
Linkage Coupler Curve Synthesis: A Historical Review—II. Developments after 1875*

H. Nolle

Received 7 July 1973

Abstract

This work has been presented in three parts. Commencing as a historical survey, the first part traces the early events relating to mechanical inventions and their applications that were instrumental in, but preceded, the eventual development of the science of linkage mechanism as originally expounded by Reuleaux. Analytical studies of the properties of coupler curves are reviewed in part two, commencing with the year 1875, when Samuel Roberts published the first accounts of the algebraic properties of the planar four-bar curve. This is followed, in part three, by a classification and critical examination of the techniques of coupler curve synthesis. However, techniques centered exclusively on graphical constructions have been placed outside the scope of this review, partly because of their limited usefulness in three-dimensional kinematics.

3. Developments after 1875

THE STUDIES of coupler curve properties and of curve synthesis proliferated to an unprecedented degree soon after 1875. It is now virtually impossible to define the limits of accumulated knowledge; it is equally difficult to identify an optimum approach for presentation of the material. The aim of this work is to identify the major areas of study, and, within these areas, to critically examine the extent of understanding. Some attention is given to purely practical aspects (applications) and to the rather elusive quantity called the “state of the art”. Both form essential parts of the historical viewpoint.

First, an outline is given of the more commonly encountered requirements recognized in the kinematic synthesis of a planar coupler curve, supplemented with selected examples of application. The second part is devoted to a discussion of the general properties of coupler curves, analytical techniques and the application of analytical and other methods in the synthesis of planar mechanism to meet particular requirements. (Spatial mechanism will be covered in Part III of this work.)

3.1 Requirements and applications

While work in the nineteenth century on synthesis of mechanisms for the generation of prescribed curves was largely of an academic character, and the resulting linkages were used merely for the drawing of curves, these linkages acquired engineering significance in the twentieth century. In the past, the kinematician's interest centered itself on

*Part I published in *Mechanism and Machine Theory* vol. 9, no 2, pp. 147–168.

the problem of generating a specific curve. This curve corresponds to the path traced by a point on the linkage coupler plane, but the motion of the coupler—and for that matter, the motion of the other links—as a whole is taken to be of secondary importance.* There are good reasons for this, as will be seen later, but the machine designer now takes a different viewpoint; he will be quick to add that synthesis of a linkage having an exceptionally close-fitting coupler curve may be of little use if, at the same time, it leads to ill-proportioned link length ratios, poor transmission characteristics or the wrong kind of linkage (e.g. double-rocker in place of a crank rocker), to name a few of the constraints frequently imposed on a solution for it to be acceptable. We find, then, numerous instances where, in the size synthesis problem, some link lengths and/or ranges of motion are specified beforehand (see also Section 3.2.4 on Multiple Generation of Coupler Curves). To meet requirements of this kind, Alt [2] exploited to the full the nine independent quantities needed to specify the four-bar coupler curve: his methods, based partly on Burmester theory, can be used to construct a five precision point curve generated by a mechanism whose input link centre of rotation and length are specified as well as its range of motion corresponding to the two extreme precision points. Other choices for specified variables are treated as variations of the method in the same work. Fortunately, it is not always essential to generate a coupler curve (or part thereof) exactly at every point in its range of interest. Such “exact” point path generation in practice is limited to relatively few special cases such as the straight-line, second order curves (particularly the conic sections) and possibly some higher order curves displaying certain simplifying features. Viewed historically, it seems that recognition of the importance or usefulness given to methods or techniques of solution (or “invention” of a particular type of linkage) has shifted from those of the “exact” type to those of the “approximate” type. The reason for this trend may be threefold, the first being the just mentioned limitation of range of curves that can reasonably be handled. Secondly, “exact” motion in practice is never exact anyway, deviating from the ideal due to clearances in link joints, deflection of links, tolerances in manufacture and so on. The question is to be asked whether link motion can be justified in applications where other developments, mainly in production techniques, provide economic alternatives. We know from experience that “exact” linkages having a large link number and inherently awkward joint locations and link configuration have not met the requirements of practical application [16]. (For example, the only four-link exact straight-line mechanism is the symmetrical crank slider, but even this has the undesirable feature that the coupler curve passes through the crank centre.) Lastly, kinematic synthesis now has reached a degree of versatility where “approximate” solutions can often be made to reach accuracies which, for all practical purposes over a substantial range, may be deemed as being equivalent to the “exact”. Nevertheless, research on the “exact” type of curve generator has continued, a comprehensive range of such mechanisms being treated exhaustively by Artobolevskii [5].

One classification (adopted in this work) of requirements of a coupler curve may be based on the order of the curve to be generated (*not* the order of coupler curve). This is our primary requirement, there being other, practical, conditions to be met in any particular case. The latter are grouped together as a set of secondary—but by no means insignificant—requirements and will be dealt with subsequently.

3.1.1 Straight-line motion. In this category we distinguish the application in which the coupler point normally moves only through *part* (the straight-line portion) of its full

*There are exceptions to this, of course. For example see the lesser known constructions proposed by Kraus [86], utilizing coupler plane positions.

circuit from that in which the point undergoes full cyclic motion in the operation of the mechanism. The former is exemplified by the limited (rocking) motion of a crank-rocker arrangement (Fig. 5), embodied in the design of a posthole borer. The motion of point G is vertical and the borer is lifted on completion of the hole by means of the hydraulic ram *d*. The oldest of the approximate straight-line four-bars, the Watt mechanism, (Fig. 4(e)), has been incorporated in a recent machine tool design. The schematic illustration of the arrangement, (Fig. 6), shows the motion, in exaggerated proportion, of the cutting tool. Although the tool cutting edge is given a slight vertical component of displacement, it is well within the limits determined by product dimensional tolerance. The patented and registered design of the linkage tool holder eliminates troublesome wear and contamination of the exposed slide surfaces of conventional tool carriages when machining raw powdering ceramics. Other advantages claimed are that removal of bulky chip coil accumulations at high cutting rates, e.g. when machining plastics, is facilitated by the all round access to the work area with the tool in the retracted position. An application of the crank-slider is seen in the tray tipping mechanism of a truck trailer (Fig. 7). The excellent straight-line characteristics of this linkage have been utilized fully in achieving a horizontal movement of the centre of gravity of the load over the range of motion. The slight upward deviation of point G, noticeable in the figure is used to advantage and ensures stability of the system. The tipping motion of the tray, therefore, requires no significant work to be done on the load. The well-known Demag crane (double rocker four-bar) is a further example of a design having horizontal load transfer capability without energy loss. In this crane the cable sheave coincides with a coupler point (on the crane truss) that is very nearly straight and horizontal. A different straight-line motion linkage used in mechanical handling is illustrated in Fig. 8. The link mechanism shown replaces the fixed mast of a mobile fork-lift truck improving man-

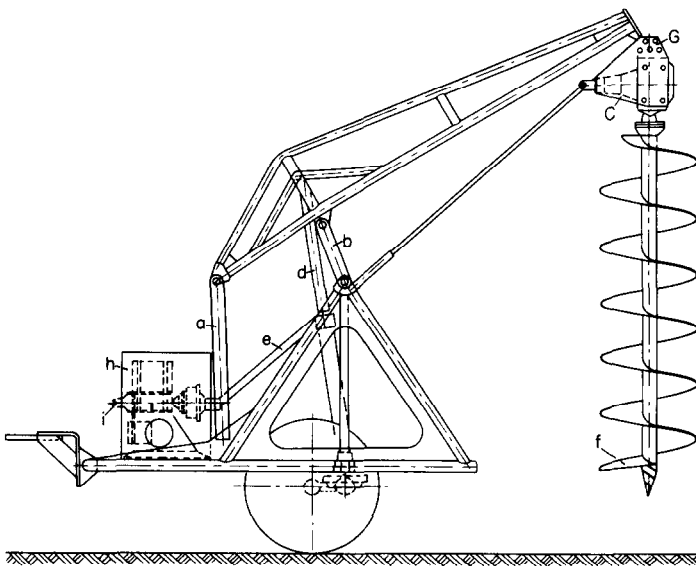


Figure 5. Schematic view of a posthole digger (Type EB-4, Wilhelm Wittenburg, Hamburg). (a), (b) links of four-bar; (c) power drive, (d) power cylinder for borer extraction, (f) drill, (h) drive transmission, (G) coupler point; (after [173]).

