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Periodic-Switched Filter Networks—A Means of Amplifying and Varying Transfer Functions

JAMES A. KAEHLER, MEMBER, IEEE

Abstract—A technique of periodically switching filter networks makes continuously variable filter parameters possible; at the same time capacitor or time-constant multiplication is obtained. With this method the time constants are multiplied by the switching period to switch aperture-time ratio. Because the aperture time is usually small compared to the switching period, the active elements can be shared.

A typical first-order system is analyzed for its step and sine response by the difference equation method. Phase and gain error expressions are established as a function of the various parameters. Limiting cases are then investigated to give a measure of a pulse-switched system performance.

Finally an application of a multiplexed system is presented where three independent transfer functions are realized using one operational amplifier.

INTRODUCTION

PRESENT methods of mechanizing analog control equations normally use continuous filters to obtain the various transfer functions required for stability. A method of switching network elements on a pulse basis to vary and amplify transfer-function constants will be described and analyzed. The constants become a function

of pulsewidth and can be changed and controlled with the pulsewidth-to-period ratio. This technique is also used to obtain a controlled capacitor multiplication for size reduction in implementing filters with long time constants. Multichannel transfer functions may be realized because amplifiers and other components are used on a pulse basis and can be shared or multiplexed.

Periodically interrupting a series RC network demonstrates the operation of a system whose single-order time constant is multiplied by the ratio of pulse period to pulsewidth.

The apparent amplification of circuit components produced results because the capacitors or storage elements are used only during a portion of a pulse period. Periodically switching these circuit components at a very rapid rate relative to the signal frequency produces quasi-continuous dynamic characteristics.

The analog filter network, in conjunction with a switching technique, has been used for some time and in varied applications. One of the first of these methods to be used and analyzed was the commutated capacitor. This has also been known as the N -path filter [1], commutated network [2], chopper network [3], reverse-switched capacitor [4], and digital filter. The characteristics of these filters have been those of a bandpass around the

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The author is with the Martin-Marietta Corporation, Denver, Colo. 80201.

commutation frequency. This is a characteristic similar to an LC-tuned circuit except the maximum obtainable Q is much higher and the notch placement can be varied as a function of the commutation frequency. The method of periodically interrupting filter branches differs from the above techniques in that it relies on operation with frequencies that are much lower than the switching frequency and it allows the multiplexing of more than one filter.

The problem of fabricating large capacitors in integrated circuits initiated an investigation by Lin *et al.* [5] on RC-time-constant multiplication. The results of this investigation show that resistors and capacitors switched on a pulse basis would yield a multiplication of those elements. This paper was followed by another by Sun and Frisch [6], including a state space analysis of RC networks containing an arbitrary number of periodically operated switches.

In this paper the solution to a typical circuit will be introduced by the difference equation method. The transient analysis of a single-order system will be presented to illustrate the operation of a typical circuit. The frequency response will then be examined by applying a sine input and solving for the output. Although only a single-order system is analyzed, the difference-equation method can be easily applied to the solution of more complex networks.

Once the expression of the output as a function of the circuit and switching parameters is found, it will be compared with an ideal continuous system whose time constants are multiplied by the pulsewidth-to-period ratio. Various limiting cases will be investigated to study the errors as a function of the system parameters. Some of the practical circuit limitations of pulse-switched systems will then be discussed.

The technique of multiplexing several filters and the sharing of active components will be presented along with some experimental circuits that have been tried.

TRANSIENT ANALYSIS

The two circuits shown in Fig. 1 represent continuous and pulse-switched systems. The output of the continuous circuit to a step input is given as

$$E_o(t) = \delta \exp(-w_1 t) + E_s[1 - \exp(-w_1 t)] \quad (1)$$

where

$$w_1 = \frac{1}{RC}$$

$$E_{in}(t) = E_s u(t)$$

δ = initial charge.

The solution (see Appendix) for the output of the switched circuit yields

$$E_o(t) = \delta \exp(-hw_1 t/T) + E_s[1 - \exp(-hw_1 t/T)] \quad (2)$$

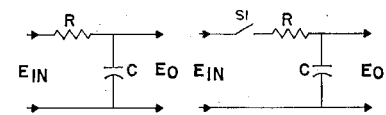


Fig. 1. Continuous and pulse-switched circuits.

