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Applied Strength of Materials for Engineering Technology

Barry Dupen

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Table of Contents

Preface....3
Purpose of the Book....3
Editors....4
Cover Photos....5
Terminology....6
Definitions....9
Chapter 1: Introduction to Strength of Materials....11
What is Strength of Materials?....11
The Factor-Label Method of Unit Conversion....12
Chapter 2: Stress and Strain....17
Normal Stress and Strain....17
Sign Convention....19
Shear Stress and Strain....20
Chapter 3: Poisson's Ratio and Thermal Expansion....23
Poisson's Ratio....23
Thermal Expansion and Thermal Stress....25
Chapter 4: Pressure Vessels and Stress Concentrations....28
Thin-Walled Pressure Vessels....28
Stress Concentration in Tension....30
Chapter 5: Bolted and Welded Joints....33
Bolted Lap Joints Loaded in Tension....33
Welded Lap Joints....38
Chapter 6: Properties of Areas....41
Dimensions and Area....41
Centroid and Centroidal Axes....41
Moment of Inertia of a Rectangle....41
Compound Beams Sharing a Centroidal Axis....42
Hollow Beams Sharing a Centroidal Axis....43
The Transfer Formula....44
Compound Beams With Different Neutral Axes....45
Hollow Beams With Different Neutral Axes....48
Moment of Inertia about the y-y Neutral Axis....51
Shortcuts....54
Radius of Gyration....54
Polar Moment of Inertia....54
Chapter 7: Torsion in Round Shafts....55
Shear Stress in a Round Shaft....55
Angle of Twist in a Round Shaft....57
Stress Concentration in Torsion....58
Chapter 8: Beam Reactions, Shear Diagrams, and Moment Diagrams....60
Loads on Beams....60
Reactions for Simply-Supported Simple Beams....61
Reactions for Overhanging and Cantilever Beams....64
Shear Diagrams....66
Moment Diagrams....72
Chapter 9: Stresses in Beams....82
Bending Stress in Beams....82
Bending Stress in Wide-Flange Steel Beams....84
Shear Stress in Beams....87
Allowable Load....92
Chapter 10: Beam Deflection....94
Radius of Curvature....94
The Formula Method for Simple Cases....95
Formula Method Hints....98
The Formula Method for Complex Cases: Superposition....98
Visualizing the Deflection Curve....100
Chapter 11: Beam Design....102
Wide-Flange Steel Beam Design in Six Easy Steps....102
Timber Beam Design in Six Easy Steps....108
All Other Beams....110
Chapter 12: Combined Stresses....112
Tension + Bending....112
Bending in Two Directions....112
Eccentric Loading....114
Chapter 13: Statically Indeterminate Beams....118
Defining Determinate and Indeterminate Beams....118
Method of Superposition....118
Chapter 14: Buckling of Columns....124
Types of Columns....124
Ideal Slender Columns....124
Structural Steel Columns....126
Steel Machine Parts....127
Chapter 15: Visualizing Stress and Strain....130
Measuring Stress....130
Stress at the Base of a Short Block....130
Mohr's Circle....131
Bibliography....148
Textbooks....148
Other Reading Material....148
Appendix A: Units....149
SI System of Units....149
US Customary System of Units....149
Appendix B: Materials Properties....150
Metals, Concrete, & Stone....150
Appendix C: Properties of Areas....154
Center of Gravity, Area, Moment of Inertia, and Radius of Gyration....154
Appendix D: Properties of Steel Beams and Pipes....157
W-beams....157
Steel Pipes....161
Copper Tubing....162
Appendix E: Mechanical and Dimensional Properties of Wood....163
Mechanical Properties of Air-Dried Boards and Timber....163
Softwood Lumber and Timber Sizes....164
Appendix F: Beam Equations....166
Index....171
Index....171

Preface

Purpose of the Book

1.9 million bachelors degrees are awarded annually in the US.[1] About 92 thousand are Engineering degrees, and about 17 thousand are Engineering Technology degrees and Technician degrees. The number of Mechanical, Civil, and Construction Engineering Technology graduates is only about 2 thousand per year, so the market for algebra-based *Strength of Materials* textbooks for Engineering Technology is a small fraction of the market for calculus-based Engineering textbooks.

Since I attended college in the 1980s, textbook prices have risen about twice as fast as inflation. The internet did not exist back then, so all textbooks were printed. Now we have another option: low-cost or free online e-books which are revised more frequently than printed books. While traditional textbooks are revised every 4 to 10 years based on input from experts in the topic, this e-book is revised every semester based on input from experts in learning: the students.

Students complain that the explanations in many Engineering Technology textbooks are too theoretical, too wordy, and too difficult to understand. They also complain about the lack of complete unit conversions in example problems, and inconsistent use of symbols between related courses. For example, some authors use s_n, s_s, and e for normal stress, shear stress, and strain, instead of the standard Greek symbols σ, τ, and ε. This use of Latin characters with multiple subscripts confuses students because the Greek symbols are used in other textbooks, and because capital S is used for section modulus later in the course. Students have trouble distinguishing between s and S on the chalkboard and in their notes.

Professors complain that too many students copy answers from online solution manuals or fraternity homework files instead of learning to solve problems from scratch, then these students fail exams. Probably 10% of the learning in *Strength of Materials* occurs in class, and 90% occurs as students solve problems. Deliberately, the problem set for this book is not available online, and is changed every semester.

I teach *Strength of Materials* to Mechanical and Construction Engineering Technology students. These students tell me they want help with algebra skills, unit conversions, and problem-solving approaches. The problem set that accompanies this book contains problems requiring an algebraic answer as well as traditional problems requiring a numerical answer. The *Factor-Label Method of Unit Conversion* is emphasized from the first chapter, and is used in *all* example problems.

Summarizing, the goals of this book are:

- Free distribution over the internet
- Frequent revisions based on student input
- Concise explanations
- Examples with complete unit conversions
- Standard Greek symbols for stress and strain
- Problems requiring algebraic answers as well as problems requiring numerical answers
- Problems requiring answers in sentences to show reasoning and understanding of the topics

This e-book is revised on an ongoing basis. Please send suggestions for improvement to me at dupenb@ipfw.edu.

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August, 2016

1 Data from 2013-2014. Current numbers are in the Digest of Educational Statistics, published by the National Center for Educational Statistics, U.S. Department of Education, at nces.ed.gov.

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Cover Photos

Cover photos by the author. Geodesic greenhouses at The Eden Project, Bodelva, England (2000); barn ceiling at Somerset Rural Life Museum, Glastonbury, England (14th century); interior of a tourist kiosk near Squamish, British Columbia; 8 mile long Confederation Bridge between New Brunswick and Prince Edward Island (1997); interior of Fitch's covered bridge, Delhi, New York (1870); Menai suspenion bridge, Menai Bridge, Wales (1826).

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Terminology

Symbols used in this book, with typical units

Because the Roman and Greek alphabets contain a finite number of letters, symbols are recycled and used for more than one term. Check the context of the equation to figure out what the unit means in that equation.

Other science and engineering disciplines use different symbols for common terms. For example, *P* is used for point load here; in Physics classes, *F* is commonly used for point load. Some older *Strength of Materials* texts use μ for Poisson's ratio, *s* for stress, and *e* for strain; the formulas are the same, but the labels differ.

Symbol	Term	U.S. Units	SI Units
α	Thermal expansion coefficient	$°F^{-1}$	$°C^{-1}$
γ	Shear strain	…	…
$\boldsymbol{\gamma}$	Specific weight	lb./in.^{3}, lb./ft.^{3}	N/m^3
δ	Change in dimension (length, diameter, etc.)	in.	mm
Δ	Change	…	…
Δ	Beam deflection	in.	mm
ε	Strain	…	…
η_{joint}	Joint efficiency	%	%
ν	Poisson's ratio	…	…
ρ	Density	slug/ft.^{3}	kg/m^3
σ	Normal (perpendicular) stress	psi, ksi	MPa
τ	Shear (parallel) stress	psi, ksi	MPa
θ	Angle of twist	(radians)	(radians)
A, a	Area	in.^{2}	mm^2, m^2
A'	Term in the General Shear Formula	in.^{2}	mm^2, m^2
b	Base dimension of a rectangle	in.	mm
c	Torsion problem: distance from centroid to outer surface Beam problem: distance from neutral axis to outer surface	in.	mm
d	Diameter	in.	mm
d	Transfer distance	in.	mm
d_i, d_o	Inside and outside diameters of a pipe	in.	mm
d_H	Hole diameter	in.	mm
e	Eccentricity	in.	mm
E	Young's modulus (a.k.a. modulus of elasticity)	psi, ksi	MPa
F.S.	Factor of safety	…	…
G	Shear modulus (a.k.a. modulus of rigidity)	psi, ksi	MPa
h	Height dimension of a rectangle	in.	mm
h	Fillet weld throat	in.	mm
I	Moment of inertia	in.^{4}	mm^4
J	Polar moment of inertia	in.^{4}	mm^4
K	Stress concentration factor	…	…
K	Effective length factor (in column analysis)	…	…
l	Fillet weld leg	in.	mm
L	Length (of a tension member or a beam)	ft.	m
L	Total weld length	in.	cm
M	Moment	lb.·ft., kip·ft.	kN·m

Terminology

Symbol	Definition	U.S. Units	SI Units
n	Number of shear planes	…	…
N	Number of bolts	…	…
N_F	Number of holes in the fracture plane	…	…
p	Fluid pressure	psi, ksi	kPa
P	Point load	lb., kip	kN
P_{cr}	Euler critical buckling load	lb., kip	kN
P_G	Bolt load – gross tensile failure of the plate	lb., kip	kN
P_N	Bolt load – net tensile failure of the plate	lb., kip	kN
P_P	Bolt load – bearing failure of the plate	lb., kip	kN
P_S	Bolt load – bolt shear failure	lb., kip	kN
P_{weld}	Weld load (lapped plates loaded in tension)	lb., kip	kN
Q	Term in the General Shear Formula	$in.^3$	mm^3
r	Radius (of a hole, fillet, or groove)	in.	mm
r_G	Radius of gyration	in.	mm
R	Reaction force	lb., kip	kN
R	Radius of curvature	in., ft.	mm, m
S	Section modulus	$in.^3$	mm^3
t	Thickness	in.	mm
T	Torque	lb.·ft., kip·ft.	kN·m
T	Temperature	°F	°C
V	Shear load	lb., kip	kN
w	Distributed load (weight per unit length)	lb./ft., kip/ft.	kN/m
W	Weight	lb., kip	kN
x, y, z	Axes in three-dimensional space: x is horizontal, y is vertical, and z is into the page.	…	…
x	Distance along the x-axis	in., ft.	mm, m
y	Distance along the y-axis, such as the distance from the neutral axis in beam problems	in., ft.	mm, m
$\overline{y}$	Distance from the reference axis to the x-x axis of a composite shape [moment of inertia problems]	in.	mm
$\overline{y}$	Term in the General Shear Formula	in.	mm
z	Distance along the z-axis	in., ft.	mm, m
Z	Plastic section modulus	$in.^3$	mm^3

Terminology

Greek Letters

Upper case	Lower case	Name	Upper case	Lower case	Name
Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ε	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

Definitions

Allowable (stress, load, etc.).....Permitted for safe design.
Bending moment, M .....Moment in a beam that is loaded in bending with transverse loads.
Bending stress, σ .....A normal stress along the length of a beam that develops due to transverse loading.
Buckling.....Collapse of a long, thin member under longitudinal compressive loading, at a load much lower than the load that causes yielding in tension.
Density, ρ .....Mass density is the mass of an object or fluid divided by its volume. See *specific weight* entry for weight density.
Distributed load, w .....Force acting over a length (such as the weight of a beam) or area (such as a snow load on a roof). Compare *point load*.
Eccentricity, e .....Distance between the neutral axis of a part and the location of an applied point load.
Effective length of a column.....Portion of the length of a column that bows like a fully pinned column.
Elastic deformation.....Temporary deformation; release the load and the part returns to its original shape. Compare *plastic deformation*.
Elastic modulus, E .....A measure of the stiffness of a material (the resistance to elastically deforming under a given load.) The slope of the linear elastic portion of the stress-strain curve. Also called *Young's modulus* or *modulus of elasticity*.
Euler critical buckling load, P_{cr} .....The load at which an ideal Euler column will fail, assuming perfect material and perfectly aligned loading.
Factor of Safety, F.S.....The material's strength (typically yield strength) divided by the actual stress in the part. Also called “factor of ignorance” because it includes unknowns such as materials defects, improper installation, abuse by the operator, lack of maintenance, corrosion or rot, temperature variations, etc.
Fillet weld.....A weld with a triangular cross section used for joining lapped plates. Unlike soldering or brazing, welding involves melting the base metal as well as the joining material.
Fracture plane.....In a bolted joint that fails, fracture occurs through the bolt holes or through a plate away from the bolt holes...the site of fracture is called the fracture plane.
General shear formula.....Equation for finding the shear stress within a beam of any shape.
Joint efficiency.....The efficiency of a bolted or welded joint is the lowest allowable load divided by the allowable load of the weaker of the two plates some distance from the joint.
Longitudinal direction.....Along the length of a part, such as a beam or shaft. Compare *transverse direction*.
Longitudinal (axial) stress, σ .....Normal stress that develops in a tensile or compressive member due to axial loading.
Modulus of elasticity, E .....See *elastic modulus*.
Moment, M .....More accurately called a force moment, the product of a length and a transversely applied force. Used in beam problems. There are other types of moment (such as area moment: the product of a length and an area).
Moment of inertia, I .....More accurately called “second moment of area”. Divide a shape into *n* tiny areas *a*, each at a distance *y* from the x-x centroidal axis, and sum the areas and distances as $I_x = \sum_1^n a_i y_i^2$. The larger the moment of inertia, the greater the bending load a beam can support, and the less bending deflection will occur.
Normal.....Perpendicular, in the mathematical sense.
Normal stress, σ .....Force divided by area, when the force acts perpendicular to the area. Tensile and compressive stresses are normal stresses.
Plastic deformation.....Permanent deformation; release the load and the part remains distorted. Compare *elastic deformation*.
Plastic section modulus, Z .....Sum of the first moments of areas above and below the neutral axis of a steel beam. Used for calculating bending stresses in structural steel beams.
Point load, P .....Force acting at a single point. Compare *distributed load*.
Poisson's ratio, ν .....A mechanical property of engineering materials equal to the negative of the transverse strain divided by longitudinal strain. A measure of how much a tensile member will thin during elastic deformation.

Definitions

Polar moment of inertia, J .....More accurately called "polar second moment of area". Divide a shape into *n* tiny areas *a*, each at a distance *r* from the centroid, and sum the areas and distances as $J=\sum_{1}^{n} a_i r_i^2$. The larger the polar moment of inertia, the greater the torque a shaft can support, and the less angular twist will be produced.

Pressure (of a fluid), p .....Fluid equivalent of normal stress. A pressurized gas produces a uniform pressure perpendicular to the walls of the pressure vessel. A pressurized liquid produces a uniform pressure in a small pressure vessel; the pressure is nonuniform in a tall vessel due to gravity (lower pressure at the top, higher at the bottom).

Radius of curvature, R .....If a beam segment is bent with a constant bending moment, the segment becomes a circular arc with a radius of curvature, *R*.

Radius of gyration, r_G .....Concentrate an area at a distance *r* from the x-x neutral axis. If the moment of inertia of the original area is the same as for the concentrated area, then r_{Gx} is the radius of gyration about the x-x axis. The larger the radius of gyration, the more resistant a column is to buckling. Calculate $r_G=\sqrt{I/A}$.

Reaction moment, M_A or M_B .....Moment at reaction point A or B which supports a transversely loaded cantilever beam.

Reaction force, R_A or R_B .....Forces at reaction points A or B which support a transversely loaded beam.

Section modulus, S .....Moment of inertia divided by the distance from the neutral axis to the surface. The larger the section modulus, the more resistant a beam is to bending.

Shear modulus, G .....The slope of the linear elastic portion of the shear stress – shear strain curve.

Shear load, V .....Transverse load on a beam.

Shear plane.....In a bolted joint with two plates pulling in opposite directions, the shear plane is the transverse plane within a bolt that lies at the interface of the two plates.

Shear strain, γ .....Shear deflection divided by original unit length

Shear stress, τ .....Force divided area, when the force acts parallel to the area.

Specific weight, $\boldsymbol{\gamma}$ .....Specific weight, a.k.a. weight density, is the weight of an object or fluid divided by its volume. The symbol, lower case gamma, is also used for shear strain. In this text, plain gamma means shear strain, while bold gamma means specific weight. See *density* entry for mass density.

Strain (normal), ε .....Change in length of a material under normal load divided by initial length.

Stress.....See *normal stress, shear stress, bending stress, torsional stress, longitudinal stress*.

Stress concentration.....A locally high stress due to a sharp discontinuity in shape, such as a hole or notch with a small radius. While the overall stress in the part may be at a safe level, the stress at the discontinuity can exceed yield or ultimate strength, causing failure.

Tensile strength, σ_{UTS} .....Maximum stress on the stress-strain diagram. Beyond this point, the material necks and soon breaks.

Thermal expansion coefficient, α ….Materials property that determines how much a material expands or contracts with changing temperature.

Torque, T .....Rotational moment applied to a shaft. Units of moment and torque are the same (force × distance).

Torsion.....Twisting of a shaft due to an applied torque.

Torsional stress, τ .....A shear stress that develops in a shaft due to torsional loading.

Transfer distance, d .....Term used in calculating moment of inertia of a compound shape.

Transverse direction.....Perpendicular (crosswise) to the length of a long part, such as a beam or shaft. Compare *longitudinal direction*.

Ultimate tensile strength, σ_{UTS} .....See *tensile strength*.

Yield strength, σ_{YS} .....Below the yield strength, a material deforms elastically; above it, the material deforms plastically (undergoes permanent deformation).

Young's modulus, E .....See *elastic modulus*.

Chapter 1: Introduction to Strength of Materials

What is Strength of Materials?

Statics is the study of forces acting in equilibrium on rigid bodies. “Bodies” are solid objects, like steel cables, gear teeth, timber beams, and axle shafts (no liquids or gases); “rigid” means the bodies do not stretch, bend, or twist; and “equilibrium” means the rigid bodies are not accelerating. Most problems in a Statics textbook also assume the rigid bodies are stationary. These assumptions do not match reality perfectly, but they make the math much easier. This model is close enough to reality to be useful for many practical problems.

In *Strength of Materials*, we keep the assumptions of bodies in equilibrium, but we drop the “rigid” assumption. Real cables stretch under tension, real floor joists bend when you walk across a wood floor, and real axle shafts twist under torsional load.

Strength of Materials is a difficult course because the topics are cumulative and highly interconnected. If you miss an early topic, you will not understand later topics. This table lists *Strength of Materials* topics, and indicates which chapters use each topic.

	Chapter number														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Algebra & some trigonometry	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
Factor-label method of unit conversion	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
Normal stress		●	●	●	●				●	●	●	●		●	●
Normal strain		●	●												●
Yield strength		●		●					●		●			●	
Tensile strength		●		●	●										
Shear stress		●			●	●			●		●				●
Shear strain		●													●
Young's modulus		●	●							●	●	●		●	
Axial deflection		●										●			
Poisson's ratio			●												
Transverse strain			●												
Transverse deflection			●							●	●	●			
Strain due to multiaxial loads			●												
Thermal expansion			●												
Thermal stress			●												
Pressure vessels				●											●
Stress concentration				●			●								
Joint efficiency				●	●										
Bolted joints					●										
Welded joints					●										
Moment of inertia of a simple shape						●			●	●	●	●	●	●	
Moment of inertia of a compound shape						●						●			
Polar moment of inertia						●	●								
Radius of gyration						●								●	
Torque							●								●
Shear modulus							●								
Beam reactions								●	●	●	●		●		
Shear diagrams								●	●	●	●		●		
Moment diagrams								●	●	●	●		●		
Deflection diagrams													●		
Section modulus									●		●	●	●		
Plastic section modulus									●		●		●		
Interpolation / Similar triangles								●	●	●	●		●		
Combined stresses												●			
Eccentric loading												●			
Statically indeterminate beams													●		
Column buckling														●	
Mohr's circle															●

If you want to be successful in a *Strength of Materials* course, you need the following:

- **Attention to Detail**. Solve every problem methodically. Make your step-by-step solution easy for the reader (the grader) to understand, with the final solution at the bottom.
- **Algebra.** Solve every problem algebraically before introducing numbers and units.
- **Unit Conversions**. Use the *Factor-Label Method of Unit Conversion*, the standard in Engineering and Chemistry. The convention in Physics is to convert everything into SI units, plug numbers into the equation, and hope the units come out OK. In engineering and chemistry, we introduce the actual units into the equation, and add the unit conversions at the end of the equation. The reason is that real objects are dimensioned in more than one unit. For example, in the U.S., steel beam lengths are in feet, depths are in inches; in Canada, steel beam lengths are in meters, depths are in millimeters. You might be able to convert inches to feet in your head, but it is easy to make mistakes when the unit is exponential (converting in.^{3} to ft.^{3} or mm^4 to m^4), so use the *Factor-Label Method* and avoid simple mistakes.

- **Strong Work Ethic**. If you copy someone else's homework solutions instead of working them out, you will fail the exams, and you will have to repeat the course. The only way to learn this material is by practicing. For every hour of class time, expect to spend at least three hours doing homework. A good estimate is 10% of the learning occurs in the classroom; 90% of the learning occurs while you are solving the problems. Start the homework the same day as class (while your memory is fresh), work with other students in study groups, go to the professor's office hours if you do not understand something, and turn in every homework assignment on time.
- **Engineering Paper**. It comes in light green or light yellow, with a grid printed on the back. Write on the front only; the printed grid on the back side helps you align graphics and text. With Engineering Paper, you can sketch beam problems to scale. The graphical result will tell you if the calculated numbers are in the ballpark.

Strength of Materials is one of the most useful courses in an Engineering Technology education. It is the foundation for advanced Structures courses in Construction Engineering Technology, and the foundation for Machine Elements in Mechanical Engineering Technology.

The Factor-Label Method of Unit Conversion

Three Simple Steps

In Engineering disciplines, we use the three-step Factor-Label Method of Unit Conversion to solve algebraic problems with mixed units.

Step 1 Write the algebraic equation so the desired quantity is on the left of the equals sign, and an algebraic expression is on the right of the equals sign.

Step 2 Draw a horizontal line on the page, and enter numbers and units above and below the line according to the algebraic expression.

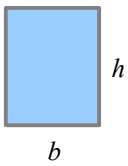
Step 3 Draw a vertical line to show the separation between each unit conversion, and enter all unit conversions necessary to solve the problem. If the unit is raised to a power, then the conversion factor **and** unit must be raised to that power. Consider memorizing the most common conversion factors, like the ones at the right. See **Appendix A** for more unit conversions and metric prefixes, and **Appendix B** for materials properties such as thermal expansion coefficient and Young's modulus. Table B1 gives these properties in U.S. Customary units, while Table B2 gives them in S.I. units.

The units in the final answer **must** appear in the equation, **and** all other units must cancel.

1 ft.	=	12 in.		
1 m	=	100 cm	=	10^3 mm
1 kip	=	10^3 lb.		
1 Pa	=	$1\,\mathrm{N/m^2}$		
1 N	=	$1\,\mathrm{kg\cdot m/s^2}$		

Metric prefixes			
milli-	(m)	=	10^{-3}
centi-	(c)	=	10^{-2}
kilo-	(k)	=	10^3
Mega-	(M)	=	10^6
Giga-	(G)	=	10^9

Example #1



The area of a rectangle is $A = b\cdot h$. Given a base $b = 83\,\text{in.}$ and a height $h = 45\,\text{ft.}$, calculate the area in square feet.

Step 1 The algebraic equation does not need to be manipulated.

Step 2 Draw a horizontal line. Enter 83 in. and 45 ft. in the numerator.

$$A = b\cdot h = \frac{83\,\text{in.}\cdot 45\,\text{ft.}}{}$$

Step 3 We want to eliminate inches to obtain a final result in square feet. Therefore, put 12 inches in the denominator of the unit conversion, and 1 ft. in the numerator.

$$A = b\cdot h = \frac{83\,\text{in.}\cdot 45\,\text{ft.}}{}\left|\frac{\text{ft.}}{12\,\text{in.}}\right. = 311.25\,\text{ft.}^2$$

Example #2

Stress is force divided by area. If the stress is 1 N/mm^2, what is the stress in MPa?

Step 1 There is no algebra to solve here because we are converting one unit to another.

Step 2 Draw a horizontal line. Enter 1 N in the numerator, and mm^2 in the denominator.

$$\frac{1\ \text{N}}{\text{mm}^2}$$

Step 3 A pascal is defined as Pa=N/m^2, so enter Pa in the numerator. Instead of writing in N/m^2 the denominator, put N in the denominator and m^2 in the numerator.

$$\frac{1\ \text{N}}{\text{mm}^2}\left|\frac{\text{Pa}\ \ \text{m}^2}{\text{N}}\right.$$

Now enter the unit conversions to eliminate the two area terms: m^2 and mm^2. There are 10^3 mm in a meter, so use parentheses to square the number and the unit.

$$\frac{1\ \text{N}}{\text{mm}^2}\left|\frac{\text{Pa}\ \ \text{m}^2}{\text{N}}\right|\frac{(10^3\text{mm})^2}{\text{m}^2}$$

Finally, 1MPa=10^{6}Pa. Put MPa in the numerator and 10^6 Pa in the denominator. If you write the equation without numbers, it looks like $\frac{\text{N}}{\text{mm}^2}\left|\frac{\text{Pa}\ \ \text{m}^2}{\text{N}}\right|\frac{\text{mm}^2}{\text{m}^2}\left|\frac{\text{MPa}}{\text{Pa}}\right.$. Cross out duplicate terms, and all terms cancel except for MPa. If you write the equation without units, it looks like $\frac{1}{}\left|\frac{(10^3)^2}{}\right|\frac{}{10^6}$.

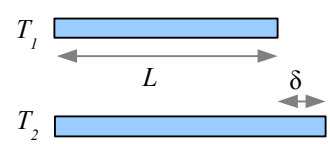
$$\frac{1\ \text{N}}{\text{mm}^2}\left|\frac{\text{Pa}\ \ \text{m}^2}{\text{N}}\right|\frac{(10^3\text{mm})^2}{\text{m}^2}\left|\frac{\text{MPa}}{10^6\,\text{Pa}}\right.$$

$$\frac{\cancel{\text{N}}}{\cancel{\text{mm}^2}}\left|\frac{\cancel{\text{Pa}}\ \ \cancel{\text{m}^2}}{\cancel{\text{N}}}\right|\frac{(\cancel{\text{mm}})^2}{\cancel{\text{m}^2}}\left|\frac{\text{MPa}}{\cancel{\text{Pa}}}\right.$$

Solving the equation with numbers and units, we get $1\,\text{N}/\text{mm}^2 = 1\,\text{MPa}$. This is a useful conversion factor in SI *Strength of Materials* problems.

$$\frac{1\ \text{N}}{\text{mm}^2}\left|\frac{\text{Pa}\ \ \text{m}^2}{\text{N}}\right|\frac{(10^3\text{mm})^2}{\text{m}^2}\left|\frac{\text{MPa}}{10^6\,\text{Pa}}\right. = 1\,\text{MPa}$$

Example #3



Deflection due to thermal expansion is $\delta = \alpha \cdot L \cdot \Delta T$. The upper-case Greek letter delta means "change", so ΔT means "change in temperature." Given a deflection $\delta = 0.06\,\text{in.}$, a length $L = 8\,\text{ft.}$, and a thermal expansion coefficient $\alpha = 5\times10^{-6}\,\text{in./in.°F}$, calculate the change in temperature in degrees Fahrenheit.

Step 1 Rewrite the equation algebraically to solve for ΔT.

$$\delta = \alpha \cdot L \cdot \Delta T \ \rightarrow\ \Delta T = \frac{\delta}{\alpha L}$$

Step 2 Draw a horizontal line. Enter 0.06 in. in the numerator. Since the units for α are a fraction, enter $5\times10^{6}\,\text{in.}$ in the denominator and in.°F in the numerator. Enter 8 ft. in the denominator.

$$\Delta T = \frac{0.06\,\text{in.}}{}\ \frac{\text{in.°F}}{5\times10^{-6}\,\text{in.}}\ \frac{}{8\,\text{ft.}}$$

Step 3 Convert feet to 12 inches so the length units to cancel, and the result is in °F.

$$\Delta T = \frac{0.06\,\text{in.}}{}\ \frac{\text{°F}}{5\times10^{-6}}\ \frac{}{8\,\text{ft.}}\left|\frac{\text{ft}}{12\,\text{in.}}\right. = 125\,\text{°F}$$

Example #4

Stress is force divided by area: $\sigma = \frac{P}{A}$. Given a force P = 7000 lb. acting on an area A = 3 ft.^{2}, calculate the stress in units of pounds per square inch (psi).

Step 1 The equation does not need to be manipulated.

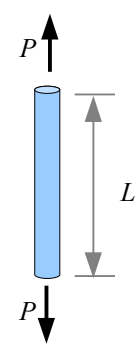
Step 2 Draw a horizontal line. Enter 7000 lb. in the numerator, and 3 ft.^{2} in the denominator.

$$\sigma = \frac{P}{A} = \frac{7000\,\text{lb.}}{3\,\text{ft.}^2}$$

Step 3 The stress is in units of pounds per square foot. There are 12 inches in a foot, but we need to convert square feet, so square the number and the unit: $(12\,\text{in.})^2$. Square feet cancel, and the answer is in pounds per square inch, also written psi.

$$\sigma = \frac{P}{A} = \frac{7000\,\text{lb.}}{3\,\text{ft.}^2}\left|\frac{\text{ft.}^2}{(12\,\text{in.})^2}\right. = 16.2\,\frac{\text{lb.}}{\text{in.}^2} = 16.2\,\text{psi}$$

Example #5



A tensile bar stretches an amount $\delta = \frac{P \cdot L}{A \cdot E}$ where P is the applied load, L is the length of the bar, A is the cross-sectional area, and E is Young's Modulus. The bar has a circular cross section. Given a load of 30 kN, a length of 80 cm, a diameter of 6 mm, and a Young's Modulus of 207 GPa, calculate the deflection in mm.

Step 1 In math class, the area of a circle is given by $A = \pi r^2$. In real life, we measure diameter using calipers; it is much easier to measure a diameter than a radius on most objects. Convert radius to diameter, and the area equation becomes more useful. This is a good equation to memorize.

$$A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4} = \frac{\pi}{4} d^2$$

Combine the two equations to obtain a single algebraic equation.

$$\delta = \frac{P \cdot L}{A \cdot E} = \frac{4\,P\,L}{\pi\,d^2\,E}$$

Step 2 Draw a horizontal line and enter the numbers and units.

$$\delta = \frac{4 \cdot 30\,\text{kN} \cdot 80\,\text{cm}}{\pi\,(6\,\text{mm})^2\,207\,\text{GPa}}$$

Step 3 The SI unit of stress or pressure is the pascal, where $\text{Pa} = \frac{\text{N}}{\text{m}^2}$, so $\text{GPa} = \frac{10^9\,\text{N}}{\text{m}^2}$. Since $1\,\text{kN} = 10^3\,\text{N}$, we can write $\text{GPa} = \frac{10^6\,\text{kN}}{\text{m}^2}$. Three conversion factors are needed: one to cancel GPa and kN; a second to cancel mm^2 and m^2; and a third to put the final answer in mm.

$$\delta = \frac{4 \cdot 30\,\text{kN} \cdot 80\,\text{cm}}{\pi\,(6\,\text{mm})^2\,207\,\text{GPa}}\left|\frac{\text{GPa}\ \ \text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^2}{\text{m}^2}\left|\frac{10\,\text{mm}}{\text{cm}}\right.$$
$$= 4.1\,\text{mm}$$

Example #6

The weight of a solid object is the specific weight of the material times the volume of the object: $W = \gamma V$. The volume of a rod, pipe, or bar is the cross-sectional area times the length: $V = A L$. Calculate the weight of a 2 inch diameter, 3 foot long bar of steel. From Appendix B, the specific weight of steel is 0.284 lb./in.^{3}

Step 1 Combine the two equations to solve for weight: $W = \gamma A L$. Since the rod is round, the cross-sectional area is $A = \frac{\pi}{4} d^2$, therefore $W = \frac{\gamma \pi d^2 L}{4}$

Step 2 Draw a horizontal line and enter the numbers and units.

$$W = \frac{\gamma \pi d^2 L}{4} = \frac{0.284\,\text{lb.}}{\text{in.}^3} \frac{\pi (2\,\text{in.})^2 3\,\text{ft.}}{4}$$

Step 3 The only unit conversion is feet to inches.

$$W = \frac{\gamma \pi d^2 L}{4} = \frac{0.284\,\text{lb.}}{\text{in.}^3} \frac{\pi (2\,\text{in.})^2 3\,\text{ft.}}{4} \left| \frac{12\,\text{in.}}{\text{ft.}} \right. = 32.1\,\text{lb.}$$

Example #7

Calculate the weight of a 5 cm diameter, 2 meter long bar of steel. From Appendix B, the density of steel is 7.85 g/cm^3

Step 1 Use $W = \frac{\gamma \pi d^2 L}{4}$ from Example #6. Specific weight is density times gravity: $\gamma = \rho g$, so $W = \frac{\rho g \pi d^2 L}{4}$.

Step 2 Draw a horizontal line and enter the numbers and units.

$$W = \frac{\rho g \pi d^2 L}{4} = \frac{7.85\,\text{g}}{\text{cm}^3} \frac{9.81\,\text{m}}{\text{s}^2} \frac{\pi (5\,\text{cm})^2 2\,\text{m}}{4}$$

Step 3 The SI unit of weight is the newton: $\text{N} = \frac{\text{kg m}}{\text{s}^2}$.

$$W = \frac{7.85\,\text{g}}{\text{cm}^3} \frac{9.81\,\text{m}}{\text{s}^2} \frac{\pi (5\,\text{cm})^2 2\,\text{m}}{4} \left| \frac{\text{kg}}{10^3\,\text{g}} \right| \frac{100\,\text{cm}}{\text{m}} \left| \frac{\text{N s}^2}{\text{kg m}} \right. = 302\,\text{N}$$

Notice the unit "g" for grams and the term "*g*" for gravity. In science and engineering, we tend to use roman type for units, and *italic type* for variables. Another example is a block sliding on an inclined plane, where "N" stands for newtons and "*N*" stands for normal force.

Example #8

A 50 mm thick wood board is planed to a thickness of 38 mm. Calculate how much material was removed, in percent.

Calculate the percent change by subtracting the initial value from the final value, then dividing by the initial value. This method works whether you are calculating thickness change, weight change, price change, or any other kind of change. The word "percent" means "per hundred", so a result of 0.36 is 36%.

