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Detection, Estimation, and Modulation Theory

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Detection, Estimation, and Modulation Theory

Part I. Detection, Estimation, and Linear Modulation Theory

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To Diane

and Stephen, Mark, Kathleen, Patricia,
Eileen, Harry, and Julia

and the next generation—
Brittany, Erin, Thomas, Elizabeth, Emily,
Dillon, Bryan, Julia, Robert, Margaret,
Peter, Emma, Sarah, Harry, Rebecca,
and Molly

Preface for Paperback Edition

In 1968, Part I of *Detection, Estimation, and Modulation Theory* [VT68] was published. It turned out to be a reasonably successful book that has been widely used by several generations of engineers. There were thirty printings, but the last printing was in 1996. Volumes II and III ([VT71a], [VT71b]) were published in 1971 and focused on specific application areas such as analog modulation, Gaussian signals and noise, and the radar–sonar problem. Volume II had a short life span due to the shift from analog modulation to digital modulation. Volume III is still widely used as a reference and as a supplementary text. In a moment of youthful optimism, I indicated in the the Preface to Volume III and in Chapter III-14 that a short monograph on optimum array processing would be published in 1971. The bibliography lists it as a reference, *Optimum Array Processing,* Wiley, 1971, which has been subsequently cited by several authors. After a 30-year delay, *Optimum Array Processing,* Part IV of *Detection, Estimation, and Modulation Theory* will be published this year.

A few comments on my career may help explain the long delay. In 1972, MIT loaned me to the Defense Communication Agency in Washington, D.C. where I spent three years as the Chief Scientist and the Associate Director of Technology. At the end of the tour, I decided, for personal reasons, to stay in the Washington, D.C. area. I spent three years as an Assistant Vice-President at COMSAT where my group did the advanced planning for the INTELSAT satellites. In 1978, I became the Chief Scientist of the United States Air Force. In 1979, Dr. Gerald Dinneen, the former Director of Lincoln Laboratories, was serving as Assistant Secretary of Defense for C3I. He asked me to become his Principal Deputy and I spent two years in that position. In 1981, I joined M/A-COM Linkabit. Linkabit is the company that Irwin Jacobs and Andrew Viterbi had started in 1969 and sold to M/A-COM in 1979. I started an Eastern operation which grew to about 200 people in three years. After Irwin and Andy left M/A-COM and started Qualcomm, I was responsible for the government operations in San Diego as well as Washington, D.C. In 1988, M/A-COM sold the division. At that point I decided to return to the academic world.

I joined George Mason University in September of 1988. One of my priorities was to finish the book on optimum array processing. However, I found that I needed to build up a research center in order to attract young research-oriented faculty and

doctoral students. The process took about six years. The Center for Excellence in Command, Control, Communications, and Intelligence has been very successful and has generated over $300 million in research funding during its existence. During this growth period, I spent some time on array processing but a concentrated effort was not possible. In 1995, I started a serious effort to write the Array Processing book.

Throughout the *Optimum Array Processing* text there are references to Parts I and III of *Detection, Estimation, and Modulation Theory*. The referenced material is available in several other books, but I am most familiar with my own work. Wiley agreed to publish Part I and III in paperback so the material will be readily available. In addition to providing background for Part IV, Part I is still useful as a text for a graduate course in Detection and Estimation Theory. Part III is suitable for a second level graduate course dealing with more specialized topics.

In the 30-year period, there has been a dramatic change in the signal processing area. Advances in computational capability have allowed the implementation of complex algorithms that were only of theoretical interest in the past. In many applications, algorithms can be implemented that reach the theoretical bounds.

The advances in computational capability have also changed how the material is taught. In Parts I and III, there is an emphasis on compact analytical solutions to problems. In Part IV, there is a much greater emphasis on efficient iterative solutions and simulations. All of the material in parts I and III is still relevant. The books use continuous time processes but the transition to discrete time processes is straightforward. Integrals that were difficult to do analytically can be done easily in Matlab®. The various detection and estimation algorithms can be simulated and their performance compared to the theoretical bounds. We still use most of the problems in the text but supplement them with problems that require Matlab® solutions.

We hope that a new generation of students and readers find these reprinted editions to be useful.

HARRY L. VAN TREES

Fairfax, Virginia
June 2001

Preface

The area of detection and estimation theory that we shall study in this book represents a combination of the classical techniques of statistical inference and the random process characterization of communication, radar, sonar, and other modern data processing systems. The two major areas of statistical inference are decision theory and estimation theory. In the first case we observe an output that has a random character and decide which of two possible causes produced it. This type of problem was studied in the middle of the eighteenth century by Thomas Bayes [1]. In the estimation theory case the output is related to the value of some parameter of interest, and we try to estimate the value of this parameter. Work in this area was published by Legendre [2] and Gauss [3] in the early nineteenth century. Significant contributions to the classical theory that we use as background were developed by Fisher [4] and Neyman and Pearson [5] more than 30 years ago. In 1941 and 1942 Kolmogoroff [6] and Wiener [7] applied statistical techniques to the solution of the optimum linear filtering problem. Since that time the application of statistical techniques to the synthesis and analysis of all types of systems has grown rapidly. The application of these techniques and the resulting implications are the subject of this book.

This book and the subsequent volume, Detection, Estimation, and Modulation Theory, Part II, are based on notes prepared for a course entitled "Detection, Estimation, and Modulation Theory," which is taught as a second-level graduate course at M.I.T. My original interest in the material grew out of my research activities in the area of analog modulation theory. A preliminary version of the material that deals with modulation theory was used as a text for a summer course presented at M.I.T. in 1964. It turned out that our viewpoint on modulation theory could best be understood by an audience with a clear understanding of modern detection and estimation theory. At that time there was no suitable text available to cover the material of interest and emphasize the points that I felt were

important, so I started writing notes. It was clear that in order to present the material to graduate students in a reasonable amount of time it would be necessary to develop a unified presentation of the three topics: detection, estimation, and modulation theory, and exploit the fundamental ideas that connected them. As the development proceeded, it grew in size until the material that was originally intended to be background for modulation theory occupies the entire contents of this book. The original material on modulation theory starts at the beginning of the second book. Collectively, the two books provide a unified coverage of the three topics and their application to many important physical problems.

For the last three years I have presented successively revised versions of the material in my course. The audience consists typically of 40 to 50 students who have completed a graduate course in random processes which covered most of the material in Davenport and Root [8]. In general, they have a good understanding of random process theory and a fair amount of practice with the routine manipulation required to solve problems. In addition, many of them are interested in doing research in this general area or closely related areas. This interest provides a great deal of motivation which I exploit by requiring them to develop many of the important ideas as problems. It is for this audience that the book is primarily intended. The appendix contains a detailed outline of the course.

On the other hand, many practicing engineers deal with systems that have been or should have been designed and analyzed with the techniques developed in this book. I have attempted to make the book useful to them. An earlier version was used successfully as a text for an in-plant course for graduate engineers.