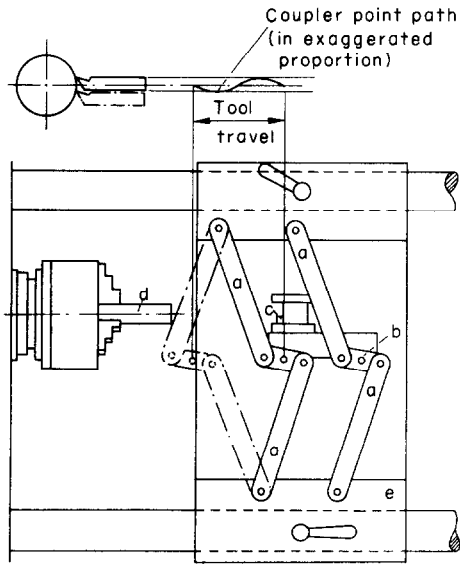


Figure 6. Elevation view of a linkage tool guide on a lathe. (a) link; (b) coupler; (c) cutting tool; (d) work piece; (e) machine frame (adjustable). This principle of tool transport may be extended to drilling operations; (after[177]).

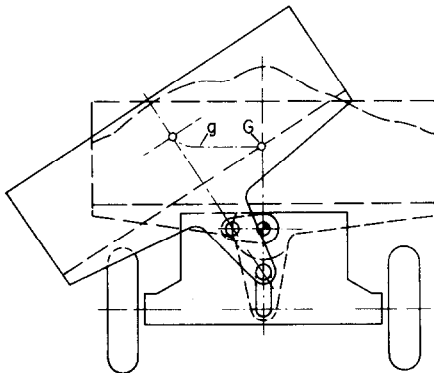


Figure 7. Truck trailer with tray tipping mechanism (crank-slider). The coupler point is chosen as the centre of gravity, G, of the tray. The path of this point is shown as g; (after[173]).

euverability, stability and visibility. The four-bar linkage is activated hydraulically at one of the links. Independent fork tilt (specified in the design) is controlled by a parallelogram linkage and secondary hydraulic ram arrangement.*

Of interest, also for its historical background, is the example of a design and development of a heavy duty oil pump in which straight-line motion, derived from a continuous rotational input, is achieved through a linkage mechanism[123]. One of the

*At the time of completion of the paper this model was no longer in production. It lost favour to lower efficiency, but also lower cost models.

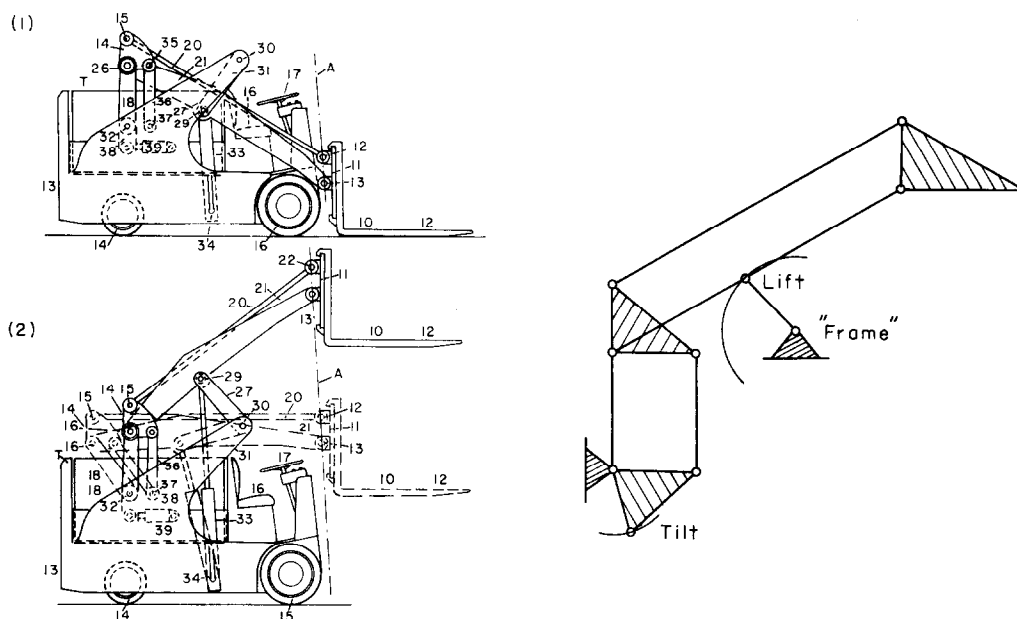


Figure 8(a).

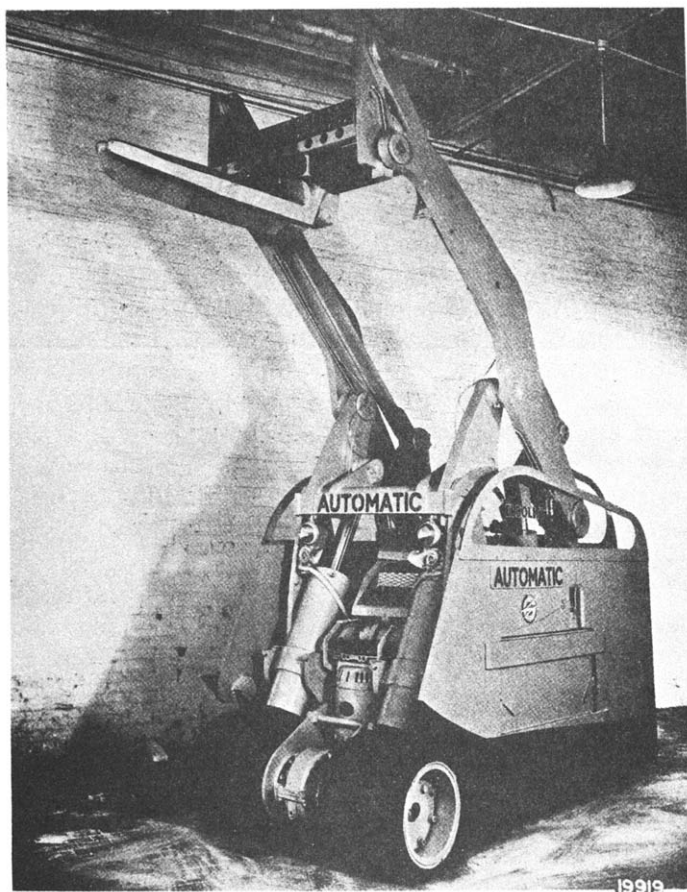


Figure 8(b).

