

Shigley's Mechanical Engineering Design

Ninth Edition

Richard G. Budynas

Professor Emeritus, Kate Gleason College of Engineering, Rochester Institute of Technology

J. Keith Nisbett

Associate Professor of Mechanical Engineering, Missouri University of Science and Technology





SHIGLEY'S MECHANICAL ENGINEERING DESIGN, NINTH EDITION

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Dedication

To my wife Joanne, my children and grandchildren, and to good friends, especially Sally and Peter.

Richard G. Budynas

To Professor T. J. Lawley, who first introduced me to Shigley's text, and who instigated in me a fascination for the details of machine design.

J. Keith Nisbett

Dedication to Joseph Edward Shigley

Joseph Edward Shigley (1909–1994) is undoubtedly one of the most known and respected contributors in machine design education. He authored or co-authored eight books, including *Theory of Machines and Mechanisms* (with John J. Uicker, Jr.), and *Applied Mechanics of Materials*. He was Coeditor-in-Chief of the well-known *Standard Handbook of Machine Design*. He began *Machine Design* as sole author in 1956, and it evolved into *Mechanical Engineering Design*, setting the model for such textbooks. He contributed to the first five editions of this text, along with co-authors Larry Mitchell and Charles Mischke. Uncounted numbers of students across the world got their first taste of machine design with Shigley's textbook, which has literally become a classic. Practically every mechanical engineer for the past half century has referenced terminology, equations, or procedures as being from "Shigley." McGraw-Hill is honored to have worked with Professor Shigley for over 40 years, and as a tribute to his lasting contribution to this textbook, its title officially reflects what many have already come to call it—*Shigley's Mechanical Engineering Design*.

Having received a Bachelor's Degree in Electrical and Mechanical Engineering from Purdue University and a Master of Science in Engineering Mechanics from The University of Michigan, Professor Shigley pursued an academic career at Clemson College from 1936 through 1954. This led to his position as Professor and Head of Mechanical Design and Drawing at Clemson College. He joined the faculty of the Department of Mechanical Engineering of The University of Michigan in 1956, where he remained for 22 years until his retirement in 1978.

Professor Shigley was granted the rank of Fellow of the American Society of Mechanical Engineers in 1968. He received the ASME Mechanisms Committee Award in 1974, the Worcester Reed Warner Medal for outstanding contribution to the permanent literature of engineering in 1977, and the ASME Machine Design Award in 1985.

Joseph Edward Shigley indeed made a difference. His legacy shall continue.

About the Authors

Richard G. Budynas is Professor Emeritus of the Kate Gleason College of Engineering at Rochester Institute of Technology. He has over 40 years experience in teaching and practicing mechanical engineering design. He is the author of a McGraw-Hill textbook, *Advanced Strength and Applied Stress Analysis*, Second Edition; and co-author of a McGraw-Hill reference book, *Roark's Formulas for Stress and Strain*, Seventh Edition. He was awarded the BME of Union College, MSME of the University of Rochester, and the Ph.D. of the University of Massachusetts. He is a licensed Professional Engineer in the state of New York.

J. Keith Nisbett is an Associate Professor and Associate Chair of Mechanical Engineering at the Missouri University of Science and Technology. He has over 25 years of experience with using and teaching from this classic textbook. As demonstrated by a steady stream of teaching awards, including the Governor's Award for Teaching Excellence, he is devoted to finding ways of communicating concepts to the students. He was awarded the BS, MS, and Ph.D. of the University of Texas at Arlington.

Brief Contents

Preface xv

Part 1	Basics	2
1	Introduction to Mechanical Engineering Design	3
2	Materials	31
3	Load and Stress Analysis	71
4	Deflection and Stiffness	147
Part 2	Failure Prevention	212
5	Failures Resulting from Static Loading	213
6	Fatigue Failure Resulting from Variable Loading	265
Part 3	Design of Mechanical Elements	358
7	Shafts and Shaft Components	359
8	Screws, Fasteners, and the Design of Nonpermanent Joints	409
9	Welding, Bonding, and the Design of Permanent Joints	475
10	Mechanical Springs	517
11	Rolling-Contact Bearings	569
12	Lubrication and Journal Bearings	617
13	Gears—General	673
14	Spur and Helical Gears	733
15	Bevel and Worm Gears	785
16	Clutches, Brakes, Couplings, and Flywheels	825
17	Flexible Mechanical Elements	879
18	Power Transmission Case Study	933

Part 4 Analysis Tools 952**19 Finite-Element Analysis 953****20 Statistical Considerations 977****Appendixes****A Useful Tables 1003****B Answers to Selected Problems 1059***Index* 1065

3

Load and Stress Analysis

Chapter Outline

3-1	Equilibrium and Free-Body Diagrams	72
3-2	Shear Force and Bending Moments in Beams	75
3-3	Singularity Functions	77
3-4	Stress	79
3-5	Cartesian Stress Components	79
3-6	Mohr's Circle for Plane Stress	80
3-7	General Three-Dimensional Stress	86
3-8	Elastic Strain	87
3-9	Uniformly Distributed Stresses	88
3-10	Normal Stresses for Beams in Bending	89
3-11	Shear Stresses for Beams in Bending	94
3-12	Torsion	101
3-13	Stress Concentration	110
3-14	Stresses in Pressurized Cylinders	113
3-15	Stresses in Rotating Rings	115
3-16	Press and Shrink Fits	116
3-17	Temperature Effects	117
3-18	Curved Beams in Bending	118
3-19	Contact Stresses	122
3-20	Summary	126

One of the main objectives of this book is to describe how specific machine components function and how to design or specify them so that they function safely without failing structurally. Although earlier discussion has described structural strength in terms of load or stress versus strength, failure of function for structural reasons may arise from other factors such as excessive deformations or deflections.

Here it is assumed that the reader has completed basic courses in statics of rigid bodies and mechanics of materials and is quite familiar with the analysis of loads, and the stresses and deformations associated with the basic load states of simple prismatic elements. In this chapter and Chap. 4 we will review and extend these topics briefly. Complete derivations will not be presented here, and the reader is urged to return to basic textbooks and notes on these subjects.

This chapter begins with a review of equilibrium and free-body diagrams associated with load-carrying components. One must understand the nature of forces before attempting to perform an extensive stress or deflection analysis of a mechanical component. An extremely useful tool in handling discontinuous loading of structures employs *Macaulay* or *singularity functions*. Singularity functions are described in Sec. 3–3 as applied to the shear forces and bending moments in beams. In Chap. 4, the use of singularity functions will be expanded to show their real power in handling deflections of complex geometry and statically indeterminate problems.

Machine components transmit forces and motion from one point to another. The transmission of force can be envisioned as a flow or force distribution that can be further visualized by isolating internal surfaces within the component. Force distributed over a surface leads to the concept of stress, stress components, and stress transformations (Mohr's circle) for all possible surfaces at a point.

The remainder of the chapter is devoted to the stresses associated with the basic loading of prismatic elements, such as uniform loading, bending, and torsion, and topics with major design ramifications such as stress concentrations, thin- and thick-walled pressurized cylinders, rotating rings, press and shrink fits, thermal stresses, curved beams, and contact stresses.

3–1 Equilibrium and Free-Body Diagrams

Equilibrium

The word *system* will be used to denote any *isolated* part or portion of a machine or structure—including all of it if desired—that we wish to study. A system, under this definition, may consist of a particle, several particles, a part of a rigid body, an entire rigid body, or even several rigid bodies.

If we assume that the system to be studied is motionless or, at most, has constant velocity, then the system has zero acceleration. Under this condition the system is said to be in *equilibrium*. The phrase *static equilibrium* is also used to imply that the system is *at rest*. For equilibrium, the forces and moments acting on the system balance such that

$$\sum \mathbf{F} = 0 \quad (3-1)$$

$$\sum \mathbf{M} = 0 \quad (3-2)$$

which states that *the sum of all force* and the *sum of all moment vectors* acting upon a system in equilibrium is zero.

Free-Body Diagrams

We can greatly simplify the analysis of a very complex structure or machine by successively isolating each element and studying and analyzing it by the use of *free-body diagrams*. When all the members have been treated in this manner, the knowledge obtained can be assembled to yield information concerning the behavior of the total system. Thus, free-body diagramming is essentially a means of breaking a complicated problem into manageable segments, analyzing these simple problems, and then, usually, putting the information together again.

Using free-body diagrams for force analysis serves the following important purposes:

- The diagram establishes the directions of reference axes, provides a place to record the dimensions of the subsystem and the magnitudes and directions of the known forces, and helps in assuming the directions of unknown forces.
- The diagram simplifies your thinking because it provides a place to store one thought while proceeding to the next.
- The diagram provides a means of communicating your thoughts clearly and unambiguously to other people.
- Careful and complete construction of the diagram clarifies fuzzy thinking by bringing out various points that are not always apparent in the statement or in the geometry of the total problem. Thus, the diagram aids in understanding all facets of the problem.
- The diagram helps in the planning of a logical attack on the problem and in setting up the mathematical relations.
- The diagram helps in recording progress in the solution and in illustrating the methods used.
- The diagram allows others to follow your reasoning, showing *all* forces.

EXAMPLE 3-1

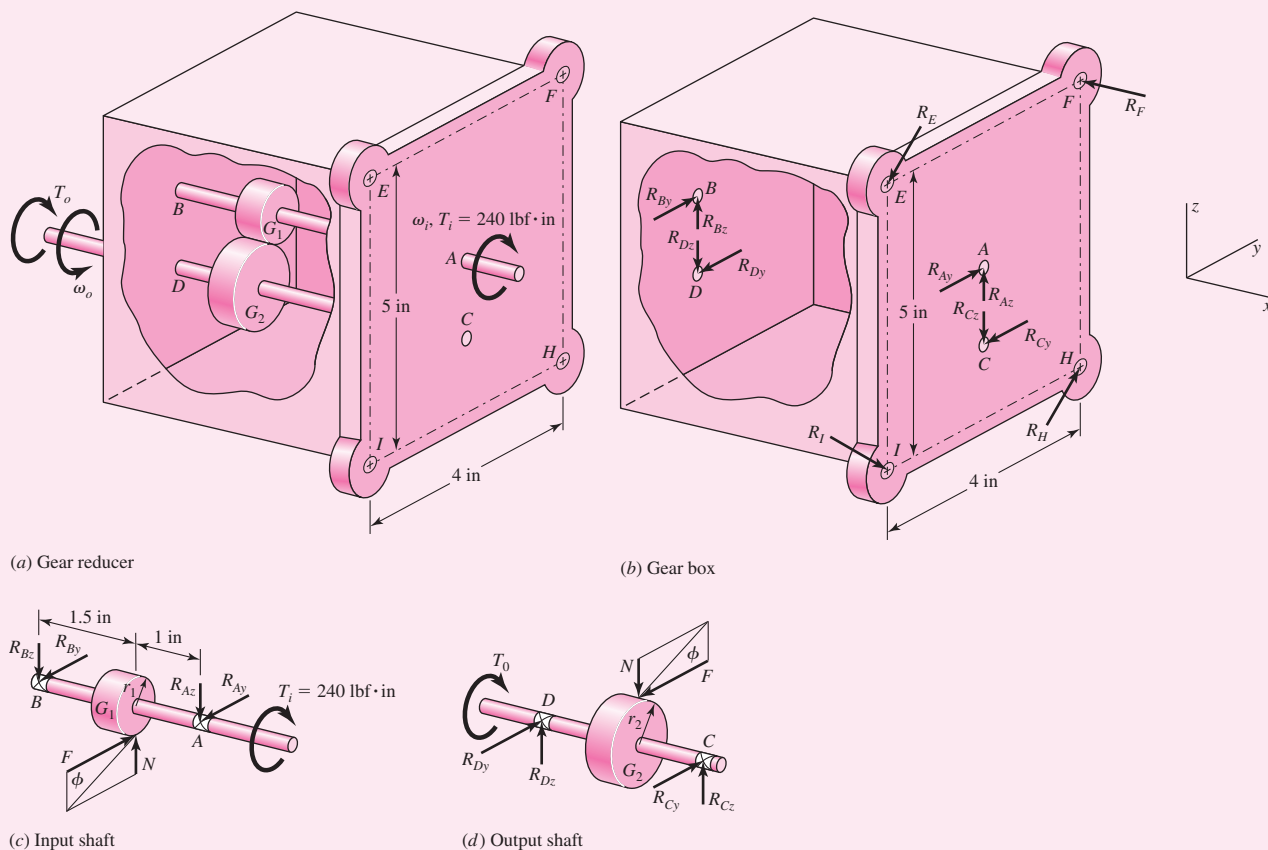
Figure 3-1a shows a simplified rendition of a gear reducer where the input and output shafts AB and CD are rotating at constant speeds ω_i and ω_o , respectively. The input and output torques (torsional moments) are $T_i = 240 \text{ lbf} \cdot \text{in}$ and T_o , respectively. The shafts are supported in the housing by bearings at A , B , C , and D . The pitch radii of gears G_1 and G_2 are $r_1 = 0.75 \text{ in}$ and $r_2 = 1.5 \text{ in}$, respectively. Draw the free-body diagrams of each member and determine the net reaction forces and moments at all points.

Solution

First, we will list all simplifying assumptions.

- 1 Gears G_1 and G_2 are simple spur gears with a standard pressure angle $\phi = 20^\circ$ (see Sec. 13-5).
- 2 The bearings are self-aligning and the shafts can be considered to be simply supported.
- 3 The weight of each member is negligible.
- 4 Friction is negligible.
- 5 The mounting bolts at E , F , H , and I are the same size.

The separate free-body diagrams of the members are shown in Figs. 3-1b–d. Note that Newton's third law, called *the law of action and reaction*, is used extensively where each member mates. The force transmitted between the spur gears is not tangential but at the pressure angle ϕ . Thus, $N = F \tan \phi$.

**Figure 3-1**

(a) Gear reducer; (b–d) free-body diagrams. Diagrams are not drawn to scale.

Summing moments about the x axis of shaft AB in Fig. 3-1d gives

$$\begin{aligned}\sum M_x &= F(0.75) - 240 = 0 \\ F &= 320 \text{ lbf}\end{aligned}$$

The normal force is $N = 320 \tan 20^\circ = 116.5 \text{ lbf}$.

Using the equilibrium equations for Figs. 3-1c and d, the reader should verify that: $R_{Ay} = 192 \text{ lbf}$, $R_{Az} = 69.9 \text{ lbf}$, $R_{By} = 128 \text{ lbf}$, $R_{Bz} = 46.6 \text{ lbf}$, $R_{Cy} = 192 \text{ lbf}$, $R_{Cz} = 69.9 \text{ lbf}$, $R_{Dy} = 128 \text{ lbf}$, $R_{Dz} = 46.6 \text{ lbf}$, and $T_o = 480 \text{ lbf} \cdot \text{in}$. The direction of the output torque T_o is opposite ω_o because it is the resistive load on the system opposing the motion ω_o .

Note in Fig. 3-1b the net force from the bearing reactions is zero whereas the net moment about the x axis is $(1.5 + 0.75)(192) + (1.5 + 0.75)(128) = 720 \text{ lbf} \cdot \text{in}$. This value is the same as $T_i + T_o = 240 + 480 = 720 \text{ lbf} \cdot \text{in}$, as shown in Fig. 3-1a. The reaction forces R_E , R_F , R_H , and R_I , from the mounting bolts cannot be determined from the equilibrium equations as there are too many unknowns. Only three equations are available, $\sum F_y = \sum F_z = \sum M_x = 0$. In case you were wondering about assumption 5, here is where we will use it (see Sec. 8-12). The gear box tends to rotate about the x axis because of a pure torsional moment of $720 \text{ lbf} \cdot \text{in}$. The bolt forces must provide

an equal but opposite torsional moment. The center of rotation relative to the bolts lies at the centroid of the bolt cross-sectional areas. Thus if the bolt areas are equal: the center of rotation is at the center of the four bolts, a distance of $\sqrt{(4/2)^2 + (5/2)^2} = 3.202$ in from each bolt; the bolt forces are equal ($R_E = R_F = R_H = R_I = R$), and each bolt force is perpendicular to the line from the bolt to the center of rotation. This gives a net torque from the four bolts of $4R(3.202) = 720$. Thus, $R_E = R_F = R_H = R_I = 56.22$ lbf.

3-2 Shear Force and Bending Moments in Beams

Figure 3-2a shows a beam supported by reactions R_1 and R_2 and loaded by the concentrated forces F_1 , F_2 , and F_3 . If the beam is cut at some section located at $x = x_1$ and the left-hand portion is removed as a free body, an *internal shear force* V and *bending moment* M must act on the cut surface to ensure equilibrium (see Fig. 3-2b). The shear force is obtained by summing the forces on the isolated section. The bending moment is the sum of the moments of the forces to the left of the section taken about an axis through the isolated section. The sign conventions used for bending moment and shear force in this book are shown in Fig. 3-3. Shear force and bending moment are related by the equation

$$V = \frac{dM}{dx} \quad (3-3)$$

Sometimes the bending is caused by a distributed load $q(x)$, as shown in Fig. 3-4; $q(x)$ is called the *load intensity* with units of force per unit length and is positive in the

Figure 3-2

Free-body diagram of simply-supported beam with V and M shown in positive directions.

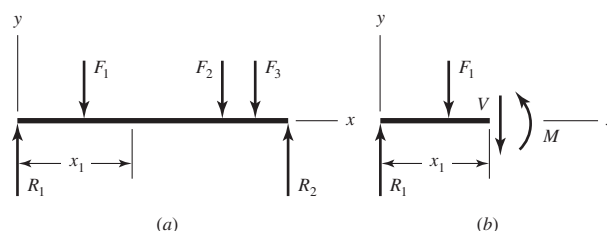


Figure 3-3

Sign conventions for bending and shear.

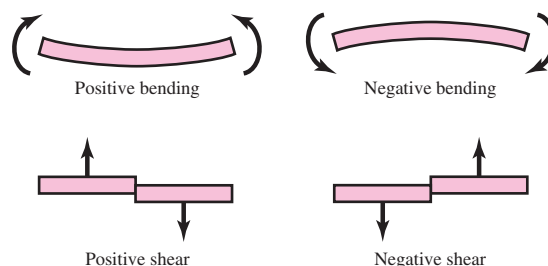
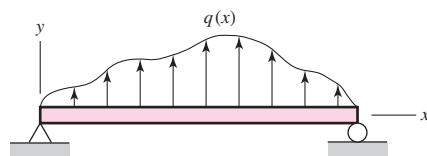


Figure 3-4

Distributed load on beam.



positive y direction. It can be shown that differentiating Eq. (3-3) results in

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q \quad (3-4)$$

Normally the applied distributed load is directed downward and labeled w (e.g., see Fig. 3-6). In this case, $w = -q$.

Equations (3-3) and (3-4) reveal additional relations if they are integrated. Thus, if we integrate between, say, x_A and x_B , we obtain

$$\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q \, dx \quad (3-5)$$

which states that *the change in shear force from A to B is equal to the area of the loading diagram between x_A and x_B .*

In a similar manner,

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V \, dx \quad (3-6)$$

which states that *the change in moment from A to B is equal to the area of the shear-force diagram between x_A and x_B .*

Table 3-1

Singularity (Macaulay[†])
Functions

Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x-a \rangle^{-2} = 0 \quad x \neq a$ $\langle x-a \rangle^{-2} = \pm\infty \quad x = a$ $\int \langle x-a \rangle^{-2} dx = \langle x-a \rangle^{-1}$
Concentrated force (unit impulse)		$\langle x-a \rangle^{-1} = 0 \quad x \neq a$ $\langle x-a \rangle^{-1} = +\infty \quad x = a$ $\int \langle x-a \rangle^{-1} dx = \langle x-a \rangle^0$
Unit step		$\langle x-a \rangle^0 = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}$ $\int \langle x-a \rangle^0 dx = \langle x-a \rangle^1$
Ramp		$\langle x-a \rangle^1 = \begin{cases} 0 & x < a \\ x-a & x \geq a \end{cases}$ $\int \langle x-a \rangle^1 dx = \frac{\langle x-a \rangle^2}{2}$

[†]W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics*, vol. 48, pp. 129-130, 1919.

3-3 Singularity Functions

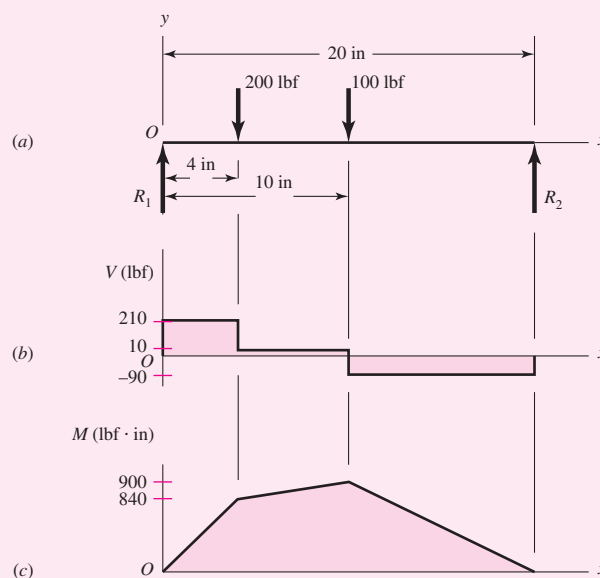
The four singularity functions defined in Table 3-1, using the *angle brackets* $\langle \rangle$, constitute a useful and easy means of integrating across discontinuities. By their use, general expressions for shear force and bending moment in beams can be written when the beam is loaded by concentrated moments or forces. As shown in the table, the concentrated moment and force functions are zero for all values of x not equal to a . The functions are undefined for values of $x = a$. Note that the unit step and ramp functions are zero only for values of x that are less than a . The integration properties shown in the table constitute a part of the mathematical definition too. The first two integrations of $q(x)$ for $V(x)$ and $M(x)$ do not require constants of integration provided *all* loads on the beam are accounted for in $q(x)$. The examples that follow show how these functions are used.

EXAMPLE 3-2

Derive the loading, shear-force, and bending-moment relations for the beam of Fig. 3-5a.

Figure 3-5

- (a) Loading diagram for a simply-supported beam.
 (b) Shear-force diagram.
 (c) Bending-moment diagram.



Solution Using Table 3-1 and $q(x)$ for the loading function, we find

$$\text{Answer} \quad q = R_1 \langle x \rangle^{-1} - 200 \langle x - 4 \rangle^{-1} - 100 \langle x - 10 \rangle^{-1} + R_2 \langle x - 20 \rangle^{-1} \quad (1)$$

Integrating successively gives

$$\text{Answer} \quad V = \int q \, dx = R_1 \langle x \rangle^0 - 200 \langle x - 4 \rangle^0 - 100 \langle x - 10 \rangle^0 + R_2 \langle x - 20 \rangle^0 \quad (2)$$

$$\text{Answer} \quad M = \int V \, dx = R_1 \langle x \rangle^1 - 200 \langle x - 4 \rangle^1 - 100 \langle x - 10 \rangle^1 + R_2 \langle x - 20 \rangle^1 \quad (3)$$

Note that $V = M = 0$ at $x = 0^-$.

The reactions R_1 and R_2 can be found by taking a summation of moments and forces as usual, *or* they can be found by noting that the shear force and bending moment must be zero everywhere except in the region $0 \leq x \leq 20$ in. This means that Eq. (2)

should give $V = 0$ at x slightly larger than 20 in. Thus

$$R_1 - 200 - 100 + R_2 = 0 \quad (4)$$

Since the bending moment should also be zero in the same region, we have, from Eq. (3),

$$R_1(20) - 200(20 - 4) - 100(20 - 10) = 0 \quad (5)$$

Equations (4) and (5) yield the reactions $R_1 = 210$ lbf and $R_2 = 90$ lbf.

The reader should verify that substitution of the values of R_1 and R_2 into Eqs. (2) and (3) yield Figs. 3-5*b* and *c*.

EXAMPLE 3-3

Figure 3-6*a* shows the loading diagram for a beam cantilevered at A with a uniform load of 20 lbf/in acting on the portion $3 \text{ in} \leq x \leq 7 \text{ in}$, and a concentrated counter-clockwise moment of 240 lbf · in at $x = 10 \text{ in}$. Derive the shear-force and bending-moment relations, and the support reactions M_1 and R_1 .

Solution

Following the procedure of Example 3-2, we find the load intensity function to be

$$q = -M_1 \langle x \rangle^{-2} + R_1 \langle x \rangle^{-1} - 20 \langle x - 3 \rangle^0 + 20 \langle x - 7 \rangle^0 - 240 \langle x - 10 \rangle^{-2} \quad (1)$$

Note that the $20 \langle x - 7 \rangle^0$ term was necessary to “turn off” the uniform load at C . Integrating successively gives

Answers

$$V = -M_1 \langle x \rangle^{-1} + R_1 \langle x \rangle^0 - 20 \langle x - 3 \rangle^1 + 20 \langle x - 7 \rangle^1 - 240 \langle x - 10 \rangle^{-1} \quad (2)$$

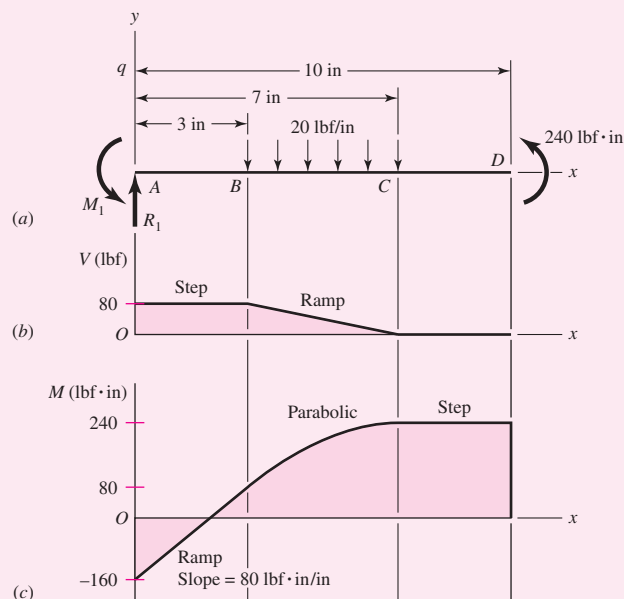
$$M = -M_1 \langle x \rangle^0 + R_1 \langle x \rangle^1 - 10 \langle x - 3 \rangle^2 + 10 \langle x - 7 \rangle^2 - 240 \langle x - 10 \rangle^0 \quad (3)$$

The reactions are found by making x slightly larger than 10 in, where both V and M are zero in this region. Noting that $\langle 10 \rangle^{-1} = 0$, Eq. (2) will then give

$$-M_1(0) + R_1(1) - 20(10 - 3) + 20(10 - 7) - 240(0) = 0$$

Figure 3-6

- (a) Loading diagram for a beam cantilevered at A .
 (b) Shear-force diagram.
 (c) Bending-moment diagram.



Answer which yields $R_1 = 80 \text{ lbf}$.
From Eq. (3) we get

$$-M_1(1) + 80(10) - 10(10 - 3)^2 + 10(10 - 7)^2 - 240(1) = 0$$

Answer which yields $M_1 = 160 \text{ lbf} \cdot \text{in}$.

Figures 3-6*b* and *c* show the shear-force and bending-moment diagrams. Note that the impulse terms in Eq. (2), $-M_1\langle x \rangle^{-1}$ and $-240\langle x - 10 \rangle^{-1}$, are physically not forces and are not shown in the V diagram. Also note that both the M_1 and $240 \text{ lbf} \cdot \text{in}$ in moments are counterclockwise and negative singularity functions; however, by the convention shown in Fig. 3-2 the M_1 and $240 \text{ lbf} \cdot \text{in}$ are negative and positive bending moments, respectively, which is reflected in Fig. 3-6*c*.

3-4 Stress

When an internal surface is isolated as in Fig. 3-2*b*, the net force and moment acting on the surface manifest themselves as force distributions across the entire area. The force distribution acting at a point on the surface is unique and will have components in the normal and tangential directions called *normal stress* and *tangential shear stress*, respectively. Normal and shear stresses are labeled by the Greek symbols σ and τ , respectively. If the direction of σ is outward from the surface it is considered to be a *tensile stress* and is a positive normal stress. If σ is into the surface it is a *compressive stress* and commonly considered to be a negative quantity. The units of stress in U.S. Customary units are pounds per square inch (psi). For SI units, stress is in newtons per square meter (N/m^2); $1 \text{ N/m}^2 = 1 \text{ pascal (Pa)}$.

3-5 Cartesian Stress Components

The Cartesian stress components are established by defining three mutually orthogonal surfaces at a point within the body. The normals to each surface will establish the x , y , z Cartesian axes. In general, each surface will have a normal and shear stress. The shear stress may have components along two Cartesian axes. For example, Fig. 3-7 shows an infinitesimal surface area isolation at a point Q within a body where the surface normal is the x direction. The normal stress is labeled σ_x . The symbol σ indicates a normal stress and the subscript x indicates the direction of the surface normal. The net shear stress acting on the surface is $(\tau_x)_{\text{net}}$ which can be resolved into components in the y and z directions, labeled as τ_{xy} and τ_{xz} , respectively (see Fig. 3-7).

