THEORY OF A LOOP REVOLVING IN AIR, WITH OBSERVATIONS ON THE SKIN-FRICTION

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[Received 22 December 1947]

SUMMARY

An investigation is made of the dynamics of a flexible loop, driven by a small pulley at a constant speed, for the case when the speed is sufficient to cause the loop to become air-borne. It does this when the air-friction is greater than the weight of the loop. The shape and position of the loop may be calculated for values of the ratio, 2α, of air-friction to weight and the angle, ϕ^*, that the tangent to the ascending portion of the loop at the pulley makes with the horizontal. This angle may be controlled by means of a second pulley so arranged as to press the loop against the driven pulley. If $\alpha = 1$, theory predicts a point of the loop at which its radius of curvature becomes zero. For values of α between $\frac{1}{2}$ and 1 the motion is stable, and the precise value of α may be found from the photographed shape of the loop. As the result of some trials, values of α have been obtained which have given a value of the air-friction drag coefficient of the right order of magnitude.

Introduction

It has long been known to engineers that a belt when driven at a high speed will occasionally be seen to stand up in the air in the form of a loop. This may happen when a shaft breaks, or when a belt slips off a driven pulley at the end of a shaft. The dynamics of such a revolving loop does not appear to have received previous consideration, nor does the more general problem of an endless flexible cord subjected to external forces which do not all pass through a fixed point.

The author's attention was drawn to this problem at the Admiralty when examining a proposal to use such a loop as a means of defence against aircraft. At that time no very precise values of the air-friction at a moving surface were available, and it occurred to the author that the revolving loop, found to be valueless for the purpose proposed, might prove useful as a means of obtaining this information. Accordingly, the theory of the revolving loop has now been rewritten. Some tests of the suitability of the loop for measurement of surface air-friction were made at the University of London Observatory in February 1947 when astronomical observations were interrupted by bad weather.

Part I. Theory of the revolving loop

Let the arc APO (Fig. 1) be the locus of any point, P, of a uniform, flexible, inextensible string of weight mg per unit length, which is moving

[Quart. Journ. Mech. and Applied Math., Vol. II, Pt. 1 (1949)]

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with constant tangential velocity V. Suppose the string is subject to the following external forces:

(i) its weight acting vertically downwards,

(ii) a constant tangential retarding force $2\alpha mg\,\delta s$ on every element of length δs,

(iii) a constant tension, T_0, at O, where the tangent to the string is horizontal,

(iv) a constant tension, T^*, at A, the tangent at A making an angle ϕ^* with the horizontal.

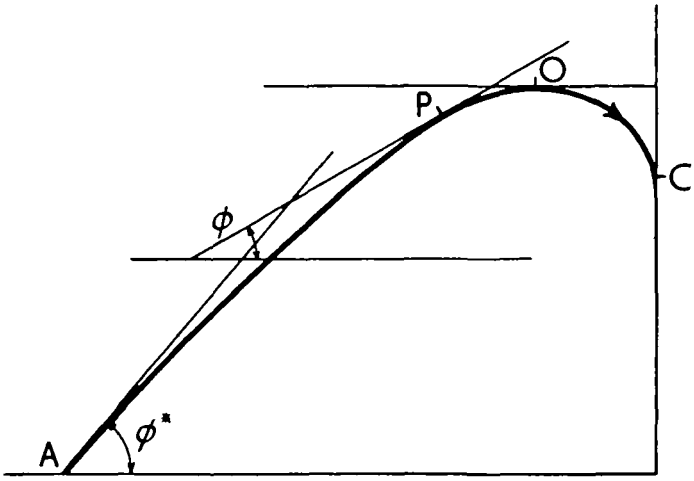

FIG. 1.

