.17a, their images can be distinguished and are said to be *resolved*. If the sources are close together, however, as in Figure 27.17b, the two central maxima may overlap and the images are *not resolved*. To decide when two images are resolved, the following condition is often used:

> When the central maximum of one image falls on the first minimum of another image, the images are said to be *just resolved*. This limiting condition of resolution is known as **Rayleigh's criterion.**

• *Rayleigh's criterion*

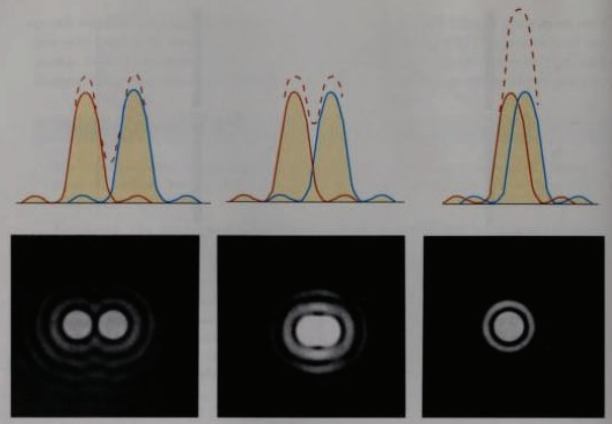
Figure 27.18 shows the diffraction patterns for three situations. When the objects are far apart, their images are well resolved (Fig. 27.18a). The images are just resolved when their angular separation satisfies Rayleigh's criterion (Fig. 27.18b). Finally, the images are not resolved in Figure 27.18c.

From Rayleigh's criterion, we can determine the minimum angular separation, θ_{min}, subtended by the sources at the slit such that their images are just resolved. In Section 27.4, we found that the first minimum in a single-slit diffraction pattern occurs at the angle that satisfies the relationship

$$\sin\theta = \frac{\lambda}{a}$$

where a is the width of the slit. According to Rayleigh's criterion, this expression gives the smallest angular separation for which the two images are resolved. Be-

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(a) (b) (c)

Figure 27.18 The diffraction patterns of two point sources (solid curves) and the resultant pattern (dashed curves), for various angular separations of the sources. In each case, the dashed curve is the sum of the two solid curves. (a) The sources are far apart, and the patterns are well resolved. (b) The sources are closer together, and the patterns are just resolved. (c) The sources are so close together that the patterns are not resolved. *(From M. Cagnet, M. Francon, and J. C. Thierr, Atlas of Optical Phenomena, Berlin, Springer-Verlag, 1962, plate 16)*

cause $\lambda \ll a$ in most situations, $\sin\theta$ is small and we can use the approximation $\sin\theta \approx \theta$. Therefore, the limiting angle of resolution for a slit of width a is

Limiting angle of resolution for a slit

$$\theta_{\min} = \frac{\lambda}{a} \qquad [27.19]$$

where $\theta_{\min}$ is expressed in radians. Hence, the angle subtended by the two sources at the slit must be *greater* than λ/a if the images are to be resolved.

Many optical systems use circular apertures rather than slits. The diffraction pattern of a circular aperture, illustrated in Figure 27.19, consists of a central circular bright disk surrounded by progressively fainter rings. Analysis shows that the limiting angle of resolution of the circular aperture is

$$\theta_{\min} = 1.22\,\frac{\lambda}{D} \qquad [27.20]$$

where D is the diameter of the aperture. Note that Equation 27.20 is similar to Equation 27.19 except for the factor of 1.22, which arises from a complex mathematical analysis of diffraction from the circular aperture.

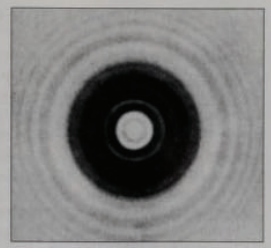


Figure 27.19 The diffraction pattern of a circular aperture consists of a central bright disk surrounded by concentric bright and dark rings. *(From M. Cagnet, M. Francon, and J. C. Thierr, Atlas of Optical Phenomena, Berlin, Springer-Verlag, 1962, plate 34)*

Thinking Physics 5

Cats' eyes have vertical pupils when in dim light. Which would cats be more successful at resolving at night—headlights on a distant car or vertically separated running lights on a distant boat's mast?

Reasoning The effective slit width in the vertical direction of the cat's eye is larger than that in the horizontal direction. Thus, it has more resolving power for lights separated in the vertical direction and would be more effective at resolving the mast lights on the boat.

CONCEPTUAL PROBLEM 5

Suppose you are observing a binary star with a telescope and are having difficulty resolving the two stars. You decide to use a colored filter to help you. Should you choose a blue filter or a red filter?

Example 27.6 Resolution of a Telescope

The Hale telescope at Mount Palomar in California has a diameter of 200 in. What is its limiting angle of resolution for 600-nm light?

Solution Because $\lambda = 6.00 \times 10^{-7}$ m and $D = 200$ in. $=$ 5.08 m, Equation 27.20 gives

$$\theta_{\min} = 1.22\,\frac{\lambda}{D} = 1.22\left(\frac{6.00 \times 10^{-7}\ \text{m}}{5.08\ \text{m}}\right)$$
$$= 1.44 \times 10^{-7}\ \text{rad} \cong 0.03\ \text{s of arc}$$

Therefore, any two stars that subtend an angle greater than or equal to this value are resolved (assuming ideal atmospheric conditions).

The Hale telescope can never reach its diffraction limit. Instead, its limiting angle of resolution is always set by atmospheric blurring. This seeing limit is usually on the order of 1 s of arc and is never smaller than 0.1 s of arc. (The Hubble Space Telescope can take higher quality photographs because it does not suffer from atmospheric limitations.)

EXERCISE 3 The large radio telescope at Arecibo, Puerto Rico, has a diameter of 305 m and is designed to detect 0.75-m radio waves. Calculate the minimum angle of resolution for this telescope and compare your answer with that of the Hale telescope. Answer 3.0×10^{-3} rad (10 min of arc), more than 10 000 times larger than the Hale minimum

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27.7 · THE DIFFRACTION GRATING

The diffraction grating, a useful device for analyzing light sources, consists of a large number of equally spaced parallel slits. A grating can be made by cutting parallel, equally spaced grooves on a glass or metal plate with a precision ruling machine. In a transmission grating, the spaces between lines are transparent to the light and hence act as separate slits. Gratings with many lines very close to each other can have very small slit spacings. For example, a diffraction grating ruled with 5000 lines/cm has a slit spacing of $d = (1/5000)\ \text{cm} = 2 \times 10^{-4}\ \text{cm}$.

Figure 27.20 is a schematic diagram of a section of a flat diffraction grating. A plane wave is incident from the left, normal to the plane of the grating. A converging lens can be used to bring the rays together at the point P. The pattern observed on the screen is the result of the combined effects of interference and diffraction. Each slit produces diffraction, and the diffracted beams interfere with each other to produce the final pattern. Moreover, each slit acts as a source of waves, with all waves starting at the slits in phase. However, for some arbitrary direction θ measured from the horizontal, the waves must travel different path lengths before reaching a particular point P on the screen. From Figure 27.20, note that the path difference between waves from any two adjacent slits is equal to $d \sin \theta$. If this path difference equals one wavelength or some integral multiple of a wavelength, waves from all slits will be in phase at P and a bright line will be observed. Therefore, when the light is incident normal to the plane of the grating, the condition for *maxima* in the interference pattern at the angle θ is

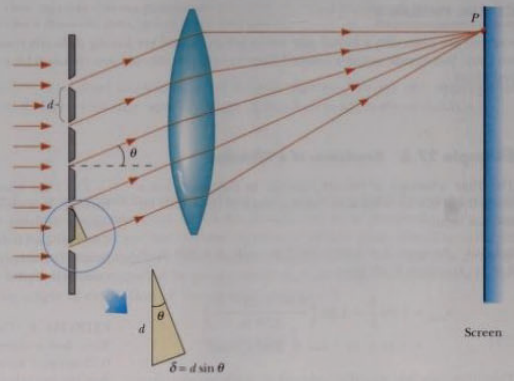


Figure 27.20 Side view of a diffraction grating. The slit separation is d, and the path difference between adjacent slits is $d \sin \theta$.

$$d \sin \theta = m\lambda \qquad (m = 0, 1, 2, 3, \ldots) \qquad [27.21]$$

This expression can be used to calculate the wavelength from a knowledge of the grating spacing and the angle of deviation, θ. If the incident radiation contains several wavelengths, the mth-order maximum for each wavelength occurs at a specific angle. All wavelengths are seen at $\theta = 0$, corresponding to $m = 0$.

The intensity distribution for a diffraction grating is shown in Figure 27.21. If the source contains various wavelengths, a spectrum of lines at different positions for different order numbers will be observed. Note the sharpness of the principal maxima and the broad range of dark areas. This is in contrast to the broad, bright fringes characteristic of the two-slit interference pattern (Fig. 27.5).

A simple arrangement for measuring angles in a diffraction pattern is shown in Figure 27.22. This is a form of a diffraction-grating spectrometer. The light to be analyzed passes through a slit, and a parallel beam of light exits from the collimator perpendicular to the grating. The diffracted light leaves the grating at angles that satisfy Equation 27.21. A telescope is used to view the image of the slit. The wavelength can be determined by measuring the precise angles at which the images of the slit appear for the various orders.

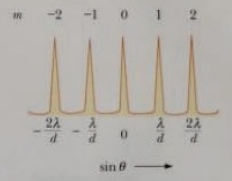


Figure 27.21 Intensity versus $\sin \theta$ for a diffraction grating. The zeroth-, first-, and second-order maxima are shown.

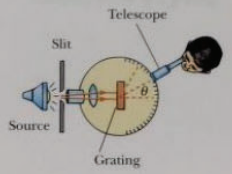


Figure 27.22 Diagram of a diffraction grating spectrometer. The collimated beam incident on the grating is diffracted into the various orders at the angles θ that satisfy the equation $d \sin \theta = m\lambda$, where $m = 1, 2, \ldots$.

Thinking Physics 6

Light reflected from the surface of a compact disc has a multicolored appearance, as shown in Figure 27.23. Furthermore, the observation depends on the orientation of the disc relative to the eye and the position of the light source. Explain how this works.

Reasoning The surface of a compact disc has a spiral grooved track (with a spacing of approximately 1 μm) that acts as a reflection grating. The light scattered by these closely spaced grooves interferes constructively only in certain directions that depend on the wavelength and on the direction of the incident light. Any one section of the disc serves as a diffraction grating for white light, sending different colors in different directions. The different colors you see when viewing one section of the disc changes as the light source, the disc, or you move to change the angles of incidence or diffraction.



Figure 27.23 (Thinking Physics 6) A compact disc acts as a diffraction grating when observed under white light. *(© Kristen Brochmann 1991, Fundamental Photographs)*

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Thinking Physics 7

White light enters through an opening in an opaque box, exits through an opening on the other side of the box, and a spectrum of colors appears on the wall. How could you determine whether the box contains a prism or a diffraction grating?

Reasoning The determination could be made by noticing the order of the colors in the spectrum relative to the direction of the original beam of white light. For a prism, in which the separation of light is a result of dispersion, the violet light will be refracted more than the red light, so the order of the spectrum will be from red, closest to the original direction, to violet. For a diffraction grating, the angle of diffraction increases with wavelength. Thus, the spectrum from the diffraction grating will have colors in the order of violet, closest to the original direction, to red.

CONCEPTUAL PROBLEM 6

If laser light is reflected from a phonograph record or a compact disc, a diffraction pattern appears. This is due to the fact that both devices contain parallel tracks of information that act as a reflection diffraction grating. Which device, record or compact disc, will result in diffraction maxima that are farther apart?

CONCEPTUAL PROBLEM 7

Astronomers often observe *occultations*, in which a star passes behind another object such as the Moon. When such an event occurs, it is found that the intensity of light from the star does not suddenly diminish as it passes behind the edge of the Moon. Instead, the intensity *fluctuates* for a short time before dropping to zero. Why would this happen?

CONCEPTUAL PROBLEM 8

When you receive a chest x-ray at a hospital, the x-rays pass through a series of parallel ribs in your chest. Do the ribs act as a diffraction grating for x-rays?

Example 27.7 The Orders of a Diffraction Grating

Monochromatic light from a helium–neon laser ($\lambda = 632.8$ nm) is incident normally on a diffraction grating containing 6000 lines/cm. Find the angles at which the first-order, second-order, and third-order maxima can be observed.

Solution First, we must calculate the slit separation, which is equal to the inverse of the number of lines per cm:

$$d = (1/6000)\ \text{cm} = 1.667 \times 10^{-4}\ \text{cm} = 1667\ \text{nm}$$

For the first-order maximum ($m = 1$), we get

$$\sin\theta_1 = \frac{\lambda}{d} = \frac{632.8\ \text{nm}}{1667\ \text{nm}} = 0.3797$$

$$\theta_1 = 22.31°$$

For $m = 2$, we find

$$\sin\theta_2 = \frac{2\lambda}{d} = \frac{2(632.8\ \text{nm})}{1667\ \text{nm}} = 0.7592$$

$$\theta_2 = 49.41°$$

For $m = 3$, we find $\sin\theta_3 = 1.139$. Because $\sin\theta$ cannot exceed unity, this does not represent a realistic solution. Hence, only zeroth-, first-, and second-order maxima are observed for this situation.

Resolving Power of the Diffraction Grating

The diffraction grating is most useful for taking accurate wavelength measurements. Like the prism, the diffraction grating can be used to disperse a spectrum into its components. Of the two devices, the grating is more precise if one wants to distinguish between two closely spaced wavelengths.

If λ_1 and λ_2 are the two nearly equal wavelengths between which the spectrometer can just barely distinguish, the **resolving power,** R, is

$$R \equiv \frac{\lambda}{\lambda_2 - \lambda_1} = \frac{\lambda}{\Delta\lambda} \qquad [27.22]$$

• *Resolving power*

where $\lambda = (\lambda_1 + \lambda_2)/2$ and $\Delta\lambda = \lambda_2 - \lambda_1$. It is left to you to show (Problem 27.52) that if N lines of the grating are illuminated, the resolving power in the mth-order diffraction equals the product Nm:

$$R = Nm \qquad [27.23]$$

• *Resolving power of a grating*

Thus, resolving power increases with increasing order number.

Furthermore, R is large for a grating with a large number of illuminated slits. Note that for $m = 0$, $R = 0$, which signifies that **all wavelengths are indistinguishable for the zeroth-order maximum.** However, consider the second-order diffraction pattern ($m = 2$) of a grating that has 5000 rulings illuminated by the light source. The resolving power of such a grating in second order is $R = 5\,000 \times 2 = 10\,000$. Therefore, the *minimum* wavelength separation between two spectral lines that can be just resolved, assuming a mean wavelength of 600 nm, is $\Delta\lambda = \lambda/R = 6 \times 10^{-2}$ nm. For the third-order principal maximum, we find that $R = 15\,000$ and $\Delta\lambda = 4 \times 10^{-2}$ nm, and so on.

Example 27.8 Resolving the Sodium Spectral Lines

Two strong lines in the spectrum of sodium have wavelengths of 589.00 nm and 589.59 nm. (a) What must the resolving power of a grating be in order to distinguish these wavelengths?

Solution

$$R = \frac{\lambda}{\Delta\lambda} = \frac{589.30\ \text{nm}}{589.59\ \text{nm} - 589.00\ \text{nm}} = \frac{589.30}{0.59} = 999$$

(b) In order to resolve these lines in the second-order spectrum ($m = 2$), how many lines of grating must be illuminated?

Solution From Equation 27.23 and the results to part (a), we find

$$N = \frac{R}{m} = \frac{999}{2} = 500\ \text{lines}$$

27.8 • DIFFRACTION OF X-RAYS BY CRYSTALS

OPTIONAL

In principle, the wavelength of any electromagnetic wave can be determined if a grating of the proper spacing (on the order of λ) is available. **X-rays,** discovered in 1895 by W. Roentgen (1845–1923), are electromagnetic waves with very short wavelengths (on the order of 10^{-10} m = 0.1 nm). Obviously, it would be impossible to construct a grating with such a small spacing. However, the atomic spacing in a solid is known to be about 10^{-10} m. In 1913, Max von Laue (1879–1960) suggested

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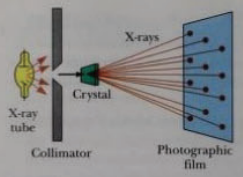


Figure 27.24 Schematic diagram of the technique used to observe the diffraction of x-rays by a crystal. The array of spots formed on the film is called a Laue pattern.

that the regular array of atoms in a crystal could act as a three-dimensional diffraction grating for x-rays. Subsequent experiments confirmed his prediction. The diffraction patterns that are observed are complicated because of the three-dimensional nature of the crystal. Nevertheless, x-ray diffraction is an invaluable technique for elucidating crystalline structures and for understanding the structure of matter.

Figure 27.24 is one experimental arrangement for observing x-ray diffraction from a crystal. A collimated beam of x-rays with a continuous range of wavelengths is incident on a crystal. The diffracted beams are very intense in certain directions, corresponding to constructive interference from waves reflected from layers of atoms in the crystal. The diffracted beams can be detected by a photographic film, and they form an array of spots known as a Laue pattern. The crystalline structure is deduced by analyzing the positions and intensities of the various spots in the pattern.

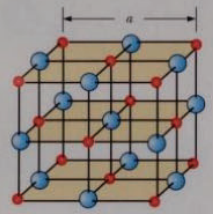


Figure 27.25 A model of the crystalline structure of sodium chloride. The blue spheres represent the Cl^- ions, and the red spheres represent the Na^+ ions. The length of the cube edge is $a = 0.562737$ nm.

The arrangement of atoms in a crystal of NaCl is shown in Figure 27.25. The red spheres represent Na^+ ions, and the blue spheres represent Cl^- ions. Each unit cell (the set of atoms that repeats through the crystal) contains four Na^+ and four Cl^- ions. The unit cell is a cube the edge length of which is a.

The ions in a crystal lie in various planes, as shown by the shaded areas in Figure 27.25. Now suppose an incident x-ray beam makes an angle of θ with one of the planes, as in Figure 27.26. The beam can be reflected from both the upper plane and the lower one. However, the geometric construction in Figure 27.26 shows that the beam reflected from the lower surface travels farther than the beam reflected from the upper surface. The path difference between the two beams is $2d \sin \theta$. The two beams will reinforce each other (constructive interference) when this path difference equals some integral multiple of the wavelength λ. The same

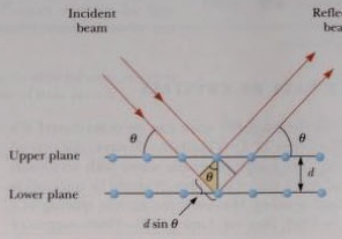


Figure 27.26 A two-dimensional description of the reflection of an x-ray beam from two parallel crystalline planes separated by a distance d. The beam reflected from the lower plane travels farther than the one reflected from the upper plane by a distance equal to $2d \sin \theta$.

is true of reflection from the entire family of parallel planes. Hence, the condition for constructive interference (maxima in the reflected wave) is

$$2d \sin \theta = m\lambda \qquad (m = 1, 2, 3, \ldots) \qquad [27.24]$$

• *Bragg's law*

This condition is known as **Bragg's law** after W. L. Bragg (1890–1971), who first derived the relationship. If the wavelength and diffraction angle are measured, Equation 27.24 can be used to calculate the spacing between atomic planes.

SUMMARY

Interference of light waves is the result of the linear superposition of two or more waves at a given point. A sustained interference pattern is observed if (1) the sources are coherent, (2) the sources have identical wavelengths, and (3) the superposition principle is applicable.

In Young's double-slit experiment, two slits separated by a distance of d are illuminated by a monochromatic light source. An interference pattern consisting of bright and dark fringes is observed on a screen that is a distance of L from the slits. The condition for **constructive interference** is

$$d \sin \theta = m\lambda \qquad (m = 0, \pm 1, \pm 2, \ldots) \qquad [27.2]$$

The condition for **destructive interference** is

$$d \sin \theta = (m + \tfrac{1}{2})\lambda \qquad (m = 0, \pm 1, \pm 2, \ldots) \qquad [27.3]$$

The number m is called the **order number** of the fringe.

The **average intensity** of the double-slit interference pattern is

$$I_{av} = I_0 \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right) \qquad [27.12]$$

where I_0 is the maximum intensity on the screen.

An electromagnetic wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change on reflection when $n_2 > n_1$. There is no phase change in the reflected wave if $n_2 < n_1$.

In a medium of refractive index n, the wavelength of light is

$$\lambda_n = \frac{\lambda}{n} \qquad [27.14]$$

where λ is the wavelength of light in free space.

The condition for constructive interference in a film of thickness t and refractive index n with the same medium on both sides of the film is given by

$$2nt = (m + \tfrac{1}{2})\lambda \qquad (m = 0, 1, 2, \ldots) \qquad [27.16]$$

In a similar way, the condition for destructive interference is

$$2nt = m\lambda \qquad (m = 0, 1, 2, \ldots) \qquad [27.17]$$

Diffraction, defined as the deviation of light from a straight-line path when the light passes through an aperture or around obstacles, arises from the interference of a large number or continuous distribution of coherent sources.

The **Fraunhofer diffraction pattern** produced by a *single slit* of width a on a distant screen consists of a central, bright maximum and alternating bright and dark regions of

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much lower intensities. The angles θ at which the diffraction pattern has *zero* intensity are given by

$$\sin\theta = m\frac{\lambda}{a} \qquad (m = \pm 1, \pm 2, \pm 3, \ldots) \qquad [27.18]$$

Rayleigh's criterion, which is a limiting condition of resolution, says that two images formed by an aperture are just distinguishable if the central maximum of the diffraction pattern for one image falls on the first minimum of the other image. The limiting angle of resolution for a slit of width a is given by $\theta_{min} = \lambda/a$, and the limiting angle of resolution for a circular aperture of diameter D is given by $\theta_{min} = 1.22\lambda/D$.

A **diffraction grating** consists of a large number of equally spaced, identical slits. The condition for intensity maxima in the interference pattern of a diffraction grating for normal incidence is

$$d\sin\theta = m\lambda \qquad (m = 0, 1, 2, 3, \ldots) \qquad [27.21]$$

where d is the spacing between adjacent slits and m is the order number of the diffraction pattern. The resolving power of a diffraction grating in the mth order of the diffraction pattern is $R = Nm$, where N is the number of rulings in the grating.