Figure 3-7

Stress components on surface normal to x direction.

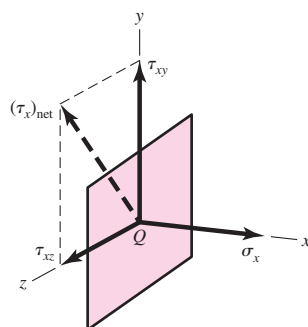
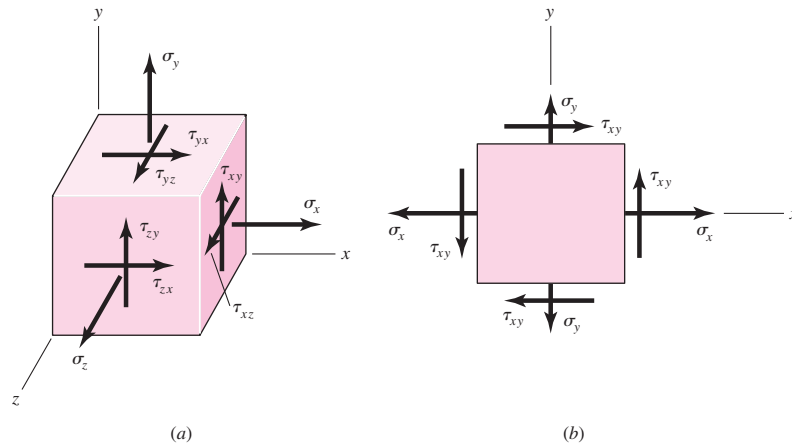


Figure 3-8

(a) General three-dimensional stress. (b) Plane stress with “cross-shears” equal.



Note that double subscripts are necessary for the shear. The first subscript indicates the direction of the surface normal whereas the second subscript is the direction of the shear stress.

The state of stress at a point described by three mutually perpendicular surfaces is shown in Fig. 3-8a. It can be shown through coordinate transformation that this is sufficient to determine the state of stress on *any* surface intersecting the point. As the dimensions of the cube in Fig. 3-8a approach zero, the stresses on the hidden faces become equal and opposite to those on the opposing visible faces. Thus, in general, a complete state of stress is defined by nine stress components, σ_x , σ_y , σ_z , τ_{xy} , τ_{xz} , τ_{yx} , τ_{yz} , τ_{zx} , and τ_{zy} .

For equilibrium, in most cases, “cross-shears” are equal, hence

$$\tau_{yx} = \tau_{xy} \quad \tau_{zy} = \tau_{yz} \quad \tau_{xz} = \tau_{zx} \quad (3-7)$$

This reduces the number of stress components for most three-dimensional states of stress from nine to six quantities, σ_x , σ_y , σ_z , τ_{xy} , τ_{yz} , and τ_{zx} .

A very common state of stress occurs when the stresses on one surface are zero. When this occurs the state of stress is called *plane stress*. Figure 3-8b shows a state of plane stress, arbitrarily assuming that the normal for the stress-free surface is the z direction such that $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. It is important to note that the element in Fig. 3-8b is still a three-dimensional cube. Also, here it is assumed that the cross-shears are equal such that $\tau_{yx} = \tau_{xy}$, and $\tau_{yz} = \tau_{zy} = \tau_{xz} = \tau_{zx} = 0$.

3-6 Mohr's Circle for Plane Stress

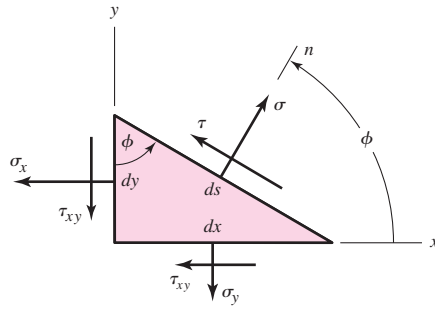
Suppose the $dx \, dy \, dz$ element of Fig. 3-8b is cut by an oblique plane with a normal n at an arbitrary angle ϕ counterclockwise from the x axis as shown in Fig. 3-9. Here, we are concerned with the stresses σ and τ that act upon this oblique plane. By summing the forces caused by all the stress components to zero, the stresses σ and τ are found to be

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi \quad (3-8)$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi \quad (3-9)$$

Equations (3-8) and (3-9) are called the *plane-stress transformation equations*.

Figure 3-9



Differentiating Eq. (3-8) with respect to ϕ and setting the result equal to zero maximizes σ and gives

$$\tan 2\phi_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad (3-10)$$

Equation (3-10) defines two particular values for the angle $2\phi_p$, one of which defines the maximum normal stress σ_1 and the other, the minimum normal stress σ_2 . These two stresses are called the *principal stresses*, and their corresponding directions, the *principal directions*. The angle between the two principal directions is 90° . It is important to note that Eq. (3-10) can be written in the form

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\phi_p - \tau_{xy} \cos 2\phi_p = 0 \quad (a)$$

Comparing this with Eq. (3-9), we see that $\tau = 0$, meaning that the perpendicular *surfaces containing principal stresses have zero shear stresses*.

In a similar manner, we differentiate Eq. (3-9), set the result equal to zero, and obtain

$$\tan 2\phi_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad (3-11)$$

Equation (3-11) defines the two values of $2\phi_s$ at which the shear stress τ reaches an extreme value. The angle between the two surfaces containing the maximum shear stresses is 90° . Equation (3-11) can also be written as

$$\frac{\sigma_x - \sigma_y}{2} \cos 2\phi_p + \tau_{xy} \sin 2\phi_p = 0 \quad (b)$$

Substituting this into Eq. (3-8) yields

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \quad (3-12)$$

Equation (3-12) tells us that the two surfaces containing the maximum shear stresses also contain equal normal stresses of $(\sigma_x + \sigma_y)/2$.

Comparing Eqs. (3-10) and (3-11), we see that $\tan 2\phi_s$ is the negative reciprocal of $\tan 2\phi_p$. This means that $2\phi_s$ and $2\phi_p$ are angles 90° apart, and thus the angles between the surfaces containing the maximum shear stresses and the surfaces containing the principal stresses are $\pm 45^\circ$.

Formulas for the two principal stresses can be obtained by substituting the angle $2\phi_p$ from Eq. (3-10) in Eq. (3-8). The result is

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (3-13)$$

In a similar manner the two extreme-value shear stresses are found to be

$$\tau_1, \tau_2 = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (3-14)$$

Your particular attention is called to the fact that an extreme value of the shear stress *may not be the same as the actual maximum value*. See Sec. 3–7.

It is important to note that the equations given to this point are quite sufficient for performing any plane stress transformation. However, extreme care must be exercised when applying them. For example, say you are attempting to determine the principal state of stress for a problem where $\sigma_x = 14$ MPa, $\sigma_y = -10$ MPa, and $\tau_{xy} = -16$ MPa. Equation (3–10) yields $\phi_p = -26.57^\circ$ and 63.43° , which locate the principal stress surfaces, whereas, Eq. (3–13) gives $\sigma_1 = 22$ MPa and $\sigma_2 = -18$ MPa for the principal stresses. If all we wanted was the principal stresses, we would be finished. However, what if we wanted to draw the element containing the principal stresses properly oriented relative to the x, y axes? Well, we have two values of ϕ_p and two values for the principal stresses. How do we know which value of ϕ_p corresponds to which value of the principal stress? To clear this up we would need to substitute one of the values of ϕ_p into Eq. (3–8) to determine the normal stress corresponding to that angle.

A graphical method for expressing the relations developed in this section, called *Mohr's circle diagram*, is a very effective means of visualizing the stress state at a point and keeping track of the directions of the various components associated with plane stress. Equations (3–8) and (3–9) can be shown to be a set of parametric equations for σ and τ , where the parameter is 2ϕ . The parametric relationship between σ and τ is that of a circle plotted in the σ, τ plane, where the center of the circle is located at $C = (\sigma, \tau) = [(\sigma_x + \sigma_y)/2, 0]$ and has a radius of $R = \sqrt{[(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2}$. A problem arises in the sign of the shear stress. The transformation equations are based on a positive ϕ being counterclockwise, as shown in Fig. 3–9. If a positive τ were plotted above the σ axis, points would rotate clockwise on the circle 2ϕ in the opposite direction of rotation on the element. It would be convenient if the rotations were in the same direction. One could solve the problem easily by plotting positive τ below the axis. However, the classical approach to Mohr's circle uses a different convention for the shear stress.

Mohr's Circle Shear Convention

This convention is followed in drawing Mohr's circle:

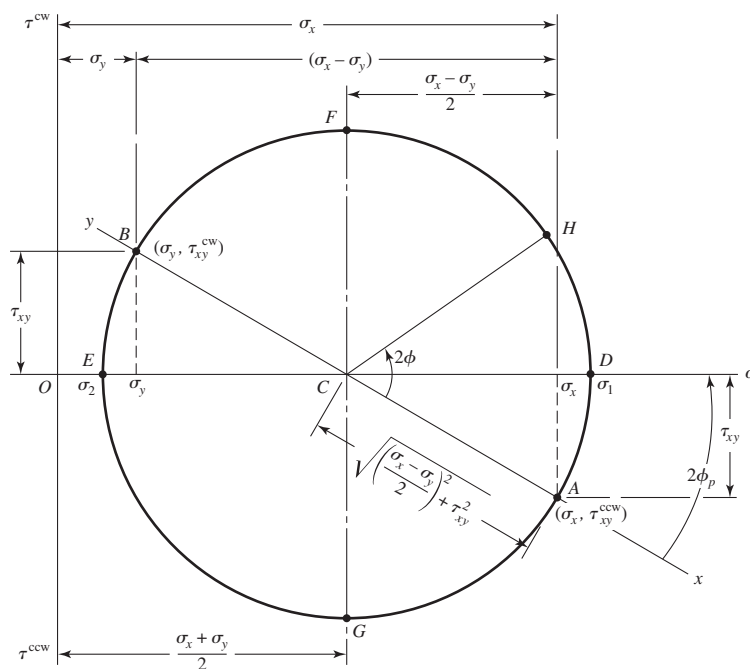
- Shear stresses tending to rotate the element clockwise (cw) are plotted *above* the σ axis.
- Shear stresses tending to rotate the element counterclockwise (ccw) are plotted *below* the σ axis.

For example, consider the right face of the element in Fig. 3–8*b*. By Mohr's circle convention the shear stress shown is plotted *below* the σ axis because it tends to rotate the element counterclockwise. The shear stress on the top face of the element is plotted *above* the σ axis because it tends to rotate the element clockwise.

In Fig. 3–10 we create a coordinate system with normal stresses plotted along the abscissa and shear stresses plotted as the ordinates. On the abscissa, tensile (positive) normal stresses are plotted to the right of the origin O and compressive (negative) normal stresses to the left. On the ordinate, clockwise (cw) shear stresses are plotted up; counterclockwise (ccw) shear stresses are plotted down.

Figure 3-10

Mohr's circle diagram.



Using the stress state of Fig. 3-8b, we plot Mohr's circle, Fig. 3-10, by first looking at the right surface of the element containing σ_x to establish the sign of σ_x and the cw or ccw direction of the shear stress. The right face is called the *x face* where $\phi = 0^\circ$. If σ_x is positive and the shear stress τ_{xy} is ccw as shown in Fig. 3-8b, we can establish point A with coordinates $(\sigma_x, \tau_{xy}^{ccw})$ in Fig. 3-10. Next, we look at the top *y face*, where $\phi = 90^\circ$, which contains σ_y , and repeat the process to obtain point B with coordinates $(\sigma_y, \tau_{xy}^{cw})$ as shown in Fig. 3-10. The two states of stress for the element are $\Delta\phi = 90^\circ$ from each other on the element so they will be $2\Delta\phi = 180^\circ$ from each other on Mohr's circle. Points A and B are the same vertical distance from the σ axis. Thus, AB must be on the diameter of the circle, and the center of the circle C is where AB intersects the σ axis. With points A and B on the circle, and center C , the complete circle can then be drawn. Note that the extended ends of line AB are labeled x and y as references to the normals to the surfaces for which points A and B represent the stresses.

The entire Mohr's circle represents the state of stress at a *single* point in a structure. Each point on the circle represents the stress state for a *specific* surface intersecting the point in the structure. Each pair of points on the circle 180° apart represent the state of stress on an element whose surfaces are 90° apart. Once the circle is drawn, the states of stress can be visualized for various surfaces intersecting the point being analyzed. For example, the principal stresses σ_1 and σ_2 are points D and E , respectively, and their values obviously agree with Eq. (3-13). We also see that the shear stresses are zero on the surfaces containing σ_1 and σ_2 . The two extreme-value shear stresses, one clockwise and one counterclockwise, occur at F and G with magnitudes equal to the radius of the circle. The surfaces at F and G each also contain normal stresses of $(\sigma_x + \sigma_y)/2$ as noted earlier in Eq. (3-12). Finally, the state of stress on an arbitrary surface located at an angle ϕ counterclockwise from the *x face* is point H .

At one time, Mohr's circle was used graphically where it was drawn to scale very accurately and values were measured by using a scale and protractor. Here, we are strictly using Mohr's circle as a visualization aid and will use a semigraphical approach, calculating values from the properties of the circle. This is illustrated by the following example.

EXAMPLE 3-4

A stress element has $\sigma_x = 80$ MPa and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3-11a.

(a) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the xy coordinates. Draw another stress element to show τ_1 and τ_2 , find the corresponding normal stresses, and label the drawing completely.

(b) Repeat part a using the transformation equations only.

Solution

(a) In the semigraphical approach used here, we first make an approximate freehand sketch of Mohr's circle and then use the geometry of the figure to obtain the desired information.

Draw the σ and τ axes first (Fig. 3-11b) and from the x face locate $\sigma_x = 80$ MPa along the σ axis. On the x face of the element, we see that the shear stress is 50 MPa in the cw direction. Thus, for the x face, this establishes point A (80, 50^{cw}) MPa. Corresponding to the y face, the stress is $\sigma = 0$ and $\tau = 50$ MPa in the ccw direction. This locates point B (0, 50^{ccw}) MPa. The line AB forms the diameter of the required circle, which can now be drawn. The intersection of the circle with the σ axis defines σ_1 and σ_2 as shown. Now, noting the triangle ACD , indicate on the sketch the length of the legs AD and CD as 50 and 40 MPa, respectively. The length of the hypotenuse AC is

Answer

$$\tau_1 = \sqrt{(50)^2 + (40)^2} = 64.0 \text{ MPa}$$

and this should be labeled on the sketch too. Since intersection C is 40 MPa from the origin, the principal stresses are now found to be

Answer

$$\sigma_1 = 40 + 64 = 104 \text{ MPa} \quad \text{and} \quad \sigma_2 = 40 - 64 = -24 \text{ MPa}$$

The angle 2ϕ from the x axis cw to σ_1 is

Answer

$$2\phi_p = \tan^{-1} \frac{50}{40} = 51.3^\circ$$

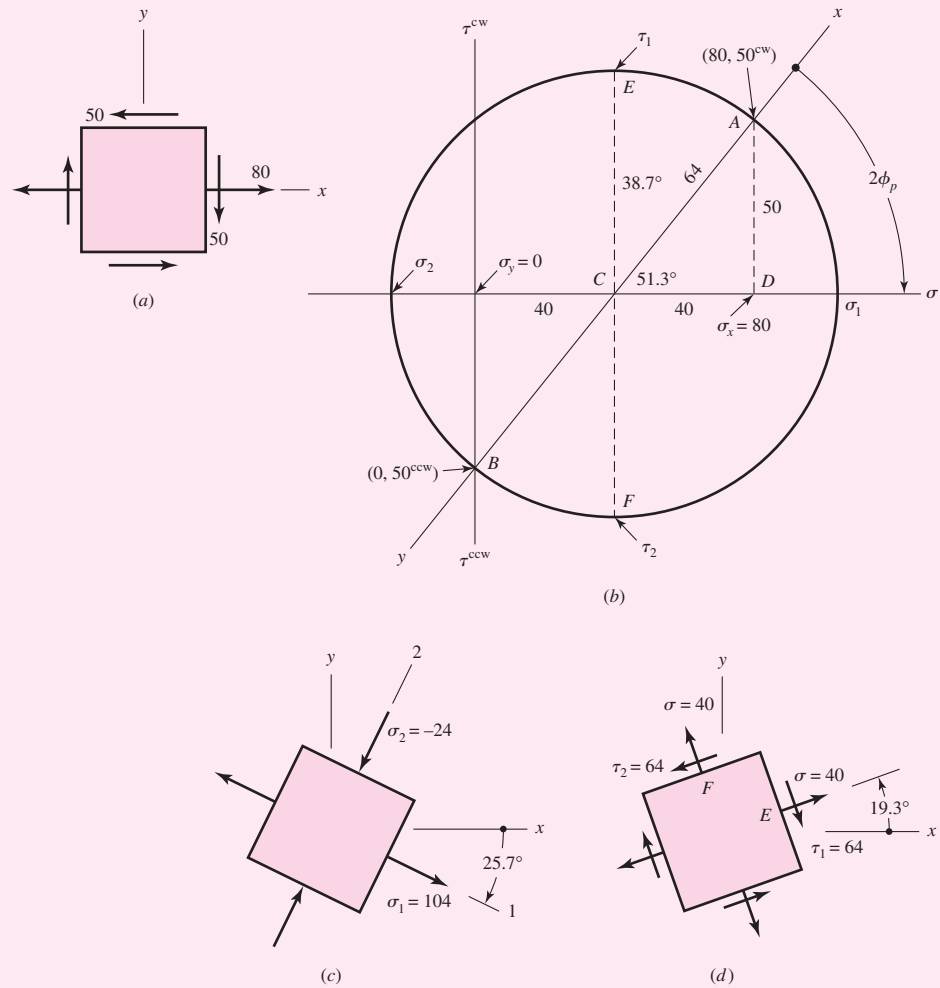
To draw the principal stress element (Fig. 3-11c), sketch the x and y axes parallel to the original axes. The angle ϕ_p on the stress element must be measured in the *same* direction as is the angle $2\phi_p$ on the Mohr circle. Thus, from x measure 25.7° (half of 51.3°) clockwise to locate the σ_1 axis. The σ_2 axis is 90° from the σ_1 axis and the stress element can now be completed and labeled as shown. Note that there are *no* shear stresses on this element.

The two maximum shear stresses occur at points E and F in Fig. 3-11b. The two normal stresses corresponding to these shear stresses are each 40 MPa, as indicated. Point E is 38.7° ccw from point A on Mohr's circle. Therefore, in Fig. 3-11d, draw a stress element oriented 19.3° (half of 38.7°) ccw from x . The element should then be labeled with magnitudes and directions as shown.

In constructing these stress elements it is important to indicate the x and y directions of the original reference system. This completes the link between the original machine element and the orientation of its principal stresses.

Figure 3-11

All stresses in MPa.

**Answer**

(b) The transformation equations are programmable. From Eq. (3-10),

$$\phi_p = \frac{1}{2} \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) = \frac{1}{2} \tan^{-1} \left(\frac{2(-50)}{80} \right) = -25.7^\circ, 64.3^\circ$$

From Eq. (3-8), for the first angle $\phi_p = -25.7^\circ$,

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(-25.7)] + (-50) \sin[2(-25.7)] = 104.03 \text{ MPa}$$

The shear on this surface is obtained from Eq. (3-9) as

$$\tau = -\frac{80 - 0}{2} \sin[2(-25.7)] + (-50) \cos[2(-25.7)] = 0 \text{ MPa}$$

which confirms that 104.03 MPa is a principal stress. From Eq. (3-8), for $\phi_p = 64.3^\circ$,

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(64.3)] + (-50) \sin[2(64.3)] = -24.03 \text{ MPa}$$

Answer Substituting $\phi_p = 64.3^\circ$ into Eq. (3-9) again yields $\tau = 0$, indicating that -24.03 MPa is also a principal stress. Once the principal stresses are calculated they can be ordered such that $\sigma_1 \geq \sigma_2$. Thus, $\sigma_1 = 104.03$ MPa and $\sigma_2 = -24.03$ MPa.

Since for $\sigma_1 = 104.03$ MPa, $\phi_p = -25.7^\circ$, and since ϕ is defined positive ccw in the transformation equations, we rotate *clockwise* 25.7° for the surface containing σ_1 . We see in Fig. 3-11c that this totally agrees with the semigraphical method.

To determine τ_1 and τ_2 , we first use Eq. (3-11) to calculate ϕ_s :

$$\phi_s = \frac{1}{2} \tan^{-1} \left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \right) = \frac{1}{2} \tan^{-1} \left(-\frac{80}{2(-50)} \right) = 19.3^\circ, 109.3^\circ$$

For $\phi_s = 19.3^\circ$, Eqs. (3-8) and (3-9) yield

Answer

$$\sigma = \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos[2(19.3)] + (-50) \sin[2(19.3)] = 40.0 \text{ MPa}$$

$$\tau = -\frac{80 - 0}{2} \sin[2(19.3)] + (-50) \cos[2(19.3)] = -64.0 \text{ MPa}$$

Remember that Eqs. (3-8) and (3-9) are *coordinate* transformation equations. Imagine that we are rotating the x, y axes 19.3° counterclockwise and y will now point up and to the left. So a negative shear stress on the rotated x face will point down and to the right as shown in Fig. 3-11d. Thus again, results agree with the semigraphical method.

For $\phi_s = 109.3^\circ$, Eqs. (3-8) and (3-9) give $\sigma = 40.0$ MPa and $\tau = +64.0$ MPa. Using the same logic for the coordinate transformation we find that results again agree with Fig. 3-11d.

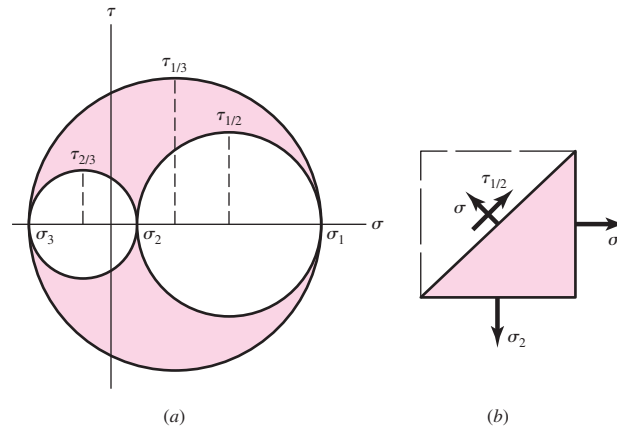
3-7 General Three-Dimensional Stress

As in the case of plane stress, a particular orientation of a stress element occurs in space for which all shear-stress components are zero. When an element has this particular orientation, the normals to the faces are mutually orthogonal and correspond to the principal directions, and the normal stresses associated with these faces are the principal stresses. Since there are three faces, there are three principal directions and three principal stresses σ_1, σ_2 , and σ_3 . For plane stress, the stress-free surface contains the third principal stress which is zero.

In our studies of plane stress we were able to specify any stress state σ_x, σ_y , and τ_{xy} and find the principal stresses and principal directions. But six components of stress are required to specify a general state of stress in three dimensions, and the problem of determining the principal stresses and directions is more difficult. In design, three-dimensional transformations are rarely performed since most maximum stress states occur under plane stress conditions. One notable exception is contact stress, which is not a case of plane stress, where the three principal stresses are given in Sec. 3-19. In fact, *all* states of stress are truly three-dimensional, where they might be described one- or two-dimensionally with respect to *specific* coordinate axes. Here it is most important to understand the relationship among the *three* principal stresses. The process in finding the three principal stresses from the six

Figure 3-12

Mohr's circles for three-dimensional stress.



stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$ and τ_{zx} , involves finding the roots of the cubic equation¹

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \quad (3-15)$$

In plotting Mohr's circles for three-dimensional stress, the principal normal stresses are ordered so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Then the result appears as in Fig. 3-12a. The stress coordinates σ, τ for any arbitrarily located plane will always lie on the boundaries or within the shaded area.

Figure 3-12a also shows the three *principal shear stresses* $\tau_{1/2}, \tau_{2/3},$ and $\tau_{1/3}$.² Each of these occurs on the two planes, one of which is shown in Fig. 3-12b. The figure shows that the principal shear stresses are given by the equations

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \quad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \quad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \quad (3-16)$$

Of course, $\tau_{\max} = \tau_{1/3}$ when the normal principal stresses are ordered ($\sigma_1 > \sigma_2 > \sigma_3$), so always order your principal stresses. Do this in any computer code you generate and you'll always generate $\tau_{\max}$.

3-8 Elastic Strain

Normal strain ϵ is defined and discussed in Sec. 2-1 for the tensile specimen and is given by Eq. (2-2) as $\epsilon = \delta/l$, where δ is the total elongation of the bar within the length l . Hooke's law for the tensile specimen is given by Eq. (2-3) as

$$\sigma = E\epsilon \quad (3-17)$$

where the constant E is called *Young's modulus* or the *modulus of elasticity*.

¹For development of this equation and further elaboration of three-dimensional stress transformations see: Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 46-78.

²Note the difference between this notation and that for a shear stress, say, τ_{xy} . The use of the shilling mark is not accepted practice, but it is used here to emphasize the distinction.

When a material is placed in tension, there exists not only an axial strain, but also negative strain (contraction) perpendicular to the axial strain. Assuming a linear, homogeneous, isotropic material, this lateral strain is proportional to the axial strain. If the axial direction is x , then the lateral strains are $\epsilon_y = \epsilon_z = -\nu\epsilon_x$. The constant of proportionality ν is called *Poisson's ratio*, which is about 0.3 for most structural metals. See Table A-5 for values of ν for common materials.

If the axial stress is in the x direction, then from Eq. (3-17)

$$\epsilon_x = \frac{\sigma_x}{E} \quad \epsilon_y = \epsilon_z = -\nu \frac{\sigma_x}{E} \quad (3-18)$$

For a stress element undergoing σ_x , σ_y , and σ_z simultaneously, the normal strains are given by

$$\begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \quad (3-19)$$

Shear strain γ is the change in a right angle of a stress element when subjected to pure shear stress, and Hooke's law for shear is given by

$$\tau = G\gamma \quad (3-20)$$

where the constant G is the *shear modulus of elasticity* or *modulus of rigidity*.

It can be shown for a linear, isotropic, homogeneous material, the three elastic constants are related to each other by

$$E = 2G(1 + \nu) \quad (3-21)$$

3-9 Uniformly Distributed Stresses

The assumption of a uniform distribution of stress is frequently made in design. The result is then often called *pure tension*, *pure compression*, or *pure shear*, depending upon how the external load is applied to the body under study. The word *simple* is sometimes used instead of *pure* to indicate that there are no other complicating effects. The tension rod is typical. Here a tension load F is applied through pins at the ends of the bar. The assumption of uniform stress means that if we cut the bar at a section remote from the ends and remove one piece, we can replace its effect by applying a uniformly distributed force of magnitude σA to the cut end. So the stress σ is said to be uniformly distributed. It is calculated from the equation

$$\sigma = \frac{F}{A} \quad (3-22)$$

This assumption of uniform stress distribution requires that:

- The bar be straight and of a homogeneous material
- The line of action of the force contains the centroid of the section
- The section be taken remote from the ends and from any discontinuity or abrupt change in cross section

For simple compression, Eq. (3–22) is applicable with F normally being considered a negative quantity. Also, a slender bar in compression may fail by buckling, and this possibility must be eliminated from consideration before Eq. (3–22) is used.³

Another type of loading that assumes a uniformly distributed stress is known as *direct shear*. This occurs when there is a shearing action with no bending. An example is the action on a piece of sheet metal caused by the two blades of tin snips. Bolts and pins that are loaded in shear often have direct shear. Think of a cantilever beam with a force pushing down on it. Now move the force all the way up to the wall so there is no bending moment, just a force trying to shear the beam off the wall. This is direct shear. Direct shear is usually assumed to be uniform across the cross section, and is given by

$$\tau = \frac{V}{A} \quad (3-23)$$

where V is the shear force and A is the area of the cross section that is being sheared. The assumption of uniform stress is not accurate, particularly in the vicinity where the force is applied, but the assumption generally gives acceptable results.

3-10 Normal Stresses for Beams in Bending

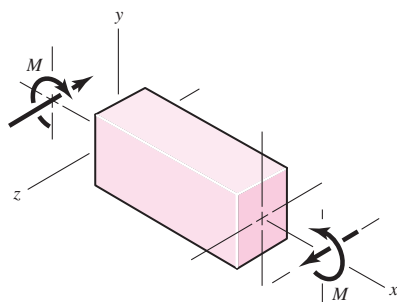
The equations for the normal bending stresses in straight beams are based on the following assumptions.

