

FIGURE 10.11 Shannon-Fano encoding

Source words	Probability	Codeword	(Length) · (Probability)
x_1	0.2500	00	2 (0.25) = 0.50
x_2	0.2500	01	2 (0.25) = 0.50
$A \text{ --- } A'$			
x_3	0.1250	100	3 (0.125) = 0.375
x_4	0.1250	101	3 (0.125) = 0.375
x_5	0.0625	1100	4 (0.0625) = 0.25
x_6	0.0625	1101	4 (0.0625) = 0.25
x_7	0.0625	1110	4 (0.0625) = 0.25
x_8	0.0625	1111	4 (0.0625) = 0.25
Average wordlength = 2.75			

Shannon-Fano Encoding

There are several methods of encoding a source output so that an instantaneous code results. We consider two such methods here. First, we consider the Shannon-Fano method, which is very easy to apply and usually yields source codes having reasonably high efficiency. In the next subsection we consider the Huffman encoding technique, which yields the source code having the shortest average word length for a given source entropy.

Assume that we are given a set of source outputs that are to be encoded. These source outputs are first ranked in order of nonincreasing probability of occurrence, as illustrated in Figure 10.11. The set is then partitioned into two sets (indicated by line $A-A'$) that are as close to equiprobable as possible, and 0's are assigned to the upper set and 1's to the lower set, as seen in the first column of the codewords. This process is continued, each time partitioning the sets with as nearly equal probabilities as possible, until further partitioning is not possible. This scheme will give a 100% efficient code if the partitioning always results in equiprobable sets; otherwise, the code will have an efficiency of less than 100%. For this particular example,

$$\text{Efficiency} = \frac{H(X)}{\bar{L}} = \frac{2.75}{2.75} = 1 \quad (10.79)$$

since equiprobable partitioning is possible.

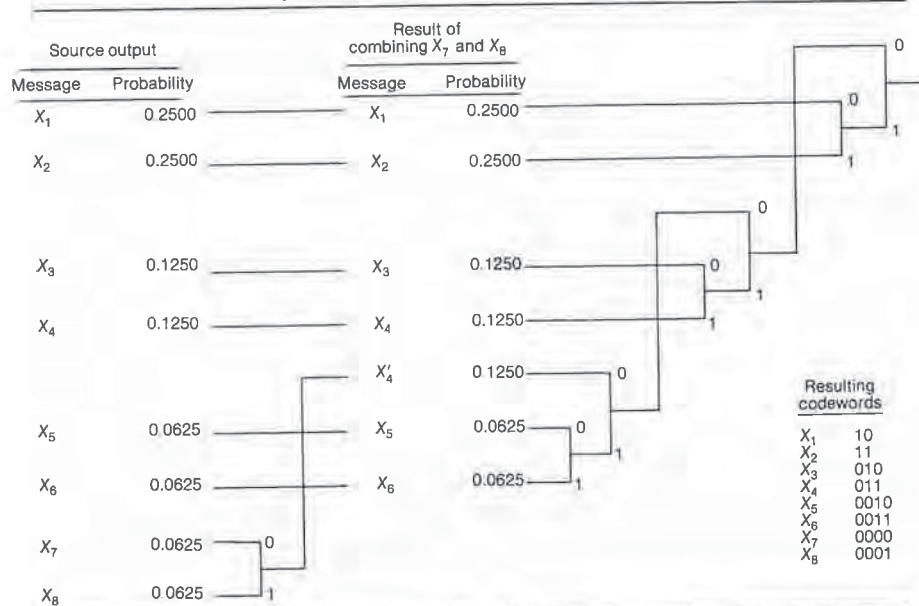
Huffman Encoding

Huffman encoding results in an optimum code in the sense that the Huffman code has the minimum average word length for a source of given entropy. Thus it is the code that has the highest efficiency. We shall illustrate the

Huffman coding procedure using the same source output of eight messages used to illustrate the Shannon-Fano encoding procedure.

Figure 10.12 illustrates the Huffman encoding procedure. The source output consists of messages $X_1, X_2, X_3, X_4, X_5, X_6, X_7$, and X_8 . They are listed in order of nonincreasing probability, as was done for Shannon-Fano encoding. The first step of the Huffman procedure is to combine the two source messages having the lowest probability, X_7 and X_8 . The upper message, X_7 , is assigned a binary 0 as the last symbol in the codeword, and the lower message, X_8 , is assigned a binary 1 as the last symbol in the codeword. The combination of X_7 and X_8 can be viewed as a composite message having a probability equal to the sum of the probabilities of X_7 and X_8 , which in this case is 0.1250, as shown. This composite message is denoted X'_4 . After this initial step, the new set of messages, denoted $X_1, X_2, X_3, X_4, X_5, X_6$, and X'_4 are arranged in order of nonincreasing probability. Note that X'_4 could be placed at any point between X_2 and X_5 , although it was given the name X'_4 because it was placed after X_4 . The same procedure is then applied once again. The messages X_5 and X_6 are combined. The resulting composite message is combined with X'_4 . This procedure is continued as far as possible. The resulting tree structure is then traced in reverse to determine the codewords. The resulting codewords are shown in Figure 10.12.

FIGURE 10.12 Example of Huffman encoding



The codewords resulting from the Huffman procedure are different from the codewords resulting from the Shannon-Fano procedure because at several points the placement of composite messages resulting from previous combinations was arbitrary. The assignment of binary 0's or binary 1's to the upper or lower messages was arbitrary. Note, however, that the average word length is the same for both procedures. This must be the case for the example chosen because the Shannon-Fano procedure yielded 100% efficiency and the Huffman procedure can be no worse. There are cases in which the two procedures do not result in equal average wordlengths.

10.3 RELIABLE COMMUNICATION IN THE PRESENCE OF NOISE

We now turn our attention to methods for achieving reliable communication in the presence of noise by combating the effects of that noise. We undertake our study with a promise from Claude Shannon of considerable success. *Shannon's theorem*, sometimes referred to as the *fundamental theorem of information theory*, is stated: *Given a discrete memoryless channel (each symbol is perturbed by noise independently of all other symbols) with capacity C and a source with positive rate R , where $R < C$, there exists a code such that the output of the source can be transmitted over the channel with an arbitrarily small probability of error.*

Thus Shannon's theorem predicts essentially error-free transmission in the presence of noise. Unfortunately, the theorem tells us only of the existence of codes and tells nothing of how to construct these codes.

Before we start our study of constructing codes for noisy channels, we will take a minute to discuss the continuous channel. This detour will yield considerable insight that will prove useful.

The Capacity of a White Additive Gaussian Noise Channel

The capacity, in bits per second, of a continuous channel with additive white Gaussian noise is given by

$$C_c = B \log_2 \left(1 + \frac{S}{N} \right) \quad (10.80)$$

where B is the channel bandwidth in hertz and S/N is the signal-to-noise power ratio. This particular formulation is known as the *Shannon-Hartley law*. The subscript is used to distinguish (10.80) from (10.43). Capacity, as expressed by (10.43) has units of bits per symbol, while (10.80) has units of bits per second.

The tradeoff between bandwidth and signal-to-noise ratio can be seen

from the Shannon-Hartley law. For infinite signal-to-noise ratio, which is the noiseless case, the capacity is infinite for any nonzero bandwidth. We will show, however, that the capacity cannot be made arbitrarily large by increasing bandwidth if noise is present.

In order to understand the behavior of the Shannon-Hartley law for the large-bandwidth case, it is desirable to place (10.80) in a slightly different form. The energy per bit E_b is equal to the bit time T_b multiplied by the signal power S . At capacity, the bit rate R_b is equal to the capacity. Thus $T_b = 1/C_c$ seconds per bit. This yields, at capacity,

$$E_b = ST_b = \frac{S}{C_c} \quad (10.81)$$

The total noise power in bandwidth B is given by

$$N = N_0 B \quad (10.82)$$

where N_0 is the single-sided noise power spectral density in watts per hertz. The signal-to-noise ratio can therefore be expressed as

$$\frac{S}{N} = \frac{E_b}{N_0} \frac{C_c}{B} \quad (10.83)$$

This allows the Shannon-Hartley law to be written in the equivalent form

$$\frac{C_c}{B} = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C_c}{B} \right) \quad (10.84)$$

Solving for E_b/N_0 yields

$$\frac{E_b}{N_0} = \frac{B}{C_c} (2^{C_c/B} - 1) \quad (10.85)$$

This expression establishes performance of the ideal system. For the case in which $B \gg C_c$,

$$2^{C_c/B} = e^{(C_c/B) \ln 2} \cong 1 + \frac{C_c}{B} \ln 2 \quad (10.86)$$

where the approximation $e^x \cong 1 + x$, $x \ll 1$, has been used. Substitution of (10.86) into (10.85) gives

$$\frac{E_b}{N_0} \cong \ln 2 = -1.6 \text{ dB}, \quad B \gg C_c \quad (10.87)$$

Thus, for the ideal system, in which $R_b = C_c$, E_b/N_0 approaches the limiting value of -1.6 dB as the bandwidth grows without bound.

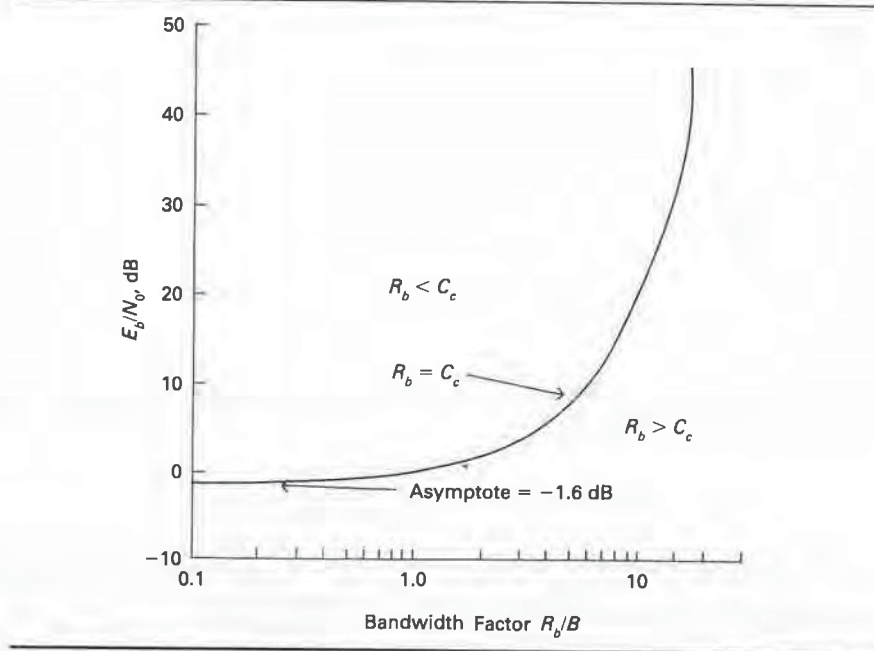
A plot of E_b/N_0 , expressed in decibels, as a function of R_b/B is illustrated in Figure 10.13. The ideal system is defined by $R_b = C_c$ and corresponds to (10.85). There are two regions of interest. The first region, for which $R_b < C_c$, is the region in which arbitrarily small error probabilities can be obtained. Clearly this is the region in which we wish to operate. The other region, for which $R_b > C_c$, does not allow the error probability to be made arbitrarily small.

An important tradeoff can be deduced from Figure 10.13. If the bandwidth factor R_b/B is large so that the bit rate is much greater than the bandwidth, then a significantly larger value of E_b/N_0 is necessary to ensure operation in the $R_b < C_c$ region than is the case if R_b/B is small. Stated another way, assume that the source bit rate is fixed at R_b bits per second and the available bandwidth is large so that $B \gg R_b$. For this case, operation in the $R_b < C_c$ region requires only that E_b/N_0 is slightly greater than -1.6 dB. The required signal power is

$$S \cong R_b (\ln 2) N_0 W \quad (10.88)$$

This is the minimum signal power for operation in the $R_b < C_c$ region. Therefore, operation in this region is desired for *power-limited operation*.

FIGURE 10.13 $R_b = C_c$ relationship for additive white Gaussian noise channel



Now assume that bandwidth is limited so that $R_b \gg B$. Figure 10.13 shows that a much larger value of E_b/N_0 is necessary for operation in the $R_b < C_c$ region. Thus the required signal power is much greater than that given by (10.88). This is referred to as *bandwidth-limited operation*.

The preceding paragraphs illustrate that, at least in the additive white Gaussian noise channel where the Shannon-Hartley law applies, a tradeoff exists between power and bandwidth. This tradeoff is of fundamental importance in the design of communication systems.

Reliable Communication Using Orthogonal Signals

Now that we realize that we can theoretically achieve perfect system performance even in the presence of noise, we start our search for system configurations that yield the performance promised by Shannon's theorem. Actually one such system was analyzed in Chapter 9. Orthogonal signals were chosen for transmission through the channel, and a correlation receiver structure was chosen for demodulation. The system performance is illustrated in Figure 9.7. Shannon's bound is clearly illustrated.

Block Coding for Binary Systems

An alternative technique for approaching Shannon's limit is to use coding schemes that allow for detection and correction of errors. One class of codes for this purpose is known as *block codes*. In order to understand the meaning of a block code, let us consider a source that produces a serial stream of binary symbols at a rate of R symbols per second. Assume that these symbols are grouped into blocks T seconds long, so that each block contains $RT = k$ source or information symbols. To each of these k -symbol blocks is added a group of redundant check symbols to produce a codeword n symbols long. The $n - k$ check symbols should supply sufficient information to the decoder to allow for the correction of most errors that might occur in the channel. An encoder that operates in this manner is said to produce an (n, k) block code.

Codes can either correct or merely detect errors, depending on the amount of redundancy contained in the check symbols. Codes that can correct errors are known as *error-correcting codes*. Codes that can only detect errors are also useful. A feedback channel can be used to request a retransmission of the codeword found to be in error, and often the error can thus be corrected. We will discuss error-detection feedback later. Often the codewords with detected errors are simply discarded by the decoder. This is practical when errors are more serious than a lost codeword.

An understanding of how codes can detect and correct errors can be gained from a geometric point of view. A binary codeword is a sequence of

1's and 0's that is n symbols in length. The *Hamming weight* $w(s_j)$ of codeword s_j is defined as the number of 1's in that codeword. The *Hamming distance* $d(s_i, s_j)$ or d_{ij} between codewords s_i and s_j is defined as the number of positions in which s_i and s_j differ. It follows that Hamming distance can be written in terms of Hamming weight as

$$d_{ij} = w(s_i \oplus s_j) \quad (10.89)$$

where the symbol $\oplus$ denotes modulo-2 addition, which is binary addition without a carry.

▼ **EXAMPLE 10.7** Compute the Hamming distance between s_1 and 101101 and $s_2 = 001100$.

SOLUTION Since

$$101101 \oplus 001100 = 100001$$

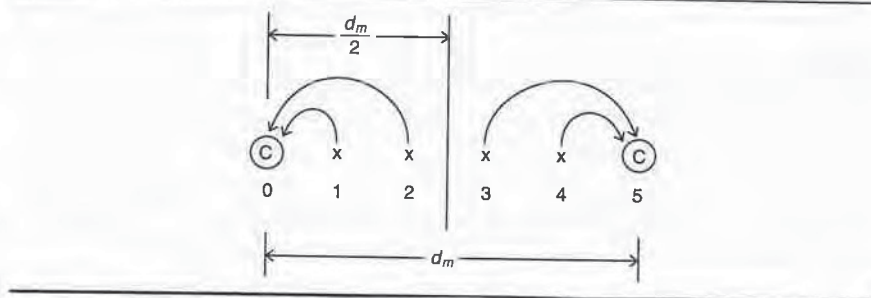
we have

$$d_{12} = w(100001) = 2$$

▲ which simply means that s_1 and s_2 differ in 2 positions.

A geometric representation of two codewords is shown in Figure 10.14. The C's represent two codewords that are distance 5 apart. The codeword on the left is the reference codeword. The first "x" to the right of the reference codeword represents a binary sequence distance 1 from the reference codeword, where distance is understood to denote the Hamming distance. The second "x" to the right of the reference codeword is distance 2

FIGURE 10.14 Geometric representation of two codewords



from the reference, and so on. Assuming that the two codewords shown are the closest in Hamming distance of all the codewords for a given code, the code is then a distance 5 code. Figure 10.14 illustrates the concept of a minimum-distance decoding, in which a given received sequence is assigned to the codeword closest, in Hamming distance, to the received sequence. A minimum-distance decoder will therefore assign the x's to the left of the vertical line to the codeword on the left and the x's to the right of the vertical line to the codeword on the right, as shown.

We can deduce that a minimum-distance decoder can always correct as many as e errors, where e is the largest integer not to exceed

$$\frac{1}{2}(d_m - 1)$$

where d_m is the minimum distance between codewords. It follows that if d_m is odd, all received words can be assigned to a codeword. However, if d_m is even, a received word can lie halfway between two codewords. For this case, errors are detected that cannot be corrected.

▼ **EXAMPLE 10.8** A code consists of codewords [0001011, 1110000, 1000110, 1111011, 0110110, 1001101, 0111101, 0000000]. If 1101011 is received, what is the decoded codeword?

SOLUTION The decoded codeword is the codeword closest in Hamming distance to 1101011. The calculations are

$$\begin{aligned} w(0001011 \oplus 1101011) &= 2 & w(0110110 \oplus 1101011) &= 5 \\ w(1110000 \oplus 1101011) &= 4 & w(1001101 \oplus 1101011) &= 3 \\ w(1000110 \oplus 1101011) &= 4 & w(0111101 \oplus 1101011) &= 4 \\ w(1111011 \oplus 1101011) &= 1 & w(0000000 \oplus 1101011) &= 5 \end{aligned}$$

▲ Thus the decoded codeword is 1111011.

We will now consider several different codes.

Single-Parity-Check Codes

A simple code that can detect single errors but does not have any error-correcting capability is formed by adding one check symbol to each block of k information symbols. This yields a $(k + 1, k)$ code. The added symbol is called a *parity-check symbol*, and it is added so that the Hamming weight of each codeword is always either odd or even. If the received word contains

an *even* number of errors, the decoder will not detect the errors. If the number of errors is *odd*, the decoder will detect that an error has been made.

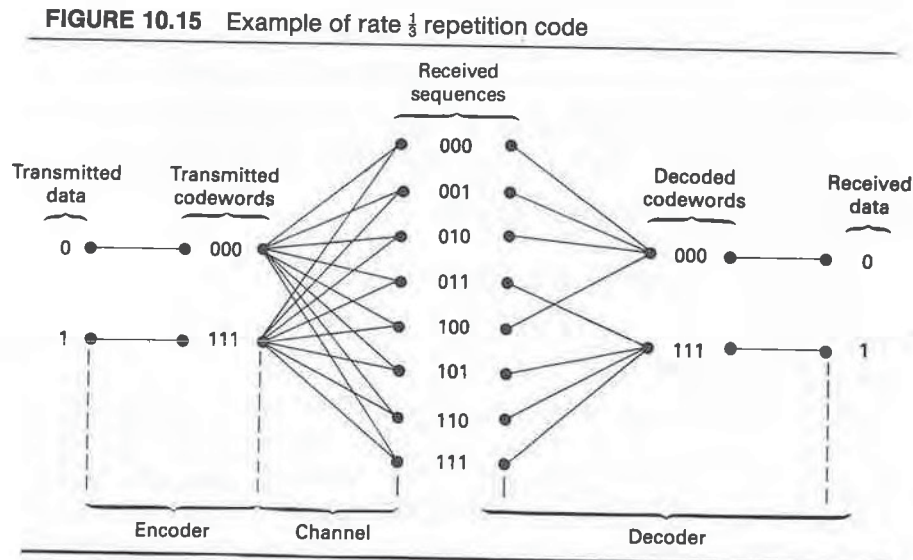
Repetition Codes

The simplest code that allows for correction of errors consists of transmitting each symbol n times, which results in $n - 1$ check symbols. This technique produces an $(n, 1)$ code having two codewords; one of all 0's and one of all 1's. A received word is decoded as a 0 if the majority of the received symbols are 0's and as a 1 if the majority are 1's. This is equivalent to minimum-distance decoding, wherein $\frac{1}{2}(n - 1)$ errors can be corrected. Repetition codes have great error-correcting capability if the symbol error probability is low but have the disadvantage of transmitting many redundant symbols. For example, if the information rate of the source is R bits per symbol, the rate R_c out of the encoder is

$$R_c = \left(\frac{k}{n}\right) R = \frac{1}{n} R \text{ bits/symbol} \quad (10.90)$$

The factor k/n is called the *code rate*.

The process of repetition coding for a rate $\frac{1}{3}$ repetition code is illustrated in detail in Figure 10.15. The encoder maps the data symbols 0 and 1 into the corresponding codewords 000 and 111. There are eight possible received sequences, as shown. The mapping from the transmitted sequence



to the received sequence is random, and the statistics of the mapping are determined by the channel characteristics derived in Chapters 7 and 8. The decoder maps the received sequence into one of the two codewords by a minimum Hamming distance decoding rule. Each decoded codeword corresponds to a data symbol, as shown.

▼ **EXAMPLE 10.9** Investigate the error-correcting capability of a repetition code having a code rate of $\frac{1}{3}$.

SOLUTION Assume that the code is used with a binary symmetric channel with a conditional error probability equal to $(1 - p)$, that is,

$$p(y_j|x_i) = 1 - p, \quad i \neq j \quad (10.91)$$

Each source 0 is encoded as 000, and each source 1 is encoded as 111. An error is made if two or three symbols undergo a change in passing through the channel. Assuming that the source outputs are equally likely, the error probability P_e becomes

$$P_e = 3(1 - p)^2p + (1 - p)^3 \quad (10.92)$$

A few simple calculations illustrate the value of the code. For $(1 - p) = 0.1$, $P_e = 0.028$, yielding an improvement factor of slightly less than 4. For $(1 - p) = 0.01$, the improvement factor is approximately 33. Thus the code performs best when $(1 - p)$ is small. Of course, performance increases as n becomes larger, but the code rate decreases. In many cases, the *information rate* must be maintained constant. Thus an increase in redundancy results in an increase in symbol rate, which changes the channel matrix. We will consider this problem in Example 10.13.

Parity-Check Codes for Correction of Single Errors

Repetition codes and single-parity-check codes are examples of codes that have either high error-correction capability or high information rate, but not both. Only codes that have a reasonable combination of these characteristics are practical for use in digital communication systems. We now examine a class of parity-check codes that satisfy these requirements.

A general codeword can be written in the form

$$a_1 a_2 \cdots a_k c_1 c_2 \cdots c_r$$

where a_i is the i th information symbol and c_j is the j th check symbol. Codes for which the first k symbols of the codeword are the information symbols

are called *systematic codes*. It should be remembered that $r = n - k$. The parity check symbols are chosen to satisfy the linear equations

$$\begin{aligned} 0 &= h_{11}a_1 \oplus h_{12}a_2 \oplus \cdots \oplus h_{1k}a_k \oplus c_1 \\ 0 &= h_{21}a_1 \oplus h_{22}a_2 \oplus \cdots \oplus h_{2k}a_k \oplus c_2 \\ &\vdots \\ 0 &= h_{r1}a_1 \oplus h_{r2}a_2 \oplus \cdots \oplus h_{rk}a_k \oplus c_r \end{aligned} \quad (10.93)$$

Equation (10.93) can be written as

$$[H][T] = [0] \quad (10.94)$$

where $[H]$ is the parity-check matrix

$$[H] = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1k} & 1 & 0 & \cdots & 0 \\ h_{21} & h_{22} & \cdots & h_{2k} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \\ h_{r1} & h_{r2} & \cdots & h_{rk} & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (10.95)$$

and $[T]$ is the codeword vector

$$[T] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ c_1 \\ \vdots \\ c_r \end{bmatrix} \quad (10.96)$$

Now let the received word be $[R]$. If

$$[H][R] \neq [0] \quad (10.97)$$

we know that $[R]$ is not a codeword and at least one error has been made. If

$$[H][R] = [0] \quad (10.98)$$

we know that $[R]$ is a codeword and that it is *most likely* the transmitted codeword.

Since $[R]$ is the received codeword, it can be written

$$[R] = [T] \oplus [E] \quad (10.99)$$

where $[E]$ represents the error pattern induced by the channel. The decoding problem essentially reduces to determining $[E]$, since the codeword can be reconstructed from $[R]$ and $[E]$.

As the first step in computing $[E]$, we multiply the received word $[R]$ by the parity-check matrix $[H]$. This yields

$$[S] = [H][R] = [H][T] \oplus [H][E] \quad (10.100)$$

or

$$[S] = [H][E] \quad (10.101)$$

The matrix $[S]$ is known as the *syndrome*.

Assuming that a single error has taken place, the error vector will be of the form

$$[E] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying $[E]$ by $[H]$ on the left-hand side shows that the syndrome is the i th column of the matrix $[H]$, where the error is in the i th position. The following example illustrates the method.

▼ **EXAMPLE 10.10** A code has the parity-check matrix

$$[H] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (10.102)$$

Assuming that 111011 is received, determine if an error has been made, and if so, determine the decoded codeword.

SOLUTION First, we compute the syndrome, remembering that all operations are modulo-2:

$$[S] = [H][R] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (10.103)$$

Since the syndrome is the third column of the parity-check matrix, the third symbol of the received word is assumed to be in error. Thus the decoded

▲ codeword is 110011. This can be proved by showing that 110011 has a zero syndrome.

