

Fundamentals of metallurgy

Related titles:

New developments in advanced welding

(ISBN-13: 978-1-85573-970-3; ISBN-10: 1-85573-970-4)

Recent developments in high-technology areas have significantly transformed the welding industry. Automation, computers, process control, sophisticated scientific instruments and advanced processing methods are all common in today's modern industry. Today's engineers and technologists have to support complex systems and apply sophisticated welding technologies. This comprehensive new book discusses the changes in advanced welding technologies preparing the reader for the modern industry.

Welding residual stress and distortion

(ISBN-13: 978-1-85573-771-6; ISBN-10: 1-85573-771-X)

Measurement techniques for characterisation of residual stress and distortion have improved significantly. More importantly the development and application of computational welding mechanics have been phenomenal. Through the collaboration of experts, this book provides a comprehensive treatment of the subject. It develops sufficient theoretical treatments on heat transfer, solid mechanics and materials behaviour that are essential for understanding and determining welding residual stress and distortion. It will outline the approach for computational analysis that engineers with sufficient background can follow and apply. The book will be useful for advanced analysis of the subject and provide examples and practical solutions for welding engineers.

The science and technology of materials in automotive engines

(ISBN-13: 978-1-85573-742-6; ISBN-10: 1-85573-742-6)

This new book provides an introductory text on the science and technology of materials used in automotive engines. It focuses on reciprocating engines, both four and two-stroke, with particular emphasis on their characteristics and the types of materials used in their construction. The book considers the engine in terms of each specific part: the cylinder, piston, camshaft, valves, crankshaft, connecting rod and catalytic converter. The materials used in automotive engines are required to fulfill a multitude of functions. It is a subtle balance between material properties, essential design and high performance characteristics. The intention here is to describe the metallurgy, surface modification, wear resistance, and chemical composition of these materials. It also includes supplementary notes that support the core text. The book will be essential reading for engineers and designers of engines, as well as lecturers and graduate students in the fields of combustion engineering, machine design and materials science looking for a concise, expert analysis of automotive materials.

Details of these and other Woodhead Publishing materials books and journals, as well as materials books from Maney Publishing, can be obtained by:

- visiting our web site at www.woodheadpublishing.com
- contacting Customer Services (e-mail: sales@woodhead-publishing.com; fax: +44 (0) 1223 893694; tel.: +44 (0) 1223 891358 ext. 30; address: Woodhead Publishing Limited, Abington Hall, Abington, Cambridge CB1 6AH, England)

If you would like to receive information on forthcoming titles, please send your address details to: Francis Dodds (address, tel. and fax as above; email: francisdd@woodhead-publishing.com). Please confirm which subject areas you are interested in.

Maney currently publishes 16 peer-reviewed materials science and engineering journals. For further information visit www.maney.co.uk/journals.

Fundamentals of metallurgy

Edited by
Seshadri Seetharaman

**Woodhead Publishing and Maney Publishing
on behalf of
The Institute of Materials, Minerals & Mining**

**CRC Press
Boca Raton Boston New York Washington, DC**

WOODHEAD PUBLISHING LIMITED
Cambridge England

Woodhead Publishing Limited and Maney Publishing Limited on behalf of
The Institute of Materials, Minerals & Mining

Published by Woodhead Publishing Limited, Abington Hall, Abington,
Cambridge CB1 6AH, England
www.woodheadpublishing.com

Published in North America by CRC Press LLC, 6000 Broken Sound Parkway, NW,
Suite 300, Boca Raton, FL 33487, USA

First published 2005, Woodhead Publishing Limited and CRC Press LLC

© Woodhead Publishing Limited, 2005

The authors have asserted their moral rights.

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. Reasonable efforts have been made to publish reliable data and information, but the authors and the publishers cannot assume responsibility for the validity of all materials. Neither the authors nor the publishers, nor anyone else associated with this publication, shall be liable for any loss, damage or liability directly or indirectly caused or alleged to be caused by this book.

Neither this book nor any part may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, microfilming and recording, or by any information storage or retrieval system, without permission in writing from Woodhead Publishing Limited.

The consent of Woodhead Publishing Limited does not extend to copying for general distribution, for promotion, for creating new works, or for resale. Specific permission must be obtained in writing from Woodhead Publishing Limited for such copying.

Trademark notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation, without intent to infringe.

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

A catalog record for this book is available from the Library of Congress.

Woodhead Publishing Limited ISBN-13: 978-1-85573-927-7 (book)

Woodhead Publishing Limited ISBN-10: 1-85573-927-5 (book)

Woodhead Publishing Limited ISBN-13: 978-1-84569-094-6 (e-book)

Woodhead Publishing Limited ISBN-10: 1-84569-094-X (e-book)

CRC Press ISBN-10: 0-8493-3443-8

CRC Press order number: WP3443

The publishers' policy is to use permanent paper from mills that operate a sustainable forestry policy, and which has been manufactured from pulp which is processed using acid-free and elementary chlorine-free practices. Furthermore, the publishers ensure that the text paper and cover board used have met acceptable environmental accreditation standards.

Project managed by Macfarlane Production Services, Dunstable, Bedfordshire
(macfarl@aol.com)

Typeset by Godiva Publishing Services Ltd, Coventry, West Midlands

Printed by TJ International Limited, Padstow, Cornwall, England

5.1 Introduction

Extraction and refining of metals involve the transfer of mass and heat within each phase and between the phases, one or more of which are in motion. For example, in blast furnace iron making, Fig. 5.1, iron oxide in the form of lumpy ore, sinter or pellet along with coke and limestone, are charged on the top of the furnace continuously. The charge slowly moves down and hot reducing gases produced by burning of coke in front of tuyeres move up through the bed of solids and finally escape through the gas outlet. The heat is transferred from hot gases to solid and at the same time the reducing gases reduce the oxides to metallic iron. At high temperature reduced iron melts and forms liquid metal and gangue in ore, coke ash and lime form the slag. The final product is molten iron containing dissolved carbon and other impurities like silicon, manganese, phosphorous and sulfur and liquid slag, which are tapped from the furnace.

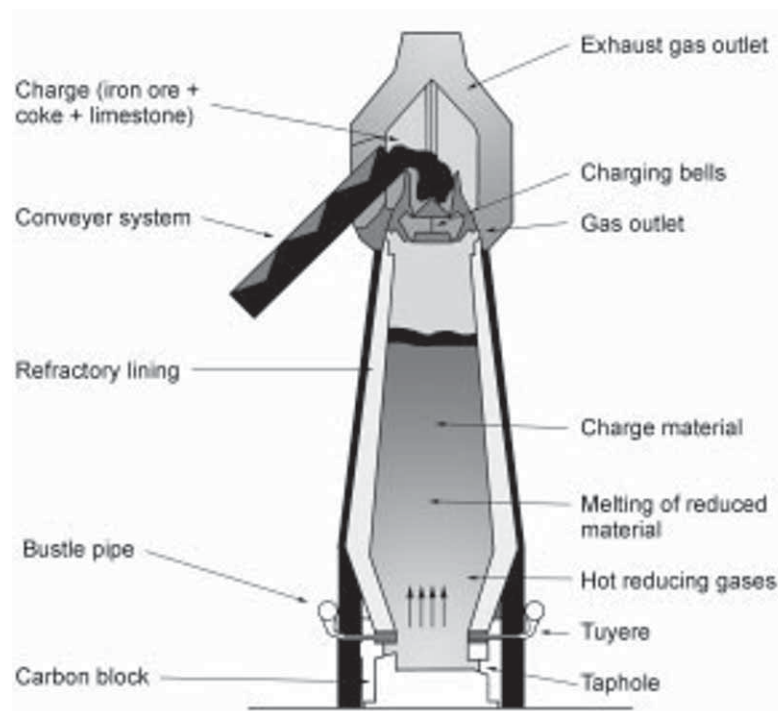
Other metallurgical operations like melting, alloying, casting, joining, heat treatment and metal deformation processes like forging, rolling, etc. also involve heat, mass and momentum transfer. So transport phenomena play a key role in all metallurgical operations.

The present chapter deals with the fundamentals of transport phenomena within a phase and between phases.

5.2 Mass transfer

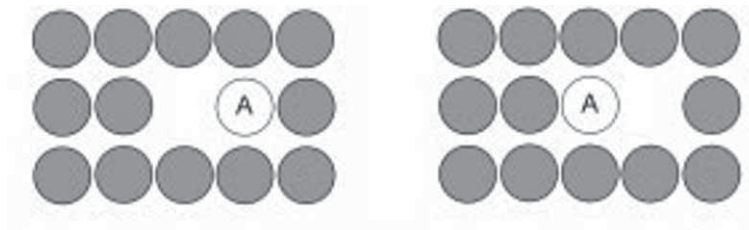
Transfer of a species from one location to another in a mixture of non-uniform composition is known as mass transfer. When a homogeneous solution flows, the species present in the solution moves from one location to another but there is no mass transfer since concentration of species is the same everywhere. So a concentration difference is an essential requirement for mass transfer.

Mass transfer takes place because of molecular motion and bulk flow. Atoms in fluids are in random motion and this causes transfer of a species from one place to another. However this could be noticed only when mixture is not



5.1 Schematic representation of blast furnace.

homogeneous or few radio active atoms are present in the system. The movement of atoms in fluid is easy because the positions of atoms are not rigid. But in solids, atoms move from one lattice site to another by exchanging their position with a neighbouring vacancy, Fig. 5.2. So the movement of atoms in solids is accompanied by movement of the vacancy in the opposite direction. Other mechanisms of movement of atoms are also possible but the vacancy mechanism is most predominant for substitutional solid solutions.



5.2 Diffusion in solid. Atom marked A jumps into vacancy and vacancy moves from left to right.

5.2.1 Mass flux

Figure 5.3(a) shows two non-reacting species A and B which are in contact with each other. The left side has species A and right side has B. Since the concentration of species A and B are not uniform throughout the system, atoms of A and B move in opposite directions. This leads to the decrease of concentration of A in the left side and corresponding increase of the concentration of A in the right side as shown in Fig. 5.3(b). The reverse is the case for concentration profile of B. The process continues till it attains the steady state when concentrations do not change with time. At steady state, the concentration of A becomes the same everywhere, Fig. 5.3(c). Let v_A and v_B be the velocities of A and B respectively. The flux of A, defined as the amount of A crossing the unit area per unit time in a direction normal to the area is

$$N_{Ax} = v_A \cdot C_A \quad (5.1)$$

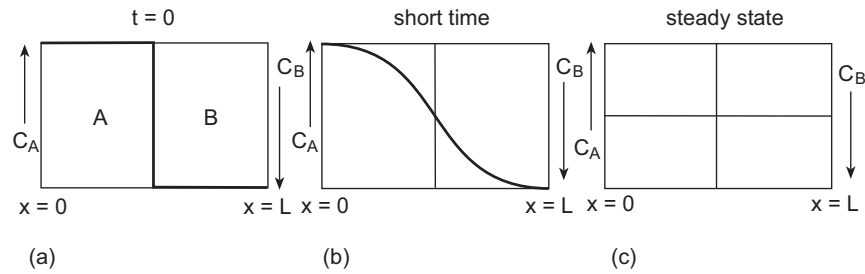
where N_{Ax} is the molar flux of A in x direction and C_A is the molar concentration of A. Equation 5.1 can be written as

$$N_{Ax} = C_A(v_A - v^{av}) + C_A \cdot v^{av} \quad (5.2)$$

where v^{av} is the average velocity and $(v_A - v^{av})$ is the velocity of A with respect to average velocity. So the first term on the right-hand side measures flux due to molecular motion with respect to average velocity and the second term is the flux due to bulk flow. The flux due to molecular motion with respect to average velocity, J_{Ax} , is given by Fick's law of diffusion as

$$J_{Ax} = C_A(v_A - v^{av}) = -D_{AB} \frac{dC_A}{dx} \quad (5.3)$$

where D_{AB} is the diffusivity of A in A–B. The negative sign in Fick's law shows that the flux is in the direction of decreasing concentration. Fick's law of diffusion gives the flux with respect to average velocity of all the species or with respect to a moving coordinate that moves with velocity v^{av} . Combining equations 5.2 and 5.3,



5.3 Schematic representation of diffusion of two species A and B. (a) Before diffusion. (b) Concentration profiles of A and B after a short time. (c) Steady state concentration profile.



5.4 Diffusion of dye in water flowing through a tube.

$$N_{Ax} = -D_{AB} \frac{dC_A}{dx} + C_A v^{av} \quad (5.4)$$

To understand the physical meaning of the diffusion and bulk flow term in equation 5.4, let us perform a simple experiment. We take a long horizontal glass tube and allow water to flow through it at a very low flow rate and inject some dye along the axis of the tube. If we follow the tip of the dye as it moves through the tube, we will find that a faintly coloured zone just ahead of the deep coloured streak, Fig. 5.4. The velocity of the tip of the streak is v^{av} and the spread of the colour beyond the tip is due to molecular diffusion.

By definition, the average velocity v^{av} is

$$v^{av} = (C_A \cdot v_A + C_B \cdot v_B) / (C_A + C_B) \quad (5.5)$$

Using equation 5.1, equation 5.5 becomes

$$v^{av} = (N_{Ax} + N_{Bx}) / (C_A + C_B) \quad (5.6)$$

Combining equations 5.4 and 5.6

$$N_{Ax} = -D_{AB} \frac{dC_A}{dx} + x_A (N_{Ax} + N_{Bx}) \quad (5.7)$$

where x_A is the mole fraction of A. Definition of average velocity v^{av} , indicates that bulk flow term can be due to molecular motion or external force or both. In solids, and stagnant fluid, there is no bulk flow due to external forces, but the diffusion of atoms and molecules itself lead to bulk flow. In these cases equation 5.7 is the appropriate form of the flux equation. When the bulk flow is primarily due to external forces as in case of Fig. 5.4, the molecular contribution in the bulk flow can be neglected and equation 5.4 is the appropriate form of the equation. The average velocity v^{av} in this case is determined by external forces and is obtained by solving fluid flow problem. Equations 5.1–5.7 have been written in terms of molar fluxes, molar concentration, and mole fraction, but these are valid when the word mole is replaced by mass, i.e. mass flux, mass concentration and mass fraction.

In the three-dimensional case, the flux of A is given by

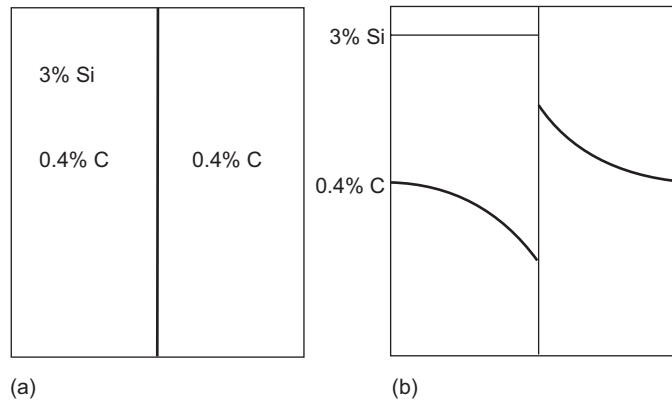
$$\mathbf{N}_A = -D_{AB} \nabla C_A + x_A (\mathbf{N}_A + \mathbf{N}_B) \quad (5.8)$$

Bold letters indicate vectors. The first term on the right-hand side of equation 5.8 is the pure diffusion flux $\mathbf{J}_A$ given by Fick's law and second term is the flux due to bulk flow. The components of the molar fluxes due to molecular motion, $\mathbf{J}_A$, in different coordinates are given in Table 5.1.

Table 5.1 Components of flux due to binary diffusion

Rectangular		Cylindrical		Spherical	
$J_x = -D_{AB} \frac{\partial C_A}{\partial x}$	(A)	$J_{Ar} = -D_{AB} \frac{\partial C_A}{\partial r}$	(D)	$J_{Ar} = -D_{AB} \frac{\partial C_A}{\partial r}$	(G)
$J_y = -D_{AB} \frac{\partial C_A}{\partial y}$	(B)	$J_{A\theta} = -\frac{D_{AB}}{r} \frac{\partial C_A}{\partial \theta}$	(E)	$J_{A\theta} = -\frac{D_{AB}}{r} \frac{\partial C_A}{\partial \theta}$	(H)
$J_z = -D_{AB} \frac{\partial C_A}{\partial z}$	(C)	$J_z = -D_{AB} \frac{\partial C_A}{\partial z}$	(F)	$J_{A\phi} = -\frac{D_{AB}}{r \sin \theta} \frac{\partial C_A}{\partial \phi}$	(I)

According to Fick's law driving force for diffusion is concentration gradient and diffusion takes place only from higher concentration to lower concentration. Darken (1949) found that when two steel bars containing the same carbon but one having silicon are welded together, carbon diffused from the bar containing silicon to the other side. Finally the bar that is free from silicon became richer in carbon than the other one. This is known as uphill diffusion. Figure 5.5 schematically shows the uphill diffusion. According to Fick's law, no diffusion of carbon should take place since carbon concentration in both the bars was initially the same. Darken's experiment clearly demonstrated that the activity gradient is the driving force for diffusion and not the concentration gradient. Silicon increases the activity of carbon and is a driving force for diffusion of carbon from the bar containing silicon to the other bar and the diffusion continues until the activity of the carbon becomes the same in both bars. However, for all usual analysis, Fick's law is used and the activity concept is invoked only to explain the anomaly.



