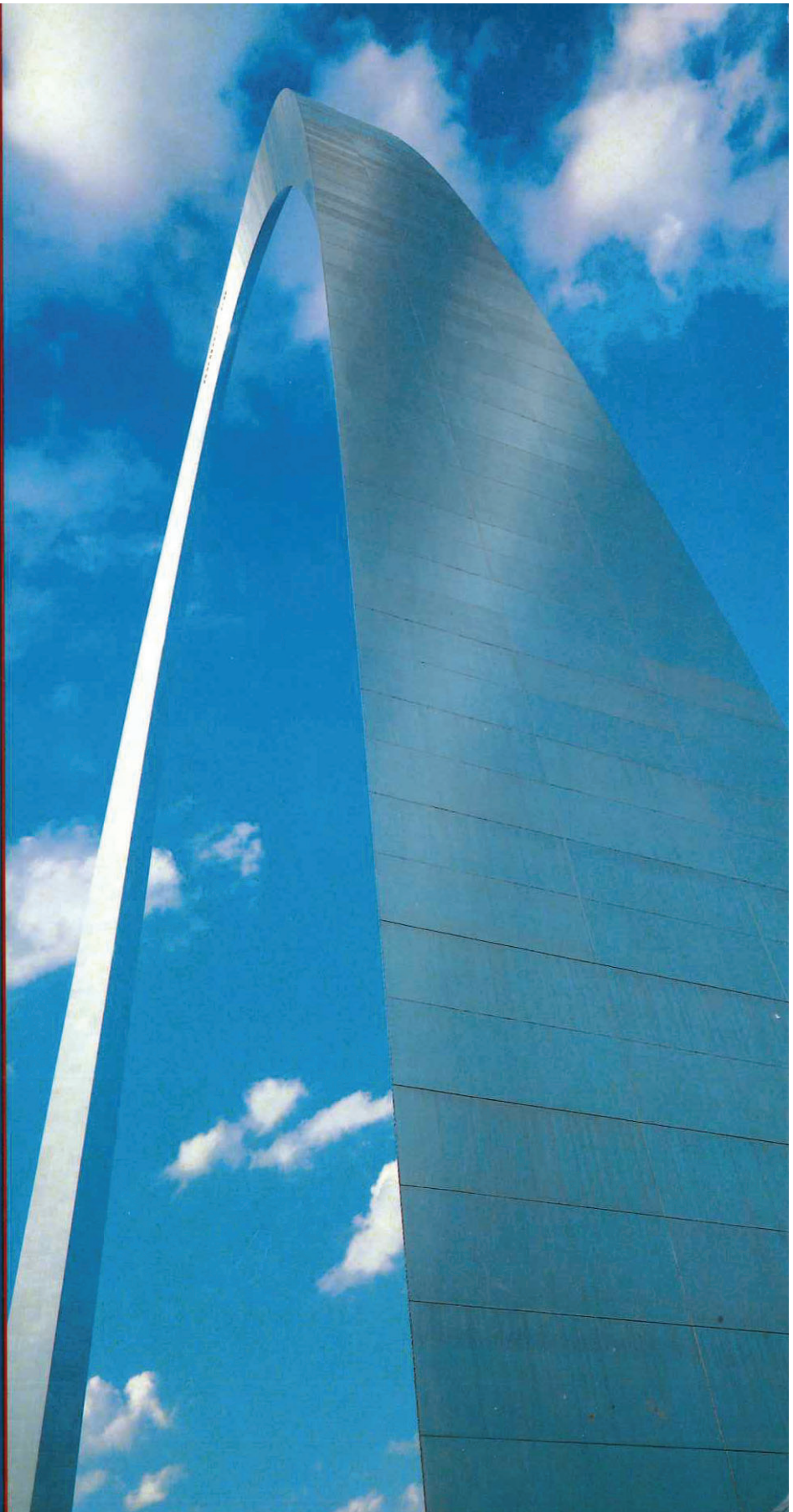


James M. Gere

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and the second for moments about point A, as follows (counterclockwise moments are positive):

$$\sum M_B = 0 \quad -R_A L + P_4(L - a) + M_1 = 0$$

$$\sum M_A = 0 \quad -P_4 a + R_B L + M_1 = 0$$

Therefore, the reactions are

$$R_A = \frac{P_4(L - a)}{L} + \frac{M_1}{L} \quad R_B = \frac{P_4 a}{L} - \frac{M_1}{L}$$

Again, summation of forces in the vertical direction provides a check on these results.

The preceding discussion illustrates how the reactions of statically determinate beams are calculated from equilibrium equations. We have intentionally used symbolic examples rather than numerical examples in order to show how the individual steps are carried out.

4.3 SHEAR FORCES AND BENDING MOMENTS

When a beam is loaded by forces or couples, stresses and strains are created throughout the interior of the beam. To determine these stresses and strains, we first must find the internal forces and internal couples that act on cross sections of the beam.

As an illustration of how these internal quantities are found, consider a cantilever beam AB loaded by a force P at its free end (Fig. 4-4a). We cut through the beam at a cross section mn located at distance x from the free end and isolate the left-hand part of the beam as a free body (Fig. 4-4b). The free body is held in equilibrium by the force P and by the stresses that act over the cut cross section. These stresses represent the action of the right-hand part of the beam on the left-hand part. At this stage of our discussion we do not know the distribution of the stresses acting over the cross section; all we know is that the resultant of these stresses must be such as to maintain equilibrium of the free body.

From statics, we know that the resultant of the stresses acting on the cross section can be reduced to a **shear force** V and a **bending moment** M (Fig. 4-4b). Because the load P is transverse to the axis of the beam, no axial force exists at the cross section. Both the shear force and the bending moment act in the plane of the beam, that is, the vector for the shear force lies in the plane of the figure and the vector for the moment is perpendicular to the plane of the figure.

Shear forces and bending moments, like axial forces in bars and internal torques in shafts, are the resultants of stresses distributed over the cross section. Therefore, these quantities are known collectively as **stress resultants**.

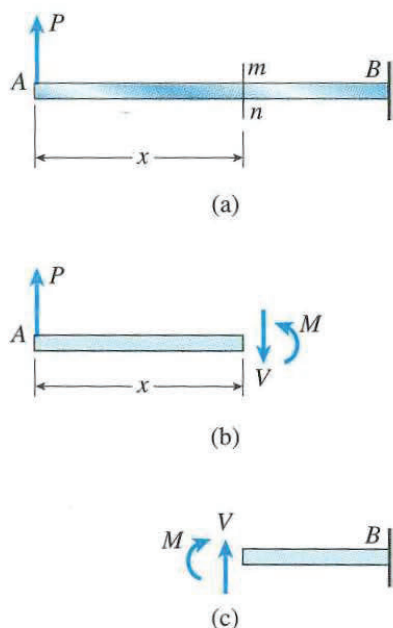


FIG. 4-4 Shear force V and bending moment M in a beam

The stress resultants in statically determinate beams can be calculated from equations of equilibrium. In the case of the cantilever beam of Fig. 4-4a, we use the free-body diagram of Fig. 4-4b. Summing forces in the vertical direction and also taking moments about the cut section, we get

$$\begin{aligned}\sum F_{\text{vert}} = 0 \quad P - V = 0 \quad \text{or} \quad V = P \\ \sum M = 0 \quad M - Px = 0 \quad \text{or} \quad M = Px\end{aligned}$$

where x is the distance from the free end of the beam to the cross section where V and M are being determined. Thus, through the use of a free-body diagram and two equations of equilibrium, we can calculate the shear force and bending moment without difficulty.

Sign Conventions

Let us now consider the sign conventions for shear forces and bending moments. It is customary to assume that shear forces and bending moments are positive when they act in the directions shown in Fig. 4-4b. Note that the shear force tends to rotate the material clockwise and the bending moment tends to compress the upper part of the beam and elongate the lower part. Also, in this instance, the shear force acts downward and the bending moment acts counterclockwise.

The action of these *same* stress resultants against the right-hand part of the beam is shown in Fig. 4-4c. The directions of both quantities are now reversed—the shear force acts upward and the bending moment acts clockwise. However, the shear force still tends to rotate the material clockwise and the bending moment still tends to compress the upper part of the beam and elongate the lower part.

Therefore, we must recognize that the algebraic sign of a stress resultant is determined by how it deforms the material on which it acts, rather than by its direction in space. In the case of a beam, *a positive shear force acts clockwise against the material* (Figs. 4-4b and c) and *a negative shear force acts counterclockwise against the material*. Also, *a positive bending moment compresses the upper part of the beam* (Figs. 4-4b and c) and *a negative bending moment compresses the lower part*.

To make these conventions clear, both positive and negative shear forces and bending moments are shown in Fig. 4-5. The forces and moments are shown acting on an element of a beam cut out between two cross sections that are a small distance apart.