We now pause to examine the parity-check code in more detail. It follows from (10.93) and (10.95) that the parity checks can be written as

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & \cdots & h_{2k} \\ \vdots & \vdots & & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rk} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \quad (10.104)$$

Thus the codeword vector $[T]$ can be written

$$[T] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ h_{11} & h_{12} & \cdots & h_{1k} \\ \vdots & \vdots & & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rk} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \quad (10.105)$$

or

$$[T] = [G][A] \quad (10.106)$$

where $[A]$ is the vector of k information symbols,

$$[A] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \quad (10.107)$$

and $[G]$, which is called the *generator matrix*, is

$$[G] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ h_{11} & h_{12} & \cdots & h_{1k} \\ \vdots & \vdots & & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rk} \end{bmatrix} \quad (10.108)$$

The relationship between the generator matrix $[G]$ and the parity-check matrix $[H]$ is apparent if we compare (10.95) and (10.108). If the m by m identity matrix is identified by $[I_m]$ and the matrix $[H_p]$ is defined by

$$[H_p] = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1k} \\ h_{21} & h_{22} & \dots & h_{2k} \\ \vdots & \vdots & \dots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rk} \end{bmatrix} \quad (10.109)$$

it follows that the generator matrix is given by

$$[G] = \begin{bmatrix} I_k & -\frac{H_p}{H_p} \end{bmatrix} \quad (10.110)$$

and that the parity-check matrix is given by

$$[H] = [H_p \mid I_r] \quad (10.111)$$

which establishes the relationship between the generator and parity-check matrices for systematic codes.

Codes defined by (10.108) are referred to as *linear codes*, since the $k + r$ codeword symbols are formed as a linear combination of the k information symbols. It is also worthwhile to note that if two different information sequences are summed to give a third sequence, then the codeword for the third sequence is the sum of the two codewords corresponding to the original two information sequences. This is easily shown. If two information sequences are summed, the resulting vector of information symbols is

$$[A_3] = [A_1] \oplus [A_2] \quad (10.112)$$

The codeword corresponding to $[A_3]$ is

$$[T_3] = [G][A_3] = [G]\{[A_1] \oplus [A_2]\} = [G][A_1] \oplus [G][A_2] \quad (10.113)$$

Since

$$[T_1] = [G][A_1] \quad (10.114a)$$

and

$$[T_2] = [G][A_2] \quad (10.114b)$$

it follows that

$$[T_3] = [T_1] \oplus [T_2] \quad (10.115)$$

Codes that satisfy this property are known as *group codes*.

Hamming Codes

A Hamming code is a particular parity-check code having distance 3. Therefore, all single errors can be corrected. The parity-check matrix for the code has dimensions $2^{n-k} - 1$ by $n - k$ and is very easy to construct. The i th column of the matrix $[H]$ is the binary representation of the number i . The code has the interesting property that, for a single error, the syndrome is the binary representation of the position in error.

▼ **EXAMPLE 10.11** Determine the parity-check matrix for a (7, 4) code and the decoded codeword if the received word is 1110001.

SOLUTION Since the i th column of the matrix $[H]$ is the binary representation of i , we have

$$[H] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (10.116)$$

For the received word 1110001, the syndrome is

$$[S] = [H][R] = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (10.117)$$

▲ Thus the error is in the seventh position, and the decoded codeword is 1110000.

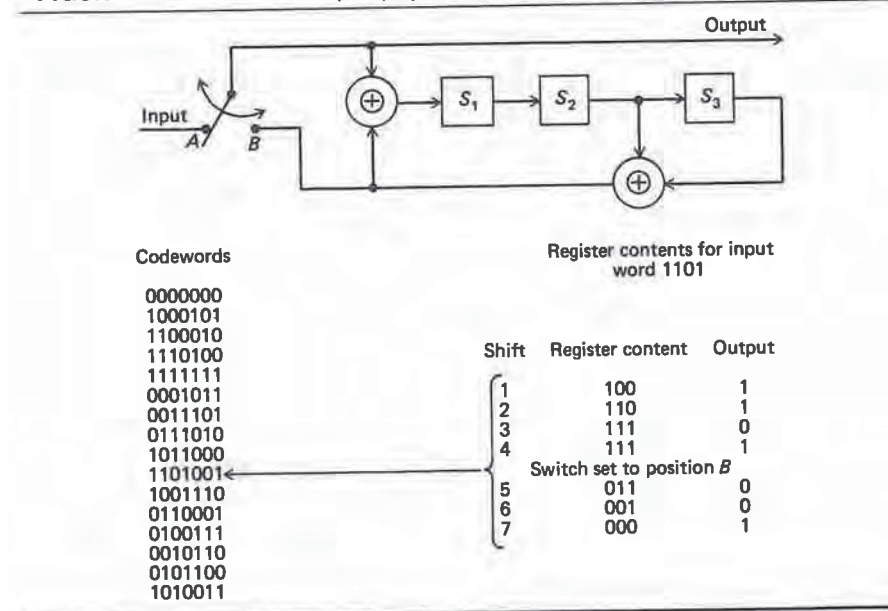
We must note in passing that for the (7, 4) Hamming code, the parity checks are in the first, second, and fourth positions in the codewords, since these are the only columns of the parity-check matrix containing only one nonzero element.

Cyclic Codes

The preceding subsections dealt primarily with the mathematical properties of parity-check codes. We have avoided any discussion of the implementation of parity-check encoders or decoders. Indeed, if we were to examine the implementation of these devices, we would find that fairly complex hardware configurations are required. However, there is a class of parity-check codes, known as *cyclic codes*, that are easily implemented using feedback shift registers. A cyclic code derives its name from the fact that a cyclic permutation of any codeword produces another codeword. For example, if $x_1x_2 \cdots x_{n-1}x_n$ is a codeword, so is $x_nx_1x_2 \cdots x_{n-1}$. In this section we examine not the underlying theory of cyclic codes but the implementation of encoders and decoders. We will accomplish this by means of an example.

An (n, k) cyclic code can easily be generated with an $n - k$ stage shift register with appropriate feedback. The register illustrated in Figure 10.16 produces a $(7, 4)$ cyclic code. The switch is initially in position *A*, and the shift register stages initially contain all zeros. The $k = 4$ information symbols are then shifted into the encoder. As each information symbol arrives, it is routed to the output and added to the value of $S_2 \oplus S_3$. The resulting sum is then placed into the first stage of the shift register. Simultaneously,

FIGURE 10.16 Coder for $(7, 4)$ cyclic code



the contents of S_1 and S_2 are shifted to S_2 and S_3 , respectively. After all information symbols have arrived, the switch is moved to position B , and the shift register is shifted $n - k = 3$ times to clear it. On each shift, the sum of S_2 and S_3 appears at the output. This sum added to itself produces a 0, which is fed into S_1 . After $n - k$ shifts, a codeword has been generated that contains $k = 4$ information symbols and $n - k = 3$ parity-check symbols. It also should be noted that at this time the register contains all 0's so that the encoder is ready to receive the next $k = 4$ information symbols.

All $2^k = 16$ codewords that can be generated with the example encoder are also illustrated in Figure 10.16. The $k = 4$ information symbols, which are the first four symbols of each codeword, were shifted into the encoder beginning with the left-hand symbol. Also shown in Figure 10.16 are the contents of the register and the output symbol after each shift for the codeword 1101.

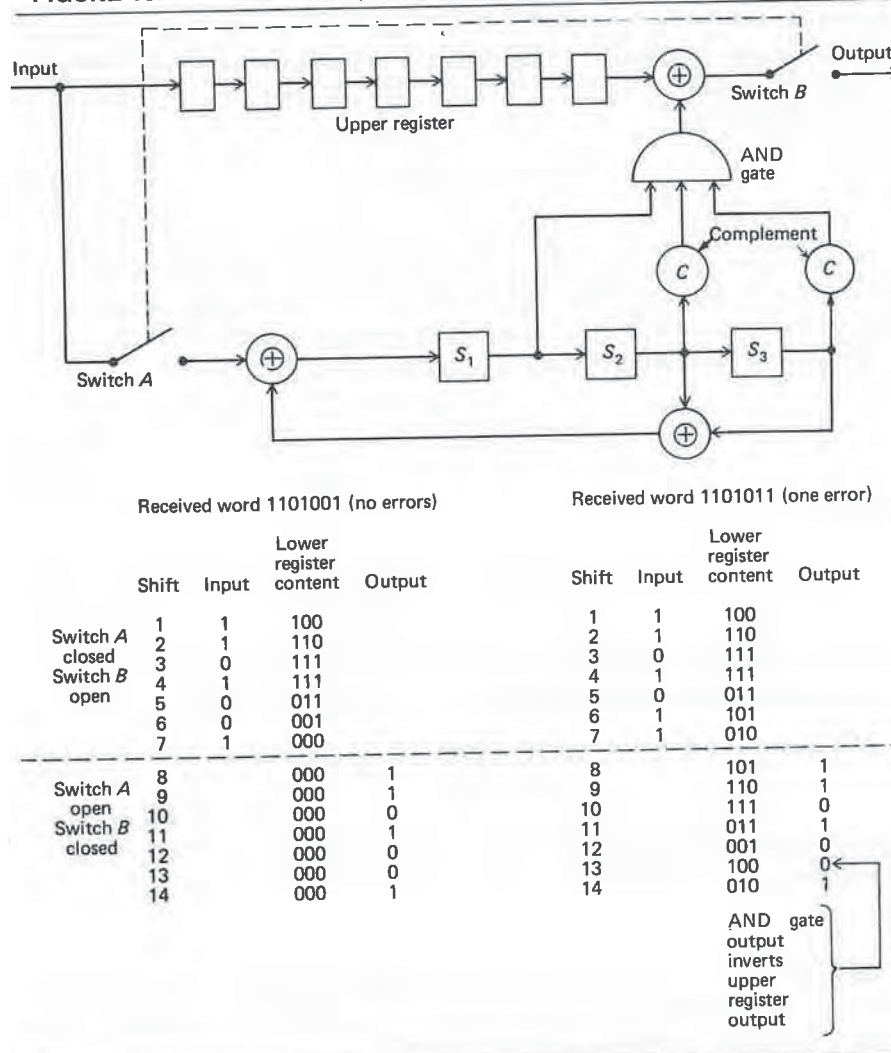
The decoder for the (7, 4) cyclic code is illustrated in Figure 10.17. The upper register is used for storage, and the lower register and feedback arrangement are identical to the feedback shift register used in the encoder. Initially, switch A is closed and switch B is open. The n received symbols are shifted into the two registers. If there are no errors, the lower register will contain all 0's when the upper register is full. The switch positions are then reversed, and the codeword that is stored in the upper register is shifted out. This operation is illustrated in Figure 10.17 for the received word 1101001.

If, after the received word is shifted into the decoder, the lower register does not contain all 0's, an error has been made. The error is corrected automatically by the decoder, since, when the incorrect symbol appears at the output of the shift register, a 1 appears at the output of the AND gate. This 1 inverts the upper register output and is produced by the sequence 100 in the lower register. The operation is illustrated in Figure 10.17.

Comparison of Errors in Block-Coded and Uncoded Systems

We will now compare the relative performance of coded and uncoded systems for block codes. The basic assumption will be that the *information rate* is the same for both systems. Since the coded and uncoded words contain the same information, the word duration T_w will be the same under the equal-information-rate assumption. Since the codeword contains more symbols than the uncoded word, as a result of the addition of parity symbols, the symbol rate will be higher for the coded system than for the uncoded system. If constant transmitter power is assumed, it follows that the energy per symbol is decreased by the use of coding, resulting in a higher proba-

FIGURE 10.17 Decoder for (7, 4) cyclic code



bility of symbol error. We must determine whether or not coding can overcome this increase in symbol error probability to the extent that a significant decrease in word error probability can be obtained.

Assume that q_u and q_c represent the probability of *symbol* error for the uncoded and coded systems, respectively. Also assume that P_{eu} and P_{ec} are the *word* error probabilities for the uncoded and coded systems. The word error probability for the uncoded system is relatively easy to compute. An

uncoded word is in error if any of the symbols in that word are in error. The probability that a symbol will be received correctly is $(1 - q_u)$, and since all symbol errors are assumed independent, the probability that all k symbols in a word are received correctly is $(1 - q_u)^k$. Thus the uncoded word error probability is

$$P_{eu} = 1 - (1 - q_u)^k \quad (10.118)$$

The probability of word error for the coded system is more difficult to compute because symbol errors can possibly be corrected by the decoder, depending on the code used. If a code is capable of correcting up to e errors, the probability of word error P_{ec} is equal to the probability that more than e errors are present in the received codeword. Thus

$$P_{ec} = \sum_{i=e+1}^n \binom{n}{i} (1 - q_c)^{n-i} q_c^i \quad (10.119)$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad (10.120)$$

is the number of combinations of n symbols taken i at a time.

For single-error-correcting codes $e = 1$, and P_{ec} becomes

$$P_{ec} = \sum_{i=2}^n \binom{n}{i} (1 - q_c)^{n-i} q_c^i \quad (10.121)$$

By comparing (10.118) and (10.121), we determine the improvement gained through the use of coding.

▼ **EXAMPLE 10.12** Let us investigate the effectiveness of a (7, 4) single-error-correcting code. Assume that the code is used with a BPSK transmission system, so the symbol error probability is

$$q_u = Q[\sqrt{2z}] \quad (10.122)$$

which is Equation (7.72) with $m = 0$. The symbol energy E is the transmitter power S times the word time T_w divided by k , since the total energy in each word is divided by k . Thus the symbol error probability without coding is

$$q_u = Q\left[\sqrt{\frac{2ST_w}{kN_0}}\right] \quad (10.123)$$

Assuming equal word rates for both the coded and uncoded system gives

$$q_c = Q \left[\sqrt{\frac{2ST_w}{nN_0}} \right] \quad (10.124)$$

for the coded symbol error probability, since the energy available for k information symbols must be spread over $n > k$ symbols when coding is used. Thus it follows that the symbol error probability is increased by the use of coding. However, we shall show that the error-correcting capability of the code can overcome the increased symbol error probability and indeed yield a net gain in word error probability for certain ranges of the signal-to-noise ratio.

From (10.121), the word error probability for the coded case is

$$P_{ec} = \sum_{i=2}^7 \binom{7}{i} (1 - q_c)^{7-i} q_c^i \quad (10.125)$$

In typical situations, the probability of two errors is much greater than the probability of three or more errors. Thus (10.125) becomes

$$P_{ec} = 21 \left\{ Q \left[\sqrt{\frac{2ST_w}{7N_0}} \right] \right\}^2 \quad (10.126)$$

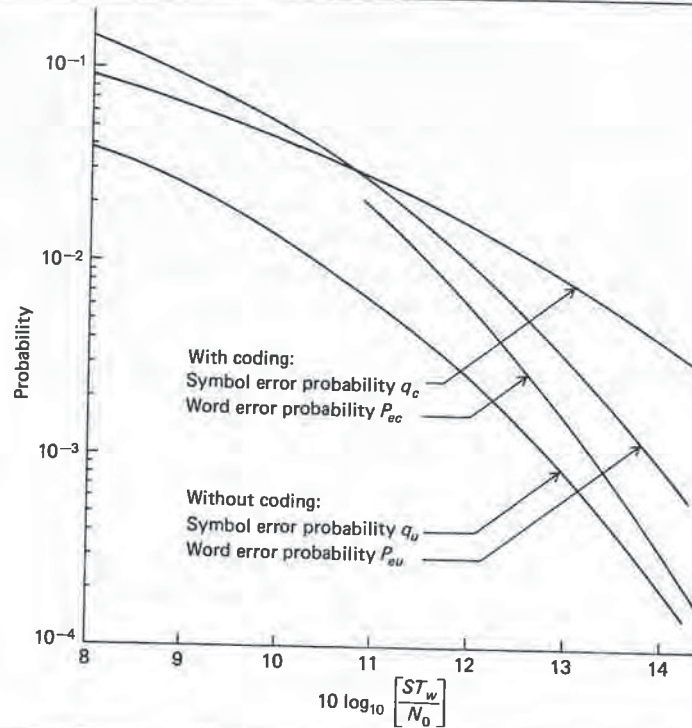
and from (10.118), the word error probability for the uncoded system is

$$P_{eu} = 1 - \left\{ 1 - Q \left[\sqrt{\frac{2ST_w}{4N_0}} \right] \right\}^4 \quad (10.127)$$

The word error probabilities for the coded and uncoded systems are illustrated in Figure 10.18. The curves are plotted as a function of ST_w/N_0 , which is word energy divided by the noise power spectral density.

You should note that coding has little effect on system performance unless the value of ST_w/N_0 is in the neighborhood of 11 dB or above. Also, the improvement afforded by a (7, 4) code is quite modest unless ST_w/N_0 is large, in which case system performance may be satisfactory without coding. However, in many systems, even small improvements are very important. Also illustrated in Figure 10.18 are the uncoded and coded symbol error probabilities q_u and q_c , respectively. The effect of spreading the available energy per word over a larger number of symbols is evident.

FIGURE 10.18 Comparison of uncoded and coded systems assuming a (7, 4) code



▼ **EXAMPLE 10.13** In this example we examine the performance of repetition codes in two different channels. Both cases utilize FSK modulation and a noncoherent receiver structure. In the first case, an *additive white Gaussian noise* (AWGN) channel is assumed. The second case assumes a Rayleigh fading channel. Distinctly different results will be obtained.

Case 1

As was shown in Chapter 7, Equation (7.110), the error probability for a noncoherent FSK system in an additive white Gaussian noise (AWGN) channel is given by

$$q_u = \frac{1}{2}e^{-0.5z} \quad (10.128)$$

where z is the ratio of signal power to noise power at the output of the receiver bandpass filter. Equivalently, if the receiver filter is matched to the received signal envelope, z is the ratio of the energy per bit to the noise

power spectral density at the receiver input. Without assuming a matched filter receiver, we can write z as

$$z = \frac{A^2}{2N_0B_T} \quad (10.129)$$

where N_0B_T is the noise power in the signal bandwidth B_T . The performance of the system is illustrated by the $n = 1$ curve in Figure 10.19. When an n -symbol repetition code is used with this system, the symbol error probability is given by

$$q_c = \frac{1}{2}e^{-z/2n} \quad (10.130)$$

This result occurs since encoding a single information symbol (bit) as n code symbols requires that the symbol duration with coding be $1/n$ times the information symbol duration without coding or, equivalently, that the signal bandwidth be increased by a factor of n with coding. Thus, with coding B_T in q_u is replaced by nB_T to give q_c . The word error probability is given by (10.119) with

$$e = \frac{1}{2}(n - 1) \quad (10.131)$$

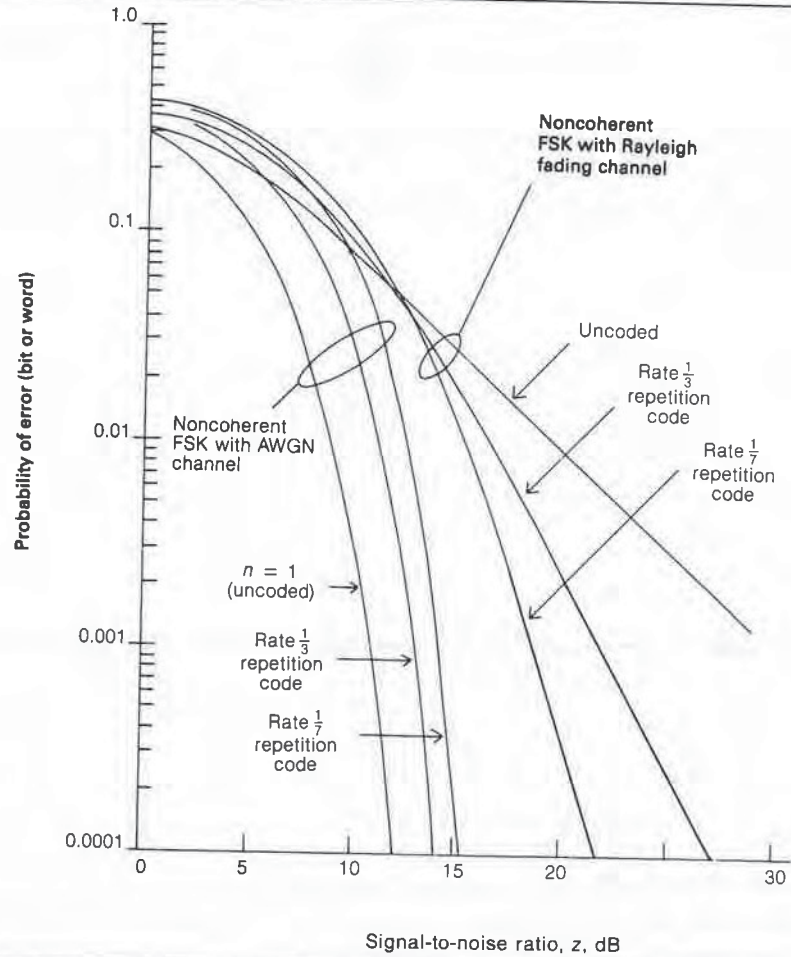
Since each codeword carries one bit of information, the word error probability is equal to the bit error probability for the repetition code.

The performance of a noncoherent FSK system with an AWGN channel with rate $\frac{1}{3}$ and $\frac{1}{7}$ repetition codes is illustrated in Figure 10.19. It should be noted that system performance is degraded through the use of repetition coding. This result occurs because the increase in symbol error probability with coding is greater than can be overcome by the error-correction capability of the code. This same result occurs with coherent FSK and BPSK as well as with ASK, illustrating that the low rate of the repetition code prohibits its effective use in systems in which the dependence on symbol error probability and signal-to-noise ratio is essentially exponential.

Case 2

An example of a system in which repetition coding can be used effectively is an FSK system operating in a Rayleigh fading environment. Such a system was analyzed in Chapter 9. We showed that the symbol error probability is given by Equation (9.133), which can be written as

$$q_u = \frac{1}{2} \frac{1}{1 + \frac{E_s}{2N_0}} \quad (10.132)$$

FIGURE 10.19 Performance of repetition codes on AWGN and Rayleigh fading channels

in which E_a is the average received energy per symbol (or bit). The use of a repetition code spreads the energy E_a over the n symbols in a codeword. Thus, with coding,

$$q_c = \frac{1}{2} \frac{1}{1 + \frac{E_a}{2nN_0}} \quad (10.133)$$

As in Case 1, the decoded bit error probability is given by (10.119) with e given by (10.131). The Rayleigh fading results are also shown in Figure 10.19 for rate 1, $\frac{1}{3}$, and $\frac{1}{7}$ repetition codes, where, for this case, the signal-to-noise ratio z is E_a/N_0 . Thus the repetition code can be used to advantage in a Rayleigh fading environment if E_a/N_0 is sufficiently large.

Repetition coding is somewhat related to diversity transmission, in which the available signal energy is divided equally among n subpaths. In Problem 9.17, it was shown that the optimal combining of the receiver outputs prior to making a decision on the transmitted information bit is as shown in Figure 10.20(a). The model for the repetition code considered in this example is shown in Figure 10.20(b). The essential difference is that a "hard decision"

FIGURE 10.20 Comparison of optimum and suboptimum systems.
(a) Optimum system. (b) Repetition code model.

