

MODERN OPTICS

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Preface

This textbook is designed for use in a standard physics course on optics. The book is the result of a one-semester elective course that has been taught to juniors, seniors, and first-year graduate students in physics and engineering at Duke University for several years. Students who take this course should have completed an introductory physics course and math courses through differential equations. Electricity and magnetism can be taken concurrently.

Modern Optics differs from the classical approach of most textbooks on this subject in that its treatment of optics includes some material that is not found in more conventional textbooks. These topics include nonlinear optics, guided waves, Gaussian beams, and light modulators. Moreover, a selection of optional material is provided for the instructor so that the course content can reflect the interest of the instructor and the students. Basic derivations are included to make the book appealing to physics departments, and design concepts are included to make the book appealing to engineering departments. Because of the material covered here, the electrical engineering and biomedical engineering departments at Duke have made the corresponding optics course a prerequisite for some of their advanced courses in optical communications and medical imaging.

Before the 1960s, the only contact that the average person had with optics was a camera lens or eyeglasses. Geometric optics was quite adequate for the design of these systems, and it was natural to emphasize this aspect of optics in a curriculum. The approach used introduced the students to the theory and to examples of the application of the theory, accomplished by a description of a large variety of optical instruments. The reason for this approach was that lens design is found to be quite tedious, and the optimization of a lens design is more easily described than accomplished.

Today the student is exposed to many more optical systems. Everyone encounters supermarket scanners, copying machines, compact disk players, holograms, and discussions of fiber optic communications. In the research environment, lasers, optical modulators, fiber optic interconnects, and nonlinear optics have become important tools. Upon graduation, many students

will be called on to participate in the use or the development of these modern optical systems. An elementary discussion of geometrical optics and a review of classical optical instruments will not adequately prepare the student for these demands.

This book was written to provide both a fundamental study of the principles of optics and an exposure to actual optics engineering problems and solutions. To include new material has meant that some of the topics covered in classical texts had to be removed. A large portion of the conventional treatment of geometrical optics was deleted, along with a discussion of classical optical systems. In their place were a geometrical optics discussion of fiber optics and a discussion of holography. Rather than describe a host of optical systems, a few optical systems, such as the Fabry–Perot interferometer, are examined using a variety of theories. This book emphasizes diffraction and the use of Fourier theory to describe the operation of an optical system.

To allow the development of a one-year course in modern optics, a number of topics have been added or expanded. A discussion of electrooptic and magneto-optic effects is used to introduce optical modulators, and a discussion of nonlinear optics is constructed around second harmonic generation. Because of the importance of birefringence in optical modulators and nonlinear optics, an expanded discussion of optical anisotropy has been included. This is a departure from most texts that ignore anisotropy because of the need to use tensors. In modern optics, anisotropy is an important design tool, and its treatment allows a discussion of the design of optical modulators and phase matching in nonlinear materials.

The first two chapters review wave theory and electromagnetic theory. Except for the section on polarization in Chapter 2, these chapters could be used as reading assignments for well-prepared students. Chapter 3 discusses reflection and refraction and utilizes the boundary conditions of Maxwell's equations to obtain the fraction of light reflected and refracted at a surface.

Chapter 4 discusses interference of waves and describes several instruments that are used to measure interference. Two of the interferometers: Young's two-slit experiment and the Fabry–Perot interferometer, should receive emphasis in discussions of this chapter because of the role they play in later discussions. An appendix to this chapter provides a brief introduction to some of the design techniques that are used to produce multilayer interference filters. All of the appendices in the book are included to fill in the gaps in students' knowledge and to provide some flexibility for the instructor. The appendices may, therefore, be ignored or used as the subject matter for special assignments.

The treatment of geometrical optics, presented in Chapter 5, is not traditional. It was through the reduction of traditional subject matter that the space was obtained to introduce more modern topics. A brief introduction to the matrix formalism used in lens design is presented, and its use is demonstrated by analyzing a confocal Fabry–Perot resonator. Geometrical optics and the concept of interference are used to analyze the propagation of light in a fiber. This introduction to fiber optics is then extended through the use of the Lagrangian formulation to propagation in a graded-index optical fiber. The first part of the chapter demonstrates the formal connection between geometric optics and wave theory. Most students would rather not cover this material and, therefore, it is usually omitted. The connection between the matrix equations and the more familiar lens equations is established in

Appendix 5-A. Because of their importance in the Graduate Records Exam, aberrations are treated in Appendix 5-B.

The Fourier theory in Chapter 6 is presented as a review of and refresher on the subject. It is an important element in the discussion of the concept of coherence in Chapter 8, and Fraunhofer diffraction in Chapter 10. The discussions of optical signal processing, Appendix 10-B, and imaging, Appendix 10-C, draw heavily on Fourier theory.

The discussion of dispersion given in Chapter 7 could be delayed and combined with the other chapters on material interactions (Chapters 13 and 15). It is included here to justify the discussion of coherence in Chapter 8. The discussion of dispersion in materials had as its objective the development by the student of a unifying view of the interaction of light and matter.

The development of coherence theory in Chapter 8 is built around applications of the theory to spectroscopy and astronomy. It is a very difficult subject, but building the theory around the methods used to measure coherence should make the subject more intelligible.

Both the Fresnel and Gaussian wave formalism of diffraction are introduced in Chapter 9. The Gaussian wave formalism is used to analyze a Fabry–Perot cavity and thin lens. This chapter can be skipped, and the material introducing the Fresnel–Huygens integral can be covered in a single lecture.

The Fresnel formalism is expanded and discussed in Chapters 10 and 11. Fraunhofer diffraction is treated from a linear-system viewpoint in Chapter 10, and applications of the theory to signal processing and imaging are presented in Appendices 10-B and 10-C. These two appendices are the most important in the book. Fresnel Diffraction is introduced in Chapter 11, where it is used to interpret Fermat’s principle and analyze zone plates and pinhole cameras. In Chapter 12, Fresnel theory is used to discuss the operation of a hologram. Chapter 12 also includes a simple, quasigeometric theory that is used to highlight the fundamental properties of a hologram.

Chapter 13 uses the introduction of polarizers and retarders as a basis for the development of the theory of the propagation of light in anisotropic materials. The treatment of anisotropic materials is expanded over the conventional presentation to allow an easy transition into the discussion of light modulators in Chapter 14. The many geometrical constructions used in the discussion of anisotropy are confusing to everyone. To try to make the material understandable, only one construction is used in Chapter 13. To provide the student with reference material to aid in reading other books and papers, the other constructions are discussed in the appendices.

The discussion of modulators in Chapter 14 provides an application-based introduction to electro- and magnetooptic interactions. The design of an electrooptic modulator provides the student with an example of the use of tensors. The material interactions presented in Chapters 14 and 15 require the use of tensors, a subject normally avoided in an undergraduate curriculum. Tensor notation has been used in this book because it is key in the understanding of many optical devices. Some familiarity with tensors removes much of the “magic” associated with the design of modulators and the application of phase matching discussed in Chapter 15.

The subject of nonlinear optics in Chapter 15 is developed by using examples based on frequency doubling. Only a few brief comments are made about third-order nonlinearities. The additional discussion of third-order processes is best presented by using a quantum mechanical viewpoint.

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It was thought that this would be best done in a separate course. The material presented in this chapter would prepare the student to immediately undertake a course in nonlinear optics.

Enough material has been included for a one-year course in optics. Chapters 2 to 4, 6, 9 (excluding Gaussian waves), and 10 contain the core material and could be used in a one-quarter course. By adding Chapter 7 and 8 along with Appendices 10-B and 10-C, a one-semester course can be created. The instructor can alter the subjects discussed from year to year by adding topics such as Appendix 4-A, the guided wave discussion of Chapter 5, or the discussion of holography in Chapter 12 in place of Appendices 10-B and 10-C. A less demanding one-semester course can be created by ignoring Chapter 8 and by substituting Chapter 5, Appendix 4-A, or possibly Chapter 12. In anticipation of developing skill and knowledge, the subject matter and problems increase in difficulty as the student moves through the book.

A number of people provided help in the preparation of this book. Those who provided photos or drawings are identified in the figure captions. Their generosity is most appreciated. Dr. Frank DeLucia provided the initial motivation for writing the book. Many ideas and concepts are the result of breakfast discussions with A. VanderLugt. The book would never have gone past the note stage without the equation writer, MacΣqn, written by Dennis Venable. Thomas Stone provided ideas, photographic skills, and encouragement during the preparation of most of the photos in this book. His enthusiasm kept me working.

A very special thanks must go to Nicholas George. He loaned me equipment and lab space to prepare many of the photos. His encouragement prevented me from shelving the project, and his technical discussions provided me with an improved understanding of optics.

Robert D. Guenther

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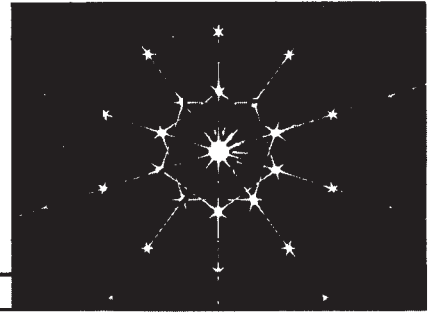
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Electromagnetic Theory

The scalar wave theory that we discussed in Chapter 1 was applied to the study of light before the development of the theory of electromagnetism. At that time, it was assumed that the light waves were longitudinal in analogy with sound waves, i.e., the wave's displacements were in the direction of propagation. A further assumption, that light propagated through some type of medium, was made because the scientists of that time approached all problems from a mechanistic point of view. The scalar theory was successful in explaining diffraction (see Chapter 9), but problems arose in interpretation of the effects of polarization in interference experiments (discussed in Chapter 4). Young was able to resolve the difficulties by suggesting that the waves could be transverse as are the waves on a vibrating string. Using this idea, Fresnel developed a mechanistic description of light that could explain the amount of reflected and transmitted light from the interface between two media (see Chapter 3).

Independent of this activity, the theory of electromagnetism was under development. **Michael Faraday (1791–1867)** observed in 1845 that a magnetic field would rotate the plane of polarization of light waves passing through the magnetized region. This observation led Faraday to associate light with electromagnetic radiation, but he was unable to quantify this association. Faraday attempted to develop electromagnetic theory by treating the field as lines pointing in the direction of the force that the field would exert on a test charge. The lines were given a mechanical interpretation with a tension along each line and a pressure normal to the line. **James Clerk Maxwell (1831–1879)** furnished a mathematical framework for Faraday's model in a paper read in 1864 and published a year later.¹ In this paper, Maxwell identified light as “an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electro-

magnetic laws” and demonstrated that the propagation velocity of light was given by the electromagnetic properties of the material.

Maxwell was not the first to recognize the connection between the electromagnetic properties of materials and the speed of light. **Kirchhoff** recognized in 1857 that the speed of light could be obtained from electromagnetic properties. **Riemann** in 1858 assumed that electromagnetic forces propagated at a finite velocity and derived a propagation velocity given by the electromagnetic properties of the medium. However, it was Maxwell who demonstrated that the electric and magnetic fields are waves that travel at the speed of light. It was not until 1887 that an experimental observation of electromagnetic waves other than light was obtained by **Heinrich Rudolf Hertz (1857–1894)**.

The classical electromagnetic theory is successful in explaining all of the experimental observations to be discussed in this book. There are, however, experiments that cannot be explained by classical wave theory, especially those conducted at short wavelengths or very low light levels. Quantum electrodynamics is capable of predicting the outcome of all optical experiments; its shortcoming is that it does not explain why or how. An excellent elementary introduction to quantum electrodynamics has been written by Richard Feynman.²

In this chapter, we will borrow, from electromagnetic theory, Maxwell’s equations and Poynting’s theorem to derive properties of light waves. Details of the origins of these fundamental electromagnetic relationships are not needed for our study of light, but can be obtained by consulting any electricity and magnetism text.³

The basic properties that will be derived are (1) the wave nature of light, (2) the fact that light is a transverse wave, (3) the velocity of light in terms of fundamental electromagnetic properties of materials, (4) the relative magnitude of the electric and magnetic fields and relationships between the two fields, and finally, (5) the momentum and energy associated with a light wave.