CONCEPTUAL QUESTIONS

1. What is the necessary condition on the path-length difference between two waves that interfere (a) constructively and (b) destructively?
2. Explain why two flashlights held close together do not produce an interference pattern on a distant screen.
3. If Young's double-slit experiment were performed under water, how would the observed interference pattern be affected?
4. In Young's double-slit experiment, why do we use monochromatic light? If white light were used, how would the pattern change?
5. As a soap bubble evaporates, it appears black just before it breaks. Explain this phenomenon in terms of the phase changes that occur on reflection from the two surfaces of the soap film.
6. An oil film on water appears brightest at the outer regions, where it is thinnest. From this information, what can you say about the index of refraction of oil relative to that of water?
7. A soap film on a wire loop and held in air appears dark in the thinnest regions when observed by reflected light and shows a variety of colors in thicker regions, as in Figure Q27.7. Explain.
8. A simple way of observing an interference pattern is to look at a distant light source through a stretched handkerchief or an opened umbrella. Explain how this works.
9. In order to observe interference in a thin film, why must the film not be very thick (on the order of a few wavelengths)?

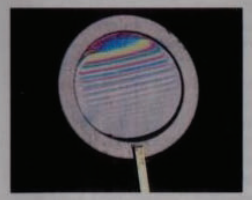


Figure Q27.7

10. A lens with outer radius of curvature R and index of refraction n rests on a flat glass plate and the combination is illuminated with white light from above. Is there a dark spot or a light spot at the center of the lens? What does it mean if the observed rings are noncircular?
11. Why is the lens on a good-quality camera coated with a thin film?
12. Would it be possible to place a nonreflective coating on an airplane to cancel radar waves of wavelength 3 cm?
13. Why is it so much easier to perform interference experiments with a laser than with an ordinary light source?
14. Observe the shadow of your book when it is held a few inches above a table with a lamp several feet above the book. Why is the shadow somewhat fuzzy at the edges?

15. Although we can hear around corners, we cannot see around corners. How can you explain this in view of the fact that sound and light are both waves?
16. Assuming that the headlights of a car are point sources, estimate the maximum distance from an observer to the car at which the headlights are distinguishable from each other.
17. A laser beam is incident at a shallow angle on a machinist's ruler that has a finely calibrated scale. The rulings on the scale give rise to a diffraction pattern on a screen. Discuss how you can use this technique to obtain a measure of the wavelength of the laser light.
18. If a coin is glued to a glass sheet and this arrangement is held in front of a laser beam, the projected shadow has diffraction rings around its edge and a bright spot in the center. How is this possible?
19. If a fine wire is stretched across the path of a laser beam, is it possible to produce a diffraction pattern?
20. Describe the change in width of the central maximum of the single-slit diffraction pattern as the width of the slit is made smaller.
21. Suppose that reflected white light is used to observe a thin, transparent coating on glass as the coating material is gradually deposited by evaporation in a vacuum. Describe possible color changes that might occur during the process of building up the thickness of the coating.
22. The brilliant colors of peacock feathers (Fig. Q27.22) are due to a phenomenon known as *iridescence*. The melanin fibers in the feathers act as a natural diffraction grating. How do you explain the different colors? Why do the colors often change as the bird moves?



Figure Q27.22 Iridescence in peacock feathers. *(© Diane Schiumo 1988, Fundamental Photographs)*

PROBLEMS

Section 27.2 Young's Double-Slit Experiment

1. A laser beam (λ = 632.8 nm) is incident on two slits 0.200 mm apart. How far apart are the bright interference lines on a screen 5.00 m away from the double slits?
2. A Young's interference experiment is performed with monochromatic light. The separation between the slits is 0.500 mm and the interference pattern on a screen 3.30 m away shows the first maximum 3.40 mm from the center of the pattern. What is the wavelength?
3. Two radio antennas separated by 300 m, as in Figure P27.3, simultaneously broadcast identical signals at the same wavelength. A radio in a car traveling due north receives the signals. (a) If the car is at the position of the second maximum, what is the wavelength of the signals? (b) How much farther must the car travel to encounter the next minimum in reception? (*Caution*: Do not use the small-angle approximation in this problem.)
4. In a location where the speed of sound is 354 m/s, a 2000 Hz sound wave impinges on two slits 30.0 cm apart. (a) At what angle is the first maximum located? (b) If the sound wave is replaced by 3.00-cm microwaves, what slit separation gives the same angle for the first maximum? (c) If the slit separation is 1.00 μm, what frequency light gives the same first maximum angle?

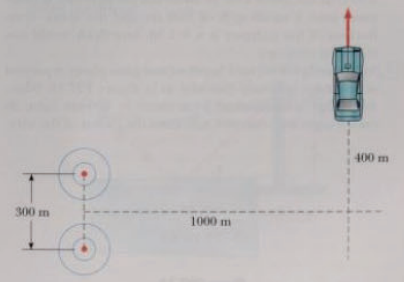


Figure P27.3

5. Young's double-slit experiment is performed with 589-nm light and a slits-to-screen distance of 2.00 m. The tenth interference minimum is observed 7.26 mm from the central maximum. Determine the spacing of the slits.
6. An oscillator drives two loudspeakers 35.0 cm apart, which vibrate in phase at a frequency of 2.00 kHz. At what angles,

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measured from the perpendicular bisector of the line joining the speakers, would a distant observer hear maximum sound intensity? Minimum? (Take the speed of sound as 340 m/s.)

7. In Figure 27.3, let $L = 120$ cm and $d = 0.250$ cm. The slits are illuminated with coherent 600-nm light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75.0% of the maximum.

8. The intensity on the screen at a certain point in a double-slit interference pattern is 64.0% of the maximum value. (a) What minimum phase difference (in radians) between sources produces this result? (b) Express this phase difference as a path difference for 486.1-nm light.

Section 27.4 Interference in Thin Films

9. A soap bubble ($n = 1.33$) is floating in air. If the thickness of the bubble wall is 115 nm, what is the wavelength of the light that is most strongly reflected?

10. A thin film of oil ($n = 1.25$) is located on a smooth wet pavement. When viewed perpendicular to the pavement, the film appears to be predominantly red (640 nm) and has no blue color (512 nm). How thick is the oil film?

11. An oil film ($n = 1.45$) floating on water is illuminated by white light at normal incidence. The film is 280 nm thick. Find (a) the dominant observed color in the reflected light and (b) the dominant color in the transmitted light. Explain your reasoning.

12. A possible means for making an airplane invisible to radar is to coat the plane with an antireflective polymer. If radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $n = 1.50$, how thick would you make the coating?

13. An air wedge is formed between two glass plates separated at one edge by a very fine wire as in Figure P27.13. When the wedge is illuminated from above by 600-nm light, 30 dark fringes are observed. Calculate the radius of the wire.

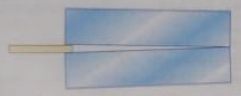


Figure P27.13

Section 27.5 Diffraction

14. Helium–neon laser light ($\lambda = 632.8$ nm) is sent through a 0.300-mm-wide single slit. What is the width of the central maximum on a screen 1.00 m from the slit?

15. A screen is placed 50.0 cm from a single slit, which is illuminated with 690-nm light. If the distance between the first and third minima in the diffraction pattern is 3.00 mm, what is the width of the slit?

16. A beam of green light is diffracted by a slit of width 0.550 mm. The diffraction pattern forms on a wall 2.06 m beyond the slit. The distance between the positions of zero intensity on both sides of the central bright fringe is 4.10 mm. Calculate the wavelength of the laser light.

17. Coherent microwaves of wavelength 5.00 cm enter a long, narrow window in a building otherwise essentially opaque to the microwaves. If the window is 36.0 cm wide, what is the distance from the central maximum to the first-order minimum along a wall 6.50 m from the window?

18. Sound of frequency 650 Hz from a distant source passes through a doorway 1.10 m wide in a sound-absorbing wall. Find the number and approximate directions of the diffraction-maximum beams radiated into the space beyond.

Section 27.6 Resolution of Single-Slit and Circular Apertures

19. The pupil of a cat's eye narrows to a slit of width 0.500 mm in daylight. What is the angular resolution? (Use 500-nm light in your calculation.)

20. Find the radius of a star image formed on the retina of the eye if the aperture diameter (the pupil) at night is 0.700 cm and the length of the eye is 3.00 cm. Assume the wavelength of starlight in the eye is 500 nm.

21. A helium–neon laser emits light that has a wavelength of 632.8 nm. The circular aperture through which the beam emerges has a diameter of 0.500 cm. Estimate the diameter of the beam 10.0 km from the laser.

22. On the night of April 18, 1775, a signal was to be sent from the Old North Church steeple to Paul Revere, who was 1.80 miles away: "One if by land, two if by sea." At what minimum separation did the sexton have to set the lanterns so that Revere could receive the correct message? Assume that Revere's pupils had a diameter of 4.00 mm at night, and that the lantern light had a predominant wavelength of 580 nm.

23. A binary star system in the constellation Orion has an angular separation between the two stars of 1.00×10^{-5} rad. If $\lambda = 500$ nm, what is the smallest diameter the telescope can have and just resolve the two stars?

24. The Impressionist painter Georges Seurat created paintings with an enormous number of dots of pure pigment approximately 2.00 mm in diameter. The idea was to have colors such as red and green next to each other to form a scintillating canvas. Outside what distance would one be unable to discern individual dots on the canvas? (Assume $\lambda = 500$ nm and a pupil diameter of 4.00 mm.)

25. A circular radar antenna on a navy ship has a diameter of 2.10 m and radiates at a frequency of 15.0 GHz. Two small boats are located 9.00 km away from the ship. How close together could the boats be and still be detected as two objects?

Section 27.7 The Diffraction Grating

In the following problems, assume that the light is incident normally on the gratings.

26. Light from an argon laser strikes a diffraction grating that has 5310 lines per centimeter. The central and first-order principal maxima are separated by 0.488 m on a wall 1.72 m from the grating. Determine the wavelength of the laser light.

27. The hydrogen spectrum has a red line at 656 nm and a violet line at 434 nm. What is the angular separation between these two spectral lines obtained with a diffraction grating that has 4500 lines/cm?

28. A helium–neon laser ($\lambda = 632.8$ nm) is used to calibrate a diffraction grating. If the first-order maximum occurs at 20.5°, what is the groove spacing, d?

29. Three discrete spectral lines occur at angles of 10.09°, 13.71°, and 14.77° in the first-order spectrum of a grating spectroscope. (a) If the grating has 3660 slits/cm, what are the wavelengths of the light? (b) At what angles are these lines found in the second-order spectra?

30. A diffraction grating has 800 rulings per millimeter. A beam of light containing wavelengths from 500 to 700 nm hits the grating. Do the spectra of different orders overlap? Explain.

31. A diffraction grating of length 4.00 cm contains 6000 rulings over a width of 2.00 cm. (a) What is the resolving power of this grating in the first three orders? (b) If two monochromatic waves incident on this grating have a mean wavelength of 400 nm, what is their wavelength separation if they are just resolved in the third order?

32. A source emits 531.62-nm and 531.81-nm light. (a) What minimum number of lines is required for a grating that resolves the two wavelengths in the first-order spectrum? (b) Determine the slit spacing for a grating 1.32 cm wide that has the required minimum number of lines.

Section 27.8 Diffraction of X-Rays by Crystals (Optional)

33. Potassium iodide (KI) has a crystalline structure with an interplanar distance of 0.353 nm. A monochromatic x-ray beam shows a diffraction maximum when the grazing angle is 7.60°. Calculate the x-ray wavelength. (Assume first-order.)

34. A wavelength of 0.129 nm characterizes K_β x-rays from zinc. When a beam of these x-rays is incident on the surface of a crystal the structure of which is similar to that of NaCl, a first-order maximum is observed at 8.15°. Calculate the interplanar spacing based on this information.

35. If the interplanar spacing of NaCl is 0.281 nm, what is the predicted angle at which 0.140-nm x-rays are diffracted in a first-order maximum?

36. The first-order diffraction is observed at 12.6° for a crystal in which the interplanar spacing is 0.240 nm. How many other orders can be observed?

Additional Problems

37. Raise your hand and hold it flat. Think of the space between your index finger and your middle finger as one slit, and think of the space between middle finger and ring finger as a second slit. (a) Consider the interference resulting from sending coherent visible light perpendicularly through this pair of openings. Compute an order-of-magnitude estimate for the angle between adjacent zones of constructive interference. (b) To make the angles in the interference pattern easy to measure with a plastic protractor, you should use an electromagnetic wave with frequency of what order of magnitude? How is this wave classified on the electromagnetic spectrum?

38. A certain crude oil has an index of refraction $n = 1.25$. A ship dumps 1.00 m^3 of this oil into the ocean, and the oil spreads into a thin slick. If the film produces a first-order maximum of light of wavelength 500 nm normally incident on it, how much surface area of the ocean does the oil slick cover? Assume the index of refraction of the ocean water is 1.34.

39. Interference effects are produced at point P on a screen as a result of direct rays from a 500-nm source and reflected rays off the mirror, as in Figure P27.39. If the source is 100 m to the left of the screen, and 1.00 cm above the mirror, find the distance y (in millimeters) to the first dark band above the mirror.

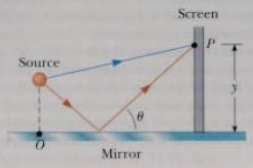


Figure P27.39

40. The waves from a radio station can reach a home receiver by two paths. One is a straight-line path from transmitter to home, a distance of 30.0 km. The second path is by reflection from the ionosphere (a layer of ionized air molecules near the top of the atmosphere). Assume this reflection takes place at a point midway between receiver and transmitter. If the wavelength broadcast by the radio station is 350 m, find the minimum height of the ionospheric layer that produces destructive interference between the direct

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and reflected beams. (Assume no phase changes on reflection.)

41. Astronomers observed a 60.0-MHz radio source both directly and by reflection from the sea. If the receiving dish is 20.0 m above sea level, what is the angle of the radio source above the horizon at first maximum?

42. Measurements are made of the intensity distribution in a Young's interference pattern (Fig. 27.5). At a particular value of y, it is found that $I/I_0 = 0.810$ when 600-nm light is used. What wavelength of light should be used to reduce the relative intensity at the same location to 64.0%?

43. In a Young's interference experiment, the two slits are separated by 0.150 mm and the incident light includes light of wavelengths $\lambda_1 = 540$ nm and $\lambda_2 = 450$ nm. The overlapping interference patterns are formed on a screen 1.40 m from the slits. Calculate the minimum distance from the center of the screen to the point at which a bright line of the λ_1 light coincides with a bright line of the λ_2 light.

44. An air wedge is formed between two glass plates in contact along one edge and slightly separated at the opposite edge. When the plates are illuminated with monochromatic light from above, the reflected light has 85 dark fringes. Calculate the number of dark fringes that would appear if water ($n = 1.33$) were to replace the air between the plates.

45. Our discussion of the techniques for determining constructive and destructive interference by reflection from a thin film in air has been confined to rays striking the film at nearly normal incidence. Assume that a ray is incident at an angle of 30.0° (relative to the normal) on a film with index of refraction 1.38. Calculate the minimum thickness for constructive interference if the light is sodium light with a wavelength of 590 nm.

46. Light from a helium–neon laser ($\lambda = 632.8$ nm) is incident on a single slit. What is the maximum width for which no diffraction minima are observed?

47. (a) Both sides of a uniform film that has index of refraction n and thickness d are in contact with air. For normal incidence of light, an intensity minimum is observed in the reflected light at λ_2 and an intensity maximum is observed at λ_1, where $\lambda_1 > \lambda_2$. If there are no intensity minima observed between λ_1 and λ_2, show that the integer m in Equations 27.16 and 27.17 is given by $m = \lambda_1/2(\lambda_1 - \lambda_2)$. (b) Determine the thickness of the film if $n = 1.40$, $\lambda_1 = 500$ nm, and $\lambda_2 = 370$ nm.

48. What are the approximate dimensions of the smallest object on Earth that astronauts can resolve by eye when they are orbiting 250 km above the Earth? Assume $\lambda = 500$ nm light and a pupil diameter of 5.00 mm.

49. A beam of 541-nm light is incident on a diffraction grating that has 400 lines/mm. (a) Determine the angle of the second-order ray. (b) If the entire apparatus is immersed in water, determine the new second-order angle of diffraction. (c) Show that the two diffracted rays of parts (a) and (b) are related through the law of refraction.

50. Consider the double-slit arrangement shown in Figure P27.50, where the separation d is 0.300 mm and the distance L is 1.00 m. A sheet of transparent plastic ($n = 1.50$) 0.0500 mm thick (about the thickness of this page) is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find this distance.

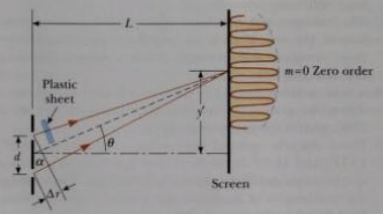


Figure P27.50

51. Consider the double-slit arrangement shown in Figure P27.50, where the slit separation is d and the slit-to-screen distance is L. A sheet of transparent plastic having an index of refraction n and thickness t is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find y'.

52. Derive Equation 27.23 for the resolving power of a grating, $R = Nm$, where N is the number of lines illuminated and m is the order in the diffraction pattern. Remember that Rayleigh's criterion (Section 27.6) states that two wavelengths will be resolved when the principal maximum for one falls on the first minimum for the other.

Spreadsheet Problems

S1. To calculate the intensity distribution of the interference pattern for equally spaced sources, it is necessary to add a series of terms such as: $E_1 = A_0 \sin \alpha$, $E_2 = A_0 \sin(\alpha + \phi)$, $E_3 = A_0 \sin(\alpha + 2\phi)$, where ϕ is the phase difference caused by different path lengths. For N sources the time-averaged relative intensity as a function of the phase angle ϕ is

$$I_{rel} = f_N^2(\phi) + g_N^2(\phi)$$

where

$$f_N(\phi) = \sum_{n=0}^{N-1} \cos(n\phi) \qquad g_N(\phi) = \sum_{n=0}^{N-1} \sin(n\phi)$$

and I_{rel} is the ratio of the average intensity for N sources to that of one source. Spreadsheet 27.1 calculates and plots I_{rel} versus phase angle ϕ for up to a maximum of six sources. The coefficients a, b, c, d, e, and f used in the spreadsheet determine the number of sources. For two sources, for example, set $a = 1$, $b = 1$, and $c = d = e = f = 0$. For three sources set $a = b = c = 1$, $d = e = f = 0$, and so on. (a) For three sources, what is the ratio of the intensity of the principal maxima to that of a single source? (b) What is the ratio of the intensity of the secondary maxima to that of the principal maximum?

S2. Use Spreadsheet 27.1 to calculate the intensity pattern for four equally spaced sources. (a) What is the ratio of the intensity of the principal maxima to that of a single source? (b) What is the ratio of the intensity of the secondary maxima to that of the principal maxima? (c) Repeat parts (a) and (b) for five and six sources. (d) Are all the secondary maxima of equal intensity?

S3. Figure S27.3 shows the relative intensity of a single-slit Fraunhofer diffraction pattern as a function of the parameter $\beta/2 = \pi a \sin \theta/\lambda$. Spreadsheet 27.2 plots the relative intensity I/I_0 as a function of θ, where θ is defined in Figure 27.16. Examine the cases where $\lambda = a$, $\lambda = 0.5a$, $\lambda = 0.1a$, and $\lambda = 0.05a$, and explain your results.

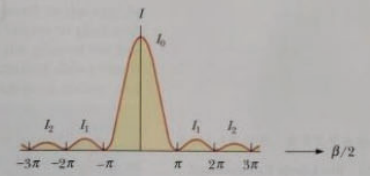


Figure S27.3 A plot of the intensity I versus $\beta/2$ for the single-slit Fraunhofer diffraction pattern.

ANSWERS TO CONCEPTUAL PROBLEMS

1. The result of the double slit is to *redistribute* the energy arriving at the screen. Although there is no energy at the location of a dark fringe, there is more energy at the location of a bright fringe than there would be without the double slit. The total amount of energy arriving at the screen is the same as without the slit, as it must be according to the conservation of energy principle.
2. One of the materials has a higher index of refraction than water, the other lower. The material with a higher index of refraction than water will appear black as it approaches zero thickness. There will be a 180° phase shift for the light reflected from the upper surface, but no such phase change from the lower surface, because the index of refraction for water on the other side is lower than that of the film. Thus, the two reflections will be out of phase and will interfere destructively. The material with index of refraction lower than water will have a phase change for the light reflected from both upper and lower surfaces, so that the reflections from the zero-thickness film will be back in phase, and the film will appear bright.
3. There will be an effect on the interference pattern—it will be distorted. The high temperature of the flame will change the index of refraction of the air for the arm of the interferometer in which the match is held. As the index of refraction varies turbulently, the wavelength of the light in that region will also vary turbulently. As a result, the effective optical path length difference between the two arms will vary, resulting in a wildly varying interference pattern.
4. For normal incidence, the extra path length followed by the reflected ray is twice the thickness of the film. For destructive interference, this must be a distance of half a wavelength of the light in the material of the film. Since no 180° phase change will occur in these reflections, the thickness of the film must be one quarter wavelength, which is the same as that for reflected destructive interference.
5. We would like to reduce the minimum angular separation for two objects below the angle subtended by the two stars in the binary system. We can do that by reducing the wavelength of the light—this in essence makes the aperture larger, relative to the light wavelength, increasing the resolving power. Thus, we would choose a blue filter.
6. The tracks of information on a compact disc are much closer together than on a phonograph record. As a result, the diffraction maxima from the compact disc will be farther apart than those from the record.
7. As the edge of the Moon cuts across the light from the star, edge diffraction effects occur. Thus, as the edge of the Moon moves relative to the star, the observed light from the star proceeds through a series of maxima and minima.
8. Strictly speaking, the ribs do act as a diffraction grating, but the separation distance of the ribs is so much larger than the wavelength of the x-rays that there are no observable effects.

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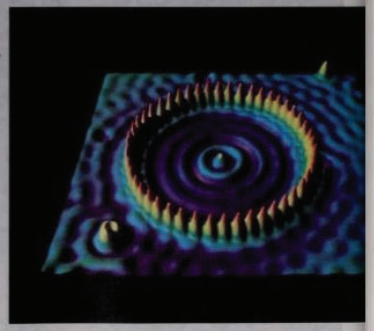
Quantum Physics

CHAPTER OUTLINE

28.1 Black-Body Radiation and Planck's Theory
28.2 The Photoelectric Effect
28.3 The Compton Effect
28.4 Photons and Electromagnetic Waves
28.5 The Wave Properties of Particles
28.6 The Double-Slit Experiment Revisited
28.7 The Uncertainty Principle
28.8 An Interpretation of Quantum Mechanics
28.9 A Particle in a Box
28.10 The Schrödinger Equation
28.11 Tunneling Through a Barrier (Optional)

Although many problems were resolved by Einstein's special theory of relativity in the early part of the 20th century, many other problems remained unanswered. Attempts to apply the laws of classical physics to the behavior of matter on the atomic scale were consistently unsuccessful. Various phenomena, such as black-body radiation, the photoelectric effect, and the emission of sharp spectral lines by atoms in a gas discharge, could not be understood within the framework of classical physics. Between 1900 and 1930, however, a modern version of mechanics called *quantum mechanics* was highly successful in explaining the behavior of atoms, molecules, and nuclei. Moreover, the quantum theory reduces to classical physics when applied to macroscopic systems. Like relativity, the quantum theory requires a modification of our ideas concerning the physical world.