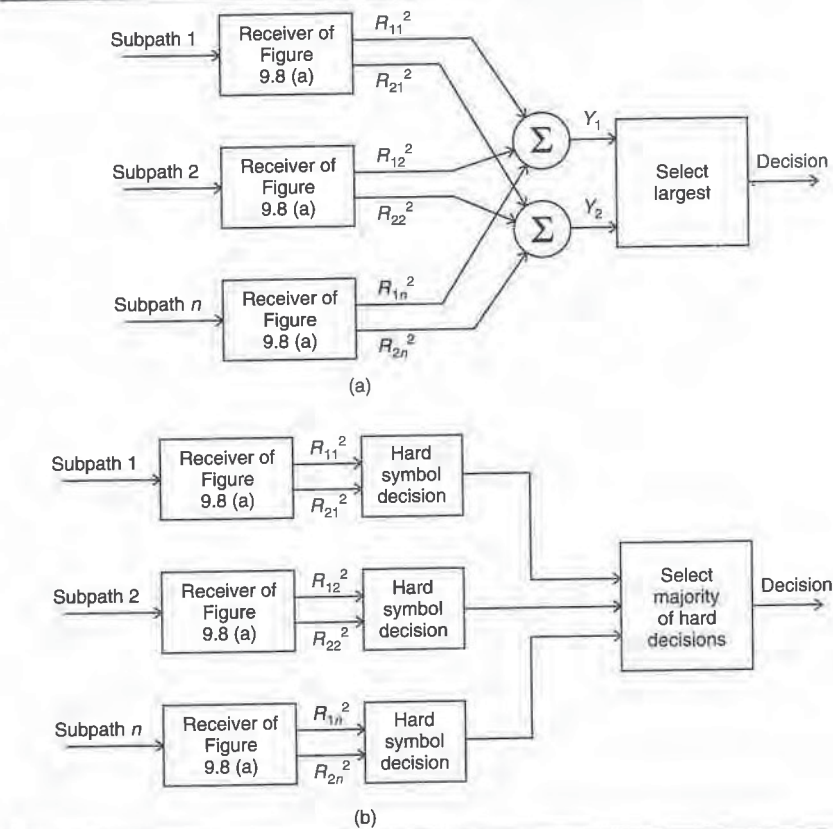
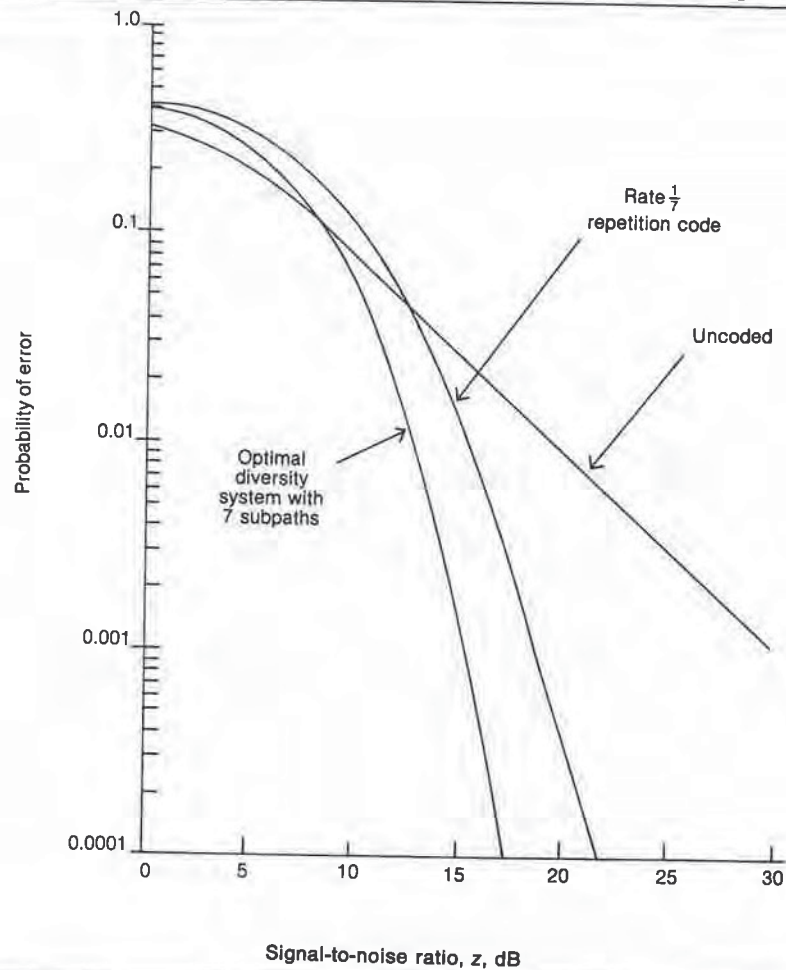


FIGURE 10.21 Performance of noncoherent FSK in Rayleigh fading channel

on each symbol of the n -symbol codeword is made at the output of the n receivers. The decoded information bit is then in favor of the majority of the n symbols of the codeword.

When a hard decision is made at the receiver output, information is clearly lost, and the result is a degradation of performance. This can be seen in Figure 10.21, which illustrates the performance of the $n = 7$ optimal system of Figure 10.20(a) and that of the rate $\frac{1}{7}$ repetition code of Figure 10.20(b). Also shown for reference is the performance of the uncoded system.

Convolutional Codes

The convolutional code is an example of a nonblock code. Rather than the parity-check symbols being calculated for a block of code symbols, the parity checks are calculated over a span of information symbols. This span, which is referred to as the *constraint span*, is shifted one information symbol each time an information symbol is input to the encoder.

A general convolutional encoder is illustrated in Figure 10.22. The encoder is rather simple and consists of three component parts. The heart of the encoder is a shift register that holds k information symbols, where k is the constraint span of the code. The shift register stages are connected to v modulo-2 adders as indicated. Not all stages are connected to all adders. In fact, the connections are “somewhat random” and can have considerable impact on the performance of the code generated. Each time a new information symbol is shifted into the encoder, the adder outputs are sampled by the commutator. Thus v output symbols are generated for each input symbol yielding a code of rate $1/v$.

A rate $\frac{1}{3}$ convolutional encoder is illustrated in Figure 10.23. For each input, the output of the encoder is the sequence $v_1 v_2 v_3$. For the encoder of Figure 10.23,

$$v_1 = S_1 \oplus S_2 \oplus S_3 \quad (10.134a)$$

$$v_2 = S_1 \quad (10.134b)$$

$$v_3 = S_1 \oplus S_2 \quad (10.134c)$$

Thus we see that the input sequence 101001 results in the output sequence 111101011101100111.

Convolutional codes can be decoded by tree-searching techniques. A portion of the code tree for the encoder of Figure 10.23 is illustrated in Figure 10.24. In the latter figure, the single binary symbols are inputs to the

FIGURE 10.22 General convolutional encoder

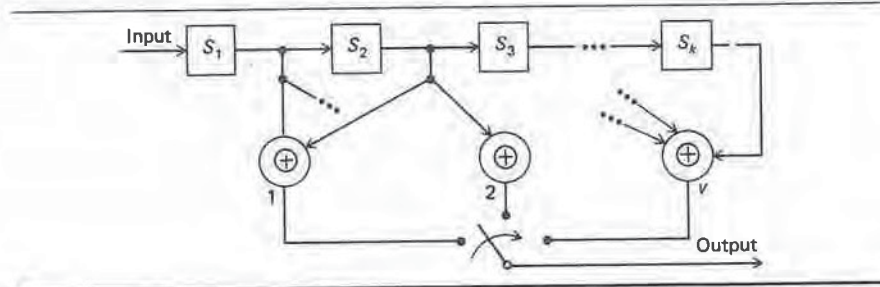
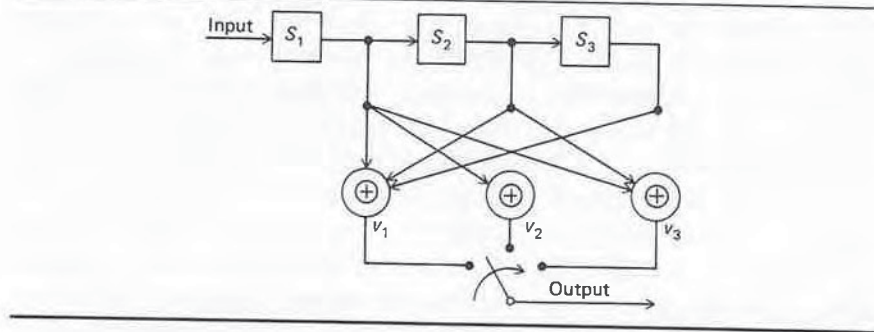


FIGURE 10.23 Example of convolutional encoder

encoder, and the three binary symbols in parentheses are the output symbols corresponding to each input symbol. For example, if 1010 is fed into the encoder, the output is 111101011101 or path *A*.

The decoding procedure also follows from Figure 10.24. To decode a received sequence, we search the code tree for the path closest in Hamming distance to the input sequence. For example, the input sequence 110101011111 is decoded as 1010, indicating an error in the third and eleventh positions of the input sequence.

The exact implementation of tree-searching techniques requires considerable hardware and is not practical for many applications. For example, N information symbols generate 2^N branches of the code tree. Storage of the entire tree is therefore impractical for large N . Several algorithms have been developed that yield excellent performance with reasonable hardware requirements. Prior to taking a brief look at the most popular of these techniques, the Viterbi algorithm, we look at the trellis diagram, which is essentially a code tree in compact form.

The key to construction of the trellis diagram is recognition that the code tree is repetitive after k branches, where k is the constraint span of the encoder. This is easily recognized from the code tree shown in Figure 10.24. After the fourth input of an information symbol, 16 branches have been generated in the code tree. The encoder outputs for the first 8 branches match exactly the encoder outputs for the second 8 branches, except for the first symbol. The encoder output therefore depends only on the latest k inputs. In this case, the constraint span, k , is 3. Thus the output corresponding to the fourth information symbol depends only on the second, third, and fourth encoder inputs. It makes no difference whether the first information symbol was a binary 0 or a binary 1. (This should clarify the meaning of a constraint span.)

The analysis is simplified by defining the state of the encoder as S_1S_2 , or

the contents of the input register prior to inputting the current information symbol. When the current information symbol is input to S_1 , S_1 is shifted to S_2 and S_2 is shifted to S_3 . The state and current input then determine S_1 , S_2 , and S_3 , which in turn determine the output. This information is summarized in Table 10.4. The outputs corresponding to given state transitions are shown in parentheses, consistent with Figure 10.24.

FIGURE 10.24 Code tree

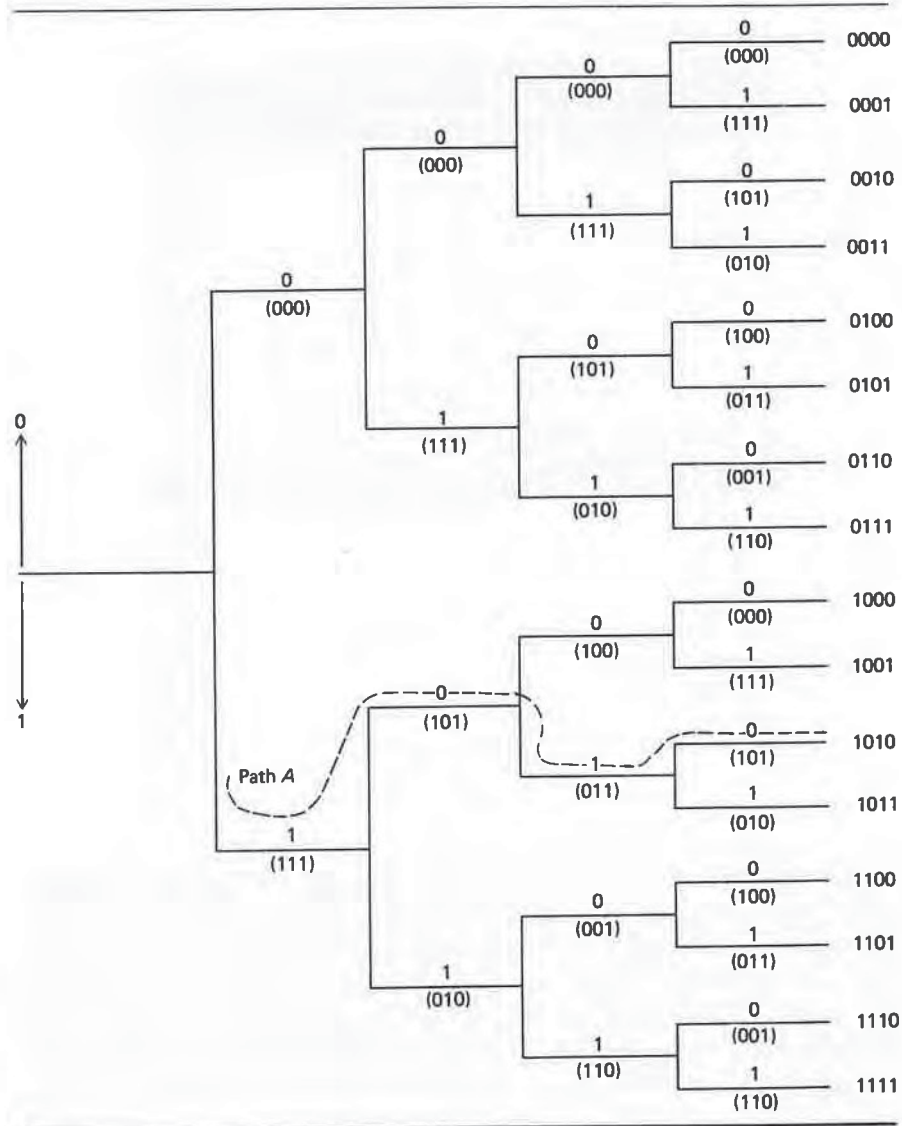


TABLE 10.4 States, Transitions, and Outputs for Convolutional Encoder Shown in Figure 10.23*(a) Definition of States*

State	S_1	S_2
<i>A</i>	0	0
<i>B</i>	0	1
<i>C</i>	1	0
<i>D</i>	1	1

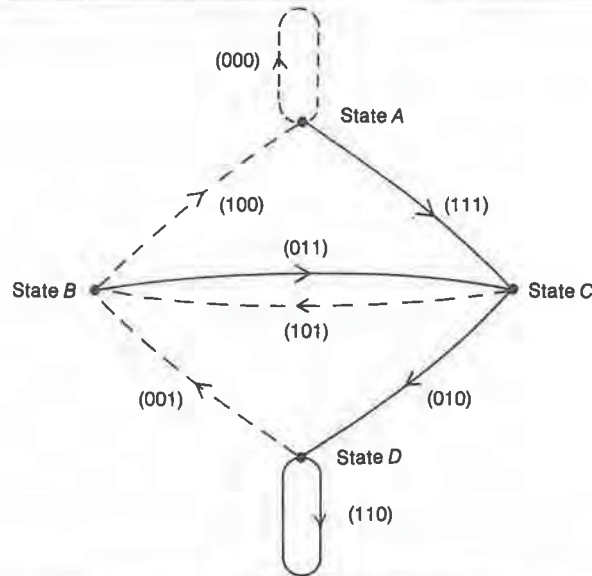
(b) State Transitions

State	Previous		Input	Current			State	Output
	S_1	S_2		S_1	S_2	S_3		
<i>A</i>	0	0	0	0	0	0	<i>A</i>	(000)
			1	1	0	0	<i>C</i>	(111)
<i>B</i>	0	1	0	0	0	1	<i>A</i>	(100)
			1	1	0	1	<i>C</i>	(011)
<i>C</i>	1	0	0	0	1	0	<i>B</i>	(101)
			1	1	1	0	<i>D</i>	(010)
<i>D</i>	1	1	0	0	1	1	<i>B</i>	(001)
			1	1	1	1	<i>D</i>	(110)

(c) Encoder Output for State Transition $x \rightarrow y$

Transition	Output
$A \rightarrow A$	(000)
$A \rightarrow C$	(111)
$B \rightarrow A$	(100)
$B \rightarrow C$	(011)
$C \rightarrow B$	(101)
$C \rightarrow D$	(010)
$D \rightarrow B$	(001)
$D \rightarrow D$	(110)

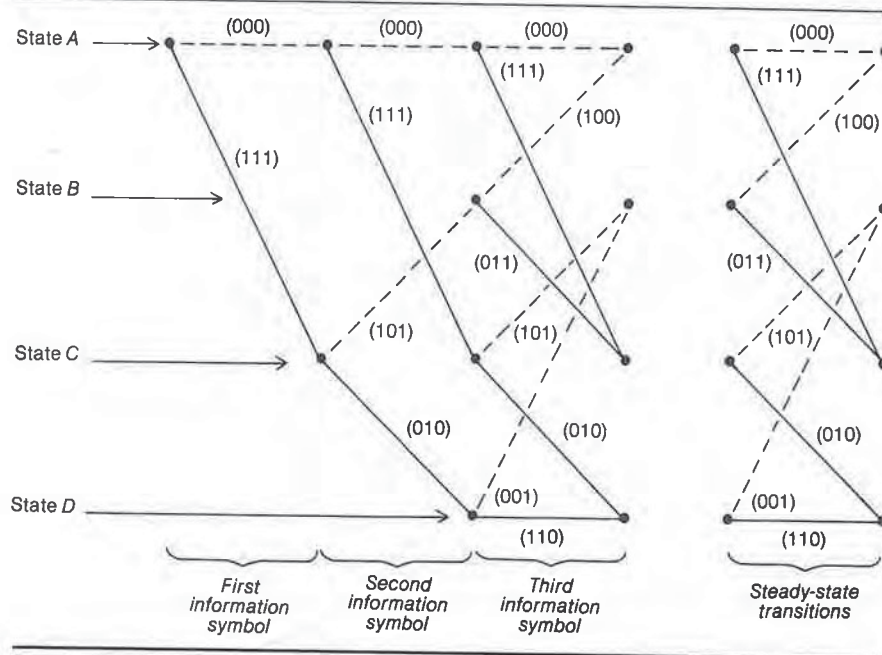
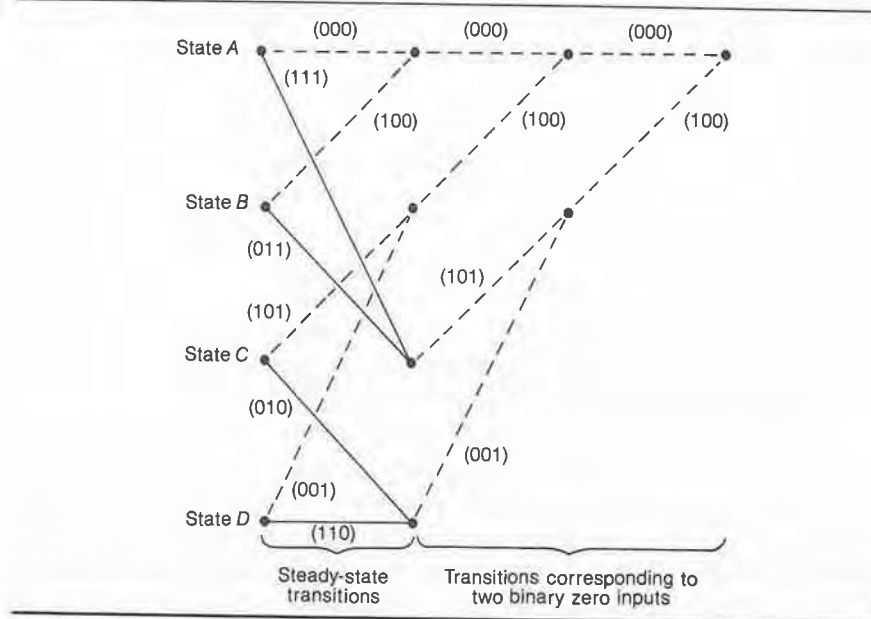
It should be noted that states *A* and *C* can only be reached from states *A* and *B*. Also, states *B* and *D* can only be reached from states *C* and *D*. The information in Table 10.4 is often shown in a state diagram, such as in Figure 10.25. In the state diagram, an input of binary 0 results in the transition denoted by a dashed line, and an input of binary 1 results in the transition designated by a solid line. The resulting encoder output is denoted by the three symbols in parentheses. For any given sequence of inputs, the resulting state transitions and encoder outputs can be traced on the state diagram.

FIGURE 10.25 State diagram for example of convolutional encoder

This is a very convenient method for determining the outputs resulting from a given sequence of inputs.

The trellis diagram (Figure 10.26) results directly from the state diagram. Initially, the encoder is assumed to be in state *A* (all contents are 0's). An input of binary 0 results in the encoder remaining in state *A*, as indicated by the dashed line, and an input of binary 1 results in a transition to state *C*, as indicated by the solid line. Any of the four states can be reached by a sequence of two inputs. The third input results in the possible transitions shown. The fourth input results in exactly the same set of possible transitions. Therefore, after the second input, the trellis becomes completely repetitive, and the possible transitions are those labeled steady-state transitions. The encoder can always be returned to state *A* by inputting two binary 0's as shown in Figure 10.27. As before, the output sequence resulting from any transition is shown by the sequence in parentheses.

In order to illustrate the Viterbi algorithm, we consider the received sequence that we previously considered to illustrate decoding using a code tree—namely, the sequence 110101011111. The first step is to compute the Hamming distances between the initial node (state *A*) and each of the four states three levels deep into the trellis. We must look three levels deep into the trellis because the constraint span of the example encoder is 3.

FIGURE 10.26 Trellis diagram**FIGURE 10.27** Termination of trellis diagram

Since each of the four nodes can be reached from only two preceding nodes, eight paths must be identified, and the Hamming distance must be computed for each path. We therefore initially look three levels deep into the trellis, and since the example encoder has rate $\frac{1}{3}$, the first nine received symbols are initially considered. Thus the Hamming distances between the input sequence 110101011 and the eight paths terminating three levels deep into the trellis are computed. These calculations are summarized in Table 10.5. After the eight Hamming distances are computed, the path having the minimum Hamming distance to *each* of the four nodes is retained. These four retained paths are known as *survivors*. The other four paths are discarded from further consideration. The four survivors are identified in Table 10.5.

The next step in the application of the Viterbi algorithm is to consider the next three received symbols, which are 111 in the example being considered. The scheme is to compute once again the Hamming distance to the four states, this time four levels deep in the trellis. As before, each of the four states can be reached from only two previous states. Thus, once again, eight Hamming distances must be computed. Each of the four previous survivors, along with their respective Hamming distances, is extended to the two states reached by each surviving path. The Hamming distance of each new segment is computed by comparing the encoder output, corresponding to each of the new segments, with 111. The calculations are summarized in Table 10.6. The path having the smallest new distance is path ACBCB. This corresponds to information sequence 1010 and is in agreement with the previous tree search.

For a general received sequence, the process identified in Table 10.6 is

TABLE 10.5 Calculations for Viterbi Algorithm: Step One
(Received Sequence = 110101011)

Path ^a	Corresponding Symbols	Hamming Distance	Survivor?
AAAA	00000000	6	No
ACBA	111101100	4	Yes
ACDB	111010001	5	Yes ^b
AACB	000111101	5	No ^b
AAAC	000000111	5	No
ACBC	111101011	1	Yes
ACDD	111010110	6	No
AACD	000111010	4	Yes

^a The initial and terminal states are identified by the first and fourth letters, respectively. The second and third letters correspond to intermediate states.

^b If two or more paths have the same Hamming distance, it makes no difference which is retained as a survivor.

TABLE 10.6 Calculations for the Viterbi Algorithm: Step Two
(Received Sequence 110101011111)

Path ^a	Previous Survivor's Distance	New Segment	Added Distance	New Distance	Survivor?
<u>ACBAA</u>	4	AA	3	7	Yes
<u>ACDBA</u>	5	BA	2	7	No
<u>ACBCB</u>	1	CB	1	2	Yes
<u>AACDB</u>	4	DB	2	6	No
<u>ACBAC</u>	4	AC	0	4	Yes
<u>ACDBC</u>	5	BC	1	6	No
<u>ACBCD</u>	1	CD	2	3	Yes
<u>AACDD</u>	4	DD	1	5	No

^a An underscore indicates the previous survivor.

continued. After each new set of calculations, involving the next three received symbols, only the four surviving paths and the accumulated Hamming distances need be retained. At the end of the process, it is necessary to reduce the number of surviving paths from four to one. This is accomplished by inserting two dummy 0's at the end of the information sequence, corresponding to the transmission of six code symbols. As shown in Figure 10.27, this forces the trellis to terminate at state *A*.

The Viterbi algorithm has found widespread application in practice, especially in satellite communications. It can be shown that the Viterbi algorithm is a maximum-likelihood decoder, and in that sense, it is optimal. Viterbi and Omura (1979) give an excellent analysis of the Viterbi algorithm. A paper by Heller and Jacobs (1971) summarizes a number of performance characteristics of the Viterbi algorithm.

Burst-Error-Correcting Codes

In many practical communication channels, such as in fading channels, errors tend to group together and occur in bursts. Much attention has been devoted to code development for improving the performance of these types of channels. Most of these codes tend to be rather complex. A code for correction of a single burst, however, is rather simple to understand.

As an example, assume that the output of a source is encoded using an (n, k) block code. The i th codeword will be of the form

$$\lambda_{i1} \quad \lambda_{i2} \quad \lambda_{i3} \quad \cdots \quad \lambda_{in}$$

Assume that l of these codewords are read into a table to yield the array

$$\begin{array}{ccccccc} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_{l1} & \lambda_{l2} & \lambda_{l3} & \cdots & \lambda_{ln} \end{array}$$

If transmission is accomplished by reading out of this table by columns, the transmitted stream of symbols will be

$$\begin{array}{ccccccccccc} \lambda_{11} & \lambda_{21} & \lambda_{31} & \cdots & \lambda_{l1} & \lambda_{12} & \lambda_{22} & \lambda_{32} & \cdots \\ \lambda_{l2} & \lambda_{13} & \lambda_{23} & \lambda_{33} & \cdots & \lambda_{ln} & & & \end{array}$$

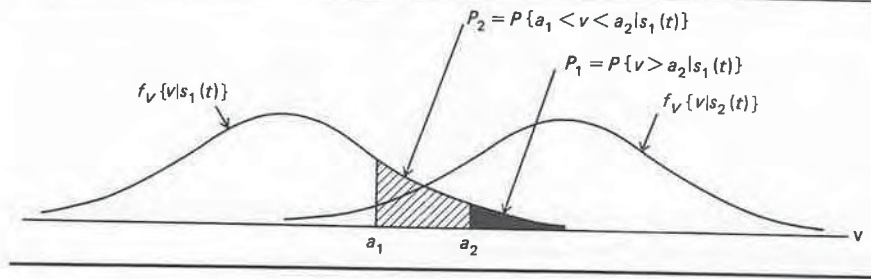
If a burst of errors affects l consecutive symbols, then each codeword will have one error, and a single error-correcting code will correct the burst if there are no other errors in the stream of ln symbols. Likewise, a double-error-correcting code can be used to correct a single burst spanning $2l$ symbols. These codes are known as *interleaved codes* since l codewords are interleaved to form the sequence of length ln .