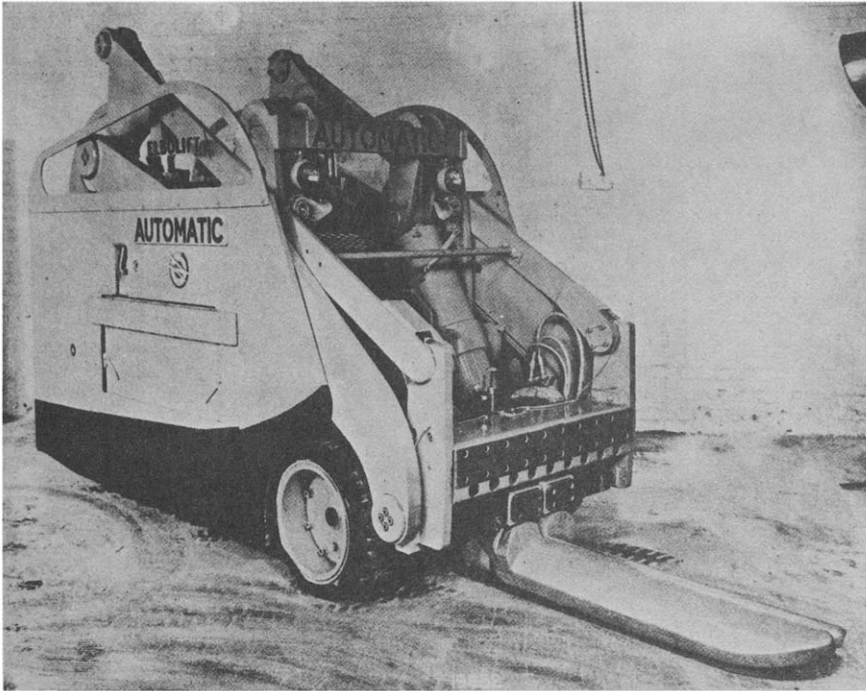
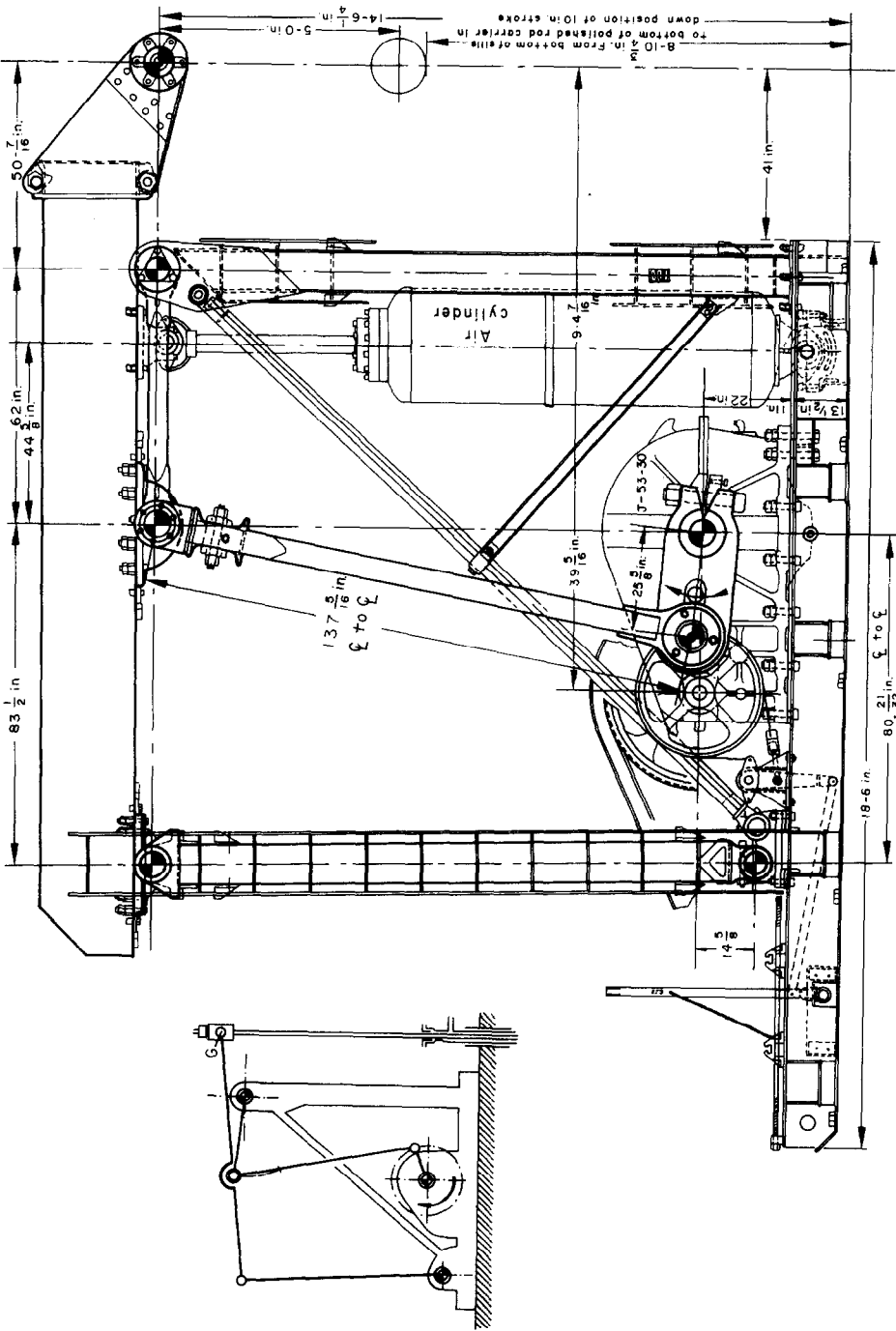


Figure 8(c).

Figure 8. Load lifting and tilting linkages of a power operated lifting truck. (a) Elevations of an industrial truck showing the forks in the “down” and “up” positions. Line A indicates slight backward tilt of path of pivots 22 and 23. The two sets of parallelogram linkages ensure rotation free lift of the fork (lift activated by main hydraulic ram 33). Independent fork tilt control is effected by secondary ram 39, acting on ternary link 38. (After Commonwealth of Australia Patent Specification 227069. Inventor: B. I. Ulinski.) The proportions of the main linkage indicate it to be an approximate straight-line mechanism of the Evans type (see Fig. 4(h)). (b) A practical design of a 25000 lb capacity heavy duty lifting truck (with transverse fork spread adjustment) used in steel mills. Fork in “up” position. (c) Same truck with fork in “down” position. Note compactness of folded linkages, visibility and improved weight distribution as compared with ordinary mast construction. The practical design was arrived at through “geometric study of sketches, and repeated modification of arrangement until the desired lift and tilt characteristics were obtained”. (Courtesy of Eaton Yale & Towne Inc., Chicago.)

first designs for this pump, manufactured by the then Parkersburg Rig & Reel Co., U.S.A., is shown in Fig. 9(a). The well polish rod is connected to a shaft which swiveled in the walking beam. Due to the circular motion of the beam connection, the polish rod would move laterally enough to wear out the stuffing box. To correct this movement the design was changed to that shown in Fig. 9(b), in which counterbalance is achieved with compressed air. Examination of the new arrangement reveals that the



120 Pumping unit — Assembly no. 11761-2 x

Figure 9(b).

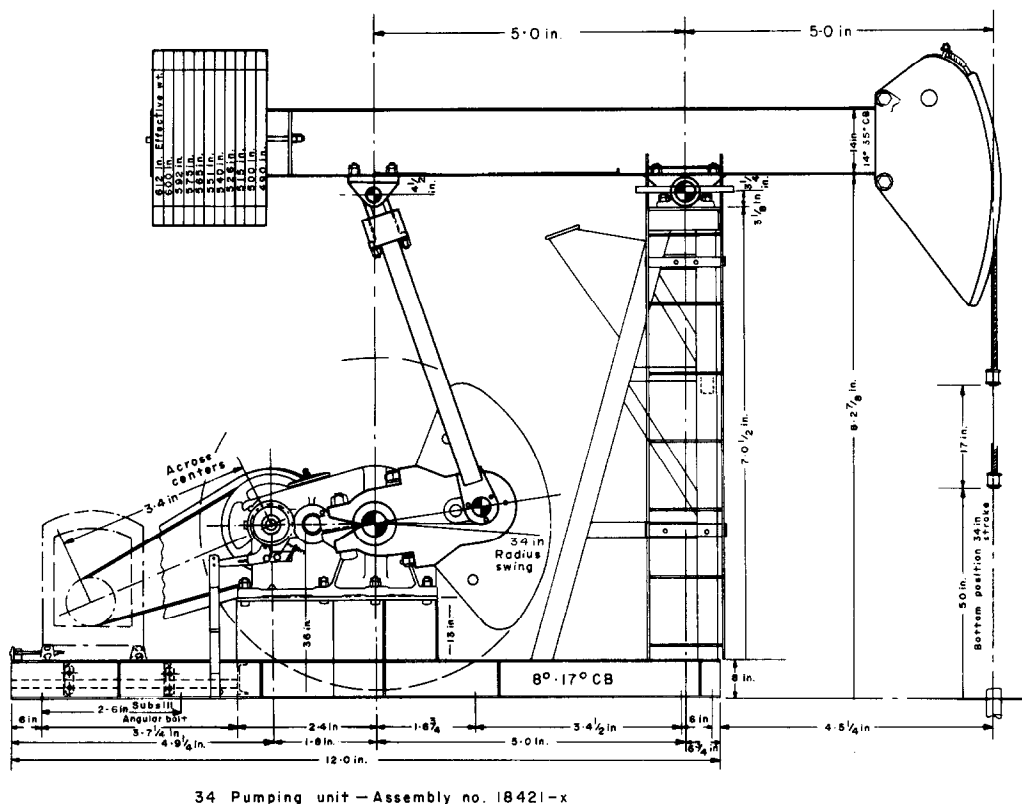


Figure 9(c).

Figure 9. Development of a heavy-duty oil pump design (Parmac, Inc.). (a) Original design of 368IU pumping unit (note lack of correction for curvilinear motion of beam connection to well polish rod: design offset is seen to be 1 in.). (b) Modified design (120BK pumping unit) incorporating a Scott–Russell type mechanism (see insert) to produce approximate straight-line motion for well rod connection, G. (c) Final design (34BY pumping unit), reverting to a “beam and quadrant” type arrangement with flexible cable connection to well rod.

of the intermittent motion.* Figure 10 shows, schematically, the construction of the table-actuating mechanism of a heavy automatic welding machine, used in the manufacture of car bodies. The table, in its lowest position, accepts the new workpiece which is then moved into its extreme upper position where the weld cycle takes place. The table motion is intermittent, the dwell period in both upper and lower positions being determined by the characteristics of the Geneva linkage. Although considered originally by the designers, the demand, in this application, of precise rest periods eventually precluded the use of a coupler-dwell linkage having two dwell periods. (It is noteworthy that the slider also provides a means of positively locking the Geneva wheel while

*The commercially favored Geneva drive has recently been modified by the replacement of the usually straight slots by circular arcs [27]. The originators of this mechanism claim improved acceleration characteristics for it. Analysis is said to be simplified by establishing kinematic equivalence between the drive and a substitute four-bar linkage. The presentation of this aspect is by no means clear in the above reference.

