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DIFFERENTIAL AND INTEGRAL CALCULUS



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DIFFERENTIAL AND INTEGRAL CALCULUS

BY

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PREFACE

THIS book presents a first course in the calculus substantially as the author has taught it at the University of Michigan for a number of years. The following points may be mentioned as more or less prominent features of the book.

In the treatment of each topic, the text is intended to contain a precise statement of the fundamental principle involved, and to insure the student's clear understanding of this principle, without distracting his attention by the discussion of a multitude of details. The accompanying exercises are intended to present the problem in hand in a great variety of forms and guises, and to train the student in adapting the general methods of the text to fit these various forms. The constant aim is to prevent the work from degenerating into mere mechanical routine, as it so often tends to do. Wherever possible, except in the purely formal parts of the course, the summarizing of the theory into rules or formulas which can be applied blindly has been avoided. For instance, in the chapter on geometric applications of the definite integral, stress is laid on the fact that the basic formulas are those of elementary geometry, and special formulas involving a coördinate system are omitted.

Where the passage from theory to practice would be too difficult for the average student, worked examples are inserted.

It seems clear that so-called applications in which the student is made to use a formula without explanation of

its meaning and derivation, are of little value. In the present text the non-geometric applications are taken systematically from one subject, mechanics, and the theory is developed as fully as in the calculus proper.

A feature of the book is its insistence on the importance of checking the results of exercises, either directly or by solving in more than one way. The latter method is largely used in the integral calculus, on account of the variety of elementary transformations possible with definite integrals.

The answers to many of the exercises are given, but seldom where a knowledge of the answer would help in the solution, or where a simple means of checking the answer exists.

Topics of minor importance are presented in such a way that they may be omitted if it is desired to give a short course.

The chapter on curve tracing is introduced as early as possible, so that the results are available for use throughout the course.

Some instructors will wish to begin the use of integral tables immediately after the chapters on formal integration. This of course can easily be done.

In spite of obvious difficulties, a chapter embodying a first treatment of centroids and moments of inertia is introduced before multiple integrals have been defined. By this arrangement the student is brought to realize the fact that in most cases of practical importance mass-moments of the first and second orders can be found by simple integration, whereas from the usual treatment he gets exactly the opposite idea.

In the chapters on differential equations, emphasis is laid on those types most likely to be met by the student of engineering or the mathematical sciences. In the last chapter the average student will doubtless require con-

siderable help from the instructor, but it is hoped that, if properly presented, the chapter may give the student some facility in writing and solving the simpler differential equations of mechanics and in interpreting the results.

To Professor Alexander Ziwet, who has read the entire manuscript, the author makes grateful acknowledgment, not only for valuable advice and criticism, but for his unfailing encouragement and support. Thanks are also due to Professor T. H. Hildebrandt, who has kindly assisted in reading the proofs, and has made a number of useful suggestions.

CLYDE E. LOVE.

ANN ARBOR,
August, 1916.

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DIFFERENTIAL AND INTEGRAL CALCULUS

CHAPTER IV

GEOMETRIC APPLICATIONS

27. Tangents and normals to curves. It is known from analytic geometry that the equation of a line through the point (x_0, y_0) with slope m is

$$(1) \quad y - y_0 = m(x - x_0).$$

Let $P : (x_0, y_0)$ be a point on the curve

$$F(x, y) = 0,$$

and denote by y'_0 the value of the derivative at the point (x_0, y_0) . The equation of the tangent to the curve at P can then be written, by (1), in the form

$$y - y_0 = y'_0(x - x_0).$$

The equation of the normal — i.e. the line perpendicular to the tangent at the point of contact — can be found at once from that of the tangent, by recalling that if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

Examples: (a) Find the tangent and normal to the ellipse

$$4x^2 + 9y^2 = 40$$

at the point $(1, -2)$.

We have

$$8x + 18yy' = 0,$$

hence

$$m = y'_0 = -\frac{4x}{9y} \Big|_{(1, -2)} = \frac{2}{9}.$$

Therefore the equation of the tangent is

$$y + 2 = \frac{2}{9}(x - 1),$$

that of the normal is

$$y + 2 = -\frac{9}{2}(x - 1).$$

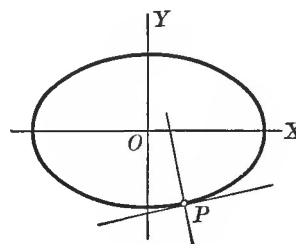


FIG. 6

(b) Find the equation of a tangent to the curve $y = x^3$ parallel to the line $y = 3x + 1$.