Feedback Channels

In many practical systems, a feedback channel is available from receiver to transmitter. When available, this channel can be utilized to achieve a specified performance with decreased complexity of the encoding scheme. Many such schemes are possible: decision feedback, error-detection feedback, and information feedback. In a decision-feedback scheme, a null-zone receiver is used, and the feedback channel is utilized to inform the transmitter either that no decision was possible on the previous symbol and to retransmit or that a decision was made and to transmit the next symbol. The null-zone receiver is usually modeled as a binary-erasure channel.

Error-detection feedback involves the combination of coding and a feedback channel. With this scheme, retransmission of codewords is requested when errors are detected.

In general, feedback schemes tend to be rather difficult to analyze. Thus only the simplest scheme, the decision-feedback channel, will be treated here. Assume a binary transmission scheme with matched-filter detection. The signaling waveforms are $s_1(t)$ and $s_2(t)$. The conditional probability density functions of the matched filter output, conditioned on $s_1(t)$ and $s_2(t)$, were derived in Chapter 7 and are illustrated in Figure 7.7, which is redrawn for our application in Figure 10.28. We shall assume that both $s_1(t)$ and $s_2(t)$

FIGURE 10.28 Decision regions for null-zone receiver

have equal *a priori* probabilities. For the null-zone receiver, two thresholds, a_1 and a_2 , are established. If the matched-filter output, sampled at time T , lies between a_1 and a_2 , no decision is made, and the feedback channel is used to request a retransmission. The probability of error, given $s_1(t)$ was transmitted, is denoted P_1 . The probability of no decision (null-zone), given $s_1(t)$ was transmitted, is denoted P_2 . By symmetry, these probabilities are the same for $s_2(t)$ transmitted.

The probability of error on the j th transmission is

$$P_{Ej} = P_2^{j-1} P_1 \quad (10.135)$$

Thus the overall probability of error P_E is

$$P_E = \sum_{j=1}^{\infty} P_2^{j-1} P_1 \quad (10.136)$$

which is

$$P_E = \frac{P_1}{1 - P_2} \quad (10.137)$$

The expected number of transmissions N is also easily derived. The result is

$$N = \frac{1}{1 - P_2} \quad (10.138)$$

which is typically only slightly greater than one.

It follows from these results that the error probability can be reduced considerably without significantly increasing N . Thus performance is improved without a great sacrifice in information rate.

10.4 MODULATION SYSTEMS

In Chapter 6, signal-to-noise ratios were computed at various points in a communication system. Of particular interest were the signal-to-noise ratio at the input to the demodulator and the signal-to-noise ratio of the demodulated output. These were referred to as the *predetection SNR*, $(\text{SNR})_T$, and the *postdetection SNR*, $(\text{SNR})_D$, respectively. The ratio of these parameters, the detection gain, has been widely used as a figure of merit for the system. In this section we will compare the behavior of $(\text{SNR})_D$ as a function of $(\text{SNR})_T$ for several systems. First, however, we investigate the behavior of an *optimum*, but *unrealizable* system. This study will provide a firm basis for comparison and also provide additional insight into the concept of the tradeoff of bandwidth for signal-to-noise ratio.

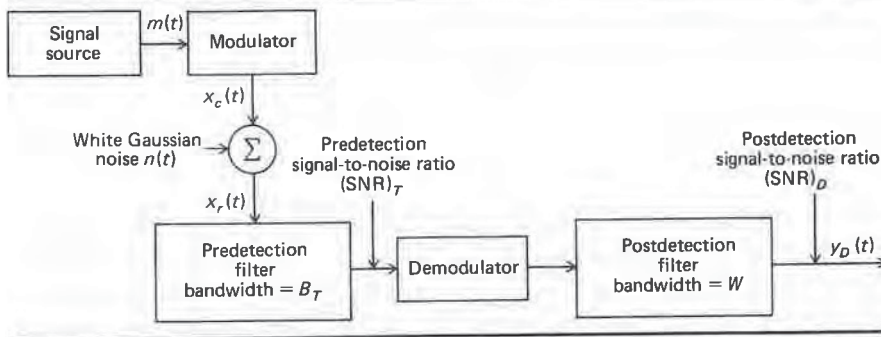
Optimum Modulation

The block diagram of a communication system is illustrated in Figure 10.29. We will focus on the receiver portion of the system. The SNR at the output of the predetection filter, $(\text{SNR})_T$ yields the *maximum* rate at which information may arrive at the receiver. From the Shannon-Hartley law, this rate, C_T , is

$$C_T = B_T \log [1 + (\text{SNR})_T] \quad (10.139)$$

where B_T , the predetection bandwidth, is typically the bandwidth of the modulated signal. Since (10.139) is based on the Shannon-Hartley law, it is valid only for additive *white* Gaussian noise cases. The SNR of the demodu-

FIGURE 10.29 Block diagram of a communication system



lated output, $(\text{SNR})_D$, yields the maximum rate at which information may leave the receiver. This rate, denoted C_D , is given by

$$C_D = W \log [1 + (\text{SNR})_D] \quad (10.140)$$

where W is the bandwidth of the message signal.

An optimum modulation system is defined as one for which $C_D = C_T$. For this system, demodulation is accomplished, in the presence of noise, without loss of information. Equating C_D to C_T yields

$$(\text{SNR})_D = [1 + (\text{SNR})_T]^{B_T/W} - 1 \quad (10.141)$$

which shows that the optimum exchange of bandwidth for SNR is *exponential*.

The ratio of transmission bandwidth B_T to the message bandwidth W is referred to as the *bandwidth expansion factor* γ . To fully understand the role of this parameter, we write the predetection SNR as

$$(\text{SNR})_T = \frac{P_T}{N_0 B_T} = \frac{W}{B_T} \frac{P_T}{N_0 W} = \frac{1}{\gamma} \frac{P_T}{N_0 W} \quad (10.142)$$

Thus (10.141) can be expressed as

$$(\text{SNR})_D = \left[1 + \frac{1}{\gamma} \left(\frac{P_T}{N_0 W} \right) \right]^\gamma - 1 \quad (10.143)$$

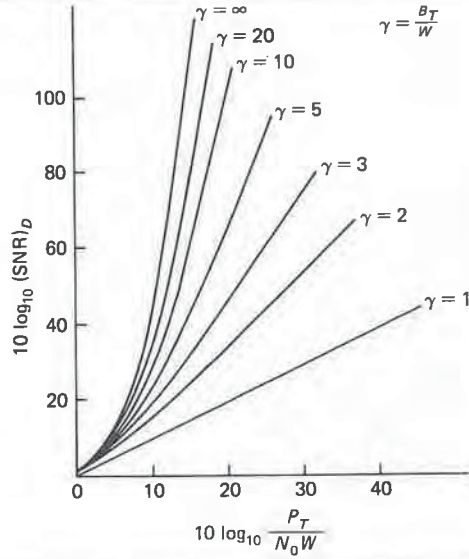
The relationship between $(\text{SNR})_D$ and $P_T/N_0 W$ is illustrated in Figure 10.30.

Comparison of Modulation Systems

The concept of an optimal modulation system provides a basis for comparing system performance. For example, an ideal single-sideband system has a bandwidth expansion factor of unity, since the transmission bandwidth is ideally equal to the message bandwidth. Thus the postdetection SNR of the optimal modulation system is, from (10.143) with γ equal to 1,

$$(\text{SNR})_D = \frac{P_T}{N_0 W} \quad (10.144)$$

This is exactly the same result as that obtained in Chapter 6 for an SSB system using coherent demodulation with a perfect phase reference. Therefore, if the transmission bandwidth B_T of an SSB system is *exactly* equal to the message bandwidth W , SSB is optimal, assuming that there are no other error sources. Of course, this can never be achieved in practice, since *ideal* filters

FIGURE 10.30 Performance of optimum modulation system

would be required in addition to *perfect* phase coherence of the demodulation carrier.

The story is quite different with DSB, AM, and QDSB. For these systems, $\gamma = 2$. In Chapter 6 we saw that the postdetection SNR for DSB and QDSB, assuming perfect coherent demodulation, is

$$(\text{SNR})_D = \frac{P_T}{N_0 W} \quad (10.145)$$

whereas for the optimal system it is given by (10.143) with $\gamma = 2$.

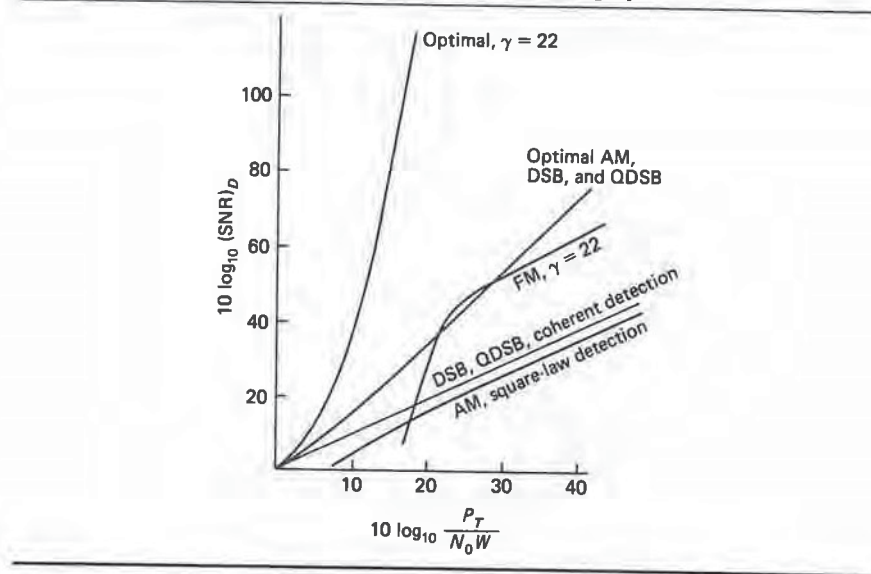
These results are shown in Figure 10.31, along with the result for AM with square-law demodulation. It can be seen that these systems are far from optimal, especially for large values of $P_T/N_0 W$.

Also shown in Figure 10.31 is the result for FM without preemphasis, with sinusoidal modulation, assuming a modulation index of 10. With this modulation index, the bandwidth expansion factor is

$$\gamma = \frac{2(\beta + 1)W}{W} = 22 \quad (10.146)$$

The realizable performance of the FM system is taken from Figure 6.18. It can be seen that realizable systems fall far short of optimal if γ and $P_T/N_0 W$ are large.

FIGURE 10.31 Performance comparison of analog systems



10.5 BANDWIDTH AND POWER-EFFICIENT MODULATION

A desirable characteristic of any modulation scheme is the simultaneous conservation of bandwidth and power. Since the late 1970's, the approach to this challenge has been to combine coding and modulation. There have been two approaches: (1) continuous phase modulation (CPM)¹ with memory extended over several modulation symbols by cyclical use of a set of modulation indices; and (2) combining coding with an M -ary modulation scheme, referred to as *trellis-coded modulation* (TCM).² We briefly explore the latter approach in this section. For an introductory discussion of the former

¹ CPM has been explored by many investigators. For introductory treatments see C.-E. Sundberg, "Continuous Phase Modulation" *IEEE Communications Magazine*, Vol. 24, Apr. 1986, pp. 25–38, and J. B. Anderson and C.-E. Sundberg, "Advances in Constant Envelope Coded Modulation," *IEEE Communications Magazine*, Vol. 29, Dec. 1991, pp. 36–45.

² Three introductory treatments of TCM can be found in G. Ungerboeck, "Channel Coding with Multilevel/Phase Signals," *IEEE Transactions on Information Theory*, Vol. IT-28, Jan. 1982, pp. 55–66, G. Ungerboeck, "Trellis-Coded Modulation with Redundant Signal Sets, Part I: Introduction," *IEEE Communications Magazine*, Vol. 25, Feb. 1987, pp. 5–11, and G. Ungerboeck, "Trellis-Coded Modulation with Redundant Signal Sets, Part II: State of the Art," *IEEE Communications Magazine*, Vol. 25, Feb. 1987, pp. 12–21.

approach, see Ziemer and Peterson (1985), Chapter 4. Sklar (1988) is a well-written reference with more examples on TCM than given in this short section.

In Chapter 8 it was illustrated through the use of signal-space diagrams that the most probable errors in an M -ary modulation scheme result from mistaking a signal point closest in Euclidian distance to the transmitted signal point as corresponding to the actual transmitted signal. Ungerboeck's solution to this problem was to use coding in conjunction with M -ary modulation to increase the minimum Euclidian distance between those signal points most likely to be confused without increasing the average power or bandwidth over an uncoded scheme transmitting the same number of bits per second. We illustrate the procedure with a specific example.

We wish to compare a TCM system and a QPSK system operating at the same data rates. Since the QPSK system transmits 2 bits per signal phase (signal-space point), we can keep that same data rate with the TCM system by employing an 8-PSK modulator, which carries 3 bits per signal phase, in conjunction with a convolutional encoder that produces three encoded symbols for every two input data bits, i.e., a rate $\frac{2}{3}$ encoder. Figure 10.32(a) shows an encoder for accomplishing this and Figure 10.32(b) shows the corresponding trellis diagram. The encoder operates by taking the first data bit as the input to a rate $\frac{1}{2}$ convolutional encoder that produces the first and second encoded symbols, and the second data bit directly as the third encoded symbol. These are then used to select the particular signal phase to be transmitted according to the following rules:

1. All parallel transitions in the trellis are assigned the maximum possible Euclidian distance. Since these transitions differ by one code symbol (the one corresponding to the uncoded bit in this example), an error in decoding these transitions amounts to a single bit error, which is minimized by this procedure.
2. All transitions emanating or converging into a trellis state are assigned the next to largest possible Euclidian distance separation.

The application of these rules to assigning the encoded symbols to a signal phase in an 8-PSK system can be done with a technique known as set partitioning, which is illustrated in Figure 10.33. If the encoded symbol c_1 is a 0, the left branch is chosen in the first tier of the tree, whereas if c_1 is a 1, the right branch is chosen. A similar procedure is followed for tiers 2 and 3 of the tree, with the result being that a unique signal phase is chosen for each possible encoded output.

To decode the TCM signal, the received signal plus noise in each signaling

FIGURE 10.32 (a) Convolutional encoder and (b) trellis diagram corresponding to 4-state, 8-PSK TCM.

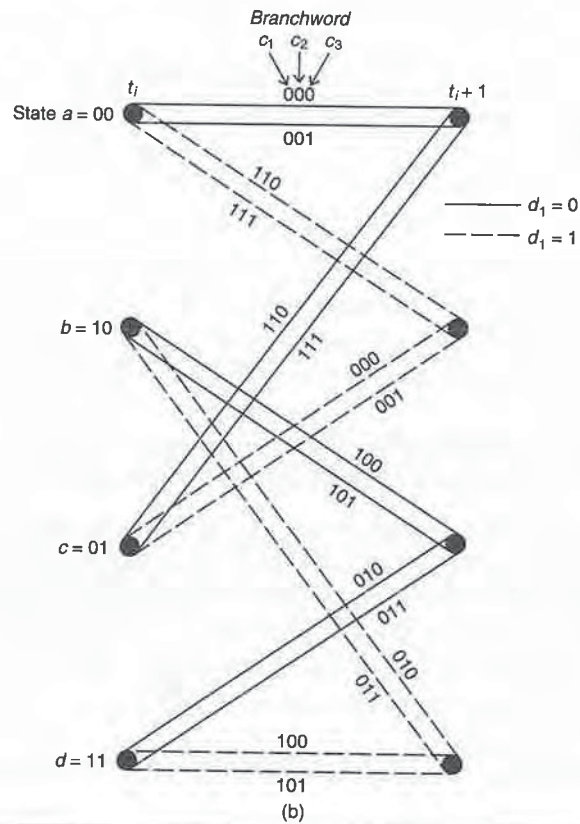
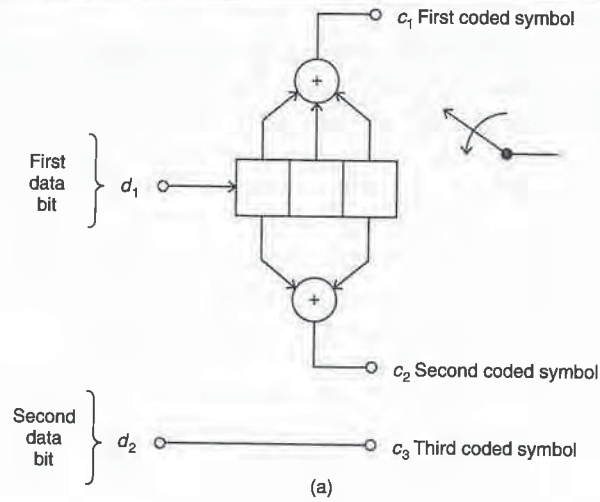
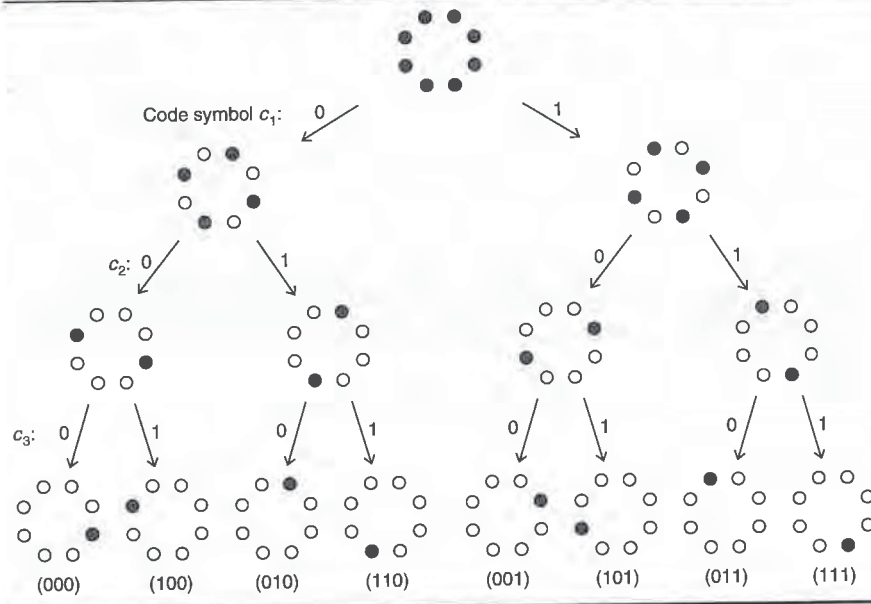


FIGURE 10.33 Set partitioning for assigning a rate $\frac{2}{3}$ encoder output to 8-PSK signal points while obeying the rules for maximizing free distance. (From G. Ungerboeck, "Channel Coding with Multilevel/Phase Signals," *IEEE Transactions on Information Theory*, Vol. IT-28, Jan. 1982, pp. 55–66).



interval is correlated with each possible transition in the trellis and a search is made through the trellis by means of a Viterbi algorithm using the sum of these cross-correlations as metrics rather than Hamming distance as discussed in conjunction with Figure 10.26 (this is called the use of a soft decision metric). Also note that the decoding procedure is twice as complicated since two branches correspond to a path from one trellis state to the next due to the uncoded bit becoming the third symbol in the code. In choosing the two decoded bits for a surviving branch, the first decoded bit of the pair corresponds to the input bit b_1 that produced the state transition of the branch being decoded. The second decoded bit of the pair is the same as the third symbol c_3 of that branch word, since c_3 is the same as the uncoded bit b_2 .

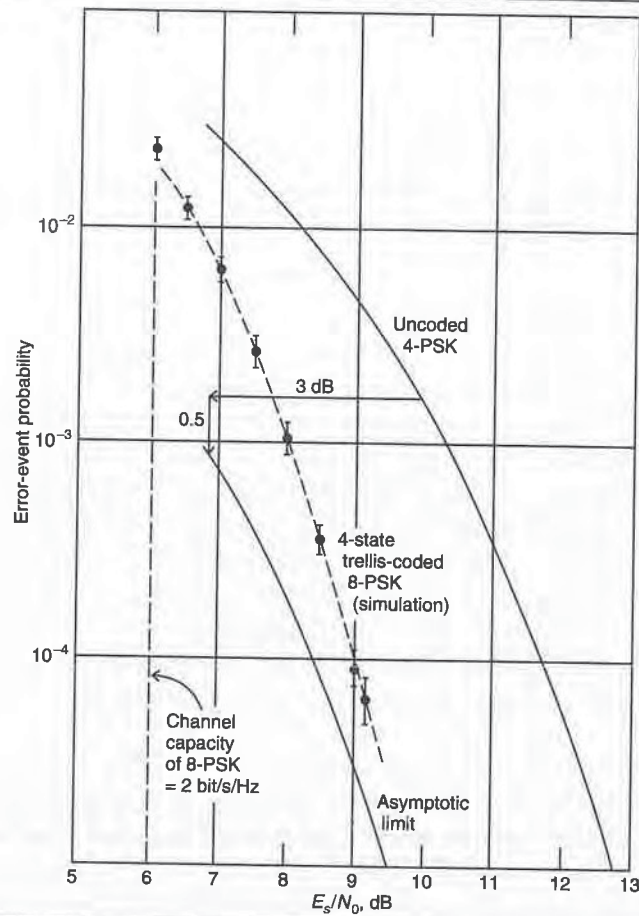
Ungerboeck has characterized the event error probability performance of a signaling method in terms of the free distance of the signal set. For high signal-to-noise ratios, the probability of an error event (i.e., the probability that at any given time the VA makes a wrong decision among the signals associated with parallel transitions, or starts to make a sequence of wrong

decisions along some path diverging from more than one transition from the correct path) is well approximated by

$$P(\text{error event}) = N_{\text{free}} Q\left(\frac{d_{\text{free}}}{2\sigma}\right) \quad (10.147)$$

where N_{free} denotes the number of nearest-neighbor signal sequences with distance d_{free} that diverge at any state from a transmitted signal sequence, and reemerge with it after one or more transitions. (The free distance is

FIGURE 10.34 Performance for a 4-state, 8-PSK TCM signaling scheme. (From G. Ungerboeck, "Trellis-Coded Modulation with Redundant Signal Set, Part I: Introduction," *IEEE Communications Magazine*, Feb. 1987, Vol. 25, pp. 5-11).



often calculated by assuming the signal energy has been normalized to unity, and that the noise standard deviation σ accounts for this normalization.)

For uncoded QPSK, we have $d_{\text{free}} = 2^{1/2}$ and $N_{\text{free}} = 2$ (there are two adjacent signal points at distance $d_{\text{free}} = 2^{1/2}$), whereas for 4-state-coded 8-PSK we have $d_{\text{free}} = 2$ and $N_{\text{free}} = 1$. Ignoring the factor N_{free} , we have an asymptotic gain due to TCM over uncoded QPSK of $2^2/(2^{1/2})^2 = 2 = 3$ dB. Figure 10.34, also from Ungerboeck, compares the asymptotic lower bound for the error event probability with simulation results.

It should be clear that the TCM encoding/modulation procedure can be generalized to higher-level M -ary schemes. Ungerboeck shows that this observation can be generalized as follows: (1) Of the m bits to be transmitted per encoder/modulator operation, $k \leq m$ bits are expanded to $k + 1$ coded symbols by a binary rate $k/(k + 1)$ convolutional encoder; (2) the $k + 1$ coded symbols select one of 2^{k+1} subsets of a redundant 2^{m+1} -ary signal set; (3) the remaining $m - k$ symbols determine one of 2^{m-k} signals within the selected subset. It should also be stated that one may use block codes or other modulation schemes, such as M -ary ASK or QASK, to implement a TCM system.

Another parameter that influences the performance of a TCM system is the constraint length of the code, ν , which is equivalent to saying that the encoder has 2^ν states. Ungerboeck has published asymptotic gains for TCM systems with various constraint lengths. These are given in Table 10.7.

TABLE 10.7 Asymptotic Coding Gains for TCM Systems

No. of States, 2^ν	k	Asymptotic $G_{8\text{PSK/QPSK}}, m = 2$	Coding Gain (dB) $G_{16\text{PSK/8PSK}}, m = 3$
4	1	3.01	—
8	2	3.60	—
16	2	4.13	—
32	2	4.59	—
64	2	5.01	—
128	2	5.17	—
256	2	5.75	—
4	1	—	3.54
8	1	—	4.01
16	1	—	4.44
32	1	—	5.13
64	1	—	5.33
128	1	—	5.33
256	2	—	5.51

SOURCE: Adapted from G. Ungerboeck, "Trellis-Coded Modulation with Redundant Signal Sets. Part II: State of the Art," *IEEE Communications Magazine*, Vol. 25, Feb. 1987, pp. 12-21.

Finally, the paper by Viterbi et al. (1989) gives a simplified scheme for M -ary PSK that uses a single rate $\frac{1}{2}$, 64-state binary convolutional code for which VLSI circuit implementations are plentiful. A technique known as puncturing converts it to rate $(n - 1)/n$.

SUMMARY

1. The information associated with the occurrence of an event is defined as the logarithm of the probability of the event. If a base 2 logarithm is used, the measure of information is the bit.
2. The average information associated with a set of source outputs is known as the entropy of the source. The entropy function has a maximum, and the maximum occurs when all source states are equally likely. Entropy is average uncertainty.
3. A channel with n inputs and m outputs is represented by the nm transition probabilities of the form $p(y_j | x_i)$. The channel model can be a diagram showing the transition probabilities or a matrix of the transition probabilities.
4. A number of entropies can be defined for a system. The entropies $H(X)$ and $H(Y)$ denote the average uncertainty of the channel input and output, respectively. The quantity $H(X|Y)$ is the average uncertainty of the channel input given the output, and $H(Y|X)$ is the average uncertainty of the channel output given the input. The quantity $H(X, Y)$ is the average uncertainty of the communication system as a whole.
5. The mutual information between the input and output of a channel is given by

$$I(X; Y) = H(X) - H(X|Y)$$

or

$$I(X; Y) = H(Y) - H(Y|X)$$

The maximum value of mutual information, where the maximization is with respect to the source probabilities, is known as the channel capacity.