From the standpoint of specific background little advanced material is required. A knowledge of elementary probability theory and second moment characterization of random processes is assumed. Some familiarity with matrix theory and linear algebra is helpful but certainly not necessary. The level of mathematical rigor is low, although in most sections the results could be rigorously proved by simply being more careful in our derivations. We have adopted this approach in order not to obscure the important ideas with a lot of detail and to make the material readable for the kind of engineering audience that will find it useful. Fortunately, in almost all cases we can verify that our answers are intuitively logical. It is worthwhile to observe that this ability to check our answers intuitively would be necessary even if our derivations were rigorous, because our ultimate objective is to obtain an answer that corresponds to some physical system of interest. It is easy to find physical problems in which a plausible mathematical model and correct mathematics lead to an unrealistic answer for the original problem.

We have several idiosyncrasies that it might be appropriate to mention. In general, we look at a problem in a fair amount of detail. Many times we look at the same problem in several different ways in order to gain a better understanding of the meaning of the result. Teaching students a number of ways of doing things helps them to be more flexible in their approach to new problems. A second feature is the necessity for the reader to solve problems to understand the material fully. Throughout the course and the book we emphasize the development of an ability to work problems. At the end of each chapter are problems that range from routine manipulations to significant extensions of the material in the text. In many cases they are equivalent to journal articles currently being published. Only by working a fair number of them is it possible to appreciate the significance and generality of the results. Solutions for an individual problem will be supplied on request, and a book containing solutions to about one third of the problems is available to faculty members teaching the course. We are continually generating new problems in conjunction with the course and will send them to anyone who is using the book as a course text. A third issue is the abundance of block diagrams, outlines, and pictures. The diagrams are included because most engineers (including myself) are more at home with these items than with the corresponding equations.

One problem always encountered is the amount of notation needed to cover the large range of subjects. We have tried to choose the notation in a logical manner and to make it mnemonic. All the notation is summarized in the glossary at the end of the book. We have tried to make our list of references as complete as possible and to acknowledge any ideas due to other people.

A number of people have contributed in many ways and it is a pleasure to acknowledge them. Professors W. B. Davenport and W. M. Siebert have provided continual encouragement and technical comments on the various chapters. Professors Estil Hoversten and Donald Snyder of the M.I.T. faculty and Lewis Collins, Arthur Baggeroer, and Michael Austin, three of my doctoral students, have carefully read and criticized the various chapters. Their suggestions have improved the manuscript appreciably. In addition, Baggeroer and Collins contributed a number of the problems in the various chapters and Baggeroer did the programming necessary for many of the graphical results. Lt. David Wright read and criticized Chapter 2. L. A. Frasco and H. D. Goldfein, two of my teaching assistants, worked all of the problems in the book. Dr. Howard Yudkin of Lincoln Laboratory read the entire manuscript and offered a number of important criticisms. In addition, various graduate students taking the course have made suggestions which have been incorporated. Most of the final draft was typed by Miss Aina Sils. Her patience with the innumerable changes is

sincerely appreciated. Several other secretaries, including Mrs. Jarmila Hrbek, Mrs. Joan Bauer, and Miss Camille Tortorici, typed sections of the various drafts.

As pointed out earlier, the books are an outgrowth of my research interests. This research is a continuing effort, and I shall be glad to send our current work to people working in this area on a regular reciprocal basis. My early work in modulation theory was supported by Lincoln Laboratory as a summer employee and consultant in groups directed by Dr. Herbert Sherman and Dr. Barney Reiffen. My research at M.I.T. was partly supported by the Joint Services and the National Aeronautics and Space Administration under the auspices of the Research Laboratory of Electronics. This support is gratefully acknowledged.

Harry L. Van Trees

Cambridge, Massachusetts
October, 1967.

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2

Classical Detection and Estimation Theory

2.1 INTRODUCTION

In this chapter we develop in detail the basic ideas of classical detection and estimation theory. The first step is to define the various terms.

The basic components of a simple decision-theory problem are shown in Fig. 2.1. The first is a *source* that generates an output. In the simplest case this output is one of two choices. We refer to them as hypotheses and label them H_0 and H_1 in the two-choice case. More generally, the output might be one of M hypotheses, which we label $H_0, H_1, \ldots, H_{M-1}$. Some typical source mechanisms are the following:

1. A digital communication system transmits information by sending ones and zeros. When "one" is sent, we call it H_1, and when "zero" is sent, we call it H_0.
2. In a radar system we look at a particular range and azimuth and try

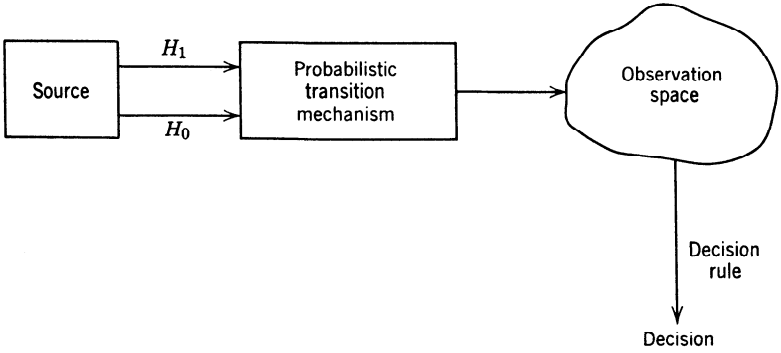

Fig. 2.1 Components of a decision theory problem.

because the details involved in constructing the likelihood ratios are the same as those in the binary case. Typical examples are given in the problems. Several important points should be emphasized.

1. The minimum dimension of the decision space is no more than $M - 1$. The boundaries of the decision regions are hyperplanes in the $(\Lambda_1, \ldots, \Lambda_{M-1})$ plane.
2. The optimum test is straightforward to find. We shall find however, when we consider specific examples that the error probabilities are frequently difficult to compute.
3. A particular test of importance is the minimum total probability of error test. Here we compute the a posteriori probability of each hypothesis $\Pr(H_i|\mathbf{R})$ and choose the largest.

These points will be appreciated more fully as we proceed through various applications.

These two sections complete our discussion of simple hypothesis tests. A case of importance that we have not yet discussed is the one in which several source outputs are combined to give a single hypothesis. To study this detection problem, we shall need some ideas from estimation theory. Therefore we defer the composite hypothesis testing problem until Section 2.5 and study the estimation problem next.

2.4 ESTIMATION THEORY

In the last two sections we have considered a problem in which one of several hypotheses occurred. As the result of a particular hypothesis, a vector random variable $\mathbf{r}$ was observed. Based on our observation, we shall try to choose the true hypothesis.

In this section we discuss the problem of *parameter estimation*. Before formulating the general problem, let us consider a simple example.

Example 1. We want to measure a voltage a at a single time instant. From physical considerations, we know that the voltage is between $-V$ and $+V$ volts. The measurement is corrupted by noise which may be modeled as an independent additive zero-mean Gaussian random variable n. The observed variable is r. Thus

$$r = a + n. \tag{116}$$

The probability density governing the observation process is $p_{r|a}(R|A)$. In this case

$$p_{r|a}(R|A) = p_n(R - A) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left(-\frac{(R - A)^2}{2\sigma_n^2}\right). \tag{117}$$

The problem is to observe r and estimate a.