The concept of polarized light and a geometrical construction used to visualize its behavior will be introduced. Both a vector and matrix notation used to describe polarization will be presented in this chapter, but details on the manipulation of light’s polarization will not be discussed until Chapter 13. The chapter will conclude with a discussion of the propagation of light in a conducting medium.

MAXWELL’S EQUATIONS

The basis of electromagnetic theory is Maxwell’s equations. They allow the derivation of the properties of light. In our study of optics, we will treat these equations as axioms but provide the reader with a source that can be consulted if information on the origin of the equations is desired. In rationalized mks units, Maxwell’s equations are the following.

Gauss’s (Coulomb’s) Law for the Electric Field *Coulomb’s law provides a means for calculating the force between two charges (see Chapter 2 of Wangsness³)*

$$\mathbf{F} = \frac{q_0}{4\pi\epsilon_0} \int \frac{dq}{r^2} \hat{\mathbf{n}}$$

where dq is the charge on an infinitesimal surface and $\hat{n}$ is a unit vector connecting charges q_0 and dq . The electric field

$$\mathbf{E} = \frac{\mathbf{F}}{q_0}$$

is obtained using Coulomb's law (see Chapter 3, Wangsness). We view this field, as Michael Faraday did, as lines of flux, called lines of force, originating on positive charges and terminating on negative charges. Gauss's law states that the quantity of charge contained within a closed surface is equal to the number of flux lines passing outward through the surface (Chapter 4, Wangsness). This view of the electric field leads to

$$\nabla \cdot \mathbf{D} = \rho \quad (2-1)$$

where ρ is the charge density and $\mathbf{D}$ is the electric displacement (Chapter 10, Section 5 of Wangsness). The use of the displacement allows the equation to be applied to any material.

Gauss's Law for the Magnetic Field Charges at rest led to (2-1). Charges in motion, that is, a current $\mathbf{i}$ or a current density $\mathbf{J}$, create a magnetic field $\mathbf{B}$ (Chapter 14, Wangsness). As we did for the electric field, we treat the magnetic field as flux lines, called lines of induction, and we assume that the current density is a constant so that $\nabla \cdot \mathbf{J} = 0$. This leads to (Chapter 16, Wangsness)

$$\nabla \cdot \mathbf{B} = 0 \quad (2-2)$$

The zero results from the fact that the magnetic equivalent of a single charge has never been observed.

Faraday's Law The previous two equations are associated with electric and magnetic fields that are constant with respect to time. The next equation, an experimentally derived equation, deals with a magnetic field that is time-varying or equivalently a conductor moving through a static magnetic field. In terms of the concept of flux, it states that an electric field around a circuit is associated with a change in the magnetic flux contained within the circuit.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (2-3)$$

Ampère's Law (Law of Biot and Savart) An electric charge in motion creates a magnetic field around its path. The law of Biot-Savart allows us to calculate the magnetic field at a point located a distance $\mathbf{R}$ from a conductor carrying a current density $\mathbf{J}$. Ampère's law is the inverse relationship used to calculate the current in a conductor due to the magnetic field contained in a loop about the conductor. Neither relationship is adequate when the current is a function of time. Maxwell's major contribution to physics was to observe that the addition of a *displacement current* to Ampère's law allowed fluctuating currents to be explained. The relationship became (see Chapter 21 of Wangsness)

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2-4)$$

As discussed in Appendix 2-B, the constants in Maxwell's equations depend on the units used. Many optics books use cgs units that result in a form for Maxwell's equations shown in Appendix 2-B.

The dynamic response of atoms and molecules in the propagation medium is taken into account through what are called the *constitutive relations*.

$$\mathbf{D} = f(\mathbf{E})$$

$$\mathbf{J} = g(\mathbf{E})$$

$$\mathbf{B} = h(\mathbf{H})$$

Here, we will assume that the functional relations are independent of space and time and we will write the constitutive relations as

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \epsilon = \text{dielectric constant}$$

$$\mathbf{J} = \sigma \mathbf{E}, \quad \sigma = \text{conductivity (Ohms law)}$$

$$\mathbf{B} = \mu \mathbf{H}, \quad \mu = \text{permeability}$$

where the constants ϵ , σ , and μ contain the description of the material. Later, we will explore the effects resulting from the constitutive relations having a temporal or spatial dependence.

Often, $\mathbf{D}$ and $\mathbf{B}$ are defined as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (2-5)$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$$

where $\mathbf{P}$ is the polarization and $\mathbf{M}$ the magnetization. This formulation emphasizes that the internal field of a material is due not only to the applied field, but also a field created by the atoms and molecules that make up the material. We will find (2-5) useful in Chapters 7 and 15. We will not use the relationship involving $\mathbf{M}$ in this book.

By manipulating Maxwell's equations, we can obtain a number of the properties of light such as its wave nature, the fact that it is a transverse wave, and the relationship between the $\mathbf{E}$ and $\mathbf{B}$ fields. We will make a number of simplifying assumptions about the medium in which light is propagating to allow a quick derivation of the properties of light. Later, we will see what happens if we modify these assumptions.

We assume that the light is propagating in a medium we will call *free space* that has the following properties:

1. Uniform: ϵ and μ have the same value at all points.
2. Isotropic: ϵ and μ do not depend on the direction of propagation.
3. Nonconducting: $\sigma = 0$, thus $\mathbf{J} = 0$.
4. Free from charge: $\rho = 0$.
5. Nondispersive: ϵ and μ are not functions of the frequency, i.e., they have no time dependence.

Our definition departs somewhat from other definitions of free space in that we include in the definition not only the vacuum, where $\epsilon = \epsilon_0$ and $\mu = \mu_0$, but also dielectrics, where $\sigma = 0$ but the other electromagnetic constants can have arbitrary values.

If we use the above assumptions, Maxwell's equations and the constitutive relations simplify to

$$\nabla \cdot \mathbf{E} = 0 \quad (2-6a)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2-6b)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2-6c)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (2-6d)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (2-6e)$$

$$\epsilon \mathbf{E} = \mathbf{D} \quad (2-6f)$$

These simplified equations can now be used to derive some of the basic properties of a light wave.

Wave Equation

To find how the electromagnetic wave described by (2-6) propagates in free space, Maxwell's equations must be rearranged to display explicitly the time and coordinate dependence. Using (2-6e) and (2-6f) we can rewrite (2-6d) as

$$\frac{1}{\mu} \nabla \times \mathbf{B} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

The curl of (2-6c) is taken and the magnetic field dependence eliminated by using the rewritten (2-6d)

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t} \left(\epsilon \mu \frac{\partial \mathbf{E}}{\partial t} \right)$$

The assumption that ϵ and μ are independent of time allows the equation to be rewritten

$$\nabla \times (\nabla \times \mathbf{E}) = -\epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity (2A-12), we can write

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Because free space is free of charge $\nabla \cdot \mathbf{E} = 0$, giving us

$$\nabla^2 \mathbf{E} = \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2-7)$$

We can use the same procedure to obtain

$$\nabla^2 \mathbf{B} = \mu \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (2-8)$$

These equations are wave equations, with the wave's velocity given by

$$v = \frac{1}{\sqrt{\mu \epsilon}} \quad (2-9)$$

The connection of the velocity of light with the electric and magnetic properties of a material was one of the most important results of Maxwell's theory. In a vacuum,

$$\begin{aligned}\mu_0\epsilon_0 &= (4\pi \times 10^{-7})(8.8542 \times 10^{-12}) \\ &= 1.113 \times 10^{-17} \frac{\text{sec}^2}{\text{m}^2} \\ \frac{1}{\sqrt{\mu_0\epsilon_0}} &= 2.998 \times 10^8 \frac{\text{m}}{\text{sec}} = c\end{aligned}\quad (2-10)$$

In a material, the velocity of light is less than c . We can characterize a material by defining the *index of refraction*, the ratio of the speed of light in a vacuum to its speed in a medium.

$$n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \quad (2-11)$$

TABLE 2.1 Representative Magnetic Permeability

Material	μ/μ_0	Class
Silver	0.99998	Diamagnetic
Copper	0.99999	Diamagnetic
Water	0.99999	Diamagnetic
Air	1.00000036	Paramagnetic
Aluminum	1.000021	Paramagnetic
Iron	5000	Ferromagnetic
Nickel	600	Ferromagnetic

The data in Table 2.1 demonstrate that if magnetic materials are not considered, then $\mu/\mu_0 \approx 1$ so that

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}}$$

The data displayed in Table 2.2 demonstrate that, at least for some materials, the theory agrees with experimental results. The materials whose indices are listed in Table 2.2 have been specially selected to demonstrate good agreement; we will see in Chapter 7 that the assumption that ϵ , μ , and σ are independent of the frequency results in a theory that neglects the response time of the system to the electromagnetic signal.

Transverse Waves

Hooke postulated, in the 17th century, that light waves might be transverse but his idea was forgotten. Young and Fresnel made the same claim in the 19th century and accompanied their postulation with a theoretical description of light based on transverse waves. Forty years later, Maxwell proved that light must be a transverse wave. We can demonstrate the transverse nature of light by substitution of the plane wave solution of the wave equation into Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

TABLE 2.2 Selected Index of Refraction

Material	n (yellow light)	$(\epsilon/\epsilon_0)^{1/2}$ (static)
Air	1.000294	1.000295
CO ₂	1.000449	1.000473
C ₆ H ₆ (benzene)	1.482	1.489
He	1.000036	1.000034
H ₂	1.000131	1.000132

To complete the demonstration, consider the divergence of the electric component of the plane wave. We will examine only the x coordinate of the divergence in detail

$$\begin{aligned}\frac{\partial E_x}{\partial x} &= \frac{\partial}{\partial x} [E_{0x} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)}] = iE_{0x} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)} \frac{\partial}{\partial x} (\omega t - k_x x - k_y y - k_z z + \phi) \\ \frac{\partial E_x}{\partial x} &= -ik_x E_x\end{aligned}$$

We easily obtain similar results for E_y and E_z , allowing the divergence of $\mathbf{E}$ to be rewritten as a dot product of $\mathbf{k}$ and $\mathbf{E}$. Gauss's law for the electric field states that the divergence of $\mathbf{E}$ is zero, which for a plane wave can be written

$$\nabla \cdot \mathbf{E} = -i\mathbf{k} \cdot \mathbf{E} = 0 \quad (2-12)$$

If the dot product of two vectors $\mathbf{E}$ and $\mathbf{k}$ is zero, then the vectors $\mathbf{E}$ and $\mathbf{k}$ must be perpendicular [see (2A-1)]. In the same manner, substituting the plane wave into $\nabla \cdot \mathbf{B} = 0$ yields $\mathbf{k} \cdot \mathbf{B} = 0$. Therefore, Maxwell's equations require light to be a transverse wave, i.e., the vector displacements $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to the direction of propagation $\mathbf{k}$.