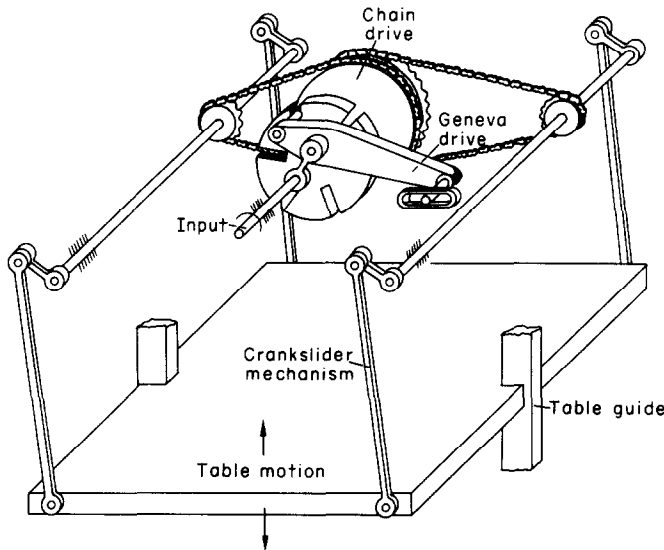


Figure 10. Table-actuating mechanism of a heavy automatic welding machine. Transmission of intermittent motion of table occurs via Geneva linkage, chain drive and crank-slider arrangement (after [11]).

disengaged.) A further example is furnished by the single dwell six-bar mechanism generally of the kind shown in Fig. 11(a). The approximate straight-line portion of the curve determines the period of dwell whereas the location of the output pivot and coupler curve “thickness” control the range of output. Applications requiring less precision in the dwell can readily be synthesized with the help of coupler curve charts [68]; in more demanding situations analytical methods suggested by Beyer [12] may be adopted.

3.1.2 Circular motion. The study of path curvature has led to numerous applications of coupler curves having approximately constant curvature over part or parts of their circuits, for example in intermittent motion mechanisms in automatic packaging and textile machinery [92] and in machine tool design [167A].

A six-bar mechanism, with the fifth link connected at one end to the coupler and the other end at the centre of the constant curvature portion of the coupler curve constitutes a dwell mechanism. The sixth link is a rocker whose range of oscillation is determined, partly, by the shape of the coupler curve. Following on from this, mechanism output is further refined—in both degree and in ability to match special complexities of curvature—by arranging in series two or more link loops. An example in excellence of such development is reproduced in Fig. 12. One six-bar linkage can be seen to drive an intermediate four-bar chain which in turn activates the needle and awl, each driven through its own four-bar linkage. The approximately circular portion of the coupler curve shown produces the phased dwells of needle and awl. This illustration represents the results, mainly, of the slow process of evolution of a design, but analytical techniques in this regard have been known for many years. Thus the necessary geometric conditions for linkages with determinate coupler points having paths that coincide with their circle of curvature in six points have been enumerated by Meyer zur Capellen [101]. The resulting mechanisms display high quality dwells. An alternative approach of achieving dwell is shown in Fig. 11(b). It shows the centric crank-slider generating a near-circular portion for the coupler curve. A practical solution to the

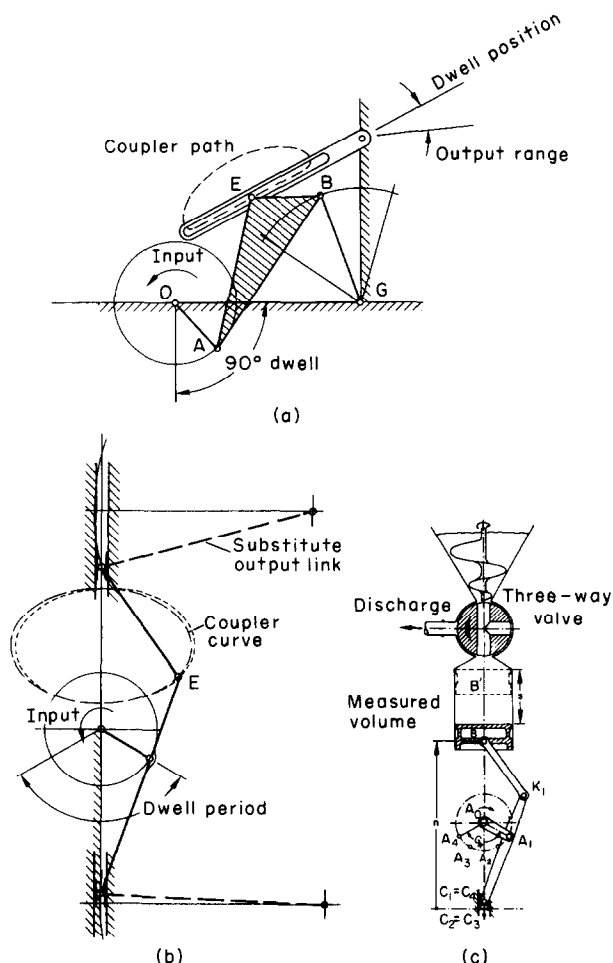


Figure 11. (a) Typical layout of a single-dwell six-bar mechanism having dwell at one extreme of the oscillatory output motion. Dwell is based on the approximately straight-line portion of the coupler curve. (b) Dwell mechanism of the centric crank-slider type. Dwell is based on the near-circular portion of the coupler curve. The slider elements may be substituted by revolute pairs and oscillating output link (substitute links and resulting coupler curve shown in dotted outline). (c) Application of the dwell linkage of Fig. 11(b) to the design of a controlled volume dispensing machine. (After Ströer H. J., *Maschinelles Verpacken*. Beuth-Verlag, Berlin (1936).)

problem of mechanizing a prescribed work cycle is offered in Fig. 11(c) in the design of a controlled volume dispensing machine. In this the measured charge is admitted, from the hopper, through the cylindrical three-way valve and into the measuring cylinder while the latter is dwelling in the bottom dead centre position. The piston subsequently discharges the product through the horizontal outlet. This illustrated example has a dwell of 120° of crank rotation.

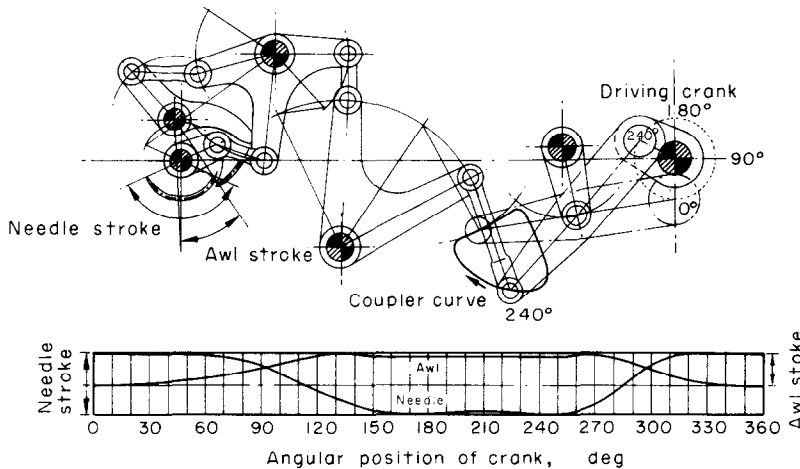


Figure 12. Crank-driven needle and awl mechanism of welt sewing machine; (after[80]).

Replacement of the slider pairs by revolute results in yet another dwell mechanism, shown in dashed line in Fig. 11(b).

3.1.3 Motion along paths describing the conic sections. Attempts to mechanize the conic sections are probably as old as those at generating the straight line. The main characteristics of the mechanisms, or conicographs as they are sometimes called, are that they consist of four or more links connected through either revolute or prism pairs.* The application of these mechanisms is limited mainly to cases requiring exact reproduction of the particular second degree profile as for example in computing devices or as profiling units in automatic machinery. The class of mechanisms generating the curve by the envelope method becomes of interest in the manufacture of profiles or surfaces by means of a worm type milling cutter or broach.

3.1.4 Curves of particular shape. The ability of the coupler curve to assume a seemingly endless variety of shapes has been put to use by designers of light and heavy machinery alike. Examples of power and motion transmission by linkages can be found in feeder mechanisms, transport mechanisms, automatic machinery, farm machinery, machine tools, locomotion devices and many other fields. Applications of curvilinear motion are seen in the straw feeding and elevating device (Fig. 13), intermittent transport mechanism (Fig. 14), a computer—designed car window linkage (Fig. 15), and walking-link drive (Fig. 16).‡ A further example is furnished in the design of a milling machine of the early 1930's. This particular model permitted the cutting tool to be guided along a coupler point path of a crank-slider mechanism (Fig. 17), thus causing a cut in the workpiece having a profile of a shape corresponding to that of the coupler curve. It was claimed[93] then that substantial savings in production time were

*See[5], ch.4; [148, p. 333]. Also Crawford W. R., The mechanical construction of the general conic section, *The Engineer*, p. 210, Aug. 28 (1936), and Dobrovolskii V. V., *Theorie der Mechanismen zur Konstruktion ebener Getriebe*. Akademie, Berlin (1957). In the latter, constructions are given also for curves of order greater than two; it makes no reference to methods of approximate synthesis.

