

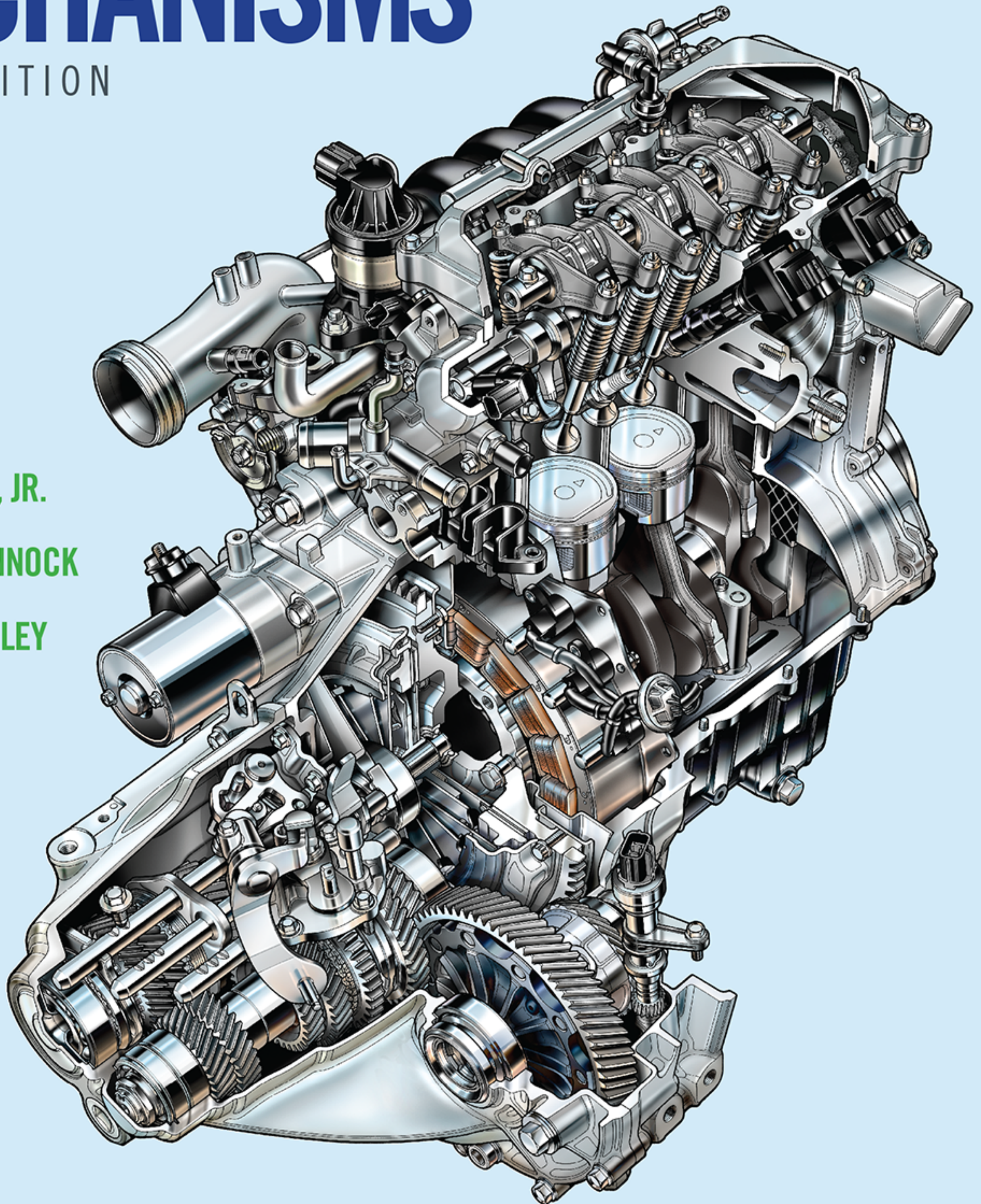
THEORY OF MACHINES AND MECHANISMS

FIFTH EDITION

JOHN J. VICKER, JR.

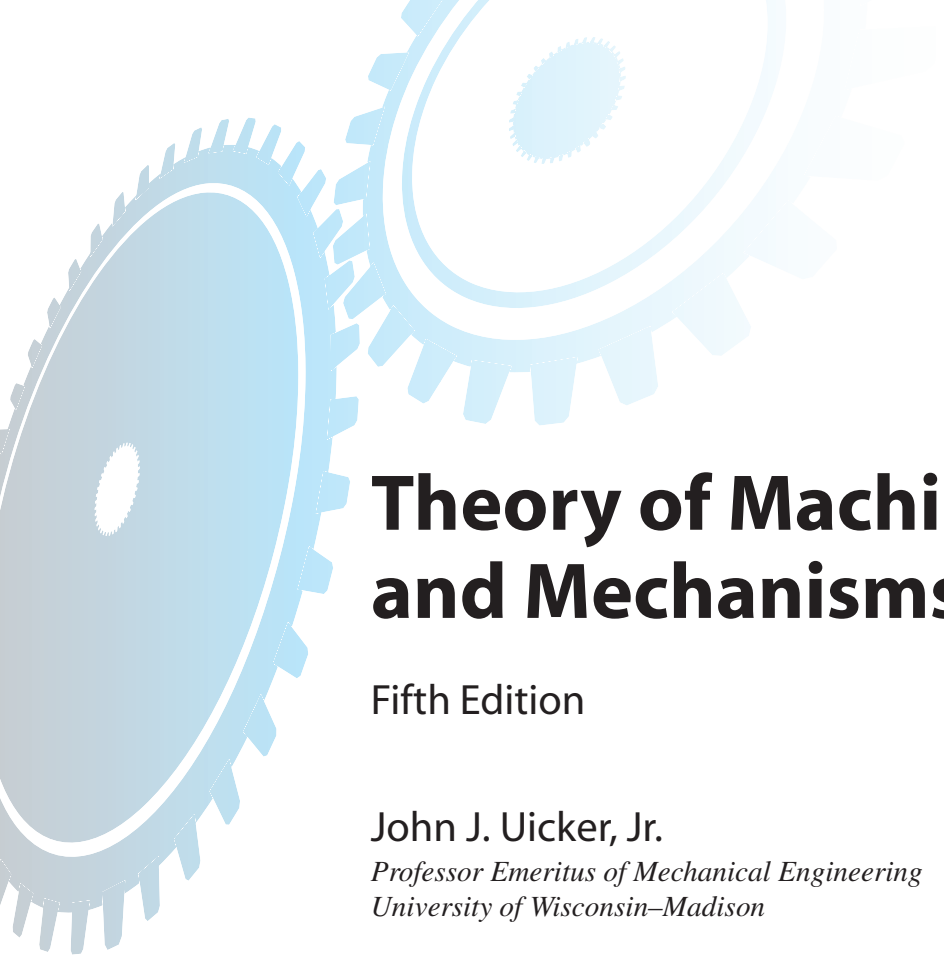
GORDON R. PENNOCK

JOSEPH E. SHIGLEY



OXFORD
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Theory of Machines and Mechanisms



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Fifth Edition

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University of Wisconsin–Madison*

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This textbook is dedicated to the memory of my parents, John J. Uicker, Emeritus Dean of Engineering, University of Detroit, Elizabeth F. Uicker, and to my six children, Theresa A. Zenchenko, John J. Uicker III, Joseph M. Uicker, Dorothy J. Winger, Barbara A. Peterson, and Joan E. Horne.

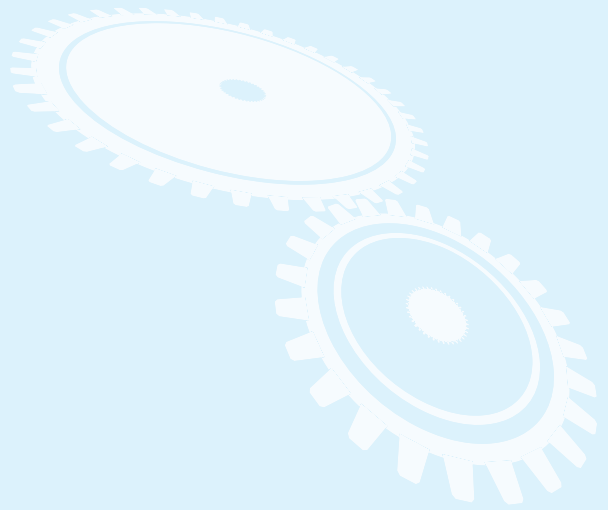
—**John J. Uicker, Jr.**

This work is also dedicated first and foremost to my wife, Mollie B., and my son, Callum R. Pennock. The work is also dedicated to my friend and mentor, the late Dr. An (Andy) Tzu Yang, and my colleagues in the School of Mechanical Engineering, Purdue University, West Lafayette, Indiana.

—**Gordon R. Pennock**

Finally, this text is dedicated to the memory of the late **Joseph E. Shigley**, Professor Emeritus, Mechanical Engineering Department, University of Michigan, Ann Arbor. Although this fifth edition contains significant changes from earlier editions, the text remains consistent with his previous writings.

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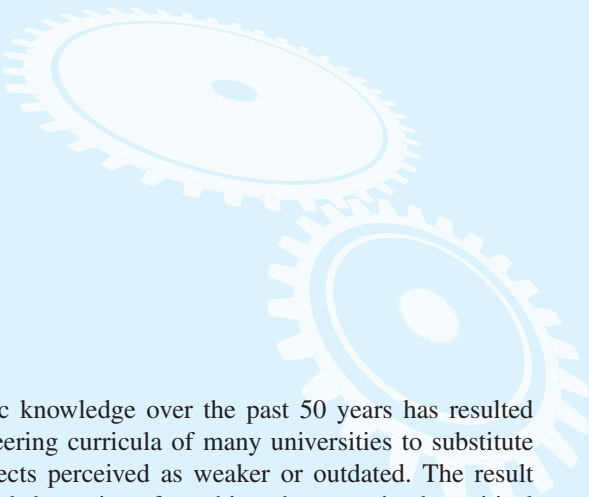
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Preface

The tremendous growth of scientific knowledge over the past 50 years has resulted in an intense pressure on the engineering curricula of many universities to substitute “modern” subjects in place of subjects perceived as weaker or outdated. The result is that, for some, the kinematics and dynamics of machines has remained a critical component of the curriculum and a requirement for all mechanical engineering students, while at others, a course on these subjects is only made available as an elective topic for specialized study by a small number of engineering students. Some schools, depending largely on the faculty, require a greater emphasis on mechanical design at the expense of depth of knowledge in analytical techniques. Rapid advances in technology, however, have produced a need for a textbook that satisfies the requirement of new and changing course structures.

Much of the new knowledge in the theory of machines and mechanisms currently exists in a large variety of technical journals and manuscripts, each couched in its own singular language and nomenclature and each requiring additional background for clear comprehension. It is possible that the individual published contributions could be used to strengthen engineering courses if the necessary foundation was provided and a common notation and nomenclature was established. These new developments could then be integrated into existing courses to provide a logical, modern, and comprehensive whole. The purpose of this book is to provide the background that will allow such an integration.

This book is intended to cover that field of engineering theory, analysis, design, and practice that is generally described as mechanisms or as kinematics and dynamics of machines. Although this text is written primarily for students of mechanical engineering, the content can also be of considerable value to practicing engineers throughout their professional careers.

To develop a broad and basic comprehension, the text presents numerous methods of analysis and synthesis that are common to the literature of the field. The authors have included graphic methods of analysis and synthesis extensively throughout the book, because they are firmly of the opinion that graphic methods provide visual feedback that enhances the student’s understanding of the basic nature of, and interplay between, the underlying equations. Therefore, graphic methods are presented as one possible solution technique, but are always accompanied by vector equations defined by the fundamental laws of mechanics, rather than as graphic “tricks” to be learned by rote and applied blindly. In addition, although graphic techniques, performed by hand, may lack accuracy, they can be performed quickly, and even inaccurate sketches can often provide reasonable estimates of a solution and can be used to check the results of analytic or numeric solution techniques.

The authors also use conventional methods of vector analysis throughout the book, both in deriving and presenting the governing equations and in their solution. Raven's methods using complex algebra for the solution of two-dimensional vector equations are included because of their compactness, because of the ease of taking derivatives, because they are employed so frequently in the literature, and because they are so easy to program for computer evaluation. In the chapter dealing with three-dimensional kinematics and robotics, the authors present a brief introduction to Denavit and Hartenberg's methods using transformation matrices.

Another feature of this text is its focus on the method of kinematic coefficients, which are derivatives of motion variables with respect to the input position variable(s) rather than with respect to time. The authors believe that this analytic technique provides several important advantages, namely: (1) Kinematic coefficients clarify for the student those parts of a motion problem that are kinematic (geometric) in their nature, and clearly separate these from the parts that are dynamic or speed dependent. (2) Kinematic coefficients help to integrate the analysis of different types of mechanical systems, such as gears, cams, and linkages, which might not otherwise seem similar.

One dilemma that all writers on the subject of this book have faced is how to distinguish between the motions of different points of the same moving body and the motions of coincident points of different moving bodies. In other texts, it has been customary to describe both of these as "relative motion"; however, because they are two distinctly different situations and are described by different equations, this causes the student confusion in distinguishing between them. We believe that we have greatly relieved this problem by the introduction of the terms *motion difference* and *apparent motion* and by using different terminology and different notation for the two cases. Thus, for example, this book uses the two terms *velocity difference* and *apparent velocity*, instead of the term "relative velocity," which will not be found when speaking rigorously. This approach is introduced beginning with position and displacement, used extensively in the chapter on velocity, and brought to fulfillment in the chapter on accelerations, where the Coriolis component *always* arises in, and *only* arises in, the apparent acceleration equation.

Access to personal computers, programmable calculators, and laptop computers is commonplace and is of considerable importance to the material of this book. Yet engineering educators have told us very forcibly that they do not want computer programs included in the text. They prefer to write their own programs, and they expect their students to do so as well. Having programmed almost all the material in the book many times, we also understand that the book should not include such programs and thus become obsolete with changes in computers or programming languages.

The authors have endeavored to use US Customary units and SI units in about equal proportions throughout the book. However, there are certain exceptions. For example, in Chapter 14 (Dynamics of Reciprocating Engines), only SI units are presented, because engines are designed for an international marketplace, even by US companies. Therefore, they are always rated in kilowatts rather than horsepower, they have displacements in liters rather than cubic inches, and their cylinder pressures are measured in kilopascals rather than pounds per square inch.

Part 1 of this book deals mostly with theory, nomenclature, notation, and methods of analysis. Serving as an introduction, Chapter 1 tells what a mechanism is, what

a mechanism can do, how mechanisms can be classified, and what some of their limitations are. Chapters 2, 3, and 4 are concerned totally with analysis, specifically with kinematic analysis, because they cover position, velocity, and acceleration analyses, respectively, of single-degree-of-freedom planar mechanisms. Chapter 5 expands this background to include multi-degree-of-freedom planar mechanisms.

Part 2 of the book goes on to demonstrate engineering applications involving the selection, the specification, the design, and the sizing of mechanisms to accomplish specific motion objectives. This part includes chapters on cam systems, gears, gear trains, synthesis of linkages, spatial mechanisms, and an introduction to robotics. Chapter 6 is a study of the geometry, kinematics, proper design of high-speed cam systems, and now includes material on the dynamics of elastic cam systems. Chapter 7 studies the geometry and kinematics of spur gears, particularly of involute tooth profiles, their manufacture, and proper tooth meshing, and then studies gear trains, with an emphasis on epicyclic and differential gear trains. Chapter 8 expands this background to include helical gears, bevel gears, worms, and worm gears. Chapter 9 is an introduction to the kinematic synthesis of planar linkages. Chapter 10 is a brief introduction to the kinematic analysis of spatial mechanisms and robotics, including the forward and inverse kinematics problems.

Part 3 of the book adds the dynamics of machines. In a sense, this part is concerned with the consequences of the mechanism design specifications. In other words, having designed a machine by selecting, specifying, and sizing the various components, what happens during the operation of the machine? What forces are produced? Are there any unexpected operating results? Will the proposed design be satisfactory in all respects? Chapter 11 presents the static force analysis of machines. This chapter also includes sections focusing on the buckling of two-force members subjected to axial loads. Chapter 12 studies the planar and spatial aspects of the dynamic force analysis of machines. Chapter 13 then presents the vibration analysis of mechanical systems. Chapter 14 is a more detailed study of one particular type of mechanical system, namely the dynamics of both single- and multi-cylinder reciprocating engines. Chapter 15 next addresses the static and dynamic balancing of rotating and reciprocating systems. Finally, Chapter 16 is on the study of the dynamics of flywheels, governors, and gyroscopes.

As with all texts, the subject matter of this book also has limitations. Probably the clearest boundary on the coverage in this text is that it is limited to the study of rigid-body mechanical systems. It does study planar multibody systems with movable connections or constraints between them. However, all motion effects are assumed to come within the connections; the shapes of the individual bodies are assumed constant, except for the dynamics of elastic cam systems. This assumption is necessary to allow the separate study of kinematic effects from those of dynamics. Because each individual body is assumed rigid, it can have no strain; therefore, except for buckling of axially loaded members, the study of stress is also outside the scope of this text. It is hoped, however, that courses using this text can provide background for the later study of stress, strength, fatigue life, modes of failure, lubrication, and other aspects important to the proper design of mechanical systems.

Despite the limitations on the scope of this book, it is still clear that it is not reasonable to expect that all of the material presented here can be covered in a single-semester

course. As stated above, a variety of methods and applications have been included to allow the instructor to choose those topics that best fit the course objectives and to still provide a reference for follow-on courses and help build the student's library. Yet, many instructors have asked for suggestions regarding a choice of topics that might fit a 3-hour per week, 15-week course. Two such outlines follow, as used by two of the authors to teach such courses at their institutions. It is hoped that these might be used as helpful guidelines to assist others in making their own parallel choices.

Tentative Schedule I

Kinematics and Dynamics of Machine Systems

Week	Topics	Sections
1	Introduction to Mechanisms	1.1–1.10
	Kutzbach and Grashof Criteria	1.6, 1.9
	Advance-to-Return Time Ratio	1.7
	Overlay Method of Synthesis	9.8
2	Vector Loop-Closure Equation	2.6, 2.7
	Velocity Difference Equation	3.1–3.3
	Velocity Polygons; Velocity Images	3.4
3	Apparent Velocity Equation	3.5, 3.6, 3.8
	Direct and Rolling Contact Velocity	3.7
4	Instantaneous Centers of Velocity	3.12
	Aronhold–Kennedy Theorem of Three Centers	3.13, 3.14
	Use of Instant Centers to Find Velocities	3.15, 3.16
5	Exam #1	
	Acceleration Difference Equation	4.1–4.3
	Acceleration Polygons; Acceleration Images	4.4
6	Apparent Acceleration Equation	4.5, 4.6
	Coriolis Component of Acceleration	
7	Direct and Rolling Contact Acceleration	4.7, 4.8
	Review of Velocity and Acceleration Analyses	
8	Raven's Method of Kinematic Analysis	2.10, 3.10, 4.10
	Kinematic Coefficients	3.11, 4.11
	Computer Methods in Kinematics	10.9

9	Exam #2	
	Static Forces	11.1–11.6
	Two-, Three-, and Four-Force Members	11.7, 11.8
	Force Polygons	
10	Coulomb Friction Forces in Machines	11.9, 11.10
11	D'Alembert's Principle	12.1–12.4
	Dynamic Forces in Machine Members	12.4, 12.5
12	Introduction to Cam Design	6.1–6.4
	Choice of Cam Profiles; Matching Displacement Curves	6.5–6.8
13	First-Order Kinematic Coefficients; Face Width; Pressure Angle	6.9
	Second-Order Kinematic Coefficients; Pointing and Undercutting	6.10
14	Exam #3	
	Introduction to Gearing	7.1–7.6
	Involute Tooth Geometry; Contact Ratio; Undercutting	7.7–7.9, 7.11
15	Epicyclic and Differential Gear Trains	7.15–7.17
	Review	
	Final Exam	

Tentative Schedule II

Machine Design I

Week	Topics	Sections
1	The World of Mechanisms	1.1–1.6
	Measures of Performance (Indices of Merit)	1.10, 3.19
	Quick Return Mechanisms	1.7
2	Position Analysis. Vector Loops	2.1–2.7
	Newton–Raphson Technique	2.8, 2.11
3	Velocity Analysis	3.1–3.9
	First-Order Kinematic Coefficients	3.11
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Supplement packages for this fifth edition have been designed to support both the student and the instructor in the kinematics and dynamics course. The Companion Website (<http://www.oup.com/us/uicker>) will include a list of any errors discovered in the text and their corrections. This website also includes over 100 animations of key figures from the text; these are marked with a  symbol in the text. These animations, created by Zhong Hu of South Dakota State University, are presented in both Working Model and .avi file formats, and are meant to help students visualize and comprehend the movement of important mechanisms.

An Ancillary Resource Center site is available for instructors only (registration is required). A complete solutions manual for all problems is available on that site. Solutions are also available on that site for 100 problems in the text worked out using MatLab software, for instructors wishing to incorporate MatLab code into their courses. Problems marked with a † signify that there is a MatLab-based solution available on that site; thank you to Bob Williams at Ohio University for his help with those solutions.

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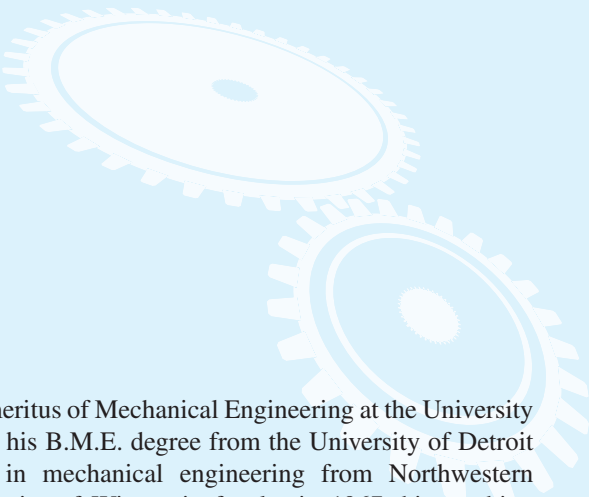
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*John J. Uicker, Jr.
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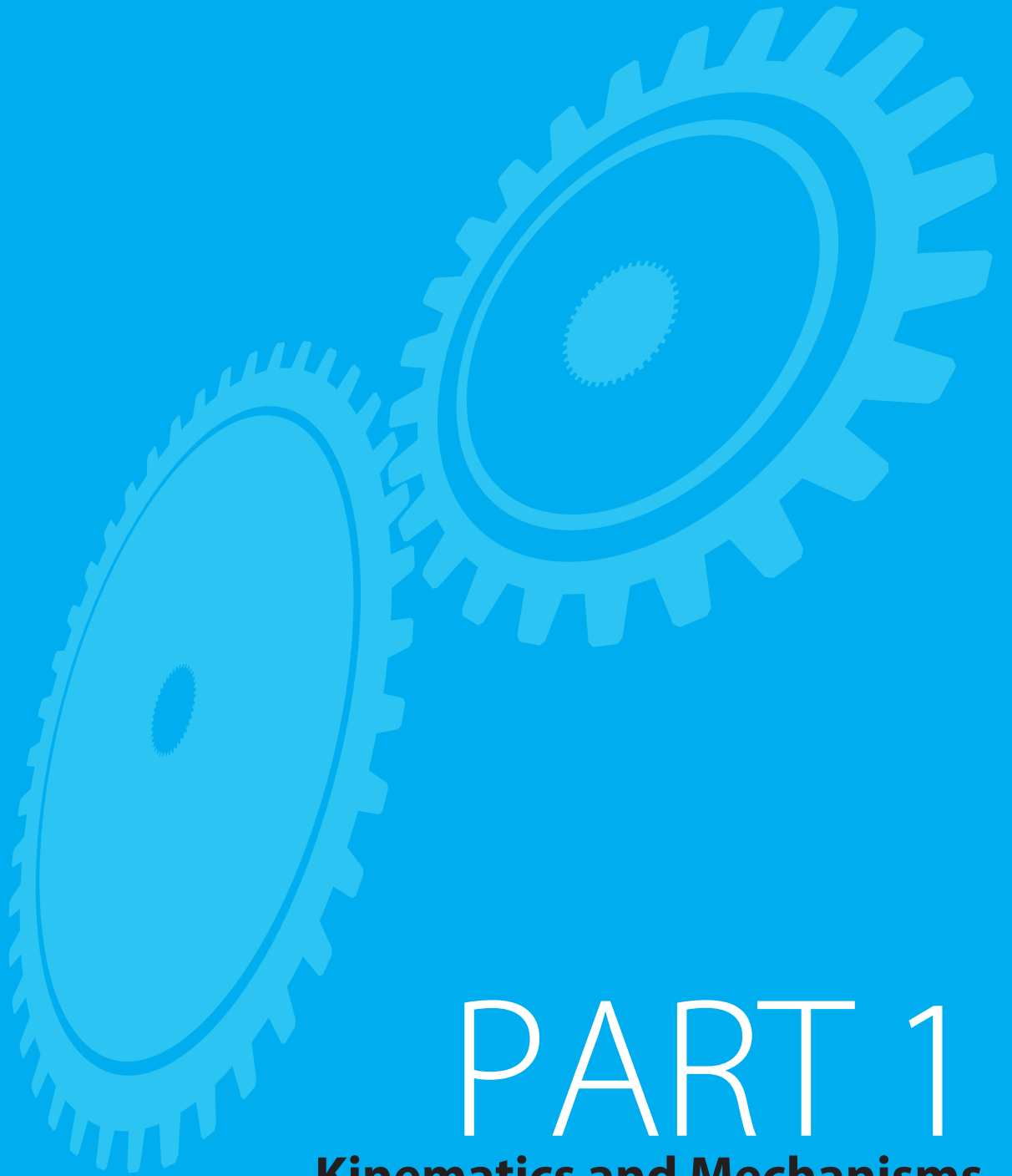
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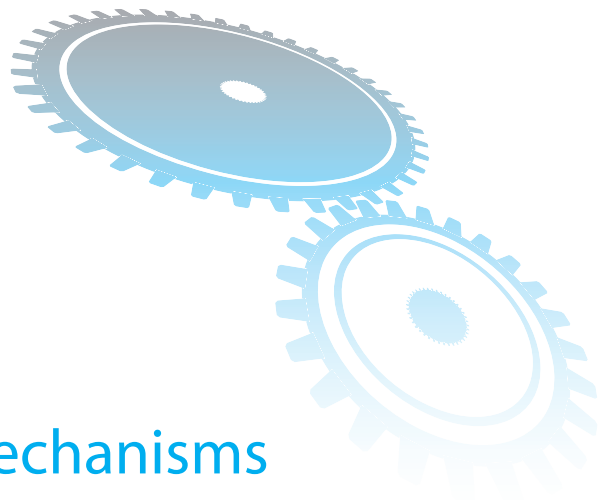
Joseph E. Shigley (deceased May 1994) was Professor Emeritus of Mechanical Engineering at the University of Michigan and a fellow in the ASME. He received the Mechanisms Committee Award in 1974, the Worcester Reed Warner medal in 1977, and the Machine Design Award in 1985. He was author of eight books, including *Mechanical Engineering Design* (with Charles R. Mischke) and *Applied Mechanics of Materials*. He was coeditor-in-chief of the *Standard Handbook of Machine Design*. He first wrote *Kinematic Analysis of Mechanisms* in 1958 and then wrote *Dynamic Analysis of Machines* in 1961, and these were published in a single volume titled *Theory of Machines* in 1961; they have evolved over the years to become the current text, *Theory of Machines and Mechanisms*, now in its fifth edition.

He was awarded the B.S.M.E. and B.S.E.E. degrees of Purdue University and received his M.S. at the University of Michigan. After several years in industry, he devoted his career to teaching, writing, and service to his profession, first at Clemson University and later at the University of Michigan. His textbooks have been widely used throughout the United States and internationally.



PART 1

Kinematics and Mechanisms



1

The World of Mechanisms

1.1 INTRODUCTION

The theory of machines and mechanisms is an applied science that is used to understand the relationships between the geometry and motions of the parts of a machine, or mechanism, and the forces that produce these motions. The subject, and therefore this book, divides itself naturally into three parts. Part 1, which includes Chaps. 1 through 5, is concerned with mechanisms and the kinematics of mechanisms, which is the analysis of their motions. Part 1 lays the groundwork for Part 2, comprising Chaps. 6 through 10, in which we study methods of designing mechanisms. Finally, in Part 3, which includes Chaps. 11 through 16, we take up the study of kinetics, the time-varying forces in machines and the resulting dynamic phenomena that must be considered in their design.

The design of a modern machine is often very complex. In the design of a new engine, for example, the automotive engineer must deal with many interrelated questions. What is the relationship between the motion of the piston and the motion of the crankshaft? What are the sliding velocities and the loads at the lubricated surfaces, and what lubricants are available for this purpose? How much heat is generated, and how is the engine cooled? What are the synchronization and control requirements, and how are they satisfied? What is the cost to the consumer, both for initial purchase and for continued operation and maintenance? What materials and manufacturing methods are used? What are the fuel economy, noise, and exhaust emissions; do they meet legal requirements? Although all these and many other important questions must be answered before the design is completed, obviously not all can be addressed in a book of this size. Just as people with diverse skills must be brought together to produce an adequate design, so too must many branches of science be brought together. This book assembles material that falls into the science of mechanics as it relates to the design of mechanisms and machines.

1.2 ANALYSIS AND SYNTHESIS

There are two completely different aspects of the study of mechanical systems: *design* and *analysis*. The concept embodied in the word “design” is more properly termed *synthesis*, the process of contriving a scheme or a method of accomplishing a given purpose. Design is the process of prescribing the sizes, shapes, material compositions, and arrangements of parts so that the resulting machine will perform the prescribed task.

Although there are many phases in the design process that can be approached in a well-ordered, scientific manner, the overall process is by its very nature as much an art as a science. It calls for imagination, intuition, creativity, judgment, and experience. The role of science in the design process is merely to provide tools to be used by designers as they practice their art.

In the process of evaluating the various interacting alternatives, designers find a need for a large collection of mathematical and scientific tools. These tools, when applied properly, provide more accurate and more reliable information for judging a design than one achieves through intuition or estimation. Thus, the tools are of tremendous help in deciding among alternatives. However, scientific tools cannot make decisions for designers; designers have every right to exert their imagination and creative abilities, even to the extent of overruling the mathematical recommendations.

Probably the largest collection of scientific methods at the designer’s disposal fall into the category called *analysis*. These are techniques that allow the designer to critically examine an already existing, or proposed, design to judge its suitability for the task. Thus, analysis in itself is not a creative science but one of evaluation and rating things already conceived.

We should bear in mind that, although most of our effort may be spent on analysis, the real goal is synthesis: the design of a machine or system. Analysis is simply a tool; however, it is a vital tool and will inevitably be used as one step in the design process.

1.3 SCIENCE OF MECHANICS

The branch of scientific analysis that deals with motions, time, and forces is called *mechanics* and is made up of two parts: statics and dynamics. *Statics* deals with the analysis of stationary systems—that is, those in which time is not a factor—and *dynamics* deals with systems that change with time.

As shown in Fig. 1.1, dynamics is also made up of two major disciplines, first recognized as separate entities by Euler* in 1765 [2]:†

The investigation of the motion of a rigid body may be conveniently separated into two parts, the one, geometrical, and the other mechanical. In the first part, the transference of the body from a given position to any other position must be investigated without respect to the causes of the motion, and must be represented by analytical formulae, which will define the position of each point of the body. This

* Leonhard Euler (1707–1783).

† Numbers in square brackets refer to references at the end of each chapter.

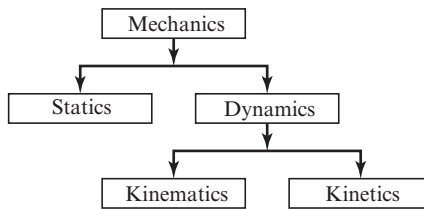


Figure 1.1

investigation will therefore be referable solely to geometry, or rather to stereotomy [the art of stonecutting, now referred to as descriptive geometry].

It is clear that by the separation of this part of the question from the other, which belongs properly to Mechanics, the determination of the motion from dynamical principles will be made much easier than if the two parts were undertaken conjointly.

These two aspects of dynamics were later recognized as the distinct sciences of *kinematics* (*cinématique* was a term coined by Ampère* and derived from the Greek word *kinema*, meaning motion) and *kinetics* and deal with motion and the forces producing the motion, respectively.

The initial problem in the design of a mechanical system, therefore, is understanding the kinematics. *Kinematics* is the study of motion, quite apart from the forces that produce the motion. In particular, kinematics is the study of position, displacement, rotation, speed, velocity, acceleration, and jerk. The study, say, of planetary or orbital motion is also a problem in kinematics, but in this book we shall concentrate our attention on kinematic problems that arise in the design and operation of mechanical systems. Thus, the kinematics of machines and mechanisms is the focus of the next several chapters of this book. In addition, statics and kinetics are also vital parts of a complete design analysis, and they are also covered in later chapters.

It should be carefully noted in the previous quotation that Euler based his separation of dynamics into kinematics and kinetics on the assumption that they deal with *rigid* bodies. It is this very important assumption that allows the two to be treated separately. For flexible bodies, the shapes of the bodies themselves, and therefore their motions, depend on the forces exerted on them. In this situation, the study of force and motion must take place simultaneously, thus significantly increasing the complexity of the analysis.

Fortunately, although all real machine parts are flexible to some degree, machines are usually designed from relatively rigid materials, keeping part deflections to a minimum. Therefore, it is common practice to assume that deflections are negligible and parts are rigid while analyzing a machine's kinematic performance and then, during dynamic analysis when loads are sought, to design the parts so that the assumption is justified. A more detailed discussion of a rigid body compared to a deformable, or flexible, body is presented in the introduction to static force analysis in Sec. 11.1.

* André-Marie Ampère (1775–1836).

1.4 TERMINOLOGY, DEFINITIONS, AND ASSUMPTIONS

Reuleaux* defines a *machine*[†] as a “combination of resistant bodies so arranged that by their means the mechanical forces of nature can be compelled to do work accompanied by certain determinate motions.” He also defines a *mechanism* as an “assemblage of resistant bodies, connected by movable joints, to form a closed kinematic chain with one link fixed and having the purpose of transforming motion.”

Some light can be shed on these definitions by contrasting them with the term *structure*. A structure is also a combination of resistant (rigid) bodies connected by joints, but the purpose of a structure (such as a truss) is not to do work or to transform motion, but to be rigid. A truss can perhaps be moved from place to place and is movable in this sense of the word; however, it has no *internal* mobility. A structure has no *relative motions* between its various links, whereas both machines and mechanisms do. Indeed, the whole purpose of a machine or mechanism is to utilize these relative internal motions in transmitting power or transforming motion.

A machine is an arrangement of parts for doing work, a device for applying power or changing the direction of motion. It differs from a mechanism in its purpose. In a machine, terms such as force, torque, work, and power describe the predominant concepts. In a mechanism, though it may transmit power or force, the predominant idea in the mind of the designer is one of achieving a desired motion. There is a direct analogy between the terms structure, mechanism, and machine and the branches of mechanics illustrated in Fig. 1.1. The term “structure” is to statics as the term “mechanism” is to kinematics and as the term “machine” is to kinetics.

We use the word *link* to designate a machine part or a component of a mechanism. As discussed in the previous section, a link is assumed to be completely rigid. Machine components that do not fit this assumption of rigidity, such as springs, usually have no effect on the kinematics of a device but do play a role in supplying forces. Such parts or components are not called links; they are usually ignored during kinematic analysis, and their force effects are introduced during force analysis (see the analysis of buckling in Secs. 11.14–11.18). Sometimes, as with a belt or chain, a machine part may possess one-way rigidity; such a body can be considered a link when in tension but not under compression.

The links of a mechanism must be connected in some manner in order to transmit motion from the *driver*, or input, to the *driven*, or *follower*, or output. The connections, the joints between the links, are called *kinematic pairs* (or simply *pairs*), because each joint consists of a pair of mating surfaces, two elements, one mating surface or element being a part of each of the joined links. Thus, we can also define a link as *the rigid connection between two or more joint elements*.

Stated explicitly, the assumption of rigidity is that there can be no relative motion (no change in distance) between two arbitrarily chosen points on the same link. In particular,

* Much of the material of this section is based on definitions originally set down by Franz Reuleaux (1829–1905), a German kinematician whose work marked the beginning of a systematic treatment of kinematics [7].

† There appears to be no agreement at all on the proper definition of a machine. In a footnote Reuleaux gives 17 definitions, and his translator gives 7 more and discusses the whole problem in detail [7].

the relative positions of joint elements on any given link do not change no matter what loads are applied. In other words, the purpose of a link is to hold a constant spatial relationship between its joint elements.

As a result of the assumption of rigidity, many of the intricate details of the actual part shapes are unimportant when studying the kinematics of a machine or mechanism. For this reason, it is common practice to draw highly simplified schematic diagrams that contain important features of the shape of each link, such as the relative locations of joint elements, but that completely subdue the real geometry of the manufactured part. The slider-crank linkage of the internal combustion engine, for example, can be simplified for purposes of analysis to the schematic diagram illustrated later in Fig. 1.3*b*. Such simplified schematics are a great help since they eliminate confusing factors that do not affect the analysis; such diagrams are used extensively throughout this text. However, these schematics also have the drawback of bearing little resemblance to physical hardware. As a result they may give the impression that they represent only academic constructs rather than real machinery. We should continually bear in mind that these simplified diagrams are intended to carry only the minimum necessary information so as not to confuse the issue with unimportant detail (for kinematic purposes) or complexity of the true machine parts.

When several links are connected together by joints, they are said to form a *kinematic chain*. Links containing only two joint elements are called *binary* links, those having three joint elements are called *ternary* links, those having four joint elements are called *quaternary* links, and so on. If every link in a chain is connected to at least two other links, the chain forms one or more closed loops and is called a *closed* kinematic chain; if not, the chain is referred to as *open*. If a chain consists entirely of binary links, it is a *simple-closed* chain. *Compound-closed* chains, however, include other than binary links and thus form more than a single closed loop.