6. Source encoding is used to remove redundancy from a source output so that the information per transmitted symbol can be maximized. If the source rate is less than the channel capacity, it is possible to encode the source output so that the information can be transmitted through the channel. This is accomplished by forming source extensions and encod-

ing the symbols of the extended source into codewords having minimum average word length. The minimum average word length $\bar{L}$ approaches $H(X^n) = nH(X)$, where $H(X^n)$ is the entropy of the n th order extension of a source having entropy $H(X)$, as n increases.

7. Two source encoding schemes are the Shannon-Fano technique and the Huffman technique. The Huffman technique yields an optimum source code, which is the source code having minimum average word length.
8. Error-free transmission on a noisy channel can be accomplished if the source rate is less than the channel capacity. This is accomplished using channel codes.
9. The capacity of an additive white Gaussian noise (AWGN) channel is

$$C_c = B \log_2 \left(1 + \frac{S}{N} \right)$$

where B is the channel bandwidth and S/N is the signal-to-noise ratio. This is known as the Shannon-Hartley law.

10. An (n, k) block code is generated by appending $r = n - k$ parity symbols to a k -symbol source sequence. This yields an n -symbol codeword.
11. Decoding is typically accomplished by computing the Hamming distance from the received n -symbol sequence to each of the possible transmitted codewords. The codeword closest in Hamming distance to the received sequence is the most likely transmitted codeword. The two codewords closest in Hamming distance determine the minimum distance of the code d_m . The code can correct $\frac{1}{2}(d_m - 1)$ errors.
12. A single-parity-check code is formed by adding a single-parity symbol to the information sequence. This code can detect single errors but provides no error-correcting capability.
13. The rate of a block code is k/n . The best codes provide powerful error-correction capabilities in combination with high rate.
14. Repetition codes are formed by transmitting each source symbol an odd number of times. Repetition codes provide powerful error-correction capabilities but have very low rate.
15. The parity-check matrix $[H]$ is defined such that $[H][T] = [0]$, where $[T]$ is the transmitted codeword written as a column vector. If the received sequence is denoted by the column vector $[R]$, the syndrome $[S]$ is determined from $[S] = [H][R]$. This can be shown to be equivalent to $[S] = [H][E]$, where $[E]$ is the error sequence. If a single error occurs

in the transmission of a codeword, the syndrome is the column of $[H]$ corresponding to the error position.

16. The generator matrix $[G]$ of a parity-check code is determined such that $[T] = [G][A]$, where $[T]$ is the n -symbol transmitted sequence and $[A]$ is the k -symbol information sequence. Both $[T]$ and $[A]$ are written as column vectors.
17. For a group code, the modulo-2 sum of any two codewords is another codeword.
18. A Hamming code is a single-error-correcting code such that the columns of the parity-check matrix correspond to the binary representation of the column index.
19. Cyclic codes are a class of block codes such a cyclic shift of the codeword symbols always yields another codeword. These codes are very useful because implementation is simple. Implementation of both the encoder and decoder is accomplished using shift registers and basic logic components.
20. The channel symbol error probability of a coded system is greater than the symbol error probability of an uncoded system because the available energy for transmission of k information symbols must be spread over the n -symbol codeword rather than just the k -information symbols. The error-correcting capability of the code often allows a net performance gain to be realized. The performance gain depends on the choice of code and the channel characteristics.
21. Convolutional codes are easily generated using simple shift registers and modulo-2 adders. Decoding is accomplished using a tree-search technique, which is often implemented using the Viterbi algorithm. The constraint span is the code parameter having the most significant impact on performance.
22. Interleaved codes are useful for burst noise environments.
23. The feedback channel system makes use of a null-zone receiver, and a retransmission is requested if the receiver decision falls within the null zone. If a feedback channel is available, the error probability can be significantly reduced with only a slight increase in the required number of transmissions.
24. Use of the Shannon-Hartley law yields the concept of optimum modulation for a system operating in an AWGN environment. The result is the performance of an optimum system in terms of predetection and

postdetection bandwidth. The tradeoff between bandwidth and signal-to-noise ratio is easily seen.

25. Trellis-coded modulation is a scheme for combining M -ary modulation with coding in a way that increases the Euclidian distance between those signal points for which errors are most likely without increasing the average power or bandwidth over an uncoded scheme having the same bit rate. Decoding is accomplished using a Viterbi decoder that accumulates decision metrics (soft decisions) rather than Hamming distances (hard decisions).

FURTHER READING

An exposition of information theory and coding that was anywhere near complete would, of necessity, be presented at a level beyond that intended for this text. The purpose in the present chapter is to present some of the basic ideas of information theory at a level consistent with the rest of this book. You should be motivated by this to further study.

The original paper by Shannon (1948) is stimulating reading at about the same level as this chapter. This paper is available as a paperback with an interesting postscript by W. Weaver (Shannon and Weaver, 1963). A number of very brief and interesting papers have traced the development of information theory through the years since Shannon's work. Examples are those by Slepian (1973) and Viterbi (1973). Many survey papers on coding theory are also available. The paper by Forney (1970), which stresses performance characteristics, and the paper by Wolf (1973) are particularly recommended.

Several standard texts on information theory are available. Those by Blahut (1987) and Mansuripur (1987) are written at a slightly higher level than this text. The volume by Gallager (1968) is a very complete exposition of the subject and is written at the graduate level. Another standard text, which is very complete, is the one by Blahut (1984). This volume is also written at the beginning graduate level. The text by Lin and Costello (1970) is less rigorous than the other two and contains much information on the implementation of decoders. An interesting perspective on coding theory is provided by Arazzi (1988). Many other well-written books and review articles are available.

Several other books deserve special mention. The book by Clark and Cain (1981) contains a wealth of practical information concerning coder and decoder design, in addition to the usual theoretical background material.

The last chapter, entitled "System Applications," contains a large number of performance curves for different channel models. The book by Viterbi and Omura (1979) is an excellent basic treatment of information theory and coding, and it contains an excellent analysis of the Viterbi algorithm. The book edited by Yuen (1983) has a very good section on coding for application to deep-space telemetry.

As mentioned in the last section of this chapter, the subject of bandwidth and power-efficient communications is very important to the implementation of modern systems. Continuous phase modulation is treated in the text by Ziemer and Peterson (1985). An introductory treatment of trellis-coded modulation, (TCM), including a discussion of coding gain, is contained in the book by Sklar (1988). The book by Biglieri, Divsalar, McLane, and Simon (1991) is a complete treatment of TCM theory, performance, and implementation.

PROBLEMS

Section 10.1

- 10.1 A message occurs with a probability of 0.6. Determine the information associated with the message in bits, nats, and hartleys.
- 10.2 Assume that you have a standard deck of 52 cards (jokers have been removed).
 - (a) What is the information associated with the drawing of a single card from the deck?
 - (b) What is the information associated with the drawing of a pair of cards, assuming that the first card drawn is replaced in the deck prior to drawing the second card?
 - (c) What is the information associated with the drawing of a pair of cards assuming that the first card drawn is *not* replaced in the deck prior to drawing the second card?
- 10.3 A source has five outputs denoted $[m_1, m_2, m_3, m_4, m_5]$ with respective probabilities $[0.3, 0.3, 0.2, 0.1, 0.1]$. Determine the entropy of the source. What is the maximum entropy of a source with five outputs?
- 10.4 A source consists of six outputs denoted $[A, B, C, D, E, F]$ with respective probabilities $[0.3, 0.2, 0.15, 0.15, 0.1, 0.1]$. Determine the entropy of the source.

10.5 A channel has the following transition matrix:

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$$

- Sketch the channel diagram showing all transition probabilities.
 - Determine the channel output probabilities assuming that the input probabilities are equal.
 - Determine the channel input probabilities that result in equally likely channel outputs.
 - Determine the joint probability matrix using (c) above.
- 10.6 Describe the channel transition probability matrix and joint probability matrix for a noiseless channel.
- 10.7 Show that $H(Y) \geq H(Y|X)$.
- 10.8 A channel is described by the transition probability matrix

$$[P(Y|X)] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine the channel capacity and the source probabilities that yield capacity.

10.9 A channel is described by the matrix

$$[P(Y|X)] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine the channel capacity and the source probabilities that yield capacity.

- 10.10 A channel has two inputs, (0, 1), and three outputs, (0, e , 1), where e indicates an erasure; that is, there is no output for the corresponding input. The channel matrix is

$$\begin{bmatrix} p & 1-p & 0 \\ 0 & 1-p & p \end{bmatrix}$$

Compute the channel capacity.

- 10.11 Determine the capacity of the channel described by the channel matrix shown below. Sketch your result as a function of p and give an intuitive argument that supports your sketch. (Note: $q = 1 - p$.)

$$\begin{bmatrix} p & q & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & q & p \end{bmatrix}$$

- 10.12 From the entropy definition given in Equations (10.22) through (10.26), derive Equations (10.27) and (10.28).
- 10.13 The input to a quantizer is a random signal having an amplitude probability density function

$$f_X(x) = \begin{cases} ae^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

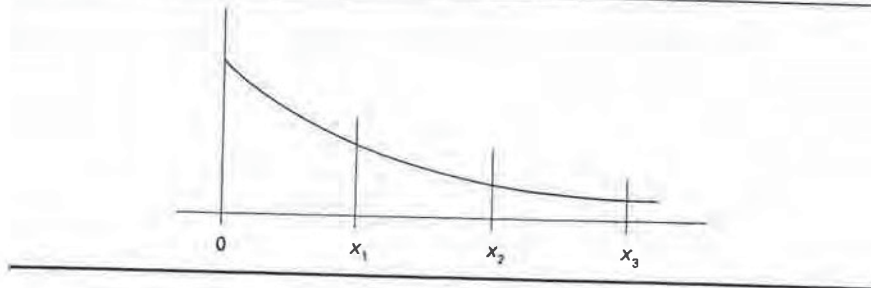
The signal is to be quantized using four quantizing levels x_i as shown in Figure 10.35. Determine the values x_i , $i = 1, 2, 3$, in terms of a so that the entropy at the quantizer output is maximized.

- 10.14 Repeat the preceding problem assuming that the input to the quantizer has the Rayleigh probability density function

$$f_X(x) = \begin{cases} \frac{x}{a^2} e^{-x^2/2a^2} & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- 10.15 A signal has a Gaussian amplitude-density function with zero mean and variance σ^2 . The signal is sampled at 750 samples per second. The samples are quantized according to the following table. Deter-

FIGURE 10.35

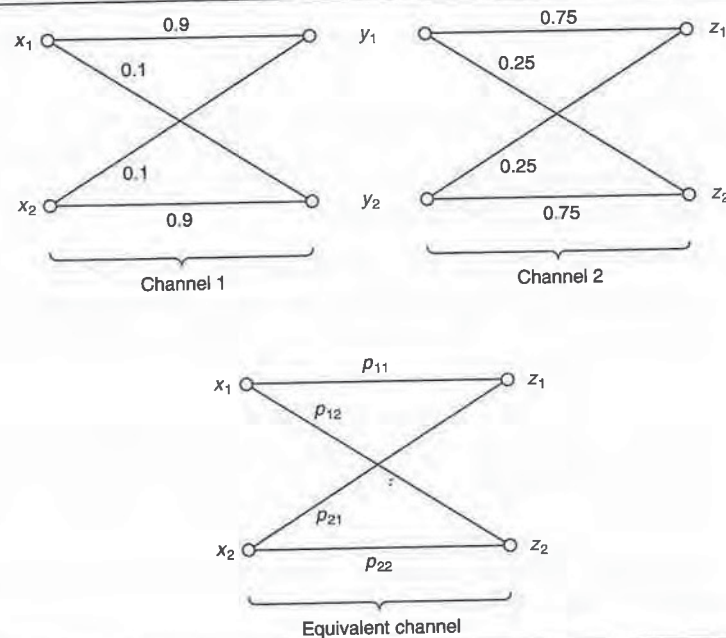


mine the entropy at the quantizer output and the information rate in samples per second.

Quantizer Input	Quantizer Output
$-\infty < x_i < -2\sigma$	m_0
$-2\sigma < x_i < -\sigma$	m_1
$-\sigma < x_i < 0$	m_2
$0 < x_i < \sigma$	m_3
$\sigma < x_i < 2\sigma$	m_4
$2\sigma < x_i < \infty$	m_5

- 10.16 Determine the quantizing levels, in terms of σ , so that the entropy at the output of a quantizer is maximized. Assume that there are six quantizing levels and that the quantizer is a zero-mean Gaussian process as in the previous problem.
- 10.17 Two binary symmetrical channels are in cascade, as shown in Figure 10.36. Determine the capacity of each channel. The overall system

FIGURE 10.36



with inputs x_1 and x_2 and outputs z_1 and z_2 can be represented as shown with p_{11} , p_{12} , p_{21} , and p_{22} properly chosen. Determine these four probabilities and the capacity of the overall system. Comment on the results.

- 10.18 A two-hop satellite communications channel uses BPSK signaling. The uplink signal-to-noise ratio is 8 dB and the downlink signal-to-noise ratio is 5 dB, where the signal-to-noise ratio is the signal power divided by the noise power in the bit-rate bandwidth. Determine the overall error probability.

Section 10.2

- 10.19 A source has two outputs $[A, B]$ with respective probabilities $[\frac{2}{3}, \frac{1}{3}]$. Determine the entropy of the fourth-order extension of this source using two different methods.
- 10.20 Calculate the entropy of the fourth-order extension of the source defined in Table 10.1. Determine L/n for $n = 4$ and add this result to those shown in Figure 10.9. Determine the efficiency of the resulting codes for $n = 1, 2, 3$, and 4.
- 10.21 A source has five equally likely output messages. Determine a Shannon-Fano code for the source and determine the efficiency of the resulting code. Repeat for the Huffman code and compare the results.
- 10.22 A source has five outputs denoted $[m_1, m_2, m_3, m_4, m_5]$ with respective probabilities $[0.40, 0.19, 0.16, 0.15, 0.10]$. Determine the codewords to represent the source outputs using both the Shannon-Fano and the Huffman techniques.
- 10.23 A binary source has output probabilities $[0.85, 0.15]$. The channel can transmit 300 binary symbols per second at the capacity of 1 bit/symbol. Determine the maximum source symbol rate if transmission is to be accomplished.
- 10.24 A source output consists of nine equally likely messages. Encode the source output using both binary Shannon-Fano and Huffman codes. Compute the efficiency of both of the resulting codes and compare the results.
- 10.25 Repeat the preceding problem assuming that the source has 12 equally likely outputs.

- 10.26 An analog source has an output described by the probability density function

$$f_x(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The output of the source is quantized into ten messages using the nine quantizing levels

$$x_i = 0.1k, \quad k = 0, 1, \dots, 10$$

The resulting messages are encoded using a binary Huffman code. Assuming that 250 samples of the source are transmitted each second, determine the resulting binary symbol rate in symbols per second. Also determine the information rate in bits per second.

- 10.27 A source output consists of four messages $[m_1, m_2, m_3, m_4]$ with respective probabilities $[0.4, 0.3, 0.2, 0.1]$. The second-order extension of the source is to be encoded using a code with a three-symbol alphabet. Determine the codewords using the Shannon-Fano technique and determine the efficiency of the resulting code.
- 10.28 It can be shown that a necessary and sufficient condition for the existence of an instantaneous binary code with wordlengths l_i , $1 \leq i \leq N$, is that

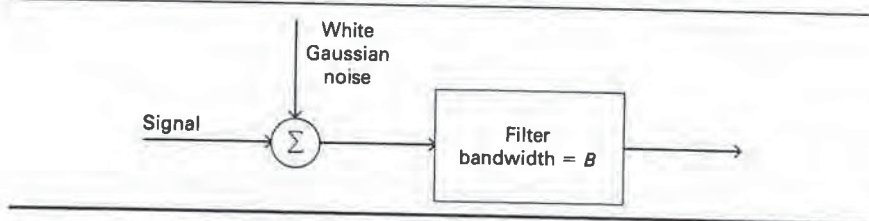
$$\sum_{i=1}^N 2^{-l_i} \leq 1$$

This is known as the Kraft inequality. Show that the Kraft inequality is satisfied by the codewords given in Table 10.3. (Note: The inequality given above must also be satisfied for uniquely decipherable codes.)

Section 10.3

- 10.29 A continuous bandpass channel can be modeled as illustrated in Figure 10.37. Assuming a signal power of 20 W and a noise power spectral density of 10^{-4} W/Hz, plot the capacity of the channel as a function of the channel bandwidth.
- 10.30 For the preceding problem determine the capacity in the limit as $B \rightarrow \infty$.

FIGURE 10.37



- 10.31 Derive an equation, similar to (10.92), that gives the error probability for a rate $\frac{1}{3}$ repetition code. Plot together, on the same set of axes, the error probability of both a rate of $\frac{1}{3}$ and rate $\frac{1}{5}$ repetition code as a function of $q = 1 - p$.
- 10.32 Show that a (7, 4) Hamming code is a distance 3 code.
- 10.33 Write the parity-check matrix and the generator matrix for a (15, 11) single error-correcting code. Assume that the code is systematic. Calculate the codeword corresponding to the all one's information sequence. Calculate the syndrome corresponding to an error in the third position assuming the codeword corresponding to the all one's input sequence.
- 10.34 A parity-check code has the parity-check matrix

$$[H] = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Determine the generator matrix and find all possible codewords.

- 10.35 For the code described in the preceding problem, find the codewords $[T_1]$ and $[T_2]$ corresponding to the information sequences

$$[A_1] = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad [A_2] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Using these two codewords, illustrate the group property.

- 10.36 Determine the generator matrix for a rate $\frac{1}{3}$ and a rate $\frac{1}{5}$ repetition code. Describe the generator matrix for a rate $1/n$ repetition code.
- 10.37 Determine the relationship between n and k for a Hamming code. Use this result to show that the code rate approaches 1 for large n .

- 10.38 Determine the generator matrix for the encoder illustrated in Figure 10.16. Use the generator matrix to generate the complete set of codewords and use the results to check the codewords shown in Figure 10.16. Show that these codewords constitute a cyclic code.
- 10.39 Use the result of the preceding problem to determine the parity-check matrix for the encoder shown in Figure 10.16. Use the parity-check matrix to decode the received sequences 1101001 and 1101011. Compare your result with that shown in Figure 10.17.
- 10.40 Modify the encoder illustrated in Figure 10.16 by using feedback taps from stages S_1 and S_3 rather than from stages S_2 and S_3 . Determine the resulting generator matrix and the complete set of 16 codewords. Is the resulting code a cyclic code?
- 10.41 Construct a set of performance curves similar to those of Figure 10.18 for a (15, 11) Hamming code.
- 10.42 Consider the coded system examined in Example 10.12. Show that the probability of three symbol errors in a codeword is negligible compared to the probability of two symbol errors in a codeword for signal-to-noise ratios above a certain level.
- 10.43 The Hamming code was defined as a code for which the i th column of the parity-check matrix is the binary representation of the number i . With a little thought it becomes clear that the columns of the parity-check matrix can be permuted without changing the distance properties, and therefore the error-correcting capabilities, of the code. Using this fact, determine a generator matrix and the corresponding parity-check matrix of the systematic code equivalent to a (7, 4) Hamming code. How many different generator matrices can be found?
- 10.44 In order to compare the performance of digital communications systems using block codes of different word lengths n , the probability of error for each code must be computed. One technique for accomplishing this task is to use the approximation¹

$$P_b = \frac{d}{n} \sum_{i=d+1}^n \binom{n}{i} p_{sc}^i (1 - p_{sc})^{n-i} + \sum_{i=d+1}^n \binom{n-1}{i-1} p_{sc}^i (1 - p_{sc})^{n-i}$$

¹ D. J. Torrieri, *Principles of Secure Communication Systems*, 2nd ed. Artech House, Norwood, Mass., 1992, p. 47.

where d is the minimum distance of the code, e is the number of correctable errors per codeword, P_{sc} is the symbol error probability for the coded system and P_b is the decoded bit-error probability. Using this approximation, compare the performance of (7, 4) and (15, 11) single-error-correcting Hamming codes in an AWGN system with BPSK modulation.

- 10.45 Repeat the previous problem for a (7, 4) Hamming code and a (23, 12) triple-error-correcting Golay code.
- 10.46 What is the constraint span of the encoder shown in Figure 10.38? How many states are required to define the state diagram and the trellis diagram? Draw the state diagram, giving the output for each state transition.
- 10.47 Determine the state diagram for the convolutional encoder shown in Figure 10.39. Draw the trellis diagram through the first set of steady-state transitions. On a second trellis diagram, show the termination of the trellis to the all-zero state.
- 10.48 A source produces binary symbols at a rate of 2000 symbols per second. The channel is subjected to error bursts lasting 0.15 sec. Devise an encoding scheme using an interleaved (n, k) Hamming code which allows full correction of the error burst. Assume that the information rate out of the encoder is equal to the information rate into the encoder. What is the minimum time between bursts if your system is to operate properly?
- 10.49 A very popular class of codes is known as BCH (Bose-Chaudhuri-Hocquenghem) codes. These allow correction of multiple errors. For

FIGURE 10.38

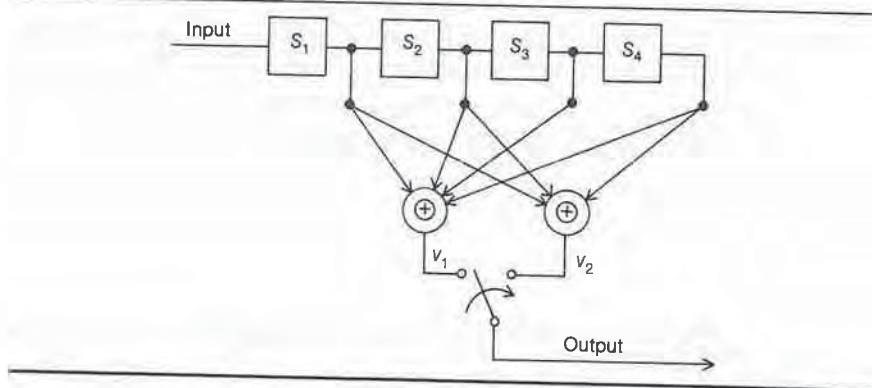
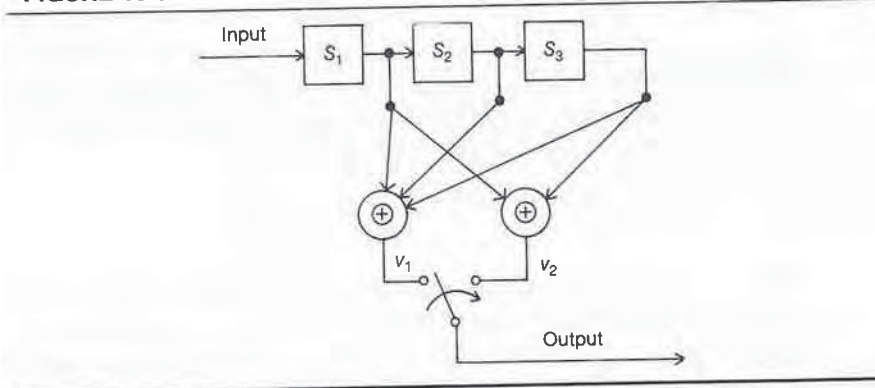


FIGURE 10.39



example, a (15, 7) BCH code can correct two errors per word. Analyze this code by deriving a set of performance curves, similar to those of Figure 10.18 for the (15, 7) BCH code.

Section 10.4

- 10.50 Compare FM with *preemphasis* to an optimal modulation system for $\beta = 1, 5$, and 10. Consider only operation above threshold, and assume 20 dB as the value of P_T/N_0W at threshold.



COMPUTER EXERCISES

- 10.1 Develop a computer program that allows you to plot the entropy of a source with variable output probabilities. We wish to observe that the maximum source entropy does indeed occur when the source outputs are equally likely. Start with a simple two-output source $[m_1, m_2]$ with respective probabilities $[a, 1-a]$ and plot the entropy as a function of the parameter a . Then consider more complex cases such as a three output source $[m_1, m_2, m_3]$ with respective probabilities $[a, b, 1-a-b]$. Be creative with the manner in which the results are displayed.
- 10.2 In this computer exercise we wish to develop a program that allows the performance of an error-correcting code to be investigated by evaluating (10.119). Use the resulting program to verify the system performance characteristics illustrated in Examples 10.12 and 10.13.