$\frac{t_f - t_o}{t_o} = \frac{38\,\text{mm} - 50\,\text{mm}}{50\,\text{mm}} = -0.24$ or -24% The minus sign means the value decreased.

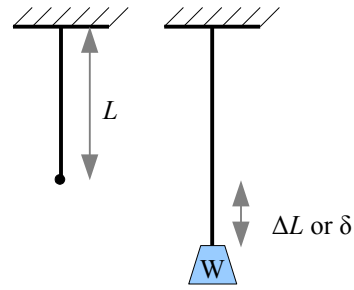
Example #9

Two pieces of carbon fiber composite are glued together as a lap joint. Shear stress is $\tau = \frac{P}{A}$. If the glue has a shear strength of 1.2 ksi and the joint is loaded with a shear load $P = 1700\,\text{lb.}$, how much glued surface area is required for a good bond?

Solve algebraically, and split ksi into $\frac{\text{kips}}{\text{in.}^2}$. Surface area $A = \frac{P}{\tau} = \frac{1700\,\text{lb.}}{1.2\,\text{kips}} \frac{\text{in.}^2}{} \left| \frac{\text{kip}}{1000\,\text{lb.}} \right. = 1.42\,\text{in.}^2$

Chapter 2: Stress and Strain

Normal Stress and Strain



The words "stress" and "strain" are used interchangeably in popular culture in a psychological sense: "I'm feeling stressed" or "I'm under a lot of strain." In engineering, these words have specific, technical meanings. If you tie a steel wire to a hook in the ceiling and hang a weight on the lower end, the wire will stretch. Divide the change in length by the original length, and you have the strain in the wire. Divide the weight hanging from the wire by the wire's cross sectional area, and you have the tensile stress in the wire. Stress and strain are ratios.

The symbol for tensile stress is σ, the lower case Greek letter sigma. If the weight is 25 lb. and the cross-sectional area of the wire is 0.002 in.^{2}, then the stress in the wire is $\sigma = \frac{W}{A} = \frac{25\text{ lb.}}{0.002\text{ in.}^2} = 12{,}700\frac{\text{lb.}}{\text{in.}^2} = 12{,}700\text{ psi}$.

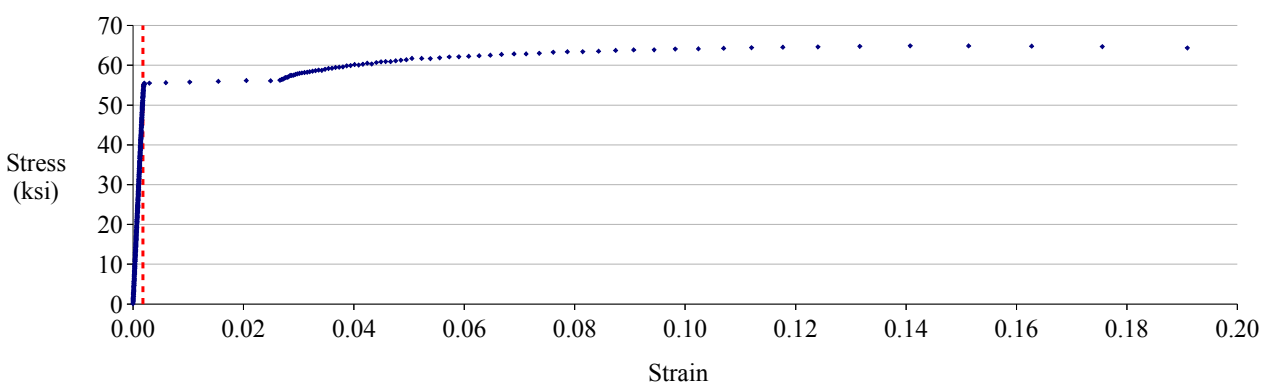
The symbol for strain is ε, the lower case Greek letter epsilon. If the original length of the wire $L = 40\text{ in.}$ and the change in length $\Delta L = 0.017\text{ in.}$ (also written $\delta = 0.017\text{ in.}$), then strain $\varepsilon = \frac{\Delta L}{L} = \frac{\delta}{L} = \frac{0.017\text{ in.}}{40\text{ in.}} = 0.000425$. This is a small number, so sometimes the strain number is multiplied by 100 and and reported as a percent: $0.000425 = 0.0425\%$. You may also see strain reported in microstrain: $0.000425 \times 10^6 = 425$ microstrain. Strain is usually reported as a percent for highly elastic materials like rubber.

Example #1

A 6 inch long copper wire is stretched to a total length of 6.05 inches. What is the strain?

Solution The change in anything is the final dimension minus the initial dimension. Here, the change in length is the final length minus the initial length: $\Delta L = L_f - L_o = 6.05\text{ in.} - 6.0\text{ in.} = 0.05\text{ in.}$. Strain is $\varepsilon = \frac{\Delta L}{L} = \frac{0.05\text{ in.}}{6.0\text{ in.}} = 0.0083$.

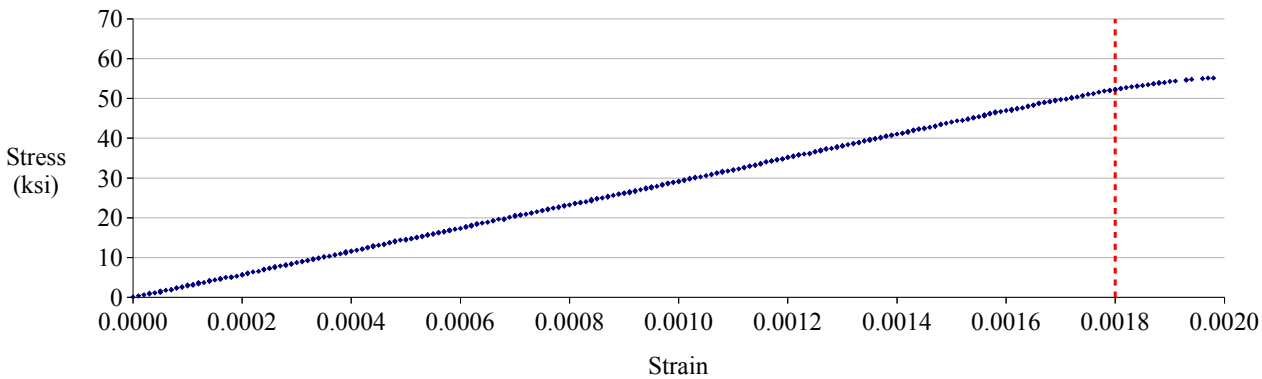
If we hang a bucket from the wire and gradually fill the bucket with water, the weight will gradually increase along with the stress and the strain in the wire, until finally the wire breaks. We can plot the stress vs. strain on an x-y scatter graph, and the result will look like this:



This graph shows the stress-strain behavior of a low-carbon steel. Stress is in units of ksi, or kips per square inch, where $1\text{ kip} = 10^3\text{ lb.}$ (1 kilopound). The points to the left of the red dashed line are so close together that they smear into a line. This straight part of the stress-strain curve is the elastic portion of the curve. If you fill the bucket with only enough water to

stretch the wire in the elastic zone, then the wire will return to its original length when you empty the bucket.

We can change the range of the strain axis from 0.0-0.2 to 0.000-0.002, to show the elastic data only:



This graph shows the leftmost 1% of the previous graph. The dashed red line is in the same position on both graphs. Now the individual data points are visible, and the curve is almost perfectly straight up to a strain of about 0.0018. The straight line has a slope, called Young's Modulus,[2] or Elastic Modulus, *E*. The slope of a straight line is the rise over run, so within this elastic zone, $E=\frac{\sigma}{\varepsilon}$. Since strain is unitless, Young's modulus has the same units as stress. Young's modulus is a mechanical property of the material being tested: 30×10^6 psi or 207 GPa for steels, 10×10^6 psi or 70 GPa for aluminum alloys. See **Appendix B** for materials properties of other materials.

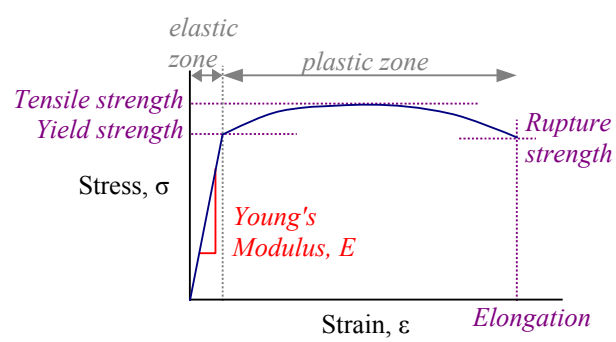
Example #2

What tensile stress is required to produce a strain of 8×10^{-5} in aluminum? Report the answer in MPa.

Solution Aluminum has a Young's modulus of E = 70 GPa. Rewrite $E=\frac{\sigma}{\varepsilon}$, solving for stress:

$$\sigma=E\,\varepsilon=\frac{8\times10^{-5}\cdot70\,\text{GPa}}{}\left|\frac{10^3\,\text{MPa}}{\text{GPa}}\right.=5.6\,\text{MPa}$$

This cartoon of a stress-strain curve illustrates the *elastic* and *plastic* zones. If you hang a light weight to the wire hanging from the ceiling, the wire stretches elastically; remove the weight and the wire returns to its original length. Apply a heavier weight to the wire, and the wire will stretch beyond the elastic limit and begins to plastically[3] deform, which means it stretches permanently. Remove the weight and the wire will be a little longer (and a little skinnier) than it was originally. Hang a sufficiently heavy weight, and the wire will break.



Two stress values are important in engineering design. The yield strength, σ_{YS}, is the limit of elastic deformation; beyond this point, the material "yields," or permanently deforms.

The ultimate tensile strength, σ_{UTS} (also called tensile strength, σ_{TS}) is the highest stress value on the stress-strain curve. The rupture strength is the stress at final fracture; this value is not particularly useful, because once the tensile strength is

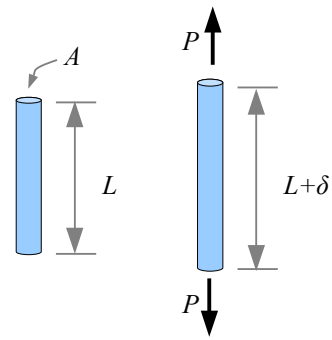
2 Named for Thomas Young, an English physics professor, who defined it in 1807.
3 Here, the word *plastic* is used in its 17th century sense "capable of being deformed" rather than the 20th century definition "polymer."

exceeded, the metal will break soon after. Young's modulus, E, is the slope of the stress-strain curve before the test specimen starts to yield. The strain when the test specimen breaks is also called the elongation.

Many manufacturing operations on metals are performed at stress levels between the yield strength and the tensile strength. Bending a steel wire into a paperclip, deep-drawing sheet metal to make an aluminum can, or rolling steel into wide-flange structural beams are three processes that permanently deform the metal, so $\sigma_{YS} < \sigma_{Applied}$. During each forming operation, the metal must not be stressed beyond its tensile strength, otherwise it would break, so $\sigma_{YS} < \sigma_{Applied} < \sigma_{UTS}$. Manufacturers need to know the values of yield and tensile strength in order to stay within these limits.

After they are sold or installed, most manufactured products and civil engineering structures are used below the yield strength, in the elastic zone.[4] In this *Strength of Materials* course, almost all of the problems are elastic, so there is a linear relationship between stress and strain.

Take an aluminum rod of length L, cross-sectional area A, and pull on it with a load P. The rod will lengthen an amount δ. We can calculate δ in three separate equations, or we can use algebra to find a simple equation to calculate δ directly. Young's modulus is defined as $E = \frac{\sigma}{\varepsilon}$. Substitute the definition of stress, $\sigma = \frac{P}{A}$, and $E = \frac{\sigma}{\varepsilon} = \frac{P}{A \cdot \varepsilon}$. Substitute the definition of strain, $\varepsilon = \frac{\delta}{L}$, and $E = \frac{P}{A \cdot \varepsilon} = \frac{PL}{A\delta}$. Rewrite this equation to solve for deflection: $\delta = \frac{PL}{AE}$. Now we have a direct equation for calculating the change in length of the rod.



Example #3

A 6 foot long aluminum rod has a cross-sectional area of 0.08 in.^{2}. How much does the rod stretch under an axial tensile load of 400 lb.? Report the answer in inches.

Solution Aluminum has a Young's modulus of $E = 10 \times 10^6 \text{ psi}$.

Deflection $\delta = \frac{PL}{AE} = \frac{400 \text{ lb.}}{0.08 \text{ in.}^2} \frac{6 \text{ ft.}}{} \frac{\text{in.}^2}{10 \times 10^6 \text{ lb.}} \left| \frac{12 \text{ in.}}{\text{ft.}} \right. = 0.036 \text{ in.}$

Note Young's modulus is in units of psi, but when you write it in an equation, split up the lb. and the in.^{2} between numerator and denominator to avoid unit confusion.

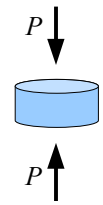
Sign Convention

A load that pulls is called a *tensile* load. If the load pushes, we call it a *compressive* load. The equations are the same: compressive stress $\sigma = P/A$, compressive strain $\varepsilon = \delta/L$, and compressive deflection $\delta = PL/AE$. We need a way to differentiate between compression and tension, so we use a sign convention. Tensile loads and stresses are positive; compressive loads and stresses are negative. Increases in length are positive; decreases in length are negative.

4 One exception is the crumple zones in a car. During an auto accident, the hood and other sheet metal components yield, preventing damage to the driver and passengers. Another exception is a shear pin in a snow blower. If a chunk of ice jams the blades, the shear pin exceeds its ultimate strength and breaks, protecting the drivetrain by working as a mechanical fuse.

Example #4

A 70 kN compressive load is applied to a 5 cm diameter, 3 cm tall, steel cylinder. Calculate stress, strain, and deflection.



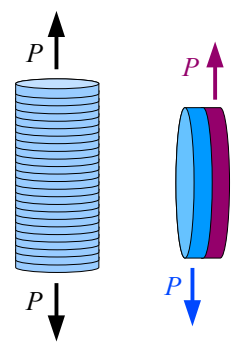
Solution The load is –70 kN, so the stress is . $\sigma=\frac{P}{A}=\frac{4P}{\pi d^2}=\frac{4(-70\,\text{kN})}{\pi(5\,\text{cm})^2}\left|\frac{\text{MPa}\quad \text{m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=-35.6\,\text{MPa}$

The negative sign tells us the stress is compressive. Young's modulus $E=\frac{\sigma}{\varepsilon}$. Rewrite the equation to solve for strain: $\varepsilon=\frac{\sigma}{E}=\frac{-35.6\,\text{MPa}}{207\,\text{GPa}}\left|\frac{\text{GPa}}{10^3\,\text{MPa}}\right.=-0.000172$

Strain is defined as $\varepsilon=\frac{\delta}{L}$. Rewrite to solve for deflection: $\delta=\varepsilon\,L=\frac{-0.000172\cdot 3\,\text{cm}}{}\left|\frac{10\,\text{mm}}{\text{cm}}\right.=-0.0052\,\text{mm}$. The negative signs tell us that the cylinder is shrinking along the direction of the load.

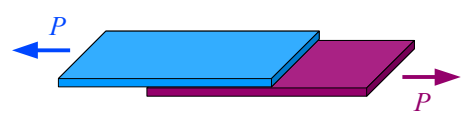
Shear Stress and Strain

The stress in the previous examples is called "normal stress" because the stress acts on an area that is normal, or perpendicular, to the direction of the applied load. Imagine a tall stack of coins glued together on their faces. If you pull on the ends of the stack, the glue will experience a stress that is normal (perpendicular) to the face of each coin. If the glue is thick and tacky, maybe it will tend to stretch, and you can see the coins gradually pull apart along the direction of the applied load. If the load is 100 lb. and the face area of each coin is 1 in.^{2}, then the normal stress is 100 psi.



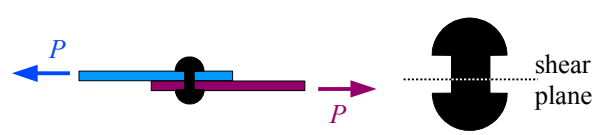
Next, imagine taking two coins that are glued together on their faces, and try to slide them apart. Now the stress is acting parallel to the glue instead of perpendicular to it. This stress is called *shear stress*, symbolized by the lower case Greek letter tau, τ. The units are the same as for normal stress because shear stress is also force divided by area. If the load is 25 lb. and the face area of each coin is 1 in.^{2}, then the shear stress is 25 psi.

Sheet metal joints are often manufactured this way, with adhesive bonding two lapped sheets to form a lap joint. The load is parallel to the area under stress (the adhesive in the shear plane between the two lapped panels). Joints can be designed to put the adhesive in either tension or in shear; typically, the shear strength of an adhesive is not the same as the strength in tension.



For example, cyanoacrylate adhesive ("superglue") is stronger in tension than in shear. An adhesive lap joint will fail when the shear strength of the adhesive is exceeded.

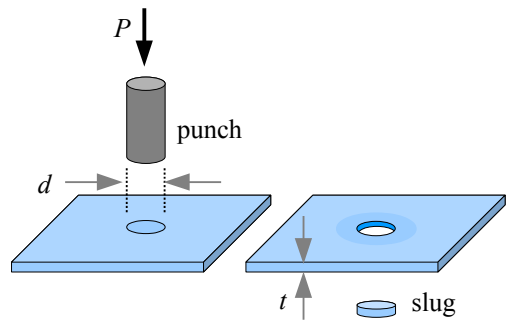
If the sheet metal is held together with rivets instead of glue, then each rivet is loaded in shear across its cross-section. The shear plane passes through the rivet where the two sheets meet. In a bolted joint, use a bolt with a smooth shank instead of a bolt that is threaded along its entire length. This way, the shear plane can pass through the smooth shank, which has a larger cross-sectional area than the root of a thread, and therefore can handle a higher applied load. Later in the book, we will see that the thread root also acts as a stress concentration site – yet another reason for keeping threads out of shear planes.



One way to produce holes in sheet metal is by punching them out with a punch and die set. The punch shears the sheet metal, so we can use shear stress calculations to figure out the stress in the sheet metal, $\tau = \frac{P}{A}$. Here, P is the force of the punch and A is the sheared area (the perimeter of the shape that is punched times the thickness of the sheet metal t).

Example #5

A 3 mm thick aluminum sheet is cut with a 4 cm diameter round punch. If the punch exerts a force of 6 kN, what is the shear stress in the sheet? Report the answer in MPa.



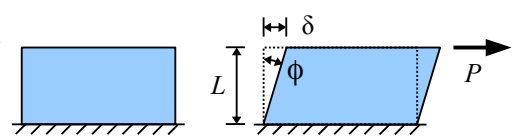
Solution The punch will create a round slug, where the cut edge is around the circumference of the slug. Think of the cut edge as the wall of a cylinder with a height of 3 mm and a diameter of 4 cm. The area equals the circumference of the circle times the thickness of the sheet metal: $A = \pi\, dt$.

$$\text{Shear stress } \tau = \frac{P}{A} = \frac{P}{\pi\, dt} = \frac{6\,\text{kN}}{\pi \cdot 4\,\text{cm} \cdot 3\,\text{mm}} \left| \frac{\text{MPa}\ \ \text{m}^2}{10^3\,\text{kN}} \right| \frac{100\,\text{cm}}{\text{m}} \left| \frac{10^3\,\text{mm}}{\text{m}} \right. = 15.9\,\text{MPa}$$

A process engineer in a stamping plant will rewrite this equation to solve for P in order to find out whether a press is capable of punching out blanks of a given size in a sheet metal of known shear strength.

Shear stress controls the design of torsion members. Think of a round shaft as a series of disks glued together on their faces. If you twist the shaft with a torque T, the glue will be loaded in shear because the load is parallel to the face of each disk.

Consider a rectangular block loaded in shear. The block will distort as a parallelogram, so the top edge moves an amount δ. Divide the distortion by length L perpendicular to the distortion, and you have the shear strain, $\gamma = \frac{\delta}{L}$. Like normal strain, shear strain is unitless. Consider the angle formed between the initial and loaded positions of the block. From trigonometry, we know that $\tan\phi = \frac{\delta}{L}$. The amount of strain in the cartoon is exaggerated. For metals, concrete, wood, and most polymers, angle ϕ is so small that $\tan\phi \approx \phi$ if we measure the angle in radians, therefore $\phi \approx \gamma = \frac{\delta}{L}$.



Key Equations

Normal stress in a tensile or compressive member is the load divided by the cross-sectional area: $\sigma = \frac{P}{A}$

Normal strain is the change in length parallel to the load divided by initial length: $\varepsilon = \frac{\Delta L}{L} = \frac{\delta}{L}$

Young's modulus is the ratio of stress over strain within the elastic zone of the stress-strain diagram: $E = \frac{\sigma}{\varepsilon}$

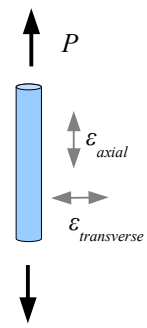
The change in length of a tensile or compressive member is derived from the three previous equations: $\delta = \frac{PL}{AE}$

Shear stress is the load divided by the area parallel to the load: $\tau = \frac{P}{A}$

Shear strain is the deformation parallel to the load divided by initial length perpendicular to the load: $\gamma = \frac{\delta}{L}$.

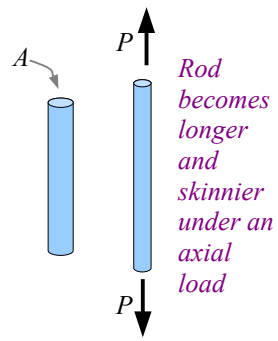
Chapter 3: Poisson's Ratio and Thermal Expansion

Poisson's Ratio



Stretch a thick rubber band, and you notice the material gets thinner as it gets longer. This effect occurs in metals, plastics, concrete, and many other materials. We can predict how much the thickness changes with a materials property called Poisson's ratio,[5] which relates the strain along the tensile axis with the strain in the transverse (crosswise) direction. The symbol for Poisson's ratio is ν, the lower case Greek letter nu, which looks similar to the lower case Roman letter v. Poisson's ratio is $\nu = \frac{-\varepsilon_{transverse}}{\varepsilon_{long}}$, where $\varepsilon_{transverse}$ is the strain in the transverse (crosswise) direction, and ε_{long} is the strain along the longitudinal axis (also called ε_{axial}). The sign convention for strain is positive for expansion, negative for shrinkage. Poisson's ratio is 0.25 for steel, 0.33 for aluminum, and 0.10 to 0.20 for concrete. We can calculate the change in length of a rod by taking the definition of strain, $\varepsilon_{long} = \frac{\delta_{long}}{L}$, and rewriting it as $\delta_{long} = \varepsilon_{long} L$. In the same way, we can calculate the change in diameter by substituting $\varepsilon_{transverse}$ for ε_{axial}, and diameter d for length L: $\delta_{transverse} = \varepsilon_{transverse} d$.

Example #1



An aluminum rod has a cross-sectional area of 0.19635 in.^{2}. An axial tensile load of 6000 lb. causes the rod to stretch along its length, and shrink across its diameter. What is the diameter before and after loading? Report the answer in inches.

Solution The rod has a circular cross section, so the cross-sectional area before the rod is loaded is $A = \frac{\pi}{4} d^2$. Rewrite to solve for the initial diameter,

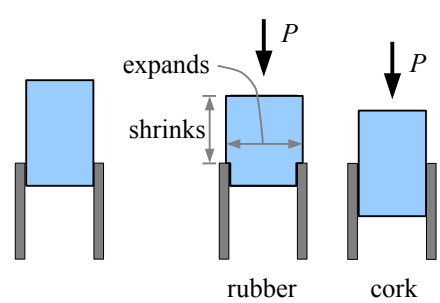
$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \cdot 0.19635\,\text{in.}^2}{\pi}} = 0.50000\,\text{in.}$. When the rod is loaded, the axial strain is

$\varepsilon_{long} = \frac{\sigma}{E} = \frac{P}{AE} = \frac{6000\,\text{lb.}}{0.19635\,\text{in.}^2} \frac{\text{in.}^2}{10 \times 10^6\,\text{lb.}} = 0.00306$.

Poisson's ratio $\nu = \frac{-\varepsilon_{transverse}}{\varepsilon_{long}}$, so $\varepsilon_{transverse} = -\nu \cdot \varepsilon_{long} = -0.33 \cdot 0.00306 = -0.00101$

The change in diameter is $\delta_{transverse} = \varepsilon_{transverse} \cdot d = -0.00101 \times 0.50000\,\text{in.} = -0.000504\,\text{in.}$

The final diameter $d_f = d_o + \delta_{transverse} = 0.50000\,\text{in.} - 0.000504\,\text{in.} = 0.4995\,\text{in.}$

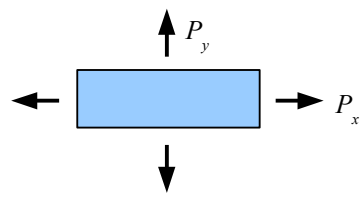


In a civil engineering structure, a dimensional change of half a thousandth of an inch is insignificant, but in a machine it could affect performance. Imagine a machine part that slides in a slot: if the part is loaded axially in compression, Poisson's effect could change a part that slides (slip fit) into a part that sticks (press fit).

You can feel the effect of Poisson's ratio when you insert a rubber stopper or a cork in a bottle. The Poisson's ratio of rubber is about 0.5, while cork is about 0.0. If you try to push a rubber stopper into the neck of a bottle, the material above the neck will shorten and thicken, making it difficult to insert into the bottle. The harder you push, the more the stopper will expand in the transverse direction, and this expansion is about 50% of the compression in

5 Named for Siméon Poisson, a French mathematician and physicist.

the axial direction. Natural cork is made from the bark of the cork oak tree, and does not expand transversely when you push on it. The only resistance comes from compressing the cork in the bottle neck, and friction.



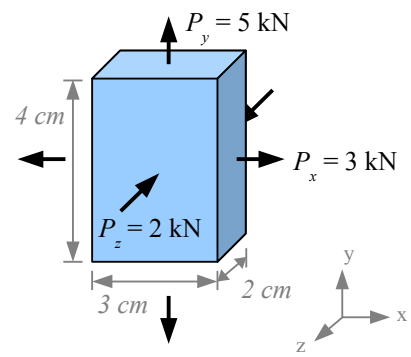
Consider a block that is pulled in two directions by forces P_x and P_y. The strain in the x direction due to the axial stress from P_x is $\varepsilon_{x\,axial}=\frac{\sigma_x}{E}$. If P_x is positive (tension), then the strain is also positive...the bar is stretching along the x axis due to the action of P_x. However, tensile load P_y is acting to shrink the bar in the x direction; the strain due to this transverse load is $\varepsilon_{x\,transverse}=-\frac{\nu\sigma_y}{E}$. Add these two strains to find the total strain in the x direction, $\varepsilon_x=\frac{\sigma_x}{E}-\frac{\nu\sigma_y}{E}=\frac{1}{E}(\sigma_x-\nu\sigma_y)$.

Similarly, the total strain in the y direction is $\varepsilon_y=\frac{\sigma_y}{E}-\frac{\nu\sigma_x}{E}=\frac{1}{E}(\sigma_y-\nu\sigma_x)$.

Real blocks, of course, are three-dimensional. If the block is loaded in all three directions, calculate the strain in the x, y, and z directions as $\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y-\nu\sigma_z)$, $\varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x-\nu\sigma_z)$, and $\varepsilon_z=\frac{1}{E}(\sigma_z-\nu\sigma_x-\nu\sigma_y)$.

Example #2

Calculate the strains in the x, y, and z directions for this steel block.



Solution First, calculate the normal stress in the x, y, and z directions as the force divided by the perpendicular surface that it acts on.

The stress in the x direction acts on the right face of the block, so the normal stress is $\sigma_x=\frac{3\,\text{kN}}{2\,\text{cm}\times4\,\text{cm}}\left|\frac{\text{MPa}\ \ \text{m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=3.75\,\text{MPa}$. The stress in the y direction acts on the top of the block, so the normal stress is

$$\sigma_y=\frac{5\,\text{kN}}{2\,\text{cm}\times3\,\text{cm}}\left|\frac{\text{MPa}\ \ \text{m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=8.33\,\text{MPa}.$$

The stress in the z direction acts on the front face of the block, so the normal stress is

$\sigma_z=\frac{-2\,\text{kN}}{3\,\text{cm}\times4\,\text{cm}}\left|\frac{\text{MPa}\ \ \text{m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=-1.67\,\text{MPa}$. The load and stress are negative because they are compressive.

Next, calculate the strains. Since the block is steel, Young's modulus is 207 GPa and Poisson's ratio is 0.25.

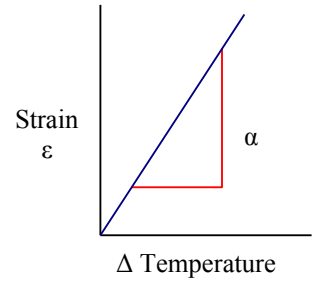
$$\varepsilon_x=\frac{1}{E}[\sigma_x-\nu\sigma_y-\nu\sigma_z]=\frac{1}{207\times10^3\,\text{MPa}}[3.75\,\text{MPa}-0.25(8.33\,\text{MPa})-0.25(-1.67\,\text{MPa})]=1.01\times10^{-5}$$

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu\sigma_x-\nu\sigma_z]=\frac{1}{207\times10^3\,\text{MPa}}[8.33\,\text{MPa}-0.25(3.75\,\text{MPa})-0.25(-1.67\,\text{MPa})]=3.77\times10^{-5}$$

$$\varepsilon_z=\frac{1}{E}[\sigma_z-\nu\sigma_x-\nu\sigma_y]=\frac{1}{207\times10^3\,\text{MPa}}[-1.67\,\text{MPa}-0.25(3.75\,\text{MPa})-0.25(8.33\,\text{MPa})]=-2.26\times10^{-5}$$

Thermal Expansion and Thermal Stress

Heat a piece of steel, wood, or concrete, and it expands. Cool the same piece, and it shrinks. Plot the strain as a function of temperature change, and for most materials, you get a relatively straight line. The slope of the line is called the thermal expansion coefficient, Greek letter α. It tells us how much strain we can expect for a given temperature change. From the graph, the slope $\alpha = \varepsilon / \Delta T$. The units of α are strain divided by temperature: $\frac{\text{in./in.}}{°\text{F}}$ or $\frac{\text{mm/mm}}{°\text{C}}$, which we can write as °F^{-1} or °C^{-1}.



Substitute the definition of strain, $\varepsilon = \delta / L$, and we have $\alpha = \frac{\delta}{L(\Delta T)}$. Rewrite the equation to solve for thermal deflection: $\delta = \alpha L(\Delta T)$.

The thermal expansion coefficient is a materials property; different materials expand at different rates. For example, aluminum expands about twice as much as steel for a given temperature change, because $\alpha_{Aluminum} = 23 \times 10^{-6} °\text{C}^{-1}$ and $\alpha_{Steel} = 12 \times 10^{-6} °\text{C}^{-1}$. One reason we use steel as a reinforcement in concrete is $\alpha_{Concrete} = 11 \times 10^{-6} °\text{C}^{-1}$, so the steel and concrete expand and contract at roughly the same rate. If the matrix and reinforcement in a composite expand at different rates, then the matrix and reinforcement may separate under repeated thermal cycles.

The change in temperature can be positive or negative, and is defined as the final temperature minus the original temperature: $\Delta T = T_f - T_o$. If the material is cooled from 70°F to 40°F, the change in temperature is $\Delta T = 40\,°\text{F} - 70\,°\text{F} = -30\,°\text{F}$. If the material is heated from 70°F to 90°F, the change in temperature is $\Delta T = 90\,°\text{F} - 70\,°\text{F} = +20\,°\text{F}$.

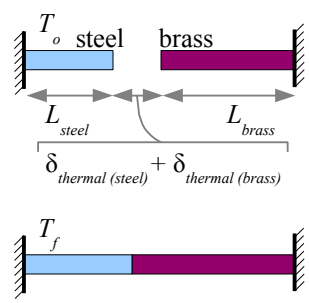
Example #3

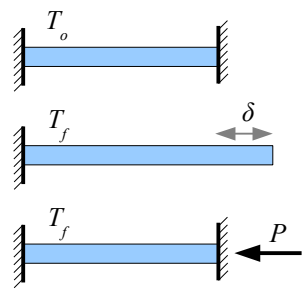
A 5 m aluminum flagpole is installed at 20°C. Overnight, the temperature drops to -5°C. How much does the height change, in millimeters? What is the final height of the flagpole, in meters?

Solution First, calculate the change in length using $\delta = \alpha L(\Delta T)$. From the Appendix, the thermal expansion coefficient for aluminum is $\alpha_{Aluminum} = 23 \times 10^{-6} °\text{C}^{-1}$. Next, calculate the final length by adding the change in length to the original length.

Change in length $\delta = \alpha L(\Delta T) = \frac{23 \times 10^{-6}}{°\text{C}} \frac{5\text{ m}(-5\,°\text{C} - 20\,°\text{C})}{} \left| \frac{10^3\text{ mm}}{\text{m}} \right. = -2.88\text{ mm}$. The negative sign indicates the flagpole is getting shorter. Final length $L_f = L + \delta = 5\text{ m} - \frac{2.88\text{ mm}}{} \left| \frac{\text{m}}{10^3\text{ mm}} \right. = 4.997\text{ m}$.

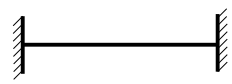
Two cantilever beams made of different materials have a measurable gap between their ends. As the bars heat up, they grow towards each other and eventually meet if the temperature rises enough. Each bar has a different thermal coefficient of expansion. How do we calculate the temperature T_f at which they meet? Consider that the gap between the two bars equals $\delta_{total} = \delta_{steel} + \delta_{brass}$. Substitute the equation for thermal expansion, and we get $\delta_{total} = \alpha_{steel} L_{steel}(\Delta T_{steel}) + \alpha_{brass} L_{brass}(\Delta T_{brass})$. The change in temperature is the same for both materials, so $\delta_{total} = \Delta T(\alpha_{steel} L_{steel} + \alpha_{brass} L_{brass})$. Rewrite the equation to solve for temperature change: $\Delta T = \frac{\delta_{total}}{\alpha_{steel} L_{steel} + \alpha_{brass} L_{brass}}$. The temperature change $\Delta T = T_f - T_o$, so we can solve for the final temperature: $T_f = T_o + \frac{\delta_{total}}{\alpha_{steel} L_{steel} + \alpha_{brass} L_{brass}}$.