The slope of the required tangent is 3. But the slope at any point (x, y) of the curve $y = x^3$ is

$$y' = 3x^2.$$

Hence the coördinates of the point of contact are found by solving the simultaneous equations

$$3x^2 = 3, \quad y = x^3.$$

This gives the points $(1, 1)$, $(-1, -1)$, and the required tangents are

$$y - 1 = 3(x - 1), \quad y + 1 = 3(x + 1).$$

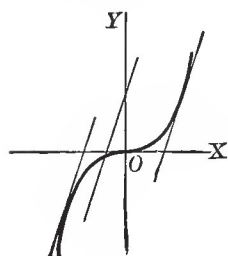


FIG. 7

28. Length of tangent, subtangent, normal, and subnormal. Let $P : (x, y)$ be a point on the curve

$$F(x, y) = 0.$$

The segment TP of the tangent intercepted between the point of tangency and the x -axis is called the *length of the tangent*; its projection TQ on OX is called the *length of the subtangent*. The segment NP of the normal intercepted between P and the x -axis is called the *length of the normal*; its projection QN on the x -axis is called the *length of the subnormal*.

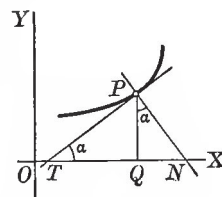


FIG. 8

It is customary to consider all these lengths as essentially positive. They are evidently determined by the coördinates (x, y) of the point P and the slope at P .

EXERCISES

Find the tangent and normal to each of the following curves at the points indicated.

1. (a) $y = 1 - x - x^2$ at $(1, -1)$; (b) $xy = 2a^2$ at $(a, 2a)$;
(c) $x^2 + y^2 = 25$ at $(-3, 4)$; (d) $y = x + \frac{1}{x}$ at $(1, 2)$.

Ans. (a) $y + 1 = -3(x - 1)$, $y + 1 = \frac{1}{3}(x - 1)$.

2. $x^2 - 2xy + 2y^2 - x = 0$ at the points where $x = 1$.

Ans. At $(1, 0)$:

$$2y = x - 1, y + 2x = 2; \text{ at } (1, 1): 2y = x + 1, y + 2x = 3.$$

3. $y = \frac{8a^3}{x^2 + 4a^2}$ at $x = 2a$. *Ans.* $x + 2y = 4a, y = 2x - 3a$.

4. Find the equation of the tangent to

(a) $y^2 = 4ax$ at (x_0, y_0) ; (b) $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ at (x_0, y_0) .

Ans. (a) $y_0y = 2a(x + x_0)$; (b) $\frac{x_0x}{a^2} \pm \frac{y_0y}{b^2} = 1$.

5. Find the subtangent, subnormal, tangent, and normal lengths in each of the cases of Ex. 1. Draw a figure in each case.

6. Find the angle between the parabolas $y^2 = x, y = x^2$ at each of their points of intersection.

7. Find the tangent and normal to the curve $y^2 = 2x^2 - x^3$ at the points $x = 1$.

Ans. At $(1, 1)$:

$$2y = x + 1, y + 2x = 3; \text{ at } (1, -1): x + 2y + 1 = 0, y = 2x - 3.$$

8. Show that the subtangent to the parabola $y^2 = 4ax$ is bisected at the vertex, and that the subnormal is constant. Hence give a geometric construction for drawing the tangent and normal; also show how to find the focus of a parabola if the axis is given.

9. Find a tangent to the parabola $y^2 = 4ax$ making an angle of 45° with the x -axis.

Ans. $y = x + a$.

10. Find the tangents to the hyperbola $4x^2 - 9y^2 + 36 = 0$ perpendicular to the line $2y + 5x = 10$.

Ans. $2x - 5y = \pm 8$.

11. Find a tangent to the curve $y = 1 - x^4$ parallel to the line $x - 2y = 5$.

12. Find a normal to the parabola $y = x^2$ perpendicular to the line $3x - 2y = 1$.

13. Show that the portion of the tangent to the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the axes is constant.

14. Find the tangents to the circle $x^2 + y^2 = 5$ which are parallel to the line $x + 3y = 0$. Draw the figure.

Ans. $y \pm \frac{3}{2}\sqrt{2} = -\frac{1}{2}(x \pm \frac{1}{2}\sqrt{2})$.

15. Find the tangents to the curve

$$y = \frac{1}{4}x^4 - x^3 + 5x$$

which make an angle of 45° with the x -axis. Plot the curve.

16. Find the angle between the line $y = -2x$ and the curve $y = x^2(1 - x)$ at each point of intersection.

17. Find the equation of a tangent to the curve $y^3 = 1 - x$ parallel to the y -axis. Trace the curve.

18. Show that the area of the triangle formed by the coördinate axes and the tangent to the hyperbola $2xy = a^2$ is constant.

19. Show that the length of the normal is constant (equal to a) in the circle

$$(x - c)^2 + y^2 = a^2,$$

where c is arbitrary, and explain geometrically.

20. Show that the sum of the intercepts on the axes of the tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is constant.