‡The original ideas of this type harvesting device and the walking-link drive both seem to be due to Chebyshev, admittedly taking the form then of a grain thrower in a threshing machine and a "marching barrow", respectively. The success of the former was due, partly, to the variation in velocity of the coupler point which would execute slow motion return during its working cycle.

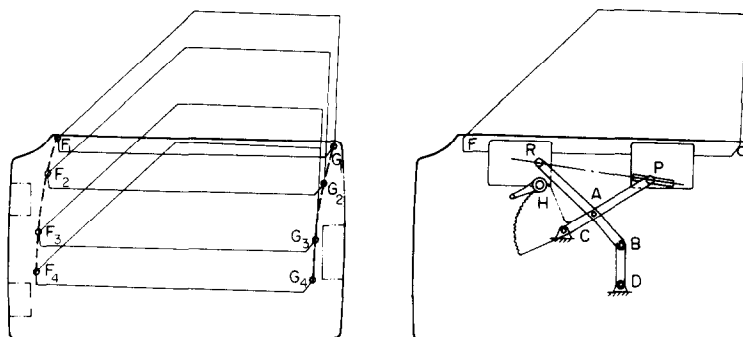


Figure 15. Computer-designed car window linkage. (a) The desired window motion is defined by trajectories of points F and G on the window. These points must stay within certain space limitations imposed by door hinges, door latches, and sheet-metal parts. And the mechanism, to accomplish this, must fit within available space. (b) The choice for the mechanism to activate the window panel is the four-bar linkage CABD. The driver arm CAP is attached directly to the sector gear which is moved by the pinion gear on the crank-handle shaft. The coupler link RAB controls the front part of the window. Since point P travels in an arc, its pivot must be a slider joint in the window support; (after [49]).

to prescribe input-output relationships at discrete points only. In applications where this relationship has to be fulfilled continuously within the range of operation of the mechanism as, for example, in a recording instrument pen mechanism requiring a linear relationship, additional restrictions are imposed. While outside the scope of this discussion of requirements, the mechanism must satisfy acceleration and force transmission criteria. For example, in a later section it is revealed that the kinematic criteria for point path synthesis (i.e. position synthesis) generally favour the achievement of a larger number of precision points. In practical design, however, this may constitute an incomplete, and sometimes undesirable, specification of mechanism characteristics. In particular, at high rotational speeds, reversals in direction of transmitted force frequently are the cause for impactive type loading, which may lead to bearing vibrations, elastic deviations from theoretical path, noise and similar other disadvantages. That an increased number of precision points can lead to less favourable dynamic performance has been shown by Broeke [14], with reference to the coupler dwell problem. For a mechanism of the type shown in Fig. 11(b) we find that in the region of greatest interest—the dwell—the output link velocity changes sign three times and acceleration two times for four precision points, while there are one and no change for the respective quantities in the case of single tangency for coupler curve and the circular arc. The two precision point construction represents an intermediate case.

Gross motion characteristics often are mandatory conditions; the same may be said about link configuration (i.e. the way the linkage is assembled) and the extent of deviation, from the desired path, of the coupler point between any two precision points.

3.2 Some coupler curve properties

The properties of curves, planar or spatial, are of interest in this work only in the context of curves defined as the loci of points of a constrained moving body (the link)

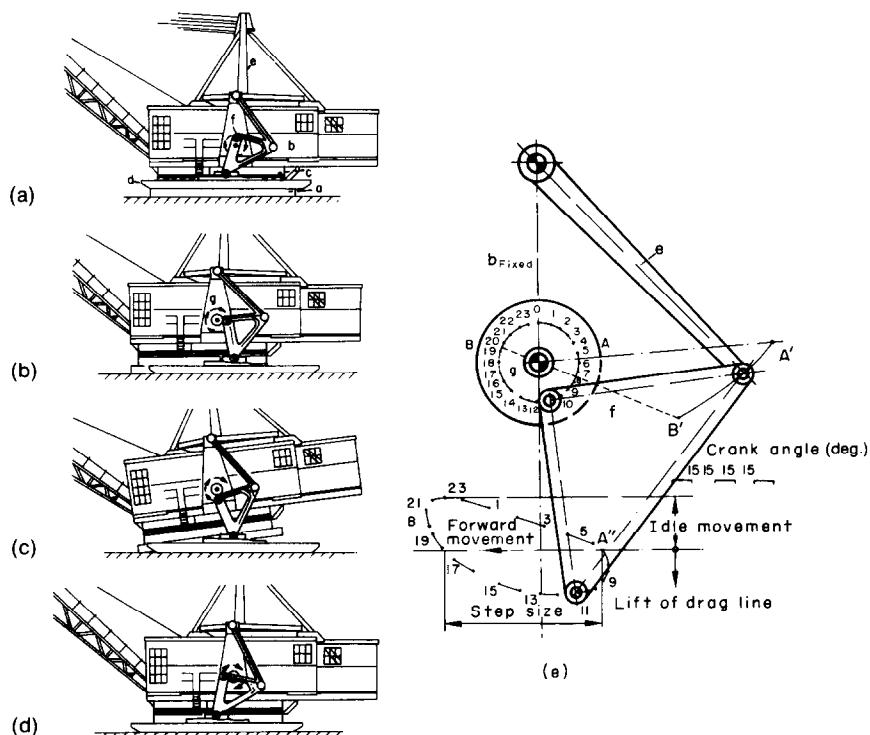


Figure 16. Linkage for walking dragline (design by Marion Power Shovel Co.), probably the largest link mechanism in use anywhere. The mechanism, shown in four positions, consists of an ordinary four-bar with the coupler pivoted to the shoe. The four phases of the stepping cycle are (a) Idling: the engine room base plate rests on the ground, the shoes are lifted. (b) Lifting: the shoes are lowered onto the ground, the backward step commences. (c) Retreat motion: the engine room tilts and drags above ground. (d) Lowering: the step motion ceases, the base plate again rests on the ground. The four phases are shown marked on the coupler curve in (e). The size and height of each step depend on the coupler point path shape and its position relative to the machine frame; (after [40] and [48]).

having one degree of freedom. Literature dealing with such properties is profuse, and need not be discussed here. A listing of the more important definitions and curve properties is given in [120, Appendix 1].*

3.2.1 Graphical vs. algebraic approaches. In this section some of the more important and, to this work, relevant properties of the planar coupler curve are reviewed. Although most of the remarks refer to the planar four-bar chain, this linkage is not treated in deliberate isolation from linkages with a greater number of links, or having slider pairs, or being three-dimensional. It will be made clear, in the discussion, when a statement expresses the characteristics of these lastmentioned linkages. It may also be stated, at this point, that because one of the objectives of this work is to explore proper-

*See ref. [17, p. 325], for some further comments on the definition of a curve.

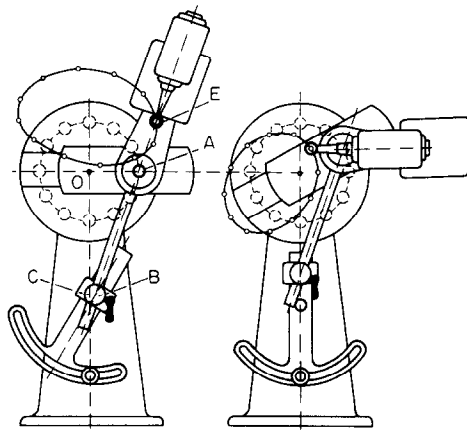


Figure 17. Profile milling machine. Adjustable proportion crank-slider mechanism OAB with oscillating block C has the cutter for coupler point E. Note that point E has two-way adjustment in the plane of the diagram; (after[93]).

ties of the coupler curve generated by three-dimensional mechanisms, geometrical-graphical techniques (constructions) in the analysis (and synthesis) of coupler curves will be de-emphasized. Graphical methods are of fundamental importance to synthesis in two dimensions. However, if extended to spatial situations, the resulting complexity of representation usually puts it beyond reasonable tractability.* Early realization of the difficulty of representation led to a prolonged period of stagnation in graphical spatial kinematics. Several attempts were made at inventing substitutes for the unwieldy orthographic projections. According to Alt [3] (his view expressed in 1929) many of the above named difficulties were removed when Federhofer [37] introduced, to the study of kinematics, the techniques of representation of Mayor [100] and von Mises [176]. These techniques are based on correspondences between space and plane vectors, thus reducing the three dimensional case to one in two dimensions. The advantages of such transformations can readily be visualized, but they apparently did not provide the incentive to popularize its wider usage, and it certainly has never enjoyed the familiarity of the conventional techniques of projection. Now, almost fifty years later, it seems certain that graphical space kinematics has gained only very limited acceptance, and will not be included in the discussion that follows.