This example illustrates the basic features of the estimation problem.

A model of the general estimation problem is shown in Fig. 2.17. The model has the following four components:

Parameter Space. The output of the source is a parameter (or variable). We view this output as a point in a parameter space. For the single-parameter case, which we shall study first, this will correspond to segments of the line $-\infty < A < \infty$. In the example considered above the segment is $(-V, V)$.

Probabilistic Mapping from Parameter Space to Observation Space. This is the probability law that governs the effect of a on the observation.

Observation Space. In the classical problem this is a finite-dimensional space. We denote a point in it by the vector $\mathbf{R}$.

Estimation Rule. After observing $\mathbf{R}$, we shall want to estimate the value of a. We denote this estimate as $\hat{a}(\mathbf{R})$. This mapping of the observation space into an estimate is called the estimation rule. The purpose of this section is to investigate various estimation rules and their implementations.

The second and third components are familiar from the detection problem. The new features are the parameter space and the estimation rule. When we try to describe the parameter space, we find that two cases arise. In the first, the parameter is a random variable whose behavior is governed by a probability density. In the second, the parameter is an unknown quantity but not a random variable. These two cases are analogous to the

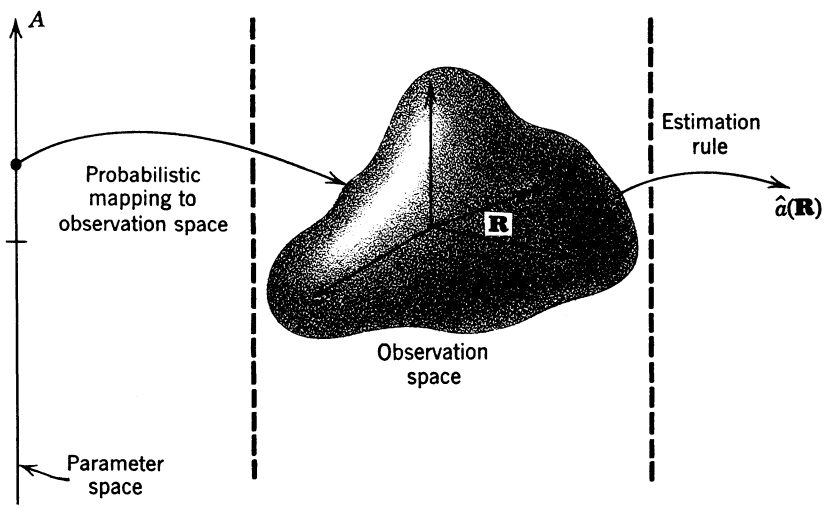

Fig. 2.17 Estimation model.

source models we encountered in the hypothesis-testing problem. To correspond with each of these models of the parameter space, we shall develop suitable estimation rules. We start with the random parameter case.

2.4.1 Random Parameters: Bayes Estimation

In the Bayes detection problem we saw that the two quantities we had to specify were the set of costs C_{ij} and the a priori probabilities P_i. The cost matrix assigned a cost to each possible course of action. Because there were M hypotheses and M possible decisions, there were M^2 costs. In the estimation problem a and $\hat{a}(\mathbf{R})$ are continuous variables. Thus we must assign a cost to all pairs $[a, \hat{a}(\mathbf{R})]$ over the range of interest. This is a function of two variables which we denote as $C(a, \hat{a})$. In many cases of interest it is realistic to assume that the cost depends only on the error of the estimate. We define this error as

$$a_\epsilon(\mathbf{R}) \triangleq \hat{a}(\mathbf{R}) - a. \tag{118}$$

The cost function $C(a_\epsilon)$ is a function of a single variable. Some typical cost functions are shown in Fig. 2.18. In Fig. 2.18*a* the cost function is simply the square of the error:

$$C(a_\epsilon) = a_\epsilon^2. \tag{119}$$

This cost is commonly referred to as the squared error cost function. We see that it accentuates the effects of large errors. In Fig. 2.18*b* the cost function is the absolute value of the error:

$$C(a_\epsilon) = |a_\epsilon|. \tag{120}$$

In Fig. 2.18*c* we assign zero cost to all errors less than $\pm\Delta/2$. In other words, an error less than $\Delta/2$ in magnitude is as good as no error. If $a_\epsilon > \Delta/2$, we assign a uniform value:

$$\begin{aligned} C(a_\epsilon) &= 0, \qquad |a_\epsilon| \le \frac{\Delta}{2}, \\ &= 1, \qquad |a_\epsilon| > \frac{\Delta}{2}. \end{aligned} \tag{121}$$

In a given problem we choose a cost function to accomplish two objectives. First, we should like the cost function to measure user satisfaction adequately. Frequently it is difficult to assign an analytic measure to what basically may be a subjective quality.

Our goal is to find an estimate that minimizes the expected value of the cost. Thus our second objective in choosing a cost function is to assign one that results in a tractable problem. In practice, cost functions are usually some compromise between these two objectives. Fortunately, in many

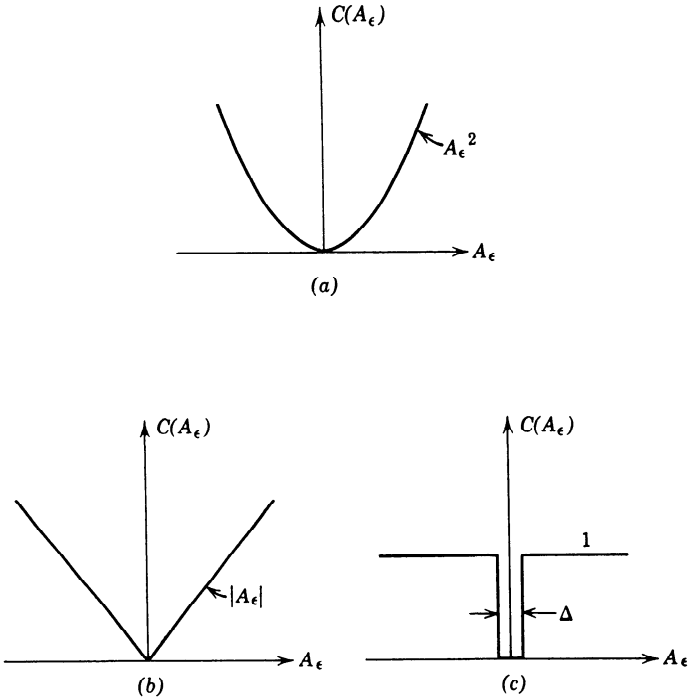

Fig. 2.18 Typical cost functions: (*a*) mean-square error; (*b*) absolute error; (*c*) uniform cost function.

problems of interest the same estimate will be optimum for a large class of cost functions.

Corresponding to the a priori probabilities in the detection problem, we have an a priori probability density $p_a(A)$ in the random parameter estimation problem. In all of our discussions we assume that $p_a(A)$ is known. If $p_a(A)$ is not known, a procedure analogous to the minimax test may be used.