- The beam is subjected to pure bending. This means that the shear force is zero, and that no torsion or axial loads are present (for most engineering applications it is assumed that these loads affect the bending stresses minimally).
- The material is isotropic and homogeneous.
- The material obeys Hooke's law.
- The beam is initially straight with a cross section that is constant throughout the beam length.
- The beam has an axis of symmetry in the plane of bending.
- The proportions of the beam are such that it would fail by bending rather than by crushing, wrinkling, or sidewise buckling.
- Plane cross sections of the beam remain plane during bending.

In Fig. 3–13 we visualize a portion of a straight beam acted upon by a positive bending moment M shown by the curved arrow showing the physical action of the moment together with a straight arrow indicating the moment vector. The x axis is coincident with the *neutral axis* of the section, and the xz plane, which contains the

Figure 3-13

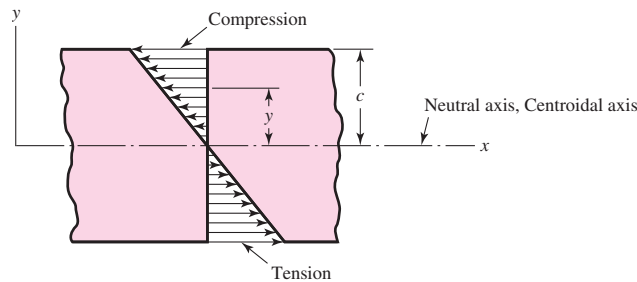
Straight beam in positive bending.



³See Sec. 4–11.

Figure 3-14

Bending stresses according to Eq. (3-24).



neutral axes of all cross sections, is called the *neutral plane*. Elements of the beam coincident with this plane have zero stress. The location of the neutral axis with respect to the cross section is coincident with the *centroidal axis* of the cross section.

The bending stress varies linearly with the distance from the neutral axis, y , and is given by

$$\sigma_x = -\frac{My}{I} \quad (3-24)$$

where I is the *second-area moment* about the z axis. That is,

$$I = \int y^2 dA \quad (3-25)$$

The stress distribution given by Eq. (3-24) is shown in Fig. 3-14. The maximum magnitude of the bending stress will occur where y has the greatest magnitude. Designating $\sigma_{\max}$ as the maximum *magnitude* of the bending stress, and c as the maximum *magnitude* of y

$$\sigma_{\max} = \frac{Mc}{I} \quad (3-26a)$$

Equation (3-24) can still be used to ascertain whether $\sigma_{\max}$ is tensile or compressive.

Equation (3-26a) is often written as

$$\sigma_{\max} = \frac{M}{Z} \quad (3-26b)$$

where $Z = I/c$ is called the *section modulus*.

EXAMPLE 3-5

A beam having a T section with the dimensions shown in Fig. 3-15 is subjected to a bending moment of $1600 \text{ N} \cdot \text{m}$, about the negative z axis, that causes tension at the top surface. Locate the neutral axis and find the maximum tensile and compressive bending stresses.

Solution

Dividing the T section into two rectangles, numbered 1 and 2, the total area is $A = 12(75) + 12(88) = 1956 \text{ mm}^2$. Summing the area moments of these rectangles about the top edge, where the moment arms of areas 1 and 2 are 6 mm and $(12 + 88/2) = 56 \text{ mm}$ respectively, we have

$$1956c_1 = 12(75)(6) + 12(88)(56)$$

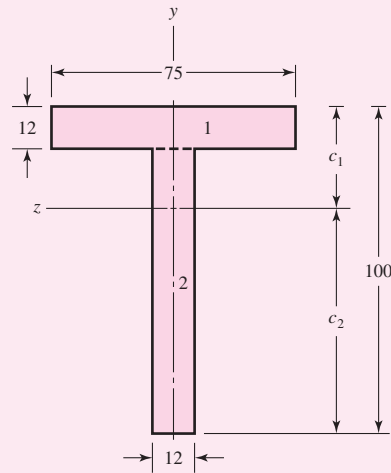
and hence $c_1 = 32.99 \text{ mm}$. Therefore $c_2 = 100 - 32.99 = 67.01 \text{ mm}$.

Next we calculate the second moment of area of each rectangle about its own centroidal axis. Using Table A-18, we find for the top rectangle

$$I_1 = \frac{1}{12}bh^3 = \frac{1}{12}(75)12^3 = 1.080 \times 10^4 \text{ mm}^4$$

Figure 3-15

Dimensions in millimeters.



For the bottom rectangle, we have

$$I_2 = \frac{1}{12}(12)88^3 = 6.815 \times 10^5 \text{ mm}^4$$

We now employ the *parallel-axis theorem* to obtain the second moment of area of the composite figure about its own centroidal axis. This theorem states

$$I_z = I_{ca} + Ad^2$$

where I_{ca} is the second moment of area about its own centroidal axis and I_z is the second moment of area about any parallel axis a distance d removed. For the top rectangle, the distance is

$$d_1 = 32.99 - 6 = 26.99 \text{ mm}$$

and for the bottom rectangle,

$$d_2 = 67.01 - \frac{88}{2} = 23.01 \text{ mm}$$

Using the parallel-axis theorem for both rectangles, we now find that

$$\begin{aligned} I &= [1.080 \times 10^4 + 12(75)26.99^2] + [6.815 \times 10^5 + 12(88)23.01^2] \\ &= 1.907 \times 10^6 \text{ mm}^4 \end{aligned}$$

Finally, the maximum tensile stress, which occurs at the top surface, is found to be

Answer
$$\sigma = \frac{Mc_1}{I} = \frac{1600(32.99)10^{-3}}{1.907(10^{-6})} = 27.68(10^6) \text{ Pa} = 27.68 \text{ MPa}$$

Similarly, the maximum compressive stress at the lower surface is found to be

Answer
$$\sigma = -\frac{Mc_2}{I} = -\frac{1600(67.01)10^{-3}}{1.907(10^{-6})} = -56.22(10^6) \text{ Pa} = -56.22 \text{ MPa}$$

Two-Plane Bending

Quite often, in mechanical design, bending occurs in both xy and xz planes. Considering cross sections with one or two planes of symmetry only, the bending stresses are given by

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \quad (3-27)$$

where the first term on the right side of the equation is identical to Eq. (3-24), M_y is the bending moment in the xz plane (moment vector in y direction), z is the distance from the neutral y axis, and I_y is the second area moment about the y axis.

For *noncircular* cross sections, Eq. (3-27) is the superposition of stresses caused by the two bending moment components. The maximum tensile and compressive bending stresses occur where the summation gives the greatest positive and negative stresses, respectively. For solid *circular* cross sections, all lateral axes are the same and the plane containing the moment corresponding to the vector sum of M_z and M_y contains the maximum bending stresses. For a beam of diameter d the maximum distance from the neutral axis is $d/2$, and from Table A-18, $I = \pi d^4/64$. The maximum bending stress for a solid circular cross section is then

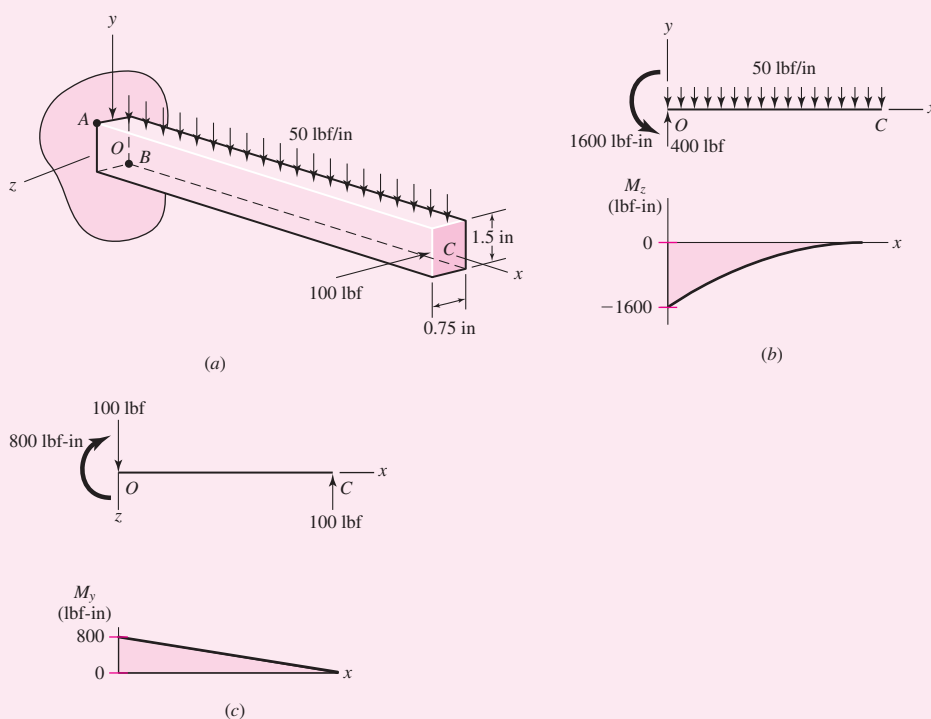
$$\sigma_m = \frac{M c}{I} = \frac{(M_y^2 + M_z^2)^{1/2} (d/2)}{\pi d^4/64} = \frac{32}{\pi d^3} (M_y^2 + M_z^2)^{1/2} \quad (3-28)$$

EXAMPLE 3-6

As shown in Fig. 3-16a, beam OC is loaded in the xy plane by a uniform load of 50 lbf/in, and in the xz plane by a concentrated force of 100 lbf at end C . The beam is 8 in long.

Figure 3-16

(a) Beam loaded in two planes; (b) loading and bending-moment diagrams in xy plane; (c) loading and bending-moment diagrams in xz plane.



(a) For the cross section shown determine the maximum tensile and compressive bending stresses and where they act.

(b) If the cross section was a solid circular rod of diameter, $d = 1.25$ in, determine the magnitude of the maximum bending stress.

Solution

(a) The reactions at O and the bending-moment diagrams in the xy and xz planes are shown in Figs. 3-16*b* and *c*, respectively. The maximum moments in both planes occur at O where

$$(M_z)_O = -\frac{1}{2}(50)8^2 = -1600 \text{ lbf-in} \quad (M_y)_O = 100(8) = 800 \text{ lbf-in}$$

The second moments of area in both planes are

$$I_z = \frac{1}{12}(0.75)1.5^3 = 0.2109 \text{ in}^4 \quad I_y = \frac{1}{12}(1.5)0.75^3 = 0.05273 \text{ in}^4$$

The maximum tensile stress occurs at point A , shown in Fig. 3-16*a*, where the maximum tensile stress is due to both moments. At A , $y_A = 0.75$ in and $z_A = 0.375$ in. Thus, from Eq. (3-27)

$$(\sigma_x)_A = -\frac{-1600(0.75)}{0.2109} + \frac{800(0.375)}{0.05273} = 11\,380 \text{ psi} = 11.38 \text{ kpsi}$$

The maximum compressive bending stress occurs at point B where, $y_B = -0.75$ in and $z_B = -0.375$ in. Thus

$$(\sigma_x)_B = -\frac{-1600(-0.75)}{0.2109} + \frac{800(-0.375)}{0.05273} = -11\,380 \text{ psi} = -11.38 \text{ kpsi}$$

(b) For a solid circular cross section of diameter, $d = 1.25$ in, the maximum bending stress at end O is given by Eq. (3-28) as

$$\sigma_m = \frac{32}{\pi(1.25)^3} [800^2 + (-1600)^2]^{1/2} = 9329 \text{ psi} = 9.329 \text{ kpsi}$$

Beams with Asymmetrical Sections⁴

The bending stress equations, given by Eqs. (3-24) and (3-27), can also be applied to beams having asymmetrical cross sections, provided the planes of bending coincide with the *area principal axes* of the section. The method for determining the orientation of the area principal axes and the values of the corresponding *principal second-area moments* can be found in any statics book. If a section has an axis of symmetry, that axis and its perpendicular axis are the area principal axes.

For example, consider a beam in bending, using an equal leg angle as shown in Table A-6. Equation (3-27) cannot be used if the bending moments are resolved about axis 1-1 and/or axis 2-2. However, Eq. (3-27) can be used if the moments are resolved

⁴For further discussion, see Sec. 5.3, Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999.

about axis 3–3 and its perpendicular axis (let us call it, say, axis 4–4). Note, for this cross section, axis 4–4 is an axis of symmetry. Table A–6 is a standard table, and for brevity, does not directly give all the information needed to use it. The orientation of the area principal axes and the values of I_{2-2} , I_{3-3} , and I_{4-4} are not given because they can be determined as follows. Since the legs are equal, the principal axes are oriented $\pm 45^\circ$ from axis 1–1, and $I_{2-2} = I_{1-1}$. The second-area moment I_{3-3} is given by

$$I_{3-3} = A(k_{3-3})^2 \quad (a)$$

where k_{3-3} is called the *radius of gyration*. The sum of the second-area moments for a cross section is invariant, so $I_{1-1} + I_{2-2} = I_{3-3} + I_{4-4}$. Thus, I_{4-4} is given by

$$I_{4-4} = 2 I_{1-1} - I_{3-3} \quad (b)$$

where $I_{2-2} = I_{1-1}$. For example, consider a $3 \times 3 \times \frac{1}{4}$ angle. Using Table A–6 and Eqs. (a) and (b), $I_{3-3} = 1.44 (0.592)^2 = 0.505 \text{ in}^4$, and $I_{4-4} = 2 (1.24) - 0.505 = 1.98 \text{ in}^4$.

3–11 Shear Stresses for Beams in Bending

Most beams have both shear forces and bending moments present. It is only occasionally that we encounter beams subjected to pure bending, that is to say, beams having zero shear force. The flexure formula is developed on the assumption of pure bending. This is done, however, to eliminate the complicating effects of shear force in the development. For engineering purposes, the flexure formula is valid no matter whether a shear force is present or not. For this reason, we shall utilize the same normal bending-stress distribution [Eqs. (3–24) and (3–26)] when shear forces are also present.

In Fig. 3–17a we show a beam segment of constant cross section subjected to a shear force V and a bending moment M at x . Because of external loading and V , the shear force and bending moment change with respect to x . At $x + dx$ the shear force and bending moment are $V + dV$ and $M + dM$, respectively. Considering forces in the x direction only, Fig. 3–17b shows the stress distribution σ_x due to the bending moments. If dM is positive, with the bending moment increasing, the stresses on the right face, for a given value of y , are larger in magnitude than the stresses on the left face. If we further isolate the element by making a slice at $y = y_1$ (see Fig. 3–17b), the net *force* in the x direction will be directed to the left with a value of

$$\int_{y_1}^c \frac{(dM)y}{I} dA$$

as shown in the rotated view of Fig. 3–17c. For equilibrium, a shear force on the bottom face, directed to the right, is required. This shear force gives rise to a shear stress τ , where, if assumed uniform, the force is $\tau b dx$. Thus

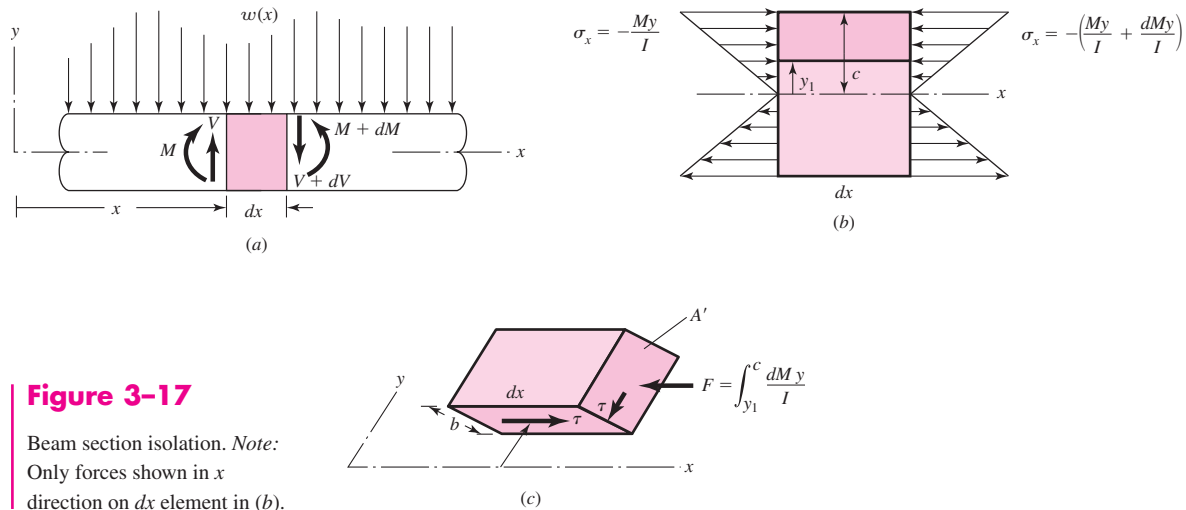
$$\tau b dx = \int_{y_1}^c \frac{(dM)y}{I} dA \quad (a)$$

The term dM/I can be removed from within the integral and $b dx$ placed on the right side of the equation; then, from Eq. (3–3) with $V = dM/dx$, Eq. (a) becomes

$$\tau = \frac{V}{Ib} \int_{y_1}^c y dA \quad (3-29)$$

In this equation, the integral is the first moment of the area A' with respect to the neutral axis (see Fig. 3–17c). This integral is usually designated as Q . Thus

$$Q = \int_{y_1}^c y dA = \bar{y}' A' \quad (3-30)$$

**Figure 3-17**

Beam section isolation. *Note:* Only forces shown in x direction on dx element in (b).

where, for the isolated area y_1 to c , $\bar{y}'$ is the distance in the y direction from the neutral plane to the centroid of the area A' . With this, Eq. (3-29) can be written as

$$\tau = \frac{VQ}{Ib} \quad (3-31)$$

This stress is known as the *transverse shear stress*. It is always accompanied with bending stress.

In using this equation, note that b is the width of the section at $y = y_1$. Also, I is the second moment of area of the entire section about the neutral axis.

Because cross shears are equal, and area A' is *finite*, the shear stress τ given by Eq. (3-31) and shown on area A' in Fig. 3-17c occurs only at $y = y_1$. The shear stress on the lateral area varies with y , normally maximum at $y = 0$ (where $\bar{y}'A'$ is maximum) and zero at the outer fibers of the beam where $A' = 0$.

The shear stress distribution in a beam depends on how Q/b varies as a function of y_1 . Here we will show how to determine the shear stress distribution for a beam with a rectangular cross section and provide results of maximum values of shear stress for other standard cross sections. Figure 3-18 shows a portion of a beam with a rectangular cross section, subjected to a shear force V and a bending moment M . As a result of the bending moment, a normal stress σ is developed on a cross section such as $A-A$, which is in compression above the neutral axis and in tension below. To investigate the shear stress at a distance y_1 above the neutral axis, we select an element of area dA at a distance y above the neutral axis. Then, $dA = b dy$, and so Eq. (3-30) becomes

$$Q = \int_{y_1}^c y dA = b \int_{y_1}^c y dy = \frac{by^2}{2} \Big|_{y_1}^c = \frac{b}{2} (c^2 - y_1^2) \quad (b)$$

Substituting this value for Q into Eq. (3-31) gives

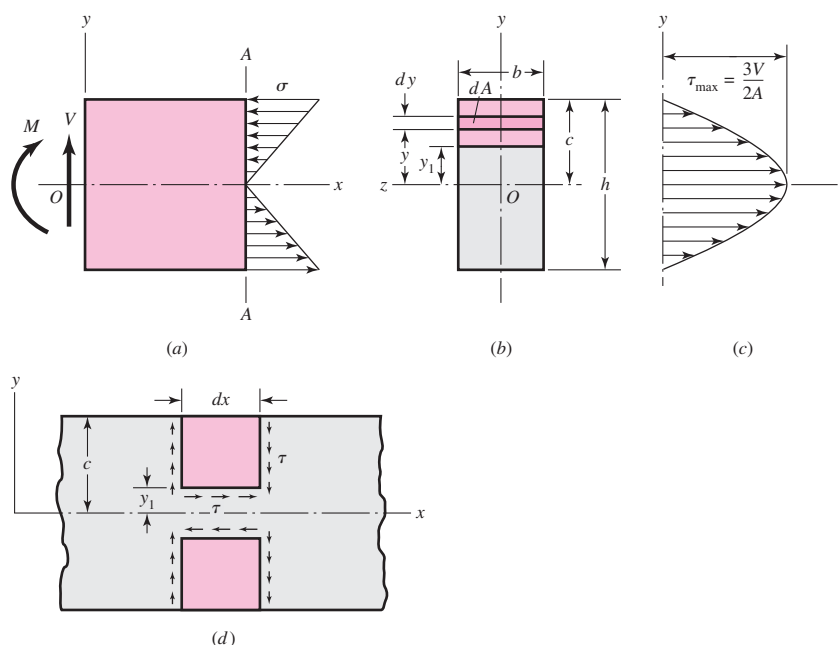
$$\tau = \frac{V}{2I} (c^2 - y_1^2) \quad (3-32)$$

This is the general equation for shear stress in a rectangular beam. To learn something about it, let us make some substitutions. From Table A-18, the second moment of area for a rectangular section is $I = bh^3/12$; substituting $h = 2c$ and $A = bh = 2bc$ gives

$$I = \frac{Ac^2}{3} \quad (c)$$

Figure 3-18

Transverse shear stresses in a rectangular beam.



If we now use this value of I for Eq. (3-32) and rearrange, we get

$$\tau = \frac{3V}{2A} \left(1 - \frac{y_1^2}{c^2} \right) \quad (3-33)$$

We note that the maximum shear stress exists when $y_1 = 0$, which is at the bending neutral axis. Thus

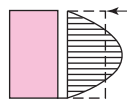
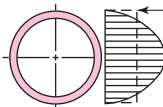
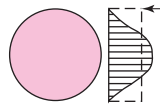
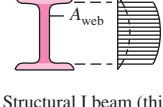
$$\tau_{\max} = \frac{3V}{2A} \quad (3-34)$$

for a rectangular section. As we move away from the neutral axis, the shear stress decreases parabolically until it is zero at the outer surfaces where $y_1 = \pm c$, as shown in Fig. 3-18c. Horizontal shear stress is always accompanied by vertical shear stress of the same magnitude, and so the distribution can be diagrammed as shown in Fig. 3-18d. Figure 3-18c shows that the shear τ on the vertical surfaces varies with y . We are almost always interested in the horizontal shear, τ in Fig. 3-18d, which is nearly uniform over dx with constant $y = y_1$. The maximum horizontal shear occurs where the vertical shear is largest. This is usually at the neutral axis but may not be if the width b is smaller somewhere else. Furthermore, if the section is such that b can be minimized on a plane not horizontal, then the horizontal shear stress occurs on an inclined plane. For example, with tubing, the horizontal shear stress occurs on a radial plane and the corresponding “vertical shear” is not vertical, but tangential.

The distributions of transverse shear stresses for several commonly used cross sections are shown in Table 3-2. The profiles represent the VQ/Ib relationship, which is a function of the distance y from the neutral axis. For each profile, the formula for the maximum value at the neutral axis is given. Note that the expression given for the I beam is a commonly used approximation that is reasonable for a standard I beam with a thin web. Also, the profile for the I beam is idealized. In reality the transition from the web to the flange is quite complex locally, and not simply a step change.

Table 3-2

Formulas for Maximum Transverse Shear Stress from VQ/Ib

Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{\max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{\max} = \frac{2V}{A}$
 Circular	$\tau_{\max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{\max} = \frac{V}{A_{\text{web}}}$

It is significant to observe that the transverse shear stress in each of these common cross sections is maximum on the neutral axis, and zero on the outer surfaces. Since this is exactly the opposite of where the bending and torsional stresses have their maximum and minimum values, the transverse shear stress is often not critical from a design perspective.

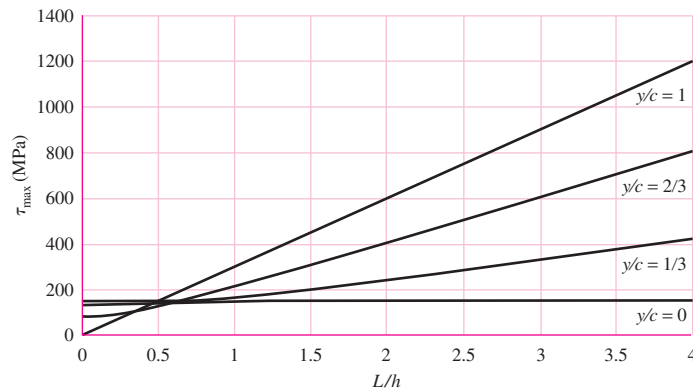
Let us examine the significance of the transverse shear stress, using as an example a cantilever beam of length L , with rectangular cross section $b \times h$, loaded at the free end with a transverse force F . At the wall, where the bending moment is the largest, at a distance y from the neutral axis, a stress element will include both bending stress and transverse shear stress. In Sec. 5-4 it will be shown that a good measure of the combined effects of multiple stresses on a stress element is the maximum shear stress. Inserting the bending stress (My/I) and the transverse shear stress (VQ/Ib) into the maximum shear stress equation, Eq. (3-14), we obtain a general equation for the maximum shear stress in a cantilever beam with a rectangular cross section. This equation can then be normalized with respect to L/h and y/c , where c is the distance from the neutral axis to the outer surface ($h/2$), to give

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{3F}{2bh} \sqrt{4(L/h)^2(y/c)^2 + [1 - (y/c)^2]^2} \quad (d)$$

To investigate the significance of transverse shear stress, we plot $\tau_{\max}$ as a function of L/h for several values of y/c , as shown in Fig. 3-19. Since F and b appear only as linear multipliers outside the radical, they will only serve to scale the plot in the vertical direction without changing any of the relationships. Notice that at the neutral axis where $y/c = 0$, $\tau_{\max}$ is constant for any length beam, since the bending stress is zero at the neutral axis and the transverse shear stress is independent of L . On the other hand, on the outer surface where $y/c = 1$, $\tau_{\max}$ increases linearly with L/h because of the bending moment. For y/c between zero and one, $\tau_{\max}$ is nonlinear for low values of L/h , but behaves linearly as L/h increases, displaying the dominance of the bending stress as the moment arm increases. We can see from the graph that the critical stress element (the largest value of $\tau_{\max}$) will always be either on the outer surface ($y/c = 1$) or at the neutral axis ($y/c = 0$), and never between. Thus, for the rectangular cross section, the transition between these two locations occurs at $L/h = 0.5$ where the line for $y/c = 1$ crosses the horizontal line for $y/c = 0$. The critical stress element is either on the outer

Figure 3-19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.



surface where the transverse shear is zero, or if L/h is small enough, it is on the neutral axis where the bending stress is zero.

The conclusions drawn from Fig. 3-19 are generally similar for any cross section that does not increase in width farther away from the neutral axis. This notably includes solid round cross sections, but not I beams or channels. Care must be taken with I beams and channels that have thin webs that extend far enough from the neutral axis that the bending and shear may both be significant on the same stress element (See Ex. 3-7). For any common cross section beam, if the beam length to height ratio is greater than 10, the transverse shear stress is generally considered negligible compared to the bending stress at any point within the cross section.

EXAMPLE 3-7

A beam 12 in long is to support a load of 488 lbf acting 3 in from the left support, as shown in Fig. 3-20a. The beam is an I beam with the cross-sectional dimensions shown. To simplify the calculations, assume a cross section with square corners, as shown in Fig. 3-20c. Points of interest are labeled (*a*), (*b*), (*c*), and (*d*) at distances y from the neutral axis of 0 in, 1.240⁻ in, 1.240⁺ in, and 1.5 in (Fig. 3-20c). At the critical axial location along the beam, find the following information.

- Determine the profile of the distribution of the transverse shear stress, obtaining values at each of the points of interest.
- Determine the bending stresses at the points of interest.
- Determine the maximum shear stresses at the points of interest, and compare them.