If the material is restrained from expanding or contracting while the temperature changes, then stress builds within the part. Consider a bar of steel embedded in two blocks of concrete. If the bar heats up, it will want to expand by an amount $\delta = \alpha L(\Delta T)$. However, the blocks of concrete prevent the bar from expanding, by exerting a force P on the bar. We know from Chapter 2 that an axial load will cause a bar to deform an amount $\delta = \frac{PL}{AE}$. However, the load P in this problem is compressive, so $\delta = \frac{-PL}{AE}$. Set the two deflections equal to each other, and $\alpha L(\Delta T) = \frac{-PL}{AE}$. Length L cancels from both sides, so $\alpha(\Delta T) = \frac{-P}{AE}$. The thermal stress in the bar is $\sigma_{thermal} = \frac{P}{A}$, so we have $\alpha(\Delta T) = \frac{-\sigma_{thermal}}{E}$. Solve for thermal stress: $\sigma_{thermal} = -\alpha E(\Delta T)$. The thermal stress does not depend on the length of the bar; it depends only on materials constants α and E and the temperature change.

Example #4



Two immovable concrete blocks are connected by a steel wire. At 72°F there is no stress in the wire. If the wire cools from 72°F to 55°F, what is the stress in the wire?

Solution Find these steel properties in the Appendix: $\alpha = 6.5\times10^{-6}\,°\text{F}^{-1}$ and $E = 30\times10^{6}$ psi.

Thermal stress $\sigma = -\alpha E(\Delta T) = \frac{-6.5\times10^{-6}}{°\text{F}}\frac{30\times10^{6}\text{ lb.}}{\text{in.}^2}(55\,°\text{F} - 72\,°\text{F}) = 3{,}315$ psi

The positive sign indicates the wire is under a tensile stress. The wire cooled and wanted to shrink, but the concrete blocks prevented it from shrinking, leaving the wire in tension.

Example #5

Two immovable concrete blocks are connected with a 2 inch by 2 inch square bar. At 72°F there is no stress in the bar. If the bar heats from 72°F to 102°F, what is the stress in the bar? How much force do the blocks exert on the bar?

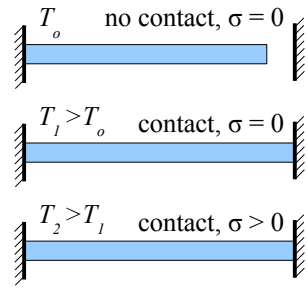
Solution Find these steel properties in the Appendix: $\alpha = 6.5\times10^{-6}\,°\text{F}^{-1}$ and $E = 30\times10^{6}$ psi.

Thermal stress $\sigma = -\alpha E(\Delta T) = \frac{-6.5\times10^{-6}}{°\text{F}}\frac{30\times10^{6}\text{ lb.}}{\text{in.}^2}(102\,°\text{F} - 72\,°\text{F}) = -5{,}850$ psi

The negative sign indicates the bar is under a compressive stress. The bar heated up and wanted to expand, but the concrete blocks prevented it from expanding, leaving the bar in compression.

We can find the force exerted by the walls from the thermal stress: $\sigma_{thermal} = \frac{P_{thermal}}{A}$. Rewrite the equation to solve for force: $P_{thermal} = \sigma_{thermal} A = \frac{-5{,}850\text{ lb.}}{\text{in.}^2}\frac{2\text{ in.}\times2\text{ in.}}{} = -23{,}400\text{ lb.}$ The negative sign shows that the force is compressive.

Some thermal expansion problems require both the deflection *and* the stress equations. For example, if this cantilever beam heats up sufficiently, it will meet the right-hand wall. If the temperature continues to rise, stress will build up in the beam. If you know the initial length, material, and temperatures T_o and T_2 (but not temperature T_1), how do you find the thermal stress? Use the thermal deflection equation and the temperature change $\Delta T = T_1 - T_o$ to figure out temperature T_1, then use the thermal stress equation and the temperature change $\Delta T = T_2 - T_1$ to figure out $\sigma_{thermal}$ at temperature T_2.



Key Equations

Poisson's ratio is the decrease in transverse strain to the increase in longitudinal strain: $\nu = \frac{-\varepsilon_{transverse}}{\varepsilon_{long}}$

Calculate the strains in an elastic block loaded in the *x*, *y*, and *z* directions as $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$, $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z)$, and $\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$. If a load does not exist in one of the directions, then the stress term for that direction is zero, and the equations become simpler.

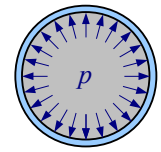
Change in length due to a change in temperature is a function of the thermal expansion coefficient, the initial length, and the change in temperature: $\delta = \alpha L(\Delta T)$

Stress due to a change in temperature is a function of the thermal expansion coefficient, Young's modulus, and the change in temperature: $\sigma_{thermal} = -\alpha E(\Delta T)$

Chapter 4: Pressure Vessels and Stress Concentrations

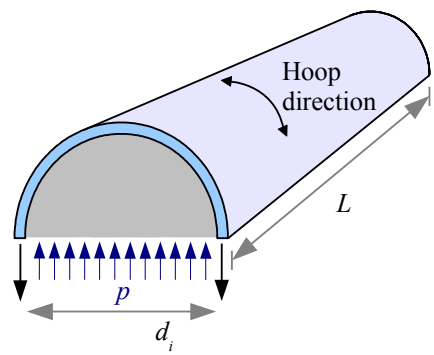
Thin-Walled Pressure Vessels

A pressure vessel is a container holding a fluid (liquid or gas) under pressure. Examples include carbonated beverage bottles, propane tanks, and water mains. Drain pipes are not pressure vessels because they are open to the atmosphere.



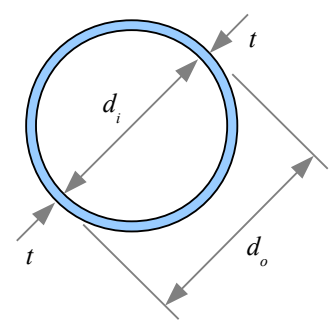
In a small pressure vessel such as a horizontal pipe, we can ignore the effects of gravity on the fluid. In the 17th century, French mathematician and physicist Blaise Pascal discovered that internal fluid pressure pushes equally against the walls of the pipe in all directions, provided the fluid is not moving. The SI pressure and stress unit, the pascal (Pa), is named after Pascal because of his work with fluid pressure. The symbol for pressure is lower-case p, not to be confused with upper-case P used for point loads.

If the thickness of the wall is less than 10% of the internal radius of the pipe or tank, then the pressure vessel is described as a thin-walled pressure vessel. Because the wall is thin, the stress in the wall is nearly the same on the inside and outside surfaces of the wall. (Thick-walled pressure vessels have a higher stress on the inner surface than on the outer surface, so cracks form from the inside out.)



Imagine cutting a thin-walled pipe lengthwise through the pressurized fluid and the pipe wall: the force exerted by the fluid must equal the force exerted by the pipe walls (sum of the forces equals zero). The force exerted by the fluid is $p \cdot A = p\, d_i\, L$ where d_i is the inside diameter of the pipe, and L is the length of the pipe. The stress in the walls of the pipe is equal to the fluid force divided by the cross-sectional area of the pipe wall. This cross-section of one wall is the thickness of the pipe, t, times its length. Since there are two walls, the total cross-sectional area of the wall is $2\,t\,L$. The stress is around the circumference, also called the hoop direction, so $\sigma_{hoop} = \frac{p\, d_i\, L}{2\, t\, L} = \frac{p\, d_i}{2\, t}$. Hoop stress is independent of the length of the pipe.

Example #1

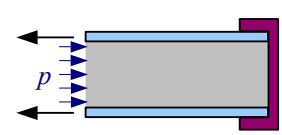


A pipe with a 15 in. outside diameter and a 14 in. inside diameter carries pressurized water at 110 psi. What is the hoop stress?

Solution First, calculate the wall thickness, t. From the diagram, see that $d_o = d_i + 2\,t$, therefore $t = \frac{d_o - d_i}{2} = \frac{15\text{ in.} = 14\text{ in.}}{2} = 0.5\text{ in.}$

Next, check if the pipe is thin-walled. The ratio of the pipe wall thickness to the internal radius is $\frac{t}{r_i} = \frac{0.5\text{ in.}}{\frac{1}{2}(14\text{ in.})} = 0.071 < 0.10$, so the pipe is thin-walled.

Hoop stress $\sigma_{hoop} = \frac{p\, d_i}{2\, t} = \frac{110\text{ psi} \cdot 14\text{ in.}}{2 \cdot 0.5\text{ in.}} = 1{,}540\text{ psi}$

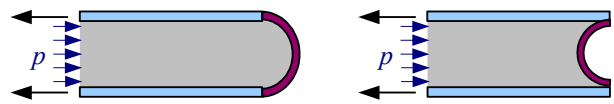


What if the pipe is capped? If the cap were loose, pressure would push the cap off the end. If the cap is firmly attached to the pipe, then a stress develops along the length of the pipe to resist pressure on the cap. Imagine cutting the pipe and pressurized fluid transversely. The force exerted by the fluid equals the force along the length of the pipe walls.

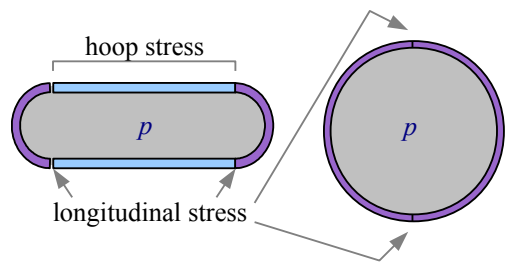
Pressure acts on a circular area of fluid, so the force exerted by the fluid is $P_{fluid}=p\cdot A=p\frac{\pi}{4}d_i^2$. The cross-sectional area of the pipe wall $A=\frac{\pi}{4}d_o^2-\frac{\pi}{4}d_i^2=\frac{\pi}{4}(d_o^2-d_i^2)$. We can estimate the cross-sectional area of a thin-walled pipe pretty closely by multiplying the wall thickness by the circumference, so $A\approx\pi d_i t$. The stress along the length of the pipe is $\sigma_{long}=\frac{P_{fluid}}{A_{pipe}}=\frac{p\pi d_i^2}{4\pi d_i t}$. Simplify by canceling π and one of the diameters: $\sigma_{long}=\frac{p d_i}{4t}$.

In a thin-walled pipe, hoop stress is twice as large as longitudinal stress. If the pressure in a pipe exceeds the strength of the material, then the pipe will split along its length (perpendicular to the hoop direction).

Does the shape of the cap affect the longitudinal stress in the pipe? No, because only the cross-sectional area of the pipe matters. Typically, pressure vessels have concave or convex domes, because flat caps tend to deflect under pressure, but the shape has no effect on longitudinal stress in the pipe walls.



A welded steel outdoor propane tank typically consists of a tube with two convex hemispherical caps (right). Hoop stress controls the design in the tube portion. Where the caps are welded to the tubes, longitudinal stress controls the design. If the steel is all the same thickness (and the welds are perfect), then the tank will fail in the tube section because hoop stress is twice the longitudinal stress. A spherical tank (far right) only has longitudinal stress, so it can handle twice the pressure of a tubular tank. Think of a spherical tank as two hemispheres welded together; the weld prevents the two halves from separating.



In conclusion, if you have a pipe or a tubular tank, use $\sigma_{hoop}=\frac{p d_i}{2t}$. If you have a spherical tank, use $\sigma_{long}=\frac{p d_i}{4t}$.

The ASME Boiler Code recommends using an allowable stress $\sigma_{allowable}=\frac{1}{4}\sigma_{UTS}$ or $\sigma_{allowable}=\frac{2}{3}\sigma_{YS}$, whichever is *smaller*.

Example #2

What is the minimum thickness of a spherical steel tank if the diameter is 10 feet, internal pressure is 600 psi, the tensile strength of the steel is 65ksi, and the yield strength of the steel is 30 ksi?

Solution First, calculate the allowable stress. Next, rewrite the stress equation for longitudinal stress (because this is a sphere, not a pipe) to solve for wall thickness.

Based on tensile strength, $\sigma_{allowable}=\frac{1}{4}\sigma_{UTS}=\frac{65\text{ ksi}}{4}=16.25\text{ ksi}$. Based on yield strength, $\sigma_{allowable}=\frac{2}{3}\sigma_{YS}=\frac{2\cdot 30\text{ ksi}}{3}=20\text{ ksi}$; therefore, use 16.25 ksi because it is the smaller number.

Longitudinal stress $\sigma_{long}=\frac{p d_i}{4t}$. Rewrite the equation for thickness, $t=\frac{p d_i}{4\sigma_{allowable}}=\frac{600\text{ psi}\cdot 10\text{ ft}}{4\cdot 16.25\text{ ksi}}\left|\frac{\text{ksi}}{10^3\text{ psi}}\right|\frac{12\text{ in.}}{\text{ft}}=1.11\text{ in.}$

The welds in real steel tanks contain defects, which reduce the strength of the welds. We can measure the strength of welded joints in the lab and compare them with the strength of the base metal. If the strengths match, we say the joint is 100% efficient. The symbol for efficiency is the Greek letter eta, η.

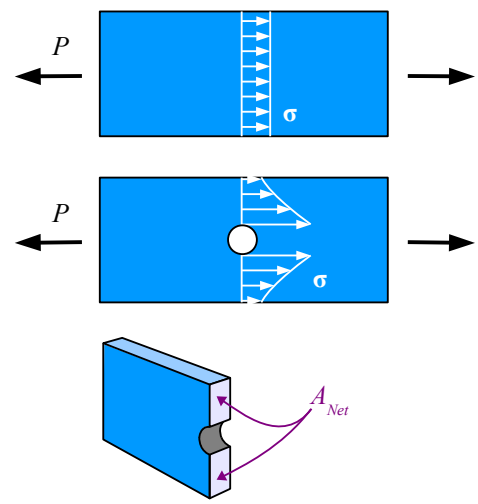
Example #3

What is the minimum thickness of the spherical steel tank in Example #2 if the joint efficiency is 80%?

Solution Use the same equations as before, except the allowable stress is multiplied by the joint efficiency.

Longitudinal stress $\eta_{joint}\sigma_{long}=\frac{p\,d_i}{4\,t}$, therefore $t=\frac{p\,d_i}{4\,\eta_{joint}\sigma_{allowable}}=\frac{600\text{ psi}\cdot 10\text{ ft}}{4\cdot 0.80\cdot 16.25\text{ ksi}}\left|\frac{\text{ksi}}{10^3\text{ psi}}\right|\frac{12\text{ in.}}{\text{ft}}=1.38\text{ in.}$

Stress Concentration in Tension



Pull a solid bar in tension, and the tensile stress in the bar is uniform within the bar: $\sigma=\frac{P}{A_{gross}}$ where A_{gross} is the gross cross-sectional area of the bar. If the bar has a hole in it, we would expect the stress to be higher because there is less material: $\sigma_{net}=\frac{P}{A_{net}}$ where A_{net} is the net cross-sectional area of the bar (gross area minus the area of the hole). Experiments show the stress is *not* uniform within the remaining solid material; instead, it is highest next to the hole, and lower as you move away from the hole. We say that the stress is *concentrated* next to the hole.

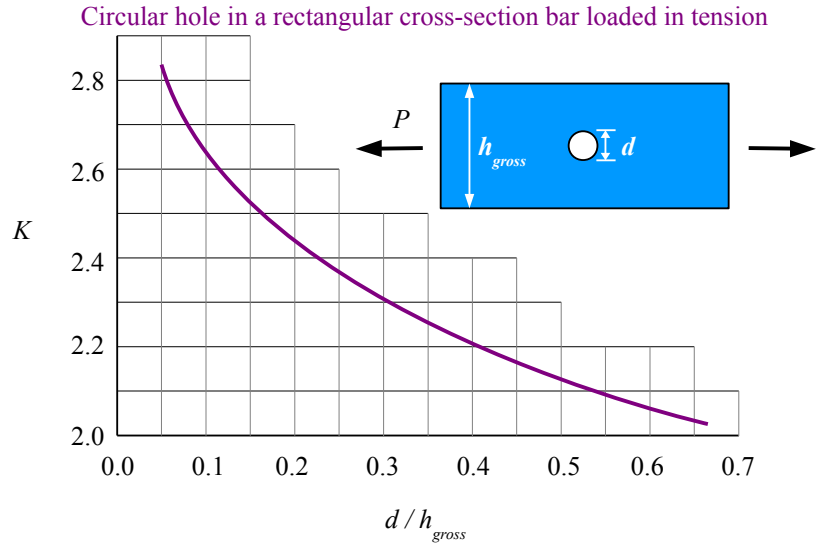
The maximum stress adjacent to the hole is $\sigma_{max}=K\,\sigma_{net}=K\frac{P}{A_{net}}$, where K is a stress concentration factor that depends on the size of the bar and the diameter of the hole. In general, the smaller the radius, the higher the stress. For example, cracks have very high stress concentrations at their tips, exceeding the tensile strength of the material, even though the average stress in the part is below the yield strength. This is why cracks grow.

One way to prevent a crack from growing in a material is to drill a hole at its tip. The drillhole has a larger radius, which reduces the stress to below the yield strength of the material, and the crack is blunted.

Stress concentrations can occur anywhere there is a change in geometry with a small radius, such as holes, fillets, and grooves. In the 1930s, M.M. Frocht[6] published a series of graphs relating K to the dimensions of the bar and hole (or fillet or groove), and R.E. Peterson published scientific papers on fatigue cracks which start at stress concentrations. Peterson's book on stress concentration factors[7] is still in print.

This graph is based on Frocht's original work. Use a four-step process to solve stress concentration problems:

Circular hole in a rectangular cross-section bar loaded in tension



6 M.M. Frocht, "Photoelastic Studies in Stress Concentration," *Mechanical Engineering*, Aug. 1936, 485-489. M.M. Frocht, "Factors of Stress Concentration Photoelastically Determined", *ASME Journal of Applied Mechanics 2,* 1935, A67-A68.

7 Walter D. Pilkey & Deborah F. Pilkey, *Peterson's Stress Concentration Factors, 3rd ed.*, Wiley, 2008.

Step 1 Divide the hole diameter by the gross width of the bar to find the ratio d/h_{gross}.

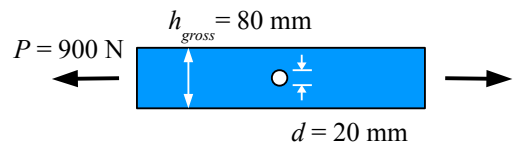
Step 2 Find the value of K from the graph.

Step 3 Calculate the net cross-sectional area (gross cross-sectional area minus the cross-sectional area of the hole). The easiest way to find this value is to multiply the net width by the thickness.

Step 4 Calculate the maximum stress using $\sigma_{max} = K\dfrac{P}{A_{net}}$

Example #4

A bar measuring 80 mm wide by 16 mm thick has a 20 mm diameter hole in the center. The tensile load on the bar is 900 N. Calculate the maximum stress in the bar due to the stress concentration. Report the result in MPa.



Solution Divide the hole diameter by the gross cross-sectional width of the bar: $\dfrac{d}{h_{gross}} = \dfrac{20\text{ mm}}{80\text{ mm}} = 0.25$. Use the graph to find $K \approx 2.36$. The net cross-sectional area of the bar is the net width times the thickness. The net width is the gross width minus the hole diameter: $A_{net} = (80\text{ mm} - 20\text{ mm})16\text{ mm} = 960\text{ mm}^2$

Maximum stress is $\sigma_{max} = K\dfrac{P}{A_{net}} = 2.36\dfrac{900\text{ N}}{960\text{ mm}^2}\left|\dfrac{\text{MPa mm}^2}{\text{N}}\right. = 2.2\text{ MPa}$

Use a similar four-step process to solve stress concentration problems in tensile bars having fillets or semicircular grooves.

Step 1 Divide the groove or fillet radius by the net width of the bar to find the ratio r/h_{net}.

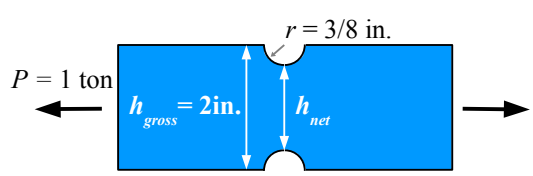
Step 2 Find the value of K from the graph on the following page. Be sure to use the right curve!

Step 3 Calculate the net cross-sectional area at the groove or fillet.

Step 4 Calculate the maximum stress using $\sigma_{max} = K\dfrac{P}{A_{net}}$

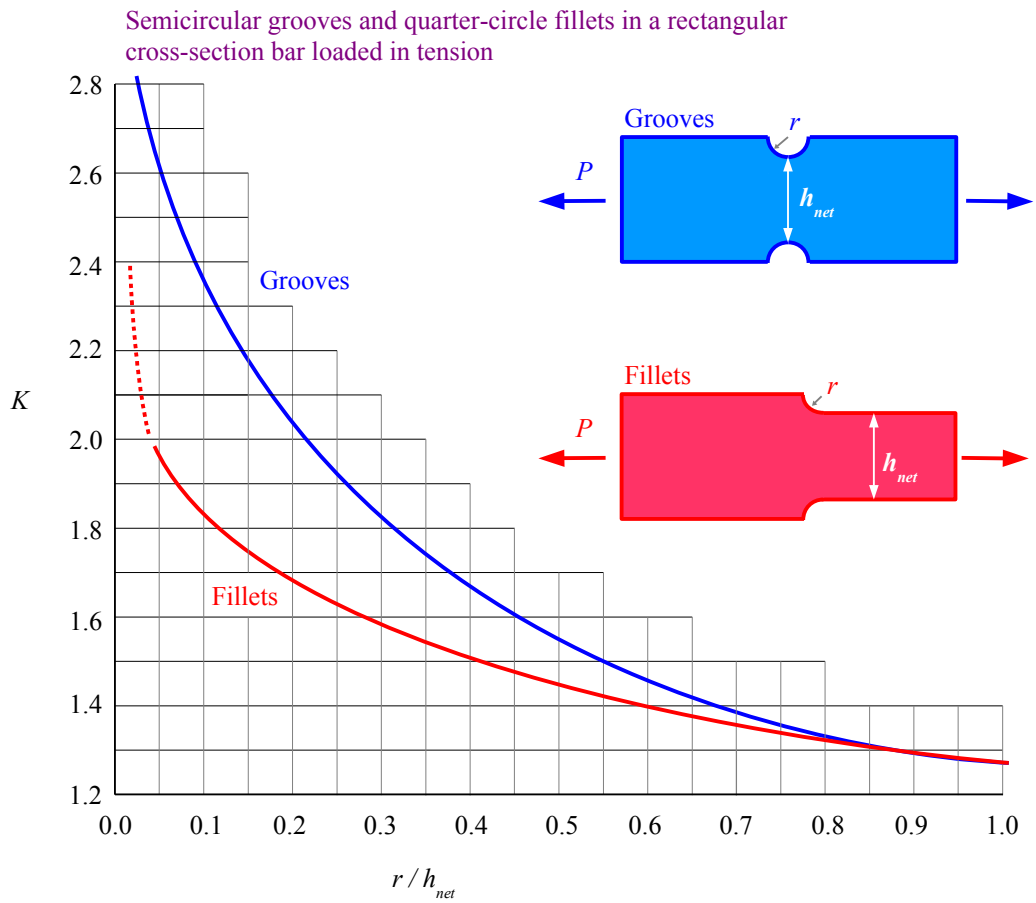
Example #5

A 0.25 in. thick, 2 in. wide bar has two semicircular 3/8 in. radius grooves. Find the maximum stress in the bar due to the stress concentration. Report the result in ksi.



Solution Net width $h_{net} = 2\text{ in.} - 2(0.375\text{ in.}) = 1.25\text{ in.}$

Calculate $\dfrac{r}{h_{net}} = \dfrac{0.375\text{ in.}}{1.25\text{ in.}} = 0.30$. From the graph on the next page, $K \approx 1.82$. The net cross-sectional area is the net width times the thickness: $A_{net} = 1.25\text{ in.} \times 0.25\text{ in.} = 0.3125\text{ in.}^2$. Max. stress is $\sigma_{max} = K\dfrac{P}{A_{net}} = 1.82\dfrac{1\text{ ton}}{0.3125\text{ in}^2}\left|\dfrac{2\text{ kips}}{\text{ton}}\right. = 11.6\text{ ksi}$



Key Equations

Hoop stress in a pressure vessel is equal to the internal pressure times the inside diameter divided by twice the wall thickness: $\sigma_{hoop} = \frac{p\,d_i}{2\,t}$. This equation is used for pipes and cylinders.

Longitudinal stress in a pressure vessel is equal to the internal pressure times the inside diameter divided by four times the wall thickness: $\sigma_{long} = \frac{p\,d_i}{4\,t}$. This equation is used for spherical pressure vessels.

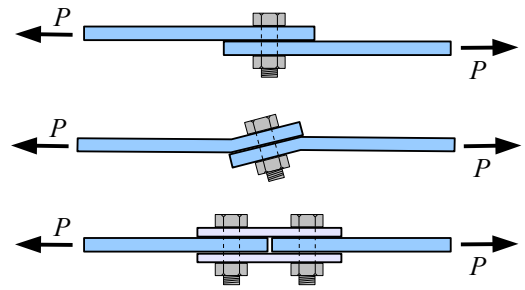
For welded tanks, the ASME Boiler Code recommends using $\sigma_{all} = \frac{1}{4}\sigma_{UTS}$ or $\sigma_{all} = \frac{2}{3}\sigma_{YS}$, whichever is smaller.

Stress at a stress concentration equal to the stress concentration factor times the average stress at the net cross-sectional area: $\sigma_{max} = K\,\sigma_{net}$. Use stress concentration graphs to find the value of K.

Chapter 5: Bolted and Welded Joints

Bolted Lap Joints Loaded in Tension

Look at a steel bridge or steel skeleton of a building, and you see plates, columns, and beams. They intersect at joints, which transmit load from one member to the next. Large steel structures are typically bolted, riveted, or welded together. The strength of the joint may determine the strength of the structure.



Lap two steel plates, and pin them with a rivet or bolt. If you pull hard enough, the plates will want to twist. You can prevent twisting by butting the two main plates, and joining them with lapped splice plates.

Bolted joints have several failure modes. We can calculate the allowable load of the joint for each failure mode; the lowest value is the limiting case, and determines the strength of the joint.

Shear Failure The bolt could shear off, preserving the integrity of the plates; this is the least expensive type of failure because bolts are cheaper and easier to replace than plates.

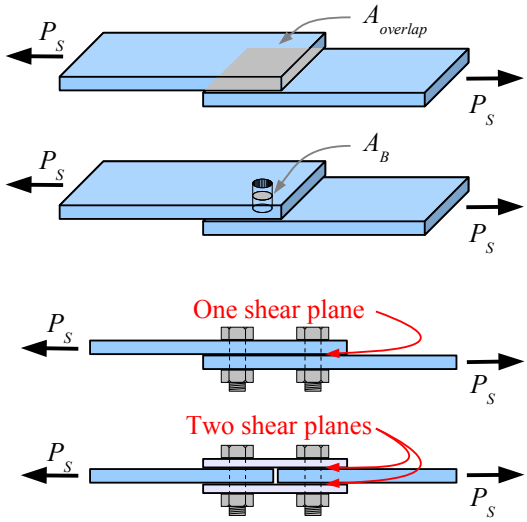
Bearing Failure The bolt could crush the plate where it bears against the plate.

Gross Tensile Failure The plate could tear across its gross cross-sectional area (thickness times overall width), away from the bolt holes.

Net Tensile Failure The plate could tear across its net cross-sectional area (thickness times actual material width) along a line of bolt holes.

Shear Failure

Imagine if we glue two plates together, so that they overlap with an area Overlap. Pull on the plates with a tensile load P_S, and the shear stress in the glue is $\tau = \frac{P_S}{A_{overlap}}$. If we know the allowable shear strength of the glue, τ_{all}, we can calculate the tensile load that can be applied to the joint as $P_S = A_{overlap}\,\tau_{all}$. Now replace the glue with a bolt. The cross-sectional area of the bolt, A_B, carries the shear stress of the joint, so we can write $P_S = A_B\,\tau_{all}$, where $A_B = \frac{\pi}{4} d^2$ and d is the bolt diameter. The bolt has a much smaller cross-sectional area than the area of the glue, which is compensated by the bolt's much higher shear strength.

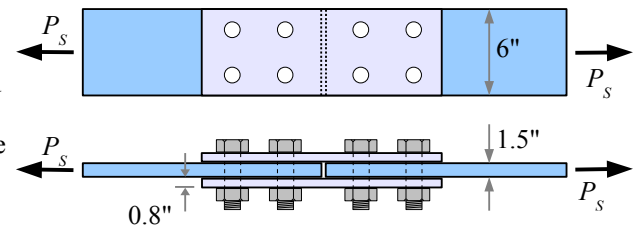


If we have more than one bolt of equal diameter and strength, then we can multiply the strength of the joint by the number of bolts, N, so that $P_S = A_B\,\tau_{all}\,N$.

A bolt can carry the load through more than one shear plane. Define n as the number of shear planes, and $P_S = n\,A_B\,\tau_{all}\,N$.

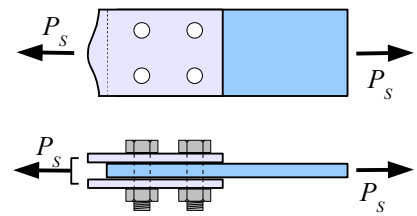
Appendix B3 lists allowable shear stresses for three bolt materials and two loading cases: either with the threads of the bolt in the shear plane, or with the smooth shank of the bolt in the shear plane. A bolt is stronger if the threads are excluded from the shear plane because the threads act as stress concentration sites.

Example #1



Two A-36 steel plates are joined with two splice plates and eight 3/4 in. diameter A325 bolts. The plates are 6 in. wide and 1.5 in. thick; the splice plates are the same width, and 0.8 in. thick. Bolt threads are excluded from the shear plane. Calculate the load that the joint can support in order to resist shear failure. Report the result in kips.

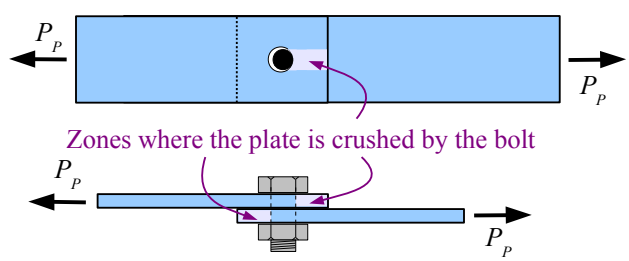
Solution From Appendix B3, A325 bolts with threads excluded from the shear plane have a shear strength of 30 ksi. The problem is symmetrical, so either the four bolts in the left plate will fail in shear first, or the four bolts in the right plate will fail first. Therefore, we can erase half of the diagram, and focus on four bolts, and $N = 4$.



The load is carried by two shear planes per bolt, so $n = 2$.

$$P_S = n\,A_B\,\tau_{all}\,N = \frac{2\text{ shear planes}}{\text{bolt}} \cdot \frac{\frac{\pi}{4}(0.75\text{ in.})^2}{\text{shear plane}} \cdot 30\frac{\text{kips}}{\text{in.}^2} \cdot 4\text{ bolts} = 106\text{ kips}$$

Bearing Failure



If one or more of the plates is crushed by the bolt, then the area that is crushed equals the bolt diameter, d, times the plate thickness, t. Multiply this area by the bearing strength of the plate, and we have the allowable bearing load for a joint with one bolt: $P_P = d\,t\,\sigma_{P-all}$.

The allowable bearing load for a joint with multiple bolts is $P_P = d\,t\,\sigma_{P-all}\,N$.

If the main plate is thicker than the sum of the thicknesses of two splice plates, then use the sum of the splice plate thicknesses for t.

The allowable bearing stress is listed in Appendix B4; it is 1.5 times the ultimate tensile strength of the plate material.

Example #2

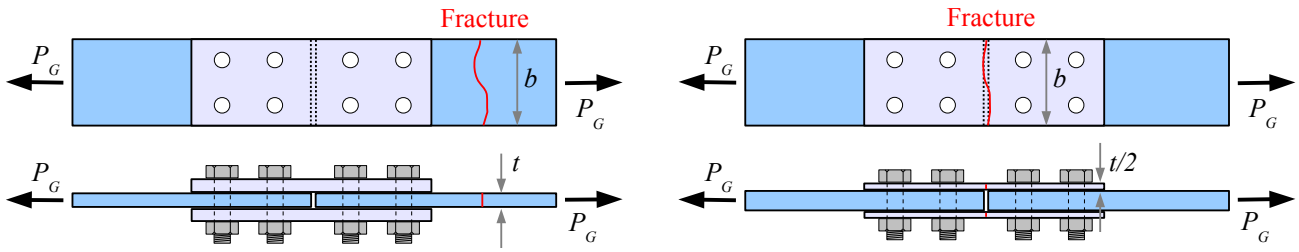
Calculate the load that the joint in Example #1 can support in order to resist bearing failure. Report the result in kips.

Solution From the Allowable Plate Stress table, A-36 steel plates have an allowable bearing strength of 87 ksi. The two splice plates have a combined thickness of 1.6 in., which is greater than the main plate thickness of 1.5 in. Using the smaller of the two numbers, $t = 1.5$ in.

$$P_P = d\,t\,\sigma_{P-all}\,N = \frac{0.75\text{ in.}}{\text{bolt}} \cdot 1.5\text{ in.} \cdot 87\frac{\text{kips}}{\text{in.}^2} \cdot 4\text{ bolts} = 392\text{ kips}$$

Gross Tensile Failure

The gross cross-sectional area of the plate some distance away from the bolt holes is the width, b, times the thickness, t. Check the gross cross-sectional areas of both the main plate and the splice plates, and use the smaller of the two. Multiply this area by the gross tensile strength of the plate to find the allowable gross tensile load for the joint: $P_G = b\,t\,\sigma_{G-allowable}$. The allowable gross tensile strength is 60% of the yield strength of the plate material, listed in Appendix B4.



Example #3

Calculate the load that the joint in Example #1 can support in order to resist gross tensile failure. Report the result in kips.

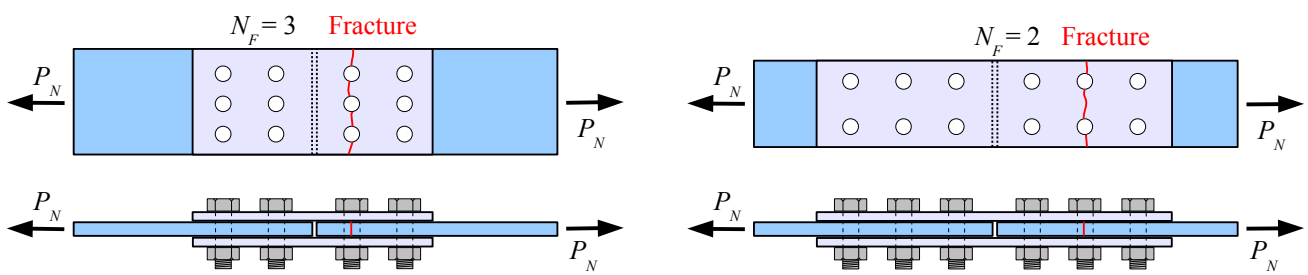
Solution From Appendix B4, A-36 steel plates have an allowable gross tensile strength of 21.6 ksi.

$$P_G = b\,t\,\sigma_{G-all} = 6.0\,\text{in.} \cdot 1.5\,\text{in.} \cdot 21.6\,\frac{\text{kips}}{\text{in.}^2} = 194\,\text{kips}$$

Net Tensile Failure

The net cross-sectional area of the plate at the bolt holes is the gross cross-sectional area, A_G, minus the cross-sectional area of the holes, A_H. Each hole has a cross-sectional area perpendicular to the direction of the load equal to the hole diameter, d_H, times the plate thickness, t: $A_H = d_H\,t$. As a rule of thumb for structural applications, use a hole diameter 1/8 inch or 3 mm larger than the nominal bolt diameter.