An extensive study of quantum theory is certainly beyond the scope of this book, and therefore this chapter is simply an introduction to its underlying ideas. We shall also discuss two simple applications of quantum theory: the photoelectric effect and the Compton effect.



This photograph is a demonstration of a "quantum-corral" consisting of a ring of 48 iron atoms located on a copper surface. The diameter of the ring is 143 nm, and the photograph was obtained using a low-temperature scanning tunneling microscope (STM). Corrals and other structures are able to confine surface-electron waves. The study of such structures will play an important role in determining the future of small electronic devices. *(IBM Corporation Research Division)*

28.1 • BLACK-BODY RADIATION AND PLANCK'S THEORY

An object at any temperature emits radiation that is sometimes referred to as **thermal radiation.** The characteristics of this radiation depend on the temperature and properties of the object. At low temperatures, the wavelengths of the thermal radiation are mainly in the infrared region and hence not observed by the eye. As the temperature of the object increases, the object eventually begins to glow red. At sufficiently high temperatures, it appears to be white, as in the glow of the hot tungsten filament of a lightbulb. A careful study of thermal radiation shows that it consists of a continuous distribution of wavelengths from the infrared, visible, and ultraviolet portions of the spectrum.

From a classical viewpoint, thermal radiation originates from accelerated charged particles near the surface of the object; those charges emit radiation much as small antennas do. The thermally agitated charges can have a distribution of accelerations, which accounts for the continuous spectrum of radiation emitted by the object. By the end of the 19th century, it had become apparent that this classical explanation of thermal radiation was inadequate. The basic problem was in understanding the observed distribution of wavelengths in the radiation emitted by a black body. As we learned in Chapter 17, Section 17.7, a black body is an ideal system that absorbs all radiation incident on it. A good approximation of a black body is a small hole leading to the inside of a hollow object, as shown in Figure 28.1. The nature of the radiation emitted through the small hole leading to the cavity depends only on the temperature of the cavity walls. Note that the hole itself is the black body.

Experimental data for the distribution of energy in black-body radiation at three temperatures are shown in Figure 28.2. The radiated energy varies with wavelength and temperature. As the temperature of the black body increases, the total

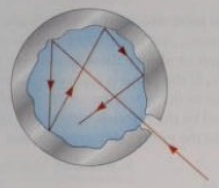


Figure 28.1 A small hole that leads to a cavity with rough walls is a good approximation of a black body. Light entering the hole strikes the far wall, where some of it is absorbed but some is reflected at some random angle. The light continues to be reflected, and each time it is, part of it is absorbed by the cavity walls. After many reflections, essentially all of the incident energy is absorbed.

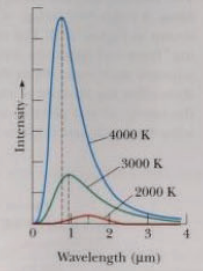


Figure 28.2 Intensity of black-body radiation versus wavelength at three temperatures. Note that the amount of radiation emitted (the area under a curve) increases with increasing temperature.

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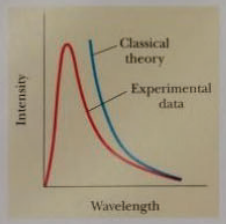


Figure 28.3 Comparison of the experimental results with the curve predicted by classical theory for the distribution of black-body radiation.



Max Planck (1858–1947)

Planck introduced the concept of a "quantum of action" (Planck's constant, h) in an attempt to explain the spectral distribution of black-body radiation, which laid the foundations for quantum theory. In 1918 he was awarded the Nobel Prize for this discovery of the quantized nature of energy. The work leading to the "lucky" black-body radiation formula was described by Planck in his Nobel Prize acceptance speech (1920): "But even if the radiation formula proved to be perfectly correct, it would after all have been only an interpolation formula found by lucky guesswork and, thus, would have left us rather unsatisfied." *(Photo courtesy of AIP Niels Bohr Library, W. F. Meggers Collection)*

amount of energy it emits increases. Also, with increasing temperatures, the peak of the distribution shifts to shorter wavelengths. This shift was found to obey the following relationship, called **Wien's displacement law:**

$$\lambda_{max} T = 0.2898 \times 10^{-2}\ \text{m}\cdot\text{K} \qquad [28.1]$$

where λ_{max} is the wavelength at which the curve peaks and T is the absolute temperature of the surface of the object emitting the radiation.

Early attempts to use classical ideas to explain the shapes of the curves in Figure 28.2 failed. Figure 28.3 shows an experimental plot of the black-body radiation spectrum (red) together with the curve predicted by classical theory (blue). At long wavelengths, classical theory is in good agreement with the experimental data. At short wavelengths, however, major disagreement exists between classical theory and experiment. This contradiction is often called the **ultraviolet catastrophe.**

In 1900 Planck developed a formula for black-body radiation that was in complete agreement with experiment at all wavelengths. Planck's analysis led to the red curve shown in Figure 28.3. In his theory, Planck made two bold and controversial assumptions concerning the nature of the oscillating charges, which he called resonators, at the surface of the black body:

- The resonators can have only certain *discrete* amounts of energy E_n,

$$E_n = nhf \qquad [28.2]$$

where, as we learned in Chapter 10, Section 10.10, n is a positive integer called a quantum number, f is the frequency of oscillation of the resonators, and h is Planck's constant. Because the energy of each resonator can have only discrete values given by Equation 28.2, we say the energy is *quantized.* Each discrete energy value represents a different *quantum state,* with each value of n representing a specific quantum state. When the resonator is in the $n = 1$ quantum state, its energy is hf; when it is in the $n = 2$ quantum state, its energy is $2hf$; and so on.

- The resonators emit energy in discrete units which were later called **photons.** A suggestive image of several photons, not to be taken too literally, is shown in Figure 28.4. The resonators emit or absorb these quanta by "jumping" from one quantum state to another. If the jump is from one state to an adjacent state—say, from the $n = 3$ state to the $n = 2$ state—Equation 28.2 shows that the amount of energy radiated by the resonator equals hf. Hence, the energy of one photon corresponds to the energy difference between two adjacent levels and is given by

$$E = hf \qquad [28.3]$$

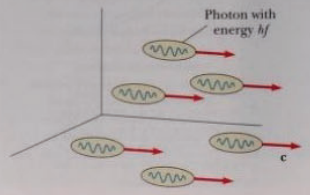


Figure 28.4 A representation of light quanta. Each photon has a discrete energy, hf.

A resonator radiates or absorbs energy only when it changes quantum states. If it remains in one quantum state, no energy is absorbed or emitted. Figure 28.5 shows the quantized energy levels and allowed transitions proposed by Planck.

The key point in Planck's theory is the radical assumption of quantized energy states. This development marked the birth of the quantum theory. When Planck presented his theory, most scientists (including Planck!) did not consider the quantum concept to be realistic. Hence, Planck and others continued to search for a more rational explanation of black-body radiation. However, subsequent developments showed that a theory based on the quantum concept (rather than on classical concepts) had to be used to explain a number of other phenomena at the atomic level.

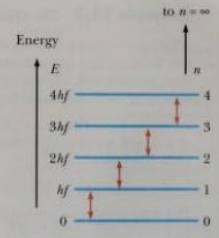


Figure 28.5 Allowed energy levels for a resonator that oscillates at a natural frequency, f. Allowed transitions are indicated by vertical arrows.

Thinking Physics 1

You are observing a yellow candle flame, and your laboratory partner claims that the light from the flame is atomic in origin. You disagree, claiming that the candle flame is hot, so the radiation must be thermal in origin. Before this disagreement leads to fisticuffs, how could you determine who is correct?

Reasoning A simple determination could be made by observing the light from the candle flame through a spectrometer, which is a slit and diffraction grating combination, and was discussed in Section 27.7. If the spectrum of the light is continuous, then it is thermal in origin. If the spectrum shows discrete lines, it is atomic in origin. The results of the experiment show that the light is indeed thermal in origin and originates in the hot particles of soot in the candle flame.

CONCEPTUAL PROBLEM 1

If you observe objects inside a very hot kiln, it is difficult to discern the shape of the objects. Why?

CONCEPTUAL PROBLEM 2

Are black bodies black?

CONCEPTUAL PROBLEM 3

All objects radiate energy. Why, then, are we not able to see all objects in a dark room?

Example 28.1 Thermal Radiation from the Human Body

The temperature of your skin is approximately 35°C. What is the peak wavelength of the radiation it emits?

Solution From Wien's displacement law (Eq. 28.1), we have

$$\lambda_{max} T = 0.2898 \times 10^{-2}\ \text{m}\cdot\text{K}$$

Solving for λ_{max}, noting that 35°C corresponds to an absolute temperature of 308 K, we have

$$\lambda_{max} = \frac{0.2898 \times 10^{-2}\ \text{m}\cdot\text{K}}{308\ \text{K}} = 9.40\ \mu\text{m}$$

This radiation is in the infrared region of the spectrum.

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Example 28.2 The Quantized Oscillator

A 2.0-kg mass is attached to a massless spring of force constant $k = 25$ N/m. The spring is stretched 0.40 m from its equilibrium position and released. (a) Find the total energy and frequency of oscillation according to classical calculations.

Solution From Chapter 12, Equation 12.22, the total energy of a simple harmonic oscillator having an amplitude of A is $\frac{1}{2}kA^2$. Therefore,

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}(25\ \text{N/m})(0.40\ \text{m})^2 = 2.0\ \text{J}$$

The frequency of oscillation is, from Equation 12.19,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{25\ \text{N/m}}{2.0\ \text{kg}}} = 0.56\ \text{Hz}$$

(b) Assume that the energy is quantized and find the quantum number, n, for the system.

Solution If the energy is quantized, we have $E_n = nhf$, and from the result of (a) we have

$$E_n = nhf = n(6.626 \times 10^{-34}\ \text{J·s})(0.56\ \text{Hz}) = 2.0\ \text{J}$$

Therefore,

$$n = 5.4 \times 10^{33}$$

(c) How much energy is carried away in a one-quantum change?

Solution The energy carried away in a one-quantum change of energy is

$$E = hf = (6.63 \times 10^{-34}\ \text{J·s})(0.56\ \text{Hz}) = 3.7 \times 10^{-34}\ \text{J}$$

The energy carried away by a one-quantum change is such a small fraction of the total energy of the oscillator that we could never expect to see such a small change in the system. Thus, even though the decrease in energy of a spring–mass system is quantized and proceeds by small quantum jumps, our senses perceive the decrease as continuous. Quantum effects become important and measurable only on the submicroscopic level of atoms and molecules.

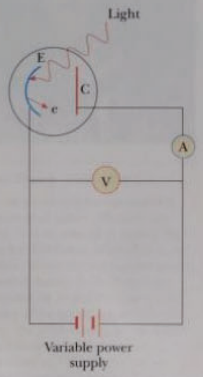


Figure 28.6 A circuit diagram for observing the photoelectric effect. When light strikes the plate, E (the emitter), photoelectrons are ejected from the plate. Electrons collected at C (the collector) constitute a current in the circuit.

EXERCISE 1 A tungsten filament is heated to 800°C. What is the wavelength of the most intense radiation? Answer 2.70 μm

28.2 · THE PHOTOELECTRIC EFFECT

In the latter part of the 19th century, experiments showed that light incident on certain metallic surfaces caused electrons to be emitted from the surfaces. As mentioned in Chapter 25, Section 25.1, this phenomenon is known as the **photoelectric effect,** and the emitted electrons are called **photoelectrons.** The first discovery of the photoelectric effect was made by Hertz.

Figure 28.6 is a schematic diagram of a photoelectric effect apparatus. An evacuated glass tube contains a metal plate, E, connected to the negative terminal of a battery. Another metal plate, C, is maintained at a positive potential by the battery. When the tube is kept in the dark, the ammeter reads zero, indicating that there is no current in the circuit. However, when monochromatic light of the appropriate wavelength shines on plate E, a current is detected by the ammeter, indicating a flow of charges across the gap between E and C. This current arises from electrons emitted from the negative plate (the emitter) and collected at the positive plate (the collector).

Figure 28.7 is a plot of the photoelectric current versus the potential difference, ΔV, between E and C for two light intensities. For large values of ΔV, the current reaches a maximum value. In addition, the current increases as the incident light intensity increases, as you might expect. Finally, when ΔV is negative—that is, when the battery polarity is reversed to make E positive and C negative—the current drops to a very low value because most of the emitted photoelectrons are repelled by the negative collecting plate, C. Only those electrons having a kinetic energy greater than $e\,\Delta V$ will reach C, where e is the charge on the electron. When ΔV is greater than or equal to ΔV_s, the **stopping potential,** no electrons reach C and the current is zero. The stopping potential is *independent* of the radiation intensity. The maximum kinetic energy of the photoelectrons is related to the stopping potential through the relation

$$K_{\max} = e\,\Delta V_s \qquad [28.4]$$

Several features of the photoelectric effect cannot be explained with classical physics or with the wave theory of light:

- No electrons are emitted if the incident light frequency falls below some **cutoff frequency,** f_c, which is characteristic of the material being illuminated. This is inconsistent with the wave theory, which predicts that the photoelectric effect should occur at any frequency, provided the light intensity is high enough.
- The maximum kinetic energy of the photoelectrons is independent of light intensity.
- The maximum kinetic energy of the photoelectrons increases with increasing light frequency.
- Electrons are emitted from the surface almost instantaneously (less than 10^{-9} s after the surface is illuminated), even at low light intensities. Classically, one would expect that the electrons would require some time to absorb the incident radiation before they acquired enough kinetic energy to escape from the metal.

A successful explanation of the photoelectric effect was given by Einstein in 1905, the same year he published his special theory of relativity. As part of a general paper on electromagnetic radiation, for which he received the Nobel Prize in 1921, Einstein extended Planck's concept of quantization to electromagnetic waves. He assumed that light (or any other electromagnetic wave) of frequency f can be considered a stream of photons. Each photon has an energy E given by Equation 28.3.

Einstein's view was that a photon of the incident light was so localized that it gave *all* its energy, hf, directly to a single electron in the metal. Electrons emitted from the surface of the metal possess the maximum kinetic energy, $K_{\max}$. According to Einstein, the maximum kinetic energy for these emitted electrons is

$$K_{\max} = hf - \phi \qquad [28.5]$$

- *Photoelectric effect equation*

where ϕ is called the **work function** of the metal. **The work function represents the minimum energy with which an electron is bound in the metal** and is on the order of a few electron volts. Table 28.1 lists selected values.

With the photon theory of light, one can explain the features of the photoelectric effect that cannot be understood using classical concepts:

- That the effect is not observed below a certain cutoff frequency follows from the fact that the photon energy must be $\geq \phi$. If the energy of the incoming photon is not $\geq \phi$, the electrons will never be ejected from the surface, regardless of light intensity.
- That $K_{\max}$ is independent of the light intensity can be understood with the following argument. If the light intensity is doubled, the number of photons is doubled, which doubles the number of photoelectrons emitted. However,

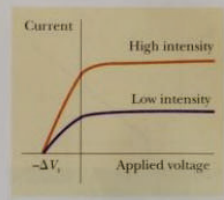


Figure 28.7 Photoelectric current versus applied voltage for two light intensities. The current increases with intensity but reaches a saturation level for large values of ΔV. At voltages equal to or less than $-\Delta V_s$, the current is zero.

TABLE 28.1 Work Functions of Selected Metals

Metal	ϕ (eV)
Na	2.46
Al	4.08
Cu	4.70
Zn	4.31
Ag	4.73
Pt	6.35
Pb	4.14
Fe	4.50

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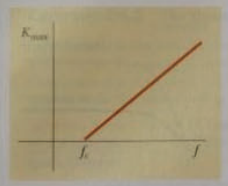


Figure 28.8 A sketch of K_{max} versus frequency of incident light for photoelectrons in a typical photoelectric effect experiment. Photons with frequency less than f_c do not have sufficient energy to eject an electron from the metal.

their kinetic energy, which equals $hf - \phi$, depends only on the light frequency and the work function, not on the light intensity.

- That K_{max} increases with increasing frequency is easily understood with Equation 28.5.
- That the electrons are emitted almost instantaneously is consistent with the particle theory of light, in which the incident energy appears in small packets and there is a one-to-one interaction between photons and electrons. This is in contrast to the energy of the photons being distributed uniformly over a large area.

Experimental observation of a linear relationship between f and K_{max} would be a final confirmation of Einstein's theory. Indeed, such a linear relationship is observed, as sketched in Figure 28.8. The slope of this curve is h. The intercept on the horizontal axis is the cutoff frequency, which is related to the work function through the relation $f_c = \phi/h$. This corresponds to a **cutoff wavelength** of

$$\lambda_c = \frac{c}{f_c} = \frac{c}{\phi/h} = \frac{hc}{\phi} \qquad [28.6]$$

where c is the speed of light. Wavelengths *greater* than λ_c incident on a material with a work function of ϕ do not result in the emission of photoelectrons.

CONCEPTUAL PROBLEM 4

A student claims that he is going to eject electrons from a piece of metal by placing a radio transmitter antenna adjacent to the metal and sending a strong AM radio signal into the antenna. The work function of a metal is typically a few electron volts. Will this work?

CONCEPTUAL PROBLEM 5

In the photoelectric effect, explain why the stopping potential depends on the frequency of the light but not on the intensity.

Example 28.3 The Photoelectric Effect for Sodium

A sodium surface is illuminated with light of wavelength 300 nm. The work function for sodium metal is 2.46 eV. Find (a) the maximum kinetic energy of the ejected photoelectrons and (b) the cutoff wavelength for sodium.

Solution (a) The energy of a photon in the illuminating light beam is

$$E = hf = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{300 \times 10^{-9}\ \text{m}}$$
$$= 6.63 \times 10^{-19}\ \text{J} = 4.14\ \text{eV}$$

Use of Equation 28.5 gives

$$K_{max} = hf - \phi = 4.14\ \text{eV} - 2.46\ \text{eV} = 1.68\ \text{eV}$$

(b) The cutoff wavelength can be calculated from Equation 28.6 after we convert ϕ from electron volts to joules:

$$\phi = 2.46\ \text{eV} = 3.94 \times 10^{-19}\ \text{J}$$
$$\lambda_c = \frac{hc}{\phi} = \frac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{3.94 \times 10^{-19}\ \text{J}}$$
$$= 5.05 \times 10^{-7}\ \text{m} = 505\ \text{nm}$$

This wavelength is in the green region of the visible spectrum.

EXERCISE 2 Calculate the maximum speed of the photoelectrons under the conditions described in this example. Answer 7.68×10^{5} m/s

EXERCISE 3 The work function for potassium is 2.24 eV. If potassium metal is illuminated with 480-nm light, find (a) the maximum kinetic energy of the photoelectrons and (b) the cutoff wavelength. Answer (a) 0.350 eV (b) 554 nm

28.3 · THE COMPTON EFFECT

In 1919 Einstein concluded that a photon of energy E travels in a single direction (unlike a spherical wave) and carries a momentum equal to $E/c = hf/c$. In his own words, "If a bundle of radiation causes a molecule to emit or absorb an energy packet hf, then momentum of quantity hf/c is transferred to the molecule, directed along the line of the bundle for absorption and opposite the bundle for emission." In 1923 Arthur Holly Compton (1892–1962) and Peter Debye (1884–1966) independently carried Einstein's idea of photon momentum farther. They realized that the scattering of x-ray photons from electrons could be explained by treating photons as point-like particles with energy hf and momentum hf/c and by assuming that the energy and momentum of the photon–electron pair are conserved in a collision.



Arthur Holly Compton (1892–1962), American physicist. *(Courtesy of AIP Niels Bohr Library)*

Prior to 1922, Compton and his coworkers had accumulated evidence that showed that the classical wave theory of light failed to explain the scattering of x-rays from electrons. According to classical theory, incident electromagnetic waves of frequency f_0 should have two effects: (1) the electrons will accelerate in the direction of propagation of the x-ray by radiation pressure (see Section 24.7), and (2) the oscillating electric field will set the electrons into oscillation at the apparent frequency of the radiation as seen by the moving electron. Both of these effects are included in Figure 28.9a. This apparent frequency will change due to the Doppler effect (see Section 13.10) since the electron absorbs as a moving particle. The electron will then re-radiate as a moving particle, exhibiting another Doppler shift in the frequency of radiation. Since different electrons will move at different speeds, depending on the amount of energy absorbed from the electromagnetic waves, the scattered wave frequency at a given angle should show a distribution of Doppler-shifted values. Contrary to this prediction, Compton's experiment showed that, at a given angle, only *one* frequency of radiation was observed. Figure 28.9b shows the quantum picture of the exchange of momentum and energy between an individual x-ray photon and an electron. Another major distinction between the classical and quantum models is shown in Figure 28.9b. In the classical model, the electron is pushed along the direction of propagation of the incident x-ray by radiation pressure. In the quantum model, the electron is scattered through an angle ϕ with respect to this direction, as if this were a billiard-ball type collision.

Figure 28.10a is a schematic diagram of the apparatus used by Compton. In his original experiment Compton measured how scattered x-ray intensity depended on wavelength at three scattering angles. The incident beam consisted of monochromatic x-rays of wavelength $\lambda_0 = 0.071$ nm. The experimental plots of intensity versus wavelength obtained by Compton for four scattering angles are shown in Figure 28.10b. They show two peaks, one at λ_0 and one at a longer wavelength λ'. The peak at λ_0 is caused by x-rays scattered from electrons that are tightly bound to the target atoms, and the shifted peak at λ' is caused by x-rays scattered from free electrons in the target. In his analysis, Compton predicted that the shifted peak should depend on scattering angle θ as

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Compton shift equation

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta) \qquad [28.7]$$

In this expression, known as the **Compton shift equation,** m_e is the mass of the electron; $h/m_e c$ is called the **Compton wavelength,** λ_C, of the electron and has a currently accepted value of

Compton wavelength

$$\lambda_C = \frac{h}{m_e c} = 0.00243 \text{ nm}$$

Compton's measurements were in excellent agreement with the predictions of Equation 28.7. It is fair to say that these were the first experimental results to convince most physicists of the fundamental validity of the quantum theory!