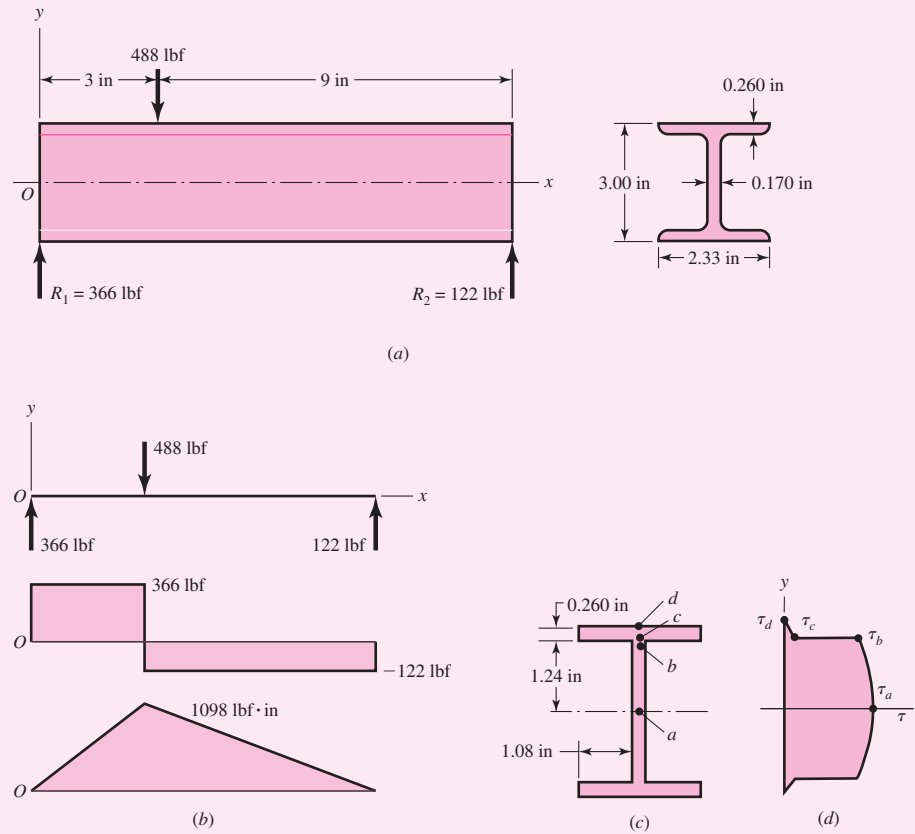
Solution

First, we note that the transverse shear stress is not likely to be negligible in this case since the beam length to height ratio is much less than 10, and since the thin web and wide flange will allow the transverse shear to be large. The loading, shear-force, and bending-moment diagrams are shown in Fig. 3-20b. The critical axial location is at $x = 3^-$ where the shear force and the bending moment are both maximum.

(a) We obtain the area moment of inertia I by evaluating I for a solid 3.0-in $\times$ 2.33-in rectangular area, and then subtracting the two rectangular areas that are not part of the cross section.

$$I = \frac{(2.33)(3.00)^3}{12} - 2 \left[\frac{(1.08)(2.48)^3}{12} \right] = 2.50 \text{ in}^4$$

Figure 3-20



Finding Q at each point of interest using Eq. (3-30) gives

$$Q_a = \left(1.24 + \frac{0.260}{2}\right) [(2.33)(0.260)] + \left(\frac{1.24}{2}\right) [(1.24)(0.170)] = 0.961 \text{ in}^3$$

$$Q_b = Q_c = \left(1.24 + \frac{0.260}{2}\right) [(2.33)(0.260)] = 0.830 \text{ in}^3$$

$$Q_d = (1.5)(0) = 0 \text{ in}^3$$

Applying Eq. (3-31) at each point of interest, with V and I constant for each point, and b equal to the width of the cross section at each point, shows that the magnitudes of the transverse shear stresses are

Answer

$$\tau_a = \frac{VQ_a}{Ib_a} = \frac{(366)(0.961)}{(2.50)(0.170)} = 828 \text{ psi}$$

$$\tau_b = \frac{VQ_b}{Ib_b} = \frac{(366)(0.830)}{(2.50)(0.170)} = 715 \text{ psi}$$

$$\tau_c = \frac{VQ_c}{Ib_c} = \frac{(366)(0.830)}{(2.50)(2.33)} = 52.2 \text{ psi}$$

$$\tau_d = \frac{VQ_d}{Ib_d} = \frac{(366)(0)}{(2.50)(2.33)} = 0 \text{ psi}$$

The magnitude of the idealized transverse shear stress profile through the beam depth will be as shown in Fig. 3–20*d*.

(*b*) The bending stresses at each point of interest are

Answer

$$\sigma_a = \frac{My_a}{I} = \frac{(1098)(0)}{2.50} = 0 \text{ psi}$$

$$\sigma_b = \sigma_c = -\frac{My_b}{I} = -\frac{(1098)(1.24)}{2.50} = -545 \text{ psi}$$

$$\sigma_d = -\frac{My_d}{I} = -\frac{(1098)(1.50)}{2.50} = -659 \text{ psi}$$

(*c*) Now at each point of interest, consider a stress element that includes the bending stress and the transverse shear stress. The maximum shear stress for each stress element can be determined by Mohr's circle, or analytically by Eq. (3–14) with $\sigma_y = 0$,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Thus, at each point

$$\tau_{\max,a} = \sqrt{0 + (828)^2} = 828 \text{ psi}$$

$$\tau_{\max,b} = \sqrt{\left(\frac{-545}{2}\right)^2 + (715)^2} = 765 \text{ psi}$$

$$\tau_{\max,c} = \sqrt{\left(\frac{-545}{2}\right)^2 + (52.2)^2} = 277 \text{ psi}$$

$$\tau_{\max,d} = \sqrt{\left(\frac{-659}{2}\right)^2 + 0} = 330 \text{ psi}$$

Answer Interestingly, the critical location is at point *a* where the maximum shear stress is the largest, even though the bending stress is zero. The next critical location is at point *b* in the web, where the thin web thickness dramatically increases the transverse shear stress compared to points *c* or *d*. These results are counterintuitive, since both points *a* and *b* turn out to be more critical than point *d*, even though the bending stress is maximum at point *d*. The thin web and wide flange increase the impact of the transverse shear stress. If the beam length to height ratio were increased, the critical point would move from point *a* to point *b*, since the transverse shear stress at point *a* would remain constant, but the bending stress at point *b* would increase. The designer should be particularly alert to the possibility of the critical stress element not being on the outer surface with cross sections that get wider farther from the neutral axis, particularly in cases with thin web sections and wide flanges. For rectangular and circular cross sections, however, the maximum bending stresses at the outer surfaces will dominate, as was shown in Fig. 3–19.

3-12 Torsion

Any moment vector that is collinear with an axis of a mechanical element is called a *torque vector*, because the moment causes the element to be twisted about that axis. A bar subjected to such a moment is also said to be in *torsion*.

As shown in Fig. 3-21, the torque T applied to a bar can be designated by drawing arrows on the surface of the bar to indicate direction or by drawing torque-vector arrows along the axes of twist of the bar. Torque vectors are the hollow arrows shown on the x axis in Fig. 3-21. Note that they conform to the right-hand rule for vectors.

The *angle of twist*, in radians, for a solid round bar is

$$\theta = \frac{Tl}{GJ} \quad (3-35)$$

where T = torque

l = length

G = modulus of rigidity

J = polar second moment of area

Shear stresses develop throughout the cross section. For a round bar in torsion, these stresses are proportional to the radius ρ and are given by

$$\tau = \frac{T\rho}{J} \quad (3-36)$$

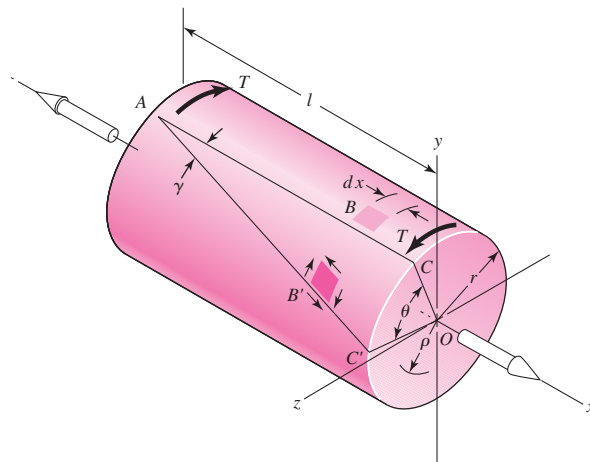
Designating r as the radius to the outer surface, we have

$$\tau_{\max} = \frac{Tr}{J} \quad (3-37)$$

The assumptions used in the analysis are:

- The bar is acted upon by a pure torque, and the sections under consideration are remote from the point of application of the load and from a change in diameter.
- The material obeys Hooke's law.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.

Figure 3-21



The last assumption depends upon the axisymmetry of the member, so it does not hold true for noncircular cross sections. Consequently, Eqs. (3–35) through (3–37) apply *only* to circular sections. For a solid round section,

$$J = \frac{\pi d^4}{32} \quad (3-38)$$

where d is the diameter of the bar. For a hollow round section,

$$J = \frac{\pi}{32} (d_o^4 - d_i^4) \quad (3-39)$$

where the subscripts o and i refer to the outside and inside diameters, respectively.

There are some applications in machinery for noncircular cross section members and shafts where a regular polygonal cross section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided. The development of equations for stress and deflection for torsional loading of noncircular cross sections can be obtained from the mathematical theory of elasticity. In general, the shear stress does not vary linearly with the distance from the axis, and depends on the specific cross section. In fact, for a rectangular section bar the shear stress is zero at the corners where the distance from the axis is the largest. The maximum shearing stress in a rectangular $b \times c$ section bar occurs in the middle of the *longest* side b and is of the magnitude

$$\tau_{\max} = \frac{T}{\alpha b c^2} \doteq \frac{T}{b c^2} \left(3 + \frac{1.8}{b/c} \right) \quad (3-40)$$

where b is the width (longer side) and c is the thickness (shorter side). They can *not* be interchanged. The parameter α is a factor that is a function of the ratio b/c as shown in the following table.⁵ The angle of twist is given by

$$\theta = \frac{Tl}{\beta b c^3 G} \quad (3-41)$$

where β is a function of b/c , as shown in the table.

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Equation (3–40) is also approximately valid for equal-sided angles; these can be considered as two rectangles, each of which is capable of carrying half the torque.⁶

It is often necessary to obtain the torque T from a consideration of the power and speed of a rotating shaft. For convenience when U. S. Customary units are used, three forms of this relation are

$$H = \frac{FV}{33\,000} = \frac{2\pi Tn}{33\,000(12)} = \frac{Tn}{63\,025} \quad (3-42)$$

⁵S. Timoshenko, *Strength of Materials*, Part I, 3rd ed., D. Van Nostrand Company, New York, 1955, p. 290.

⁶For other sections see W. C. Young and R. G. Budynas, *Roark's Formulas for Stress and Strain*, 7th ed., McGraw-Hill, New York, 2002.

where H = power, hp
 T = torque, lbf · in
 n = shaft speed, rev/min
 F = force, lbf
 V = velocity, ft/min

When SI units are used, the equation is

$$H = T\omega \quad (3-43)$$

where H = power, W
 T = torque, N · m
 ω = angular velocity, rad/s

The torque T corresponding to the power in watts is given approximately by

$$T = 9.55 \frac{H}{n} \quad (3-44)$$

where n is in revolutions per minute.

EXAMPLE 3-8

Figure 3-22 shows a crank loaded by a force $F = 300$ lbf that causes twisting and bending of a $\frac{3}{4}$ -in.-diameter shaft fixed to a support at the origin of the reference system. In actuality, the support may be an inertia that we wish to rotate, but for the purposes of a stress analysis we can consider this a statics problem.

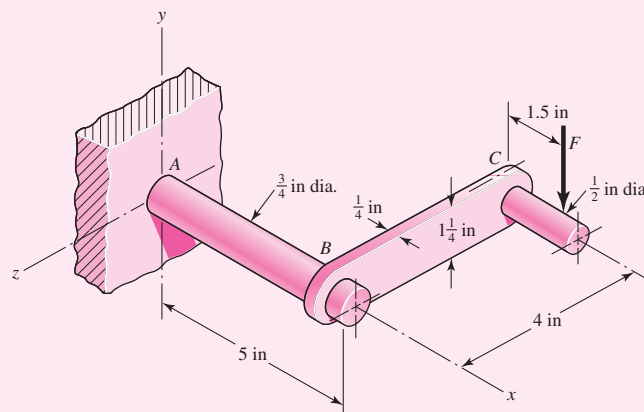
(a) Draw separate free-body diagrams of the shaft AB and the arm BC , and compute the values of all forces, moments, and torques that act. Label the directions of the coordinate axes on these diagrams.

(b) Compute the maxima of the torsional stress and the bending stress in the arm BC and indicate where these act.

(c) Locate a stress element on the top surface of the shaft at A , and calculate all the stress components that act upon this element.

(d) Determine the maximum normal and shear stresses at A .

| Figure 3-22



Solution (a) The two free-body diagrams are shown in Fig. 3–23. The results are

At end *C* of arm *BC*: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_C = -450\mathbf{k}$ lbf · in

At end *B* of arm *BC*: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_1 = 1200\mathbf{i}$ lbf · in, $\mathbf{T}_1 = 450\mathbf{k}$ lbf · in

At end *B* of shaft *AB*: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_2 = -1200\mathbf{i}$ lbf · in, $\mathbf{M}_2 = -450\mathbf{k}$ lbf · in

At end *A* of shaft *AB*: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_A = 1950\mathbf{k}$ lbf · in, $\mathbf{T}_A = 1200\mathbf{i}$ lbf · in

(b) For arm *BC*, the bending moment will reach a maximum near the shaft at *B*. If we assume this is 1200 lbf · in, then the bending stress for a rectangular section will be

Answer
$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(1200)}{0.25(1.25)^2} = 18\,400 \text{ psi} = 18.4 \text{ kpsi}$$

Of course, this is not exactly correct, because at *B* the moment is actually being transferred into the shaft, probably through a weldment.

For the torsional stress, use Eq. (3–43). Thus

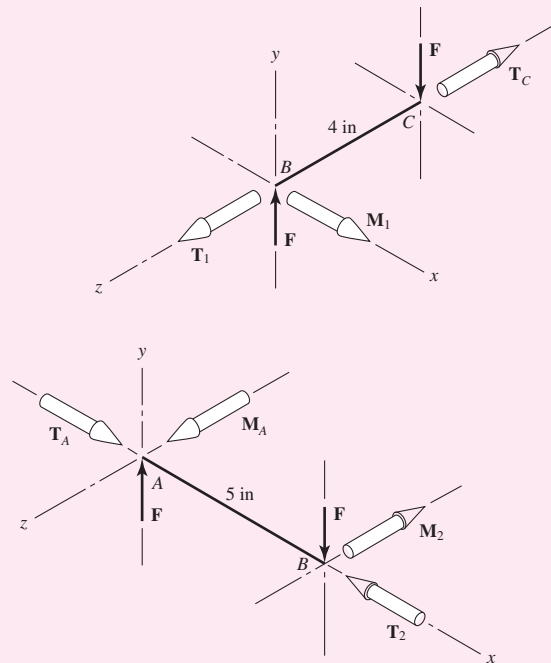
Answer
$$\tau_{\max} = \frac{T}{bc^2} \left(3 + \frac{1.8}{b/c} \right) = \frac{450}{1.25(0.25^2)} \left(3 + \frac{1.8}{1.25/0.25} \right) = 19\,400 \text{ psi} = 19.4 \text{ kpsi}$$

This stress occurs at the middle of the $1\frac{1}{4}$ -in side.

(c) For a stress element at *A*, the bending stress is tensile and is

Answer
$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(1950)}{\pi(0.75)^3} = 47\,100 \text{ psi} = 47.1 \text{ kpsi}$$

| Figure 3–23



The torsional stress is

Answer
$$\tau_{xz} = \frac{-T}{J/c} = \frac{-16T}{\pi d^3} = \frac{-16(1200)}{\pi(0.75)^3} = -14\,500 \text{ psi} = -14.5 \text{ kpsi}$$

where the reader should verify that the negative sign accounts for the direction of τ_{xz} .

(d) Point A is in a state of plane stress where the stresses are in the xz plane. Thus the principal stresses are given by Eq. (3-13) with subscripts corresponding to the x, z axes.

Answer The maximum normal stress is then given by

$$\begin{aligned}\sigma_1 &= \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} \\ &= \frac{47.1 + 0}{2} + \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 51.2 \text{ kpsi}\end{aligned}$$

Answer The maximum shear stress at A occurs on surfaces different than the surfaces containing the principal stresses or the surfaces containing the bending and torsional shear stresses. The maximum shear stress is given by Eq. (3-14), again with modified subscripts, and is given by

$$\tau_1 = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 27.7 \text{ kpsi}$$

EXAMPLE 3-9

The 1.5-in-diameter solid steel shaft shown in Fig. 3-24a is simply supported at the ends. Two pulleys are keyed to the shaft where pulley B is of diameter 4.0 in and pulley C is of diameter 8.0 in. Considering bending and torsional stresses only, determine the locations and magnitudes of the greatest tensile, compressive, and shear stresses in the shaft.

Solution

Figure 3-24b shows the net forces, reactions, and torsional moments on the shaft. Although this is a three-dimensional problem and vectors might seem appropriate, we will look at the components of the moment vector by performing a two-plane analysis. Figure 3-24c shows the loading in the xy plane, as viewed down the z axis, where bending moments are actually vectors in the z direction. Thus we label the moment diagram as M_z versus x . For the xz plane, we look down the y axis, and the moment diagram is M_y versus x as shown in Fig. 3-24d.

The net moment on a section is the vector sum of the components. That is,

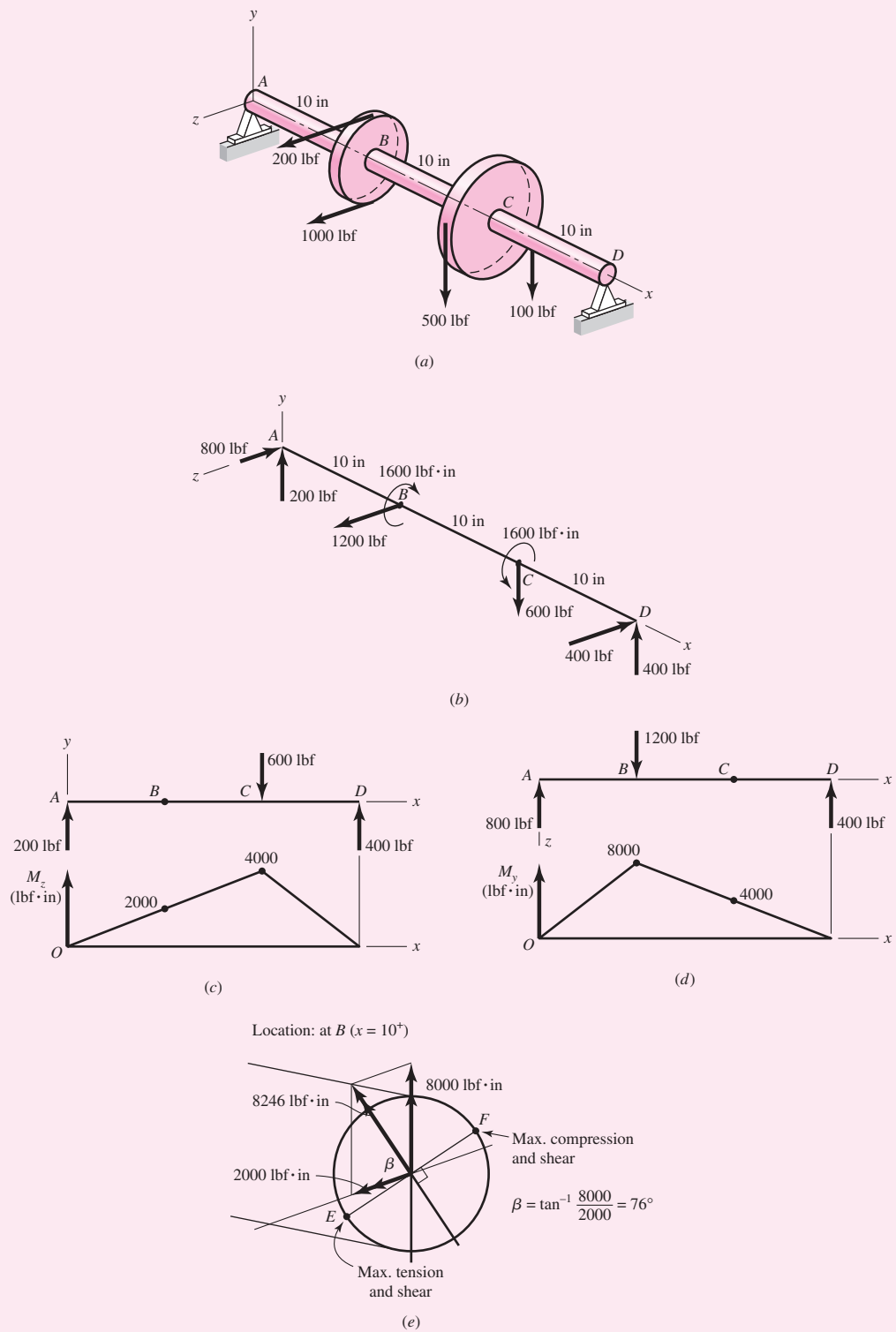
$$M = \sqrt{M_y^2 + M_z^2} \quad (1)$$

At point B,

$$M_B = \sqrt{2000^2 + 8000^2} = 8246 \text{ lbf} \cdot \text{in}$$

At point C,

$$M_C = \sqrt{4000^2 + 4000^2} = 5657 \text{ lbf} \cdot \text{in}$$



| Figure 3-24

Thus the maximum bending moment is 8246 lbf · in and the maximum bending stress at pulley *B* is

$$\sigma = \frac{M d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(8246)}{\pi(1.5^3)} = 24\,890 \text{ psi} = 24.89 \text{ kpsi}$$

The maximum torsional shear stress occurs between *B* and *C* and is

$$\tau = \frac{T d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(1600)}{\pi(1.5^3)} = 2414 \text{ psi} = 2.414 \text{ kpsi}$$

The maximum bending and torsional shear stresses occur just to the right of pulley *B* at points *E* and *F* as shown in Fig. 3–24*e*. At point *E*, the maximum tensile stress will be σ_1 given by

Answer
$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{24.89}{2} + \sqrt{\left(\frac{24.89}{2}\right)^2 + 2.414^2} = 25.12 \text{ kpsi}$$

At point *F*, the maximum compressive stress will be σ_2 given by

Answer
$$\sigma_2 = \frac{-\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{-24.89}{2} - \sqrt{\left(\frac{-24.89}{2}\right)^2 + 2.414^2} = -25.12 \text{ kpsi}$$

The extreme shear stress also occurs at *E* and *F* and is

Answer
$$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm 24.89}{2}\right)^2 + 2.414^2} = 12.68 \text{ kpsi}$$

Closed Thin-Walled Tubes ($t \ll r$)⁷

In closed thin-walled tubes, it can be shown that the product of shear stress times thickness of the wall τt is constant, meaning that the shear stress τ is inversely proportional to the wall thickness t . The total torque T on a tube such as depicted in Fig. 3–25 is given by

$$T = \int \tau t r ds = (\tau t) \int r ds = \tau t (2A_m) = 2A_m t \tau$$

where A_m is the area enclosed by the section median line. Solving for τ gives

$$\tau = \frac{T}{2A_m t} \quad (3-45)$$

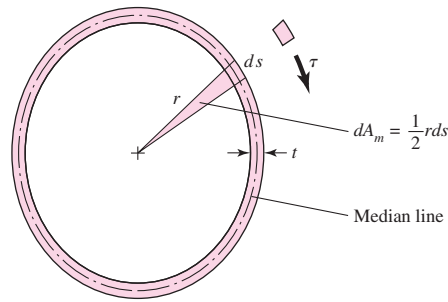
For constant wall thickness t , the angular twist (radians) per unit of length of the tube θ_1 is given by

$$\theta_1 = \frac{TL_m}{4GA_m^2 t} \quad (3-46)$$

⁷See Sec. 3–13, F. P. Beer, E. R. Johnston, and J. T. De Wolf, *Mechanics of Materials*, 5th ed., McGraw-Hill, New York, 2009.

Figure 3-25

The depicted cross section is elliptical, but the section need not be symmetrical nor of constant thickness.



where L_m is the length of the section median line. These equations presume the buckling of the tube is prevented by ribs, stiffeners, bulkheads, and so on, and that the stresses are below the proportional limit.

EXAMPLE 3-10

A welded steel tube is 40 in long, has a $\frac{1}{8}$ -in wall thickness, and a 2.5-in by 3.6-in rectangular cross section as shown in Fig. 3-26. Assume an allowable shear stress of 11 500 psi and a shear modulus of $11.5(10^6)$ psi.

- Estimate the allowable torque T .
- Estimate the angle of twist due to the torque.

Solution

- Within the section median line, the area enclosed is

$$A_m = (2.5 - 0.125)(3.6 - 0.125) = 8.253 \text{ in}^2$$

and the length of the median perimeter is

$$L_m = 2[(2.5 - 0.125) + (3.6 - 0.125)] = 11.70 \text{ in}$$

Answer

From Eq. (3-45) the torque T is

$$T = 2A_mt\tau = 2(8.253)(0.125)(11\,500) = 23\,730 \text{ lbf} \cdot \text{in}$$

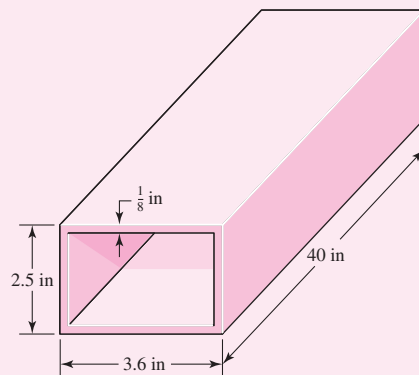
Answer

- The angle of twist θ from Eq. (3-46) is

$$\theta = \theta_1 l = \frac{TL_m}{4GA_m^2 t} l = \frac{23\,730(11.70)}{4(11.5 \times 10^6)(8.253^2)(0.125)} (40) = 0.0284 \text{ rad} = 1.62^\circ$$

Figure 3-26

A rectangular steel tube produced by welding.



EXAMPLE 3-11

Compare the shear stress on a circular cylindrical tube with an outside diameter of 1 in and an inside diameter of 0.9 in, predicted by Eq. (3-37), to that estimated by Eq. (3-45).

Solution

From Eq. (3-37),

$$\tau_{\max} = \frac{Tr}{J} = \frac{Tr}{(\pi/32)(d_o^4 - d_i^4)} = \frac{T(0.5)}{(\pi/32)(1^4 - 0.9^4)} = 14.809T$$

From Eq. (3-45),

$$\tau = \frac{T}{2A_mt} = \frac{T}{2(\pi 0.95^2/4)0.05} = 14.108T$$

Taking Eq. (3-37) as correct, the error in the thin-wall estimate is -4.7 percent.

Open Thin-Walled Sections

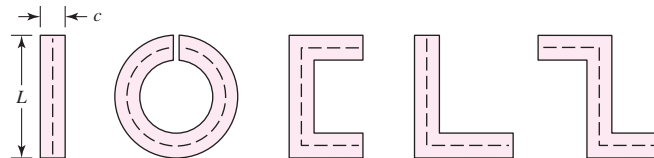
When the median wall line is not closed, the section is said to be an *open section*. Figure 3-27 presents some examples. Open sections in torsion, where the wall is thin, have relations derived from the membrane analogy theory⁸ resulting in:

$$\tau = G\theta_1 c = \frac{3T}{Lc^2} \quad (3-47)$$

where τ is the shear stress, G is the shear modulus, θ_1 is the angle of twist per unit length, T is torque, and L is the length of the median line. The wall thickness is designated c (rather than t) to remind you that you are in open sections. By studying the table that follows Eq. (3-41) you will discover that membrane theory presumes $b/c \rightarrow \infty$. Note that open thin-walled sections in torsion should be avoided in design. As indicated in Eq. (3-47), the shear stress and the angle of twist are inversely proportional to c^2 and c^3 , respectively. Thus, for small wall thickness, stress and twist can become quite large. For example, consider the thin round tube with a slit in Fig. 3-27. For a ratio of wall thickness of outside diameter of $c/d_o = 0.1$, the open section has greater magnitudes of stress and angle of twist by factors of 12.3 and 61.5, respectively, compared to a closed section of the same dimensions.

Figure 3-27

Some open thin-wall sections.



⁸See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970, Sec. 109.

EXAMPLE 3-12

A 12-in-long strip of steel is $\frac{1}{8}$ in thick and 1 in wide, as shown in Fig. 3-28. If the allowable shear stress is 11 500 psi and the shear modulus is $11.5(10^6)$ psi, find the torque corresponding to the allowable shear stress and the angle of twist, in degrees, (a) using Eq. (3-47) and (b) using Eqs. (3-40) and (3-41).