5.5 Schematic diagram of uphill diffusion of carbon. (a) Before diffusion. (b) After diffusion.

Relation between fluxes

Equation 5.7 for N_{Ax} , contains two unknowns N_{Ax} and N_{Bx} , but in fluids a similar equation for N_{Bx} is not an independent equation (see Example 5.2, page 185). Therefore to obtain N_{Ax} , the relationship between the N_{Ax} and N_{Bx} must be known from other physical considerations or laws.

(1) Graham's Law

The ratio of diffusion fluxes in gases are inversely related to the square root of molecular weight or

$$N_{Ax}/N_{Bx} = (M_B/M_A)^{1/2} \quad (5.9)$$

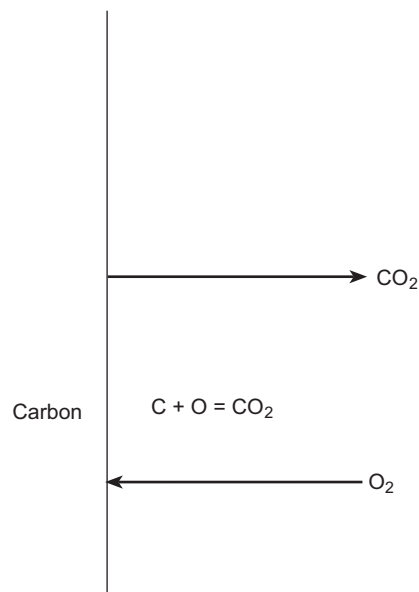
This equation applies to effusion of gases in a vacuum and equal pressure counter diffusion.

(2) Chemical reaction

Let us consider that oxygen is reacting with carbon according to the reaction



Figure 5.6 shows schematically the fluxes on the surface of the carbon block. Oxygen diffuses at the carbon–gas interface for reaction and carbon dioxide



5.6 C-O reaction on the surface of carbon.

formed diffuses away from the reaction interface. Since one molecule of oxygen forms one molecule of CO_2

$$N_{\text{O}_2,x} = -N_{\text{CO}_2,x} \quad (5.11)$$

The above condition indicates that per unit time the number of molecules of oxygen going in one direction is equal to number of molecules of CO_2 going in the opposite direction. This type of diffusion is termed as equimolar counter diffusion.

In the previous case one mole of reactant gas formed one mole of product gas. Let us consider, carbon reacts with oxygen to form CO



If one mole of O_2 comes to the surface of carbon and reacts with it, two moles of CO is produced and go out in the opposite direction. Or the flux of CO is two times that of O_2 and the relationship between the fluxes is

$$N_{\text{O}_2,x} = -\frac{1}{2}N_{\text{CO},x} \quad (5.13)$$

For a reaction of the following type



the fluxes are related by

$$N_{\text{Ax}} = -(n/m) N_{\text{Bx}} \quad (5.15)$$

(3) Flux of inert species

Let us consider that C- O_2 reaction, equations 5.10 or 5.12, takes place in air. Obviously nitrogen present in air will not take part in the reaction so the flux of nitrogen will be zero. In other words, flux of an inert species is always zero.

(4) Dilute solution

If the concentration of the diffusing species A is very small (x_A is very small), the bulk flow term can be neglected since its contribution is very small. In dilute liquid and solid solutions the bulk flow term is neglected.

Example 5.1

Show that for a binary mixture of gases diffusivity of A is same as that of B.

Solution

Let D_{AB} and D_{BA} be the diffusivities of A and B respectively in the mixture. Let us consider that A and B shown in Fig. 5.3 are two gases and the system is isothermal and isobaric. So $(C_A + C_B)$ is constant, or

$$\frac{\partial C_A}{\partial x} + \frac{\partial C_B}{\partial x} = 0 \quad (a)$$

From equation 5.7

$$N_{Ax} = -D_{AB} \frac{\partial C_A}{\partial x} + x_A(N_A + N_B) \quad (b)$$

$$N_{Bx} = -D_{BA} \frac{\partial C_B}{\partial x} + x_B(N_A + N_B) \quad (c)$$

Since $x_A + x_B = 1$, adding equations (b) and (c)

$$D_{AB} \frac{\partial C_A}{\partial x} + D_{BA} \frac{\partial C_B}{\partial x} = 0 \quad (d)$$

From equations (a) and (d) $D_{AB} = D_{BA}$ ■■■

In the case of diffusion in binary substitutional solid solution, there are three fluxes, that of A, B and vacancies so the above derivation is not applicable. Although concentration of vacancy is very small, the flux is not small. It can be shown that $J_{vx} + J_{Ax} + J_{Bx} = 0$.

Example 5.2

Show that the following two equations are not independent for diffusion in fluids.

$$N_{Ax} = -D_{AB} \frac{dC_A}{dx} + x_A(N_{Ax} + N_{Bx}) \quad (5.7)$$

$$N_{Ax} = -D_{AB} \frac{dC_B}{dx} + x_A(N_{Ax} + N_{Bx}) \quad (5.7a)$$

Solution

Let

$$C_A + C_B = C \quad (a)$$

where C is the total concentration, a constant. Substituting the value of C_A from equation (a) in equation 5.7

$$N_{Ax} = -D_{AB} \frac{d(C - C_B)}{dx} + \frac{C - C_B}{C_A + C_B} (N_{Ax} + N_{Bx}) \quad (b)$$

$$= D_{AB} \frac{dC_B}{dx} - \frac{C_B}{C_A + C_B} (N_{Ax} + N_{Bx}) + N_{Ax} + N_{Bx} \quad (c)$$

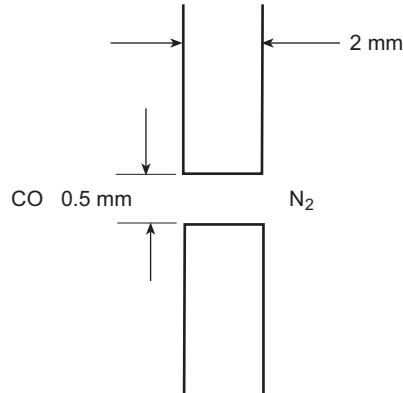
Rearranging

$$N_{Bx} = -D_{AB} \frac{dC_B}{dx} + x_B(N_{Ax} + N_{Bx}) \quad (5.7a)$$

Hence equations 5.7 and 5.7a are not independent equations. ■■■

Example 5.3

Carbon monoxide and nitrogen at 2 atm pressure and 300K are separated by a wall of 2 mm thickness, Fig. 5.7. The separating wall has a small hole of 0.5 mm. Calculate the maximum leakage rate of carbon monoxide and nitrogen. The profile of oxygen and nitrogen can be assumed to be linear in the hole. At 300K and 2 atm pressure $D_{\text{CO-N}_2} = 1.03 \times 10^{-5} \text{ m}^2/\text{s}$.



5.7 Leakage of CO through a hole.

Solution

The diffusion of N_2 and CO are in opposite directions so it is counter diffusion. Furthermore, the pressure and concentration of gases are the same and the flow of gas inside the hole is only due to molecular motion. So the ratio of fluxes can be calculated by Graham's law, equation 5.9. Since the molecular weights of both CO and N_2 are same, the ratio of molar fluxes is unity. Or $N_{\text{N}_2} = -N_{\text{CO}}$. So equation 5.7 reduces to

$$N_{\text{N}_2} = -D_{\text{CO-N}_2} \frac{dC_{\text{N}_2}}{dx} \quad (\text{a})$$

The concentration of N_2 in nitrogen side = $P/RT = 2/(0.082300) = 0.081 \text{ kmol.m}^{-3}$. Leakage rate is maximum when concentration of nitrogen on CO side is zero. Since concentration profile is linear, $dC_{\text{N}_2}/dx = (C_{\text{N}_2}|_x - C_{\text{N}_2}|_0)/x$. Taking the face on the nitrogen side as $x = 0$,

$$\begin{aligned} N_{\text{N}_2} &= -1.03 \times 10^{-5} [(0 - 0.081)/(2 \times 10^{-3})] \\ &= 4.17 \times 10^{-4} \text{ kmol.m}^{-2}\text{s}^{-1} \end{aligned}$$

So the leakage rate of nitrogen

$$\begin{aligned} &= N_{\text{N}_2} \times \text{area of the hole} = 4.17 \times 10^{-4} \times \pi \times (0.5 \times 10^{-3}/2)^2 \\ &= 8.19 \times 10^{-11} \text{ kmole.s}^{-1} \end{aligned}$$

The leakage rate of carbon monoxide will be the same. ■■■

5.2.2 Diffusivity of gas, liquid and solid

Evaluation of fluxes requires the knowledge of diffusivities. Over the years, considerable efforts have been made to calculate these values based on first principles. But the success has been rather limited. So the primary source of diffusivity is the measured values. However, in the absence of experimentally measured values, empirical and semi-empirical co-relations are used to calculate these.

Diffusivity of gases

According to the kinetic theory of gases, diffusivity of gases are proportional, $T^{1.5}/p$ where T is temperature and p is pressure. This pressure dependence is valid for less than 10 atmospheres but the exponents of temperature for real gases are higher. Chapman–Enskog equation gives a good estimation of binary diffusivity of gases. For details see Reid *et al.* (1977). Fuller, Schettler and Giddings (1966) proposed the following empirical relationship:

$$D_{AB} = \frac{10^{-7} T^{1.75} [1/M_A + 1/M_B]^{1/2}}{p[(V_A)^{1/3} + (V_B)^{1/3}]^2} \quad (5.16)$$

where D_{AB} is in m^2/s , temperature T is in K and p is pressure in atmospheres. V_A and V_B are diffusion volumes of A and B respectively. Diffusion volumes of simple molecules are given in Table 5.2. The above equation predicts diffusivity within 10% in most of the cases. Diffusivity of gas phases is usually in the range of 10^{-5} to $10^{-3} m^2/s$.

Diffusivity through porous solid

Porous solid consists of a number of interconnecting pores (Fig. 5.8). If we take unit area on the surface of the solid, the diffusion takes place only through the area occupied by the pores. Again, gas cannot move in a straight line in the porous solid. It takes a tortuous path. To take into account these two special features of porous solid, diffusivity of gas in porous solid is defined as effective diffusivity

Table 5.2 Diffusion volumes for simple molecules (Fuller *et al.* 1966)

Gas	V_A	Gas	V_A	Gas	V_A
H ₂	7.07	Ne	5.59	NH ₃	14.9
He	2.88	Ar	16.1	H ₂ O	12.7
N ₂	17.9	CO	18.9	Cl ₂	37.7
O ₂	16.6	CO ₂	26.9	Br ₂	67.2
Air	20.1	N ₂ O	35.9	SO ₂	41.1



5.8 Schematic diagram of a porous solid.

$$D_{\text{eff}} = \epsilon D_{\text{AB}} / \tau \quad (5.17)$$

where ϵ is porosity of solid and τ is the tortuosity factor. τ measures the ratio of actual path travelled by gas to geometrical diffusion distance.

When gases diffuse through pores it diffuses either by ordinary gas diffusion or Knudsen diffusion or a combination of both. When the pore diameter is much larger than the mean free path, diffusion is ordinary gas diffusion. But if the pore size is smaller than the mean free path, then gas molecules have a higher probability of collision with the pore wall than with each other. Gas flow under this condition is known as Knudsen diffusion. Knudsen diffusion coefficient for diffusion through a cylindrical pore is

$$D_K = 97 r (T/M)^{1/2} \text{ m}^2/\text{s} \quad (5.18)$$

where r is the radius of pore in m, T is the temperature in K and M is the molecular weight. In general, pores are not cylindrical so a correction of tortuosity is required for Knudsen diffusion. Since in porous solid, nature of diffusion depends on the size of pores, effective diffusivity of porous solid is often taken as

$$1/D'_{\text{eff}} = 1/D_{\text{AB}} + 1/D_K \quad (5.19)$$

$$D_{\text{eff}} = D'_{\text{eff}} \cdot \epsilon / \tau \quad (5.20)$$

Example 5.4

Find out the pore diameter where molecular and Knudsen diffusivity are comparable for diffusion of H₂-H₂O mixture through a porous solid at 1000K and 1 atm pressure.

Solution

We use equation 5.16, to calculate molecular diffusivity of H₂-H₂O. From Table 5.2, $V_{H_2} = 7.07$ and $V_{H_2O} = 12.7$ and molecular weights of H₂ and H₂O are 2 and 18 respectively. Substituting in equation 5.16

$$D_{H_2-H_2O} = \frac{10^{-7}(1000)^{1.75}[1/2 + 1/18]^{1/2}}{[7.07^{1/3} + 12.7^{1/3}]^2} = 7.3 \times 10^{-4} \text{ m}^2/\text{s} \quad (\text{a})$$

Equation 5.18 shows that Knudsen diffusivity of H₂ is greater than that of H₂O. So we should compare the molecular diffusivity of H₂-H₂O mixture with Knudsen diffusivity of H₂O. Knudsen diffusivity of H₂O is

$$D_{H_2O} = 97 r (1000/18)^{1/2} = 723r \text{ m}^2/\text{s} \quad (\text{b})$$

Both the diffusivities will be same if the pore radius r is

$$r + 7.3 \times 10^{-4}/723 = 1.01 \times 10^{-6} \text{ m}$$

So if pore radius is about $1 \mu\text{m}$, the contribution of both Knudsen and molecular diffusion will be almost same. If the pore radius is $10 \mu\text{m}$, equation 5.19 indicates that the contribution of Knudsen diffusion will be only about 10% of total diffusion. On the other hand, if pore radius is $0.1 \mu\text{m}$, the contribution of molecular diffusion will be only 10%. ■■■

Diffusion in liquids

The experimentally measured values of diffusivity are often expressed in the form of the Arrhenius equation

$$D = D_0 \exp(-Q/RT) \quad (5.21)$$

D_0 is the frequency factor, Q is the activation energy for diffusion and R is gas constant. The value of Q for metallic system is mostly less than 16 kJ mol^{-1} . Table 5.3 shows the diffusivity of different solutes in liquid iron at 1873K.

Table 5.3 Diffusivity of solutes in liquid iron at 1873K (Morita 1996)

Solute	C	Si	Mn	S	O	H
Diffusivity $\times 10^9 \text{ m}^2/\text{s}$	4–20	2.5–12	3.5–20	4.5–20	2.5–20	80–200
Solute	N	Ni	Cr	V	Mo	
Diffusivity $\times 10^9 \text{ m}^2/\text{s}$	6–20	4.5–5.6	3–5	4–5	3.8–4.1	

Because of difficulties in measurement at high temperature, uncertainty in measured values is often large. Note that diffusivities of different solutes except hydrogen in liquid iron are of the same order of magnitude irrespective of size of atoms. These values compare quite well with diffusivity of CO_2 in water, $1.9 \times 10^{-9} \text{ m}^2/\text{s}$ at 298K.

Diffusion in solids

In solid solutions we define both intrinsic and inter or mutual diffusivities. The intrinsic diffusivity of an element is the diffusivity of that element in the solution. In a mixture of gases, intrinsic diffusivities of A and B in A-B is denoted by D_{AB} and D_{BA} respectively, but for solids these are denoted by D_A and D_B respectively. In a gas mixture, $D_{AB} = D_{BA}$ (see Example 5.1 on pages 184–5) but in a solid (in general), $D_A \neq D_B$. Inter or mutual diffusivity in solid is the diffusivity of A in B or vice versa and is denoted by D_{AB} . For example, in Cu-Zn alloy, intrinsic diffusivities of Cu and Zn are denoted by D_{Cu} and D_{Zn} respectively and inter or mutual diffusivity by D_{CuZn} . Inter and intrinsic diffusivities are related by

$$D_{AB} = x_B D_A + x_A D_B \quad (5.22)$$

x_A and x_B are mole fractions of A and B in the solution.

Diffusivity in solids follow the Arrhenius equation 5.21 and activation energy of diffusion is quite large. Table 5.4 gives diffusivity of different solutes in iron. It shows that both activation energy for diffusion and diffusivity are strongly related to size of atoms.