The plate will fail along the line of one set of holes. Defining the number of holes in the fracture plane as N_F, the net cross-sectional area is $A_N = A_G - A_H = b\,t - N_F\,d_H\,t$. The allowable net tensile load is this area times the allowable net tensile strength: $P_N = (b\,t - N_F\,d_H\,t)\,\sigma_{N-all}$. The allowable net tensile strength, listed in Appendix B4, is half of the ultimate tensile strength of the plate material.



Both joints have six bolts per symmetrical half, but the joint on the left has three holes in the fracture plane, while the joint on the right has two holes in the fracture plane.

Example #4

Calculate the load that the joint in Example #1 can support in order to resist net tensile failure. Report the result in kips.

Solution From Appendix B4, A-36 steel plates have an allowable net tensile strength of 29 ksi. The hole diameter is 1/8 in. larger than the nominal bolt diameter, so $d_H = 3/4\,\text{in} + 1/8\,\text{in.} = 7/8\,\text{in.}$

$$P_N = (b\,t - N_F\,d_H\,t)\,\sigma_{N-all} = \left(6\,\text{in.} \cdot 1.5\,\text{in.} - 2\,\text{holes}\,\frac{7/8\,\text{in.} \cdot 1.5\,\text{in.}}{\text{hole}}\right) 29\,\frac{\text{kips}}{\text{in.}^2} = 185\,\text{kips}$$

Joint Efficiency

Punching holes in steel causes it to weaken, so a joint may be weaker than the base material. Analysis of the roof of the Hartford Civic Center, which collapsed in January 1978 under a load of heavy wet snow, showed that some joint plates consisted of 85% holes, 15% steel! This flaw was one of several design and construction errors that led to the roof collapse.

We can calculate the *efficiency* of the joint by dividing the minimum joint strength by the gross strength of the plate some distance away from the joint.

Example #5

Calculate the efficiency of the joint in the previous Examples. Report the result in percent.

Solution The minimum joint strength is 106 kips (bolt shear failure). The gross tensile load of the plate is 194 kips. Joint efficiency is $\eta_{joint} = \frac{106\,\text{kips}}{194\,\text{kips}} = 0.55 = 55\,\%$.

$\eta_{joint} < 100\,\%$ indicates an overloaded joint will fail before the structure does. This is the better failure mode for repairability, because it is easier to replace a bolt than to replace an I-beam. $\eta_{joint} \ll 100\,\%$ indicates a poorly designed joint, which could be improved with stronger bolts, more bolts, or a better hole pattern.

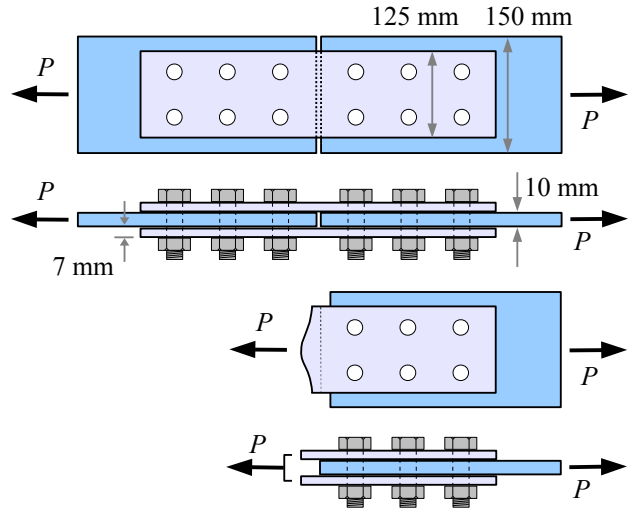
$\eta_{joint} > 100\,\%$ indicates an overdesigned joint – the structure will fail before the joint.

Solution Method for Bolted Joints

Use all of the previous steps. Solve for P based on bolt shear failure, bearing failure, gross tensile failure, and net tensile failure. The lowest value of P is the limiting case. Divide this result by P_G to find the efficiency of the joint.

Example #6

Two A-36 steel plates are joined with two splice plates and twelve 25 mm diameter A325-X high strength bolts. The main plates are 150 mm wide and 10 mm thick; the splice plates are 125 mm wide and 7 mm thick. Calculate the load that the joint can support; report the result in kN. Calculate the joint efficiency; report the result in percent.



Solution Solve for P based on shear, bearing, gross tensile, and net tensile; the lowest value is the limiting case. Divide this result by P_G to find the efficiency of the joint.

Shear The "X" in A325 -X means bolts with threads excluded from the shear plane. From the Bolt Shear Strength table, these bolts have a shear strength of 207 MPa.

The problem is symmetrical, so we can erase half of the diagram, and $N = 6$. The load is carried by two shear planes per bolt, so $n = 2$.

$$P_S = n\,A_B\,\tau_{allowable}\,N = \frac{2\text{ shear planes}}{\text{bolt}}\,\frac{\frac{\pi}{4}(25\text{ mm})^2}{\text{shear plane}}\,\frac{207\text{ MPa}\cdot 6\text{ bolts}}{}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 1220\text{ kN}$$

Bearing From Appendix B4, the allowable bearing stress in the plates is 600 MPa. The two splice plates have a total thickness of 14 mm, which is thicker than the 10 mm thick main plate, therefore $t = 10$ mm.

$$P_P = d\,t\,\sigma_{P-allowable}\,N = \frac{25\text{ mm}}{\text{bolt}}\,\frac{10\text{ mm}\cdot 600\text{ MPa}}{}\,\frac{6\text{ bolts}}{}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 900\text{ kN}$$

Gross Tensile From Appendix B4, the allowable gross tensile stress is 150 MPa. The splice and main plates have different thicknesses and widths, so calculate P_G both ways. Note the splice plate total thickness $t = 7\text{ mm} + 7\text{ mm} = 14\text{ mm}$.

Splice plates: $P_G = b\,t\,\sigma_{G-allowable} = \frac{125\text{ mm}\cdot 14\text{ mm}\cdot 150\text{ MPa}}{}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 263\text{ kN}$

Main plate: $P_G = b\,t\,\sigma_{G-allowable} = \frac{150\text{ mm}\cdot 10\text{ mm}\cdot 150\text{ MPa}}{}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 225\text{ kN}$. Use this lower value for P_G.

Net Tensile From Appendix B4, the allowable net tensile stress is 200 MPa. Calculate P_N both ways.

Splice plates:

$$P_N = (b\,t - N_F\,d_H\,t)\,\sigma_{N-allowable} = \left(125\text{ mm}\cdot 14\text{ mm} - 2\text{ holes}\,\frac{28\text{ mm}\cdot 14\text{ mm}}{\text{hole}}\right)200\text{ MPa}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 193\text{ kN}$$

Main plate:

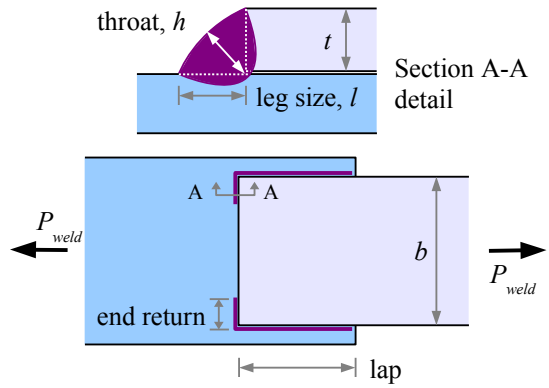
$$P_N = (b\,t - N_F\,d_H\,t)\,\sigma_{N-allowable} = \left(150\text{ mm}\cdot 10\text{ mm} - 2\text{ holes}\,\frac{28\text{ mm}\cdot 10\text{ mm}}{\text{hole}}\right)200\text{ MPa}\left|\frac{10^3\text{ kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\text{ mm})^2} = 188\text{ kN}$$

The lowest value is 188 kN, so the joint will fail first by fracturing across a line of bolt holes.

Joint Efficiency $P_{min} = 188\text{ kN}$ and $P_{G-main\ plate} = 225\text{ kN}$. $\eta_{joint} = \frac{188\text{ kN}}{225\text{ kN}} = 0.84 = 84\%$

Welded Lap Joints

Bolted and riveted joints contain holes which may act as stress concentration sites. If joined plates have to be watertight, such as the plates in a tank or a ship, then welding is a better choice. Welding is also a good choice for parts that will not be disassembled in the future. If you have ever soldered copper pipes together, you have melted the solder but not the pipes. In welding, the weld metal *and* the base metal melt together. In electric arc welding, an electric arc is struck between an electrode and the base metal. The arc melts the metals, and the electrode is gradually consumed. Welding problems in this chapter focus on continuous electric-arc fillet welds in lapped steel joints.



In cross-section, a fillet weld looks roughly like a right triangle. The size l of a fillet weld is the length of a leg of the triangle. Overloaded fillet welds fail by shearing along the throat h.

Using trigonometry, $h \approx l \sin 45°$. The weld supports shear load P_S over area $h \times L$, where L is the total length of the weld. The allowable shear strength of the weld metal is $\tau_{allowable} = 0.3\,\sigma_{UTS}$, where σ_{UTS} is the ultimate tensile strength of the weld metal (not the plate metal). Combining these equations, the load that the welded joint can support is $P_{weld} = \tau_{allowable} \times area = 0.3\sigma_{UTS}\, l \sin 45°\, L = 0.212 \cdot l \cdot L \cdot \sigma_{UTS}$

Weld strength is proportional to weld size, weld length, and the strength of the electrode. Mild steel electrodes are classified by their strength when welded, as E*xx* where *xx* is the ultimate tensile strength of the weld in ksi. For example, an E60 electrode has an ultimate tensile strength of 60 ksi, while an E70 electrode has a tensile strength of 70 ksi. Appendix B5 shows joint strengths for common plate thicknesses, using E60 and E70 electrodes. The values come from $P_{weld} = 0.212 \cdot l \cdot L \cdot \sigma_{UTS}$. Use Appendix B5 for common weld and plate sizes; use the equation for a different electrode such as E90, or for a nonstandard plate size.

The strengths in the table are given as load per unit length; multiply this value by the total length of the weld to find the shear load capacity of the welded joint. For example, a 6 in. long, 7/16 in. weld, using an E60 electrode, has a strength of 5.57 kips/in. The weld can support a load of $P_{weld} = \frac{6\text{ in.}}{} \frac{5.57\text{ kips}}{\text{in.}} = 33.4\text{ kips}$.

Weld strength is not the only consideration when you design a welded joint. AISC recommends the following practices:

End Return The end of a fillet weld creates a stress concentration. Run the weld around the end of the top plate to minimize this effect. The length of the end return should be at least twice the nominal weld size:, a 3/8 in. weld should wrap around the end at least 0.75 in. End returns are included in the total weld length, L.

Lap Based on Thickness Two plates in a lap joint should lap at least five times the thickness of the thinner plate, and no less than 1 inch. For example, a 3/8 in. thick plate welded to a 1/2 in. thick plate should lap at least $3/8\text{ in.} \times 5 = 1.875\text{ in.}$; an 1/8 in. plate welded to a 1/4 in. plate should lap at least 1 in. because $1/8\text{ in.} \times 5 = 0.625\text{ in.}$, which is less than an inch.

Lap Based on Width This rule applies to welds that are parallel to the loading direction. Two plates in a lap joint should lap at least the width b of the narrower plate, unless the weld wraps fully around the end of the plate (*i.e.*, the end returns meet at the centerline). For example, a 3/8 in. thick, 6 in. wide plate welded along its sides to a 1/2 in. thick, 8 in. wide plate should lap at least 6 in.

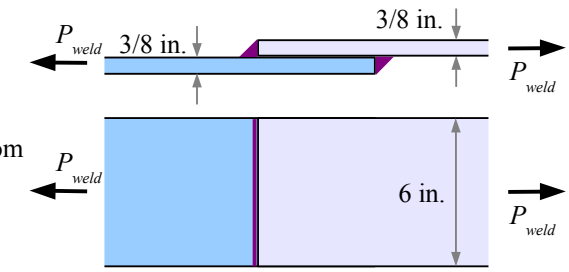
Weld Size The leg of a fillet weld can be made as large as the thickness of the top plate, or smaller; AISC recommends ranges of size according to Appendix B6. If the top plate is less than 1/4 in. thick, then the leg can be as large as the thickness; for thicker plates, the leg should be no larger than 1/16 in. less than the plate thickness.

Joint Efficiency

Calculate the efficiency of a welded joint the same way as for a bolted joint: divide the joint load by the gross tensile load in the plate with the smaller cross-section.

Example #7

Two 3/8 in. thick, 6 in. wide plates made of A36 steel are welded as shown. Select the maximum size weld, and using an E70 electrode, determine the joint strength, reporting the result in kips. Also, calculate the efficiency of the joint.

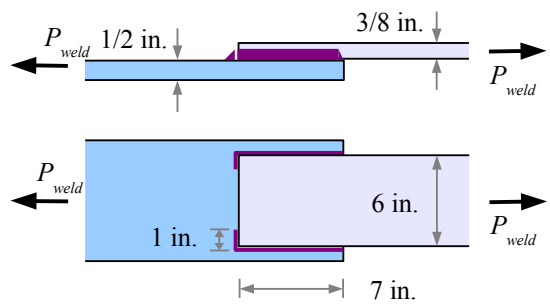


Solution From Appendix B6, the maximum weld size is 5/16 in. From Appendix B5, a 5/16 in. weld made with an E70 electrode can support a load of 4.63 kips/in. Each weld can support a load of $P_{weld}=\frac{4.63\text{ kips}}{\text{in.}}\frac{6\text{ in.}}{}=27.78\text{ kips}$. Since there are two welds, the joint strength is $P_S=2\times 27.78\text{ kips}=55.56\text{ kips}$

From Appendix B4, A36 steel has an allowable gross tensile strength of 21.6 ksi. The gross plate strength is $P_G=b\,t\,\sigma_{G-allowable}=6\text{ in.}\cdot 3/8\text{ in.}\cdot 21.6\text{ ksi}=48.6\text{ kips}$. $\eta_{joint}=\frac{55.56\text{ kips}}{48.60\text{ kips}}=1.14=114\%$, which means that the joint is stronger than the base metal. If the structure fails, the plate will crack while the joint remains intact.

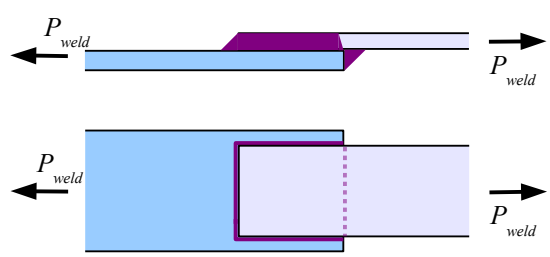
Example #8

A 3/8 in. thick, 6 in. wide plate made of A36 steel is welded to a 1/2 in. thick, 8 in. wide plate using a 3/16 in. weld with an E80 electrode. Determine the joint strength, reporting the result in kips. Also, calculate the efficiency of the joint.



Solution Appendix B5 does not include data for the E80 electrode; instead, use the weld strength equation. The length of each weld is 7 in. plus the 1 in. end return, or 8 in. The total weld length $L=2(8\text{ in.})=16\text{ in.}$ The tensile strength of an E80 weld is 80 ksi. The joint can support a load of

$$P_{weld}=0.212\cdot l\cdot L\cdot\sigma_{UTS}=0.212\cdot\frac{3}{16}\text{in.}\cdot 16\text{ in.}\cdot 80\frac{\text{kips}}{\text{in.}^2}=50.9\text{ kips}.$$



From Appendix B4, A36 steel has an allowable gross tensile strength of 21.6 ksi. Use the smaller plate cross-sectional area (the top plate). The gross plate strength is

$$P_G=b\,t\,\sigma_{G-allowable}=6\text{ in.}\cdot\frac{7}{16}\text{in.}\cdot 21.6\frac{\text{kips}}{\text{in.}^2}=56.7\text{ kips}.$$

$\eta_{joint}=\frac{50.9\text{ kips}}{56.7\text{ kips}}=0.90=90\%$, which means that the joint is 10% weaker than the 6 in. plate. The joint could be strengthened in several ways: a larger weld bead, a stronger weld electrode, a continuous bead around the end of the top plate, or a weld bead underneath.

Key Equations

Check bolted joints for shear, bearing, gross tensile, and net tensile failure; the lowest result is the limiting case. With a symmetrical spliced joint, erase half of the joint and solve.

The allowable shear load for a bolted joint is $P_S = n A_B \tau_{allowable} N$ where n is the number of shear planes per bolt, A_B is the cross-sectional area of the bolt (calculated from its nominal diameter), $\tau_{allowable}$ is the allowable shear stress of the bolt (see Appendix B3), and N is the number of bolts being considered.

The allowable bearing load for a bolted joint is $P_P = d t \sigma_{P-allowable} N$ where d is the nominal bolt diameter, t is the plate thickness (either the thickness of the main plate, or the combined thicknesses of the splice plates, whichever is smaller), and $\sigma_{P\text{-}allowable}$ is the allowable bearing stress in the plate (see Appendix B4).

The allowable gross tensile load for a bolted joint is $P_G = b t \sigma_{G-allowable}$ where b is the plate width and $\sigma_{G\text{-}allowable}$ is the allowable gross tensile stress in the plate.

The allowable net tensile load for a bolted joint is $P_N = (b t - N_F d_H t) \sigma_{N-allowable}$ where N_F is the number of holes in the fracture plane, d_H is the nominal hole diameter (1/8 inch or 3 mm larger than d), and $\sigma_{N\text{-}allowable}$ is the allowable net tensile stress in the plate.

The allowable tensile load of a fillet-welded joint is $P_S = 0.212 \cdot l \cdot L \cdot \sigma_{UTS}$. If standard plates are used with E60 or E70 electrodes, then use Appendix B5 to find the allowable tensile load.

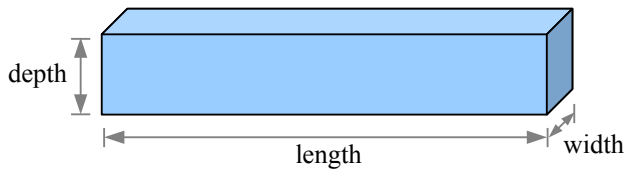
The efficiency of a bolted or welded joint equals the lowest allowable load divided by the allowable load of the weaker of the two plates some distance from the joint.

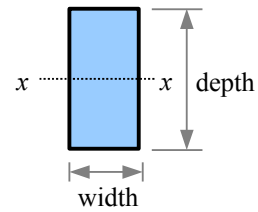
Chapter 6: Properties of Areas

Cross-sectional areas of beams, shafts, and columns have seven properties that we need in order to calculate stresses, deflections, angles of twist, and buckling resistance in these structures. These properties are dimensions, area, centroid, centroidal *x-x* and *y-y* axes, moment of inertia, radius of gyration, and polar moment of inertia.

Dimensions and Area

The *depth* of a beam is the distance from the top to the bottom (the height of the beam). The *width* is the distance from the front to the back, as you face the long length of the beam (the thickness of the beam).



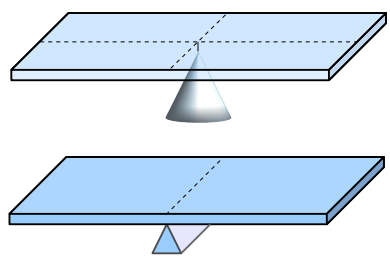


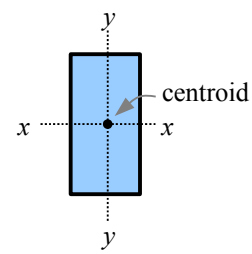
The *area* is the cross-sectional area. For a rectangular beam, the area is the width times the depth.

Centroid and Centroidal Axes

Carefully balance a cardboard rectangle on a sharp point. The location of the point on the rectangle is called the *centroid* of the rectangle. The centroid of a two-dimensional shape is analogous to the center of gravity of a three-dimensional object.

Carefully balance the rectangle on a straightedge so that two ends of the rectangle are parallel to the straightedge. The location of the balance line is a *centroidal axis* of the rectangle.

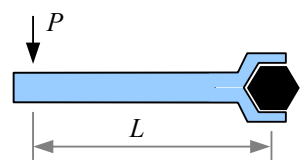




Every shape has a centroid and centroidal axes. We will use these properties in beam problems. Set a beam on edge, define the *x* and *y* axes as horizontal and vertical, and we have *x-x* and *y-y* centroidal axes of the beam cross-sectional area.

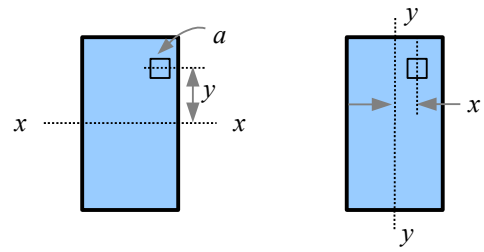
Moment of Inertia of a Rectangle

In everyday speech, the word "moment" means a short amount of time. In engineering mechanics, moment is the product of a quantity and the distance from that quantity to a given point or axis. For example, in *Statics*, a force acting on a wrench handle produces a torque, or moment, about the axis of a bolt: $M = P \times L$. This is the moment of a force.



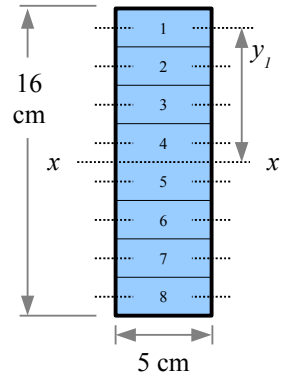
We can also describe moments of areas. Consider a beam with a rectangular cross-section. The horizontal centroidal axis of this beam is the *x-x* axis in the drawing.

Take a small area *a* within the cross-section at a distance *y* from the *x-x* centroidal axis of the beam. The first moment of this area is $a \times y$. The second moment of this area is $I_x = (a \times y) \times y = ay^2$. In *Strength of Materials*, "second moment of area" is usually abbreviated "moment of inertia".[8] If we divide the total area into many little areas, then the moment of inertia of the entire cross-section is the sum of the moments of inertia of all of the little areas. We can also calculate the moment of inertia about the vertical *y-y* centroidal axis: $I_y = (a \times x) \times x = ax^2$. The *x* and *y* in I_x and I_y refer to the centroidal axis.



8 Engineers use the term "moment of inertia" out of habit. As A.P. Poorman states in *Applied Mechanics* in 1940, "the term moment of inertia has been in use too long to be changed," as it was first used in this sense by Swiss mathematician Leonhard Euler in the 18th century.

This rectangular beam cross-section has a depth of 16 cm and a width of 5 cm. We can divide the beam into 8 equal segments 2 cm deep, 5 cm wide, so that each segment has an area $a = 2\,\text{cm} \times 5\,\text{cm} = 10\,\text{cm}^2$. The centroidal axis of segment #1 is 7 cm from the x-x axis $(y_1 = 7\,\text{cm})$; the centroidal axis of segment #2 is 5 cm from the x-x axis $(y_2 = 5\,\text{cm})$; and so on. We can estimate the moment of inertia for the entire area as the sum of the moments of inertia of the segments, written as $I_x = \sum_1^n a_i y_i^2$ where n = the total number of segments, and i = the number of each segment (from 1 to n), or:

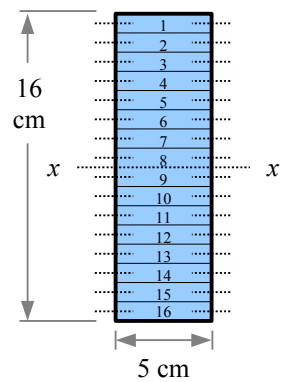


$$I_x = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_4 y_4^2 + a_5 y_5^2 + a_6 y_6^2 + a_7 y_7^2 + a_8 y_8^2$$

With numbers and units, we have:

$$I_x = 10\,\text{cm}^2(7\,\text{cm})^2 + 10\,\text{cm}^2(5\,\text{cm})^2 + 10\,\text{cm}^2(3\,\text{cm})^2 + 10\,\text{cm}^2(1\,\text{cm})^2$$
$$+ 10\,\text{cm}^2(1\,\text{cm})^2 + 10\,\text{cm}^2(3\,\text{cm})^2 + 10\,\text{cm}^2(5\,\text{cm})^2 + 10\,\text{cm}^2(7\,\text{cm}) = 1{,}680\,\text{cm}^4$$

We can take the same beam and split it into 16 segments 1 cm deep.

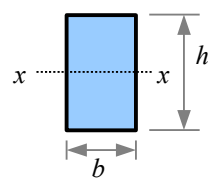


$$I_x = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_4 y_4^2 + a_5 y_5^2 + a_6 y_6^2 + a_7 y_7^2 + a_8 y_8^2$$
$$+ a_9 y_9^2 + a_{10} y_{10}^2 + a_{11} y_{11}^2 + a_{12} y_{12}^2 + a_{13} y_{13}^2 + a_{14} y_{14}^2 + a_{15} y_{15}^2 + a_{16} y_{16}^2$$

We can estimate the moment of inertia as:

$$I_x = 5\,\text{cm}^2(7.5\,\text{cm})^2 + 5\,\text{cm}^2(6.5\,\text{cm})^2 + 5\,\text{cm}^2(5.5\,\text{cm})^2 + 5\,\text{cm}^2(4.5\,\text{cm})^2$$
$$+ 5\,\text{cm}^2(3.5\,\text{cm})^2 + 5\,\text{cm}^2(2.5\,\text{cm})^2 + 5\,\text{cm}^2(1.5\,\text{cm})^2 + 5\,\text{cm}^2(0.5\,\text{cm})$$
$$+ 5\,\text{cm}^2(0.5\,\text{cm})^2 + 5\,\text{cm}^2(1.5\,\text{cm})^2 + 5\,\text{cm}^2(2.5\,\text{cm})^2 + 5\,\text{cm}^2(3.5\,\text{cm})^2$$
$$+ 5\,\text{cm}^2(4.5\,\text{cm})^2 + 5\,\text{cm}^2(5.5\,\text{cm})^2 + 5\,\text{cm}^2(6.5\,\text{cm})^2 + 5\,\text{cm}^2(7.5\,\text{cm}) = 1{,}704.5\,\text{cm}^4$$

As the segment size drops, the estimates converge on a solution. Split the beam into an infinite number of infinitely-small segments to get the actual solution, derived from calculus: $I_x = \frac{bh^3}{12}$, where b is the width and h is the depth. See Appendix C for moments of inertia of other simple shapes.



Example #1

A rectangular beam has a depth of 16 cm and a width of 5 cm. What is the moment of inertia about the *x-x* centroidal axis? Report the answer in cm^4.

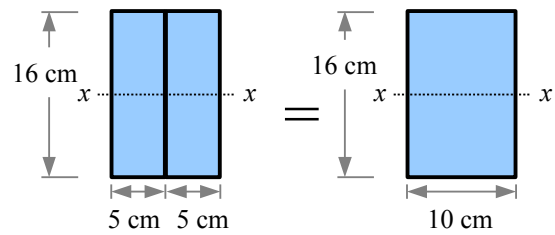
Solution $I_x = \frac{bh^3}{12} = \frac{5\,\text{cm}(16\,\text{cm})^3}{12} = 1{,}706.7\,\text{cm}^4$ The estimated I_x based on 8 segments was within 2% of the exact answer; the estimated I_x based on 16 segments is within 0.1%. Estimates can get us close to the exact solution, but in this case, the math for the exact solution is less time-consuming (one equation instead of 16).

Compound Beams Sharing a Centroidal Axis

Laminated structural wood beams are manufactured by glueing planks together. If we stand the planks vertically and glue them along their facing surfaces, we can increase the moment of inertia. Since bending strength is directly proportional I, if you double the moment of inertia you double the strength of the beam.

Example #2

Two $5\,\text{cm}\times 16\,\text{cm}$ rectangular planks are glued together as shown. What is the moment of inertia about the *x*-*x* centroidal axis? Report the answer in cm^4.

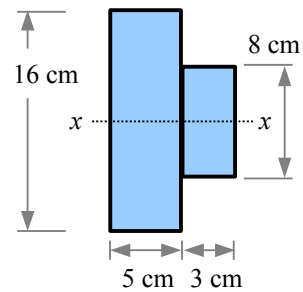


Solution The moment of inertia is $I_x=\frac{bh^3}{12}=\frac{10\,\text{cm}(16\,\text{cm})^3}{12}=3{,}413.3\,\text{cm}^4$, which is twice the moment of inertia of a single beam. Therefore, we can add the moments of inertia of two cross-sections as long as they share the same centroidal axis: $I_{xtotal}=I_{x1}+I_{x2}$. A laminated beam made of *n* parallel planks would have a moment of inertia of $I_{xtotal}=\sum_1^n I_{xn}$.

You can use this method even if the planks are not the same size, as long as they share the same centroidal axis.

Example #3

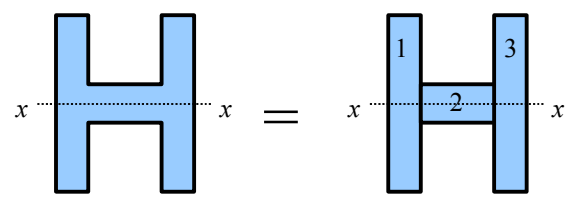
A $5\,\text{cm}\times 16\,\text{cm}$ rectangular plank is glued to an $8\,\text{cm}\times 3\,\text{cm}$ as shown, so they share the same neutral *x*-*x* axis. What is the moment of inertia about the centroidal axis? Report the answer in cm^4.



Solution Add the moments of inertia of the individual planks:

$$I_x=\frac{b_1h_1^3}{12}+\frac{b_2h_2^3}{12}=\frac{b_1h_1^3+b_2h_2^3}{12}=\frac{5\,\text{cm}(16\,\text{cm})^3+3\,\text{cm}(8\,\text{cm})^3}{12}=1{,}834.7\,\text{cm}^4$$

A compound beam shape may be made by welding or glueing two beams together, or it may be made by extruding or rolling. Aluminum extrusions are widely used in architecture, machine parts, furniture, automotive components, and other consumer goods. Steel is hot-rolled into railroad rails and wide-flange structural steel beams, channels, and angles. Draw construction lines on a cross-section and analyze it as if it were glued together with rectangles sharing a common centroidal axis.

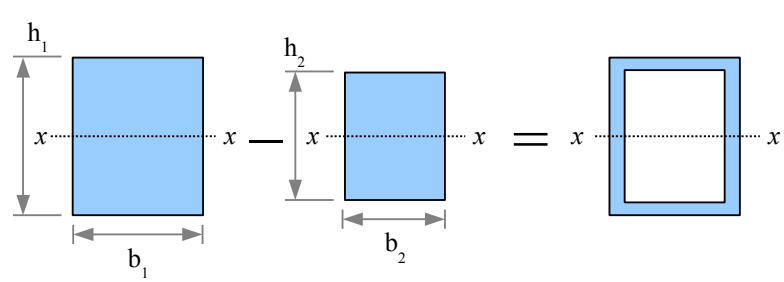


The moment of inertia for the entire shape is $I_{xtotal}=I_{x1}+I_{x2}+I_{x3}=\frac{b_1h_1^3+b_2h_2^3+b_3h_3^3}{12}$.

Hollow Beams Sharing a Centroidal Axis

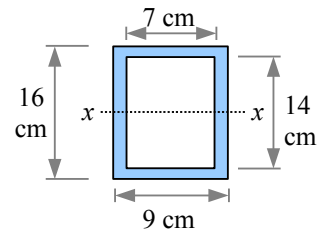
If a beam is hollow *and* the hollow space shares the same centroidal axis as the beam, then we can subtract the moment of inertia of the hollow area from the moment of inertia of an equivalent solid beam. For a rectangular beam with a rectangular hollow section,

$$I_{xtotal}=I_{x1}-I_{x2}=\frac{b_1h_1^3}{12}-\frac{b_2h_2^3}{12}=\frac{b_1h_1^3-b_2h_2^3}{12}$$



Example #4

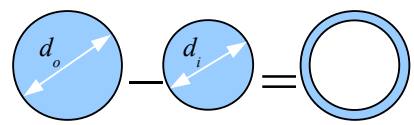
A hollow rectangular beam has 1 cm thick walls, as shown. What is the moment of inertia about the *x-x* centroidal axis? Report the answer in cm^4.



Solution Subtract the moment of inertia of the hollow space from the moment of inertia of the outside dimensions.

$$I_x = \frac{b_o h_o^3}{12} - \frac{b_i h_i^3}{12} = \frac{9\,\text{cm}(16\,\text{cm})^3 - 7\,\text{cm}(14\,\text{cm})^3}{12} = 1{,}471\,\text{cm}^4$$

We can use the same technique for finding the moment of inertia of a hollow tube. From calculus, the moment of inertia of a circle is $I_x = \frac{\pi d^4}{64}$, therefore the moment of inertia of a hollow circle is $I_x = \frac{\pi d_o^4}{64} - \frac{\pi d_i^4}{64} = \frac{\pi(d_o^4 - d_i^4)}{64}$.



Example #5

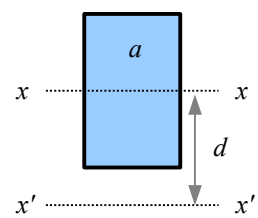
A standard 2" steel pipe has an outside diameter $d_o = 2.375\,\text{in.}$ and an inside diameter $d_i = 2.067\,\text{in.}$ What is the moment of inertia about the *x-x* centroidal axis? Report the answer in in.^{4}.