Once we have specified the cost function and the a priori probability, we may write an expression for the risk:

$$\mathcal{R} \triangleq E\{C[a, \hat{a}(\mathbf{R})]\} = \int_{-\infty}^{\infty} dA \int_{-\infty}^{\infty} C[A, \hat{a}(\mathbf{R})] p_{a,\mathbf{r}}(A, \mathbf{R})\, d\mathbf{R}. \quad (122)$$

The expectation is over the random variable a and the observed variables $\mathbf{r}$. For costs that are functions of one variable only (122) becomes

$$\mathcal{R} = \int_{-\infty}^{\infty} dA \int_{-\infty}^{\infty} C[A - \hat{a}(\mathbf{R})] p_{a,\mathbf{r}}(A, \mathbf{R})\, d\mathbf{R}. \quad (123)$$

The Bayes estimate is the estimate that minimizes the risk. It is straightforward to find the Bayes estimates for the cost functions in Fig. 2.18. For the cost function in Fig. 2.18*a*, the risk corresponds to mean-square error. We denote the risk for the mean-square error criterion as $\mathcal{R}_{\text{ms}}$. Substituting (119) into (123), we have

$$\mathcal{R}_{\text{ms}} = \int_{-\infty}^{\infty} dA \int_{-\infty}^{\infty} d\mathbf{R}[A - \hat{a}(\mathbf{R})]^2 p_{a,\mathbf{r}}(A, \mathbf{R}). \tag{124}$$

The joint density can be rewritten as

$$p_{a,\mathbf{r}}(A, \mathbf{R}) = p_{\mathbf{r}}(\mathbf{R}) p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{125}$$

Using (125) in (124), we have

$$\mathcal{R}_{\text{ms}} = \int_{-\infty}^{\infty} d\mathbf{R}\, p_{\mathbf{r}}(\mathbf{R}) \int_{-\infty}^{\infty} dA[A - \hat{a}(\mathbf{R})]^2 p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{126}$$

Now the inner integral and $p_{\mathbf{r}}(\mathbf{R})$ are non-negative. Therefore we can minimize $\mathcal{R}_{\text{ms}}$ by minimizing the inner integral. We denote this estimate $\hat{a}_{\text{ms}}(\mathbf{R})$. To find it we differentiate the inner integral with respect to $\hat{a}(\mathbf{R})$ and set the result equal to zero:

$$\begin{aligned} &\frac{d}{d\hat{a}} \int_{-\infty}^{\infty} dA[A - \hat{a}(\mathbf{R})]^2 p_{a|\mathbf{r}}(A|\mathbf{R}) \\ &\qquad = -2 \int_{-\infty}^{\infty} A p_{a|\mathbf{r}}(A|\mathbf{R})\, dA + 2\hat{a}(\mathbf{R}) \int_{-\infty}^{\infty} p_{a|\mathbf{r}}(A|\mathbf{R})\, dA. \end{aligned} \tag{127}$$

Setting the result equal to zero and observing that the second integral equals 1, we have

$$\hat{a}_{\text{ms}}(\mathbf{R}) = \int_{-\infty}^{\infty} dA\, A p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{128}$$

This is a unique minimum, for the second derivative equals two. The term on the right side of (128) is familiar as the mean of the a posteriori density (or the conditional mean).

Looking at (126), we see that if $\hat{a}(\mathbf{R})$ is the conditional mean the inner integral is just the a posteriori variance (or the conditional variance). Therefore the minimum value of $\mathcal{R}_{\text{ms}}$ is just the average of the conditional variance over all observations $\mathbf{R}$.

To find the Bayes estimate for the absolute value criterion in Fig. 2.18*b* we write

$$\mathcal{R}_{\text{abs}} = \int_{-\infty}^{\infty} d\mathbf{R}\, p_{\mathbf{r}}(\mathbf{R}) \int_{-\infty}^{\infty} dA[|A - \hat{a}(\mathbf{R})|] p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{129}$$

To minimize the inner integral we write

$$I(\mathbf{R}) = \int_{-\infty}^{\hat{a}(\mathbf{R})} dA[\hat{a}(\mathbf{R}) - A]\, p_{a|\mathbf{r}}(A|\mathbf{R}) + \int_{\hat{a}(\mathbf{R})}^{\infty} dA[A - \hat{a}(\mathbf{R})]\, p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{130}$$

Differentiating with respect to $\hat{a}(\mathbf{R})$ and setting the result equal to zero, we have

$$\int_{-\infty}^{\hat{a}_{\mathrm{abs}}(\mathbf{R})} dA\, p_{a|\mathbf{r}}(A|\mathbf{R}) = \int_{\hat{a}_{\mathrm{abs}}(\mathbf{R})}^{\infty} dA\, p_{a|\mathbf{r}}(A|\mathbf{R}). \tag{131}$$

This is just the definition of the median of the a posteriori density.

The third criterion is the uniform cost function in Fig. 2.18*c*. The risk expression follows easily:

$$\mathcal{R}_{\mathrm{unf}} = \int_{-\infty}^{\infty} d\mathbf{R}\, p_{\mathbf{r}}(\mathbf{R})\left[1 - \int_{\hat{a}_{\mathrm{unf}}(\mathbf{R})-\Delta/2}^{\hat{a}_{\mathrm{unf}}(\mathbf{R})+\Delta/2} p_{a|\mathbf{r}}(A|\mathbf{R})\, dA\right]. \tag{132}$$

To minimize this equation we maximize the inner integral. Of particular interest to us is the case in which Δ is an arbitrarily small but nonzero number. A typical a posteriori density is shown in Fig. 2.19. We see that for small Δ the best choice for $\hat{a}(\mathbf{R})$ is the value of A at which the a posteriori density has its maximum. We denote the estimate for this special case as $\hat{a}_{\mathrm{map}}(\mathbf{R})$, the *maximum a posteriori* estimate. In the sequel we use $\hat{a}_{\mathrm{map}}(\mathbf{R})$ without further reference to the uniform cost function.

To find $\hat{a}_{\mathrm{map}}$ we must have the location of the maximum of $p_{a|\mathbf{r}}(A|\mathbf{R})$. Because the logarithm is a monotone function, we can find the location of the maximum of $\ln p_{a|\mathbf{r}}(A|\mathbf{R})$ equally well. As we saw in the detection problem, this is frequently more convenient.