Recalling Reuleaux's definition of a mechanism, we see that it is necessary to have a closed kinematic chain *with one link fixed*. When we say that one link is fixed, we mean that it is chosen as the frame of reference for all other links; that is, the motions of all points on the links of the mechanism are measured with respect to the fixed link. This link, in a practical machine, usually takes the form of a stationary platform or base (or a housing rigidly attached to such a base) and is commonly referred to as the *ground*, *frame*, or *base link*.^{*} The question of whether this reference frame is truly stationary (in the sense of being an inertial reference frame) is immaterial in the study of kinematics, but becomes important in the investigation of kinetics, where forces are considered. In either case, once a frame link is designated (and other conditions are met), the kinematic chain becomes a mechanism and, as the driver is moved through various positions, all other links have well-defined motions with respect to the chosen frame of reference. We use the term *kinematic chain* to specify a particular arrangement of links and joints when it is not clear which link is to be treated as the frame. When the frame link is specified, the kinematic chain is called a *mechanism*.

For a mechanism to be useful, the motions between links cannot be completely arbitrary; they too must be constrained to produce the proper relative motions—those chosen by the designer for the particular task to be performed. These desired relative

^{*} In this text, the ground, frame, or base of the mechanism is commonly numbered 1.

motions are achieved by proper choice of the number of links and the kinds of joints used to connect them. Thus we are led to the concept that, in addition to the distances between successive joints, the nature of the joints themselves and the relative motions they permit are essential in determining the kinematics of a mechanism. For this reason, it is important to look more closely at the nature of joints in general terms, and in particular at several of the more common types.

The controlling factors that determine the relative motions allowed by a given joint are the shapes of the mating surfaces or elements. Each type of joint has its own characteristic shapes for the elements, and each allows a given type of motion, which is determined by the possible ways in which these elemental surfaces can move with respect to each other. For example, the pin joint in Fig. 1.2*a*, has cylindric elements, and, assuming that the links cannot slide axially, these surfaces permit only relative rotational motion. Thus a pin joint allows the two connected links to experience relative rotation about the pin center. So, too, other joints each have their own characteristic element shapes and relative motions. These shapes restrict the totally arbitrary motion of two unconnected links to some prescribed type of relative motion and form constraining conditions (constraints) on the mechanism's motion.

It should be pointed out that the element shapes may often be subtly disguised and difficult to recognize. For example, a pin joint might include a needle bearing, so that two mating surfaces, as such, are not distinguishable. Nevertheless, if the motions of the individual rollers are not of interest, the motions allowed by the joints are equivalent, and the joints are of the same generic type. Thus the criterion for distinguishing different joint types is the relative motions they permit and not necessarily the shapes of the elements, though these may provide vital clues. The diameter of the pin used (or other dimensional data) is also of no more importance than the exact sizes and shapes of the connected links. As stated previously, the kinematic function of a link is to hold a fixed geometric relationship between the joint elements. Similarly, the only kinematic function of a joint, or pair, is to determine the relative motion between the connected links. All other features are determined for other reasons and are unimportant in the study of kinematics.

When a kinematic problem is formulated, it is necessary to recognize the type of relative motion permitted in each of the joints and to assign to it some variable parameter(s) for measuring or calculating the motion. There will be as many of these parameters as there are degrees of freedom of the joint in question, and they are referred to as *joint variables*. Thus, the joint variable of a pinned joint will be a single angle measured between reference lines fixed in the adjacent links, while a spheric joint will have three joint variables (all angles) to specify its three-dimensional rotation.

Reuleaux separated kinematic pairs into two categories: namely, *higher pairs* and *lower pairs*, with the latter category consisting of the six prescribed types to be discussed next. He distinguished between the categories by noting that lower pairs, such as the pin joint, have surface contact between the joint elements, while higher pairs, such as the connection between a cam and its follower, have line or point contact between the elemental surfaces. This criterion, however, can be misleading (as noted in the case of a needle bearing). We should rather look for distinguishing features in the relative motion(s) that the joint allows between the connected links.

Lower pairs consist of the six prescribed types shown in Fig. 1.2.

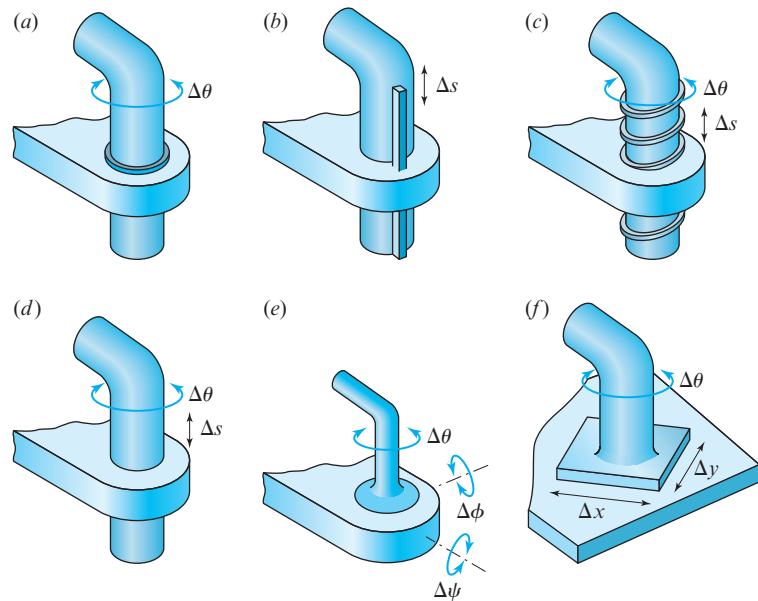


Figure 1.2 (a) Revolute; (b) prism; (c) screw; (d) cylinder; (e) sphere; (f) flat pairs.

The names and the symbols (Hartenberg and Denavit [4]) that are commonly employed for the six lower pairs are presented in Table 1.1. The table also includes the number of degrees of freedom and the joint variables that are associated with each lower pair.

The *revolute* or *turning pair*, R (Fig. 1.2a), permits only relative rotation and is often referred to as a pin joint. This joint has one degree of freedom.

The *prism* or *prismatic pair*, P (Fig. 1.2b), permits only relative sliding motion and therefore is often called a sliding joint. This joint also has one degree of freedom.

The *screw* or *helical pair*, H (Fig. 1.2c), permits both rotation and sliding motion. However, it only has one degree of freedom, since the rotation and sliding motions are related by the helix angle of the thread. Thus, the joint variable may be chosen as either Δs or $\Delta \theta$, but not both. Note that the helical pair reduces to a revolute if the helix angle is made zero, and to a prism if the helix angle is made 90° .

Table 1.1 Lower Pairs

Pair	Symbol	Pair Variable	Degrees of Freedom	Relative Motion
Revolute	R	$\Delta \theta$	1	Circular
Prism	P	Δs	1	Rectilinear
Screw	H	$\Delta \theta$ or Δs	1	Helical
Cylinder	C	$\Delta \theta$ and Δs	2	Cylindric
Sphere	S	$\Delta \theta$, $\Delta \phi$, $\Delta \psi$	3	Spheric
Flat	F	Δx , Δy , $\Delta \theta$	3	Planar

The *cylinder* or *cylindric pair*, C (Fig. 1.2d), permits both rotation and an independent sliding motion. Thus, the cylindric pair has two degrees of freedom.

The *sphere* or *globular pair*, S (Fig. 1.2e), is a ball-and-socket joint. It has three degrees of freedom, sometimes taken as rotations about each of the coordinate axes.

The *flat* or *planar pair*, sometimes called an *ebene pair* (German), F (Fig. 1.2f), is seldom found in mechanisms in its undisguised form, except at a support point. It has three degrees of freedom, that is, two translations and a rotation.

All other joint types are called higher pairs. Examples include mating gear teeth, a wheel rolling and/or sliding on a rail, a ball rolling on a flat surface, and a cam contacting its follower. Since an unlimited variety of higher pairs exist, a systematic accounting of them is not a realistic objective. We shall treat each separately as it arises.

Among the higher pairs is a subcategory known as *wrapping pairs*. Examples are the connections between a belt and a pulley, a chain and a sprocket, or a rope and a drum. In each case, one of the links has only one-way rigidity.

The treatment of various joint types, whether lower or higher pairs, includes another important limiting assumption. Throughout the book, we assume that the actual joint, as manufactured, can be reasonably represented by a mathematical abstraction having perfect geometry. That is, when a real machine joint is assumed to be a spheric joint, for example, it is also assumed that there is no “play” or clearance between the joint elements and that any deviation from spheric geometry of the elements is negligible. When a pin joint is treated as a revolute, it is assumed that no axial motion takes place; if it is necessary to study the small axial motions resulting from clearances between real elements, the joint must be treated as cylindric, thus allowing the axial motion.

The term “mechanism,” as defined earlier, can refer to a wide variety of devices, including both higher and lower pairs. A more limited term, however, refers to those mechanisms having only lower pairs; such a mechanism is commonly called a *linkage*. A linkage, then, is connected only by the lower pairs shown in Fig. 1.2.

1.5 PLANAR, SPHERIC, AND SPATIAL MECHANISMS

Mechanisms may be categorized in several different ways to emphasize their similarities and differences. One such grouping divides mechanisms into planar, spheric, and spatial categories. All three groups have many things in common; the criterion that distinguishes the groups, however, is to be found in the characteristics of the motions of the links.

A *planar mechanism* is one in which all particles describe planar curves in space, and all these curves lie in parallel planes; that is, the loci of all points are planar curves parallel to a single common plane. This characteristic makes it possible to represent the locus of any chosen point of a planar mechanism in its true size and shape in a single drawing or figure. The motion transformation of any such mechanism is called *coplanar*. The planar four-bar linkage, the slider-crank linkage, the plate cam-and-follower mechanism, and meshing gears are familiar examples of planar mechanisms.

Planar mechanisms utilizing only lower pairs are called *planar linkages*; they include only revolute and prismatic joints. Although the planar pair might theoretically be included in a planar linkage, this would impose no constraint on the motion. Planar motion also

requires that all revolute axes be normal to the plane of motion, and that all prismatic joint axes be parallel to the plane.

As already pointed out, it is possible to observe the motions of all particles of a planar mechanism in true size and shape from a single direction. In other words, all motions can be represented graphically in a single view. Thus, graphic techniques are well suited to their analysis, and this background is beneficial to the student once mastered. Since spheric and spatial mechanisms do not have this special geometry, visualization becomes more difficult and more powerful techniques must be used for their study.

A *spheric mechanism* is one in which each moving link has a point that remains stationary as the mechanism moves. Also, arbitrary points fixed in each moving link travel on spheric surfaces; the spheric surfaces must all be *concentric*. Therefore, the motions of all these points can be completely described by their radial projections (or shadows) on the surface of a sphere with a properly chosen center. Note that the only lower pairs (Table 1.1) that allow spheric motion are the revolute pair and the spheric pair. In a spheric linkage, the axes of all revolute pairs must intersect at a single point. In addition, a spheric pair center must be concentric with this point, and, then, it would not produce any constraint on the motions of the other links. Therefore, a *spheric linkage* must consist of only revolute pairs, and the axes of all such pairs must intersect at a single point. A familiar example of a spheric mechanism is the Hooke universal joint (also referred to as the Cardan joint) shown in Fig. 1.21*b*.

Spatial mechanisms include *no* restrictions on the relative motions of the links. For example, a mechanism that contains a screw joint (Fig. 1.2*c*) must be a spatial mechanism, since the relative motion within a screw joint is helical. An example of a spatial mechanism is the differential screw shown in Fig. 1.11. Because of the more complex motion characteristics of spatial mechanisms, and since these motions can not be analyzed graphically from a single viewing direction, more powerful techniques are required for their analysis. Such techniques are introduced in Chap. 10 for a detailed study of spatial mechanisms and robotics.

Since the majority of mechanisms in modern machinery are planar, one might question the need to study these complex mathematical techniques. However, even though the simpler graphic techniques for planar mechanisms may have been mastered, an understanding of the more complex techniques is of value for the following reasons:

1. They provide new, alternative methods that can solve problems in a different way. Thus, they provide a means for checking results. Certain problems by their nature may also be more amenable to one method than another.
2. Methods that are analytic in nature are better suited to solution by a calculator or a digital computer than by graphic techniques.
3. One reason why planar mechanisms are so common is that good methods for the analysis of spatial mechanisms have not been available until relatively recently. Without these methods, the design and application of spatial mechanisms has been hindered, even though they may be inherently better suited to certain applications.
4. We will discover that spatial mechanisms are, in fact, much more common in practice than their formal description indicates.

Consider the planar four-bar linkage (Fig. 1.3c), which has four links connected by four revolute pairs whose axes are parallel. This “parallelism” is a mathematical hypothesis; it is not a reality. The axes, as produced in a machine shop—in any machine shop, no matter how precise the machining—are only approximately parallel. If the axes are far out of parallel, there is binding in no uncertain terms, and the linkage moves only because the “rigid” links flex and twist, producing loads in the bearings. If the axes are nearly parallel, the linkage operates because of looseness of the running fits of the bearings or flexibility of the links. A common way of compensating for small nonparallelism is to connect the links with self-aligning bearings, actually spheric joints allowing three-dimensional rotation. Such a “planar” linkage is thus a low-grade spatial linkage.

Thus, the overwhelmingly large category of planar mechanisms and the category of spheric mechanisms are special cases, or subsets, of the all-inclusive category of spatial mechanisms. They occur as a consequence of the special orientations of their joint axes.

1.6 MOBILITY

One of the first concerns in either the design or the analysis of a mechanism is the number of degrees of freedom, also called the *mobility* of the device. The mobility* of a mechanism is the number of input parameters (usually joint variables) that must be controlled independently to bring the device into a particular posture. Ignoring, for the moment, certain exceptions to be mentioned later, it is possible to determine the mobility of a mechanism directly from a count of the number of links and the number and types of joints comprising the mechanism.

To develop this relationship, consider that—before they are connected together—each link of a planar mechanism has three degrees of freedom when moving with planar motion relative to the fixed link. Not counting the fixed link, therefore, an n -link planar mechanism has $3(n - 1)$ degrees of freedom before any of the joints are connected. Connecting two of the links by a joint that has one degree of freedom, such as a revolute, has the effect of providing two constraints between the connected links. If the two links are connected by a two-degree-of-freedom joint, it provides one constraint. When the constraints for all joints are subtracted from the total degrees of freedom of the unconnected links, we find the resulting mobility of the assembled mechanism.

If we denote the number of single-degree-of-freedom joints as j_1 and the number of two-degree-of-freedom joints as j_2 , then the resulting mobility, m , of a planar n -link mechanism is given by

$$m = 3(n - 1) - 2j_1 - j_2. \quad (1.1)$$

Written in this form, Eq. (1.1) is called the *Kutzbach criterion* for the mobility of a planar mechanism [8]. Its application is illustrated for several simple examples in Fig. 1.3.

* The German literature distinguishes between *movability* and *mobility*. Movability includes the six degrees of freedom of the device as a whole, as though the ground link were not fixed, and thus applies to a kinematic chain. Mobility neglects these degrees of freedom and considers only the internal relative motions, thus applying to a mechanism. The English literature seldom recognizes this distinction, and the terms are used somewhat interchangeably.

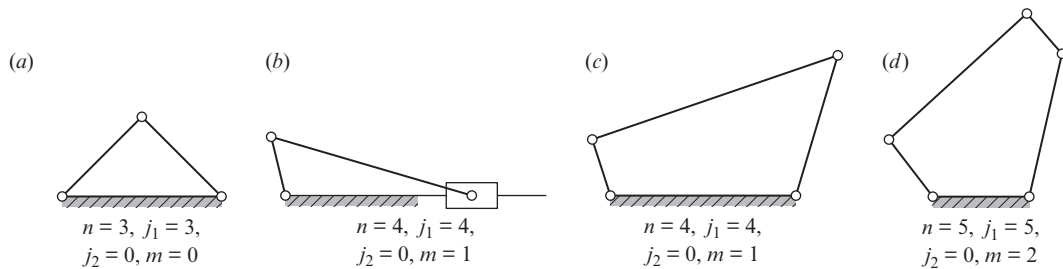


Figure 1.3 Applications of the Kutzbach criterion.

If the Kutzbach criterion yields $m > 0$, the mechanism has m degrees of freedom. If $m = 1$, the mechanism can be driven by a single input motion to produce constrained (uniquely defined) motion. Two examples are the slider-crank linkage and the four-bar linkage, shown in Figs. 1.3b and 1.3c, respectively. If $m = 2$, then two separate input motions are necessary to produce constrained motion for the mechanism; such a case is the five-bar linkage shown in Fig. 1.3d.

If the Kutzbach criterion yields $m = 0$, as in Figs. 1.3a and 1.4a, motion is impossible and the mechanism forms a structure.

If the criterion yields $m < 0$, then there are redundant constraints in the chain and it forms a statically indeterminate structure. An example is illustrated in Fig. 1.4b. Note in the examples of Fig. 1.4 that when three links are joined by a single pin, such a connection is treated as two separate but concentric joints; two j_1 joints must be counted.

Figure 1.5 shows two examples of the Kutzbach criterion applied to mechanisms with two-degree-of-freedom joints—that is, j_2 joints. Particular attention should be paid to the contact (joint) between the wheel and the fixed link in Fig. 1.5b. Here it is assumed that slipping is possible between the two links. If this contact prevents slipping, the joint would be counted as a one-degree-of-freedom joint—that is, a j_1 joint—because only one relative motion would then be possible between the links. Recall that, in this case, the mechanism is generally referred to as a “linkage.”

It is important to realize that the Kutzbach criterion can give an incorrect result. For example, note that Fig. 1.6a represents a structure and that the criterion properly predicts $m = 0$. However, if link 5 is arranged as in Fig. 1.6b, the result is a double-parallelgram linkage with a mobility of $m = 1$, even though Eq. (1.1) indicates that it is a structure. The actual mobility of $m = 1$ results only if the parallelogram geometry is achieved. In the development of the Kutzbach criterion, no consideration was given to the lengths of the

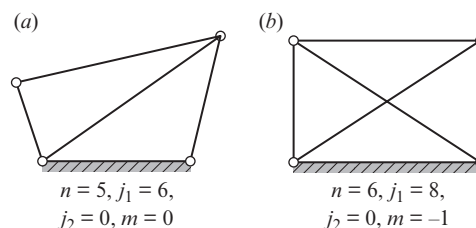


Figure 1.4 Applications of the Kutzbach criterion to structures.

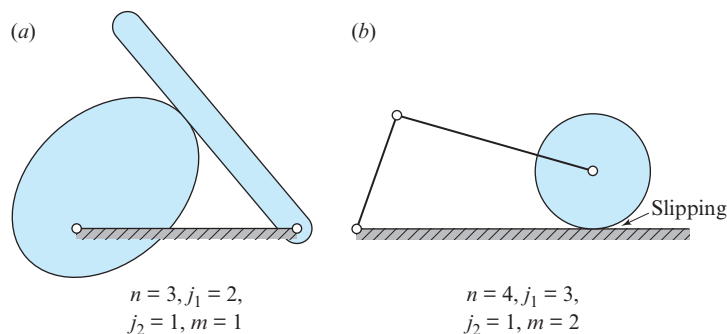


Figure 1.5

links or other dimensional properties. Therefore, it should not be surprising that exceptions to the criterion are found for particular cases with equal link lengths, parallel links, or other special geometric features.

Although there are exceptions, the Kutzbach criterion remains useful, because it is so easily applied during mechanism design. To avoid exceptions, it would be necessary to include all the dimensional properties of the mechanism. The resulting criterion would be very complex and would be useless at the early stages of design when dimensions may not be known.

An earlier mobility criterion, named after Grübler [3], applies to a planar linkage where the overall mobility is $m = 1$. Substituting $j_2 = 0$ and $m = 1$ into Eq. (1.1) and rearranging, we find that Grübler's criterion for planar linkages can be written as

$$3n - 2j_1 - 4 = 0. \quad (1.2)$$

Rearranging this equation, the number of links is

$$n = \frac{2j_1 + 4}{3}. \quad (1.3)$$

From this equation, we see that a planar linkage with a mobility of $m = 1$ cannot have an odd number of links. Also, the simplest possible linkage with all binary links has $n = j_1 = 4$. This explains one reason why the slider-crank linkage (Fig. 1.3b) and the four-bar linkage (Fig. 1.3c) appear so commonly in machines.

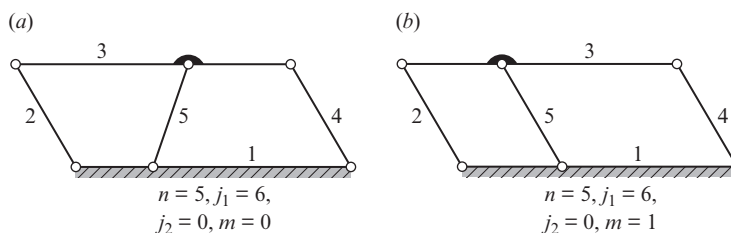


Figure 1.6

The Kutzbach criterion, Eq. (1.1), and the Grübler criterion, Eq. (1.2), were derived for the case of planar mechanisms and linkages, respectively. If similar criteria are developed for spatial mechanisms and linkages, which is the subject of Chap. 10, we must recall that each unconnected link has six degrees of freedom and each single-degree-of-freedom joint provides five constraints, each two-degree-of-freedom joint provides four constraints, and so on. Similar arguments then lead to the Kutzbach criterion for spatial mechanisms,

$$m = 6(n - 1) - 5j_1 - 4j_2 - 3j_3 - 2j_4 - j_5,$$

and the Grübler criterion for spatial linkages,

$$6n - 5j_1 - 7 = 0. \quad (1.4)$$

Therefore, the simplest form of a spatial linkage* with a mobility of $m = 1$ is $n = j_1 = 7$.

EXAMPLE 1.1

Determine the mobility of the planar mechanism shown in Fig. 1.7a.

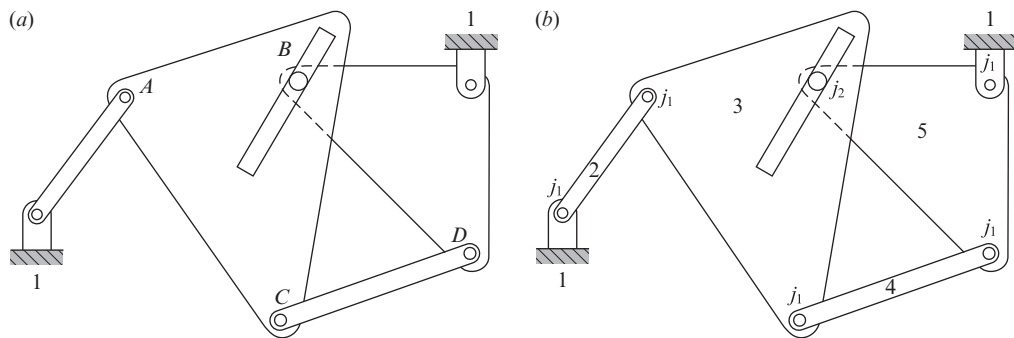


Figure 1.7 Planar mechanism.

SOLUTION

The link numbers and the joint types for the mechanism are shown in Fig. 1.7b. The number of links is $n = 5$, the number of lower pairs is $j_1 = 5$, and the number of higher pairs is $j_2 = 1$. Substituting these values into the Kutzbach criterion, Eq. (1.1), the mobility of the mechanism is

$$m = 3(5 - 1) - 2(5) - 1(1) = 1. \quad \text{Ans.}$$

Note that the Kutzbach criterion gives the correct answer for the mobility of this mechanism; that is, a single input motion is required to give a unique output motion.

* Note that all planar linkages are exceptions to the spatial mobility criterion. They have the special geometric characteristics that all revolute axes are parallel and perpendicular to the plane of motion and that all prismatic axes lie in the plane of motion.

For example, rotation of link 2 could be used as the input and rotation of link 5 could be used as the output.

EXAMPLE 1.2

For the mechanism shown in Fig. 1.8*a*, determine: (a) the number of lower pairs (j_1 joints) and the number of higher pairs (j_2 joints); and (b) the mobility of the mechanism using the Kutzbach criterion. Treating rolling contact to mean rolling with no slipping. Does this criterion provide the correct answer for the mobility of this mechanism? Briefly explain why or why not.

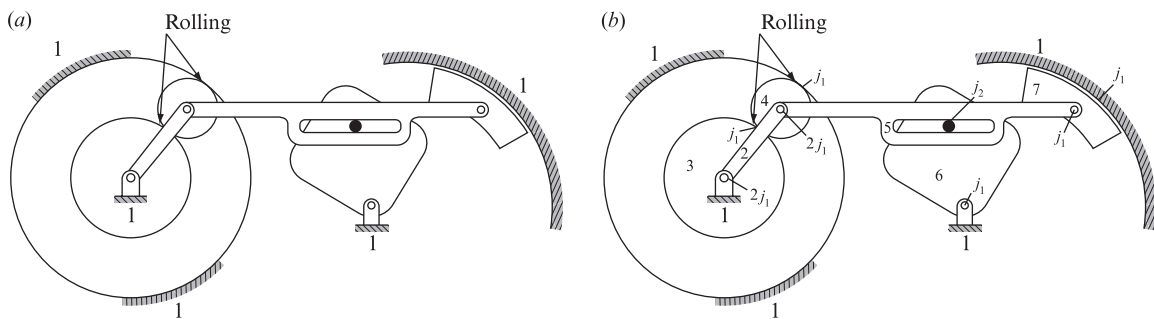


Figure 1.8 Planar mechanism.

SOLUTION

- (a) The links and the joint types of the mechanism are labeled in Fig. 1.8*b*. The number of links is $n = 7$, the number of lower pairs is $j_1 = 9$, and the number of higher pairs is

$j_2 = 1.$ *Ans.*

- (b) Substituting these values into the Kutzbach criterion, Eq. (1.1), the mobility of the mechanism is

$$m = 3(7 - 1) - 2(9) - 1(1) = -1. \quad \text{Ans.}$$

However, this answer is not correct; that is, the Kutzbach criterion does not give the correct mobility for this mechanism. The mobility of this mechanism is, in fact, $m = 1$; that is, a single input motion gives a unique output motion.

Reasoning: Links 3 and 4 are superfluous to the constraints of the mechanism. If links 3 and 4 were removed, the motion of the remaining links would be unaffected. With links 3 and 4 removed, the mobility of the mechanism using the Kutzbach criterion is $m = 1$. Note that if links 3 and 4 were attached with no special conditions—that is, not pinned at their centers, for example—then the mechanism would indeed be locked and the answer $m = -1$ would be correct.

EXAMPLE 1.3

For the mechanism shown in Fig. 1.9a, determine: (a) the number of lower pairs and the number of higher pairs; and (b) the mobility of the mechanism predicted by the Kutzbach criterion. Does this criterion provide the correct answer for this mechanism? Briefly explain why or why not.

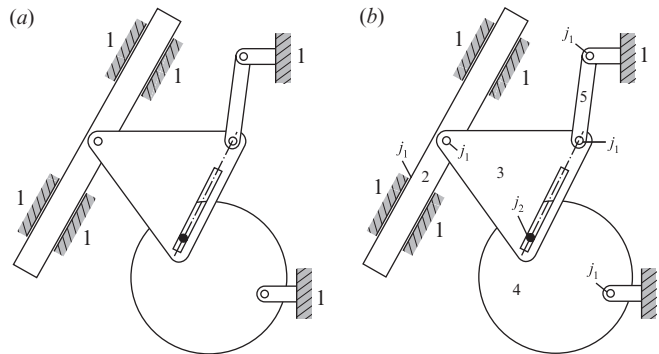


Figure 1.9 Planar mechanism.

SOLUTION

- (a) The links and the joints of the mechanism are labeled as shown in Fig. 1.9b. The number of links is $n = 5$, the number of lower pairs is $j_1 = 5$, and the number of higher pairs is

$$j_2 = 1. \quad \text{Ans.}$$

- (b) Substituting these values into the Kutzbach criterion, Eq. (1.1), the mobility of the mechanism is

$$m = 3(5 - 1) - 2(5) - 1(1) = 1. \quad \text{Ans.}$$

For this mechanism, the mobility is indeed 1, which indicates that the Kutzbach criterion gives the correct answer for this mechanism.

For a mechanism, or a linkage, with a mobility of $m = 1$, the input or driving link will, in general, be numbered as 2 in this text.

1.7 CHARACTERISTICS OF MECHANISMS

An ideal system for the classification of mechanisms would be a system that allows a designer to enter the system with a set of specifications and leave with one or more mechanisms that satisfy those specifications. Although history* demonstrates that

* For an excellent short history of the kinematics of mechanisms, see [4, Chap. 1].

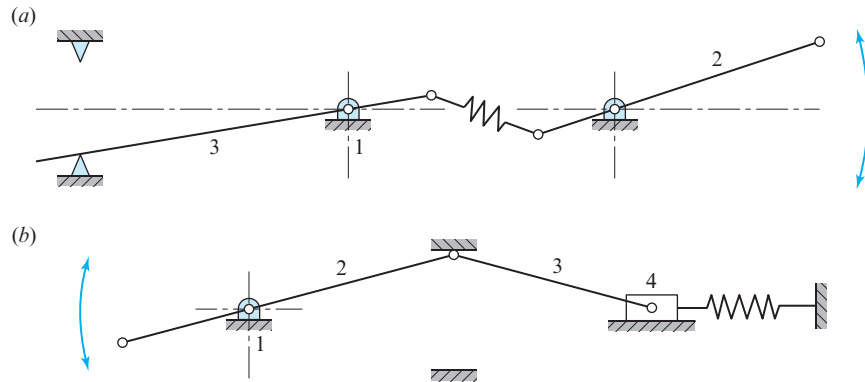


Figure 1.10 (a) Bistable mechanism; (b) true toggle mechanism.

many attempts have been made, few have been particularly successful in devising a satisfactory classification system. In view of the fact that the purpose of a mechanism is the transformation of motion, we will follow Torfason's lead [9] and classify mechanisms according to the type of motion transformation. In total, Torfason displays 262 mechanisms, each of which can have a variety of dimensions. His categories are as follows:

Snap-Action Mechanisms Snap-action, toggle, or flip-flop mechanisms are used for switches, clamps, or fasteners. Torfason also includes spring clips and circuit breakers. Fig. 1.10 shows examples of bistable and true toggle mechanisms.

Linear Actuators Linear actuators include stationary screws with traveling nuts, stationary nuts with traveling screws, and single-acting and double-acting hydraulic and pneumatic cylinders.

Fine Adjustments Fine adjustments may be obtained with screws, including differential screws, worm gearing, wedges, levers, levers in series, and various motion-adjusting mechanisms. For the differential screw shown in Fig. 1.11, you should be able to determine that the translation of the carriage resulting from one turn of the handle is 0.0069 in to the left (see Prob. 1.17).

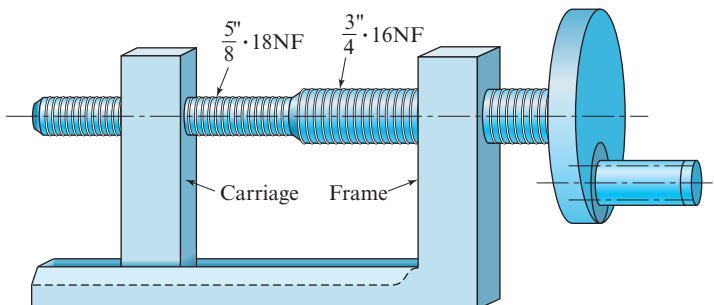


Figure 1.11 Differential screw.

Clamping Mechanisms Typical clamping mechanisms are the C-clamp, the wood worker's screw clamp, cam-actuated and lever-actuated clamps, vises, presses (such as the toggle press shown in Fig. 1.10*b*), collets, and stamp mills.

Locational Devices Torfason [9] shows 15 locational mechanisms. These are usually self-centering and locate either axially or angularly using springs and detents.

Ratchets and Escapements There are many different forms of ratchets and escapements, some quite clever. They are used in locks, jacks, clockwork, and other applications requiring some form of intermittent motion. Figure 1.12 shows four typical applications.

The ratchet in Fig. 1.12*a* allows only one direction of rotation of wheel 2. Pawl 3 is held against the wheel by gravity or by a spring. A similar arrangement is used for lifting jacks, which then employ a toothed rack for rectilinear motion.

Figure 1.12*b* is an escapement used for rotary adjustments.

Graham's escapement shown in Fig. 1.12*c* is used to regulate the movement of clockwork. Anchor 3 drives a pendulum whose oscillating motion is caused by the two clicks engaging wheel 2. One is a push click, the other is a pull click. The lifting and engaging of each click caused by oscillation of the pendulum results in a wheel motion

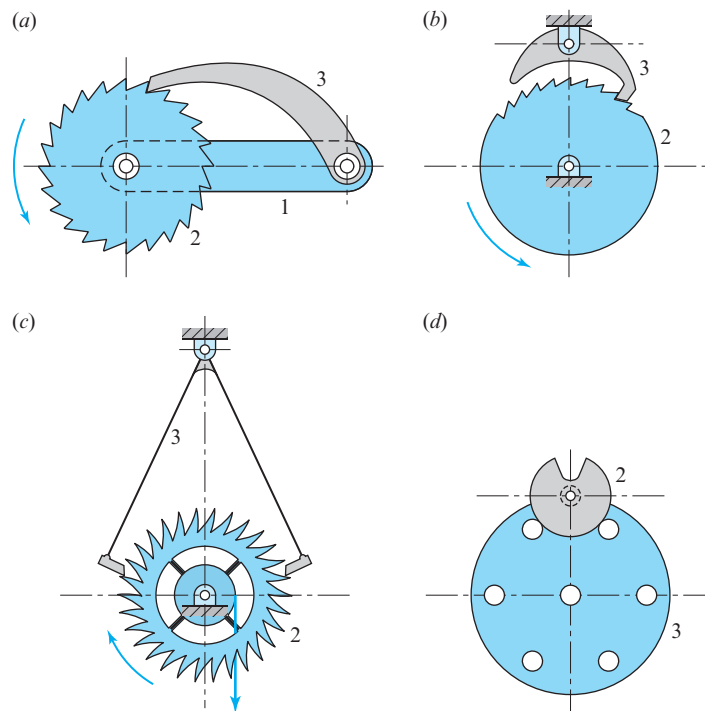


Figure 1.12 Ratchets and escapements.

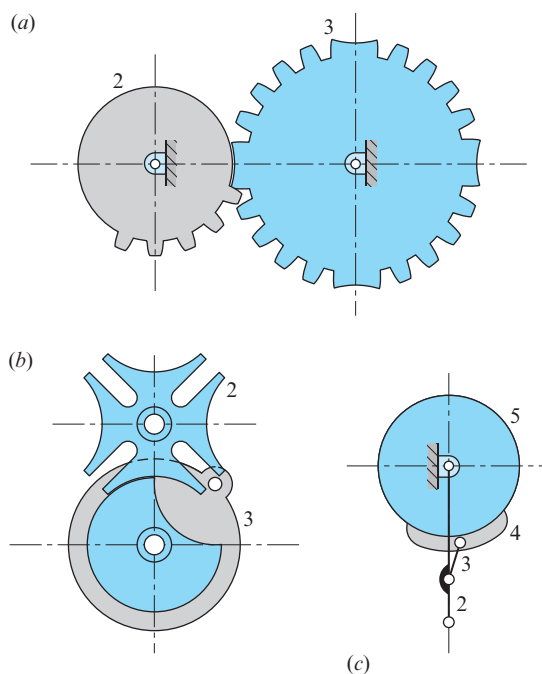


 **Figure 1.13** Indexing mechanisms.

that, at the same time, presses each respective click and adds a gentle force to the motion of the pendulum.

The escapement shown in Fig. 1.12*d* has a control wheel, 2, that may rotate continuously to allow wheel 3 to be driven (by another source) in either direction.

Indexing Mechanisms The indexer shown in Fig. 1.13*a* uses standard gear teeth; for light loads, pins can be used in the input wheel 2 with corresponding slots in wheel 3, but neither form should be used if the shaft inertias are large.

Figure 1.13*b* shows a Geneva-wheel, sometimes called a “Maltese-cross,” indexer. Three or more slots may be used in the driven link, 2, which can be attached to, or geared to, the output to be indexed. High speeds and large inertias may cause problems with this indexer.

Toothless ratchet 5 in Fig. 1.13*c* is driven by the oscillating crank, 2, of variable throw. Note the similarity of this indexing mechanism to the ratchet of Fig. 1.12*a*.

Torfason [9] lists nine different indexing mechanisms, and many variations are possible.

Swinging or Rocking Mechanisms The category of swinging or rocking mechanisms is often termed *oscillators*; in each case, the output rocks or swings through an angle that is generally less than 360° . However, the output shaft can be geared to a successor shaft to produce a larger angle of oscillation.

Figure 1.14*a* is a mechanism consisting of a rotating crank 2 and a coupler 3 containing a rack, which meshes with output gear 4 to produce the oscillating motion.