Interdependence of $\mathbf{E}$ and $\mathbf{B}$

The electric and magnetic fields are not independent as we can see by continuing our examination of the plane wave solutions of Maxwell's equations. First, let us calculate several derivatives of the plane wave. We will need

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= \frac{\partial}{\partial t} \mathbf{B}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)} = i\mathbf{B} \frac{\partial}{\partial t} (\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \\ \frac{\partial \mathbf{B}}{\partial t} &= i\omega \mathbf{B}\end{aligned} \quad (2-13)$$

and in a similar manner

$$\frac{\partial \mathbf{E}}{\partial t} = i\omega \mathbf{E} \quad (2-14)$$

A simple expression for the curl of $\mathbf{E}$, $\nabla \times \mathbf{E}$, can be obtained when we use the derivatives just calculated. The expression for the curl of $\mathbf{E}$ is given by (2A-7) and is rewritten here

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{\mathbf{k}}$$

The terms making up the x component of the curl are

$$\begin{aligned}\frac{\partial E_z}{\partial y} &= E_{0z} \frac{\partial}{\partial y} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)} = -ik_y E_z \\ \frac{\partial E_y}{\partial z} &= -ik_z E_y\end{aligned}$$

By the evaluation of each component, we find that the curl of $\mathbf{E}$ for a plane wave is

$$\nabla \times \mathbf{E} = -i\mathbf{k} \times \mathbf{E} \quad (2-15)$$

A similar derivation leads to the curl of $\mathbf{B}$ for a plane wave

$$\nabla \times \mathbf{B} = -i \mathbf{k} \times \mathbf{B} \quad (2-16)$$

With these vector operations on a plane wave defined, we can evaluate (2-6c) for a plane wave. The left side of

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

is replaced with (2-15) and the right side by (2-13), resulting in an equation connecting the electric and magnetic field

$$-i \mathbf{k} \times \mathbf{E} = -i \omega \mathbf{B}$$

Using the relationship between ω and $\mathbf{k}$ given by (1-2) and the relationship for the wave velocity in terms of the electromagnetic properties of the material (2-9), we can write

$$\frac{\sqrt{\mu\epsilon}}{k} \mathbf{k} \times \mathbf{E} = \mathbf{B} \quad (2-17)$$

A second relationship between the magnetic and electric fields can be generated by using the same procedure to rewrite

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}$$

for a plane wave as

$$\begin{aligned} -i \mathbf{k} \times \mathbf{B} &= i\epsilon\mu\omega \mathbf{E} \\ \frac{1}{k\sqrt{\mu\epsilon}} \mathbf{k} \times \mathbf{B} &= -\mathbf{E} \end{aligned} \quad (2-18)$$

From the definition of the cross product given in Appendix 2-A (2A-2), we see that the electric and magnetic fields are perpendicular to each other, in phase, and form a right-handed coordinate system with the propagation direction $\mathbf{k}$; see Figure 2-1.

If we are only interested in the magnitude of the two fields, we can use (2-11) to write

$$n|\mathbf{E}| = c|\mathbf{B}| \quad (2-19)$$

In a vacuum, $n = 1$ so that for a vacuum

$$|\mathbf{E}| = c|\mathbf{B}| \quad (2-20)$$

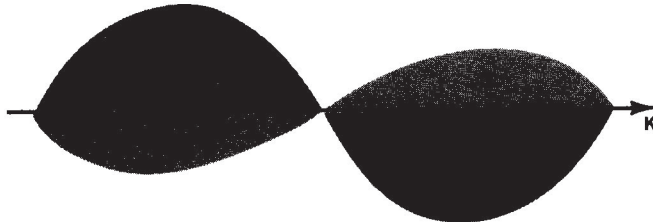


FIGURE 2-1. Graphical representation of an electromagnetic plane wave. Note $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other, individually perpendicular to the propagation vector $\mathbf{k}$, in phase and form a right-handed coordinate system as required by (2-17) and (2-18).

For our plane wave, the ratio of the field magnitudes is

$$\frac{|E|}{|H|} = \sqrt{\frac{\mu}{\epsilon}}$$

This ratio has the units of ohms (Ω) ($\mu \rightarrow ml/Q^2$, $\epsilon \rightarrow Q^2 t^2/ml^3$, and $\Omega \rightarrow ml^2/Q^2 t$) and is called the impedance of the medium. In a vacuum,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \Omega$$

When the ratio is a real quantity, as it is here, then $\mathbf{E}$ and $\mathbf{H}$ are in phase.

We saw in our discussion of waves propagating along strings that the power transmitted by a wave is proportional to the square of the amplitude of the wave. Any text on electromagnetic theory (see Chapter 21 of Wangsness, for example) demonstrates that the energy density (in J/m^3) associated with an electromagnetic wave is given by

$$U = \frac{(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H})}{2} \quad (2-21)$$

We can simplify (2-21) by using the simple constitutive relations $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, if they apply to the propagation medium

$$U = \frac{1}{2} \left(\epsilon E^2 + \frac{B^2}{\mu} \right) = \frac{1}{2} \left(\epsilon + \frac{1}{\mu c^2} \right) E^2$$

In a vacuum, further simplification is possible

$$U = \epsilon_0 E^2 = \frac{B^2}{\mu_0}$$

John Henry Poynting (1852–1914) demonstrated that the presence of both an electric and magnetic field at the same point in space results in a flow of the field energy. This fact is called the Poynting theorem and the *Poynting vector* completely describes the flow

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad (2-22)$$

The units of the Poynting vector are $J/(m^2 \cdot \text{sec})$. We will use a plane wave to determine some of the properties of this vector. Since $\mathbf{S}$ will involve terms quadratic in $\mathbf{E}$, it will be necessary to use the real form of $\mathbf{E}$ (see Problem 1-4).

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{\sqrt{\mu \epsilon}}{\mu k} \mathbf{k} \times \mathbf{E}$$

where

$$\begin{aligned} \mathbf{E} &= E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \\ \mathbf{S} &= \frac{\sqrt{\mu \epsilon}}{\mu k} E_0 \times (\mathbf{k} \times \mathbf{E}_0) \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \\ &= \frac{n}{\mu c} |E_0|^2 \frac{\mathbf{k}}{k} \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) \end{aligned} \quad (2-23)$$

ENERGY DENSITY AND FLOW

Note that the energy is flowing in the direction of propagation (denoted by the unit vector $\mathbf{k}/k$).

We normally do not detect $\mathbf{S}$ at the very high frequencies associated with light ($\approx 10^{15}$ Hz) but rather detect a temporal average of $\mathbf{S}$ with the average taken over a time T determined by the response time of the detector used. We must obtain the time average of $\mathbf{S}$ to relate theory to actual measurements. The time average of $\mathbf{S}$ is called the *flux density* and has units of W/m^2 . We will call this quantity the *intensity* of the light wave

$$I = \langle \mathbf{S} \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{A} \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) dt \quad (2-24)$$

where we have defined

$$\mathbf{A} = \frac{n}{\mu c} |\mathbf{E}_0|^2 \frac{\mathbf{k}}{k}$$

to simplify the notation.

The units used for the flux density are a confusing mess in optics. One area of optics is interested in measuring the physical effects of light and the measurement of energy is called *radiometry*. In radiometry, the flux density is called the *irradiance* with units of W/m^2 . Another area of optics is interested in the psychophysical effects of light and the measurement of energy is called *photometry*. For this group, the flux density is called *illuminance* with units of lumen/ m^2 or lux. Each of these two group has its own set of units to measure the energy flow of a field that is not well defined in frequency or phase. Much of the research in modern optics belongs to a third area of optics that is associated with the use of a light source that has both a well-defined frequency and phase—the laser. For this area of optics, common usage defines the flux density as the intensity. In this book, all of the waves discussed are uniquely defined in terms of the electric field and the electromagnetic properties of the material in which the wave is propagating. To emphasize that the results of our theory are only immediately applicable to a light source with a well-defined frequency and phase, we will use the term intensity for the magnitude of the Poynting vector.

We will assume that $\mathbf{k}$ is independent of time over the period T

$$\langle \mathbf{S} \rangle = \frac{\mathbf{A}}{\omega T} \int_{t_0 \omega}^{(t_0+T)\omega} \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi) d(\omega t)$$

Using the trig identity

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

and evaluating the integral result in the expression

$$\langle \mathbf{S} \rangle = \frac{\mathbf{A}}{2} + \frac{\mathbf{A}}{4\omega T} [\sin 2(\omega t_0 + \omega T - \mathbf{k} \cdot \mathbf{r} + \phi) - \sin 2(\omega t_0 - \mathbf{k} \cdot \mathbf{r} + \phi)] \quad (2-25)$$

The largest value the term in brackets can assume is 2. The period T is the response time of the detector to the light wave. Normally, it is much longer

than the period of light oscillations so that $\omega T \gg 1$ and we can neglect the second term of (2-25). As an example, suppose our detection system has a 1 GHz bandwidth yielding a response time of $T = 10^{-9}$ sec (the reciprocal of the bandwidth). Green light has a frequency of $\nu = 6 \times 10^{14}$ Hz or $\omega \approx 4 \times 10^{15}$. With these values, $\omega T = 4 \times 10^5$ and the neglected term would be no larger than 10^{-6} of the first term. Therefore, in optics the assumption that $\omega T \gg 1$ is reasonable and allows the average Poynting vector to be written as

$$\langle \mathbf{S} \rangle = \frac{\mathbf{A}}{2} = \frac{n}{2\mu c} |\mathbf{E}_0|^2 \frac{\mathbf{k}}{k} \quad (2-26)$$

Just as we saw in our discussion of the vibrating string, the energy per unit time per unit area depends on the square of the amplitude of the wave.

The calculation was made with plane waves of $\mathbf{E}$ and $\mathbf{H}$ that are in phase. We will later see that materials, where the conductivity $\sigma \neq 0$, will yield a complex impedance because $\mathbf{E}$ and $\mathbf{H}$ are no longer in phase. If the two waves are 90° out of phase, then the integral in (2-24) will contain $\sin x \cos x$ as its integrand, resulting in $\langle \mathbf{S} \rangle = 0$. Therefore, no energy is transmitted.

The energy crossing a unit area A in time Δt is contained in a volume $A(v\Delta t)$ (in a vacuum $v = c$) as shown in Figure 2-2. To find the magnitude of this energy, we must multiply this volume by the average energy density $\langle U \rangle$. Thus we expect the energy flow to be given by

$$|\langle \mathbf{S} \rangle| = \frac{\text{energy}}{A\Delta t} \propto \frac{Av\Delta t \langle U \rangle}{A\Delta t} = v \langle U \rangle$$

We may use the definitions of the wave velocity

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

and index of refraction $n = c/v$ to rewrite (2-26)

$$|\langle \mathbf{S} \rangle| = \frac{\epsilon v E_0^2}{2} = v \langle U \rangle \quad (2-27)$$

giving the expected result that the energy is flowing through space at the speed of light in the medium. The relationship defined by (2-27),

$$(\text{energy flow}) = (\text{wave velocity}) \cdot (\text{energy density})$$

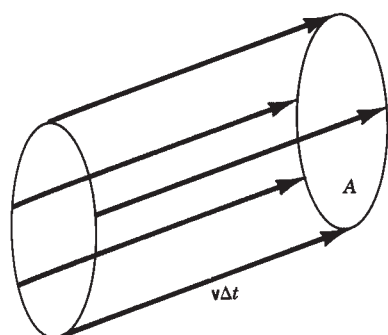


FIGURE 2-2. The energy or momentum of a wave crossing a unit area A in a time Δt .

In quantum mechanics, the energy in light is carried by discrete particles called photons. If the light has a frequency of ν , then the photon's energy is $h\nu$. The intensity of the light is equal to the number of photons striking a unit area, in a unit time N , multiplied by the energy of an individual photon

$$I = Nh\nu$$

is a general property of waves and could have been derived using the string model of Chapter 1.

At the Earth's surface, the flux density of full sunlight is 1.34×10^3 J/(m²·sec). It is not completely correct, but we will associate this flux with the time average of the Poynting vector; the electric field associated with the sunlight is then $E_0 = 10^3$ V/m.

Our discussion of the time average of the Poynting vector provides an opportunity to discover one of the advantages of the use of complex notation. To obtain the time average of the product of two waves A and B , where

$$A = \mathcal{R}_e \{ \mathcal{A} \} = \mathcal{R}_e \{ A_0 e^{i(\omega t + \phi_1)} \}$$

$$B = \mathcal{R}_e \{ \mathcal{B} \} = \mathcal{R}_e \{ B_0 e^{i(\omega t + \phi_2)} \}$$

use (1B-6) to write the average over one period as

$$\langle AB \rangle = \frac{1}{T} \int_0^T \left(\frac{\mathcal{A} + \mathcal{A}^*}{2} \right) \left(\frac{\mathcal{B} + \mathcal{B}^*}{2} \right) dt$$

$$(\mathcal{A} + \mathcal{A}^*)(\mathcal{B} + \mathcal{B}^*) = \mathcal{A}\mathcal{B} + \mathcal{A}^*\mathcal{B}^* + \mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}$$

where

$$\mathcal{A}\mathcal{B} = A_0 B_0 e^{i(2\omega t + \phi_1 + \phi_2)}$$

and

$$\mathcal{A}^*\mathcal{B}^* = A_0 B_0 e^{-i(2\omega t + \phi_1 + \phi_2)}$$

The time averages of the latter two terms are zero and we are left with

$$\langle AB \rangle = \frac{1}{T} \int_0^T \frac{\mathcal{A}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}}{4} dt$$

From (1B-6), we may rewrite this as

$$\langle AB \rangle = \frac{1}{2} \mathcal{R}_e \{ \mathcal{A}\mathcal{B}^* \} \quad (2-28)$$

The reader may find this quite general relation easier to use than performing an integration such as (2-24).