The *deformations* of an element caused by both positive and negative shear forces and bending moments are sketched in Fig. 4-6. We see that a positive shear force tends to deform the element by causing the right-hand face to move downward with respect to the left-hand face, and, as already mentioned, a positive bending moment compresses the upper part of a beam and elongates the lower part.

Sign conventions for stress resultants are called **deformation sign conventions** because they are based upon how the material is deformed.

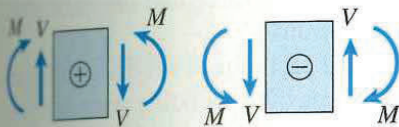
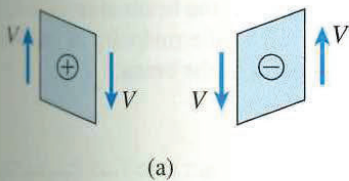
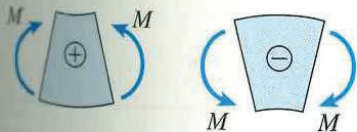


FIG. 4-5 Sign conventions for shear force V and bending moment M



(a)



(b)

FIG. 4-6 Deformations (highly exaggerated) of a beam element caused by (a) shear forces, and (b) bending moments

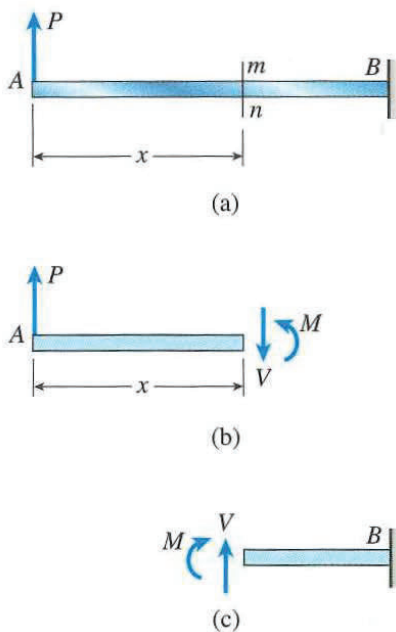


FIG. 4-4 (Repeated)

For instance, we previously used a deformation sign convention in dealing with axial forces in a bar. We stated that an axial force producing elongation (or tension) in a bar is positive and an axial force producing shortening (or compression) is negative. Thus, the sign of an axial force depends upon how it deforms the material, not upon its direction in space.

By contrast, when writing equations of equilibrium we use **static sign conventions**, in which forces are positive or negative according to their directions along the coordinate axes. For instance, if we are summing forces in the y direction, forces acting in the positive direction of the y axis are taken as positive and forces acting in the negative direction are taken as negative.

As an example, consider Fig. 4-4b, which is a free-body diagram of part of the cantilever beam. Suppose that we are summing forces in the vertical direction and that the y axis is positive upward. Then the load P is given a positive sign in the equation of equilibrium because it acts upward. However, the shear force V (which is a *positive* shear force) is given a negative sign because it acts downward (that is, in the negative direction of the y axis). This example shows the distinction between the deformation sign convention used for the shear force and the static sign convention used in the equation of equilibrium.

The following examples illustrate the techniques for handling sign conventions and determining shear forces and bending moments in beams. The general procedure consists of constructing free-body diagrams and solving equations of equilibrium.

Example 4-1

A simple beam AB supports two loads, a force P and a couple M_0 , acting as shown in Fig. 4-7a.

Find the shear force V and bending moment M in the beam at cross sections located as follows: (a) a small distance to the left of the midpoint of the beam, and (b) a small distance to the right of the midpoint of the beam.

Solution

Reactions. The first step in the analysis of this beam is to find the reactions R_A and R_B at the supports. Taking moments about ends B and A gives two equations of equilibrium, from which we find, respectively,

$$R_A = \frac{3P}{4} - \frac{M_0}{L} \quad R_B = \frac{P}{4} + \frac{M_0}{L} \quad (1)$$

(a) *Shear force and bending moment to the left of the midpoint.* We cut the beam at a cross section just to the left of the midpoint and draw a free-body diagram of either half of the beam. In this example, we choose the left-hand half of

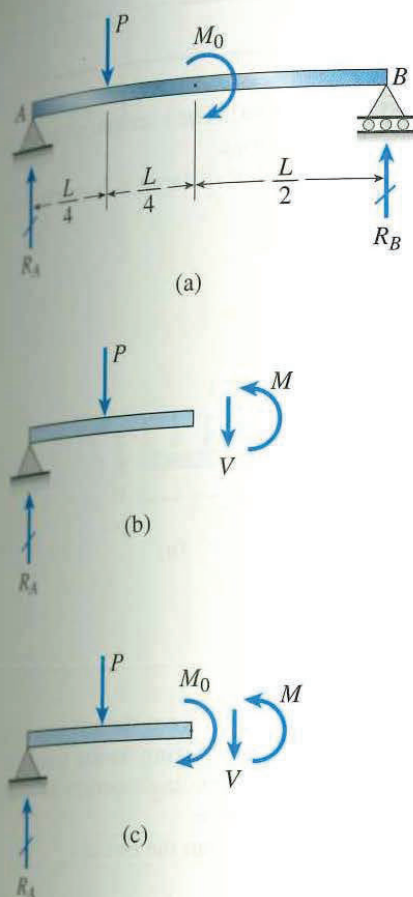


FIG. 4-7 Example 4-1. Shear forces and bending moments in a simple beam

the beam as the free body (Fig. 4-7b). This free body is held in equilibrium by the load P , the reaction R_A , and the two unknown stress resultants—the shear force V and the bending moment M , both of which are shown in their positive directions (see Fig. 4-5). The couple M_0 does not act on the free body because the beam is cut to the left of its point of application. Summing forces in the vertical direction (upward is positive) gives

$$\sum F_{\text{vert}} = 0 \quad R_A - P - V = 0$$

from which we get the shear force:

$$V = R_A - P = -\frac{P}{4} - \frac{M_0}{L} \quad (b) \quad \leftarrow$$

This result shows that when P and M_0 act in the directions shown in Fig. 4-7a, the shear force (at the selected location) is negative and acts in the opposite direction to the positive direction assumed in Fig. 4-7b.

Taking moments about an axis through the cross section where the beam is cut (see Fig. 4-7b) gives

$$\sum M = 0 \quad -R_A\left(\frac{L}{2}\right) + P\left(\frac{L}{4}\right) + M = 0$$

in which counterclockwise moments are taken as positive. Solving for the bending moment M , we get

$$M = R_A\left(\frac{L}{2}\right) - P\left(\frac{L}{4}\right) = \frac{PL}{8} - \frac{M_0}{2} \quad (c) \quad \leftarrow$$

The bending moment M may be either positive or negative, depending upon the magnitudes of the loads P and M_0 . If it is positive, it acts in the direction shown in the figure; if it is negative, it acts in the opposite direction.

(b) *Shear force and bending moment to the right of the midpoint.* In this case we cut the beam at a cross section just to the right of the midpoint and again draw a free-body diagram of the part of the beam to the left of the cut section (Fig. 4-7c). The difference between this diagram and the former one is that the couple M_0 now acts on the free body. From two equations of equilibrium, the first for forces in the vertical direction and the second for moments about an axis through the cut section, we obtain

$$V = -\frac{P}{4} - \frac{M_0}{L} \quad M = \frac{PL}{8} + \frac{M_0}{2} \quad (d,e) \quad \leftarrow$$

These results show that when the cut section is shifted from the left to the right of the couple M_0 , the shear force does not change (because the vertical forces acting on the free body do not change) but the bending moment increases algebraically by an amount equal to M_0 (compare Eqs. c and e).

Example 4-2

A cantilever beam that is free at end A and fixed at end B is subjected to a distributed load of linearly varying intensity q (Fig. 4-8a). The maximum intensity of the load occurs at the fixed support and is equal to q_0 .

Find the shear force V and bending moment M at distance x from the free end of the beam.

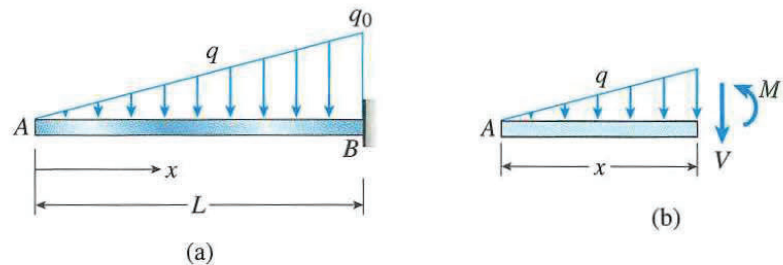


FIG. 4-8 Example 4-2. Shear force and bending moment in a cantilever beam

Solution

Shear force. We cut through the beam at distance x from the left-hand end and isolate part of the beam as a free body (Fig. 4-8b). Acting on the free body are the distributed load q , the shear force V , and the bending moment M . Both unknown quantities (V and M) are assumed to be positive.