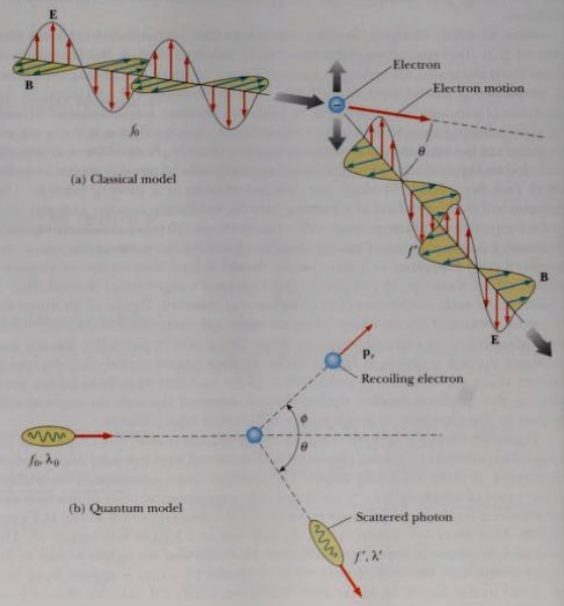


Figure 28.9 X-ray scattering from an electron: (a) the classical model and (b) the quantum model.

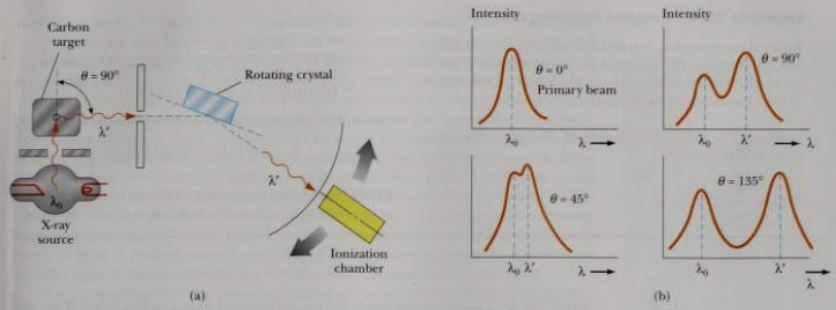


Figure 28.10 (a) Schematic diagram of Compton's apparatus. The wavelength was measured with a rotating crystal spectrometer using carbon as the target. The intensity was determined by a movable ionization chamber that generated a current proportional to the x-ray intensity. (b) Scattered intensity versus wavelength of Compton scattering at $\theta = 0°$, 45°, 90°, and 135°.

Thinking Physics 2

The Compton effect involves a change in wavelength as photons are scattered through different angles. Suppose we illuminate a piece of material with a beam of light and then view the material from different angles relative to the beam of light. Will we see a *color change* corresponding to the change in wavelength of the scattered light?

Reasoning There will be a wavelength change for visible light scattered by the material, but the change will be far too small to detect as a color change. The largest possible wavelength change, at 180° scattering, will be twice the Compton wavelength, about 0.005 nm. This represents a change of less than 0.001% of the wavelength of red light. The Compton effect is only detectable for wavelengths that are very short to begin with, so that the Compton wavelength is an appreciable fraction of the incident wavelength. As a result, the usual radiation for observing the Compton effect is in the x-ray range of the electromagnetic spectrum.

CONCEPTUAL PROBLEM 6

Suppose a photoelectric effect experiment is attempted, but the frequency of the radiation is less than the cutoff frequency for the target material. In such a situation, the photon energy is not sufficient to eject electrons. Where does the photon energy go in this case?

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Example 28.4 **Compton Scattering at 45°**

X-rays of wavelength $\lambda_0 = 0.200\ 000$ nm are scattered from a block of material. The scattered x-rays are observed at an angle of 45.0° to the incident beam. Calculate the wavelength of the x-rays scattered at this angle.

Solution The shift in wavelength of the scattered x-rays is given by Equation 28.7:

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

$$= \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})}(1 - \cos 45°)$$

$$= 7.10 \times 10^{-13}\ \text{m} = 0.000\ 710\ \text{nm}$$

Hence, the wavelength of the scattered x-rays at this angle is

$$\lambda' = \Delta\lambda + \lambda_0 = 0.200\ 710\ \text{nm}$$

EXERCISE 4 Find the fraction of energy lost by the photon in this collision. Answer Fraction = $\Delta E/E = 0.003\ 54$

EXERCISE 5 X-rays of wavelength 0.200 nm are scattered from a carbon target. If the scattered radiation is detected at 60° to the incident beam, find (a) the Compton shift $\Delta\lambda$ and (b) the kinetic energy imparted to the recoiling electron.
Answer (a) 0.001 21 nm (b) 37.4 eV

28.4 • PHOTONS AND ELECTROMAGNETIC WAVES

Explanations of phenomena such as the photoelectric effect and the Compton effect offer ironclad evidence that when light and matter interact, the light behaves as if it were composed of particles with energy hf and momentum h/λ. An obvious question at this point is, "How can light be considered a photon when it exhibits wave-like properties?" On the one hand, we describe light in terms of photons having energy and momentum. On the other hand, we recognize that light and other electromagnetic waves exhibit interference and diffraction effects, which are consistent only with a wave interpretation.

Which model is correct? Is light a wave or a particle? The answer depends on the phenomenon being observed. Some experiments can be explained better, or solely, with the photon concept, whereas others are best described, or can be described only, with a wave model. The end result is that **we must accept both models and admit that the true nature of light is not describable in terms of any single classical picture.** However, you should recognize that the same light beam that can eject photoelectrons from a metal can also be diffracted by a grating. In other words, **the photon theory and the wave theory of light complement each other.**

The dual nature of light •

All forms of electromagnetic radiation can be described from two points of view. At one extreme, electromagnetic waves describe the overall interference pattern or probability distribution, formed by photons. At the other extreme, the photon description is natural in the context of a highly energetic photon of very short wavelength. Both views are necessary and complementary. Hence, **light has a dual nature: It exhibits both wave and photon characteristics.**

The success of the particle model of light in explaining the photoelectric effect and the Compton effect raises many other questions. Because the photon is a particle, what is the meaning of its "frequency" and "wavelength," and which determines its energy and momentum? Is light in some sense simultaneously a wave and a particle? Although photons have no rest energy (a nonobservable quantity), is there a simple expression for the effective mass-energy of a "moving" photon? If a "moving" photon has mass-energy, do photons experience gravitational attraction? What is the spatial extent of a photon, and how does an electron absorb or scatter one photon? Some of these questions can be answered, but others demand a view of atomic processes that is too pictorial and literal. Furthermore, many of these questions stem from classical analogies such as colliding billiard balls and water waves breaking on a shore. Quantum mechanics gives light a more fluid and flexible nature by treating the particle model and wave model of light as both necessary and complementary. Neither model can be used exclusively to describe all properties of light. A complete understanding of the observed behavior of light can be attained only if the two models are combined in a complementary manner.

We can perhaps understand why photons are compatible with electromagnetic waves in the following manner. We may suspect that long-wavelength waves do not exhibit particle characteristics. Consider, for instance, radio waves at a frequency of 2.5 MHz. The energy of a photon having this frequency is only about 10^{-8} eV. From a practical viewpoint, this energy is too small to be detected as a single photon. A sensitive radio receiver might require as many as 10^{10} of these photons to produce a detectable signal. Such a large number of photons would appear, on the average, as a continuous wave. With so many photons reaching the detector every second, it is unlikely that any graininess would appear in the detected signal. That is, with 2.5-MHz waves one would not be able to detect the individual photons striking the antenna.

Now consider what happens with shorter wavelengths. In the visible region, it is possible to observe both the photon and the wave characteristics of light. As we mentioned earlier, a beam of visible light shows interference phenomena (thus, it is a wave) and at the same time can produce photoelectrons (thus, it is a particle). At even shorter wavelengths, the momentum and energy of the photon increase. As a consequence, the photon nature of light becomes more evident than its wave nature. For example, absorption of an x-ray photon is easily detected as a single event. However, as the wavelength decreases, wave effects become more difficult to observe.

Thinking Physics 3

Suppose you are observing a photon while moving toward its source at a high speed. According to the Doppler effect, you should detect a higher frequency. According to the equation for the energy of the photon, $E = hf$, this should result in an increased energy. How can the energy of a photon be increased simply by moving toward it? Does this violate the conservation of energy principle?

Reasoning There is no violation of conservation of energy, because energy had to be added to the *system of the photon and the observer* to represent the motion of the observer. According to the observer, the increased energy appears in the form of a higher frequency of the photon. According to a second observer, external to the system, the increased energy appears as kinetic energy of the first observer.

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28.5 • THE WAVE PROPERTIES OF PARTICLES



Louis de Broglie (1892–1987), a French physicist, was awarded the Nobel Prize in 1929 for his discovery of the wave nature of electrons. "It would seem that the basic idea of quantum theory is the impossibility of imaging an isolated quantity of energy without associating with it a certain frequency." *(AIP Niels Bohr Library)*

Students who are being introduced to the dual nature of light often find the concept very difficult to accept. Even more disconcerting, however, is that *matter* has a dual nature as well!

In 1923, in his doctoral dissertation, Louis Victor de Broglie postulated that, **because photons have wave and particle characteristics and electrons have particle characteristics and properties typical of waves, perhaps all forms of matter have wave as well as particle properties.** This was a highly revolutionary idea with no experimental confirmation at that time. According to de Broglie, electrons have a dual particle–wave nature. An electron in motion exhibits wave character or wave properties. De Broglie explained the source of this assertion in his 1929 Nobel Prize acceptance speech:

> On the one hand the quantum theory of light cannot be considered satisfactory since it defines the energy of a light corpuscle by the equation $E = hf$ containing the frequency f. Now a purely corpuscular theory contains nothing that enables us to define a frequency; for this reason alone, therefore, we are compelled, in the case of light, to introduce the idea of a corpuscle and that of periodicity simultaneously. On the other hand, determination of the stable motion of electrons in the atom introduces integers, and up to this point the only phenomena involving integers in physics were those of interference and of normal modes of vibration. This fact suggested to me the idea that electrons too could not be considered simply as corpuscles, but that periodicity must be assigned to them also.

In Chapter 9 we found that the relationship between energy and momentum for a photon is $p = E/c$. We also know from Equation 28.3 that the energy of a photon is $E = hf = hc/\lambda$. Thus, the momentum of a photon can be expressed as

Momentum of a photon •

$$p = \frac{E}{c} = \frac{hc}{c\lambda} = \frac{h}{\lambda} \qquad [28.8]$$

From this equation we see that the photon wavelength can be specified by its momentum, or $\lambda = h/p$. De Broglie suggested that material particles of momentum p should also have wave properties and a corresponding wavelength. Because the momentum of a particle of mass m and speed[1] v is $p = mv$, the **de Broglie wavelength** of a particle is

De Broglie wavelength •

$$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad [28.9]$$

Furthermore, in analogy with photons, de Broglie postulated that the frequencies of matter waves obey the Einstein relation $E = hf$, so that

Frequency of matter waves •

$$f = \frac{E}{h} \qquad [28.10]$$

[1]This speed, v, the speed of the particle, is the group speed of the wave (the speed at which a signal may travel). It is not the same as the phase speed of the wave, related to the index of refraction and wavelength. See L. de Broglie, *Matter and Light*, Dover, New York, pp. 168–171.

The dual nature of matter is apparent in these two equations because each contains both particle concepts (mv and E) and wave concepts (λ and f). The fact that these relationships are established experimentally for photons makes the de Broglie hypothesis that much easier to accept.

Quantization of Angular Momentum in the Bohr Model

Bohr's model of the atom, discussed in Chapter 11, had many shortcomings and problems. For example, as the electrons revolve around the nucleus, how can one understand the fact that only certain electron energies are allowed? Why do all atoms of a given element have precisely the same physical properties regardless of the infinite variety of starting velocities and positions of the electrons in each atom?

De Broglie's great insight was to recognize that wave theories of matter use interference phenomena to explain these observations. Recall from Chapter 14 that a plucked guitar string, although initially subjected to a wide range of wavelengths, will support only those standing wave patterns that have nodes at its ends. Any free vibration of the string consists of a superposition of varied amounts of many standing waves. This same reasoning can be applied to electron matter waves bent into a circle around the nucleus. All of the possible states of the electron are standing wave states, each with its own wavelength, speed, and energy. Figure 28.11a illustrates this point when three complete wavelengths are contained in one circumference of the orbit. Similar patterns can be drawn for orbits containing two wavelengths, four wavelengths, and so forth. This situation is analogous to that of standing waves on a string that has preferred (resonant) frequencies of vibration. Figure 28.11b shows a standing wave pattern containing three wavelengths for a string that is fixed at each end. Now imagine that the vibrating string is removed from its supports at A and B and bent into a circular shape such that points A and B are connected. The end result is a pattern like that shown in Figure 28.11a.

Another aspect of the Bohr theory that is easier to visualize by using de Broglie's hypothesis is the quantization of angular momentum. We simply assume that the allowed Bohr orbits arise because the electron matter waves form standing waves when an integral number of wavelengths exactly fits into the circumference of a circular orbit. Thus,

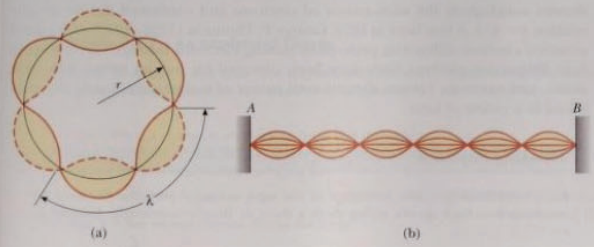


Figure 28.11 (a) The standing wave pattern for an electron wave in a stable orbit of hydrogen. There are three full wavelengths in this orbit. (b) The standing wave pattern for a vibrating, stretched string fixed at its ends. This pattern has three full wavelengths.

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$$n\lambda = 2\pi r$$

where r is the radius of the orbit and $n = 1, 2, 3, \ldots$. Because the de Broglie wavelength of an electron is $\lambda = h/m_e v$, and because $\hbar = h/2\pi$, we can write the preceding condition as $n(h/m_e v) = 2\pi r$, or

$$m_e v r = n\hbar$$

Note that this is precisely the Bohr condition of quantization of angular momentum (see Chapter 11, Section 11.5).[2] Thus, electron waves that fit the orbits are standing waves because of the boundary conditions imposed. These standing waves have discrete frequencies, corresponding to the allowed wavelengths. If $n\lambda \neq 2\pi r$, a standing-wave pattern can never form a closed circular orbit, as in Figure 28.11a.

The Davisson–Germer Experiment

De Broglie's proposal that any kind of particle exhibits both wave and particle properties was first regarded as pure speculation. If particles such as electrons had wave-like properties, then under the correct conditions they should exhibit diffraction effects. In 1927, 3 years after de Broglie published his work, Clinton J. Davisson (1881–1958) and Lester H. Germer (1896–1971) of the United States succeeded in measuring the wavelength of electrons. Their important discovery provided the first experimental confirmation of the matter waves proposed by de Broglie.

It is interesting to note that the intent of the initial Davisson–Germer experiment was not to confirm the de Broglie hypothesis. In fact, their discovery was made by accident (as is often the case). The experiment involved the scattering of low-energy electrons (about 54 eV) shot at a nickel target in a vacuum. During one experiment, the nickel surface was badly oxidized because of an accidental break in the vacuum system. After the nickel target was heated in a flowing stream of hydrogen to remove the oxide coating, electrons scattered by it exhibited intensity maxima and minima at specific angles. The experimenters finally realized that the nickel had formed large crystal regions on heating and that the regularly spaced planes of atoms in the crystalline regions served as a diffraction grating for electron matter waves.

Shortly thereafter, Davisson and Germer performed more extensive diffraction measurements on electrons scattered from single-crystal targets. Their results showed conclusively the wave nature of electrons and confirmed the de Broglie relation $p = h/\lambda$. A year later in 1928, George P. Thomson (1892–1975) of Scotland observed electron diffraction patterns by passing electrons through very thin gold foils. Diffraction patterns have since been observed for helium atoms, hydrogen atoms, and neutrons. Hence, the universal nature of matter waves has been established in a variety of ways.

Thinking Physics 4

Electron microscopes take advantage of the wave nature of particles. Electrons are accelerated to high speeds, giving them a short de Broglie wavelength. Imagine an electron microscope using electrons with a de Broglie wavelength of 0.2 nm. Why don't we design a microscope using 0.2 nm *photons* to do the same thing?

Reasoning Because electrons are charged particles, they will interact with the sample in the microscope and scatter according to the shape and density of various portions of the sample, providing a means of viewing the sample. Photons of wavelength 0.2 nm are uncharged and in the x-ray region of the spectrum. They will tend to simply pass through the sample without interacting. It is also much easier to image electrons because, unlike x-rays, electrons can be guided by electrostatic focusing fields.

[2]Note that de Broglie's analysis still failed to explain the fact that states having an orbital angular momentum quantum number of zero do exist. Furthermore, the actual states of the hydrogen atom are not planar but have spherical symmetry.

CONCEPTUAL PROBLEM 7

Why is an electron microscope more suitable than an optical microscope for "seeing" objects of an atomic size?

Example 28.5 The Wavelength of an Electron

Calculate the de Broglie wavelength for an electron ($m_e = 9.11 \times 10^{-31}$ kg) moving at 1.00×10^7 m/s.

Solution Equation 28.9 gives

$$\lambda = \frac{h}{m_e v} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\ \text{kg})(1.00 \times 10^7\ \text{m/s})} = 7.28 \times 10^{-11}\ \text{m}$$

This wavelength corresponds to that of typical x-rays.

EXERCISE 6 Find the de Broglie wavelength of a proton moving with a speed of 1.00×10^7 m/s.
Answer 3.97×10^{-14} m

Example 28.6 The Wavelength of a Rock

A rock of mass 50 g is thrown with a speed of 40 m/s. What is its de Broglie wavelength?

Solution From Equation 28.9 we have

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(50 \times 10^{-3}\ \text{kg})(40\ \text{m/s})} = 3.32 \times 10^{-34}\ \text{m}$$

This wavelength is much smaller than any aperture through which the rock could possibly pass. This means that we could not observe diffraction effects, and as a result the wave properties of large-scale objects cannot be observed.

Example 28.7 An Accelerated Charge

A particle of charge q and mass m is accelerated from rest through a potential difference ΔV. Find its de Broglie wavelength.

Solution When a charge is accelerated from rest through a potential difference of ΔV, its gain in kinetic energy, $\frac{1}{2}mv^2$, must equal the loss in potential energy, $q\,\Delta V$:

$$\tfrac{1}{2}mv^2 = q\,\Delta V$$

Because $p = mv$, we can express this in the form

$$\frac{p^2}{2m} = q\,\Delta V \quad \text{or} \quad p = \sqrt{2mq\,\Delta V}$$

Substituting this expression for p in the de Broglie relation $\lambda = h/p$ gives

$$\lambda = \frac{h}{\sqrt{2mq\,\Delta V}}$$

EXERCISE 7 Calculate the de Broglie wavelength of an electron accelerated through a potential difference of 50 V.
Answer 0.174 nm

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28.6 · THE DOUBLE-SLIT EXPERIMENT REVISITED

One way to crystallize our ideas about the electron's wave–particle duality is to consider a double-slit electron diffraction experiment. This experiment shows the impossibility of *simultaneously* measuring both wave and particle properties and illustrates the use of the wave function in the determination of interference effects.

Consider a parallel beam of monoenergetic electrons that is incident on a double slit, as in Figure 28.12. Because the slit openings are much smaller than the slit separation D, no structure is expected from single-slit diffraction. An electron detector is positioned far from the slits at a distance much greater than D. If the detector collects electrons for a long enough time, one finds a typical wave interference pattern for the counts per minute or probability of arrival of electrons. Such an interference pattern would not be expected if the electrons behaved as classical particles.

If the experiment is carried out at lower beam intensities, the interference pattern is still observed if the time of the measurement is sufficiently long. This is illustrated by the computer-simulated patterns in Figure 28.13. Note that the interference pattern becomes clearer as the number of electrons reaching the screen increases.

If one imagines a single electron producing in-phase "wavelets" as it reaches one of the slits, standard wave theory can be used to find the angular separation, θ, between the central probability maximum and its neighboring minimum. The minimum occurs when the path length difference between A and B is half a wavelength, or

$$D \sin \theta = \frac{\lambda}{2}$$

As an electron's wavelength is given by $\lambda = h/p_x$, we see that, for small θ,

$$\sin \theta \approx \theta = \frac{h}{2p_x D}$$

Thus, the dual nature of the electron is clearly shown in this experiment: **The electrons are detected as particles, each at a localized spot at some instant of time, but the probability of arrival at that spot is determined by finding the intensity of two interfering matter waves.**

And there is more. If one slit is covered during the experiment, and the slit width is small compared to the wavelength of the electrons, a symmetric curve peaked around the center of the open slit is observed. Plots of the counts per minute (probability of arrival of electrons) with the lower or upper slit closed are shown as blue curves in the central part of Figure 28.14. These are expressed as the square of the absolute value of some wave function, $|\psi_1|^2 = \psi_1^* \psi_1$ or $|\psi_2|^2 = \psi_2^* \psi_2$, where ψ_1 and ψ_2 represent the electron passing through slit 1 and slit 2, respectively.

If an experiment is now performed with slit 2 blocked half of the time and then slit 1 blocked during the remaining time, the accumulated pattern of counts/min shown by the blue curve on the right side of Figure 28.14 is completely different

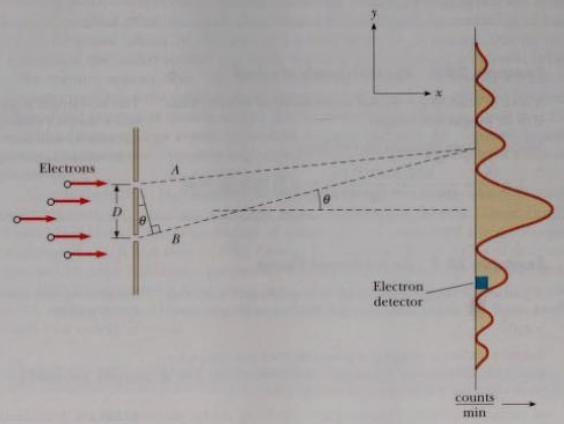


Figure 28.12 Electron diffraction. The slit separation D is much greater than the individual slit widths and much less than the distance between the slit and the detector. The electron detector is movable along the y-direction in the drawing and so can detect electrons diffracted at different values of θ. The detector acts like the "viewing screen" of Young's double-slit experiment with light, discussed in Chapter 27.