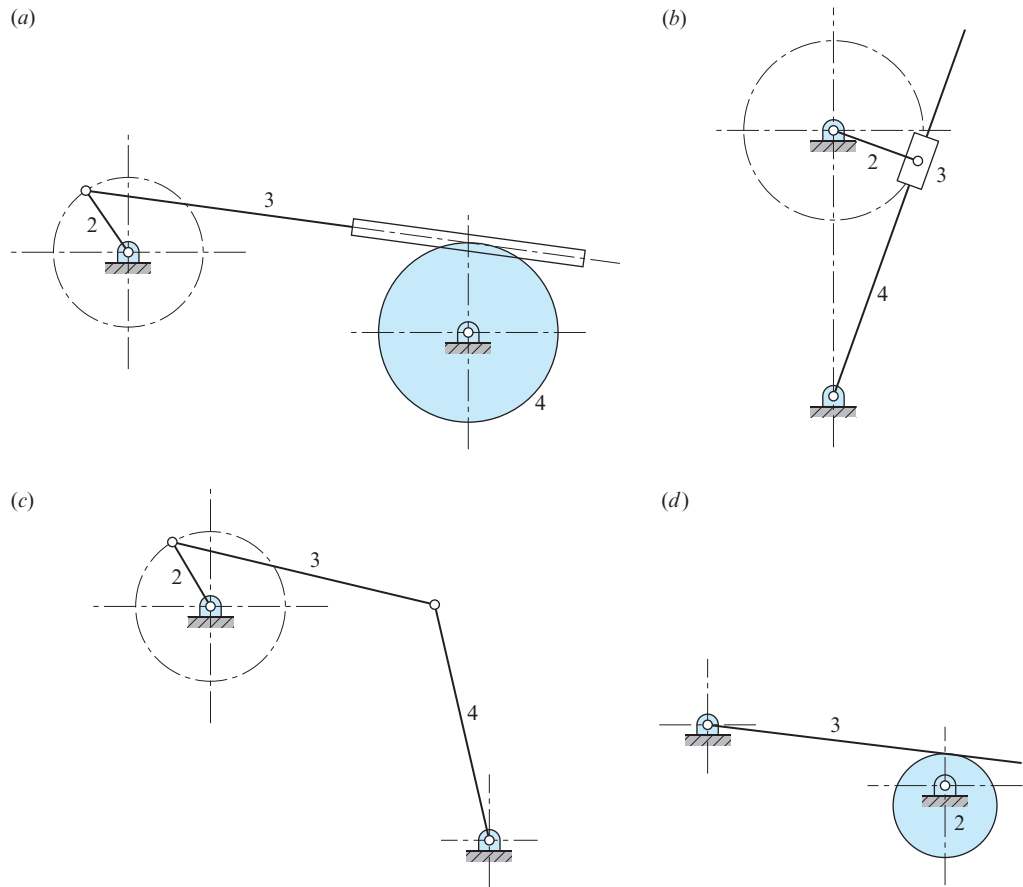


Figure 1.14 Oscillating mechanisms.

In Fig. 1.14*b*, crank 2 drives link 3, which slides on output link 4, producing a rocking motion. This mechanism is described as a *quick-return mechanism*, because crank 2 rotates through a larger angle on the forward stroke of link 4 than on the return stroke.

Figure 1.14*c* is a *four-bar linkage* called the *crank-rocker linkage* (Sec. 1.9). Crank 2 drives rocker 4 through the coupler 3. Of course, link 1 is the frame. The characteristics of the rocking motion depend on the dimensions of the links and the placement of the frame pivots.

Figure 1.14*d* shows a *cam-and-follower mechanism*, in which the rotating link 2, called the *cam*, drives link 3, called the *follower*, in a rocking motion. An endless variety of cam-and-follower mechanisms are possible, many of which are discussed in Chap. 6. In each case, the cam can be designed to produce an output motion with the desired characteristics.

Reciprocating Mechanisms Repeating straight-line motion is commonly obtained using either a pneumatic or hydraulic cylinder, a stationary screw with a traveling nut, a rectilinear drive using a reversible motor or reversing gears, or a cam-and-follower

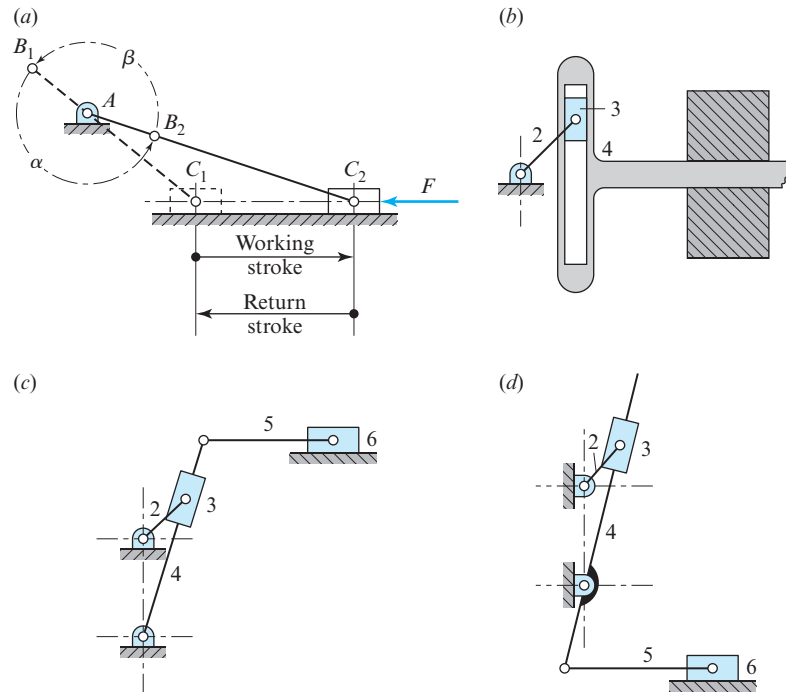


 **Figure 1.15** Reciprocating linkages.

mechanism. A variety of typical linkages for obtaining reciprocating motion are shown in Figs. 1.15 and 1.16 [5].

The *offset slider-crank linkage* shown in Fig. 1.15a has kinematic characteristics that differ from the in-line (or on-center) slider-crank, shown in Fig. 1.13b. If the length of connecting rod 3 is long compared to the length of crank 2, then the resulting motion is nearly harmonic. Exact harmonic motion can be obtained from link 4 of the *Scotch-yoke linkage* shown in Fig. 1.15b.

The six-bar linkage shown in Fig. 1.15c is often called the *shaper linkage*, after the name of the machine tool in which it is used. Note that it is obtained from Fig. 1.14b by adding coupler 5 and slider 6. The stroke of the slider has a quick-return characteristic.

Figure 1.15d shows another version of the shaper linkage, which is termed the *Whitworth quick-return linkage*. The linkage is presented in an upside-down posture to illustrate its similarity to Fig. 1.15c.

Another example of a six-bar linkage is the *Wanzer needle-bar linkage* [5] shown in Fig. 1.16.

Figure 1.17a shows a six-bar linkage derived from the crank-rocker linkage of Fig. 1.14c by expanding coupler 3 and adding coupler 5 and slider 6. Coupler point C should be located to produce the desired motion characteristic for slider 6.

A crank-driven toggle linkage is shown in Fig. 1.17b. With this linkage, a high mechanical advantage is obtained at one end of the stroke of slider 6. (For a detailed discussion of the mechanical advantage of a mechanism, see Secs. 1.10 and 3.20).

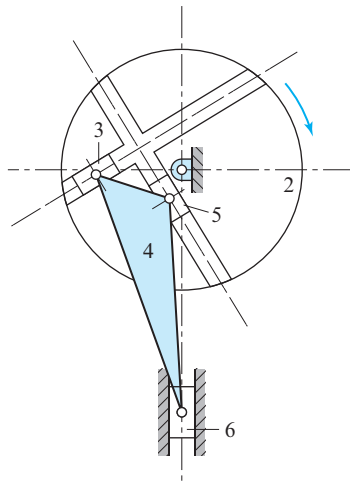


Figure 1.16 Wanzer needle-bar linkage.
(Richard Mott Wanzer, 1812–1900).

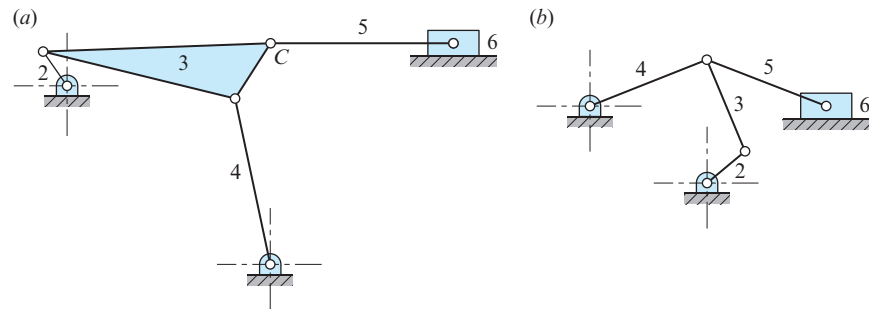


Figure 1.17 Additional six-bar reciprocating linkages.

In many applications, mechanisms are used to perform repetitive operations, such as pushing parts along an assembly line, clamping parts together while they are welded, or folding cardboard boxes in an automated packaging machine. In such applications it is often desirable to use a constant-speed motor; this leads us to a discussion of Grashof's law in Sec. 1.9. In addition, however, we should give some consideration to the power and timing requirements.

In such repetitive operations, there is usually a part of the cycle when the mechanism is under load, called the *advance* or *working stroke*, and a part of the cycle, called the *return stroke*, when the mechanism is not working but simply returning to repeat the operation. For example, consider the offset slider-crank linkage shown in Fig. 1.15a. Work may be required to overcome the load, F , while the piston moves to the right from position C_1 to position C_2 but not during its return to position C_1 , since the load may have been removed. In such situations, in order to keep the power requirement of the motor to a minimum and to avoid wasting valuable time, it is desirable to design a mechanism so that the piston moves much faster through the return stroke than it does during the advance (or working) stroke—that is, to use a higher portion of the cycle time for doing work than for returning.

A measure of the suitability of a mechanism from this viewpoint, called the *advance-to-return ratio* is defined as

$$Q = \frac{\text{cycle fraction for advance stroke}}{\text{cycle fraction for return stroke}}. \quad (a)$$

A mechanism for which the value of Q is high is more desirable for such repetitive operations than one in which the value of Q is lower. Certainly, any such operation would call for a mechanism for which Q is greater than unity. Because of this, mechanisms with Q greater than unity are called *quick-return mechanisms*.

As shown in Fig. 1.15a, the first step is to determine the two crank postures, AB_1 and AB_2 , that mark the beginning and the end of the working stroke. Next, noting the direction of rotation of the crank, we determine the crank angle α traveled through during the advance stroke and the remaining crank angle β of the return stroke. Then,

$$\text{cycle fraction for advance stroke} = \frac{\alpha}{2\pi}, \quad (b)$$

and

$$\text{cycle fraction for return stroke} = \frac{\beta}{2\pi}. \quad (c)$$

Finally, substituting Eqs. (b) and (c) into Eq. (a), the advance-to-return ratio can be written as

$$Q = \frac{\alpha}{\beta}. \quad (1.5)$$

Note that the advance-to-return ratio depends only on geometry (that is, on changes in the crank position); this ratio does not depend on the amount of work being done or on the speed of the driving motor. It is a kinematic property of the mechanism itself. Therefore, this ratio can be used for either design or analysis totally by graphic constructions. The following two examples illustrate applications in design.

EXAMPLE 1.4

The rocker of a crank-rocker four-bar linkage is required to have a length of 4 in and swing through a total angle of 45° . Also, the advance-to-return ratio of the linkage is required to be 2.0. Determine a suitable set of link lengths for the remaining three links.

SOLUTION

Equation (1.5) requires

$$Q = \frac{\alpha}{\beta} = 2.0, \quad (1)$$

where

$$\alpha = 180^\circ + \phi \quad (2)$$

and

$$\beta = 180^\circ - \phi. \quad (3)$$

Substituting Eqs. (2) and (3) into Eq. (1) allows us to solve for

$$\phi = 60^\circ, \quad \alpha = 240^\circ, \quad \text{and} \quad \beta = 120^\circ.$$

Now, referring to Fig. 1.18, we apply the following graphic procedure:

- Draw the rocker ($r_4 = 4.0$ in) to a suitable scale in its two extreme postures; that is, show the swing angle of the rocker of 45° . Label the ground pivot O_4 , and label pin B in the two positions B_1 and B_2 .
- Through point B_1 , draw an arbitrary line (labeled the X -line). Through B_2 , draw a line parallel to the X -line.
- Measure the angle $\phi = 60^\circ$ counterclockwise from the X -line through point B_1 . The intersection of this line with the line parallel to the X -line is the required position of the input crank pivot O_2 .
- The length O_2O_4 of the ground link can be measured from the drawing—that is

$$r_1 = 1.50 \text{ in.} \quad \text{Ans.}$$

- The lengths of crank r_2 and the coupler r_3 can be determined from the measurements

$$O_2B_1 = r_3 + r_2 = 3.50 \text{ in} \quad \text{and} \quad O_2B_2 = r_3 - r_2 = 2.50 \text{ in.}$$

That is,

$$r_2 = 0.5(O_2B_1 - O_2B_2) = 0.50 \text{ in} \quad \text{and} \quad r_3 = 0.5(O_2B_2 + O_2B_1) = 3.00 \text{ in.} \quad \text{Ans.}$$

The solution for the synthesized four-bar linkage is shown in Fig. 1.18.

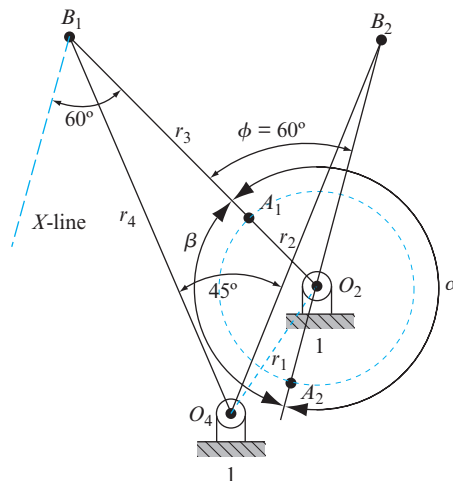


Figure 1.18 Synthesized four-bar linkage.

In the two limiting positions, B_1 and B_2 , the output is momentarily stopped, and, for this reason, these two postures of the linkage are referred to as *dead-center postures* (see Prob. 1.35). Also, note that this problem is an example of two-posture synthesis. For a detailed discussion on the general problems of two-, three-, and four-posture synthesis, see Chap. 9.

EXAMPLE 1.5

Determine a suitable set of link lengths for a slider-crank linkage such that the stroke is 2.50 in and the advance-to-return ratio is 1.4.

SOLUTION

Equation (1.5) requires

$$Q = \frac{\alpha}{\beta} = 1.40, \quad (1)$$

where

$$\alpha = 180^\circ + \phi \quad (2)$$

and

$$\beta = 180^\circ - \phi. \quad (3)$$

Substituting Eqs. (2) and (3) into Eq. (1) allows us to solve for

$$\phi = 30^\circ, \quad \alpha = 210^\circ, \quad \text{and} \quad \beta = 150^\circ.$$

Now, referring to Fig. 1.19, we apply the following graphic procedure:

- (a) Draw the stroke (shown horizontal) of 2.50 in of the slider-crank linkage to a suitable scale. Label pin B in its two extreme positions B_1 and B_2 .
- (b) Through point B_2 , draw an arbitrary line (labeled the X -line). Through point B_1 , draw a line parallel to the X -line.
- (c) Measure the angle $\phi = 30^\circ$ clockwise from the X -line. The intersection of this line with the line parallel to the X -line is the ground pivot O_2 .
- (d) The length of the ground link—that is, the offset or eccentricity (the perpendicular distance from the ground pivot O_2 to the line of travel of the slider)—can be measured from the drawing. That is,

$$r_1 = 2.17 \text{ in.} \quad \text{Ans.}$$

- (e) The lengths of crank r_2 and coupler r_3 can be determined from the measurements

$$O_2B_1 = r_3 + r_2 = 4.33 \text{ in} \quad \text{and} \quad O_2B_2 = r_3 - r_2 = 2.50 \text{ in.}$$

That is,

$$r_2 = 0.5(O_2B_1 - O_2B_2) = 0.92 \text{ in} \quad \text{and} \quad r_3 = 0.5(O_2B_2 + O_2B_1) = 3.42 \text{ in.}$$

Ans.

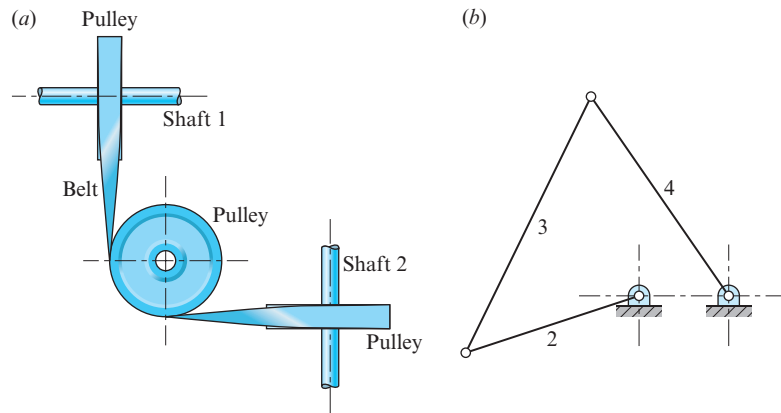


Figure 1.20 Two-shaft coupling mechanisms.

4 is the output. This is a very interesting linkage; you should try to construct one using cardboard strips and thumbtacks for joints to observe its motion. Can you devise a working model that allows complete rotation of both links 2 and 4 (see Prob. 1.14)?

The *Reuleaux coupling*, shown in Fig. 1.21a, for intersecting shafts is recommended only for light loads. The *Hooke joint*, shown in Fig. 1.21b, is also used for intersecting shafts. However, this joint can withstand heavy loads and is commonly used with a driveshaft in rear-wheel-drive vehicles. It is customary to use two of these joints in series for connecting parallel shafts.

Sliding Connectors Sliding connectors are used when one slider (the input) is to drive another slider (the output). The usual problem is that the two sliders operate in the same plane but in different directions. The possible solutions are:

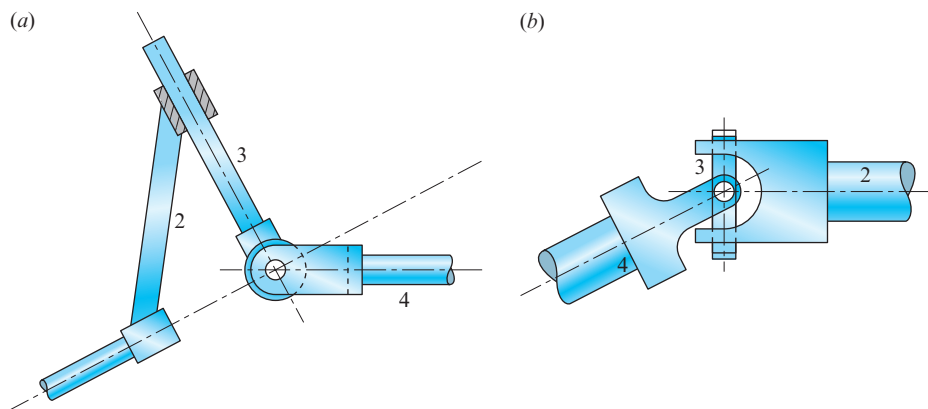


Figure 1.21 Coupling mechanisms for intersecting shafts.

1. A rigid link pivoted at each end to a slider.
2. A belt or chain connecting the two sliders with the use of a guide pulley or sprocket.
3. Rack gear teeth cut on each slider and the connection completed using one or more gears.
4. A flexible cable connector.

Stop, Pause, and Hesitation Mechanisms In an automotive engine a valve must open, remain open for a short period of time, and then close. A conveyor line may need to halt for an interval of time while an operation is being performed and then continue its motion. Many similar requirements occur in the design of machines. Torfason [9] classifies these as *stop-and-dwell*, *stop-and-return*, *stop-and-advance*, and so on. Such requirements can often be met using cam-and-follower mechanisms (Chap. 6), indexing mechanisms, including those of Fig. 1.13, ratchets, linkages at the limits of their motion, and gear-and-clutch mechanisms.

The six-bar linkage of Fig. 1.22 is a clever method to obtain a rocking motion (of link 6) containing a dwell. This linkage, an extension of the four-bar linkage, consisting of frame 1, crank 2, coupler 3, and rocker 4, can be designed such that point C on the coupler generates the curve shown by dashed lines. A portion of this curve will then fit closely to a circular arc whose radius is equal to the length of link 5—that is, distance DC . Thus, when point C traverses this portion of the coupler curve, link 6, the output rocker, is stationary.

Curve Generators The connecting rod, or coupler, of a planar four-bar linkage may be imagined as an infinite plane extending in all directions but pin-connected to the input and output cranks. Then, during motion of the linkage, any point attached to the plane of the coupler generates a path with respect to the fixed link; this path is called a *coupler curve*. Two of these paths, namely those generated by the pin connectors of the coupler, are true circles with centers at the two fixed pivots. However, other points can be found that trace much more complex curves.

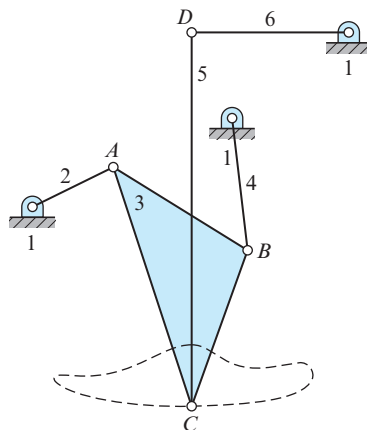


 **Figure 1.22** Six-bar stop-and-dwell linkage.

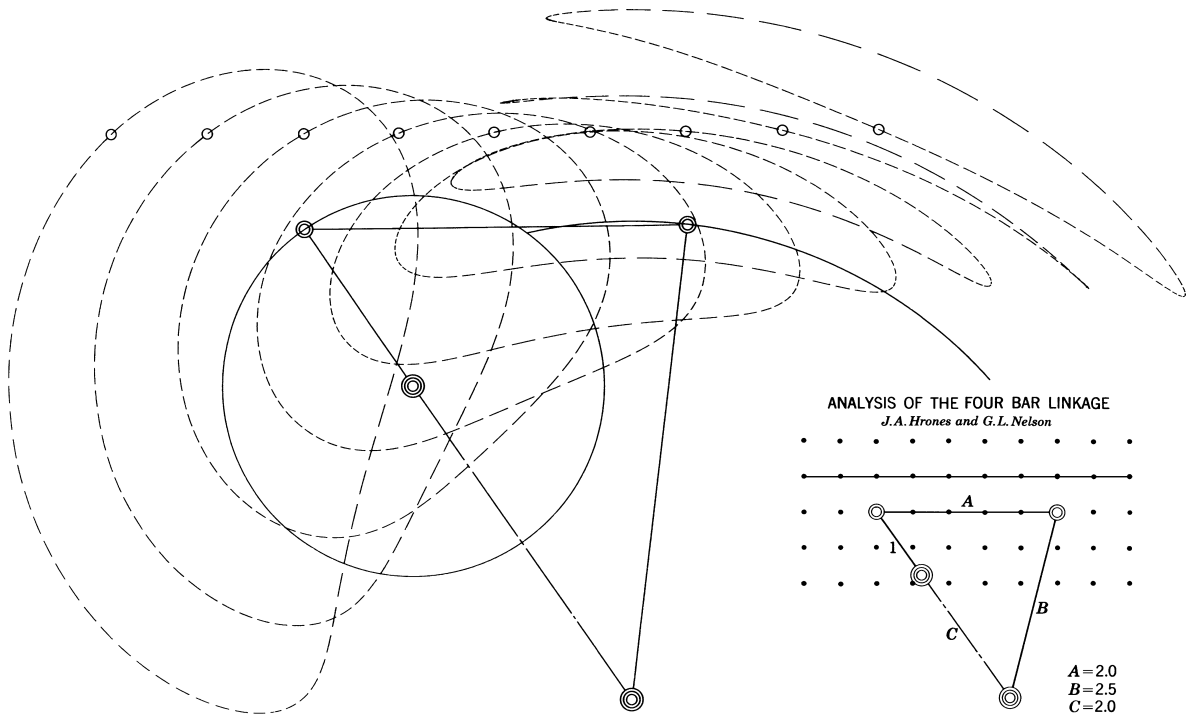


 **Figure 1.23** A set of coupler curves [6].

One of the best sources of coupler curves for the four-bar linkage is the Hrones and Nelson atlas [6]. This book consists of a set of 11 in $\times$ 17 in drawings containing over 7,000 coupler curves of crank-rocker linkages. Figure 1.23 is a reproduction of a typical page of this atlas (by permission of the publishers). In each case, the crank has unit length, and the lengths of the remaining links vary from page to page to produce the different combinations. On each page a number of different coupler points are chosen, and their coupler curves are shown. This atlas of coupler curves has proven to be invaluable to the designer who needs a linkage to generate a curve with specified characteristics.

The algebraic equation of a four-bar linkage coupler curve is, in general, a sixth-order polynomial [1]; thus, it is possible to find coupler curves with a wide variety of shapes and many interesting features. Some coupler curves have sections that are nearly straight line segments; others have almost exact circular arc segments; still others have one or more cusps or cross over themselves like a figure eight. Therefore, it is often not necessary to use a mechanism with a large number of links to obtain a complex motion of a coupler point.

Yet the complexity of the coupler-curve equation is also a hinderance; it means that hand-calculation methods can become very cumbersome. Thus, over the years, many mechanisms have been designed by strictly intuitive procedures and proven with cardboard models, without the use of kinematic principles or procedures. Until quite recently,

those techniques that did offer a rational approach have been graphic, avoiding tedious computations. Finally, with the availability of digital computers, and particularly with computer graphics, useful design methods are now emerging that can deal directly with the complex calculations required without burdening the designer with the computational drudgery (Sec. 10.9 has details on some of these).

One of the more curious and interesting facts about the coupler-curve equation is that the same curve can always be generated by three different four-bar linkages. These are called *cognate linkages*, and the theory is developed in Sec. 9.10.

Straight-Line Generators In the late 17th century, before the development of the milling machine, it was extremely difficult to machine straight, flat surfaces. For this reason, good prismatic joints with close clearances were not available. During that era, much thought was given to the problem of attaining a straight line as a part of the coupler curve of a linkage having only revolute connections. Probably the best-known result of this search is the straight-line mechanism developed by Watt for guiding the piston of early steam engines. Figure 1.24a shows a four-bar linkage, known as *Watt's linkage*, which generates an approximate straight line as a part of its coupler curve. Although the coupler point (tracing point P) does not generate an exact straight line, a good approximation is achieved over a considerable distance of travel.

Another four-bar linkage in which the tracing point P generates an approximate straight-line coupler-curve segment is Roberts' linkage (Fig. 1.24b). The dashed lines BP and CP in the figure indicate that the linkage is defined by forming three congruent isosceles triangles; thus, $BC = AP = PD = AD/2$.

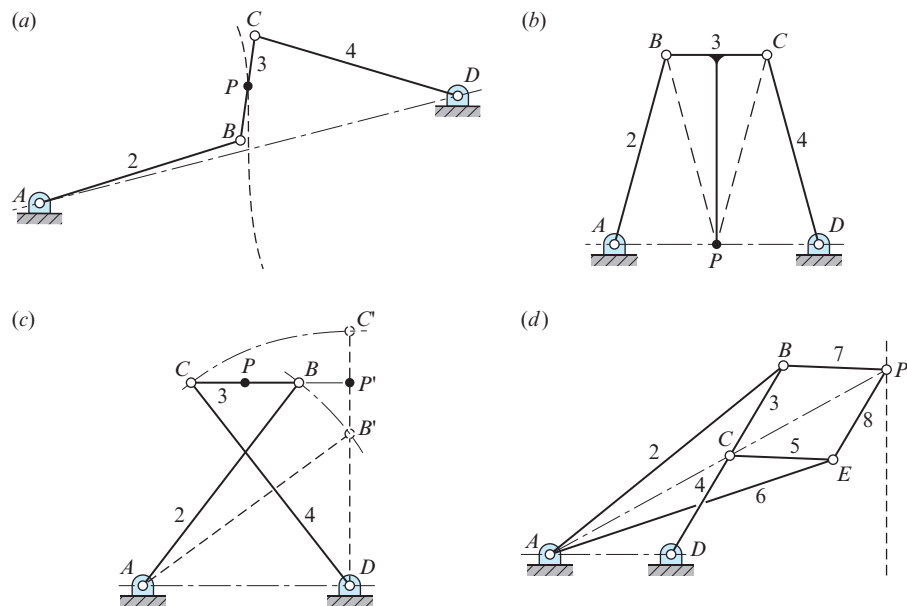


Figure 1.24 (a) Watt's linkage; (b) Roberts' linkage; (c) Chebyshev linkage; (d) Peaucellier invensor.

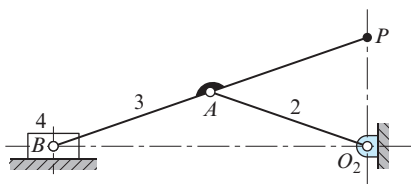


Figure 1.25 $AB = AP = O_2A$.

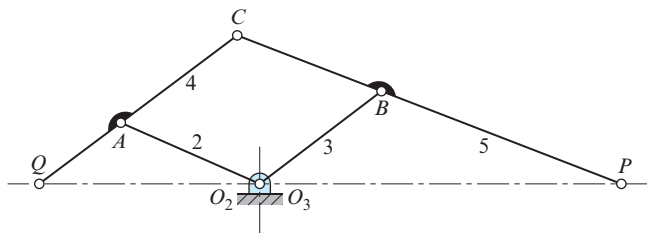


Figure 1.26 Pantograph linkage.

The tracing point P (the midpoint of coupler link BC) of the Chebychev linkage shown in Fig. 1.24c also generates an approximate straight line. The linkage forms a 3:4:5 triangle when link 4 is in the vertical posture, as shown by the dashed lines; thus, $DB' = 3$ units, $AD = 4$ units, and $AB' = 5$ units. Note that $AB = DC$, $DC' = 5$ units, and tracing point P' is the midpoint of dashed line $B'C'$. Also, note that $DP'C$ forms another 3:4:5 triangle, and hence the line containing P and P' is parallel to the ground link AD .

A linkage that generates an exact straight line is the Peaucillier invisor shown in Fig. 1.24d. The conditions describing its geometry are that $BC = BP = EC = EP$ and $AB = AE$ such that, by symmetry, points A , C , and P always lie on a straight line passing through A . Under these conditions $AC \cdot AP = k$, a constant, and the curves generated by C and P are said to be *inverses* of each other. If we place the other fixed pivot D such that $AD = CD$, then point C must trace a circular arc while point P follows an exact straight line. Another interesting property is that if AD is not equal to CD , then point P traces a true circular arc of very large radius. This was the first straight-line generator, and it was important in the development of the steam engine.

Figure 1.25 shows another linkage that generates exact straight-line motion: the Scott-Russell linkage. However, note that it employs a slider.

The *pantograph* shown in Fig. 1.26 is used to trace figures at a larger or smaller size. If, for example, point P traces a map, then a pen at Q will draw a similar map at a smaller scale. The dimensions O_2A , AC , CB , and BO_3 must conform to an equal-sided parallelogram.

Torfason [9] also includes robots, speed-changing devices, computing mechanisms, function generators, loading mechanisms, and transportation devices in his classification. Many of these utilize arrangements of mechanisms already presented. Others appear in some of the chapters to follow.

1.8 KINEMATIC INVERSION

In Sec. 1.4 we noted that every mechanism has a fixed link called the frame. When different links are chosen as the frame, the *relative* motions between the various links are not altered, but their *absolute* motions (those measured with respect to the frame) may be changed significantly. The process of choosing different links for the frame is known as *kinematic inversion*. In an n -link mechanism, choosing each link in turn as the frame yields n distinct kinematic inversions—that is, n different mechanisms.

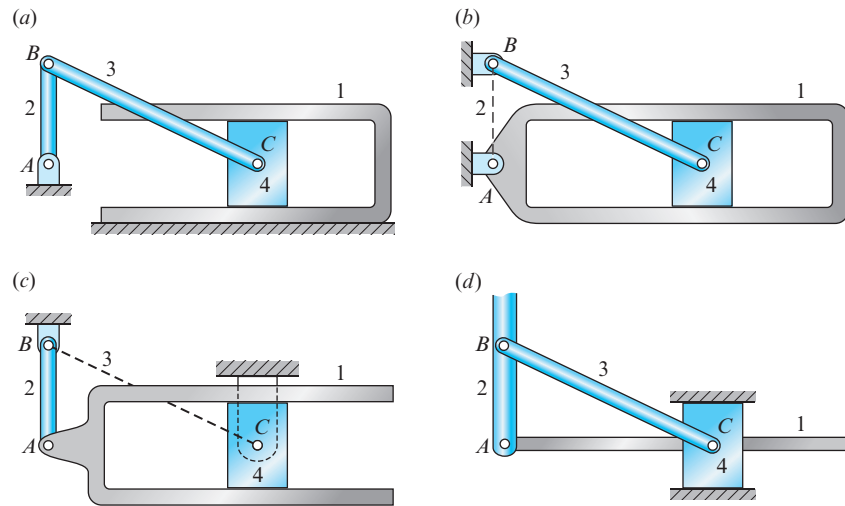


Figure 1.27 Four inversions of the slider-crank linkage.

For illustration, consider the slider-crank linkage. Figure 1.27a shows the first inversion, which is found in most internal combustion engines, where the frame is the cylinder block, link 1. Link 4, the piston, is driven by the expanding gases, and this movement provides the input. Link 3, the connecting rod, connects link 2, the crank, which rotates as the driven output. By reversing the roles of the input and output, this linkage can be used as a compressor.

Figure 1.27b shows another kinematic inversion, where link 2 is stationary. Link 1, formerly the frame, now rotates about revolute A. This inversion of the slider-crank linkage was the basis of the rotary engine found in many early aircraft.

Yet another inversion is shown in Fig. 1.27c; it has link 3, formerly the connecting rod, as the fixed link. This linkage was used to drive the wheels of early steam locomotives, with link 2 being attached to a wheel.

The final inversion, shown in Fig. 1.27d, has the piston, link 4, stationary. Although it is not found in engines, by rotating the figure 90° clockwise, this linkage is recognized as part of a garden water pump. Note in this inversion that the prismatic joint connecting links 1 and 4 is also inverted—that is, the “inside” and “outside” elements of the joint are reversed.

1.9 GRASHOF'S LAW

A very important consideration when designing a mechanism to be driven by a motor, obviously, is to ensure that the input crank can make a complete revolution. Mechanisms in which no link can make a complete revolution would not be useful in such applications. For the four-bar linkage, there is a very simple test of whether this is the case.

Grashof's law states that, *for a planar four-bar linkage, the sum of the shortest and longest link lengths cannot be greater than the sum of the remaining two link lengths if*

there is to be continuous relative rotation between two links. This is shown in Fig. 1.28, where the longest link has length l , the shortest link has length s , and the other two links have lengths p and q . In this notation, Grashof's law states that one of the links, in particular the shortest link, will rotate continuously relative to the other three links if and only if

$$s + l \leq p + q. \quad (1.6)$$

If this inequality is not satisfied, no link will make a complete revolution relative to another.

Attention is called to the fact that nothing in Grashof's law specifies the order in which the links are connected or which link of the four-bar linkage is fixed. We are free, therefore, to fix any of the four links. When we do so, we create variations of the four-bar linkage, four of which are shown in Fig. 1.28. All of these fit Grashof's law, and, in each, link s makes a complete revolution relative to the other links. The different variations are distinguished by the location of link s relative to the fixed link.

If the shortest link, s , is adjacent to the fixed link, as shown in Fig. 1.28a and Fig. 1.28b, we obtain what is called a *crank-rocker linkage*. Link s is, of course, the crank, since it is able to rotate continuously, and link p , which can only oscillate between limits, is the rocker.

The drag-link (or double-crank linkage) is obtained by fixing the shortest link, s , as the frame. In this variation, shown in Fig. 1.28c, both links adjacent to s can rotate continuously, and both are properly described as cranks; the shorter of the two is generally used as the input. Although this is a very common linkage, you will find it an interesting challenge to devise a practical working model that can operate through the full cycle.

By fixing the link opposite to link s , we obtain the final variation, the double-rocker linkage of Fig. 1.28d. Note that although link s is able to make a complete revolution, neither link adjacent to the frame can do so; both must oscillate between limits and, therefore, are rockers.

In each of the variations shown, the shortest link, s , is adjacent to the longest link, l . However, exactly the same types of linkages will occur if the longest link, l , is opposite the shortest link, s ; you should demonstrate this for your own satisfaction.

Reuleaux approaches the problem somewhat differently but, of course, obtains the same results. In his approach, and using Fig. 1.29, the links are labeled

s the crank, p the lever,
 l the coupler, q the frame,

where l need not be the longest link. Then the following conditions apply:

$$s + l + p \geq q, \quad (1.7a)$$

$$s + l - p \leq q, \quad (1.7b)$$

$$s + q + p \geq l, \quad (1.7c)$$

$$s + q - p \leq l. \quad (1.7d)$$

If these four conditions are not satisfied, then the results are as illustrated in Fig. 1.29.

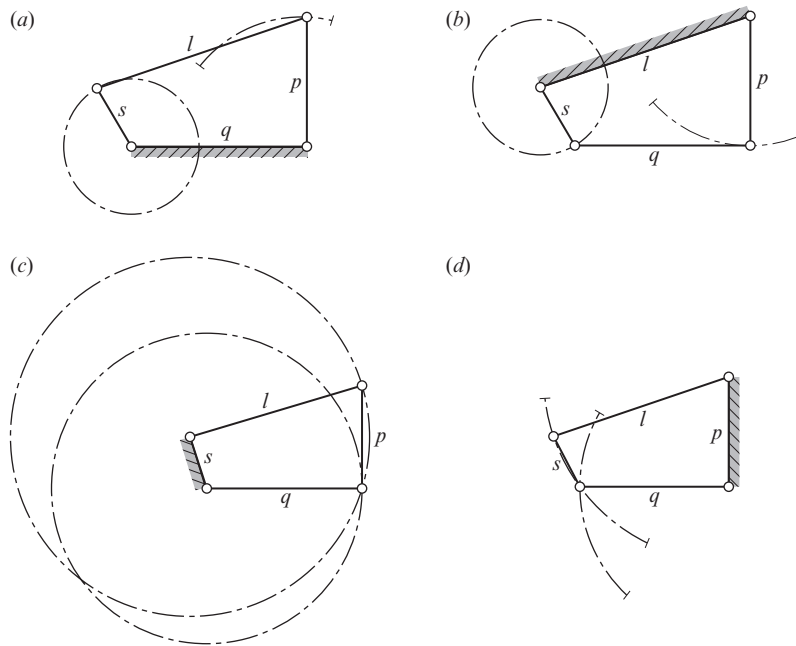


Figure 1.28 (a, b) Crank-rocker linkages; (c) drag-link linkage; and (d) double-rocker linkage.