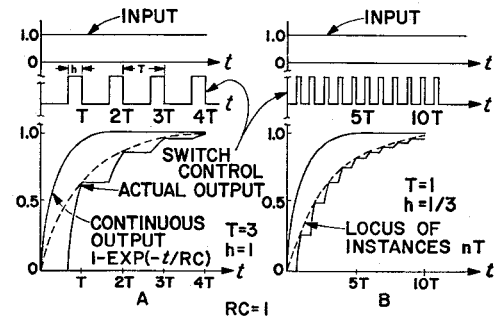


Fig. 2. Comparison of the step responses of a pulsed RC network with different switching periods.

where

h = switch-on time

T = switch period.

Bearing in mind that this equation is only valid at instances nT , it can be seen that the coefficient of t is hw_1/T , while the coefficient of t in (1) is just w_1 . The pulse-switched circuit shows an apparent multiplication of T/h . The output of the switched and unswitched circuits is shown in Fig. 2, and it can be seen that the output of the switched circuit is discontinuous but a continuous locus is formed at instances nT . This locus has the RC time constant multiplied by T/h . Fig. 2(b) shows the effect of increasing the switching frequency and keeping the T/h ratio constant. It can be seen that the same locus was obtained; only the magnitude of the discontinuities was reduced.

The frequency-response analysis will prove that a continuous function is approached by letting the switching frequency go to infinity in the limit.

FREQUENCY RESPONSE

To obtain an insight into how this system behaves in the frequency domain, we will apply a sine input,

$$E_{in}(t) = A \sin w_0 t \quad (3)$$

to a pulse-switched system. Before the response of this system is found, it is necessary to know the response of a continuous first-order system to a translated sine input function

$$E_{in}(t) = A \sin w_0(t + \Delta), \quad (4)$$

where Δ = delay time.

Considering the continuous circuit in Fig. 1, which represents the transfer function, $T(s) = w_1/(s + w_1)$, the output corresponding to the translated input (4) is given by

$$\begin{aligned}
E_0(t) = & \frac{Aw_1}{(w_0^2 + w_1^2)^{1/2}} \\
& \cdot \left\{ -\sin \left[w_0 \Delta - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \exp(-w_1 t) \right. \\
& + \sin \left[w_0 t - w_0 \Delta - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \left. \right\} \\
& + \delta \exp(-w_1 t)
\end{aligned} \quad (5)$$

where

$$w_1 = \frac{1}{RC}$$

δ = initial charge on the capacitor.

This was obtained by taking the LaPlace transform of the input [4], multiplying by the transfer function, and then taking the inverse transform. Now that the translated function [5] is developed, we can proceed with the analysis of the pulse-switched circuit. Considering the input and switching function shown in Fig. 2, we see that the action of the circuit is to hold the output when switch S1 is open. This output is considered the initial charge the next time S1 closes.

Let us now go through a few cycles of operation to derive a general output expression. The output at time $t = T$ is given by

$$\begin{aligned}
E_1 = & \frac{Aw_1}{(w_0^2 + w_1^2)^{1/2}} \\
& \cdot \left\{ -\exp(-w_1 h) \sin \left[w_0 T - w_0 h - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \right. \\
& + \sin \left[w_0 T - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \left. \right\} + \delta \exp(-w_1 h)
\end{aligned} \quad (6)$$

where the input is given by (3).

This expression was obtained by letting the time variable in [5] equal h , the time S1 be actually closed, and $\Delta = T - h$ be the translation time. Solving for the output at $t = 2T$ gives

$$\begin{aligned}
E_2 = & \frac{Aw_1}{(w_0^2 + w_1^2)^{1/2}} \\
& \cdot \left\{ -\exp(-w_1 h) \sin \left[2w_0 T - w_1 h - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \right. \\
& + \sin \left[2w_0 T - \tan^{-1} \left(\frac{w_0}{w_1} \right) \right] \left. \right\} + E_1 \exp(-w_1 h).
\end{aligned} \quad (7)$$

Writing the general output expression at any time nT in terms of the circuit constants, switching parameters, and the output at $(n-1)T$ yields

$$E_n + A_2 E_{n-1} = A_3 \sin nw_0 T + A_4 \cos nw_0 T \quad (8)$$

where the constants A_2 , A_3 , and A_4 are defined in the Appendix. This is a first-order difference equation whose

solution is obtained by equating the right side of (8) to zero and solving the characteristic equation. From the solution to the characteristic equation, we construct a complementary function

$$E_n = B_2(-A_2)^n, \quad (9)$$

where B_2 is an arbitrary constant.

For the nonhomogeneous case, a particular solution is added, and the output obtained is

$$E_n = B_2(-A_2)^n + C \cos nw_0 T + D \sin nw_0 T. \quad (10)$$

By evaluating (10) at instances $n = 0, 1, 2$, and equating it to δ (the output at $n = 0$), (6), and (7), respectively, we have three equations and three unknowns (B_2, C, D). The solution for the quantities B_2, C, D is given in the Appendix.