3.2.2 The Algebraic order of the coupler curve. The foundations of algebraic properties of the coupler curve were laid in the last century. Studies of the “mechanical” generation of curves (i.e. by linkwork) have been reported before 1875 by a number of workers. In 1875 Roberts [136] analyzed the general planar coupler curve. His discussion opens with the remark that the locus of a point on the coupler is a tricircular sextic, having three nodes besides the triple circular points. Therefore, the order of the curve must have been known before that time.‡ (This particular paper contains other important results on cognate linkages, to be mentioned later, and straight-line motion.) Par-

*In this regard, views contrary to those expressed here seem to exist. See, for example, Tavkhelidze D. S., *J. Mechanisms* 2, 117, 118 (1967).

‡With reference to the Watt linkage, Prony (1755–1839) had, in fact, derived the equation of the coupler curve in about 1796, mainly with a view of examining the straight-line properties of this curve. The coupler point coordinates were expressed in parametric form, however.

ticular analytical expressions for the locus of a point on the coupler centre line have been derived by Johnson [75]. This same work treats the conditions under which factorization into lower degree polynomials is feasible. Later, Somov [153, 154] develops expressions for the coupler curve whose forms allow further insight into the influence of linkage parameters on possible changes in the coupler curve. Hippisley [67] establishes a theorem relating a geometrical property of the pedal triangle to the triangle of foci, Bennett [9] points out a property of the triple-generation mechanism of Roberts, drawing also on relevant results and analyses of Cayley and Darboux. Morley [105] treats the coupler curve from the point of view of inversive geometry.

Intrigued by the problem of conversion of circular into rectilinear motion,* Kempe [79] set out to show that any plane curve of the n^{th} order may be generated by selectively combining particular link assemblies with turning pairs only, in which a link (or point on a link) could be shown to mechanize prescribed angular or rectilinear motion. This result, stated in a different way, proves the possibility (theoretically anyway) of a coupler curve to generate a curve of the n^{th} order.† Although generally attributed to Kempe, this discovery may also have been known to Sylvester‡ who also claimed to have been in possession of a proof. Conversely, the turning pair, or revolute-pair, system does not permit it to generate transcendental curves [159, 83, 104]. A more complete proof of this latter statement has recently been given by Freudenstein [43].§ Since the point path locus is, in general, established through use of displacement equations, a transcendental curve requires at least one expression in the displacement equations to be also transcendental.¶

The complexity of the algebraic expressions governing the point paths of coupler (or floating) links of plane mechanisms has restricted investigations of the algebraic, geometric and kinematic properties of the curve to cases with relatively few links, or to special geometric configurations. Thus, the six-link mechanism can assume two different forms depending on the arrangement of binary and ternary links.** Either can be inverted in a number of ways, each inversion producing different coupler (or floating link) curves. The named particular cases (Fig. 18), lead to curves of the order 14 (for points on link 1) and 16 (for points on link 2) for the Watt mechanism, and also of order 16 (for points on links 1 or 2) for the Stephenson mechanisms [131]. The highest order is 18 (point G), for the general 4-bar and dyad†† arrangement, (Fig. 18). When we examine the eight-link mechanism, different combinations of binary, ternary and quaternary links lead to sixteen kinematic chains.‡‡ By holding different links fixed, altogether

*Ref. [78]; How to draw a straight line *Nature* 16, 65–67, 86–89, 125–127, 145–146 (1877); On a general method of producing exact rectilinear motion by linkwork. *Van Nostrand's Engng Mag.* 16, 103–108.

†See also Bricard R., *Lecons de Cinematique* 2, Paris (1927), for a different proof of Kempe's theorem.

‡See [159], *Mathematical Papers*, pp. 13, 14. Sylvester's discourse appears vague and does not lend itself to precise interpretation as to the nature of the basis for his proof. The ideas of Kempe go back to the year 1872, at least. In a short note [77] a method is given of obtaining, by mechanical means, the real roots of an algebraic polynomial equation. Kempe claims that there are many ways of achieving this, employing a "machine", without, however providing further details of how it could be constructed.

§Freudenstein, in his proof, considers also the question of multiplicity of link angle combinations, that, each, lead to real roots for the point path polynomial. Since the number of such roots is finite, it is shown that provided path selection at a branch point follows a definite cyclic law, the curve generated will repeat itself after a finite integer number of cycles.

¶See [5, ch. 7], for examples of transcendental curve generators.

**A third form, consisting of two four-bar mechanisms coupled by a ternary is a trivial case, producing only sixth order curves.

††We define the dyad as two links joined by a revolute pair and each having provision for one other revolute pair connection. Dyads can be connected to other linkages to form new link loops.

‡‡One of these may be shown to generate coupler curves no more complicated than the six-bar curve, and therefore does not represent an advance on the latter.

71 mechanisms are derivable from these chains, excluding multiplicities due to symmetry [53]. The method for finding the order of the coupler curve may, in principle, be developed further by adopting the results of the Watt or Stephenson mechanism. Primrose *et al.* [132] have illustrated the procedure by establishing the order of one particular eight-bar curve (in this instance of order 30), and, ultimately, that of a special $2n$ -bar curve, namely the Stephenson-1 mechanism, modified by removing link BD (Fig. 18), and connecting B. and D by $n - 2$ four-bar mechanisms. The order of the curve is then $2(2^n - 1)$. Earlier, Levitskii [88] examined the symmetrical coupler curve for any number of links built up of a series of Chebyshev dyads. While not of a general form, his expressions are of interest because they lead to a concise formulation of the degree of polynomial expressing the coupler point trajectory. Thus, if the input link makes a revolute pair with the stationary link, the addition of each Chebyshev dyad increases the order of the coupler curve three times. With zero dyads, the curve being second order (circle), one dyad gives the four-bar curve (sixth order), two dyads an eighteenth order curve, three dyads a 54th order curve and so on. This same result was derived by Wunderlich [186], in a different manner, fifteen years later. He refers to work and ideas by Rischen [135], but for reasons unstated makes no reference to Levitskii. It is interesting to note, however, that Wunderlich's derivation in no way assumes conditions of symmetry, thus generalizing the result of Levitskii. The particular solution of the six-bar appears to have been derived also by Meyer zur Capellen (see [135, p. 381, footnote]). If the revolute-pair input is replaced by a slider-pair, the corresponding or-

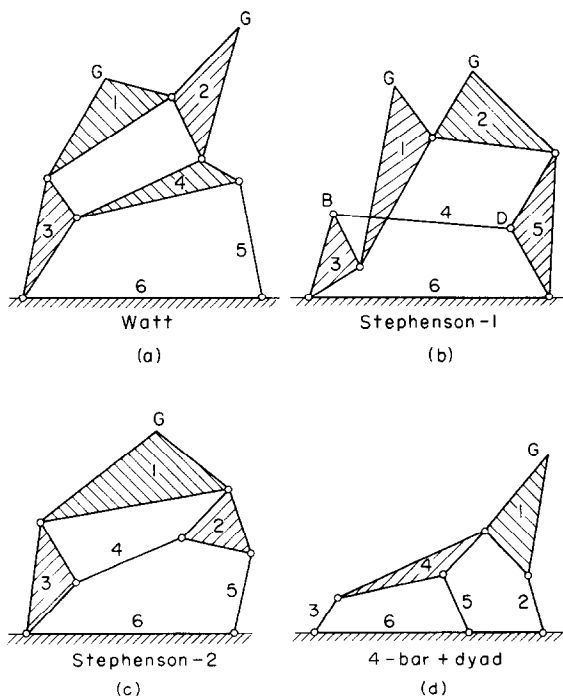


Figure 18. Six-bar linkages with binary and ternary links. In all cases input is at link 3; coupler point(s) is (are) G. (a) Watt mechanism; (b) Stephenson-1 mechanism; (c) Stephenson-2 mechanism; (d) Four-bar and dyad mechanism.

ders become 1, 4, 14, 46, etc., or, in general

$$m = 2 \cdot 3^{n-2/2} - 2^{n-2/2},$$

where m and n are the order of the curve and the number of links, respectively. Under special conditions, however, the order of the curve of a particular linkage may be less than that given, and in this sense Levitskii's expressions must be considered as giving the upper bound for order only. Properties other than the order of the curve have also been studied[131, 132], such as the genus, number of double points, number of real circuits, number of cusps, number of intersections with other simple curves (such as straight line, circle) and others.* It is clear, also, that, while versatility in complying with specified shape requirements may, in general, be the outstanding feature of a higher order coupler curve, other favourable kinematic characteristics have been recognized in mechanisms having a higher number of links. Thus the range of crank-rocker motion may be extended to a full 360° for the output with suitable choice for link configurations of two four-bars in series[53]. Furthermore, we know that six-bar linkages are possible in which the output link completes two full revolutions for each revolution of input crank, and eight-bar linkages completing a corresponding four full-revolutions at the output link.† High quality dwells with rest periods exceeding 180° crank input have been constructed by Müller[106], who found it necessary to use as many as ten links, and Wetzel[179] (constructions with 8-link arrangements). Whether or not it is possible to translate such kinematic agility into practical forms of mechanism is, of course a question touching on machine design; Hain's[57] treatment of practical problems in this regard provides some of the answers, particularly in connection with the transmission of motion of drag-link mechanisms in series arrangement. Other practical limitations of motion and force transmissibility in mechanisms with more than four links are discussed by Rössner[138, 139] and Nerge[117].