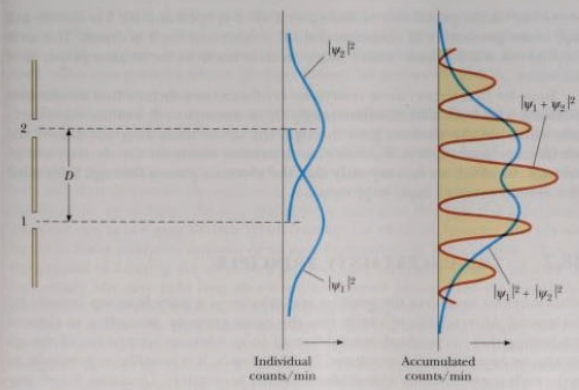


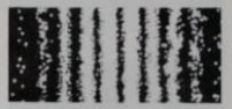
Figure 28.14 Accumulated results of the two-slit electron diffraction experiment with each slit closed half the time. The result with both slits open is shown in red.



(a) After 28 electrons



(b) After 1000 electrons



(c) After 10,000 electrons



(d) Two-slit electron pattern

Figure 28.13 (a, b, c) Computer-simulated interference patterns for a beam of electrons incident on a double slit. *(From E. R. Huggins, Physics I, New York, W. A. Benjamin, 1968)* (d) Photograph of a double-slit interference pattern produced by electrons. *(From C. Jönsson, Zeitschrift fur Physik 161:454, 1961; used with permission)*

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from the case with both slits open (red curve). There is no longer a maximum probability of arrival of an electron at $\theta = 0$. In fact, **the interference pattern has been lost and the accumulated result is simply the sum of the individual results.** When only one slit is open at a time, we know the electron has the same localizability and indivisibility at the slits as we measure at the detector, because the electron clearly goes through slit 1 or slit 2. Thus, the total must be analyzed as the sum of those electrons that come through slit 1, $|\psi_1|^2$, and those that come through slit 2, $|\psi_2|^2$.

When both slits are open, it is tempting to assume that the electron goes through either slit 1 or slit 2, and that the counts/min are again given by $|\psi_1|^2 + |\psi_2|^2$. We know, however, that the experimental results indicated by the red interference pattern in Figure 28.14 contradict this. Hence, our assumption that the electron is localized and goes through only one slit when both slits are open must be wrong (a painful conclusion!). Because it exhibits interference, we must conclude that—somehow—*the electron must be simultaneously present at both slits.* In order to find the probability of detecting the electron at a particular point on the screen with both slits open, we may say that the electron is in a *superposition state* given by

$$\psi = \psi_1 + \psi_2$$

Thus, the probability of detecting the electron at the screen is equal to the quantity $|\psi_1 + \psi_2|^2$ and not $|\psi_1|^2 + |\psi_2|^2$. Because matter waves that start out in phase at the slits in general travel different distances to the screen (see Fig. 28.12), ψ_1 and ψ_2 possess a relative phase difference of ϕ at the screen. Using a vector diagram (Fig. 28.15) to find $|\psi_1 + \psi_2|^2$ immediately yields

$$|\psi|^2 = |\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2| \cos \phi$$

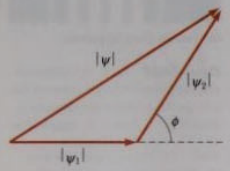


Figure 28.15 A vector diagram representing the addition of two complex quantities, ψ_1 and ψ_2.

where $|\psi_1|^2$ is the probability of detection if slit 1 is open and slit 2 is closed, and $|\psi_2|^2$ is the probability of detection if slit 2 is open and slit 1 is closed. The term $2|\psi_1||\psi_2| \cos \phi$ is the interference term, which arises from the relative phase, ϕ, of the waves.

In order to interpret these results, we are forced to conclude that **an electron interacts with both slits simultaneously.** If we attempt to determine experimentally which slit the electron goes through, the act of measuring will destroy the interference pattern. It is impossible to determine which slit the electron will go through. In effect, **we can say only that the electron passes through both slits!** The same arguments apply to photons.

28.7 · THE UNCERTAINTY PRINCIPLE

Whenever one measures the position and velocity of a particle at any instant, experimental uncertainties are built into the measurements. According to classical mechanics, there is no fundamental barrier to an ultimate refinement of the apparatus or experimental procedures. In other words, it is possible, in principle, to make such measurements with arbitrarily small uncertainty. Quantum theory predicts, however, that **it is fundamentally impossible to make simultaneous measurements of a particle's position and momentum with infinite accuracy.**

In 1927, Werner Heisenberg (1901–1976) introduced this notion, which is now known as the **Heisenberg uncertainty principle:**

> If a measurement of position is made with precision Δx and a simultaneous measurement of momentum in the x direction is made with precision Δp_x, then the product of the two uncertainties can never be smaller than $\hbar/2$, where $\hbar = h/2\pi$.
>
> $$\Delta x \, \Delta p_x \geq \frac{\hbar}{2} \qquad [28.11]$$

• *Uncertainty principle*

That is, **it is physically impossible to simultaneously measure the exact position and exact momentum of a particle.** Heisenberg was careful to point out that the inescapable uncertainties Δx and Δp_x do not arise from imperfections in practical measuring instruments. Rather, they arise from the quantum structure of matter, from effects such as the unpredictable recoil of an electron when struck by an indivisible photon, or the diffraction of light or electrons passing through a small opening.

Another important uncertainty relation involves the uncertainty in the energy of a particle ΔE and the time Δt taken to measure that energy. This energy–time uncertainty principle is

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \qquad [28.12]$$

Equation 28.12 states that the precision with which we can know the energy of some system is limited by the time available for measuring the energy. A common application of the energy–time uncertainty is in calculating the lifetimes of very short-lived subatomic particles whose lifetimes cannot be measured directly, but whose uncertainty in energy or mass can be measured.

To understand the uncertainty principle, consider the following thought experiment introduced by Heisenberg. Suppose you wish to measure the position and momentum of an electron as accurately as possible. You might be able to do this by viewing the electron with a powerful light microscope. In order to see the electron and thus determine its position, at least one photon of light must bounce off the electron, as in Figure 28.16a, and then it must pass through the microscope into your eye, as in Figure 28.16b. When it strikes the electron, however, the photon transfers some unknown amount of its own momentum to the electron. Thus, in the process of locating the electron's position very accurately (that is, making Δx very small), the very light that allows you to succeed changes the electron's momentum to some undeterminable extent (making Δp_x very large).

Let us analyze the collision by first noting that the incoming photon has momentum h/λ. As a result of the collision, the photon transfers all of its momentum along the x axis to the electron. Thus, the uncertainty in the electron's momentum after the collision is as large as the momentum of the incoming photon. That is, $\Delta p_x = h/\lambda$. Furthermore, because light also has wave properties, we would expect to be able to determine the position of the electron to within one wavelength of



Werner Heisenberg (1901–1976)

A German theoretical physicist, Heisenberg obtained his PhD in 1923 at the University of Munich, where he studied under Arnold Sommerfeld. Although physicists such as de Broglie and Schrödinger tried to develop physical models of the atom, Heisenberg developed an abstract mathematical model called *matrix mechanics* to explain the wavelengths of spectral lines. The more successful wave mechanics by Schrödinger, announced a few months later, was shown to be equivalent to Heisenberg's approach. Heisenberg made many other significant contributions to physics, including his famous uncertainty principle, for which he received the Nobel Prize in 1932: the prediction of two forms of molecular hydrogen; and theoretical models of the nucleus. *(Courtesy of University of Hamburg)*

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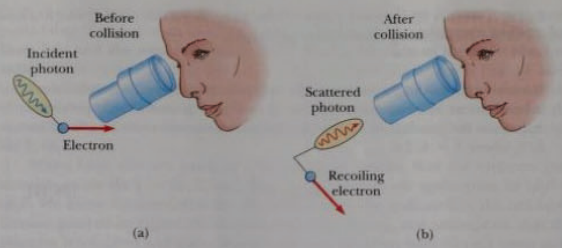


Figure 28.16 A thought experiment for viewing an electron with a powerful microscope. (a) The electron is viewed before it collides with the photon. (b) The electron recoils (is disturbed) as a result of the collision with the photon.

the light being used to view it, so that $\Delta x = \lambda$. Multiplying these two uncertainties gives

$$\Delta x\,\Delta p_x = \lambda\left(\frac{h}{\lambda}\right) = h$$

This represents the minimum in the products of the uncertainties. Because the uncertainty can always be greater than this minimum, we have

$$\Delta x\,\Delta p_x \geq h$$

This agrees with Equation 28.11 (apart from a small numerical factor introduced by Heisenberg's more precise analysis).

Heisenberg's uncertainty principle enables us to better understand the dual wave–particle nature of light and matter. We have seen that the wave description is quite different from the particle description. Therefore, if an experiment (such as the photoelectric effect) is designed to reveal the particle character of an electron, its wave character will become less apparent. If the experiment (such as diffraction from a crystal) is designed to accurately measure the electron's wave properties, its particle character will become less apparent.

Thinking Physics 5

A common, but erroneous, description of the absolute zero of temperature is *that temperature at which all molecular motion ceases.* How can the uncertainty principle be used to argue against this description?

Reasoning Let us imagine molecules in a finite piece of material. The molecules are constrained to be within the piece of material, so there is a fixed uncertainty in their position along one axis, Δx, which is the size of the piece of material along this axis. If the molecular motion were to cease at absolute zero, we would be claiming that the molecular velocity is *zero*, with no uncertainty, $\Delta v = 0$. The product of zero uncertainty in velocity and a nonzero uncertainty in position will be zero, violating the uncertainty principle. Thus, even at absolute zero, there must be some molecular motion.

Thinking Physics 6

According to the Bohr model of the hydrogen atom, the electron in the ground state moves in a circular orbit of radius 0.529×10^{-10} m, and the speed of the electron in this state is 2.2×10^6 m/s. What is the approximate uncertainty in the electron's radial position when the electron is in the ground state?

Reasoning Because the Bohr model assumes that the electron is located exactly at $r = 0.0529$ nm, the uncertainty Δr in its radial position in this model is zero. But according to the uncertainty principle, the product $\Delta p_r\,\Delta r \geq \hbar/2$, where Δp_r is the uncertainty in the momentum of the electron in the radial direction. Because the momentum of the electron is $m_e v$, we can assume that the uncertainty in its momentum is less than this value. That is,

$$\Delta p_r < m_e v = (9.11 \times 10^{-31}\ \text{kg})(2.2 \times 10^6\ \text{m/s}) = 2.0 \times 10^{-24}\ \text{kg}\cdot\text{m/s}$$

From the uncertainty principle, we can now estimate the minimum uncertainty in the radial position:

$$\Delta r_{\text{min}} = \frac{\hbar}{2\,\Delta p_r} = \frac{1.05 \times 10^{-34}\ \text{J}\cdot\text{s}}{2 \times 2 \times 10^{-24}\ \text{kg}\cdot\text{m/s}} = 0.26 \times 10^{-10}\ \text{m}$$

This uncertainty in the radial position of the electron is half the size of the Bohr radius, and is a measure of the fuzziness of the orbit.

Example 28.8 Locating an Electron

The speed of an electron is measured to be 5.00×10^5 m/s $\pm$ 0.003%. Within what limits could one determine the position of this electron?

Solution The momentum of the electron is

$$p = m_e v = (9.11 \times 10^{-31}\ \text{kg})(5.00 \times 10^5\ \text{m/s}) = 4.56 \times 10^{-27}\ \text{kg}\cdot\text{m/s}$$

Because the uncertainty is 0.003% of this value, we get

$$\Delta p = 0.00003p = 1.37 \times 10^{-31}\ \text{kg}\cdot\text{m/s}$$

The uncertainty in position can now be calculated by using this value of Δp and Equation 28.11:

$$\Delta x \geq \frac{\hbar}{2\,\Delta p_x} = \frac{1.05 \times 10^{-34}\ \text{J}\cdot\text{s}}{2.74 \times 10^{-31}\ \text{kg}\cdot\text{m/s}} = 0.384 \times 10^{-3}\ \text{m} = 0.384\ \text{mm}$$

This is the *lower limit* on the uncertainty with which we could have determined the electron's position.

EXERCISE 8 A monoenergetic beam of electrons is incident on a single slit of width 0.500 nm. A diffraction pattern is formed on a screen 20.0 cm from the slit. If the distance between successive minima of the diffraction pattern is 2.10 cm, what is the energy of the incident electrons? Answer 546 eV

28.8 • AN INTERPRETATION OF QUANTUM MECHANICS

Matter waves are described by the complex-valued wave function ψ, introduced in Section 28.6. The absolute square $|\psi|^2 = \psi^*\,\psi$ gives the probability density of finding a particle at a given point at some instant, where ψ^* is the complex conjugate of ψ. The wave function contains within it all the information that can be known about the particle.

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This interpretation of de Broglie's matter waves was first suggested by Max Born (1882–1970) in 1928. In 1926, Erwin Schrödinger (1887–1961) proposed a wave equation that described the manner in which matter waves change in space and time. The *Schrödinger wave equation*, which we will examine in Section 28.10, represents a key element in the theory of quantum mechanics.

The concepts of quantum mechanics, strange as they sometimes may seem, developed from classical ideas. In fact, when the techniques of quantum mechanics are applied to macroscopic systems, the results are essentially identical to those of classical physics. This blending of the two theories occurs when the de Broglie wavelength is small compared with the dimensions of the system. The situation is similar to the agreement between relativistic mechanics and classical mechanics when $v \ll c$.

A number of experiments show that matter has both a wave nature and a particle nature. A question that arises quite naturally in this regard is the following: If we are describing a particle, how do we view what is waving? In the cases of waves on strings, water waves, and sound waves, the wave is represented by some quantity that varies with time and position. In a similar manner, matter waves (de Broglie waves) are represented by the **wave function,** Ψ. In general, Ψ depends on the positions of all the particles in a system and on time, and therefore is often written $\Psi\ (x, y, z, t)$. If Ψ is known for a particle, then the particular properties of that particle can be described. In fact, a fundamental problem of quantum mechanics is this: Given the wave function at some instant, find the wave function at some later time, t.

In Section 28.5, we found that the de Broglie equation relates the momentum of a particle to its wavelength through the relation $p = h/\lambda$. If a free particle has a precisely known momentum, p_x, its wave function is an uninterrupted sinusoidal wave of wavelength $\lambda = h/p_x$, and the particle has equal probability of being at any point along the x axis. The wave function for such a free particle moving along the x axis can be written as

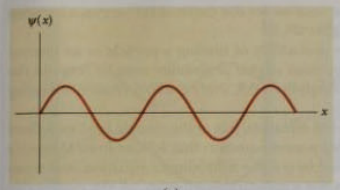
$$\psi(x) = A \sin\left(\frac{2\pi x}{\lambda}\right) = A \sin(kx) \qquad [28.13]$$

where $k = 2\pi/\lambda$ is the angular wave number and A is a constant amplitude.[3] As we mentioned earlier, the wave function is generally a function of both position and time. Equation 28.13 represents the part of the wave function that depends on position only. For this reason, one can view $\psi(x)$ as a snapshot of the wave function at a given instant, as shown in Figure 28.17a. The wave function for a particle the wavelength of which is not precisely defined is shown in Figure 28.17b. Because the wavelength is not precisely defined, it follows that the momentum is only approximately known. That is, if one were to measure the momentum of the particle, the result would have any value over some range, determined by the spread in wavelength.

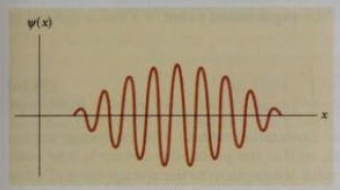
Although we cannot measure ψ, we can measure the quantity $|\psi|^2$, which can be interpreted as follows. If ψ represents a single particle, then $|\psi|^2$—called the **probability density**—is the relative probability per unit volume that the particle will be found at any point in the volume. This interpretation, first suggested by

• *$|\psi|^2$ equals the probability density*

[3] In general, ψ has a real part and an imaginary part, and we would write the wave function for the particle in the form Ae^{ikx}, where the imaginary part of the function describes the phase of the wave.



(a)



(b)

Figure 28.17 (a) A portion of the wave function for a particle the wavelength of which is precisely known. (b) The wave function for a particle the wavelength of which is not precisely known and hence the momentum of which is known only over some range of values. This pattern is formed by a sum of functions as in (a) having a continuous distribution of wavelengths.

Born in 1928, can also be stated in the following manner. If dV is a small volume element surrounding some point, then the probability of finding the particle in that volume element is $|\psi|^2\ dV$. In this section we deal only with one-dimensional systems, in which the particle must be located along the x axis, and thus we replace dV with dx. In this case, the probability, $P(x)\ dx$, that the particle will be found in the infinitesimal interval dx around the point x is

$$P(x)\ dx = |\psi|^2\ dx$$

Because the particle must be somewhere along the x axis, the sum of the probabilities over all values of x must be 1:

$$\int_{-\infty}^{\infty} |\psi|^2\ dx = 1 \qquad [28.14]$$

• *Normalization condition on ψ*

Any wave function satisfying Equation 28.14 is said to be **normalized.** Normalization is simply a statement that the particle exists at some point at all times. If the probability were zero, the particle would not exist. Therefore, although it is not possible to specify the position of a particle with complete certainty, it is possible, through $|\psi|^2$, to specify the probability of observing it. Furthermore, **the probability of finding the particle in the interval $a \leq x \leq b$ is**

$$P_{ab} = \int_a^b |\psi|^2\ dx \qquad [28.15]$$

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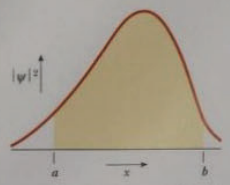


Figure 28.18 The probability that a particle is in the interval $a \leq x \leq b$ is represented by the area under the curve from a to b of the probability density function $|\psi(x, t)|^2$.

That is, the probability P_{ab} is just the area under the curve of $|\psi|^2$ versus x between the points $x = a$ and $x = b$, as in Figure 28.18.

Experimentally, there is a finite probability of finding a particle in an interval near some point at some instant. The value of that probability must lie between the limits 0 and 1. For example, if the probability is 0.3, there is a 30% chance of finding the particle.

The wave function ψ satisfies a wave equation, just as the electric field associated with an electromagnetic wave satisfies a wave equation that follows from Maxwell's equations. The wave equation satisfied by ψ is the Schrödinger equation, and ψ can be computed from it. Although ψ is not a measurable quantity, all the measurable quantities of a particle, such as its energy and momentum, can be derived from a knowledge of ψ. For example, once the wave function for a particle is known, it is possible to calculate the average position, x, of the particle after many experimental trials. This average position is called the **expectation value** of x and is defined by the equation

Expectation value of x •

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2 \, dx \qquad [28.16]$$

(Brackets $\langle \ldots \rangle$ are used to denote expectation values.) This expression implies that the particle is in a definite state, so that the probability density is time independent. Note that the expectation value is equivalent to the average value of x that one would obtain when dealing with a large number of particles in the same state. Furthermore, one can find the expectation value of any function $f(x)$ by using Equation 28.16 with the integrand replaced by $\psi^* f(x) \psi$.

28.9 • A PARTICLE IN A BOX

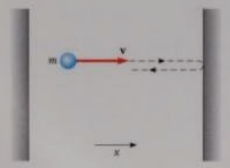


Figure 28.19 A particle of mass m and velocity $\mathbf{v}$, confined to moving parallel to the x axis and bouncing between two impenetrable walls.

From a classical viewpoint, if a particle is confined to moving along the x axis and to bouncing back and forth between two impenetrable walls (Fig. 28.19), its motion is easy to describe. If the speed of the particle is v, then the magnitude of its momentum (mv) remains constant, as does its kinetic energy. Furthermore, classical physics places no restrictions on the values of a particle's momentum and energy. The quantum mechanics approach to this problem is quite different and requires that we find the appropriate wave function consistent with the conditions of the situation.[4] As we shall see, this description leads to the interesting phenomenon of energy quantization.

Because the walls are impenetrable, the particle can *never* be found outside the box, which requires $\psi(x)$ to be zero at the walls and outside the walls. That is, $\psi(x) = 0$ for $x \leq 0$ and for $x \geq L$, where L is the distance between the two walls. Only those wave functions that satisfy this condition are allowed. In analogy with standing waves on a string (Eq. 14.3), the allowed wave functions for the particle in the box are sinusoidal and are given by

Allowed wave functions for a particle in a box •

$$\psi(x) = A \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots \qquad [28.17]$$

[4]Before continuing, you should review Chapter 14, Sections 14.2 and 14.3 on standing mechanical waves.

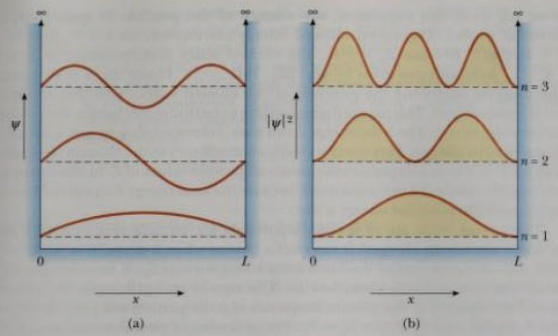


Figure 28.20 The first three allowed stationary states for a particle confined to a one-dimensional box. (a) The wave functions for $n = 1$, 2, and 3. (b) The probability distributions for $n = 1$, 2, and 3.

where A is the maximum value of the wave function. This expression shows that, for a particle confined to a box and having a well-defined de Broglie wavelength, ψ is represented by a sinusoidal wave. The allowed wavelengths are those for which the length L is equal to an integral number of half-wavelengths—that is, $L = n\lambda/2$. These allowed states of the systems are called **stationary states** because they are constant with time.

Figures 28.20a and b are graphs of ψ versus x and $|\psi|^2$ versus x for $n = 1$, 2, and 3. As we shall soon see, these correspond to the three lowest allowed energies for the particle. Note that although ψ can be positive or negative, $|\psi|^2$ is always positive. From any viewpoint, a negative value for $|\psi|^2$ is meaningless.

Further inspection of Figure 28.20b shows that $|\psi|^2$ is always zero at the boundaries, indicating that it is impossible to find the particle at these points. In addition, $|\psi|^2$ is zero at other points, depending on the value of n. For $n = 2$, $|\psi|^2 = 0$ at $x = L/2$; for $n = 3$, $|\psi|^2 = 0$ at $x = L/3$ and at $x = 2L/3$. For $n = 2$, $|\psi|^2$ has maxima at $x = L/4$ and at $x = 3L/4$, and so on.