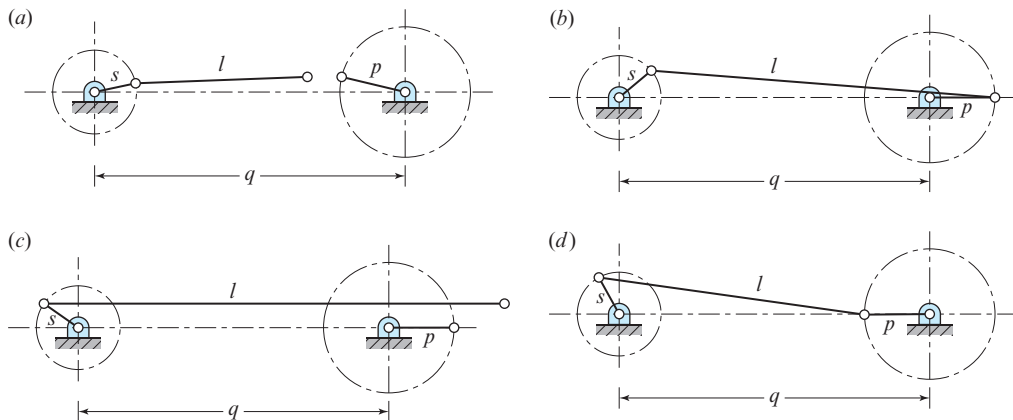


Figure 1.29 (a) Eq. (1.7a); $s + l + p < q$, and the links cannot be connected; (b) Eq. (1.7b); $s + l - p > q$, and s is incapable of rotation; (c) Eq. (1.7c); $s + q + p < l$, and the links cannot be connected; (d) Eq. (1.7d); $s + q - p > l$, and s is incapable of rotation.

EXAMPLE 1.6

Is the linkage shown in Fig. 1.30 a crank-rocker, a double-rocker, or a drag-link four-bar linkage?

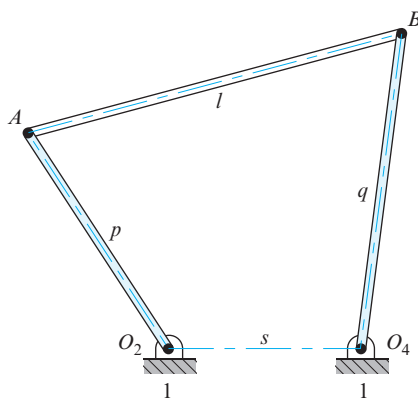


Figure 1.30 $p = 4$ in, $l = 6$ in, $q = 5$ in, and $s = 3$ in.

SOLUTION

Substituting the link lengths into Eq. (1.6) gives

$$3 \text{ in} + 6 \text{ in} \leq 4 \text{ in} + 5 \text{ in}$$

or

$$9 \text{ in} \leq 9 \text{ in}.$$

Therefore, the linkage satisfies Grashof's law; that is, the linkage is a Grashof four-bar linkage. Since the shortest link is the frame, the two links adjacent to the shortest link can both rotate continuously (Fig. 1.28c), and both are properly described as cranks. Therefore, Fig. 1.30 shows a drag-link linkage. *Ans.*

1.10 MECHANICAL ADVANTAGE

In general, the *mechanical advantage* of a mechanism is defined as the ratio of the force or torque exerted by the driven link to the necessary force or torque required at the driver. With the widespread use of the four-bar linkage, a few remarks are in order here that will help us judge the quality of such a linkage for its intended application.

Consider the crank-rocker four-bar linkage shown in Fig. 1.31, where link 2 is the driver and link 4 is the follower.

In Sec. 3.19, we will show that the mechanical advantage of the four-bar linkage can be written as

$$MA = \frac{R_{CD} \sin \gamma}{R_{BA} \sin \beta}. \quad (1.8)$$

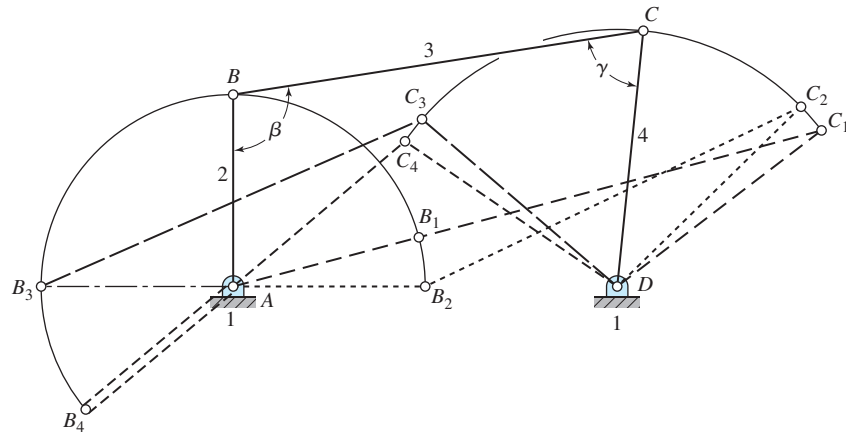


Figure 1.31 Crank-rocker four-bar linkage.

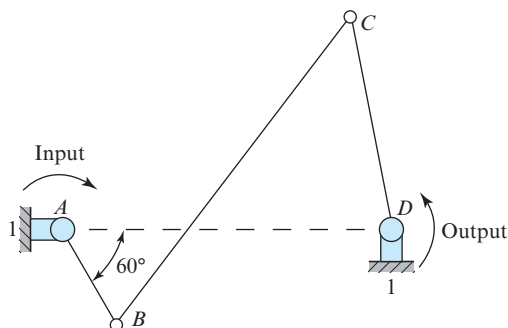
Note that this is directly proportional to the sine of the angle γ between the coupler and the follower, and is inversely proportional to the sine of angle β between the coupler and the driver. Of course, both these angles, and therefore the mechanical advantage, are continuously changing as the linkage moves.

When the sine of angle β becomes zero, the mechanical advantage becomes infinite; thus, at such a posture, only a small input torque is necessary to produce a very large output torque load. This is the case when the driver AB is directly in line with the coupler BC as shown in Fig. 1.31; it occurs when the crank is in posture AB_1 and again when the crank is in posture AB_4 . Note that these also define the extreme postures of travel of the rocker DC_1 and DC_4 . When the four-bar linkage is in either of these postures, the mechanical advantage is infinite—that is, $\beta = 0^\circ$ or $\beta = 180^\circ$ —and the linkage is said to be in a *toggle* (or *limit*) posture.

The angle γ between the coupler and the follower is called the *transmission angle*. As this angle becomes smaller, the mechanical advantage decreases and even a small amount of friction might cause the mechanism to lock or jam. A common rule of thumb is that a four-bar linkage should not be used to overcome a load in a region where the transmission angle is less than, say, 45° or 50° . The extreme values of the transmission angle occur when crank AB lies along the line of the frame, AD . The transmission angle is minimal when the crank is in posture AB_2 and is maximal when the crank is in posture AB_3 (see Fig. 1.31). Because of the ease with which it can be visually inspected, the transmission angle has become a commonly accepted measure of the quality of a design of the four-bar linkage. A double-rocker four-bar linkage has a *dead-center* posture when links 3 and 4 lie along a straight line. In a dead-center posture, the transmission angle is $\gamma = 0^\circ$ or $\gamma = 180^\circ$, and the linkage is locked. The designer must either avoid such a posture or provide an external force, such as a spring, to unlock the linkage.

EXAMPLE 1.7

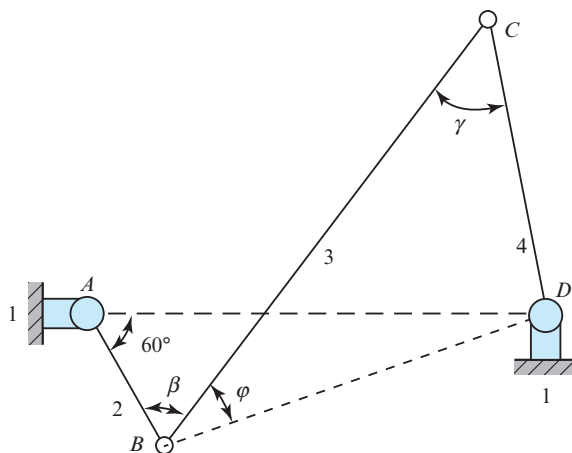
Determine the mechanical advantage of the four-bar linkage in the posture shown in Fig. 1.32.

**Figure 1.32**

$R_{DA} = 180$ mm, $R_{BA} = 60$ mm, $R_{CB} = 210$ mm, and $R_{CD} = 120$ mm.

SOLUTION

Angles γ and β for Eq. (1.8) are as shown in Fig. 1.33 and can be obtained from trigonometry.

**Figure 1.33** Angles γ and β .

Using the law of cosines,

$$R_{DB} = \sqrt{(60 \text{ mm})^2 + (180 \text{ mm})^2 - 2(60 \text{ mm})(180 \text{ mm}) \cos 60^\circ} = 158.745 \text{ mm.} \quad (1)$$

Also, the angles

$$\gamma = \cos^{-1} \left[\frac{(158.745 \text{ mm})^2 - (210 \text{ mm})^2 - (120 \text{ mm})^2}{-2(210 \text{ mm})(120 \text{ mm})} \right] = 48.65^\circ, \quad (2)$$

and

$$\phi = \cos^{-1} \left[\frac{(120 \text{ mm})^2 - (210 \text{ mm})^2 - (158.745 \text{ mm})^2}{-2(210 \text{ mm})(158.745 \text{ mm})} \right] = 34.57^\circ. \quad (3)$$

Finally, the sum of the angles

$$\beta + \phi = \cos^{-1} \left[\frac{(180 \text{ mm})^2 - (60 \text{ mm})^2 - (158.745 \text{ mm})^2}{-2(60 \text{ mm})(158.745 \text{ mm})} \right] = 100.90^\circ. \quad (4)$$

Subtracting Eq. (3) from Eq. (4) gives

$$\beta = 100.90^\circ - 34.57^\circ = 66.33^\circ. \quad (5)$$

Then, substituting Eqs. (2) and (5) into Eq. (1.8), the mechanical advantage of the four-bar linkage in the given posture is

$$MA = \frac{(120 \text{ mm}) \sin 48.65^\circ}{(60 \text{ mm}) \sin 66.33^\circ} = 1.64. \quad \text{Ans.}$$

Note that mechanical advantage, toggle posture, transmission angle, and dead-center posture depend on the choice of the driver and driven links. For example, in Fig. 1.31, if link 4 is used as the driver and link 2 as the driven link, then the roles of γ and β are reversed. In this case, the linkage has no toggle posture, and its mechanical advantage becomes zero when link 2 is in posture AB_1 or AB_4 , because the transmission angle is zero. These and other methods of rating the suitability of the four-bar linkage or other mechanisms are discussed more thoroughly in Sec. 3.19.

1.11 REFERENCES

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PROBLEMS

1.1 Sketch at least six different examples of the use of a planar four-bar linkage in practice. These can be found in the workshop, in domestic appliances, on vehicles, on agricultural machines, and so on.

†1.2 The link lengths of a planar four-bar linkage are 1 in, 3 in, 5 in, and 5 in. Assemble the links in all possible combinations and sketch the four inversions of each. Do these linkages satisfy Grashof's law? Describe each inversion by name (for example, a crank-rocker linkage or a drag-link linkage).

†1.3 A crank-rocker linkage has a 100-mm frame, a 25-mm crank, a 90-mm coupler, and a 75-mm rocker. Draw the linkage and find the maximum and minimum values of the transmission angle. Locate both toggle postures and record the corresponding crank angles and transmission angles.

†1.4 Plot the complete path of coupler point C.

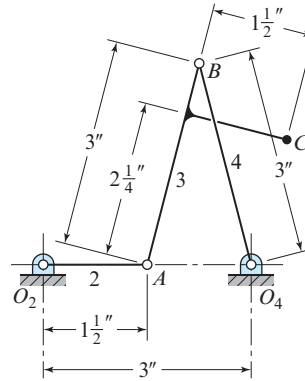


Figure P1.4

†1.5 Find the mobility of each mechanism.

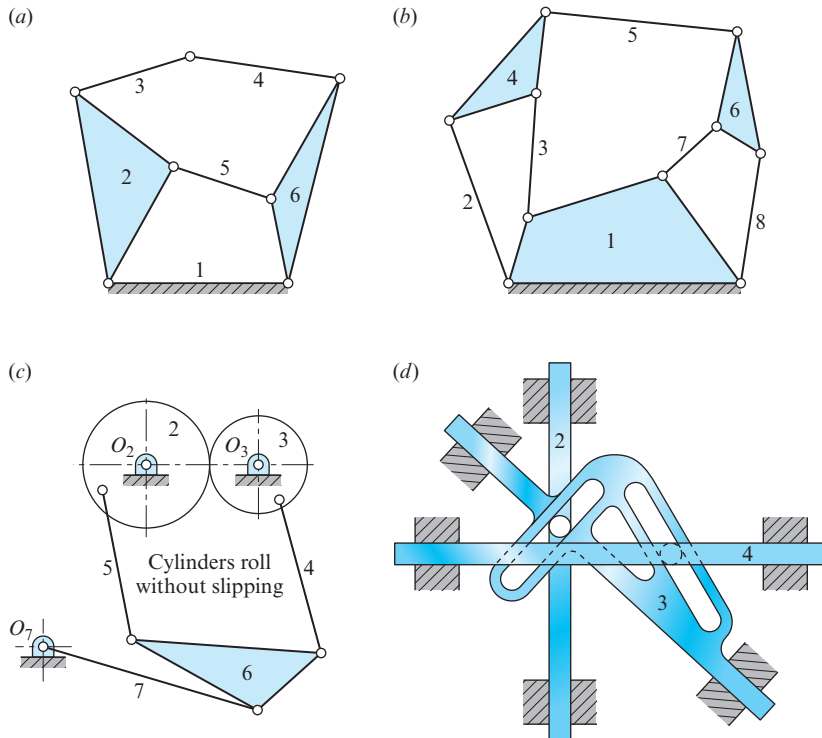


Figure P1.5

- 1.6** Use the Kutzbach criterion to determine the mobility of the mechanism.

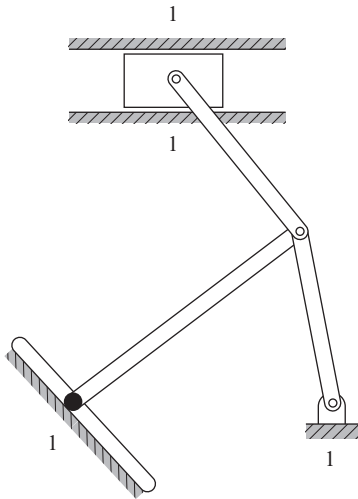


Figure P1.6

- 1.7** Sketch a planar linkage with only revolute joints and a mobility of $m = 1$ that contains a quaternary link. How many distinct variations of this linkage can you find?
- †1.8** Use the Kutzbach criterion to determine the mobility of the mechanism. Clearly number each link, and label the lower pairs (j_1 joints) and higher pairs (j_2 joints).

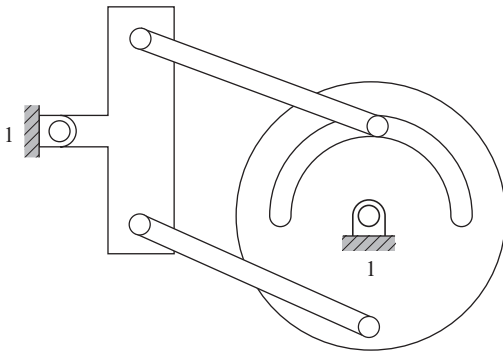


Figure P1.8

- †1.9** Determine the number of links, the number of lower pairs, and the number of higher pairs. Use

the Kutzbach criterion to determine the mobility of the mechanism. Is the answer correct? Briefly explain.

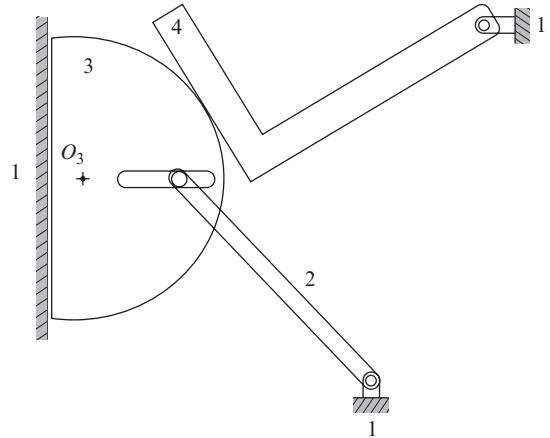


Figure P1.9

- 1.10** Use the Kutzbach criterion to determine the mobility of the mechanism. Clearly number each link, and label the lower pairs and higher pairs.

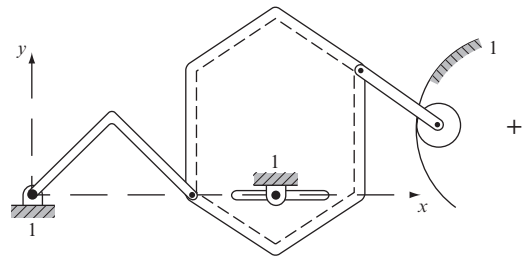


Figure P1.10

- 1.11** Determine the number of links, the number of lower pairs, and the number of higher pairs. Treat rolling contact to mean rolling with no slipping.

Using the Kutzbach criterion, determine the mobility. Is the answer correct? Briefly explain.

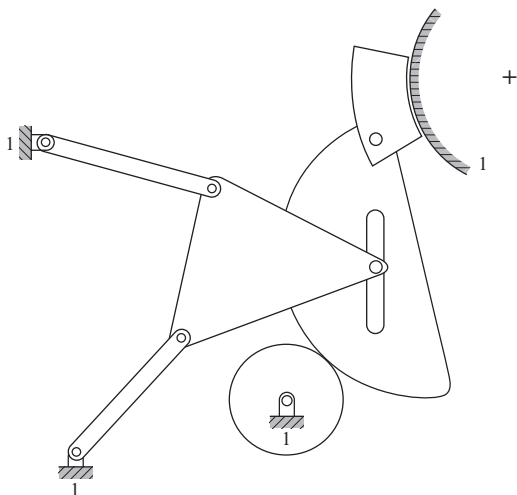


Figure P1.11

- 1.12** Does the Kutzbach criterion provide the correct result for this mechanism? Briefly explain why or why not.

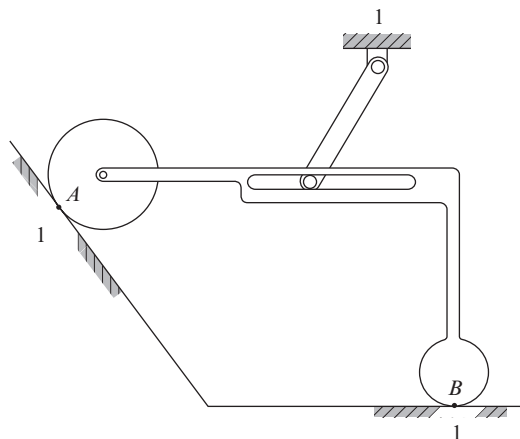


Figure P1.12

- 1.13** The mobility of the mechanism is $m = 1$. Use the Kutzbach criterion to determine the number of lower pairs and the number of higher pairs. Is the wheel rolling without slipping, or rolling and slipping, at point A on the wall?

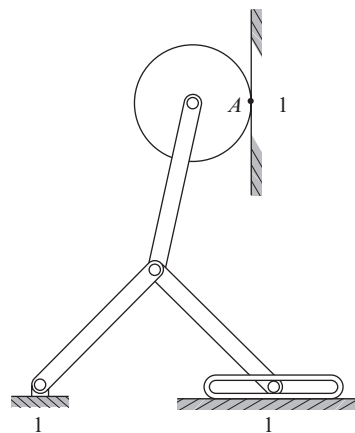


Figure P1.13

- 1.14** Devise a practical working model of the drag-link linkage.
- †1.15** Find the advance-to-return ratio of the linkage of Prob. 1.3.
- †1.16** Plot the complete coupler curve of the Roberts' linkage shown in Fig. 1.24b. Use $AB = CD = AD = 2.5$ in and $BC = 1.25$ in.
- 1.17** If the handle of the differential screw in Fig. 1.11 is turned 15 revolutions clockwise, how far and in what direction does the carriage move?
- †1.18** Show how the linkage of Fig. 1.15b can be used to generate a sine wave.
- †1.19** Devise a crank-rocker four-bar linkage, as in Fig. 1.14c, having a rocker angle of 60° . The rocker length is to be 0.50 m.
- 1.20** A crank-rocker four-bar linkage is required to have an advance-to-return ratio $Q = 1.2$. The rocker is to have a length of 2.5 in and oscillate through a total angle of 60° . Determine a suitable set of link lengths for the remaining three links of the four-bar linkage.

- 1.21** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

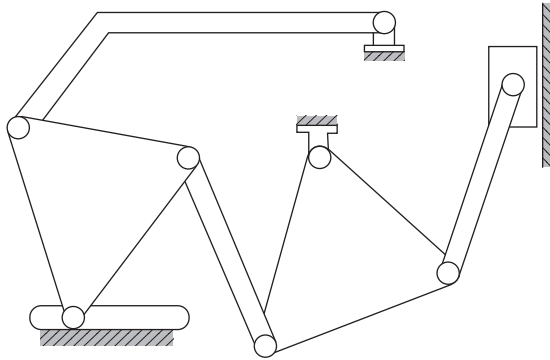


Figure P1.21

- 1.22** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

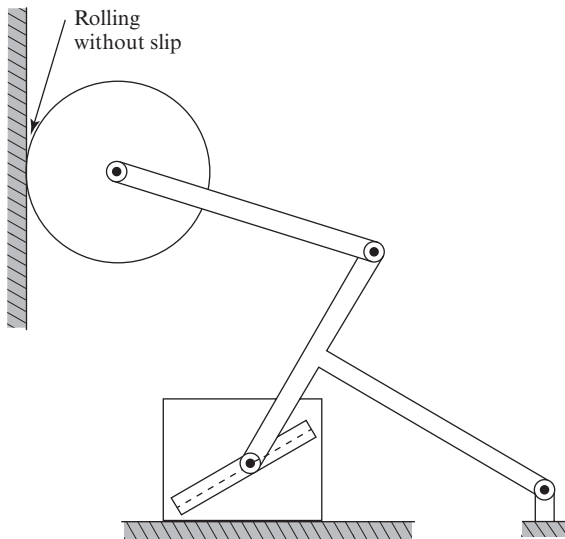


Figure P1.22

- 1.23** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

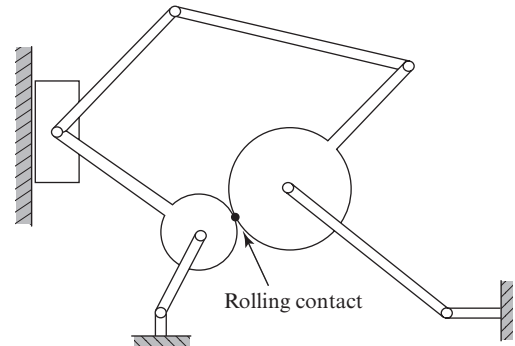


Figure P1.23

- 1.24** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.



Figure P1.24

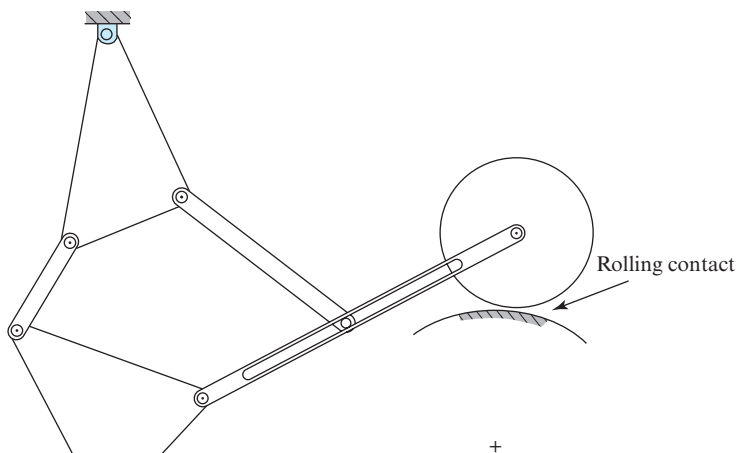


Figure P1.24

1.25 Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

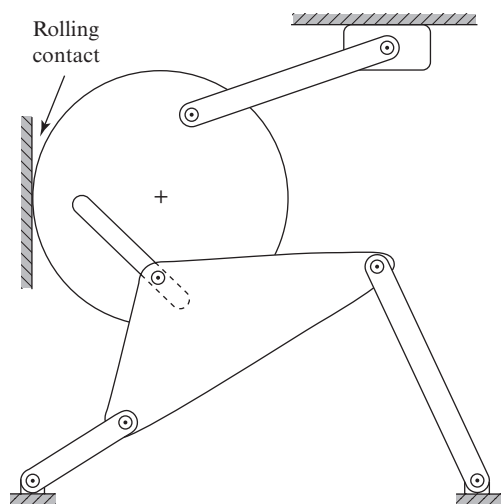


Figure P1.25

1.26 Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

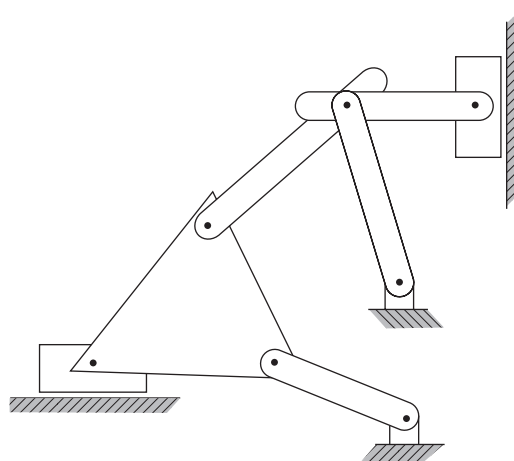


Figure P1.26

- 1.27** Determine the mechanical advantage of the four-bar linkage in the posture shown.

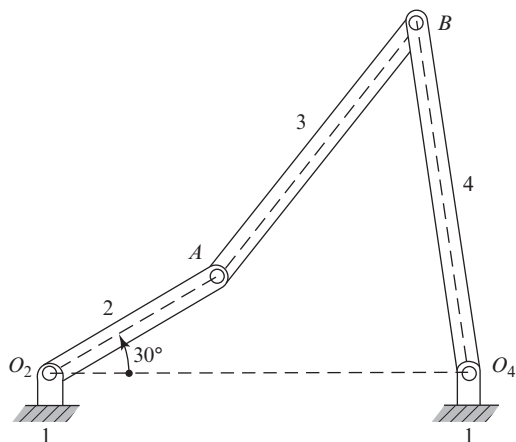


Figure P1.27 $O_2O_4 = 120$ mm, $O_2A = 60$ mm, $AB = 100$ mm, and $O_4B = 130$ mm.

- 1.28** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

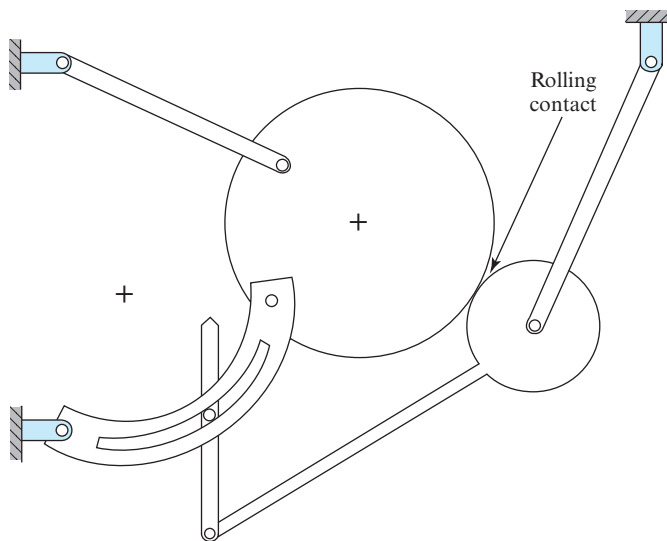


Figure P1.28

- 1.29** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

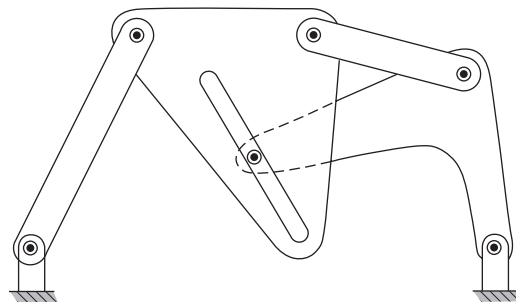


Figure P1.29

- 1.30** The rocker of a crank-rocker four-bar linkage is required to have a length of 6 in and swing through a total angle of 30° . Also, the advance-to-return ratio of the linkage is required to be 1.75. Determine a suitable set of link lengths for the remaining three links.
- 1.31** Determine a suitable set of link lengths for a slider-crank linkage such that the stroke will be 500 mm and the advance-to-return ratio will be 1.8.

- 1.32** Determine the transmission angle and the mechanical advantage of the four-bar linkage in the posture shown. What type of four-bar linkage is this?

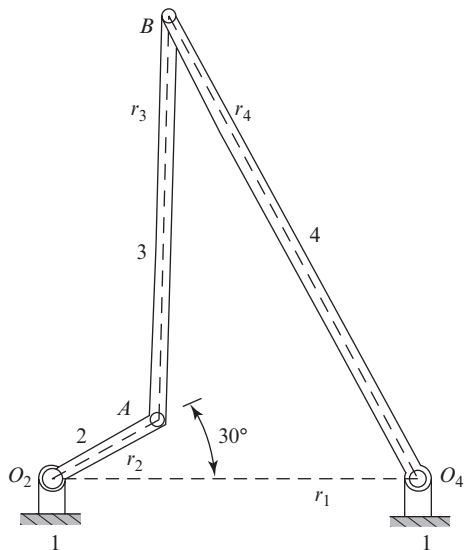


Figure P1.32 $R_{AO_2} = 20 \text{ mm}$, $R_{BA} = 70 \text{ mm}$, $R_{BO_4} = 90 \text{ mm}$, and $R_{O_4O_2} = 60 \text{ mm}$.

- 1.33** Determine the mobility of the mechanism. Number each link and label the lower pairs and the higher pairs. Identify a suitable input, or inputs, for the mechanism.

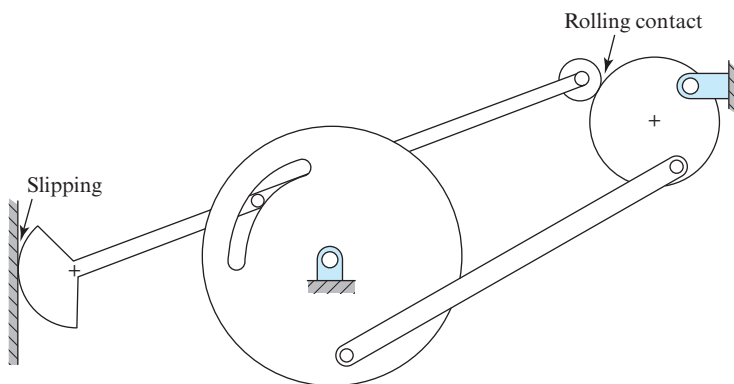


Figure P1.33

- 1.34** A crank-rocker four-bar linkage is shown in one of its two toggle postures. Find θ_2 and θ_4 corresponding to each toggle posture. What is the total rocking angle of link 4? What are the transmission angles at the extremes?

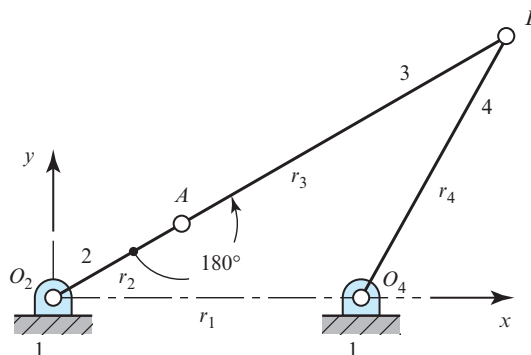


Figure P1.34 $R_{AO_2} = 8 \text{ in}$, $R_{BA} = 20 \text{ in}$, $R_{BO_4} = 16 \text{ in}$, and $R_{O_4O_2} = 16 \text{ in}$.

- 1.35** Find θ_2 and θ_4 corresponding to a dead-center posture. Is there a toggle posture?
- 1.36** Determine the advance-to-return ratio for the slider-crank linkage with the offset e . Also,

determine in which direction the crank should rotate to provide quick return.

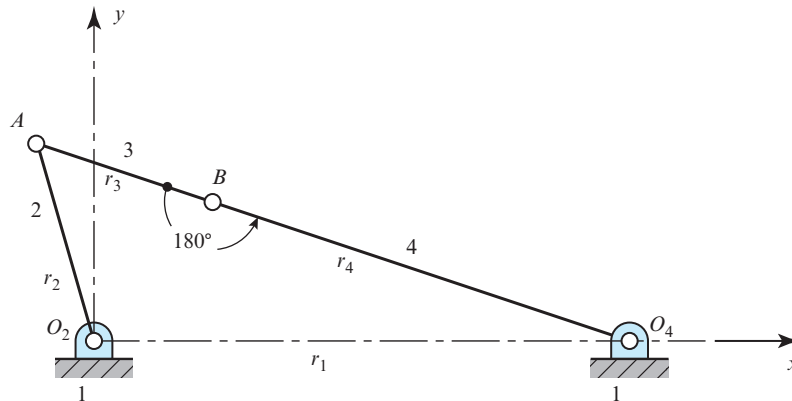


Figure P1.35 $R_{AO_2} = 110$ mm, $R_{BA} = 100$ mm, $R_{BO_4} = 240$ mm, and $R_{O_4O_2} = 280$ mm.

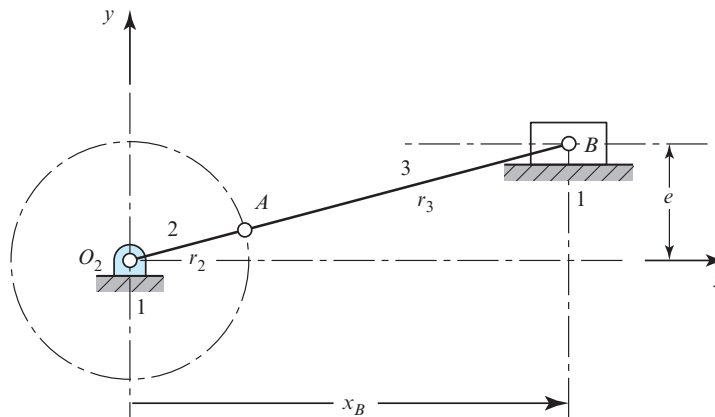
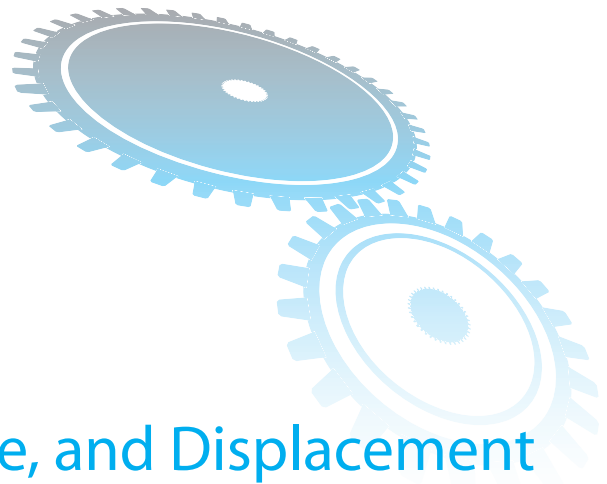


Figure P1.36 Offset slider-crank linkage in the two dead-center postures.

2

Position, Posture, and Displacement



In analyzing motion, the first and most basic problem encountered is that of defining and dealing with the concepts of position, posture, and displacement. Since motion can be thought of as a time series of displacements between successive positions of a point or postures of a body, it is important to understand exactly the meaning of the terms *position* and *posture*. Rules or conventions will be established here to make the definitions precise.

2.1 LOCUS OF A MOVING POINT

In speaking of the position of a point, or particle, we are really answering the question: Where is the point, or what is its location? We are speaking of something that exists in nature, and we are considering the question of how to express this (in words or symbols or numbers) in such a way that the meaning is clear. We soon discover that position cannot be defined on a truly absolute basis. We must define the position of a point in terms of some agreed-upon frame of reference (that is, some reference frame or reference coordinate system).

Once we have agreed upon the right-handed three-dimensional coordinate system xyz as the frame of reference, as shown in Fig. 2.1, we can say that point P is located x units along the x axis, y units along the y axis, and z units along the z axis *from the origin* O . In this statement, we see that three vitally important parts of the definition depend on the existence of the reference coordinate system:

1. The *origin* O of the coordinate system provides an agreed-upon location from which to measure the location of point P .

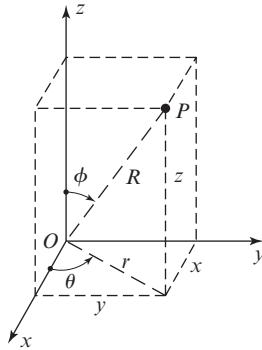


Figure 2.1 Right-handed three-dimensional coordinate system illustrating how point P is located algebraically.