3.2.3 Curvature theory. A vast field of study is represented by what is known as Burmester theory. It is concerned with the kinematic and geometric properties of point path curvature and specific points having a special significance in curvature theory. Although not central to the present work, its importance in both mechanism analysis and synthesis is sufficiently far reaching to warrant an appraisal, here, of some of the most significant results and discoveries bearing also on coupler curve synthesis.

As mentioned earlier, Burmester's work concerned itself with finite, coplanar displacement. His motivation however, was not so much in establishing curvature theory as such as in approximating straight-line motion. Several years later Müller extended Burmester's findings to infinitesimal motion with particular reference to four-bar motion. He showed that there could be up to four real Burmester points of which two must of course be the input and output link pin centres respectively,‡ Müller was able to establish a number of other important theorems relating to collinearity, order of contact, symmetry, and location of Burmester points, again in the context of infinitesimal motion[108, 109, 110]. Allievi[1] may have been the first to record the analytical de-

*Occasionally geared and revolute-pair linkages have been studied conjointly. Geared link motion may, in some instances, be likened to revolute-pair linkages. For example, the point of connection of the two floating links of a geared five-bar mechanism with gear ratio minus one traces an algebraic bicircular sextic (note similarity with the four-bar curve). Other gear ratios lead to higher order curves[44], also algebraic.

†See Ref.[53, pp. 86, 87].

‡Points on the coupler plane whose first and second derivatives of the radius of curvature of the point path with respect to the distance along the path are instantaneously zero are called Burmester points. Five infinitesimally separated positions of such points lie on a circle.

rivation of the Burmester point positions for a four-bar mechanism. The solution of Allievi was later confirmed by Wolford[181A], who also compares the analytical technique with the corresponding graphical solution of Müller[108, 109]. Of particular significance appear also the results of Müller, concerning path-curvature analysis by means of what he names to be the “return-pole” of higher order (Rückkehrpol). This fundamental study establishes the algebraic relationships existing between the evolutes of the evolute of a point path, and their respective centres of curvature[111]. By kinematic inversion, similar relationships can be shown to exist for motion of the fixed (reference) plane with respect to the first moving plane. If applied to the instantaneous motion of a four-bar mechanism, the results distinguish between all special cases applicable to the motion of a point on the coupler plane. Thus Müller identifies, in terms of special conditions for his parameters, the location of Ball’s point and the conditions for Burmester’s points. Further, Müller in this work discovers that (i) provided certain simple geometric constraints are satisfied, the four-bar curve can be made to possess sixth-order contact with a straight line; no higher order contact being possible. (ii) The degeneration of the mechanism generating a sixth-order contact into the exact straight-line motion crank-slider is a gradual process in which the crank-rocker coupler curve accuracy improves. (iii) The three moving links of the sixth-order approximating mechanism may be interchanged in any possible order. (iv) Chebyshev’s straight-line mechanism (Fig. 4(i)) appears as a special case of a sixth-order straight-line approximation (a not very favourable case at that, according to Müller). The existence or otherwise of coupler curves with sixth-order straight-line tangency for the different classes of four-bar linkages was established several years later[112]. More recently the question as to the relationship between Müller’s results in finitestimal motion to those obtained from finite motion theory was treated by Primrose *et al.*[130]. Based, partly, on work by Freudenstein and Sandor[42] some years earlier, this work presents a unified treatment of some aspects of Burmester’s theory. Here we find confirmation of the possibility of extending a six-point approximation to a circle,* as well as the necessary conditions for three centrepoin to be collinear (cf. Müller[108, p. 149]). Other theorems regarding number and position of centrepoin are established for special conditions of position symmetry for the coupler plane. The authors conclude that the results of that study apply both to finite, as well as to infinitesimal and “mixed” motions.

For mainly practical reasons the question of path curvature has manifested itself largely in kinematic studies leading ultimately to one of two special conditions for the path radius of curvature, ρ , one being the instantaneous condition of linear motion, $\rho = \infty$, the other motion in a circular path, $\rho = \text{constant}$ ‡. But since we are interested also in the instantaneous motion of *any* point on the coupler plane, it is essential to know, in order to identify the point instantaneous motion, not only the path curvature (with reference to a single position) but also the rate of change of path curvature and the second rate of change of path curvature. In this regard, earlier work, again by Müller[107, 108], on the analyses of the evolutes of point path trajectories has substantially been supplemented by Freudenstein[45]. Freudenstein’s contribution locates the Burmester points in the whole spectrum of points having prescribed values for parame-

*It should be noted, however, that Müller’s statement, Ref.[111, p. 261], concerning sixth-order approximation is not a generalization, but was made with particular reference to contact with a straight line. The possibility of sixth-order approximation to a circle should have been known to Müller.

‡We define ρ by $ds/d\alpha$, where s = arc length along curve; α = angle of inclination of the normal to the curve with respect to a coordinate axis. Then rate of change of path curvature is $d\rho/d\alpha$, second rate of change of path curvature is $d^2\rho/d\alpha^2$, etc. Relationships between $d\rho/d\alpha$ and $d\rho/ds$ may be established if required in the derivation of particular results (e.g. Ref.[45, p. 185]).

ters λ_1 and λ_2 ($\lambda_1 = \rho_E/\rho$, $\lambda_2 = \rho_E^{(2)}/\rho$; ρ = path radius of curvature, ρ_E = radius of curvature of the evolute of the path, $\rho_E^{(2)}$ = radius of curvature of the evolute of the evolute of the path), i.e. having fifth-order contact with an arbitrary curve. There are, in general, five such points corresponding to an arbitrary position of a plane for each set of λ_1 and λ_2 . The special case when $\lambda_1 = \lambda_2 = 0$ reduces the total number of such points to four which can be identified as being the Burmester points. Due to the existence of an odd number of these generalized Burmester points at least one of them must be real, leading to the result that different points on a *single* link in a *single* position can be used to generate practically any continuous plane curve with at least fifth-order contact.* Most of Freudenstein's formulations are algebraic and appear in relatively concise form in cases of symmetric four-bar motion. However, it is conceded by the author (Freudenstein) that at extension of the above mentioned analysis to finite motion would be of little practical value because of the difficulties of arriving at a meaningful specification of change of average curvature. In this regard alternative methods[144] have been referred to, seemingly holding greater promise.

It is clear that not all developments in Burmester theory, bearing directly on coupler point path properties and path synthesis, can be covered. Significant advances were made also in the period between Müller's earlier work and the above discussed advances of the 1960's. In this regard further studies of a general nature or studies emphasizing algebraic-analytical treatment may be found elsewhere [85, 151, 161, 169, 26].

3.2.4 Multiple generation of coupler curves. It is now a well-known fact that in many instances the same coupler curve may be generated by more than one linkage. Such linkages, additional to the original one, are called cognates† and we find the first result, concerning cognate linkages, in Roberts's theorem [136]. This theorem states that there are three different four-bar linkages which generate identical coupler curves, and that from any given four-bar, the two cognate four-bars may be uniquely determined. Below, it will be shown that there are exceptions to this theorem when pairs other than revolutes are introduced. Three years after Roberts announced his discovery (proof in geometrical form), Chebyshev [22], unaware of this, provided a different proof (analytical) for the same result. Since then the same theorem has been re-proved by Hart [62] (quaternion proof, based on some constructions and figures of Cayley who, already, had confirmed Roberts's theorem), Schor [149] (complex numbers), Schmid [148] (complex numbers), Meyer zur Capellen [102] (complex numbers), Wunderlich [185, 186] (complex numbers), De Jonge [28] (geometrical) and probably others.

From a curve synthesis point of view such (exact) multiple line generation means a choice for fixed link dimension and generally of mechanism geometry and type.§ In this

*Ref. [45, p. 189]. The possibility of sixth-order contact is also examined. This is realized in cases of position symmetry, e.g. for parabolic arc at origin (p. 188), and elliptic arc at the end of a principal diameter. (Compare with results of Müller [111], pp. 260–265].

†In literature, the term "cognate" is used ambiguously, particularly with reference to the number of such linkages that may exist in any one case. Thus expressions like "there are three cognates for a planar four-bar", "for any given four-bar, two cognate four-bars may be found", "investigation of the existence of cognates" abound, but never make it clear whether here a set of two linkages is identified by referring to two cognates or one cognate.