Because the wavelengths of the particle are restricted by the condition $\lambda = 2L/n$, the magnitude of the momentum is also restricted to specific values:

$$p = \frac{h}{\lambda} = \frac{h}{2L/n} = \frac{nh}{2L}$$

Using $p = mv$, we find that the allowed values of the kinetic energy are

$$E_n = \tfrac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{(nh/2L)^2}{2m}$$

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2 \qquad n = 1, 2, 3 \ldots \qquad [28.18]$$

• *Allowed energies for a particle in a box*

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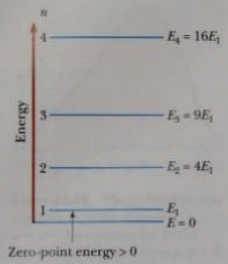


Figure 28.21 An energy level diagram for a particle confined to a one-dimensional box of width L. The lowest allowed energy is E_1 and has the value $E_1 = h^2/8mL^2$.

As we see from this expression, **the energy of the particle is quantized,** as we would expect. The lowest allowed energy corresponds to $n = 1$, for which $E_1 = h^2/8mL^2$. Because $E_n = n^2E_1$, the excited states corresponding to $n = 2, 3, 4, \ldots$ have energies given by $4E_1, 9E_1, 16E_1, \ldots$. Figure 28.21 is an energy level diagram describing the positions of the allowed states. Note that the state $n = 0$ is not allowed. This means that according to quantum mechanics, the particle can never be at rest. The least energy it can have, corresponding to $n = 1$, is called the **zero-point energy.** This result is clearly contradictory to the classical viewpoint, in which $E = 0$ is an acceptable state, as are all positive values of E. In this situation, only positive values of E are considered, because the total energy E equals the kinetic energy, and the potential energy is zero.

The energy levels are of special importance for the following reason. If the particle is electrically charged, it can emit a photon when it drops from an excited state, such as E_3, to one of the lower-lying states, such as E_2. It can also absorb a photon the energy of which matches the difference in energy between two allowed states. For example, if the photon frequency is f, the particle will jump from the state E_1 to the state E_2 if $hf = E_2 - E_1$. The processes of photon emission and absorption can be observed by spectroscopy, in which spectral wavelengths are a direct measure of such energy differences.

Example 28.9 A Bound Electron

An electron is confined between two impenetrable walls 0.200 nm apart. Determine the energy levels for $n = 1$, 2, and 3.

Solution We can apply Equation 28.18, taking $n = 1$ and using $m_e = 9.11 \times 10^{-31}$ kg. This gives

$$E_1 = \frac{h^2}{8m_eL^2} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11 \times 10^{-31}\ \text{kg})(2 \times 10^{-10}\ \text{m})^2}$$
$$= 1.51 \times 10^{-18}\ \text{J} = 9.42\ \text{eV}$$

For $n = 2$ and $n = 3$, we find that $E_2 = 4E_1 = 37.7$ eV and $E_3 = 9E_1 = 84.8$ eV. Although this is a rather primitive model, it can be used to describe an electron trapped in a vacant crystal site.

Example 28.10 Energy Quantization for a Macroscopic Object

A 1.00-mg object is confined to moving between two rigid walls separated by 1.00 cm. (a) Calculate its minimum speed.

Solution The minimum speed corresponds to the state for which $n = 1$. Using Equation 28.18 with $n = 1$ gives the zero-point energy:

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(1.00 \times 10^{-6}\ \text{kg})(1.00 \times 10^{-2}\ \text{m})^2}$$
$$= 5.49 \times 10^{-58}\ \text{J}$$

Because $E = \frac{1}{2}mv^2$, we can find v as follows:

$$\tfrac{1}{2}mv^2 = 5.49 \times 10^{-58}\ \text{J}$$
$$v = \left[\frac{2(5.49 \times 10^{-58}\ \text{J})}{1.00 \times 10^{-6}\ \text{kg}}\right]^{1/2} = 3.31 \times 10^{-26}\ \text{m/s}$$

This result is so small that the object can be considered to be at rest, which is what one would expect for a macroscopic object.

(b) If the speed of the object is 3.00×10^{-2} m/s, find the corresponding value of n.

Solution The kinetic energy of the object is

$$E = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(1.00 \times 10^{-6}\ \text{kg})(3.00 \times 10^{-2}\ \text{m/s})^2$$
$$= 4.50 \times 10^{-10}\ \text{J}$$

Because $E_n = n^2E_1$ and $E_1 = 5.49 \times 10^{-58}$ J, we find that

$$n^2E_1 = 4.50 \times 10^{-10}\ \text{J}$$
$$n = \left(\frac{4.50 \times 10^{-10}\ \text{J}}{E_1}\right)^{1/2} = \left(\frac{4.50 \times 10^{-10}\ \text{J}}{5.49 \times 10^{-58}\ \text{J}}\right)^{1/2}$$
$$\approx 9.05 \times 10^{23}$$

This value of n is so large that we would never be able to distinguish the quantized nature of the energy levels. That is, the difference in energy between the two states $n_1 = 9.05 \times 10^{23}$ and $n_2 = (9.05 \times 10^{23}) + 1$ is too small to be detected experimentally. This is another example that illustrates the working of the correspondence principle; that is, as m and/or L become large, the quantum description must agree with the classical result.

EXERCISE 9 An alpha particle in a nucleus can be modeled as a particle moving in a box of width 1.0×10^{-14} m (the approximate diameter of a nucleus). Using this model, estimate (a) the energy and (b) the momentum of an alpha particle in its lowest energy state. ($m_\alpha = 4 \times 1.66 \times 10^{-27}$ kg)
Answer (a) 0.52 MeV (b) 3.3×10^{-20} kg·m/s

28.10 · THE SCHRÖDINGER EQUATION

As we mentioned earlier, the wave function for de Broglie waves must satisfy an equation developed by Schrödinger in 1926. One of the methods of quantum mechanics is to determine a solution to this equation, which in turn yields the allowed wave functions and energy levels of the system under consideration. Proper manipulation of the wave functions enables one to calculate all measurable features of the system.

The Schrödinger equation as it applies to a particle confined to moving along the x axis is

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - U)\psi \qquad [28.19]$$

Because this equation is independent of time, it is commonly referred to as the **time-independent Schrödinger equation.** (We shall not discuss the time-dependent Schrödinger equation in this text.)

In principle, if the potential energy $U(x)$ for the system is known, one can solve Equation 28.19 and obtain the wave functions and energies for the allowed states. Because U may vary discontinuously with position, it may be necessary to solve the equation in pieces. In the process, the wave functions for the different regions must join smoothly at the boundaries. In the language of mathematics, we require that $\psi(x)$ be *continuous.* Furthermore, in order that $\psi(x)$ obey the normalization condition, we require that $\psi(x)$ approach zero as x approaches $\pm\infty$. Finally, $\psi(x)$ must be *single-valued* and $d\psi/dx$ must also be continuous for finite values of $U(x)$.

The task of solving the Schrödinger equation may be very difficult, depending on the form of the potential energy function. As it turns out, the Schrödinger equation has been extremely successful in explaining the behavior of atomic and nuclear systems, whereas classical physics has failed to do so. Furthermore, when quantum mechanics is applied to macroscopic objects, the results agree with classical physics, as required by the correspondence principle.



Erwin Schrödinger (1887–1961)

An Austrian theoretical physicist, Schrödinger is best known as the creator of wave mechanics. He also produced important papers in the fields of statistical mechanics, color vision, and general relativity. Schrödinger did much to hasten the universal acceptance of quantum theory by demonstrating the mathematical equivalence between his wave mechanics and the more abstract matrix mechanics developed by Heisenberg. In 1927, Schrödinger accepted the chair of theoretical physics at the University of Berlin, where he formed a close friendship with Max Planck. In 1933 he left Germany and eventually settled at the Dublin Institute of Advanced Study, where he spent 17 creative years working on problems in general relativity, cosmology, and the application of quantum physics to biology. In 1956 he returned home to Austria, where he died in 1961.

The Particle in a Box

Let us solve the Schrödinger equation for our particle in a one-dimensional box of width L (Fig. 28.22). The walls are infinitely high, corresponding to $U(x) = \infty$ for $x = 0$ and $x = L$. The potential energy is constant within the box, and it is conve-

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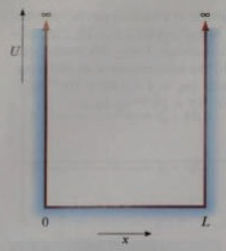


Figure **28.22** Diagram of a one-dimensional box of width L and infinitely high walls.

nient to choose $U = 0$ as its value. Hence, in the region $0 < x < L$, we can express the Schrödinger equation in the form

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad [28.20]$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Because the walls are infinitely high, the particle cannot exist outside the box. Consequently, $\psi(x)$ must be zero outside the box and at the walls. The solution of Equation 28.20 that meets the boundary conditions $\psi(x) = 0$ at $x = 0$ and $x = L$ is

$$\psi(x) = A\sin(kx) \qquad [28.21]$$

This can easily be verified by substitution into Equation 28.20. Note that the first boundary condition, $\psi(0) = 0$, is satisfied by Equation 28.21 because $\sin 0° = 0$. The second boundary condition, $\psi(L) = 0$, is satisfied only if kL is an integral multiple of π, that is, if $kL = n\pi$, where n is an integer. Because $k = \sqrt{2mE}/\hbar$, we get

$$kL = \frac{\sqrt{2mE}}{\hbar}L = n\pi$$

Solving for the allowed energies E gives

$$E_n = \left(\frac{\hbar^2}{8mL^2}\right)n^2 \qquad [28.22]$$

Likewise, the allowed wave functions are given by

$$\psi_n(x) = A\sin\left(\frac{n\pi x}{L}\right) \qquad [28.23]$$

These results agree with those obtained in the preceding section. Normalizing this relationship shows that $A = (2/L)^{1/2}$ (see Problem 28.36).

OPTIONAL

28.11 • TUNNELING THROUGH A BARRIER

A very interesting and peculiar phenomenon occurs when a particle strikes a barrier of finite height and width. Consider a particle of energy E that is incident on a rectangular barrier of height U and width L, where $E < U$ (Fig. 28.23). Classically, the particle is reflected by the barrier, because it does not have sufficient energy to cross or even penetrate it. Thus, regions II and III are classically *forbidden* to the particle. According to quantum mechanics, however, **all regions are accessible to the particle, regardless of its energy,** because the amplitude of the matter wave associated with the particle is nonzero everywhere (except at certain points). A typical wavefunction for this case, illustrated in Figure 28.23, shows the penetration of the wave into the barrier and beyond. The wave functions are sinusoidal to the left (region I) and right (region III) of the barrier and join smoothly with an exponentially decaying function within the barrier (region II). Because the probability of locating the particle is proportional to $|\psi|^2$, we conclude that the chance of finding the particle beyond the barrier in region III is nonzero. This barrier penetration is in complete disagreement with classical physics. The possibility of finding

the particle on the far side of the barrier is called **tunneling or barrier penetration.** Any attempt to observe the particle inside the barrier and confirm the value of its energy is frustrated by the uncertainty principle. If tunneling is to take place, the barrier must be sufficiently narrow and low, to allow a measurement of the particle's position and momentum consistent with expectations.

The probability of tunneling can be described with a **transmission coefficient, *T*,** and a **reflection coefficient, *R*. The transmission coefficient measures the probability that the particle penetrates to the other side of the barrier, and the reflection coefficient is the probability that the particle is reflected by the barrier.** Because the incident particle is either reflected or transmitted, we must require that $T + R = 1$. An approximate expression for the transmission coefficient that is obtained when $T \ll 1$ (a high or wide barrier) is

$$T \approx e^{-2KL} \qquad [28.24]$$

where

$$K = \frac{\sqrt{2m(U-E)}}{\hbar} \qquad [28.25]$$

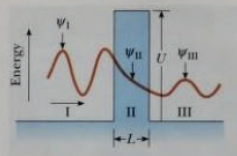


Figure 28.23 Wave function for a particle incident from the left on a barrier of height U. Note that the wave function is sinusoidal in regions I and III but is exponentially decaying in region II. Both amplitude of ψ and energy are plotted along the vertical axis.

Example 28.11 Transmission Coefficient for an Electron

A 30-eV electron is incident on a square barrier of height 40 eV. What is the probability that the electron will tunnel through the barrier if its width is (a) 1.0 nm? (b) 0.10 nm?

Solution (a) In this situation, the quantity $U - E$ has the value

$$U - E = (40\text{ eV} - 30\text{ eV}) = 10\text{ eV} = 1.6 \times 10^{-18}\text{ J}$$

Using Equation 28.25, and given that $L = 1.0$ nm, the quantity $2KL$ is

$$2KL = 2\frac{\sqrt{2(9.11 \times 10^{-31}\text{ kg})(1.6 \times 10^{-18}\text{ J})}}{1.054 \times 10^{-34}\text{ J}\cdot\text{s}}(1 \times 10^{-9}\text{ m}) = 32.4$$

Thus, the probability of tunneling through the barrier is

$$T \approx e^{-2KL} = e^{-32.4} = 8.5 \times 10^{-15}$$

That is, the electron has only about 1 chance in 10^{14} to tunnel through the 1.0-nm-wide barrier.

(b) For $L = 0.10$ nm, we find $2KL = 3.24$, and

$$T \approx e^{-2KL} = e^{-3.24} = 0.039$$

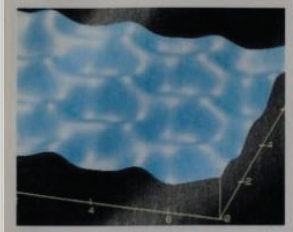
This result shows that the electron has a high probability (4% chance) of penetrating the 0.10-nm barrier. Thus, reducing the width of the barrier by only one order of magnitude has increased the probability of tunneling by about 12 orders of magnitude!

Applications of Tunneling

As we have seen, tunneling is a quantum phenomenon, a manifestation of the wave nature of matter. There are many examples in nature on the atomic and nuclear scales that may be understood only on the basis of tunneling.

- *Tunnel diode.* The tunnel diode is a semiconductor device consisting of two oppositely charged regions separated by a very narrow neutral region. The current in this device is largely due to tunneling of electrons through the neutral region. The current, or rate of tunneling, can be controlled over a wide range by varying the bias voltage, which changes the height of the barrier.

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The surface of graphite as "viewed" with a scanning tunneling microscope. This technique enables scientists to see small details on surfaces with a lateral resolution of about 0.2 nm and a vertical resolution of 0.001 nm. The contours seen here represent the arrangement of individual carbon atoms on the crystal surface.

- *Josephson junction.* A Josephson junction consists of two superconductors separated by a thin insulating oxide layer, 1 to 2 nm thick. Under appropriate conditions, electrons in the superconductors travel as pairs and tunnel from one superconductor to the other through the oxide layer. Several effects have been observed in this type of junction. For example, **a dc current is observed across the junction in the absence of electric and magnetic fields.** The current is proportional to $\sin \phi$, where ϕ is the phase difference between the wave functions in the two superconductors. When a bias voltage, ΔV, is applied across the junction, one observes the current oscillating with a frequency $f = 2e\,\Delta V/h$, where e is the charge on the electron.
- *Alpha decay.* One form of radioactive decay is the emission of alpha particles (the nuclei of helium atoms) by unstable, heavy nuclei. In order for the alpha particle to escape from the nucleus, it must penetrate a barrier several times larger than its energy. The barrier is due to a combination of the attractive nuclear force and the Coulomb repulsion between the alpha particle and the rest of the nucleus. Occasionally an alpha particle tunnels through the barrier, which explains the basic mechanism for this type of decay and the large variations in the mean lifetimes of various radioactive nuclei.
- *Scanning tunneling microscope.* The scanning tunneling microscope, or STM, is a remarkable device that uses tunneling to create images of surfaces with resolution comparable to the size of a single atom. A small probe with a very fine tip is made to scan close to the surface of a specimen. A tunneling current is maintained between the probe and specimen; the current is sensitive to the separation between the tip and specimen. With the maintenance of a constant tunneling current, a feedback signal is obtained that is used to raise and lower the probe as the surface is scanned. Because the vertical motion of the probe follows the contour of the specimen's surface, an image of the surface is obtained.
- *Decay of black holes.* Once inside the event horizon, nothing—not even light—can escape the gravitational pull of a black hole. That was the view held until 1974, when the brilliant British astrophysicist Stephen Hawking proposed that black holes are indeed radiant objects, emitting a variety of particles by a mechanism involving tunneling through the (gravitational) potential barrier surrounding the black hole. The thickness of this barrier is proportional to the size of the black hole, so that the likelihood of a tunneling event initially may be extremely small. As the black hole emits particles, however, its mass and size steadily decrease, making it easier for particles to tunnel out. In this way emission continues at an ever increasing rate, until eventually the black hole radiates itself out of existence in an explosive climax! Thus, Hawking's scenario leads to the decay and eventual demise of any black hole. Calculations indicate that a black hole with the mass of our Sun would survive against decay by tunneling for about 10^{66} years. On the other hand, a black hole with the mass of only a billion tons and roughly the size of a proton should have almost completely evaporated in the 10 billion years that have elapsed since the time of creation. (Such mini black holes are believed to have been formed just after the Big Bang origin of the Universe.)

SUMMARY

The characteristics of *black-body radiation* cannot be explained by classical concepts. Planck introduced the *quantum concept* when he assumed that the atomic oscillators responsible for this radiation existed only in discrete states.

In the **photoelectric effect,** electrons are ejected from a metallic surface when light is incident on that surface. Einstein provided a successful explanation of this effect by extending Planck's quantum theory to the electromagnetic field. In this model, light is viewed as a stream of particles called *photons,* each with energy $E = hf$, where f is the frequency and h is Planck's constant. The maximum kinetic energy of the ejected photoelectron is given by

$$K_{max} = hf - \phi \qquad [28.5]$$

where ϕ is the work function of the metal.

X-rays striking a target are scattered at various angles by electrons in the target. A shift in wavelength is observed for the scattered x-rays, and the phenomenon is known as the **Compton effect.** Classical physics does not explain this effect. If the x-ray is treated as a photon, conservation of energy and momentum applied to the photon–electron collisions yield for the Compton shift the expression

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta) \qquad [28.7]$$

where m_e is the mass of the electron, c is the speed of light, and θ is the scattering angle.

Every object of mass m and momentum p has wave-like properties, with a wavelength given by the de Broglie relation

$$\lambda = \frac{h}{p} \qquad [28.9]$$

By applying this wave theory of matter to electrons in atoms, de Broglie was able to explain the appearance of quantization in the Bohr model of hydrogen as a standing wave phenomenon.

The **uncertainty principle** states that if a measurement of position is made with precision Δx and a *simultaneous* measurement of momentum is made with precision Δp_x, then the product of the two uncertainties can never be less than a number on the order of $\hbar$.

$$\Delta x\,\Delta p_x \geq \frac{\hbar}{2} \qquad [28.11]$$

Matter waves are represented by the wave function $\Psi(x, y, z, t)$. The probability per unit volume that a particle will be found at a point is $|\psi|^2$. If the particle is confined to moving along the x axis, then the probability that it will be located in an interval dx is given by $|\psi|^2\,dx$. Furthermore,

$$\int_{-\infty}^{\infty} |\psi|^2\,dx = 1 \qquad [28.14]$$

The measured position x of the particle, averaged over many trials, is called the **expectation value** of x and is defined by

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2\,dx \qquad [28.16]$$

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If a particle of mass m is confined to moving in a one-dimensional box of width L the walls of which are perfectly rigid, the **allowed wave functions** for the particle are

$$\psi(x) = A \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots \qquad [28.17]$$

where A is the maximum value of ψ. The particle has a well-defined wavelength λ the values of which are such that the width of the box L is equal to an integral number of half wavelengths—that is, $L = n\lambda/2$. These allowed states are called **stationary states** of the system. The energies of a particle in a box are quantized and are given by

$$E_n = \left(\frac{h^2}{8mL^2}\right) n^2 \qquad n = 1, 2, 3, \ldots \qquad [28.18]$$

The wave function must satisfy the **Schrödinger equation.** The time-independent Schrödinger equation for a particle confined to moving along the x axis is

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - U)\psi \qquad [28.19]$$

where E is the total energy of the system and U is the potential energy.

When a particle of energy E meets a barrier of height U, where $E < U$, the particle has a finite probability of penetrating the barrier. Part of the incident wave is transmitted through the barrier and part is reflected. This process, called **tunneling,** is the basic mechanism that explains the operation of the Josephson junction and the phenomenon of alpha decay in some radioactive nuclei.

CONCEPTUAL QUESTIONS

1. Why is it impossible to simultaneously measure with infinite accuracy the position and speed of a particle?
2. If matter has a wave nature, why is this wave-like characteristic not observable in our daily experiences?
3. An electron and a proton are accelerated from rest through the same potential difference. Which particle has the longer wavelength?
4. In describing the passage of electrons through a slit and arriving at a screen, the physicist Richard Feynman said that "electrons arrive in lumps, like particles, but the probability of arrival of these lumps is determined as the intensity of the waves would be. It is in this sense that the electron behaves sometimes like a particle and sometimes like a wave." Elaborate on this point in your own words. (For a further discussion of this point, see R. Feynman, *The Character of Physical Law,* Cambridge, Mass., MIT Press, 1980, chap. 6.)
5. In the photoelectric effect, explain why the photocurrent depends on the intensity of the light source but not on the frequency.
6. Why does the existence of a cutoff frequency in the photoelectric effect favor a particle theory of light rather than a wave theory?
7. Is light a wave or a particle? Support your answer by citing specific experimental evidence.
8. Suppose a photograph were made of a person's face using only a few photons. Would the result be simply a very faint image of the face? Discuss.
9. Which has more energy, a photon of ultraviolet radiation or a photon of yellow light?
10. The brightest star in the constellation Lyra is the bluish star Vega, whereas the brightest star in Bootes is the reddish star Arcturus. How do you account for the difference in color of the two stars?
11. An x-ray photon is scattered by an electron. What happens to the frequency of the scattered photon relative to that of the incident photon?
12. Using Wien's displacement law, estimate the wavelength of highest intensity of radiation given off by a human body. Using this information, explain why an infrared detector would be a useful alarm for security work.
13. Why was the Davisson–Germer diffraction of electrons an important experiment?
14. What is the significance of the wave function ψ?
15. What is the Schrödinger equation? How is it useful in describing atomic phenomena?
16. If the photoelectric effect is observed for one metal, can you conclude that the effect will also be observed for another metal under the same conditions? Explain.