Simplifying (10) and letting the exponential part $(-A_2)^n$ go to zero, the output is obtained as

$$E_n = (C^2 + D^2)^{1/2} \sin(nw_0 T + \beta_1) \quad (11)$$

where

$$\beta_1 = \tan^{-1} \left(\frac{C}{D} \right). \quad (12)$$

Thus the phase and gain of the pulse-switched system is obtained as

$$\text{phase} = \beta_1 \quad (13)$$

$$\text{gain} = (C^2 + D^2)^{1/2}.$$

Ideally it would be desirable to obtain the transfer function

$$T(s) = \frac{Aw_2}{(w_2^2 + w_0^2)^{1/2}}, \quad (14)$$

where $w_2 = w_1 h / T$. The difference between the ideal function's gain and phase and the actual phase and gain given by (12) and (13) will be used to set up error functions to study various limits.

SINE INPUT-OUTPUT COMPARISON

Before the frequency characteristics are investigated, let us look at the input-output comparison in the time domain. Fig. 3 shows the input, ideal output, and the actual output plotted for some typical parameter values. Actually we have solved only the circuit at instances nT , but because the output is held at a constant value from instances nT to $(n+1)T - h$, we are able to describe two loci that enclose the actual function. The second locus is of the output at instances $(nT - h)$. This function is characterized by the same gain that is given by (11), except the phase is displaced by an amount $w_0(T - h)$ in time from the phase given by (11). The total phase lag of the second locus is given by

$$\beta_3 = \beta_1 - w_0(T - h).$$

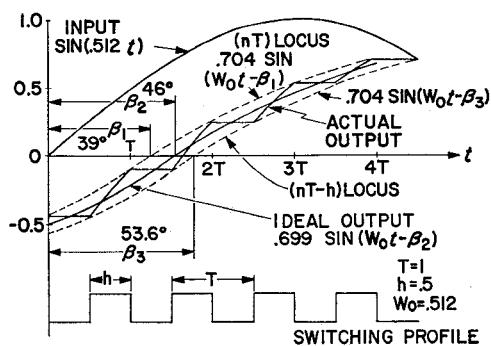


Fig. 3. Sine input-output comparison.

Since the gain of the two loci is the same, we have only one error and it is given by

$$\text{percent error gain} = (100) \frac{\text{ideal gain} - \text{actual gain}}{\text{ideal gain}}. \quad (15)$$

The two phase errors are given by

$$\text{percent error in phase 1} = ER_1 = (100) \frac{\beta_2 - \beta_1}{\beta_2} \quad (16)$$

$$\text{percent error in phase 2} = ER_2 = (100) \frac{\beta_2 - \beta_3}{\beta_2}. \quad (17)$$

These errors are based on differences between the ideal function and the loci of points described by the actual output equations. The phase errors are a measure of the extreme differences or the worst-case deviations and do not necessarily indicate errors that would be seen when applied in a system, because the definition of a phase error can be interpreted in several ways.

Now that the expressions for the errors have been developed, they can be studied as various limits are taken.

Pulse Period Approaches Zero

Fig. 4 shows a Bode plot of the system for some typical switching frequencies. It is seen that for low input frequencies the differences in the actual and ideal systems are undetectable. This case is duplicated by letting the pulse period approach zero, which is the same as letting the switching frequency approach infinity. Indeed, it is found that in the limit as $T \rightarrow 0$, the errors ER_1 , ER_2 , and the gain error all approach zero.

Input Frequency Approaches Zero

From Fig. 4 it appears as if the only requirement for zero errors is to have the input frequency much lower than the other parameters of the system, but by taking the limit of the error functions as the input frequency (w_0) approaches zero, the limits shown in Table I are obtained. This demonstrates that finite phase errors are obtained even though the input frequency is infinitely lower than the other parameters. Taking the iterated limit of the phase errors as $w_0 \rightarrow 0$ and then as $w_1 \rightarrow \infty$ gave the

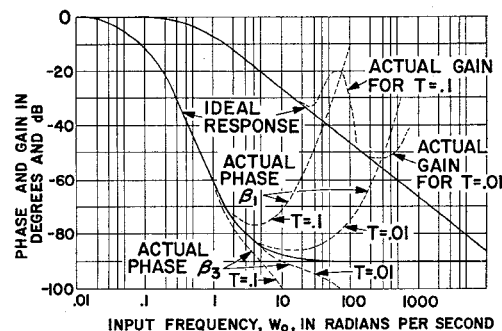


Fig. 4. Comparison of the frequency response of ideal and actual systems.

results shown in the fourth column of Table I. The variations of the phase errors as a function of w_1 for very low input frequencies are shown in Fig. 5. An evaluation of the limits in Table I would show a good correlation to the phase errors in Fig. 5.