For uniform motion the forces (i) to (iv) must balance the centripetal force $(mV^2\,\delta s)/R$, where R is the radius of curvature at P. If T is the tension at P, and the tangent at P makes an angle ϕ with the horizontal, then

$$\delta T = -mg(2\alpha+\sin\phi)\,\delta s,$$

$$T\,\delta\phi = mg\left(\frac{V^2}{gR}-\cos\phi\right)\delta s,$$

where the motion is in the direction APO, and s is measured from O along the curve of the string to P. The differential equations to be solved are thus

$$\left.\begin{aligned} \frac{dT}{ds} &= -mg(2\alpha+\sin\phi), && (a)\\ mV^2-T &= mgR\cos\phi, && (b)\\ \text{where}\qquad R &= \frac{ds}{d\phi}. && (c) \end{aligned}\right\} \qquad (1)$$

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From 1(b) and 1(c),

$$\frac{dT}{ds} = -mg\frac{dR}{ds}\cos\phi + mg\sin\phi,$$

so that, by 1(a), $$\frac{dR}{ds}\cos\phi = 2(\alpha+\sin\phi),$$

or, $$\frac{dR}{R} = 2(\alpha\sec\phi+\tan\phi)\,d\phi.$$

Hence, we obtain $$R = k\{\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\}^{2\alpha}\sec^2\phi,$$

where k is a constant of integration, and by 1(b),

$$mV^2 - T = mgk\{\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\}^{2\alpha}\sec\phi.$$

Consideration of the conditions at $\phi = 0$ leads finally to the expression

$$mV^2 - T = (mV^2 - T_0)\{\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\}^{2\alpha}\sec\phi. \tag{2}$$

Since change of sign of α corresponds to reversal of the direction of V, the above equation, when written in the form

$$mV^2 - T = (mV^2 - T_0)\{\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\}^{-2\alpha}\sec\phi,$$

represents, for negative values of ϕ, the descending branch of the curve from O to the point C, where the tangent is vertical.

Writing t for $\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)$ and K for $(V^2/g - T_0/mg)$, (2) becomes

$$mV^2 - T = \tfrac{1}{2}Kmg(t^{2\alpha+1}+t^{2\alpha-1}). \tag{3}$$

Since $T > T_0$ at C, it follows that

$$2\alpha > 1. \tag{4}$$

We also have, from (1)

$$R = K\{\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi)\}^{2\alpha}\sec^2\phi = \tfrac{1}{4}K(t^{2\alpha-2}+2t^{2\alpha}+t^{2\alpha+2}), \tag{5}$$

and since $$ds/dt = R\,d\phi/dt = \tfrac{1}{2}K(t^{2\alpha-2}+t^{2\alpha}),$$

it follows that $$s = \tfrac{1}{2}K[t^{2\alpha-1}/(2\alpha-1)+t^{2\alpha+1}/(2\alpha+1)]. \tag{6}$$

Taking axes through C, x to the left and y downwards, the coordinates of P are

$$x = \tfrac{1}{2}Kt^{2\alpha}/\alpha, \qquad y = \tfrac{1}{2}K[t^{2\alpha+1}/(2\alpha+1) - t^{2\alpha-1}/(2\alpha-1)]. \tag{7}$$

We consider now the equations of motion of an element $\delta s'$ of a similar string at the point P' and moving with the same velocity, V, the tangent at C being vertical and the tension at C being mV^2 (Fig. 2). These equations are

$$\left.\begin{aligned} \delta T' &= -mg(2\alpha - \sin\phi')\,\delta s', \\ T'\,\delta\phi' &= mg\left(\frac{V^2}{gR'} + \cos\phi'\right)\delta s'. \end{aligned}\right\} \tag{8}$$

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Using a similar method to that given above we obtain

$$T'-mV^2 = \tfrac{1}{2}K'mg(t'^{2\alpha+1}+t'^{2\alpha-1}), \tag{9}$$

where ϕ' is the angle which the tangent at P' makes with the horizontal,

$$K' = (T^{*\prime}-mV^2)/\tfrac{1}{2}mg(t^{*\prime 2\alpha+1}+t^{*\prime 2\alpha-1}).$$

$$t' = \tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi'), \qquad t^{*\prime} = \tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi^{*\prime}).$$

$T^{*\prime}$ and $\phi^{*\prime}$ are, respectively, the string tension and the angle of the tangent at a point, B, at which the constant driving force, $T^{*\prime}$, is applied