21. Show that, in the curve $y^n = ax$, the subtangent is n times the abscissa of the point of contact. Hence show how to draw the tangent at any point of the curve $y = ax^n$.

22. Find the length of the tangent, subtangent, normal, and subnormal to the curve $y = f(x)$ at the point (x, y) . *Ans.* Tangent,

$$\frac{y}{y'}\sqrt{1 + y'^2}; \text{ subtangent, } \frac{y}{y'}; \text{ normal, } y\sqrt{1 + y'^2}; \text{ subnormal, } yy'.$$

29. **Increasing and decreasing functions.** In studying the properties of a function

$$y = f(x),$$

it is usually of great assistance to represent the function graphically. In tracing a curve, it is well to begin by locating several points, *e.g.* the intersections with the axes, and finding the slope at those points; it is also useful to note the behavior of y for large positive and negative values of x .

In addition to giving the slope at any point, the differential calculus is of assistance in a variety of other ways, as will be shown in the next few articles.

We shall assume as usual that the function in question is one-valued, continuous, and differentiable.

We note first that, as x increases, *the curve rises if the slope is positive*, as on the arc AB (Fig. 9); *it falls if the slope is negative*, as along BD :

If $y' > 0$, y increases;

If $y' < 0$, y decreases.

Of course this also appears at once from the fact that y' is the rate of change of y .

30. Maxima and minima. At a point such as B (Fig. 9), where the function is algebraically greater than at any neighboring point, the function is said to have a *maximum value*, and the point is called a *maximum point*. Similarly, at a point such as D the function has a *minimum value*. It is evident that at such a point the tangent is parallel to OX ; i.e.

$$y' = 0.$$

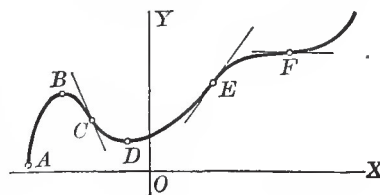


FIG. 9

But the vanishing of the derivative does not mean that the function is necessarily a maximum or a minimum; the tangent is parallel to OX at F , yet the function is neither a maximum nor a minimum there. It appears from the figure that the test is as follows:

At a point where $y' = 0$, if y' changes from positive to negative (as x increases), y is a maximum; if y' changes from negative to positive, y is a minimum; if y' does not change sign, y is neither a maximum nor a minimum.

Since the function is continuous, the maxima and minima must alternate: between two maxima there must be a minimum, and vice versa.

The points at which $y' = 0$ are called *critical points*, and the corresponding values of x are the *critical values* of x .

31. Concavity. The second derivative is the rate of change of the first derivative. It follows that when y'' is positive, y' is increasing: as x increases the tangent turns in counterclockwise sense and the curve is *concave upward*.

When y'' is negative, y' decreases: the curve is *concave downward*.

At a maximum point the curve is concave downward, and hence y'' , if it is not 0, must be negative. At a minimum y'' , if not 0, must be positive. If the second derivative is easily obtained and if it does not happen to be 0 at the critical point in question, it is usually more convenient to determine whether we have a maximum or a minimum by finding the sign of y'' ; but the test of § 30 has the advantage of being perfectly general. However, in practice other considerations usually enable us to distinguish between maxima and minima *without the application of either of these tests*.

Example: Find the maximum and minimum values of the function

$$y = x^3 - 3x,$$

and trace the curve.

This curve crosses the x -axis at $x = 0, \pm\sqrt{3}$. Since

$$y' = 3x^2 - 3,$$

the slope at $(0, 0)$ is -3 , at $(\pm\sqrt{3}, 0)$ it is 6.

Setting

$$y' = 0,$$

we find the critical points $(-1, 2), (1, -2)$. When x is large and negative, y is large and negative; when x is large and positive, y is large and positive. It is therefore

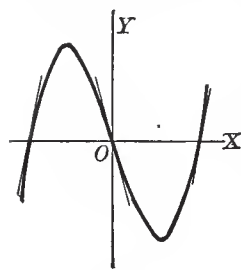


FIG. 10

clear that the curve must rise to a maximum at $(-1, 2)$, fall to a minimum at $(1, -2)$, and then rise indefinitely. These conclusions may be verified as follows. The second derivative,

$$y'' = 6x,$$

is negative at $x = -1$, so that the point $(-1, 2)$ is a maximum; y'' is

positive at $x = 1$, hence $(1, -2)$ is a minimum. The curve is shown in the figure.

EXERCISES

Examine the following functions for maxima and minima, and trace the curves.