- 10.3 Using your favorite high-level language, develop a program that allows you to investigate the performance of a digital communication system that utilizes a null-zone receiver and a noiseless feedback channel. Use the resulting program to evaluate the performance of a baseband system with an integrate-and-dump detector. The output of the integrate-and-dump detector is given by

$$V = \begin{cases} +AT + N, & \text{if } +A \text{ is sent} \\ -AT + N, & \text{if } -A \text{ is sent} \end{cases}$$

where N is a random variable representing the noise at the detector output at the sampling instant. The detector uses two thresholds, a_1 and a_2 , where $a_1 = \gamma AT$ and $a_2 = \gamma AT$. A retransmission occurs if $a_1 < V < a_2$. Choose several values of γ , such as $\gamma = 0.05, 0.10$ and 0.15 , and plot the probability of retransmission, and the expected number of retransmissions, as a function of A^2T/N_0 with γ a parameter of the family of curves. Also plot the probability of error and compare the results to the probability of error for a standard baseband transmission system. Comment on your results.

APPENDIX A

PHYSICAL NOISE SOURCES AND NOISE CALCULATIONS IN COMMUNICATION SYSTEMS

As discussed in Chapter 1, noise originates in a communication system from two broad classes of sources: those external to the system, such as atmospheric, solar, cosmic, or manmade sources, and those internal to the system. The degree to which external noise sources influence system performance depends heavily upon system location and configuration. Consequently, the reliable analysis of their effect on system performance is difficult and depends largely on empirical formulas and on-site measurements. Their importance in the analysis and design of communication systems depends on their intensity relative to the internal noise sources. In this Appendix, we are concerned with techniques of characterization and analysis of internal noise sources.

Noise internal to the subsystems that compose a communication system arises as a result of the random motion of charge carriers within the devices composing those subsystems. We now discuss several mechanisms that give rise to internal noise, and suitable models for these mechanisms.

A.1 PHYSICAL NOISE SOURCES

Thermal Noise

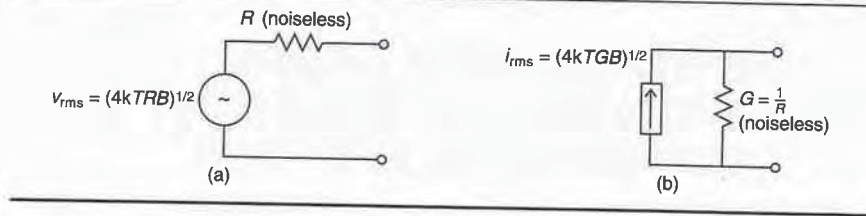
Thermal noise is the noise arising from the random motion of charge carriers in a conducting, or semiconducting, medium. Such random agitation at the atomic level is a universal characteristic of matter at temperatures other than absolute zero. Nyquist was one of the first to have studied thermal noise. *Nyquist's theorem* states that the mean-square noise voltage appearing across the terminals of a resistor of R ohms at temperature T Kelvin in a frequency band B hertz is given by

$$v_{\text{rms}}^2 = \langle v_n^2(t) \rangle = 4kTRB \text{ V}^2 \quad (\text{A.1})$$

where

$$k = \text{Boltzmann's constant} = 1.38 \times 10^{-23} \text{ J/K}$$

FIGURE A.1 Equivalent circuits for a noisy resistor. (a) Thevenin. (b) Norton.



Thus a *noisy* resistor can be represented by an equivalent circuit consisting of a *noiseless* resistor in series with a noise generator of rms voltage v_{rms} as shown in Figure A.1(a). Short circuiting the terminals of Figure A.1(a) results in a short-circuit noise current of mean-square value

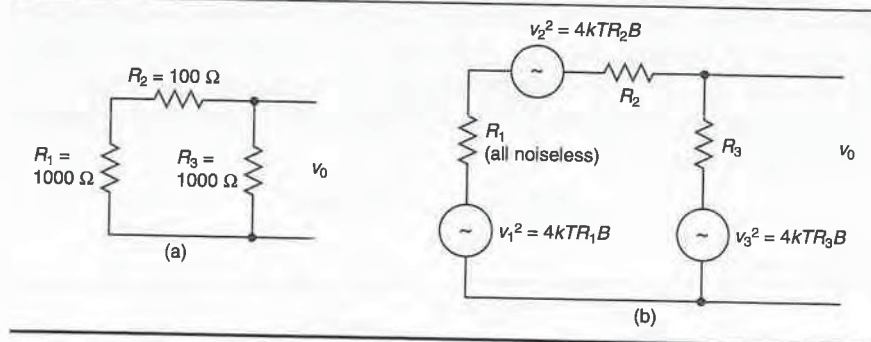
$$i_{rms}^2 = \langle i_n^2(t) \rangle = \frac{\langle v_n^2(t) \rangle}{R^2} = \frac{4kTB}{R} = 4kTGB \text{ A}^2 \quad (\text{A.2})$$

where $G = 1/R$ is the conductance of the resistor. The Thevenin equivalent in Figure A.1(a) can therefore be transformed to the Norton equivalent in Figure A.1(b).

▼ **EXAMPLE A.1** Consider the resistor network shown in Figure A.2. Assuming room temperature of $T = 290 \text{ K}$, find the rms noise voltage appearing at the output terminals in a 100-kHz bandwidth.

SOLUTION We use voltage division to find the noise voltage due to each resistor across the output terminals. Then, since powers due to independent

FIGURE A.2 Circuits for noise calculation. (a) Resistor network. (b) Noise equivalent circuit.



sources add, we find the rms output voltage v_0 by summing the square of the voltages due to each resistor, which gives the total mean-square voltage, and take the square root to get rms voltage. The calculation yields

$$v_0^2 = v_{01}^2 + v_{02}^2 + v_{03}^2 \quad (\text{A.3})$$

where

$$v_{01} = \frac{(4kTR_1B)^{1/2}}{R_1 + R_2 + R_3} \quad (\text{A.4a})$$

$v_1 \text{ (rms)}$

$$v_{02} = \frac{(4kTR_2B)^{1/2}}{R_1 + R_2 + R_3} \quad (\text{A.4b})$$

$v_2 \text{ (rms)}$

$$v_{03} = \frac{(4kTR_3B)^{1/2}}{R_1 + R_2 + R_3} \quad (\text{A.4c})$$

$v_3 \text{ (rms)}$

Thus

$$\begin{aligned} v_0^2 &= (4kTB) \left[\frac{(R_1 + R_2)R_3^2}{(R_1 + R_2 + R_3)^2} + \frac{(R_1 + R_2)^2 R_3}{(R_1 + R_2 + R_3)^2} \right] \\ &= (4 \times 1.38 \times 10^{-23} \times 290 \times 10^5) \\ &\quad \times \left[\frac{(1100)(1000)^2}{(2100)^2} + \frac{(1100)^2(1000)}{(2100)^2} \right] \\ &\cong 8.39 \times 10^{-13} \text{ V}^2 \end{aligned} \quad (\text{A.5})$$

Therefore,

$$v_0 = 9.16 \times 10^{-7} \text{ V (rms)} \quad (\text{A.6})$$

Nyquist's Formula

Although Example A.1 is instructive from the standpoint of illustrating noise computations involving several noisy resistors, it also illustrates that such computations can be exceedingly long if many resistors are involved. *Nyquist's formula*, which can be proved from thermodynamic arguments, simplifies such computations considerably. It states: The mean-square noise voltage produced at the output terminals of any one-port network containing only resistors, capacitors, and inductors is given by

$$\langle v_n^2(t) \rangle = 2kT \int_{-\infty}^{\infty} R(f) df \quad (\text{A.7a})$$

where $R(f)$ is real part of the complex impedance seen looking back into the terminals. If the network contains only resistors, the mean-square noise voltage in a bandwidth B is

$$\langle v_n^2 \rangle = 4kTR_{eq}B \text{ V}^2 \quad (\text{A.7b})$$

where R_{eq} is the Thevenin equivalent resistance of the network.

▼ **EXAMPLE A.1 (continued)** If we look back into the terminals of the network shown in Figure A.2, the equivalent resistance is

$$R_{eq} = R_3 \parallel (R_1 + R_2) = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} \quad (\text{A.8})$$

Thus

$$v_0^2 = \frac{4kTBR_3(R_1 + R_2)}{(R_1 + R_2 + R_3)} \quad (\text{A.9})$$

▲ which can be shown to be equivalent to the result obtained previously.

Shot Noise

Shot noise arises from the discrete nature of current flow in electronic devices. For example, the *electron* flow in a saturated thermionic diode is due to the sum total of electrons emitted from the cathode, which arrive randomly at the anode, thus providing an average current flow I_d (from anode to cathode when taken as positive), plus a randomly fluctuating component of mean-square value

$$i_{rms}^2 = \langle i_n^2(t) \rangle = 2eI_dB \text{ A}^2 \quad (\text{A.10})$$

where e = charge of the electron = 1.6×10^{-19} C. Equation (A.10) is known as *Schottky's theorem*.

Since powers from independent sources add, it follows that the squares of noise voltages or noise currents from independent sources, such as two resistors or two currents originating from independent sources, add. Thus, when applying Schottky's theorem to a *p-n* junction, we recall that the current flowing in a *p-n* junction diode

$$I = I_s \left[\exp \left(\frac{eV}{kT} \right) - 1 \right] \quad (\text{A.11})$$

where V is the voltage across the diode and I_s is the reverse saturation current, can be considered as being caused by two independent currents

$I_s \exp(eV/kT)$ and $-I_s$. Both currents fluctuate independently, producing a mean-square shot noise current given by

$$\begin{aligned} i_{\text{rms,tot}}^2 &= \left[2eI_s \exp\left(\frac{eV}{kT}\right) + 2eI_s \right] B \\ &= 2e(I + 2I_s)B \end{aligned} \quad (\text{A.12})$$

For normal operation, $I \gg I_s$ and the differential conductance is $g_0 = dI/dV = eI/kT$, so that (A.12) may be approximated as

$$i_{\text{rms,tot}}^2 \cong 2eIB = 2kT \left(\frac{eI}{kT} \right) B = 2kTg_0B \quad (\text{A.13})$$

which can be viewed as *half-thermal noise* of the differential conductance g_0 since there is a factor of 2 rather than a factor of 4 as in (A.2).

Other Noise Sources

In addition to thermal and shot noise, there are three other noise mechanisms that contribute to internally generated noise in electronic devices. We summarize them briefly here. A fuller treatment of their inclusion in the noise analysis of electronic devices is given by Van der Ziel (1970).

Generation-Recombination Noise Generation-recombination noise is the result of free carriers being generated and recombining in semiconductor material. One can consider these generation and recombination events to be random. Therefore, this noise process can be treated as a shot noise process.

Temperature-Fluctuation Noise Temperature-fluctuation noise is the result of the fluctuating heat exchange between a small body, such as a transistor, and its environment due to the fluctuations in the radiation and heat-conduction processes. If a liquid or gas is flowing past the small body, fluctuation in heat convection also occurs.

Flicker Noise Flicker noise is due to various causes. It is characterized by a spectral density that increases with decreasing frequency. The dependence of the spectral density on frequency is often found to be proportional to the inverse first power of the frequency. Therefore, flicker noise is sometimes referred to as *one-over-f noise*. More generally, flicker noise phenomena are characterized by power spectra that are of the form $\text{constant}/f^\alpha$, where α is close to unity. The physical mechanism that gives rise to flicker noise is not well understood.

Available Power

Since calculations involving noise involve transfer of power, the concept of maximum power available from a source of fixed internal resistance is useful. Figure A.3 illustrates the familiar theorem regarding maximum power transfer, which states that a source of internal resistance R delivers maximum power to a resistive load R_L if $R = R_L$ and that, under these conditions, the power P produced by the source is evenly split between source and load resistances. If $R = R_L$, the load is said to be *matched* to the source, and the power delivered to the load is referred to as the *available power* P_a . Thus $P_a = \frac{1}{2} P$, which is delivered to the load only if $R = R_L$. Consulting Figure A.3(a), in which v_{rms} is the rms voltage of the source, we see that the voltage across $R_L = R$ is $\frac{1}{2}v_{rms}$, giving

$$P_a = \frac{(\frac{1}{2}v_{rms})^2}{R} = \frac{v_{rms}^2}{4R} \quad (\text{A.14})$$

Similarly, when dealing with a Norton equivalent circuit as shown in Figure A.3(b), we can write the available power as

$$P_a = (\frac{1}{2}i_{rms})^2 R = \frac{i_{rms}^2}{4G} \quad (\text{A.15})$$

where $i_{rms} = v_{rms}/R$ is the rms noise current.

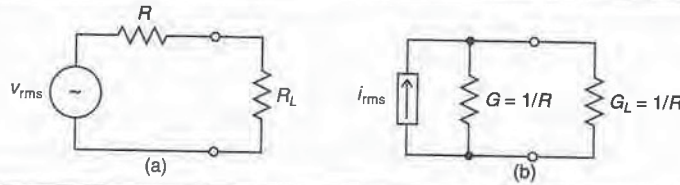
Returning to (A.1) or (A.2) and using (A.14) or (A.15), we see that a noisy resistor produces the available power

$$P_{a,R} = \frac{4kTRB}{4R} = kTB \text{ W} \quad (\text{A.16})$$

Similarly, from (A.13), a diode with load resistance matched to its differential conductance produces the available power

$$P_{a,D} = \frac{1}{2}kTB \text{ W}, \quad I \gg I_s \quad (\text{A.17})$$

FIGURE A.3 Circuits pertinent to maximum power transfer theorem. (a) Thevenin equivalent for a source with load resistance R_L . (b) Norton equivalent for a source with load conductance G_L .



▼ **EXAMPLE A.2** Calculate the available power per hertz of bandwidth for a resistance at room temperature, taken to be $T_0 = 290$ K. Express in decibels referenced to 1 watt (dBW) and decibels referenced to 1 milliwatt (dBm).

SOLUTION

$$\text{Power/hertz} = \frac{P_{a,R}}{B} = (1.38 \times 10^{-23})(290) = 4.002 \times 10^{-21} \text{ W/Hz}$$

$$\text{Power/hertz in dBW} = 10 \log_{10} (4.002 \times 10^{-21}/1) \cong -204 \text{ dBW}$$

$$\text{Power/hertz in dBm} = 10 \log_{10} (4.002 \times 10^{-21}/10^{-3}) \cong -174 \text{ dBm} \quad (\text{A.18})$$

▲

Frequency Dependence

In Example A.2, available power per hertz for a noisy resistor at $T_0 = 290$ K was computed and found, to good approximation, to be -174 dBm/Hz, independent of the frequency of interest. Actually, Nyquist's theorem, as stated by (A.1), is a simplification of a more general result. The proper quantum mechanical expression for available power per hertz, or available power spectral density $S_a(f)$, is

$$S_a(f) \triangleq \frac{P_a}{B} = \frac{hf}{\exp(hf/kT) - 1} \text{ W/Hz} \quad (\text{A.19})$$

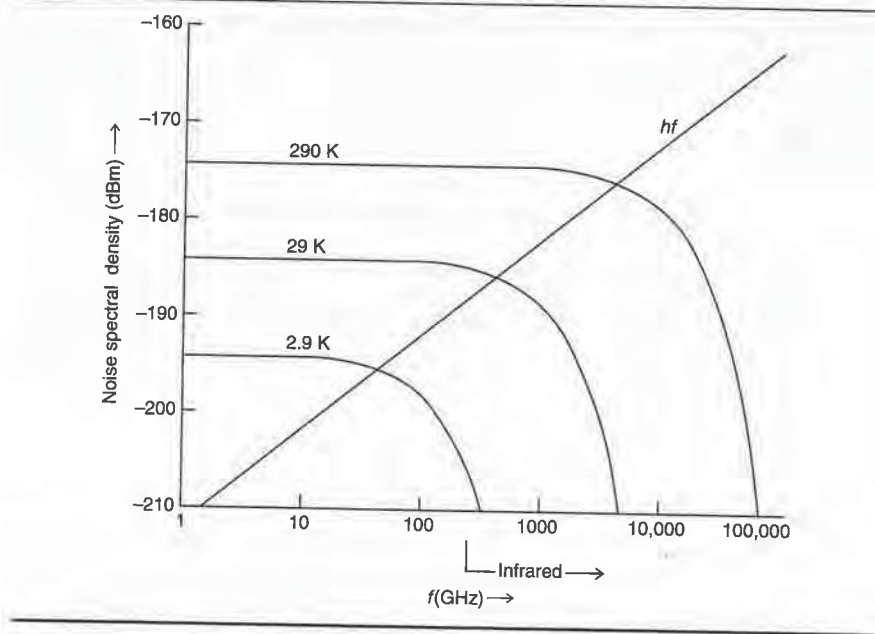
where $h = \text{Planck's constant} = 6.6254 \times 10^{-34}$ J-s.

This expression is plotted in Figure A.4, where it is seen that for all but very low temperatures and very high frequencies, the approximation is good that $S_a(f)$ is constant (that is, P_a is proportional to bandwidth B).

Quantum Noise

Taken by itself, (A.19) might lead to the false assumption that for very high frequencies where $hf \gg kT$, such as those used in optical communication, the noise would be negligible. However, it can be shown that a quantum noise term equal to hf must be added to (A.19) in order to account for the discrete nature of the electron energy. This is shown in Figure A.4 as the straight line, which permits the transition frequency between the thermal noise and quantum noise regions to be estimated. This transition frequency is seen to be above 20 GHz even for $T = 2.9$ K.

FIGURE A.4 Noise power spectral density versus frequency for thermal resistors



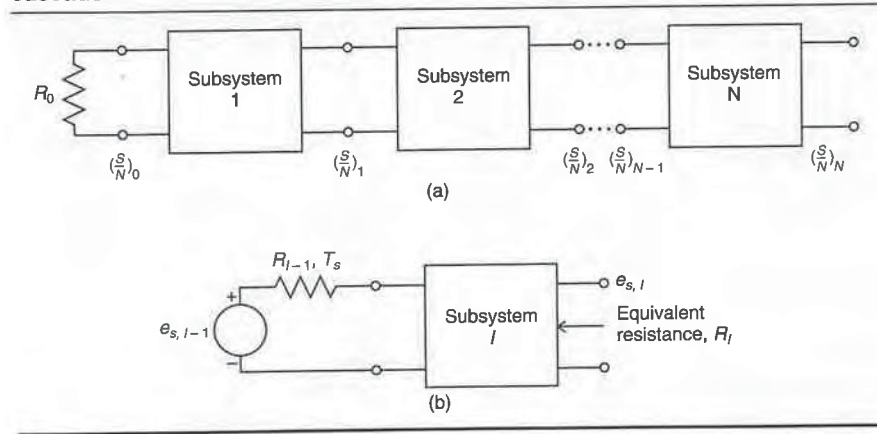
A.2 CHARACTERIZATION OF NOISE IN SYSTEMS—NOISE FIGURE AND NOISE TEMPERATURE

Having considered several possible sources of internal noise in communication systems, we now wish to discuss convenient methods for characterization of the noisiness of the subsystems that make up a system, as well as overall noisiness of the system. Figure A.5 illustrates a cascade of N stages or subsystems that make up a system. For example, if this block diagram represents a superheterodyne receiver, subsystem 1 would be the RF amplifier, subsystem 2 the mixer, subsystem 3 the IF amplifier, and subsystem 4 the detector. At the output of each stage, we wish to be able to relate the signal-to-noise power ratio to the signal-to-noise power ratio at the input. This will allow us to pinpoint those subsystems that contribute significantly to the output noise of the overall system, thereby enabling us to implement designs that minimize the noise.

Noise Figure of a System

One useful measure of system noisiness is the so-called *noise figure* F , defined as the ratio of the SNR at the system input to the SNR at the system output.

FIGURE A.5 Cascade of subsystems making up a system. (a) N -subsystem cascade with definition of SNR's at each point. (b) The l th subsystem in the cascade.



In particular, for the l th subsystem in Figure A.5, the noise figure F_l is defined by the relation

$$\left(\frac{S}{N}\right)_l = \frac{1}{F_l} \left(\frac{S}{N}\right)_{l-1} \quad (\text{A.20})$$

For an ideal, noiseless subsystem, $F_l = 1$; that is, the subsystem introduces no additional noise. For physical devices, $F_l > 1$.

Noise figures for devices and systems are often stated in terms of decibels (dB). Specifically,

$$F_{\text{dB}} = 10 \log_{10} F_{\text{ratio}} \quad (\text{A.21})$$

Table A.1 lists several devices along with typical noise figures for them.* For extremely low-noise devices, such as a maser amplifier, it is more convenient to work with noise temperature, to be discussed shortly.

The definition of noise figure given by (A.20) requires the calculation of both signal and noise powers at each point of the system. An alternative definition, equivalent to (A.20), involves the calculation of noise powers only. Although signal and noise powers at any point in the system depend on the loading of a subsystem on the preceding one, SNRs are independent of load, since both signal and noise appear across the same load. Hence any

* No temperature has yet been specified for the equivalent input resistance of the device. To standardize the noise figures of devices, $T_0 = 290 \text{ K}$ is specified as a standard temperature by standards of the Institute of Electrical and Electronic Engineers.

TABLE A.1 Typical Noise Figures and Temperatures for Several Devices^a

Device	F (Ratio)	F (dB)	Noise Temp. (°K)	Gain (dB)	Operating Frequency (GHz)
Parametric amp. (uncooled)	1.26	1.0	75	10–20	0.5
(liq. N ₂ cooled)	1.45	1.61	130		9
(liq. He cooled)	1.17	0.69	50		3 & 6
	1.03	0.13	9		4
Traveling wave tube amp.	1.59	2.00	170	20–30	2.6
	1.86	2.70	250		3
	2.69	4.30	490		9
Tunnel diode amp. ger.	2.38	3.77	400	20–40	
gal. ars.	1.69	2.28	200		
Low-noise superhet. rec'r (ant. noise and image response excl.)	2.38	3.77	400		
Integrated circuit IF ampl.	5.01	7	1163	50	≤11 MHz

^a Information in column 4 from W. W. Mumford and E. H. Scheibe, *Noise: Performance Factors in Communication Systems*, Horizon House—Microwave, Inc., Dedham, Mass. (1968), pages 38, 39. Also Motorola Semiconductor Products Division, *Linear Integrated Circuits Data Book*, Motorola, Inc. (1974).

convenient load impedance may be used in making signal and noise calculations. In particular, we will use load impedances matched to the output impedance, thereby working with available signal and noise powers.

Consider the l th subsystem in the cascade of the system shown in Figure A.5. If we represent its input by a Thevenin equivalent circuit with rms signal voltage $e_{s,l-1}$ and equivalent resistant R_{l-1} , the available signal power is

$$P_{sa,l-1} = \frac{e_{s,l-1}^2}{4R_{l-1}} \quad (\text{A.22})$$

If we assume that only thermal noise is present, the available noise power for a source temperature of T_s is

$$P_{na,l-1} = kT_s B \quad (\text{A.23})$$

giving an input SNR of

$$\left(\frac{S}{N}\right)_{l-1} = \frac{e_{s,l-1}^2}{4kT_s R_{l-1} B} \quad (\text{A.24})$$

The available output signal power, from Figure A.5(b), is

$$P_{sa,l} = \frac{e_{s,l}^2}{4R_l} \quad (\text{A.25})$$

We can relate $P_{sa,l}$ to $P_{sa,l-1}$ by the *available power gain* G_a of subsystem l , defined to be

$$P_{sa,l} = G_a P_{sa,l-1} \quad (\text{A.26})$$

which is obtained if all resistances are matched. The output SNR is

$$\left(\frac{S}{N}\right)_l = \frac{P_{sa,l}}{P_{na,l}} = \frac{1}{F_l} \frac{P_{sa,l-1}}{P_{na,l-1}} \quad (\text{A.27})$$

or

$$\begin{aligned} F_l &= \frac{P_{sa,l-1}}{P_{sa,l}} \frac{P_{na,l}}{P_{na,l-1}} = \frac{P_{sa,l-1}}{G_a P_{sa,l-1}} \frac{P_{na,l}}{P_{na,l-1}} \\ &= \frac{P_{na,l}}{G_a P_{na,l-1}} \end{aligned} \quad (\text{A.28})$$

where any mismatches may be ignored, since they affect signal and noise the same.* Thus the noise figure is the ratio of the output noise power to the noise power that would have resulted had the system been noiseless. Noting that $P_{na,l} = G_a P_{na,l-1} + P_{\text{int},l}$, where $P_{\text{int},l}$ is the available internally generated noise power of subsystem l , and that $P_{na,l-1} = kT_s B$, we may write (A.28) as

$$F_l = 1 + \frac{P_{\text{int},l}}{G_a kT_s B} \quad (\text{A.29a})$$

or, setting $T_s = T_0 = 290$ K to standardize the noise figure,† we obtain

$$F_l = 1 + \frac{P_{\text{int},l}}{G_a kT_0 B} \quad (\text{A.29b})$$

For $G_a \gg 1$, $F \cong 1$, which shows that the internally generated noise becomes inconsequential for systems with large gains. Conversely, systems with low gain enhance the importance of internal noise.

* This assumes that the gains for noise power and signal power are the same. If gain varies with frequency, then a spot noise figure can be defined, where signal power and noise power are measured in a small bandwidth Δf .

† If this were not done, the manufacturer of a receiver could claim superior noise performance of its product over that of a competitor simply by choosing T_s larger than the competitor. See Mumford and Scheibe (1968), pages 53–56, for a summary of the various definitions of noise figure used in the past.