Solution

(a) The length of the median line is 1 in. From Eq. (3-47),

$$T = \frac{Lc^2\tau}{3} = \frac{(1)(1/8)^2 11\,500}{3} = 59.90 \text{ lbf} \cdot \text{in}$$

$$\theta = \theta_1 l = \frac{\tau l}{Gc} = \frac{11\,500(12)}{11.5(10^6)(1/8)} = 0.0960 \text{ rad} = 5.5^\circ$$

A torsional spring rate k_t can be expressed as T/θ :

$$k_t = 59.90/0.0960 = 624 \text{ lbf} \cdot \text{in/rad}$$

(b) From Eq. (3-40),

$$T = \frac{\tau_{\max} bc^2}{3 + 1.8/(b/c)} = \frac{11\,500(1)(0.125)^2}{3 + 1.8/(1/0.125)} = 55.72 \text{ lbf} \cdot \text{in}$$

From Eq. (3-41), with $b/c = 1/0.125 = 8$,

$$\theta = \frac{Tl}{\beta bc^3 G} = \frac{55.72(12)}{0.307(1)(0.125)^3 (11.5)10^6} = 0.0970 \text{ rad} = 5.6^\circ$$

$$k_t = 55.72/0.0970 = 574 \text{ lbf} \cdot \text{in/rad}$$

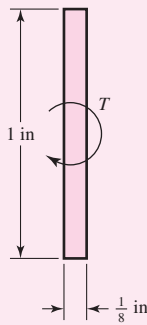


Figure 3-28

The cross-section of a thin strip of steel subjected to a torsional moment T .

The cross section is not thin, where b should be greater than c by at least a factor of 10. In estimating the torque, Eq. (3-47) provides a value of 7.5 percent higher than Eq. (3-40), and is 8.5 percent higher than when the table on page 102 is used.

3-13 Stress Concentration

In the development of the basic stress equations for tension, compression, bending, and torsion, it was assumed that no geometric irregularities occurred in the member under consideration. But it is quite difficult to design a machine without permitting some changes in the cross sections of the members. Rotating shafts must have shoulders designed on them so that the bearings can be properly seated and so that they will take thrust loads; and the shafts must have key slots machined into them for securing pulleys and gears. A bolt has a head on one end and screw threads on the other end, both of which account for abrupt changes in the cross section. Other parts require holes, oil grooves, and notches of various kinds. Any discontinuity in a machine part alters the stress distribution in the neighborhood of the discontinuity so that the elementary stress equations no longer describe the state of stress in the part at these locations. Such discontinuities are called *stress raisers*, and the regions in which they occur are called areas of *stress concentration*. Stress concentrations can also arise from some irregularity not inherent in the member, such as tool marks, holes, notches, grooves, or threads.

A *theoretical*, or *geometric*, *stress-concentration factor* K_t or K_{ts} is used to relate the actual maximum stress at the discontinuity to the *nominal stress*. The factors are defined by the equations

$$K_t = \frac{\sigma_{\max}}{\sigma_0} \quad K_{ts} = \frac{\tau_{\max}}{\tau_0} \quad (3-48)$$

where K_t is used for normal stresses and K_{ts} for shear stresses. The nominal stress σ_0 or τ_0 is the stress calculated by using the elementary stress equations and the net area, or net cross section. Sometimes the gross cross section is used instead, and so it is always wise to double check the source of K_t or K_{ts} before calculating the maximum stress.

The stress-concentration factor depends for its value only on the *geometry* of the part. That is, the particular material used has no effect on the value of K_t . This is why it is called a *theoretical* stress-concentration factor.

The analysis of geometric shapes to determine stress-concentration factors is a difficult problem, and not many solutions can be found. Most stress-concentration factors are found by using experimental techniques.⁹ Though the finite-element method has been used, the fact that the elements are indeed finite prevents finding the true maximum stress. Experimental approaches generally used include photoelasticity, grid methods, brittle-coating methods, and electrical strain-gauge methods. Of course, the grid and strain-gauge methods both suffer from the same drawback as the finite-element method.

Stress-concentration factors for a variety of geometries may be found in Tables A-15 and A-16.

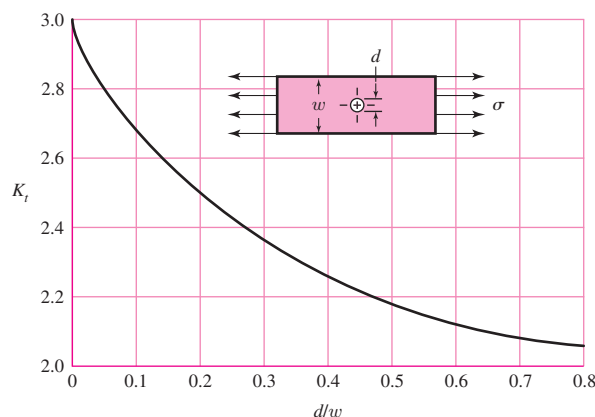
An example is shown in Fig. 3-29, that of a thin plate loaded in tension where the plate contains a centrally located hole.

In *static loading*, stress-concentration factors are applied as follows. In *ductile materials* ($\epsilon_f \geq 0.05$), the stress-concentration factor *is not* usually applied to predict the critical stress, because plastic strain in the region of the stress is localized and has a strengthening effect. In *brittle materials* ($\epsilon_f < 0.05$), the geometric stress-concentration factor K_t is applied to the nominal stress before comparing it with strength. Gray cast iron has so many inherent stress raisers that the stress raisers introduced by the designer have only a modest (but additive) effect.

Figure 3-29

Thin plate in tension or simple compression with a transverse central hole. The net tensile force is $F = \sigma wt$, where t is the thickness of the plate. The nominal stress is given by

$$\sigma_0 = \frac{F}{(w-d)t} = \frac{w}{(w-d)}\sigma$$



⁹The best source book is W. D. Pilkey and D. F. Pilkey, *Peterson's Stress Concentration Factors*, 3rd ed., John Wiley & Sons, New York, 2008.

Consider a part made of a ductile material and loaded by a gradually applied static load such that the stress in an area of a stress concentration goes beyond the yield strength. The yielding will be restricted to a very small region, and the permanent deformation as well as the residual stresses after the load is released will be insignificant and normally can be tolerated. If yielding does occur, the stress distribution changes and tends toward a more uniform distribution. In the region where yielding occurs, there is little danger of fracture of a ductile material, but if the possibility of a brittle fracture exists, the stress concentration must be taken seriously. Brittle fracture is not just limited to brittle materials. Materials often thought of as being ductile can fail in a brittle manner under certain conditions, e.g., any single application or combination of cyclic loading, rapid application of static loads, loading at low temperatures, and parts containing defects in their material structures (see Sec. 5–12). The effects on a ductile material of processing, such as hardening, hydrogen embrittlement, and welding, may also accelerate failure. Thus, care should always be exercised when dealing with stress concentrations.

For *dynamic loading*, the stress concentration effect is significant for *both* ductile and brittle materials and must always be taken into account (see Sec. 6–10).

EXAMPLE 3-13

The 2-mm-thick bar shown in Fig. 3–30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?

Solution

Since the material is brittle, the effect of stress concentrations near the discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, the nominal stress is

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w - d)t} = \frac{10\,000}{(40 - 4)2} = 139 \text{ MPa}$$

The theoretical stress concentration factor, from Fig. A–15–1, with $d/w = 4/40 = 0.1$, is $K_t = 2.7$. The maximum stress is

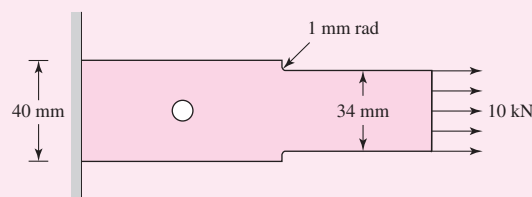
$$\sigma_{\max} = K_t \sigma_0 = 2.7(139) = 380 \text{ MPa}$$

Similarly, for an 8-mm hole,

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w - d)t} = \frac{10\,000}{(40 - 8)2} = 156 \text{ MPa}$$

With $d/w = 8/40 = 0.2$, then $K_t = 2.5$, and the maximum stress is

| Figure 3–30



Answer $\sigma_{\max} = K_t \sigma_0 = 2.5(156) = 390 \text{ MPa}$

Though the stress concentration is higher with the 4-mm hole, in this case the increased nominal stress with the 8-mm hole has more effect on the maximum stress.

For the fillet,

$$\sigma_0 = \frac{F}{A} = \frac{10\,000}{(34)2} = 147 \text{ MPa}$$

From Table A-15-5, $D/d = 40/34 = 1.18$, and $r/d = 1/34 = 0.026$. Then $K_t = 2.5$.

Answer $\sigma_{\max} = K_t \sigma_0 = 2.5(147) = 368 \text{ MPa}$

Answer The crack will most likely occur with the 8-mm hole, next likely would be the 4-mm hole, and least likely at the fillet.

3-14 Stresses in Pressurized Cylinders

Cylindrical pressure vessels, hydraulic cylinders, gun barrels, and pipes carrying fluids at high pressures develop both radial and tangential stresses with values that depend upon the radius of the element under consideration. In determining the radial stress σ_r and the tangential stress σ_t , we make use of the assumption that the longitudinal elongation is constant around the circumference of the cylinder. In other words, a right section of the cylinder remains plane after stressing.

Referring to Fig. 3-31, we designate the inside radius of the cylinder by r_i , the outside radius by r_o , the internal pressure by p_i , and the external pressure by p_o . Then it can be shown that tangential and radial stresses exist whose magnitudes are¹⁰

$$\begin{aligned}\sigma_t &= \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2} \\ \sigma_r &= \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2}\end{aligned}\quad (3-49)$$

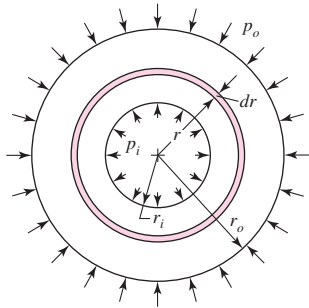


Figure 3-31

A cylinder subjected to both internal and external pressure.

As usual, positive values indicate tension and negative values, compression.

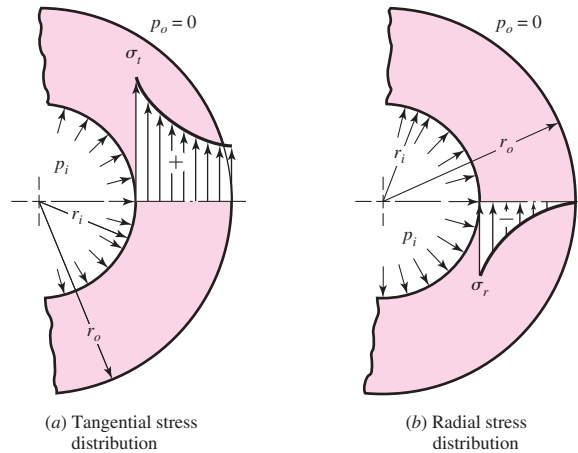
For the special case of $p_o = 0$, Eq. (3-49) gives

$$\begin{aligned}\sigma_t &= \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r^2} \right) \\ \sigma_r &= \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 - \frac{r_o^2}{r^2} \right)\end{aligned}\quad (3-50)$$

¹⁰See Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 348–352.

Figure 3-32

Distribution of stresses in a thick-walled cylinder subjected to internal pressure.



The equations of set (3-50) are plotted in Fig. 3-32 to show the distribution of stresses over the wall thickness. It should be realized that longitudinal stresses exist when the end reactions to the internal pressure are taken by the pressure vessel itself. This stress is found to be

$$\sigma_l = \frac{p_i r_i^2}{r_o^2 - r_i^2} \quad (3-51)$$

We further note that Eqs. (3-49), (3-50), and (3-51) apply only to sections taken a significant distance from the ends and away from any areas of stress concentration.

Thin-Walled Vessels

When the wall thickness of a cylindrical pressure vessel is about one-tenth, or less, of its radius, the radial stress that results from pressurizing the vessel is quite small compared with the tangential stress. Under these conditions the tangential stress can be obtained as follows: Let an internal pressure p be exerted on the wall of a cylinder of thickness t and inside diameter d_i . The force tending to separate two halves of a unit length of the cylinder is $p d_i$. This force is resisted by the tangential stress, also called the *hoop stress*, acting uniformly over the stressed area. We then have $p d_i = 2t \sigma_t$, or

$$(\sigma_t)_{av} = \frac{p d_i}{2t} \quad (3-52)$$

This equation gives the *average* tangential stress and is valid regardless of the wall thickness. For a thin-walled vessel an approximation to the maximum tangential stress is

$$(\sigma_t)_{max} = \frac{p(d_i + t)}{2t} \quad (3-53)$$

where $d_i + t$ is the average diameter.

In a closed cylinder, the longitudinal stress σ_l exists because of the pressure upon the ends of the vessel. If we assume this stress is also distributed uniformly over the wall thickness, we can easily find it to be

$$\sigma_l = \frac{p d_i}{4t} \quad (3-54)$$

EXAMPLE 3-14

An aluminum-alloy pressure vessel is made of tubing having an outside diameter of 8 in and a wall thickness of $\frac{1}{4}$ in.

(a) What pressure can the cylinder carry if the permissible tangential stress is 12 kpsi and the theory for thin-walled vessels is assumed to apply?

(b) On the basis of the pressure found in part (a), compute the stress components using the theory for thick-walled cylinders.

Solution

(a) Here $d_i = 8 - 2(0.25) = 7.5$ in, $r_i = 7.5/2 = 3.75$ in, and $r_o = 8/2 = 4$ in. Then $t/r_i = 0.25/3.75 = 0.067$. Since this ratio is less than 0.1, the theory for thin-walled vessels should yield safe results.

We first solve Eq. (3-53) to obtain the allowable pressure. This gives

Answer

$$p = \frac{2t(\sigma_t)_{\max}}{d_i + t} = \frac{2(0.25)(12)(10)^3}{7.5 + 0.25} = 774 \text{ psi}$$

(b) The maximum tangential stress will occur at the inside radius, and so we use $r = r_i$ in the first equation of Eq. (3-50). This gives

Answer

$$(\sigma_t)_{\max} = \frac{r_i^2 p_i}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r_i^2} \right) = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 774 \frac{4^2 + 3.75^2}{4^2 - 3.75^2} = 12\,000 \text{ psi}$$

Similarly, the maximum radial stress is found, from the second equation of Eq. (3-50) to be

Answer

$$\sigma_r = -p_i = -774 \text{ psi}$$

The stresses σ_t and σ_r are principal stresses, since there is no shear on these surfaces. Note that there is no significant difference in the stresses in parts (a) and (b), and so the thin-wall theory can be considered satisfactory for this problem.

3-15 Stresses in Rotating Rings

Many rotating elements, such as flywheels and blowers, can be simplified to a rotating ring to determine the stresses. When this is done it is found that the same tangential and radial stresses exist as in the theory for thick-walled cylinders except that they are caused by inertial forces acting on all the particles of the ring. The tangential and radial stresses so found are subject to the following restrictions:

- The outside radius of the ring, or disk, is large compared with the thickness $r_o \geq 10t$.
- The thickness of the ring or disk is constant.
- The stresses are constant over the thickness.

The stresses are¹¹

$$\begin{aligned} \sigma_t &= \rho\omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right) \\ \sigma_r &= \rho\omega^2 \left(\frac{3+\nu}{8} \right) \left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2 \right) \end{aligned} \quad (3-55)$$

¹¹Ibid, pp. 348–357.

where r is the radius to the stress element under consideration, ρ is the mass density, and ω is the angular velocity of the ring in radians per second. For a rotating disk, use $r_i = 0$ in these equations.

3-16 Press and Shrink Fits

When two cylindrical parts are assembled by shrinking or press fitting one part upon another, a contact pressure is created between the two parts. The stresses resulting from this pressure may easily be determined with the equations of the preceding sections.

Figure 3-33 shows two cylindrical members that have been assembled with a shrink fit. Prior to assembly, the outer radius of the inner member was larger than the inner radius of the outer member by the *radial interference* δ . After assembly, an interference contact pressure p develops between the members at the nominal radius R , causing radial stresses $\sigma_r = -p$ in each member at the contacting surfaces. This pressure is given by¹²

$$p = \frac{\delta}{R \left[\frac{1}{E_o} \left(\frac{r_o^2 + R^2}{r_o^2 - R^2} + \nu_o \right) + \frac{1}{E_i} \left(\frac{R^2 + r_i^2}{R^2 - r_i^2} - \nu_i \right) \right]} \quad (3-56)$$

where the subscripts o and i on the material properties correspond to the outer and inner members, respectively. If the two members are of the same material with $E_o = E_i = E$, $\nu_o = \nu_i$, the relation simplifies to

$$p = \frac{E\delta}{2R^3} \left[\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{r_o^2 - r_i^2} \right] \quad (3-57)$$

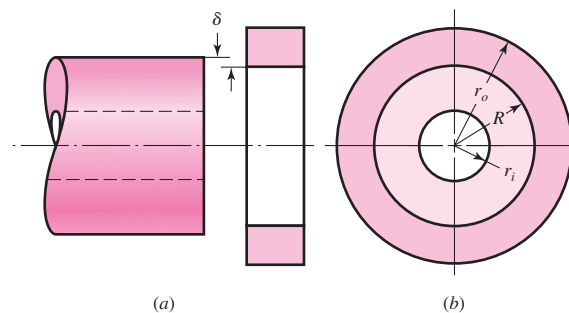
For Eqs. (3-56) or (3-57), diameters can be used in place of R , r_i , and r_o , provided δ is the diametral interference (twice the radial interference).

With p , Eq. (3-49) can be used to determine the radial and tangential stresses in each member. For the inner member, $p_o = p$ and $p_i = 0$. For the outer member, $p_o = 0$ and $p_i = p$. For example, the magnitudes of the tangential stresses at the transition radius R are maximum for both members. For the inner member

$$(\sigma_t)_i \Big|_{r=R} = -p \frac{R^2 + r_i^2}{R^2 - r_i^2} \quad (3-58)$$

Figure 3-33

Notation for press and shrink fits. (a) Unassembled parts; (b) after assembly.



¹²Ibid, pp. 348–354.

and, for the outer member

$$(\sigma_r)_o \Big|_{r=R} = p \frac{r_o^2 + R^2}{r_o^2 - R^2} \quad (3-59)$$

Assumptions

It is assumed that both members have the same length. In the case of a hub that has been press-fitted onto a shaft, this assumption would not be true, and there would be an increased pressure at each end of the hub. It is customary to allow for this condition by employing a stress-concentration factor. The value of this factor depends upon the contact pressure and the design of the female member, but its theoretical value is seldom greater than 2.

3-17 Temperature Effects

When the temperature of an unrestrained body is uniformly increased, the body expands, and the normal strain is

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(\Delta T) \quad (3-60)$$

where α is the coefficient of thermal expansion and ΔT is the temperature change, in degrees. In this action the body experiences a simple volume increase with the components of shear strain all zero.

If a straight bar is restrained at the ends so as to prevent lengthwise expansion and then is subjected to a uniform increase in temperature, a compressive stress will develop because of the axial constraint. The stress is

$$\sigma = -\epsilon E = -\alpha(\Delta T)E \quad (3-61)$$

In a similar manner, if a uniform flat plate is restrained at the edges and also subjected to a uniform temperature rise, the compressive stress developed is given by the equation

$$\sigma = -\frac{\alpha(\Delta T)E}{1-\nu} \quad (3-62)$$

The stresses expressed by Eqs. (3-61) and (3-62) are called *thermal stresses*. They arise because of a temperature change in a clamped or restrained member. Such stresses, for example, occur during welding, since parts to be welded must be clamped before welding. Table 3-3 lists approximate values of the coefficients of thermal expansion.

Table 3-3

Coefficients of Thermal Expansion (Linear Mean Coefficients for the Temperature Range 0–100°C)

Material	Celsius Scale (°C ⁻¹)	Fahrenheit Scale (°F ⁻¹)
Aluminum	23.9(10) ⁻⁶	13.3(10) ⁻⁶
Brass, cast	18.7(10) ⁻⁶	10.4(10) ⁻⁶
Carbon steel	10.8(10) ⁻⁶	6.0(10) ⁻⁶
Cast iron	10.6(10) ⁻⁶	5.9(10) ⁻⁶
Magnesium	25.2(10) ⁻⁶	14.0(10) ⁻⁶
Nickel steel	13.1(10) ⁻⁶	7.3(10) ⁻⁶
Stainless steel	17.3(10) ⁻⁶	9.6(10) ⁻⁶
Tungsten	4.3(10) ⁻⁶	2.4(10) ⁻⁶

3-18 Curved Beams in Bending¹³

The distribution of stress in a curved flexural member is determined by using the following assumptions:

- The cross section has an axis of symmetry in the plane of bending.
- Plane cross sections remain plane after bending.
- The modulus of elasticity is the same in tension as in compression.

We shall find that the neutral axis and the centroidal axis of a curved beam, unlike the axes of a straight beam, are not coincident and also that the stress does not vary linearly from the neutral axis. The notation shown in Fig. 3-34 is defined as follows:

r_o = radius of outer fiber

r_i = radius of inner fiber

h = depth of section

c_o = distance from neutral axis to outer fiber

c_i = distance from neutral axis to inner fiber

r_n = radius of neutral axis

r_c = radius of centroidal axis

e = distance from centroidal axis to neutral axis

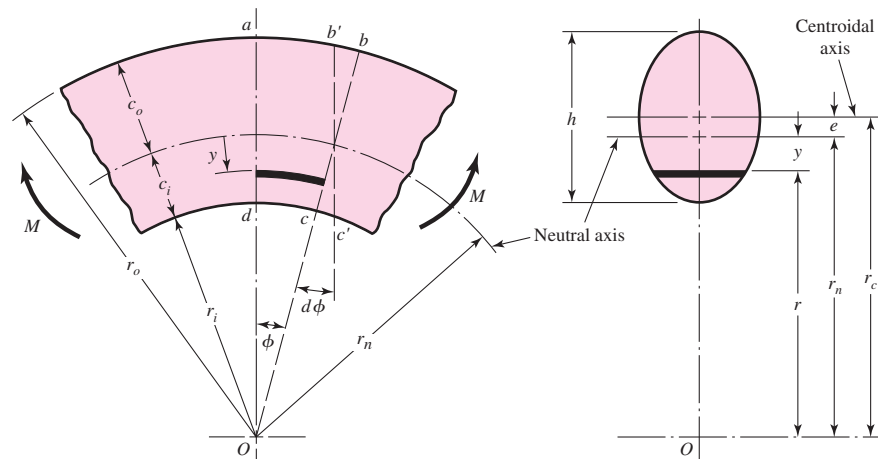
M = bending moment; positive M decreases curvature

Figure 3-34 shows that the neutral and centroidal axes are not coincident. The location of the neutral axis with respect to the center of curvature O is given by the equation

$$r_n = \frac{A}{\int \frac{dA}{r}} \quad (3-63)$$

Figure 3-34

Note that y is positive in the direction toward the center of curvature, point O .



¹³For a complete development of the relations in this section, see Richard G. Budynas, *Advanced Strength and Applied Stress Analysis*, 2nd ed., McGraw-Hill, New York, 1999, pp. 309–317.

Furthermore, it can be shown that the stress distribution is given by

$$\sigma = \frac{My}{Ae(r_n - y)} \quad (3-64)$$

where M is positive in the direction shown in Fig. 3–34. The stress distribution given by Eq. (3–64) is *hyperbolic* and not linear as is the case for straight beams. The critical stresses occur at the inner and outer surfaces where $y = c_i$ and $y = -c_o$, respectively, and are

$$\sigma_i = \frac{Mc_i}{Aer_i} \quad \sigma_o = -\frac{Mc_o}{Aer_o} \quad (3-65)$$

These equations are valid for pure bending. In the usual and more general case, such as a crane hook, the U frame of a press, or the frame of a C clamp, the bending moment is due to a force acting at a distance from the cross section under consideration. Thus, the cross section transmits a bending moment *and* an axial force. The axial force is located at the *centroidal axis* of the section and the bending moment is then computed at this location. The tensile or compressive stress due to the axial force, from Eq. (3–22), is then added to the bending stresses given by Eqs. (3–64) and (3–65) to obtain the resultant stresses acting on the section.

EXAMPLE 3-15

Plot the distribution of stresses across section A–A of the crane hook shown in Fig. 3–35a. The cross section is rectangular, with $b = 0.75$ in and $h = 4$ in, and the load is $F = 5000$ lbf.

Solution

Since $A = bh$, we have $dA = b dr$ and, from Eq. (3–63),

$$r_n = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r} dr} = \frac{h}{\ln \frac{r_o}{r_i}} \quad (1)$$

From Fig. 3–35b, we see that $r_i = 2$ in, $r_o = 6$ in, $r_c = 4$ in, and $A = 3$ in². Thus, from Eq. (1),

$$r_n = \frac{h}{\ln(r_o/r_i)} = \frac{4}{\ln \frac{6}{2}} = 3.641 \text{ in}$$

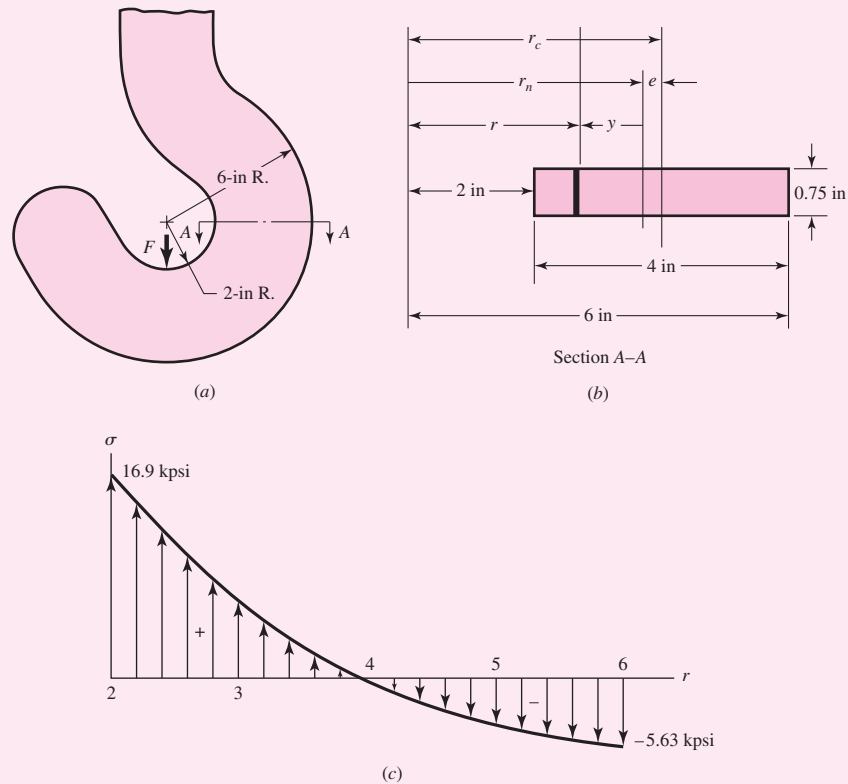
and the eccentricity is $e = r_c - r_n = 4 - 3.641 = 0.359$ in. The moment M is positive and is $M = Fr_c = 5000(4) = 20\,000$ lbf · in. Adding the axial component of stress to Eq. (3–64) gives

$$\sigma = \frac{F}{A} + \frac{My}{Ae(r_n - y)} = \frac{5000}{3} + \frac{(20\,000)(3.641 - r)}{3(0.359)r} \quad (2)$$

Substituting values of r from 2 to 6 in results in the stress distribution shown in Fig. 3–35c. The stresses at the inner and outer radii are found to be 16.9 and -5.63 kpsi, respectively, as shown.

Figure 3-35

(a) Plan view of crane hook;
 (b) cross section and notation;
 (c) resulting stress distribution.
 There is no stress concentration.



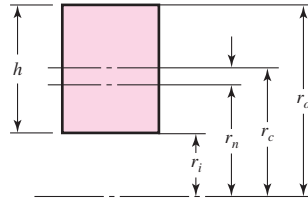
Note in the hook example, the symmetrical rectangular cross section causes the maximum tensile stress to be 3 times greater than the maximum compressive stress. If we wanted to design the hook to use material more effectively we would use more material at the inner radius and less material at the outer radius. For this reason, trapezoidal, T, or unsymmetric I, cross sections are commonly used. Sections most frequently encountered in the stress analysis of curved beams are shown in Table 3-4.

Alternative Calculations for e

Calculating r_n and r_c mathematically and subtracting the difference can lead to large errors if not done carefully, since r_n and r_c are typically large values compared to e . Since e is in the denominator of Eqs. (3-64) and (3-65), a large error in e can lead to an inaccurate stress calculation. Furthermore, if you have a complex cross section that the tables do not handle, alternative methods for determining e are needed. For a quick and simple approximation of e , it can be shown that¹⁴

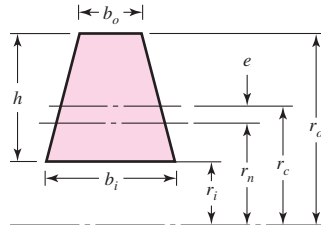
$$e \doteq \frac{I}{r_c A} \quad (3-66)$$

¹⁴Ibid., pp. 317–321. Also presents a numerical method.