Limits as T/h Ratio Approaches Infinity

It has been shown earlier that the T/h ratio multiplied the RC time constant, and therefore it would be desirable to know if there is any analytical limitation to having large ratios. First it is instructive to note that for $T/h = 1$, there are zero errors (see Fig. 6). This is the continuous case where the switch would be closed for the entire period.

By letting $T/h \rightarrow \infty$ in the error equations, the results shown in the fifth column of Table I are obtained. By using some elementary trigonometric substitutions in the gain error limit, it is seen that this function is very small for any w_0T less than one. Accordingly the gain errors are much less sensitive to the relative values of w_0 and T than the phase errors.

Variations in the phase errors as a function of the T/h ratio are plotted in Fig. 6. A numeric substitution of values into the error limits in Table I would correlate well with the errors for large T/h ratios in Fig. 6.

MULTIPLEXING

An illustration of an experimental multiplexed filter circuit is shown in Fig. 7. This particular configuration realizes three independent transfer functions. One operational amplifier is used with input, feedback, and output switches. This amplifier is required to have a fast response relative to the time constants of the various filters. The switches were mechanized using MOSFET's and periodic pulse trains were applied to the FET gates.

The transfer function $T_1(s)$ shown in Fig. 7 is obtained through the multiplied time constant R_4C_1T/h_1 . The analysis of this system is the same as performed earlier on the series RC circuit. The dynamics of function $T_2(s)$ is realized through the combinations of R_4 , R_5 , and C_2 . Finally the third-channel function is obtained through combinations of C_3 , R_4 , and R_6 . A transient analysis of channels 2 and 3 would yield the results that a multiplication of the time constants by the ratio T/h is obtained,

TABLE I
LIMITS OF GAIN AND PHASE ERRORS FOR VARIOUS CASES*

Error Limits				
Error Function	$T \rightarrow 0$	$w_0 \rightarrow 0$	$w_0 \rightarrow 0$ $w_1 \rightarrow \infty$	$\frac{T}{h} \rightarrow \infty$
$100 \frac{\beta_2 - \beta_1}{\beta_2}$	0	$\frac{100 (T - h)}{T} \left[1 - \frac{w_1 h}{\exp(w_1 h) - 1} \right]$	$\frac{100 (T - h)}{T}$	$\frac{100 w_0 T}{\pi}$
$100 \frac{\beta_2 - \beta_3}{\beta_2}$	0	$\frac{100 (T - h)}{T} \left[1 - \frac{w_1 h}{1 - \exp(-w_1 h)} \right]$	$-\infty$	$\frac{-100 w_0 T}{\pi}$
$100 \frac{(\text{ideal gain} - \text{actual gain})}{\text{ideal gain}}$	0	0	0	$\frac{w_0 T}{2 \sin\left(\frac{w_0 T}{2}\right)}$

* w_1 = natural RC frequency, $1/RC$.

w_0 = input frequency.

T = sampling period; h = switch-on time.

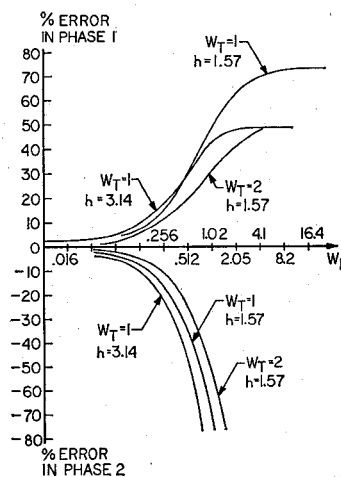


Fig. 5. Percent errors in phase versus w_1 .

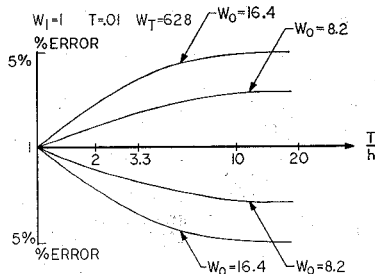


Fig. 6. Phase errors as a function of the T/h ratio.