MOMENTUM

The origin of momentum, associated with an electromagnetic wave, is easier to understand than is the source of momentum associated with an abstract wave (1-9). The electric field of the electromagnetic wave acts on a charged particle in the material with a force

$$\mathbf{F}_E = q\mathbf{E} \quad (2-29)$$

This force accelerates the charged particle to a velocity $\mathbf{v}$ in a direction transverse to the direction of light propagation and parallel to the electric field. The moving charges interact with the magnetic field of the electromagnetic wave with a force, parallel to the propagation vector, of

$$\mathbf{F}_H = q(\mathbf{v} \times \mathbf{B}) \quad (2-30)$$

The combined action of these two forces creates a radiation pressure.

Although the actual derivation will not be carried out here, it is possible, by the comparison of the sum of these two forces $\mathbf{F}_E + \mathbf{F}_H$ (called the *Lorentz force*) with results that can be derived from Maxwell's theory,³ to postulate a momentum density associated with the electromagnetic wave, given by

$$\mathbf{g} = \frac{\mathbf{S}}{c^2} \quad (2-31)$$

A dimensional analysis can be used to verify that (2-31) is a momentum density; the units of $\mathbf{g}$ are

$$\frac{\text{J}/(\text{m}^2 \cdot \text{sec})}{(\text{m}/\text{sec})^2} = \frac{\text{kgm} \cdot \text{m}/\text{sec}}{\text{m}^3}$$

The pressure on a surface of area A is defined as

$$P = \frac{\mathbf{F} \cdot \hat{\mathbf{n}}}{A} = \frac{\frac{\Delta \mathbf{p}}{\Delta t} \cdot \hat{\mathbf{n}}}{A}$$

We assume the light is totally absorbed, i.e., the momentum change is equal to the total momentum contained in the light wave, $\Delta \mathbf{p} = \mathbf{p}$. The total momentum in the light wave is given by the momentum density $\mathbf{g}$, multiplied by a unit volume V , $\mathbf{p} = \mathbf{g}V$.

$$P = \frac{\frac{V \mathbf{g}}{\Delta t} \cdot \hat{\mathbf{n}}}{A}$$

We will choose a volume $c \Delta t$ long with a cross-sectional area of A (see Figure 2-2), enabling the pressure to be expressed as

$$P = \frac{\left(\frac{\mathbf{g} \cdot \hat{\mathbf{n}}}{\Delta t} \right) c \cdot \Delta t \cdot A}{A} = \frac{\mathbf{S} \cdot \hat{\mathbf{n}}}{c} = \frac{I}{c} \quad (2-32)$$

At the Earth's surface and normal to it, sunlight has a flux density of $1.34 \times 10^3 \text{ J}/(\text{m}^2 \cdot \text{sec})$. We will again make the incorrect assumption that the flux density of sunlight is equal to the Poynting vector, allowing the use of (2-32) to estimate the pressure of sunlight to be

$$P = 4.46 \times 10^{-6} \frac{\text{N}}{\text{m}^2}$$

As a point of reference, atmospheric pressure is about $10^5 \text{ N}/\text{m}^2$.

Substituting (2-27) into (2-32), we discover that the radiation pressure is equal to the energy density of the incident radiation

$$P = \langle U \rangle$$

Combining (2-27) with (2-31) suggests that it is proper to associate momentum with the ratio of wave energy to velocity. This is consistent with relativistic principles. In the theory of relativity, the energy is given by

$$U = mc^2$$

which implies a mass of U/c^2 and a momentum of U/c . The idea is also consistent with quantum theory, where $U = h\nu$ so that

$$p = \frac{h}{\lambda} = \frac{h\nu}{c} = \frac{U}{c}$$

POLARIZATION

The displacement of a transverse wave is a vector quantity. We must therefore specify not only the frequency, phase, and direction of the wave but also the magnitude and direction of the displacement. The direction of the displacement vector is called the *direction of polarization* and the plane containing the direction of polarization and the propagation vector is called the *plane of polarization*. This quantity has the same name as the field quantity introduced in (2-5). Because the two terms describe completely different physical phenomena, there should be no danger of confusion.

From our study of Maxwell's equations, we know that $\mathbf{E}$ and $\mathbf{H}$, for a plane wave in free space, are mutually perpendicular and lie in a plane normal to the direction of propagation $\mathbf{k}$. We also know that, given one of the two vectors, we can use (2-17) to obtain the other. Convention requires that we use the electric vector to label the direction of the electromagnetic wave's polarization. The selection of the electric field is not completely arbitrary. From (2-29) and (2-30), we can write the ratio of the forces on a moving charge in an electromagnetic field due to the electric and magnetic fields as

$$\frac{F_E}{F_H} = \frac{eE}{evB}$$

We can replace B , using (2-19) to obtain

$$\frac{F_E}{F_H} = \frac{c}{nv} \quad (2-33)$$

where v is the velocity of the moving charge. Assume that a charged particle is traveling in air at the speed of sound so that $v = 335$ m/sec; then the force due to the electric field of a light wave on that particle would be 8.9×10^5 times larger than the force due to the magnetic field. The size of these numbers demonstrates that except in relativistic situations, when $v \approx c$, the interaction of the electromagnetic wave with matter will be dominated by the electric field.

A conventional vector notation is used to describe the polarization of a light wave; however, to visualize the behavior of the electric field vector as light propagates, a geometrical construction is useful. The geometrical construction, called a Lissajous' figure, describes the path followed by the tip of the electric field vector.

Polarization Ellipse

Assume that a plane wave is propagating in the z direction and the electric field, determining the direction of polarization, is oriented in the x, y plane. In complex notation, the plane wave is given by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi)} = \mathbf{E}_0 e^{i(\omega t - kz + \phi)}$$

This wave can be written in terms of the x and y components of $\mathbf{E}_0$

$$\mathbf{E} = E_{0x} e^{i(\omega t - kz + \phi_1)} \hat{\mathbf{i}} + E_{0y} e^{i(\omega t - kz + \phi_2)} \hat{\mathbf{j}} \quad (2-34)$$

(We will use only the real part of $\mathbf{E}$ for manipulation to prevent errors.) We divide each component of the electric field by its maximum value so that the problem is reduced to one of the following two sinusoidally varying unit vectors:

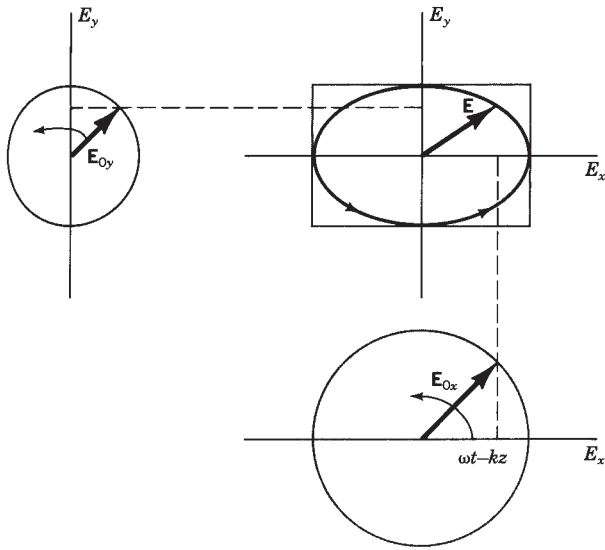


FIGURE 2-3. Geometrical construction showing how the Lissajous' figures are constructed from harmonic motion along the x and y coordinate axes. The harmonic motion along each coordinated axis is created by projecting a vector rotating around a circle onto the axis according to the technique discussed in Appendix 1B-1.

$$\frac{E_x}{E_{0x}} = \cos(\omega t - kz + \phi_1) = \cos(\omega t - kz) \cos \phi_1 - \sin(\omega t - kz) \sin \phi_1$$

$$\frac{E_y}{E_{0y}} = \cos(\omega t - kz) \cos \phi_2 - \sin(\omega t - kz) \sin \phi_2$$

When these unit vectors are added together, the result will be a set of figures called *Lissajous' figures*. The geometrical construction shown in Figure 2-3 can be used to visualize the generation of the Lissajous' figure. The harmonic motion along the x axis is found by projecting a vector rotating around a circle of diameter E_{0x} onto the x axis. The harmonic motion along the y axis is generated the same way using a circle of diameter E_{0y} . The resulting x and y components are added to obtain $\mathbf{E}$. In Figure 2-3, the two harmonic oscillators both have the same frequency $(\omega t - kz)$, but differ in phase by

$$\delta = \phi_2 - \phi_1 = -\frac{\pi}{2}$$

The tip of the electric field $\mathbf{E}$ in Figure 2-3 traces out an ellipse, with its axes aligned with the coordinate axes. To determine the direction of the rotation of the vector, assume that $\phi_1 = 0$, $\phi_2 = -\pi/2$, and $z = 0$ so that

$$\frac{E_x}{E_{0x}} = \cos \omega t \quad \frac{E_y}{E_{0y}} = \sin \omega t$$

$$\mathbf{E} = \left(\frac{E_x}{E_{0x}} \right) \hat{\mathbf{i}} + \left(\frac{E_y}{E_{0y}} \right) \hat{\mathbf{j}}$$

The normalized vector $\mathbf{E}$ can easily be evaluated at a number of values of ωt to discover the direction of rotation. Table 2.3 shows the value of the vector

TABLE 2.3 Rotating E-Field Vector

ωt	$\mathbf{E}$
0	$\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$
$\frac{\pi}{2}$	$\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}})$
π	$-\hat{\mathbf{i}}$

as ωt increases. The rotation of the vector $\mathbf{E}$ in Figure 2-3 is seen to be in a counterclockwise direction, moving from the positive x direction, to the y direction, and finally to the negative x direction.

To obtain the equation for the Lissajous' figure, we eliminate the dependence of the unit vectors on $(\omega t - kz)$. First, multiply the equations by $\sin \phi_2$ and $\sin \phi_1$, respectively, and then subtract the resulting equations. Second, multiply the two equations by $\cos \phi_2$ and $\cos \phi_1$, respectively, and then subtract the new equations. These two operations yield the following pair of equations:

$$\frac{E_x}{E_{0x}} \sin \phi_2 - \frac{E_y}{E_{0y}} \sin \phi_1 = \cos(\omega t - kz)(\cos \phi_1 \sin \phi_2 - \sin \phi_1 \cos \phi_2)$$

$$\frac{E_x}{E_{0x}} \cos \phi_2 - \frac{E_y}{E_{0y}} \cos \phi_1 = \sin(\omega t - kz)(\cos \phi_1 \sin \phi_2 - \sin \phi_1 \cos \phi_2)$$

The term in parens can be simplified using the trig identity

$$\sin \delta = \sin(\phi_2 - \phi_1) = \cos \phi_1 \sin \phi_2 - \sin \phi_1 \cos \phi_2$$

After replacing the term in parens by $\sin \delta$, the two equations are squared and added, yielding the equation for the Lissajous' figure

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - \left(\frac{2E_x E_y}{E_{0x} E_{0y}}\right) \cos \delta = \sin^2 \delta \quad (2-35)$$

The trig identity

$$\cos \delta = \cos(\phi_2 - \phi_1) = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2$$

was also used to further simplify (2-35).