Table 3-4Formulas for Sections
of Curved Beams

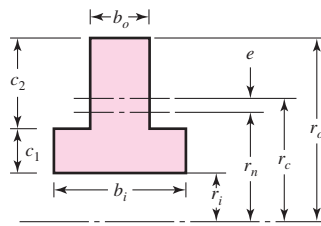
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



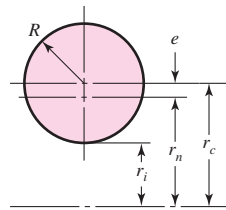
$$r_c = r_i + \frac{h}{3} \frac{b_i + 2b_o}{b_i + b_o}$$

$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h] \ln(r_o/r_i)}$$



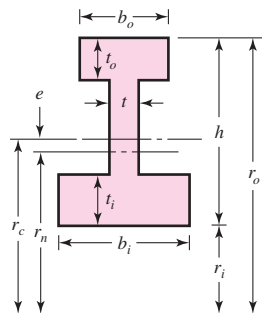
$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$



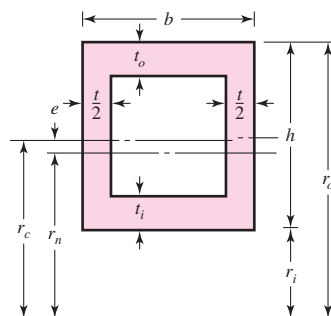
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2(r_c - \sqrt{r_c^2 - R^2})}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln \frac{r_i + t}{r_i} + t \ln \frac{r_o - t_o}{r_i + t_i} + b_o \ln \frac{r_o}{r_o - t_o}}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b \left(\ln \frac{r_i + t_i}{r_i} + \ln \frac{r_o}{r_o - t_o} \right) + t \ln \frac{r_o - t_o}{r_i + t_i}}$$

This approximation is good for a large curvature where e is small with $r_n \doteq r_c$. Substituting Eq. (3-66) into Eq. (3-64), with $r_n - y = r$, gives

$$\sigma \doteq \frac{My r_c}{I r} \quad (3-67)$$

If $r_n \doteq r_c$, which it should be to use Eq. (3-67), then it is only necessary to calculate r_c , and to measure y from this axis. Determining r_c for a complex cross section can be done easily by most CAD programs or numerically as shown in the before-mentioned reference. Observe that as the curvature increases, $r \rightarrow r_c$, and Eq. (3-67) becomes the straight-beam formulation, Eq. (3-24). Note that the negative sign is missing because y in Fig. 3-34 is vertically downward, opposite that for the straight-beam equation.

EXAMPLE 3-16

Consider the circular section in Table 3-4 with $r_c = 3$ in and $R = 1$ in. Determine e by using the formula from the table and approximately by using Eq. (3-66). Compare the results of the two solutions.

Solution Using the formula from Table 3-4 gives

$$r_n = \frac{R^2}{2(r_c - \sqrt{r_c^2 - R^2})} = \frac{1^2}{2(3 - \sqrt{3^2 - 1})} = 2.914\ 21 \text{ in}$$

This gives an eccentricity of

$$\text{Answer} \quad e = r_c - r_n = 3 - 2.914\ 21 = 0.085\ 79 \text{ in}$$

The approximate method, using Eq. (3-66), yields

$$\text{Answer} \quad e \doteq \frac{I}{r_c A} = \frac{\pi R^4/4}{r_c(\pi R^2)} = \frac{R^2}{4r_c} = \frac{1^2}{4(3)} = 0.083\ 33 \text{ in}$$

This differs from the exact solution by -2.9 percent.

3-19 Contact Stresses

When two bodies having curved surfaces are pressed together, point or line contact changes to area contact, and the stresses developed in the two bodies are three-dimensional. Contact-stress problems arise in the contact of a wheel and a rail, in automotive valve cams and tappets, in mating gear teeth, and in the action of rolling bearings. Typical failures are seen as cracks, pits, or flaking in the surface material.

The most general case of contact stress occurs when each contacting body has a double radius of curvature; that is, when the radius in the plane of rolling is different from the radius in a perpendicular plane, both planes taken through the axis of the contacting force. Here we shall consider only the two special cases of contacting spheres and contacting cylinders.¹⁵ The results presented here are due to Hertz and so are frequently known as *Hertzian stresses*.

¹⁵A more comprehensive presentation of contact stresses may be found in Arthur P. Boresi and Richard J. Schmidt, *Advanced Mechanics of Materials*, 6th ed., Wiley, New York, 2003, pp. 589–623.

Spherical Contact

When two solid spheres of diameters d_1 and d_2 are pressed together with a force F , a circular area of contact of radius a is obtained. Specifying E_1, ν_1 and E_2, ν_2 as the respective elastic constants of the two spheres, the radius a is given by the equation

$$a = \sqrt[3]{\frac{3F}{8} \frac{(1 - \nu_1^2)/E_1 + (1 - \nu_2^2)/E_2}{1/d_1 + 1/d_2}} \quad (3-68)$$

The pressure distribution within the contact area of each sphere is hemispherical, as shown in Fig. 3-36*b*. The maximum pressure occurs at the center of the contact area and is

$$p_{\max} = \frac{3F}{2\pi a^2} \quad (3-69)$$

Equations (3-68) and (3-69) are perfectly general and also apply to the contact of a sphere and a plane surface or of a sphere and an internal spherical surface. For a plane surface, use $d = \infty$. For an internal surface, the diameter is expressed as a negative quantity.

The maximum stresses occur on the z axis, and these are principal stresses. Their values are

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -p_{\max} \left[\left(1 - \left| \frac{z}{a} \right| \tan^{-1} \frac{1}{|z/a|} \right) (1 + \nu) - \frac{1}{2 \left(1 + \frac{z^2}{a^2} \right)} \right] \quad (3-70)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{1 + \frac{z^2}{a^2}} \quad (3-71)$$

Figure 3-36

(a) Two spheres held in contact by force F ; (b) contact stress has a hemispherical distribution across contact zone diameter $2a$.

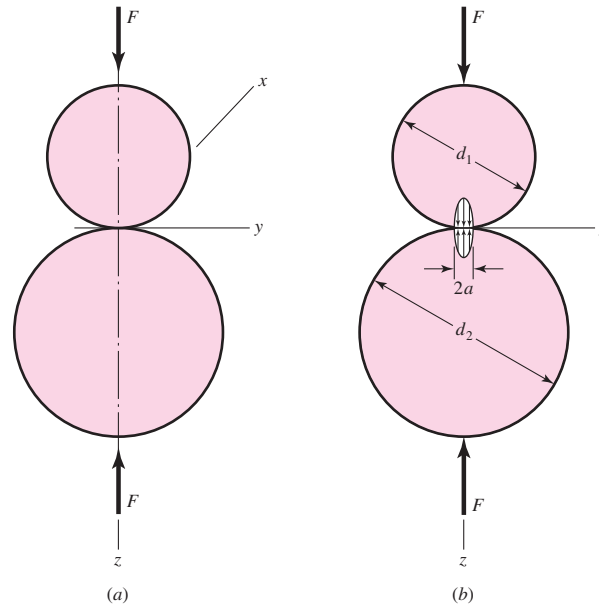
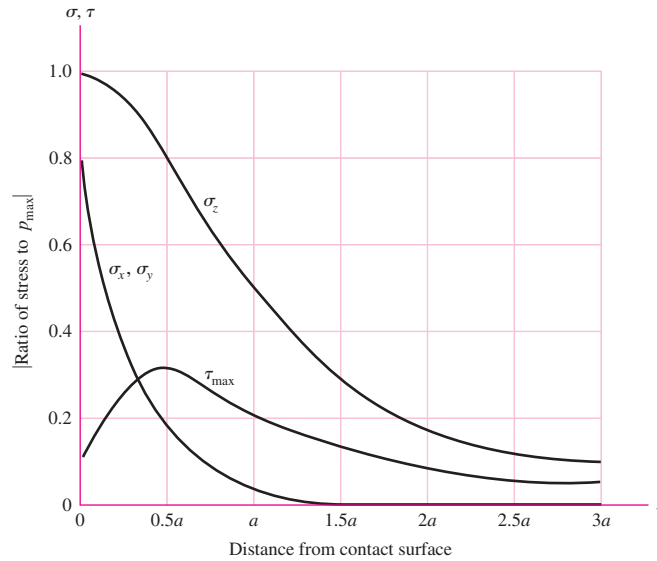


Figure 3-37

Magnitude of the stress components below the surface as a function of the maximum pressure of contacting spheres. Note that the maximum shear stress is slightly below the surface at $z = 0.48a$ and is approximately $0.3p_{\max}$. The chart is based on a Poisson ratio of 0.30. Note that the normal stresses are all compressive stresses.



These equations are valid for either sphere, but the value used for Poisson's ratio must correspond with the sphere under consideration. The equations are even more complicated when stress states off the z axis are to be determined, because here the x and y coordinates must also be included. But these are not required for design purposes, because the maxima occur on the z axis.

Mohr's circles for the stress state described by Eqs. (3-70) and (3-71) are a point and two coincident circles. Since $\sigma_1 = \sigma_2$, we have $\tau_{1/2} = 0$ and

$$\tau_{\max} = \tau_{1/3} = \tau_{2/3} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_2 - \sigma_3}{2} \quad (3-72)$$

Figure 3-37 is a plot of Eqs. (3-70), (3-71), and (3-72) for a distance to $3a$ below the surface. Note that the shear stress reaches a maximum value slightly below the surface. It is the opinion of many authorities that this maximum shear stress is responsible for the surface fatigue failure of contacting elements. The explanation is that a crack originates at the point of maximum shear stress below the surface and progresses to the surface and that the pressure of the lubricant wedges the chip loose.

Cylindrical Contact

Figure 3-38 illustrates a similar situation in which the contacting elements are two cylinders of length l and diameters d_1 and d_2 . As shown in Fig. 3-38b, the area of contact is a narrow rectangle of width $2b$ and length l , and the pressure distribution is elliptical. The half-width b is given by the equation

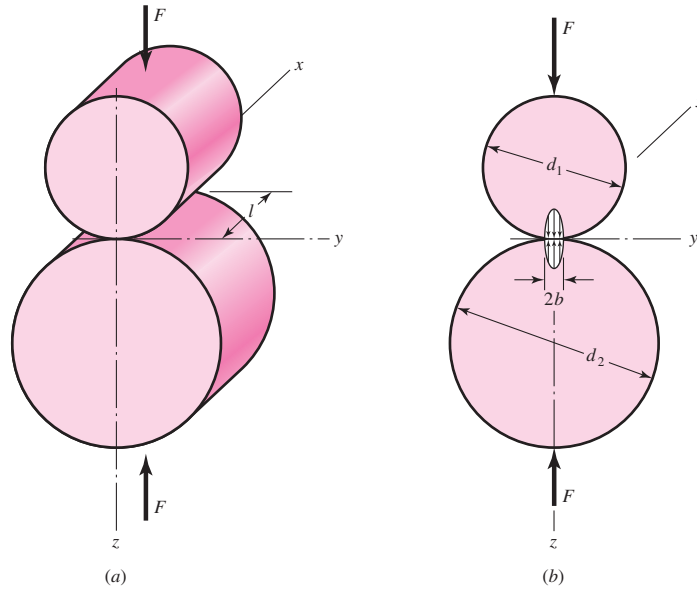
$$b = \sqrt{\frac{2F(1 - \nu_1^2)/E_1 + (1 - \nu_2^2)/E_2}{\pi l (1/d_1 + 1/d_2)}} \quad (3-73)$$

The maximum pressure is

$$p_{\max} = \frac{2F}{\pi bl} \quad (3-74)$$

Figure 3-38

(a) Two right circular cylinders held in contact by forces F uniformly distributed along cylinder length l . (b) Contact stress has an elliptical distribution across the contact zone width $2b$.



Equations (3-73) and (3-74) apply to a cylinder and a plane surface, such as a rail, by making $d = \infty$ for the plane surface. The equations also apply to the contact of a cylinder and an internal cylindrical surface; in this case d is made negative for the internal surface.

The stress state along the z axis is given by the equations

$$\sigma_x = -2\nu p_{\max} \left(\sqrt{1 + \frac{z^2}{b^2}} - \left| \frac{z}{b} \right| \right) \quad (3-75)$$

$$\sigma_y = -p_{\max} \left(\frac{1 + 2\frac{z^2}{b^2}}{\sqrt{1 + \frac{z^2}{b^2}}} - 2\left| \frac{z}{b} \right| \right) \quad (3-76)$$

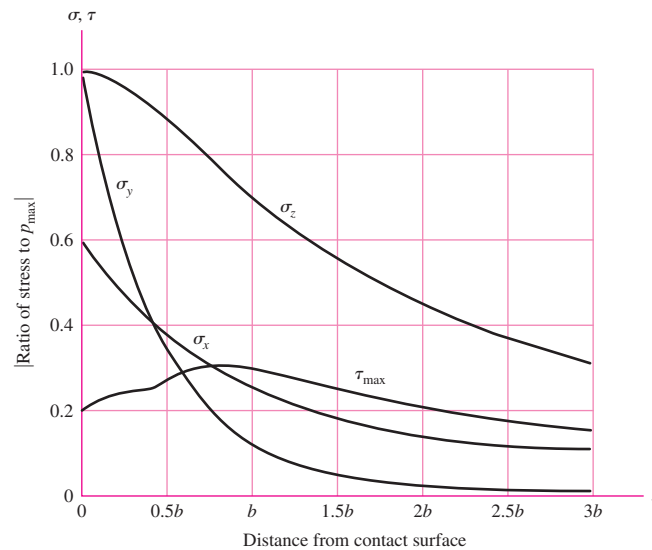
$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{\sqrt{1 + z^2/b^2}} \quad (3-77)$$

These three equations are plotted in Fig. 3-39 up to a distance of $3b$ below the surface. For $0 \leq z \leq 0.436b$, $\sigma_1 = \sigma_x$, and $\tau_{\max} = (\sigma_1 - \sigma_3)/2 = (\sigma_x - \sigma_z)/2$. For $z \geq 0.436b$, $\sigma_1 = \sigma_y$, and $\tau_{\max} = (\sigma_y - \sigma_z)/2$. A plot of $\tau_{\max}$ is also included in Fig. 3-39, where the greatest value occurs at $z/b = 0.786$ with a value of $0.300 p_{\max}$.

Hertz (1881) provided the preceding mathematical models of the stress field when the contact zone is free of shear stress. Another important contact stress case is *line of contact* with friction providing the shearing stress on the contact zone. Such shearing stresses are small with cams and rollers, but in cams with flatfaced followers, wheel-rail contact, and gear teeth, the stresses are elevated above the Hertzian field. Investigations of the effect on the stress field due to normal and shear stresses in the contact zone were begun theoretically by Lundberg (1939), and continued by Mindlin (1949), Smith-Liu (1949), and Poritsky (1949) independently. For further detail, see the reference cited in Footnote 15, p. 122.

Figure 3-39

Magnitude of the stress components below the surface as a function of the maximum pressure for contacting cylinders. The largest value of $\tau_{\max}$ occurs at $z/b = 0.786$. Its maximum value is $0.30p_{\max}$. The chart is based on a Poisson ratio of 0.30. Note that all normal stresses are compressive stresses.



3-20 Summary

The ability to quantify the stress condition at a critical location in a machine element is an important skill of the engineer. Why? Whether the member fails or not is assessed by comparing the (damaging) stress at a critical location with the corresponding material strength at this location. This chapter has addressed the description of stress.

Stresses can be estimated with great precision where the geometry is sufficiently simple that theory easily provides the necessary quantitative relationships. In other cases, approximations are used. There are numerical approximations such as finite element analysis (FEA, see Chap. 19), whose results tend to converge on the true values. There are experimental measurements, strain gauging, for example, allowing *inference* of stresses from the measured strain conditions. Whatever the method(s), the goal is a robust description of the stress condition at a critical location.

The nature of research results and understanding in any field is that the longer we work on it, the more involved things seem to be, and new approaches are sought to help with the complications. As newer schemes are introduced, engineers, hungry for the improvement the new approach *promises*, begin to use the approach. Optimism usually recedes, as further experience adds concerns. Tasks that promised to extend the capabilities of the nonexpert eventually show that expertise is not optional.

In stress analysis, the computer can be helpful if the necessary equations are available. Spreadsheet analysis can quickly reduce complicated calculations for parametric studies, easily handling “what if” questions relating trade-offs (e.g., less of a costly material or more of a cheaper material). It can even give insight into optimization opportunities.

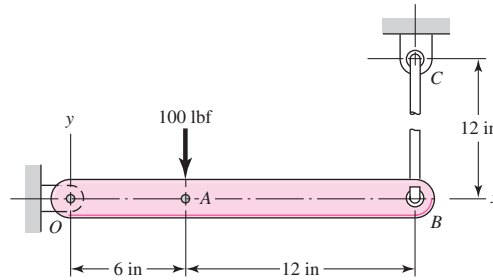
When the necessary equations are not available, then methods such as FEA are attractive, but cautions are in order. Even when you have access to a powerful FEA code, you should be near an expert while you are learning. There are nagging questions of convergence at discontinuities. Elastic analysis is much easier than elastic-plastic analysis. The results are no better than the modeling of reality that was used to formulate the problem. Chapter 19 provides an idea of what finite-element analysis is and how it can be used in design. The chapter is by no means comprehensive in finite-element theory and the application of finite elements in practice. Both skill sets require much exposure and experience to be adept.

PROBLEMS

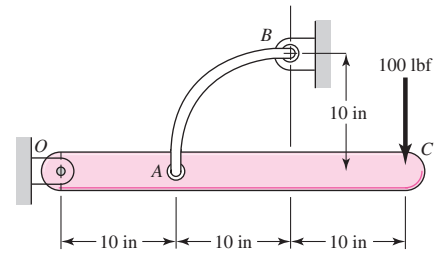
Problems marked with an asterisk (*) are linked with problems in other chapters, as summarized in Table 1-1 of Sec. 1-16, p. 24.

3-1* to
3-4

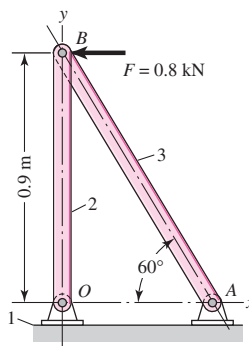
Sketch a free-body diagram of each element in the figure. Compute the magnitude and direction of each force using an algebraic or vector method, as specified.



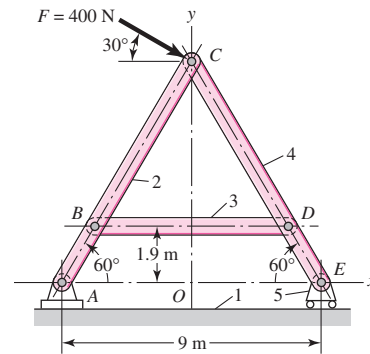
Problem 3-1*



Problem 3-2



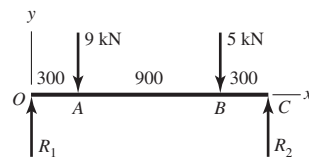
Problem 3-3



Problem 3-4

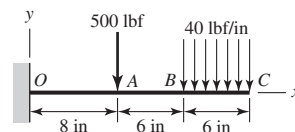
3-5 to
3-8

For the beam shown, find the reactions at the supports and plot the shear-force and bending-moment diagrams. Label the diagrams properly and provide values at all key points.

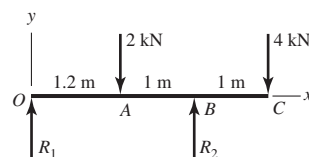


Problem 3-5

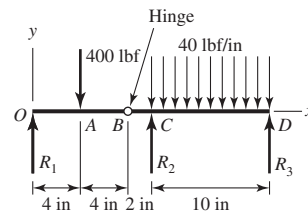
Dimensions in millimeters



Problem 3-6



Problem 3-7



Problem 3-8

- 3-9** Repeat Prob. 3-5 using singularity functions exclusively (including reactions).
- 3-10** Repeat Prob. 3-6 using singularity functions exclusively (including reactions).
- 3-11** Repeat Prob. 3-7 using singularity functions exclusively (including reactions).
- 3-12** Repeat Prob. 3-8 using singularity functions exclusively (including reactions).
- 3-13** For a beam from Table A-9, as specified by your instructor, find general expressions for the loading, shear-force, bending-moment, and support reactions. Use the method specified by your instructor.
- 3-14** A beam carrying a uniform load is simply supported with the supports set back a distance a from the ends as shown in the figure. The bending moment at x can be found from summing moments to zero at section x :

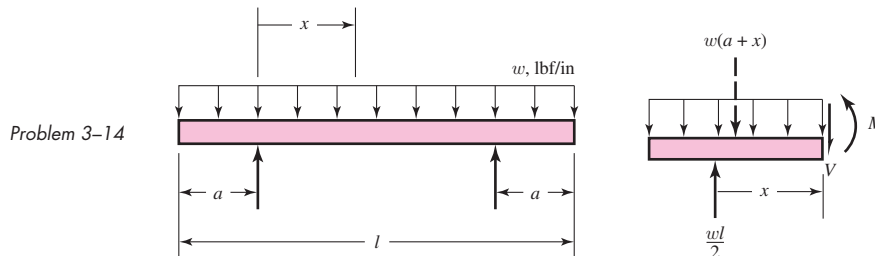
$$\sum M = M + \frac{1}{2}w(a+x)^2 - \frac{1}{2}wlx = 0$$

or

$$M = \frac{w}{2}[lx - (a+x)^2]$$

where w is the loading intensity in lbf/in. The designer wishes to minimize the necessary weight of the supporting beam by choosing a setback resulting in the smallest possible maximum bending stress.

- (a) If the beam is configured with $a = 2.25$ in, $l = 10$ in, and $w = 100$ lbf/in, find the magnitude of the severest bending moment in the beam.
- (b) Since the configuration in part (a) is not optimal, find the optimal setback a that will result in the lightest-weight beam.

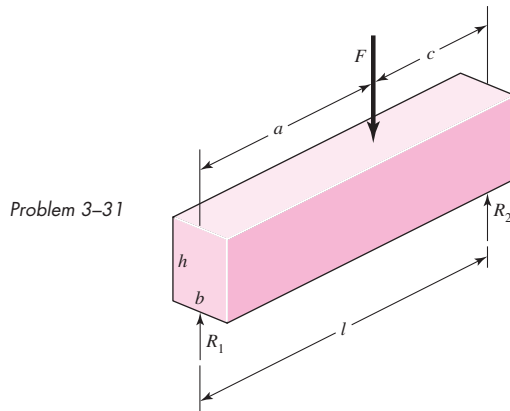


- 3-15** For each of the plane stress states listed below, draw a Mohr's circle diagram properly labeled, find the principal normal and shear stresses, and determine the angle from the x axis to σ_1 . Draw stress elements as in Fig. 3-11c and d and label all details.
- (a) $\sigma_x = 20$ kpsi, $\sigma_y = -10$ kpsi, $\tau_{xy} = 8$ kpsi cw
- (b) $\sigma_x = 16$ kpsi, $\sigma_y = 9$ kpsi, $\tau_{xy} = 5$ kpsi ccw
- (c) $\sigma_x = 10$ kpsi, $\sigma_y = 24$ kpsi, $\tau_{xy} = 6$ kpsi ccw
- (d) $\sigma_x = -12$ kpsi, $\sigma_y = 22$ kpsi, $\tau_{xy} = 12$ kpsi cw
- 3-16** Repeat Prob. 3-15 for:
- (a) $\sigma_x = -8$ MPa, $\sigma_y = 7$ MPa, $\tau_{xy} = 6$ MPa cw
- (b) $\sigma_x = 9$ MPa, $\sigma_y = -6$ MPa, $\tau_{xy} = 3$ MPa cw
- (c) $\sigma_x = -4$ MPa, $\sigma_y = 12$ MPa, $\tau_{xy} = 7$ MPa ccw
- (d) $\sigma_x = 6$ MPa, $\sigma_y = -5$ MPa, $\tau_{xy} = 8$ MPa ccw

- 3-17** Repeat Prob. 3-15 for:
- (a) $\sigma_x = 12$ kpsi, $\sigma_y = 6$ kpsi, $\tau_{xy} = 4$ kpsi cw
 - (b) $\sigma_x = 30$ kpsi, $\sigma_y = -10$ kpsi, $\tau_{xy} = 10$ kpsi ccw
 - (c) $\sigma_x = -10$ kpsi, $\sigma_y = 18$ kpsi, $\tau_{xy} = 9$ kpsi cw
 - (d) $\sigma_x = 9$ kpsi, $\sigma_y = 19$ kpsi, $\tau_{xy} = 8$ kpsi cw
- 3-18** For each of the stress states listed below, find all three principal normal and shear stresses. Draw a complete Mohr's three-circle diagram and label all points of interest.
- (a) $\sigma_x = -80$ MPa, $\sigma_y = -30$ MPa, $\tau_{xy} = 20$ MPa cw
 - (b) $\sigma_x = 30$ MPa, $\sigma_y = -60$ MPa, $\tau_{xy} = 30$ MPa cw
 - (c) $\sigma_x = 40$ MPa, $\sigma_z = -30$ MPa, $\tau_{xy} = 20$ MPa ccw
 - (d) $\sigma_x = 50$ MPa, $\sigma_z = -20$ MPa, $\tau_{xy} = 30$ MPa cw
- 3-19** Repeat Prob. 3-18 for:
- (a) $\sigma_x = 10$ kpsi, $\sigma_y = -4$ kpsi
 - (b) $\sigma_x = 10$ kpsi, $\tau_{xy} = 4$ kpsi ccw
 - (c) $\sigma_x = -2$ kpsi, $\sigma_y = -8$ kpsi, $\tau_{xy} = 4$ kpsi cw
 - (d) $\sigma_x = 10$ kpsi, $\sigma_y = -30$ kpsi, $\tau_{xy} = 10$ kpsi ccw
- 3-20** The state of stress at a point is $\sigma_x = -6$, $\sigma_y = 18$, $\sigma_z = -12$, $\tau_{xy} = 9$, $\tau_{yz} = 6$, and $\tau_{zx} = -15$ kpsi. Determine the principal stresses, draw a complete Mohr's three-circle diagram, labeling all points of interest, and report the maximum shear stress for this case.
- 3-21** Repeat Prob. 3-20 with $\sigma_x = 20$, $\sigma_y = 0$, $\sigma_z = 20$, $\tau_{xy} = 40$, $\tau_{yz} = -20\sqrt{2}$, and $\tau_{zx} = 0$ kpsi.
- 3-22** Repeat Prob. 3-20 with $\sigma_x = 10$, $\sigma_y = 40$, $\sigma_z = 40$, $\tau_{xy} = 20$, $\tau_{yz} = -40$, and $\tau_{zx} = -20$ MPa.
- 3-23** A $\frac{3}{4}$ -in-diameter steel tension rod is 5 ft long and carries a load of 15 kip. Find the tensile stress, the total deformation, the unit strains, and the change in the rod diameter.
- 3-24** Repeat Prob. 3-23 except change the rod to aluminum and the load to 3000 lbf.
- 3-25** A 30-mm-diameter copper rod is 1 m long with a yield strength of 70 MPa. Determine the axial force necessary to cause the diameter of the rod to reduce by 0.01 percent, assuming elastic deformation. Check that the elastic deformation assumption is valid by comparing the axial stress to the yield strength.
- 3-26** A diagonal aluminum alloy tension rod of diameter d and initial length l is used in a rectangular frame to prevent collapse. The rod can safely support a tensile stress of σ_{allow} . If $d = 0.5$ in, $l = 8$ ft, and $\sigma_{\text{allow}} = 20$ kpsi, determine how much the rod must be stretched to develop this allowable stress.
- 3-27** Repeat Prob. 3-26 with $d = 16$ mm, $l = 3$ m, and $\sigma_{\text{allow}} = 140$ MPa.
- 3-28** Repeat Prob. 3-26 with $d = \frac{5}{8}$ in, $l = 10$ ft, and $\sigma_{\text{allow}} = 15$ kpsi.
- 3-29** Electrical strain gauges were applied to a notched specimen to determine the stresses in the notch. The results were $\epsilon_x = 0.0019$ and $\epsilon_y = -0.00072$. Find σ_x and σ_y if the material is carbon steel.
- 3-30** Repeat Prob. 3-29 for a material of aluminum.
- 3-31** The Roman method for addressing uncertainty in design was to build a copy of a design that was satisfactory and had proven durable. Although the early Romans did not have the intellectual tools to deal with scaling size up or down, you do. Consider a simply supported, rectangular-cross-section beam with a concentrated load F , as depicted in the figure.
- (a) Show that the stress-to-load equation is

$$F = \frac{\sigma b h^2 l}{6ac}$$

- (b) Subscript every parameter with m (for model) and divide into the above equation. Introduce a scale factor, $s = a_m/a = b_m/b = c_m/c$ etc. Since the Roman method was to not “lean on” the material any more than the proven design, set $\sigma_m/\sigma = 1$. Express F_m in terms of the scale factors and F , and comment on what you have learned.