- | | |
|----------------------------------|-----------------------------------|
| 1. $y = x(x + 5)$. | 2. $y = x^3 - 2x^2 + x$. |
| 3. $y = x^2(x^2 - 8)$. | 4. $y = (x^2 - 4)^2$. |
| 5. $y = x^4 + 1$. | 6. $y = x^3$. |
| 7. $y = \frac{a^3}{a^2 + x^2}$. | 8. $y = \frac{a^2x}{a^2 + x^2}$. |
| 9. $s = \frac{t^2}{1 + t^2}$. | 10. $x = (t + 1)^3$. |

32. Points of inflection. A point at which the curve changes from concave upward to concave downward, or vice versa, is called a *point of inflection*. At a point of inflection the tangent reverses the sense in which it turns, so that y'' changes sign. Hence at such a point y'' , if it is continuous, must vanish. In Fig. 9 the points C, E, F are points of inflection.

Since y'' —i.e. the rate of change of the slope—vanishes at a point of inflection, the tangent is sometimes said to be *stationary for an instant* at such a point, and in the neighborhood of the point it turns very slowly. Hence the inflectional tangent agrees more closely with the curve near its point of contact than does an ordinary tangent; it is therefore especially useful in tracing the curve to *draw the tangent at each point of inflection*.

33. Summary of tests for maxima and minima, etc. The results of §§ 29–32 may be summarized as follows:

Let the function $y = f(x)$ and its first and second derivatives be one-valued and continuous.

(a) In an interval where $y' > 0$, the curve rises; where $y' < 0$, the curve falls.

(b) A point where $y' = 0$ is a maximum or a minimum point, unless at the same point $y'' = 0$, in which case see (c)

below. If $y' > 0$ at the left* of this point and $y' < 0$ at the right, y is a maximum; if $y' < 0$ at the left and $y' > 0$ at the right, y is a minimum. Or, if $y'' < 0$ at the point, y is a maximum; if $y'' > 0$, y is a minimum.

(c) In an interval where $y'' > 0$, the curve is concave upward; where $y'' < 0$, the curve is concave downward.

(d) A point at which $y'' = 0$ is a point of inflection, provided y'' changes sign as the curve passes through the point.

(e) A point at which both $y' = 0$ and $y'' = 0$ is a maximum or a minimum if y' changes sign as the curve passes through the point; it is a point of inflection with a horizontal tangent if y' does not change sign.

Example: Trace the curve

$$y = x(x-1)^3.$$

This curve crosses the x -axis at $(0, 0)$, $(1, 0)$. For large positive or negative values of x , y is large and positive. The derivatives are

$$y' = (x-1)^3 + 3x(x-1)^2 = (x-1)^2(4x-1),$$

$$y'' = 2(x-1)(4x-1) + 4(x-1)^2 = 6(x-1)(2x-1).$$

The slope at $(0, 0)$ is -1 ; at $(1, 0)$ it is 0 .

The critical values are $x = 1, \frac{1}{4}$. When $x = \frac{1}{4}$, y'' is positive; hence $(\frac{1}{4}, -\frac{27}{56})$ is a minimum point. When

$x = 1$, $y'' = 0$ and the test by the second derivative fails. Since y' does not change sign as x passes through 1 , the function has neither a maximum nor a minimum at that point.

The second derivative vanishes at $(\frac{1}{2}, -\frac{1}{16})$, $(1, 0)$; these are points of inflection. The slope at $(\frac{1}{2}, -\frac{1}{16})$ is $\frac{1}{4}$.

The curve is shown in the figure.



FIG. 11

* That is, immediately at the left.

EXERCISES

Trace the following curves. Where possible, find the points of intersection with the axes, determine the behavior of y for large values of x , find the maxima and minima and points of inflection, and draw the tangent at each point of inflection.

- | | |
|--------------------------------|------------------------------------|
| 1. $y = x^3 - 6x^2 + 9x + 3.$ | 2. $y = 4 + 3x - x^2.$ |
| 3. $y = x^3 - 3x^2 + 6x + 10.$ | 4. $y = (x - 3)^2(x - 2).$ |
| 5. $y = (1 - x^2)^3.$ | 6. $y = (4 - x^2)^2.$ |
| 7. $y = (x - 1)^3(x + 2)^2.$ | 8. $y = x^3 - 3x^2 - 9x + 5.$ |
| 9. $y = x^4.$ | 10. $y = x^5.$ |
| 11. $y = x(x - 1)(x - 2).$ | 12. $y = \frac{8a^3}{x^2 + 4a^2}.$ |
| 13. $y = \frac{1}{1 + x^4}.$ | 14. $y = \frac{1}{(1 + x^2)^2}.$ |

15. Show that the curve $y = \frac{a^2x}{x^2 + a^2}$ has three points of inflection lying on a straight line. Trace the curve.

16. Show that, for the curve $y = x^n$, where $n = 2, 3, 4, \dots$, the origin is a minimum or a point of inflection according as n is even or odd.