2. The *coordinate axes* provide agreed-upon *directions* along which the measurements are to be made; they also provide known lines and planes for the definition and measurement of angles.
3. The unit distance along any of the axes provides a scale for quantifying distances.

These observations are not restricted to the Cartesian coordinates (x, y, z) of point P . All three properties of the coordinate system are also necessary in defining cylindric coordinates (r, θ, z) , spheric coordinates (R, θ, ϕ) , or any other coordinates of point P . The same properties are also required if point P is restricted to remain in a single plane and a two-dimensional coordinate system is used. No matter how it is defined, the concept of the position of a point cannot be realized without the definition of a reference coordinate system.

The direction cosines for locating point P are defined as

$$\cos \alpha = \frac{x}{R}, \quad \cos \beta = \frac{y}{R}, \quad \text{and} \quad \cos \gamma = \frac{z}{R}, \quad (2.1)$$

where α, β , and γ are the angles measured from the positive x -, y -, and z -coordinate axes, respectively, to the directed line OP .

One means of expressing the motion of a point, or particle, is to define its components along the reference axes as functions of some parameter, such as time:

$$x = x(t), \quad y = y(t), \quad z = z(t). \quad (2.2)$$

If these relations are known, then the position of the point can be found for any time t . This is the general case for the motion of a point, or particle, and is illustrated in the following example.

EXAMPLE 2.1

Describe the motion of a particle, P whose position changes with time according to the equations $x = a \cos 2\pi t$, $y = a \sin 2\pi t$, and $z = bt$.

SOLUTION

Substitutng values for t from 0 to 2 s gives the coordinates listed in Table 2.1. As shown in Fig. 2.2, the particle moves with *helical motion* with radius a around the z axis and with a lead of b . Note that if $b = 0$, then $z(t) = 0$, the moving particle is confined to the xy plane, and the motion is a circle with its center at the origin.

Table 2.1 Helical Motion of a Particle

t	x	y	z
0	a	0	0
1/4	0	a	$b/4$
1/2	$-a$	0	$b/2$
3/4	0	$-a$	$3b/4$
1	a	0	b
5/4	0	a	$5b/4$
3/2	$-a$	0	$3b/2$
7/4	0	$-a$	$7b/4$
2	a	0	$2b$

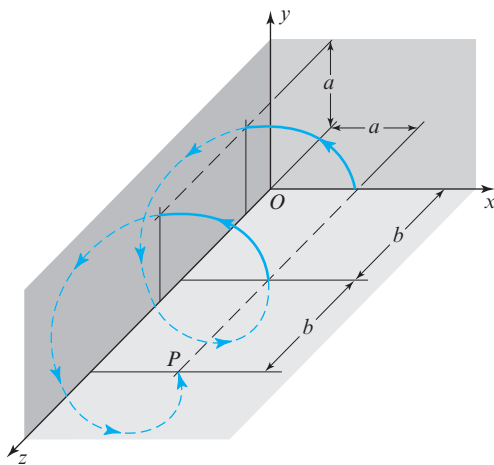


Figure 2.2 Helical motion of a particle, P .

So far, we have been using the words *point* and *particle* interchangeably. Henceforth, when the word *point* is used, we have in mind something that has no dimensions—that is, something with zero length, zero width, and zero thickness. When the word *particle* is used, we have in mind something whose dimensions are unimportant—that is, a material body whose dimensions are negligible or, in other words, a body whose dimensions have no effect on the analysis to be performed.

The successive positions of a moving point define a line or curve. This curve has no thickness, since the point has no dimensions. However, the curve does have length,

since the point occupies different positions as time changes. This curve, representing the successive positions of the point, is called the *path* or *locus* of the moving point in the reference coordinate system.

If three coordinates are necessary to describe the path of a moving point, the point is said to have *spatial motion*. If the path can be described by only two coordinates—that is, if the coordinate axes can be chosen such that one coordinate is always zero or constant—the path lies in a single plane, and the point is said to have *planar motion*. Sometimes it happens that the path of a point can be described by a single coordinate—that is, two of the spatial position coordinates are zero or constant. In this case, the path of the point is a straight line, and the point is said to have *rectilinear motion*.

In each of these three cases, it is assumed that the coordinate system is chosen so as to obtain the least number of coordinates necessary to describe the motion of the point. Thus, a point whose locus is a *spatial curve*, sometimes called a *skew curve*, requires three position coordinates, a point whose locus is a *planar curve* requires two coordinates, and a point whose locus is a straight line (rectilinear motion) requires only one coordinate.

2.2 POSITION OF A POINT

The physical process involved in observing the position of a point, as shown in Fig. 2.3, implies that the observer is actually keeping track of the relative location of two points, P and O . The observer looks at both points, performs a mental comparison, and recognizes that point P has a certain location *with respect to point O* . In this determination, two properties are noted: the distance from O to P (based on the unit distance or grid size of the reference coordinate system), and the *relative angular orientation* of line OP in the coordinate system. Note that these properties, magnitude and direction, are precisely those required for a vector. Therefore, we define the *position of a point* as the *vector from the origin of a specified reference coordinate system to the point*. We choose the symbol $\mathbf{R}_{PO}$

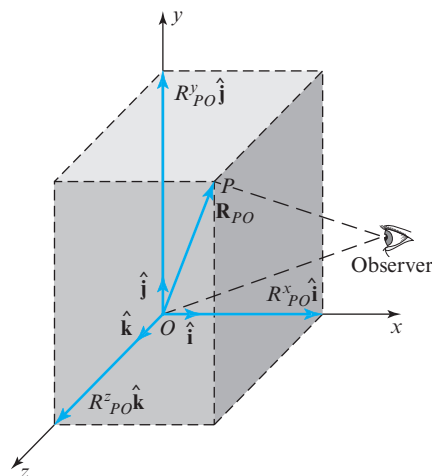


Figure 2.3 Position of a point defined by a vector.

to denote the vector position of point P relative to point O , which points from O to P and is read as *the position of P with respect to O* .*

The coordinate system is related in a very special way to what is seen by a specific observer. This raises the important question: What properties must this coordinate system have to ensure that position measurements made in this system are actually those of the observer? The key to this question is that the coordinate system *must* be stationary with respect to this particular observer. To phrase this another way, the observer is always *stationary* in this coordinate system. This means that if the observer moves, the coordinate system also moves—through a rotation, a distance, or both. If points or bodies are fixed in this coordinate system, then they always appear stationary to the observer, regardless of what movements the observer (and the coordinate system) may execute. Their positions with respect to the observer do not change, and hence their position vectors remain unchanged. The actual location of the observer within the frame of reference has no meaning because the positions of observed points are always defined with respect to the origin of the coordinate system.

Often it is convenient to express the position vector in terms of its components along the coordinate axes:

$$\mathbf{R}_{PO} = R_{PO}^x \hat{\mathbf{i}} + R_{PO}^y \hat{\mathbf{j}} + R_{PO}^z \hat{\mathbf{k}}, \quad (2.3)$$

where superscripts are used to denote the direction of each component. As in the remainder of this text, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are used to designate unit vectors in the directions of the x -axis, y -axis, and z -axis, respectively.

Whereas vectors are denoted throughout this text by boldface symbols, the scalar magnitude of a vector is signified by the same symbol in italics, without boldface.

For example, the magnitude of the position vector is

$$R_{PO} = |\mathbf{R}_{PO}| = \sqrt{\mathbf{R}_{PO} \cdot \mathbf{R}_{PO}} = \sqrt{(R_{PO}^x)^2 + (R_{PO}^y)^2 + (R_{PO}^z)^2}. \quad (2.4)$$

The unit vector in the direction of $\mathbf{R}_{PO}$ is denoted by the same boldface symbol with a caret; that is

$$\hat{\mathbf{R}}_{PO} = \frac{\mathbf{R}_{PO}}{R_{PO}}. \quad (2.5)$$

A distinction can be made between the direction of a line and the orientation of a directed line—that is, a line that is assigned a positive or a negative sense. In general, a vector has a magnitude, a direction, and a sense. The sense defines the positive or negative attribute of the vector and can be used to distinguish between the direction of a vector and its orientation.

* Note that we do not use the slash notation of some other texts; that is, $\mathbf{R}_{PO} \neq \mathbf{R}_{P/O}$. The slash notation is reserved for a different meaning and is described in Sec. 2.4.

2.3 POSITION DIFFERENCE BETWEEN TWO POINTS

We now investigate the relationship between the position vectors of two different points. For purposes of illustration, consider points P and Q in Fig. 2.4. An observer fixed in the xyz coordinate system would observe the positions of P and Q by comparing each with the position of the origin O ; see Eq. (2.3). The positions of the two points are defined by vectors $\mathbf{R}_{PO}$ and $\mathbf{R}_{QO}$. These two vectors are related by a third vector, $\mathbf{R}_{PQ}$, the *position difference* to point P from point Q ; that is,

$$\mathbf{R}_{PQ} = \mathbf{R}_{PO} - \mathbf{R}_{QO}. \quad (2.6)$$

The physical interpretation of this equation is different from that of the position vector itself. The observer is no longer defining the position of P with respect to O but is now defining the position of P with respect to Q . Put another way, the position of P is being defined as if it were in the $x'y'z'$ coordinate system with origin at Q and axes directed parallel to the xyz coordinate system. Either point of view can be used for the interpretation, but we should understand both of them since both are used in future developments.

Finally, it is worth remarking that having the $x'y'z'$ axes parallel to the x,y,z axes is only a matter of convenience and not a necessary condition; it is only necessary that $x'y'z'$ coordinate system does not rotate with respect to the xyz coordinate system. This parallelism causes no loss of generality and it simplifies the visualization when the coordinate systems are in motion.

Having now generalized our concept of position to include the position difference between any two points, we reflect again on the previous discussion of the position vector itself. We note that it is merely the special case where we agree to use the origin of coordinates as the second point. Thus, to be consistent in notation, we have denoted the position vector of a single point P by the dual subscripted symbol $\mathbf{R}_{PO}$. However, in the interest of brevity, we will henceforth agree that when the second subscript is not given explicitly, it is understood to be the origin of the observer's coordinate system; that is,

$$\mathbf{R}_P = \mathbf{R}_{PO}. \quad (2.7)$$

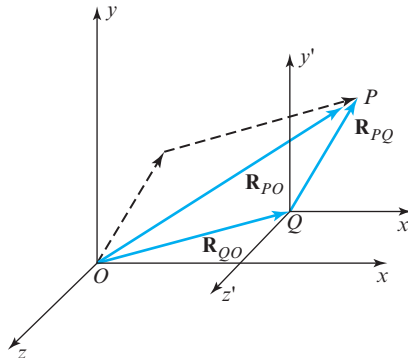


Figure 2.4 Definition of the position-difference vector $\mathbf{R}_{PQ}$.

2.4 APPARENT POSITION OF A POINT

In discussing the position vector, our point of view, up to now, has been entirely that of a single observer in a single coordinate system. However, it is sometimes desirable to make observations in a secondary coordinate system—that is, as seen by a second observer in a different coordinate system, and then to convert this information into the primary coordinate system. Such a situation is shown in Fig. 2.5.

If observer 1, using the primary coordinate system $x_1y_1z_1$, and observer 2, using the secondary coordinate system $x_2y_2z_2$, were asked to give the location of a point at P , they would report different results. Observer 1 would observe the vector $\mathbf{R}_{PO_1}$, whereas observer 2 would report the position vector $\mathbf{R}_{PO_2}$. These position vectors are related by

$$\mathbf{R}_{PO_1} = \mathbf{R}_{O_2O_1} + \mathbf{R}_{PO_2}. \quad (2.8)$$

The difference in the positions of the two origins is not the only discrepancy between the two observations of the position of point P . Since the two coordinate systems are not aligned, the two observers are using different reference lines for their measurements of direction; observer 1 measures along the x_1, y_1, z_1 axes while observer 2 measures along the x_2, y_2, z_2 axes.

Another very important distinction between these two observations arises when we consider that the two coordinate systems could be moving with respect to each other. Whereas point P may appear stationary with respect to one observer, while it is in motion with respect to the other observer; that is, position vector $\mathbf{R}_{PO_1}$ may appear constant to observer 1 while $\mathbf{R}_{PO_2}$ appears to vary as seen by observer 2.

When any of these conditions exists, it is convenient to add an additional subscript to our notation that will distinguish which observer is being considered. When we are considering the position of P as seen by observer 1 using coordinate system $x_1y_1z_1$, we denote this by the symbol $\mathbf{R}_{PO_1/1}$, or, since O_1 is the origin for this observer,* by $\mathbf{R}_{P/1}$. The observations made by observer 2, in coordinate system $x_2y_2z_2$, are denoted as $\mathbf{R}_{PO_2/2}$

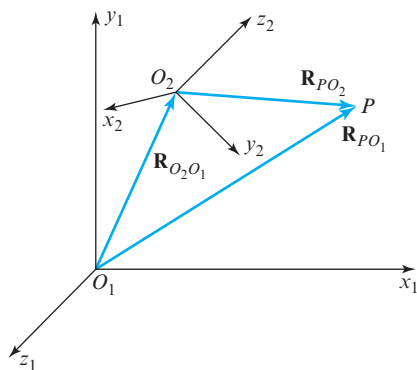


Figure 2.5 Definition of the apparent-position vector $\mathbf{R}_{PO_2}$ of point P .

* Note that $\mathbf{R}_{PO_2/1}$ cannot be abbreviated as $\mathbf{R}_{P/1}$, since O_2 is not the origin used by observer 1.

or $\mathbf{R}_{P/2}$. With this extension of the notation, Eq. (2.8) can be written as

$$\mathbf{R}_{P/1} = \mathbf{R}_{O_2/1} + \mathbf{R}_{P/2}. \quad (2.9)$$

We refer to $\mathbf{R}_{P/2}$ as *the apparent position* of point P to an *observer in coordinate system 2*, and we note that it is by no means equal to the apparent position vector $\mathbf{R}_{P/1}$ seen by observer 1.

We have now made note of certain intrinsic differences between $\mathbf{R}_{P/1}$ and $\mathbf{R}_{P/2}$ and found Eq. (2.9) to relate them. However, there is no reason why components of either vector must be taken along the natural axes of the observer's coordinate system. As with all vectors, components can be found along any desired set of axes; in applying Eq. (2.9), we must use a single consistent set of axes during the numeric evaluation. Although the observer in coordinate system 2 may find it natural to measure the components of $\mathbf{R}_{P/2}$ along the $x_2y_2z_2$ axes, these components must be transformed into the equivalent components along the $x_1y_1z_1$ axes before the addition is actually performed; that is,

$$\begin{aligned} \mathbf{R}_{P/1} &= \mathbf{R}_{O_2/1} + \mathbf{R}_{P/2} \\ &= R_{O_2/1}^{x_1} \hat{\mathbf{i}}_1 + R_{O_2/1}^{y_1} \hat{\mathbf{j}}_1 + R_{O_2/1}^{z_1} \hat{\mathbf{k}}_1 + R_{P/2}^{x_1} \hat{\mathbf{i}}_1 + R_{P/2}^{y_1} \hat{\mathbf{j}}_1 + R_{P/2}^{z_1} \hat{\mathbf{k}}_1 \\ &= (R_{O_2/1}^{x_1} + R_{P/2}^{x_1}) \hat{\mathbf{i}}_1 + (R_{O_2/1}^{y_1} + R_{P/2}^{y_1}) \hat{\mathbf{j}}_1 + (R_{O_2/1}^{z_1} + R_{P/2}^{z_1}) \hat{\mathbf{k}}_1 \\ &= R_{P/1}^{x_1} \hat{\mathbf{i}}_1 + R_{P/1}^{y_1} \hat{\mathbf{j}}_1 + R_{P/1}^{z_1} \hat{\mathbf{k}}_1. \end{aligned}$$

The addition can be performed equally well if all vector components are transformed into the $x_2y_2z_2$ coordinate system or, for that matter, into any other consistent coordinate system. However, they cannot be added algebraically when they have been evaluated along inconsistent coordinate systems. The additional subscript in the apparent-position vector, therefore, does not specify a set of directions to be used in the evaluation of components; it merely states the coordinate system in which the vector is defined, the coordinate system in which the observer is stationary.

2.5 ABSOLUTE POSITION OF A POINT

In Sec. 2.2 we learned that every position vector is defined relative to a second point, the origin of the observer's coordinate reference frame. It is one particular case of the position-difference vector studied in Sec. 2.3, where the reference point is the origin of the coordinate system. Then in Sec. 2.4, we noted that, for certain problems, it may be convenient to consider the apparent positions of a single point as viewed by more than one observer using different coordinate systems. When a particular problem leads us to consider multiple coordinate systems, the application will lead us to choose one of the coordinate systems as primary. This primary coordinate system is commonly referred to as the *absolute coordinate system*. Most often this is the coordinate system in which the final result is to be expressed, and this coordinate system is usually stationary. The absolute position of a point is defined as the apparent position of the point as seen by an observer in the *absolute coordinate system*.

Which coordinate system is designated absolute is an arbitrary decision and unimportant in the study of kinematics. Whether the absolute coordinate system is truly stationary is also a moot point, since, as we have seen, all position (and motion) information is measured relative to something else; nothing is truly absolute in the strict sense. When analyzing the kinematics of an automobile suspension, for example, it may be convenient to choose an “absolute” coordinate system attached to the frame of the car and to study the motion of the suspension relative to this coordinate system. It is unimportant whether the car is moving; motions of the suspension relative to the frame are then defined as absolute.

It is a common convention to number the absolute coordinate system 1 and to use other numbers for other coordinate systems. Since we adopt this convention throughout this text, absolute-position vectors are those apparent-position vectors viewed by an observer in coordinate system 1 and carry symbols of the form $\mathbf{R}_{P/1}$. In the interest of brevity and to reduce complexity, we will agree that when the coordinate system number is not given explicitly, it is assumed to be 1; thus, $\mathbf{R}_{P/1}$ can be abbreviated as $\mathbf{R}_P$. Similarly, the apparent-position equation, Eq. (2.9), can be written* as

$$\mathbf{R}_P = \mathbf{R}_{O_2} + \mathbf{R}_{P/2}. \quad (2.10)$$

EXAMPLE 2.2

The path of a moving point is defined by the equation $y = 2x^2 - 28$. Find the position difference from point P to point Q on the path where $R_P^x = 4$ and $R_Q^x = -3$.

SOLUTION

The y components of the two vectors can be written as

$$R_P^y = 2(4)^2 - 28 = 4 \quad \text{and} \quad R_Q^y = 2(-3)^2 - 28 = -10.$$

Therefore, the two vectors can be written as

$$\mathbf{R}_P = 4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{R}_Q = -3\hat{\mathbf{i}} - 10\hat{\mathbf{j}}.$$

The position difference from point P to point Q is

$$\mathbf{R}_{QP} = \mathbf{R}_Q - \mathbf{R}_P = -7\hat{\mathbf{i}} - 14\hat{\mathbf{j}} = 15.65^\circ \angle -116.6^\circ. \quad \text{Ans.}$$

2.6 POSTURE OF A RIGID BODY

Consider the term *position* when applied to something other than a point. In order to specify the location of a rigid body, for example, it is necessary to specify more than just three coordinates. It is necessary to specify coordinates that uniquely determine the location

* Reviewing Secs. 2.1 through 2.3 will verify that the position-difference vector $\mathbf{R}_{PQ}$ was treated entirely from the absolute coordinate system and is an abbreviation of the notation $\mathbf{R}_{PQ/1}$. We have no need to treat the completely general case $\mathbf{R}_{PQ/2}$, the apparent-position-difference vector.

of every point of the body. If all of these coordinates are grouped into a single quantity according to some agreed-upon set of conventions, then the result describes the location of the body.

The location and orientation of a coordinate system fixed to the body with respect to a stationary reference, or world coordinate system, describes the *posture* of that body. Posture is described in terms of the location of the origin of the coordinate system fixed to the body as well as a description of the orientation of this coordinate system, both specified with respect to the world coordinate system.

In the robotics literature, for example, a matrix is often used to describe the location and orientation of a coordinate system attached to the end-effector with respect to a base or world coordinate system (Chap. 10). In some robotics literature, the word *position* is used loosely to describe the location of only a single point (such as the origin) of a coordinate system attached to the end-effector. In such literature, the end-effector is said to have a certain position; the orientation may then be added and the term *pose* is sometimes used for the combination of the two. It should be pointed out, however, that the term *posture* is more appropriate* and is utilized throughout this text.

The term *posture* becomes even more suitable when dealing with a mechanism or multibody mechanical system, because we are not concerned with the position of only a single point or the position and orientation of only a single rigid body, but we wish to describe the positions and orientations of the assembly of rigid bodies. We use the term *posture* to describe the location and orientation of a rigid body or for the configuration of a mechanism, including both the locations and orientations of every link, all at a particular instant in time.

The problem of posture analysis is to determine the values of all position variables (the positions of all points and joints) and the postures of all links, given the dimensions of each link and the value(s) of the independent variable(s)—that is, the variables chosen to represent the degree(s) of freedom of the mechanism.

2.7 LOOP-CLOSURE EQUATIONS

Our discussion of the position-difference and apparent-position vectors has been somewhat abstract so far, the intent being to develop a rigorous foundation for the analysis of motion in mechanical systems. Certainly, precision is not without merit, since it is rigor that permits science to predict a correct result despite the personal prejudices and emotions of the analyst. However, tedious developments are not interesting unless they lead to applications in real-life problems. Although many fundamental principles are yet to be discovered, it may be worthwhile at this point to show the relationship between the relative-position vectors discussed earlier and some typical linkages encountered in real machines.

* The *Webster Comprehensive Dictionary: International Edition*, states under its definition of attitude, “Synonyms: pose, position, posture. A posture is assumed without any special reference to expression of feeling. . . . A pose is a position studied for artistic effect or considered with reference to such an effect.”

As pointed out in Chap. 1, one of the most common and most useful of all mechanisms is the planar four-bar linkage. A practical example of this linkage is the clamping device shown in Figs 2.6 and 2.7. A brief study of the assembly drawing indicates that, as the handle of the clamp is lifted, the clamping bar swings away from the clamping surface, thereby opening the clamp. As the handle is pressed down, the clamping bar swings down and the clamp closes again. If we wish to design such a clamp, however, things are not quite this simple. It may be desirable, for example, for the clamp to open at a given rate for a specified rate of lift of the handle. Such relationships are not obvious; they depend on the exact dimensions of the various parts and the relationships, or interactions, between the parts. To discover these relationships, a rigorous description of the essential features of the device is required. The position-difference and apparent-position vectors can be used to provide a description of any posture of the clamp.

Figure 2.7 shows the detail drawings of the individual links of the disassembled clamp. Although not shown here, the detail drawings would be completely dimensioned, thus fixing the complete geometry of each link. The assumption that each link is rigid ensures

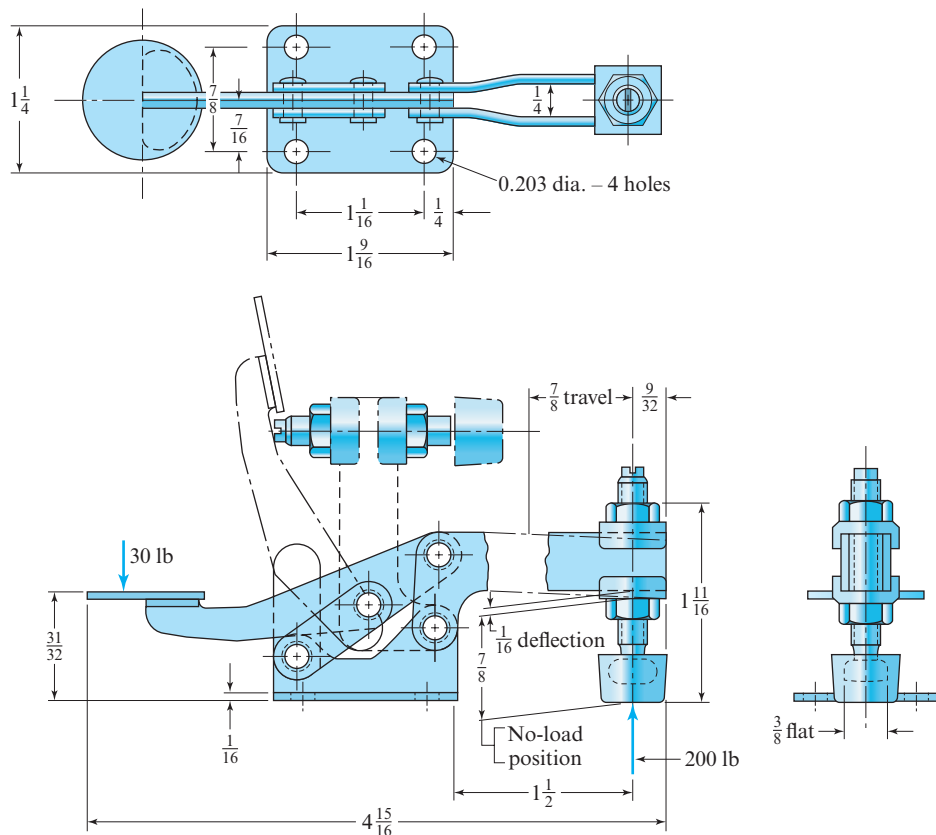


Figure 2.6 Assembly drawing of a hand-operated clamp.

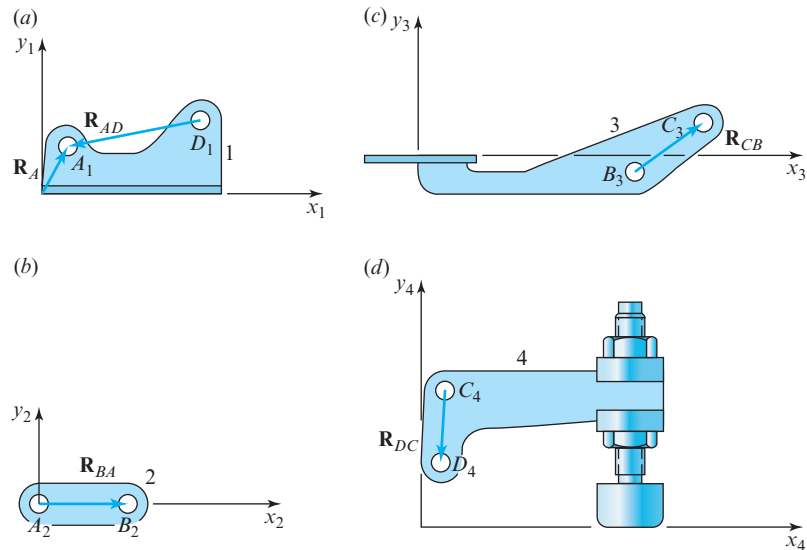


Figure 2.7 (a) Frame link; (b) connecting link; (c) handle; (d) clamping bar.

that the position of any point is determined precisely relative to any other point on the same link. However, the detail drawings do not provide the interrelationships between the individual parts—that is, the constraints that ensure each link moves relative to its neighboring links in a prescribed fashion. These constraints are, of course, provided by the four pinned joints. Anticipating that they will be of importance in any description of the linkage, we label the pin centers as A , B , C , and D , and we identify the appropriate points on link 1 as A_1 and D_1 , those on link 2 as A_2 and B_2 , and so on. We also choose a coordinate system fixed to each link, as shown in Fig. 2.7.

Since it is necessary to relate the positions of the successive pin centers, we define the position-difference vectors $\mathbf{R}_{AD}$ on link 1, $\mathbf{R}_{BA}$ on link 2, $\mathbf{R}_{CB}$ on link 3, and $\mathbf{R}_{DC}$ on link 4. We note that each vector appears constant to an observer fixed in the coordinate system of that link; the magnitude and direction of each vector can be obtained from the constant dimensions of that link.

A vector equation can also be written to describe the constraints provided by each of the pinned joints. Note that no matter which position or which observer is chosen, the two points describing each pin center (for example, A_1 and A_2) remain coincident. Thus,

$$\mathbf{R}_{A_2A_1} = \mathbf{R}_{B_3B_2} = \mathbf{R}_{C_4C_3} = \mathbf{R}_{D_1D_4} = \mathbf{0}. \quad (2.11)$$

Let us now develop vector equations for the absolute position of each pin center. Since link 1 is the frame, absolute positions are those defined relative to an observer in coordinate system 1. Point A_1 is, of course, at the position described by $\mathbf{R}_A$. Next, we connect link 2

to link 1 (mathematically) by writing

$$\mathbf{R}_{A_2} = \mathbf{R}_{A_1} + \mathbf{R}_{A_2A_1}^0 = \mathbf{R}_A. \quad (a)$$

Transferring to the other end of link 2, we attach link 3. Therefore,

$$\mathbf{R}_B = \mathbf{R}_A + \mathbf{R}_{BA}. \quad (b)$$

Connecting joints C and D in the same manner, we obtain

$$\mathbf{R}_C = \mathbf{R}_B + \mathbf{R}_{CB} = \mathbf{R}_A + \mathbf{R}_{BA} + \mathbf{R}_{CB}, \quad (c)$$

$$\mathbf{R}_D = \mathbf{R}_C + \mathbf{R}_{DC} = \mathbf{R}_A + \mathbf{R}_{BA} + \mathbf{R}_{CB} + \mathbf{R}_{DC}. \quad (d)$$

Then, transferring back across link 1 to point A , we have

$$\mathbf{R}_A = \mathbf{R}_D + \mathbf{R}_{AD} = \mathbf{R}_A + \mathbf{R}_{BA} + \mathbf{R}_{CB} + \mathbf{R}_{DC} + \mathbf{R}_{AD}. \quad (e)$$

Finally, rearranging Eq. (e), we obtain

$$\mathbf{R}_{BA} + \mathbf{R}_{CB} + \mathbf{R}_{DC} + \mathbf{R}_{AD} = \mathbf{0}. \quad (2.12)$$

This important equation is called the *loop-closure* equation, or the *vector loop* equation, for the clamp. As shown in Fig. 2.8, it expresses the fact that this linkage forms a closed loop. Therefore, the polygon formed by the position-difference vectors through successive links and joints must remain closed as the linkage moves. The constant lengths of these vectors ensure that the joint centers remain separated by constant distances, the requirement for rigid links. The rotations between successive vectors indicate the motions within the pinned joints. Recall that the rotation of a position-difference vector shows the rotation of a particular link. Thus, the loop-closure equation holds within it all the important constraints that illustrate the operation of the clamp. This equation forms a mathematical description, or *model*, of the linkage, and many of the later developments are based on this loop-closure model as a starting point.

Of course, the form of the loop-closure equation depends on the type of mechanism. This is demonstrated by another example, the *Geneva* mechanism or *Maltese cross*, shown in Fig. 2.9. One early application of this mechanism was to prevent overwinding a watch.

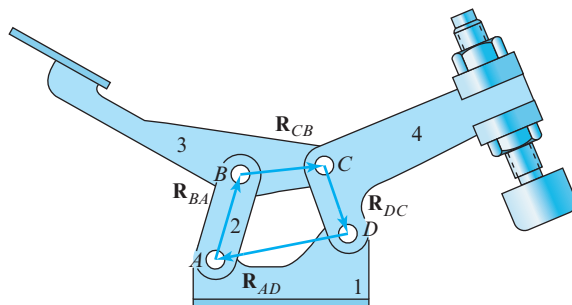


Figure 2.8 Loop-closure equation.

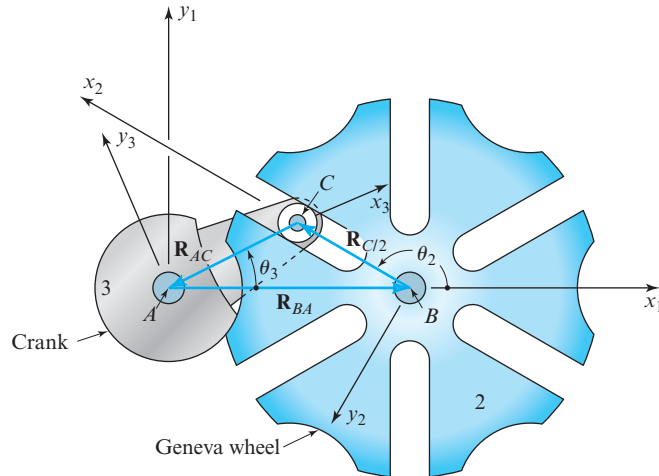


Figure 2.9 Geneva mechanism or Maltese cross.

Today the mechanism finds wide use as an indexing device, for example, in a milling machine with an automatic tool changer.

First, we define vectors for the absolute posture of each link. Although the frame of the mechanism, link 1, is not shown in Fig. 2.9, it is an important part of the mechanism, since it holds the two shafts, with centers A and B, a constant distance apart. Thus we define the vector $\mathbf{R}_{BA}$ to show this dimension. The left crank, link 3, is attached to a shaft, usually rotating at a constant speed, and carries a roller at C, running in the slot of the Geneva wheel. The vector $\mathbf{R}_{AC}$ has a constant magnitude equal to the crank length, the distance from the center of the roller C to the shaft center A. The rotation of this vector relative to link 1 is used later to describe the angular speed of the crank. The x_2 axis is aligned along one slot of the wheel; thus the roller is constrained to ride along this slot. The vector $\mathbf{R}_{C/2}$ has the same rotation as the wheel, link 2; also, the change in length, $\Delta \mathbf{R}_{C/2}$, demonstrates the relative sliding motion taking place between the roller C and the slot in link 2. Note that $\mathbf{R}_{C/2}$ is equivalent to $\mathbf{R}_{CB/2}$, since point B is the origin of coordinate system 2. Therefore, the loop-closure equation for this mechanism can be written as

$$\mathbf{R}_{BA} + \mathbf{R}_{C/2} + \mathbf{R}_{AC} = \mathbf{0}. \quad (2.13)$$

This form of the loop-closure equation is a valid mathematical model only while roller C remains in the slot along the x_2 axis. However, this condition does not hold throughout the entire cycle of motion. Once the roller leaves the slot, the motion is controlled by the two mating circular arcs on links 2 and 3. A new form of the loop-closure equation rules that portion of the cycle.

A mechanism can, of course, be composed of a multiple-loop kinematic chain. In such a case, more than one loop-closure equation is required to completely model the mechanism. The procedures for obtaining the loop-closure equations, however, are identical to those shown in the previous two examples.

2.8 GRAPHIC POSTURE ANALYSIS

When the paths of the points in the moving links of a mechanism lie in a single plane or in parallel planes, the mechanism is called a *planar* mechanism. Since a substantial portion of the study in this book deals with planar mechanisms, the development of methods specific to the analysis of planar mechanisms is justified. As we will see in the following section, the nature of the loop-closure equation approached analytically often leads to the solution of simultaneous nonlinear equations and can become quite cumbersome. Yet, particularly for planar mechanisms, the solution is usually straightforward when approached graphically.

First, let us briefly review the graphic process of vector addition. Any two known vectors **A** and **B** can be added graphically as shown in Fig. 2.10a. After a scale is chosen, the vectors are drawn tip to tail in either order and their vector sum **C** is identified:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \quad (2.14)$$

Note that the magnitudes and orientations of both vectors **A** and **B** are used in performing the addition, and that both the magnitude and orientation of the sum **C** are found as a result.

The operation of graphic vector subtraction is shown in Fig. 2.10b, where the vectors are drawn tip to tip in solving the equation

$$\mathbf{A} = \mathbf{C} - \mathbf{B}. \quad (2.15)$$

These graphic vector operations should be studied carefully and understood, since they are used extensively throughout the book.

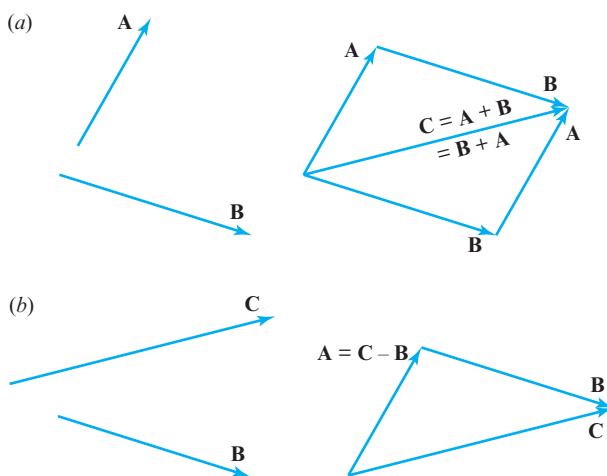


Figure 2.10 (a) Vector addition; (b) vector subtraction.

A spatial (three-dimensional) vector equation, such as

$$\mathbf{C} = \mathbf{D} + \mathbf{E} + \mathbf{B}, \quad (a)$$

can be divided into components along any convenient axes, leading to the three scalar equations

$$C^x = D^x + E^x + B^x, \quad C^y = D^y + E^y + B^y, \quad \text{and} \quad C^z = D^z + E^z + B^z. \quad (b)$$

Since they are components of the same vector equation, these three scalar equations must be consistent. If the three equations are also linearly independent, they can be solved simultaneously for three unknowns, which may be three magnitudes, three directions, or any combination of three magnitudes and directions. For some combinations, however, the problem is highly nonlinear and quite difficult to solve. Therefore, we shall delay consideration of the three-dimensional case until it is needed in Chap. 10.

A planar (two-dimensional) vector equation can be solved for two unknowns—namely, two magnitudes, two directions, or one magnitude and one direction. In general, it is desirable to indicate the known ($\surd$) and unknown (?) quantities above each vector in an equation like this:

$${}^? \surd \mathbf{C} = \surd \surd \mathbf{D} + \surd \surd \mathbf{E} + {}^? \surd \mathbf{B}, \quad (c)$$

where the first symbol ($\surd$ or $?$) above each vector indicates the state of its magnitude and the second symbol indicates the state of its direction. Another, equivalent form is

$${}^? \hat{\surd} \mathbf{C} = \surd \hat{\surd} \mathbf{D} + \surd \hat{\surd} \mathbf{E} + {}^? \hat{\surd} \mathbf{B}. \quad (d)$$

Either of Eqs. (c) or (d) clearly identifies the unknowns and indicates whether a solution is possible. In Eq. (c), vectors $\mathbf{D}$ and $\mathbf{E}$ are completely defined and can be replaced by their sum

$$\mathbf{A} = \mathbf{D} + \mathbf{E}, \quad (e)$$

giving

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \quad (2.16)$$

Note that any planar vector equation, if it is solvable, can be reduced to a three-term equation with two unknowns. Depending on the forms of the two unknowns, four distinct cases occur. These cases and the corresponding unknowns are presented in Table 2.2.