Equation (2-35) has the same form as the equation of a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Geometry defines the conic as an ellipse because from (2-35),

$$B^2 - 4AC = \frac{4}{E_{0x}^2 E_{0y}^2} (\cos^2 \delta - 1) < 0$$

This ellipse is called the *polarization ellipse*. The orientation of the ellipse with respect to the x axis is

$$\tan 2\theta = \frac{B}{A - C} = \frac{2E_{0x} E_{0y} \cos \delta}{E_{0x}^2 - E_{0y}^2} \quad (2-36)$$

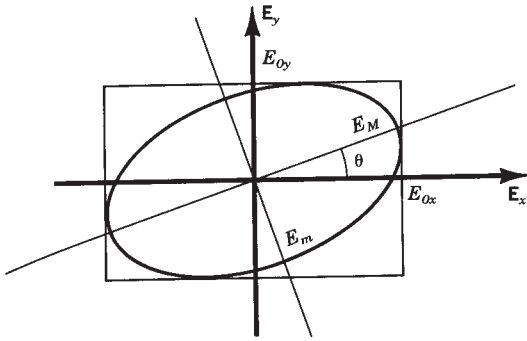


FIGURE 2-4. General form of the ellipse described by (2-35).

If $A = C$ and $B \neq 0$, then $\theta = 45^\circ$. When $\delta = \pm\pi/2$, then $\theta = 0^\circ$ as shown in Figure 2-3.

The tip of the resultant electric field vector obtained from (2-34) traces out the polarization ellipse in the plane normal to $\mathbf{k}$, as predicted by (2-35). A generalized polarization ellipse is shown in Figure 2-4. The x and y coordinates of the electric field are bounded by $\pm E_{0x}$ and $\pm E_{0y}$. The rectangle in Figure 2-4 illustrates those limits. The component of the electric field along the major axis of the ellipse is

$$E_M = E_x \cos \theta + E_y \sin \theta$$

and along the minor axis of the ellipse is

$$E_m = -E_x \sin \theta + E_y \cos \theta$$

where θ is obtained from (2-36). The ratio of the length of the minor to the major axis of the ellipse is equal to the ellipticity φ , i.e., the amount of deviation of the ellipse from a circle

$$\tan \varphi = \pm \left(\frac{E_m}{E_M} \right) = \frac{E_{0x} \sin \phi_1 \sin \theta - E_{0y} \sin \phi_2 \cos \theta}{E_{0x} \cos \phi_1 \cos \theta + E_{0y} \cos \phi_2 \sin \theta} \quad (2-37)$$

To find the time dependence of the vector $\mathbf{E}$, rewrite (2-34) in complex form

$$\mathbf{E} = e^{i(\omega t - kx)} (\hat{\mathbf{i}} E_{0x} e^{i\phi_1} + \hat{\mathbf{j}} E_{0y} e^{i\phi_2}) \quad (2-38)$$

This equation shows explicitly that the electric vector moves about the ellipse in a sinusoidal motion.

By specifying the parameters that characterize the polarization ellipse (θ and φ), we completely characterize a wave's polarization. A review of two special cases will aid in understanding the polarization ellipse.

Linear Polarization

First consider when $\delta = 0$ or π ; then (2-35) becomes

$$\left(\frac{E_x}{E_{0x}} \right)^2 + \left(\frac{E_y}{E_{0y}} \right)^2 \pm \left(\frac{2E_x E_y}{E_{0x} E_{0y}} \right) = 0$$

The ellipse collapses into a straight line with slope E_{0y}/E_{0x} . The equation of the straight line is

$$\frac{E_x}{E_{0x}} = \pm \frac{E_y}{E_{0y}}$$

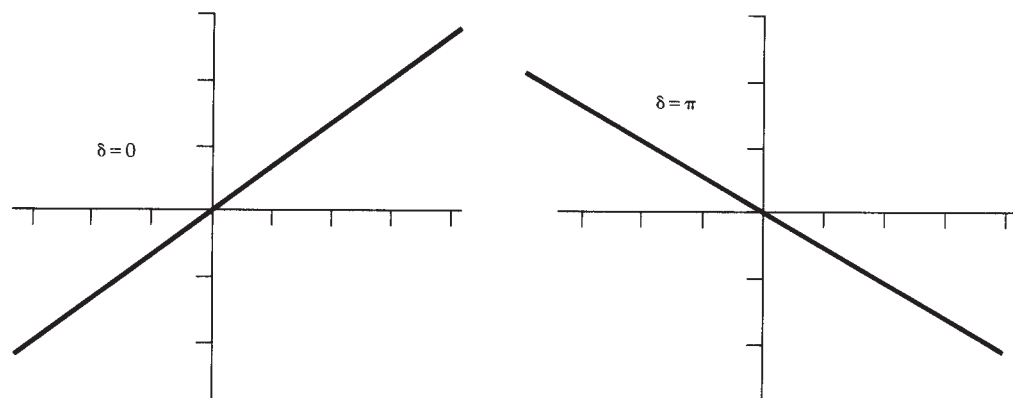


FIGURE 2-5. Lissajous' figures for phase differences between the y and x components of oscillation of 0 and π .

Figure 2-5 displays the straight-line Lissajous' figures for the two phase differences. The θ parameter of the ellipse is the slope of the straight line

$$\tan \theta = \frac{E_{0y}}{E_{0x}}$$

resulting in the value of (2-36) being given by

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}$$

The φ parameter is given by (2-37) as $\tan \varphi = 0$.

The time dependence of the $\mathbf{E}$ vector shown in Figure 2-5 is given by (2-38). The real component is

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}}) \cos(\omega t - kz)$$

At a fixed point in space, the x and y components oscillate in phase (or 180° out of phase) according to the equation

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}}) \cos(\omega t - \phi)$$

The electric vector undergoes simple harmonic motion along the line defined by E_{0x} and E_{0y} . At a fixed time, the electric field varies sinusoidally along the propagation path (the z axis) according to the equation

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}}) \cos(\phi - kz)$$

This light is said to be *linearly polarized*.

Circular Polarization

The second case occurs when $E_{0x} = E_{0y} = E_0$ and $\delta = \pm\pi/2$. From (2-35),

$$\left(\frac{E_x}{E_0}\right)^2 + \left(\frac{E_y}{E_0}\right)^2 = 1$$

The ellipse becomes a circle as shown in Figure 2-6. For this polarization, $\tan 2\theta$ is indeterminate and $\tan \varphi = 1$.

From (2-38), the temporal behavior is given by

$$\mathbf{E} = E_0[\cos(\omega t - kz)\hat{\mathbf{i}} \pm \sin(\omega t - kz)\hat{\mathbf{j}}]$$

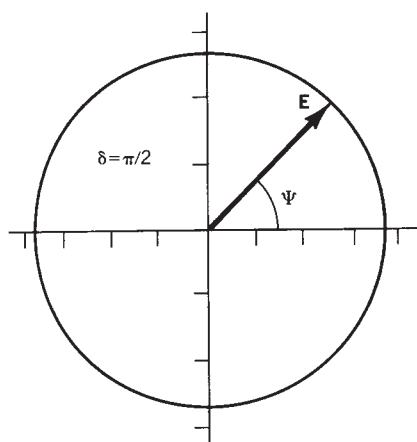


FIGURE 2-6. Lissajous' figures for the case when the phase difference between the y and x components of oscillation differ by $\pm(\pi/2)$ and the amplitudes of the two components are equal. The tip of the electric field vector shown moves along the circle.

The time dependence of the angle Ψ that the $\mathbf{E}$ field makes with the x axis in Figure 2-6 can be obtained by finding the tangent of the angle Ψ .

$$\tan \Psi = \frac{E_y}{E_x} = \pm \frac{\sin(\omega t - kz)}{\cos(\omega t - kz)} = \pm \tan(\omega t - kz)$$

The interpretation of this result is that at a fixed point in space, the $\mathbf{E}$ vector rotates in a clockwise direction if $\delta = \pi/2$ and a counterclockwise direction if $\delta = -\pi/2$.

In particle physics, the light would be said to have a negative helicity if it rotated in a clockwise direction. If we look at the source, the electric vector seems to follow the threads of a left-handed screw, agreeing with the nomenclature that left-handed quantities are negative. However, in optics the light that rotates clockwise as we view it traveling toward us from the source is said to be *right-circularly polarized*. The counterclockwise rotating light is *left-circularly polarized*.

The association of right-circularly polarized light with "right handedness" in optics came about by looking at the path of the electric vector in space at a fixed time; then, $\tan \Psi = \tan(\phi - kz)$. See Figure 2-7. As shown in Figure 2-7, right-circular polarized light at a fixed time seems to spiral in a counterclockwise fashion along the z direction, following the threads of a right-handed screw.

This motion can be generalized to include elliptical polarized light when $E_{0x} \neq E_{0y}$. Figure 2-3 schematically displays the generation of the Lissajous' figure for the case of $\delta = \pi/2$, but with unequal values of E_{0x} and E_{0y} . Figure 2-8 shows two calculated Lissajous' figures. If the electric vector moves around the ellipse in a clockwise direction, as we face the source, then the phase difference and ellipticity are

$$0 \leq \delta \leq \pi \quad \text{and} \quad 0 < \varphi < \frac{\pi}{4}$$

and the polarization is right-handed. If the motion of the electric vector is moving in a counterclockwise direction, then the phase difference and ellipticity are

$$-\pi \leq \delta \leq 0 \quad \text{and} \quad -\frac{\pi}{4} < \varphi < 0$$

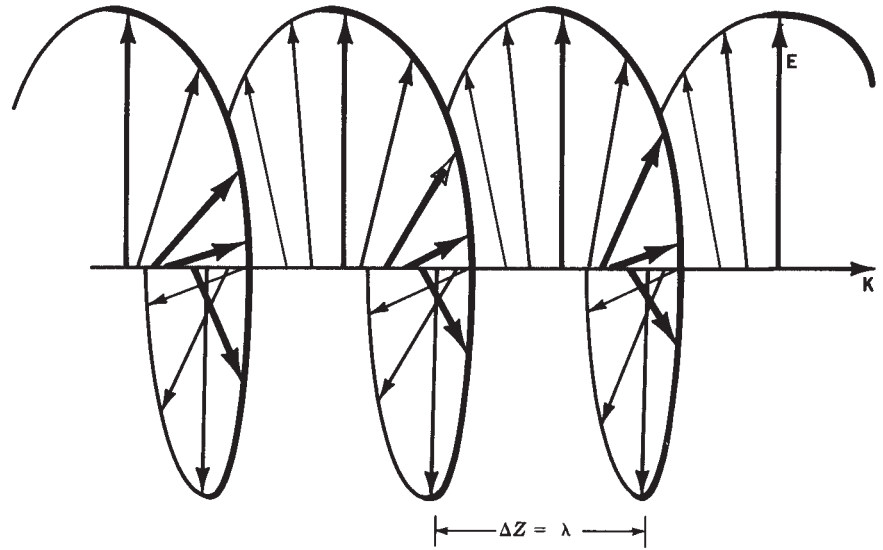


FIGURE 2-7. The path of the electric vector of right-circular polarized light at a fixed time.

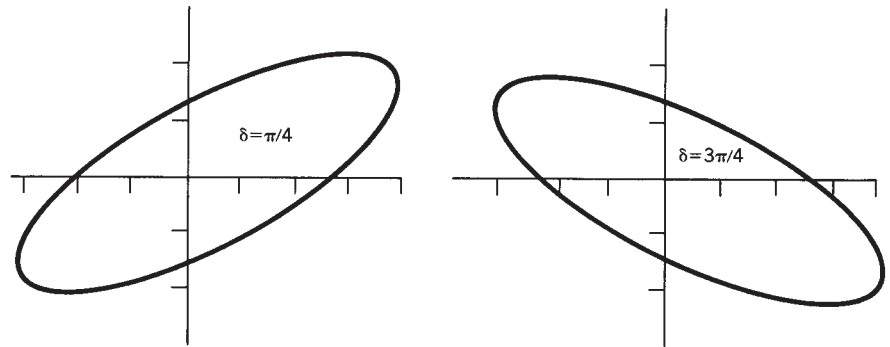


FIGURE 2-8. Lissajous' figures for elliptical polarized light. They were calculated with $E_{0x} = 0.75$ and $E_{0y} = 0.25$.

The orientation of either ellipse with respect to the x axis will be given by (2-36) and will depend upon the relative magnitudes of E_{0x} and E_{0y} .