34. Applications of maxima and minima. The theory of maxima and minima finds application in a great variety of problems. In the applications it is rarely necessary to use either of the tests of § 33 to distinguish between maxima and minima; the critical value that gives the desired result can usually be selected by inspection of the conditions of the problem.

It frequently happens that the function to be tested for maxima or minima can be most simply expressed in terms of two variables. When this is done, a relation between the two variables must be found from the conditions of the problem. By means of this relation one of the variables can be eliminated, after which the maxima and minima can be found in the usual way. However, it is often more convenient not to eliminate, but to proceed as in example (b) below.

Examples: (a) A box is to be made of a piece of cardboard 4 in. square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Let x be the length of the side of each of the squares cut out. Then the volume of the box is

$$(1) \quad V = x(4 - 2x)^2.$$

The derivative is

$$\begin{aligned} \frac{dV}{dx} &= (4 - 2x)^2 - 4x(4 - 2x) \\ &= (4 - 2x)(4 - 6x). \end{aligned}$$

Setting

$$\frac{dV}{dx} = 0,$$

we find

$$x = 2 \text{ or } \frac{2}{3}.$$

Since V vanishes when $x = 0$ and again when $x = 2$, it must reach a maximum at some intermediate point; it therefore follows without the application of further tests that the critical value $x = \frac{2}{3}$ gives the required maximum volume:

$$V_{\max.} = \frac{2}{3}(4 - \frac{4}{3})^2 = \frac{128}{27} \text{ cu. in.}$$

The graph of equation (1) is shown in the figure, the V -scale being one fourth as large as the x -scale. Since by the conditions of the problem x is restricted to values between 0 and 2, the dotted portions of the curve have no meaning in the present case.

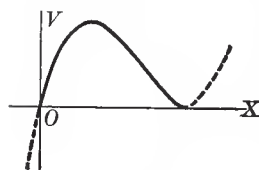


FIG. 12

(b) Find the dimensions of the largest rectangle that can be inscribed in a given circle.

Take the coördinate axes parallel to the sides of the rectangle. The area of the rectangle is

$$(2) \quad A = 4xy.$$

This can be expressed in terms of x (or y) by means of the relation

$$(3) \quad x^2 + y^2 = a^2,$$

which gives

$$y = \sqrt{a^2 - x^2},$$

$$A = 4x\sqrt{a^2 - x^2}.$$

From this point the method is the same as in (a).

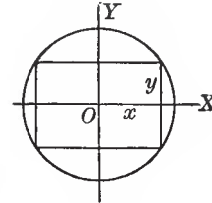


FIG. 13

The problem can be solved without eliminating x or y , as follows: Differentiating equation (2) with respect to x and setting the derivative equal to 0, we have, since y is a function of x ,

$$\frac{dA}{dx} = 4\left(x\frac{dy}{dx} + y\right) = 0,$$

or

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Differentiating equation (3), we get

$$2x + 2y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Equating values of $\frac{dy}{dx}$, we find

$$-\frac{y}{x} = -\frac{x}{y},$$

whence

$$y = x;$$

the maximum rectangle is a square.

EXERCISES

1. What is the largest rectangular area that can be inclosed by 800 yd. of fencing?

2. For a rectangle of given area, what shape has the minimum perimeter?

3. Find the most economical proportions for a cylindrical tin cup of given volume. *Ans.* Radius = height.

4. A rectangular field is to be fenced off along the bank of a straight river. If no fence is needed along the river, what is the shape of the rectangle requiring the least amount of fencing?

5. The equal sides of an isosceles triangle are 10 in. long. Find the length of the base if the area is a maximum.

6. Find the rectangle of maximum perimeter inscribed in a given circle.

7. Find the most economical proportions for a box with an open top and a square base.

8. Find the most economical proportions for a covered box whose base is a rectangle with one side twice the other.

Ans. Altitude = $\frac{4}{3} \times$ shorter side.

9. Find the dimensions of the largest right circular cylinder that can be inscribed in a given sphere. *Ans.* Diameter = $\sqrt{2} \times$ height.

10. In Ex. 9, find the form of the cylinder if its convex surface is a maximum.

11. Find the dimensions of the largest rectangle that can be inscribed in a given right triangle. *Ans.* $x = \frac{1}{2} a$.

12. Find the most economical proportions for a conical tent of given capacity. *Ans.* $h = \sqrt{2} r$.

13. A man in a rowboat 6 miles from shore desires to reach a point on the shore at a distance of 10 miles from his present position. If he can walk 4 miles per hour and row 3 miles per hour, where should he land in order to reach his destination in the shortest possible time?

Ans. 1.2 miles from his destination.

14. A rectangular field of given area is to be inclosed, and divided into two lots by a parallel to one of the sides. What must be the shape of the field if the amount of fencing is to be a minimum?

15. A Norman window consists of a rectangle surmounted by a semicircle. What shape gives the most light for a given perimeter?