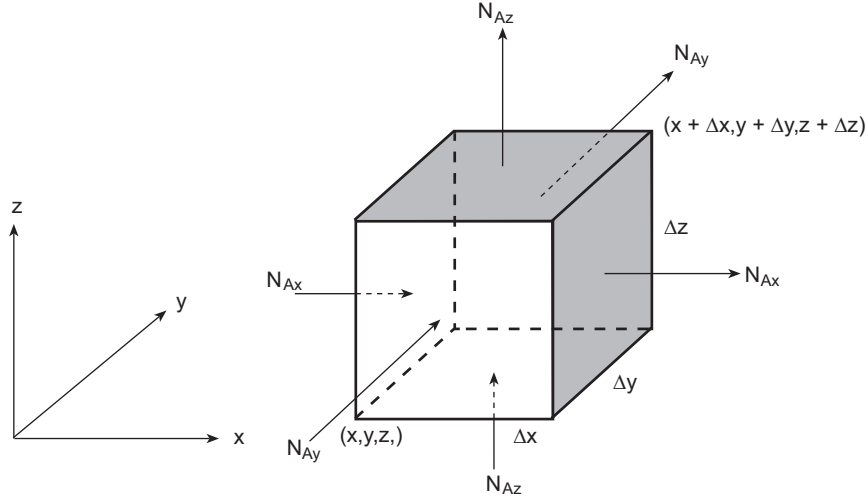
5.2.3 Conservation of mass

Let us consider an elemental volume $\Delta x \Delta y \Delta z$ as shown in Fig. 5.9. Since the mass is conserved, the balance equation is

$$\begin{aligned} \text{Rate of accumulation of species A in } \Delta x \Delta y \Delta z = \\ \text{Rate in} - \text{Rate out} + \text{Rate of generation of A} \\ \text{in } \Delta x \Delta y \Delta z \text{ volume by reaction} \end{aligned}$$

Table 5.4 Diffusivity of solutes in γ iron (Kucera and Stransky 1982)

Diffusing elements	$D_0 \times 10^7 \text{ m}^2.\text{s}^{-1}$	$Q \text{ kJ mol}^{-1}$	$D \text{ at } 1200\text{K } \text{m}^2.\text{s}^{-1}$
Hydrogen	8.1	43.2	1.07×10^{-8}
Boron	2.0	87.92	2.98×10^{-11}
Carbon	738	158.98	8.86×10^{-12}
Nitrogen	480	159.1	5.70×10^{-12}
Chromium	4080	286.8	1.34×10^{-16}
Nickel	1090	296.8	1.31×10^{-17}



5.9 Mass balance in a control volume.

In this case there are six faces. Mass flux enters through three faces and goes out through the opposite three faces. N_{Ax} enters at x through the face of area $\Delta y \Delta z$ and goes out through the face at $x + \Delta x$. Similarly, N_{Az} enters at z through the face of area $\Delta x \Delta y$ and goes out at $z + \Delta z$ and N_{Ay} enters at y through the face of area $\Delta x \Delta z$ and goes out at $y + \Delta y$. Hence mass balance equation can be written as

$$\begin{aligned} \frac{\partial(\Delta x \Delta y \Delta z C_A)}{\partial t} = & (\Delta y \Delta z N_{Ax} | _x - \Delta y \Delta z N_{Ax} | _{x+\Delta x}) \\ & + (\Delta x \Delta z N_{Ay} | _y - \Delta x \Delta z N_{Ay} | _{y+\Delta y}) \\ & + (\Delta x \Delta y N_{Az} | _z - \Delta x \Delta y N_{Az} | _{z+\Delta z}) \\ & + \Delta x \Delta y \Delta z R_A \end{aligned} \quad (5.23)$$

where R_A is the rate of generation of A per unit volume by reaction.

Dividing throughout by $\Delta x \Delta y \Delta z$ and taking the limit $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$,

$$\frac{\partial C_A}{\partial t} = -\frac{\partial N_{Ax}}{\partial x} - \frac{\partial N_{Ay}}{\partial y} - \frac{\partial N_{Az}}{\partial z} + R_A \quad (5.24)$$

Equation 5.24 is the general mass balance equation of a species in rectangular coordinate. Table 5.5 gives the equation in different coordinate systems. These differential equations along with appropriate boundary conditions are solved to find out the concentration profiles and fluxes in a system. If we are interested in finding out the concentration profile in a cylinder or a system having cylindrical symmetry we use equation B of Table 5.5. Equation C of Table 5.5 is used for

Table 5.5 Equation of continuity of A in different coordinate systems

Rectangular coordinates

$$\frac{\partial C_A}{\partial t} + \frac{\partial N_{Ax}}{\partial x} + \frac{\partial N_{Ay}}{\partial y} + \frac{\partial N_{Az}}{\partial z} = R_A \quad [A]$$

Cylindrical coordinates

$$\frac{\partial C_A}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r N_{Ar}) + \frac{1}{r} \frac{\partial N_{Az}}{\partial \theta} + \frac{\partial N_{Az}}{\partial z} = R_A \quad [B]$$

Spherical coordinates

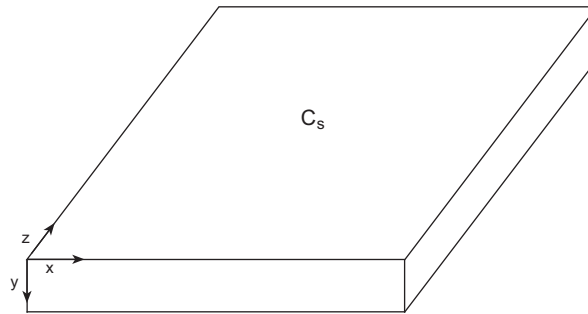
$$\frac{\partial C_A}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 N_{Ar}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (N_{A\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial N_{A\phi}}{\partial \phi} = R_A \quad [C]$$

concentration profiles in a sphere. Although any problem can be solved in a rectangular coordinate system, appropriate choice of coordinate system makes the final equation simpler.

Diffusion in solid

Let us consider carburizing of steel as an application of diffusion in solid. A steel plate containing uniform carbon concentration, C_0 , is exposed in a carburizing gas atmosphere. The carburizing atmosphere maintains carbon concentration on the top surface of plate at C_s , Fig. 5.10. We want to find out the carbon concentration profile in the plate.

Carbon diffuses from the top to the interior of the plate in y direction. So concentration gradient of carbon in x and z directions is zero. Furthermore, carbon does not take part in any reaction in the steel plate, so $R_A = 0$. Thereby, the problem involves diffusion of carbon only in y direction and equation 5.24 simplifies to



5.10 Carburizing of steel.

$$\frac{\partial C_C}{\partial t} = -\frac{\partial N_{Cy}}{\partial y} \quad (5.25)$$

where C_C is the concentration of carbon and N_{Cy} is the carbon flux in steel in y direction. Since concentration of carbon in steel is small, we neglect the bulk flow term in the definition of flux given by equation 5.7 and simplify it as

$$N_{Cy} = -D_C \frac{\partial C}{\partial y} \quad (5.26)$$

where D_C is intrinsic diffusivity of carbon in steel.

Substituting equation 5.26 in equation 5.25 and assuming that diffusivity of carbon is constant

$$\frac{\partial C_C}{\partial t} = D_C \frac{\partial^2 C_C}{\partial y^2} \quad (5.27)$$

Equation 5.27 is known as Fick's second law. The initial and boundary conditions are

$$\text{At } t = 0, \quad y > 0, \quad C_C = C_0 \quad (5.28a)$$

$$\text{At } t \geq 0, \quad y = 0, \quad C_C = C_S \quad (5.28b)$$

Normally, the thickness of carburized layer is a fraction of a millimetre only and carbon concentration far away from the top surface remains at C_0 . Thereby, although the plate is of finite thickness, it can be considered as infinity and another boundary condition can be written as

$$\text{At } t > 0, \quad y = \infty, \quad C_C = C_0 \quad (5.28c)$$

To solve equation 5.27 along with equations 5.28a, b and c, we define two new variables

$$\eta = y/2(D_C t)^{1/2} \quad (5.29)$$

$$p = \frac{dC_C}{d\eta} \quad (5.30)$$

We know,

$$\begin{aligned} \frac{\partial C_C}{\partial t} &= \frac{\partial C_C}{\partial \eta} \frac{\partial \eta}{\partial t} = -\frac{y}{4t(D_C t)^{1/2}} \frac{\partial C_C}{\partial \eta} = -\frac{\eta}{2t} \frac{\partial C_C}{\partial \eta^2} \\ \frac{\partial C_C}{\partial y} &= \frac{1}{2(D_C t)^{1/2}} \frac{\partial C_C}{\partial \eta} \quad \text{and} \quad \frac{\partial^2 C_C}{\partial y^2} = \frac{1}{4D_C t} \frac{\partial^2 C_C}{\partial \eta^2} \end{aligned}$$

Using the above relationships and equation 5.30, equation 5.27 can be written as

$$-2\eta p = \frac{dp}{d\eta} \quad (5.31)$$

The variable η given by equation 5.29 has converted the partial differential equation into an ordinary differential equation. This transformation, which reduces the number of independent variables, is known as the similarity transform. The boundary conditions given by equations 5.28 become

$$\text{At } \eta = 0, \quad C_C = C_S \quad (5.32a)$$

$$\text{At } \eta = \infty, \quad C_C = C_0 \quad (5.32b)$$

Integrating equation 5.31,

$$p = A \exp(-\eta^2)$$

where A is a constant. Substituting p from equation 5.30

$$\frac{dC_C}{d\eta} = A \exp(-\eta^2)$$

Integrating in the limit $\eta = 0$ to η and using the boundary condition equation 5.32a,

$$C_C - C_S = A \int_0^\eta \exp(-\eta^2) d\eta \quad (5.33)$$

Using the boundary condition equation 5.32b

$$C_0 - C_S = A \int_0^\infty \exp(-\eta^2) d\eta = A \frac{\sqrt{\pi}}{2}$$

Or,

$$A = 2(C_0 - C_S)/\sqrt{\pi}$$

Hence equation 5.33 becomes

$$(C_C - C_S) = (C_0 - C_S) \text{erf}(\eta) \quad (5.34)$$

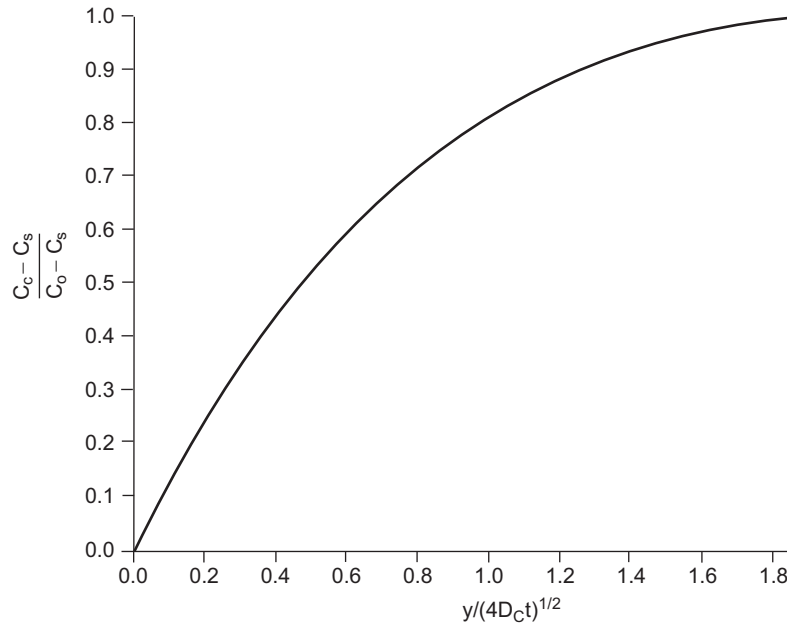
where

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\eta^2) d\eta$$

Figure 5.11 shows the dimensionless concentration $(C_C - C_S)/(C_0 - C_S)$ as a function of $y/(4D_c t)^{1/2}$. This is essentially a plot of $\text{erf}(\eta)$ with η . The figure shows that $\text{erf}(\eta) \approx 1$ when $\eta = 2$. This indicates that if $y/(4D_c t)^{1/2} > 2$, $C_C = C_0$ or that carbon concentration is the same as the initial concentration. So if maximum time of interest $t_{\max}$ and length in the direction of diffusion, L , are such that $L/(4D_c t_{\max})^{1/2} > 2$, the system can be considered as semi-infinite.

Example 5.5

A steel plate with 0.2% carbon is exposed to a carburizing atmosphere at 1223K. Carburizing atmosphere maintains 0.5% carbon on the surface of steel. Calculate


 5.11 Variation of $(C_C - C_S)/(C_0 - C_S)$ with $y/(4D_Ct)^{1/2}$.

- (a) the concentration of carbon at 1 mm away from the surface after 1 hour, (b) the layer thickness where carbon is greater than 0.3% after 1 hour and (c) what should be the carburizing time if the layer thickness of 0.3% carbon is to be doubled?

Solution

Initial carbon concentration $C_0 = 0.2$, and surface concentration $C_S = 0.5\%$. From Table 5.4, at 1223K, diffusivity of carbon $D_C = 1.2 \times 10^{-11} \text{ m}^2/\text{s}$.

- (a) After 1 hour or 3600s, at $y = 1 \text{ mm}$

$$\eta = y/2(D_Ct)^{1/2} = 10^{-3}/\{2(1.2 \times 10^{-11} \times 3600)^{1/2}\} = 2.4$$

$$\text{erf}(2.4) = 1$$

From equation 5.34, $C_C = 0.5 + (0.2 - 0.5) \times 1 = 0.2\%$ or no change in % carbon.

- (b) From equation 5.34, $(0.3 - 0.5) = (0.2 - 0.5)\text{erf}(\eta)$ or $\text{erf}(\eta) = 0.666$. Using Fig. 5.11, $\eta = 0.67$, from the definition of η , $y = 2 \times 0.67 \times (1.2 \times 10^{-11} \times 3600)^{1/2} = 0.28 \text{ mm}$.
- (c) Thickness of the layer having $\%C \geq 0.3\%$ is twice that of (b) or $y = 0.56 \text{ mm}$. Obviously the value of $y/(4t)^{1/2}$ must be same both for (b) and (c). $0.28/(3600)^{1/2} = 0.56/t^{1/2}$ or $t = (0.56/0.28)^2 \times 3600$, or 4 hours. ■■■

Evaporation of liquid

Figure 5.12 shows a volatile liquid A in a cylindrical container of cross-sectional area S . A gas stream of A and B is flowing over container. The gas B is insoluble in liquid A. The system is isothermal and the rate of evaporation is so slow that the liquid level in the container does not change appreciably. We want to find out the rate of evaporation and concentration profile of A in the container at steady state.

The problem involves diffusion in a cylindrical container, so equation B of Table 5.5 is the appropriate equation. But in this case diffusion of A is in z direction only, so the final equation will be the same both in rectangular and cylindrical coordinate systems. If we consider rectangular coordinates, fluxes N_{Ax} and N_{Ay} are zero and for cylindrical coordinate, fluxes N_{Ar} and $N_{A\theta}$ are zero. Table 5.5 shows that equations A and B become identical for this condition. There is no reaction, so rate of generation of A, $R_A = 0$. We are interested in a concentration profile at steady state so $\partial C_A / \partial t = 0$. Hence the conservation equation of species A simplifies to

$$-\frac{dN_{Az}}{dz} = 0 \quad (5.35)$$

Integrating

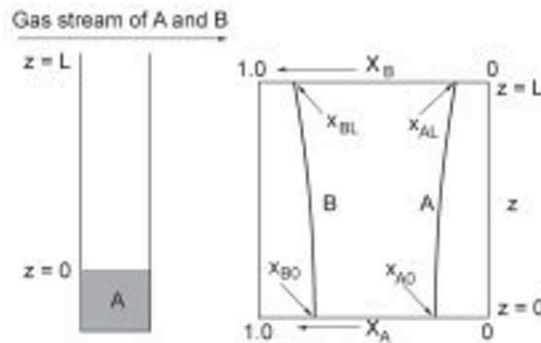
$$N_{Az} = C_1 \quad (5.36)$$

From the definition of flux, equation 5.7

$$N_{Az} = -D_{AB} \frac{dC_A}{dz} + x_A(N_{Az} + N_{Bz}) \quad (5.37)$$

Since the gas B is insoluble in liquid A, the flux of B is zero.

$$N_{Bz} = 0 \quad (5.38)$$



5.12 Evaporation of a liquid A from a container and concentration profiles of A and B.

Using the relation $C_A = C \cdot x_A$ where C is the total concentration of A and B and equation 5.38, equation 5.37 can be written as

$$N_{Az} = -\frac{D_{AB}C}{1-x_A} \frac{dx_A}{dz} \quad (5.39)$$

Substituting the above relationship in equation 5.36

$$-D_{AB}C \cdot \frac{dx_A}{1-x_A} = C_1 dz$$

Integration of the above equation gives

$$D_{AB}C \ln(1-x_A) = C_1 z + C_2 \quad (5.40)$$

Taking the top surface of the liquid as $z = 0$, the boundary conditions are

$$\text{At } z = 0, \quad x_A = x_{A0} \quad (5.41a)$$

$$\text{At } z = L, \quad x_A = x_{AL} \quad (5.41b)$$

The values of x_{A0} and x_{AL} are determined by the vapour pressure of liquid A and the concentration of A in the gas stream that is flowing above the container respectively. From equations 5.40 and 5.41

$$C_2 = D_{AB}C \ln(1-x_{A0}) \quad (5.42)$$

$$C_1 = C \frac{D_{AB}}{L} \ln \frac{1-x_{AL}}{1-x_{A0}} \quad (5.43)$$

Substituting equations 5.42 and 5.43 in equation 5.40 and rearranging

$$\frac{1-x_A}{1-x_{A0}} = \left(\frac{1-x_{AL}}{1-x_{A0}} \right)^{z/L} \quad (5.44)$$

In terms of mole fraction of B, the above equation can be written as

$$x_B/x_{B0} = (x_{BL}/x_{B0})^{z/L} \quad (5.45)$$

Although the flux of B is zero, the above equation shows that the concentration gradient of B is not zero. Diffusive flux due to concentration gradient is exactly balanced by the flux due to bulk flow. Figure 5.12 shows the concentration profiles of A and B.

$$\text{The rate of evaporation of A} = S \cdot N_{Az} \big|_{z=0} \quad (5.46)$$

Using equations 5.36 and 5.43

$$\text{Rate of evaporation of A} = S \cdot C \frac{D_{AB}}{L} \ln \frac{1-x_{AL}}{1-x_{A0}} \quad (5.47)$$

When concentration of A in the gas phase is very small, $\ln(1-x_{AL}) \approx -x_{AL}$ and $\ln(1-x_{A0}) \approx -x_{A0}$ and the above equation simplifies to

$$\text{Rate of evaporation} = SD_{AB}C \frac{x_{A0} - x_{AL}}{L} \quad (5.48)$$

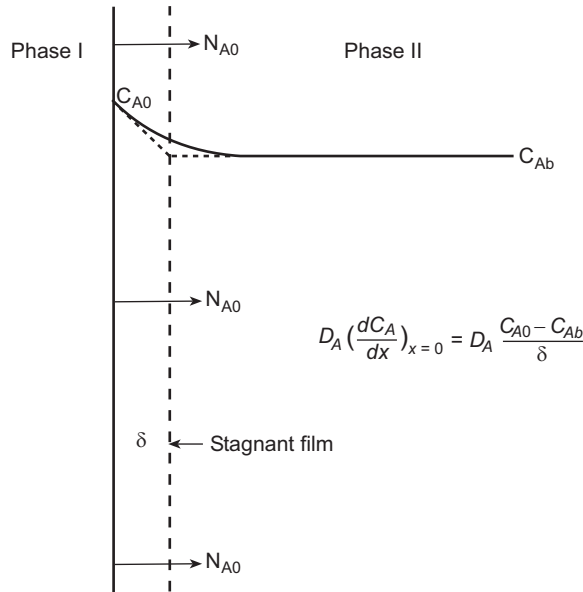
The above equation can also be obtained from equations 5.36 and 5.37 by neglecting the bulk flow term from the later equation.