The intensity of the distributed load at distance x from the end is

$$q = \frac{q_0 x}{L} \quad (4-1)$$

Therefore, the total downward load on the free body, equal to the area of the triangular loading diagram (Fig. 4-8b), is

$$\frac{1}{2} \left(\frac{q_0 x}{L} \right) (x) = \frac{q_0 x^2}{2L}$$

From an equation of equilibrium in the vertical direction we find

$$V = -\frac{q_0 x^2}{2L} \quad (4-2a) \quad \leftarrow$$

At the free end A ($x = 0$) the shear force is zero, and at the fixed end B ($x = L$) the shear force has its maximum value:

$$V_{\max} = -\frac{q_0 L}{2} \quad (4-2b)$$

which is numerically equal to the total downward load on the beam. The minus signs in Eqs. (4-2a) and (4-2b) show that the shear forces act in the opposite direction to that pictured in Fig. 4-8b.

Bending moment. To find the bending moment M in the beam (Fig. 4-8b), we write an equation of moment equilibrium about an axis through the cut section. Recalling that the moment of a triangular load is equal to the area of the loading diagram times the distance from its centroid to the axis of moments, we obtain the following equation of equilibrium (counterclockwise moments are positive):

$$\sum M = 0 \quad M + \frac{1}{2} \left(\frac{q_0 x}{L} \right) (x) \left(\frac{x}{3} \right) = 0$$

from which we get

$$M = -\frac{q_0 x^3}{6L} \quad (4-3a) \quad \leftarrow$$

At the free end of the beam ($x = 0$), the bending moment is zero, and at the fixed end ($x = L$) the moment has its numerically largest value:

$$M_{\max} = -\frac{q_0 L^2}{6} \quad (4-3b)$$

The minus signs in Eqs. (4-3a) and (4-3b) show that the bending moments act in the opposite direction to that shown in Fig. 4-8b.

5.7 NONPRISMATIC BEAMS

The beam theories described in this chapter were derived for prismatic beams, that is, straight beams having the same cross sections throughout their lengths. However, nonprismatic beams are commonly used to reduce weight and improve appearance. Such beams are found in automobiles, airplanes, machinery, bridges, buildings, tools, and many other applications (Fig. 5-23). Fortunately, the flexure formula (Eq. 5-13) gives reasonably accurate values for the bending stresses in nonprismatic beams whenever the changes in cross-sectional dimensions are gradual, as in the examples shown in Fig. 5-23.

The manner in which the bending stresses vary along the axis of a nonprismatic beam is not the same as for a prismatic beam. In a prismatic beam the section modulus S is constant, and therefore the stresses vary in direct proportion to the bending moment (because $\sigma = M/S$). However, in a nonprismatic beam the section modulus also varies along the axis. Consequently, we cannot assume that the maximum stresses occur at the cross section with the largest bending moment—sometimes the maximum stresses occur elsewhere, as illustrated in Example 5-9.

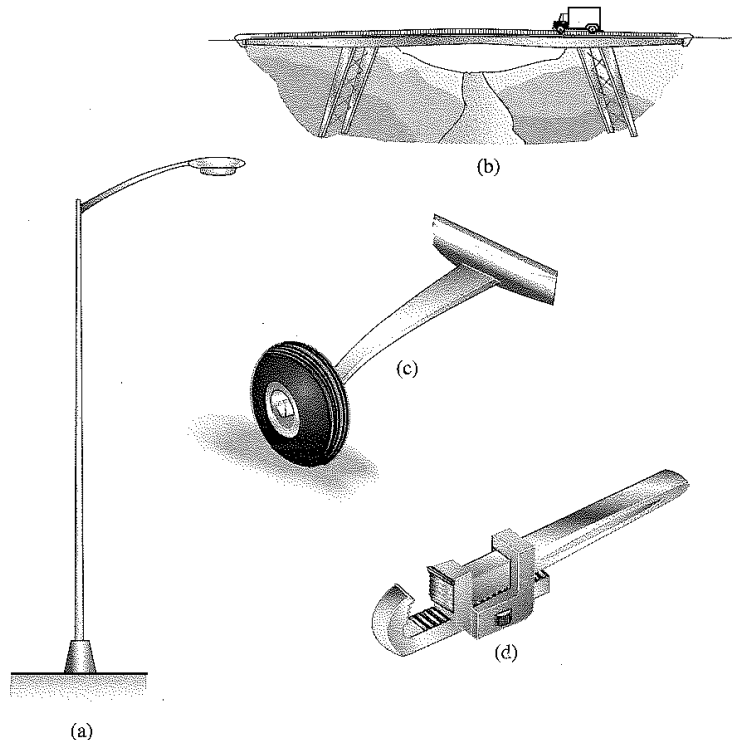


FIG. 5-23 Examples of nonprismatic beams: (a) street lamp, (b) bridge with tapered girders and piers, (c) wheel strut of a small airplane, and (d) wrench handle

Fully Stressed Beams

To minimize the amount of material and thereby have the lightest possible beam, we can vary the dimensions of the cross sections so as to have the maximum allowable bending stress at every section. A beam in this condition is called a **fully stressed beam**, or a *beam of constant strength*. Of course, these ideal conditions are seldom attained because of practical problems in constructing the beam and the possibility of the loads being different from those assumed in design. Nevertheless, knowing the properties of a fully stressed beam can be an important aid to the engineer when designing structures for minimum weight. Familiar examples of structures designed to maintain nearly constant maximum stress are leaf springs in automobiles, bridge girders that are tapered or have cover plates, and some of the structures shown in Fig. 5-23.

The determination of the shape of a fully stressed beam is illustrated in Example 5-10.

Example 5-9

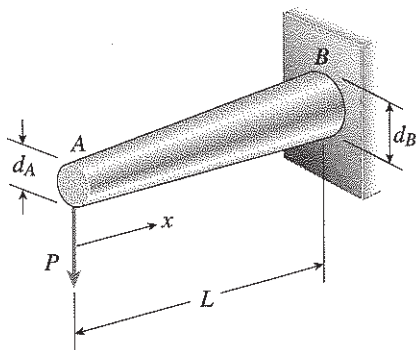


FIG. 5-24 Example 5-9. Tapered cantilever beam of circular cross section

A tapered cantilever beam AB of solid circular cross section supports a load P at the free end (Fig. 5-24). The diameter d_B at the large end is twice the diameter d_A at the small end:

$$\frac{d_B}{d_A} = 2$$

Determine the bending stress σ_B at the fixed support and the maximum bending stress $\sigma_{\max}$.

Solution

If the angle of taper of the beam is small, the bending stresses obtained from the flexure formula will differ only slightly from the exact values. As a guideline concerning accuracy, we note that if the angle between line AB (Fig. 5-24) and the longitudinal axis of the beam is about 20° , the error in calculating the normal stresses from the flexure formula is about 10%. Of course, as the angle of taper decreases, the error becomes smaller.

Section modulus. The section modulus at any cross section of the beam can be expressed as a function of the distance x measured along the axis of the beam. Since the section modulus depends upon the diameter, we first must express the diameter in terms of x , as follows:

$$d_x = d_A + (d_B - d_A) \frac{x}{L} \quad (5-30)$$

in which d_x is the diameter at distance x from the free end. Therefore, the section modulus at distance x from the end (Eq. 5-19b) is

$$S_x = \frac{\pi d_x^3}{32} = \frac{\pi}{32} \left[d_A + (d_B - d_A) \frac{x}{L} \right]^3 \quad (5-31)$$

Bending stresses. Since the bending moment equals Px , the maximum normal stress at any cross section is given by the equation

$$\sigma_1 = \frac{M_x}{S_x} = \frac{32Px}{\pi[d_A + (d_B - d_A)(x/L)]^3} \quad (5-32)$$

We can see by inspection of the beam that the stress σ_1 is tensile at the top of the beam and compressive at the bottom.

Note that Eqs. (5-30), (5-31), and (5-32) are valid for any values of d_A and d_B , provided the angle of taper is small. In the following discussion, we consider only the case where $d_B = 2d_A$.

Maximum stress at the fixed support. The maximum stress at the section of largest bending moment (end B of the beam) can be found from Eq. (5-32) by substituting $x = L$ and $d_B = 2d_A$; the result is

$$\sigma_B = \frac{4PL}{\pi d_A^3} \quad (a) \quad \leftarrow$$

Maximum stress in the beam. The maximum stress at a cross section at distance x from the end (Eq. 5-32) for the case where $d_B = 2d_A$ is

$$\sigma_1 = \frac{32Px}{\pi d_A^3(1 + x/L)^3} \quad (b)$$

To determine the location of the cross section having the largest bending stress in the beam, we need to find the value of x that makes σ_1 a maximum. Taking the derivative $d\sigma_1/dx$ and equating it to zero, we can solve for the value of x that makes σ_1 a maximum; the result is

$$x = \frac{L}{2} \quad (c)$$

The corresponding maximum stress, obtained by substituting $x = L/2$ into Eq. (b), is

$$\sigma_{\max} = \frac{128PL}{27\pi d_A^3} = 4.741 \frac{PL}{\pi d_A^3} \quad (d) \quad \leftarrow$$

In this particular example, the maximum stress occurs at the midpoint of the beam and is 19% greater than the stress σ_B at the built-in end.