Ans. Breadth = height.

16. Find the most economical proportions for a quart can.

Ans. Diameter = length.

17. The strength of a rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of given diameter.

Ans. Depth = $\sqrt{2} \times$ breadth.

18. Find the volume of the largest box that can be made by cutting equal squares out of the corners of a piece of cardboard 6×16 in. and turning up the sides. *Ans.* $1\frac{690}{27}$ cu. in.

19. A gutter is to be made of a strip of tin 12 in. wide, the cross section having the form shown in the figure. What depth gives a maximum carrying capacity?

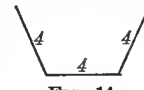


FIG. 14

20. Find the most economical proportions for a cylindrical cup of given capacity, if the bottom is to be three times as thick as the sides.

21. Find the most economical proportions for an A-tent of given volume, whose sides slope at 45° to the horizontal.

22. Find the dimensions of the largest right circular cylinder that can be inscribed in a given right circular cone. *Ans.* Altitude = $\frac{1}{3}h$.

23. Solve Ex. 22 if the convex surface of the cylinder is to be a maximum.

24. Find the right circular cone of maximum volume inscribed in a given sphere.

25. Find the cone of minimum volume circumscribed about a given sphere.

26. The base of a box is a rectangle with one side twice the other. The top and front are to be made of oak, the remainder of pine. If oak is twice as valuable as pine, find the most economical proportions.

27. A wall tent 12×16 ft. is to contain a given volume. Find the most economical proportions. *Ans.* Height above eaves = $2\sqrt{3}$ ft.

28. An oil can is made in the shape of a cylinder surmounted by a cone. If the radius of the cone is three fourths of its height, find the most economical proportions.

Ans. Altitude of cylinder = altitude of cone.

29. A cupboard 5 ft. high and having shelves 1 ft. apart is to be made from a given amount of material. If a front, but no back, is required, what shape gives the greatest amount of shelf room?

Ans. Width = twice depth.

30. A silo is made in the form of a cylinder, with a hemispherical roof; there is a floor of the same thickness as the wall and roof. Find the most economical shape. *Ans.* Diameter = total height.

31. Solve Ex. 30 if the floor is twice as thick as the wall and roof.

Ans. Height of cylinder = diameter.

32. The cost of erecting an office building is \$100,000 for the first floor, \$105,000 for the second, \$110,000 for the third, etc.; other expenses (lot, plans, excavation, etc.) are \$700,000. The net annual income is \$10,000 for each story. How high should the building be, to return the maximum rate of interest on the investment?

Ans. 17 stories.

35. **Derived curves.** The curves $y = f'(x)$, $y = f''(x)$, $y = f'''(x)$, ... are called the *first, second, third, ... derived curves*, corresponding to the curve $y = f(x)$.

The relations between the given function and its first and second derivatives, which have been formulated analytically in § 33, are well brought out graphically by drawing the original curve and its first and second derived curves. This is shown in Fig. 15 for the curve

$$y = \frac{x^3}{3} - x^2.$$

The work should be arranged with the several axes of ordinates lying in the same vertical line.

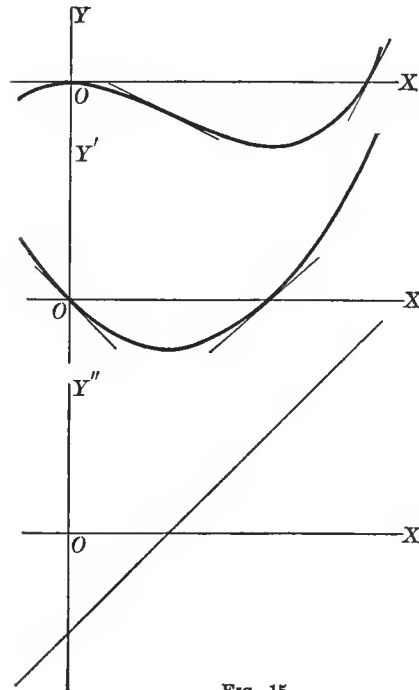


FIG. 15

It often happens in practical work that a function is defined in such a way—for instance by experimental data

— that no mathematical expression for it is known. If then we wish to examine the behavior of one of the derivatives, as is frequently the case, we plot the graph of the original function from the given data, and construct the required derived curve graphically. The process is obvious. We can measure the slope at any point of the original curve; the number thus obtained is the ordinate of the corresponding point on the first derived curve. In this way as many points as desired may be plotted on the first derived curve and a smooth curve drawn through them, after which the second derived curve may be constructed in a similar way; etc.