The procedure used to decompose an arbitrary polarization into polarizations parallel to two axes of a Cartesian coordinate system is a technique used extensively in vector algebra to simplify mathematical calculations. According to the mathematical formalism associated with this technique, the polarization is described in terms of a set of basis vectors $\mathbf{e}_i$. An arbitrary polarization would be expressed as

$$\mathbf{E} = \sum_{i=1}^2 a_i \mathbf{e}_i \quad (2-39)$$

The set of basis vectors $\mathbf{e}_i$ is orthonormal, i.e.,

$$\mathbf{e}_i \mathbf{e}_j^* = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

where we have assumed that the basis vectors could be complex. We mention this mathematical formalism because an identical formalism is encountered in elementary particle physics in which it is used to describe spin.⁴

In a Cartesian coordinate system, the $\mathbf{e}_i$'s are the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The summation in (2-39) extends over only two terms because the electromagnetic wave is transverse, confining $\mathbf{E}$ to a plane normal to the direction of propagation (according to the coordinate convention we have selected, the $\mathbf{E}$ field is in the x, y plane).

The polarization could also be described in terms of a right-circularly polarized component

$$\mathbf{E}_{\mathcal{R}} = E_{0\mathcal{R}} [\hat{\mathbf{i}} \cos(\omega t - kz) - \hat{\mathbf{j}} \sin(\omega t - kz)]$$

and a left-circularly polarized component

$$\mathbf{E}_{\mathcal{L}} = E_{0\mathcal{L}} [\hat{\mathbf{i}} \cos(\omega t - kz) + \hat{\mathbf{j}} \sin(\omega t - kz)]$$

An arbitrary elliptical polarization would then be written as

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\mathcal{R}} + \mathbf{E}_{\mathcal{L}} \\ &= \hat{\mathbf{i}} (E_{0\mathcal{R}} + E_{0\mathcal{L}}) \cos(\omega t - kz) - \hat{\mathbf{j}} (E_{0\mathcal{R}} - E_{0\mathcal{L}}) \sin(\omega t - kz) \quad (2-40) \end{aligned}$$

The geometrical construction that demonstrates the expression of an arbitrary elliptical polarized light wave in terms of right and left circularly polarized waves is shown in Figure 2-9. The use of circular polarized waves as the basis set for describing polarization is discussed by Klein.⁵

In the formalism associated with (2-39), the expansion coefficients a_i can be used to form a 2-2 matrix, which in statistical mechanics is called the *density matrix* and in optics the *coherency matrix*.⁶ The elements of the matrix are formed by the rule

$$\rho_{ij} = \mathbf{a}_i \mathbf{a}_j^*$$

We will not develop the theory of polarization using the coherency matrix, but simply use the coherency matrix to justify the need for four independent measurements to characterize polarization. There is no unique set of measurements required by theory but normally measurements made are of the *Stokes parameters*, which are directly related to the polarization ellipse of Figure 2-4. (We will see in a few moments that only three of the four measurements are independent. This will be in agreement with the definition of the coherency matrix where $\rho_{ij} = \rho_{ji}^*$, i.e., the matrix is Hermitian.)

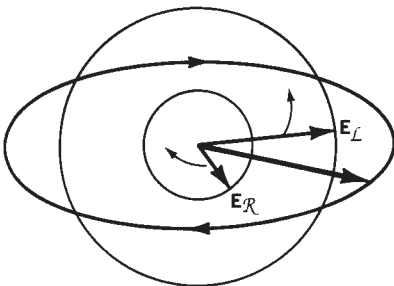


FIGURE 2-9. Construction of elliptical polarized light from two circularly polarized waves.

STOKES PARAMETERS

The Stokes parameters⁷ of a light wave are measurable quantities, defined as

$s_0 \rightarrow$ Total flux density.

$s_1 \rightarrow$ Difference between flux density transmitted by a linear polarizer oriented parallel to the x axis and one oriented parallel to the y axis. The x and y axes are usually selected to be parallel to the horizontal and vertical directions in the laboratory.

$s_2 \rightarrow$ Difference between flux density transmitted by a linear polarizer oriented at 45° to the x axis and one oriented at 135° .

$s_3 \rightarrow$ Difference between flux density transmitted by a right-circular polarizer and a left-circular polarizer.

The physical instruments that can be used to measure the Stokes parameters will be discussed in Chapter 13.

If the Stokes parameters are to characterize the polarization of a wave, they must be related to the parameters of the polarization ellipse. It is therefore important to establish that the Stokes parameters are variables of the polarization ellipse (2-35).

In its current form, (2-35) contains no measurable quantities and thus must be modified if it is to be associated with the Stokes parameters. In the discussion of the Poynting vector, it was pointed out that the time average of the Poynting vector is the quantity observed when measurements are made of light waves. We must, therefore, find the time average of (2-35) if we wish to relate its parameters to observable quantities. To simplify the discussion, assume that the amplitudes of the orthogonally polarized waves, E_{0x} and E_{0y} and their relative phase δ are constants. We will also use the shorthand notation for a time average introduced in (2-24)

$$\langle E_x^2 \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} E_{0x}^2 [\cos(\omega t - kz) \cos \phi_1 - \sin(\omega t - kz) \sin \phi_1]^2 dt$$

The time average of (2-35) can now be written

$$\frac{\langle E_x^2 \rangle}{E_{0x}^2} + \frac{\langle E_y^2 \rangle}{E_{0y}^2} - 2 \frac{\langle E_x E_y \rangle}{E_{0x} E_{0y}} \cos \delta = \sin^2 \delta \quad (2-41)$$

Multiplying both sides of (2-41) by $(2E_{0x}E_{0y})^2$ removes the terms in the denominators of (2-41)

$$4E_{0y}^2 \langle E_x^2 \rangle + 4E_{0x}^2 \langle E_y^2 \rangle - 8E_{0x}E_{0y} \langle E_x E_y \rangle \cos \delta = (2E_{0x}E_{0y} \sin \delta)^2$$

The same argument that was used to simplify (2-25) can be used to obtain the time averages for the first two terms

$$\langle E_x^2 \rangle = \frac{E_{0x}^2}{2}, \quad \langle E_y^2 \rangle = \frac{E_{0y}^2}{2}$$

The calculation of the time average in the third term

$$\langle E_x E_y \rangle = \frac{1}{2} E_{0x} E_{0y} \cos \delta \quad (2-42)$$

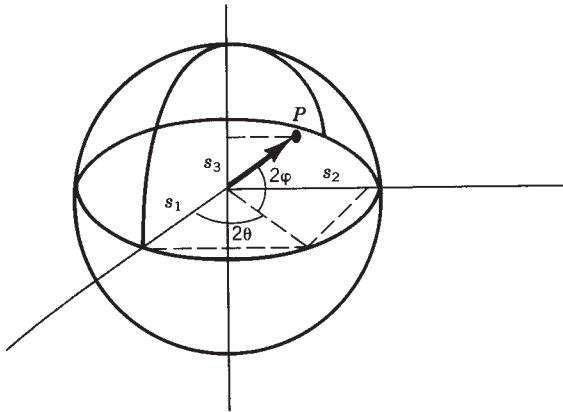


FIGURE 2-10. Poincaré's sphere.

is left as a problem, Problem 2-12. With these time averages, (2-41) can be written as

$$4E_{0x}^2 E_{0y}^2 - (2E_{0x}E_{0y} \cos \delta)^2 = (2E_{0x}E_{0y} \sin \delta)^2$$

If $E_{0x}^2 + E_{0y}^2$ is added to both sides of this equation, it can be rewritten

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y} \cos \delta)^2 = (2E_{0x}E_{0y} \sin \delta)^2 \quad (2-43)$$

Each term in this equation can be identified with a Stokes parameter.

In our derivation, we required that the amplitudes and relative phase of the two orthogonally polarized waves be a constant, but we can relax this requirement and instead define the Stokes parameters as temporal averages. With this modification, the terms of (2-43) become

$$\begin{aligned} s_0 &= \langle E_{0x}^2 \rangle + \langle E_{0y}^2 \rangle, & s_1 &= \langle E_{0x}^2 \rangle - \langle E_{0y}^2 \rangle \\ s_2 &= \langle 2E_{0x}E_{0y} \cos \delta \rangle, & s_3 &= \langle 2E_{0x}E_{0y} \sin \delta \rangle \end{aligned} \quad (2-44)$$

Equation (2-43) can now be written as

$$s_0^2 - s_1^2 - s_2^2 = s_3^2 \quad (2-45)$$

For a polarized wave, only three of the Stokes parameters are independent. This agrees with the requirement placed on elements of the Hermitian coherency matrix introduced above.

With this demonstration of the connection between the Stokes parameters and the polarization ellipse, the Stokes parameters can be written in terms of the parameters of the polarization ellipse in Figure 2-4.

$$\begin{aligned} s_1 &= s_0 \cos 2\varphi \cos 2\theta \\ s_2 &= s_0 \cos 2\varphi \sin 2\theta \\ s_3 &= s_0 \sin 2\varphi \end{aligned} \quad (2-46)$$

It is this close relationship between the Stokes parameters and the polarization ellipse that makes the Stokes parameters a useful characterization of polarization.

The Stokes parameters can be used to describe the *degree of polarization* defined as

Before it was discovered that the Stokes parameters could be treated as elements of a column matrix, a geometric construction was used to determine the effect of an anisotropic medium on polarized light. The parameters s_1, s_2, s_3 are viewed as the Cartesian coordinates of a point on a sphere of radius s_0 . This sphere is called the *Poincaré sphere*⁸ and is shown in Figure 2-10.

On the sphere, right-hand polarized light is represented by points on the upper half-surface. Linear polarization is represented by points on the equator. Circular polarization is represented by the poles. With the development of the matrix view of polarization, the usefulness of the Poincaré's sphere has decreased and it is now, for many people, only of historical interest.

$$V = \frac{1}{s_0} \sqrt{s_1^2 + s_2^2 + s_3^2} \quad (2-47)$$

[The equality of (2-45) applies to completely polarized light when $V = 1$.] The degree of polarization can be used to characterize any light source that is physically realizable. If the time averages used in the definition of the Stokes parameters s_2 and s_3 are zero,

$$\langle E_{0x}^2 \rangle = \langle E_{0y}^2 \rangle \quad \text{and} \quad s_0 = 2\langle E_{0x}^2 \rangle$$

then the light wave is said to be unpolarized and $V = 0$.

H. Mueller⁷ pointed out that the Stokes parameters can be thought of as elements of a column matrix or a 4-vector; see Table 2.4.

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

TABLE 2.4 Jones and Stokes Vectors

Horizontal Polarization	
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
Vertical Polarization	
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$
+ 45° Polarization	
$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
- 45° Polarization	
$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$
Right-Circular Polarization	
$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
Left-Circular Polarization	
$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

This view will allow us to follow a polarized wave through a series of optical devices through the use of matrix algebra as we will see later.

There is one other representation of polarized light, complementary to the Stokes parameters, developed by **R. Clark Jones** in 1941 and called the *Jones vector*. It is superior to the Stokes vector in that it handles light of a known phase and amplitude with a reduced number of parameters. It is inferior to the Stokes vector in that, unlike the Stokes representation, that is experimentally determined, the Jones representation cannot handle unpolarized or partially polarized light. The Jones vector is a theoretical construct that can only describe light with a well-defined phase and frequency. The density matrix formalism can be used to correct the shortcomings of the Jones vector, but then the simplicity of the Jones representation is lost.

JONES VECTOR

If we assume that the coordinate system is such that the electromagnetic wave is propagating along the z axis, it was shown earlier that any polarization could be decomposed into two orthogonal $\mathbf{E}$ vectors, say for this discussion, parallel to the x and y directions. The Jones vector is defined as a two-row, column matrix consisting of the complex components in the x and y direction

$$\mathbf{E} = \begin{bmatrix} E_{0x} \exp \{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_1)\} \\ E_{0y} \exp \{i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_2)\} \end{bmatrix} \quad (2-48)$$

If absolute phase is not an issue, then we may normalize the vector by dividing by that number (real or complex) that simplifies the components but keeps the sum of the square of the components equal to 1. For example,

$$\mathbf{E} = \frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_1)] \begin{bmatrix} 1 \\ \frac{E_{0y}}{E_{0x}} e^{i\delta} \end{bmatrix}$$

The normalized vector would be the terms contained within the bracket, each divided by $1/\sqrt{2}$ if $E_{0x} = E_{0y}$. The general form of the Jones vector is

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}, \quad \mathbf{E}^* = [\mathbf{A}^* \ \mathbf{B}^*]$$

Some examples of Jones vectors (on the left) and Stokes vectors (on the right) are shown in Table 2.4.