PROBLEMS

Section 28.1 Black-Body Radiation and Planck's Theory

1. The human eye is most sensitive to 560-nm light. What temperature black body would radiate most intensely at this wavelength?
2. An FM radio transmitter has a power output of 150 kW and operates at a frequency of 99.7 MHz. How many photons per second does the transmitter emit?
3. Calculate the energy, in electron volts, of a photon the frequency of which is (a) 6.20×10^{14} Hz, (b) 3.10 GHz, (c) 46.0 MHz. (d) Determine the corresponding wavelengths for these photons.
4. The Sun's light reaches the Earth at an average intensity of 1340 W/m^2. Estimate the number of photons that reach the Earth per second if the temperature of the Sun is 6000 K.
5. The average threshold of dark-adapted (scotopic) vision is 4.00×10^{-11} W/m^2 at a central wavelength of 500 nm. If light having this intensity and wavelength enters the eye and the pupil is open to its maximum diameter of 8.50 mm, how many photons per second enter the eye?
6. A simple pendulum has a length of 1.00 m and a mass of 1.00 kg. If the amplitude of oscillations of the pendulum is 3.00 cm, estimate the quantum number for the pendulum.

Section 28.2 The Photoelectric Effect

7. Molybdenum has a work function of 4.20 eV. (a) Find the cutoff wavelength and threshold frequency for the photoelectric effect. (b) Calculate the stopping potential if the incident light has a wavelength of 180 nm.
8. Electrons are ejected from a metallic surface with speeds ranging up to 4.60×10^5 m/s when light with a wavelength of $\lambda = 625$ nm is used. (a) What is the work function of the surface? (b) What is the cutoff frequency for this surface?
9. A student studying the photoelectric effect from two different metals records the following information: (a) the stopping potential for photoelectrons released from metal 1 is 1.48 V larger than that for metal 2, and (b) the threshold frequency for metal 1 is 40% smaller than that for metal 2. Determine the work function for each metal.
10. From the scattering of sunlight, Thomson calculated the classical radius of the electron as having a value of 2.82×10^{-15} m. If sunlight with an intensity of 500 W/m^2 falls on a disk with this radius, estimate the time required to accumulate 1.00 eV of energy. Assume that light is a classical wave and that the light striking the disk is completely absorbed. How does your estimate compare with the observation that photoelectrons are emitted promptly (within 10^{-9} s)?

Section 28.3 The Compton Effect

11. Calculate the energy and momentum of a photon of wavelength 700 nm.
12. X-rays having an energy of 300 keV undergo Compton scattering from a target. If the scattered rays are detected at 37.0° relative to the incident rays, find (a) the Compton shift at this angle, (b) the energy of the scattered x-ray, and (c) the energy of the recoiling electron.
13. A 0.00160-nm photon scatters from a free electron. For what (photon) scattering angle will the kinetic energy of the recoiling electron and the energy of the scattered photon be the same?
14. A 0.110-nm photon collides with a stationary electron. After the collision, the electron moves forward and the photon recoils backward. Find the momentum and kinetic energy of the electron.
15. A 0.880-MeV photon is scattered by a free electron initially at rest such that the scattering angle of the scattered electron equals that of the scattered photon ($\phi = \theta$ in Fig. 28.9b). (a) Determine the common value of these angles. (b) Determine the energy and momentum of the scattered photon. (c) Determine the kinetic energy and momentum of the scattered electron.

Section 28.4 Photons and Electromagnetic Waves

Section 28.5 The Wave Properties of Particles

16. Calculate the de Broglie wavelength for a proton moving with a speed of 1.00×10^6 m/s.
17. Calculate the de Broglie wavelength for an electron that has kinetic energy (a) 50.0 eV and (b) 50.0 keV.
18. (a) An electron has kinetic energy 3.00 eV. Find its wavelength. (b) A photon has energy 3.00 eV. Find its wavelength.
19. For an electron to be confined to a nucleus, its de Broglie wavelength would have to be less than 1.00×10^{-14} m. (a) What would be the kinetic energy of an electron confined to this region? (b) On the basis of this result, would you expect to find an electron in a nucleus? Explain.
20. In the Davisson–Germer experiment, 54.0-eV electrons were diffracted from a nickel lattice. If the first maximum in the diffraction pattern was observed at $\phi = 50.0°$ (Fig. P28.20), what was the lattice spacing a?
21. Robert Hofstadter won the 1961 Nobel Prize in physics for his pioneering work in scattering 20-GeV electrons from nuclei. (a) What is the γ-factor for a 20.0-GeV electron, where $\gamma = (1 - v^2/c^2)^{-1/2}$? What is the momentum of the electron in kg·m/s? (b) What is the wavelength of a 20.0-GeV electron and how does it compare with the size of a nucleus?

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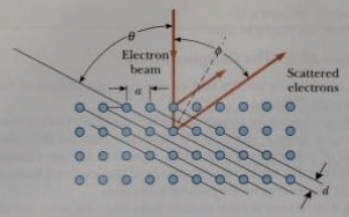


Figure P28.20

22. (a) Show that the frequency f, and wavelength λ, of a particle are related by the expression

$$\left(\frac{f}{c}\right)^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda_C^2}$$

where $\lambda_C = h/mc$ is the Compton wavelength of the particle. (b) Is it ever possible for a photon and a particle (having nonzero mass) to have the same wavelength *and* frequency? Explain.

23. The resolving power of a microscope depends on the wavelength used. If one wished to use a microscope to "see" an atom, a resolution of approximately 1.00×10^{-11} m would have to be obtained. (a) If electrons are used (in an electron microscope), what minimum kinetic energy is required for the electrons? (b) If photons are used, what minimum photon energy is needed to obtain the required resolution?

Section 28.6 The Double-Slit Experiment Revisited

24. A modified oscilloscope is used to perform an electron interference experiment. Electrons are incident on a pair of narrow slits 0.0600 μm apart. The bright bands in the interference pattern are separated by 0.400 mm on a screen 20.0 cm from the slits. Determine the potential difference through which the electrons were accelerated to give this pattern.

25. Neutrons traveling at 0.400 m/s are directed through a double slit having a 1.00-mm separation. An array of detectors is placed 10.0 m from the slit. (a) What is the de Broglie wavelength of the neutrons? (b) How far off axis is the first zero-intensity point on the detector array? (c) When a neutron reaches a detector, can we say which slit the neutron passed through? Explain.

Section 28.7 The Uncertainty Principle

26. Suppose Fuzzy, a quantum-mechanical duck, lives in a world in which $h = 2\pi$ J·s. Fuzzy has a mass of 2.00 kg and is initially known to be within a pond 1.00 m wide. (a) What is the minimum uncertainty in his speed? (b) Assuming this uncertainty in speed to prevail for 5.00 s, determine the uncertainty in position after this time.

27. An electron ($m_e = 9.11 \times 10^{-31}$ kg) and a bullet ($m = 0.0200$ kg) each have a speed of 500 m/s, accurate to within 0.0100%. Within what limits could we determine the position of the objects?

28. A woman on a ladder drops small pellets toward a spot on the floor. (a) Show that, according to the uncertainty principle, the average miss distance must be at least

$$\Delta x = \left(\frac{\hbar}{2m}\right)^{1/2}\left(\frac{H}{2g}\right)^{1/4}$$

where H is the initial height of each pellet above the floor and m is the mass of each pellet. (b) If $H = 2.00$ m and $m = 0.500$ g, what is Δx?

29. Use the uncertainty principle to show that an electron confined inside a nucleus of diameter 2×10^{-15} m is moving relativistically, whereas a proton confined to the same nucleus can be moving non-relativistically.

Section 28.8 An Interpretation of Quantum Mechanics

30. An electron in an infinite square well has a wave function given by

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

for $0 \leq x \leq L$ and zero elsewhere. What are the most probable positions of the electron?

31. A free electron has a wave function

$$\psi(x) = A\sin(5.00 \times 10^{10}\, x)$$

where x is in meters. Find (a) the de Broglie wavelength, (b) the momentum, and (c) the energy in electron volts.

32. The wave function for a particle in an infinite square well is

$$\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

for $0 \leq x \leq L$ and zero elsewhere. (a) Determine the expectation value of x. (b) Determine the probability of finding the particle near $L/2$, by calculating the probability the particle lies in the range $0.49L \leq x \leq 0.51L$. (c) Determine the probability of finding the particle near $L/4$ by calculating the probability the particle lies in the range $0.24L \leq x \leq 0.26L$. (d) Reconcile these probabilities with the result for the average value of x found in part (a).

33. The wave function for a particle is

$$\psi(x) = \sqrt{\frac{a}{\pi(x^2 + a^2)}}$$

for $a > 0$ and $-\infty < x < +\infty$. Determine the probability that the particle is located somewhere between $x = -a$ and $x = +a$.

Section 28.9 A Particle in a Box

34. An electron that has an energy of approximately 6 eV moves between rigid walls 1.00 nm apart. Find (a) the quantum number n for the energy state that the electron occupies and (b) the precise energy of the electron.

35. An electron is contained in a one-dimensional box of width 0.100 nm. (a) Draw an energy-level diagram for the electron for levels up to $n = 4$. (b) Find the wavelengths of all photons that can be emitted by the electron in making transitions that will eventually get it from the $n = 4$ state to the $n = 1$ state.

36. The wave function for a particle confined to moving in a one-dimensional box is

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

Use the normalization condition on ψ to show that

$$A = \sqrt{\frac{2}{L}}$$

Hint: Because the box width is L, the normalization condition (Eq. 28.14) is

$$\int_0^L |\psi|^2\, dx = 1$$

37. A ruby laser emits 694.3-nm light. If this light were due to a transition of an electron in a box from the $n = 2$ state to the $n = 1$ state, find the width of the box.

38. A laser emits light of wavelength λ. If this light were due to a transition of an electron in a box from the $n = 2$ state to the $n = 1$ state, find the width of the box.

39. The nuclear potential energy that binds protons and neutrons in a nucleus is often approximated by a square well. Imagine a proton confined in an infinitely high square well of width 1.00×10^{-5} nm, a typical nuclear diameter. Calculate the wavelength and energy associated with the photon emitted when the proton moves from the $n = 2$ state to the ground state. In what region of the electromagnetic spectrum does this wavelength belong?

Section 28.10 The Schrödinger Equation

40. The wave function of a particle is

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

where A, B, and k are constants. Show that ψ is a solution of the Schrödinger equation (Eq. 28.19), assuming the particle is free ($U = 0$), and find the corresponding energy E of the particle.

41. Show that the wave function $\psi = Ae^{i(kx - \omega t)}$ is a solution to the Schrödinger equation (Eq. 28.19), where $k = 2\pi/\lambda$ and $U = 0$.

42. A particle of mass m moves in a potential well of width $2L$ (from $x = -L$ to $x = +L$), and in this well is a potential given by

$$U(x) = \frac{-\hbar^2 x^2}{mL^2(L^2 - x^2)}$$

In addition, the particle is in a stationary state described by the wave function, $\psi(x) = A(1 - x^2/L^2)$ for $-L < x < +L$, and $\psi(x) = 0$ elsewhere. (a) Determine the energy of the particle in terms of $\hbar$, m, and L. (*Hint:* Use the Schrödinger equation, Eq. 28.19.) (b) Show that $A = (15/16L)^{1/2}$. (c) Determine the probability that the particle is located between $x = -L/3$ and $x = +L/3$.

28.11 Tunneling Through a Barrier (Optional)

43. An electron with kinetic energy $E = 5.00$ eV is incident on a barrier with thickness $L = 0.200$ nm and height $U = 10.0$ eV (Fig. P28.43). What is the approximate probability that the electron (a) will tunnel through the barrier and (b) will be reflected?

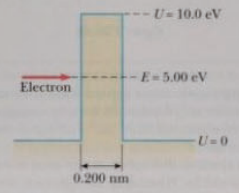


Figure P28.43

44. An electron having total energy $E = 4.50$ eV approaches a rectangular energy barrier with $U = 5.00$ eV and $L = 950$ pm, as in Figure P28.44. Classically, the electron could not pass through the barrier because $E < U$. However, quan-

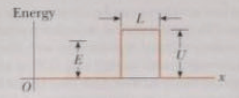


Figure P28.44

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tum-mechanically there is a finite probability of tunneling. Calculate this probability, which is the transmission coefficient.

45. In Problem 44, by how much would the width L of the potential barrier have to be increased for the chance of an incident 4.50-eV electron tunneling through the barrier to be one in one million?

Additional Problems

46. Figure P28.46 shows the stopping potential versus incident photon frequency for the photoelectric effect for sodium. Use these data points to find (a) the work function, (b) the ratio h/e, and (c) the cutoff wavelength. (Data taken from R. A. Millikan, *Phys. Rev.* 7:362, 1916.)

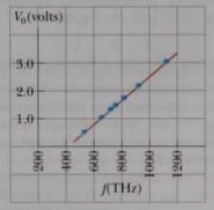


Figure P28.46

47. Photons of wavelength 450 nm are incident on a metal. The most energetic electrons ejected from the metal are bent into a circular arc of radius 20.0 cm by a magnetic field the strength of which is 2.00×10^{-5} T. What is the work function of the metal?

48. A two-slit electron diffraction experiment is done with slits of *unequal* widths. When only slit 1 is open, the number of electrons reaching the screen per second is 25.0 times the number of electrons reaching the screen per second when only slit 2 is open. When both slits are open, an interference pattern results in which the destructive interference is not complete. Find the ratio of the probability of an electron arriving at an interference maximum to the probability of an electron arriving at an adjacent interference minimum. (*Hint:* Use the superposition principle.)

49. The neutron has a mass of 1.67×10^{-27} kg. Neutrons emitted in nuclear reactions can be slowed down via collisions with matter. They are referred to as thermal neutrons once they come into thermal equilibrium with their surroundings. The average kinetic energy ($3k_BT/2$) of a thermal neutron is approximately 0.04 eV. Calculate the de Broglie wavelength of a neutron with a kinetic energy of 0.0400 eV. How does it compare with the characteristic atomic spacing in a crystal? Would you expect thermal neutrons to exhibit diffraction effects when scattered by a crystal?

50. A particle of mass 2.00×10^{-28} kg is confined to a one-dimensional box of width 1.00×10^{-10} m (1 Å). For $n = 1$, (a) What is the particle wavelength? (b) its momentum? (c) its ground-state energy?

51. The table below shows data obtained in a photoelectric experiment. (a) Using these data, make a graph similar to Figure 28.8 that plots as a straight line. From the graph, determine (b) an experimental value for Planck's constant (in joule-seconds) and (c) the work function (in electron volts) for the surface. (Two significant figures for each answer are sufficient.)

Wavelength (nm)	Maximum Kinetic Energy of Photoelectrons (eV)
588	0.67
505	0.98
445	1.35
399	1.63

52. A particle that has kinetic energy $E = 7.00$ eV moves from a region where the potential energy is zero into one in which $U_0 = 5.00$ eV (Fig. P28.52). Classically, one would expect the particle to continue on, although with less kinetic energy. According to quantum mechanics, the particle has a probability of being transmitted and a probability of being reflected. What are these probabilities?

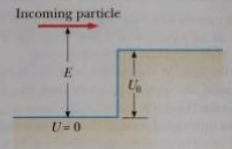


Figure P28.52

53. Particles incident from the left are confronted with a step potential energy shown in Figure P28.52. The step has a height U_0, and the particles have energy $E > U_0$. Classically, all the particles would pass into the region of higher potential energy at the right. However, according to quantum mechanics, a fraction of the particles are reflected at the barrier. The reflection coefficient R for this case is

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

where $k_1 = 2\pi/\lambda_1$ and $k_2 = 2\pi/\lambda_2$ are the wave numbers for the incident and transmitted particles, respectively. If $E = 2U_0$, what fraction of the incident particles are reflected? (This situation is analogous to the partial reflection and transmission of light striking an interface between two different media.)

54. An electron is confined to one-dimensional motion between two rigid walls separated by a distance L. (a) What is the probability of finding the electron within the interval $x = 0$ to $x = L/3$ from one wall if the electron is in its $n = 1$ state? (b) Compare this value with the classical probability.

55. An electron is represented by the time-independent wave function

$$\psi(x) = \begin{cases} Ae^{-\alpha x} & \text{for } x > 0 \\ Ae^{+\alpha x} & \text{for } x < 0 \end{cases}$$

(a) Sketch the wave function as a function of x. (b) Sketch the probability that the electron is found between x and $x + dx$. (c) Why could this be a physically reasonable wave function? (d) Normalize the wave function. (e) Determine the probability of finding the electron somewhere in the range

$$x_1 = -\frac{1}{2\alpha} \quad \text{to} \quad x_2 = \frac{1}{2\alpha}$$

56. An electron is trapped at a defect in a crystal. The defect may be modeled as a one-dimensional, rigid-walled box of width 1.00 nm. (a) Sketch the wave functions and probability densities for the $n = 1$ and $n = 2$ states. (b) For the $n = 1$ state, calculate the probability of finding the electron between $x_1 = 0.150$ nm and $x_2 = 0.350$ nm, where $x = 0$ is the left side of the box. (c) Repeat part (b) for the $n = 2$ state. (d) Calculate the energies in electron volts of the $n = 1$ and $n = 2$ states. *Hint:* For parts (b) and (c), use Equation 28.15 and note that

$$\int \sin^2 ax\, dx = \tfrac{1}{2}x - \frac{1}{4a}\sin 2ax$$

Spreadsheet Problem

S1. Compton scattering involves the scattering of an incident photon of wavelength λ_0 from a free electron. The scattered photon has a decreased wavelength λ' that depends on its scattering angle θ. The electron recoils with kinetic energy K_e at a scattering angle ϕ relative to the incident direction. Spreadsheet 28.1 calculates and plots both K_e and ϕ as functions of θ for a given λ. (a) For $\lambda = 1 \times 10^{-10}$ m, examine the values of K_e and ϕ at $\theta = 30°$, 90°, 120°, and 180°. Do your results make sense? (Consider conservation of energy and momentum.) (b) Repeat part (a) for $\lambda = 1 \times 10^{-9}$ m, 1×10^{-11} m, and 1×10^{-12} m. Explain the overall variation with incident wavelength.

ANSWERS TO CONCEPTUAL PROBLEMS

1. The shape of an object is determined by observing the light reflecting from its surface. In a kiln, the objects will be very hot and will be glowing red. The emitted radiation is far stronger than the reflected radiation, and the thermal radiation emitted is only slightly dependent on the material from which the objects are made. Thus, we have a collection of objects glowing equally with emitted radiation, and only weak reflected light compared to the emitted light. This will result in indistinct outlines of the objects when viewed.
2. The "blackness" of a black body refers to its ideal property of absorbing all radiation incident on it. If an observed room-temperature object in everyday life absorbs all radiation, we describe it as (visibly) black. The black appearance, however, is due to the fact that our eyes are sensitive only to visible light. If we could detect infrared light with our eyes, we would see the object emitting radiation. If the temperature of the black body is raised, Wien's law tells us that the emitted radiation will move into the visible. Thus, the black body could appear possibly as red, white, or blue, depending on its temperature.
3. All objects do radiate energy, but at room temperature, this energy is primarily in the infrared region of the electromagnetic spectrum, which our eyes cannot detect. (Pit vipers have sensory organs that are sensitive to infrared radiation; thus they can seek out their warm-blooded prey in what we would consider complete darkness.)
4. Most metals have cutoff frequencies corresponding to photons in or near the visible range of the electromagnetic spectrum. AM radio wave photons will have far too little energy to eject electrons from the metal.
5. We can picture higher frequency light as a stream of photons of higher energy. In a collision, one photon can give all of its energy to a single electron. The kinetic energy of such an electron is measured by the stopping potential. The reverse voltage (stopping potential) required to stop the current is proportional to the frequency of the incoming light. More intense light consists of more photons striking a unit area each second, but atoms are so small that one emitted electron never gets a "kick" from more than one photon. Increasing the light intensity will increase the size of the current but will not change the energy of the individual ejected electrons. Thus, the stopping potential remains constant.

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6. There are several possibilities for the fate of the photon energy. The photon could simply pass through the material and carry the energy away. It could also reflect from the surface of the material and carry the energy away. The photon could be absorbed into an atom, raising it to a higher energy state. The photon could Compton scatter off many electrons, dissipating its energy as it moves. The photon could also scatter off an atom, exciting it to a higher level and carry a reduced amount of energy off toward another interaction.

7. A microscope can see details no smaller than the wavelength of the waves it uses to produce images. Electrons with kinetic energies of several electron volts have wavelengths of less than a nanometer, which is much smaller than the wavelength of visible light (having wavelengths ranging from about 400 nm to 700 nm.) Therefore, an electron microscope can resolve details of much smaller size as compared to an optical microscope.

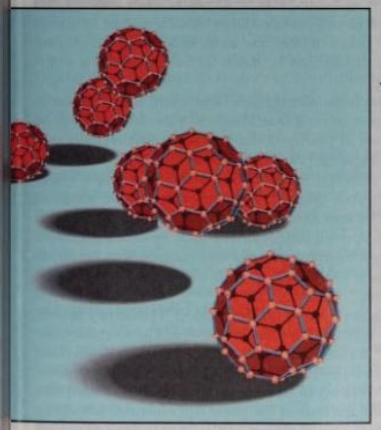
29

Atomic Physics

In Chapter 28 we introduced some of the basic concepts and techniques used in quantum mechanics, along with their applications to various simple systems. This chapter describes the application of quantum mechanics to the real world of atomic structure.

A large portion of this chapter is concerned with the study of the hydrogen atom from the viewpoint of quantum mechanics. Although the hydrogen atom is the simplest atomic system, it is an especially important system to understand, for several reasons:

- Much of what we learn about the hydrogen atom, with its single electron, can be extended to such single-electron ions as He^+ and Li^{2+}.
- The hydrogen atom is an ideal system for performing precise tests of theory against experiment and for improving our overall understanding of atomic structure.



◀ **An artist's rendition of "buckyballs," short for the molecule buckminsterfullerene. These nearly spherical molecular structures that look like soccer balls were named for R. Buckminster Fuller, inventor of the geodesic dome. This new form of carbon, C_{60}, was discovered by astrophysicists while investigating the carbon gas that exists between stars. Scientists are actively studying the properties and potential uses of buckminsterfullerene and related molecules. For example, samples containing C_{60} doped with rubidium exhibit superconductivity at approximately 30 K.**

CHAPTER OUTLINE

29.1 Early Models of the Atom
29.2 The Hydrogen Atom Revisited
29.3 The Spin Magnetic Quantum Number
29.4 The Wave Functions for Hydrogen
29.5 The "Other" Quantum Numbers
29.6 The Exclusion Principle and the Periodic Table
29.7 Atomic Spectra: Visible and X-Ray
29.8 Atomic Transitions
29.9 Lasers and Holography (Optional)

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- The quantum numbers used to characterize the allowed states of hydrogen can be used to describe approximately the allowed states of more complex atoms. This characterization enables us to understand the periodic table of the elements, which is one of the greatest triumphs of quantum mechanics.
- The basic ideas about atomic structure must be well understood before we attempt to deal with the complexities of molecular structures and the electronic structures of solids.