Solution Subtract the moment of inertia of the hollow space from the moment of inertia of the outside dimensions.

$$I_x = \frac{\pi[(2.375\,\text{in.})^4 - (2.067\,\text{in.})^4]}{64} = 0.6657\,\text{in.}^4$$

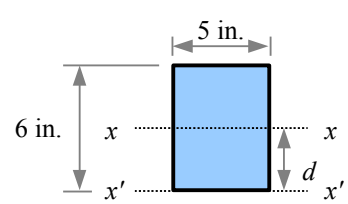
The Transfer Formula

The moments of inertia calculated in the previous examples were evaluated about the centroidal axis of each shape. Sometimes we need to calculate the moment of inertia of a beam about a different, noncentroidal axis. The Transfer Formula is $I = I_o + ad^2$, where I_o = moment of inertia about the *x-x* centroidal axis, I = moment of inertia about a parallel *x′-x′* axis, a = the area of the shape, and d = the distance between the *x-x* centroidal axis and the parallel *x′-x′* axis (the transfer distance). Note: the symbol d is also used for the diameter of a circle; these quantities are different, even though they share the same symbol.



Example #6

A beam has a width of 5 inches and a depth of 6 inches. What is the moment of inertia about the base of the beam, marked as the *x′-x′* axis? Report the answer in in.^{4}.



Solution The moment of inertia about the centroidal axis is

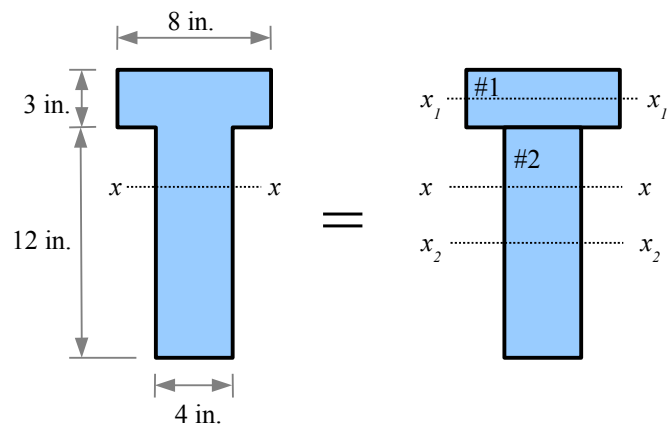
$I_o = \frac{bh^3}{12} = \frac{5\,\text{in.}(6\,\text{in.})^3}{12} = 90\,\text{in.}^4$ and the cross-sectional area is $a = 5\,\text{in.} \times 6\,\text{in.} = 30\,\text{in.}^2$.

The distance from the *x-x* centroidal axis to the *x′-x′* axis $d = 3\,\text{in.}$ Using the Transfer Formula,

$I_{x'} = I_o + ad^2 = 90\,\text{in.}^4 + (30\,\text{in.}^2)(3\,\text{in.})^2 = 360\,\text{in.}^4$

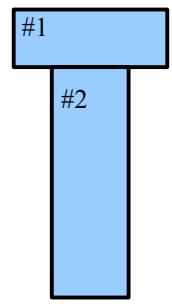
Compound Beams With Different Neutral Axes

Some compound cross-sections are made of segments which do not share the same centroidal axis. As long as the neutral axes are parallel, we can use the Transfer Formula to find the moment of inertia of the compound beam. For example, this beam cross-section consists of two rectangular segments, and the *x*-*x* centroidal axis of the beam is different from the x_1-x_1 and x_2-x_2 neutral axes of segments #1 and #2. Using the Transfer Formula, we can calculate the moment of inertia of each segment about the *x*-*x* centroidal axis of the compound shape, then add the results to obtain the total moment of inertia.



Using a 10-step process, we can calculate the moment of inertia of the compound beam. This approach is sometimes called a *tabular method* because you enter all of the required numbers in a table as you solve the problem.

Step 1 Divide the compound beam into simple shapes, and label the segments. This compound beam can be divided into two segments; the method also works for complex shapes made up of many simple shapes.



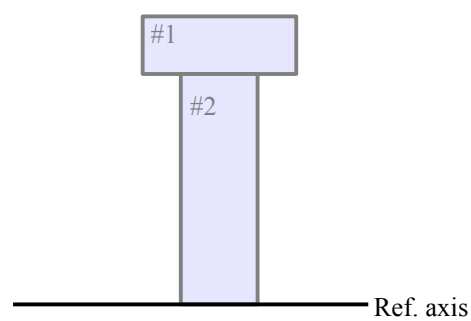
Step 2 Calculate the area *a* of each segment. Enter the areas and their sum into a table. Be sure to list the units, because in some problems, you may need to include a conversion factor in the calculation.

Seg-ment	a (in.^{2})
#1	24
#2	48
Sum	72

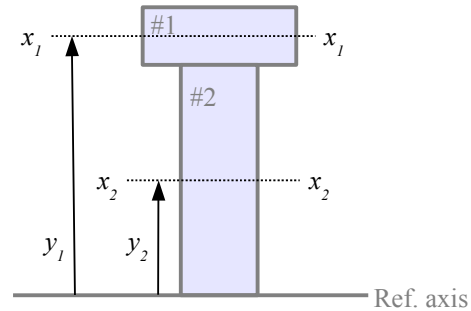
$$a_1 = 8\,\text{in.} \times 3\,\text{in.} = 24\,\text{in.}^2$$

$$a_2 = 4\,\text{in.} \times 12\,\text{in.} = 48\,\text{in.}^2$$

Step 3 Pick a Reference Axis, and label it on the diagram. In theory, you can select any axis, but in practice, the math is usually easier if you pick an axis along the top or bottom of the complex shape, or along the centroidal axis of one of the segments.



Step 4 Draw the distance from the Reference Axis to the centroidal axes of the segments, x_1-x_1 and x_2-x_2. Label these distances y_1, y_2, etc. Enter these values into the table.



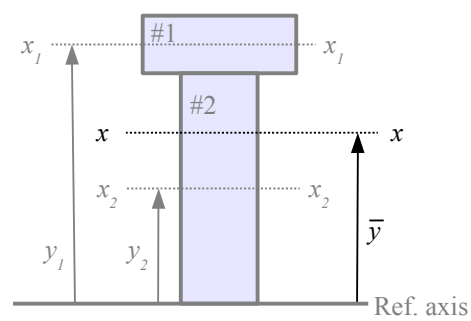
Seg-ment	a (in.^{2})	y (in.)
#1	24	13.5
#2	48	6
Sum	72	

Step 5 Calculate the product $a \times y$ for each segment, and enter these values and their sum in the table.

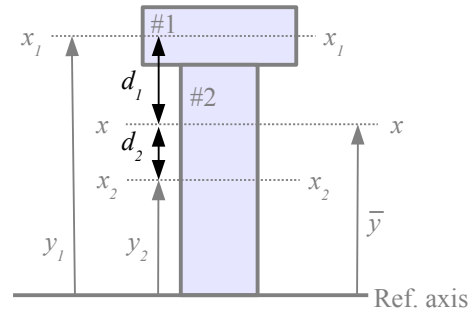
Seg-ment	a (in.^{2})	y (in.)	ay (in.^{3})
#1	24	13.5	324
#2	48	6	288
Sum	72		612

Step 6 Draw the distance from the Reference Axis to the *x-x* centroidal axis of the complex shape. Calculate this distance as

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{612\,\text{in.}^3}{72\,\text{in.}^2} = 8.5\,\text{in.}$$



Step 7 Draw the Transfer Distance *d* for each segment. This is the distance from the centroidal axis of the segment to the centroidal axis of the complex shape. Given the way this beam is drawn, $d_1 = y_1 - \bar{y}$ and $d_2 = \bar{y} - y_2$. For other compound beams, you will have to figure out the formulas for d_1, d_2, d_3, etc. based on the drawing. Enter the results into the table.



Seg-ment	a (in.^{2})	y (in.)	ay (in.^{3})	d (in.)
#1	24	13.5	324	5
#2	48	6	288	2.5
Sum	72		612	

Step 8 Calculate the product $a \times d^2$ for each segment, and enter the results and their sum in the table. Be sure to calculate $a \times d^2$, not $a \times d$ ...it's an easy error to make.

Segment 1: $a \times d^2 = 24\text{ in.}^2 (5\text{ in.})^2 = 600\text{ in.}^4$

Segment 2: $a \times d^2 = 48\text{ in.}^2 (2.5\text{ in.})^2 = 300\text{ in.}^4$

Enter these values and their sum into the table.

Seg-ment	a (in.^{2})	y (in.)	ay (in.^{3})	d (in.)	ad^2 (in.^{4})
#1	24	13.5	324	5	600
#2	48	6	288	2.5	300
Sum	72		612		900

Step 9 Calculate *I* for each segment about its centroidal axis:

$$I_{o1} = \frac{bh^3}{12} = \frac{8\text{ in.}(3\text{ in.})^3}{12} = 18\text{ in.}^4$$

$$I_{o2} = \frac{bh^3}{12} = \frac{4\text{ in.}(12\text{ in.})^3}{12} = 576\text{ in.}^4$$

Enter these values and their sum into the table.

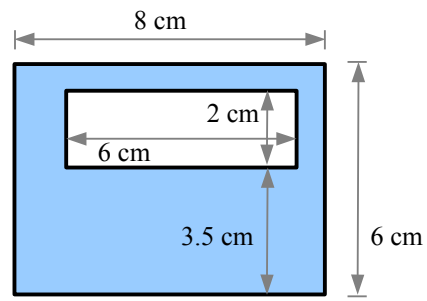
Seg-ment	a (in.^{2})	y (in.)	ay (in.^{3})	d (in.)	ad^2 (in.^{4})	I_o (in.^{4})
#1	24	13.5	324	5	600	18
#2	48	6	288	2.5	300	576
Sum	72		612		900	594

Step 10 Use the Transfer Formula to calculate *I* for the compound shape.

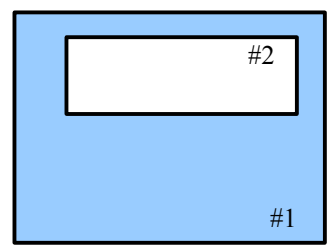
$$I = \sum I_o + \sum ad^2 = 594\text{ in.}^4 + 900\text{ in.}^4 = 1494\text{ in.}^4$$

Hollow Beams With Different Neutral Axes

If the beam is hollow and the cavity does not share the same neutral axis as the outline of the solid shape, then we need the Transfer Formula. Let Segment #1 be the solid shape (with no hole), and Segment #2 be the hole. In all calculations, the area of the hole and the moment of inertia of the hole are *negative* numbers. Thus a_1, a_1y_1, $a_1d_1^2$, and I_1 are positive numbers; a_2, a_2y_2, $a_2d_1^2$, and I_2 are negative numbers.



Step 1 Divide the compound beam into simple shapes, and label the segments. Segment #1 is a solid rectangle measuring 8 cm wide by 6 cm deep; Segment #2 is a hole measuring 6 cm wide by 2 cm deep.



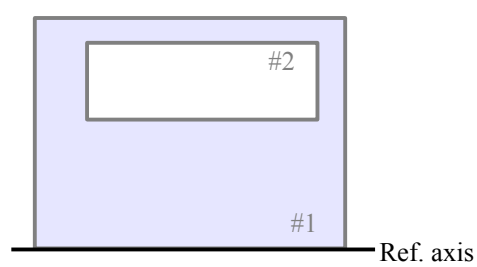
Step 2 Calculate the area a of each segment.

$$a_1 = 8\,\text{cm} \times 6\,\text{cm} = 48\,\text{cm}^2$$

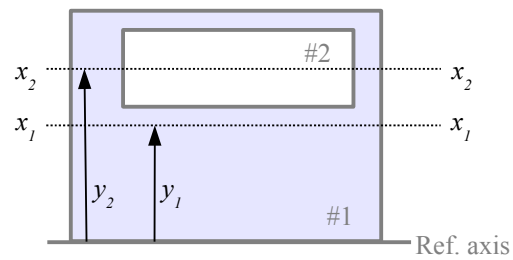
$$a_2 = -(6\,\text{cm} \times 2\,\text{cm}) = -12\,\text{cm}^2$$

Seg-ment	a (cm^2)
#1	48
#2	-12
Sum	36

Step 3 Pick a Reference Axis, and label it on the diagram.



Step 4 Draw the distance from the Reference Axis to the centroidal axes of the segments, x_1-x_1 and x_2-x_2. Label these distances y_1, y_2, etc. Enter these values into the table.



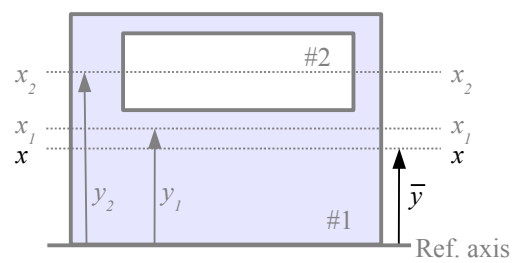
Seg-ment	a (cm^2)	y (cm)
#1	48	3
#2	-12	4.5
Sum	36	

Step 5 Calculate the product $a \times y$ for each segment, and enter these values and their sum in the table.

Seg-ment	a (cm^2)	y (cm)	ay (cm^3)
#1	48	3	144
#2	-12	4.5	-54
Sum	36		90

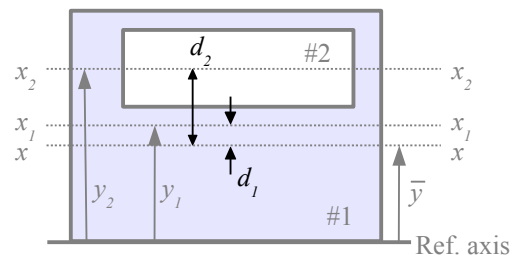
Step 6 Draw the distance from the Reference Axis to the *x-x* centroidal axis of the complex shape. Calculate this distance as

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{90\,\text{cm}^3}{36\,\text{cm}^2} = 2.5\,\text{cm}$$



Step 7 Draw the Transfer Distance *d* for each segment. This is the distance from the centroidal axis of the segment to the centroidal axis of the complex shape. Enter the results into the table.

Placing a hole in the upper part of this beam shifts the centroidal axis downward, to where more of the material lies. Since $d_1 = 0.5\,\text{cm}$, the centroidal axis of this hollow beam is 0.5 cm below the centroidal axis of a solid 6 cm × 8 cm solid beam.



Seg-ment	a (cm^2)	y (cm)	ay (cm^3)	d (cm)
#1	48	3	144	0.5
#2	-12	4.5	-54	2
Sum	36		90	

Step 8 Calculate the product $a \times d^2$ for each segment, and enter the results and their sum in the table. Be sure to calculate $a \times d^2$, not $a \times d$ ...it's an easy error to make.

Segment 1: $a \times d^2 = 48\,\text{cm}^2 (0.5\,\text{cm})^2 = 12\,\text{cm}^4$

Segment 2: $a \times d^2 = -12\,\text{cm}^2 (2\,\text{cm})^2 = -48\,\text{cm}^4$

Seg-ment	a (cm^2)	y (cm)	ay (cm^3)	d (cm)	ad^2 (cm^4)
#1	48	3	144	0.5	12
#2	-12	4.5	-54	2	-48
Sum	36		90		-36

Step 9 Calculate *I* for each segment about its centroidal axis. Note that *I* is negative for the hole.

$$I_{o1} = \frac{bh^3}{12} = \frac{8\,\text{cm}(6\,\text{cm})^3}{12} = 144\,\text{cm}^4$$

$$I_{o2} = -\frac{bh^3}{12} = -\frac{6\,\text{cm}(2\,\text{cm})^3}{12} = -4\,\text{cm}^4$$

Enter these values and their sum into the table.

Seg-ment	a (cm^2)	y (cm)	ay (cm^3)	d (cm)	ad^2 (cm^4)	I_o (cm^4)
#1	48	3	144	0.5	12	144
#2	-12	4.5	-54	2	-48	-4
Sum	36		90		-36	140

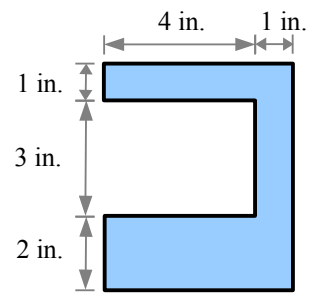
Step 10 Use the Transfer Formula to calculate *I* for the compound shape.

$$I = \sum I_o + \sum ad^2 = 140\,\text{cm}^4 - 36\,\text{cm}^4 = 104\,\text{cm}^4$$

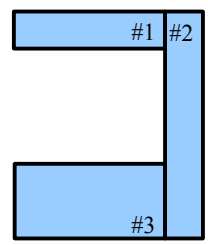
Some compound beams have more than two segments; the 10-step procedure is the same, with more rows in the table.

Moment of Inertia about the y–y Neutral Axis

The solution method for I_y is the same as for I_x, turned 90°. Instead of y terms in the table, we have x terms.



Step 1 Divide the compound beam into three rectangles.



Step 2 Calculate the area a of each segment.

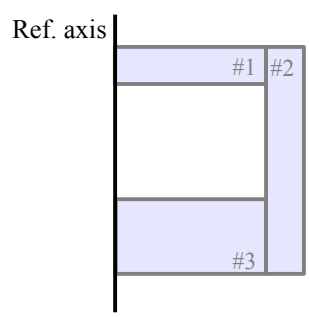
$a_1 = 4\,\text{in.} \times 1\,\text{in.} = 4\,\text{in.}^2$

$a_2 = 6\,\text{in.} \times 1\,\text{in.} = 6\,\text{in.}^2$

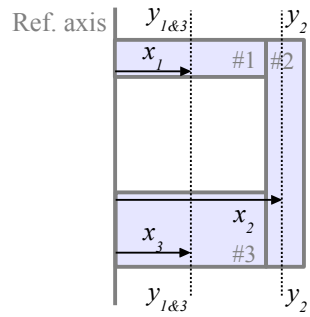
$a_3 = 4\,\text{in.} \times 2\,\text{in.} = 8\,\text{in.}^2$

Segment	a (in.^{2})
#1	4
#2	6
#3	8
Sum	18

Step 3 Pick a Reference Axis (easiest along the left edge or along the right edge).



Step 4 Draw the distance from the Reference Axis to the centroidal axes of the segments, y_1-y_1, y_2-y_2., and y_3-y_3. Label these distances x_1, x_2, etc. In this problem, the y_1-y_1 and y_3-y_3 centroidal axes are the same.



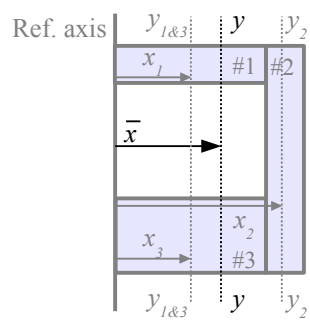
Segment	a (in.^{2})	x (in.)
#1	4	2
#2	6	4.5
#3	8	2
Sum	18	

Step 5 Calculate the product $a \times x$ for each segment, and enter these values and their sum in the table.

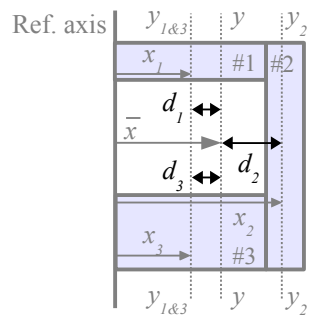
Segment	a (in.^{2})	x (in.)	ax (in.^{3})
#1	4	2	8
#2	6	4.5	27
#3	8	2	16
Sum	18		51

Step 6 Draw the distance from the Reference Axis to the y-y centroidal axis of the complex shape. Calculate this distance as

$$\bar{x} = \frac{\sum ax}{\sum a} = \frac{51\,\text{in.}^3}{18\,\text{in.}^2} = 2.833\,\text{in.}$$



Step 7 Draw the Transfer Distance d for each segment. This is the distance from the centroidal axis of the segment to the centroidal axis of the complex shape. For this drawing, $d_1 = \bar{x} - x_1 = d_3$ and $d_2 = x_2 - \bar{x}$.



Segment	a (in.^{2})	x (in.)	ax (in.^{3})	d (in.)
#1	4	2	8	0.83
#2	6	4.5	27	1.67
#3	8	2	16	0.83
Sum	18		51	

Step 8 Calculate the product $a \times d^2$ for each segment.

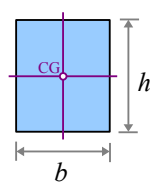
Segment	a (in.^{2})	x (in.)	ax (in.^{3})	d (in.)	ad^2 (in.^{4})
#1	4	2	8	0.83	2.78
#2	6	4.5	27	1.67	16.67
#3	8	2	16	0.83	5.56
Sum	18		51		25.00

Step 9 Calculate I for each segment about its centroidal axis:

$$I_{o1} = \frac{hb^3}{12} = \frac{1\text{ in.}(4\text{ in.})^3}{12} = 5.333\text{ in.}^4$$

$$I_{o2} = \frac{hb^3}{12} = \frac{6\text{ in.}(1\text{ in.})^3}{12} = 0.5\text{ in.}^4$$

$$I_{o3} = \frac{hb^3}{12} = \frac{2\text{ in.}(4\text{ in.})^3}{12} = 10.667\text{ in.}^4$$



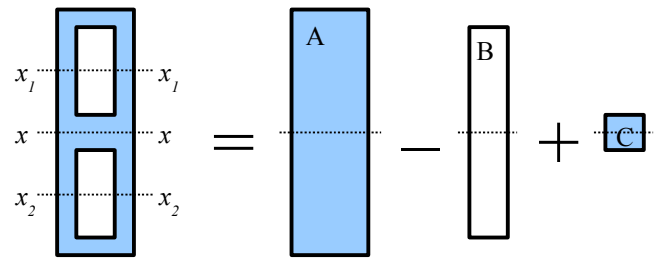
Segment	a (in.^{2})	x (in.)	ax (in.^{3})	d (in.)	ad^2 (in.^{4})	I_o (in.^{4})
#1	4	2	8	0.83	2.78	5.33
#2	6	4.5	27	1.67	16.67	0.5
#3	8	2	16	0.83	5.56	10.67
Sum	18		51		25.00	16.50

Step 10 Use the Transfer Formula to calculate I_y for the compound shape.

$$I_y = \sum I_o + \sum ad^2 = 25.0\text{ in.}^4 + 16.5\text{ in.}^4 = 41.5\text{ in.}^4$$

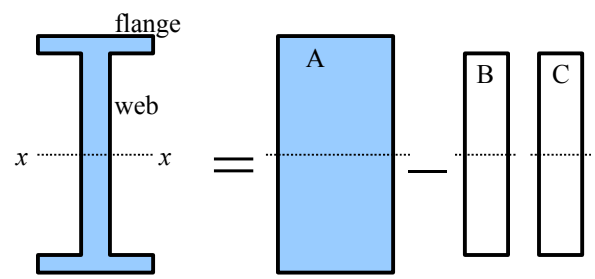
Shortcuts

Some shapes look like they require the Transfer Formula, but creative segment choices make the problem easy. Consider a beam with two hollow sections. The moment of inertia could be calculated using the 10-step Transfer Formula method with segments #1 (large rectangle), #2 (upper cavity), and #3 (lower cavity), but it is easier to break it into rectangles sharing the same centroidal axis.



Calculate the moment of inertia of segment A (large rectangle), subtract the moment of inertia of segment B (two cavities joined together) and add the moment of inertia of segment C (material between the two cavities).

The moment of inertia of a wide-flange beam made of welded rectangular plates is easy to solve: subtract the moments of inertia of the spaces to the left and right of the web from the moment of inertia of a large rectangle.



Hot-rolled steel wide-flange beams, channels, and angles have rounded internal and external corners, which makes hand calculation of I_x and I_y difficult. Instead of calculating these values, you can look them up in tables (see Appendix D).

Radius of Gyration

Columns are tall, thin structures loaded in compression which fail at stresses below the expected yield strength of the material. Column analysis uses the radius of gyration to calculate failure loads. The radius of gyration is defined as $r_G = \sqrt{I/A}$; the greater the radius of gyration, the more resistant the column is to buckling failure.

Polar Moment of Inertia

The moment of inertia of a cross-sectional area is $I_x = \sum_1^n a_i y_i^2$ relative to the x-x centroidal axis and $I_y = \sum_1^n a_i x_i^2$ relative to the y-y centroidal axis. In torsion problems, we need a moment of inertia relative to the centroid of the shape, so we use the radius of each elemental area from the center of gravity, *r*, to define the polar moment of inertia $J = \sum_1^n a_i r_i^2$. The units are the same as for moment of inertia: in.^{4} or mm^4. From calculus, the exact equation for a circle is $J = \frac{\pi}{32} d^4$; for a hollow circle, $J = \frac{\pi}{32}(d_o^4 - d_i^4)$.

Key Equations

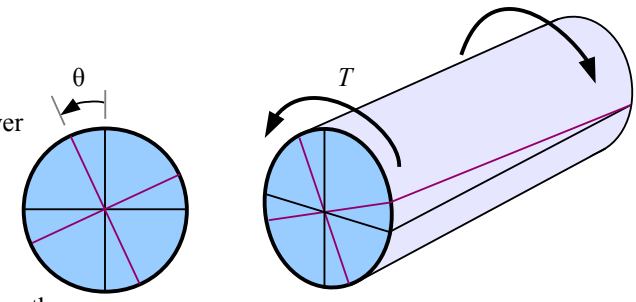
Moments of inertia of simple shapes about the *x-x* centroidal axis are in Appendix C.

Use the Transfer Formula to find the moment of inertia about a noncentroidal axis, or for calculating the moment of inertia of a compound shape: $I = I_o + ad^2$.

The polar moment of inertia for a solid circle is $J = \frac{\pi}{32} d^4$, and for a hollow circle is $J = \frac{\pi}{32}(d_o^4 - d_i^4)$.

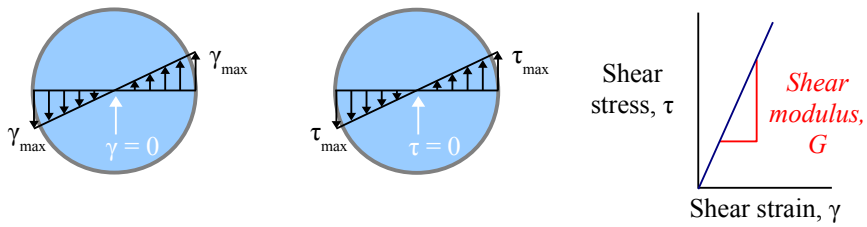
Chapter 7: Torsion in Round Shafts

Shear Stress in a Round Shaft

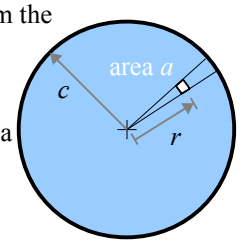


Many machine parts are loaded in torsion, either to transmit power (like a driveshaft or an axle shaft in a vehicle) or to support a dynamic load (like a coil spring or a torsion bar). Power transmission parts are typically circular solid shafts or circular hollow shafts because these shapes are easy to manufacture and balance, and because the outermost material carries most of the stress. For a given maximum size, more material is available along the entire surface of a circle than at the four corners of a square.

Apply a torque T to a round shaft, and the shaft will twist through an angle θ. Twisting means the material is deforming, so we have strain in the material. The greatest strain, γ_{max}, is at the surface, while strain is zero at the center of the shaft. The strain varies linearly from the center to the surface of the shaft.



We saw in Chapter 2 that materials like steel and aluminum follow Hooke's law: the ratio of normal stress to normal strain is Young's modulus: $E=\sigma/\varepsilon$. Young's modulus is a materials property (see Appendix B). In shear, the ratio of shear stress to shear strain is the shear modulus: $G=\tau/\gamma$ (also in Appendix B). Therefore, the stress in the shaft also varies linearly from the center to the surface of the shaft. We also saw in Chapter 2 that this stress is a shearing stress, so $\tau=0$ at the center, and $\tau=\tau_{max}$ at the surface.



Consider a small area a at a distance r from the center of the circle. If we define c as the distance from the centroid to the surface of the circle, then the shear stress at r is $\tau=\left(\frac{r}{c}\right)\tau_{surface}$. Since shear stress is force divided by area, the shear force acting on area a is $P_{shear(a)}=\tau a=\frac{r\tau_{surface}a}{c}$. The torque on area a is the force times the distance from the centroid: $T_{(a)}=P_{shear}r=\frac{r^2\tau_{surface}a}{c}=\frac{\tau_{surface}}{c}a r^2$. The total torque on the entire circular area about the centroid is the sum of the torques on all the small areas that comprise the circle, so $T=\frac{\tau_{surface}}{c}\sum a r^2$. We've seen the last part of this equation before: the polar moment of inertia of a circle is $J=\sum a r^2$, therefore $T=\frac{\tau_{surface}J}{c}$. For design purposes, only the maximum stress matters, so we usually drop the subscript from the shear stress, understanding that we mean the stress at the surface, so we write $T=\frac{\tau J}{c}$. In many problems, we know the applied torque and dimensions, but need the stress. Solving for stress, $\tau=\frac{Tc}{J}$.

Example #1

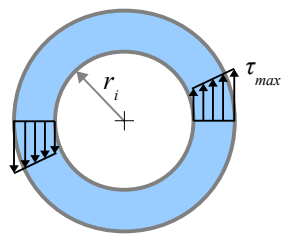
The torque on a ¼ inch diameter circular shaft is 5 ft.·lb. What is the shear stress? Report the answer in ksi.

Solution The distance from the centroid to the surface is $c=\frac{d}{2}=\frac{0.25\,\text{in.}}{2}=0.125\,\text{in.}$. The polar moment of inertia of a circle is $J=\frac{\pi d^4}{32}$. Shear stress $\tau=\frac{Tc}{J}=\frac{5\,\text{ft.}\cdot\text{lb.}\;\;0.125\,\text{in.}}{}\frac{32}{\pi(0.25\,\text{in.})^4}\left|\frac{12\,\text{in.}}{1\,\text{ft.}}\right|\frac{\text{kip}}{10^3\,\text{lb.}}=19.6\,\text{ksi}$

Once we have the shear stress, we can compare it with the allowable shear stress for this material. If the calculated stress is higher than the allowable stress, then we can either select a stronger material or use a larger diameter shaft.

The shear stress in a hollow shaft varies linearly from the inside surface to the outside surface. Calculate the shear stress on the inside surface as $\tau=\frac{Tr_i}{J}$ where r_i is the inside radius.

Example #2



A torque of 800 N·m is applied to a hollow shaft. The inside diameter is 100 mm; the outside diameter is 120 mm. What is the shear stress on the inside and outside surfaces? Report the answer in MPa.

Solution The distance from the centroid to the inside surface is $r_i=\frac{d_i}{2}=\frac{100\,\text{mm}}{2}=50\,\text{mm}$.

The distance from the centroid to the outside surface is $c=\frac{d_o}{2}=\frac{120\,\text{mm}}{2}=60\,\text{mm}$.

The polar moment of inertia of a hollow circle is $J=\frac{\pi(d_o^4-d_i^4)}{32}$. We use the same J for both the inside and the outside shear stress calculations.

Shear stress on the inside surface:

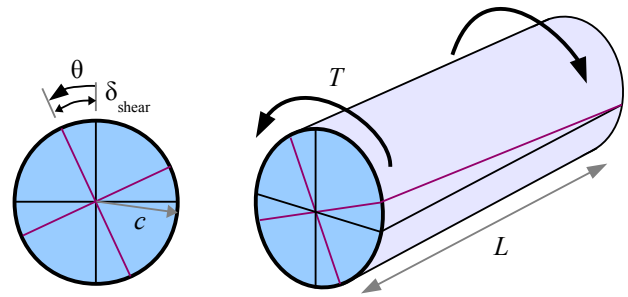
$$\tau_i=\frac{Tr_i}{J}=\frac{800\,\text{N}\cdot\text{m}\;\;50\,\text{mm}}{}\frac{32}{\pi[(120\,\text{mm})^4-(100\,\text{mm})^4]}\left|\frac{\text{MPa}\;\;\text{m}^2}{10^6\,\text{N}}\right|\frac{(10^3\,\text{mm})^3}{\text{m}^3}=3.80\,\text{MPa}\,.$$

Shear stress on the outside surface:

$$\tau_o=\frac{Tc}{J}=\frac{800\,\text{N}\cdot\text{m}\;\;60\,\text{mm}}{}\frac{32}{\pi[(120\,\text{mm})^4-(100\,\text{mm})^4]}\left|\frac{\text{MPa}\;\;\text{m}^2}{10^6\,\text{N}}\right|\frac{(10^3\,\text{mm})^3}{\text{m}^3}=4.55\,\text{MPa}$$

In some problems, we know the applied torque and the shear strength of the material, and we need to calculate the minimum diameter that will work. Write the shear stress equation algebraically, substituting $\frac{\pi d^4}{32}$ for J and $\frac{d}{2}$ for c. Now rewrite the equation to solve for diameter: $d=\sqrt[3]{\text{algebraic expression}}$.

Angle of Twist in a Round Shaft



Normal strain is defined as the change in length of tensile member divided by its original length: $\varepsilon = \delta / L$. We can define shear strain on a torsion member as the change in location of a point on the surface of the shaft divided by the length of the shaft: $\gamma = \delta_{shear} / L$.

The angle of twist, θ, is measured in radians, so we can substitute $\delta_{shear} = c\,\theta$. Now the shear strain is $\gamma = c\,\theta / L$. Substitute this value in the equation for shear modulus, and $G = \frac{\tau}{\gamma} = \frac{\tau L}{c\,\theta}$. Rewrite the equation to find the angle of twist $\theta = \frac{\tau L}{G c}$.

Since shear stress $\tau = \frac{Tc}{J}$, we can substitute this expression into the angle of twist equation to get $\theta = \frac{\tau L}{G c} = \frac{Tc}{J}\frac{L}{Gc} = \frac{TL}{JG}$.

Example #3

A solid round aluminum shaft has a maximum allowable shear stress of 80 MPa. The shaft is 10 cm in diameter and 1 m long. What is the angle of twist? Report the answer in radians and in degrees.

Solution From Appendix B, the shear modulus of aluminum is $G = 28 \times 10^3\,\text{MPa}$. The angle of twist is

$$\theta = \frac{\tau L}{G c} = \frac{80\,\text{MPa}\;\;1\,\text{m}}{28 \times 10^3\,\text{MPa}\,\frac{1}{2}(10\,\text{cm})}\left|\frac{100\,\text{cm}}{\text{m}}\right. = 0.057\,\text{rad}$$

Convert to degrees using the conversion button on your calculator, or use the Factor-Label Method of Unit Conversion:

$$\frac{0.057\,\text{rad}}{}\left|\frac{180\,°}{\pi\,\text{rad}}\right. = 3.3\,°$$

Example #4

A hollow round steel shaft has a torque of 600 ft.lb. The shaft has an outside diameter of 1.25 in., an inside diameter of 1.10 in., and a length of 36 in. What is the angle of twist? Report the answer in radians and in degrees.