Measurement of Noise Figure

Using (A.28), with the available noise power at the output $P_{no,out}$ referred to the device input, and representing this noise by a current generator $\overline{i_n^2}$ in parallel with the source resistance R_s or a voltage generator e_n^2 in series with it, we can determine the noise figure by changing the input noise a known amount and measuring the change in noise power at the device output. In particular, if we assume a current source represented by a saturated thermionic diode, so that

$$\overline{i_n^2} = 2eI_d B A^2 \quad (\text{A.30})$$

and sufficient current is passed through the diode so the noise power at the output is double the amount that appeared without the diode, then the noise figure is

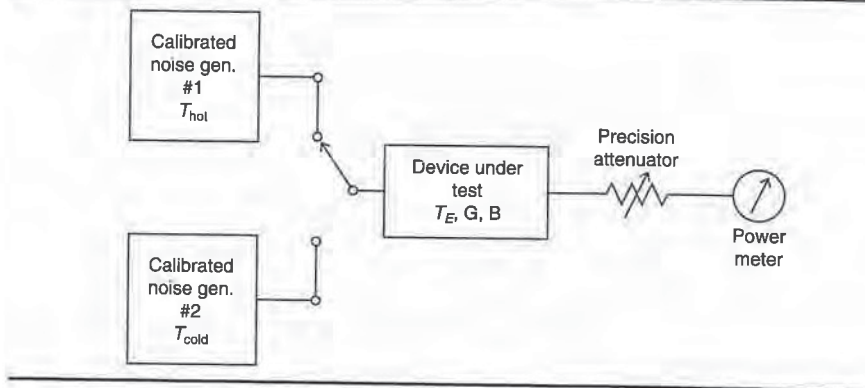
$$F = \frac{eI_d R_s}{2kT_0} \quad (\text{A.31})$$

where e is the charge of the electron in coulombs, I_d is the diode current in amperes, R_s is the input resistance, k is Boltzmann's constant, and T_0 is the standard temperature in degrees Kelvin.

A variation of the preceding method is the Y-factor method, which is illustrated in Figure A.6. Assume that two calibrated noise sources are available, one at effective temperature T_{hot} and the other at T_{cold} . With the first at the input of the unknown system with unknown temperature T_e , the available output noise power is

$$P_h = k(T_{hot} + T_e)BG \quad (\text{A.32})$$

FIGURE A.6 Y-factor method for measuring effective noise temperature



where B the noise bandwidth of the device under test and G is its available power gain. With the cold noise source present, the available output noise power is

$$P_c = k(T_{\text{cold}} + T_e)BG \quad (\text{A.33})$$

The two unknowns in these two equations are T_e and BG . Dividing the first by the second, we obtain

$$\frac{P_h}{P_c} = Y = \frac{T_{\text{hot}} + T_e}{T_{\text{cold}} + T_e} \quad (\text{A.34})$$

When solved for T_e , this equation becomes

$$T_e = \frac{T_{\text{hot}} - YT_{\text{cold}}}{Y - 1} \quad (\text{A.35})$$

which involves the two known source temperatures and the measured Y factor. The Y factor can be measured with the aid of the precision attenuator shown in Figure A.6 as follows: (1) connect the hot noise source to the system under test and adjust the attenuator for a convenient meter reading, (2) switch to the cold noise source and adjust the attenuator for the same meter reading as before, (3) noting the change in attenuator setting ΔA in dB, calculate $Y = 10^{-\Delta A/10}$, and (4) calculate the effective noise temperature using (A.35).

Noise Temperature

Equation (A.16) states that the available noise power of a resistor at temperature T is kTB W, independent of the value of R . We may use this result to define the *equivalent noise temperature* T_n of any noise source:

$$T_n = \frac{P_{n,\text{max}}}{kB} \quad (\text{A.36})$$

where $P_{n,\text{max}}$ is the maximum noise power the source can deliver in bandwidth B .

▼ **EXAMPLE A.3** Two resistors R_1 and R_2 at temperatures T_1 and T_2 are connected in series to form a white-noise source. Find the equivalent noise temperature of the combination.

SOLUTION The mean-square voltage generated by the combination is

$$\langle v_n^2 \rangle = 4kBR_1T_1 + 4kBR_2T_2 \quad (\text{A.37})$$

Since the equivalent resistance is $R_1 + R_2$, the available noise power is

$$P_{na} = \frac{\langle v_n^2 \rangle}{4(R_1 + R_2)} = \frac{4k(T_1 R_1 + T_2 R_2)B}{4(R_1 + R_2)} \quad (\text{A.38})$$

The equivalent noise temperature is therefore

$$T_n = \frac{P_{na}}{kB} = \frac{R_1 T_1 + R_2 T_2}{R_1 + R_2} \quad (\text{A.39})$$

▲ Note that T_n is *not* a physical temperature unless both resistors are at the same temperature.

Effective Noise Temperature

Returning to Equation (A.29), we note that the second term, $P_{\text{int},l}/G_a k T_0 B$, which is dimensionless, is due solely to the internal noise of the system. Noting that $P_{\text{int},l}/G_a k B$ has the dimensions of temperature, we may write the noise figure as

$$F_l = 1 + \frac{T_e}{T_0} \quad (\text{A.40})$$

where

$$T_e = \frac{P_{\text{int},l}}{G_a k B} \quad (\text{A.41a})$$

Thus,

$$T_e = (F_l - 1)T_0 \quad (\text{A.41b})$$

T_e is the *effective noise temperature* of the system and depends only on the parameters of the system. It is a measure of noisiness of the system referred to the input, since it is the temperature required of a thermal resistance, placed at the input of a noiseless system, in order to produce the same available noise power at the output as is produced by the internal noise sources of the system. Recalling that $P_{na,l} = G_a P_{na,l-1} + P_{\text{int},l}$ and that $P_{na,l-1} = k T_s B$, we may write the available noise power at the subsystem output as

$$\begin{aligned} P_{na,l} &= G_a k T_s B + G_a k T_e B \\ &= G_a k (T_s + T_e) B \end{aligned} \quad (\text{A.42})$$

where the actual temperature of the source T_s is used. Thus the available noise power at the output of a system can be found by adding the effective

noise temperature of the system to the temperature of the source and multiplying by $G_a k B$, where the term G_a appears because all noise is referred to the input.

Noise Temperature and Noise Figure of a Cascade of Subsystems

Considering the first two stages in Figure A.5, we see that noise appears at the output due to the following sources:

1. Amplified source noise, $G_{a_1} G_{a_2} k T_s B$
2. Internal noise from the first stage amplified by the second stage,
 $G_{a_2} P_{a, \text{int}_1} = G_{a_2} (G_{a_1} k T_{e_1} B)$
3. Internal noise from the second stage, $P_{a, \text{int}_2} = G_{a_2} k T_{e_2} B$

Thus the total available noise power at the output of the cascade is

$$P_{na,2} = G_{a_1} G_{a_2} k \left(T_s + T_{e_1} + \frac{T_{e_2}}{G_{a_1}} \right) B \quad (\text{A.43})$$

Noting that the available gain for the cascade is $G_{a_1} G_{a_2}$ and comparing with (A.42), we see that the effective temperature of the cascade is

$$T_e = T_{e_1} + \frac{T_{e_2}}{G_{a_1}} \quad (\text{A.44})$$

From (A.40), the overall noise figure is

$$\begin{aligned} F &= 1 + \frac{T_e}{T_0} = 1 + \frac{T_{e_1}}{T_0} + \frac{1}{G_{a_1}} \frac{T_{e_2}}{T_0} \\ &= F_1 + \frac{F_2 - 1}{G_{a_1}} \end{aligned} \quad (\text{A.45})$$

where F_1 is the noise figure of stage 1 and F_2 is the noise figure of stage 2. The generalization of this result to an arbitrary number of stages is known as Friis's formula and is given by

$$F = F_1 + \frac{F_2 - 1}{G_{a_1}} + \frac{F_3 - 1}{G_{a_1} G_{a_2}} + \dots \quad (\text{A.46})$$

whereas the generalization of (A.44) is

$$T_e = T_{e_1} + \frac{T_{e_2}}{G_{a_1}} + \frac{T_{e_3}}{G_{a_1} G_{a_2}} + \dots \quad (\text{A.47})$$

▼ **EXAMPLE A.4** A parabolic dish antenna is pointed up into the sky not directly at the sun. Noise due to atmospheric radiation is equivalent to a source temperature of 70 K. A low-noise preamplifier with noise figure of 2 dB and an available power gain of 20 dB over a bandwidth of 20 MHz is mounted at the antenna feed (focus of the parabolic reflector). (a) Find the effective noise temperature at the input of the preamplifier. (b) Find the available noise power at the preamplifier output.

SOLUTION (a) From (A.42), we have

$$T_{\text{eff,in}} = T_s + T_{e,\text{preamp}} \quad (\text{A.48})$$

But (A.41b) gives

$$\begin{aligned} T_{e,\text{preamp}} &= T_0(F_{\text{preamp}} - 1) \\ &= 290(10^{2/10} - 1) \\ &= 169.6 \text{ K} \end{aligned} \quad (\text{A.49})$$

(b) From (A.42), the available output noise power is

$$\begin{aligned} P_{na,\text{out}} &= G_a k(T_s + T_e)B \\ &= 10^{20/10} (1.38 \times 10^{-23}) (239.6) (20 \times 10^6) \\ &= 6.61 \times 10^{-12} \text{ W} \end{aligned} \quad (\text{A.50})$$

▼ **EXAMPLE A.5** A preamplifier with power gain to be found and a noise figure of 2.5 dB is cascaded with a mixer with a gain of 2 dB and a noise figure of 8 dB. Find the preamplifier gain such that the overall noise figure of the cascade is at most 4 dB.

SOLUTION From Friis's formula

$$F = F_1 + \frac{F_2 - 1}{G_1} \quad (\text{A.51})$$

Solving for G_1 ,

$$G_1 = \frac{F_2 - 1}{F - F_1} = \frac{10^{8/10} - 1}{10^{4/10} - 10^{2.5/10}} = 7.24 \text{ (ratio)} = 8.6 \text{ dB} \quad (\text{A.52})$$

▲ Note that the gain of the mixer is immaterial.

Attenuator Noise Temperature and Noise Figure

Consider a purely resistive attenuator that imposes a loss of a factor of L in available power between input and output; thus the available power at its output $P_{a,\text{out}}$ is related to the available power at its input $P_{a,\text{in}}$ by

$$P_{a,\text{out}} = \frac{1}{L} P_{a,\text{in}} = G_a P_{a,\text{in}} \quad (\text{A.53})$$

However, since the attenuator is resistive and assumed to be at the *same* temperature T_s as the equivalent resistance at its input, the available output power is

$$P_{na,\text{out}} = kT_s B \quad (\text{A.54})$$

Characterizing the attenuator by an effective temperature T_e and employing (A.42), we may also write $P_{na,\text{out}}$ as

$$\begin{aligned} P_{na,\text{out}} &= G_a k(T_s + T_e) B \\ &= \frac{1}{L} k(T_s + T_e) B \end{aligned} \quad (\text{A.55})$$

Equating (A.54) and (A.55) and solving for T_e , we obtain

$$T_e = (L - 1)T_s \quad (\text{A.56})$$

for the effective temperature of a noise resistance at temperature T_s followed by an attenuator. From Equation (A.40), the noise figure of the cascade of source resistance and attenuator is

$$F = 1 + \frac{(L - 1)T_s}{T_0} \quad (\text{A.57a})$$

or

$$F = 1 + \frac{(L - 1)T_0}{T_0} = L \quad (\text{A.57b})$$

for an attenuator at *room temperature*, T_0 .

▼ **EXAMPLE A.6** Consider a receiver system consisting of an antenna with lead-in cable having a loss factor of $L = 1.5 \text{ dB} = F_1$, an RF preamplifier with a noise figure of $F_2 = 7 \text{ dB}$ and a gain of 20 dB, followed by a mixer with a noise figure of $F_3 = 10 \text{ dB}$ and a conversion gain of 8 dB, and finally

an integrated-circuit IF amplifier with a noise figure of $F_4 = 6$ dB and a gain of 60 dB.

- (a) Find the overall noise figure and noise temperature of the system.
- (b) Find the noise figure and noise temperature of the system with preamplifier and cable interchanged.

SOLUTION (a) Converting decibel values to ratios and employing (A.46), we obtain

$$\begin{aligned} F &= 1.41 + \frac{5.01 - 1}{1/1.41} + \frac{10 - 1}{100/1.41} + \frac{3.98 - 1}{(100)(6.3)/1.41} \\ &= 1.41 + 5.65 + 0.13 + 6.7 \times 10^{-3} = 7.19 = 8.57 \text{ dB} \end{aligned} \quad (\text{A.58})$$

Note that the cable and RF amplifier essentially determine the noise figure of the system and that the noise figure of the system is enhanced because of the loss of the cable.

If we solve (A.40) for T_e , we have the effective noise temperature

$$T_e = T_0(F - 1) = 290(7.19 - 1) = 1796 \text{ K} \quad (\text{A.59})$$

- (b) Interchanging the cable and RF preamplifier, we obtain the noise figure

$$\begin{aligned} F &= 5.01 + \frac{1.41 - 1}{100} + \frac{10 - 1}{100/1.41} + \frac{3.98 - 1}{(100)(6.3)/1.41} \\ &= 5.01 + (4.1 \times 10^{-3}) + 0.127 + (6.67 \times 10^{-3}) \\ &= 5.15 = 7.12 \text{ dB} \end{aligned} \quad (\text{A.60})$$

The noise temperature is

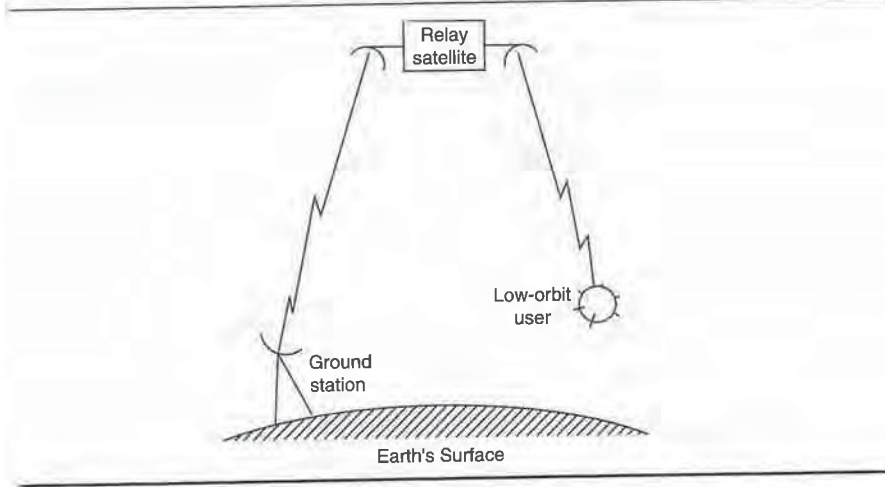
$$T_e = 290(4.15) = 1203 \text{ K} \quad (\text{A.61})$$

Now the noise figure and noise temperature are essentially determined by the noise level of the RF preamplifier.

We have omitted one possibly important source of noise—namely, the antenna. If the antenna is directive and pointed at an intense source of thermal noise, such as the sun, its equivalent temperature may also be of importance in the calculation. This is particularly true when a low-noise preamplifier is employed (see Table A.1).

A.3 FREE-SPACE PROPAGATION EXAMPLE

As a final example of noise calculation, we consider a free-space electromagnetic-wave propagation channel. For the sake of illustration, suppose the

FIGURE A.7 A satellite-relay communication link

communication link of interest is between a synchronous-orbit relay satellite and a low-orbit satellite or aircraft, as shown in Figure A.7. This might represent part of a relay link between a ground station and a small scientific satellite or an aircraft. Since the ground station is high power, we assume the ground-station-to-relay-satellite link is noiseless and focus our attention on the link between the two satellites.

Assume a relay satellite transmitted signal power of P_T W. If radiated isotropically, the power density at a distance d from the satellite is given by

$$p_t = \frac{P_T}{4\pi d^2} \text{ W/m}^2 \quad (\text{A.62})$$

If the satellite antenna has directivity, with the radiated power being directed toward the low-orbit vehicle, the antenna can be described by an antenna power gain G_T over the isotropic radiation level. For aperture-type antennas with aperture area A_T large compared with the square of the transmitted wavelength λ^2 , it can be shown that the maximum gain is given by $G_T = 4\pi A_T/\lambda^2$. The power P_R intercepted by the receiving antenna is given by the product of the receiving aperture area A_R and the power density at the aperture:

$$P_R = p_t A_R = \frac{P_T G_T}{4\pi d^2} A_R \quad (\text{A.63})$$

However, we may relate the receiving aperture antenna to its maximum gain by the expression $G_R = 4\pi A_R/\lambda^2$, giving

$$P_R = \frac{P_T G_T G_R \lambda^2}{(4\pi d)^2} \quad (\text{A.64})$$

Equation (A.64) includes only the loss in power from isotropic spreading of the transmitted wave. If other losses, such as atmospheric absorption, are important, they may be included as a loss factor L_0 in (A.64) to yield

$$P_R = \left(\frac{\lambda}{4\pi d} \right)^2 \frac{P_T G_T G_R}{L_0} \quad (\text{A.65})$$

The factor $(4\pi d/\lambda)^2$ is sometimes referred to as the *free-space loss*.*

In the calculation of received power, it is convenient to work in terms of decibels. Taking $10 \log_{10} P_R$, we obtain

$$\begin{aligned} 10 \log_{10} P_R &= 20 \log_{10} (\lambda/4\pi d) + 10 \log_{10} P_T \\ &\quad + 10 \log_{10} G_T + 10 \log_{10} G_R - 10 \log_{10} L_0 \end{aligned} \quad (\text{A.66})$$

Now $10 \log_{10} P_R$ can be interpreted as the received power in decibels referenced to 1 W; it is commonly referred to as power in dBW. Similarly, $10 \log_{10} P_T$ is commonly referred to as the transmitted signal power in dBW. The terms $10 \log_{10} G_T$ and $10 \log_{10} G_R$ are the transmitter and receiver antenna gains (above isotropic) in decibels, while the term $10 \log_{10} L_0$ is the loss factor in decibels. When $10 \log_{10} P_T$ and $10 \log_{10} G_T$ are taken together, this sum is referred to as the *effective radiated power* in decibel watts (ERP, or sometimes EIRP, for effective radiated power referenced to isotropic). The negative of the first term is the free-space loss in decibels. For $d = 10^6$ mi (1.6×10^9 m) and a frequency of 500 MHz ($\lambda = 0.6$ m),

$$20 \log_{10} \left(\frac{\lambda}{4\pi d} \right) = -210 \text{ dB} \quad (\text{A.67})$$

If λ or d change by a factor of 10, this value changes by 20 dB. We now make use of (A.66) and the results obtained for noise figure and temperature to compute the signal-to-noise ratio for a typical satellite link.

▼ **EXAMPLE A.7** We are given the following parameters for a relay-satellite-to-user link:

Relay satellite effective radiated power ($G_T = 35$ dB; $P_T = 100$ W): 55 dBW
Transmit frequency: 2 GHz ($\lambda = 0.15$ m)

* We take the convention here that a *loss* is a factor in the denominator of P_R ; i.e., a *loss* in decibels is a positive quantity (a negative gain).

Receiver noise temperature of user (includes noise figure of receiver and background temperature of antenna): 700 K

User satellite antenna gain: 0 dB

Total system losses: 3 dB

Relay-user separation: 41,000 km

Find the signal-to-noise power ratio in a 50 kHz bandwidth at the user satellite receiver IF amplifier output.

SOLUTION The received signal power is computed using (A.65) as follows (+ and - signs in parentheses indicate whether the quantity is added or subtracted):

Free-space loss: $-20 \log_{10} (0.15/4\pi \times 41 \times 10^6)$: 190.7 dB (-)

Effective radiated power: 55 dBW (+)

Receiver antenna gain: 0 dB (+)

System losses: 3 dB (-)

Received signal power: -138.7 dBW

The noise power level is calculated from (A.41a), written as

$$P_{int} = G_a k T_e B \quad (\text{A.68})$$

where P_{int} is the receiver output noise power due to internal sources. Since we are calculating the signal-to-noise ratio, the available gain of the receiver does not enter the calculation because both signal and noise are multiplied by the same gain. Hence we may set G_a to unity, and the noise level is

$$\begin{aligned} P_{int|dBW} &= 10 \log_{10} \left[k T_0 \left(\frac{T_e}{T_0} \right) B \right] \\ &= 10 \log_{10}(k T_0) + 10 \log_{10} \left(\frac{T_e}{T_0} \right) + 10 \log_{10} B \\ &= -204 + 10 \log_{10}(700/290) + 10 \log_{10}(50,000) \\ &= -153.2 \text{ dBW} \end{aligned} \quad (\text{A.69})$$

Hence the signal-to-noise ratio at the receiver output is

$$\text{SNR}_0 = -138.7 + 153.2 = 14.5 \text{ dB} \quad (\text{A.70})$$

▼ **EXAMPLE A.8** To interpret the result obtained in the previous example in terms of the performance of a digital communication system, we must convert the signal-to-noise ratio obtained to energy-per-bit-to-noise-spectral density ratio E_b/N_0 (see Chapter 7). By definition of SNR_0 , we have

$$\text{SNR}_0 = \frac{P_R}{kT_e B} \quad (\text{A.71})$$

Multiplying numerator and denominator by the duration of a data bit T_b , we obtain

$$\text{SNR}_0 = \frac{P_R T_b}{kT_e B T_b} = \frac{E_b}{N_0 B T_b} \quad (\text{A.72})$$

where $P_R T_b = E_b$ and $kT_e = N_0$ are the signal energy per bit and the noise power spectral density, respectively. Thus, to obtain E_b/N_0 from SNR_0 , we calculate

$$E_b/N_0|_{\text{dB}} = (\text{SNR}_0)_{\text{dB}} + 10 \log_{10}(B T_b) \quad (\text{A.73})$$

For example, from Chapter 8, we recall that the null-to-null bandwidth of a phase-shift keyed carrier is $2/T_b$ Hz. Suppose the data rate in Example A.7 is 2,500 bps. Therefore, $B T_b$ for BPSK is 3 dB, and

$$E_b/N_0|_{\text{dB}} = 14.5 + 3 = 17.5 \text{ dB} \quad (\text{A.74})$$

The probability of error for a binary BPSK digital communication system was derived in Chapter 7 as

$$\begin{aligned} P_E &= Q(\sqrt{2E_b/N_0}) \cong \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}} \\ &\cong 1.4 \times 10^{-26} \text{ for } E_b/N_0 = 17.5 \text{ dB} \end{aligned} \quad (\text{A.75})$$

which is a very small probability of error indeed. It appears that the system may have been overdesigned. However, no margin has been included as a safety factor. Components degrade, or the system may be operated in an environment for which it was not intended. With only 6 dB allowed for margin, the performance in terms of error probability becomes 5.5×10^{-8} .

FURTHER READING

Treatments of internal noise sources and calculations oriented toward communication systems comparable to the scope and level of the presentation here may be found in most of the books on communications referenced in Chapters 2 and 3, for instance, Carlson (1986), Schwartz (1990), and Taub and Schilling (1986). A concise, but thorough, treatment at an elementary level is available in Mumford and Scheibe (1968). An in-depth treatment of noise in solid-state devices is available in Van der Ziel (1970). A complete treatment of noise theory and calculation principles is found in Van der Ziel

(1954), which is the classical reference on noise, although noise characteristics of some of the newer devices are not included since the book was published before the invention of many of them. For discussion of satellite-link power budgets, see Ziemer and Peterson (1992).

PROBLEMS

Section A.1

- A.1 A true rms voltmeter (assumed noiseless) with an effective noise bandwidth of 30 MHz is used to measure the noise voltage produced by the following devices. Calculate the meter reading in each case.
- A 10-k Ω resistor at room temperature, $T_0 = 290$ K
 - A 10-k Ω resistor at 29 K
 - A 10-k Ω resistor at 2.9 K
 - What happens to all of the above results if the bandwidth is decreased by a factor of 4? a factor of 10? a factor of 100?
- A.2 Given a junction diode with reverse saturation current $I_s = 15 \mu\text{A}$.
- At room temperature (290 K), find V such that $I > 20I_s$, thus allowing Equation (A.12) to be approximated by Equation (A.13). Find the rms noise current.
 - Repeat part (a) for $T = 29$ K.
- A.3
- Obtain an expression for the mean-square noise voltage appearing across R_3 in Figure A.8.
 - If $R_1 = 2000 \Omega$, $R_2 = R_L = 300 \Omega$, and $R_3 = 500 \Omega$, find the mean-square noise voltage per hertz.
- A.4. Referring to the circuit of Figure A.8, consider R_L to be a load resistance, and find it in terms of R_1 , R_2 , and R_3 so that the maximum available noise power available from R_1 , R_2 , and R_3 is delivered to it.

FIGURE A.8

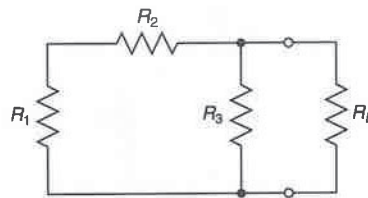
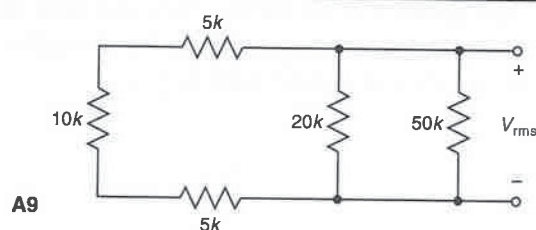


FIGURE A.9



- A.5. Assuming a bandwidth of 2 Mhz, find the rms noise voltage across the output terminals of the circuit shown in Figure A.9 if it is at a temperature of 400 K.