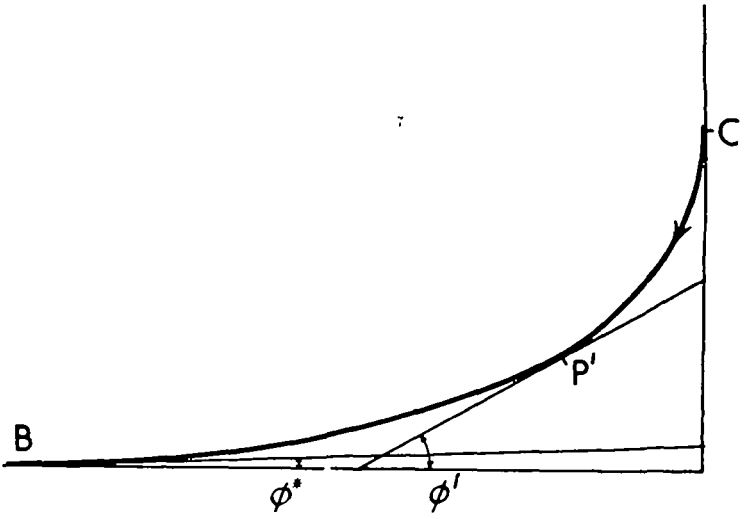

FIG. 2.

to the string. Accordingly, P' lies on a curve similar to the portion, OC, of the curve in Fig. 1, except that it is inverted and the scale is increased by the factor K'/K.

We may now suppose that the two portions of string form one continuous loop $COPAP'C$, by joining the curves P and P' at C and making B coincide with A. [In practice this may be done, approximately, by using a large loop constrained to bend around a small driving pulley at A.] Equating the x-coordinates of A and B, we have

$$K'/K = (t^*/t^{*\prime})^{2\alpha}, \tag{10}$$

where t^* is the value of t at A. Equating the y-coordinates of A and B, and using (10), we obtain

$$t^*t^{*\prime} = (2\alpha+1)/(2\alpha-1); \tag{11}$$

also,
$$S' = \text{arc}\,CP'A = \tfrac{1}{2}K'\left[\frac{t^{*\prime 2\alpha+1}}{2\alpha+1}+\frac{t^{*\prime 2\alpha-1}}{2\alpha-1}\right] \tag{12}$$

and, from (6),
$$S = \text{arc}\,COPA = \tfrac{1}{2}K\left[\frac{t^{*2\alpha+1}}{2\alpha+1}+\frac{t^{*2\alpha-1}}{2\alpha-1}\right]. \tag{13}$$

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Equations (10) to (13) give

$$S = S', \quad \text{or} \quad \text{arc}\,COPA = \text{arc}\,CP'A. \tag{14}$$

From (9),

$$T^{*\prime} - mV^2 = \tfrac{1}{2}K'mg(t^{*\prime} + t^{*\prime -1})t^{*\prime 2\alpha}$$
$$= \tfrac{1}{2}Kmg[(2\alpha-1)t^{*2\alpha+1}/(2\alpha+1) + (2\alpha+1)t^{*2\alpha-1}/(2\alpha-1)]$$

and from (3), $mV^2 - T^* = \tfrac{1}{2}Kmg(t^* + t^{*-1})t^{*2\alpha}$,

whence
$$T^{*\prime} - T^* = 4\alpha mgS, \tag{15}$$

and also

$$\left.\begin{aligned} T^{*\prime} + T^* &= 2mV^2 - K[t^*/(2\alpha+1) - t^{*-1}/(2\alpha-1)]t^{*2\alpha}g \\ &= 2mV^2 - 2y^*mg, \end{aligned}\right\} \tag{16}$$

where x^*, y^* are the coordinates of A. Accordingly,

$$T^{*\prime} = mV^2 + 2\alpha mgS - mgy^* \tag{17}$$

and
$$T^* = mV^2 - 2\alpha mgS - mgy^*. \tag{18}$$

Also, if x', y' be the coordinates of P',

$$x' = \tfrac{1}{2}K't'^{2\alpha}/\alpha, \tag{19}$$

$$y' = \tfrac{1}{2}K'\left[\frac{t'^{2\alpha-1}}{2\alpha-1} - \frac{t'^{2\alpha+1}}{2\alpha+1}\right]. \tag{20}$$