Note: If the taper of the beam is reduced, the cross section of maximum normal stress moves from the midpoint toward the fixed support. For very small angles of taper, the maximum stress occurs at end B (see Problem 5.7-6).

11

Columns

11.1 INTRODUCTION

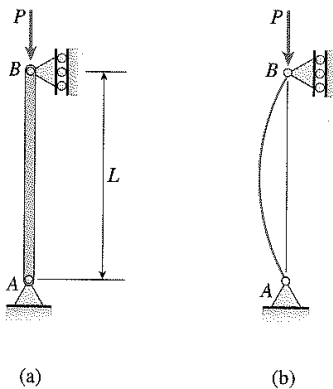


FIG. 11-1 Buckling of a column due to an axial compressive load P

Load-carrying structures may fail in a variety of ways, depending upon the type of structure, the conditions of support, the kinds of loads, and the materials used. For instance, an axle in a vehicle may fracture suddenly from repeated cycles of loading, or a beam may deflect excessively, so that the structure is unable to perform its intended functions. These kinds of failures are prevented by designing structures so that the maximum stresses and maximum displacements remain within tolerable limits. Thus, **strength** and **stiffness** are important factors in design, as discussed throughout the preceding chapters.

Another type of failure is **buckling**, which is the subject matter of this chapter. We will consider specifically the buckling of **columns**, which are long, slender structural members loaded axially in compression (Fig. 11-1a). If a compression member is relatively slender, it may deflect laterally and fail by bending (Fig. 11-1b) rather than failing by direct compression of the material. You can demonstrate this behavior by compressing a plastic ruler or other slender object. When lateral bending occurs, we say that the column has *buckled*. Under an increasing axial load, the lateral deflections will increase too, and eventually the column will collapse completely.

The phenomenon of buckling is not limited to columns. Buckling can occur in many kinds of structures and can take many forms. When you step on the top of an empty aluminum can, the thin cylindrical walls buckle under your weight and the can collapses. When a large bridge collapsed a few years ago, investigators found that failure was caused by the buckling of a thin steel plate that wrinkled under compressive stresses. Buckling is one of the major causes of failures in structures, and therefore the possibility of buckling should always be considered in design.

11.2 BUCKLING AND STABILITY

To illustrate the fundamental concepts of buckling and stability, we will analyze the *idealized structure*, or *buckling model*, shown in Fig. 11-2a. This hypothetical structure consists of two rigid bars AB and BC , each of length $L/2$. They are joined at B by a pin connection and held in a vertical position by a rotational spring having stiffness β_R .* This structure is analogous to the column of Fig. 11-1a, because both structures have simple supports at the ends and are compressed by an axial load P . However, the elasticity of the idealized structure is “concentrated” in the rotational spring, whereas a real column can bend throughout its length (Fig. 11-1b).

In the idealized structure, the two bars are perfectly aligned and the axial load P has its line of action along the longitudinal axis (Fig. 11-2a). Consequently, the spring is unstressed and the bars are in direct compression.

Now suppose that the structure is disturbed by some external force that causes point B to move a small distance laterally (Fig. 11-2b). The rigid bars rotate through small angles θ and a moment develops in the spring. The direction of this moment is such that it tends to return the structure to its original straight position, and therefore it is called a *restoring moment*. At the same time, however, the tendency of the axial

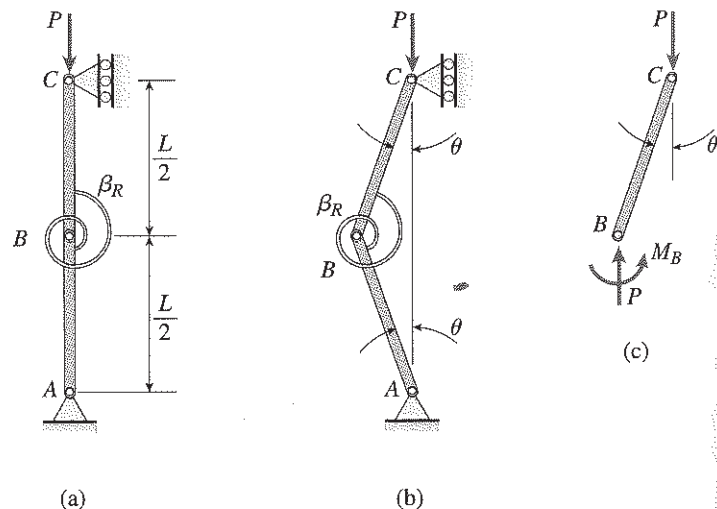


FIG. 11-2 Buckling of an idealized structure consisting of two rigid bars and a rotational spring

*The general relationship for a rotational spring is $M = \beta_R \theta$, where M is the moment acting on the spring, β_R is the rotational stiffness of the spring, and θ is the angle through which the spring rotates. Thus, rotational stiffness has units of moment divided by angle, such as lb-in./rad or N-m/rad. The analogous relationship for a translational spring is $F = \beta \delta$, where F is the force acting on the spring, β is the translational stiffness of the spring (or spring constant), and δ is the change in length of the spring. Thus, translational stiffness has units of force divided by length, such as lb/in. or N/m.

compressive force is to increase the lateral displacement. Thus, these two actions have opposite effects—the restoring moment tends to *decrease* the displacement and the axial force tends to *increase* it.

Now consider what happens when the disturbing force is removed. If the axial force P is relatively small, the action of the restoring moment will predominate over the action of the axial force and the structure will return to its initial straight position. Under these conditions, the structure is said to be **stable**. However, if the axial force P is large, the lateral displacement of point B will increase and the bars will rotate through larger and larger angles until the structure collapses. Under these conditions, the structure is **unstable** and fails by lateral buckling.

Critical Load

The transition between the stable and unstable conditions occurs at a special value of the axial force known as the **critical load** (denoted by the symbol P_{cr}). We can determine the critical load by considering the structure in the disturbed position (Fig. 11-2b) and investigating its equilibrium.

First, we consider the entire structure as a free body and sum moments about support A . This step leads to the conclusion that there is no horizontal reaction at support C . Second, we consider bar BC as a free body (Fig. 11-2c) and note that it is subjected to the action of the axial forces P and the moment M_B in the spring. The moment M_B is equal to the rotational stiffness β_R times the angle of rotation 2θ of the spring; thus,

$$M_B = 2\beta_R\theta \quad (a)$$

Since the angle θ is a small quantity, the lateral displacement of point B is $\theta L/2$. Therefore, we obtain the following equation of equilibrium by summing moments about point B (Fig. 11-2c):

$$M_B - P\left(\frac{\theta L}{2}\right) = 0 \quad (b)$$

or, upon substituting from Eq. (a),

$$\left(2\beta_R - \frac{PL}{2}\right)\theta = 0 \quad (11-1)$$

One solution of this equation is $\theta = 0$, which means that the structure is in equilibrium when it is perfectly straight, regardless of the magnitude of the force P .

A second solution is obtained by setting the term in parentheses equal to zero and solving for the load P , which is the critical load:

$$P_{cr} = \frac{4\beta_R}{L} \quad (11-2)$$

At this value of the load the structure is in equilibrium regardless of the magnitude of the angle θ (provided the angle remains small, because we made that assumption when deriving Eq. b).

From the preceding analysis we see that the critical load is the *only* load for which the structure will be in equilibrium in the disturbed position. At this value of the load, the restoring effect of the moment in the spring just matches the buckling effect of the axial load. Therefore, the critical load represents the boundary between the stable and unstable conditions.

If the axial load is less than P_{cr} , the effect of the moment in the spring predominates and the structure returns to the vertical position after a slight disturbance; if the axial load is larger than P_{cr} , the effect of the axial force predominates and the structure buckles:

If $P < P_{cr}$, the structure is stable

If $P > P_{cr}$, the structure is unstable

From Eq. (11-2) we see that the stability of the structure is increased either by *increasing its stiffness* or by *decreasing its length*. Later in this chapter, when we determine critical loads for various types of columns, we will see that these same observations apply.

Summary

Let us now summarize the behavior of the idealized structure (Fig. 11-2a) as the axial load P increases from zero to a large value.

When the axial load is less than the critical load ($0 < P < P_{cr}$), the structure is in equilibrium when it is perfectly straight. Because the equilibrium is *stable*, the structure returns to its initial position after being disturbed. Thus, the structure is in equilibrium *only* when it is perfectly straight ($\theta = 0$).