Solution From Appendix B, the shear modulus of steel is $G = 12{,}000\,\text{ksi}$. The angle of twist is

$$\theta = \frac{T L}{JG} = \frac{32\,T L}{\pi d^4 G} = \frac{32 \cdot 600\,\text{ft.lb.} \cdot 36\,\text{in.}\qquad\qquad \text{in.}^2}{\pi[(1.25\,\text{in.})^4 - (1.10\,\text{in.})^4]\,12{,}000\,\text{kip}}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\;\frac{12\,\text{in.}}{\text{ft.}}\right. = 0.225\,\text{rad}$$

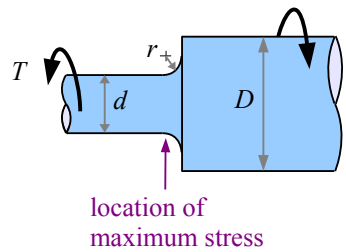
Convert to degrees using the conversion button on your calculator, or use the Factor-Label Method of Unit Conversion:

$$\frac{0.225\,\text{rad}}{}\left|\frac{180\,°}{\pi\,\text{rad}}\right. = 12.9\,°$$

Looking at the equations, you can see that the angle of twist is directly proportional to length: if you double the length of the shaft, the angle of twist doubles. In a tensile member, if you double the length the elongation doubles. Shear stress is independent on length, just as normal stress is independent of the length of a tensile member.

Stress Concentration in Torsion

If a shaft has a uniform diameter along its length, then the maximum shear stress at the surface is $\tau = \frac{Tc}{J}$. Many shafts in machines have different diameters in different locations, such as at the flange of an axle shaft, or adjacent to a bearing surface. A sudden change in diameter creates a stress concentration. The maximum stress in the smaller diameter section, at the base of the fillet, is $\tau = K\frac{Tc}{J}$ where K is the stress concentration factor. Other dimensional changes, such as keyways and transverse through-holes, also create stress concentrations in shafts.



Look at the diagram of the shaft: the fillet radius meets the flat face of the large diameter portion of the shaft. The two shaft diameters and fillet radius are all independent variables, so the stress concentration graph has several curves, each for a different ratio of shaft diameters. Use a five-step process for calculating the maximum shear stress at the stress concentration site:

Step 1 Divide the fillet radius by the smaller shaft diameter find the ratio r/d.

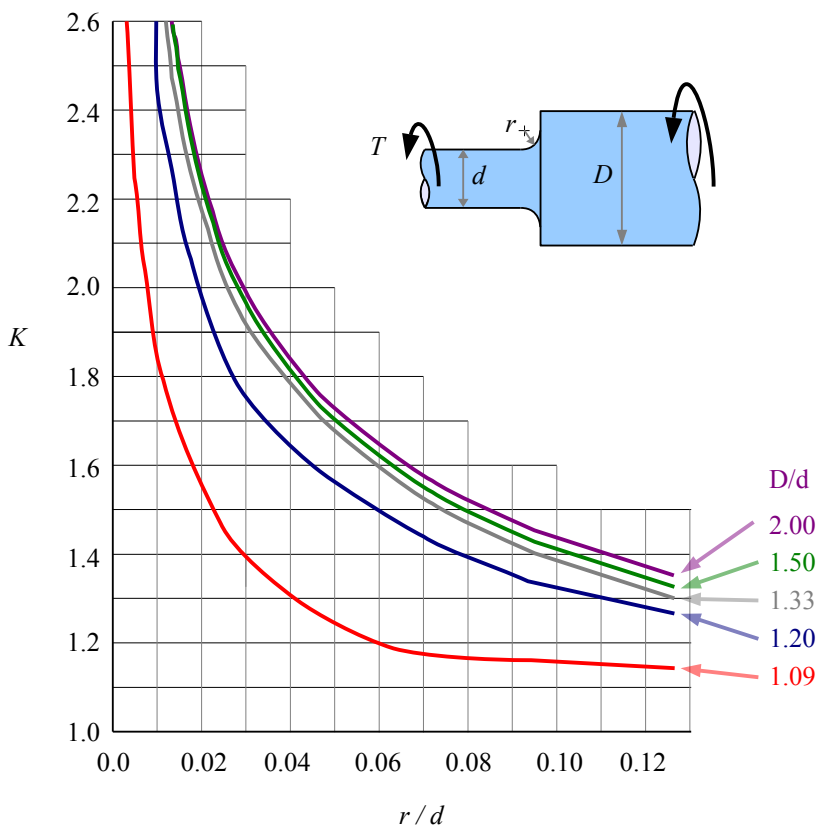
Step 2 Divide the larger shaft diameter by the smaller shaft diameter find the ratio D/d.

Step 3 Using these two values, find the value of K from the graph. If your calculated D/d does not match the values on the graph, interpolate. For example, if $r/d = 0.06$ and $D/d = 1.15$, then $K \approx 1.35$.

Step 4 Calculate c and J using the smaller shaft diameter.

Step 5 Calculate the maximum stress using $\tau = K\frac{Tc}{J}$.

Fillet in a stepped shaft loaded in torsion



Example #5

A 2¼ inch diameter shaft is welded to a 3 inch diameter shaft. The weld bead is ground to a radius of 0.125 inches. If the torque is 600 ft.lb., what is the maximum shear stress at the fillet? Report the answer in ksi.

Solution Calculate the ratios $\frac{D}{d} = \frac{3\,\text{in.}}{2.25\,\text{in.}} = 1.33$ and $\frac{r}{d} = \frac{0.125\,\text{in.}}{2.25\,\text{in.}} = 0.056$. Use the 1.33 line on the graph to find the stress concentration factor $K \approx 1.65$. $\tau_{max} = K\frac{Tc}{J} = K\,T\frac{d}{2}\frac{32}{\pi d^4} = \frac{16\,K\,T}{\pi d^3} = \frac{16 \cdot 1.65 \cdot 600\,\text{ft.}\cdot\text{lb.}}{\pi(2.25\,\text{in.})^3}\left|\frac{12\,\text{in.}}{\text{ft.}}\right|\frac{\text{kip}}{10^3\,\text{lb.}} = 5.31\,\text{ksi}$

Key Equations

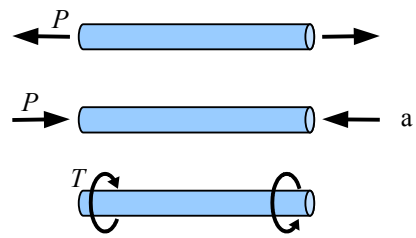
Shear stress on the surface of a round solid or hollow shaft is the torque times the distance from the center to the outside surface, divided by the polar moment of inertia: $\tau = \frac{Tc}{J}$

The angle of twist of a round solid or hollow shaft can be calculated with $\theta = \frac{\tau L}{G c}$ or with $\theta = \frac{TL}{JG}$, depending on the available input variables. With a stepped shaft, the maximum stress is $\tau = K \frac{Tc}{J}$ where K is the stress concentration factor.

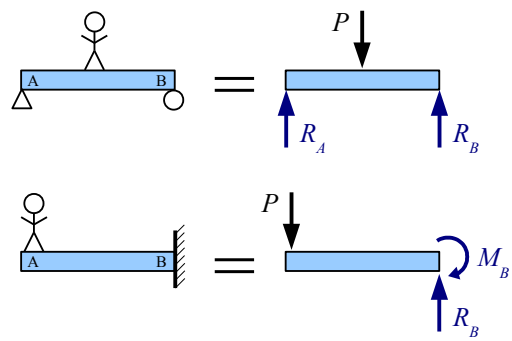
Chapter 8: Beam Reactions, Shear Diagrams, and Moment Diagrams

Loads on Beams

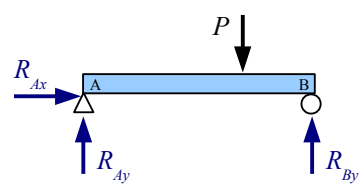
The way a part is loaded determines whether it is called a tensile or compressive member, a torsional shaft, or a beam. If you take a ½ inch diameter steel rod and pull it lengthwise with a load P, the rod will develop a tensile stress $\sigma = P/A$ where A is the cross-sectional area of the rod. Loading the rod in tension parallel to its axis makes the rod a tensile member; loading it in compression parallel to its axis makes it compressive member. If you twist the steel rod with torque T, then we call it a torsional shaft.



If loading is perpendicular (transverse) to its axis so that the rod bends, then the rod is called a *beam*. You can load a beam with point loads, uniformly distributed loads, or nonuniformly distributed loads. A swimmer standing on the end of a diving board is an example of a point load: a force applied at a single point on the beam. This point load P could be the weight of an object on the beam, or it could be a load applied by a cable or rod attached to the beam at that point. Typically, the left end of the beam is marked "A" and the right end is marked "B". For each loaded beam we draw beam reactions: reaction forces R_A and R_B for a simply supported beam; reaction force R_B and reaction moment M_B for a cantilever beam.

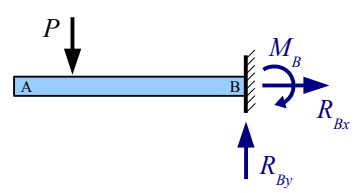


The symbols for supports indicate the reactions that develop at the support. For example, support "A" is pinned, like a hinge, so the symbol for the support is a triangle. A pinned support may have vertical and horizontal reaction forces. The beam at the right has no applied horizontal loads, therefore $R_{Ax} = 0$. In the beam problems in this chapter, there are no applied horizontal forces, so the horizontal reaction force is zero, and the vertical reaction forces R_{Ay} and R_{By} are abbreviated R_A and R_B.

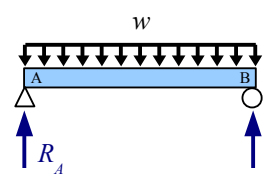


A roller support allows the beam to move freely horizontally; the symbol is a circle. A roller support has only a vertical reaction force. A beam supported by a pin at one end and a roller at the other end is called a *simply-supported beam*.

A *cantilever beam* is embedded in a wall, so the beam has reaction forces as well as a reaction moment. The horizontal reaction force $R_{Bx} = 0$ if there are no horizontal applied forces, so the vertical reaction force R_{By} is abbreviated R_B.



The weight of a beam is a uniform distributed load. The weight per unit length, w, typically has units of lb./ft., kips/ft., or kN/m. Consider a wide-flange beam, or "W-beam," having a cross-section that looks like a Courier font capital letter `I` The U.S. Customary W-beam designation system has two numbers: the first is the nominal depth, and the second is the weight per unit length. For example, a W24×162 beam has a nominal depth of 24 inches and a weight per unit length $w = 162$ lb./ft. . If the beam is 10 feet long, then the total weight, W, of the beam is $W = wL = \frac{162\,\text{lb.}}{\text{ft.}}\frac{10\,\text{ft.}}{} = 1{,}620\,\text{lb.}$



In Canada, W-beams are specified in SI (metric) units. These beams are designated by mass, not weight: a W250×115 wide flange beam has a nominal depth of 250 mm and a mass per unit length of 115 kg/m. From Newton's Second Law, $\text{force} = \text{mass} \times \text{acceleration}$, or in this case, $\text{weight} = \text{mass} \times \text{acceleration of gravity}$. The SI unit of force and weight is the newton (N), defined as $1\,\text{N} = 1\,\frac{\text{kg m}}{\text{s}^2}$, and the acceleration of gravity is $9.81\,\text{m/s}^2$. The weight per unit length of a W250×115 wide-flange beam is $w = \frac{115\,\text{kg}}{\text{m}}\frac{9.81\,\text{m}}{\text{s}^2}\left|\frac{\text{N s}^2}{\text{kg m}}\right|\frac{\text{kN}}{10^3\,\text{N}} = 1.13\,\text{kN/m}$. If the beam is 4 m long, then the total

weight of the beam is $W = wL = \frac{1.13\,\text{kN}}{\text{m}}\frac{4\,\text{m}}{} = 4.51\,\text{kN}$.

You can also calculate the weight per unit length from the cross-sectional area and the specific weight of the material. Specific weight is weight divided by volume: $\gamma = W/V$. The volume of a beam of uniform cross-section is the cross-sectional area times the length: $V = A \cdot L$. Combining, $\gamma = \frac{W}{V} = \frac{W}{A \cdot L}$ or $\gamma \cdot A = \frac{W}{L}$. Weight per unit length $w = \frac{W}{L} = \gamma \cdot A$.

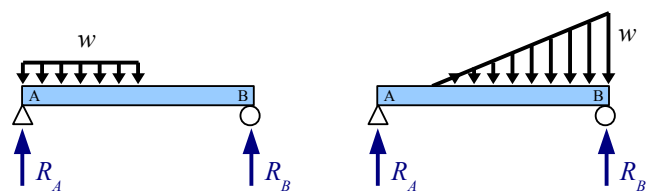
Example #1

What is the weight per unit length of a 1 inch diameter steel rod? Report the answer in lb./ft.

Solution The cross-sectional area of a circle $A = \frac{\pi}{4} d^2$. The specific weight of steel is 490 lb./ft.^{3}.

Weight per unit length $w = \rho \cdot A = \frac{\gamma \pi d^2}{4} = \frac{490\,\text{lb.}}{\text{ft.}^3}\frac{\pi(1\,\text{in.})^2}{4}\left|\frac{\text{ft.}^2}{(12\,\text{in.})^2}\right. = 2.67\,\text{lb./ft.}$

A distributed load may run the length of the beam (like the beam's weight), may run along a portion of the beam, or may be nonuniform.



Reactions for Simply-Supported Simple Beams

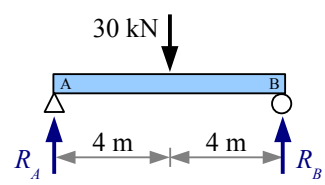
You can calculate the reaction forces for a symmetrically-loaded, simply-supported beam by dividing the total load by 2, because each end of the beam carries half the load. The reactions for the beam with a point load are $R_A = R_B = P/2$.

Example #2

Calculate the reaction forces R_A and R_B for a beam with a 30 kN load at the midspan.

Solution Divide the total load by 2 to obtain the reaction forces,

$R_A = R_B = \frac{P}{2} = \frac{30\,\text{kN}}{2} = 15\,\text{kN}$



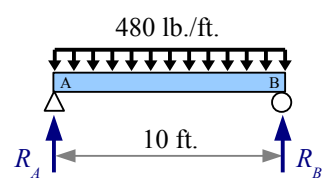
A simply-supported beam with a uniform distributed load also has a symmetrical loading pattern. Divide the total load on the beam by 2 to find the reaction forces.

Example #3

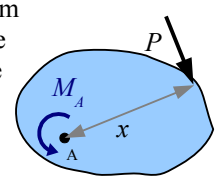
Calculate the reaction forces R_A and R_B for a 10-ft. beam with a 480 lb./ft. uniformly distributed load. Report the answer in kips.

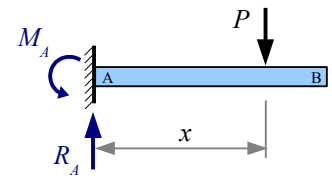
Solution Multiply the uniform distributed load by the length to find the total load on the beam: $W = wL$. Divide the total load by 2 to obtain the reaction forces,

$R_A = R_B = \frac{W}{2} = \frac{wL}{2} = \frac{480\,\text{lb.}}{\text{ft.}}\frac{10\,\text{ft.}}{}\frac{1}{2}\left|\frac{\text{kip}}{1000\,\text{lb.}}\right. = 2.4\,\text{kips}$

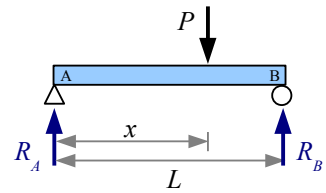


For beams with nonsymmetrical loading, we need two equations from *Statics*: the sum of the vertical forces equals zero, and the sum of the moments about a point equals zero. The moment about a point is the force acting on an object times the perpendicular distance from the force to the pivot point. Whether the object is a blob or a beam, the moment about point A is $M_A = P \cdot x$.



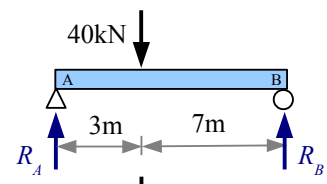


You can pick a pivot point at either end of a simply-supported beam. Most students find it easier to select the left end of the beam, point A. Since moment has a magnitude and a direction (clockwise or counterclockwise), we need to establish a convention for positive and negative moments. We'll select counterclockwise as positive, symbolized as ↺+, and start adding up the moments about point A. The load acts at a distance x from point A. Think of point A as a hinge point…the load causes the beam to rotate clockwise about point A, so the moment is negative. The reaction force R_B acts at a distance L from point A, and causes the beam to rotate counterclockwise about point A, so the moment is positive. The moment about point A is ↺+ $\sum M_A = 0 = -Px + R_B L$. Now solve for the reaction force $R_B = \frac{Px}{L}$. Use the sum of the forces in the vertical direction to calculate the other reaction force. Forces have magnitude and direction; pick upwards as positive, so ↑+ $\sum F_y = 0 = R_A - P + R_B$. Now solve for the reaction force $R_A = P - R_B$.



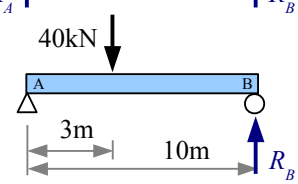
Example #4

Calculate the reaction forces R_A and R_B for this simply-supported beam.



Solution Redraw the diagram, marking the distances to all loads and reactions from point A.

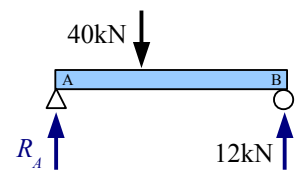
The moment about point A is ↺+ $\sum M_A = 0 = -40\,\text{kN} \cdot 3\,\text{m} + R_B \cdot 10\,\text{m}$.



Rewrite the equation to find the reaction force $R_B = \frac{40\,\text{kN} \cdot 3\,\text{m}}{10\,\text{m}} = 12\,\text{kN}$.

Use the sum of the forces in the vertical direction to calculate the other reaction force:

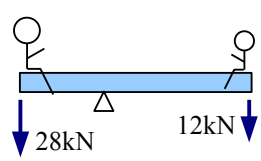
↑+ $\sum F_y = 0 = R_A - 40\,\text{kN} + 12\,\text{kN}$.

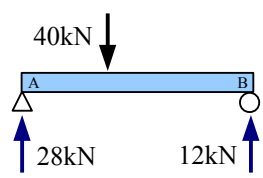


Rewrite the equation to find the reaction force $R_A = 40\,\text{kN} - 12\,\text{kN} = 28\,\text{kN}$.

You can check the answer by solving the sum of the moments about point B.

Most of the applied load is supported by the left end of the beam. Intuitively, this makes sense because the load is closer to the left end of the beam. Flip the beam upside down and it looks like two children on a see-saw: the pivot point has to be closer to the heavier child in order to balance the see-saw.

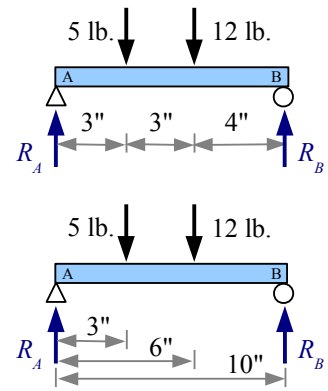




Use the same technique for a simply-supported beam with multiple point loads.

Example #5

Calculate the reaction forces R_A and R_B for a beam with two point loads.



Solution Redraw the diagram, marking the distances to all loads and reactions from point A. This step may seem to be a waste of time, but as the loading conditions become more complicated, it becomes more important to redimension the drawing, in order to keep track of the distances used in the Sum of the Moments calculation.

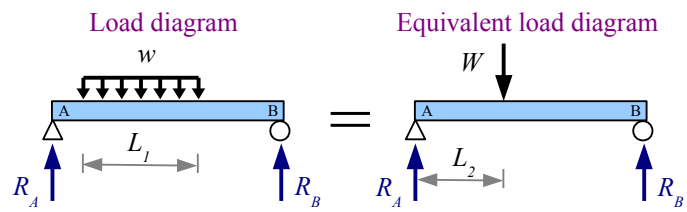
The moment about point A is $\circlearrowleft + \sum M_A = 0 = -5\,\text{lb.}\cdot 3\,\text{in.} - 12\,\text{lb.}\cdot 6\,\text{in.} + R_B \cdot 10\,\text{in.}$

Rewrite the equation to find the reaction force $R_B = \dfrac{5\,\text{lb.}\cdot 3\,\text{in.} + 12\,\text{lb.}\cdot 6\,\text{in.}}{10\,\text{in.}} = 8.7\,\text{lb.}$

Sum of the forces $\uparrow +\ \sum F_y = 0 = R_A - 5\,\text{lb.} - 12\,\text{lb.} + 8.7\,\text{lb.}$

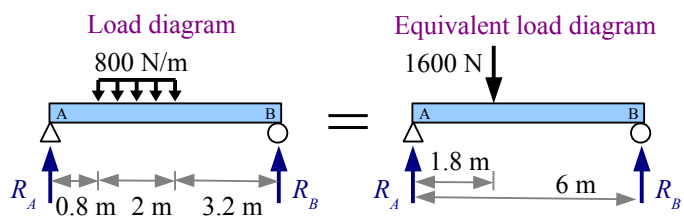
Rewrite the equation to find the reaction force $R_A = 5\,\text{lb.} + 12\,\text{lb.} - 8.7\,\text{lb.} = 8.3\,\text{lb.}$

If a uniformly distributed load is not symmetrical, then we need to convert the distributed load into a point load equivalent to the total load $W = wL_1$ where L_1 is the length of the distributed load. The equivalent point load is located at the centroid of the distributed load… the center of the rectangle. Use the *equivalent load diagram* for calculating the reaction forces.



Example #6

Calculate the reaction forces R_A and R_B for a beam with a uniform distributed load of 800 N/m. Report the result in N.



Solution Draw an equivalent load diagram, placing the equivalent load at the centroid of the distributed load. Use the equivalent load diagram to find the reaction forces.

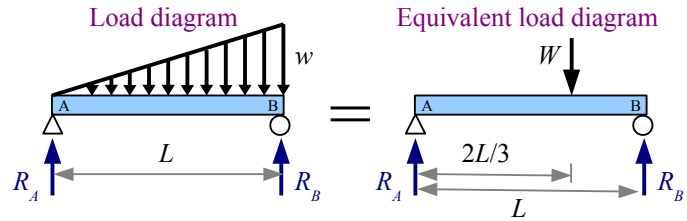
The distributed load runs for 2m, so the location of the equivalent load is 1m from the left end of the distributed load, or 1.8m from point A. The equivalent load $W = wL = \dfrac{800\,\text{N}\cdot 2\,\text{m}}{\text{m}} = 1600\,\text{N}$.

The moment about point A is $\circlearrowleft + \sum M_A = 0 = -1600\,\text{N}\cdot 1.8\,\text{m} + R_B \cdot 6\,\text{m}$.

Rewrite the equation to find the reaction force $R_B = \dfrac{1600\,\text{N}\cdot 1.8\,\text{m}}{6\,\text{m}} = 480\,\text{N}$.

Sum of the forces $\uparrow +\ \sum F_y = 0 = R_A - 1600\,\text{N} + 480\,\text{N}$. Solve for the reaction force $R_A = 1600\,\text{N} - 480\,\text{N} = 1120\,\text{N}$.

Use the same approach for a nonuniformly distributed load. Again, the location of the equivalent load is at the centroid of the distributed load. The centroid of a triangle is one third of the distance from the wide end of the triangle, so the location of the equivalent load is one third of the distance from the right end of this beam, or two thirds of the distance from the left end.



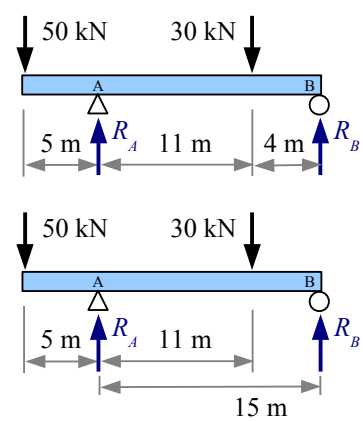
The load varies from 0 at the left end to w at the right end; therefore, the total load is the average of these loads times the beam length: $W=\left(\frac{0+w}{2}\right)L=\frac{wL}{2}$. If the beam has point loads *and* distributed loads, draw an equivalent load diagram with the applied point loads and the equivalent point loads, then solve like Example #5.

Reactions for Overhanging and Cantilever Beams

A simply-supported beam is supported by a pinned connection at one end and a roller support at the other; all applied loads lie between these two points. An overhanging beam extends beyond one or both supports. The solution method is the same as for simply-supported beams: use the sum of the moments about one of the support points to find the reaction at the other support point.

Example #7

Calculate the reaction forces R_A and R_B for an overhanging beam with two point loads.



Solution Redraw the diagram, marking the distances to all loads and reactions from point A.

The moment about point A is $\circlearrowleft+\ \sum M_A=0=50\,\text{kN}\cdot 5\,\text{m}-30\,\text{kN}\cdot 11\,\text{m}+R_B\cdot 15\,\text{m}$. Notice the 50 kN load produces a positive (counterclockwise) moment about point A, while the 30 kN load produces a negative (clockwise) moment about point A.

Rewrite the equation to find the reaction force

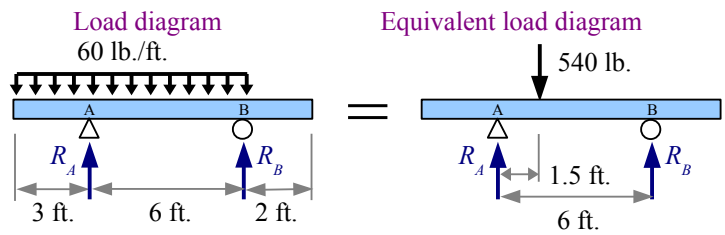
$$R_B=\frac{-50\,\text{kN}\cdot 5\,\text{m}+30\,\text{kN}\cdot 11\,\text{m}}{15\,\text{m}}=5.33\,\text{kN}$$

Sum of the forces $\uparrow+\ \sum F_y=0=R_A-50\,\text{kN}-30\,\text{kN}+5.33\,\text{kN}$. Solve for the reaction force $R_A=50\,\text{kN}+30\,\text{kN}-5.33\,\text{kN}=74.67\,\text{kN}$.

Solve for the reactions to a distributed load on an overhanging beam the same way as for a distributed load on a simply-supported beam: draw an equivalent load diagram, then use the sum of the moments and the sum of the forces to find the reactions.

Example #8

Calculate the reaction forces R_A and R_B for an overhanging beam with a uniform distributed load.



Solution Draw an equivalent load diagram, marking the distances to all loads and reactions from point A. We do not need the dimensions to the ends of the overhangs, because there are no loads outside of the two supports.

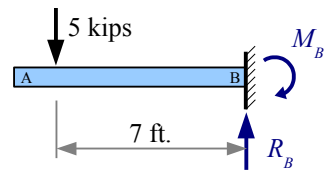
The distributed load runs for 9 ft., so the location of the equivalent load is 4.5 ft. from the left end of the distributed load, or 1.5 ft. to the right of point A. The equivalent load $W = wL = \frac{60\,\text{lb.}\cdot 9\,\text{ft.}}{\text{ft.}} = 540\,\text{lb.}$

The moment about point A is $\circlearrowleft + \sum M_A = 0 = -540\,\text{lb.}\cdot 1.5\,\text{ft.} + R_B \cdot 6\,\text{ft.}$

Rewrite the equation to find the reaction force $R_B = \frac{540\,\text{lb.}\cdot 1.5\,\text{ft.}}{6\,\text{ft.}} = 135\,\text{lb.}$

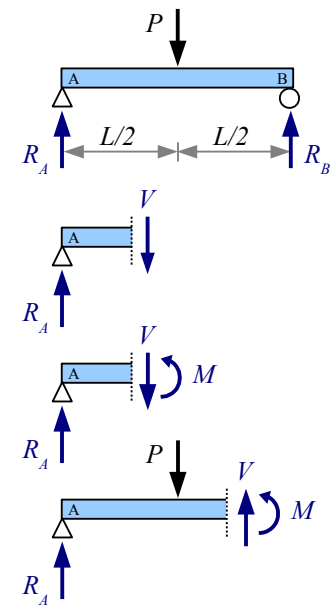
Sum of the forces $\uparrow + \sum F_y = 0 = R_A - 540\,\text{lb.} + 135\,\text{lb.}$. Solve for the reaction force $R_A = 540\,\text{lb.} - 135\,\text{lb.} = 405\,\text{lb.}$.

A cantilever beam with a single support has a reaction force and a reaction moment. The reaction force R_B equals the sum of the applied forces on the beam, so $R_B = P = 5\,\text{kips}$.



The moment reaction equals the sum of the moments about point B – the applied load times its distance from the wall – so $M_B = P \cdot x = 5\,\text{kips}\cdot 7\,\text{ft.} = 35\,\text{kip ft.}$.

Shear Diagrams



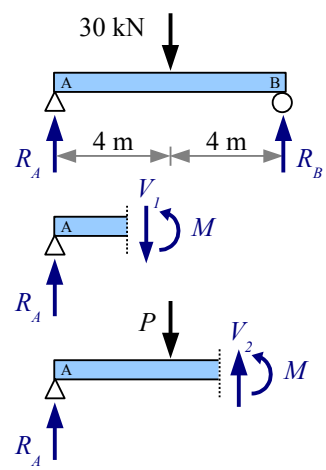
When we calculate reaction forces and torques on tension members, torsion members, and beams, we are calculating external forces and torques. Unless the material has no strength at all, the material resists these external loads by developing internal loads. Apply a torque of 25 ft.lb. to each end of a ½ inch diameter rod, and a resisting torque of 25 ft.lb. exists within the rod all along its length. Apply a tensile force of 400 N to each end of a 2 cm diameter rod, and a resisting tensile force of 400N exists within the rod all along its length. Beams in bending also develop internal forces to resist external forces. Since the external forces on beams are transverse (perpendicular to the axis of the beam), the internal resisting forces are also transverse forces.

Imagine a simply-supported beam with a point load at the mid-span. Cut the beam to the left of the point load, and draw a free-body diagram of the beam segment. In a free-body diagram, forces must balance. Therefore, a downward force at the cut edge balances the support reaction R_A. We call this shear force V. It is a shear force because the force acts parallel to a surface (the cut edge of the beam).

The forces R_A and V are in balance (equal in value; opposite in sign), but our segment wants to spin clockwise about point A. To counteract this tendency to spin, a moment M develops within the beam to prevent this rotation. The moment equals the shear force times its distance from point A.

Cut the beam to the right of the point load, and draw the free-body diagram. Since P is larger than R_A, force V points upwards.

Example #9



Calculate the shear forces in this beam to the left and to the right of the 30 kN point load.

Solution The loading is symmetrical, so $R_A = R_B = \frac{P}{2} = \frac{30\,\text{kN}}{2} = 15\,\text{kN}$.

Use the sum of the forces to find V.

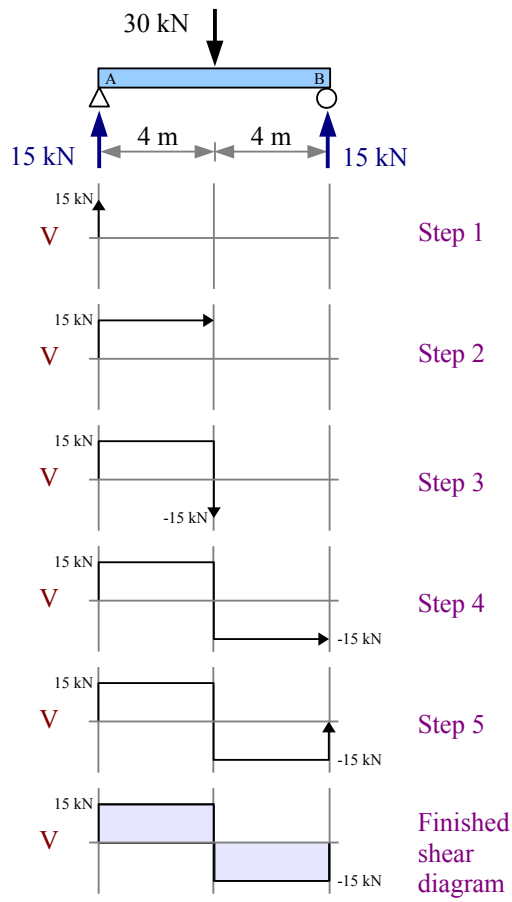
Between support A and point load P, $\uparrow + \quad \sum F_y = 0 = R_A - V_1$.

Solving for shear load, $V_1 = -R_A = -15\,\text{kN}$.

Between point load P and support B, $\uparrow + \quad \sum F_y = 0 = R_A - P + V_2$.

Solving for shear load, $V_2 = -R_A + P = -15\,\text{kN} + 30\,\text{kN} = 15\,\text{kN}$.

We can sketch V as a function of location along the beam using a *Shear Diagram*. Draw vertical construction lines below the load diagram wherever the applied loads and reactions occur. Draw a horizontal construction line, indicating zero shear load. Next, draw the value of V along the length of the beam, as follows:



Step 1 Starting at the left side of the shear diagram, go up 15 kN, because R_A is 15 kN upwards.

Step 2 There are no additional loads on the beam until you get to the midspan, so the shear value remains at 15 kN.

Step 3 The applied load at the midspan is 30 kN downwards, therefore the shear load is $15\,\text{kN} - 30\,\text{kN} = -15\,\text{kN}$.

Step 4 There are no additional loads on the beam until you get to point B, so the shear value remains at -15kN.

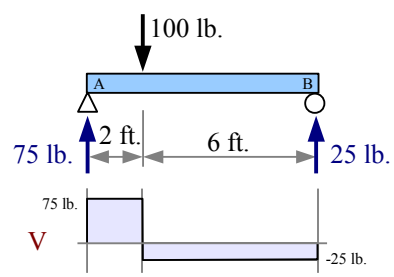
Step 5 At point B, the reaction force R_B = 15 kN upwards, therefore the shear load is $-15\,\text{kN} + 15\,\text{kN} = 0$. If you don't get to 0, you know you made a mistake someplace.

Finish the shear diagram by shading the areas between your line and the horizontal zero shear line. Mark all significant points (anywhere the shear line changes direction). In the next chapter, we will use the maximum absolute value of shear load, $|V|_{max}$, to calculate the maximum shear stress in the beam.

A point load at the midspan of a simply-supported beam produces identical reaction forces and a symmetric shear diagram with two rectangles. If the point load is not at the midspan, use sum of the moments and sum of the forces to calculate the reaction forces. Draw vertical construction lines below the applied loads and reaction forces, draw a horizontal line at zero shear, then draw the shear value along the length of the beam.

Example #10

Draw a complete shear diagram for a simply-supported 8 ft. beam with a 100 lb. point load 2 ft. to the right of point A.



Solution Use sum of the moments and sum of the forces to find the reaction forces R_A and R_B.