If the maximum is interior to the allowable range of A and $\ln p_{a|\mathbf{r}}(A|\mathbf{R})$ has a continuous first derivative then a necessary, but not sufficient, condition for a maximum can be obtained by differentiating $\ln p_{a|\mathbf{r}}(A|\mathbf{R})$ with respect to A and setting the result equal to zero:

$$\left.\frac{\partial \ln p_{a|\mathbf{r}}(A|\mathbf{R})}{\partial A}\right|_{A=\hat{a}(\mathbf{R})} = 0. \tag{133}$$

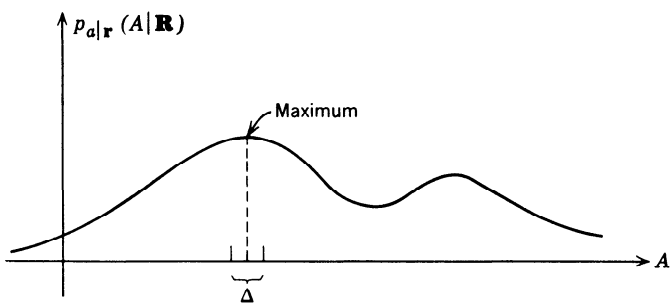

Fig. 2.19 An a posteriori density.

We refer to (133) as the MAP equation. In each case we must check to see if the solution is the absolute maximum.

We may rewrite the expression for $p_{a|\mathbf{r}}(A|\mathbf{R})$ to separate the role of the observed vector $\mathbf{R}$ and the a priori knowledge:

$$p_{a|\mathbf{r}}(A|\mathbf{R}) = \frac{p_{\mathbf{r}|a}(\mathbf{R}|A)p_a(A)}{p_{\mathbf{r}}(\mathbf{R})}. \tag{134}$$

Taking logarithms,

$$\ln p_{a|\mathbf{r}}(A|\mathbf{R}) = \ln p_{\mathbf{r}|a}(\mathbf{R}|A) + \ln p_a(A) - \ln p_{\mathbf{r}}(\mathbf{R}). \tag{135}$$

For MAP estimation we are interested only in finding the value of A where the left-hand side is maximum. Because the last term on the right-hand side is not a function of A, we can consider just the function

$$l(A) \triangleq \ln p_{\mathbf{r}|a}(\mathbf{R}|A) + \ln p_a(A). \tag{136}$$

The first term gives the probabilistic dependence of $\mathbf{R}$ on A and the second describes a priori knowledge.

The MAP equation can be written as

$$\left.\frac{\partial l(A)}{\partial A}\right|_{A=\hat{a}(\mathbf{R})} = \left.\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right|_{A=\hat{a}(\mathbf{R})} + \left.\frac{\partial \ln p_a(A)}{\partial A}\right|_{A=\hat{a}(\mathbf{R})} = 0. \tag{137}$$

Our discussion in the remainder of the book emphasizes minimum mean-square error and maximum a posteriori estimates.

To study the implications of these two estimation procedures we consider several examples.

Example 2. Let

$$r_i = a + n_i, \qquad i = 1, 2, \ldots, N. \tag{138}$$

We assume that a is Gaussian, $N(0, \sigma_a)$, and that the n_i are each independent Gaussian variables $N(0, \sigma_n)$. Then

$$\begin{aligned} p_{\mathbf{r}|a}(\mathbf{R}|A) &= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left(-\frac{(R_i - A)^2}{2\sigma_n^2}\right), \\ p_a(A) &= \frac{1}{\sqrt{2\pi}\,\sigma_a} \exp\left(-\frac{A^2}{2\sigma_a^2}\right). \end{aligned} \tag{139}$$

To find $\hat{a}_{\mathrm{ms}}(\mathbf{R})$ we need to know $p_{a|\mathbf{r}}(A|\mathbf{R})$. One approach is to find $p_{\mathbf{r}}(\mathbf{R})$ and substitute it into (134), but this procedure is algebraically tedious. It is easier to observe that $p_{a|\mathbf{r}}(A|\mathbf{R})$ is a probability density with respect to a for any $\mathbf{R}$. Thus $p_{\mathbf{r}}(\mathbf{R})$ just contributes to the constant needed to make

$$\int_{-\infty}^{\infty} p_{a|\mathbf{r}}(A|\mathbf{R})\, dA = 1. \tag{140}$$

(In other words, $p_{\mathbf{r}}(\mathbf{R})$ is simply a normalization constant.) Thus

$$p_{a|\mathbf{r}}(A|\mathbf{R}) = \left[\frac{\left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_n}\right)\frac{1}{\sqrt{2\pi}\,\sigma_a}}{p_{\mathbf{r}}(\mathbf{R})}\right] \exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^{N}(R_i - A)^2}{\sigma_n^2} + \frac{A^2}{\sigma_a^2}\right]\right\}. \tag{141}$$

Rearranging the exponent, completing the square, and absorbing terms depending only on R_i^2 into the constant, we have

$$p_{a|\mathbf{r}}(A|\mathbf{R}) = k(\mathbf{R}) \exp\left\{-\frac{1}{2\sigma_p^2}\left[A - \frac{\sigma_a^2}{\sigma_a^2 + \sigma_n^2/N}\left(\frac{1}{N}\sum_{i=1}^{N} R_i\right)\right]^2\right\}, \tag{142}$$

where

$$\sigma_p^2 \triangleq \left(\frac{1}{\sigma_a^2} + \frac{N}{\sigma_n^2}\right)^{-1} = \frac{\sigma_a^2\sigma_n^2}{N\sigma_a^2 + \sigma_n^2} \tag{143}$$

is the a posteriori variance.

We see that $p_{a|\mathbf{r}}(A|\mathbf{R})$ is just a Gaussian density. The estimate $\hat{a}_{\mathrm{ms}}(\mathbf{R})$ is just the conditional mean

$$\hat{a}_{\mathrm{ms}}(\mathbf{R}) = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_n^2/N}\left(\frac{1}{N}\sum_{i=1}^{N} R_i\right). \tag{144}$$

Because the a posteriori variance is not a function of $\mathbf{R}$, the mean-square risk equals the a posteriori variance (see (126)).

Two observations are useful:

1. The R_i enter into the a posteriori density only through their sum. Thus

$$l(\mathbf{R}) = \sum_{i=1}^{N} R_i \tag{145}$$

is a *sufficient statistic*. This idea of a sufficient statistic is identical to that in the detection problem.

2. The estimation rule uses the information available in an intuitively logical manner. If $\sigma_a^2 \ll \sigma_n^2/N$, the a priori knowledge is much better than the observed data and the estimate is very close to the a priori mean. (In this case, the a priori mean is zero.) On the other hand, if $\sigma_a^2 \gg \sigma_n^2/N$, the a priori knowledge is of little value and the estimate uses primarily the received data. In the limit $\hat{a}_{\mathrm{ms}}$ is just the arithmetic average of the R_i.

$$\lim_{\frac{\sigma_n^2}{N\sigma_a^2} \to 0} \hat{a}_{\mathrm{ms}}(\mathbf{R}) = \frac{1}{N}\sum_{i=1}^{N} R_i. \tag{146}$$

The MAP estimate for this case follows easily. Looking at (142), we see that because the density is Gaussian the maximum value of $p_{a|\mathbf{r}}(A|\mathbf{R})$ occurs at the conditional mean. Thus

$$\hat{a}_{\mathrm{map}}(\mathbf{R}) = \hat{a}_{ms}(\mathbf{R}). \tag{147}$$

Because the conditional median of a Gaussian density occurs at the conditional mean, we also have

$$\hat{a}_{\mathrm{abs}}(\mathbf{R}) = \hat{a}_{ms}(\mathbf{R}). \tag{148}$$

Thus we see that for this particular example all three cost functions in Fig. 2.18 lead to the same estimate. This invariance to the choice of a cost function is obviously a useful feature because of the subjective judgments that are frequently involved in choosing $C(a_\epsilon)$. Some conditions under which this invariance holds are developed in the next two properties.†

† These properties are due to Sherman [20]. Our derivation is similar to that given by Viterbi [36].