When the axial load is greater than the critical load ($P > P_{cr}$), the structure is still in equilibrium when $\theta = 0$ (because it is in direct compression and there is no moment in the spring), but the equilibrium is *unstable* and cannot be maintained. The slightest disturbance will cause the structure to buckle.

At the critical load ($P = P_{cr}$), the structure is in equilibrium even when point B is displaced laterally by a small amount. In other words, the structure is in equilibrium for *any* small angle θ , including $\theta = 0$. However, the structure is neither stable nor unstable—it is at the boundary between stability and instability. This condition is referred to as **neutral equilibrium**.

The equilibrium conditions for the idealized structure are shown in the graph of axial load P versus angle of rotation θ (Fig. 11-3). The two heavy lines, one vertical and one horizontal, represent the equilibrium conditions. Point B , where the equilibrium diagram branches, is called a **bifurcation point**.

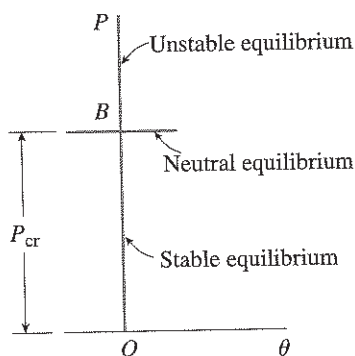


FIG. 11-3 Equilibrium diagram for buckling of an idealized structure

The horizontal line for neutral equilibrium extends to the left and right of the vertical axis because the angle θ may be clockwise or counterclockwise. The line extends only a short distance, however, because our analysis is based upon the assumption that θ is a small angle. (This assumption is quite valid, because θ is indeed small when the structure first departs from its vertical position. If buckling continues and θ becomes large, the line labeled "Neutral equilibrium" curves upward, as shown later in Fig. 11-11 of Section 11.3.)

The three equilibrium conditions represented by the diagram of Fig. 11-3 are analogous to those of a ball placed upon a smooth surface (Fig. 11-4). If the surface is concave upward, like the inside of a dish, the equilibrium is stable and the ball always returns to the low point when disturbed. If the surface is convex upward, like a dome, the ball can theoretically be in equilibrium on top of the surface, but the equilibrium is unstable and in reality the ball rolls away. If the surface is perfectly flat, the ball is in neutral equilibrium and remains wherever it is placed.

As we will see in the next section, the behavior of an ideal elastic column is analogous to that of the buckling model shown in Fig. 11-2. Furthermore, many other kinds of structural and mechanical systems fit this model.

11.3 COLUMNS WITH PINNED ENDS

We begin our investigation of the stability behavior of columns by analyzing a slender column with pinned ends (Fig. 11-5a). The column is loaded by a vertical force P that is applied through the centroid of the end cross section. The column itself is perfectly straight and is made of a linearly elastic material that follows Hooke's law. Since the column is assumed to have no imperfections, it is referred to as an **ideal column**.

For purposes of analysis, we construct a coordinate system with its origin at support A and with the x axis along the longitudinal axis of the column. The y axis is directed to the left in the figure, and the z axis (not shown) comes out of the plane of the figure toward the viewer. We assume that the xy plane is a plane of symmetry of the column and that any bending takes place in that plane (Fig. 11-5b). The coordinate system is identical to the one used in our previous discussions of beams, as can be seen by rotating the column clockwise through an angle of 90° .

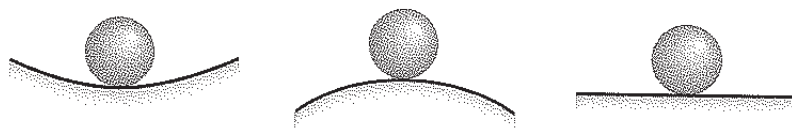
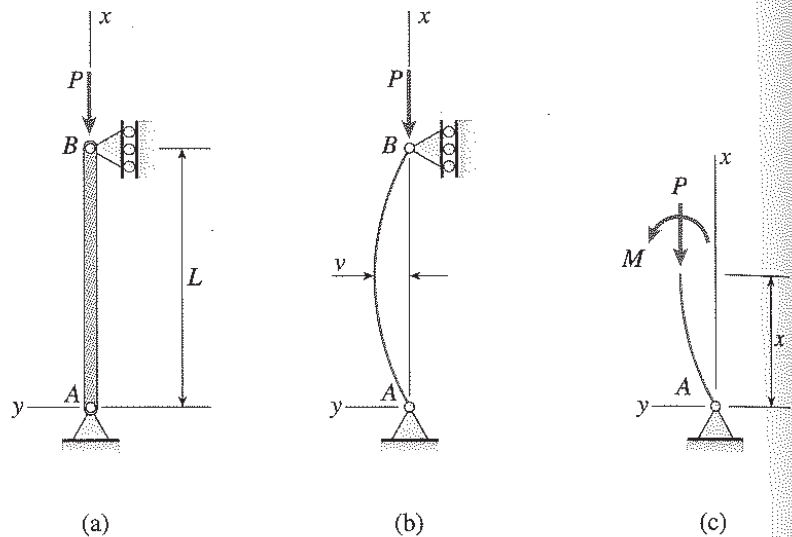


FIG. 11-4 Ball in stable, unstable, and neutral equilibrium

FIG. 11-5 Column with pinned ends:
 (a) ideal column, (b) buckled shape, and
 (c) axial force P and bending moment M
 acting at a cross section



When the axial load P has a small value, the column remains perfectly straight and undergoes direct axial compression. The only stresses are the uniform compressive stresses obtained from the equation $\sigma = P/A$. The column is in **stable equilibrium**, which means that it returns to the straight position after a disturbance. For instance, if we apply a small lateral load and cause the column to bend, the deflection will disappear and the column will return to its original position when the lateral load is removed.

As the axial load P is gradually increased, we reach a condition of **neutral equilibrium** in which the column may have a bent shape. The corresponding value of the load is the **critical load** P_{cr} . At this load the column may undergo small lateral deflections with no change in the axial force. For instance, a small lateral load will produce a bent shape that does not disappear when the lateral load is removed. Thus, the critical load can maintain the column in equilibrium *either* in the straight position or in a slightly bent position.

At higher values of the load, the column is **unstable** and may collapse by buckling, that is, by excessive bending. For the ideal case that we are discussing, the column will be in equilibrium in the straight position even when the axial force P is greater than the critical load. However, since the equilibrium is unstable, the smallest imaginable disturbance will cause the column to deflect sideways. Once that happens, the deflections will immediately increase and the column will fail by buckling. The behavior is similar to that described in the preceding section for the idealized buckling model (Fig. 11-2).

The behavior of an ideal column compressed by an axial load P (Figs. 11-5a and b) may be summarized as follows:

If $P < P_{cr}$, the column is in stable equilibrium in the straight position.

If $P = P_{cr}$, the column is in neutral equilibrium in either the straight or a slightly bent position.

If $P > P_{cr}$, the column is in unstable equilibrium in the straight position and will buckle under the slightest disturbance.

Of course, a real column does not behave in this idealized manner because imperfections are always present. For instance, the column is not *perfectly* straight, and the load is not *exactly* at the centroid. Nevertheless, we begin by studying ideal columns because they provide insight into the behavior of real columns.

Differential Equation for Column Buckling

To determine the critical loads and corresponding deflected shapes for an ideal pin-ended column (Fig. 11-5a), we use one of the differential equations of the deflection curve of a beam (see Eqs. 9-12a, b, and c in Section 9.2). These equations are applicable to a buckled column because the column bends as though it were a beam (Fig. 11-5b).

Although both the fourth-order differential equation (the load equation) and the third-order differential equation (the shear-force equation) are suitable for analyzing columns, we will elect to use the second-order equation (the bending-moment equation) because its general solution is usually the simplest. The bending-moment equation (Eq. 9-12a) is

$$EIv'' = M \quad (11-3)$$

in which M is the bending moment at any cross section, v is the lateral deflection in the y direction, and EI is the flexural rigidity for bending in the xy plane.

The bending moment M at distance x from end A of the buckled column is shown acting in its positive direction in Fig. 11-5c. Note that the bending moment sign convention is the same as that used in earlier chapters, namely, positive bending moment produces positive curvature (see Figs. 9-3 and 9-4). The axial force P acting at the cross section is also shown in Fig. 11-5c. Since there are no horizontal forces acting at the supports, there are no shear forces in the column. Therefore, from equilibrium of moments about point A, we obtain

$$M + Pv = 0 \quad \text{or} \quad M = -Pv \quad (11-4)$$

where v is the deflection at the cross section.