We will show the solutions of these four cases graphically.

In case 1, the two unknowns are the magnitude and the direction of the same vector (say, C and θ_C). This case can be solved by graphic addition or subtraction of the remaining two vectors, which are completely defined. This case was illustrated in Fig. 2.10.

Table 2.2 Unknowns in Planar Vector Equations

Case	Unknowns		
1	C, θ_C	Fig. 2.10	Eq. (2.14)
2	A, θ_B	Fig. 2.11	Eq. (2.17)
3	A, B	Fig. 2.12	Eq. (2.18)
4	θ_A, θ_B	Fig. 2.13	Eq. (2.19)

In case 2, the two unknowns are a magnitude and a direction from different vectors (say, A and θ_B). The vector equation is

$$\overset{\sqrt{\sqrt{}}}{\mathbf{C}} = \overset{? \sqrt{}}{\mathbf{A}} + \overset{\sqrt{?}}{\mathbf{B}}. \quad (2.17)$$

The solution, illustrated in Fig. 2.11, is obtained as follows:

1. Choose a convenient scale factor and draw vector $\mathbf{C}$.
2. Construct a line through the origin of $\mathbf{C}$ parallel to $\hat{\mathbf{A}}$.
3. Adjust a compass to the scaled magnitude B and construct a circular arc with the center at the terminus of $\mathbf{C}$.
4. The two intersections of the line and the arc define the two sets of solutions $A, \hat{\mathbf{B}}$ and $A', \hat{\mathbf{B}}'$.

In case 3, the two unknowns are two magnitudes (say, A and B). The vector equation is

$$\overset{\sqrt{\sqrt{}}}{\mathbf{C}} = \overset{? \sqrt{}}{\mathbf{A}} + \overset{? \sqrt{}}{\mathbf{B}}. \quad (2.18)$$

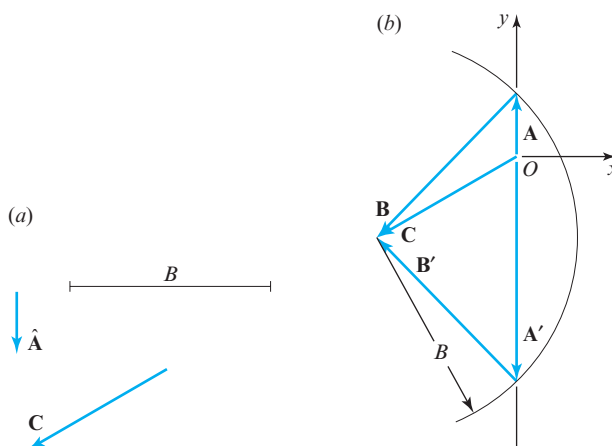


Figure 2.11 Case 2: (a) given $\mathbf{C}$, θ_A , and B ; (b) solutions for A, θ_B , and A', θ_B' .

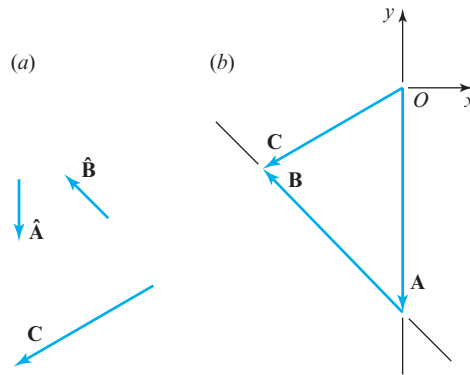


Figure 2.12 Case 3: (a) given $\mathbf{C}$, θ_A , and θ_B . (b) solution for A and B .

The solution, illustrated in Fig. 2.12, is obtained as follows:

1. Choose a convenient scale factor and draw vector $\mathbf{C}$.
2. Construct a line through the origin of $\mathbf{C}$ parallel to $\hat{\mathbf{A}}$.
3. Construct another line through the terminus of $\mathbf{C}$ parallel to $\hat{\mathbf{B}}$.
4. The intersection of these two lines defines both magnitudes, A and B , each of which may be either positive or negative.

Note that case 3 has a unique solution unless the lines are collinear or parallel. If the lines are collinear, the magnitudes A and B are both indeterminate. If the lines are parallel but distinct, magnitudes A and B are both infinite.

Finally, in case 4, the two unknowns are the directions of two vectors, $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$. The vector equation is

$$\sqrt{\sqrt{\mathbf{C}}} = \sqrt{?} \sqrt{?} \mathbf{B}. \quad (2.19)$$

The solution, illustrated in Fig. 2.13, is obtained as follows:

1. Choose a convenient scale factor and draw vector $\mathbf{C}$.
2. Construct a circular arc of radius A centered at the origin of $\mathbf{C}$.
3. Construct a circular arc of radius B centered at the terminus of $\mathbf{C}$.
4. The two intersections of these arcs define the two sets of solutions $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ and $\hat{\mathbf{A}}', \hat{\mathbf{B}}'$.

Note that a real solution is possible only if $A + B \geq C$.

The graphic procedures of this section are now applied to obtain solutions of the loop-closure equations for the slider-crank linkage and the four-bar linkage.

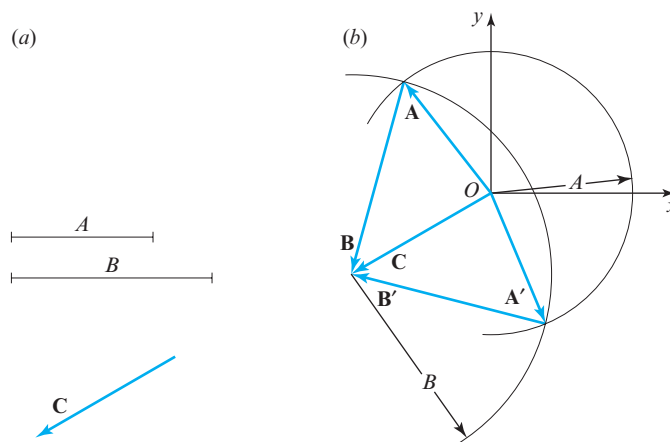


Figure 2.13 Case 4: (a) given C, A, and B; (b) solution for θ_A , θ_B and θ'_A , θ'_B .

Slider-Crank Linkage Consider the slider-crank linkage shown in Fig. 2.14a. Given the position of link 4, the problem is to determine the postures of links 2 and 3.

Replace the link centerlines with vectors $\mathbf{R}_A$ for the crank, $\mathbf{R}_B$ for the slider, and $\mathbf{R}_{BA}$ for the connecting rod, as shown in Fig. 2.14b. With a known location of the slider (that is, the distance R_B) and the angles θ_2 and θ_3 unknown, the loop-closure equation for the linkage is

$$\overset{\sqrt{?}}{\mathbf{R}}_A + \overset{\sqrt{?}}{\mathbf{R}}_{BA} - \overset{\sqrt{I}}{\mathbf{R}}_B = \mathbf{0},$$

which can be written as

$$\overset{\sqrt{I}}{\mathbf{R}}_B = \overset{\sqrt{?}}{\mathbf{R}}_A + \overset{\sqrt{?}}{\mathbf{R}}_{BA}, \quad (a)$$

where the “I” symbol, for the direction of vector $\mathbf{R}_B$, denotes that the location of the slider is the given input.

We recognize Eq. (a) as case 4 in Table 2.2.

The graphic solution procedure explained in Fig. 2.13 is now carried out in Fig. 2.14c. We note that there are two possible solutions (θ_2 and θ_3 , and θ'_2 and θ'_3), which correspond to two different postures of the linkage (that is, two configurations of the links), both of which are consistent with the given position of the slider. We know in advance which of the two solutions is desired; both sets of results are equally valid roots to the loop-closure equation, and the choice is based on the application.

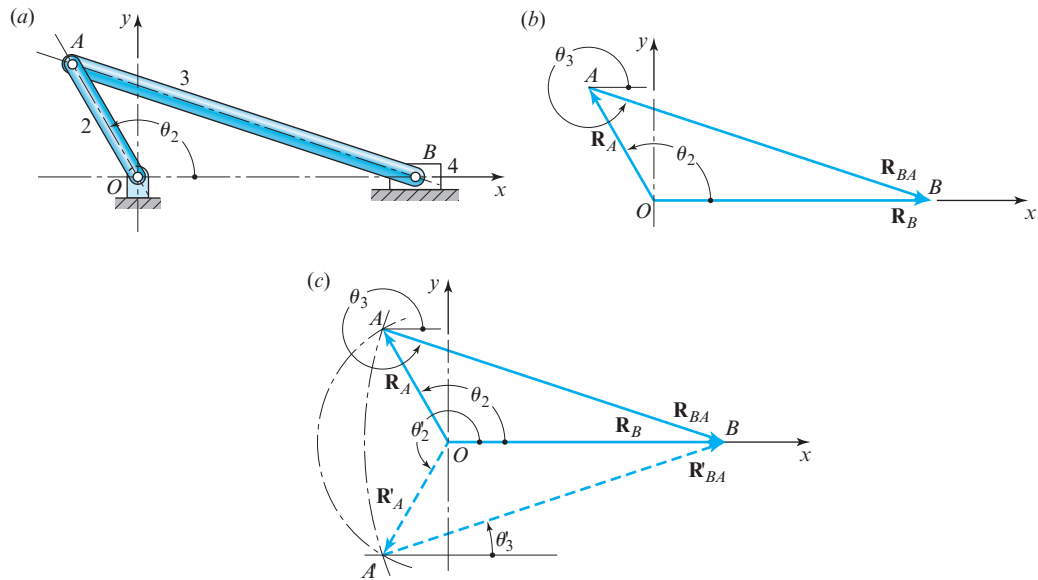


Figure 2.14 (a) Slider-crank linkage; (b) vectors replace the link centerlines; (c) graphic posture analysis.

Four-Bar Linkage Consider the four-bar linkage shown in Fig. 2.15. Given the posture of link 2, the problem is to define the postures of links 3 and 4 and the position of coupler point P .

Replace the link centerlines with vectors $\mathbf{R}_A$ for the crank, $\mathbf{R}_{BA}$ for the connecting rod, $\mathbf{R}_{BC}$ for the output link, and $\mathbf{R}_C$ for the frame, as shown in Fig. 2.16a. With a known posture of link 2, that is, the angle θ_2 , and the angles θ_3 and θ_4 unknown, the loop-closure equation for the linkage is

$$\sqrt{I} \mathbf{R}_A + \sqrt{?} \mathbf{R}_{BA} - \sqrt{?} \mathbf{R}_{BC} - \sqrt{?} \mathbf{R}_C = \mathbf{0}, \quad (a)$$

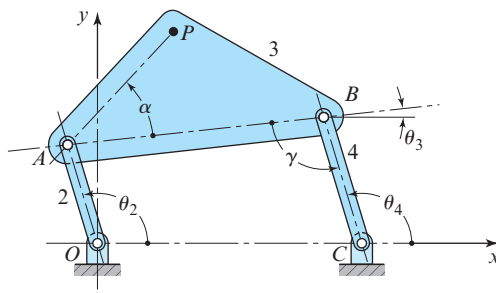


Figure 2.15 Four-bar linkage and coupler point P .

which can be rearranged as

$$\overset{\sqrt{I}}{\mathbf{R}}_A + \overset{\sqrt{?}}{\mathbf{R}}_{BA} = \overset{\sqrt{\sqrt{?}}}{\mathbf{R}}_C + \overset{\sqrt{?}}{\mathbf{R}}_{BC}, \quad (b)$$

where the “ I ” symbol, for the direction of vector $\mathbf{R}_A$, denotes that the orientation of the crank is the given input.

Also, the position of coupler point P is given by the position-difference equation

$$\overset{??}{\mathbf{R}}_P = \overset{\sqrt{I}}{\mathbf{R}}_A + \overset{\sqrt{?}}{\mathbf{R}}_{PA}. \quad (c)$$

Note that this equation contains three unknowns, but this can be reduced to two after Eq. (b) is solved by noting the constant angular relationship between $\hat{\mathbf{R}}_{PA}$ and $\hat{\mathbf{R}}_{BA}$ —namely,

$$\theta_5 = \theta_3 + \alpha, \quad (d)$$

where the constant coupler angle $\alpha = \angle BAP$. Therefore, Eq. (c) reduces to

$$\overset{??}{\mathbf{R}}_P = \overset{\sqrt{I}}{\mathbf{R}}_A + \overset{\sqrt{C}}{\mathbf{R}}_{PA}, \quad (e)$$

where the “ C ” symbol, for the direction of $\mathbf{R}_{PA}$, denotes that this direction is known from the constraint equation—that is, Eq. (d)—and, therefore, is not a third unknown.

We begin the graphic solution by combining the two unknown terms in Eq. (b), thus locating the positions of points A and C as shown in Figs. 2.16a and 2.16b:

$$\overset{\sqrt{\sqrt{?}}}{\mathbf{S}} = \overset{\sqrt{\sqrt{?}}}{\mathbf{R}}_C - \overset{\sqrt{I}}{\mathbf{R}}_A = \overset{\sqrt{?}}{\mathbf{R}}_{BA} - \overset{\sqrt{?}}{\mathbf{R}}_{BC}. \quad (f)$$

We recognize this equation as case 4 in Table 2.2. The solution procedure (that is, two unknown orientations) is then used to locate point B . Note that two solutions are possible, which indicates that the four-bar linkage can be assembled in two different postures. The two postures will be referred to here as (a) the open posture shown in Fig. 2.16a and (b) the crossed posture shown in Fig. 2.16b.

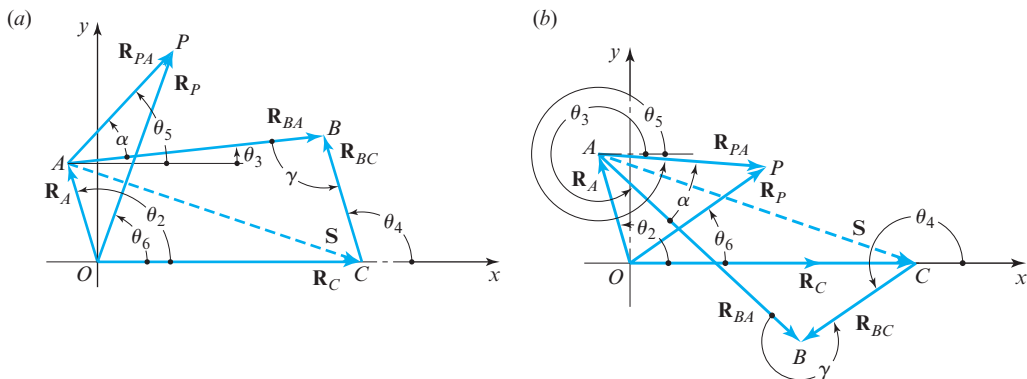


Figure 2.16 Vector diagrams illustrating the graphic solutions for (a) open posture; (b) crossed posture.

To determine the position of coupler point P , first we obtain the two orientations of $\hat{\mathbf{R}}_{PA}$ from Eq. (d). Then we solve Eq. (c) by the procedure for case 1. Two solutions are obtained for the position of point P , as shown in Figs. 2.16a and 2.16b. Both are valid solutions to Eqs. (b) through (d), although, in this instance, the crossed posture shown could not be reached from the open posture without first disassembling the linkage.

From the two examples presented here it is clear that graphic posture analysis usually requires precisely the same constructions that would be chosen naturally in drafting scale drawings of a mechanism at the posture under consideration. For this reason, the procedure may seem trivial and not truly worthy of the title “analysis.” Yet this is highly misleading. As we shall see in later sections of Chap. 2, the posture analysis of a mechanism is a nonlinear algebraic problem when approached by analytic or numeric methods. It is, in fact, the most difficult problem in kinematic analysis, and this is one primary reason why graphic solution techniques have retained their attraction.

2.9 ALGEBRAIC POSTURE ANALYSIS

To illustrate the classic algebraic approach for the posture analysis of planar mechanisms, this section presents the same two examples of the previous section—namely, the slider-crank linkage and the four-bar linkage.

Slider-Crank Linkage For generality, the offset version of this linkage, shown in Fig. 2.17, is chosen for the analysis. By making the offset or eccentricity $e = R_{O_2O} = 0$, the resulting equations can be used for the in-line (on-center or symmetric) version. The notation in Fig. 2.17 shows that the input crank angle θ_2 is measured around the origin of vector $\mathbf{r}_2 = \mathbf{R}_{AO_2}$, and the connecting rod angle θ_3 is measured around the origin of vector $\mathbf{r}_3 = \mathbf{R}_{AB}$. Both measurements are taken as positive (counterclockwise) angles.

There are two problems that occur in the posture analysis of the slider-crank linkage—namely:

Problem 1: Given the input crank angle θ_2 , find the connecting rod angle θ_3 and the position x_B .

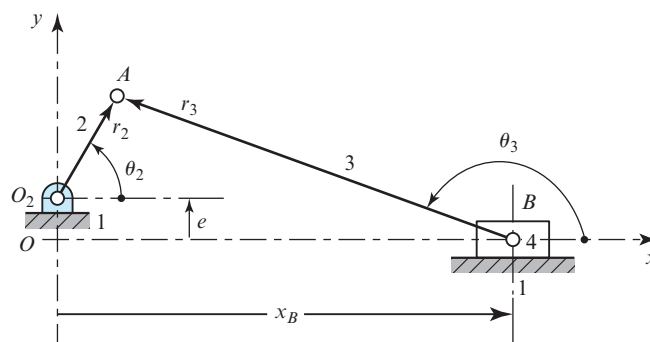


Figure 2.17 Notation for the offset slider-crank linkage.

Problem 2: Given the position x_B , find the input crank angle θ_2 and connecting rod angle θ_3 . This is the problem that we addressed in the previous section.

Solution to Problem 1: First, we define the position of point A by the equations

$$x_A = r_2 \cos \theta_2 \quad \text{and} \quad y_A = e + r_2 \sin \theta_2. \quad (2.20)$$

Next, we note that

$$e + r_2 \sin \theta_2 = r_3 \sin \theta_3, \quad (a)$$

so that

$$\sin \theta_3 = \frac{1}{r_3} (e + r_2 \sin \theta_2). \quad (2.21)$$

From the geometry of Fig. 2.17, we see that

$$x_B = r_2 \cos \theta_2 - r_3 \cos \theta_3. \quad (b)$$

Substituting Eq. (2.21) into the trigonometric identity $\cos \theta_3 = \pm \sqrt{1 - \sin^2 \theta_3}$ gives

$$\cos \theta_3 = -\frac{1}{r_3} \sqrt{r_3^2 - (e + r_2 \sin \theta_2)^2}. \quad (c)$$

Here the negative sign corresponds to an obtuse angle for θ_3 when measured around point B, as shown in Fig. 2.17. (A positive sign would imply an acute angle for θ_3 .)

Finally, substituting Eq. (c) into Eq. (b) gives

$$x_B = r_2 \cos \theta_2 + \sqrt{r_3^2 - (e + r_2 \sin \theta_2)^2}. \quad (2.22)$$

Thus, with the angle θ_2 given, the unknowns θ_3 and x_B can be obtained by solving Eqs. (2.21), (c), and (2.22).

Solution to Problem 2: Given x_B , we must solve Eq. (2.22) for the angle θ_2 . This requires the use of a calculator or a computer together with a root-finding technique. Here we select the well-known Newton-Raphson method.* This method can be explained by reference to Fig. 2.18, which is a graph of some function $f(x)$ versus x .

Let x_n be a first approximation (a rough estimate) of the root that we wish to find and for which $f(x) = 0$. A tangent line to the curve at $x = x_n$ intersects the x axis at x_{n+1} , which is a better approximation to the root. The slope of the tangent line is equal to the derivative of the function at $x = x_n$ and is

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}. \quad (d)$$

* See, for example, [1].

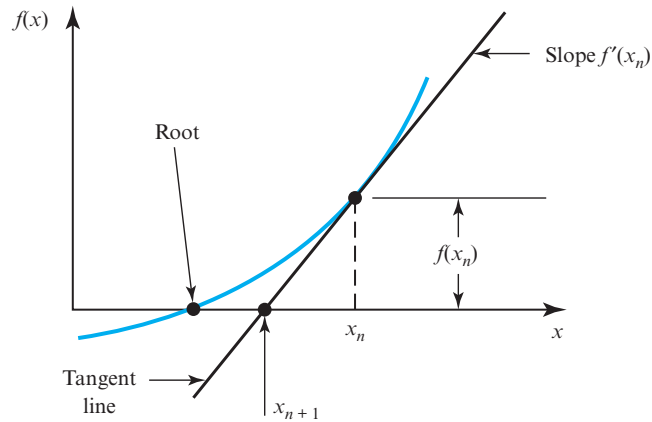


Figure 2.18 Newton-Raphson method.

Solving this for x_{n+1} gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (2.23)$$

Using a calculator, for example, to start a solution, we enter an estimate of x_n , solve Eq. (2.23) for x_{n+1} , use this as the next estimate, and repeat this process as many times as needed to obtain the result with satisfactory accuracy. The accuracy is evaluated by comparing $(x_{n+1} - x_n)$ with a small number ε after each repetition and continuing until $(x_{n+1} - x_n) < \varepsilon$.

We note that the root-finding programs built into many calculators utilize an approximation to obtain $f'(x_n)$. These will sometimes be of little value in solving Eq. (2.22), so we proceed as follows. In Eq. (2.22), we replace the angle θ_2 with the symbol x , and let r_2 , r_3 , e , and x_B be given constants. Then

$$f(x) = r_2 \cos x + \sqrt{r_3^2 - (e + r_2 \sin x)^2} - x_B, \quad (e)$$

and

$$f'(x) = -r_2 \sin x - \frac{(e + r_2 \sin x)r_2 \cos x}{\sqrt{r_3^2 - (e + r_2 \sin x)^2}}. \quad (f)$$

These two equations can now be programmed with Eq. (2.23) to solve for the unknown value of the angle θ_2 when x_B is given. We note that the angle θ_2 will have two possible values. These may be found separately by using appropriate initial estimates.

A closed-form algebraic solution is possible if the eccentricity e is zero—that is, if the linkage is centered. For this case, we take Eqs. (a) and (b), square them, and add them together. Noting a trigonometric identity, the result is

$$x_B^2 - 2x_B r_2 \cos \theta_2 + r_2^2 = r_3^2. \quad (g)$$

Solving for θ_2 gives

$$\theta_2 = \cos^{-1} \frac{x_B^2 + r_2^2 - r_3^2}{2x_B r_2}. \quad (2.24)$$

The solution of Problem 1 for the centered version is, of course, obtained directly from Eq. (2.22) by setting $e = 0$.

Four-Bar Linkage To obtain the analytic solution, the distance R_{AO_4} is designated as S in Fig. 2.19 (as it was in Figs. 2.16a and 2.16b). The cosine law for each of the two triangles AO_2O_4 and ABO_4 can then be written in terms of the angles and link lengths; that is,

$$S = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta_2}, \quad (2.25)$$

$$\beta = \cos^{-1} \frac{r_1^2 + S^2 - r_2^2}{2r_1 S}, \quad (2.26)$$

$$\psi = \cos^{-1} \frac{r_3^2 + S^2 - r_4^2}{2r_3 S}, \quad (2.27)$$

$$\lambda = \cos^{-1} \frac{r_4^2 + S^2 - r_3^2}{2r_4 S}. \quad (2.28)$$

Note that Eqs. (2.26) through (2.28) can yield double values, since they involve inverse cosines. Here we will take β , ψ , and λ as positive numeric values. Then a careful study of the figure shows that, when θ_2 is in the range $0 \leq \theta_2 \leq \pi$, the unknown angles are

$$\theta_3 = -\beta \pm \psi, \quad (2.29)$$

$$\theta_4 = \pi - \beta \mp \lambda. \quad (2.30)$$

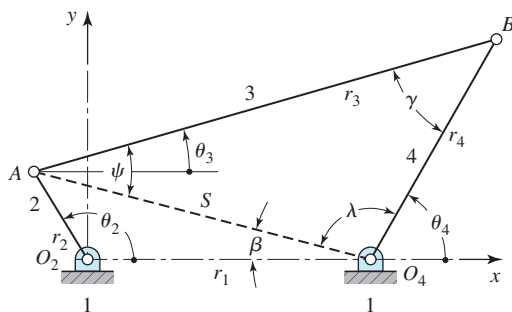


Figure 2.19 Four-bar linkage.

However, when θ_2 is in the range $\pi \leq \theta_2 \leq 2\pi$, then

$$\theta_3 = \beta \pm \psi, \quad (2.31)$$

$$\theta_4 = \pi + \beta \mp \lambda. \quad (2.32)$$

In these results, the upper signs correspond to the open posture, whereas the lower signs correspond to the crossed posture.

In Sec. 1.10, the concept of transmission angle was discussed in connection with the subject of mechanical advantage. Now, with Fig. 2.19 and using the cosine law, this angle can be written as

$$\gamma = \pm \cos^{-1} \frac{r_3^2 + r_4^2 - S^2}{2r_3r_4}. \quad (2.33)$$

2.10 COMPLEX-ALGEBRAIC SOLUTIONS OF PLANAR VECTOR EQUATIONS

In planar problems, it is often desirable to express a vector by specifying its magnitude and orientation in *polar notation*:

$$\mathbf{R} = R \angle \theta. \quad (a)$$

In Fig. 2.20a, the two-dimensional vector

$$\mathbf{R} = R^x \hat{\mathbf{i}} + R^y \hat{\mathbf{j}} \quad (2.34)$$

has two rectangular components of magnitudes,

$$R^x = R \cos \theta \quad \text{and} \quad R^y = R \sin \theta, \quad (2.35)$$

with

$$R = \sqrt{(R^x)^2 + (R^y)^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{R^y}{R^x}. \quad (2.36)$$

Note that we have made the arbitrary choice here of accepting the positive square root for magnitude R when calculating from the components of $\mathbf{R}$. Therefore, we must be careful

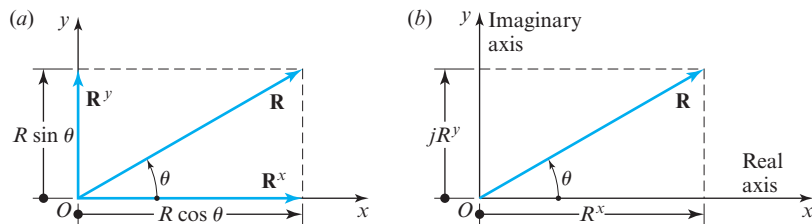


Figure 2.20 Correlation of planar vectors and complex values.

to interpret the signs of R^x and R^y individually when deciding upon the quadrant of the angle θ . Note that θ is defined as the angle from the positive x axis to the positive end of vector $\mathbf{R}$, measured about the origin of the vector, and this angle is positive when measured counterclockwise.

EXAMPLE 2.3

Express vectors $\mathbf{A} = 10\angle 30^\circ$ and $\mathbf{B} = 8\angle -15^\circ$ in rectangular notation* and find their sum.

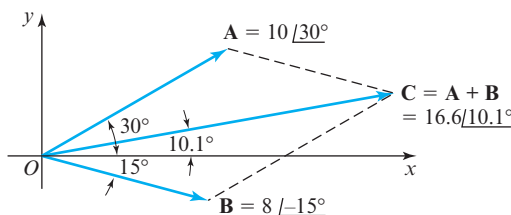


Figure 2.21 Addition of vectors.

SOLUTION

The vectors are shown in Fig. 2.21 and can be written as

$$\mathbf{A} = 10\cos 30^\circ\hat{\mathbf{i}} + 10\sin 30^\circ\hat{\mathbf{j}} = 8.66\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}} \quad \text{Ans.}$$

$$\mathbf{B} = 8\cos(-15^\circ)\hat{\mathbf{i}} + 8\sin(-15^\circ)\hat{\mathbf{j}} = 7.73\hat{\mathbf{i}} - 2.07\hat{\mathbf{j}} \quad \text{Ans.}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (8.66 + 7.73)\hat{\mathbf{i}} + (5.00 - 2.07)\hat{\mathbf{j}} = 16.39\hat{\mathbf{i}} + 2.93\hat{\mathbf{j}}.$$

The magnitude of the resultant is determined from Eq. (2.36),

$$C = \sqrt{16.39^2 + 2.93^2} = 16.65,$$

and the angle is

$$\theta = \tan^{-1} \frac{2.93}{16.39} = 10.1^\circ.$$

The final result in polar notation is

$$\mathbf{C} = 16.6\angle 10.1^\circ. \quad \text{Ans.}$$

2.11 COMPLEX POLAR ALGEBRA

Another method of treating two-dimensional vector problems analytically makes use of complex algebra. Although complex numbers are not vectors, they can be used to represent

*Many calculators are equipped to perform polar-rectangular and rectangular-polar conversions directly.

vectors in a plane by choosing an origin and real and imaginary axes. In two-dimensional kinematic problems, these axes can conveniently be chosen coincident with the x_1y_1 axes of the absolute coordinate system.

As shown in Fig 2.20*b*, the location of any point in the plane can be specified either by its absolute-position vector or by its corresponding real and imaginary coordinates,

$$\mathbf{R} = R^x + jR^y, \quad (2.37)$$

where the operator j is defined as the unit imaginary number,

$$j = \sqrt{-1}. \quad (2.38)$$

Complex numbers are commonly used in planar analysis because of the ease with which they can be changed into polar form. Employing complex rectangular notation for vector $\mathbf{R}$, we can write

$$\mathbf{R} = R\angle\theta = R\cos\theta + jR\sin\theta. \quad (2.39)$$

But using the well-known Euler equation from trigonometry,

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad (2.40)$$

we can also write $\mathbf{R}$ in complex polar form as

$$\mathbf{R} = Re^{j\theta}, \quad (2.41)$$

where the magnitude and direction of the vector appear explicitly. As we will see in Chaps. 3 and 4, expressing a vector in this form is also especially useful when differentiation is required.

Some familiarity with useful manipulation techniques for vectors written in complex polar forms can be gained by again solving the four cases of the loop-closure equation of Table 2.2. Writing Eq. (2.16) in complex polar form, we have

$$Ce^{j\theta_C} = Ae^{j\theta_A} + Be^{j\theta_B}. \quad (2.42)$$

In case 1, the two unknowns are C and θ_C . We begin the solution by substituting Euler's equation, Eq. (2.40), into Eq. (2.42), which gives

$$C(\cos\theta_C + j\sin\theta_C) = A(\cos\theta_A + j\sin\theta_A) + B(\cos\theta_B + j\sin\theta_B). \quad (a)$$

Then, equating the real terms and the imaginary terms separately, we obtain two real equations corresponding to the horizontal and vertical components of the two-dimensional vector equation:

$$C\cos\theta_C = A\cos\theta_A + B\cos\theta_B, \quad (b)$$

$$C\sin\theta_C = A\sin\theta_A + B\sin\theta_B. \quad (c)$$

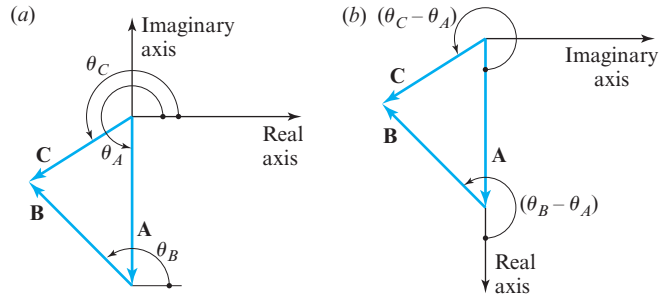


Figure 2.22 (a) Original axes; (b) rotated axes.

The unknown angle θ_C can be eliminated by squaring and adding these two equations. The result is

$$C = \sqrt{A^2 + B^2 + 2AB\cos(\theta_B - \theta_A)}. \quad (2.43)$$

The positive square root is chosen arbitrarily; the negative square root would yield a negative solution for C with θ_C differing by 180° . Dividing Eq. (c) by Eq. (b), and rearranging, the angle θ_C can be written as

$$\theta_C = \tan^{-1} \frac{A\sin\theta_A + B\sin\theta_B}{A\cos\theta_A + B\cos\theta_B}, \quad (2.44)$$

where the signs of the numerator and the denominator must be considered separately in determining the proper quadrant of θ_C .* Only a single solution is found for case 1, as previously illustrated in Fig. 2.10.

In case 2, the two unknowns are A and θ_B . One convenient way of solving this case, in complex polar form, is to first divide Eq. (2.42) by $e^{j\theta_A}$. The resulting equation can be written as

$$Ce^{j(\theta_C - \theta_A)} = A + Be^{j(\theta_B - \theta_A)}. \quad (d)$$

Comparing this equation with Fig. 2.22, we see that division by the complex polar form of a unit vector $e^{j\theta_A}$ has the effect of rotating the real and imaginary axes by the angle θ_A *clockwise* such that the real axis lies along vector **A**.

* When writing computer programs it will be noted that most programming languages, including both ANSI/ISO standard FORTRAN and C, provide a library function named ATAN2(y, x) that accepts the numerator and denominator separately and provides the solution (in radians) in the proper quadrant. If such a function is not available, then a solution (in radians) in either the first or fourth quadrant is usually provided, and π radians (180°) should be added to the angle if the denominator is negative.

We can now use Euler's equation, Eq. (2.40), to separate the real and imaginary components; that is,

$$C\cos(\theta_C - \theta_A) = A + B\cos(\theta_B - \theta_A), \quad (e)$$

$$C\sin(\theta_C - \theta_A) = B\sin(\theta_B - \theta_A). \quad (f)$$

The solutions are then obtained directly from Eqs. (f) and (e), respectively, as

$$\theta_B = \theta_A + \sin^{-1} \frac{C\sin(\theta_C - \theta_A)}{B}. \quad (2.45)$$

$$A = C\cos(\theta_C - \theta_A) - B\cos(\theta_B - \theta_A). \quad (2.46)$$

The solutions given by Eqs. (2.45) and (2.46) are intentionally presented in this order since Eq. (2.46) cannot be evaluated numerically until after θ_B is found. We also note that the arcsine term in Eq. (2.45) is double-valued. Therefore, case 2 has two distinct solutions, A, θ_B and A', θ'_B ; these are shown in Fig. 2.11b.

In case 3, the two unknowns are the magnitudes A and B . As in case 2, we obtain Eqs. (e) and (f). Therefore, from Eq. (f), the solution for B is

$$B = C \frac{\sin(\theta_C - \theta_A)}{\sin(\theta_B - \theta_A)}. \quad (2.47)$$

The solution for the unknown magnitude, A , is obtained by dividing Eq. (2.42) by $e^{j\theta_B}$. The resulting equation is then separated into real and imaginary parts and yields

$$A = C \frac{\sin(\theta_C - \theta_B)}{\sin(\theta_A - \theta_B)}. \quad (2.48)$$

Note that this case yields a unique solution, as illustrated in Fig. 2.12.

In case 4 the two unknowns are θ_A and θ_B . We begin by dividing Eq. (2.42) by $e^{j\theta_C}$ to align the real axis along vector $\mathbf{C}$:

$$C = Ae^{j(\theta_A - \theta_C)} + Be^{j(\theta_B - \theta_C)}. \quad (g)$$

Using Euler's equation to separate components and then rearranging terms, we obtain

$$A\cos(\theta_A - \theta_C) = C - B\cos(\theta_B - \theta_C), \quad (h)$$

$$A\sin(\theta_A - \theta_C) = -B\sin(\theta_B - \theta_C). \quad (i)$$

Squaring both equations and adding the results gives

$$A^2 = C^2 + B^2 - 2BC\cos(\theta_B - \theta_C). \quad (j)$$

We recognize this as the law of cosines for the vector triangle. Rearranging this equation, the angle θ_B can be written as

$$\theta_B = \theta_C \pm \cos^{-1} \frac{C^2 + B^2 - A^2}{2BC}. \quad (2.49)$$

Moving C to the left-hand side of Eq. (h) before the squaring and adding operations results in another form of the law of cosines, from which

$$\theta_A = \theta_C \pm \cos^{-1} \frac{C^2 + A^2 - B^2}{2AC}. \quad (2.50)$$

The plus or minus signs in these two equations are a reminder that the arccosines are each double-valued and, therefore, θ_B and θ_A each have two solutions. These two pairs of angles can be paired naturally together as θ_A, θ_B and θ'_A, θ'_B under the restriction of Eq. (i). Therefore, case 4 has two distinct solutions, as shown in Fig. 2.13.

2.12 POSTURE ANALYSIS TECHNIQUES

There are many different approaches to the posture analysis of mechanisms. Here, we identify the following five approaches:

- (a) Graphic approach.
- (b) Analytic approach.
- (c) Complex-algebraic approach.
- (d) Vector-algebraic approach.
- (e) Numeric approach.

These approaches are illustrated in the following two examples.

EXAMPLE 2.4

Consider a sliding-block linkage in the posture illustrated in Fig. 2.23. Perform the posture analysis; that is, find θ_4 and distance R_{AO_4} .

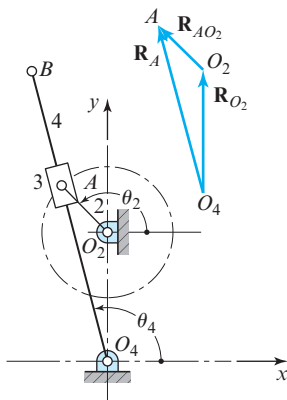


Figure 2.23 $R_{O_2O_4} = 9.0$ in, $R_{AO_2} = 4.5$ in, and $\theta_2 = 135^\circ$.