Section A.2

- A.6 Obtain an expression for F and T_e for the two-port resistive matching network shown in Figure A.10, assuming a source at $T_0 = 290$ K.
- A.7 A source with equivalent noise temperature $T_s = 1000$ K is followed by a cascade of three amplifiers having the specifications shown in Table A.2. Assume a bandwidth of 50 kHz.
- Find the noise figure of the cascade.
 - Suppose amplifiers 1 and 2 are interchanged. Find the noise figure of the cascade.
 - Find the noise temperature of the systems of parts (a) and (b).
 - Assuming the configuration of part (a), find the required input signal power to give an output signal-to-noise ratio of 40 dB. Perform the same calculation for the system of part (b).

FIGURE A.10

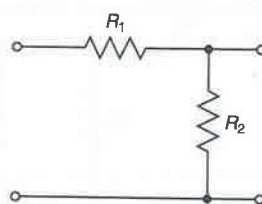


TABLE A.2

Amplifier No.	F	T_e	Gain
1		300 K	10 dB
2	6 dB		30 dB
3	11 dB		30 dB

- A.8 (a) An attenuator with loss $L \gg 1$ is followed by an amplifier with noise figure F and gain $G_a = L$. Find the noise figure of the cascade.
- (b) Consider the cascade of two identical attenuator-amplifier stages as in part (a). Consider the noise figure of the cascade.
- (c) Generalize these results to N identical attenuators and amplifiers. How many decibels does the noise figure increase as a result of doubling the number of attenuators and amplifiers?
- A.9 Given a cascade of a preamplifier, mixer, and amplifier with the specifications shown in Table A.3.
- (a) Find the minimum gain of the preamplifier such that the overall noise figure of the cascade is 5 dB or greater.
- (b) The preamplifier is fed by an antenna with noise temperature of 300 K (this is the temperature of the earth viewed from space). Find the temperature of the overall system using a preamplifier gain of 15 dB; for the preamplifier gain found in part (a).
- (c) Find the noise power at the amplifier output for the two cases of part (b).
- (d) Repeat part (b) except now assume that a transmission line with loss of 2 dB connects the antenna to the preamplifier.
- A.10 An antenna with temperature of 300 K is fed into a receiver with total gain of 80 dB, $T_e = 1500$ K, and a bandwidth of 3 MHz.
- (a) Find the available noise power at the output of the receiver.
- (b) Find the necessary signal power P_r in dBm at the antenna terminals such that the output signal-to-noise ratio is 50 dB.
- A.11 Referring to (A.35) and the accompanying discussion, suppose that two calibrated noise sources have effective temperatures of 600 K and 300 K. (a) Obtain the noise temperature of an amplifier with these noise sources used as inputs in turn if the difference in attenuator settings to get the same power meter reading at the amplifier's

TABLE A.3

	Noise Figure, dB	Gain, dB	Bandwidth
Preamplifier	2	?	*
Mixer	8	1.5	*
Amplifier	5	30	10 MHz

* The bandwidth of this stage is much greater than the amplifier bandwidth.

output is 1 dB; 1.5 dB; 2 dB. (b) Obtain the corresponding noise figures.

Section A.3

A.12 Given a relay-user link as described in Section A.3 with the following parameters:

Average transmit power of relay satellite: 35 dBW

Transmit frequency: 7.7 GHz

Effective antenna aperture of relay satellite: 1 m^2

Noise temperature of user receiver (including antenna): 1000 K

Antenna gain of user: 6 dB

Total system losses: 5 dB

System bandwidth: 1 MHz

Relay-user separation: 41,000 km

- Find the received signal power level at the user in dBW.
- Find the receiver noise level in dBW.
- Compute the signal-to-noise ratio at the receiver in decibels.
- *Find the average probability of error for the following digital signaling methods: (1) BPSK, (2) binary DPSK, (3) binary noncoherent FSK, (4) QPSK.

* This part of the problem requires results from Chapters 7 and 8.

APPENDIX B

JOINTLY GAUSSIAN RANDOM VARIABLES

B.1 JOINT pdf AND CHARACTERISTIC FUNCTION

The joint probability density function of N jointly Gaussian random variables is*

$$f_{\mathbf{x}}(\mathbf{x}) = (2\pi)^{-N/2} |\det \mathbf{C}|^{-1/2} \exp \left[-\frac{1}{2}(\mathbf{x} - \mathbf{m})' \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m}) \right] \quad (\text{B.1})$$

where $\mathbf{x}$ and $\mathbf{m}$ are column matrices defined as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad \text{and} \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{bmatrix} \quad (\text{B.2})$$

respectively, and $\mathbf{C}$ is the positive definite matrix of correlation coefficients with elements

$$C_{ij} = \frac{E[(X_i - m_i)(X_j - m_j)]}{\sigma_{X_i} \sigma_{X_j}} \quad (\text{B.3})$$

The joint characteristic function of the Gaussian random variables $X_1, X_2, \dots, X_N$ is

$$M_{\mathbf{x}}(\mathbf{v}) = \exp \left[j\mathbf{m}'\mathbf{v} - \frac{1}{2}\mathbf{v}'\mathbf{C}\mathbf{v} \right] \quad (\text{B.4})$$

From the power-series expansion of (B.4), it follows that for any four zero-mean Gaussian random variables,

$$\begin{aligned} E(X_1 X_2 X_3 X_4) &= E(X_1 X_2) E(X_3 X_4) + E(X_1 X_3) E(X_2 X_4) \\ &\quad + E(X_1 X_4) E(X_2 X_3) \end{aligned} \quad (\text{B.5})$$

This is a rule that is useful enough to be worth memorizing.

* See C. W. Helstrom, *Probability and Stochastic Processes for Engineers* (New York: Macmillan, 1984), p. 208.

B.2 LINEAR TRANSFORMATIONS OF GAUSSIAN RANDOM VARIABLES

If a set of jointly Gaussian random variables is transformed to a new set of random variables by a linear transformation, the resulting random variables are jointly Gaussian. To show this, consider the linear transformation

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad (\text{B.6})$$

where $\mathbf{y}$ and $\mathbf{x}$ are column matrices of dimension N and $\mathbf{A}$ is a nonsingular $N \times N$ square matrix with elements $[a_{ij}]$. From (B.6), the Jacobian is

$$J \begin{pmatrix} x_1, x_2, \dots, x_N \\ y_1, y_2, \dots, y_N \end{pmatrix} = \det(\mathbf{A}^{-1}) \quad (\text{B.7})$$

where $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$. But $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$. Using this in (B.1), along with

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \quad (\text{B.8})$$

gives

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-N/2} |\det \mathbf{C}|^{-1/2} |\det \mathbf{A}|^{-1} \times \exp\left[-\frac{1}{2}(\mathbf{A}^{-1}\mathbf{y} - \mathbf{m})^t \mathbf{C}^{-1}(\mathbf{A}^{-1}\mathbf{y} - \mathbf{m})\right] \quad (\text{B.9})$$

Now $\det \mathbf{A} = \det \mathbf{A}^t$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, the identity matrix, so that (B.9) can be written as

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-N/2} |\det \mathbf{A}\mathbf{C}\mathbf{A}^t|^{-1/2} \times \exp\left\{-\frac{1}{2}[\mathbf{A}^{-1}(\mathbf{y} - \mathbf{A}\mathbf{m})]^t \mathbf{C}^{-1}[\mathbf{A}^{-1}(\mathbf{y} - \mathbf{A}\mathbf{m})]\right\} \quad (\text{B.10})$$

But the equalities $(\mathbf{A}\mathbf{B})^t = \mathbf{B}^t\mathbf{A}^t$ and $(\mathbf{A}^{-1})^t = (\mathbf{A}^t)^{-1}$ allow the term inside the braces in (B.10) to be written as

$$-\frac{1}{2}[(\mathbf{y} - \mathbf{A}\mathbf{m})^t (\mathbf{A}^t)^{-1} \mathbf{C}^{-1} \mathbf{A}^{-1} (\mathbf{y} - \mathbf{A}\mathbf{m})]$$

Finally, the equality $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ allows the above term to be rearranged to

$$-\frac{1}{2}[(\mathbf{y} - \mathbf{A}\mathbf{m})^t (\mathbf{A}\mathbf{C}\mathbf{A}^t)^{-1} (\mathbf{y} - \mathbf{A}\mathbf{m})]$$

Thus (B.10) becomes

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-N/2} |\det \mathbf{A}\mathbf{C}\mathbf{A}^t| \times \exp\left\{-\frac{1}{2}(\mathbf{y} - \mathbf{A}\mathbf{m})^t (\mathbf{A}\mathbf{C}\mathbf{A}^t)^{-1} (\mathbf{y} - \mathbf{A}\mathbf{m})\right\} \quad (\text{B.11})$$

We recognize this as a joint Gaussian density function for a random vector $\mathbf{Y}$ with mean vector $E[\mathbf{Y}] = \mathbf{A}\mathbf{m}$ and covariance matrix $\mathbf{A}\mathbf{C}\mathbf{A}^t$.

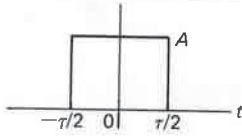
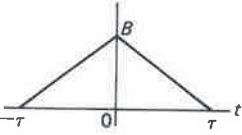
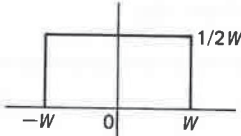
APPENDIX C

MATHEMATICAL AND NUMERICAL TABLES

This Appendix contains several tables pertinent to the material contained in this book. The tables are as follows:

- C.1 Fourier Transform Pairs
- C.2 Fourier Transform Theorems
- C.3 Trigonometric Identities
- C.4 Series Expansions
- C.5 Definite Integrals
- C.6 The Q -Function

C.1 Fourier Transform Pairs

Signal	Transform
 $\triangleq A\Pi(t/\tau)$	$A\tau \frac{\sin \pi f \tau}{\pi f \tau} \triangleq A\tau \operatorname{sinc} f\tau$
 $\triangleq B\Lambda(t/\tau)$	$B\tau \frac{\sin^2 \pi f \tau}{(\pi f \tau)^2} \triangleq B\tau \operatorname{sinc}^2 f\tau$
$e^{-\alpha t} u(t)$	$\frac{1}{\alpha + j2\pi f}$
$\exp(- t /\tau)$	$\frac{2\tau}{1 + (2\pi f \tau)^2}$
$\exp[-\pi(t/\tau)^2]$	$\tau \exp[-\pi(f\tau)^2]$
$\frac{\sin 2\pi Wt}{2\pi Wt} \triangleq \operatorname{sinc} 2Wt$	 $\triangleq \frac{\Pi(f/2W)}{2W}$
$\exp[j(\omega_c t + \phi)]$	$\exp(j\phi) \delta(f - f_c), \omega_c = 2\pi f_c$
$\cos(\omega_c t + \phi)$	$\frac{1}{2} \delta(f - f_c) \exp(j\phi) + \frac{1}{2} \delta(f + f_c) \exp(-j\phi)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right)$
$\operatorname{sgn} t = \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$-\frac{j}{\pi f}$
$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{j2\pi f} + \frac{1}{2} \delta(f)$
$\hat{x}(t)$	$-j \operatorname{sgn}(f) X(f)$

C.2 Fourier Transform Theorems

Name of Theorem	Signal*	Fourier Transform
1. Superposition	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j\omega t_0}$
3a. Scale change	$x(at)$	$ a ^{-1}X(f/a)$
b. Time reversal	$x(-t)$	$X(-f) = X^*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0), \omega_0 = 2\pi f_0$
b. Modulation	$x(t) \cos \omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^t x(t') dt'$	$(j2\pi f)^{-1} X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t') dt'$ $= \int_{-\infty}^{\infty} x_1(t')x_2(t - t') dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f') df'$ $= \int_{-\infty}^{\infty} X_1(f')X_2(f - f') df'$

*All signals are assumed to be real.

C.3 Trigonometric Identities

Euler's theorem: $e^{\pm j\omega} = \cos \omega \pm j \sin \omega$

$$\cos \omega = \frac{1}{2}(e^{j\omega} + e^{-j\omega})$$

$$\sin \omega = (e^{j\omega} - e^{-j\omega})/2j$$

$$\sin^2 \omega + \cos^2 \omega = 1$$

$$\cos^2 \omega - \sin^2 \omega = \cos 2\omega$$

$$2 \sin \omega \cos \omega = \sin 2\omega$$

$$\cos^2 \omega = \frac{1}{2}(1 + \cos 2\omega)$$

$$\sin^2 \omega = \frac{1}{2}(1 - \cos 2\omega)$$

$$\cos^{2n} \omega = \left[\sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)\omega + \binom{2n}{n} \right] / 2^{2n}$$

$$\cos^{2n-1} \omega = \left[\sum_{k=0}^{n-1} \binom{2n-1}{k} \cos (2n-2k-1)\omega \right] / 2^{2n-2}$$

$$\sin^{2n} \omega = \left[\sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)\omega + \binom{2n}{n} \right] / 2^{2n}$$

$$\sin^{2n-1} \omega = \left[\sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin (2n-2k-1)\omega \right] / 2^{2n-2}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\sin (u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos (u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\sin u \sin v = \frac{1}{2} [\cos (u - v) - \cos (u + v)]$$

$$\cos u \cos v = \frac{1}{2} [\cos (u - v) + \cos (u + v)]$$

$$\sin u \cos v = \frac{1}{2} [\sin (u - v) + \sin (u + v)]$$

C.4 Series Expansions

$$(u + v)^n = \sum_{k=0}^n \binom{n}{k} u^{n-k} v^k, \text{ where } \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Letting $u = 1$ and $v = x$, where $|x| \ll 1$, results in the following approximations:

$$(1 + x)^n \cong 1 + nx$$

$$(1 + x)^{1/2} \cong 1 + \frac{1}{2}x$$

$$(1 + x)^{-n} \cong 1 - nx$$

$$\ln(1 + u) = \sum_{k=1}^{\infty} (-1)^{k+1} u^k / k$$

$$\log_a u = \log_e u \log_a e; \log_e u = \ln u = \log_a u \log_a a$$

$$e^u = \sum_{k=0}^{\infty} u^k / k! \cong 1 + u, \quad |u| \ll 1$$

$$a^u = e^{u \ln a}$$

$$\sin u = \sum_{k=0}^{\infty} (-1)^k u^{2k+1} / (2k+1)! \cong u - u^3/3!, \quad |u| \ll 1$$

$$\cos u = \sum_{k=0}^{\infty} (-1)^k u^{2k} / (2k)! \cong 1 - u^2/2!, \quad |u| \ll 1$$

$$\tan u = u + \frac{1}{3} u^3 + \frac{2}{15} u^5 + \dots$$

$$\sin^{-1} u = u + \frac{1}{6} u^3 + \frac{3}{40} u^5 + \dots$$

$$\tan^{-1} u = u - \frac{1}{3} u^3 + \frac{1}{5} u^5 - \dots$$

$$\operatorname{sinc} u = 1 - (\pi u)^2/3! + (\pi u)^4/5! - \dots$$

$$J_n(u) \cong \begin{cases} \frac{u^n}{2^n n!} \left[1 - \frac{u^2}{2^2(n+1)} + \frac{u^4}{2 \cdot 2^4(n+1)(n+2)} - \dots \right], \\ \sqrt{\frac{2}{\pi u}} \cos \left(u - \frac{n\pi}{2} - \frac{\pi}{2} \right), \quad u \gg 1 \end{cases}$$

$$I_0(u) \cong \begin{cases} 1 + \frac{u^2}{2^2} + \frac{u^4}{2^2 4^2} + \dots \cong e^{u^2/4}, & 0 \leq u \ll 1 \\ \frac{e^u}{\sqrt{2\pi u}}, & u \gg 1 \end{cases}$$

C.5 Definite Integrals

$$\int_0^\infty \frac{x^{m-1}}{1+x^n} dx = \frac{\pi/n}{\sin(m\pi/n)}, \quad n > m > 0$$

$$\int_0^\infty x^{\alpha-1} e^{-x} dx = \Gamma(\alpha), \quad \alpha > 0$$

where

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$$

$$\Gamma(1) = 1; \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n) = (n-1)! \quad \text{if } n \text{ is an integer}$$

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}, \quad a > 0$$

$$\int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}, \quad a > 0$$

$$\int_0^\infty e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi} e^{-b^2/4a^2}}{2a}, \quad a > 0$$

$$\int_0^\infty x^{\alpha-1} \cos bx dx = \frac{\Gamma(\alpha)}{b^\alpha} \cos \frac{1}{2} \pi \alpha, \quad 0 < \alpha < 1, \quad b > 0$$

$$\int_0^\infty x^{\alpha-1} \sin bx dx = \frac{\Gamma(\alpha)}{b^\alpha} \sin \frac{1}{2} \pi \alpha, \quad 0 < \alpha < 1, \quad b > 0$$

$$\int_0^\infty x e^{-ax^2} I_k(bx) dx = \frac{1}{2a} e^{-b^2/4a}$$

$$\int_0^\infty \operatorname{sinc} x dx = \int_0^\infty \operatorname{sinc}^2 x dx = \frac{1}{2}$$

$$\int_0^\infty \frac{\cos ax}{b^2 + x^2} dx = \frac{\pi}{2b} e^{-ab}, \quad a > 0, b > 0$$

$$\int_0^\infty \frac{x \sin ax}{b^2 + x^2} dx = \frac{\pi}{2} e^{-ab}, \quad a > 0, b > 0$$

C.6 The Q-Function*

The Gaussian probability density function of unit variance and zero mean is

$$Z(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad (\text{C.1})$$

and the corresponding cumulative distribution function is

$$P(x) = \int_{-\infty}^x Z(t) dt \quad (\text{C.2})$$

The Q-function is defined as†

$$Q(x) = 1 - P(x) = \int_x^{\infty} Z(t) dt, \quad x \geq 0 \quad (\text{C.3})$$

An asymptotic expansion for $Q(x)$ valid for large x is

$$Q(x) = \frac{Z(x)}{x} \left[1 - \frac{1}{x^2} + \frac{1 \cdot 3}{x^4} - \dots + \frac{(-1)^n \cdot 3 \cdot \dots \cdot (2n-1)}{x^{2n}} \right] + R_n \quad (\text{C.4})$$

where the remainder is given by

$$R_n = (-1)^{n+1} 1 \cdot 3 \cdot \dots \cdot (2n+1) \int_x^{\infty} \frac{Z(t)}{t^{2n+2}} dt \quad (\text{C.5})$$

which is less in absolute value than the first neglected term. For $x \geq 3$, less than 10% error results if only the first term in (C.4) is used to approximate the Q-function.

For moderate values of x , several rational approximations are available. One such approximation is

$$Q(x) = Z(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t + b_5 t^5 + \epsilon(x)) \quad (\text{C.6})$$

where

$$t = \frac{1}{1 + px} \quad (\text{C.7})$$

* The information given in this appendix is extracted from M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, New York: Dover, 1972 (Originally published in 1964 as part of the National Bureau of Standards Applied Mathematics Series 55).

† For $x < 0$, $Q(x) = 1 - Q(|x|)$.

The error term is bounded by $|\epsilon(x)| < 7.5 \times 10^{-8}$, and the constants are given by

$$\begin{aligned} p &= 0.2316419 \\ b_1 &= 0.319381530 & b_2 &= -0.356563782 & b_3 &= 1.781477937 \\ b_4 &= -1.821255978 & b_5 &= 1.330274429 \end{aligned}$$

The error function can be related to the Q -function by

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 1 - 2Q(\sqrt{2}x) \quad (\text{C.8})$$

The complementary error function is $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$, and can be approximated similarly to the Q -function as in (C.4) and (C.6). These approximations will not be given here.

A short table of values for $Q(x)$ and $Z(x)$ follows. Extensive tables of $P(x)$, $Q(x)$, and $Z(x)$ and its derivatives can be found in Abramowitz and Stegun (1972).

x	$Q(x)$	$Z(x)$	x	$Q(x)$	$Z(x)$
0.0	0.50000	0.39894	2.2	0.01390	0.03547
0.1	0.46017	0.39695	2.3	0.01072	0.02833
0.2	0.42074	0.39104	2.4	0.00820	0.02239
0.3	0.38209	0.38138	2.5	0.00621	0.01753
0.4	0.34458	0.36827	2.6	0.00466	0.01358
0.5	0.30854	0.35206	2.7	0.00347	0.01042
0.6	0.27425	0.33322	2.8	0.00256	0.00792
0.7	0.24196	0.31225	2.9	0.00187	0.00595
0.8	0.21186	0.28969	3.0	0.00135	0.00443
0.9	0.18406	0.26608	3.1	0.00097	0.00327
1.0	0.15866	0.24197	3.2	0.00069	0.00238
1.1	0.13567	0.21785	3.3	0.00048	0.00172
1.2	0.11507	0.19419	3.4	0.00035	0.00123
1.3	0.09680	0.17137	3.5	0.00023	0.00087
1.4	0.08076	0.14973	3.6	0.00016	0.00061
1.5	0.06681	0.12952	3.7	0.00011	0.00042
1.6	0.05480	0.11092	3.8	7.24×10^{-5}	0.00029
1.7	0.04457	0.09405	3.9	4.81×10^{-5}	0.00020
1.8	0.03593	0.07895	4.0	3.17×10^{-5}	0.00013
1.9	0.02872	0.06562			
2.0	0.02275	0.05399			
2.1	0.01786	0.04398			

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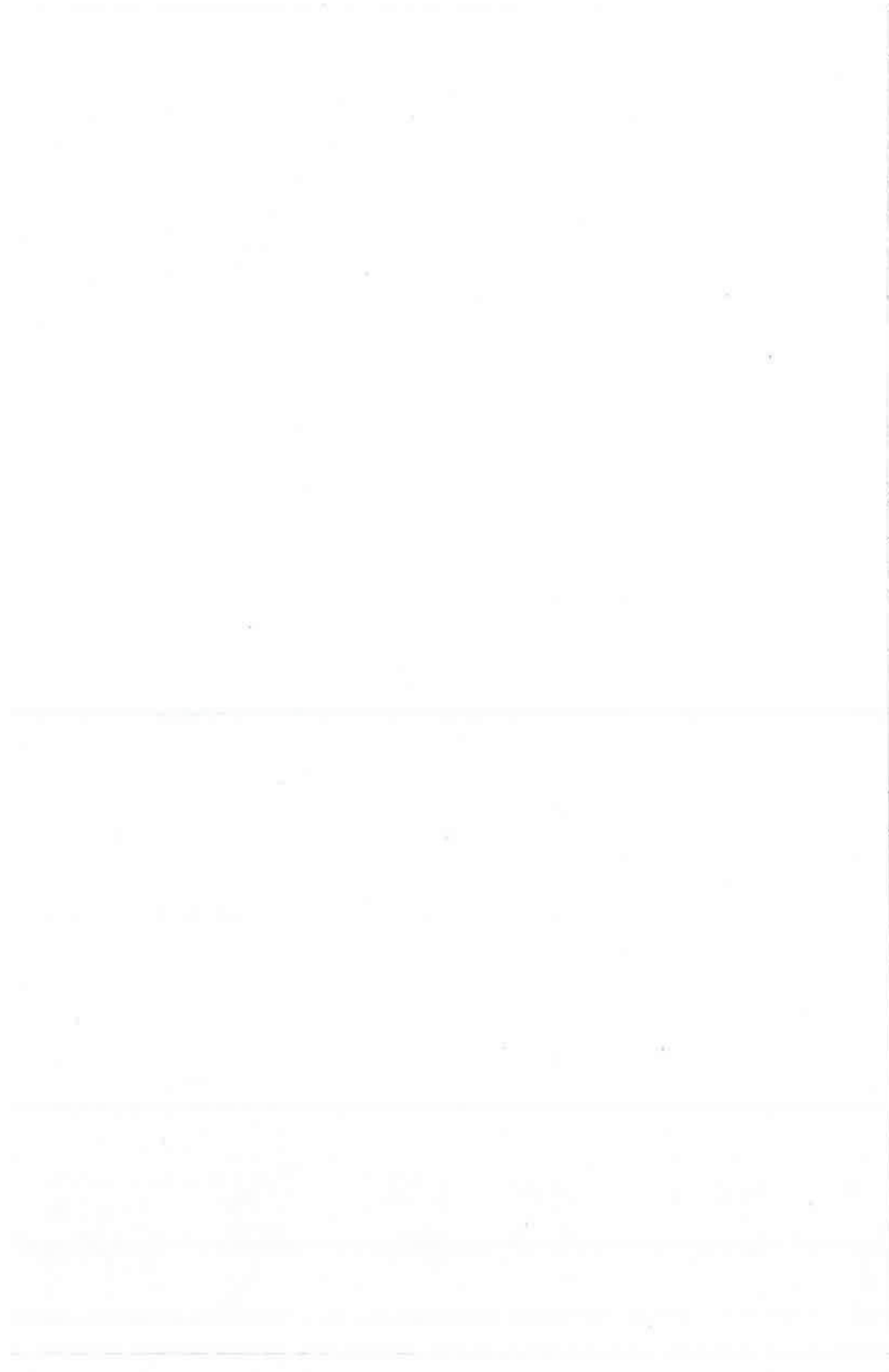
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