This same expression for the bending moment is obtained if we assume that the column buckles to the right instead of to the left (Fig. 11-6a). When the column deflects to the right, the deflection itself is $-v$ but the moment of the axial force about point A also changes

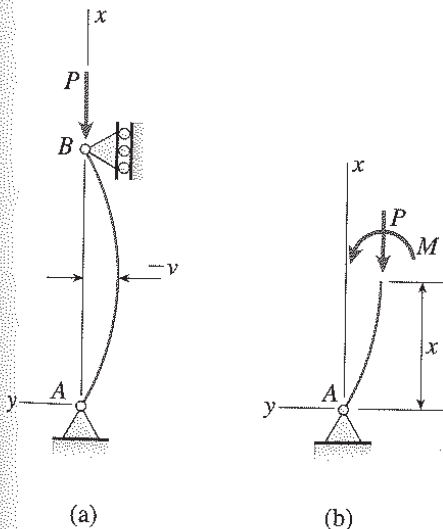


FIG. 11-6 Column with pinned ends (alternative direction of buckling)

sign. Thus, the equilibrium equation for moments about point A (see Fig. 11-6b) is

$$M - P(-v) = 0$$

which gives the same expression for the bending moment M as before.

The differential equation of the deflection curve (Eq. 11-3) now becomes

$$EIv'' + Pv = 0 \quad (11-5)$$

By solving this equation, which is a *homogeneous, linear, differential equation of second order with constant coefficients*, we can determine the magnitude of the critical load and the deflected shape of the buckled column.

Note that we are analyzing the buckling of columns by solving the same basic differential equation as the one we solved in Chapters 9 and 10 when finding beam deflections. However, there is a fundamental difference in the two types of analysis. In the case of beam deflections, the bending moment M appearing in Eq. (11-3) is a function of the loads only—it does not depend upon the deflections of the beam. In the case of buckling, the bending moment is a function of the deflections themselves (Eq. 11-4). Thus, we now encounter a new aspect of bending analysis. In our previous work, the deflected shape of the structure was not considered, and the equations of equilibrium were based upon the geometry of the *undeformed* structure. Now, however, the geometry of the *deformed* structure is taken into account when writing equations of equilibrium.

Solution of the Differential Equation

For convenience in writing the general solution of the differential equation (Eq. 11-5), we introduce the notation

$$k^2 = \frac{P}{EI} \quad \text{or} \quad k = \sqrt{\frac{P}{EI}} \quad (11-6a,b)$$

in which k is always taken as a positive quantity. Note that k has units of the reciprocal of length, and therefore quantities such as kx and kL are nondimensional.

Using this notation, we can rewrite Eq. (11-5) in the form

$$v'' + k^2v = 0 \quad (11-7)$$

From mathematics we know that the solution of this equation is

$$v = C_1 \sin kx + C_2 \cos kx \quad (11-8)$$

in which C_1 and C_2 are constants of integration (to be evaluated from the boundary conditions, or end conditions, of the column). Note that

the number of arbitrary constants in the solution (two in this case) agrees with the order of the differential equation. Also, note that we can verify the solution by substituting the expression for v (Eq. 11-8) into the differential equation (Eq. 11-7) and reducing it to an identity.

To evaluate the constants of integration appearing in the solution (Eq. 11-8), we use the boundary conditions at the ends of the column; namely, the deflection is zero when $x = 0$ and $x = L$ (see Fig. 11-5b):

$$v(0) = 0 \quad \text{and} \quad v(L) = 0 \quad (\text{a,b})$$

The first condition gives $C_2 = 0$, and therefore

$$v = C_1 \sin kx \quad (\text{c})$$

The second condition gives

$$C_1 \sin kL = 0 \quad (\text{d})$$

From this equation we conclude that either $C_1 = 0$ or $\sin kL = 0$. We will consider both of these possibilities.

Case 1. If the constant C_1 equals zero, the deflection v is also zero (see Eq. c), and therefore the column remains straight. In addition, we note that when C_1 equals zero, Eq. (d) is satisfied for *any* value of the quantity kL . Consequently, the axial load P may also have any value (see Eq. 11-6b). This solution of the differential equation (known in mathematics as the *trivial solution*) is represented by the vertical axis of the load-deflection diagram (Fig. 11-7). It gives the behavior of an ideal column that is in equilibrium (either stable or unstable) in the straight position (no deflection) under the action of the compressive load P .

Case 2. The second possibility for satisfying Eq. (d) is given by the following equation, known as the **buckling equation**:

$$\sin kL = 0 \quad (11-9)$$

This equation is satisfied when $kL = 0, \pi, 2\pi, \dots$. However, since $kL = 0$ means that $P = 0$, this solution is not of interest. Therefore, the solutions we will consider are

$$kL = n\pi \quad n = 1, 2, 3, \dots \quad (\text{e})$$

or (see Eq. 11-6a):

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad n = 1, 2, 3, \dots \quad (11-10)$$

This formula gives the values of P that satisfy the buckling equation and provide solutions (other than the trivial solution) to the differential equation.

The equation of the deflection curve (from Eqs. c and e) is

$$v = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \dots \quad (11-11)$$

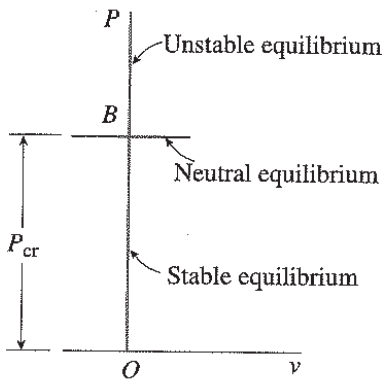


FIG. 11-7 Load-deflection diagram for an ideal, linearly elastic column

Only when P has one of the values given by Eq. (11-10) is it theoretically possible for the column to have a bent shape (given by Eq. 11-11). For all other values of P , the column is in equilibrium only if it remains straight. Therefore, the values of P given by Eq. (11-10) are the **critical loads** for this column.

Critical Loads

The lowest critical load for a column with pinned ends (Fig. 11-8a) is obtained when $n = 1$:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad (11-12)$$

The corresponding buckled shape (sometimes called a *mode shape*) is

$$v = C_1 \sin \frac{\pi x}{L} \quad (11-13)$$

as shown in Fig. 11-8b. The constant C_1 represents the deflection at the midpoint of the column and may have any small value, either positive or negative. Therefore, the part of the load-deflection diagram corresponding to P_{cr} is a horizontal straight line (Fig. 11-7). Thus, the deflection at the critical load is *undefined*, although it must remain small for our equations to be valid. Above the bifurcation point B the equilibrium is unstable, and below point B it is stable. Buckling of a pinned-end column in the first mode is called the **fundamental case** of column buckling.

The type of buckling described in this section is called **Euler buckling**, and the critical load for an ideal elastic column is often called

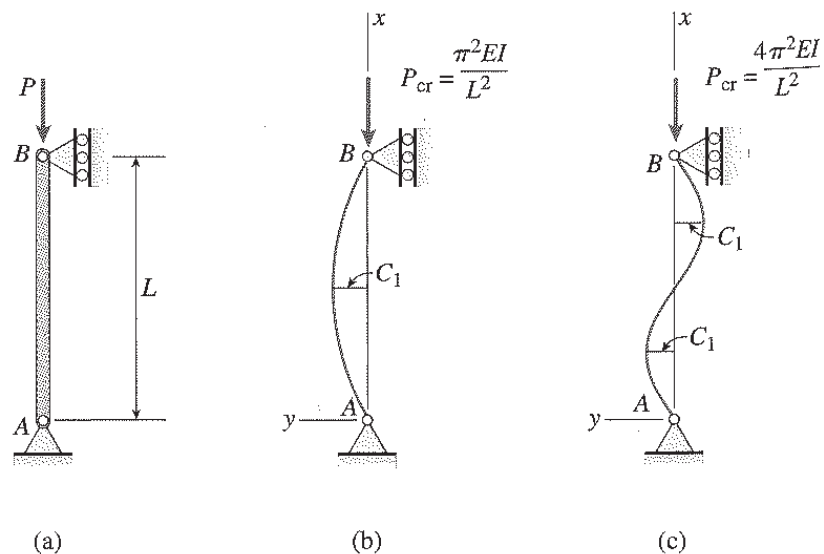


FIG. 11-8 Buckled shapes for an ideal column with pinned ends: (a) initially straight column, (b) buckled shape for $n = 1$, and (c) buckled shape for $n = 2$

the **Euler load**. The famous mathematician Leonhard Euler (1707–1783), generally recognized as the greatest mathematician of all time, was the first person to investigate the buckling of a slender column and determine its critical load (Euler published his results in 1744); see Ref. 11-1.

By taking higher values of the index n in Eqs. (11-10) and (11-11), we obtain an infinite number of critical loads and corresponding mode shapes. The mode shape for $n = 2$ has two half-waves, as pictured in Fig. 11-8c. The corresponding critical load is four times larger than the critical load for the fundamental case. The magnitudes of the critical loads are proportional to the square of n , and the number of half-waves in the buckled shape is equal to n .

Buckled shapes for the higher modes are often of no practical interest because the column buckles when the axial load P reaches its lowest critical value. The only way to obtain modes of buckling higher than the first is to provide lateral support of the column at intermediate points, such as at the midpoint of the column shown in Fig. 11-8 (see Example 11-1 at the end of this section).