Property 1. We assume that the cost function $C(a_\epsilon)$ is a symmetric, convex-upward function and that the a posteriori density $p_{a|\mathbf{r}}(A|\mathbf{R})$ is symmetric about its conditional mean; that is,

$$C(a_\epsilon) = C(-a_\epsilon) \qquad \text{(symmetry)}, \quad (149)$$

$$C(bx_1 + (1-b)x_2) \leq bC(x_1) + (1-b)\,C(x_2) \qquad \text{(convexity)} \quad (150)$$

for any b inside the range (0, 1) and for all x_1 and x_2. Equation 150 simply says that all chords lie above or on the cost function.

This condition is shown in Fig. 2.20*a*. If the inequality is strict whenever $x_1 \neq x_2$, we say the cost function is strictly convex (upward). Defining

$$z \triangleq a - \hat{a}_{\text{ms}} = a - E[a|\mathbf{R}] \quad (151)$$

the symmetry of the a posteriori density implies

$$p_{z|\mathbf{r}}(Z|\mathbf{R}) = p_{z|\mathbf{r}}(-Z|\mathbf{R}). \quad (152)$$

The estimate $\hat{a}$ that minimizes any cost function in this class is identical to $\hat{a}_{\text{ms}}$ (which is the conditional mean).

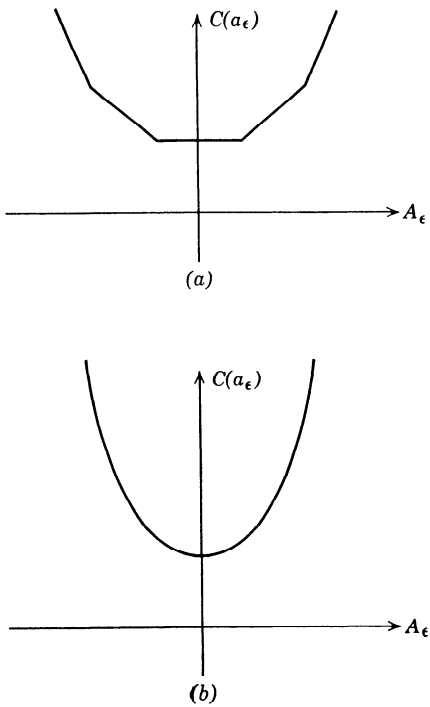

Fig. 2.20 Symmetric convex cost functions: (*a*) convex; (*b*) strictly convex.

Proof. As before we can minimize the conditional risk [see (126)]. Define

$$\mathcal{R}_B(\hat{a}|\mathbf{R}) \triangleq E_a[C(\hat{a} - a)|\mathbf{R}] = E_a[C(a - \hat{a})|\mathbf{R}], \tag{153}$$

where the second equality follows from (149). We now write four equivalent expressions for $\mathcal{R}_B(\hat{a}|\mathbf{R})$:

$$\mathcal{R}_B(\hat{a}|\mathbf{R}) = \int_{-\infty}^{\infty} C(\hat{a} - \hat{a}_{\text{ms}} - Z)p_{z|\mathbf{r}}(Z|\mathbf{R})\, dZ \tag{154}$$

[Use (151) in (153)]

$$= \int_{-\infty}^{\infty} C(\hat{a} - \hat{a}_{\text{ms}} + Z)p_{z|\mathbf{r}}(Z|\mathbf{R})\, dZ \tag{155}$$

[(152) implies this equality]

$$= \int_{-\infty}^{\infty} C(\hat{a}_{\text{ms}} - \hat{a} - Z)p_{z|\mathbf{r}}(Z|\mathbf{R})\, dZ \tag{156}$$

[(149) implies this equality]

$$= \int_{-\infty}^{\infty} C(\hat{a}_{\text{ms}} - \hat{a} + Z)p_{z|\mathbf{r}}(Z|\mathbf{R})\, dZ \tag{157}$$

[(152) implies this equality].

We now use the convexity condition (150) with the terms in (155) and (157):

$$\begin{aligned}\mathcal{R}_B(\hat{a}|\mathbf{R}) &= \tfrac{1}{2}E(\{C[Z + (\hat{a}_{\text{ms}} - \hat{a})] + C[Z - (\hat{a}_{\text{ms}} - \hat{a})]\}|\mathbf{R}) \\ &\geq E\{C[\tfrac{1}{2}(Z + (\hat{a}_{\text{ms}} - \hat{a})) + \tfrac{1}{2}(Z - (\hat{a}_{\text{ms}} - \hat{a}))]|\mathbf{R}\} \\ &= E[C(Z)|\mathbf{R}].\end{aligned} \tag{158}$$

Equality will be achieved in (158) if $\hat{a}_{\text{ms}} = \hat{a}$. This completes the proof. If $C(a_\epsilon)$ is strictly convex, we will have the additional result that the minimizing estimate $\hat{a}$ is unique and equals $\hat{a}_{\text{ms}}$.

To include cost functions like the uniform cost functions which are not convex we need a second property.

Property 2. We assume that the cost function is a symmetric, nondecreasing function and that the a posteriori density $p_{a|\mathbf{r}}(A|\mathbf{R})$ is a symmetric (about the conditional mean), unimodal function that satisfies the condition

$$\lim_{x \to \infty} C(x)p_{a|\mathbf{r}}(x|\mathbf{R}) = 0.$$

The estimate $\hat{a}$ that minimizes any cost function in this class is identical to $\hat{a}_{\text{ms}}$. The proof of this property is similar to the above proof [36].

The significance of these two properties should not be underemphasized. Throughout the book we consider only minimum mean-square and maximum a posteriori probability estimators. Properties 1 and 2 ensure that whenever the a posteriori densities satisfy the assumptions given above the estimates that we obtain will be optimum for a large class of cost functions. Clearly, if the a posteriori density is Gaussian, it will satisfy the above assumptions.

We now consider two examples of a different type.

Example 3. The variable a appears in the signal in a nonlinear manner. We denote this dependence by $s(A)$. Each observation r_i consists of $s(A)$ plus a Gaussian random variable n_i, $N(0, \sigma_n)$. The n_i are statistically independent of each other and a. Thus

$$r_i = s(A) + n_i. \tag{159}$$

Therefore

$$p_{a|\mathbf{r}}(A|\mathbf{R}) = k(\mathbf{R}) \exp\left(-\frac{1}{2}\left\{\frac{\sum_{i=1}^{N} [R_i - s(A)]^2}{\sigma_n^2} + \frac{A^2}{\sigma_a^2}\right\}\right). \tag{160}$$

This expression cannot be further simplified without specifying $s(A)$ explicitly.