SOLUTION

Using the vector diagram shown in Fig. 2.23 we recognize this as a case 1 problem, and the vector loop equation can be written as

$$\mathbf{R}_{AO_4} = \mathbf{R}_{O_2O_4}^{\sqrt{\sqrt{}}} + \mathbf{R}_{AO_2}^{\sqrt{I}}, \quad (1)$$

where the “I” symbol, for the direction of vector $\mathbf{R}_{AO_2}$, denotes that the orientation of link 2 is the given input.

(a) *Graphic approach* When Fig. 2.23 is drawn to a suitable scale (for example, 6 in/in), then, by direct measurement of this figure, we find that

$$\theta_4 = 105.3^\circ \quad \text{Ans.}$$

$$R_{AO_4} = 2.08 \text{ in (6 in/in)} = 12.48 \text{ in.} \quad \text{Ans.}$$

(b) *Analytic approach* For case 1, we employ Eqs. (2.43) and (2.44); therefore,

$$\begin{aligned} R_{AO_4} &= \sqrt{R_{O_2O_4}^2 + R_{AO_2}^2 + 2R_{O_2O_4}R_{AO_2}\cos(\theta_2 - 90^\circ)} \\ &= \sqrt{(9 \text{ in})^2 + (4.5 \text{ in})^2 + 2(9 \text{ in})(4.5 \text{ in})\cos(135^\circ - 90^\circ)} \\ &= 12.59 \text{ in,} \end{aligned} \quad \text{Ans.}$$

$$\begin{aligned} \theta_4 &= \tan^{-1} \frac{R_{O_2O_4}\sin 90^\circ + R_{AO_2}\sin \theta_2}{R_{O_2O_4}\cos 90^\circ + R_{AO_2}\cos \theta_2} \\ &= \tan^{-1} \frac{(9 \text{ in})\sin 90^\circ + (4.5 \text{ in})\sin 135^\circ}{(9 \text{ in})\cos 90^\circ + (4.5 \text{ in})\cos 135^\circ} = \tan^{-1} \left(\frac{12.182 \text{ in}}{-3.182 \text{ in}} \right) = -75.36^\circ. \end{aligned}$$

However, note that the calculator used here did not recognize for the arctangent function that the negative sign in the denominator indicates an angle in the second or third quadrant. Therefore, the proper value is

$$\theta_4 = -75.36^\circ \pm 180^\circ = 104.64^\circ. \quad \text{Ans.}$$

(c) *Complex-algebraic approach* The terms for Eq. (1) are

$$\begin{aligned} \mathbf{R}_{AO_4} &= R_{AO_4} \angle \theta_4 = R_{AO_4} \cos \theta_4 + j R_{AO_4} \sin \theta_4, \\ \mathbf{R}_{O_2O_4} &= 9 \text{ in} \angle 90^\circ = (9 \text{ in}) \cos 90^\circ + j(9 \text{ in}) \sin 90^\circ = 0 + j9 \text{ in}, \\ \mathbf{R}_{AO_2} &= 4.5 \text{ in} \angle 135^\circ = (4.5 \text{ in}) \cos 135^\circ + j(4.5 \text{ in}) \sin 135^\circ \\ &= -3.182 \text{ in} + j3.182 \text{ in}. \end{aligned}$$

Substituting these into Eq. (1) yields

$$\mathbf{R}_{AO_4} = (0 + j9 \text{ in}) + (-3.182 \text{ in} + j3.182 \text{ in}) = -3.182 \text{ in} + j12.182 \text{ in}.$$

Thus,

$$R_{AO_4} = \sqrt{(-3.182 \text{ in})^2 + (12.182 \text{ in})^2} = 12.59 \text{ in} \quad \text{Ans.}$$

and

$$\theta_4 = \tan^{-1} \frac{R_{AO_4}^y}{R_{AO_4}^x} = \tan^{-1} \frac{12.182 \text{ in}}{-3.182 \text{ in}} = 104.64^\circ. \quad \text{Ans.}$$

Again, note the comment in (b) regarding arctangent results from the calculator.

(d) Vector-algebraic approach Here we employ a scientific hand calculator that will add and subtract complex numbers in rectangular notation and will convert from rectangular to polar notation or vice versa. Thus,

$$\mathbf{R}_{O_2O_4} = 9 \text{ in} \angle 90^\circ = 0 + j9 \text{ in},$$

$$\mathbf{R}_{AO_2} = 4.5 \text{ in} \angle 135^\circ = -3.182 \text{ in} + j3.182 \text{ in},$$

$$\begin{aligned} \mathbf{R}_{AO_4} &= \mathbf{R}_{O_2O_4} + \mathbf{R}_{AO_2} = (0 + j9 \text{ in}) + (-3.182 \text{ in} + j3.182) \\ &= -3.182 \text{ in} + j12.182 \text{ in} = 12.59 \text{ in} \angle 104.64^\circ. \end{aligned} \quad \text{Ans.}$$

(e) Numeric approach Some scientific hand calculators permit equations containing complex numbers to be solved in either rectangular or polar form or both, with the results displayed in either mode. Using such a calculator, we enter

$$\mathbf{R}_{AO_4} = 9 \text{ in} \angle 90^\circ + 4.5 \text{ in} \angle 135^\circ = 12.59 \text{ in} \angle 104.64^\circ. \quad \text{Ans.}$$

Note that all algebraic methods give results that are in exact agreement and are more accurate than graphic results obtained manually.

EXAMPLE 2.5

A crank-rocker four-bar linkage with the given input angle, is shown in both the open and crossed postures in Fig. 2.24. Perform a posture analysis of the linkage; that is, find θ_3 and θ_4 for both postures.

SOLUTION

Note that the line connecting the ground pins, O_2 and O_4 (that is, the ground link), is not horizontal. In the previous examples, the ground link was chosen to be coincident with the x axis; that is, $\theta_1 = 0$. In this example, however, the angle $\theta_1 = 15^\circ$ and must be accounted for when using Eqs. (2.25) through (2.30).

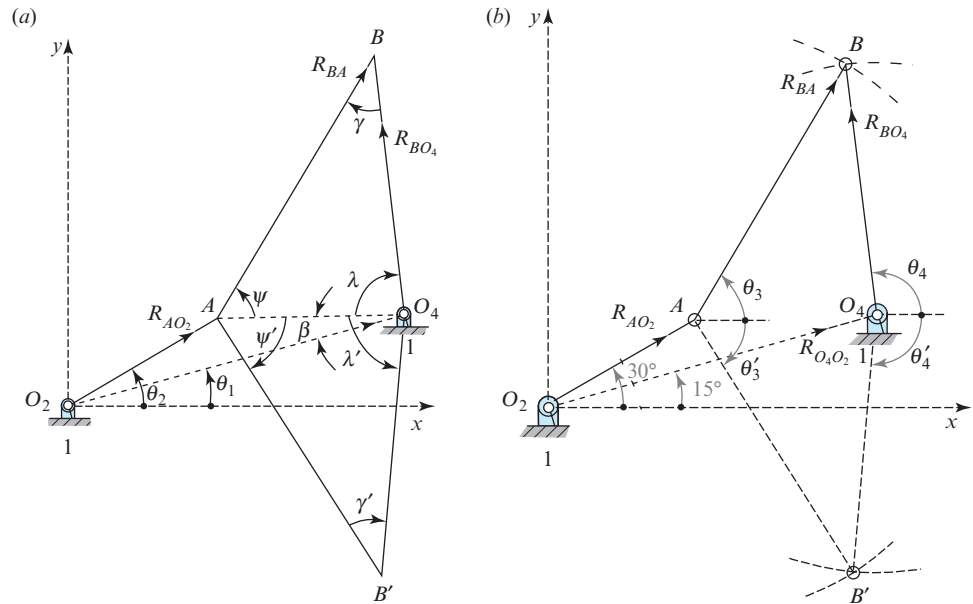


Figure 2.24 $\mathbf{R}_{AO_2} = 200 \text{ mm} \angle 30^\circ$, $R_{BA} = 350 \text{ mm}$, $R_{BO_4} = 300 \text{ mm}$, and $\mathbf{R}_{O_4O_2} = 400 \text{ mm} \angle 15^\circ$.

For the input angle $\theta_2 = 30^\circ$, we observe that

$$\mathbf{R}_{AO_2} = 0.200 \text{ m} \angle 30^\circ \quad \text{and} \quad \mathbf{R}_{O_4O_2} = 0.400 \text{ m} \angle 15^\circ.$$

Therefore,

$$\mathbf{S} = \mathbf{R}_{O_4O_2} - \mathbf{R}_{AO_2} = 0.400 \text{ m} \angle 15^\circ - 0.200 \text{ m} \angle 30^\circ = 0.213 \text{ m} \angle 0.95^\circ.$$

Note that the vectors in triangle ABO_4 are related by the equation

$$\overset{\sqrt{\sqrt{}}}{\mathbf{S}} = \overset{\sqrt{?}}{\mathbf{R}_{BA}} - \overset{\sqrt{?}}{\mathbf{R}_{BO_4}}. \quad (1)$$

There are two unknown orientations in this equation, and so we identify this as case 4 (Table 2.2).

(a) Graphic approach When Fig. 2.24 is drawn at a chosen scale, then, by direct measurement of the figure in the open posture, we find

$$\theta_3 = 60^\circ \quad \text{and} \quad \theta_4 = 97^\circ, \quad \text{Ans.}$$

and in the crossed posture, we find

$$\theta'_3 = -57^\circ \quad \text{and} \quad \theta'_4 = -95^\circ. \quad \text{Ans.}$$

(b) *Analytic approach* Comparing the notation in Fig. 2.24 with Fig. 2.19, we set $r_1 = R_{O_4O_2} = 0.400$ m, $r_2 = R_{AO_2} = 0.200$ m, $r_3 = R_{BA} = 0.350$ m, and $r_4 = R_{BO_4} = 0.300$ m. Also, note that the ground link of Fig. 2.19 must be rotated by $\theta_1 = 15^\circ$ ccw, and $\theta_2 = 30^\circ - 15^\circ = 15^\circ$; therefore, Eqs. (2.25) through (2.30) give

$$S = \sqrt{(0.400 \text{ m})^2 + (0.200 \text{ m})^2 - 2(0.400 \text{ m})(0.200 \text{ m})\cos(30^\circ - 15^\circ)}$$

$$= 0.213 \text{ m},$$

$$\beta = \cos^{-1} \frac{(0.400 \text{ m})^2 + (0.213 \text{ m})^2 - (0.200 \text{ m})^2}{2(0.400 \text{ m})(0.213 \text{ m})} = 14.05^\circ,$$

$$\psi = \cos^{-1} \frac{(0.350 \text{ m})^2 + (0.213 \text{ m})^2 - (0.300 \text{ m})^2}{2(0.350 \text{ m})(0.213 \text{ m})} = 58.51^\circ,$$

$$\lambda = \cos^{-1} \frac{(0.300 \text{ m})^2 + (0.213 \text{ m})^2 - (0.350 \text{ m})^2}{2(0.300 \text{ m})(0.213 \text{ m})} = 84.19^\circ.$$

Then, in the open posture, remembering that $\theta_1 = 15^\circ$,

$$\theta_3 = 15^\circ - 14.05^\circ + 58.51^\circ = 59.46^\circ, \quad \text{Ans.}$$

$$\theta_4 = 15^\circ + 180^\circ - 14.05^\circ - 84.19^\circ = 96.76^\circ, \quad \text{Ans.}$$

and, in the crossed posture,

$$\theta'_3 = 15^\circ - 14.05^\circ - 58.51^\circ = -57.56^\circ, \quad \text{Ans.}$$

$$\theta'_4 = 15^\circ + 180^\circ - 14.05^\circ + 84.19^\circ = -94.86^\circ. \quad \text{Ans.}$$

(c) *Complex-algebraic approach* Substituting S for C , R_{BA} for A , $-R_{BO_4}$ for B , θ_S for θ_C , and θ_4 for θ_B , into Eq. (2.49) gives

$$\theta_4 = \theta_S \pm \cos^{-1} \frac{S^2 + R_{BO_4}^2 - R_{BA}^2}{2(-R_{BO_4})S} = (\theta_1 - \beta) \pm \cos^{-1} \frac{S^2 + R_{BO_4}^2 - R_{BA}^2}{2(-R_{BO_4})S}$$

$$= 0.95^\circ \pm \cos^{-1} \frac{(0.213 \text{ m})^2 + (-0.300 \text{ m})^2 - (0.350 \text{ m})^2}{2(-0.300 \text{ m})(0.213 \text{ m})}$$

$$= 0.95^\circ \pm 95.81^\circ = 96.76^\circ \text{ or } -94.86^\circ. \quad \text{Ans.}$$

In addition, substituting θ_3 for θ_A into Eq. (2.50) gives

$$\theta_3 = \theta_S \pm \cos^{-1} \frac{S^2 + R_{BA}^2 - R_{BO_4}^2}{2R_{BA}S}$$

$$= 0.95^\circ \pm \cos^{-1} \frac{(0.213 \text{ m})^2 + (0.350 \text{ m})^2 - (-0.300 \text{ m})^2}{2(0.350 \text{ m})(0.213 \text{ m})}$$

$$= 0.95^\circ \pm 58.51^\circ = 59.46^\circ \text{ or } -57.56^\circ, \quad \text{Ans.}$$

where the first result corresponds to the open posture and the second result is for the crossed posture. We note that the answers in Parts (b) and (c) are in good agreement with those of Part (a), but are of higher accuracy than measurements taken from the scaled drawing.

(d) *Vector-algebraic approach* The vector loop equation for the four-bar-linkage can be written as

$$\mathbf{R}_{BA} = \mathbf{R}_{O_4O_2} + \mathbf{R}_{BO_4} - \mathbf{R}_{AO_2}. \quad (2)$$

Separating this equation into horizontal and vertical components,

$$R_{BA} \cos \theta_3 = R_{O_4O_2} \cos \theta_1 + R_{BO_4} \cos \theta_4 - R_{AO_2} \cos \theta_2, \quad (3a)$$

$$R_{BA} \sin \theta_3 = R_{O_4O_2} \sin \theta_1 + R_{BO_4} \sin \theta_4 - R_{AO_2} \sin \theta_2. \quad (3b)$$

By squaring and adding these two equations, we can eliminate the variable θ_3 :

$$\begin{aligned} R_{BA}^2 = & R_{O_4O_2}^2 + R_{BO_4}^2 + R_{AO_2}^2 + 2R_{O_4O_2}R_{BO_4}[\cos \theta_1 \cos \theta_4 + \sin \theta_1 \sin \theta_4] \\ & - 2R_{O_4O_2}R_{AO_2}[\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2] - 2R_{BO_4}R_{AO_2}[\cos \theta_4 \cos \theta_2 + \sin \theta_4 \sin \theta_2]. \end{aligned}$$

Then we can write this equation as

$$A \cos \theta_4 + B \sin \theta_4 = C, \quad (4)$$

where

$$A = 2R_{O_4O_2}R_{BO_4} \cos \theta_1 - 2R_{BO_4}R_{AO_2} \cos \theta_2, \quad (5a)$$

$$B = 2R_{O_4O_2}R_{BO_4} \sin \theta_1 - 2R_{BO_4}R_{AO_2} \sin \theta_2, \quad (5b)$$

$$C = R_{BA}^2 - R_{O_4O_2}^2 - R_{BO_4}^2 - R_{AO_2}^2 + 2R_{O_4O_2}R_{AO_2} \cos(\theta_2 - \theta_1). \quad (5c)$$

Equation (4) is one form of Freudenstein's equation (Sec. 9.11 shows another variation that is more suitable for kinematic synthesis).

Substituting the known data into Eqs. (5) gives

$$A = 2(0.400 \text{ m})(0.300 \text{ m}) \cos 15^\circ - 2(0.300 \text{ m})(0.200 \text{ m}) \cos 30^\circ = 0.12790 \text{ m}^2,$$

$$B = 2(0.400 \text{ m})(0.300 \text{ m}) \sin 15^\circ - 2(0.300 \text{ m})(0.200 \text{ m}) \sin 30^\circ = 0.00212 \text{ m}^2,$$

$$\begin{aligned} C = & (0.350 \text{ m})^2 - (0.400 \text{ m})^2 - (0.300 \text{ m})^2 - (0.200 \text{ m})^2 \\ & + 2(0.400 \text{ m})(0.200 \text{ m}) \cos(30^\circ - 15^\circ) \\ = & -0.01295 \text{ m}^2. \end{aligned}$$

Therefore, Eq. (4) can be written as

$$(0.12790 \text{ m}^2) \cos \theta_4 + (0.00212 \text{ m}^2) \sin \theta_4 = -0.01295 \text{ m}^2. \quad (6)$$

To determine the output angle θ_4 , this transcendental equation can be written as a quadratic equation. The procedure is to use the half-angle relationships; that is, to define

$$Z = \tan(\theta_4/2), \quad (7a)$$

which gives

$$\sin \theta_4 = \frac{2Z}{1+Z^2} \quad \text{and} \quad \cos \theta_4 = \frac{1-Z^2}{1+Z^2}. \quad (7b)$$

Substituting Eqs. (7b) into Eq. (6), multiplying through by $(1+Z^2)$, and rearranging, gives the quadratic equation

$$(0.114\,95\,\text{m}^2)Z^2 + (-0.004\,24\,\text{m}^2)Z + (-0.140\,85\,\text{m}^2) = 0.$$

The solutions to this equation can be written as

$$\begin{aligned} Z &= \frac{-(-0.004\,24\,\text{m}) \pm \sqrt{(-0.004\,24\,\text{m})^2 - 4(0.114\,95\,\text{m})(-0.140\,85\,\text{m})}}{2(0.114\,95\,\text{m})} \\ &= 0.018\,41 \pm 1.106\,80 = 1.125\,21 \text{ or } -1.088\,39. \end{aligned} \quad (8)$$

Substituting Eq. (8) into Eq. (7a), gives the two output angles corresponding to the open and crossed postures, namely

$$\theta_4 = 96.76^\circ \quad \text{and} \quad \theta'_4 = -94.86^\circ. \quad \text{Ans.}$$

Substituting these results into Eqs. (3a) and (3b) gives the corresponding coupler angles

$$\theta_3 = 59.46^\circ \quad \text{and} \quad \theta'_3 = -57.56^\circ. \quad \text{Ans.}$$

(e) *Numeric approach* Rearrange the vector loop equation, Eq. (2), into the form

$$\mathbf{f} = \mathbf{R}_{AO_2} + \mathbf{R}_{BA} - \mathbf{R}_{O_4O_2} - \mathbf{R}_{BO_4} = \mathbf{0}, \quad (9)$$

and separate the horizontal and vertical components:

$$f^x = R_{AO_2} \cos \theta_2 + R_{BA} \cos \theta_3 - R_{O_4O_2} \cos \theta_1 - R_{BO_4} \cos \theta_4 = 0, \quad (10a)$$

$$f^y = R_{AO_2} \sin \theta_2 + R_{BA} \sin \theta_3 - R_{O_4O_2} \sin \theta_1 - R_{BO_4} \sin \theta_4 = 0. \quad (10b)$$

We recognize these as two equations in two unknowns, θ_3 and θ_4 , and we solve them by the Newton-Raphson method introduced in Sec. 2.9 to an accuracy of 0.01° . To do this, we expand Eqs. (10) to first-order in Taylor series in the two unknowns as follows:

$$R_{AO_2} \cos \theta_2 + R_{BA} \cos \theta_3 - R_{BA} \sin \theta_3 \Delta \theta_3 - R_{O_4O_2} \cos \theta_1 - R_{BO_4} \cos \theta_4 + R_{BO_4} \sin \theta_4 \Delta \theta_4 = 0,$$

$$R_{AO_2} \sin \theta_2 + R_{BA} \sin \theta_3 + R_{BA} \cos \theta_3 \Delta \theta_3 - R_{O_4O_2} \sin \theta_1 - R_{BO_4} \sin \theta_4 - R_{BO_4} \cos \theta_4 \Delta \theta_4 = 0.$$

Then, writing these equations in matrix form,

$$\begin{bmatrix} R_{BA} \sin \theta_3 & -R_{BO_4} \sin \theta_4 \\ -R_{BA} \cos \theta_3 & R_{BO_4} \cos \theta_4 \end{bmatrix} \begin{bmatrix} \Delta \theta_3 \\ \Delta \theta_4 \end{bmatrix} = \begin{bmatrix} R_{AO_2} \cos \theta_2 + R_{BA} \cos \theta_3 - R_{O_4O_2} \cos \theta_1 - R_{BO_4} \cos \theta_4 \\ R_{AO_2} \sin \theta_2 + R_{BA} \sin \theta_3 - R_{O_4O_2} \sin \theta_1 - R_{BO_4} \sin \theta_4 \end{bmatrix}.$$

Substituting the given data, these matrices become

$$\begin{bmatrix} (0.350 \text{ m}) \sin \theta_3 & -(0.300 \text{ m}) \sin \theta_4 \\ -(0.350 \text{ m}) \cos \theta_3 & (0.300 \text{ m}) \cos \theta_4 \end{bmatrix} \begin{bmatrix} \Delta \theta_3 \\ \Delta \theta_4 \end{bmatrix} = \begin{bmatrix} (-0.213 \text{ m}) + (0.350 \text{ m}) \cos \theta_3 - (0.300 \text{ m}) \cos \theta_4 \\ (-0.003 \text{ m}) + (0.350 \text{ m}) \sin \theta_3 - (0.300 \text{ m}) \sin \theta_4 \end{bmatrix}. \quad (11)$$

Note that $\Delta \theta_3$ and $\Delta \theta_4$ are called the *corrections* and are dimensionless values, meaning values in radians. For the iteration procedure, we must be careful when combining the corrections with θ_3 and θ_4 if the two angles have units of degrees; therefore,

$$(\theta_3^\circ)_{\text{new}} = (\theta_3^\circ)_{\text{old}} + \frac{180^\circ}{\pi \text{ rad}} \Delta \theta_3, \quad (12a)$$

$$(\theta_4^\circ)_{\text{new}} = (\theta_4^\circ)_{\text{old}} + \frac{180^\circ}{\pi \text{ rad}} \Delta \theta_4. \quad (12b)$$

To use the iteration procedure, we begin with initial estimates of the unknowns θ_3 and θ_4 . From a scale drawing of the linkage, such that, for the open posture, our starting estimates are

$$\theta_3 = 60^\circ \quad \text{and} \quad \theta_4 = 100^\circ. \quad (13a)$$

Inserting these estimates into Eq. (11) gives

$$\begin{bmatrix} 0.303 \text{ m} & -0.295 \text{ m} \\ -0.175 \text{ m} & -0.052 \text{ m} \end{bmatrix} \begin{bmatrix} \Delta \theta_3 \\ \Delta \theta_4 \end{bmatrix} = \begin{bmatrix} 0.013 \text{ m} \\ 0.004 \text{ m} \end{bmatrix}.$$

Using Cramer's rule, the corrections are

$$\Delta \theta_3 = -0.007 \text{ rad} \quad \text{and} \quad \Delta \theta_4 = -0.005 \text{ rad}.$$

Substituting these corrections into Eqs. (12) gives

$$\theta_3 = 59.58^\circ \quad \text{and} \quad \theta_4 = 96.87^\circ. \quad (13b)$$

Inserting these improved estimates into Eq. (11) gives

$$\begin{bmatrix} 0.301 \text{ m} & -0.297 \text{ m} \\ -0.177 \text{ m} & -0.035 \text{ m} \end{bmatrix} \begin{bmatrix} \Delta \theta_3 \\ \Delta \theta_4 \end{bmatrix} = \begin{bmatrix} -0.577 \text{ m} \cdot 10^{-4} \\ 0.441 \text{ m} \cdot 10^{-3} \end{bmatrix}.$$

Solving again by Cramer's rule, the updated corrections are

$$\Delta \theta_3 = -0.002 \text{ rad} \quad \text{and} \quad \Delta \theta_4 = -0.001 \text{ rad}.$$

Substituting these corrections into Eqs. (12), gives

$$\theta_3 = 59.46^\circ \quad \text{and} \quad \theta_4 = 96.76^\circ. \quad (13c)$$

Inserting these improved estimates into Eq. (11) gives us

$$\begin{bmatrix} 0.301 \, 45 \, \text{m} & -0.297 \, 91 \, \text{m} \\ -0.177 \, 85 \, \text{m} & -0.035 \, 31 \, \text{m} \end{bmatrix} \begin{bmatrix} \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = \begin{bmatrix} 0.213 \, 51(10)^{-5} \, \text{m} \\ 0.173 \, 94(10)^{-5} \, \text{m} \end{bmatrix}.$$

Solving again by Cramer's rule, the updated corrections are

$$\Delta\theta_3 = -0.000 \, 069 \, 59 \, \text{rad} \quad \text{and} \quad \Delta\theta_4 = -0.000 \, 014 \, 2 \, \text{rad}.$$

Substituting the corrections into Eqs. (12), gives

$$\theta_3 = 59.46^\circ \quad \text{and} \quad \theta_4 = 96.76^\circ. \quad \text{Ans.}$$

Since the corrections are now both smaller than the desired accuracy of $0.00017 \, \text{rad} = 0.01^\circ$, we accept these values as final results for the open posture.

For the crossed posture, we begin with initial estimates from our drawing of $\theta_3 = -60^\circ$ and $\theta_4 = -100^\circ$. Then, using these as starting estimates with Eq. (11), we repeat the same process and iterate until we converge to a result of

$$\theta_3 = -57.56^\circ \quad \text{and} \quad \theta_4 = -94.86^\circ. \quad \text{Ans.}$$

A computer code could be written using a commercially available language such as MATLAB or JAVA.

2.13 COUPLER-CURVE GENERATION

In Chap. 1, Fig. 1.23, we learned of the vast variety of useful coupler curves that can be generated by a planar four-bar linkage. These curves are quite easy to obtain graphically, but computer-generated curves can be obtained more quickly and are easier to vary to obtain desired curve characteristics. Here we present the basic equations but omit the programming details required for display of a curve on an electronic monitor.

Considering the four-bar linkage of Fig. 2.16, we first write the vector equation

$$\overset{??}{\mathbf{S}} = \overset{\sqrt{\sqrt{}}}{\mathbf{R}}_C - \overset{\sqrt{\sqrt{}}}{\mathbf{R}}_A.$$

This is case 1, where S and θ_S are the two unknowns. The solutions are given by Eqs. (2.43) and (2.44). Substituting gives

$$S = \sqrt{R_C^2 + R_A^2 - 2R_C R_A \cos(\theta_A - \theta_C)}, \quad (2.51)$$

$$\theta_S = \tan^{-1} \frac{R_C \sin \theta_C - R_A \sin \theta_A}{R_C \cos \theta_C - R_A \cos \theta_A}. \quad (2.52)$$

Then we note that

$$\sqrt{\sqrt{}} \mathbf{S} = \mathbf{R}_{BA} - \mathbf{R}_{BC}.$$

This is case 4, where θ_3 and θ_4 are the two unknowns. The solutions are given by Eqs. (2.49) and (2.50). Substituting gives

$$\theta_4 = \theta_S \pm \cos^{-1} \frac{S^2 + R_{BC}^2 - R_{BA}^2}{2R_{BC}S}, \quad (2.53)$$

$$\theta_3 = \theta_S \mp \cos^{-1} \frac{S^2 + R_{BA}^2 - R_{BC}^2}{2R_{BA}S}. \quad (2.54)$$

The solution for the open posture of the linkage shown in Fig. 2.16a corresponds to the lower set of signs; the crossed posture of the linkage shown in Fig. 2.16b corresponds to the upper set.

Coupler point P generates a coupler curve when crank 2 is rotated. From Fig. 2.16, we see that

$$\mathbf{R}_P = R_P e^{j\theta_6} = R_A e^{j\theta_2} + R_{PA} e^{j(\theta_3 + \alpha)}. \quad (2.55)$$

We recognize this as case 1, where R_P and θ_6 are the two unknowns. The solutions are given by Eqs. (2.43) and (2.44); that is,

$$R_P = \sqrt{R_A^2 + R_{PA}^2 + 2R_A R_{PA} \cos(\theta_3 + \alpha - \theta_2)} \quad (2.56)$$

and

$$\theta_6 = \tan^{-1} \frac{R_A \sin \theta_2 + R_{PA} \sin(\theta_3 + \alpha)}{R_A \cos \theta_2 + R_{PA} \cos(\theta_3 + \alpha)}. \quad (2.57)$$

Note that each of these equations give two solutions because of the double values for θ_3 corresponding to the two postures of the linkage. The following example uses Eqs. (2.56) and (2.57) to plot the coupler curve of a four-bar linkage.

EXAMPLE 2.6

Consider the crank-rocker four-bar linkage shown in Fig. 2.25 with $R_{BA} = 100$ mm, $R_{CB} = 250$ mm, $R_{CD} = 300$ mm, and $R_{DA} = 200$ mm. The paths of coupler pins B and C are shown by the circle and the circular arc, respectively. The location of coupler point P is given by $R_{PB} = 150$ mm and $\alpha = \angle CBP = -45^\circ$. Calculate the coordinates of coupler

point P and plot the path of this point (the coupler curve) for a complete rotation of the crank.

SOLUTION

For each value of crank angle θ_2 , the angles β , ψ , and γ are calculated from Eqs. (2.26), (2.27), and (2.33), respectively. This notation corresponds to that of Figs. 2.15 and 2.16. Next, Eqs. (2.29) and (2.31) are applied to give the posture of link 3, θ_3 , as displayed in Table 2.3. Note that the same result can be obtained from Eq. (2.50).

Finally, the polar coordinates of coupler point P are calculated from Eqs. (2.56) and (2.57). From these, we can obtain the Cartesian coordinates of point P . The solutions for crank angles from 0° to 90° for the open posture are displayed in Table 2.3. The coupler curve is shown in Fig. 2.25. The interested reader can repeat the solution to this problem for the crossed posture.

Table 2.3 Posture of Link 3 and Coordinates of Coupler Point P

θ_2, deg	θ_3, deg	R_P, mm	θ_6, deg	R_P^x, mm	R_P^y, mm
0.0	110.5	212.0	40.1	162.2	136.5
10.0	99.4	232.2	36.9	185.8	139.3
20.0	87.8	245.3	33.7	204.0	136.1
30.0	77.5	249.9	31.5	213.1	130.6
40.0	69.2	247.7	30.5	213.4	125.8
50.0	62.9	240.7	30.7	207.0	122.7
60.0	58.3	230.4	31.7	196.0	121.1
70.0	55.1	218.0	33.5	181.9	120.3
80.0	53.0	204.4	35.7	165.9	119.4
90.0	51.8	189.9	38.3	148.9	117.8

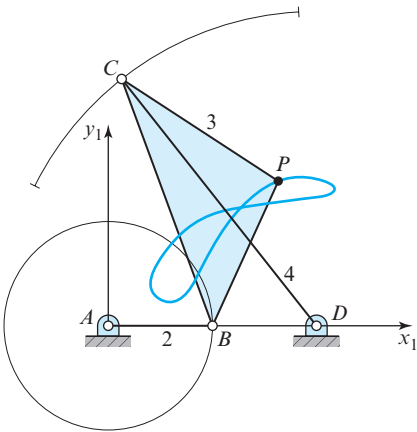


Figure 2.25 The coupler curve traced by point P .

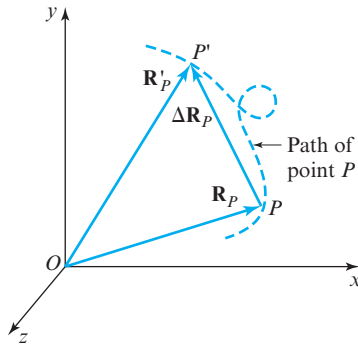


Figure 2.26 Displacement of a moving point.

2.14 DISPLACEMENT OF A MOVING POINT

In the study of motion, we must be concerned with the relationship between successive positions of a point and postures of a link. In Fig. 2.26, a point, originally at position P , is moving along the path shown and, sometime later, arrives at position P' . The *displacement* of the point during the time interval is defined as the *net change in position*; that is,

$$\Delta \mathbf{R}_P = \mathbf{R}'_P - \mathbf{R}_P. \quad (2.58)$$

Note that this displacement is a vector quantity having the magnitude and direction of the vector from point P to point P' .

It is important to note that the displacement $\Delta \mathbf{R}_P$ is the net change in position and does not depend on the particular path taken from P to P' . The magnitude of this vector is *not* necessarily equal to the length of the path (the distance traveled), and its direction is *not* necessarily along the tangent to the path, although both are true when the displacement is infinitesimally small. Knowledge of the path actually traveled from P to P' is not even necessary to find the displacement vector, providing the initial and final positions of the point are known.

2.15 DISPLACEMENT DIFFERENCE BETWEEN TWO POINTS

In this section we consider the difference in the displacements of two moving points. In particular, we are concerned with the case where the two moving points are both fixed in the same rigid body. The situation is shown in Fig. 2.27, where a rigid body moves from an initial posture defined by $x_2y_2z_2$ to a later posture defined by $x'_2y'_2z'_2$.

From Eq. (2.6), the position difference between the two points P and Q at the initial instant is

$$\mathbf{R}_{PQ} = \mathbf{R}_P - \mathbf{R}_Q. \quad (a)$$

After the displacement of this body, the two points are located at P' and Q' . At that time, the position difference is

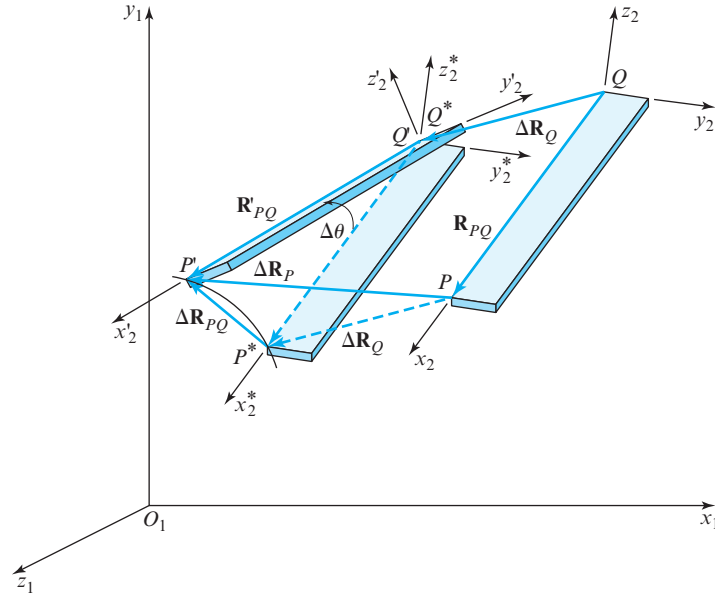


Figure 2.27 Displacement difference between two points fixed in the same rigid body.

$$\mathbf{R}'_{PQ} = \mathbf{R}'_P - \mathbf{R}'_Q. \quad (b)$$

During the time interval of the movement, the two points have undergone individual displacements of $\Delta\mathbf{R}_P$ and $\Delta\mathbf{R}_Q$, respectively.

As the name implies, the *displacement difference* between the two points is defined as the net difference between their respective displacements and can be written as

$$\Delta\mathbf{R}_{PQ} = \Delta\mathbf{R}_P - \Delta\mathbf{R}_Q. \quad (2.59)$$

Note that this equation corresponds to the vector triangle PP^*P' in Fig. 2.27. As stated in the previous section, the displacement depends only on the net change in position and not on the particular path taken. Therefore, no matter how the body (containing points P and Q) was *actually* displaced, we are free to visualize the path as we choose. Equation (2.59) leads us to visualize the displacement as taking place in two stages. First, the body translates (slides without rotation) from $x_2y_2z_2$ to $x_2^*y_2^*z_2^*$; during this movement, all points, including P and Q , have the same displacement $\Delta\mathbf{R}_Q$. Second, the body rotates about point Q' to its final posture $x_2'y_2'z_2'$.

A different interpretation can be obtained by manipulating Eq. (2.59) as follows:

$$\begin{aligned} \Delta\mathbf{R}_{PQ} &= (\mathbf{R}'_P - \mathbf{R}_P) - (\mathbf{R}'_Q - \mathbf{R}_Q) \\ &= (\mathbf{R}'_P - \mathbf{R}'_Q) - (\mathbf{R}_P - \mathbf{R}_Q), \end{aligned} \quad (c)$$

and then, from Eqs. (a) and (b),

$$\Delta \mathbf{R}_{PQ} = \mathbf{R}'_{PQ} - \mathbf{R}_{PQ}. \quad (2.60)$$

This equation corresponds to the vector triangle $Q'P^*P'$ in Fig. 2.27 and demonstrates that the displacement difference, defined as the difference between two displacements, is equal to the net change between the position-difference vectors.

For either interpretation, we are illustrating that *any displacement of a rigid body is equivalent to the sum of a net translation of one point and a net rotation of the body about that point*. We also see that only the rotation contributes to the displacement difference between two points fixed in the same rigid body; that is, *there is no difference between the displacements of any two points fixed in the same rigid body as the result of a translation*. (The following section gives the definition of the term *translation*.)

In view of the previous discussion, we can visualize the displacement difference $\Delta \mathbf{R}_{PQ}$ as the displacement that would be seen for point P by a moving observer who travels along, always staying coincident with point Q but *not rotating* with the moving body—that is, always using the absolute coordinate axes $x_1y_1z_1$ for the measurement of direction. It is important to understand the difference between the interpretation of an observer moving with point Q but not rotating and the case of the observer *on* the moving body. To an observer *on* the moving body, both points P and Q would appear stationary; neither would be seen to have a displacement because they do not move relative to the observer, and the displacement difference seen by such an observer would be zero.

2.16 TRANSLATION AND ROTATION

Using the concept of displacement difference between two points fixed in the same rigid body, we are now able to define translation and rotation.