From Eq. (11-12) we see that the critical load of a column is proportional to the flexural rigidity EI and inversely proportional to the square of the length. Of particular interest is the fact that the strength of the material itself, as represented by a quantity such as the proportional limit or the yield stress, does not appear in the equation for the critical load. Therefore, increasing a strength property does not raise the critical load of a slender column. It can only be raised by increasing the flexural rigidity, reducing the length, or providing additional lateral support.

The flexural rigidity can be increased by using a “stiffer” material (that is, a material with larger modulus of elasticity E) or by distributing the material in such a way as to increase the moment of inertia I of the cross section, just as a beam can be made stiffer by increasing the moment of inertia. The moment of inertia is increased by distributing the material farther from the centroid of the cross section. Hence, a hollow tubular member is generally more economical for use as a column than a solid member having the same cross-sectional area.

Reducing the wall thickness of a tubular member and increasing its lateral dimensions (while keeping the cross-sectional area constant) also increases the critical load because the moment of inertia is increased. This process has a practical limit, however, because eventually the wall itself will become unstable. When that happens, localized buckling occurs in the form of small corrugations or wrinkles in the walls of the column. Thus, we must distinguish between *overall buckling* of a column, which is illustrated in Fig. 11-8, and *local buckling* of its parts. The latter requires more detailed investigations, such as those described in Refs. 11-4, 11-5, and 11-6 and in current textbooks on buckling and stability. In this chapter we consider only overall buckling of columns.

In the preceding analysis we assumed that the xy plane was a plane of symmetry of the column and that buckling took place in that plane

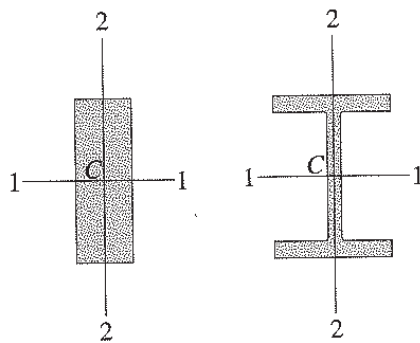


FIG. 11-9 Cross sections of columns showing principal centroidal axes with $I_1 > I_2$

(Fig. 11-8). The latter assumption will be met if the column has lateral supports perpendicular to the plane of the figure, so that the column is constrained to buckle in the xy plane. If the column is supported only at its ends and is free to buckle in *any* direction, then bending will occur about the principal centroidal axis having the smaller moment of inertia.

For instance, consider the rectangular and wide-flange cross sections shown in Fig. 11-9. In each case, the moment of inertia I_1 is greater than the moment of inertia I_2 ; hence the column will buckle in the 1-1 plane, and the smaller moment of inertia I_2 should be used in the formula for the critical load. If the cross section is square or circular, all centroidal axes have the same moment of inertia and buckling may occur in any longitudinal plane.

Critical Stress

After finding the critical load for a column, we can calculate the corresponding **critical stress** by dividing the load by the cross-sectional area. For the fundamental case of buckling (Fig. 11-8b), the critical stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2} \quad (11-14)$$

in which I is the moment of inertia for the principal axis about which buckling occurs. This equation can be written in a more useful form by introducing the notation

$$r = \sqrt{\frac{I}{A}} \quad (11-15)$$

in which r is the **radius of gyration** of the cross section in the plane of bending.* Then the equation for the critical stress becomes

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \quad (11-16)$$

in which L/r is a nondimensional ratio called the **slenderness ratio**:

$$\text{Slenderness ratio} = \frac{L}{r} \quad (11-17)$$

Note that the slenderness ratio depends only on the dimensions of the column. A column that is long and slender will have a high slenderness ratio and therefore a low critical stress. A column that is short and stubby will have a low slenderness ratio and will buckle at a high stress.

*Radius of gyration is described in Section 12.4.

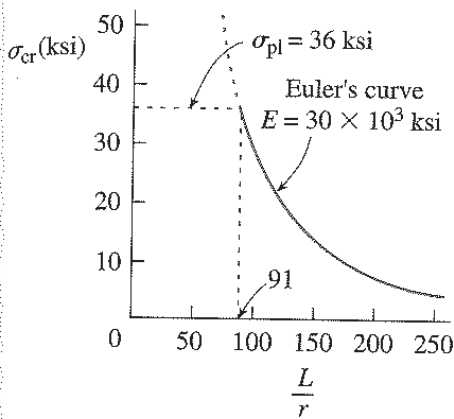


FIG. 11-10 Graph of Euler's curve (from Eq. 11-16) for structural steel with $E = 30 \times 10^3$ ksi and $\sigma_{pl} = 36$ ksi

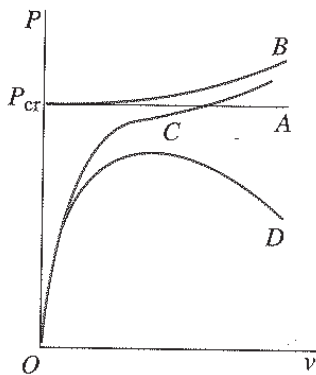


FIG. 11-11 Load-deflection diagram for columns: Line A, ideal elastic column with small deflections; Curve B, ideal elastic column with large deflections; Curve C, elastic column with imperfections; and Curve D, inelastic column with imperfections

Typical values of the slenderness ratio for actual columns are between 30 and 150.

The critical stress is the average compressive stress on the cross section at the instant the load reaches its critical value. We can plot a graph of this stress as a function of the slenderness ratio and obtain a curve known as **Euler's curve** (Fig. 11-10). The curve shown in the figure is plotted for a structural steel with $E = 30 \times 10^3$ ksi. The curve is valid only when the critical stress is less than the proportional limit of the steel, because the equations were derived using Hooke's law. Therefore, we draw a horizontal line on the graph at the proportional limit of the steel (assumed to be 36 ksi) and terminate Euler's curve at that level of stress.*

Effects of Large Deflections, Imperfections, and Inelastic Behavior

The equations for critical loads were derived for ideal columns, that is, columns for which the loads are precisely applied, the construction is perfect, and the material follows Hooke's law. As a consequence, we found that the magnitudes of the small deflections at buckling were undefined.** Thus, when $P = P_{cr}$, the column may have any small deflection, a condition represented by the horizontal line labeled A in the load-deflection diagram of Fig. 11-11. (In this figure, we show only the right-hand half of the diagram, but the two halves are symmetric about the vertical axis.)

The theory for ideal columns is limited to small deflections because we used the second derivative v'' for the curvature. A more exact analysis, based upon the exact expression for curvature (Eq. 9-13 in Section 9.2), shows that there is no indefiniteness in the magnitudes of the deflections at buckling. Instead, for an ideal, linearly elastic column, the load-deflection diagram goes upward in accord with curve B of Fig. 11-11. Thus, after a linearly elastic column begins to buckle, an increasing load is required to cause an increase in the deflections.

Now suppose that the column is not constructed perfectly; for instance, the column might have an imperfection in the form of a small initial curvature, so that the unloaded column is not perfectly straight. Such imperfections produce deflections from the onset of loading, as shown by curve C in Fig. 11-11. For small deflections, curve C approaches line A as an asymptote. However, as the deflections become

*Euler's curve is not a common geometric shape. It is sometimes mistakenly called a hyperbola, but hyperbolas are plots of polynomial equations of the second degree in two variables, whereas Euler's curve is a plot of an equation of the third degree in two variables.

**In mathematical terminology, we solved a *linear eigenvalue problem*. The critical load is an *eigenvalue* and the corresponding buckled mode shape is an *eigenfunction*.

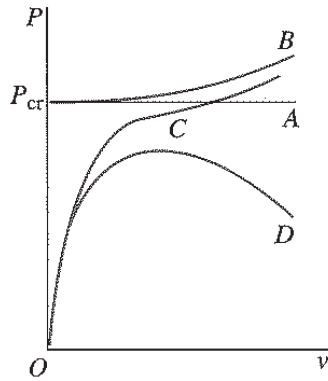


FIG. 11-11 (Repeated)

large, it approaches curve *B*. The larger the imperfections, the further curve *C* moves to the right, away from the vertical line. Conversely, if the column is constructed with considerable accuracy, curve *C* approaches the vertical axis and the horizontal line labeled *A*. By comparing lines *A*, *B*, and *C*, we see that for practical purposes the critical load represents the maximum load-carrying capacity of an elastic column, because large deflections are not acceptable in most applications.

Finally, consider what happens when the stresses exceed the proportional limit and the material no longer follows Hooke's law. Of course, the load-deflection diagram is unchanged up to the level of load at which the proportional limit is reached. Then the curve for inelastic behavior (curve *D*) departs from the elastic curve, continues upward, reaches a maximum, and turns downward.

The precise shapes of the curves in Fig. 11-11 depend upon the material properties and column dimensions, but the general nature of the behavior is typified by the curves shown.