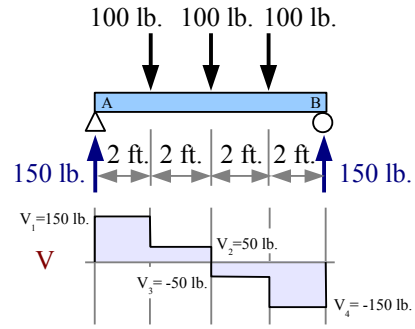
Starting the shear diagram at zero shear, go up $R_A = 75\,\text{lb.}$ at point A. There are no loads between point A and the applied point load, so the shear load does not change. Draw a horizontal line to the right, until you reach the point load. Draw a vertical line down 100 lb., reaching a value $V = -25\,\text{lb.}$ There are no loads between the point load and point B, so the shear load does not change.

Draw a horizontal line to the right, until you reach point B. The reaction force at B is 25 lb. upwards, so draw a vertical line up 25 lb., reaching a value $V = 0$. Again we have two rectangles, but they are not symmetric; the beam carries three times as much shear load to the left of the point load than it does to the right.

Multiple point loads produce a stepped shear diagram.

Example #11

Draw a complete shear diagram for a simply-supported 8 ft. beam with 100 lb. point loads every 2 ft. along the length.



Solution The loading is symmetrical, so the reaction forces equal half the total applied load.

Calculate the shear values as:

$V_1 = R_A = 150\,\text{lb.}$

$V_2 = V_1 - 100\,\text{lb.} = 50\,\text{lb.}$

$V_3 = V_2 - 100\,\text{lb.} = -50\,\text{lb.}$

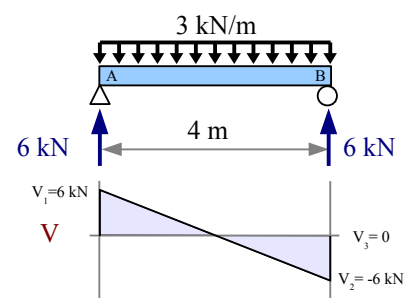
$V_4 = V_3 - 100\,\text{lb.} = -150\,\text{lb.}$

$V_5 = V_4 + R_B = -150\,\text{lb.} + 150\,\text{lb.} = 0$ In this problem, $|V|_{max} = 150\,\text{lb.}$

A uniformly distributed load is like an infinite number of small point loads along the length of the beam, so the shear diagram is like a stepped multiple point load shear diagram with infinitely small steps.

Example #12

Draw a complete shear diagram for a simply-supported 4 m beam with a uniformly distributed load of 3 kN/m.



Solution The loading is symmetrical, so the reaction forces equal half the total applied load. $R_A = R_B = \frac{W}{2} = \frac{wL}{2} = \frac{3\,\text{kN} \cdot 4\,\text{m}}{\text{m}\ \ 2} = 6\,\text{kN}$.

A complete shear diagram includes the values of the shear at locations of applied point loads and reaction forces.

$V_1 = R_A = 6\,\text{kN}$

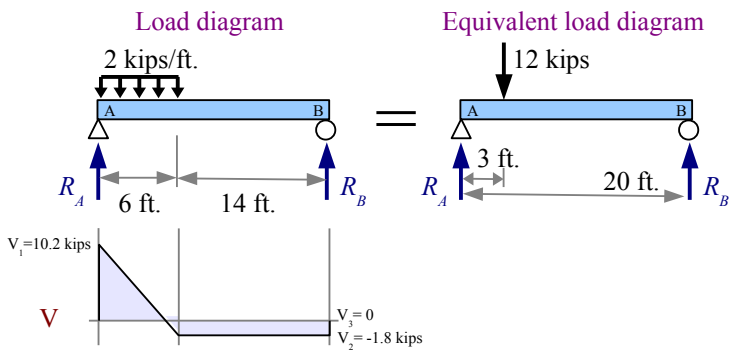
$V_2 = V_1 - \frac{3\,\text{kN}}{\text{m}} \frac{4\,\text{m}}{} = -6\,\text{kN}$

$V_3 = V_2 + R_B = -6\,\text{kN} + 6\,\text{kN} = 0$ In this problem, $|V|_{max} = 6\,\text{kN}$

If the uniformly distributed load does not extend along the entire length of the beam, then draw an equivalent load diagram to find the reaction forces. Go back to the *original* load diagram to draw the shear diagram; do *not* use the equivalent load diagram to draw the shear diagram.

Example #13

Draw a complete shear diagram for a simply-supported 20 ft. beam which has a uniform distributed load of 2 kips/ft. running from the left end for 6 feet.



Solution Draw an equivalent load diagram, placing the equivalent load at the centroid of the distributed load. Use the equivalent load diagram to find the reaction forces.

The distributed load runs for 6 ft., so the location of the equivalent load is 3 ft. from the left end of the beam.

The equivalent load $W = wL = \frac{2\text{ kips}\cdot 6\text{ ft.}}{\text{ft.}} = 12\text{ kips}$.

The moment about point A is $\circlearrowleft + \sum M_A = 0 = -12\text{ kips}\cdot 3\text{ ft.} + R_B \cdot 20\text{ ft.}$.

Rewrite the equation to find the reaction force $R_B = \frac{12\text{ kips}\cdot 3\text{ ft.}}{20\text{ ft.}} = 1.8\text{ kips}$.

Sum of the forces $\uparrow + \sum F_y = 0 = R_A - 12\text{ kips} + 1.8\text{ kips}$. Solve for the reaction force $R_A = 12\text{ kips} - 1.8\text{ kips} = 10.2\text{ kips}$.

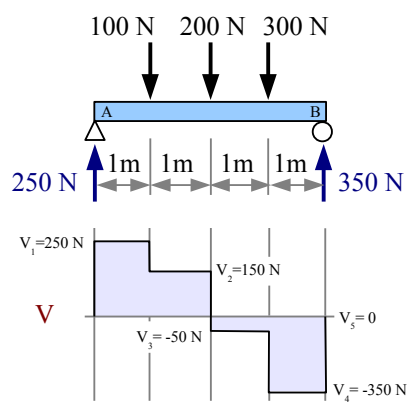
Draw construction lines down from the original load diagram wherever a point load or reaction exists, and wherever a distributed load starts or stops. Calculate the shear loads at these points.

$V_1 = R_A = 10.2\text{ kips}$ $\quad V_2 = V_1 - \frac{2\text{ kips}}{\text{ft.}}\frac{6\text{ ft.}}{} = -1.8\text{ kips}$ $\quad V_3 = V_2 + R_B = -1.8\text{ kips} + 1.8\text{ kips} = 0$ $\quad |V|_{max} = 10.2\text{ kips}$

Multiple point loads of increasing load will give you an asymmetric stepped shear diagram.

Example #14

Draw a complete shear diagram for a simply-supported 4 m beam which has a three different point loads as shown in the load diagram.



Solution Use sum of the moments and sum of the forces to find R_A and R_B.

Calculate the shear loads as follows:

$V_1 = R_A = 250\text{ N}$

$V_2 = V_1 - 100\text{ N} = 150\text{ N}$

$V_3 = V_2 - 200\text{ lb.} = -50\text{ N}$

$V_4 = V_3 - 300\text{ N} = -350\text{ N}$

$V_5 = V_4 + R_B = -350\text{ N} + 350\text{ N} = 0$

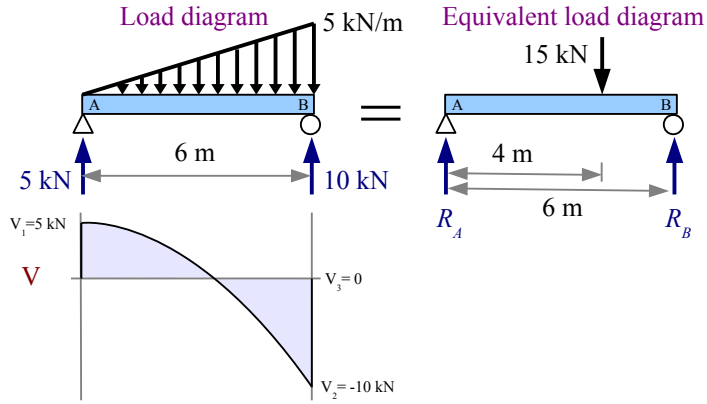
In this problem, $|V|_{max} = 350\text{ kN}$

A nonuniformly distributed load produces two different reaction forces. Draw an equivalent load diagram to find the reaction forces. Go back to the *original* load diagram to draw the shear diagram. A wedge-shaped nonuniformly distributed load is like an infinite number of small point loads of increasing magnitude along the length of the beam, so the shear

diagram is like the previous stepped multiple point load shear diagram in which the slope of the curve increases parabolically.

Example #15

Draw a complete shear diagram for a simply-supported 6 m beam which has a wedge-shaped nonuniformly distributed load of 0 kN/m at the left end of the beam to 5 kN/m at the right end of the beam.



Solution Draw an equivalent load diagram, placing the equivalent load at the centroid of the distributed load. The centroid of a triangle lies two-thirds of the distance from the point of the triangle. Use the equivalent load diagram to find the reaction forces.

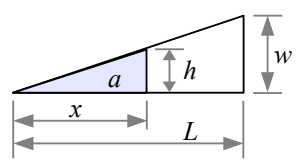
The centroid is located at $x=\frac{2}{3}L=\frac{2\cdot 6\text{ m}}{3}=4\text{ m}$,

measuring from the left end of the beam. The equivalent load is the average of the minimum and maximum distributed loads times the length of the distributed load: $W=\frac{(0+5)\text{ kN}}{2}\frac{6\text{ m}}{\text{m}}=15\text{ kN}$.

The moment about point A is $\circlearrowleft+\ \sum M_A=0=-15\text{ kN}\cdot 4\text{ m}+R_B\cdot 6\text{ m}$. Rewrite the equation to find the reaction force $R_B=\frac{15\text{ kN}\cdot 4\text{ m}}{6\text{ m}}=10\text{ kN}$. Sum of the forces $\uparrow+\ \sum F_y=0=R_A-15\text{ kN}+10\text{ kN}$. Solve for $R_A=15\text{ kN}-10\text{ kN}=5\text{ kN}$.

Draw construction lines down from the original load diagram at the two reaction forces. Calculate the shear loads at these points.

$V_1=R_A=5\text{ kN}$ $\qquad V_2=V_1-15\text{ kN}=-10\text{ kN}$ $\qquad V_3=V_2+R_B=-10\text{ kN}+10\text{ kN}=0$



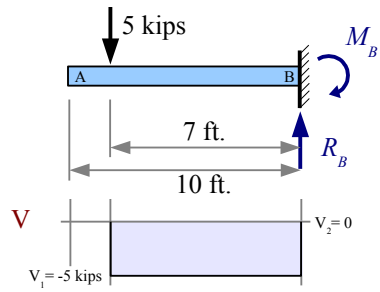
The shear line crosses the axis where the area of the distributed load equals R_A. The area of the distributed load is one half the base times the height of the triangle, or $a=\frac{xh}{2}$. By similar triangles, the height of the little triangle $h=\frac{x}{L}w$, so $a=\frac{x^2 w}{2L}=R_A$. Solving, $x=\sqrt{\frac{2LR_A}{w}}=\sqrt{\frac{2\cdot 6\text{ m}\cdot 5\text{ kN}}{5\text{ kN}}\frac{\text{m}}{}}=3.46\text{ m}$. The shear line crosses the axis 3.46 m from the left end of the beam.

Look at the shear diagrams, and you can see that point loads create rectangles, uniform distributed loads create triangles, and wedge-shaped (triangular) distributed loads create parabolas.

Shear diagrams for cantilever beams follow the same rules as for simply supported beams. First, find the reactions; next, draw the shear diagram using construction lines wherever a point load occurs or a distributed load starts or stops.

Example #16

Draw a complete shear diagram for a 10 foot cantilever beam having a 5 kip point load 7 feet from the wall.



Solution Since there is no reaction force at the left end, there is no shear load until we get to the applied load. Then, the shear load is negative (downward) until we get to the support, where R_B is positive (upward).

The moment about point B is $\circlearrowleft + \sum M_B = 0 = 5\text{ kips} \cdot 7\text{ ft.} = 35\text{ kip ft.}$. Sum of the forces $\uparrow + \sum F_y = 0 = R_B - 5\text{ kips}$ therefore $R_B = 5\text{ kips}$.

Draw construction lines down from the original load diagram at the point load and the reaction force. Calculate the shear loads at these points.

$V_1 = P = -5\text{ kips}$ $\qquad$ $V_2 = V_1 + 5\text{ kips} = 0$

With a shear diagram, we can identify the location and size of the largest shear load in a beam. Therefore, we know the location of the largest shear stress, and we can calculate the value of this stress. Once we know the actual stress in the material, we can compare this values with the shear strength of the material, and we can know whether the beam will fail in shear. Shear diagrams are necessary for drawing bending moment diagrams ("moment diagrams", for short), which we can use to identify the location and size of bending stresses that develop within beams. We can compare the actual bending stresses with the yield strength of the material, and we can know whether the beam will fail in bending.

Moment Diagrams

The moment about a point along a beam is defined as the distance from that point to a force acting perpendicular to the beam, so the units are force × distance: $\text{lb.}\cdot\text{ft.}$ (or $\text{ft.}\cdot\text{lb.}$ – the order does not matter), $\text{lb.}\cdot\text{in.}$, $\text{kip}\cdot\text{ft.}$, $\text{N}\cdot\text{m}$, or $\text{kN}\cdot\text{m}$. We can graph the value of the bending moment along a beam by drawing a moment diagram.

To draw a moment diagram, sketch the value of the moment produced by the shear force V times the distance from the left end of the beam. At the first meter, $V = 15$ kN, so moment $M_1 = 15\,\text{kN} \times 1\,\text{m} = 15\,\text{kN}\cdot\text{m}$.

At 2 m, $M_2 = M_1 + 15\,\text{kN} \times 1\,\text{m} = 30\,\text{kN}\cdot\text{m}$

At 3 m, $M_3 = M_2 + 15\,\text{kN} \times 1\,\text{m} = 45\,\text{kN}\cdot\text{m}$

At 4 m, $M_4 = M_3 + 15\,\text{kN} \times 1\,\text{m} = 60\,\text{kN}\cdot\text{m}$

At 5 m, $M_5 = M_4 - 15\,\text{kN} \times 1\,\text{m} = 45\,\text{kN}\cdot\text{m}$

At 6 m, $M_6 = M_5 - 15\,\text{kN} \times 1\,\text{m} = 30\,\text{kN}\cdot\text{m}$

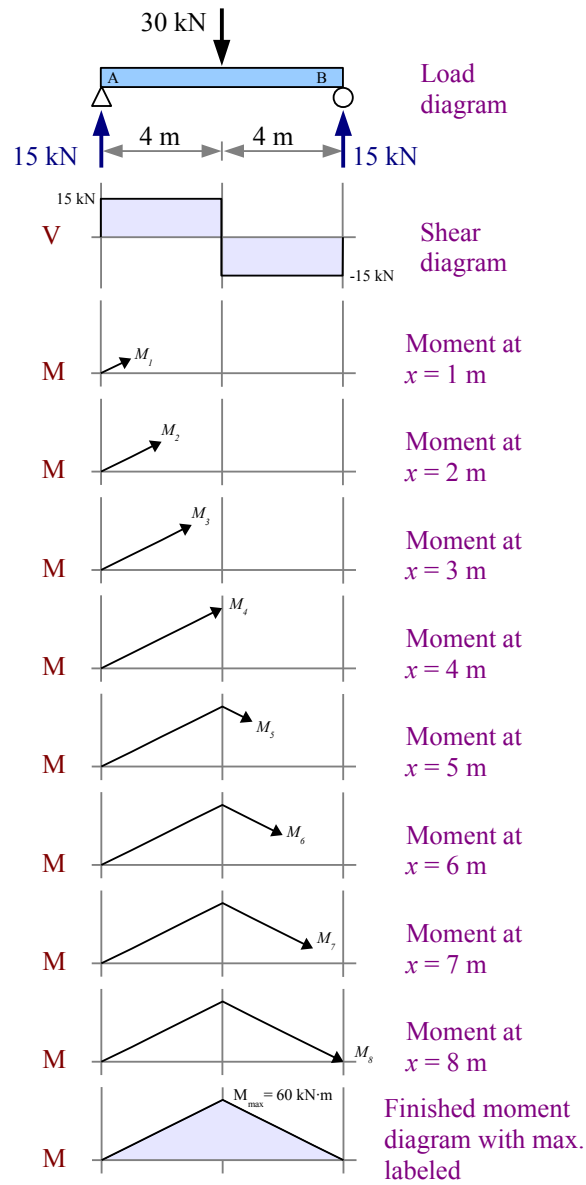
At 7 m, $M_7 = M_6 - 15\,\text{kN} \times 1\,\text{m} = 15\,\text{kN}\cdot\text{m}$

At 8 m, $M_8 = M_7 - 15\,\text{kN} \times 1\,\text{m} = 0\,\text{kN}\cdot\text{m}$

Shade the finished diagram, marking all significant points (places where the moment line changes direction). The most important point is the maximum absolute value of M; this is where the maximum bending stress occurs in the beam.

The value of the moment diagram at any point equals the area of the shear diagram up to that point. The shear diagram has positive areas above the zero line, and negative areas below the zero line. When the shear area is positive, the moment increases; when the shear area is negative, the moment decreases.

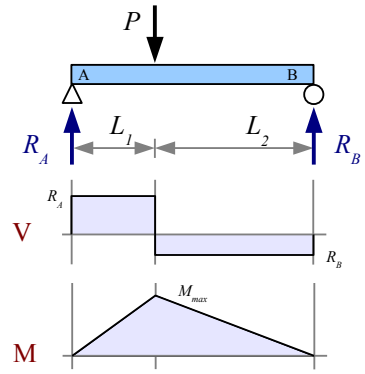
You can draw the moment diagram faster by calculating the area of the left-hand rectangle in the shear diagram: $M_{max} = 15\,\text{kN} \times 4\,\text{m} = 60\,\text{kN}\cdot\text{m}$. Verify that $M_8 = 0$ by subtracting the area of the second rectangle in the shear diagram: $M_8 = M_{max} - 15\,\text{kN} \times 4\,\text{m} = 0\,\text{kN}\cdot\text{m}$.



It is a good idea to check the moment at point B. If you end up with a value other than 0, then you know there is a mistake someplace. Most likely, the error is in the reaction forces. You can draw a shear diagram that works with the wrong reaction forces, but you cannot draw a good moment diagram if R_A and R_B are wrong. This is an amazing feature of moment diagrams – they tell you if your solution is right!

If the point load is not at the midspan, then the maximum moment will also be offset. In this example, the maximum moment is the area of the shear diagram up to the point load: $M_{max} = R_A \cdot L_1$.

Multiple point loads will give you multiple rectangles on the shear diagram, and multiple triangles on the moment diagram.



Example #17

Draw complete shear and moment diagrams for an 8 foot long, simply-supported beam having 100 lb. point loads every 2 feet.

Solution The loading is symmetrical, therefore the reaction forces R_A and R_B are each equal to half the total applied load.
$R_A = R_B = (100\,\text{lb.} + 100\,\text{lb.} + 100\,\text{lb.})/2 = 150\,\text{lb.}$

Draw construction lines down from the load diagram at each reaction force and applied load. The shear diagram is a series of rectangles.

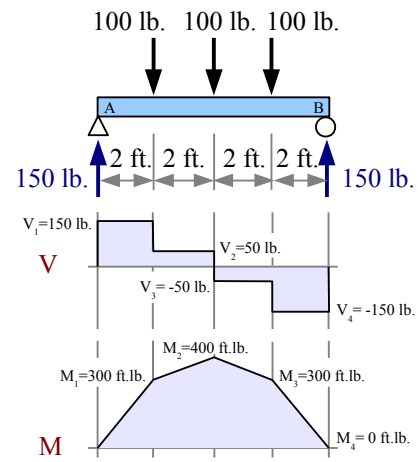
Calculate the values on the moment diagram as follows:

$M_1 = 2\,\text{ft.} \times 150\,\text{lb.} = 300\,\text{ft.}\cdot\text{lb.}$

$M_2 = M_1 + 2\,\text{ft.} \times 50\,\text{lb.} = 400\,\text{ft.}\cdot\text{lb.}$

$M_3 = M_2 - 2\,\text{ft.} \times 50\,\text{lb.} = 300\,\text{ft.}\cdot\text{lb.}$

$M_4 = M_3 - 2\,\text{ft.} \times 150\,\text{lb.} = 0\,\text{ft.}\cdot\text{lb.}$

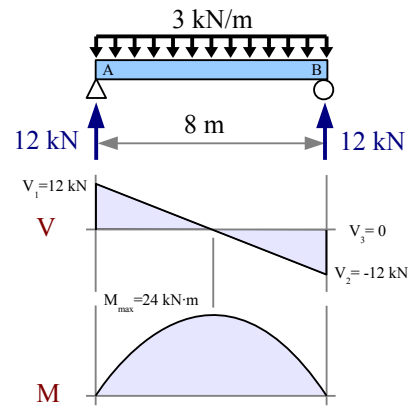


A uniformly distributed load produces a parabolic moment diagram. Close to point A, a large shear produces a steep slope in the moment diagram. As you approach the midspan, the smaller shear produces a shallower slope in the moment diagram. Beyond the midspan, an increasingly negative shear produces an increasingly steeper slope downwards.

The maximum moment equals the area of the left-hand triangle. Subtract the area of the right-hand triangle to get the moment at point B.

Example #18

Draw complete shear and moment diagrams for an 8 meter long, simply-supported beam having a uniform distributed load of 3 kN/m.



Solution The loading is symmetrical, therefore the reaction forces R_A and R_B are each equal to half the total applied load. $R_A = R_B = \frac{3\,\text{kN}}{\text{m}}\frac{8\,\text{m}}{2} = 12\,\text{kN}$

Draw construction lines down from the load diagram at each reaction force . The shear diagram is two triangles. Draw a construction line down from where the shear line crosses the zero axis.

The max. moment is equal to the area of the left-hand triangle: the base times the height divided by 2. The base of the triangle is half the beam length, 4 m.

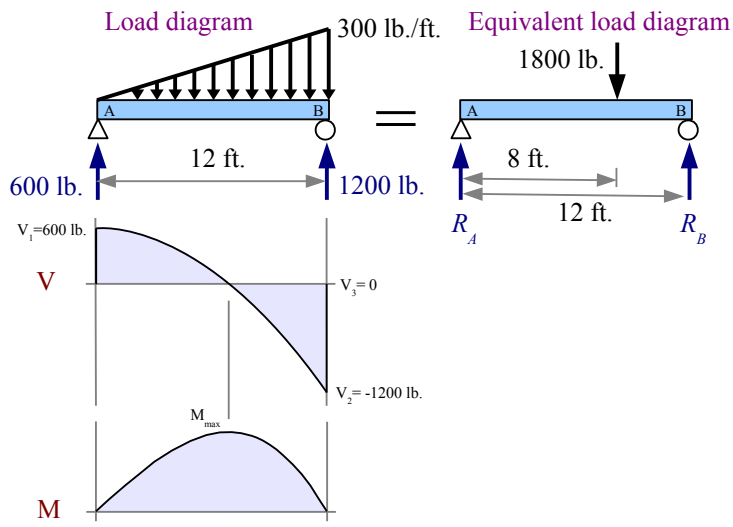
$M_{max} = \frac{12\,\text{kN} \times 4\,\text{m}}{2} = 24\,\text{kN}\cdot\text{m}$ The moment at point B is the moment at the midspan minus the area of the right-hand triangle.

$$M_B = M_{max} - \frac{12\,\text{kN} \times 4\,\text{m}}{2} = 0\,\text{kN}\cdot\text{m}$$

We've seen that a nonuniform wedge-shaped distributed load produces a parabolic shear diagram. The moment diagram looks like a parabola skewed to the right, with the maximum moment at the point where the shear diagram crosses the zero-load axis.

Example #19

Draw complete shear and moment diagrams for a 12 foot long, simply-supported beam having a wedge-shaped nonuniform distributed load of 0 lb./ft. at the left end of the beam and 300 lb./ft. at the right end of the beam.



Solution Draw an equivalent load diagram, placing the equivalent point load at the centroid of the distributed load. The centroid of a triangle lies 2/3 of the distance from the tip of the triangle. Use the equivalent load diagram to find the reaction forces.

The centroid of the triangular load profile lies at $x=\frac{2}{3}L=\frac{2\cdot 12\text{ ft.}}{3}=8\text{ ft.}$, measuring from point A.

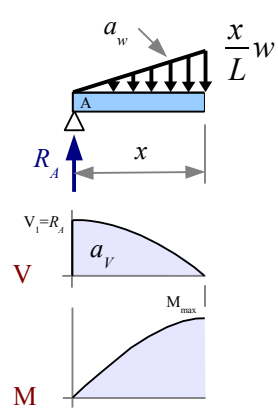
The equivalent load is the average of the minimum and maximum distributed loads times the length of the distributed load: $W=\frac{(0+300)\text{ lb.}}{2}\frac{12\text{ ft.}}{\text{ft.}}=1800\text{ lb.}$

The moment about point A is $\circlearrowleft+\ \sum M_A=0=-1800\text{ lb.}\cdot 8\text{ ft.}+R_B\cdot 12\text{ ft.}$. Rewrite the equation to find the reaction force $R_B=\frac{1800\text{ lb.}\cdot 8\text{ ft.}}{12\text{ ft.}}=1200\text{ lb.}$ Sum of the forces $\uparrow+\ \sum F_y=0=R_A-1800\text{ lb.}+1200\text{ lb.}$ Solve for $R_A=1800\text{lb.}-1200\text{lb.}=600\text{lb.}$

Draw construction lines down from the original load diagram at the two reaction forces. Calculate the shear loads at these points. $V_1=R_A=600\text{ lb.}$, $V_2=V_1-1800\text{ lb.}=-1200\text{ lb.}$, $V_3=V_2+R_B=-1200\text{ lb.}+1200\text{ lb.}=0$

Draw a construction line down from the shear diagram where the shear line crosses the zero axis – this construction line marks the location of the max. moment. We need to find the location of this point in order to calculate the max. moment.

Consider that the value of V at any point along the shear diagram equals the reaction force R_A minus the area of the distributed load, a_w: $V=R_A-a_w$. The area of the triangle-shaped distributed load at any point equals one-half the base times the height of the triangle. Let the base be x, the distance from the left end of the beam. The height is a fraction of the distributed load w at point B. By similar triangles, height $=\frac{x}{L}w$, so area $a_w=\frac{1}{2}x\frac{x}{L}w=\frac{x^2 w}{2L}$ and shear force $V=R_A-\frac{x^2 w}{2L}$. Where the shear line crosses the zero line, $V=0$, therefore $x=\sqrt{\frac{2LR_A}{w}}=\sqrt{\frac{2\cdot 12\text{ ft.}\cdot 600\text{ lb.}}{300\text{ lb.}}\frac{\text{ft.}}{}}=6.928\text{ ft.}$



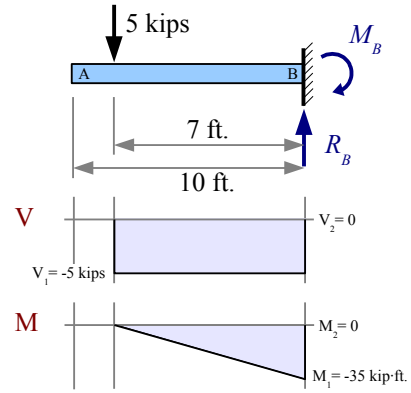
The maximum moment is equal to the area of the shear diagram to the left of this construction line. Since the shear diagram is a parabola, the area of the shear diagram to the left of the construction line is 2/3 of the base of the parabola times its height.

$$M_{max}=a_V=\frac{2}{3}V_1 x=\frac{2\cdot 600\text{ lb.}\cdot 6.928\text{ ft.}}{3}=2770\text{ ft.}\cdot\text{lb.}$$

So far, all of the examples have shown moment diagrams with positive areas. A cantilever beam with a point load has a negative area in the moment diagram.

Example #20

Draw complete shear and moment diagrams for a cantilever beam having a point load of 5 kips, located 7 feet from the wall.



Solution By inspection, the reaction force $R_B = 5\,\text{kips}$. The reaction moment $M_B = 5\,\text{kips} \times 7\,\text{ft.} = 35\,\text{kip}\cdot\text{ft.}$ The shear diagram is a rectangle, so the moment diagram will be a triangle. The area of the shear diagram equals the value of the moment diagram, so

$$M_1 = -5\,\text{kips} \times 7\,\text{ft.} = -35\,\text{kip}\cdot\text{ft.}$$

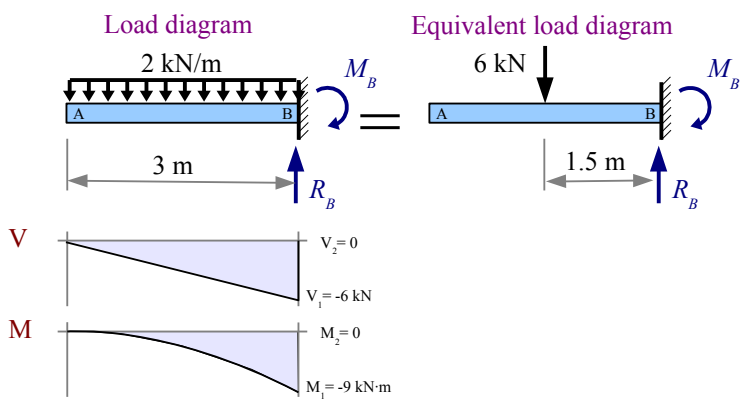
$$M_2 = -35\,\text{kip}\cdot\text{ft.} + M_B = 0\,\text{kip}\cdot\text{ft.}$$

In this problem, $|M|_{max} = 35\,\text{kip}\cdot\text{ft.}$

The shear diagram of a cantilever beam with a uniform distributed load is a triangle. Use an equivalent load diagram to find the reaction force and reaction moment, then draw shear and moment diagrams below the original load diagram.

Example #21

Draw complete shear and moment diagrams for a 3 m long cantilever beam having a uniform distributed load of 2 kN/m along its length.



Solution Draw an equivalent load diagram, placing the equivalent point load at the centroid of the distributed load. The centroid of a rectangle lies at the halfway point. Use the equivalent load diagram to find the reaction force and reaction moment.

$$R_B = \frac{2\,\text{kN}}{\text{m}} \frac{3\,\text{m}}{} = 6\,\text{kN}$$

$$M_B = 6\,\text{kN} \times 1.5\,\text{m} = 9\,\text{kN}\cdot\text{m}$$

The shear diagram is a triangle, so the moment diagram is a parabola with $|M|_{max} = 9\,\text{kN}\cdot\text{m}$.

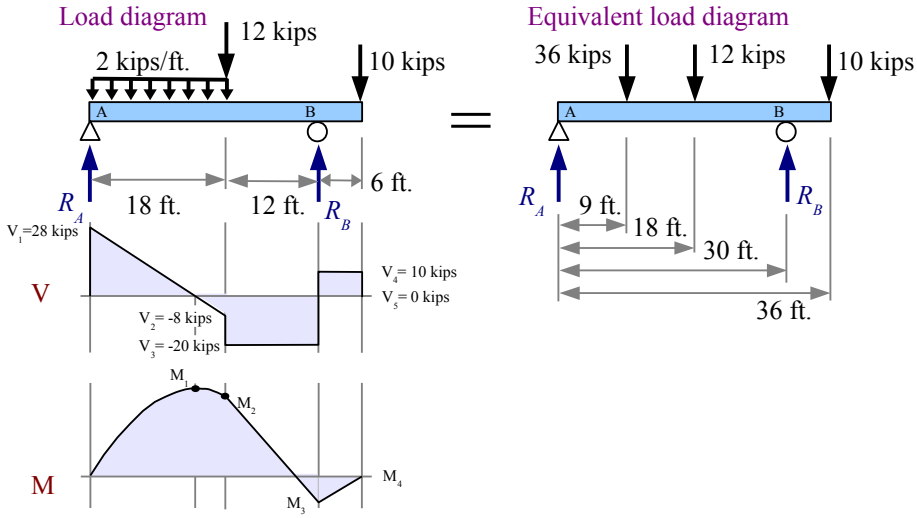
Follow the same procedures to draw the shear and moment diagrams of an overhanging beam with a distributed load and a two point loads.

Example #22

Draw complete shear and moment diagrams for a 36 foot long overhanging beam having a uniform distributed load and two point loads as shown.

Solution Draw an equivalent load diagram, placing the equivalent point load at the centroid of the distributed load, which is 9 ft. from point A.

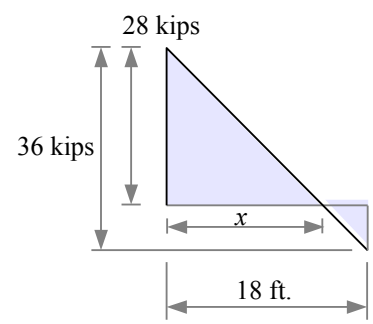
Use the equivalent load diagram to find the reaction forces, then draw the shear and moment diagrams below the original load diagram.



$$R_B = \frac{36\text{ kips}\cdot 9\text{ ft.} + 12\text{ kips}\cdot 18\text{ ft.} + 10\text{ kips}\cdot 36\text{ ft.}}{30\text{ ft.}} = 30\text{ kips}\ ,\quad R_A = 36\text{ kips} + 12\text{ kips} + 10\text{ kips} - 30\text{ kips} = 28\text{ kips}$$

Calculate the shear values: $V_1 = R_A = 28\text{ kips}$, $V_2 = V_1 - \frac{2\text{ kips}}{\text{ft.}}\frac{18\text{ ft.}}{} = -8\text{ kips}$, $V_3 = V_2 - 12\text{ kips} = -20\text{ kips}$, $V_4 = V_3 + 30\text{ kips} = 10\text{ kips}$, $V_5 = V_4 - 10\text{ kips} = 0\text{ kips}$

The moment curve starts with a parabola going up until the shear curve crosses zero; once the shear is negative, the moment curve drops parabolically until the end of the distributed load. In order to calculate the moment values, we need to know where the shear curve crosses the zero line. Use similar triangles to find *x*, then calculate the area of the left-hand triangle in the shear diagram: $x = 18\text{ ft.}\frac{28\text{ kips}}{36\text{ kips}} = 14\text{ ft.}$



The area of the left-hand triangle is $M_1 = \frac{28\text{ kips}\cdot 14\text{ ft.}}{2} = 196\text{ kip}\cdot\text{ft.}$

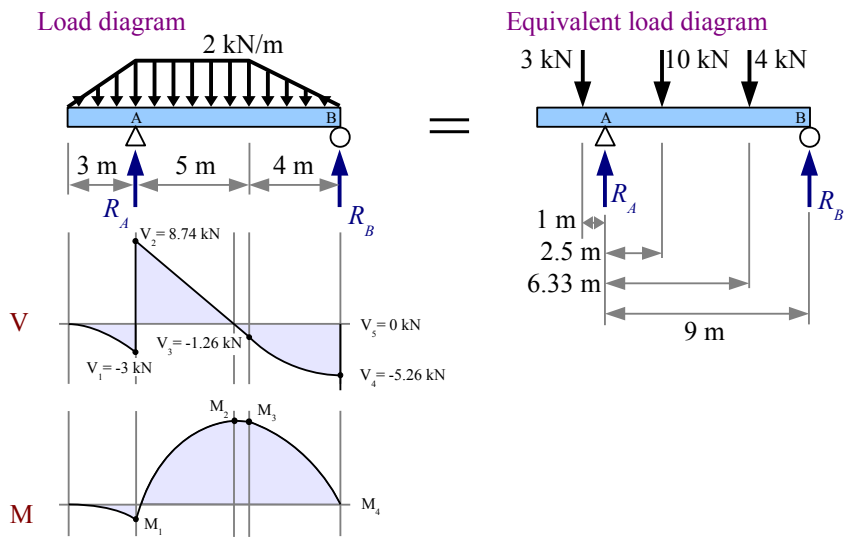
Subtract the area of the right-hand triangle to find $M_2 = M_1 - \frac{8\text{ kips}\cdot 4\text{ ft.}}{2} = 180\text{ kip}\cdot\text{ft.}$

Subtract the lower rectangle to find $M_3 = M_2 - 20\text{ kips}\cdot 12\text{ ft.} = -60\text{ kip}\cdot\text{ft.}$ Add the upper rectangle to find $M_4 = M_3 + 10\text{ kips}\cdot 6\text{ ft.} = 0\text{ kip}\cdot\text{ft.}$

In this problem, $|M|_{max} = 196\text{ kip}\cdot\text{ft.}$

Example #23

Draw complete shear and moment diagrams for a 12 meter long overhanging beam having a uniform distributed load and two nonuniform distributed loads.