The area of the loop is given by

$$\Delta = \int_O^A y'\,dx' - \int_O^A y\,dx$$
$$= \tfrac{1}{2}K'^2\left[\int_0^{t^{*\prime}} \frac{t'^{4\alpha-2}}{2\alpha-1}\,dt - \int_0^{t^{*\prime}} \frac{t'^{4\alpha}}{2\alpha+1}\,dt\right] - \tfrac{1}{2}K^2\left[\int_0^{t^*} \frac{t^{4\alpha}}{2\alpha+1}\,dt - \int_0^{t^*} \frac{t^{4\alpha-2}}{2\alpha-1}\,dt\right]$$
$$= \frac{K^2t^{*4\alpha}}{4}\left[\frac{t^*}{(2\alpha+1)(4\alpha-1)} - \frac{t^{*-1}}{(2\alpha-1)(4\alpha+1)} - \frac{t^*}{(2\alpha+1)(4\alpha+1)} + \right.$$
$$\left. + \frac{t^{*-1}}{(2\alpha-1)(4\alpha-1)}\right]$$
$$= \frac{2\alpha Sx^*}{16\alpha^2-1}. \tag{21}$$

The left-handed couple about A, required to balance the weight of S and S' together (considered as a rigid, stationary loop held at the point A), is given by

$$G = mg\left[\int_O^A (x^* - x)\,ds + \int_O^A (x^* - x')\,ds'\right]$$
$$= 2mgx^*S - mg\int_O^A x\,ds - mg\int_O^A x'\,ds'.$$

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Since

$x = Kt^{2\alpha}/2\alpha$, $x' = K't'^{2\alpha}/2\alpha$, $ds = \frac{1}{2}K(t^{2\alpha}+t^{2\alpha-2})dt$, $ds' = \frac{1}{2}K'(t'^{2\alpha}+t'^{2\alpha-2})dt$,

it follows from (21) that

$$G = \frac{16\alpha^2}{16\alpha^2-1} mgSx^* = 4\alpha mg\Delta. \tag{22}$$

Here G is the left-handed couple about A, arising from air skin-friction when the loop is rotating with constant speed V. Air-friction will exactly balance gravity, and the loop becomes airborne provided $\alpha > \frac{1}{2}$, as required by (4).

In order to determine whether the radius of curvature, R, of the loop always remains finite, we have, from (5)

$$dR/dt = \tfrac{1}{2}K[(\alpha+1)t^{2\alpha+1}+2\alpha t^{2\alpha-1}+(\alpha-1)t^{2\alpha-3}]$$

$$= 0 \quad \text{when} \quad t^2 = 0 \text{ (when } 2\alpha > 3), \ -1, \text{ or } \frac{1-\alpha}{1+\alpha}.$$

$t^2 = \dfrac{1-\alpha}{1+\alpha}$ gives values of ϕ from $-30°$ to $-90°$ for values of α from $\frac{1}{2}$ to 1 respectively. The corresponding values of R, given by

$$R_{\min} = K(1-\alpha)^{\alpha-1}(1+\alpha)^{-\alpha-1},$$

are finite for $\alpha < 1$. For $\alpha \geqslant 1$, $R_{\min} = 0$ at C, where $-\phi = \frac{1}{2}\pi$. [In an actual case some distortion of the loop occurs when $\alpha > 1$, since no material loop is infinitely flexible; the rotational effect of air-friction, in this case, causes the top of the loop to bend back somewhat.]

From (21) it will be seen that, when V (and consequently α) is constant, the area of the loop, Δ, varies as x^*; the locus of A when C is supposed fixed may be found as follows.

Let the two tangents at A meet the vertical through C in E and D (Fig. 3), then

$$AE = x^* \sec\phi^* = Kt^{*2\alpha}(t^*+t^{*-1})/4\alpha$$

$$= (mV^2-T^*)/2\alpha mg = (S+y^*)/2\alpha,$$

from (18), $$AD = x^* \sec\phi^{*\prime}$$

$$= Kt^{*2\alpha}\left[\frac{2\alpha+1}{2\alpha-1}t^{*-1}+\frac{2\alpha-1}{2\alpha+1}t^*\right]\Big/4\alpha.$$