Only extremely slender columns remain elastic up to the critical load. Stockier columns behave inelastically and follow a curve such as *D*. Thus, the maximum load that can be supported by an inelastic column may be considerably less than the Euler load for that same column. Furthermore, the descending part of curve *D* represents sudden and catastrophic collapse, because it takes smaller and smaller loads to maintain larger and larger deflections. By contrast, the curves for elastic columns are quite stable, because they continue upward as the deflections increase, and therefore it takes larger and larger loads to cause an increase in deflection. (Inelastic buckling is described in more detail in Sections 11.7 and 11.8.)

Optimum Shapes of Columns

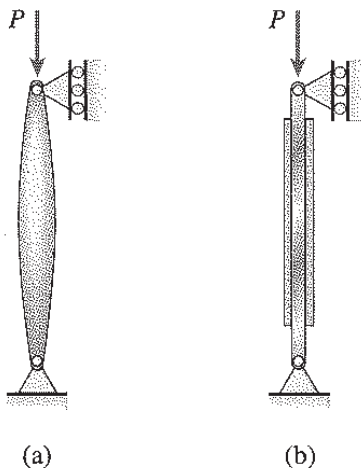


FIG. 11-12 Nonprismatic columns

Compression members usually have the same cross sections throughout their lengths, and therefore only prismatic columns are analyzed in this chapter. However, prismatic columns are not the optimum shape if minimum weight is desired. The critical load of a column consisting of a given amount of material may be increased by varying the shape so that the column has larger cross sections in those regions where the bending moments are larger. Consider, for instance, a column of solid circular cross section with pinned ends. A column shaped as shown in Fig. 11-12a will have a larger critical load than a prismatic column made from the same volume of material. As a means of approximating this optimum shape, prismatic columns are sometimes reinforced over part of their lengths (Fig. 11-12b).

Now consider a prismatic column with pinned ends that is free to buckle in *any* lateral direction (Fig. 11-13a). Also, assume that the column has a solid cross section, such as a circle, square, triangle,

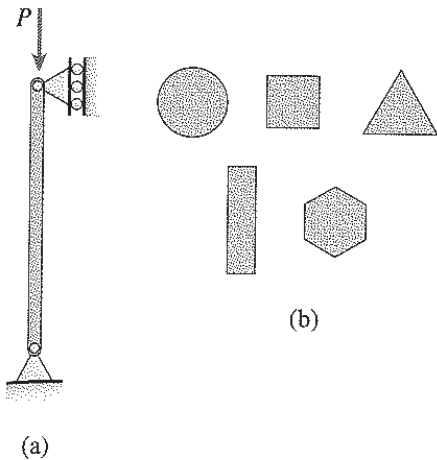


FIG. 11-13 Which cross-sectional shape is the optimum shape for a prismatic column?

rectangle, or hexagon (Fig. 11-13b). An interesting question arises: For a given cross-sectional area, which of these shapes makes the most efficient column? Or, in more precise terms, which cross section gives the largest critical load? Of course, we are assuming that the critical load is calculated from the Euler formula $P_{cr} = \pi^2 EI/L^2$ using the smallest moment of inertia for the cross section.

While a common answer to this question is “the circular shape,” you can readily demonstrate that a cross section in the shape of an equilateral triangle gives a 21% higher critical load than does a circular cross section of the same area (see Problem 11.3-10). The critical load for an equilateral triangle is also higher than the loads obtained for the other shapes; hence, an equilateral triangle is the optimum cross section (based only upon theoretical considerations). For a mathematical analysis of optimum column shapes, including columns with varying cross sections, see Ref. 11-7.

Example 11-1

A long, slender column *ABC* is pin-supported at the ends and compressed by an axial load *P* (Fig. 11-14). Lateral support is provided at the midpoint *B* in the plane of the figure. However, lateral support perpendicular to the plane of the figure is provided only at the ends. The column is constructed of a steel wide-flange section (*W* 8 × 28) having modulus of elasticity $E = 29 \times 10^3$ ksi and proportional limit $\sigma_{pl} = 42$ ksi. The total length of the column is $L = 25$ ft.

Determine the allowable load P_{allow} using a factor of safety $n = 2.5$ with respect to Euler buckling of the column.

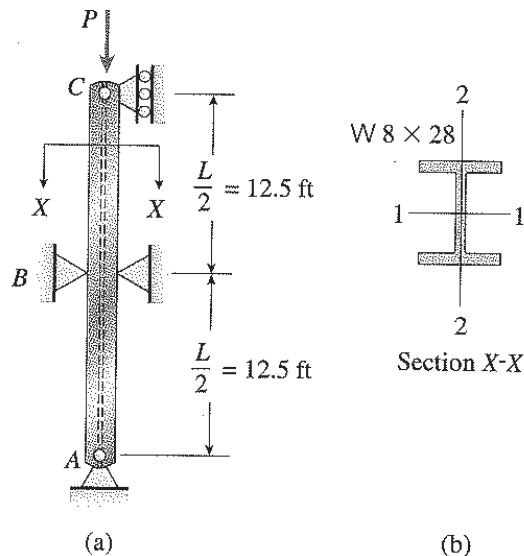


FIG. 11-14 Example 11-1. Euler buckling of a slender column

continued

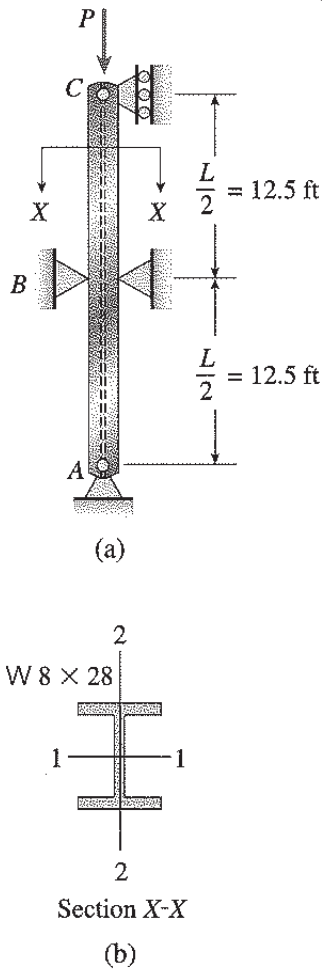


FIG. 11-14 (Repeated)

Solution

Because of the manner in which it is supported, this column may buckle in either of the two principal planes of bending. As one possibility, it may buckle in the plane of the figure, in which case the distance between lateral supports is $L/2 = 12.5$ ft and bending occurs about axis 2-2 (see Fig. 11-8c for the mode shape of buckling). As a second possibility, it may buckle perpendicular to the plane of the figure with bending about axis 1-1. Because the only lateral support in this direction is at the ends, the distance between lateral supports is $L = 25$ ft (see Fig. 11-8b for the mode shape of buckling).

Column properties. From Table E-1, Appendix E, we obtain the following moments of inertia and cross-sectional area for a W 8 × 28 column:

$$I_1 = 98.0 \text{ in.}^4 \quad I_2 = 21.7 \text{ in.}^4 \quad A = 8.25 \text{ in.}^2$$

Critical loads. If the column buckles in the plane of the figure, the critical load is

$$P_{cr} = \frac{\pi^2 EI_2}{(L/2)^2} = \frac{4\pi^2 EI_2}{L^2}$$

Substituting numerical values, we obtain

$$P_{cr} = \frac{4\pi^2 EI_2}{L^2} = \frac{4\pi^2 (29 \times 10^3 \text{ ksi})(21.7 \text{ in.}^4)}{[(25 \text{ ft})(12 \text{ in./ft})]^2} = 276 \text{ k}$$

If the column buckles perpendicular to the plane of the figure, the critical load is

$$P_{cr} = \frac{\pi^2 EI_1}{L^2} = \frac{\pi^2 (29 \times 10^3 \text{ ksi})(98.0 \text{ in.}^4)}{[(25 \text{ ft})(12 \text{ in./ft})]^2} = 312 \text{ k}$$

Therefore, the critical load for the column (the smaller of the two preceding values) is

$$P_{cr} = 276 \text{ k}$$

and buckling occurs in the plane of the figure.

Critical stresses. Since the calculations for the critical loads are valid only if the material follows Hooke's law, we need to verify that the critical stresses do not exceed the proportional limit of the material. In the case of the larger critical load, we obtain the following critical stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{312 \text{ k}}{8.25 \text{ in.}^2} = 37.8 \text{ ksi}$$

Since this stress is less than the proportional limit ($\sigma_{pl} = 42$ ksi), both critical-load calculations are satisfactory.

Allowable load. The allowable axial load for the column, based on Euler buckling, is

$$P_{allow} = \frac{P_{cr}}{n} = \frac{276 \text{ k}}{2.5} = 110 \text{ k}$$

in which $n = 2.5$ is the desired factor of safety.