EXERCISES

1. What can be said of the first derived curve *ordinate is*
 - (a) in an interval where the original curve is $\begin{cases} \text{rising} \\ \text{falling} \end{cases}$ $\begin{cases} + \\ - \end{cases}$
 - (b) in an interval where the original curve is concave $\begin{cases} \text{upward} \\ \text{downward} \end{cases}$ $\begin{cases} ? \\ ? \end{cases}$
 - (c) at a point where the original curve has a $\begin{cases} \text{maximum} \\ \text{minimum} \end{cases}$ *reverse*
 - (d) at a point where the original curve has a point of inflection? *on a*
2. What can be said of the second derived curve
 - (a) in an interval where the original curve is concave $\begin{cases} \text{upward} \\ \text{downward} \end{cases}$ $\begin{cases} + \\ - \end{cases}$
 - (b) at a point where the original curve has a point of inflection? *0*
 - (c) in an interval where the first derived curve is $\begin{cases} \text{rising} \\ \text{falling} \end{cases}$ $\begin{cases} + \\ - \end{cases}$
 - (d) at a point where the first derived curve has a $\begin{cases} \text{maximum} \\ \text{minimum} \end{cases}$ $\begin{cases} - \\ + \end{cases}$
3. What can be said of the original curve at a point where the second derived curve touches the x -axis without crossing it? *concave*
4. Plot the curve $y = \frac{x^4}{4} - x^3 + x^2$ and its first, second, and third derived curves.
5. Plot the curve $y = \sin x$, and construct the first derived curve. What well-known curve does the latter resemble?

6. Draw a smooth curve, on a large scale, through the points

x	-4	-2	0	2	4	6	8	10	12	14	16	18	20	22	24
y	0	-1	$-\frac{3}{2}$	0	6	10	3	0	-1	-1	$-\frac{1}{2}$	1	9	20	35

and construct the first and second derived curves.

7. The national debt of the United States at the indicated dates is given in the accompanying table, the unit being \$100,000,000. Construct the curve showing the rate at which the debt has increased or decreased.

Date . . .	1850	'55	'60	'61	'62	'63	'64	'65	'68	'70
Debt . . .	0.6	0.4	0.6	0.9	5.0	11.1	17.1	26.8	24.8	23.3
Date . . .	'75	'80	'85	'90	'95	'99	1900	'05	'10	
Debt . . .	20.9	19.2	13.8	8.9	9.0	11.6	11.1	9.9	10.5	

CHAPTER VII

CURVATURE

52. Differential of arc. Let s denote the length of the arc of the plane curve

$$y = f(x)$$

counted from some initial point P_0 up to the point $P: (x, y)$, and suppose for definiteness that s increases as x increases. The arc s can be regarded as a function of x . Its derivative $\frac{ds}{dx}$ may be found as follows :

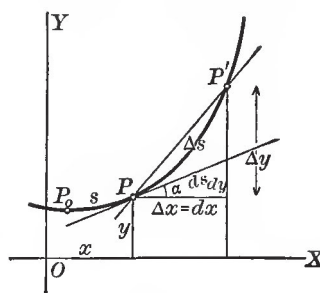


FIG. 27

$$\begin{aligned} \frac{\Delta s}{\Delta x} &= \frac{\Delta s}{PP'} \cdot \frac{PP'}{\Delta x} = \frac{\Delta s}{PP'} \cdot \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} \\ &= \frac{\Delta s}{PP'} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}, \end{aligned}$$

where Δs is the length of the arc, PP' the length of the chord, from $P: (x, y)$ to $P': (x + \Delta x, y + \Delta y)$. Since, as pointed out in § 38,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta s}{PP'} = 1,$$

we have

$$(1) \quad \frac{ds}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

If s increases as x decreases, then

$$\frac{\Delta s}{\Delta x} = - \frac{\Delta s}{PP'} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

and

$$(2) \quad \frac{ds}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

After squaring and clearing of fractions, equation (1) (or (2)), becomes

$$ds^2 = dx^2 + dy^2;$$

i.e. ds is the hypotenuse of the right triangle whose sides are dx and dy .

If the tangent to the curve at P makes an angle α with OX , then

$$\cos \alpha = \frac{dx}{ds}, \quad \sin \alpha = \frac{dy}{ds}.$$

53. Curvature. We say in ordinary language that a curve whose direction changes rapidly has great *curvature*, or is sharply curved. Thus a circular arc is said to have greater curvature when the radius is small than when it is large. This somewhat vague idea may be made precise as follows.

Consider, first, two points P, P' on a circle, and denote the arc PP' by Δs , the angle between the tangents at P, P' by $\Delta \alpha$. The quotient $\frac{\Delta \alpha}{\Delta s}$ is evidently the change in the direction of the curve, per unit of arc*; it is called the *curvature* of the circle.