5.2.4 Interface mass transfer

Quite often we are interested in finding out the rate of transport of a species from one phase to another one. For example, evaporation of a liquid, dissolution of a solid in a liquid, absorption of a gas by liquid, etc. involves transport of a species from one phase to another. These are determined by the flux at the interface between two phases $N_{Ax}|_{x=0}$ or N_{A0} . Figure 5.13 shows the flux at interface. This interphase transport between phases I and II, is calculated using the concept of mass transfer coefficient.

$$\begin{aligned} N_{A0} &= -D_{AB} \left(\frac{dC_A}{dx} \right)_{x=0} + x_{A0}(N_{A0} + N_{B0}) \\ &= k_{MA}(C_{A0} - C_{Ab}) + x_{A0}(N_{A0} + N_{B0}) \end{aligned} \quad (5.49)$$

where N_{A0} is the flux of A from phase I to II, k_{MA} is the mass transfer coefficient of A, C_{A0} and C_{Ab} are concentrations of A in phase II at the interface and bulk respectively and x_{A0} is the mole fraction of A in phase II at the interface. The interface concentration of A in phase II, C_{A0} , is related to the concentration of A



5.13 Transfer of a species from phase I to II.

in phase I. For example: for evaporation of water at 300K, C_{A0} is determined by vapour pressure of water (phase I) at 300K. For removal of nitrogen from liquid steel by argon purging, C_{A0} is determined by the partial pressure of nitrogen in equilibrium with nitrogen dissolved in steel (phase I).

In a number of practical systems x_{A0} is very small and equation 5.49 simplifies to

$$N_{A0} = k_{MA}(C_{Aw} - C_{Ab}) \quad (5.50)$$

A physical picture of mass transfer coefficient can be obtained from film theory. According to this model, at the interface there is a stagnant film of thickness δ . The concentration of A varies linearly in the film as shown by dotted line in Fig. 5.13 and beyond the film; concentration of A is the same as the bulk concentration. So from equation 5.49 we get $k_{MA} = D_{AB}/\delta$. Using this concept quite often we say that at the interface there is a stagnant film.

Mass transfer coefficient is a system property; in the sense it depends on the shape and size of the system from which mass transfer takes place. Besides, it is determined by density, viscosity, diffusivity and velocity of fluid. So

$$k_{MA} = f(\rho, \mu, D_{AB}, v, L) \quad (5.51)$$

where ρ , μ and v are respectively density, viscosity and velocity of fluid and L is the characteristic length of the system. Dimensionless analysis shows that equation 5.51 can be expressed as

$$Sh = f(Re, Sc) \quad (5.52)$$

where Sh , Re and Sc are respectively the Sherwood number, Reynolds number and Schmidt number and are defined as

$$\text{Sherwood number} \quad Sh = k_{MA}L/D_{AB} \quad (5.53)$$

$$\text{Reynolds number} \quad Re = \rho v L / \mu \quad (5.54)$$

$$\text{Schmidt number} \quad Sc = \mu / (\rho D_{AB}) \quad (5.55)$$

The Sherwood number is the ratio of interface or convective mass transfer to diffusive mass transfer rates. The Reynolds number is the ratio inertial force to viscous force and the Schmidt number is the ratio of momentum diffusivity (μ/ρ) to mass diffusivity

Correlations for mass transfer coefficient

The mass transfer from a sphere is mostly calculated by Ranz and Marshall correlation

$$Sh = 2 + 0.6(Re)^{1/2}(Sc)^{1/3} \quad (5.56)$$

The characteristic length for both Sh and Re is the particle diameter. So Re appearing in equation 5.56 is called the particle Reynolds number. When there is

no fluid flow, the above relation predicts $Sh = 2$ which is obtained by theoretical calculation. This correlation is often used for calculation of mass transfer in a packed bed.

Example 5.6

A spherical drop of water of 0.5 mm diameter is falling through dry still air at a velocity of 0.75 ms^{-1} . Assuming that the temperature of water droplet and air is at 300K, calculate the instantaneous evaporation rate. At 300K vapour pressure of water is 0.035 bar. Density and viscosity of air at 300K are 1.177 kg m^{-3} and $18.53 \times 10^{-6} \text{ kg m}^{-1} \text{ s}^{-1}$ respectively. $1 \text{ bar} = 10^5 \text{ Pa}$ and $R = 8.314 \text{ Pa.m}^3.\text{mol}^{-1} \text{ K}^{-1}$, Atmospheric pressure = $1.013 \times 10^5 \text{ Pa}$.

Solution

Rate of evaporation = surface area of droplet $\times N_{A0} = \pi(0.5 \times 10^{-3})^2 N_{A0}$

From equation 5.49, $N_{A0} = [k_M(C_{A0} - C_{Ab}) + x_{A0}(N_{A0} + N_{B0})]$

In the present case A is H_2O and B is air. Since air is not soluble in water, $N_{B0} = 0$,

$$N_{A0} = k_M(C_{A0} - C_{Ab})/(1 - x_{A0})$$

Rate of evaporation = $25\pi \times 10^{-8} [k_M(C_{A0} - C_{Ab})]/(1 - x_{A0})$

$$C_{A0} = p_{A0}/RT = 0.035 \times 10^5/(8.314 \times 300) = 1.4 \text{ mole/m}^3$$

$$C_{Ab} = 0$$

$$x_{A0} = p_{A0}/\text{atmospheric pressure} = 0.035 \times 10^5/1.013 \times 10^5 = 0.0345$$

$$D_{\text{H}_2\text{O-air at 300K}} = 0.255 \times 10^{-4} \text{ m}^2/\text{s from equation 5.16}$$

$$Re = \rho vL/\mu = 1.177 \times 0.75 \times 0.5 \times 10^{-3}/18.53 \times 10^{-6} = 23.8$$

$$Sc = \mu/(\rho D_{AB}) = 18.53 \times 10^{-6}/(1.177 \times 0.255 \times 10^{-4}) = 0.617$$

From equation 5.56, $Sh = 2 + 0.6 \times (23.8)^{1/2} \times (0.617)^{1/3} = 4.49$

$$K_M = Sh.D_{AB}/L = 4.49 \times 0.255 \times 10^{-4}/0.5 \times 10^{-3} = 0.229 \text{ m}^2/\text{s}$$

So rate of evaporation = $25\pi \times 10^{-8} \times 0.229 \times 1.4/(1 - 0.0345)$

$$= 26.1 \times 10^{-8} \text{ mole/s}$$



5.3 Heat transfer

When a hot object is kept in atmosphere, heat is transferred from the surface of the object to the surroundings by convection and radiation. As the surface is cooled, heat is transferred from the interior of the object to the surface by conduction. These processes continue until the temperature of the object becomes the same as that of the surroundings. In all the three modes of heat transfer namely, conduction, convection and radiation; the driving force is temperature difference.

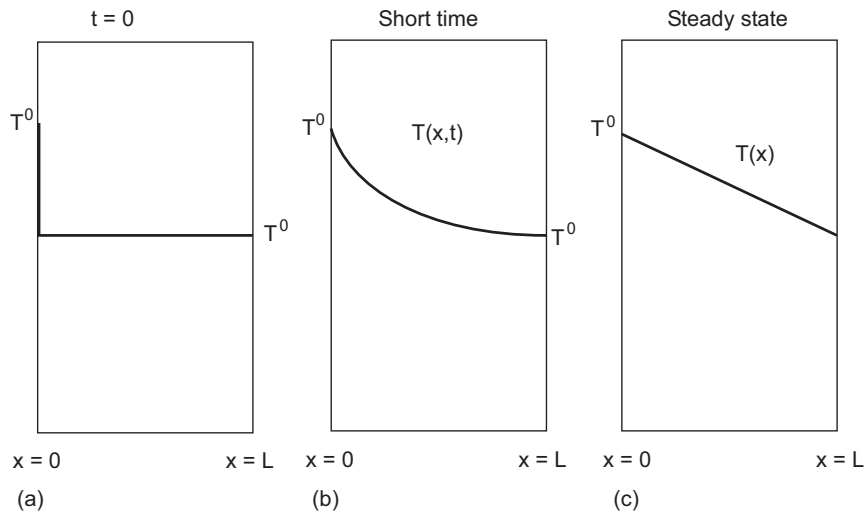
The conduction of heat in solid and liquid takes place due to molecular vibration and that in gas by molecular collision. Convective heat transfer has

two limiting cases: forced and natural. In forced convection the flow is due to external forces whereas in natural convection, it is due to density difference. When a hot object is kept in stagnant air, the heat transfer is by natural convection. The air in contact with the hot object becomes hot and thereby lighter and moves up and cold air from the surroundings occupies the space. This leads to fluid motion that is termed as natural convection. On the other hand, if the hot object is cooled by blowing air over it, the mechanism of heat transfer is forced convection. Both conduction and convection of heat takes place through a material medium, but energy is transported through empty space by radiation.

5.3.1 Conduction

Let us consider a plate that is at a uniform temperature T^0 . At time, $t = 0$, the left face is suddenly raised to a temperature T_0 , Fig. 5.14(a). Due to difference in temperature, heat flows from left to right by conduction and temperature along the length increases with time as shown in Fig. 5.14(b). After sufficiently long time, temperature profile attains the steady state Fig. 5.14(c). Both during steady and unsteady state, heat flows from left to right. This flow of heat is measured in terms of heat flux defined as the amount of heat flowing through unit area per unit time in a direction normal to the area. The heat flux by conduction is given by

$$q_x = -k \frac{dT}{dx} \quad (5.57)$$



5.14 Temperature profile in a slab. (a) The left face of the slab is suddenly raised to T_0 at $t = 0$. (b) Temperature profile after some time. (c) Temperature profile at steady state.

where q_x is the heat flux in x direction W/m^2 , k is the thermal conductivity of material $\text{Wm}^{-1}\text{K}^{-1}$ and dT/dx is the temperature gradient K/m . The negative sign in equation 5.57 indicates that heat flux and temperature gradient are in opposite direction. In Fig. 5.14 temperature decreases with increase of x , dT/dx is negative, and heat flux is positive or it is in direction of increasing x . It should be noted that the flux is a vector and its sign gives the direction of flow. Equation 5.57 is the one-dimensional form of Fourier's law of heat conduction.

In the example above, temperature depends only on its distance from the face or T is a function of x only. But if temperature varies from point to point, i.e., if T is a function of x , y and z , heat will flow in all directions. It is given by

$$\mathbf{q} = -k\nabla T \quad (5.58)$$

The components of heat flux in different coordinate systems are similar to that for molar flux $\mathbf{J}_A$ given in Table 5.1. The above equation is the three-dimensional form of Fourier's law of heat conduction and is valid for a medium whose conductivity is the same in all direction. These types of materials are known as isotropic material. But a number of materials, for example laminated composites, unidirectional fibrous composite material like bamboo etc., are not isotropic, i.e., conductivity is different in different directions. For non-isotropic materials equation 5.58 takes the form

$$\mathbf{q}^m = -\left[\mathbf{i}k_x \frac{\partial T}{\partial x} + \mathbf{j}k_y \frac{\partial T}{\partial y} + \mathbf{k}k_z \frac{\partial T}{\partial z}\right] \quad (5.59)$$

where k_x , k_y and k_z are thermal conductivity of the material and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in x , y , and z directions respectively.

Example 5.7

Two faces of a stainless steel plate of 0.1 m^2 area and 4 mm thickness are kept at 723K and 323K respectively. The temperature profile in the plate is linear. Calculate the heat flux and total heat transferred in one minute through the plate. Thermal conductivity of stainless steel is $19 \text{ Wm}^{-1}\text{K}^{-1}$.

Solution

The problem involves heat transfer only in one direction and heat flux is given by equation 5.57. Let us assume that the face at temperature 723K as $x = 0$ and that at 323K as $x = \Delta x$. Since the temperature profile is linear, $dT/dx = (T|_{\Delta x} - T|_0)/\Delta x = (323 - 723)/(4 \times 10^{-3}) = -10^5 \text{ K/m}$. So heat flux $= 19 \times 10^5 \text{ Wm}^{-2}$. The direction of heat flux is +ve direction of x .

Total heat transferred in one minute

$$\begin{aligned} &= \text{area of the plate} \times \text{heat flux} \times \text{time in seconds} \\ &= 0.1 \times 19 \times 10^5 \times 60 = 1.14 \times 10^7 \text{ J} \end{aligned}$$



5.3.2 Thermal conductivity

According to kinetic theory of gases, thermal conductivity of monatomic gases is independent of pressure and is proportional to square root of temperature. The predicted pressure dependence is valid up to 10 atmospheres but temperature dependent is too weak. The temperature dependence of the thermal conductivity of gas can be expressed as

$$k = k_0(T/T_0)^n \quad (5.60)$$

where k_0 is thermal conductivity at T_0 K. Eucken's equation

$$k = (C_p + 1.25R/M)\mu \quad (5.61)$$

is widely used for estimation of thermal conductivity of gases. C_p , R , M and μ are respectively specific heat, gas constant, molecular weight and viscosity. Table 5.6 gives the thermal conductivity of some common gases. It shows that thermal conductivity of hydrogen is much higher than other gases.

Thermal conductivity of liquid

Thermal conductivity of liquid depends on the nature of the liquid. Liquid metals have a much higher thermal conductivity compared with water or slag. Table 5.7 shows the thermal conductivity of different liquids.

Thermal conductivity of solids

Energy is transferred due to elastic vibrations of the lattice in solids. In the case of metal, besides the above mechanism, free electrons moving through the lattice carry energy. Heat transferred by the latter mechanism is greater than that

Table 5.6 Thermal conductivity of some common gases ($\text{Wm}^{-1} \text{K}^{-1}$)

Gases	H ₂	H ₂ O	CO	CO ₂	Air
$k \times 10^3$ at 400K	226	26.1	31.8	24.3	33.8
$k \times 10^3$ at 800K	378	59.2	55.5	55.1	57.3

Table 5.7 Thermal conductivity of liquids ($\text{Wm}^{-1} \text{K}^{-1}$)

Material	Temp. K	k	Material	Temp. K	k
Water	293	0.59	Aluminum	933	91
Glycerol	293	0.29	Copper	1600	174
Slag	1873	4.0	Iron	1809	40.3

Table 5.8 Thermal conductivity of solids at room temperature

Material	Al	Cu	Brass	Fe	Steel	Concrete	Brick
$k \text{ Wm}^{-1} \text{ K}^{-1}$	237	398	127	79	52	0.9	0.6

by the former. So thermal conductivity of metal is much higher than that of non-metals. Thermal conductivity of pure metal decreases with temperature. Table 5.8 gives thermal conductivity of some solids.

Thermal conductivity of porous solid is given by

$$k_{\text{eff}} = k(1 - \epsilon) \quad (5.62)$$

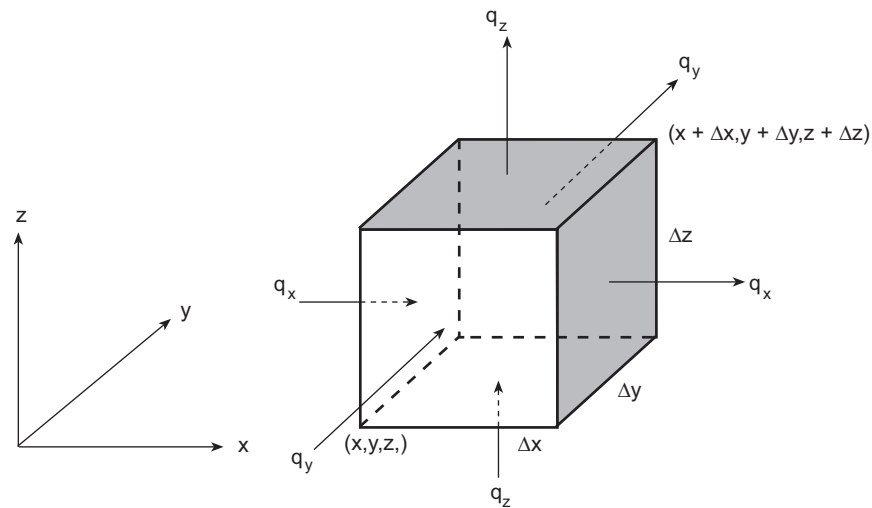
where k_{eff} and k are the thermal conductivity of porous solid and solid respectively and ϵ is the void fraction in solid.