The full mathematical solution of the Schrödinger equation as applied to the hydrogen atom gives a complete and beautiful description of the atom's properties. However, the mathematical procedures that make up the solution are beyond the scope of this book, and so the details are omitted. The solutions for some states of hydrogen are discussed, together with the quantum numbers used to characterize allowed stationary states. We also discuss the physical significance of the quantum numbers and the effect of a magnetic field on certain quantum states.

The *exclusion principle*, also presented in this chapter, is extremely important to an understanding of the properties of multielectron atoms and the arrangement of elements in the periodic table. The implications of the exclusion principle are almost as far reaching as those of the Schrödinger equation. Finally, we apply our knowledge of atomic structure to describe the mechanisms involved in the production of x-rays and in the operation of a laser.

29.1 • EARLY MODELS OF THE ATOM

The model of the atom in Newton's day was a tiny, hard, indestructible sphere. Although this model was a good basis for the kinetic theory of gases, new models had to be devised when later experiments revealed the electrical nature of atoms. J. J. Thomson suggested a model that describes the atom as a volume of positive charge with electrons embedded throughout it, much like the seeds in a watermelon (Fig. 29.1).

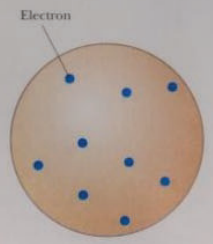


Figure 29.1 Thomson's model of the atom with the electrons embedded inside the positive charge like seeds in a watermelon.

In 1911 Ernest Rutherford and his students Hans Geiger and Ernest Marsden performed a critical experiment that showed that Thomson's model could not be correct. In this experiment, a beam of positively charged alpha particles was projected into a thin metal foil, as in Figure 29.2a. The results of the experiment were astounding. Most of the particles passed through the foil as if it were empty space. But many particles deflected from their original direction of travel were scattered through large angles. Some particles were even deflected backward, reversing their direction of travel. When Geiger informed Rutherford of these results, Rutherford wrote, "It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

Such large deflections were not expected on the basis of Thomson's model. According to this model, a positively charged alpha particle would never come close enough to a large positive charge to cause any large-angle deflections. Rutherford explained his astounding results by assuming that the positive charge was concentrated in a region that was small relative to the size of the atom. He called this concentration of positive charge the **nucleus** of the atom. Any electrons belonging to the atom were assumed to be in the relatively large volume outside the nucleus. In order to explain why these electrons were not pulled into the nucleus, the electrons were viewed as moving in orbits about the positively charged nucleus in the same manner as the planets orbit the Sun, as in Figure 29.2b.

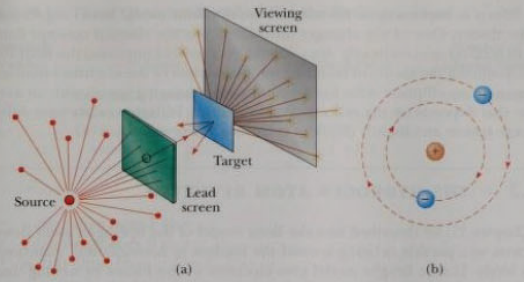


Figure 29.2 (a) Rutherford's technique for observing the scattering of alpha particles from a thin foil target. The source is a naturally occurring radioactive substance, such as radium. (b) Rutherford's planetary model of the atom.

There are two basic difficulties with Rutherford's planetary model. As we saw in Chapter 11, an atom emits certain characteristic frequencies of electromagnetic radiation and no others; the Rutherford model is unable to explain this phenomenon. A second difficulty is that Rutherford's electrons are undergoing a centripetal acceleration. According to Maxwell's theory of electromagnetism, centripetally accelerated charges revolving with frequency f should radiate electromagnetic waves having the same frequency. Unfortunately, this classical model leads to disaster when applied to the atom. As the electron radiates energy, the radius of its orbit steadily decreases and its frequency of revolution increases. This leads to an ever increasing frequency of emitted radiation and a rapid collapse of the atom as the electron plunges into the nucleus (Fig. 29.3).

Now the stage was set for Bohr. In order to circumvent the erroneous deductions of electrons falling into the nucleus and a continuous emission spectrum from elements, Bohr postulated that classical radiation theory does not hold for atomic-sized systems. He overcame the problem of a classical electron that continuously loses energy by applying Planck's ideas of quantized energy levels to orbiting atomic electrons. Thus, as described in Section 11.5, Bohr postulated that electrons in atoms are generally confined to stable, nonradiating energy levels and orbits called stationary states. Furthermore, he applied Einstein's concept of the photon to arrive at an expression for the frequency of light emitted when the electron jumps from one stationary state to another.

One of the first indications that there was a need to modify the Bohr theory arose when improved spectroscopic techniques were used to examine the spectral lines of hydrogen. It was found that many of the lines in the Balmer and other series were not single lines at all. Instead, each was a group of lines spaced very close together. An additional difficulty arose when it was observed that, in some situations, certain single spectral lines were split into three closely spaced lines when the atoms were placed in a strong magnetic field.

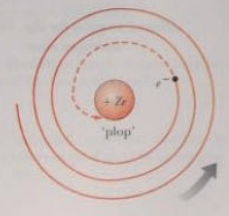


Figure 29.3 The classical model of the nuclear atom.

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Efforts to explain these deviations from the Bohr model led to improvements in the theory. One of the changes introduced was the classical concept that the electron could spin on its axis. Also, Arnold Sommerfeld improved the Bohr theory by introducing the theory of relativity into the analysis of the electron's motion. An electron in an elliptical orbit has a continuously changing speed, with an average value that depends on the eccentricity of the orbit. Different orbits have different average speeds and slightly different relativistic energies.

29.2 • THE HYDROGEN ATOM REVISITED

In Chapter 11, we described how the Bohr model of the hydrogen atom views the electron as a particle orbiting around the nucleus in nonradiating, quantized energy levels. The de Broglie model gave electrons a wave nature by viewing them as standing waves in allowed orbits. This standing wave description removed the objections to Bohr's postulates, which were quite arbitrary. However, the de Broglie model created other problems. The exact nature of the de Broglie wave was unspecified, and the model implied unlikely electron densities at great distances from the nucleus. Fortunately, these difficulties are removed when the methods of quantum mechanics are used to describe atoms.

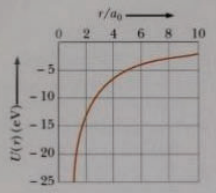


Figure 29.4 The potential energy $U(r)$ versus the ratio r/a_0 for the hydrogen atom. The constant a_0 is the Bohr radius, and r is the electron–proton separation.

The potential energy function for the hydrogen atom is

$$U(r) = -k_e \frac{e^2}{r} \qquad [29.1]$$

where k_e is the Coulomb constant and r is the radial distance from the proton (situated at $r = 0$) to the electron. Figure 29.4 is a plot of this function versus r/a_0 where a_0 is the Bohr radius, 0.0529 nm.

The formal procedure for solving the problem of the hydrogen atom would be to substitute $U(r)$ into the Schrödinger equation and find appropriate solutions to the equation, as we did for the particle in the box in Chapter 28. The current problem is more complicated, however, because it is three-dimensional and because U depends on the radial coordinate r. We shall not attempt to carry out these solutions. Rather, we shall simply describe their properties and some of their implications with regard to atomic structure.

According to quantum mechanics, the energies of the allowed states for the hydrogen atom are

Allowed energies for the hydrogen atom •

$$E_n = -\left(\frac{k_e e^2}{2a_0}\right)\frac{1}{n^2} = -\frac{13.6}{n^2}\text{ eV} \qquad n = 1, 2, 3, \ldots \qquad [29.2]$$

This result is in exact agreement with the Bohr theory. Note that the allowed energies depend only on the quantum number n.

In one-dimensional problems, only one quantum number is needed to characterize a stationary state. In the three-dimensional hydrogen atom, three quantum numbers are needed for each stationary state, corresponding to three independent degrees of freedom for the electron. The three quantum numbers that emerge from the theory are represented by the symbols n, ℓ, and m_ℓ. The quantum number n is called the **principal quantum number,** ℓ is called the **orbital quantum number,** and m_ℓ is called the **orbital magnetic quantum number,** where all three quantum numbers are integers and

TABLE 29.1 Three Quantum Numbers for the Hydrogen Atom

Quantum Number	Name	Allowed Value	Number of Allowed States
n	Principal quantum number	$1, 2, 3, \ldots$	Any number
ℓ	Orbital quantum number	$0, 1, 2, \ldots, n-1$	n
m_ℓ	Orbital magnetic quantum number	$-\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$	$2\ell+1$

- n can range from 1 to ∞
- ℓ can range from 0 to $n-1$
- m_ℓ can range from $-\ell$ to ℓ

Table 29.1 summarizes the rules for determining the allowed values of ℓ and m_ℓ for a given value of n.

For historical reasons, all states with the same principal quantum number are said to form a shell. Shells are identified by the letters $K, L, M, \ldots$, which designate the states for which $n = 1, 2, 3, \ldots$. Likewise, **all states with given values of n and ℓ are said to form a subshell.** The letters $s, p, d, f, g, h, \ldots$ are used to designate the subshells for which $\ell = 0, 1, 2, 3, \ldots$. For example, the state designated by $3p$ has the quantum numbers $n = 3$ and $\ell = 1$; the $2s$ state has the quantum numbers $n = 2$ and $\ell = 0$. These notations are summarized in Table 29.2.

States that violate the rules given in Table 29.1 cannot exist. For instance, the $2d$ state, which would have $n = 2$ and $\ell = 2$, cannot exist because the highest allowed value of ℓ is $n - 1$, or 1 in this case. Thus, for $n = 2$, $2s$ and $2p$ are allowed states but $2d, 2f, \ldots$ are not. For $n = 3$, the allowed states are $3s$, $3p$, and $3d$.

TABLE 29.2 Atomic Shell and Subshell Notations

n	Shell Symbol	ℓ	Subshell Symbol
1	K	0	s
2	L	1	p
3	M	2	d
4	N	3	f
5	O	4	g
6	P	5	h
...		...	

CONCEPTUAL PROBLEM 1

In the hydrogen atom, the quantum number n can increase without limit. Because of this, does the frequency of possible spectral lines from hydrogen also increase without limit?

Example 29.1 The $n = 2$ Level of Hydrogen

For a hydrogen atom, determine the number of orbital states corresponding to the principal quantum number $n = 2$, and calculate the energies of these states.

Solution When $n = 2$, ℓ can be 0 or 1. For $\ell = 0$, m_ℓ can only be 0; for $\ell = 1$, m_ℓ can be -1, 0, or 1. Hence, we have a state designated as the $2s$ state associated with the quantum numbers $n = 2$, $\ell = 0$, and $m_\ell = 0$, and three orbital states designated as $2p$ states for which the quantum numbers are $n = 2$, $\ell = 1$, $m_\ell = -1$; $n = 2$, $\ell = 1$, $m_\ell = 0$; and $n = 2$, $\ell = 1$, $m_\ell = 1$.

Because all these states have the same principal quantum number, they also have the same energy, which can be calculated with Equation 29.2, taking $n = 2$:

$$E_2 = -\frac{13.6}{2^2}\text{ eV} = -3.40\text{ eV}$$

EXERCISE 1 How many possible orbital states are there for the $n = 3$ level of hydrogen? For the $n = 4$ level?
Answer 9 and 16

EXERCISE 2 When the principal quantum number is $n = 4$, how many different values of (a) ℓ and (b) m_ℓ are possible? Answer (a) 4 (b) 7

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29.3 • THE SPIN MAGNETIC QUANTUM NUMBER

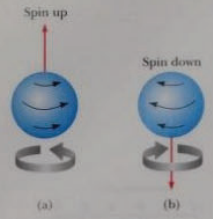


Figure 29.5 The spin of an electron can be either (a) up or (b) down.

Example 29.1 was presented to give you practice in manipulating quantum numbers, but, as we shall see in this section, there are *eight* electron states for $n = 2$ rather than the four we found. These extra states can be explained by requiring a fourth quantum number for each state; the **spin magnetic quantum number, m_s.**

The need for this new quantum number came about because of an unusual feature in the spectra of certain gases such as sodium vapor. Close examination of one of the prominent lines of sodium shows that it is, in fact, two very closely spaced lines called a *doublet*. The wavelengths of these lines occur in the yellow region at 589.0 nm and 589.6 nm. In 1925, when this doublet was first noticed, atomic theory could not explain it. To resolve this dilemma, Samuel Goudsmidt and George Uhlenbeck, following a suggestion by the Austrian physicist Wolfgang Pauli, proposed a new quantum number, called the *spin quantum number.*

In order to describe the spin quantum number, it is convenient (but incorrect) to think of the electron as spinning on its axis as it orbits the nucleus, just as the Earth spins on its axis as it orbits the Sun. There are only two directions that the electron can spin as it orbits the nucleus, as shown in Figure 29.5. If the direction of spin is as shown in Figure 29.5a, the electron is said to have "spin up." If the direction of spin is reversed, as in Figure 29.5b, the electron is said to have "spin down." In the presence of a magnetic field, the energy of the electron is slightly different for the two spin directions, and this energy difference accounts for the sodium doublet. The quantum numbers associated with electron spin are $m_s = \frac{1}{2}$ for the spin-up state and $m_s = -\frac{1}{2}$ for the spin-down state. As we shall see in Example 29.2, this added quantum number doubles the number of allowed states specified by the quantum numbers n, ℓ, and m_ℓ.

The classical description of electron spin as due to a spinning electron is incorrect, because quantum mechanics tells us that a rotational degree of freedom would require too many quantum numbers, and more recent theory indicates that the electron is a point particle, without spatial extent. The electron cannot be considered to be spinning, as pictured in Figure 29.5. In spite of this conceptual difficulty, all experimental evidence supports the fact that the electron does have some intrinsic property that can be described by the spin quantum number. The origin of this fourth quantum number was shown by Sommerfeld and Dirac to lie in the relativistic properties of the electron, which requires four quantum numbers to describe its location in four-dimensional space–time.



Wolfgang Pauli and Niels Bohr watch a spinning top. *(Courtesy of AIP Niels Bohr Library, Margarethe Bohr Collection)*

Example 29.2 Adding Some Spin on Hydrogen

For a hydrogen atom, determine the quantum numbers associated with the possible states that correspond to the principal quantum number $n = 2$.

Solution With the addition of the spin quantum number, we have the possibilities given in the table below.

n	ℓ	m_ℓ	m_s	Subshell	Shell	Maximum Number of Electrons in Subshell
2	0	0	$\frac{1}{2}$	$2s$	L	2
2	0	0	$-\frac{1}{2}$			
2	1	1	$\frac{1}{2}$	$2p$	L	6
2	1	1	$-\frac{1}{2}$			
2	1	0	$\frac{1}{2}$			
2	1	0	$-\frac{1}{2}$			
2	1	−1	$\frac{1}{2}$			
2	1	−1	$-\frac{1}{2}$			

EXERCISE 3 Show that for $n = 3$, there are 18 possible states.

29.4 • THE WAVE FUNCTIONS FOR HYDROGEN

Neglecting electron spin for the present, the potential energy of the hydrogen atom depends only on the radial distance r between nucleus and electron. We therefore expect that some of the allowed states for this atom can be represented by wave functions that depend only on r. This indeed is the case. The simplest wave function for hydrogen is the one that describes the $1s$ ground state and is designated $\psi_{1s}(r)$:

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \qquad [29.3]$$

• *Wave function for hydrogen in its ground state*

where a_0 is the Bohr radius. This wave function satisfies the condition that it approach zero as r approaches ∞ and is normalized as presented. Furthermore, because ψ_{1s} depends only on r, it is spherically symmetric. In fact, **all *s* states have spherical symmetry.**

Recall that the probability of finding the electron in any region is equal to an integral of $|\psi|^2$ over the region, if ψ is normalized. The probability density for the $1s$ state is

$$|\psi_{1s}|^2 = \left(\frac{1}{\pi a_0^3}\right) e^{-2r/a_0} \qquad [29.4]$$

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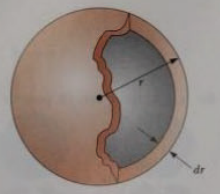


Figure 29.6 A spherical shell of radius r and thickness dr has a volume equal to $4\pi r^2\,dr$.

The probability of finding the electron in a volume element, dV, is $|\psi|^2\,dV$. It is convenient to define the **radial probability density function,** $P(r)$, as the probability per unit radial distance of finding the electron in a spherical shell of radius r and thickness dr. The volume of such a shell equals its surface area, $4\pi r^2$, multiplied by the shell thickness, dr (Fig. 29.6), so that we get

$$P(r)\,dr = |\psi|^2\,dV = |\psi|^2 4\pi r^2\,dr \qquad [29.5]$$

$$P(r) = 4\pi r^2|\psi|^2 \qquad [29.6]$$

Substituting Equation 29.4 into Equation 29.6 gives the radial probability density function for the hydrogen atom in its ground state:

$$P_{1s}(r) = \left(\frac{4r^2}{a_0^3}\right)e^{-2r/a_0} \qquad [29.7]$$

A plot of the function $P_{1s}(r)$ versus r is presented in Figure 29.7a. The peak of the curve corresponds to the most probable value of r for this particular state. The spherical symmetry of the distribution function is shown in Figure 29.7b.

Example 29.3 The Ground State of Hydrogen

Calculate the most probable value of r for an electron in the ground state of the hydrogen atom.

Solution The most probable value of r corresponds to the peak of the plot of $P(r)$ versus r. The slope of the curve at this point is zero, and so we can evaluate the most probable value of r by setting $dP/dr = 0$ and solving for r. Using Equation 29.7, we get

$$\frac{dP}{dr} = \frac{d}{dr}\left[\left(\frac{4r^2}{a_0^3}\right)e^{-2r/a_0}\right] = 0$$

Carrying out the derivative operation and simplifying the expression, we get

$$e^{-2r/a_0}\frac{d}{dr}(r^2) + r^2\frac{d}{dr}(e^{-2r/a_0}) = 0$$

$$2re^{-2r/a_0} + r^2(-2/a_0)e^{-2r/a_0} = 0$$

$$2r[1 - (r/a_0)]e^{-2r/a_0} = 0$$

This expression is satisfied if

$$1 - \frac{r}{a_0} = 0$$

$$r = a_0$$

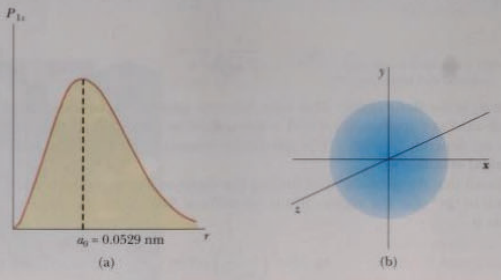


Figure 29.7 (a) The probability of finding the electron as a function of distance from the nucleus for the hydrogen atom in the 1s (ground) state. Note that the probability has its maximum value when r equals the Bohr radius, a_0. (b) The spherical electron cloud for the hydrogen atom in its 1s state.

Example 29.4 Probabilities for the Electron in Hydrogen

Calculate the probability that the electron in the ground state of hydrogen will be found outside the Bohr radius.

Solution The probability is found by integrating the radial probability density for this state, $P_{1s}(r)$, from the Bohr radius, a_0, to ∞. Using Equation 29.7, we get

$$P = \int_{a_0}^{\infty} P_{1s}(r)\,dr = \frac{4}{a_0^3}\int_{a_0}^{\infty} r^2 e^{-2r/a_0}\,dr$$

We can put the integral in dimensionless form by changing variables from r to $z = 2r/a_0$. Noting that $z = 2$ when $r = a_0$, and that $dr = (a_0/2)\,dz$, we get

$$P = \tfrac{1}{2}\int_2^{\infty} z^2 e^{-z}\,dz = -\tfrac{1}{2}(z^2 + 2z + 2)e^{-z}\Big|_2^{\infty}$$

$$P = 5e^{-2} = 0.677 \quad \text{or} \quad 67.7\%$$

Example 29.3 shows that, for the ground state of hydrogen, the most probable value of r equals the Bohr radius, a_0. It turns out that the average value of r for the ground state of hydrogen is $\frac{3}{2}a_0$, which is 50% larger than the most probable value of r (see Problem 33). The reason for this is the large asymmetry in the radial distribution function shown in Figure 29.7a. According to quantum mechanics, there is no sharply defined boundary to the atom. The probability distribution for the electron can be viewed as being a diffuse region of space, commonly referred to as an "electron cloud."

The next-simplest wave function for the hydrogen atom is the one corresponding to the 2s state ($n = 2$, $\ell = 0$). The normalized wave function for this state is

$$\psi_{2s}(r) = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left[2 - \frac{r}{a_0}\right]e^{-r/2a_0} \qquad [29.8]$$

• *Wave function for hydrogen in the 2s state*

Like the ψ_{1s} function, ψ_{2s} depends only on r and is spherically symmetric. The energy corresponding to this state is $E_2 = -(13.6/4)$ eV $= -3.4$ eV. This energy level represents the first excited state of hydrogen. Plots of the radial distribution function for this state and several other states of hydrogen are shown in Figure 29.8. The plot for the 2s state, which applies to a single electron, has two peaks. In this case, the most probable value corresponds to that value of r with the higher value of $P(\approx 5a_0)$. An electron in the 2s state would be much farther from the nucleus (on the average) than an electron in the 1s state. The average value of r is even greater for the 3d, 3p, and 4d states.

As we have mentioned, all s states have spherically symmetric wave functions. The other states are not spherically symmetric. For example, the three wave functions corresponding to the states for which $n = 2$, $\ell = 1$ ($m_\ell = 1$, 0, or -1) can be expressed as appropriate linear combinations of the three p states. Although quantum mechanics limits our knowledge of angular momentum to the projection along any one axis at a time, these p states may be described mathematically as linear combinations of mutually perpendicular functions p_x, p_y, and p_z, as represented in Figure 29.9, where *only* the angular dependence of these functions is shown. Note that the three clouds have identical structure but differ in orientation with respect to the x, y, and z axes. The nonspherical wave functions for these states are

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