- 3-32** Using our experience with concentrated loading on a simple beam, Prob. 3-31, consider a uniformly loaded simple beam (Table A-9-7).

- (a) Show that the stress-to-load equation for a rectangular-cross-section beam is given by

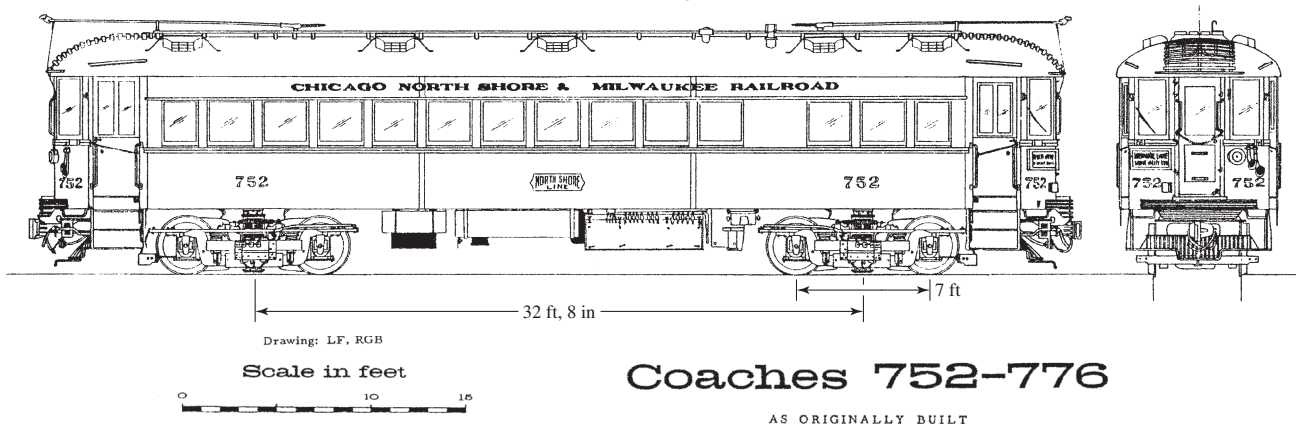
$$W = \frac{4}{3} \frac{\sigma b h^2}{l}$$

where $W = wl$.

- (b) Subscript every parameter with m (for model) and divide the model equation into the prototype equation. Introduce the scale factor s as in Prob. 3-31, setting $\sigma_m/\sigma = 1$. Express W_m and w_m in terms of the scale factor, and comment on what you have learned.

- 3-33** The Chicago North Shore & Milwaukee Railroad was an electric railway running between the cities in its corporate title. It had passenger cars as shown in the figure, which weighed 104.4 kip, had 32-ft, 8-in truck centers, 7-ft-wheelbase trucks, and a coupled length of 55 ft, $3\frac{1}{4}$ in. Consider the case of a single car on a 100-ft-long, simply supported deck plate girder bridge.

- (a) What was the largest bending moment in the bridge?
 (b) Where on the bridge was the moment located?
 (c) What was the position of the car on the bridge?
 (d) Under which axle is the bending moment?

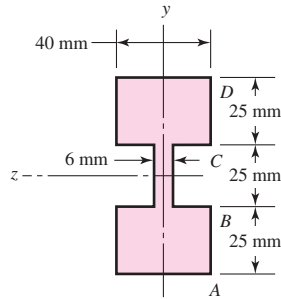


Problem 3-33

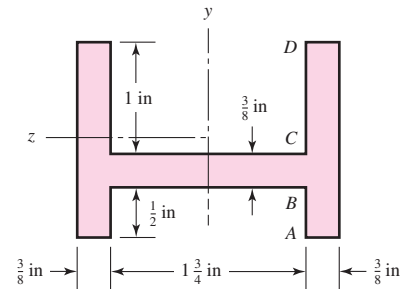
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3-34

For each section illustrated, find the second moment of area, the location of the neutral axis, and the distances from the neutral axis to the top and bottom surfaces. Consider that the section is transmitting a positive bending moment about the z axis, M_z , where $M_z = 10 \text{ kip} \cdot \text{in}$ if the dimensions of the section are given in ips units, or $M_z = 1.13 \text{ kN} \cdot \text{m}$ if the dimensions are in SI units. Determine the resulting stresses at the top and bottom surfaces and at every abrupt change in the cross section.

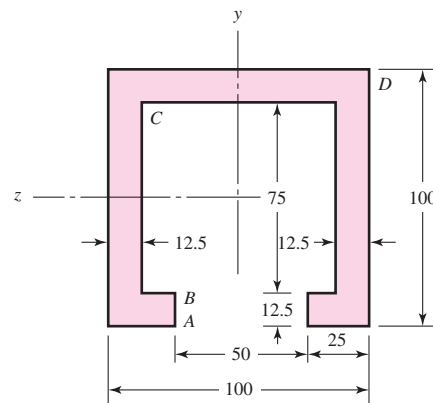


(a)

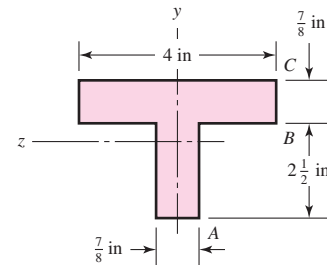


(b)

Problem 3-34



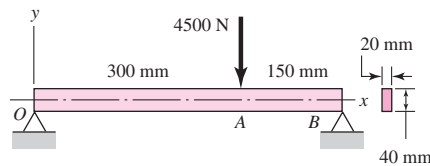
(c) Dimensions in mm



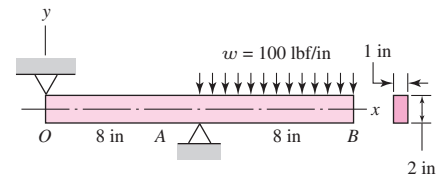
(d)

3-35 to 3-38

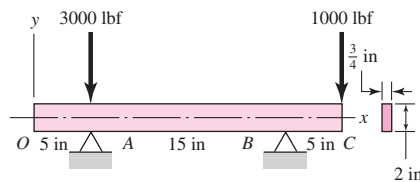
For the beam illustrated in the figure, find the locations and magnitudes of the maximum tensile bending stress due to M and the maximum shear stress due to V .



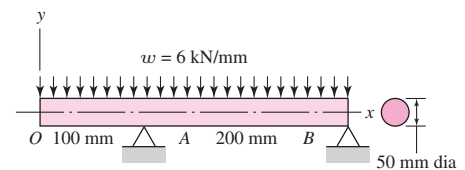
Problem 3-35



Problem 3-36



Problem 3-37



Problem 3-38

3-39

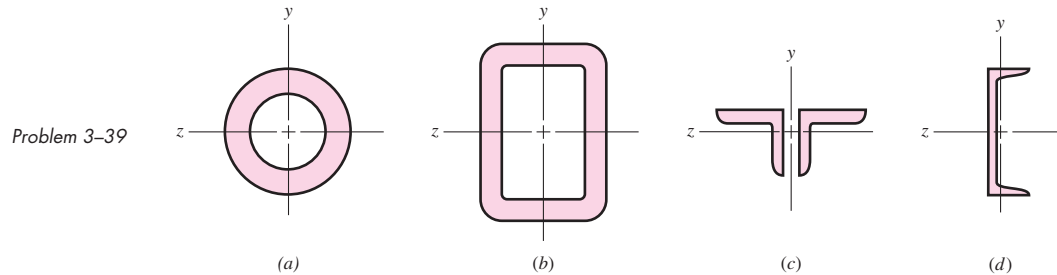
The figure illustrates a number of beam sections. Use an allowable bending stress of 12 kpsi for steel and find the maximum safe uniformly distributed load that each beam can carry if the given lengths are between simple supports.

(a) Standard 2-in $\times$ $\frac{1}{4}$ -in tube, 48 in long

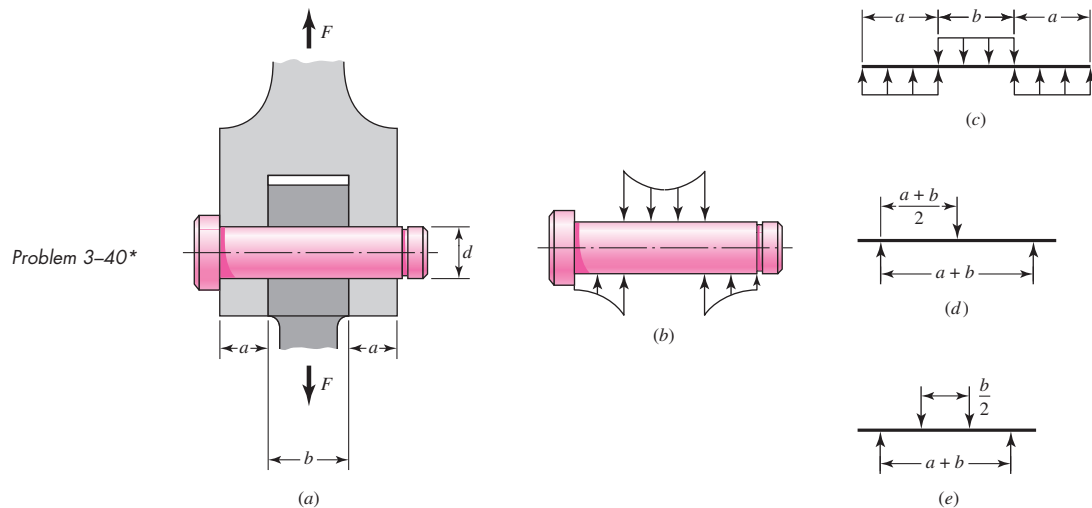
(b) Hollow steel tube 3 by 2 in, outside dimensions, formed from $\frac{3}{16}$ -in material and welded, 60 in long

(c) Steel angles $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$ in and 60 in long

(d) A 6.0 lbf/ft, 3-in steel channel, 60 in long

**3-40***

A pin in a knuckle joint carrying a tensile load F deflects somewhat on account of this loading, making the distribution of reaction and load as shown in part (b) of the figure. A common simplification is to assume uniform load distributions, as shown in part (c). To further simplify, designers may consider replacing the distributed loads with point loads, such as in the two models shown in parts d and e. If $a = 0.5$ in, $b = 0.75$ in, $d = 0.5$ in, and $F = 1000$ lbf, estimate the maximum bending stress and the maximum shear stress due to V for the three simplified models. Compare the three models from a designer's perspective in terms of accuracy, safety, and modeling time.

**3-41**

Repeat Prob. 3-40 for $a = 6$ mm, $b = 18$ mm, $d = 12$ mm, and $F = 4$ kN.

3-42

For the knuckle joint described in Prob. 3-40, assume the maximum allowable tensile stress in the pin is 30 kpsi and the maximum allowable shearing stress in the pin is 15 kpsi. Use the model

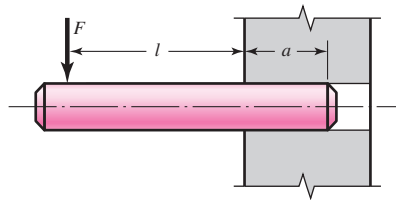
shown in part *c* of the figure to determine a minimum pin diameter for each of the following potential failure modes.

- (a) Consider failure based on bending at the point of maximum bending stress in the pin.
- (b) Consider failure based on the average shear stress on the pin cross section at the interface plane of the knuckle and clevis.
- (c) Consider failure based on shear at the point of the maximum transverse shear stress in the pin.

3-43

The figure illustrates a pin tightly fitted into a hole of a substantial member. A usual analysis is one that assumes concentrated reactions R and M at distance l from F . Suppose the reaction is distributed linearly along distance a . Is the resulting moment reaction larger or smaller than the concentrated reaction? What is the loading intensity q ? What do you think of using the usual assumption?

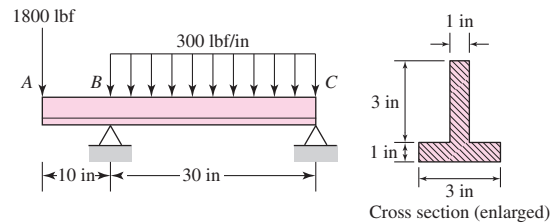
Problem 3-43



3-44

For the beam shown, determine (a) the maximum tensile and compressive bending stresses, (b) the maximum shear stress due to V , and (c) the maximum shear stress in the beam.

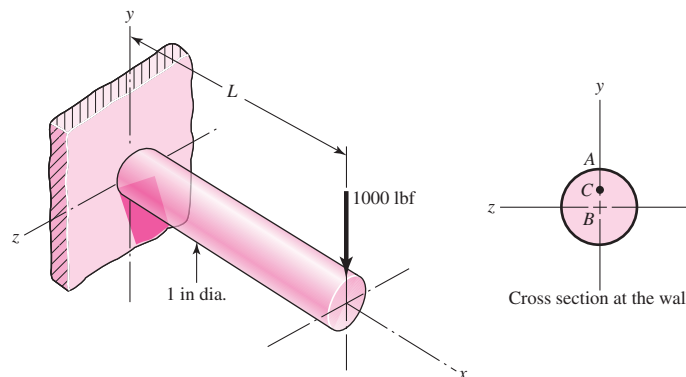
Problem 3-44



3-45

A cantilever beam with a 1-in-diameter round cross section is loaded at the tip with a transverse force of 1000 lbf, as shown in the figure. The cross section at the wall is also shown, with labeled points A at the top, B at the center, and C at the midpoint between A and B . Study the

Problem 3-45



significance of the transverse shear stress in combination with bending by performing the following steps.

- Assume $L = 10$ in. For points A , B , and C , sketch three-dimensional stress elements, labeling the coordinate directions and showing all stresses. Calculate magnitudes of the stresses on the stress elements. Do not neglect transverse shear stress. Calculate the maximum shear stress for each stress element.
- For each stress element in part (a), calculate the maximum shear stress if the transverse shear stress is neglected. Determine the percent error for each stress element from neglecting the transverse shear stress.
- Repeat the problem for $L = 4$, 1, and 0.1 in. Compare the results and state any conclusions regarding the significance of the transverse shear stress in combination with bending.

- 3-46** Consider a simply supported beam of rectangular cross section of constant width b and variable depth h , so proportioned that the maximum stress σ_x at the outer surface due to bending is constant, when subjected to a load F at a distance a from the left support and a distance c from the right support. Show that the depth h at location x is given by

$$h = \sqrt{\frac{6Fcx}{lb\sigma_{\max}}} \quad 0 \leq x \leq a$$

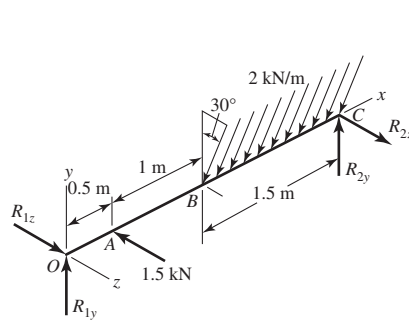
- 3-47** In Prob. 3-46, $h \rightarrow 0$ as $x \rightarrow 0$, which cannot occur. If the maximum shear stress $\tau_{\max}$ due to direct shear is to be constant in this region, show that the depth h at location x is given by

$$h = \frac{3}{2} \frac{Fc}{lb\tau_{\max}} \quad 0 \leq x \leq \frac{3}{8} \frac{Fc\sigma_{\max}}{lb\tau_{\max}^2}$$

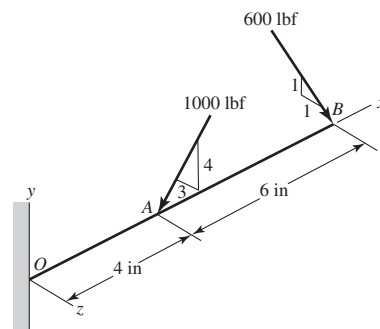
**3-48 and
3-49**

The beam shown is loaded in the xy and xz planes.

- Find the yz components of the reactions at the supports.
- Plot the shear-force and bending-moment diagrams for the xy and xz planes. Label the diagrams properly and provide the values at key points.
- Determine the net shear-force and bending-moment at the key points of part (b).
- Determine the maximum tensile bending stress. For Prob. 3-48, use the cross section given in Prob. 3-34, part (a). For Prob. 3-49, use the cross section given in Prob. 3-39, part (b).



Problem 3-48



Problem 3-49

- 3-50** Two steel thin-wall tubes in torsion of equal length are to be compared. The first is of square cross section, side length b , and wall thickness t . The second is a round of diameter b and wall thickness t . The largest allowable shear stress is τ_{all} and is to be the same in both cases. How does the angle of twist per unit length compare in each case?

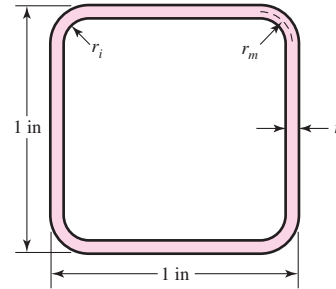
3-51

Consider a 1-in-square steel thin-walled tube loaded in torsion. The tube has a wall thickness $t = \frac{1}{16}$ in, is 36 in long, and has a maximum allowable shear stress of 12 kpsi. Determine the maximum torque that can be applied and the corresponding angle of twist of the tube.

(a) Assume that the internal radius at the corners $r_i = 0$.

(b) Assume that the internal radius at the corners is more realistically $r_i = \frac{1}{8}$ in.

Problem 3-51

**3-52**

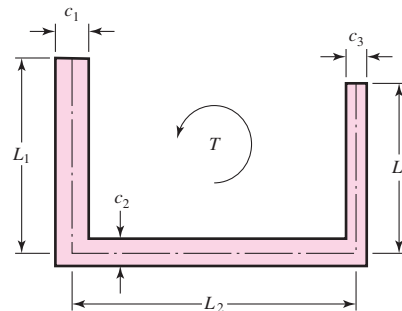
The thin-walled open cross-section shown is transmitting torque T . The angle of twist per unit length of each leg can be determined separately using Eq. (3-47) and is given by

$$\theta_1 = \frac{3T_i}{GL_i c_i^3}$$

where for this case, $i = 1, 2, 3$, and T_i represents the torque in leg i . Assuming that the angle of twist per unit length for each leg is the same, show that

$$T = \frac{G\theta_1}{3} \sum_{i=1}^3 L_i c_i^3 \quad \text{and} \quad \tau_{\max} = G\theta_1 c_{\max}$$

Problem 3-52

**3-53 to 3-55**

Using the results from Prob. 3-52, consider a steel section with $\tau_{\text{allow}} = 12$ kpsi.

(a) Determine the torque transmitted by each leg and the torque transmitted by the entire section.

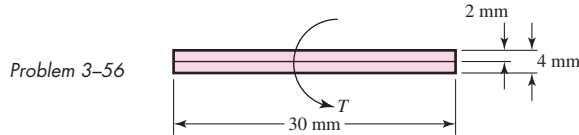
(b) Determine the angle of twist per unit length.

Problem Number	c_1	L_1	c_2	L_2	c_3	L_3
3-53	2 mm	20 mm	3 mm	30 mm	0	0
3-54	$\frac{1}{16}$ in	$\frac{3}{4}$ in	$\frac{1}{8}$ in	1 in	$\frac{1}{16}$ in	$\frac{5}{8}$ in
3-55	2 mm	20 mm	3 mm	30 mm	2 mm	25 mm

3-56

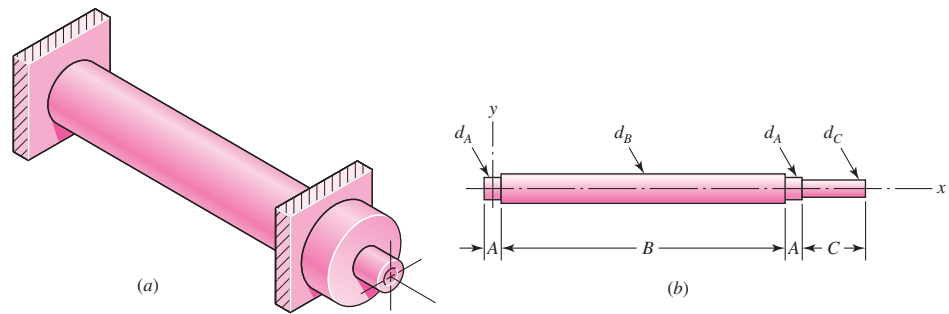
Two 300-mm-long rectangular steel strips are placed together as shown. Using a maximum allowable shear stress of 80 MPa, determine the maximum torque and angular twist, and the torsional

spring rate. Compare these with a single strip of cross section 30 mm by 4 mm. Solve the problem two ways: (a) using Eqs. (3–40) and (3–41), and (b) using Eq. (3–47). Compare and discuss your results



- 3–57** Using a maximum allowable shear stress of 70 MPa, find the shaft diameter needed to transmit 40 kW when
 (a) The shaft speed is 2500 rev/min.
 (b) The shaft speed is 250 rev/min.
- 3–58** Repeat Prob. 3–57 with an allowable shear stress of 20 kpsi and a power of 50 hp.
- 3–59** Using an allowable shear stress of 50 MPa, determine the power that can be transmitted at 2000 rpm through a shaft with a 30-mm diameter.
- 3–60** A 20-mm-diameter steel bar is to be used as a torsion spring. If the torsional stress in the bar is not to exceed 110 MPa when one end is twisted through an angle of 15° , what must be the length of the bar?
- 3–61** A 2-ft-long steel bar with a $\frac{3}{4}$ -in diameter is to be used as a torsion spring. If the torsional stress in the bar is not to exceed 30 kpsi, what is the maximum angle of twist of the bar?
- 3–62** A 40-mm-diameter solid steel shaft, used as a torque transmitter, is replaced with a hollow shaft having a 40-mm OD and a 36-mm ID. If both materials have the same strength, what is the percentage reduction in torque transmission? What is the percentage reduction in shaft weight?
- 3–63** Generalize Prob. 3–62 for a solid shaft of diameter d replaced with a hollow shaft of the same material with an outside diameter d_o and an inside diameter that is a fraction of the outside diameter, $x \times d_o$, where x is any fraction between zero and one. Obtain expressions for percentage reduction in torque transmission and percentage reduction in weight in terms of only x . Notice that the length and diameter of the shaft, and the material, are not needed for this comparison. Plot both results on the same axis for the range $0 < x < 1$. From the plot, what is the approximate value of x to obtain the greatest difference between the percent decrease in weight and the percent decrease in torque?
- 3–64** A hollow steel shaft is to transmit 4200 N · m of torque and is to be sized so that the torsional stress does not exceed 120 MPa.
 (a) If the inside diameter is 70 percent of the outside diameter, what size shaft should be used? Use preferred sizes.
 (b) What is the stress on the inside of the shaft when full torque is applied?
- 3–65** The figure shows an endless-belt conveyor drive roll. The roll has a diameter 120 mm and is driven at 10 rev/min by a geared-motor source rated at 1.5 kW. Determine a suitable shaft diameter d_c for an allowable torsional stress of 80 MPa.
 (a) What would be the stress in the shaft you have sized if the motor starting torque is twice the running torque?
 (b) Is bending stress likely to be a problem? What is the effect of different roll lengths B on bending?

Problem 3-65

**3-66**

The conveyer drive roll in the figure for Prob. 3-65 is 5 in in diameter and is driven at 8 rev/min by a geared-motor source rated at 1 hp. Find a suitable shaft diameter d_C based on an allowable torsional stress of 15 kpsi.

3-67

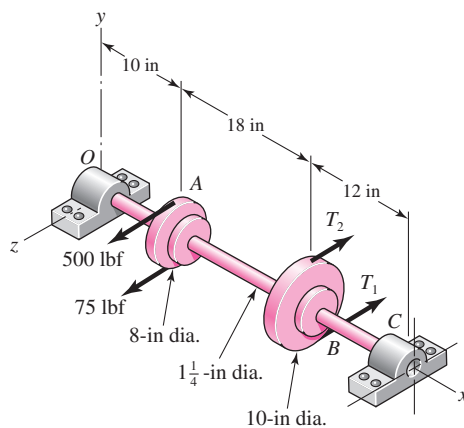
Consider two shafts in torsion, each of the same material, length, and cross-sectional area. One shaft has a solid square cross section and the other shaft has a solid circular section.

- Which shaft has the greater maximum shear stress and by what percentage?
- Which shaft has the greater angular twist θ and by what percentage?

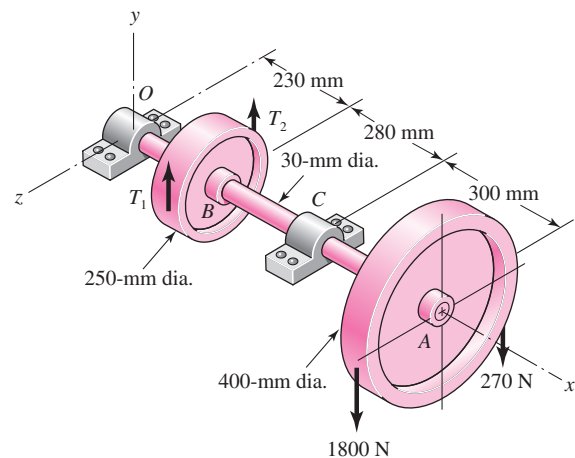
3-68* to 3-71*

A countershaft carrying two V-belt pulleys is shown in the figure. Pulley A receives power from a motor through a belt with the belt tensions shown. The power is transmitted through the shaft and delivered to the belt on pulley B. Assume the belt tension on the loose side at B is 15 percent of the tension on the tight side.

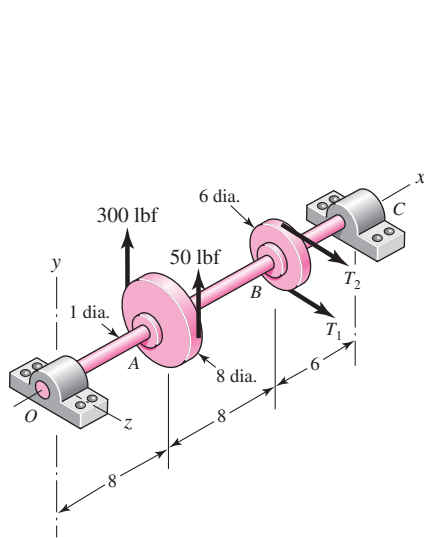
- Determine the tensions in the belt on pulley B, assuming the shaft is running at a constant speed.
- Find the magnitudes of the bearing reaction forces, assuming the bearings act as simple supports.
- Draw shear-force and bending-moment diagrams for the shaft. If needed, make one set for the horizontal plane and another set for the vertical plane.
- At the point of maximum bending moment, determine the bending stress and the torsional shear stress.
- At the point of maximum bending moment, determine the principal stresses and the maximum shear stress.



Problem 3-68*

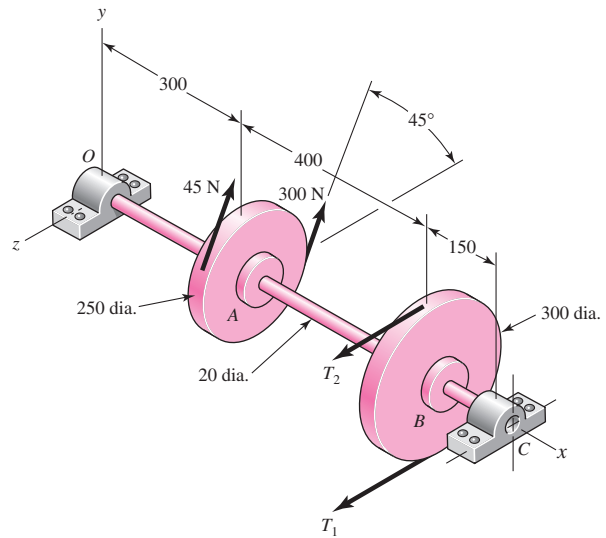


Problem 3-69*



Problem 3-70*

Dimensions in inches.



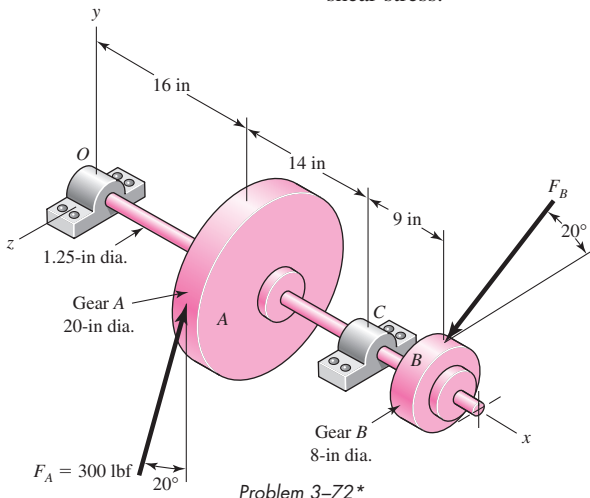
Problem 3-71*

Dimensions in millimeters.