5.3.3 Conservation equation

Let us consider an elemental volume $\Delta x \Delta y \Delta z$ as shown in Fig. 5.15. Since energy is conserved, the balance equation is

$$\begin{aligned} \text{Rate of accumulation of energy in } \Delta x \Delta y \Delta z &= \text{rate in} - \text{rate out} \\ &+ \text{rate of generation of energy in } \Delta x \Delta y \Delta z \text{ volume} \end{aligned} \quad (5.63)$$

In Fig. 5.15, heat enters through three faces and goes out through the corresponding opposite three faces. In the case of stationary solid, heat flux is only by conduction. So following the procedure given in Section 5.2.3, we get



5.15 Heat balance in a control volume.

$$\frac{\partial}{\partial t}(\rho C_p T) = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + H_G \quad (5.64)$$

where $\rho C_p T$ is the heat content per unit volume and H_G is the rate of heat generation per unit volume. Assuming ρ , C_p and k are constant and using the definition of flux equation 5.58, the above equation can be written as

$$\rho C_p \frac{\partial T}{\partial t} = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + H_G \quad (5.65)$$

In the case of fluid, the terms ‘rate in’ and ‘rate out’ in equation 5.63 include heat transfer due to bulk flow as well. Similarly if the solid is in motion, in the direction of motion, the terms ‘rate in’ and ‘rate out’ include heat flux due to bulk motion. Table 5.9 gives the general heat balance equation in different coordinate systems. These equations are valid for solids and incompressible fluids.

Heat generation within a solid can be due to phase transformation, chemical reaction or electrical heating. In the case of fluid besides the above mechanisms, heat generation can be due to viscous dissipation as well. But normally heat generation due to viscous dissipation is very small and is neglected.

Steady state heat conduction

Let us consider heat transfer in the plate shown in Fig. 5.14. The appropriate energy balance equation is equation A in Table 5.9. In this case heat transfer takes place only along the x axis that is normal to the plate face. So, heat flux in y and z directions is zero. The plate is stationary so v_x , v_y and v_z are also zero. At the steady state, $\partial T / \partial t = 0$ and there is no heat generation in the plate, $H_G = 0$. Hence equation (A) of Table 5.9 becomes

Table 5.9 Energy equation for incompressible media in different coordinate systems

Rectangular coordinates

$$\rho C_p \left[\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right] = - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] + H_G \quad [A]$$

Cylindrical coordinates

$$\rho C_p \left[\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right] = - \left[\frac{1}{r} \frac{\partial}{\partial r} (r q_r) + \frac{1}{r} \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} \right] + H_G \quad [B]$$

Spherical coordinates

$$\begin{aligned} \rho C_p \left[\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right] \\ = - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial q_\phi}{\partial \phi} \right] + H_G \end{aligned} \quad [C]$$

$$\frac{dq}{dx} = 0$$

Integrating,

$$q_x = C_1$$

where C_1 is the integration constant.

Using the definition of heat flux, equation 5.57, the above equation becomes

$$-k \frac{dT}{dx} = C_1 \quad (5.66)$$

Integrating,

$$-kT = C_1x + C_2 \quad (5.67)$$

Integration constants C_1 and C_2 are evaluated from the boundary conditions.

$$\text{At } x = 0, T = T_0 \quad \text{and} \quad x = L, T = T_L$$

Using these conditions in equation 5.67

$$C_1 = k(T_0 - T_L)/L \quad (5.68)$$

$$C_2 = -kT_0 \quad (5.69)$$

and

$$T = T_0 - (T_0 - T_L) \frac{x}{L} \quad (5.70)$$

Quite often we are interested in the case where one face is maintained at temperature T_0 but the other face is exposed to the atmosphere. In this case the boundary conditions are

$$\text{At } x = 0, T = T_0 \quad \text{and} \quad x = L, -k \frac{dT}{dx} = h(T - T_a) \quad (5.71)$$

where h is the heat transfer coefficient and T_a is the ambient temperature. C_2 is same as that given by equation 5.69. Using equations 5.67 and 5.71

$$C_1 = h \left[\frac{-C_1L + kT_0}{k} - T_a \right]$$

Rearranging

$$C_1 = \frac{T_0 - T_a}{\frac{1}{h} + \frac{L}{k}} \quad (5.72)$$

So

$$T = T_0 - \frac{x}{k} \frac{T_0 - T_a}{\frac{1}{h} + \frac{L}{k}} \quad (5.73)$$



5.16 Electrical analogy for heat loss through a plate. $T_{\text{outer face}}$ is the temperature of outer face exposed to atmosphere.

Heat is lost through the face at $x = L$, so the rate of heat lost per unit area is

$$H_{\text{Loss}} = -k \left(\frac{dT}{dx} \right)_{x=L}$$

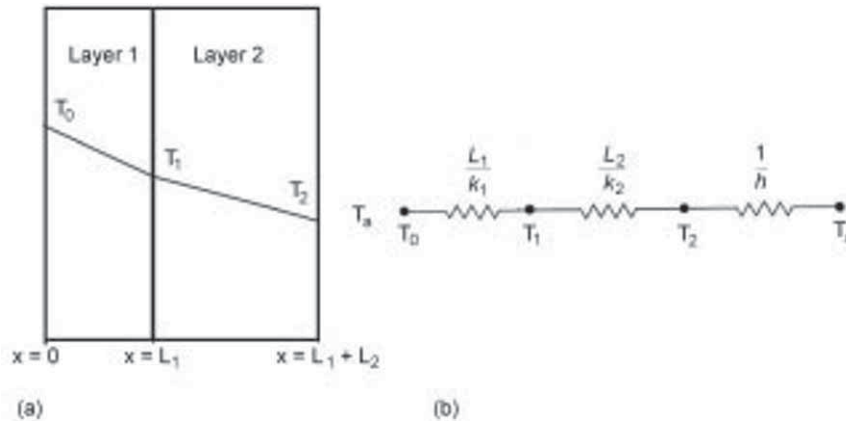
At steady state dT/dx is constant so heat loss per unit area is same as heat flux and is equal to C_1 , see equation 5.67, hence

$$H_{\text{Loss}} = \frac{T_0 - T_a}{\frac{1}{h} + \frac{L}{k}} \quad (5.74)$$

Using an electrical analogue, one can say that heat loss is similar to current, the numerator of above equation is the driving force and the denominator is the resistance for heat loss. Two resistances, one due to plate, L/k , and other one due to the interface, $1/h$, are in series. Figure 5.16 shows the electrical analogue.

Heat loss through composite wall

Let us consider heat flux at the steady state through a composite wall made up of two layers shown in Fig. 5.17(a). The thickness of layers 1 and 2 and their thermal conductivities are L_1 , L_2 and k_1 , k_2 , respectively. Obviously, at the steady state temperature profiles of the layers are given by equation 5.67 and can be written as:



5.17 (a) Temperature distribution in composite wall and (b) electrical analogue of heat flux.

$$\text{For layer 1} \quad T^I = C_1x + C_2 \quad (5.75a)$$

$$\text{For layer 2} \quad T^{II} = C_3x + C_4 \quad (5.75b)$$

The integration constants, C_1 , C_2 , C_3 , C_4 , are determined by the following boundary conditions

$$\text{At } x = 0, \quad T^I = T_0 \quad (5.76a)$$

The outer face is exposed to atmosphere, so

$$\text{At } x = L_1 + L_2, \quad -k_2(dT^{II}/dx) = h(T^{II} - T_a) \quad (5.76b)$$

The walls are in perfect contact, so temperature and heat flux at the interface must be the same. Or

$$\text{At } x = L_1 \quad T^I = T^{II} \quad (5.76c)$$

$$k_1(dT^I/dx) = k_2(dT^{II}/dx) \quad (5.76d)$$

From equations 5.75 and 5.76,

$$C_1 = (k_2/k_1)C_3$$

$$C_2 = T_0$$

$$C_3 = -\frac{T_0 - T_a}{\frac{k_2}{h} + L_2 + \frac{k_2}{k_1}L_1}$$

$$C_4 = L_1C_3(k_2/k_1 - 1) + T_0$$

$$\text{Heat loss} = (-k_2dT^{II}/dx)_{x=L_1+L_2} = -k_2C_3 = -\frac{T_0 - T_a}{\frac{1}{h} + \frac{L_1}{k_1} + \frac{L_2}{k_2}} \quad (5.77)$$

Figure 5.17(b) shows the electrical analogue for heat loss through the composite wall. Using the electrical analogue, we could directly obtain equation 5.77.

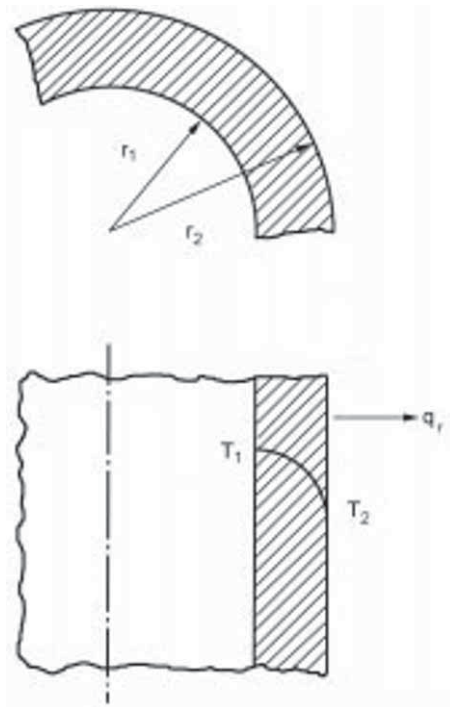
Temperature distribution in a hollow cylinder

Let us consider a hollow cylinder with inner radius r_1 and outer radius r_2 , Fig. 5.18. Inner and outer faces are at T_1 and T_2 respectively. Obviously, the appropriate equation for energy conservation is equation B in Table 5.9. In the hollow cylinder, heat conduction is in the radial direction only; thereby q_z and q_θ are zero. Furthermore, there is no heat generation and all velocity components in equation B are zero, so at steady state the equation simplifies to

$$\frac{d}{dr}(rq_r) = 0 \quad (5.78)$$

Integrating

$$rq_r = C_1$$



5.18 Schematic diagram of a hollow cylinder.

Since

$$q_r = -k \frac{dT}{dr},$$

$$-k \frac{dT}{dr} = C_1/r$$

Integrating

$$-kT = C_1 \ln r + C_2$$

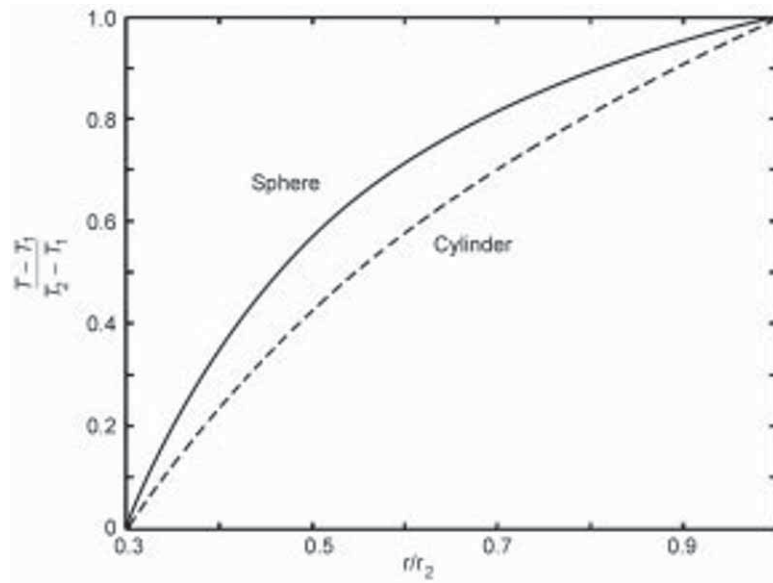
The boundary conditions are

$$\text{At } r = r_1, \quad T = T_1$$

$$\text{At } r = r_2, \quad T = T_2$$

Using the above boundary conditions

$$= T_1 - (T_1 - T_2) \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}$$



5.19 Temperature distribution in a hollow cylinder and sphere. Ratio of inner to outer radius (r_1/r_2) is 0.3.

The above equation can be written in a dimensionless form as

$$\frac{T - T_1}{T_2 - T_1} = \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \quad (5.79)$$

Figure 5.19 shows the dimensionless temperature profile in a hollow cylinder. It shows that temperature profile deviates considerably from linearity.

Steady state temperature distribution in a spherical shell

Let us consider a hollow spherical shell of inner radius r_1 at temperature T_1 and outer radius r_2 at T_2 . Heat conduction takes place only in the radial direction. The appropriate equation is equation C in Table 5.9 which simplifies for the steady state conduction problem without heat generation to

$$\frac{d}{dr}(r^2 q_r) = 0$$

and boundary conditions are

$$\begin{aligned} \text{At } r = r_1, \quad T &= T_1 \\ \text{At } r = r_2, \quad T &= T_2 \end{aligned}$$

The solution of the differential equation along with the above boundary

condition is

$$\frac{T - T_1}{T_2 - T_1} = \frac{\frac{1}{r_1} - \frac{1}{r}}{\frac{1}{r_1} - \frac{1}{r_2}} \quad (5.80)$$

Figure 5.19 shows the dimensionless temperature profile in a hollow sphere. Comparison of temperature profiles in a hollow cylinder and sphere shows that the curvature of the latter is greater.

5.3.4 Heat transfer coefficient

At solid–fluid interface, heat flux from solid to fluid or fluid to solid is defined as

$$q = h(T_s - T_b) \quad (5.81)$$

where T_s is the temperature of solid surface in contact with fluid, T_b is the bulk fluid temperature and h is the heat transfer coefficient, $\text{Wm}^{-2}\text{K}^{-1}$. Equation 5.81 is also used for heat transfer between two fluids. h is often termed as convective heat transfer coefficient to differentiate it from radiative heat transfer coefficient. The value of h depends on characteristic length of body, fluid properties like thermal conductivity, density and specific heat and velocity of fluid. By dimensional analysis it can be shown that

$$\text{Nu} = f(\text{Re}, \text{Pr}) \quad (5.82)$$

where

$$\begin{aligned} \text{Nusselt number} \quad \text{Nu} &= hL/k \\ \text{Prandtl number} \quad \text{Pr} &= C_p\mu/k = \nu/\alpha \end{aligned}$$

where L is the characteristic length of the system and $\alpha = k/\rho C_p$ is thermal diffusivity and $\nu = \mu/\rho$ is momentum diffusivity or kinematic viscosity. The Nusselt number measures the ratio of interface or convective heat flux to conductive heat flux and the Prandtl number is the ratio of thermal diffusivity to momentum diffusivity.

The heat transfer from a sphere is often calculated by Ranz and Marshall correlation

$$\text{Nu} = 2 + 0.6(\text{Re}_f)^{1/2}(\text{Pr}_f)^{1/3} \quad (5.83)$$

The characteristic length for both Nu and Re is the particle diameter. The suffix f in Re and Pr indicates that these should be evaluated at the film temperature which is taken as average of surface and bulk temperature. This relationship is also often used for calculation of heat transfer in the packed bed. Comparison of equations 5.56 and 5.83 shows that the relationship between Sh , Re and Sc

number for mass transfer is the same as that between Nu, Re and Pr for heat transfer. This is true not only for a sphere but for other systems as well.

Example 5.8

Show that when a hot sphere is placed in stagnant air, in the absence of natural convection, $Nu = 2$.

Solution

Let T_0 be the temperature of the sphere or temperature of air in contact with the sphere and T_∞ is the temperature of air far away from it. Stagnant air around the sphere is an infinitely large hollow sphere whose inner radius is the radius of the sphere r_0 and the outer radius $r_2 = \infty$. So equation 5.80 simplifies to

$$T = T_0 - (T_0 - T_\infty) \left(\frac{1}{r_0} - \frac{1}{r} \right) r_0$$

$$T = T_\infty + (T_0 - T_\infty) \frac{r_0}{r}$$

Heat flux at the outer face of the sphere is

$$-k \left(\frac{dT}{dr} \right)_{r=r_0} = k \frac{T_0 - T_\infty}{r_0}$$

Equating the above equation with equation 5.81,

$$h = k/r_0 = 2k/D$$

since in the case of the sphere the characteristic length is the diameter of the sphere, D ,

$$Nu = hD/k = 2$$

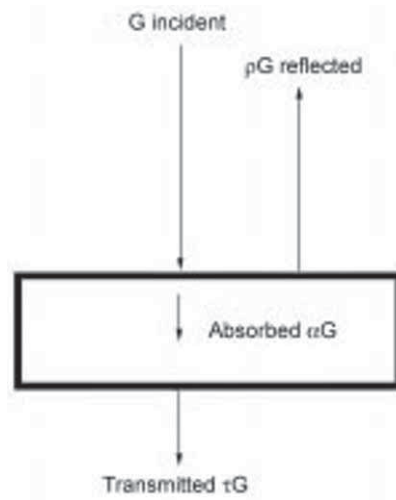


5.3.5 Radiation

A hot body emits radiation in a continuous band of wavelength in the range of 0.1 to 100 microns. This is known as thermal radiation and it includes ultra violet, visible range and infra red of the electromagnetic radiation spectrum. The proportion of different frequencies of radiation emitted by a hot body depends on its temperature and thereby its colour changes with temperature. Like all electromagnetic waves, heat radiation travels through space at the velocity of light.