In Chapter 1, we discussed the propagation of a wave with attenuation. In our discussion of the propagation of light, however, we have ensured that we would experience no loss by assuming $\sigma = 0$. We now relax that assumption and allow $\sigma \neq 0$. Maxwell's equations become

PROPAGATION IN A CONDUCTING MEDIUM

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

We continue to neglect dynamic or resonant effects so that we may use the simple constitutive relations

$$\mathbf{J} = \sigma \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}$$

where ϵ , μ , and σ are independent of time. Maxwell's equations in a medium with dissipation can be rewritten using these constitutive relations as

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, & \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{H} &= \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}, & \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t}\end{aligned}\quad (2-49)$$

We now apply the same procedure used to derive the wave equation for free space

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \\ &= -\mu \frac{\partial}{\partial t} \left(\sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \right) \\ \nabla \times (\nabla \times \mathbf{E}) &= \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}\end{aligned}$$

yielding the wave equation in a conducting medium

$$\nabla^2 \mathbf{E} = \mu \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2-50)$$

This wave equation is of the same form as (1-19). We can derive a similar equation for the magnetic field

$$\nabla^2 \mathbf{B} = \mu \sigma \frac{\partial \mathbf{B}}{\partial t} + \mu \epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (2-51)$$

Equations (2-50) and (2-51) are called the *telegraph equations*. They are wave equations derived to explain the propagation of pulses on telegraph lines.

We see that the wave equation (2-50) contains a damping term $\partial \mathbf{E} / \partial t$ when we allow $\sigma \neq 0$. By comparing (2-50) to (1-19), we can state that the solutions of (2-50) will be electromagnetic waves that will experience attenuation proportional to $\mu \sigma$ as it propagates. Using (1-21) and (1-22), we may rewrite (2-49), for plane wave solutions, as

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \quad \nabla \times \mathbf{E} = -i\omega \mu \mathbf{H} \\ \nabla \times \mathbf{H} &= \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \nabla \times \mathbf{H} = i\omega \left(\epsilon - \frac{i\sigma}{\omega} \right) \mathbf{E}\end{aligned}\quad (2-52)$$

We rewrite (2-50) in terms of these expressions for the curl of $\mathbf{E}$ and $\mathbf{H}$

$$\nabla^2 \mathbf{E} + \omega^2 \mu \left(\epsilon - \frac{i\sigma}{\omega} \right) \mathbf{E} = 0 \quad (2-53)$$

This has the form of the Helmholtz equation (1-12) if we replace k^2 by the complex function

$$\zeta^2 = \omega^2 \mu \left(\epsilon - \frac{i\sigma}{\omega} \right) \quad (2-54)$$

We use the identity

$$k = \frac{n\omega}{c} = \omega \sqrt{\mu \epsilon}$$

to demonstrate that the equations for conducting media are identical to those derived for nonconducting media if the dielectric constant ϵ is replaced by a complex dielectric constant

$$\tilde{\epsilon} = \epsilon - i \left(\frac{\sigma}{\omega} \right) \quad (2-55)$$

This equation suggests that σ may contain a frequency dependence (in fact, in the cgs system, the units of σ are sec^{-1} ; for copper in cgs units, $\sigma = 5.14 \times 10^{17}/\text{sec}$). In solid-state physics, one finds that the mobility of the electrons creates a frequency dependence that shows up in σ .

Since we have replaced k by the complex quantity,

$$\kappa = \omega \sqrt{\mu \left(\epsilon - i \frac{\sigma}{\omega} \right)}$$

we must replace the index of refraction by a complex index. In the literature, this is accomplished in two ways

$$\mathcal{N} = n(1 - i\kappa) \quad (2-56)$$

$$\mathcal{N} = n_1 - in_2$$

We will use the notation displayed in (2-56).

To find out how the plane wave propagates in this conductive medium, we simply replace the propagation constant k by

$$\kappa = \mathcal{N} \frac{\omega}{c} = \left(\frac{\omega n}{c} \right) (1 - i\kappa)$$

as we did in Chapter 2. κ is called the *extinction coefficient* and $n\kappa$ the *absorption coefficient*.

If we assume $\mathbf{k}$ is parallel to the z axis, then the plane wave is

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i\omega t} \exp \{ -i\omega(n/c)(1 - i\kappa)z \} \\ \mathbf{E} &= \mathbf{E}_0 \exp \{ -(\omega n\kappa/c)z \} \exp \{ i\omega(t - nz/c) \} \end{aligned} \quad (2-57)$$

$$\Re \{ \mathbf{E} \} = \mathbf{E}_0 \exp \{ -(\omega n\kappa/c)z \} \cos(\omega t - kz) \quad (2-58)$$

The wave described by (2-58) is a plane wave, attenuated by the exponent

$$\exp \{ -(\omega/c)n\kappa z \} \quad (2-59)$$

Figure 2-11 displays the exponential decay of a light wave propagating in an absorbing medium.

In Figure 2-11, a layer of xylene floats on water containing the dye Rhodamine 6G in solution. The Rhodamine strongly absorbs a beam of blue light from a HeCd laser (442 nm). As can be seen in Figure 2-11, the blue light is rapidly attenuated once it enters the water. Some of the energy absorbed by the Rhodamine is reemitted at longer wavelengths. The reemitted light travels in all directions, as it has no memory of the direction traveled by the blue light. For this reason, the beam of light in the water appears diffuse and, as can be seen in the color insert, is orange in color.

To evaluate the absorption coefficient $n\kappa$ in terms of electromagnetic properties of the medium, we will derive a relationship between $n\kappa$ and σ . We rewrite (2-56) as

$$\mathcal{N}^2 = n^2(1 - \kappa^2 - 2i\kappa) = \frac{c^2}{\omega^2} \kappa^2$$

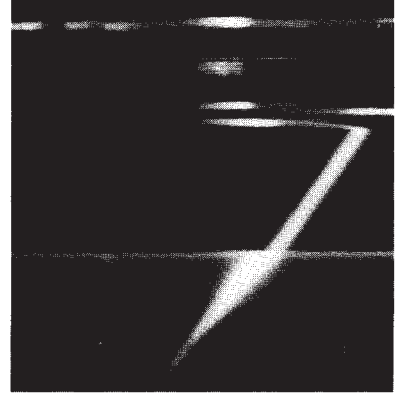


FIGURE 2-11. Blue laser light is shown propagating in xylene (above) and water (below). The water contains the dye Rhodamine 6G in solution. The red Rhodamine dye absorbs the blue light and the beam rapidly decays to zero. Some of the energy absorbed by the dye is reemitted in the yellow to red region of the spectrum. This reemitted light caused the diffuse appearance of the light as it propagates in the water. A color image of this figure can be seen in the insert.

Equation (2-54) can be used to express $\mathcal{N}^2$ in terms of the constants of the material

$$\mathcal{N}^2 = c^2 \mu \left(\epsilon - i \frac{\sigma}{\omega} \right) \quad (2-60)$$

Equating real and imaginary terms, we obtain

$$n^2(1 - \kappa^2) = c^2 \mu \epsilon, \quad 2n^2 \kappa = c^2 \frac{\mu \sigma}{\omega}$$

We can use these two relationships to find

$$n^2 = \frac{c^2}{2} \left[\sqrt{\mu^2 \epsilon^2 + \left(\frac{\mu \sigma}{\omega} \right)^2} + \mu \epsilon \right] \quad (2-61)$$

$$n^2 \kappa^2 = \frac{c^2}{2} \left[\sqrt{\mu^2 \epsilon^2 + \left(\frac{\mu \sigma}{\omega} \right)^2} - \mu \epsilon \right] \quad (2-62)$$

Note that when $\sigma = 0$, $\kappa = 0$, and we obtain the free space result (2-11)

$$n^2 = \frac{\mu \epsilon}{\mu_0 \epsilon_0}$$

An estimate of the magnitude of the quantities under the radicals in (2-61) and (2-62) can be obtained by using values for copper, where in the mks units $\sigma = 5.8 \times 10^7$ mhos/m and $n = 0.62$ at $\lambda = 589.3$ nm. (The index of refraction is less than 1, which implies that the phase velocity is greater than the speed of light. This apparent contradiction of a fundamental postulate of the theory of relativity will be discussed during the study of dispersion in Chapter 7.) The two terms under the radical are

$$\frac{\mu \sigma}{\omega} = \frac{[(4\pi \times 10^{-7}) (5.8 \times 10^7) (5.893 \times 10^{-7})]}{(2\pi) (3 \times 10^8)} = 2.3 \times 10^{-14} \frac{\text{sec}^2}{\text{m}^2}$$

$$\mu \epsilon = \mu_0 \epsilon_0 n^2 = (4\pi \times 10^{-7}) (8.8542 \times 10^{-12}) (0.62)^2 = 4.3 \times 10^{-18} \frac{\text{sec}^2}{\text{m}^2}$$

By comparing the relative magnitude of these two terms, we are justified in assuming that $\sigma/\omega \gg \epsilon$ and can make the approximation

$$\begin{aligned} n^2 \kappa^2 &\approx \frac{c^2 \mu \sigma}{2\omega} \\ n \kappa &= c \sqrt{\frac{\mu \sigma}{2\omega}} \end{aligned} \quad (2-63)$$

We use (2-63) to find the depth at which an electromagnetic wave is attenuated to $1/e$ of its original energy when propagating into a conductor. At that depth, denoted by d , the exponent in (2-59) will equal 1; thus,

$$\frac{\omega}{c} n \kappa d = \frac{2\pi}{\lambda_0} n \kappa d = 1$$

$$d = \frac{\lambda_0}{2\pi n \kappa} \approx \frac{\lambda_0}{2\pi c} \sqrt{\frac{2\omega}{\mu \sigma}}$$

$$d = \sqrt{\frac{2}{\mu \sigma \omega}} \quad (2-64)$$

TABLE 2.5 Skin Depth for Copper

λ_0	d
10^{-7} m	6.2×10^{-10} m
10^{-5}	6.2×10^{-9} m
10^{-1}	6.2×10^{-7} m
10^3	6.2×10^{-5} m

The depth d is called the *skin depth*. The skin depth for copper at a number of wavelengths is shown in Table 2.5.

In this chapter, Maxwell's equations were used to obtain the wave equation for free space

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and a conductive medium

$$\nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Although the form of the two equations appears quite different, we demonstrated that plane wave solutions existed for both equations when the dielectric constant of the conductive medium was replaced by a complex constant

$$\tilde{\epsilon} = \epsilon - i \left(\frac{\sigma}{\omega} \right)$$

This replacement means that the optical properties of a conductive material are described by a complex index of refraction

$$\mathcal{N} = n(1 - i\kappa)$$

The plane wave propagating in the conductive material has an amplitude attenuated by the exponent

$$e^{-(\omega/c)n\kappa z}$$

By manipulation of Maxwell's equations, we were able to show that the propagation velocity of a light wave is governed by the electrical properties of the medium. The index of refraction was used to indicate the propagation velocity in the medium, relative to the propagation velocity in a vacuum

$$n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

Also, we were able to demonstrate that the light wave must be a transverse wave and that the magnitude of the electric and magnetic fields is related by

$$n|\mathbf{E}| = c|\mathbf{B}|$$

By comparing the forces experienced by a charged particle in an electromagnetic field, we found that we could describe the polarization of an electromagnetic wave by the electric field vector. We developed the formalism necessary for discussion of polarization, but delayed a discussion of the manipulation of a light wave's polarization until Chapter 13.