Thus $$AE+AD = Kt^{*2\alpha}[(2\alpha+1)^{-1}t^*+(2\alpha-1)^{-1}t^{*-1}]$$

$$= 2S = \text{constant},$$

$$AD = S-y^*/2\alpha,$$

$$CE = x^* \tan\phi^*-y^* = S/2\alpha,$$

and $$CD = y^*-x^* \tan\phi^{*\prime} = S/2\alpha = CE,$$

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hence the locus of A is the ellipse

$$y^{*2}+\frac{4\alpha^2}{4\alpha^2-1}x^{*2}=S^2, \tag{23}$$

D and E being the foci, C the centre, and S and $S(4\alpha^2-1)^{\frac{1}{2}}/2\alpha$ the semi-axes.

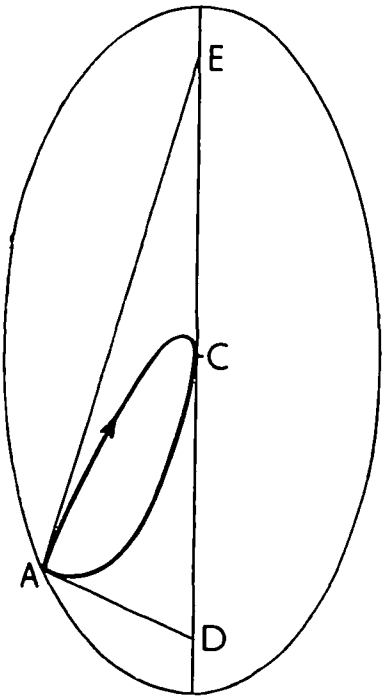

FIG. 3.

Part II. The airborne loop in practice, and the determination of skin-friction at various speeds

The determination of air-resistance to a surface, when in uniform motion relative to still air parallel to the surface, has always proved difficult from an experimental point of view on account of end-effects of the material body which provides the moving surface. An endless cord or belt of uniform cross-section driven by a pair of wheels mounted on fixed parallel axes, and made to revolve at a constant known speed would, indeed, overcome the difficulty of 'end effects', but other difficulties would arise. Only at very low speeds would such a driven belt be free from 'flutter' unless it were stretched very tight, in which case the presence of eddy currents might be concealed. Another difficulty with such an arrangement would be the accurate determination of the rather small additional force required to drive the belt against air resistance, over and above that

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required to overcome the friction and air resistance of the driving wheels and other moving parts of the apparatus.

When a belt is driven by a single small wheel or pulley, however, these difficulties do not arise, the motion of the 'loop' being observably steady so long as the speed is such that the value of α lies between $\frac{1}{2}$ and 1. The precise value of α can be found from the curvature of the loop in the

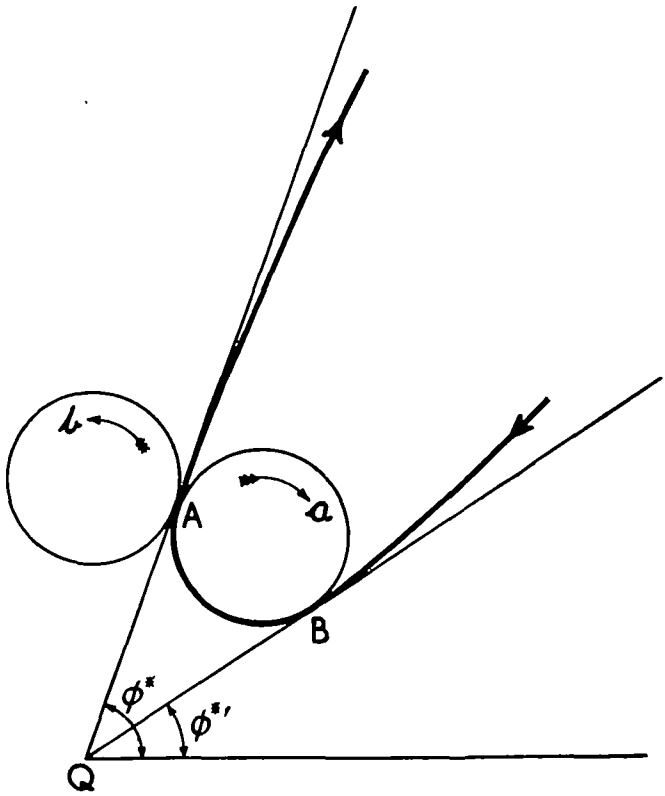

FIG. 4.

vicinity of the point C (Fig. 1), no measurement of force being required. Both the shape of the loop (giving α) and its rotational speed, V, may be determined by a cinematograph camera working at a known speed, a short length of the (white) loop being blackened so as to be visible.