Emissivity and absorptivity

When radiation falls on an object, a part of it is reflected, a part is transmitted through the object and the rest is absorbed, Fig. 5.20. If G is the flux of radiant



5.20 Schematic illustration of absorption, reflection and transmission of radiation falling on a body.

energy Wm^{-2} , then

$$G = \alpha G + \rho G + \tau G$$

or

$$\alpha + \rho + \tau = 1 \quad (5.84)$$

where α is the fraction of incident radiation absorbed or absorptivity, ρ is the fraction reflected or reflectivity and τ is the fraction transmitted or transmissivity. Most of the solids and liquids are opaque to thermal radiation, so the above equation simplifies to

$$\alpha + \rho = 1 \quad (5.85)$$

For any real body absorptivity α is less than unity and varies with the frequency of radiation. A body for which α is a constant over the entire range of frequency is known as a *grey body*. This is a hypothetical body but we idealize all real bodies as grey bodies for thermal radiation calculations. A limiting case of grey body is $\alpha = 1$ for all frequencies and temperature. This is known as a *black body*. A cavity absorbs all radiation falling on it and hence is a black body.

Let q and q_b be, respectively, total radiant energy emitted per unit area per unit time (Wm^{-2}) by a real surface and a black body when they are at the same temperature. The ratio q/q_b is known as *emissivity* ϵ of the real surface. Or

$$\epsilon = q/q_b \quad (5.86)$$

Let us consider that a grey body of surface area A is enclosed in a cavity and both grey body and cavity are in thermal equilibrium. Obviously, for thermal equilibrium, the energy absorbed by the grey body must be equal to that emitted

by it. Since the cavity is a black body, radiation emitted by it is $q_b \text{ Wm}^{-2}$ and energy absorbed by the grey body is $A\alpha q_b$. Hence

$$Aq = A\alpha q_b \quad (5.87)$$

Comparing equations 5.86 and 5.87

$$\alpha = \epsilon \quad (5.88)$$

So, at a given temperature emissivity and absorptivity are equal for any solid surface. This is *Kirchhoff's law*. Highly polished metal surfaces reflect most of the radiations falling on them so their emissivity is very low. It is in the range of 0.015–0.06. Emissivity of refractory bricks lies between 0.85 and 0.95.

The total energy emitted per unit area per unit time by a black surface is given by the *Stefan–Boltzmann law*.

$$q_b = \sigma T^4 \quad (5.89)$$

The Stefan–Boltzmann constant $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$. For non-black bodies, the radiant energy emitted per unit area per unit time is

$$q = \sigma \epsilon T^4 \quad (5.90)$$

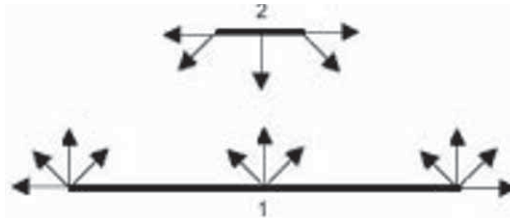
View factor

Let us consider exchange of radiation between two black surfaces 1 and 2 shown in Fig. 5.21. Obviously only a fraction of radiation emitted by surface 1 will be intercepted by surface 2 and vice versa. So radiation energy transferred from 1 to 2 and 2 to 1 can be written as

$$Q_{1 \rightarrow 2} = q_{b1} A_1 F_{12}$$

$$Q_{2 \rightarrow 1} = q_{b2} A_2 F_{21}$$

where F_{12} is the fraction of radiation energy emitted by the unit area of surface 1 which is intercepted or viewed by surface 2 and F_{21} is the fraction of radiation energy emitted by the unit area of surface 2 which is intercepted by surface 1. These are known as the *view factor*. The view factor is a geometric factor and can be calculated analytically. The plots of these values for different geometries



5.21 Radiation exchange between two black bodies.

are available in textbooks on heat transfer. It can easily be proved that the fraction of total energy emitted by surface 1 and intercepted by surface 2 is equal to fraction of total energy emitted by surface 2 and intercepted by surface 1. Or $A_1 F_{12} = A_2 F_{21}$. This is known as the *reciprocity relation*. Hence net exchange of radiation energy between 1 and 2 is

$$Q_{1 \rightarrow 2} = Q_{1 \rightarrow 2} - Q_{2 \rightarrow 1} = A_1 F_{12}(q_{b1} - q_{b2}) = A_2 F_{21}(q_{b1} - q_{b2}) \quad (5.91)$$

The definition of the view factor indicates that the view factor for two large parallel plates of equal dimension separated by a small distance is unity since radiation emitted by one plate is fully intercepted by the other. Also, if an object 1 with surface area A_1 is surrounded by object 2 having surface area A_2 , $F_{12} = 1$ since all radiations emitted by surface 1 are intercepted by 2, but $F_{21} < 1$. Using the reciprocity relation, $F_{21} = A_1/A_2$. Since the view factor is the fraction of radiation emitted by a surface and intercepted by another surface, then if a surface 1 is enclosed by surfaces 2, 3, ... n

$$F_{11} + F_{12} + F_{13} + \dots F_{1n} = 1 \quad (5.92)$$

If the surface 1 is convex, it will not intercept any radiation emitted by itself and $F_{11} = 0$.

Heat exchange between grey bodies

Real bodies are considered as grey bodies where $\alpha < 1$. So when radiation falls on a grey body, a part is reflected back and treatment becomes complex because of multiple reflection of radiation between surfaces. To overcome this problem, we define G as total incident radiation flux on a surface (Wm^{-2}) and J as the total radiation flux leaving the surface (Wm^{-2}). Obviously J has two components, namely radiation emitted by the surface, ϵq_b , and radiation reflected by the surface, ρG . J is known as *radiosity*. So

$$J = \epsilon q_b + \rho G$$

Using equation 5.85 and 5.88,

$$\rho = 1 - \epsilon$$

Hence,

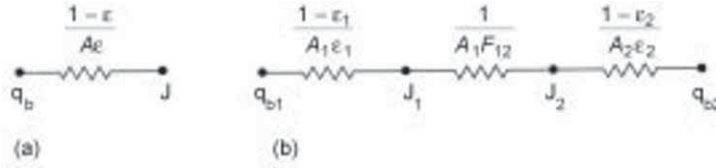
$$J = \epsilon q_b + (1 - \epsilon)G \quad (5.93)$$

Net heat flux leaving the surface is

$$q_{\text{net}} = J - G$$

Substituting the value of G from equation 5.93

$$q_{\text{net}} = \frac{\epsilon}{1 - \epsilon}(q_b - J)$$



5.22 Electrical analogue of radiation heat exchange between grey bodies. (a) Net heat flow from a surface of area A. (b) Heat exchange between two grey bodies.

Therefore net heat flow from a surface of area A is

$$Q_{\text{net}} = \frac{\epsilon}{1-\epsilon} A(q_b - J) \quad (5.94)$$

Using an electrical analogue, net heat flow can be considered as current. The driving force for net heat flow from a surface is $(q_b - J)$ and the resistance is $(1 - \epsilon)/(A\epsilon)$. Figure 5.22(a) shows the electrical analogue for a grey surface.

Now let us consider radiative energy exchange between two grey bodies 1 and 2.

$$Q_{1-2} = Q_{1 \rightarrow 2} - Q_{2 \rightarrow 1} = A_1 F_{12} J_1 - A_2 F_{21} J_2$$

Using the reciprocity theorem, the above equation can be written as

$$Q_{1-2} = \frac{J_1 - J_2}{\frac{1}{A_1 F_{12}}} \quad (5.95)$$

Using the electrical analogue, the numerator of the above equation can be identified as the driving force and the denominator as the resistance for heat exchange. To evaluate J_1 and J_2 and thereby Q_{1-2} , we draw the electrical analogue of heat exchange between two grey bodies, shown in Fig. 5.22(b). It shows that the driving force heat flow from surface 1 is $(q_{b1} - J_1)$ and resistance is $(1 - \epsilon_1)/(A_1 \epsilon_1)$ (equation 5.94). The driving force for heat exchange between surfaces 1 and 2 is $(J_1 - J_2)$ and resistance is $(1/A_1 F_{12})$ (equation 5.95) and finally the driving force for heat flow to surface 2 is $(J_2 - q_{b2})$ and the resistance is $(1 - \epsilon_2)/(A_2 \epsilon_2)$. So

$$Q_{1-2} = \frac{q_{b1} - q_{b2}}{\frac{1-\epsilon_1}{A_1 \epsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{A_2 \epsilon_2}} \quad (5.96)$$

It can be noted that the above equation reduces to equation 5.91 for black bodies.

Let us consider two large parallel plates of equal size separated by a short distance. Here $A_1 = A_2 = A$ and $F_{12} = 1$, hence equation 5.96 simplifies to

$$Q_{1-2} = \frac{A(q_{b1} - q_{b2})}{\frac{1}{\epsilon_1} - 1 + \frac{1}{\epsilon_2}} \quad (5.97)$$

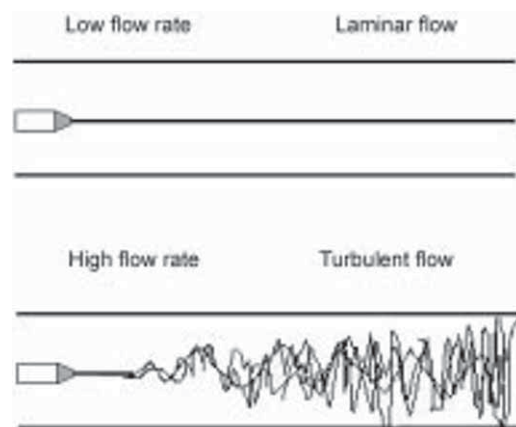
Quite often, we encounter the problem where a small convex grey body is enclosed in an isothermal enclosure. The latter can be treated as a cavity or a black body. Considering the grey body as surface 1 and the enclosure as surface 2, $F_{12} = 1$ and $\epsilon_2 = 1$, so equation 5.96 simplifies to

$$Q_{1-2} = A_1 \epsilon_1 (q_{b1} - q_{b2}) \quad (5.98)$$

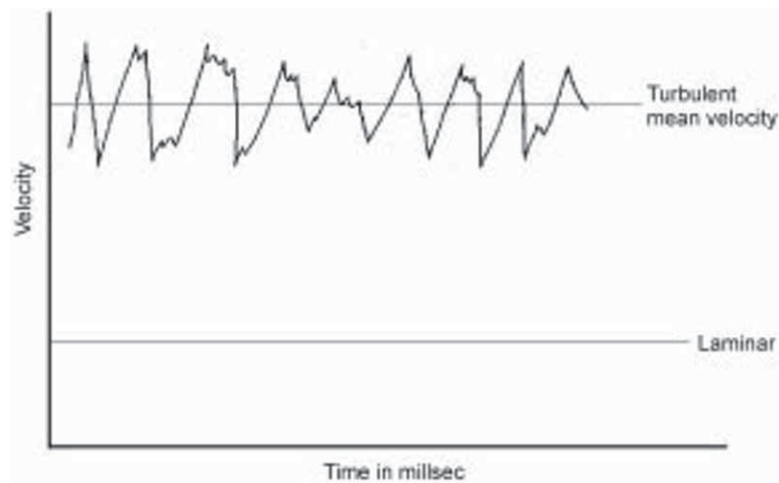
In a real system, gases are mostly present. So heat exchange by radiation between surfaces takes place through them. Gases with symmetric molecules like He, H₂, O₂, N₂, etc. are transparent to thermal radiation so their presence does not affect heat exchange by radiation. But CO, CO₂, H₂O, SO₂, NH₃, HCl, etc. are not transparent to thermal radiation. They absorb and emit thermal radiation in some narrow bands of wavelengths. For example, CO₂ absorbs radiation in wavelength ranges of 2.4–3, 4–4.8 and 12.5–16.5 μm . So if these gases are present in significant amount as in the case of fuel-fired furnaces, their effect on radiation heat exchange cannot be neglected.

5.4 Fluid flow

If we allow water to flow through a long horizontal glass tube and inject a dye into it, we will notice that the dye moves in a straight line at low flow rates of water. But the dye gets dispersed throughout the cross-section of the tube at a short distance from the entry point at high flow rates. Figure 5.23 schematically shows behaviour of dye. This experiment was first performed by Osborne Reynolds in 1883 to show that there are two types of flow behaviour of a fluid. At low flow rates, fluid elements maintain their own path. But at high flow rate, swirling or circular motion is generated within small packets of fluids, known as eddies. These eddies move in random fashion within the fluid and are continuously formed and destroyed by interaction with the surrounding fluid.



5.23 Behaviour of dye streak injected with water in a horizontal tube.



5.24 Schematic diagram of laminar and turbulent velocity at a point at steady state.

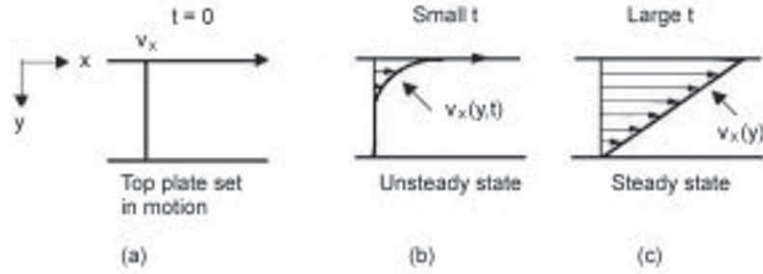
This leads to rapid mixing of dye. The size of eddies can vary from fractions of a millimetre to fairly large values. The first type of flow is known as laminar flow and the second type is known as turbulent flow. Figure 5.24 shows schematically the velocity at a point in laminar and turbulent flow at steady state. Turbulent velocity at a point fluctuates about a mean value because of movements of eddies. This fluctuation is often 20–30% of the mean value.

The transition from laminar to turbulent does not take place suddenly. As velocity is increased a transition region occurs, when the flow becomes unstable and forms local turbulent spots or eddies. With further increases in velocity, local turbulent spots spread and finally the flow becomes fully developed turbulent flow. In the transition region, the dye streak in the pipe flow becomes wavy.

Although most of the flows of metallurgical systems are turbulent, in the present section we will primarily discuss laminar flow.

5.4.1 Newton's law of viscosity

Let us consider a liquid between two parallel plates as shown in Fig. 5.25(a). Initially the liquid is at rest and at time $t = 0$, the top plate is given a constant velocity v_x . The fluid, which is in contact with the top plate, moves with the velocity v_x . The top layer of liquid drags the layer just below it. Similarly this layer in turn drags the layer below it and this continues. Figure 5.25(b) shows the velocity profile after a short time where velocity profile is changing with time. Figure 5.25(c) shows the steady state velocity profile. The velocity of any layer is less than that of the upper one and the velocity of the bottom layer,



5.25 Velocity profile of fluid between two parallel flat plates. (a) At $t = 0$, top plate is set to motion with velocity v_x . (b) Velocity profile after a short time, unsteady state. (c) Velocity profile at steady state.

which is in contact with the stationary plate, is zero. The motion of the top plate has resulted in x momentum (or x component of momentum) which travelled in y direction due to viscous drag. This momentum flux (amount of momentum transferred per unit area per unit time in a direction normal to the area) is given by:

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \quad (5.99)$$

where μ is the viscosity of fluid. The negative sign in the equation shows that the direction of momentum transfer is from high velocity to low velocity. Equation 5.99 is Newton's law of viscosity. The above equation has another meaning. To move the top plate with velocity v_x , a shear force has to be applied. This shear force per unit area or shear stress is τ_{yx} . The subscript yx indicates that the shear force is in x direction and is acting on a plane of constant y . The unit of shear stress is Pa or $\text{kgm}^{-1}\text{s}^{-2}$, the same as that of momentum flux, and the unit of viscosity is $\text{kgm}^{-1}\text{s}^{-1}$ or Pa.s. The fluids that obey this law, i.e. the fluids for which μ is independent of shear rate, (dv_x/dy) , is known as Newtonian fluid. All gases and most of the simple liquids follow Newton's law.

When τ_{yx} is considered as momentum flux, y gives the direction of momentum flux and x the component of momentum. On the other hand when τ_{yx} is considered as shear stress, it is in the x direction and is acting on a fluid surface of constant y . So the direction of momentum flux and shear stress are not the same. Shear stress is always in the direction of velocity or momentum component.

Although equation 5.99 has the same form as that of mass flux given by equation 5.3 or heat flux (equation 5.57), there is a basic difference between this equation and other two. Velocity is a vector but temperature and concentration are scalar. So the heat and mass fluxes are vectors and have three components but the double suffix in τ_{yx} indicates that in three-dimensional problem τ has nine components and it is the stress tensor.

Example 5.9

Two flat plates of area 0.06 m^2 are 0.05 mm apart (Fig. 5.25). The space between the plates is filled by a lubricating oil of viscosity $0.2 \text{ kgm}^{-1}\text{s}^{-1}$. The lower plate is stationary and the upper plate moves with a velocity of 20 mm/s . Calculate the momentum flux from the upper to lower plate at the steady state. What is the force required to keep the upper plate moving?