The MAP equation is obtained by substituting (160) into (137)

$$\hat{a}_{\mathrm{map}}(\mathbf{R}) = \frac{\sigma_a^2}{\sigma_n^2} \sum_{i=1}^{N} [R_i - s(A)] \frac{\partial s(A)}{\partial A}\bigg|_{A = \hat{a}_{\mathrm{map}}(\mathbf{R})}. \tag{161}$$

To solve this explicitly we must specify $s(A)$. We shall find that an analytic solution is generally not possible when $s(A)$ is a nonlinear function of A.

Another type of problem that frequently arises is the estimation of a parameter in a probability density.

Example 4. The number of events in an experiment obey a Poisson law with mean value a. Thus

$$\Pr(n \text{ events} \mid a = A) = \frac{A^n}{n!} \exp(-A), \qquad n = 0, 1, \ldots. \tag{162}$$

We want to observe the number of events and estimate the parameter a of the Poisson law. We shall assume that a is a random variable with an exponential density

$$p_a(A) = \begin{cases} \lambda \exp(-\lambda A), & A > 0, \\ 0, & \text{elsewhere.} \end{cases} \tag{163}$$

The a posteriori density of a is

$$p_{a|n}(A|N) = \frac{\Pr(n = N \mid a = A) p_a(A)}{\Pr(n = N)}. \tag{164}$$

Substituting (162) and (163) into (164), we have

$$p_{a|n}(A|N) = k(N)[A^N \exp(-A(1 + \lambda))], \qquad A \geq 0, \tag{165}$$

where

$$k(N) = \frac{(1 + \lambda)^{N+1}}{N!} \tag{166}$$

in order for the density to integrate to 1. (As already pointed out, the constant is unimportant for MAP estimation but is needed if we find the MS estimate by integrating over the conditional density.)

The mean-square estimate is the conditional mean:

$$\hat{a}_{\mathrm{ms}}(N) = \frac{(1+\lambda)^{N+1}}{N!}\int_0^\infty A^{N+1}\exp[-A(1+\lambda)]\,dA$$

$$= \frac{(1+\lambda)^{N+1}}{(1+\lambda)^{N+2}}(N+1) = \left(\frac{1}{\lambda+1}\right)(N+1). \tag{167}$$

To find $\hat{a}_{\mathrm{map}}$ we take the logarithm of (165)

$$\ln p_{a|n}(A|N) = N\ln A - A(1+\lambda) + \ln k(N). \tag{168}$$

By differentiating with respect to A, setting the result equal to zero, and solving, we obtain

$$\hat{a}_{\mathrm{map}}(N) = \frac{N}{1+\lambda}. \tag{169}$$

Observe that $\hat{a}_{\mathrm{map}}$ is not equal to $\hat{a}_{\mathrm{ms}}$.

Other examples are developed in the problems. The principal results of this section are the following:

1. The minimum mean-square error estimate (MMSE) is always the mean of the a posteriori density (the conditional mean).
2. The maximum a posteriori estimate (MAP) is the value of A at which the a posteriori density has its maximum.
3. For a large class of cost functions the optimum estimate is the conditional mean whenever the a posteriori density is a unimodal function which is symmetric about the conditional mean.

These results are the basis of most of our estimation work. As we study more complicated problems, the only difficulty we shall encounter is the actual evaluation of the conditional mean or maximum. In many cases of interest the MAP and MMSE estimates will turn out to be equal.

We now turn to the second class of estimation problems described in the introduction.

2.4.2 Real (Nonrandom) Parameter Estimation†

In many cases it is unrealistic to treat the unknown parameter as a random variable. The problem formulation on pp. 52–53 is still appropriate. Now, however, the parameter is assumed to be nonrandom, and we want to design an estimation procedure that is good in some sense.

† The beginnings of classical estimation theory can be attributed to Fisher [5, 6, 7, 8]. Many discussions of the basic ideas are now available (e.g., Cramer [9]), Wilks [10], or Kendall and Stuart [11]).

A logical first approach is to try to modify the Bayes procedure in the last section to eliminate the average over $p_a(A)$. As an example, consider a mean-square error criterion,

$$\mathcal{R}(A) \triangleq \int_{-\infty}^{\infty} [\hat{a}(\mathbf{R}) - A]^2 \, p_{\mathbf{r}|a}(\mathbf{R}|A) \, d\mathbf{R}, \tag{170}$$

where the expectation is only over $\mathbf{R}$, for it is the only random variable in the model. Minimizing $\mathcal{R}(A)$, we obtain

$$\hat{a}_{\mathrm{ms}}(\mathbf{R}) = A. \tag{171}$$

The answer is correct, but not of any value, for A is the unknown quantity that we are trying to find. Thus we see that this direct approach is not fruitful. A more useful method in the nonrandom parameter case is to examine other possible measures of quality of estimation procedures and then to see whether we can find estimates that are good in terms of these measures.

The first measure of quality to be considered is the expectation of the estimate

$$E[\hat{a}(\mathbf{R})] \triangleq \int_{-\infty}^{+\infty} \hat{a}(\mathbf{R}) \, p_{\mathbf{r}|a}(\mathbf{R}|A) \, d\mathbf{R}. \tag{172}$$

The possible values of the expectation can be grouped into three classes

1. If $E[\hat{a}(\mathbf{R})] = A$, for all values of A, we say that the estimate is *unbiased.* This statement means that the average value of the estimates equals the quantity we are trying to estimate.
2. If $E[\hat{a}(\mathbf{R})] = A + B$, where B is not a function of A, we say that the estimate has a *known bias.* We can always obtain an unbiased estimate by subtracting B from $\hat{a}(\mathbf{R})$.
3. If $E[\hat{a}(\mathbf{R})] = A + B(A)$, we say that the estimate has an *unknown bias.* Because the bias depends on the unknown parameter, we cannot simply subtract it out.

Clearly, even an unbiased estimate may give a bad result on a particular trial. A simple example is shown in Fig. 2.21. The probability density of the estimate is centered around A, but the variance of this density is large enough that big errors are probable.

A second measure of quality is the variance of estimation error:

$$\mathrm{Var}\,[\hat{a}(\mathbf{R}) - A] = E\{[\hat{a}(\mathbf{R}) - A]^2\} - B^2(A). \tag{173}$$

This provides a measure of the spread of the error. In general, we shall try to find unbiased estimates with small variances. There is no straightforward minimization procedure that will lead us to the minimum variance unbiased estimate. Therefore we are forced to try an estimation procedure to see how well it works.

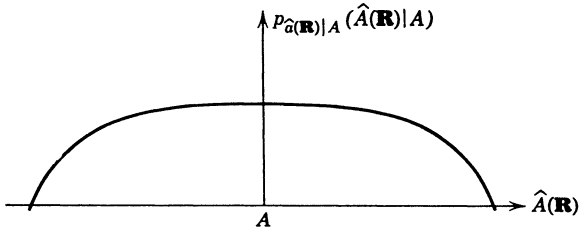

Fig. 2.21 Probability density for an estimate.