Translation is defined as *a motion of a body for which the displacement difference between any two points P and Q in the body is zero*; that is, from the displacement-difference equation, Eq. (2.60),

$$\Delta \mathbf{R}_{PQ} = \Delta \mathbf{R}_P - \Delta \mathbf{R}_Q = \mathbf{0}.$$

That is,

$$\Delta \mathbf{R}_P = \Delta \mathbf{R}_Q, \quad (2.61)$$

which states that *the displacements of any two points in the body are equal*.

Rotation is a motion of a body for which different points of the body exhibit different displacements.

Figure 2.28a shows a situation where the body has moved along a curved path from posture x_2y_2 to posture $x'_2y'_2$. Despite the fact that the point paths are curved,* $\Delta \mathbf{R}_P$ is still

* Translation in which the point paths are straight lines is called *rectilinear* translation; translation in which the point paths are identical curves is referred to as *curvilinear* translation.

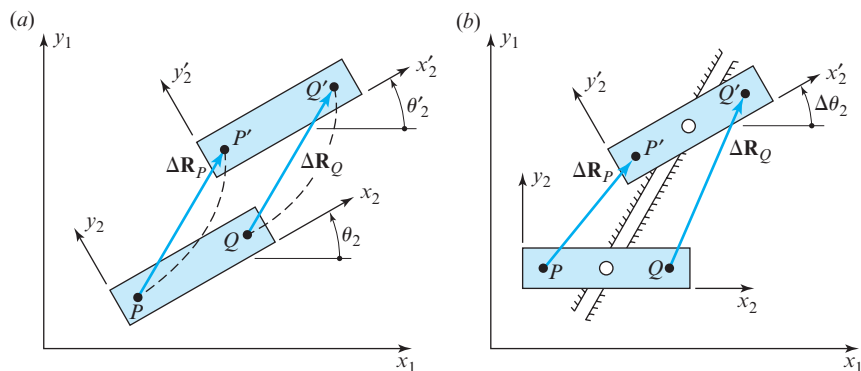


Figure 2.28 (a) Translation: $\Delta \mathbf{R}_P = \Delta \mathbf{R}_Q$, $\Delta \theta_2 = 0$; (b) rotation: $\Delta \mathbf{R}_P \neq \Delta \mathbf{R}_Q$, $\Delta \theta_2 \neq 0$.

equal to $\Delta \mathbf{R}_Q$, and the body has undergone a translation. Note that in translation, the point paths described by any two points on the body are identical, and there is no change of angular orientation between the moving coordinate system and the coordinate system of the observer; that is, $\Delta \theta_2 = \theta'_2 - \theta_2 = 0$.

Now consider the midpoint of a body that is *constrained* to move along a straight-line path, as shown in Fig. 2.28b. As it does so, the body rotates so that $\Delta \theta_2 = \theta'_2 - \theta_2 \neq 0$ and the displacements $\Delta \mathbf{R}_P$ and $\Delta \mathbf{R}_Q$ are not equal. Although there is no obvious point on the body about which it has rotated, the coordinate system $x'_2y'_2$ has changed angular orientation relative to x_1y_1 , and the body is said to have undergone a rotation.

In conclusion, we note that translation and rotation of a body cannot be defined from the motion of a single point. These are characteristics of a body or a coordinate system. It is improper to speak of “rotation of a point” since there is no meaning for angular orientation of a point. It is also improper to associate the terms “translation” and “rotation” with the path of a single point of a moving body. Although it does not matter which points of the body are chosen, the motions of two or more points must be compared before meaningful definitions exist for these terms.

2.17 APPARENT DISPLACEMENT

We have stated in previous sections that the displacement of a point does not depend on the particular path on which it travels. However, since displacement is computed from the position vectors of the endpoints of the path, knowledge of the coordinate system of the observer is essential.

In Fig. 2.29 we identify three bodies: body 1 is a fixed or stationary body containing the absolute reference system $x_1y_1z_1$; body 2 is a moving body containing the reference system $x_2y_2z_2$; and body 3, which moves with respect to body 2. We also identify two observers. We designate that HE is an observer fixed to body 1, the stationary system, and we designate that SHE is another observer on body 2, fixed to the moving system $x_2y_2z_2$.

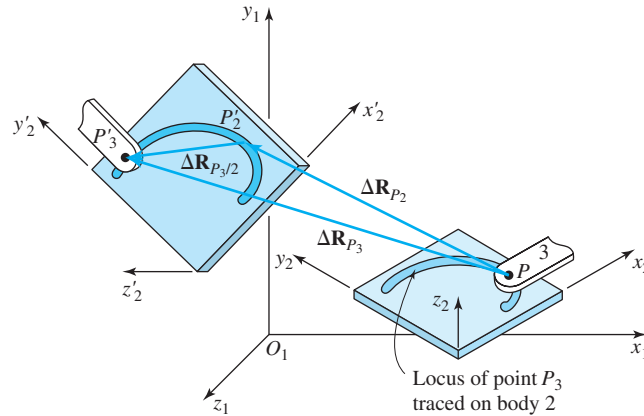


Figure 2.29 Apparent displacement of a point.

Thus, we might say that HE is on the ground observing whereas SHE is going for a ride on body 2.

Now consider a point P_3 , fixed in body 3 and moving along a known path on body 2. HE and SHE are both observing the motion of P_3 , and we want to compare what they see. We must define another point, P_2 , which is fixed in (or rigidly attached to) body 2 and is instantaneously coincident with P_3 .

Now let body 2 and the axes $x_2y_2z_2$ be displaced to a new posture $x'_2y'_2z'_2$. While this motion is taking place, let P_3 move to another position on body 2, which we identify as P'_3 . But P_2 moves with body 2 and is now in the new position identified as P'_2 . SHE, on body 2, reports the motion of P_3 as vector $\Delta\mathbf{R}_{P_3/2}$, which is read as the displacement of P_3 as it appears to an observer on body 2. This is called the apparent-displacement vector. Note that SHE sees no motion of P_2 , since it appears to HER to be stationary on body 2. Therefore, $\Delta\mathbf{R}_{P_2/2} = \mathbf{0}$.

However, HE, on the stationary body, reports the displacement of P_3 as vector $\Delta\mathbf{R}_{P_3}$. Note that HE also reports the displacement of P_2 as vector $\Delta\mathbf{R}_{P_2}$. From the vector triangle in Fig. 2.29, we see that the observations of the two observers are related by the *apparent-displacement equation*

$$\Delta\mathbf{R}_{P_3} = \Delta\mathbf{R}_{P_2} + \Delta\mathbf{R}_{P_3/2}. \quad (2.62)$$

We can take this equation as the definition of the apparent-displacement vector, although it is important also to understand the physical concepts involved. Note that the apparent-displacement vector relates the absolute displacements of two *coincident points* that are points of *different* moving bodies. Note also that there is no restriction on the actual location of the observer moving with coordinate system 2, only that SHE be fixed in that coordinate system so that SHE senses no displacement for point P_2 .

One primary use of the apparent displacement is to determine an absolute displacement. It is not uncommon in machines to find a point, such as P_3 , which is constrained to move along a known slot or path or guideway defined by the shape of another moving

link 2. In such cases, it may be much more convenient to measure or calculate $\Delta \mathbf{R}_{P_2}$ and $\Delta \mathbf{R}_{P_3/2}$ and to use Eq. (2.62) than to measure the absolute displacement $\Delta \mathbf{R}_{P_3}$ directly.

2.18 ABSOLUTE DISPLACEMENT

In reflecting on the definition and concept of the apparent-displacement vector, we conclude that the absolute displacement of a moving point, say $\Delta \mathbf{R}_{P_3/1}$, is the special case of an apparent displacement where the observer is fixed in the absolute coordinate system. As explained for the position vector, the notation is often abbreviated to read $\Delta \mathbf{R}_{P_3}$ or just $\Delta \mathbf{R}_P$ and an absolute observer is implied whenever this is not noted explicitly.

Perhaps a better physical understanding of apparent displacement can be achieved by relating it to absolute displacement. Imagine a car, P_3 , traveling along a roadway and under observation by an absolute observer some distance to one side. Consider how this observer visually senses the motion of the car. Although HE may not be conscious of all of the following steps, the idea here is that the observer first imagines a fixed point P_1 coincident with P_3 , which HE defines in his mind as stationary; HE may relate to a fixed point of the roadway or a nearby tree or road sign, for example. HE then compares his later observation of the car P_3 with that of P_1 to sense displacement. Note that HE does not compare with his own location but with the initially coincident point P_1 . In this instance, the apparent-displacement equation becomes an identity:

$$\Delta \mathbf{R}_{P_3} = \Delta \mathbf{R}_{P_1}^0 + \Delta \mathbf{R}_{P_3/1}.$$

2.19 APPARENT ANGULAR DISPLACEMENT

In general, rotations cannot be treated as vectors (this is explained in some detail in Sec. 3.2). However, for planar motion, the concept of apparent displacement also extends to include rotation. For example, suppose we consider the rotations of the two gears 2 and 3 connected by an arm, body 4, as shown in Fig. 2.30. In Fig. 2.30a, we see the rotations as they would appear in HER coordinate system attached to the arm. In Fig. 2.30b, we see the same situation, but as it might appear to HIM in the absolute frame of reference.

Although it may not be easy to write equations relating the absolute rotations of the gears shown in Fig. 2.30, it is quite easy to relate their apparent rotations if we take HER point of view as an observer riding on the moving coordinate system attached to the arm. From HER vantage point it is obvious that, if there is no slipping between bodies 2 and 3, then the arc lengths of the two gear sectors that pass the arm at the point of contact must be equal. That is, if ρ_2 is the radius of gear 2 and ρ_3 is the radius of gear 3, then

$$\rho_2 \Delta \theta_{2/4} = -\rho_3 \Delta \theta_{3/4}, \quad (a)$$

where $\Delta \theta_{2/4}$ and $\Delta \theta_{3/4}$ are the angular displacements of gears 2 and 3 as they appear to HER in coordinate system 4, and the negative sign accounts for the difference in the senses of the two apparent rotations.

When these angular displacements are replaced by those seen by HIM from the absolute coordinate system, Eq. (a) becomes

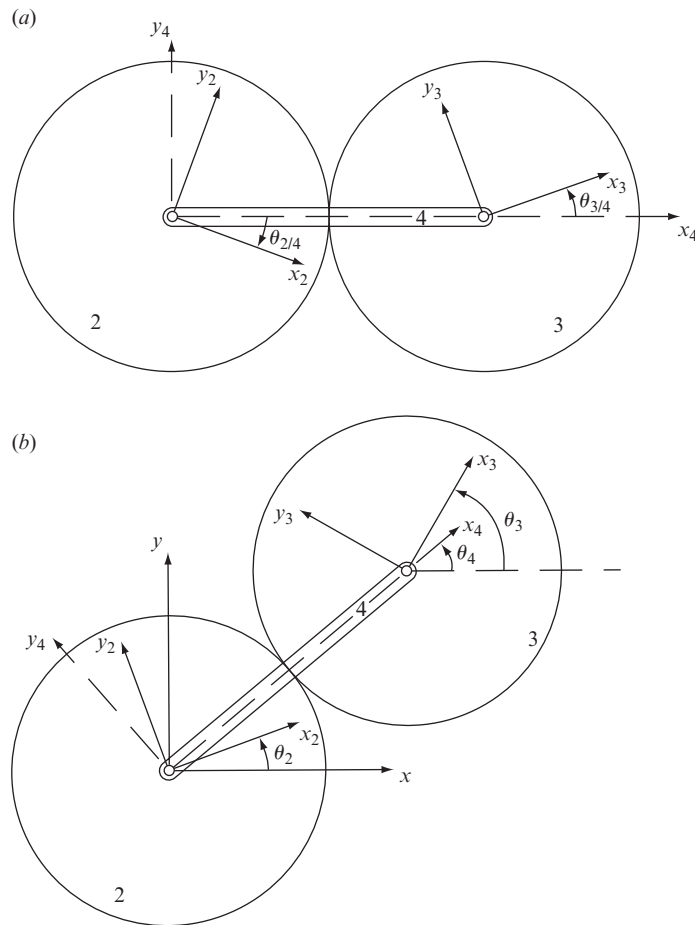


Figure 2.30 (a) Apparent rotations observed from the arm; (b) absolute rotations.

$$\rho_2(\Delta\theta_2 - \Delta\theta_4) = -\rho_3(\Delta\theta_3 - \Delta\theta_4). \quad (b)$$

Using these ideas, the loop-closure equations for mechanisms having rolling contact without slipping can be solved. The following two examples will make the procedure clear.

EXAMPLE 2.7

For the cam-and-follower mechanism in the posture shown in Fig. 2.31, define a set of vectors that is suitable for a complete kinematic analysis. Label and show the sense and orientation of each vector on the mechanism. Write the vector loop equation(s) and identify suitable input(s) for the mechanism. Identify the known quantities, the unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

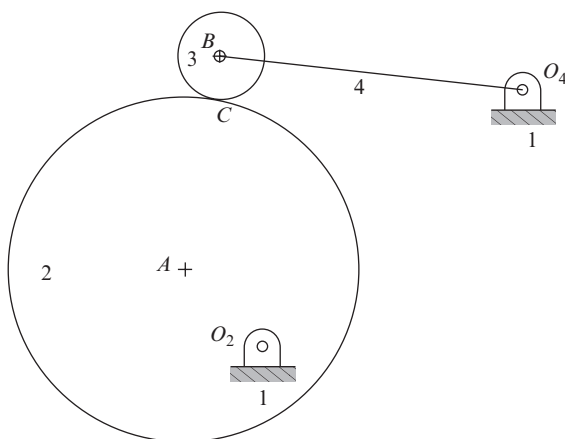


Figure 2.31
Cam-and-follower mechanism.

SOLUTION

Since the circular cam, link 2, and the roller follower, link 3, are in rolling contact, let us consider them linked together by a (fictitious) arm, link 23. This leads us to consider the set of vectors shown in Fig. 2.32.

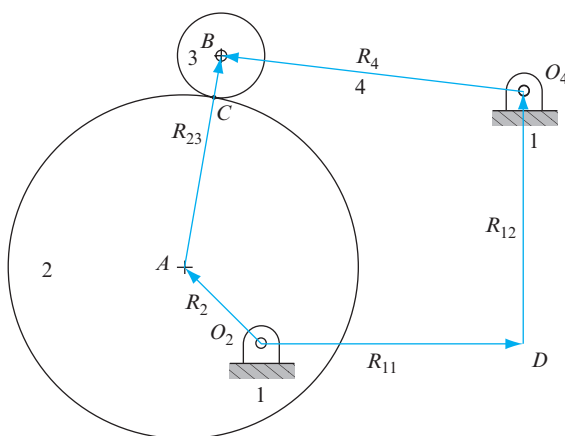


Figure 2.32 Vectors chosen for analysis of the mechanism.

The vector loop equation for the mechanism can be written as

$$\sqrt{I} \mathbf{R}_2 + \sqrt{?} \mathbf{R}_{23} - \sqrt{?} \mathbf{R}_4 - \sqrt{?} \mathbf{R}_{12} - \sqrt{?} \mathbf{R}_{11} = \mathbf{0}. \quad \text{Ans. (1)}$$

This is a valid loop-closure equation and, given the angle θ_2 as input, it can be solved for the postures of link 4 and of the fictitious link 23. However, this does not yet allow a solution for the rotation of the roller, link 3.

Since there is rolling contact between cam 2 and roller 3 at point C, let us consider the apparent angular displacements of links 2 and 3 as they would appear to an observer riding on the fictitious arm, link 23. Such an observer would observe

$$\rho_3 \Delta\theta_{3/23} = -\rho_2 \Delta\theta_{2/23}, \quad (2)$$

which gives the following relationship between the absolute angular displacements:

$$\rho_3(\Delta\theta_3 - \Delta\theta_{23}) = -\rho_2(\Delta\theta_2 - \Delta\theta_{23}). \quad (3)$$

Rearranging this equation gives

$$\rho_3 \Delta\theta_3 = (\rho_2 + \rho_3) \Delta\theta_{23} - \rho_2 \Delta\theta_2. \quad \text{Ans. (4)}$$

Since the angular displacement of link 2 is known (the input) and $\Delta\theta_{23}$ is determined from Eq. (1), the angular displacement, $\Delta\theta_3$, can be determined from Eq. (4). It is worth noting that we can obtain a solution for the posture of link 3 even though no vector is attached to or rotates with link 3.

EXAMPLE 2.8

For the rack-and-pinion mechanism shown in Fig. 2.33, define a set of vectors that is suitable for a complete kinematic analysis. Label and show the sense and orientation of each vector on the mechanism. Write the vector loop equation(s) for the mechanism and identify suitable input(s) for the mechanism. Identify the known quantities, the unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

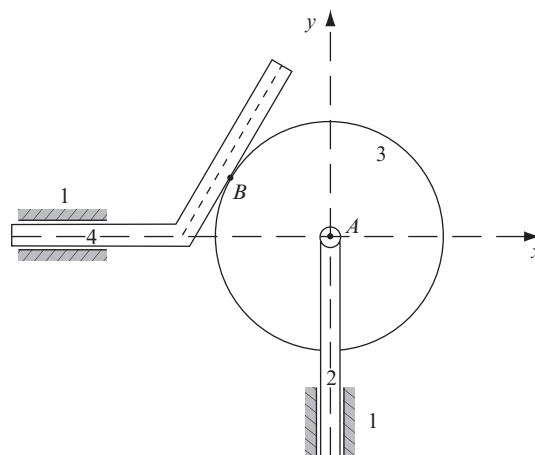


Figure 2.33 Rack-and-pinion mechanism.

SOLUTION

Let us consider the set of vectors shown in Fig. 2.34.

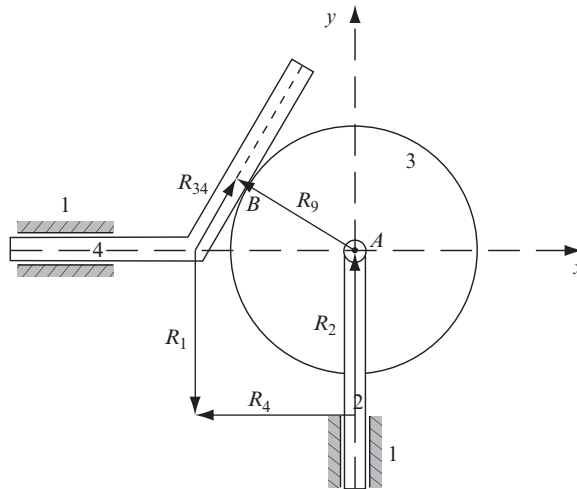


Figure 2.34 Vectors chosen for analysis of the mechanism.

The length of vector $\mathbf{R}_2$ represents the input, and the length of vector $\mathbf{R}_4$ indicates the output. The loop-closure equation is

$$\overset{I\sqrt{}}{\mathbf{R}_2} + \overset{\sqrt{C}}{\mathbf{R}_9} - \overset{?\sqrt{}}{\mathbf{R}_{34}} + \overset{\sqrt{\sqrt{}}}{\mathbf{R}_1} - \overset{?\sqrt{}}{\mathbf{R}_4} = \mathbf{0}. \quad \text{Ans.}$$

The constraint that gives the orientation of vector $\mathbf{R}_9$ is

$$\theta_9 = \theta_{34} + 90^\circ. \quad \text{Ans.}$$

The angular displacement of pinion 3 can be determined by considering the apparent rotation of the pinion to an observer riding on the rack, link 4. As such an observer watches the changing position of the point of contact B , SHE must see the same increment of surface on both links 3 and 4. The constraint relationship can be written as

$$\Delta R_{34} = \rho_3 \Delta \theta_{3/4} = \rho_3 (\Delta \theta_3 - \overset{0}{\Delta \theta_4}) = \rho_3 \Delta \theta_3.$$

Therefore, the angular displacement of the pinion is

$$\Delta \theta_3 = \Delta R_{34} / \rho_3. \quad \text{Ans.}$$

2.20 REFERENCES

- [1] Mischke, C. R., 1980. *Mathematical Model Building*, Ames, IA: Iowa State University Press, 86.

PROBLEMS*

- †2.1** Describe and sketch the locus of a point A , which moves according to the equations $R_A^x = at\cos(2\pi t)$, $R_A^y = at\sin(2\pi t)$, and $R_A^z = 0$.
- †2.2** Find the position difference from point P to point Q on the curve $y = x^2 + x - 16$, where $R_P^x = 2$ and $R_Q^x = 4$.
- †2.3** The path of a moving point is defined by the equation $y = 2x^2 - 28$. Find the position difference from point P to point Q if $R_P^x = 4$ and $R_Q^x = -3$.
- †2.4** The path of a moving point P is defined by the equation $y = 60 - x^3/3$. What is the displacement of the point if its motion begins at $R_P^x = 0$ and ends at $R_P^x = 3$?
- †2.5** If point A moves on the locus of Problem 2.1, find its displacement from $t = 2$ to $t = 2.5$.
- †2.6** The position of a point is given by the equation $\mathbf{R} = 100e^{j2\pi t}$. What is the path of the point? Determine the displacement of the point from $t = 0.10$ to $t = 0.40$.
- †2.7** The equation $\mathbf{R} = (t^2 + 4)e^{-j\pi t/10}$ defines the position of a point. In which direction is the position vector rotating? Where is the point located when $t = 0$? What is the next value t can have if the orientation of the position vector is to be the same as it is when $t = 0$? What is the displacement from the first position of the point to the second?
- †2.8** The location of a point is defined by the equation $\mathbf{R} = (4t + 2)e^{j\pi t^2/30}$. Motion of the point is initiated when $t = 0$. What is the displacement until $t = 3$? Find the change in angular orientation of the position vector during the same time interval.
- †2.9** Link 2 rotates according to the equation $\theta = \pi t/4$. Block 3 slides outward on link 2 according to the equation $r = t^2 + 2$. What is the absolute displacement $\Delta \mathbf{R}_{P_3}$ from $t = 1$ to $t = 2$? What is the apparent displacement $\Delta \mathbf{R}_{P_{3/2}}$?

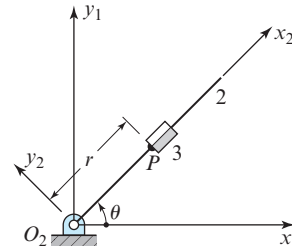


Figure P2.9

- †2.10** A wheel with center at O rolls without slipping on the ground at point P . If point O is displaced 10 in to the right, determine the displacement of point P during this interval.

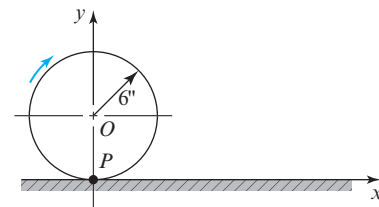
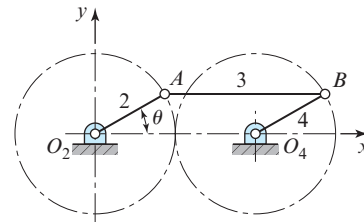


Figure P2.10 Rolling wheel.

- †2.11** A point Q moves from A to B along link 3 while link 2 rotates from $\theta_2 = 30^\circ$ to $\theta_2' = 120^\circ$. Find the absolute displacement of Q .

Figure P2.11 $R_{AO_2} = R_{BO_4} = 3$ in and $R_{BA} = R_{O_4O_2} = 6$ in.

- †2.12** The double-slider linkage is driven by moving sliding block 2. Write the loop-closure equation. Solve analytically for the position of sliding block 4. Check the result graphically for the position where $\phi = -45^\circ$.

* When assigning problems, the instructor may wish to specify the method of solution to be used, since a variety of approaches are presented in the text.

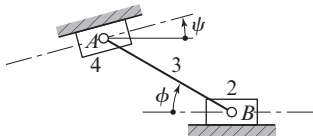


Figure P2.12 $R_{AB} = 200$ mm and $\psi = 15^\circ$.

- †2.13 The offset slider-crank linkage is driven by crank 2. Write the loop-closure equation. Solve for the position of slider 4 as a function of θ_2 .

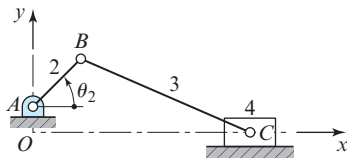


Figure P2.13 $R_{AO} = 1$ in, $R_{BA} = 2.5$ in, and $R_{CB} = 7$ in.

- 2.14 Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

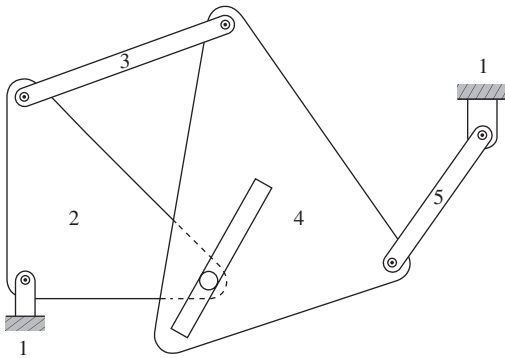


Figure P2.14

- 2.15 Define a set of vectors that is suitable for a complete kinematic analysis of the rack-and-pinion mechanism. Label and show the sense and orientation of each vector. Assuming rolling with no slip between rack 4 and pinion 5, write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown

variables, and any constraints. If you identify constraints, then write the constraint equation(s).

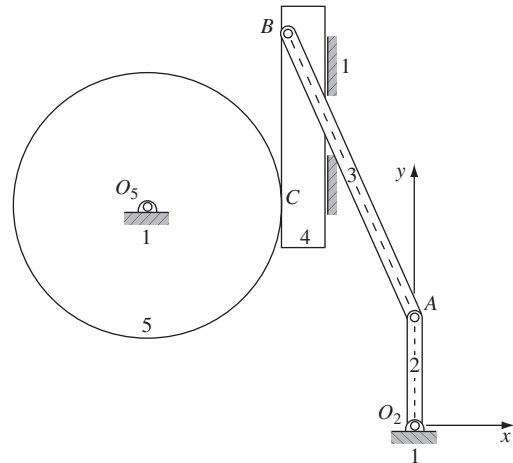


Figure P2.15 Rack-and-pinion mechanism.

- 2.16 Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Assuming rolling with no slipping between gears 2 and 5, write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

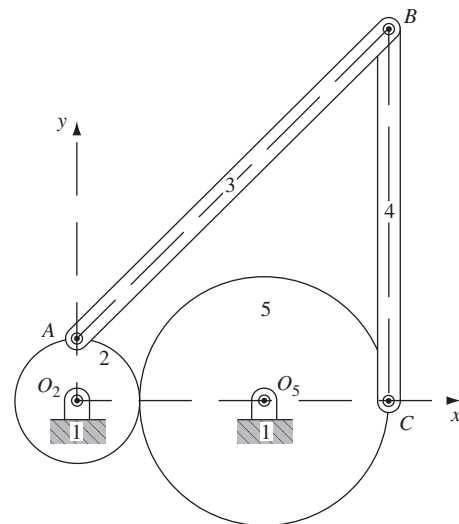


Figure P2.16 Geared five-bar mechanism.

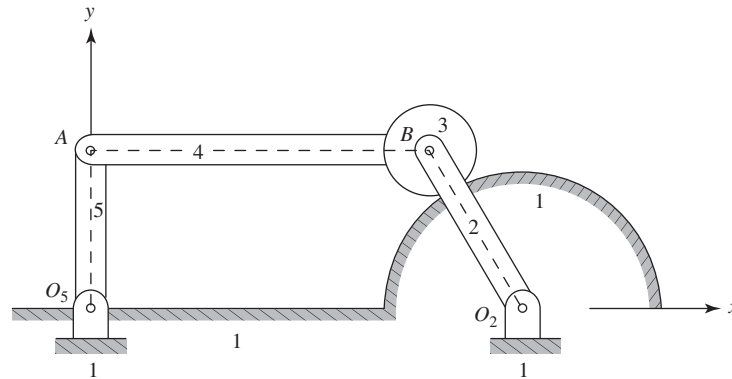


Figure P2.17

- 2.17** Gear 3, which is pinned to link 4 at point B , is rolling without slipping on semicircular ground link 1. The radius of gear 3 is ρ_3 , and the radius of the ground link is ρ_1 . Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).
- 2.18** For the mechanism in Fig. P1.6, define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).
- 2.19** For the mechanism in Fig. P1.8, define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).
- 2.20** For the mechanism in Fig. P1.9, define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).
- 2.21** For the mechanism in Fig. P1.10, define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).
- †2.22** Write a calculator program to find the sum of any number of two-dimensional vectors expressed in mixed rectangular or polar forms. The result should be obtainable in either form with the magnitude and angle of the polar form having only positive values.
- †2.23** Write a computer program to plot the coupler curve of any crank-rocker or double-crank form of the four-bar linkage. The program should accept four link lengths and either rectangular or polar coordinates of the coupler point with respect to the coupler.

- †2.24 Plot the path of point P for: (a) inverted slider-crank linkage; (b) second inversion of the slider-crank linkage; (c) Scott-Russell straight-line linkage; and (d) drag-link linkage.

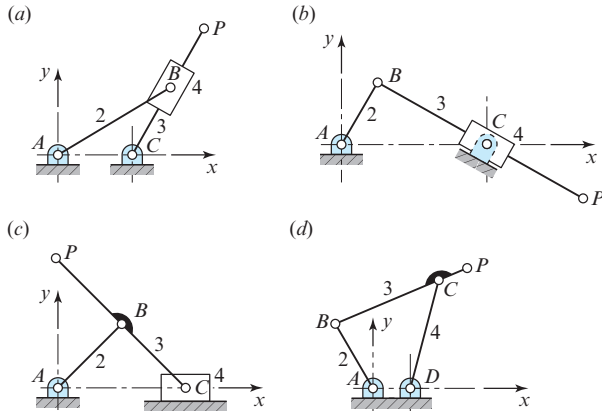


Figure P2.24 (a) $R_{CA} = 2$ in, $R_{BA} = 3.5$ in, and $R_{PC} = 4$ in; (b) $R_{CA} = 40$ mm, $R_{BA} = 20$ mm, and $R_{PB} = 65$ mm; (c) $R_{BA} = R_{CB} = R_{PB} = 25$ mm; (d) $R_{DA} = 1$ in, $R_{BA} = 2$ in, $R_{CB} = R_{CD} = 3$ in, and $R_{PB} = 4$ in.

- †2.25 Using the offset slider-crank linkage in Fig. P2.13, find the crank angles corresponding to the extreme values of the transmission angle.

- 2.26 Section 1.10 states that the transmission angle reaches an extreme value for the four-bar linkage when the crank lies on the line between the fixed pivots. Referring to Fig. 2.19, this means that γ reaches a maximum or minimum when crank 2 is collinear with line O_2O_4 . Show analytically that this statement is true.

- 2.27 Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for

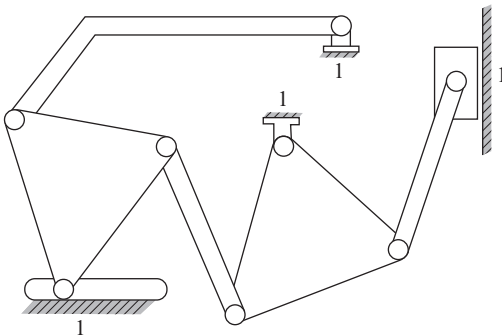


Figure P2.27

the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

- 2.28 Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraints, then write the constraint equation(s).

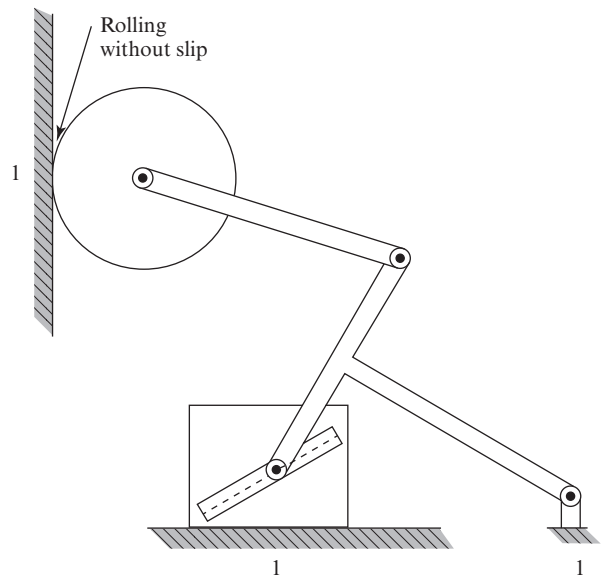


Figure P2.28

- 2.29 Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraint(s), then write the constraint equation(s).

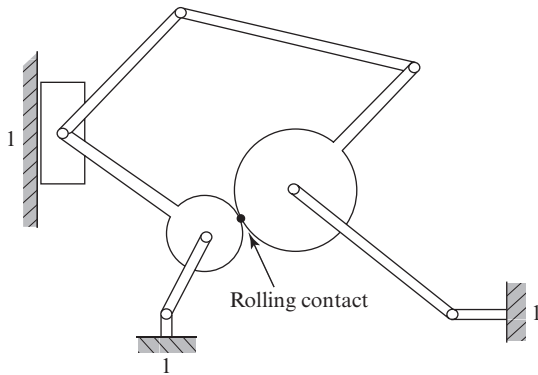


Figure P2.29

- 2.30** Define a set of vectors that is suitable for a complete kinematic analysis of the mechanism. Label and show the sense and orientation of each vector. Write the vector loop equation(s) for the mechanism. Identify suitable input(s), known quantities, unknown variables, and any constraints. If you identify constraint(s), then write the constraint equation(s).

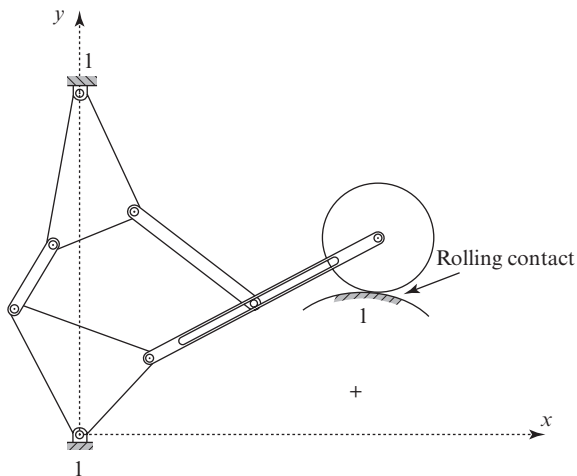


Figure P2.30

- 2.31** For the input angle $\theta_2 = 300^\circ$, measured counterclockwise from the x axis, determine the two postures of link 4.

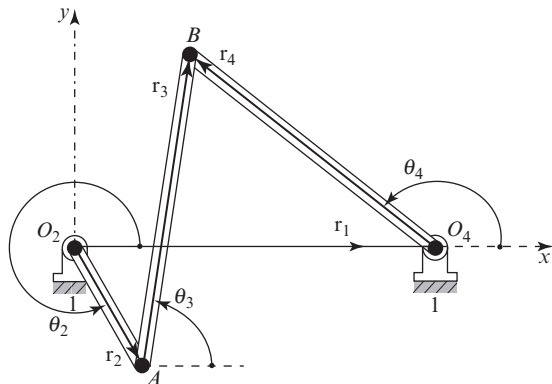


Figure P2.31 $r_2 = 60$ mm, $r_3 = 140$ mm, $r_4 = 140$ mm, and $r_1 = 160$ mm.

- 2.32** For the input angle $\theta_2 = 60^\circ$, measured counterclockwise from the x axis, determine the two postures of link 4.

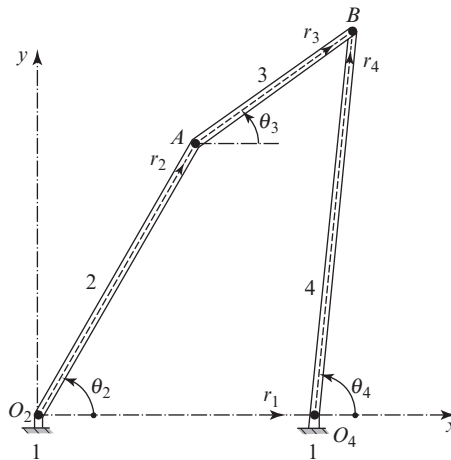


Figure P2.32 $r_2 = 80$ mm, $r_3 = 50$ mm, $r_4 = 100$ mm, and $r_1 = 70$ mm.

2.33 Consider a four-bar linkage for which ground link 1 is 14 in, input link 2 is 7 in, coupler link 3 is 10 in, and output link 4 is 8 in. The fixed x and y axes are specified as horizontal and vertical, respectively. The origin of this reference frame is coincident with the ground pivot of link 2, and the ground link is aligned with the x axis. For input angle $\theta_2 = 60^\circ$ (counterclockwise from the x axis): (a) Using a suitable scale, draw the linkage in the open and crossed postures and measure the values of variables θ_3 and θ_4 for each posture. (b) Use trigonometry (that is, the laws of sines and cosines) to determine θ_3 and θ_4 for the open posture. (c) Use Freudenstein's equation to determine θ_3 and θ_4 for both postures. (d) Use the Newton-Raphson iteration procedure to determine θ_3 and θ_4 for the open posture. Using the measurements in (a) as initial estimates for θ_3 and θ_4 , iterate until the two variables converge to within 0.01° .

2.34 A crank-rocker four-bar linkage is shown in two different postures for which $\theta_2 = 150^\circ$ and $\theta'_2 = 240^\circ$. Determine θ_3 and θ_4 for the open posture and θ'_3 and θ'_4 for the crossed posture.

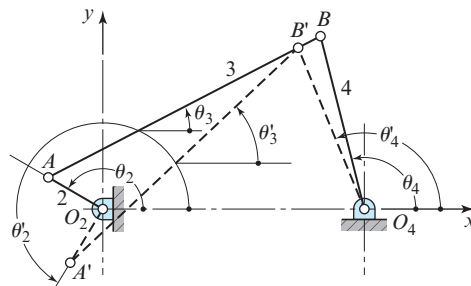


Figure P2.34 $r_{O_4O_2} = 600$ mm, $r_{AO_2} = 140$ mm, $r_{BA} = 690$ mm, and $r_{BO_4} = 400$ mm.