§The following being the possible revolute pair mechanism and its cognate combinations [99]: crank rocker (2) and double rocker (1); double rocker (3); drag link (3).

It is of interest to note that in the first alternative, the two crank-rockers possess, always, the same minimum value of transmission angle (with inputs at the cranks). The location of the coupler point does, of course, not enter this condition which is related solely to the geometry of the link loop. We conclude, therefore, that application of Robert's theorem in this regard cannot help to improve the geometric transmission characteristics in mechanism design (Ref. [140, p. 106]). If the mechanisms are draglinks, the extremes of transmission angle are identical in pairs, when, for each pair, the input is from the same revolute at the fixed link.

regard Roberts's theorem furnishes only one specific answer to the wider question of multiple generation with any link number mechanisms. The planar mechanism following the four-bar with the least number of links is the six-bar. Although shown to trace, in general, an 18th order coupler curve,* through application of Roberts theorem and by choosing one of the dyad revolute axes to coincide with the four-bar output revolute fixed axis, Rössner[140] was able to show that a six-bar coupler curve can be made to coincide with a prescribed four-bar curve. His special conditions in the construction must thus lead to conditions of degeneracy (or factorization) of the general algebraic expression for the curve. In fact, the construction assumes an arbitrary position of the six-bar input axis, leading to the conclusion also that there is possibly a double infinitude of solutions of this kind.‡

The complete set of cognates of this type of six-bar linkage was determined by Rischen[135] in 1962. The total number of cognates is five. This particular result was generalized in 1963 by Wunderlich[186]. His results, derived, in part, on the basis of some characteristics of Sylvester's "Plagiograph" (skew pantograph), may be summarized in the statement that the coupler curve generated by the last dyad in a series of s dyads (with one pivot grounded) has $3 \cdot 2^{s-2} - 1$ cognates. The same result has been derived by Roth[43], who also furnished the corresponding expressions for number of cognates for the cases where the base linkage is a crank-slider: $2 \cdot 2^{(n-4)/2} - 1$, and oscillating block mechanism: $2^{(n-4)/2} - 1$, where n is the total number of links.§

Before leaving the question of multiple generation of coupler curves two further points should be noted. The first concerns the curve of the five-bar having two input links and rotating with equal angular velocity in the same sense. This curve is identical to that of a four-bar. It should be possible, therefore, to substitute the latter by a five-bar of the kind just mentioned. Following the method of Luck[99], we find that there is a double infinitude of such five-bar solutions and that the two inputs possess, always, full rotational mobility. This abovementioned two-input condition is identical to the geared five-bar linkage having gear ratio of plus-one; the corresponding linkage with gear ratio minus-one, however, having no cognates[143]. The next higher odd-numbered link arrangement (seven links) again has one cognate provided both inputs are made to rotate with equal angular velocity[135].¶

*We are referring here to the four-bar plus dyad case (Fig. 18).

‡Earlier, Hartenberg and Denavit[63] extended the Roberts theorem by showing the existence of two six-bar cognates corresponding to a given four-bar. The former are constructed on the fixed link of the four-bar and turn out to be special cases of both the Watt and Stephenson mechanisms, (Fig. 18). These cognate curves are confirmed by the results of Primrose[131] (see case VIII, p. 26), but they also throw doubts on the general validity of a further conclusion in this same work (case XIII, p. 26). In a more recent paper Dijksman[31A] shows that the Watt-I six-bar curve (of degree 14) can be produced by a doubly infinite number of cognates of the same form.

§In Roth's paper these results are described in ambiguous form (p. 178), and could lead the reader to the wrong conclusion. The uncertainty as to the number of linkages is introduced by the loose definition of "cognate". See footnote, p. 345.

¶The seven-bar is derived by combining two crank-rocker four-bars (cognates with respect to each other) and one dyad, then removing from each of the four-bars the respective "output" links. Rischen shows that a second construction of this kind is possible, yielding the seven-bar cognate. When the above four-bars are drag-links, Rischen's conclusion becomes less certain: although assurance is given by Rischen that his developments on the crank-rocker do lead to the same results for the drag-link (see footnote, p. 382), it has not been made clear as to how one arrives at the correct choice for the input cranks. Since the above combination of four-bars yields three available crank inputs, it would appear that five cognates exist for this case. It is doubtful, however, whether one of the two cognates (drag-links) of a four-bar does in fact possess the required input crank-coupler point relationship. If this latter relationship is not satisfied, the drag-link case reverts back to the result for the crank-rocker and Rischen is right (compare with findings of Freudenstein as regards crank-spacing and coupler point position relationships in cognates[41]).

If one or two revolute pairs of a four-bar are replaced by a slider pair, Roberts's theorem, in its original form, does not hold. Under these conditions we may distinguish three cases. The solution to the first, the centric or offset crank-slider, has been known since the time of Roberts who has stated (although without proof) that for this linkage only one accessible cognate exists (the second vanishing at infinity). Case two constitutes the mechanism where the revolute and slider pairs are on the coupler of fixed length. This mechanism now lacks any useful cognate (in the sense of Roberts's theorem, one cognate coinciding with the original linkage and the other collapsing into a straight line), and the result was proved by Schmid[148] in 1950. Ten years later, De Jonge[28], in his exhaustive study on the correlation of hinged four-bar straight-line motion devices, establishes that the oscillating block crank-slider (centric or offset) also has no cognate. De Jonge claims* to be the first to appreciate and to prove that there are four-bars with one sliding pair that have no cognates. That this claim cannot be substantiated is shown by Schmid's publication. Furthermore, De Jonge's and Schmid's mechanisms can be shown, by inversion of the sliding pair, to be kinematically equivalent. Hence, the result of one immediately follows that of the other.‡ Lastly, we have the group of mechanisms with two slider pairs (three variations exist: either one revolute, two revolutes, or no revolutes on the stationary link) where again only one cognate exists[28, 148].

The converse to the general question of duplication of a coupler curve is the condition for uniqueness of a particular curve. Clearly, where cognates exist there will be two or three linkage solutions for a given coupler curve. However, apart from the case of duplication or triplication by cognates, the necessary and sufficient conditions for uniqueness of a curve may be of interest. In this regard, relevant properties of the general four-bar coupler curve have been established by Roberts[136], Hippisley[67], Bennett[9], Haarbleicher[50, 51] and Eckhardt[35], and may be summarised as follows. The coupler curve of the general four-bar is a tricircular sextic, whose three focal points and three double points lie on the same circle. Between these points exist the relationship that the triangles of foci and double points, respectively, are in isogonal relationship with respect to each other[178],§ and the coupler curve is in self-correspondence in the isogonal inversion with respect to its double point triangle[9]. Since there are nine independent parameters determining the coupler curve, fixing of two of the three foci (i.e. selecting one of the three linkages that are cognates with respect to each other) and the three double points permits the generation, within the imposed constraints, of a single infinitude of coupler curves having a common set of double points.¶ The only conditions to be satisfied by link lengths is that $(F_1A)^2 - (AE)^2$ and $(F_2B)^2 - (BE)^2$ are constants, where F_1 , F_2 are the foci, A , B are the coupler revolutes and E is the coupler point[35]. (It should be noted that although the location of the coupler point requires two independent quantities to be specified, one of these is contained in the condition of similarity between the triangle of foci and the coupler triangle, leaving only one independent unknown.) The above conditions imply also that two coupler curves cannot be identical unless their respective triangles of foci and double points coincide.

*See(28, p. 99).

‡Ibid., p. 103.

§See also[175, p. 105]; [174, pp. 492–494].

¶The relative positions of the five points require specification of eight independent coordinate positions to locate four points with respect to the fifth. This leaves one parameter unspecified.

Synthese der Gelenk-Koppelkurve: Ein historischer Überblick. Teil II—Entwicklungen nach 1875.

H. Nolle

Kurzfassung—Die vorliegende Abhandlung besteht aus drei Teilen. Geschichtliche mechanische Erfindungen and deren Anwendungen werden besprochen, und ihr Einfluss auf die Entwicklung der Reuleaux'schen Kinematik wird angedeutet. Im zweiten Teil Werden die analytischen Eigenschaften der Koppelkurve zusammengefasst, angefangen mit dem Jahr 1875, in dem Samuel Roberts die algebraischen Eigenschaften der Koppelkurve des zwei-dimensionalen Gelenkvierecks veroeffentlichte. Der dritte Teil besteht aus einer Klassifizierung der Methoden fuer die Synthese der Koppelkurve. Auf rein graphischem Wege erhaltene Konstruktionen werden jedoch, teilweise aufgrund ihrer begrenzten Brauchbarkeit in der drei-dimensionalen Kinematik, in diesem Teil nicht weiter untersucht.