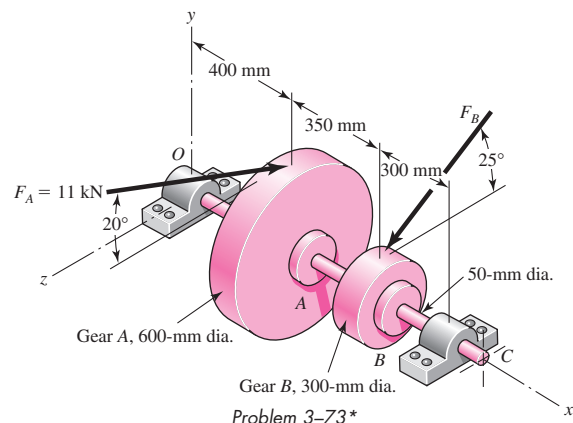
3-72* to 3-73*

A gear reduction unit uses the countershaft shown in the figure. Gear A receives power from another gear with the transmitted force F_A applied at the 20° pressure angle as shown. The power is transmitted through the shaft and delivered through gear B through a transmitted force F_B at the pressure angle shown.

- Determine the force F_B , assuming the shaft is running at a constant speed.
- Find the magnitudes of the bearing reaction forces, assuming the bearings act as simple supports.
- Draw shear-force and bending-moment diagrams for the shaft. If needed, make one set for the horizontal plane and another set for the vertical plane.
- At the point of maximum bending moment, determine the bending stress and the torsional shear stress.
- At the point of maximum bending moment, determine the principal stresses and the maximum shear stress.



Problem 3-72*

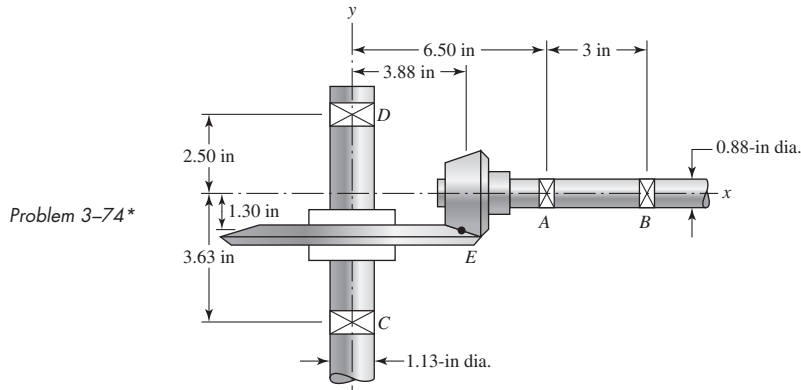


Problem 3-73*

3-74*

In the figure, shaft AB transmits power to shaft CD through a set of bevel gears contacting at point E. The contact force at E on the gear of shaft CD is determined to be $(\mathbf{F}_E)_{CD} = -92.8\mathbf{i} - 362.8\mathbf{j} + 808.0\mathbf{k}$ lbf. For shaft CD: (a) draw a free-body diagram and determine the reactions at C and D

assuming simple supports (assume also that bearing C carries the thrust load), (b) draw the shear-force and bending-moment diagrams, (c) for the critical stress element, determine the torsional shear stress, the bending stress, and the axial stress, and (d) for the critical stress element, determine the principal stresses and the maximum shear stress.



3-75

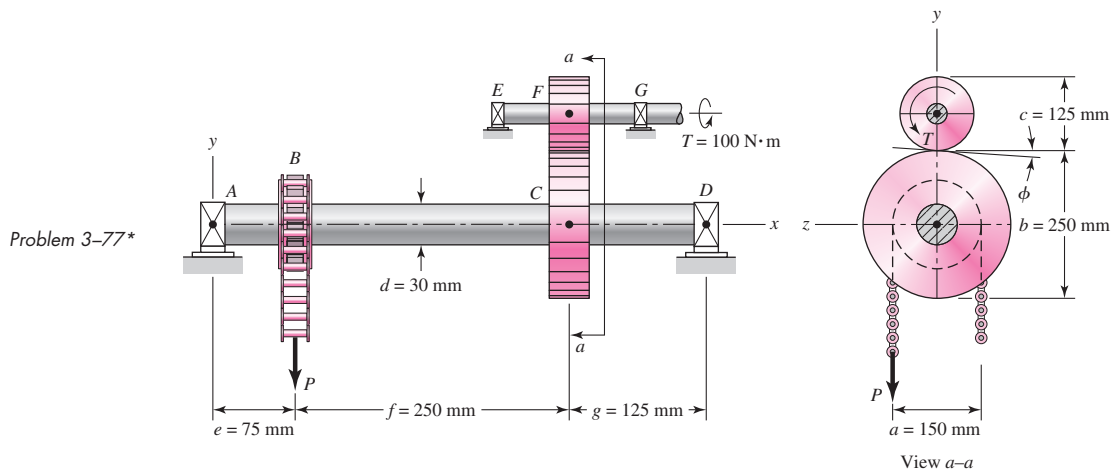
Repeat Prob. 3-74 except for a contact force at E of $(\mathbf{F}_E)_{CD} = -46.6\mathbf{i} - 140\mathbf{j} + 406\mathbf{k}$ lbf and a shaft diameter of 1.0 in.

3-76*

Repeat the analysis of Prob. 3-74 for shaft AB . Assume that bearing A carries the thrust load.

3-77*

A torque $T = 100 \text{ N} \cdot \text{m}$ is applied to the shaft EFG , which is running at constant speed and contains gear F . Gear F transmits torque to shaft $ABCD$ through gear C , which drives the chain sprocket at B , transmitting a force P as shown. Sprocket B , gear C , and gear F have pitch diameters of $a = 150$, $b = 250$, and $c = 125$ mm, respectively. The contact force between the gears is transmitted through the pressure angle $\phi = 20^\circ$. Assuming no frictional losses and considering the bearings at A , D , E , and G to be simple supports, locate the point on shaft $ABCD$ that contains the maximum tensile bending and maximum torsional shear stresses. Combine these stresses and determine the maximum principal normal and shear stresses in the shaft.



3-78

Repeat Prob. 3-77 with the chain parallel to the z axis with P in the positive z direction.

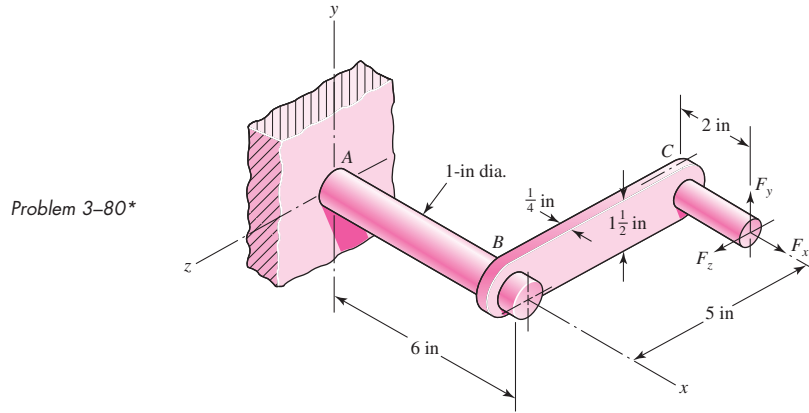
3-79*

Repeat Prob. 3-77 with $T = 900 \text{ lbf} \cdot \text{in}$, $a = 6 \text{ in}$, $b = 5 \text{ in}$, $c = 10 \text{ in}$, $d = 1.375 \text{ in}$, $e = 4 \text{ in}$, $f = 10 \text{ in}$, and $g = 6 \text{ in}$.

3-80*

The cantilevered bar in the figure is made from a ductile material and is statically loaded with $F_y = 200 \text{ lbf}$ and $F_x = F_z = 0$. Analyze the stress situation in rod AB by obtaining the following information.

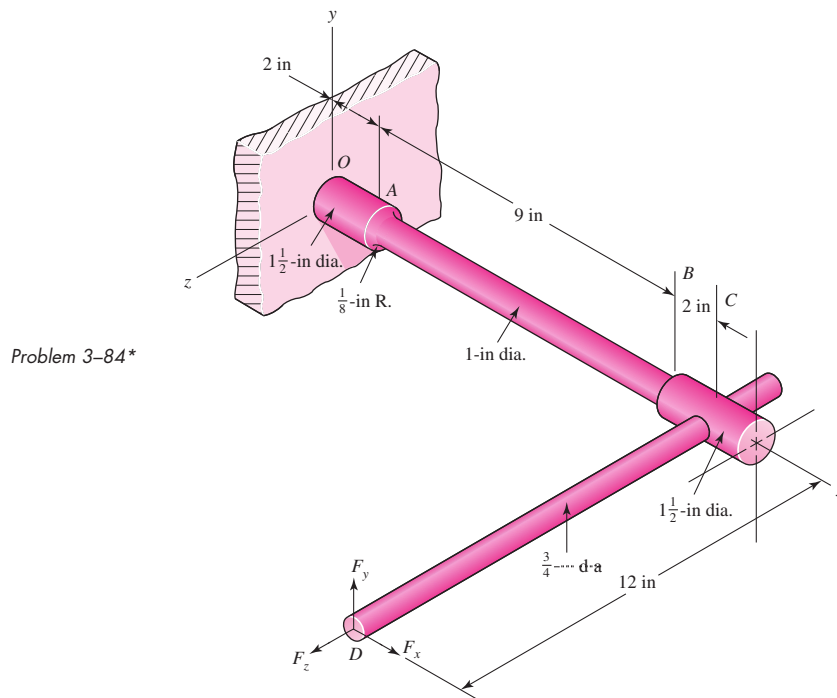
- (a) Determine the precise location of the critical stress element.
 (b) Sketch the critical stress element and determine magnitudes and directions for all stresses acting on it. (Transverse shear may only be neglected if you can justify this decision.)
 (c) For the critical stress element, determine the principal stresses and the maximum shear stress.

**3-81***Repeat Prob. 3-80 with $F_x = 0$, $F_y = 175$ lbf, and $F_z = 100$ lbf.**3-82***Repeat Prob. 3-80 with $F_x = 75$ lbf, $F_y = -200$ lbf, and $F_z = 100$ lbf.**3-83***

For the handle in Prob. 3-80, one potential failure mode is twisting of the flat plate BC . Determine the maximum value of the shear stress due to torsion in the main section of the plate, ignoring the complexities of the interfaces at B and C .

3-84*

The cantilevered bar in the figure is made from a ductile material and is statically loaded with $F_y = 250$ lbf and $F_x = F_z = 0$. Analyze the stress situation in the small diameter at the shoulder at A by obtaining the following information.



- (a) Determine the precise location of the critical stress element at the cross section at A.
 (b) Sketch the critical stress element and determine magnitudes and directions for all stresses acting on it. (Transverse shear may be neglected if you can justify this decision.)
 (c) For the critical stress element, determine the principal stresses and the maximum shear stress.

3-85* Repeat Prob. 3-84 with $F_x = 300$ lbf, $F_y = 250$ lbf, and $F_z = 0$.

3-86* Repeat Prob. 3-84 with $F_x = 300$ lbf, $F_y = 250$ lbf, and $F_z = -100$ lbf.

3-87* Repeat Prob. 3-84 for a brittle material, requiring the inclusion of stress concentration in the fillet radius.

3-88 Repeat Prob. 3-84 with $F_x = 300$ lbf, $F_y = 250$ lbf, and $F_z = 0$, and for a brittle material, requiring the inclusion of stress concentration in the fillet radius.

3-89 Repeat Prob. 3-84 with $F_x = 300$ lbf, $F_y = 250$ lbf, and $F_z = -100$ lbf, and for a brittle material, requiring the inclusion of stress concentration in the fillet radius.

3-90 The figure shows a simple model of the loading of a square thread of a power screw transmitting an axial load F with an application of torque T . The torque is balanced by the frictional force F_f acting along the top surface of the thread. The forces on the thread are considered to be distributed along the circumference of the *mean diameter* d_m over the number of engaged threads, n_t . From the figure, $d_m = d_r + p/2$, where d_r is the *root diameter* of the thread and p is the *pitch* of the thread.

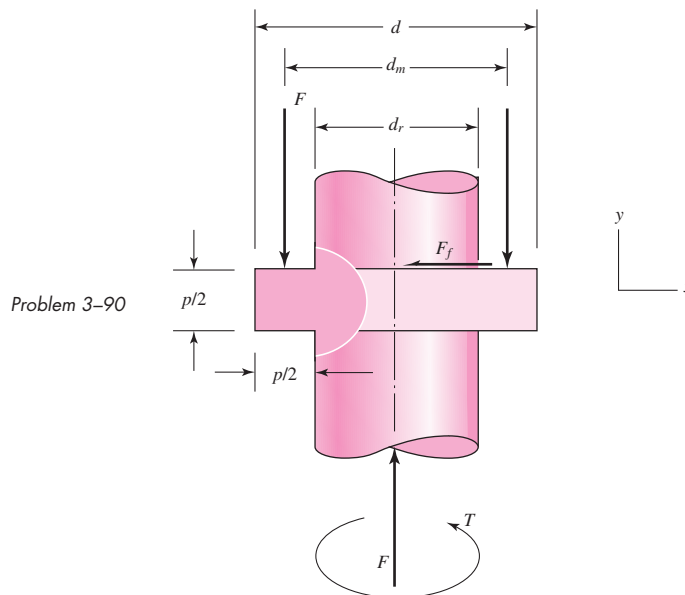
- (a) Considering the thread to be a cantilever beam as shown in the cutaway view, show that the bending stress at the root of the thread can be approximated by

$$\sigma_b = \pm \frac{6F}{\pi d_r n_t p}$$

- (b) Show that the axial and maximum torsional shear stresses in the body of the shaft can be approximated by

$$\sigma_a = -\frac{4F}{\pi d_r^2} \quad \text{and} \quad \tau_t = \frac{16T}{\pi d_r^3}$$

- (c) For the stresses of parts (a) and (b) show a three-dimensional representation of the state of stress on an element located at the intersection of the lower thread root base and the



thread body. Using the given coordinate system label the stresses using the notation given in Fig. 3–8a.

- (d) A square-thread power screw has an outside diameter $d = 1.5$ in, pitch $p = 0.25$ in, and transmits a load $F = 1\,500$ lbf through the application of a torque $T = 235$ lbf · in. If $n_t = 2$, determine the key stresses and the corresponding *principal stresses* (normal and shear).

3–91 Develop the formulas for the maximum radial and tangential stresses in a thick-walled cylinder due to internal pressure only.

3–92 Repeat Prob. 3–91 where the cylinder is subject to external pressure only. At what radii do the maximum stresses occur?

3–93 Develop the equations for the principal stresses in a thin-walled spherical pressure vessel of inside diameter d_i , thickness t , and with an internal pressure p_i . You may wish to follow a process similar to that used for a thin-walled cylindrical pressure vessel on p. 114.

3–94 to 3–96 A pressure cylinder has an outer diameter d_o , wall thickness t , internal pressure p_i , and maximum allowable shear stress $\tau_{\max}$. In the table given, determine the appropriate value of x .

Problem Number	d_o	t	p_i	$\tau_{\max}$
3–94	6 in	0.25 in	$x_{\max}$	10 kpsi
3–95	200 mm	$x_{\min}$	4 MPa	25 MPa
3–96	8 in	0.25 in	500 psi	x

3–97 to 3–99 A pressure cylinder has an outer diameter d_o , wall thickness t , external pressure p_o , and maximum allowable shear stress $\tau_{\max}$. In the table given, determine the appropriate value of x .

Problem Number	d_o	t	p_o	$\tau_{\max}$
3–97	6 in	0.25 in	$x_{\max}$	10 kpsi
3–98	200 mm	$x_{\min}$	4 MPa	25 MPa
3–99	8 in	0.25 in	500 psi	x

3–100 An AISI 1040 cold-drawn steel tube has an OD = 50 mm and wall thickness 6 mm. What maximum external pressure can this tube withstand if the largest principal normal stress is not to exceed 80 percent of the minimum yield strength of the material?

3–101 Repeat Prob. 3–100 with an OD of 2 in and wall thickness of 0.25 in.

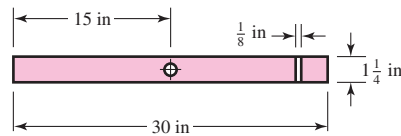
3–102 Repeat Prob. 3–100 with an internal pressure.

3–103 Repeat Prob. 3–101 with an internal pressure.

3–104 A thin-walled cylindrical steel water storage tank 30 ft in diameter and 60 ft long is oriented with its longitudinal axis vertical. The tank is topped with a hemispherical steel dome. The wall thickness of the tank and dome is 0.75 in. If the tank is unpressurized and contains water 55 ft above its base, and considering the weight of the tank, determine the maximum state of stress in the tank and the corresponding principal stresses (normal and shear). The weight density of water is 62.4 lbf/ft³.

- 3-105** Repeat Prob. 3-104 with the tank being pressurized to 50 psig.
- 3-106** Find the maximum shear stress in a $5\frac{1}{2}$ -in-diameter circular saw blade if it runs idle at 5000 rev/min. The saw is 14 gauge (0.0747 in) steel and is used on a $\frac{5}{8}$ -in-diameter arbor. The thickness is uniform. What is the maximum radial component of stress?
- 3-107** The maximum recommended speed for a 250-mm-diameter abrasive grinding wheel is 2000 rev/min. Assume that the material is isotropic; use a bore of 20 mm, $\nu = 0.24$, and a mass density of 3320 kg/m^3 , and find the maximum tensile stress at this speed.
- 3-108** An abrasive cutoff wheel has a diameter of 5 in, is $\frac{1}{16}$ in thick, and has a $\frac{3}{4}$ -in bore. It weighs 5 oz and is designed to run at 12 000 rev/min. If the material is isotropic and $\nu = 0.20$, find the maximum shear stress at the design speed.
- 3-109** A rotary lawnmower blade rotates at 3500 rev/min. The steel blade has a uniform cross section $\frac{1}{8}$ in thick by $1\frac{1}{4}$ in wide, and has a $\frac{1}{2}$ -in-diameter hole in the center as shown in the figure. Estimate the nominal tensile stress at the central section due to rotation.

Problem 3-109



**3-110 to
3-115**

The table lists the maximum and minimum hole and shaft dimensions for a variety of standard press and shrink fits. The materials are both hot-rolled steel. Find the maximum and minimum values of the radial interference and the corresponding interface pressure. Use a collar diameter of 100 mm for the metric sizes and 4 in for those in inch units.

Problem Number	Fit Designation [†]	Basic Size	Hole		Shaft	
			$D_{\max}$	$D_{\min}$	$d_{\max}$	$d_{\min}$
3-110	50H7/p6	50 mm	50.025	50.000	50.042	50.026
3-111	(2 in)H7/p6	2 in	2.0010	2.0000	2.0016	2.0010
3-112	50H7/s6	50 mm	50.025	50.000	50.059	50.043
3-113	(2 in)H7/s6	2 in	2.0010	2.0000	2.0023	2.0017
3-114	50H7/u6	50 mm	50.025	50.000	50.086	50.070
3-115	(2 in)H7/u6	2 in	2.0010	2.0000	2.0034	2.0028

[†]Note: See Table 7-9 for description of fits.

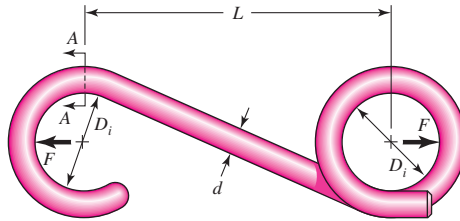
**3-116 to
3-119**

The table gives data concerning the shrink fit of two cylinders of differing materials and dimensional specification in inches. Elastic constants for different materials may be found in Table A-5. Identify the radial interference δ , then find the interference pressure p , and the tangential normal stress on both sides of the fit surface. If dimensional tolerances are given at fit surfaces, repeat the problem for the highest and lowest stress levels.

Problem Number	Inner Cylinder			Outer Cylinder		
	Material	d_i	d_o	Material	D_i	D_o
3-116	Steel	0	2.002	Steel	2.000	3.00
3-117	Steel	0	2.002	Cast iron	2.000	3.00
3-118	Steel	0	1.002/1.003	Steel	1.001/1.002	2.00
3-119	Aluminum	0	2.003/2.006	Steel	2.000/2.002	3.00

- 3-120** A utility hook was formed from a round rod of diameter $d = 20$ mm into the geometry shown in the figure. What are the stresses at the inner and outer surfaces at section A-A if $F = 4$ kN, $L = 250$ mm, and $D_i = 75$ mm?

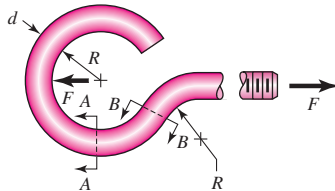
Problem 3-120



- 3-121** Repeat Prob. 3-120 with $d = 0.75$ in, $F = 750$ lbf, $L = 10$ in, and $D_i = 2.5$ in.

- 3-122** The steel eyebolt shown in the figure is loaded with a force $F = 300$ N. The bolt is formed from wire of diameter $d = 6$ mm to a radius $R = 10$ mm in the eye and at the shank. Estimate the stresses at the inner and outer surfaces at section A-A.

Problem 3-122



- 3-123** For Prob. 3-122 estimate the stresses at the inner and outer surfaces at section B-B.

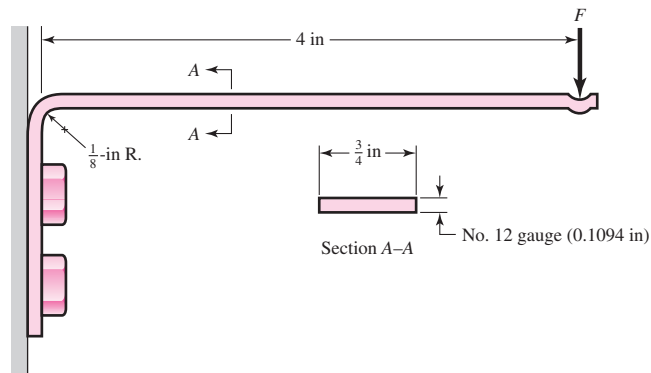
- 3-124** Repeat Prob. 3-122 with $d = \frac{1}{4}$ in, $R = \frac{1}{2}$ in, and $F = 75$ lbf.

- 3-125** Repeat Prob. 3-123 with $d = \frac{1}{4}$ in, $R = \frac{1}{2}$ in, and $F = 75$ lbf.

- 3-126** Shown in the figure is a 12-gauge (0.1094-in) by $\frac{3}{4}$ -in latching spring that supports a load of $F = 3$ lbf. The inside radius of the bend is $\frac{1}{8}$ in.

- Using straight-beam theory, determine the stresses at the top and bottom surfaces immediately to the right of the bend.
- Using curved-beam theory, determine the stresses at the inner and outer surfaces at the bend.
- By comparing the stresses at the bend with the nominal stresses before the bend, estimate effective stress concentration factors for the inner and outer surfaces.

Problem 3-126

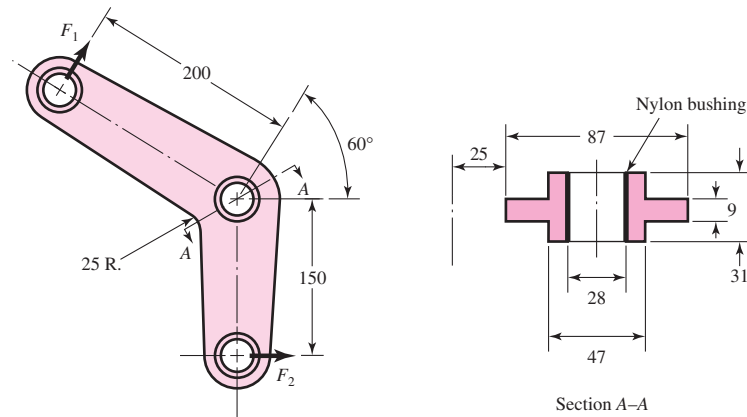


- 3-127** Repeat Prob. 3-126 with a 10-gauge (0.1406-in) material thickness.

3-128 Repeat Prob. 3-126 with a bend radius of $\frac{1}{4}$ in.

3-129 The cast-iron bell-crank lever depicted in the figure is acted upon by forces F_1 of 2.4 kN and F_2 of 3.2 kN. The section A-A at the central pivot has a curved inner surface with a radius of $r_i = 25$ mm. Estimate the stresses at the inner and outer surfaces of the curved portion of the lever.

Problem 3-129

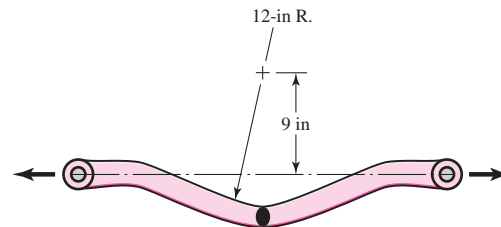


Dimensions in mm

3-130 The crane hook depicted in Fig. 3-35 has a $\frac{3}{4}$ -in-diameter hole in the center of the critical section. For a load of 6 kip, estimate the bending stresses at the inner and outer surfaces at the critical section.

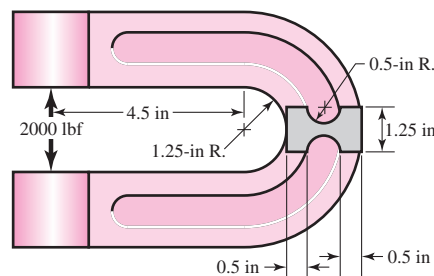
3-131 An offset tensile link is shaped to clear an obstruction with a geometry as shown in the figure. The cross section at the critical location is elliptical, with a major axis of 3 in and a minor axis of 1.5 in. For a load of 20 kip, estimate the stresses at the inner and outer surfaces of the critical section.

Problem 3-131



3-132 A cast-steel C frame as shown in the figure has a rectangular cross section of 1.25 in by 2 in, with a 0.5-in-radius semicircular notch on both sides that forms midflank fluting as shown. Estimate A , r_c , r_n , and e , and for a load of 2000 lbf, estimate the inner and outer surface stresses at the throat C . Note: Table 3-4 can be used to determine r_n for this section. From the table, the integral $\int dA/r$ can be evaluated for a rectangle and a circle by evaluating A/r_n for each shape [see Eq. (3-64)]. Subtracting A/r_n of the circle from that of the rectangle yields $\int dA/r$ for the C frame, and r_n can then be evaluated.

Problem 3-132



- 3-133** Two carbon steel balls, each 30 mm in diameter, are pressed together by a force F . In terms of the force F , find the maximum values of the principal stress, and the maximum shear stress, in MPa.
- 3-134** A carbon steel ball with 25-mm diameter is pressed together with an aluminum ball with a 40-mm diameter by a force of 10 N. Determine the maximum shear stress, and the depth at which it will occur for the aluminum ball. Assume Fig. 3-37, which is based on a typical Poisson's ratio of 0.3, is applicable to estimate the depth at which the maximum shear stress occurs for these materials.
- 3-135** Repeat Prob. 3-134 but determine the maximum shear stress and depth for the steel ball.
- 3-136** A carbon steel ball with a 30-mm diameter is pressed against a flat carbon steel plate with a force of 20 N. Determine the maximum shear stress, and the depth in the plate at which it will occur.
- 3-137** An AISI 1018 steel ball with 1-in diameter is used as a roller between a flat plate made from 2024 T3 aluminum and a flat table surface made from ASTM No. 30 gray cast iron. Determine the maximum amount of weight that can be stacked on the aluminum plate without exceeding a maximum shear stress of 20 kpsi in any of the three pieces. Assume Fig. 3-37, which is based on a typical Poisson's ratio of 0.3, is applicable to estimate the depth at which the maximum shear stress occurs for these materials.
- 3-138** An aluminum alloy cylindrical roller with diameter 1.25 in and length 2 in rolls on the inside of a cast-iron ring having an inside radius of 6 in, which is 2 in thick. Find the maximum contact force F that can be used if the shear stress is not to exceed 4000 psi.
- 3-139** A pair of mating steel spur gears with a 0.75-in face width transmits a load of 40 lbf. For estimating the contact stresses, make the simplifying assumption that the teeth profiles can be treated as cylindrical with instantaneous radii at the contact point of interest of 0.47 in and 0.62 in, respectively. Estimate the maximum contact pressure and the maximum shear stress experienced by either gear.
- 3-140 to 3-142** A wheel of diameter d and width w carrying a load F rolls on a flat rail. Assume that Fig. 3-39, which is based on a Poisson's ratio of 0.3, is applicable to estimate the depth at which the maximum shear stress occurs for these materials. At this critical depth, calculate the Hertzian stresses σ_x , σ_y , σ_z , and $\tau_{\max}$ for the wheel.

Problem Number	d	w	F	Wheel Material	Rail Material
3-140	5 in	2 in	600 lbf	Steel	Steel
3-141	150 mm	40 mm	2 kN	Steel	Cast iron
3-142	3 in	1.25 mm	250 lbf	Cast iron	Cast iron