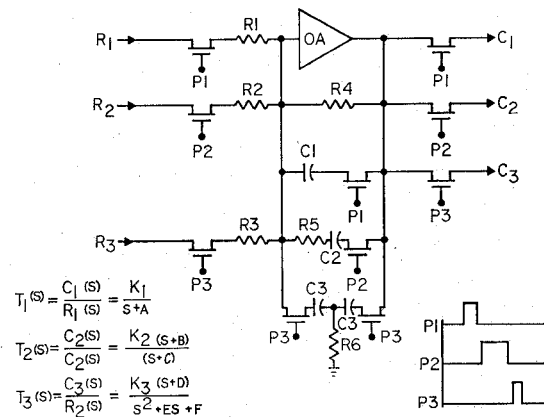


Fig. 7. Multiplexed filter circuit.

CONCLUSIONS

The pulse-switched single-order lag has been analyzed for the response to a step and sine input. The results of a step input showed that the RC time constant was multiplied by the pulse period-width ratio. The sine response shows that a quasi-continuous system was approached when the pulse repetition frequency was an order of magnitude larger than the input and RC frequencies. The sine analysis produced expressions for the phase and gain errors as functions of the various parameters. It was shown that the pulse frequency being larger than just the input frequency would not necessarily be sufficient for minimum errors. The relative values of the pulse and RC frequencies are also important in considering errors.

For large pulse period-width ratios, the phase errors become directly proportional to the product of the input frequency and the pulse period; the gain errors are generally negligible for any respectable phase errors. Large pulse period-width ratios mean large RC time constant multiplication factors, and therefore, probably the most practical condition to be analyzed. Also this condition allows maximum sharing of the nonstorage components since the on time is the smallest.

where h is the aperture time of the particular switches identified with the channel.

Various pulse-switched networks have been built that realize independent transfer functions whose break frequencies or time constants are controlled by the T/h ratio. The characteristics of these channels have been similar to those of a continuous system and the phase- and gain-measuring devices have not been able to detect any applicable errors.

Most transfer functions can be synthesized using the single-order lags in conjunction with feedback loops. Thus, we have a way of mechanizing a general transfer function whose characteristics are pulse controlled and errors analytic.

It has been found empirically that periodically switching components in higher order filter networks produce the same multiplication effects as in single-order systems.

There are three distinct advantages to periodically switching filter network components: 1) capacitor multiplication; 2) transfer-function control; and 3) multiplexing or sharing of components.

APPENDIX

Definitions

$$A_1 = \frac{Aw_1}{(w_0^2 + w_1^2)^{1/2}}$$

$$A_2 = -\exp(-w_1 h)$$

$$A_3 = (A_2 \cos \phi + \cos \theta) A_1$$

$$A_4 = (A_2 \sin \phi + \sin \theta) A_1$$

$$\theta = -\tan^{-1} \frac{w_0}{w_1}$$

$$\phi = -hw_0 - \tan^{-1} \left(\frac{w_0}{w_1} \right)$$

$$w_T = \frac{2\pi}{T}$$

$$C = \frac{A_4 + A_2 A_3 \sin w_0 T + A_2 A_4 \cos w_0 T}{1 + A_2^2 + 2A_2 \cos w_0 T}$$

$$D = \frac{A_3 + A_2 A_3 \cos w_0 T - A_2 A_4 \sin w_0 T}{1 + A_2^2 + 2A_2 \cos w_0 T}$$

Transient Analysis

Referring to Fig. 1, it can be seen that the output of the switched circuit is discontinuous. By inspection the output can be expressed at any instance nT as a function of the previous instance $(n-1)T$ as

$$E_n - E_{n-1} \exp(-hw_1) = E_s[1 - \exp(-hw_1)] \quad (18)$$

where h and T are defined in Fig. 2. This is a first-order nonhomogeneous difference equation whose solution is ob-

tained by equating the left side of (18) to zero and solving the characteristic equation. From the solution to the characteristic equation we construct a complementary function,

$$E_n = B_1 \exp(-hw_1 n). \quad (19)$$

Since the right-hand side of (18) is a constant, a constant is added to the complementary function to obtain a particular solution,

$$E_n = B_1 \exp(-hw_1 n) + C. \quad (20)$$

Now all that needs to be done is to solve for the arbitrary constants (B_1 , C) by equating (20) evaluated at instances $n = 0, 1$ to (18) evaluated at $n = 0, 1$. This gives two equations and two unknowns from which we obtain the final solution as

$$E_n = \delta \exp(-hw_1 n) + E_s[1 - \exp(-hw_1 n)]. \quad (21)$$

Recalling that the time variable t is related to n by

$$t = nt \quad \text{or} \quad n = t/T. \quad (22)$$

Substituting (22) into (21) for n we have

$$E_o(t) = \delta \exp(-hw_1 t/T) + E_s[1 - \exp(-hw_1 t/T)]. \quad (23)$$

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