Maximum Likelihood Estimation. There are several ways to motivate the estimation procedure that we shall use. Consider the simple estimation problem outlined in Example 1. Recall that

$$r = A + n, \tag{174}$$

$$p_{r|a}(R|A) = (\sqrt{2\pi}\,\sigma_n)^{-1} \exp\left[-\frac{1}{2\sigma_n^2}(R - A)^2\right]. \tag{175}$$

We choose as our estimate the value of A that most likely caused a given value of R to occur. In this simple additive case we see that this is the same as choosing the most probable value of the noise ($N = 0$) and subtracting it from R. We denote the value obtained by using this procedure as a maximum likelihood estimate.

$$\hat{a}_{ml}(R) = R. \tag{176}$$

In the general case we denote the function $p_{\mathbf{r}|a}(\mathbf{R}|A)$, viewed as a function of A, as the *likelihood function.* Frequently we work with the logarithm, $\ln p_{\mathbf{r}|a}(\mathbf{R}|A)$, and denote it as the *log likelihood function.* The maximum likelihood estimate $\hat{a}_{ml}(\mathbf{R})$ is that value of A at which the likelihood function is a maximum. If the maximum is interior to the range of A, and $\ln p_{\mathbf{r}|a}(\mathbf{R}|A)$ has a continuous first derivative, then a necessary condition on $\hat{a}_{ml}(\mathbf{R})$ is obtained by differentiating $\ln p_{\mathbf{r}|a}(\mathbf{R}|A)$ with respect to A and setting the result equal to zero:

$$\left.\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right|_{A=\hat{a}_{ml}(\mathbf{R})} = 0. \tag{177}$$

This equation is called the *likelihood equation.* Comparing (137) and (177), we see that the ML estimate corresponds mathematically to the limiting case of a MAP estimate in which the a priori knowledge approaches zero.

In order to see how effective the ML procedure is we can compute the bias and the variance. Frequently this is difficult to do. Rather than approach the problem directly, we shall first derive a lower bound on the variance on *any* unbiased estimate. Then we shall see how the variance of $\hat{a}_{ml}(\mathbf{R})$ compares with this lower bound.

Cramér-Rao Inequality: Nonrandom Parameters. We now want to consider the variance of *any* estimate $\hat{a}(\mathbf{R})$ of the real variable A. We shall prove the following statement.

Theorem. (a) If $\hat{a}(\mathbf{R})$ is *any* unbiased estimate of A, then

$$\operatorname{Var}[\hat{a}(\mathbf{R}) - A] \geq \left(E\left\{\left[\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right]^2\right\}\right)^{-1} \tag{178}$$

or, equivalently,

(b)

$$\operatorname{Var}[\hat{a}(\mathbf{R}) - A] \geq \left\{-E\left[\frac{\partial^2 \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A^2}\right]\right\}^{-1}, \tag{179}$$

where the following conditions are assumed to be satisfied:

(c)

$$\frac{\partial p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} \quad \text{and} \quad \frac{\partial^2 p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A^2}$$

exist and are absolutely integrable.

The inequalities were first stated by Fisher [6] and proved by Dugué [31]. They were also derived by Cramér [9] and Rao [12] and are usually referred to as the Cramér-Rao bound. Any estimate that satisfies the bound with an equality is called an *efficient* estimate.

The proof is a simple application of the Schwarz inequality. Because $\hat{a}(\mathbf{R})$ is unbiased,

$$E[\hat{a}(\mathbf{R}) - A] \triangleq \int_{-\infty}^{\infty} p_{\mathbf{r}|a}(\mathbf{R}|A)[\hat{a}(\mathbf{R}) - A]\, d\mathbf{R} = 0. \tag{180}$$

Differentiating both sides with respect to A, we have

$$\frac{d}{dA}\int_{-\infty}^{\infty} p_{\mathbf{r}|a}(\mathbf{R}|A)[\hat{a}(\mathbf{R}) - A]\, d\mathbf{R} = \int_{-\infty}^{\infty} \frac{\partial}{\partial A}\{p_{\mathbf{r}|a}(\mathbf{R}|A)[\hat{a}(\mathbf{R}) - A]\}\, d\mathbf{R} = 0, \tag{181}$$

where condition (c) allows us to bring the differentiation inside the integral. Then

$$-\int_{-\infty}^{\infty} p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R} + \int_{-\infty}^{\infty} \frac{\partial p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}[\hat{a}(\mathbf{R}) - A]\, d\mathbf{R} = 0. \tag{182}$$

The first integral is just $+1$. Now observe that

$$\frac{\partial p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} = \frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} p_{\mathbf{r}|a}(\mathbf{R}|A). \tag{183}$$

Substituting (183) into (182), we have

$$\int_{-\infty}^{\infty} \frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} p_{\mathbf{r}|a}(\mathbf{R}|A)[\hat{a}(\mathbf{R}) - A]\, d\mathbf{R} = 1. \tag{184}$$

Rewriting, we have

$$\int_{-\infty}^{\infty} \left[\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} \sqrt{p_{\mathbf{r}|a}(\mathbf{R}|A)}\right]\left[\sqrt{p_{\mathbf{r}|a}(\mathbf{R}|A)}\,[\hat{a}(\mathbf{R}) - A]\right] d\mathbf{R} = 1, \tag{185}$$

and, using the Schwarz inequality, we have

$$\left\{\int_{-\infty}^{\infty} \left[\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right]^2 p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R}\right\} \times \left\{\int_{-\infty}^{\infty} [\hat{a}(\mathbf{R}) - A]^2 p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R}\right\} \geq 1, \tag{186}$$

where we recall from the derivation of the Schwarz inequality that equality holds if and only if

$$\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} = [\hat{a}(\mathbf{R}) - A]\, k(A), \tag{187}$$

for all $\mathbf{R}$ and A. We see that the two terms of the left side of (186) are the expectations in statement (*a*) of (178). Thus,

$$E\{[\hat{a}(\mathbf{R}) - A]^2\} \geq \left\{E\left[\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right]^2\right\}^{-1}. \tag{188}$$

To prove statement (b) we observe

$$\int_{-\infty}^{\infty} p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R} = 1. \tag{189}$$

Differentiating with respect to A, we have

$$\int_{-\infty}^{\infty} \frac{\partial p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\, d\mathbf{R} = \int_{-\infty}^{\infty} \frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A} p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R} = 0. \tag{190}$$

Differentiating again with respect to A and applying (183), we obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A^2} p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R} + \int_{-\infty}^{\infty} \left(\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right)^2 p_{\mathbf{r}|a}(\mathbf{R}|A)\, d\mathbf{R} = 0 \tag{191}$$

or

$$E\left[\frac{\partial^2 \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A^2}\right] = -E\left[\frac{\partial \ln p_{\mathbf{r}|a}(\mathbf{R}|A)}{\partial A}\right]^2, \tag{192}$$

which together with (188) gives condition (b).