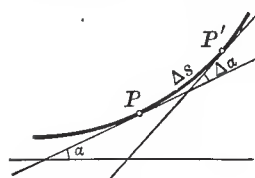


FIG. 28

If now the curve in question is not a circle, the direction of the curve no longer changes uniformly, and the quotient $\frac{\Delta \alpha}{\Delta s}$ represents

merely the *average curvature* of the arc Δs . But if P' be made to approach P along the curve, so that Δs and $\Delta \alpha$ approach 0, the quantity $\frac{\Delta \alpha}{\Delta s}$ in general approaches a limit $\frac{d\alpha}{ds}$, which

* It is easily seen that, in the case of the circle, this quotient is constant.

is called the *curvature at the point P* :

$$\kappa = \lim_{\Delta s \rightarrow 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}.$$

The definition is of course independent of the particular coördinate system used; the angle α is the angle made by the tangent at P with *any* fixed line in the plane of the curve. When the equation of the curve is given in cartesian coördinates, it is convenient to take α as the slope-angle of the tangent — *i.e.* the angle between the tangent and the x -axis. The curvature κ is then easily expressed in terms of the coördinates. For,

$$\begin{aligned}\tan \alpha &= \frac{dy}{dx} = y', \\ \alpha &= \arctan y', \\ d\alpha &= \frac{dy'}{1 + y'^2} = \frac{y'' dx}{1 + y'^2}.\end{aligned}$$

Also, by § 52,

$$ds = \sqrt{1 + y'^2} dx.$$

Hence

$$(1) \quad \kappa = \frac{d\alpha}{ds} = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}.$$

It is customary to consider κ as essentially positive, so that, strictly speaking, we should write

$$\kappa = \left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}},$$

where the symbol $|a|$ means the absolute or numerical value of a .

It should be noted that when $y' = 0$, formula (1) reduces to

$$\kappa = y''.$$

Thus the value of the second derivative at any point is equal to the curvature at that point when the coördinate axes are so chosen that the first derivative is 0.

54. Radius of curvature. The reciprocal of the curvature is called the *radius of curvature*, and is denoted by ρ :

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\alpha} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}.$$

This quantity is also to be considered as essentially positive.

If a length equal to the radius of curvature ρ at the point P be laid off on the normal from P toward the con-

cave side of the curve, the extremity Q of this segment is called the *center of curvature*. It can be shown that the circle with radius ρ and center Q represents the curve near P more closely than any other circle. This circle is called the *osculating circle*, or *circle of curvature*.

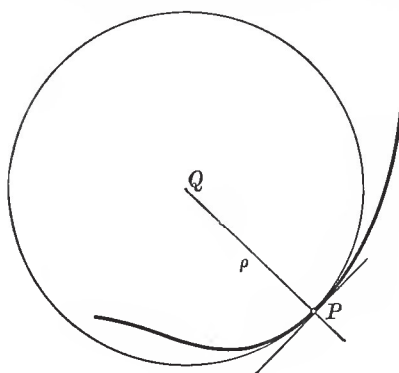


FIG. 29

In general, the circle of curvature crosses the curve at P , as is the case in Fig. 29.

EXERCISES

1. Show that the curvature of a straight line is everywhere 0.
2. Show that the radius of curvature of a circle is the radius of the circle.

Find the radius of curvature of the following curves.

3. $y = x^2$ (a) at any point; (b) at the vertex.

$$4. \quad y^2 = 4ax. \quad \text{Ans.} \quad \frac{(4a^2 + y^2)^{\frac{3}{2}}}{4a^2}.$$

$$5. \quad \text{The equilateral hyperbola } 2xy = a^2 \text{ at } (a, \frac{1}{2}a). \quad \text{Ans.} \quad \frac{5}{8}\sqrt{5}a.$$

$$6. \quad y = x^3 + 5x^2 + 6x \text{ at } (0, 0). \quad \text{Ans.} \quad 22.51.$$

$$7. \quad \text{The ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{Ans.} \quad \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

8. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
9. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $3(axy)^{\frac{1}{3}}$.
10. The ellipse $x = a \cos \phi$, $y = b \sin \phi$.
11. The curve $x = t^2$, $y = 1 - t^4$.
12. The curve $x = 3t^2$, $y = 3t - t^3$. *Ans.* $\frac{3}{2}(1 + t^2)^2$.
13. The catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ at the point $(0, a)$. *Ans.* a .
14. Show that the curvature at a point of inflection is 0.
15. Find the point of maximum curvature on the curve $y = e^x$.
Ans. $(-0.347, 0.707)$.
16. At what points of the curve $y = x^3$ is the curvature greatest?
17. Plot the parabola $x^2 = 4y$ accurately, on a large scale, in the interval from $x = -\frac{1}{2}$ to $x = \frac{3}{2}$, and draw the osculating circles at the points $x = 0$, $x = \frac{1}{2}$, $x = 1$.

$$k = \frac{\frac{d^2 y}{dx^2}}{\sec^3 \alpha}$$