SUMMARY

PROBLEMS

- 2-1. Light is traveling in glass ($n = 1.5$). If the amplitude of the electric field of the light is 100 V/m (volts/meter), what is the amplitude of the magnetic field? What is the magnitude of the Poynting vector?
- 2-2. A 60 W monochromatic point source is radiating equally in all directions in a vacuum. What is the electric field amplitude 2 m from the source?
- 2-3. The flux density at the Earth's surface due to sunlight is $I = 1.34 \times 10^3 \text{ J}/(\text{m}^2 \cdot \text{sec})$. Calculate the electric and magnetic fields at the Earth's surface by assuming that the average Poynting vector is equal to this flux density.
- 2-4. What is the flux density of light needed to keep a glass sphere of 10^{-8} g and $2 \times 10^{-5} \text{ m}$ in diameter floating in midair?
- 2-5. An 85 kg astronaut has only a flashlight to propel him in space. If the flashlight emits 1 W of light in a parallel beam for 1 hr, how fast will the astronaut be going at the end of the hour, assuming he started at rest?
- 2-6. What is the polarization of the following waves?

$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(\omega t - kz) + \hat{\mathbf{j}} \cos\left(\omega t - kz + \frac{5\pi}{4}\right) \right]$$

$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(\omega t + kz) + \hat{\mathbf{j}} \cos\left(\omega t + kz - \frac{\pi}{4}\right) \right]$$

$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(\omega t - kz) - \hat{\mathbf{j}} \cos\left(\omega t - kz + \frac{\pi}{6}\right) \right]$$

- 2-7. Show that the addition of two elliptical polarized waves propagating along the z axis results in another elliptical polarized wave.
- 2-8. Write an expression in mks units for a plane electromagnetic wave, with a wavelength of 500 nm and an intensity of $53.2 \text{ W}/\text{m}^2$, propagating in the z direction. Assume that the wave is linearly polarized at an angle of 45° to the x axis.
- 2-9. Using conventional vector notation, prove that a right- and a left-circularly polarized wave can combine to yield a linearly polarized wave. Carry out the same demonstration using the Jones vector notation. What requirement must be placed on the two circularly polarized waves? Sketch the geometrical construction that demonstrates the combination of circularly polarized waves to generate a linearly polarized wave.
- 2-10. Write the equation for a plane wave propagating in the positive z direction that has right-elliptical polarization with the major axis of the ellipse parallel to the x axis. Use both the conventional vector and the Jones vector notation.
- 2-11. Describe the polarization of a wave with the Jones vector

$$\begin{pmatrix} -i \\ 2 \end{pmatrix}$$

Write the Jones vector that is orthogonal to this vector and describe its polarization.

- 2-12. Prove that (2-42) is correct using the expressions

$$E_x = E_{0x} [\cos(\omega t - kz) \cos \phi_1 - \sin(\omega t - kz) \sin \phi_1]$$

$$E_y = E_{0y} [\cos(\omega t - kz) \cos \phi_2 - \sin(\omega t - kz) \sin \phi_2]$$

for the two orthogonally polarized electric fields.

- 2-13. Demonstrate using the Jones vector notation that right- and left-circularly polarized light waves are orthogonal.

- 2-14. Find the skin-depth for sea water with a resistivity of $\rho = 0.20 \Omega/\text{m}$ for $\nu = 30 \text{ KHz}$ and 30 MHz . What frequency should we use to communicate with a submarine that will not be deeper than 100 m ?
- 2-15. At what frequency would the approximation used to obtain (2-26) produce a 10% error?
- 2-16. If a 1 kw laser beam is focused to a spot with an area of 10^{-9} m^2 , what is the amplitude of the electric field at the focus?
- 2-17. The human eye is sensitive to light of wavelengths from approximately 600 nm (red) to 400 nm (blue). (a) Calculate the frequency of both wavelengths. (b) Find the energy of the photons associated with the red and blue wavelength limits.
- 2-18. If green light, 500 nm , could be frequency-modulated to 0.1% of the light wave's frequency, calculate the number of 6MHz bandwidth TV channels that could be carried by the modulation.
- 2-19. Given the following Stokes vector:

$$\begin{pmatrix} 1 \\ 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

- (a) Calculate the degree of polarization, (b) determine the orthogonal vector, and (c) draw the polarization ellipse.
- 2-20. How thin must a sheet of iron be if it is one skin-depth thick? How many atoms thick is such a sheet?

Appendix 2-A

VECTORS

We will review a few properties of vectors that will be of use in our discussion of light. A vector is a quantity with both magnitude and direction; it can be defined in terms of unit vectors along the three orthogonal axes of a Cartesian coordinate system

$$\mathbf{E} = \hat{\mathbf{i}}E_x + \hat{\mathbf{j}}E_y + \hat{\mathbf{k}}E_z$$

To add two vectors, we add like components

$$\mathbf{E}_1 + \mathbf{E}_2 = (E_{1x} + E_{2x})\hat{\mathbf{i}} + (E_{1y} + E_{2y})\hat{\mathbf{j}} + (E_{1z} + E_{2z})\hat{\mathbf{k}}$$

There are two ways to multiply vectors.

Scalar (Dot) Product

$$\mathbf{E} \cdot \mathbf{H} = EH \cos \theta = E_x H_x + E_y H_y + E_z H_z \quad (2A-1)$$

where θ is the angle between $\mathbf{E}$ and $\mathbf{H}$. This product is a scalar quantity that gives the projection of one vector onto the second vector. If $\mathbf{E}$ and $\mathbf{H}$ are perpendicular, then the dot product is zero.

Vector (Cross) Product

$$\mathbf{E} \times \mathbf{H} = (EH \sin \theta) \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane formed by $\mathbf{E}$ and $\mathbf{H}$. The cross product is a vector with a magnitude equal to the area of the parallelogram formed by $\mathbf{E}$ and $\mathbf{H}$; it is zero if the two vectors are parallel. The calculation of the components of this new vector is as follows:

$$\mathbf{E} \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$\mathbf{E} \times \mathbf{H} = (E_y H_z - E_z H_y) \hat{\mathbf{i}} - (E_x H_z - E_z H_x) \hat{\mathbf{j}} + (E_x H_y - E_y H_x) \hat{\mathbf{k}} \quad (2A-2)$$

We will find use for the vector triple-product relationship

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \quad (2A-3)$$

Derivatives

A vector operator called the *del operator* is defined in Cartesian notation as

$$\nabla = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \quad (2A-4)$$

We can treat the operator as a vector and calculate three products of use in optics.

Gradient of a Scalar

$$\nabla V = \hat{\mathbf{i}} \frac{\partial V}{\partial x} + \hat{\mathbf{j}} \frac{\partial V}{\partial y} + \hat{\mathbf{k}} \frac{\partial V}{\partial z} \quad (2A-5)$$

The gradient is a vector giving the magnitude and direction of the fastest rate of change of the scalar quantity V .

Divergence of a Vector

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (2A-6)$$

The divergence gives the amount of flux flowing toward (negative) or away from (positive) a point. If the divergence is zero, there are no sources or sinks in the volume.

Curl of a Vector

$$\nabla \times \mathbf{E} = \hat{\mathbf{i}} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \quad (2A-7)$$

A physical interpretation of this operation can be made easily if the vector is a velocity; then when the curl of the velocity is nonzero, rotation is also occurring.

If we calculate the divergence of a gradient of a scalar, we obtain the *Laplacian*

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad (2A-8)$$

for a scalar function and for a vector quantity

$$\nabla^2 \mathbf{E} = \hat{\mathbf{i}} \nabla^2 E_x + \hat{\mathbf{j}} \nabla^2 E_y + \hat{\mathbf{k}} \nabla^2 E_z \quad (2A-9)$$

Several other identities involving the del operator will come in handy in our study of optics

$$\nabla \times \nabla V = 0 \quad (2A-10)$$

$$\nabla \cdot \nabla \times \mathbf{E} = 0 \quad (2A-11)$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (2A-12)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\nabla \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{A}) \quad (2A-13)$$

We have displayed all of the above relations in a Cartesian coordinate system; similar expressions can be derived using spherical coordinates. We list them in Table 2A.1 for the reader's convenience.

TABLE 2A.1 Vector Operations in Spherical Coordinates

Operator	r comp	θ comp	ϕ comp
∇V	$\frac{\partial V}{\partial r}$	$\frac{1}{r} \left(\frac{\partial V}{\partial \theta} \right)$	$\frac{1}{r \sin \theta} \left(\frac{\partial V}{\partial \phi} \right)$
$\nabla \cdot \mathbf{E}$	$\frac{1}{r} \left(\frac{\partial r E_r}{\partial r} \right)$	$\frac{1}{r \sin \theta} \left(\frac{\partial [E_\theta \sin \theta]}{\partial \theta} \right)$	$\frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi}$
$\nabla^2 V$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right)$	$\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right)$	$\frac{1}{(r \sin \theta)^2} \frac{\partial^2 V}{\partial \phi^2}$
$(\nabla \times \mathbf{E})_r$	$\frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (E_\phi \sin \theta) - \frac{\partial E_\theta}{\partial \phi} \right]$		
$(\nabla \times \mathbf{E})_\theta$	$\frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \phi} - \frac{\partial}{\partial r} (r E_\phi) \right]$		
$(\nabla \times \mathbf{E})_\phi$	$\frac{1}{r} \left[\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right]$		

Appendix 2-B

Of all the topics in physics, the one that introduces the most confusion is the subject of electromagnetic units. In this book, we have used mks units, but the reader will find most older optics books use cgs units.

In the rationalized mks units, ϵ and μ in a vacuum have the following values:

$$\epsilon_0 = 8.8542 \times 10^{-12} \frac{\text{C}^2}{\text{Nm}^2} (\text{F/m})$$

$$\epsilon_0 \approx \left(\frac{1}{36\pi} \right) \times 10^{-9}$$

$$\mu_0 = 4\pi \times 10^{-7} \frac{\text{Ns}^2}{\text{C}^2} (\text{H/m})$$

In cgs units, $\epsilon_0 = \mu_0 = 1$.

Maxwell's equations in cgs units are written

$$\nabla \cdot \mathbf{D} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$$

A good discussion of the subject of units can be found in Jackson.³ Here, we provide Table 2B.1 to aid the reader in converting from one system to the other.

TABLE 2B.1 Electromagnetic Units

cgs	Units	mks	Units
c	$\frac{\text{cm}}{\text{sec}}$	$\frac{1}{\sqrt{\mu_0 \epsilon_0}}$	$\frac{\text{m}}{\text{sec}}$
D	$12\pi \times 10^{-5} \left(\frac{\text{statC}}{\text{cm}^2} \right)$	$\sqrt{4\pi \epsilon_0} \mathbf{D}$	$\frac{\text{C}}{\text{m}^2}$
B	10^4 G	$\sqrt{\frac{4\pi}{\mu_0}} \mathbf{B}$	$\frac{\text{weber}}{\text{m}^2}$

ELECTROMAGNETIC UNITS

TABLE 2B.1 (continued)

cgs	Units	mks	Units
H	$4\pi \times 10^{-3} \text{ O}$	$\sqrt{4\pi\mu_0}\mathbf{H}$	$\frac{\text{A-tum}}{\text{m}}$
E	$\frac{1}{3} \times 10^{-4} \frac{\text{statC}}{\text{cm}}$	$\sqrt{4\pi\epsilon_0}\mathbf{E}$	$\frac{\text{V}}{\text{m}}$
J	$3 \times 10^5 \frac{\text{statA}}{\text{cm}^2}$	$\frac{\mathbf{J}}{\sqrt{4\pi\epsilon_0}}$	$\frac{\text{A}}{\text{m}^2}$
σ	$\frac{9 \times 10^9}{\text{sec}}$	$\frac{\sigma}{4\pi\epsilon_0}$	$\frac{\text{mho}}{\text{m}}$
ρ	$3 \times 10^3 \frac{\text{statC}}{\text{cm}^3}$	$\frac{\rho}{\sqrt{4\pi\epsilon_0}}$	$\frac{\text{C}}{\text{m}^3}$
ϵ	ϵ_r	$\frac{\epsilon}{\epsilon_0}$	
μ	μ_r	$\frac{\mu}{\mu_0}$	