Solution

At the steady state, the velocity profile is linear as shown in Fig. 5.24. Taking the top plate as $y = 0$,

$$\frac{dv_x}{dy} = (0 - 20 \times 10^{-3}) / (0.05 \times 10^{-3}) = -400 \text{ s}^{-1}$$

Momentum flux

$$\tau_{yx} = -0.2 \times (-400) = 80 \text{ kgm}^{-1}\text{s}^{-2}$$

τ_{yx} is the shear force on the top plate. So the force required to keep the plate moving is

$$\begin{aligned} F &= \tau_{yx} \times \text{area of plate} \\ &= 80 \times 0.06 = 4.8 \text{ N} \end{aligned}$$

**5.4.2 Viscosity of gases and liquid**

The kinetic theory of gases shows that viscosity of gases is directly proportional to the square root of temperature and independent of pressure. The latter is found to be true when pressure is less than 10 atmospheres but exponents of temperature for real gases are higher. The temperature dependence of viscosity is

$$\mu = \mu_0 (T/T_0)^n \quad (5.100)$$

where μ and μ_0 are viscosity at temperature T and T_0 respectively. The exponent n lies in the range 0.6–1.0.

Table 5.10 gives the viscosity of some common gases

Table 5.10 Viscosity of gases ($\text{kgm}^{-1} \text{ s}^{-1}$ or Pa.s)

Gas	H ₂	H ₂ O	CO	CO ₂	Air
$\mu \times 10^5$ at 400K	1.09	1.32	2.21	1.94	2.29
$\mu \times 10^5$ at 800K	1.77	2.95	3.54	3.39	3.64

Viscosity of liquids

Viscosities of liquids vary widely depending on the nature of liquid. Viscosities of water and liquid metal are very low but that of slag and some of the organic liquids are very high. Viscosity of liquid decreases with temperature. Table 5.11 shows the viscosity of several liquids.

Table 5.11 Viscosity of liquids ($\text{kgm}^{-1} \text{s}^{-1}$ or Pa.s)

Material	Viscosity
Water at 293K	0.86×10^{-3}
Glycerin at 293K	1.49
Lead at 700K	2.15×10^{-3}
Iron at 2000K	5.6×10^{-3}
Slags	0.1–10

5.4.3 Conservation of momentum

Quite often, we need to find out the velocity distribution in a system. This is obtained by solving the equation of continuity or conservation of mass and conservation of momentum or equation of motion. These equations are

Rate of mass accumulation = rate of mass in – rate of mass out

Rate of momentum accumulation = rate in – rate out
+ sum of forces acting on system

In Sections 5.2.3 and 5.3.3 we have derived the conservation of mass and heat in a differential volume. Following the same procedure, the equation of continuity and conservation of momentum can be derived. The final form of these equations for incompressible Newtonian fluids in different coordinate systems is given in Tables 5.12 and 5.13. The equation of motion given in Table 5.13 is known as *Navier–Stokes* equation. Since all liquids are incompressible and even gases can be considered as incompressible if the velocity is much less than that of sound, these equations have a very wide applicability. The equation of motion contains four unknowns, three components of velocity and pressure. The three equations of motion given in Table 5.13 along with the equation of continuity given in Table 5.12 are the required four equations for the four unknowns.

Flow through a pipe

Let us consider steady state liquid flow through a horizontal pipe as an example of the application of the equation of motion. The flow is laminar and we are

Table 5.12 Equation of continuity for incompressible fluid

Rectangular coordinates

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad [A]$$

Cylindrical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad [B]$$

Spherical coordinates

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} = 0 \quad [C]$$

interested in the velocity profile away from the entrance zone where the entrance effect is absent and the flow is fully developed. Since the system is cylindrical, we use a cylindrical coordinate system. Fluid flows in the z direction, so velocity components in the radial and θ direction, v_r and v_θ are zero. Using the above considerations, the equation of continuity in Table 5.12, equation B, simplifies to

$$\frac{\partial v_z}{\partial z} = 0 \quad (5.101)$$

The above equation indicates that v_z is independent of z .

The relevant equation of motion is equation F in Table 5.13. This equation gets simplified for the following reasons:

- (a) Cylindrical symmetry suggests that v_z is independent of θ and the equation of continuity, equation 5.101, shows that v_z is independent of z . So all derivatives of v_z with respect to z and θ are zero.

Table 5.13 Equation for motion for Newtonian fluid with constant ρ and μ

Rectangular coordinates

x component

$$\rho \left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right] = \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] - \frac{\partial p}{\partial x} + \rho g_x \quad [A]$$

y component

$$\rho \left[\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right] = \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] - \frac{\partial p}{\partial y} + \rho g_y \quad [B]$$

z component

$$\rho \left[\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right] = \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z \quad [C]$$

Table 5.13 Continued

Cylindrical coordinates

r component

$$\begin{aligned} & \rho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right] \\ &= \mu \left[\frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right\} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] - \frac{\partial p}{\partial r} + \rho g_r \end{aligned} \quad [D]$$

 θ component

$$\begin{aligned} & \rho \left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right] \\ &= \mu \left[\frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right\} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \end{aligned} \quad [E]$$

z component

$$\begin{aligned} & \rho \left[\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right] \\ &= \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial v_z}{\partial r} \right\} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z \end{aligned} \quad [F]$$

Spherical coordinates

r component

$$\begin{aligned} & \rho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{(v_\theta^2 + v_\phi^2)}{r} \right] \\ &= \mu \left[\nabla^2 v_r - \frac{2}{r^2} v_r - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2} v_\theta \cot \theta - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] - \frac{\partial p}{\partial r} + \rho g_r \end{aligned} \quad [G]$$

 θ component

$$\begin{aligned} & \rho \left[\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right] \\ &= \mu \left[\nabla^2 v_\theta + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\phi}{\partial \phi} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \end{aligned} \quad [H]$$

 ϕ component

$$\begin{aligned} & \rho \left[\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right] \\ &= \mu \left[\nabla^2 v_\phi + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2 \sin^2 \theta} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \phi} \right] - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \rho g_\phi \end{aligned} \quad [I]$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

- (b) Since we are considering steady state, the time derivative is zero.
- (c) For a horizontal pipe, the component of acceleration due to gravity in z direction g_z is zero.
- (d) Velocity components in radial and θ direction, v_r and v_θ are zero.

So the z component of the equation of motion in the cylindrical coordinate simplifies to

$$\mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{dp}{dz} \quad (5.102)$$

Since v_z is a function of r only, the partial derivative is replaced by the total. Integrating,

$$\mu r \frac{dv_z}{dr} = \frac{r^2}{2} \frac{dp}{dz} + C_1$$

Integrating again,

$$\mu v_z = \frac{r^2}{4} \frac{dp}{dz} + C_1 \ln r + C_2$$

The boundary conditions are

At $r = 0$, v_z is finite

At $r = r_0$, $v_z = 0$

Using the above conditions,

$$C_1 = 0 \quad \text{and} \quad C_2 = -\frac{r_0^2}{4} \frac{dp}{dz}$$

Hence

$$v_z = \frac{r^2 - r_0^2}{4\mu} \frac{dp}{dz} \quad (5.103)$$

If the length of pipe is L, and inlet and outlet pressures are P_0 and P_L respectively, $dp/dz = (P_L - P_0)/L$. So the above equation can be written as

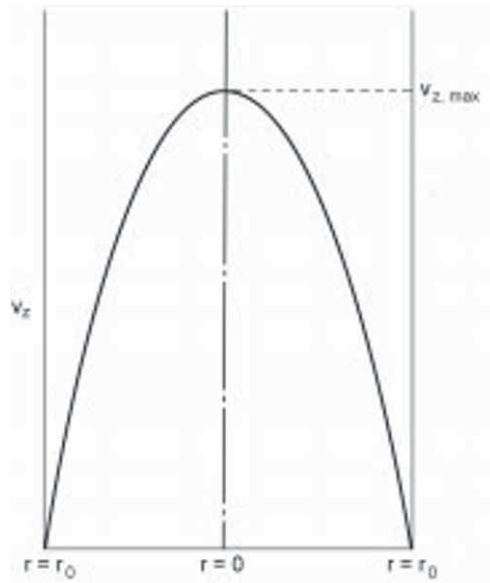
$$v_z = \frac{(r_0^2 - r^2)(P_0 - P_L)}{4\mu L} \quad (5.104)$$

Figure 5.26 shows the parabolic velocity profile. The maximum velocity is at $r = 0$ and has the value

$$v_{z,\max} = \frac{r_0^2(P_0 - P_L)}{4\mu L} \quad (5.105)$$

Volumetric flow rate is given by

$$Q = \int_0^{r_0} 2\pi r v_z dr = \frac{\pi(P_0 - P_L)r_0^4}{8\mu L} \quad (5.106)$$



5.26 Velocity profile of laminar flow in a pipe.

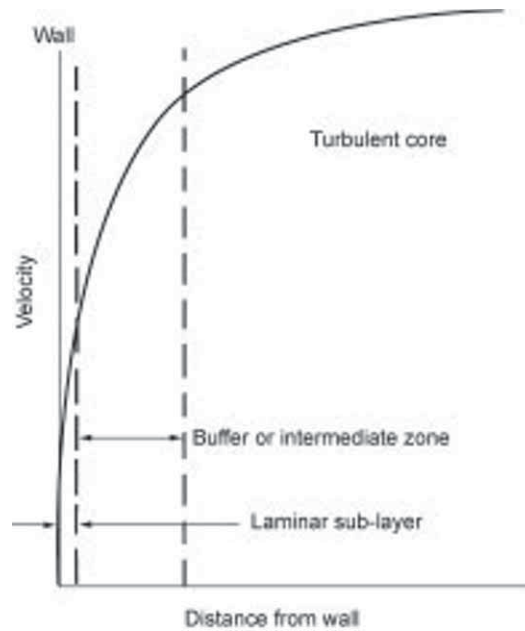
Equation 5.106 is known as the Hagen–Poiseuille law.

The shear force of fluid on the pipe wall is

$$F_z = 2\pi r_0 L \left(-\mu \frac{dv_z}{dr} \right)_{r=r_0} = \pi r_0^2 (P_0 - P_L) \quad (5.107)$$

This shows that viscous force counterbalances the force acting on the fluid. When fluid flows through a vertical pipe, viscous forces counterbalance force due to gravity and pressure.

The above equations are valid for laminar flow, which is obtained for Reynolds number $Re = \rho v_{z,av} D / \mu$, where $D = 2r_0$ and average velocity $v_{z,av} = Q / (\pi r_0^2)$, is less than 2100. Besides, there is no entrance effect. A distance of $L_e = 0.0567 D Re$ is required for the development of the parabolic profile given by equation 5.104. Although the critical Reynolds number for transition from laminar flow is normally taken as 2100, pipe flow can be maintained laminarly even at much higher Reynolds numbers in controlled experiments where all external disturbances are avoided. When the Reynolds number is more than 10,000 the flow is fully turbulent. However, even when the flow is turbulent, it is not turbulent near the wall. Figure 5.27 schematically shows the velocity profile near the wall. Close to the wall, the viscous forces dominate and the flow is laminar. This layer is known as the viscous sub-layer or laminar sub-layer. Some distance away from the wall, the velocity profile is relatively flat. This region is the turbulent core or fully developed turbulent flow. The intermediate region is known as the buffer or intermediate zone. Eddies are generated here. The



5.27 Velocity profile near wall in turbulent flow.

thickness of buffer layer is about six times more than that of viscous sub-layer and total thickness of these two layers is less than a millimetre.

When the flow is fully turbulent velocity distribution follows the relation

$$\frac{v_z}{v_{z,\max}} = \left(1 - \frac{r}{r_0}\right)^{1/7} \quad (5.108)$$

In the case of laminar flow $v_{z,\max}/v_{z,\text{av}} = 2$ and for turbulent flow it is $5/4$.

Example 5.9

Calculate the maximum flow rate of water through a pipe of diameter 20 mm when the parabolic velocity profile given by equation 5.104 is applicable. What is the corresponding pressure drop per metre length of pipe. Assume viscosity and density of water as $0.86 \times 10^{-3} \text{ kgm}^{-1}\text{s}^{-1}$ and 1000 kgm^{-3} respectively.

Solution

Equation 5.104 is valid if Re is less than 2100 or $\rho v_{z,\text{av}} D / \mu = 2100$,

$$\begin{aligned} v_{z,\text{av}} &= (2100 \times 0.86 \times 10^{-3}) / (1000 \times 20 \times 10^{-3}) \\ &= 0.09 \text{ m/s} = 90 \text{ mm/s} \end{aligned}$$

$$\text{Flow rate } Q = \pi r_0^2 v_{z,\text{av}} = 28.3 \times 10^{-6} \text{ m}^3/\text{s}$$

From equation 5.106

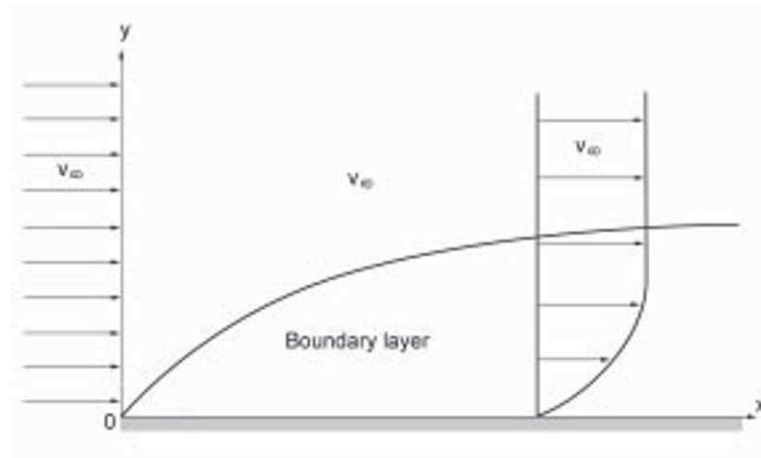
$$\begin{aligned}(P_0 - P_L)/L &= (8\mu Q)/(\pi r_0^4) \\ &= (8 \times 0.86 \times 10^{-3} \times 28.3 \times 10^{-6})/(\pi \times 0.01^4) = 6.2 \text{ Pa}\end{aligned}$$

Since kinematic viscosity of water μ/ρ is very low about $10^{-6} \text{ m}^2/\text{s}$, flow is turbulent at a very low flow rate. Liquid metals also have similar low values of kinematic viscosity, so the flow of liquid metal is mostly turbulent. ■■■

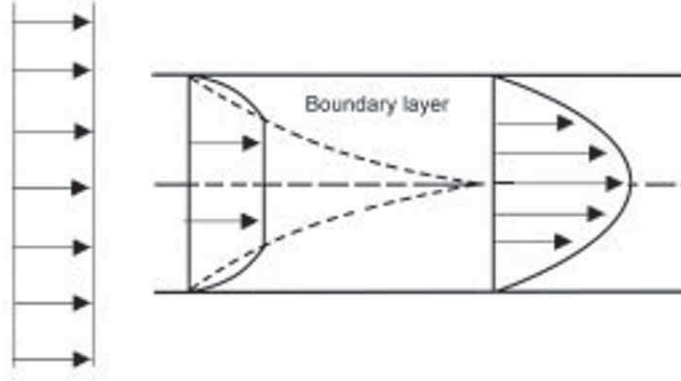
Flow over a flat plate

When a fluid flows over an immersed body, such as a flat plate, the velocity of fluid at the solid–fluid interface is same as that of the immersed body. Viscous forces affect the flow near the solid surface but further away the flow is almost inviscid and velocity is same as that of the free stream. This layer where the velocity of fluid changes from that of the solid to the free stream is known as the boundary layer and this type of flow is known as the boundary layer flow. Figure 5.28 schematically shows the velocity profile over a flat plate. The boundary layer thickness is zero initially and it grows as the fluid moves along the plate. In the case of flow through a duct, the viscous boundary layers grow from the walls and fill the entire duct. So, except at the entrance region, there is no boundary layer separating viscous and inviscid flow, as shown in Fig. 5.29. Flow beyond the entrance region is fully developed flow.

Let us assume that the fluid stream is approaching the stationary flat plate at a uniform velocity v_∞ . The flow is in the x direction. The z direction is normal to the paper. On the left side of the plate the fluid velocity is uniform, $v_x = v_\infty$. But over the entire surface of the plate ($y = 0$), v_x is zero because there is no slip condition and in the entire boundary layer $v_x < v_\infty$. The fluid whose velocity has decreased moves up and this gives rise to the small v_y component in the



5.28 Flow over a flat plate.



5.29 Velocity profile at the entrance region in a duct.

boundary layer. Thereby within the boundary layer there are two components of velocity v_x and v_y and outside the boundary layer there is only v_x . There is no velocity gradient in the z direction, so $\partial v_x / \partial z = \partial v_y / \partial z = 0$. Since it is a steady state problem, the time derivatives are zero. Besides, in the x direction gravity force is zero and we neglect gravity in the y direction. With these simplifications, the equation of continuity, equation A in Table 5.12, and the equations of motion for the x and y components, equations A and B in Table 5.13, in the boundary layer become

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (5.109a)$$

$$\rho \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right] = \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right] - \frac{\partial p}{\partial x} \quad (5.109b)$$

$$\rho \left[v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right] = \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right] - \frac{\partial p}{\partial y} \quad (5.109c)$$

These equations should be solved for the unknowns v_x , v_y and p subject to no slip boundary condition on the surface of the plate and inlet and exit boundary conditions. These equations are too complex for analytical solution. However, the equations can be simplified from the following considerations:

1. $v_y \ll v_x$, so the y component of the momentum balance equation, 5.109c can be neglected.
2. The terms $\rho v_x \frac{\partial v_x}{\partial x}$ and $\mu \frac{\partial^2 v_x}{\partial x^2}$ in equation 5.109b are the net rate of x component of momentum transfer in the x direction due to bulk flow and viscous flow respectively. The term $\mu \frac{\partial^2 v_x}{\partial y^2}$ can be neglected since viscous diffusion is much smaller than the bulk flow.
3. $\partial p / \partial x$ is negligible.