From the foregoing theoretical discussion the necessary and sufficient conditions for the maintenance of an air-borne loop in a steady state can readily be seen.

Let a (Fig. 4) be a cylindrical pulley wheel driven at a constant peripheral speed, V, by an electric motor, say. Around a is placed the loop of

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length $2S$ and weight $2Smg$. A loose pulley, b, on an axle parallel to that of a, keeps the loop in contact with a at A.

Let QA be the common tangent of a and b at A, and let QA make an angle ϕ^* with the plane of the horizon. On first starting the motor, the part of the loop in the vicinity of A is projected along this tangent QA with velocity V, and, if air resistance is neglected, would attain a vertical height equal to $V^2/2g$. Taking air-resistance into account, V must be sufficiently great to make T^* greater than zero in (18).

Once the loop is airborne, the whole family of possible curves given by (23) for all positions of A in Fig. 3, and subject to the restriction $\phi^* > 90°$, may be obtained by rotating the line of centres of a and b, and thus altering ϕ^*. However large V may be, the loop will fall as soon as $\phi^* = 90°$, as may be seen from (2).

Should it be possible to measure with accuracy the constant couple which must be applied to a to maintain the loop at any (sufficient) speed, α would at once be obtained from (15), the mass ($2Smg$) of the loop being readily measured. This value of α could then be compared with that derived from the radius of curvature of the loop at C, as mentioned in the second paragraph.

It is worth noting that measurements of surface air-friction over a considerable range of speed can be made by employing loops of very different cross-section, V being greatest (for $\alpha = 0{\cdot}6$, say) when the cross-section is circular. If the cross-section be rectangular, of thickness a and breadth b much greater than a, then

$$\alpha mg = cV_1^2(a+b),$$

where c is assumed constant.

If r be the radius of the circular section of a loop of similar material such that $\pi r^2 = ab$, the ratio of the speed, V_2, which it must have to give the same α, to V_1 is given by

$$V_2/V_1 = \frac{\pi^{\frac{1}{2}} r^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}} = (\pi b/a)^{\frac{1}{2}} \text{ (approx.)}$$

With $b/a = 800$, the ratio of the speeds is about seven.

A few trials were made at the University of London Observatory in February 1947. Loops from 7 to 20 feet in length were maintained in the air by an electric motor driving pulleys of 0·298 and 0·255 ft. diameter respectively at about 50 r.p.s. The loops were made from strips of chronograph paper 0·0833 ft. wide and 0·00031 ft. thick, m being 0·0012 lb./ft. α was found to vary from about 0·5 to 0·7 with speeds of about 40 and 46 ft./sec. and was obtained from photographs of the loop by measuring $R_{\min}$. It was found that the loop could be kept steady for any length of time so

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long as it was subjected to a slight draught of air perpendicularly to its plane. This was probably necessary in order to prevent circulatory air currents forming and so reducing the air-speed of the loop. When driven at constant speed in still air the loop was observed slowly to drop a little and then more quickly recover its original height, this 'hunting' continuing in a somewhat irregular manner.

Regarding the accepted value of the skin-friction drag coefficient K_f, the value suggested by M.A.P. for a steel wire is between 0·003 and 0·0015.

Assuming $$K_f \rho V^2(a+b) = \alpha m,$$

where $\rho = 0{\cdot}00238$, $a = 0{\cdot}00031$, $b = 0{\cdot}0833$, and $m = 0{\cdot}0012$, we deduce that

$$K_f = \alpha/0{\cdot}166V^2,$$

whence, (i) for $\alpha = 0{\cdot}5$ and $V^2 = 1600$, we obtain $K_f = 0{\cdot}0019$,

and (ii) for $\alpha = 0{\cdot}7$ and $V^2 = 2100$, we obtain $K_f = 0{\cdot}0020$.

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