So the above equations can be simplified to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (5.110a)$$

$$\rho \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right] = \mu \frac{\partial^2 v_x}{\partial y^2} \quad (5.110b)$$

The boundary conditions for all x are

$$\text{At } y = 0, \quad v_x = v_y = 0 \quad (5.111a)$$

$$\text{At } y = \infty, \quad v_x = v_\infty \quad (5.111b)$$

$$\text{At } x = 0 \text{ for all } y, \quad v_x = v_\infty \quad (5.111c)$$

To solve the above equations let us define 'stream function' ψ as

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}$$

The substitution of v_x and v_y , defined above, in equation 5.110a shows that the stream function satisfies the equation of continuity so we need to solve only equation 5.110b, which becomes

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} \quad (5.112)$$

where $\nu = (\mu/\rho)$ is the kinematic viscosity. To solve the above equation, let us define two dimensionless variables

$$f = \psi / (x \nu v_\infty)^{1/2} \quad (5.113)$$

$$\eta = y (v_\infty / x \nu)^{1/2} \quad (5.114)$$

In terms of these new variables, equation 5.112 becomes an ordinary differential equation.

$$ff'' + 2f''' = 0 \quad (5.115)$$

where

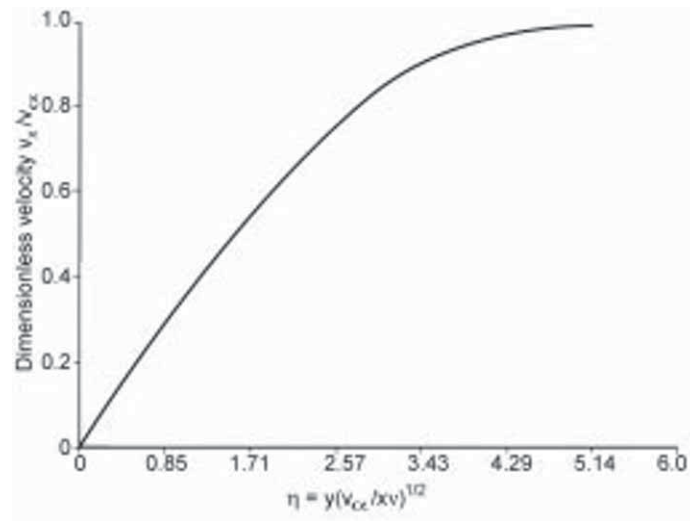
$$f' = \frac{df}{d\eta}$$

The boundary conditions given by equations 5.111(a, b, c) become

$$\text{At } \eta = 0, \quad f = f' = 0$$

$$\text{At } \eta = \infty, \quad f' = 1$$

Equation 5.115 indicates that f and thereby velocity is a function of η only. The equation was first solved by Blasius in 1908 by power series expansion. Figure 5.30 shows the results of numerical solution of the equation. The dimensionless velocity v_x/v_∞ rapidly increases with η and becomes almost unity at $\eta = 5$. The

5.30 Dimensionless velocity vs η .

boundary layer thickness δ is taken as the point where velocity is 99% of the free stream velocity or $v_x/v_\infty = 0.99$. This corresponds to $\eta \approx 5$, so

$$\delta(x) = 5(\nu x/v_\infty)^{1/2} \quad (5.116)$$

The boundary layer thickness grows along the length of flat plate following the square root law and the thickness is inversely proportional to the square root of the free stream velocity. In terms of the Reynolds number, equation 5.116 can be written as

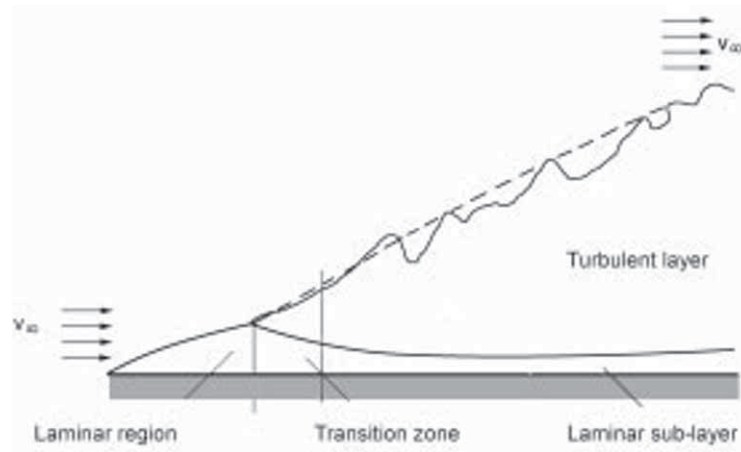
$$\delta(x)/x = 5/\text{Re}_x^{1/2} \quad (5.117)$$

where $\text{Re}_x = \rho v_\infty x/\mu$.

The above equation is valid when the Reynolds number is smaller than 5×10^5 and the flat surface is very smooth. As the fluid moves along the flat surface the Reynolds number increases and the flow becomes unstable which finally leads to a turbulent boundary layer when $\text{Re} > 5 \times 10^6$. Figure 5.31 shows the turbulent boundary layer. The turbulent boundary layer grows much faster along the length of the plate. Because of the fluctuating nature of turbulent flow the edge of boundary layer is not smooth.

Example 5.10

Calculate the boundary layer thickness at the trailing edge of a 0.5 m long plate, for (a) air, (b) water, and (c) mercury at 20°C when the free stream velocity is 0.2 m/s. ν for air, water and mercury are 1.5×10^{-5} , 0.9×10^{-6} and 1.1×10^{-7} m²/s, respectively.



5.31 Turbulent boundary layer.

Solution

- (a) For air, $Re_x = v_\infty x / \nu = 0.2 \times 0.5 / 1.5 \times 10^{-5} = 0.67 \times 10^4$

Since $Re_x < 5 \times 10^5$, equation 5.117 is applicable and

$$\delta = 5 \times 0.5 / (0.67 \times 10^4)^{1/2} \text{ m} = 30.5 \text{ mm}$$

- (b) For water, $Re_x = 0.2 \times 0.5 / 0.9 \times 10^{-6} = 1.11 \times 10^5$

Since this is less than 10^6 , $\delta = 5 \times 0.5 / (1.11 \times 10^5)^{1/2} \text{ m} = 7.5 \text{ mm}$

- (c) For mercury, $Re_x = 0.2 \times 0.5 / 1.1 \times 10^{-7} = 0.91 \times 10^6$

This value lies in the transition range, so the calculated value may not be correct if the plate is not very smooth and the flow is not quiet. However, assuming the equation is valid

$$\delta = 5 \times 0.5 / (0.91 \times 10^6)^{1/2} \text{ m} = 2.6 \text{ mm}$$

The boundary layer thickness is highest for air and lowest for mercury. δ for air is about four times more than that of water and about 11.5 times more than that of mercury. This is because $\nu_{\text{air}}/\nu_{\text{water}}$ is 16.6 and $\nu_{\text{air}}/\nu_{\text{mercury}}$ is 136. ■■■

Example 5.11

A 5 mm thick film of water is flowing over a flat plate of 1 m long. At what distance from the leading edge will the boundary layer thickness be the same as the film thickness. The free stream velocity at the leading edge is 0.5 m/s. ν for water is $0.9 \times 10^{-6} \text{ m}^2/\text{s}$

Solution

We want to know the value of x when $\delta = 5 \text{ mm} = 5 \times 10^{-3} \text{ m}$ and $v_\infty = 0.5 \text{ m/s}$. From equation 5.116

$$\delta(x) = 5(\nu x / v_\infty)^{1/2}$$

So

$$5 \times 10^{-3} = 5(0.9 \times 10^{-6} x/0.5)^{1/2}$$

or

$$x = 0.5/0.9 = 0.55 \text{ m}$$

So 0.55 m is the length of entrance zone. After this the flow is fully developed.



5.4.4 Friction factor and drag coefficient

When a fluid flows through a pipe, shear force acts on the pipe wall. This is called friction force and is defined as

$$F_k = \frac{1}{2} \rho v_{av}^2 f A \quad (5.118)$$

where v_{av} is the average velocity of the fluid, f is the friction factor and A is the surface area on which shear force is acting. For flow through a pipe, $A = \pi DL$. Since F_z defined by equation 5.107 and F_k defined by equation 5.118 are the same,

$$f = \frac{D}{4L} \frac{(P_0 - P_L)}{\frac{1}{2} \rho v_{av}^2} \quad (5.119)$$

By dimensional analysis it can be shown that for fully developed flow, friction factor f depends only on the Reynolds number. For flow in a long tube

$$f = 16/\text{Re} \quad \text{Re} < 2.1 \times 10^3 \quad (5.120a)$$

$$f = 0.079/(\text{Re})^{1/4} \quad 2.1 \times 10^3 < \text{Re} < 10^5 \quad (5.120b)$$

Equation 5.120(a) is obtained from the Hagen–Poiseuille equation.

When the cross-section of pipe is not circular, the effective diameter is calculated using the concept of hydraulic radius. Hydraulic radius is defined as

$$R_h = (\text{cross-sectional area})/(\text{wetted perimeter}) \quad (5.121)$$

For circular tubing hydraulic radius is $D/4$. For non-circular tube equation 5.119 becomes

$$f = \frac{R_h}{L} \frac{(P_0 - P_L)}{\frac{1}{2} \rho v_{av}^2} \quad (5.122)$$

The concept of hydraulic radius is not applicable for laminar flow.

Flow around a sphere

When fluid flows over an immersed object, shear force acting on the fluid is called drag force and is defined as

$$F_k = \frac{1}{2} \rho v_\infty^2 C_D A_p \quad (5.123)$$

where v_∞ is the velocity far away from the immersed object, C_D is the drag coefficient and A_p is the projected area of the solid on the fluid. When fluid flows over a sphere of radius r at a very low flow rate such that the Reynolds number defined as $\rho v_\infty d_p / \mu < 0.1$

$$F_k = 6\pi\mu r v_\infty \quad (5.124)$$

This is known as Stokes law. From equations 5.123 and 5.124 and using $A_p = \pi d_p^2/4$ where d_p is the diameter of the sphere, we get

$$C_D = 24/\text{Re} \quad \text{Re} < 0.1 \quad (5.125)$$

For higher flow rates,

$$C_D = 18.5/(\text{Re})^{3/5} \quad 2 < \text{Re} < 5 \times 10^2 \quad (5.126a)$$

$$C_D = 0.44 \quad 5 \times 10^2 < \text{Re} < 2 \times 10^5 \quad (5.126b)$$

When the Reynolds number lies in the range 0.1 to 2 equation 5.125 is often used for estimation of drag force. Equation 5.126b is termed as Newton's Law.

Terminal velocity

Stokes law is often used to calculate the terminal velocity of particles or bubbles in a fluid medium. Let us assume that the density of the particles is greater than that of the fluid medium, so the particle falls down. The downward force acting on the particle is due to gravity and upward forces are buoyant force and drag force. Initially the velocity of the particle will go on increasing because of net downward force. But as velocity increases, the drag force also increases since it is directly proportional to velocity (see equation 5.124), and finally the upward and downward forces are balanced. This velocity, which is attained when the total force acting on the particle is zero, is the terminal velocity. The force balance equation is

$$\frac{4}{3} \pi r^3 (\rho - \rho_f) g = 6\pi\mu r v_t$$

ρ and ρ_f are, respectively, density of particle and fluid and v_t is the terminal velocity. The left-hand side of the above equation is the difference between gravity and buoyant force and the right-hand side is the drag force. So the terminal velocity is

$$v_t = \frac{2}{9} \frac{r^2 (\rho - \rho_f) g}{\mu} \quad (5.127)$$

Example 5.12

Calculate the terminal velocity when a sand particle of $10\ \mu\text{m}$ diameter is falling through a water column. Density of sand and water are respectively 2500 , $1000\ \text{kgm}^{-3}$ respectively and viscosity of water is $9.6 \times 10^{-4}\ \text{kgm}^{-1}\ \text{s}^{-1}$.

Solution

Using equation 5.127 and $r = 0.5 \times 10^{-5}\ \text{m}$

$$v_t = [2 \times (0.5 \times 10^{-5})^2 \times (2500 \times 1000) \times 9.8] / (9 \times 9.6 \times 10^{-4})\ \text{m/s} \\ = 0.085\ \text{mm/s}$$

$$\text{Re} = 2500 \times 0.85 \times 10^{-4} \times 10^{-4} / 9.6 \times 10^{-4} = 0.022$$

Since Re is less than 0.1 , calculated value of terminal velocity is correct.

*Correlations for packed bed*

The packed bed is made up of solids of different shapes and sizes. The fluid flows through the voids in the bed. In general, the distributions of voids and the particle sizes are not uniform throughout the bed. But for our discussion we assume that distributions of particles, voids and gas are uniform. If the bed is made of spheres of diameter d_p , the contact area between the fluid and solid per unit bed volume is

$$S_p = n \ 4\pi(d_p/2)^2 \quad (5.128)$$

where S_p is the specific surface area (m^2/m^3) and n is the number of particles per unit volume. Let ϵ be the void fraction in the bed. Then out of unit bed volume $(1 - \epsilon)$ is occupied by the particles and the rest is void. So

$$1 - \epsilon = n \ 4/3\pi(d_p/2)^3 \quad (5.129)$$

From equations 5.128 and 5.129

$$S_p = 6(1 - \epsilon)/d_p \quad (5.130)$$

If the bed is made up of mixed particle size, d_p is the mean particle size.

In the case of the packed bed, we define superficial velocity as the velocity of gas in the empty tube. Superficial velocity v_0 is related to true average velocity v_b of the gas through the void as

$$v_b = v_0/\epsilon \quad (5.131)$$

If we consider that gas flow through the packed bed is similar to gas flow through a pipe of variable cross-section, the pressure drop and friction factor are related by equation 5.122. Since the cross-section of tube is not constant, hydraulic radius can be defined as the ratio void volume per unit bed volume to

surface area of particles per unit bed volume,

$$R_h = \frac{\epsilon}{S_p} = \frac{d_p \epsilon}{6(1 - \epsilon)} \quad (5.132)$$

Substituting the value of R_h and v_b in equation 5.122

$$f = \frac{(P_0 - P_L)}{\rho v_0^2} \frac{d_p \epsilon^3}{3(1 - \epsilon)L}$$

The bed friction is defined as

$$f_b = \frac{f}{3} = \frac{(P_0 - P_L)}{\rho v_0^2} \frac{d_p \epsilon^3}{(1 - \epsilon)L} \quad (5.133)$$

In the laminar and turbulent flow region, f_b is related to the Reynolds number by

$$f_b = \frac{150}{Re_b} \quad Re_b < 20 \quad (5.134a)$$

$$f_b = 1.75 \quad 10^3 < Re_b < 10^4 \quad (5.134b)$$

where

$$Re_b = \frac{d_b \rho v_0}{\mu(1 - \epsilon)} \text{ is the Reynolds number for the packed bed.}$$

Equations 5.134a and 5.134b are respectively Kozeny–Carman and Burke–Plummer equations. The former is applicable in laminar flow. Ergun combined the above two equations:

$$f_b = \frac{150}{Re_b} + 1.75 \quad 1 < Re < 10^4 \quad (5.135)$$

This equation is known as Ergun's equation.

Combining equation 5.133 and Ergun's equation 5.135 and substituting for Re_b ,

$$\frac{P_0 - P_L}{L} = \left[\frac{150\mu(1 - \epsilon)}{d_b \rho v_0} + 1.75 \right] \frac{\rho v_0^2}{d_p} \frac{1 - \epsilon}{\epsilon^3} \quad (5.136)$$

The above equation shows that the pressure drop in a packed bed is very sensitive to bed porosity and particle size. A decrease in bed porosity or particle size increases pressure drop very significantly.

5.5 Further reading

R.I.L. Guthrie (1989) *Engineering in Process Metallurgy*, Oxford.

F.P. Incropera and D.P. DeWitt (1990) *Fundamentals of Heat and Mass Transfer*, John Wiley, New York.

D.R. Poirier and G.H. Geiger (1994) *Transport Phenomena in Materials Processing*, Publication of TMS.

5.6 References

- Darken, L.S. (1949) *Trans. A.I.M.E.*, vol. 180, 430.
- Fuller, E.N., Schettler, P.D. and Giddings, J.C. (1966) *Ind. Eng. Chem.* Vol. 58 no. 5, 18.
- Kucera, J. and Stransky, K. (1982) *Mater. Sc. Eng.*, Vol. 52, 1.
- Morita, Zen-ichiro (1996) *ISIJ Int.*, Vol. 36, Supplement, S6.
- Reid, R.C., Prausnitz, J.M. and Sherwood, T.K. (1977) *Properties of Gases and Liquids*, 3rd edn. McGraw-Hill, New York.