Solution Draw an equivalent load diagram, placing the equivalent point loads at the centroids of the distributed loads (two triangular distributed loads and one rectangular distributed load).

Use the equivalent load diagram to find the reaction forces, then draw the shear and moment diagrams below the original load diagram.

The equivalent loads are

$$\frac{2\,\text{kN}}{\text{m}}\frac{3\,\text{m}}{2}=3\,\text{kN}\ ,$$

$$\frac{2\,\text{kN}}{\text{m}}\frac{5\,\text{m}}{}=10\,\text{kN}\ ,\ \text{and}$$

$$\frac{2\,\text{kN}}{\text{m}}\frac{4\,\text{m}}{2}=4\,\text{kN}\ .$$

$$R_B=\frac{-3\,\text{kN}\cdot 1\,\text{m}+10\,\text{kN}\cdot 2.5\,\text{m}+4\,\text{kN}\cdot 6.33\,\text{m}}{9\,\text{m}}=5.26\,\text{kN}\ ,\quad R_A=3\,\text{kN}+10\,\text{kN}+4\,\text{kN}-5.26\,\text{kN}=11.74\,\text{kN}$$

The shear diagram goes downward as a concave second-degree parabola under the nonuniform distributed load, straight up at R_A, downward at a diagonal under the uniform distributed load, then downward as a convex second-degree parabola under the nonuniform distributed load, and straight up at R_B.

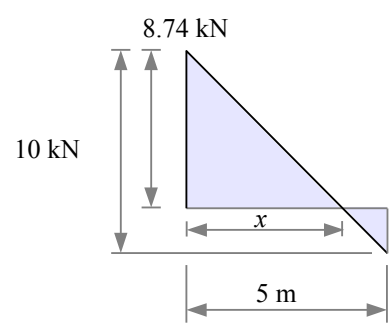
Calculate the shear values: $V_1=\frac{-2\,\text{kN}}{\text{m}}\frac{3\,\text{m}}{2}=-3\,\text{kN}$, $V_2=V_1+R_A=8.74\,\text{kN}$, $V_3=V_2-\frac{2\,\text{kN}}{\text{m}}\frac{5\,\text{m}}{}=-1.26\,\text{kN}$,

$V_4=V_3-\frac{-2\,\text{kN}}{\text{m}}\frac{4\,\text{m}}{2}=-5.26\,\text{kN}$, $V_5=V_4+R_B=0\,\text{kN}$

The moment curve goes downward as a concave third-degree parabola. To find the value of M_1, calculate the area under the concave second-degree parabola at the left end of the shear diagram. From Appendix C, the area under a concave second-degree parabola is $\frac{bh}{3}$. The base is 3 m, and the height is -3 kN. Therefore

$$M_1=\frac{-3\,\text{kN}\cdot 3\,\text{m}}{3}=-3\,\text{kN}\cdot\text{m}$$

In order to calculate M_2, we need to know where the shear curve crosses the zero line. Use similar triangles to find x, then calculate the area of the left-hand triangle in the shear diagram: $x=5\,\text{m}\frac{8.74\,\text{kN}}{10\,\text{kN}}=4.37\,\text{m}$



Example #23, continued

$$M_2 = M_1 + \frac{8.74\,\text{kN}\cdot 4.37\,\text{m}}{2} = 16.1\,\text{kN}\cdot\text{m}$$

The base of the small triangle in the shear diagram is 5 m – 4.37 m = 0.63 m.

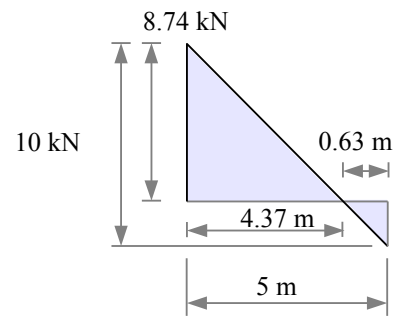
$$M_3 = M_2 - \frac{1.26\,\text{kN}\cdot 0.63\,\text{m}}{2} = 15.7\,\text{kN}\cdot\text{m}$$

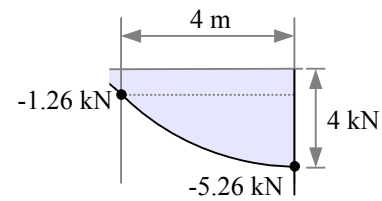
Although it is tempting to assume that $M_4 = 0\,\text{kN}\cdot\text{m}$ to finish the moment diagram, it's best to calculate it, to be sure that there are no errors in the reaction force calculations.

The rightmost segment in the shear diagram is a rectangle and a convex second degree parabola. From Appendix C, the area of a convex second degree parabola is $\frac{2bh}{3}$.

$$M_4 = M_3 - 1.26\,\text{kN}\cdot 4\,\text{m} - \frac{2\cdot 4\,\text{m}\cdot 4\,\text{kN}}{3} = 0\,\text{kN}\cdot\text{m}$$

In this problem, $V_{max} = 8.74\,\text{kN}$ and $M_{max} = 16.1\,\text{kN}\cdot\text{m}$

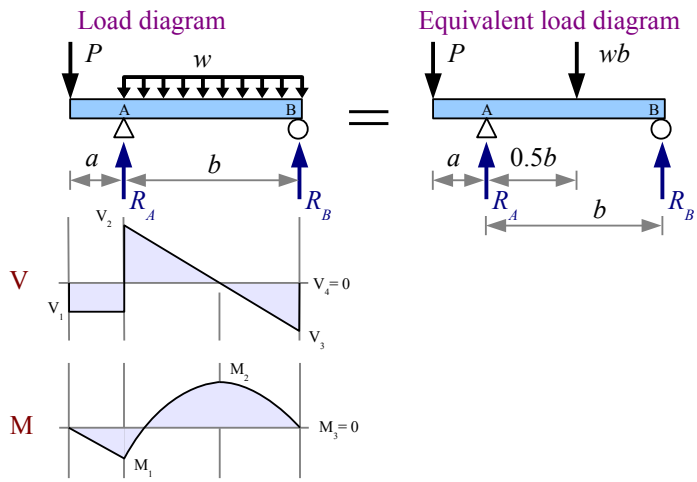




Some engineering solutions require an algebraic solution. Imagine you are designing a set of beams using the same general loading case, but the loads change from one case to the next. If you can develop an algebraic solution, then you can plug in the numbers for each case (perhaps with a computer program) instead of solving every case from scratch. Example #24 shows how to create shear and moment diagrams for an algebraic problem.

Example #24

Draw complete algebraic shear and moment diagrams for an overhanging beam loaded as shown.



Solution Draw an equivalent load diagram, placing the equivalent point load at the centroid of the distributed load.

Use Sum of the Moments and Sum of the Forces to find the reactions at points A and B.

$$R_B = \frac{-Pa + wb \cdot 0.5b}{b} = \frac{-Pa}{b} + \frac{wb}{2}$$

$$R_A = P + wb - R_B = P + \frac{Pa}{b} + \frac{wb}{2}$$

$$V_1 = -P$$

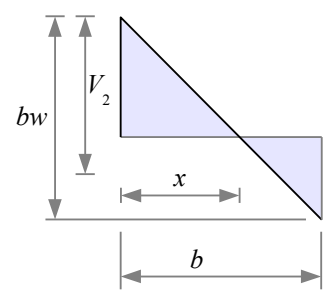
$$V_2 = V_1 + R_A = \frac{Pa}{b} + \frac{wb}{2}$$

$$V_3 = V_2 - wb = \frac{Pa}{b} - \frac{wb}{2}$$

$$V_4 = V_3 + R_B = 0$$

$$M_1 = -V_1 a = -Pa$$

Use Similar Triangles to determine the base of the upper triangle. The overall height is the weight per unit length times the distance it acts upon, *bw*.



$$x = b\frac{V_2}{bw} = \frac{V_2}{w} = \frac{\frac{Pa}{b} + \frac{wb}{2}}{w} = \frac{Pa}{bw} + \frac{b}{2}$$

$$M_2 = M_1 + \frac{V_2 x}{2} = -Pa + \frac{\left(\frac{Pa}{b} + \frac{wb}{2}\right)\left(\frac{Pa}{bw} + \frac{b}{2}\right)}{2} = \frac{P^2a^2}{2b^2w} - \frac{Pa}{2} + \frac{wb^2}{8}$$

The base of the lower triangle is $b - x = b - \left(\frac{Pa}{bw} + \frac{b}{2}\right) = \frac{b}{2} - \frac{Pa}{bw}$

$$M_3 = M_2 + \frac{V_3(b-x)}{2} = M_2 + \frac{\left(\frac{Pa}{b} - \frac{wb}{2}\right)\left(\frac{b}{2} - \frac{Pa}{bw}\right)}{2} = M_2 + \frac{\frac{Pab}{2b} - \frac{P^2a^2}{b^2w} - \frac{wb^2}{4} + \frac{Pabw}{2bw}}{2}$$

$$= M_2 + \frac{\frac{Pa}{2} - \frac{P^2a^2}{b^2w} - \frac{wb^2}{4} + \frac{Pa}{2}}{2} = \frac{P^2a^2}{2b^2w} - \frac{Pa}{2} + \frac{wb^2}{8} + \frac{Pa}{2} - \frac{Pa^2}{2b^2w} - \frac{wb^2}{8} = 0$$

We do not know in advance whether $M_{max} = M_1$ or $M_{max} = M_2$ ...it depends on the values of *a*, *b*, *w*, and *P*.

In summary, the value of the moment diagram at a given point equals the area of the shear diagram up to that point. The slope of the moment diagram at a given point equals the value of the shear load at that point.

Load type	**Shear diagram shape**	**Moment diagram shape**
Point	Rectangles	Triangles
Uniform distributed	Triangles	Parabolas (second degree)
Wedge-shaped nonuniform distributed	Parabolas (second degree)	Parabolas (third degree)

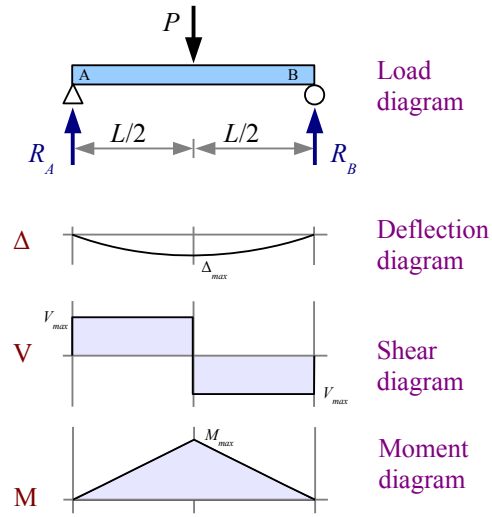
Key Equations

Use $\circlearrowleft + \sum M_A = 0$ and $\uparrow + \sum F_y = 0$ to find reaction forces for simply-supported beams.

Use $\circlearrowleft + \sum M_{support}$ and $\uparrow + \sum F_y = 0$ to find the reaction moment and reaction force for cantilever beams.

Chapter 9: Stresses in Beams

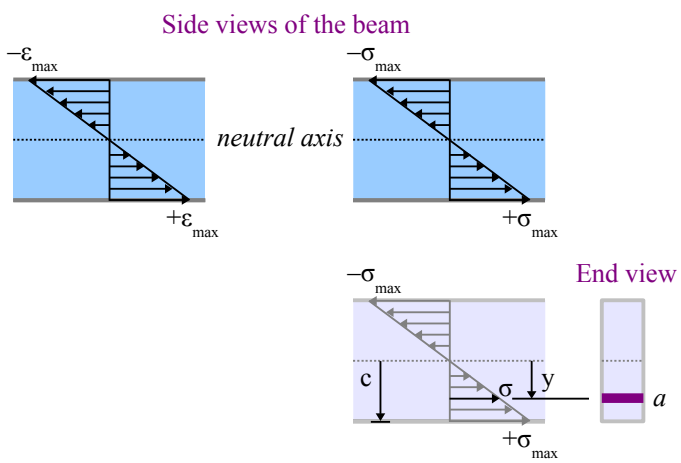
In the last chapter, we learned how to draw shear and bending moment diagrams for beams. These diagrams tell us the location and magnitude of the maximum shear load and maximum bending moment. We can use V_{max} and M_{max} to calculate the maximum shear stress and max. bending stress in a beam, then we can compare these results with the allowable shear stress and bending stress of the material. If the actual value is less than the allowable value, then the beam is safe; if the actual value is greater than the allowable, then we need to select a different beam.



Bending Stress in Beams

A point load at the midspan of a beam makes the beam bend. We can sketch a *deflection diagram* to show this bending. The deflection diagram shows the beam as if it had no depth, because it is easier to draw a curve than to draw a double curve with shading. A real beam has depth, and when it is bent, the top surface shortens while the bottom surface lengthens...the top surface has a negative strain, while the bottom surface has a positive strain. Plot strain vs. depth: the strain varies linearly from top to bottom, and is zero at the centroidal axis.

We saw in Chapter 2 that materials like steel and aluminum follow Hooke's law: the ratio of stress/strain is Young's modulus, a constant. Therefore, the stress in the beam also varies linearly from top to bottom, and is zero at the centroidal axis of the beam. We call this axis the *neutral axis*, where stress is zero.[9] This bending stress acts perpendicular to the cross-sectional area of the beam, so the stress is a normal stress; it is negative on the top and positive on the bottom.



We can calculate the bending stress at any position y from the neutral axis: the stress is proportional to the distance from the neutral axis, so $\sigma = \sigma_{max}\frac{y}{c}$ where c is the distance from the neutral axis to the surface of the beam. This is the same meaning of c that we used in torsion problems.

Think about a very small area a at a distance y from the neutral axis. Since stress is force divided by area, the force acting on this area is equal to the stress times the area: $P = \sigma \cdot a = \frac{\sigma_{max}\, a\, y}{c}$. The moment of this force with respect to the neutral axis is $M = P \cdot y = \frac{\sigma_{max}\, a\, y^2}{c}$. If we sum up the moment for all values of a and y, $M = \frac{\sigma_{max}}{c}\sum_1^n a_i y_i^2$. Recall from Chapter 5, the moment of inertia of a cross-sectional area relative to the x-x centroidal axis is $I_x = \sum_1^n a_i y_i^2$, therefore $M = \frac{\sigma_{max} I_x}{c}$.

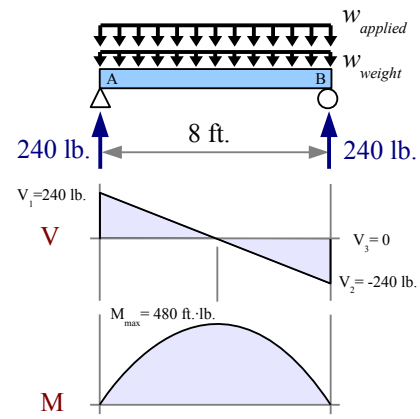
Rewrite this equation to solve for bending stress at the surface: $\sigma = \frac{Mc}{I_x}$. This equation is called the *Flexure Formula*.

9 The centroidal axis and the neutral axis are the same if the beam is made of a single material that obeys Hooke's law. The centroidal and neutral axes are often different in beams made of composite materials, like steel-reinforced concrete, or glass/epoxy composite.

We can use the moment equation if we want to know how much bending moment the beam can support. The allowable moment $M_{allowable} = \frac{\sigma_{allowable} I_x}{c}$.

Example #1

A 4×6 timber beam, 8 feet long, is loaded with a uniform distributed load of 54.65 lb./ft. The beam itself has a weight per unit length of 5.35 lb./ft. What is the maximum bending stress, and where does it occur? Report the answer in psi.



Solution The total uniform distributed load is the applied load plus the weight of the beam, $w_{total} = w_{applied} + w_{weight} = \frac{54.65\,\text{lb.}}{\text{ft.}} + \frac{5.35\,\text{lb.}}{\text{ft.}} = \frac{60\,\text{lb.}}{\text{ft.}}$

The load is symmetrical, so the reaction forces are equal to half of the applied load. $R_A = R_B = \frac{W}{2} = \frac{wL}{2} = \frac{60\,\text{lb.}}{\text{ft.}} \frac{8\,\text{ft.}}{2} = 240\,\text{lb.}$

The maximum moment equals the area of the left-hand triangle in the shear diagram: one half the base times the height. $M_{max} = \frac{240\,\text{lb.} \cdot 4\,\text{ft.}}{2} = 480\,\text{ft.} \cdot \text{lb.}$

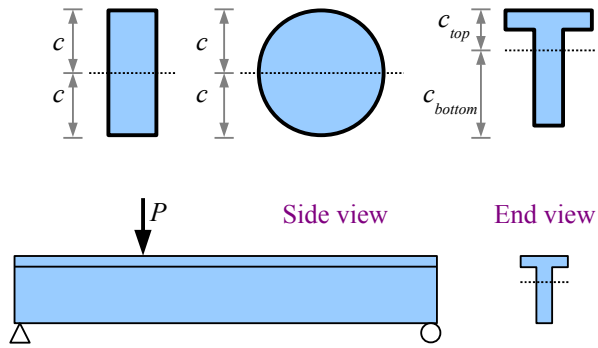
From Appendix E, the dimensions of a 4×6 timber are 3.5"×5.5". Since the cross-section is a rectangle, the moment of inertia is $I_x = \frac{bh^3}{12} = \frac{3.5\,\text{in.}(5.5\,\text{in.})^3}{12} = 48.53\,\text{in.}^4$ The value of c is $c = \frac{h}{2} = \frac{5.5\,\text{in.}}{2} = 2.75\,\text{in.}$ The maximum bending stress occurs at the midspan, and is $\sigma = \frac{Mc}{I_x} = \frac{480\,\text{ft.} \cdot \text{lb.}(2.75\,\text{in.})}{48.53\,\text{in.}^4} \left| \frac{12\,\text{in.}}{\text{ft.}} \right. = 326\,\text{psi}$. Bending stress diminishes to zero at the ends.

What happens if you use the nominal dimensions instead of actual dimensions for the timber? $I_x = \frac{4\,\text{in.}(6\,\text{in.})^3}{12} = 72\,\text{in.}^4$, $c = \frac{6\,\text{in.}}{2} = 3\,\text{in.}$, and $\sigma = \frac{480\,\text{ft.} \cdot \text{lb.}(3\,\text{in.})}{72\,\text{in.}} \left| \frac{12\,\text{in.}}{\text{ft.}} \right. = 240\,\text{psi}$, which is 26% lower than the actual stress.

Tables in the Appendix list values for I_x and I_y for many shapes, but no values for c. Instead, the tables give the section modulus, defined as $S_x = \frac{I_x}{c}$ and $S_y = \frac{I_y}{c}$. Section modulus makes the stress calculation easier, because it combines two terms into one. Bending stress $\sigma = \frac{Mc}{I_x} = \frac{M}{S_x}$, and allowable moment $M_{allowable} = \frac{\sigma_{allowable} I_x}{c} = \sigma_{allowable} S_x$.

In Example #1, the section modulus of a 4×6 timber is 17.6 in.^{3}. Bending stress is $\sigma = \frac{M}{S_x} = \frac{480\,\text{ft.} \cdot \text{lb.}}{17.6\,\text{in.}^3} \left| \frac{12\,\text{in.}}{\text{ft.}} \right. = 327\,\text{psi}$.

Simple shapes like rectangles and circles are symmetrical about the neutral axis, so the distance c from the neutral axis to the top surface is the same as the distance c from the neutral axis to the bottom surface. Compound shapes like a **T** are not symmetrical about the x-x neutral axis, so there are two values to consider: c_{top} and c_{bottom}. From the flexure formula, $\sigma = \frac{Mc}{I_x}$, the largest stress occurs on the surface with the largest value of c. In the case of a **T**-shaped beam loaded on the top flange as shown, the largest stress is the tensile stress along the bottom surface of the beam: $\sigma = \frac{M\ c_{bottom}}{I_x}$.



Example #2

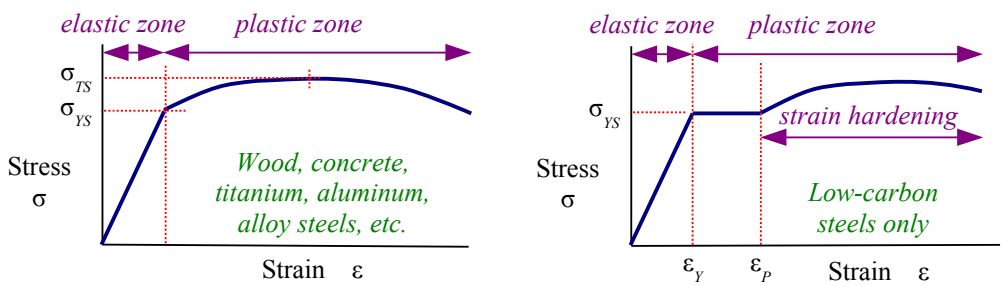
What is the allowable moment of a 2×4 Douglas fir beam? Report the answer in ft.·lb.

Solution The allowable moment is the largest bending moment that a beam can support. From Appendix E, the section modulus of a 2×4 timber is 3.06 in.^{3}, and the allowable bending stress is 900 psi.

The allowable moment is $M_{allowable} = \sigma_{allowable} S_x = \frac{900\,\text{lb.}}{\text{in.}^2} \frac{3.06\,\text{in.}^3}{} \left| \frac{\text{ft.}}{12\,\text{in.}} \right. = 230\ \text{ft.}\cdot\text{lb.}$

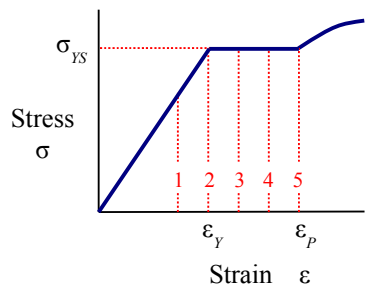
Bending Stress in Wide-Flange Steel Beams

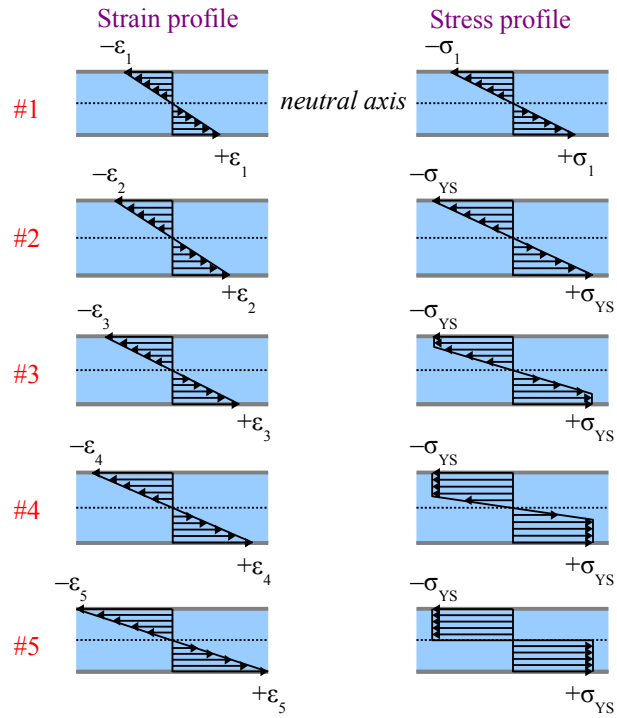
Stress-strain diagrams for many materials are simplified as a cartoon showing a straight line segment for elastic behavior followed by an arc for plastic behavior. In the elastic zone, stress is proportional to strain; the ratio is Young's modulus $E = \frac{\sigma}{\varepsilon}$.



Almost all equations in this book apply only to the elastic zone. In the plastic zone, the stress required to stretch the material increases due to *strain hardening* (also called *work hardening*) until the stress reaches a peak at the tensile strength of the material. After the peak, the material begins to thin, and the cross-sectional area shrinks, so there is less material to support the load, and eventually the material breaks.

Low-carbon steel used for wide-flange beams has an elastic zone followed by a horizontal plastic zone in which the stress does not change, so there is no strain hardening. The stress-strain curve for low-carbon sheet steel in Chapter 2 shows this effect clearly. As the strain builds up, eventually strain hardening begins, and the stress-strain curve arcs upwards. In steel beam design, we can use the horizontal part of the stress-strain curve to extend the load-carrying capacity of the beam. Let's look at the strain and stress profiles in a steel beam as the load is increased:





Point #1 Strain and stress are in the elastic zone. They are both zero at the neutral axis, and we have maximum compressive and tensile values at the top and bottom surfaces.

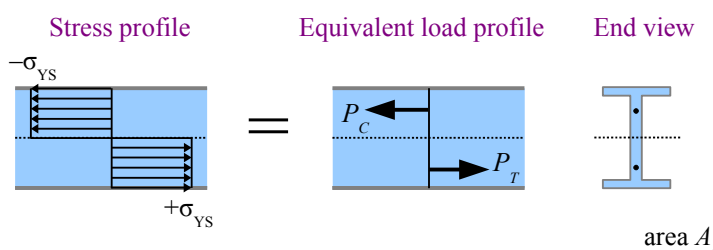
Point #2 Strain and stress have reached the yield point, so $\varepsilon_2 = \varepsilon_Y$ and $\sigma_2 = \sigma_{YS}$, the yield strength of the material.

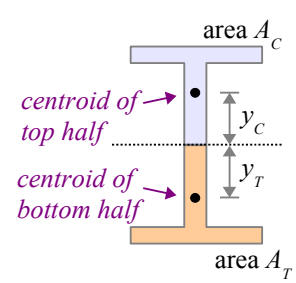
Point #3 Strain continues to increase, but the stress does not exceed the yield strength. Material below the surface reaches the yield point, down to a certain depth. Below this depth, the interior bending stress remains in the elastic zone.

Point #4 Strain continues to increase. More of the material below the surface reaches the yield point, but the core is in the elastic zone.

Point #5 Strain reaches the point of strain hardening, $\varepsilon_5 = \varepsilon_P$. All of the material has reached the yield point, becoming fully plastic. Any additional load on the beam will cause the beam to fail by bending at this location, like two hinged bars.

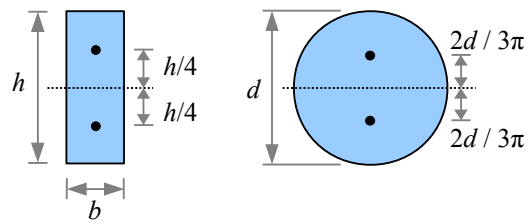
We can draw an equivalent load diagram for the fully-plastic beam at Point #5. The compressive force acting above the neutral axis equals the yield strength times the cross-sectional area of the beam in compression (the area above the neutral axis): $P_C = \sigma_{YS} \cdot A_C$. Likewise, the tensile force below the neutral axis is $P_T = \sigma_{YS} \cdot A_T$. The beam develops an internal moment to counteract these forces, equal to each force times the distance from the neutral axis of the beam to the centroidal axis of each area. Since this moment is counteracting fully-plastic loading, we call it the *plastic moment*, M_P., given as $M_P = P_C y_C + P_T y_T = \sigma_{YS} A_C y_C + \sigma_{YS} A_T y_T$.





If we define a new variable $Z = A_C y_C + A_T y_T$, then $M_P = \sigma_{YS} Z$. This new variable is the sum of the first moments of the areas above and below the neutral axis; we call it the *plastic section modulus*. Like section modulus, *S*, the plastic section modulus is calculated with reference to either the *x-x* or the *y-y* neutral axis. The example in the diagram is Z_x. Beams loaded in the strong direction will use Z_x; beams that are either loaded from the side or tipped on the side will use Z_y.

We can easily calculate Z for simple shapes; you can find Z_x and Z_y for complex shapes like wide-flanged beams in the Appendix.



Rectangle Each area is half the area of the rectangle. Each moment arm is half the height of each area, or a quarter of the height of the entire rectangle.

$$A_C = A_T = bh/2$$

$$y_C = y_T = h/4$$

$$Z_x = A_C\, y_C + A_T\, y_T = \frac{bh}{2}\frac{h}{4} + \frac{bh}{2}\frac{h}{4} = \frac{bh^2}{4} \quad \text{and} \quad Z_y = \frac{hb^2}{4}$$

Circle Each area is half the area of a circle. Find the moment arm in the Appendix.

$$A_C = A_T = \frac{\pi d^2}{4} \times \frac{1}{2} = \frac{\pi d^2}{8}, \quad y_C = y_T = \frac{2d}{3\pi}, \quad \text{and} \quad Z_x = Z_y = A_C\, y_C + A_T\, y_T = \frac{\pi d^2}{8}\frac{2d}{3\pi} + \frac{\pi d^2}{8}\frac{2d}{3\pi} = \frac{d^3}{6}$$

While ideal beams follow the ideal stress-strain curves in the preceding section, real beams vary in strength due to defects in manufacturing and handling. AISC recommends using a factor of safety of 1.67 in steel beam design, so $\sigma_{allowable} = \frac{\sigma_{YS}}{1.67} = 0.6\,\sigma_{YS}$. If we let the moment in a beam equal the plastic moment, then $M_{allowable} = \sigma_{allowable} Z = 0.6\,\sigma_{YS} Z$.

Example #3

What is the allowable moment of a W200×59 wide-flange beam? Report the answer in kN·m.

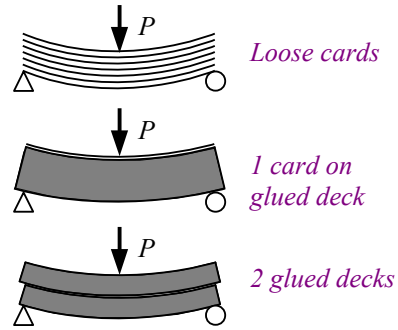
Solution From Appendix D, the plastic section modulus is 652×10^3 mm^3. From Appendix B, the yield strength is 345 MPa.

The allowable moment is $M_{allowable} = 0.6\,\sigma_{YS} Z_x = \frac{0.6 \;\; 345\,\text{MPa} \;\; 652\times10^3\,\text{mm}^3}{} \left|\frac{\text{m}^3}{(10^3\,\text{mm})^3}\right| \frac{10^3\,\text{kN}}{\text{MPa} \;\; \text{m}^2} = 135\,\text{kN}\cdot\text{m}$

The bending stress in a low-carbon steel beam is $\sigma = \frac{M}{Z_x}$.

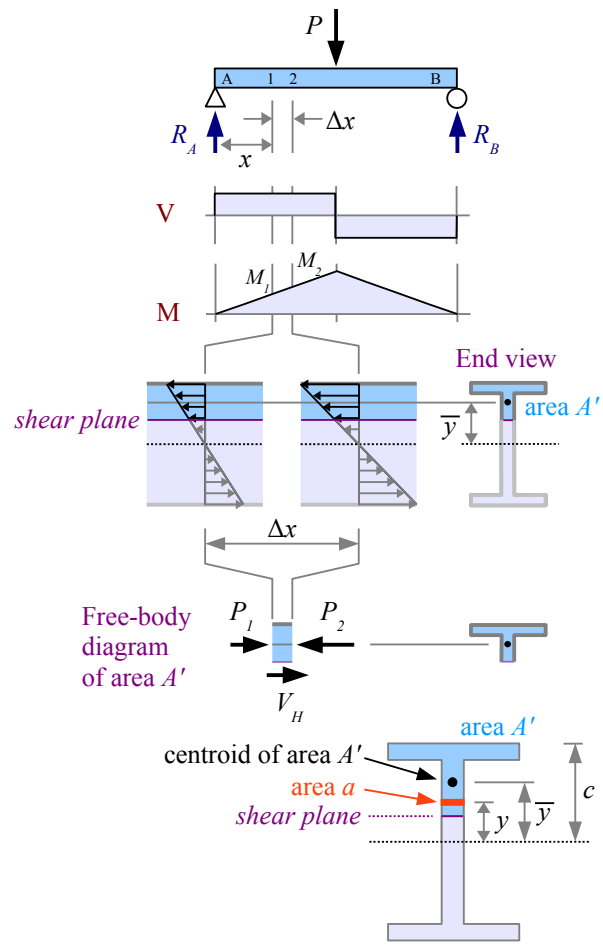
Shear Stress in Beams

Bending stress is a normal stress, acting perpendicularly to the cross-sectional area of the beam. Apply a downward load on top of a simply-supported beam, and the bending stress on the top surface is horizontal and compressive; along the bottom surface it is horizontal and tensile.



Beams also have a horizontal shear stress which acts parallel to the neutral axis of the beam. Take a deck of playing cards and flex it in your hand; the cards slide past each other. Think of this sliding as strain; if the cards had been glued together, then a corresponding shear stress would have developed in the adhesive.

Glue all but one of the cards on their faces, then place the free card on top of the glued deck. Bend the deck, and the top card slides very little with respect to the rest of the deck. Now take a second deck having the same number of cards as the first deck, and glue all of the cards in the second deck together. Place one deck on the other and bend them; the two decks slide with respect to each other – more so than if the decks were different thicknesses. This experiment shows that the shear stress is zero at the surface of a beam, and maximum at the neutral axis.



Bend a simply-supported beam with a point load at the midspan, and the moment diagram is an isosceles triangle. Select two points on the beam that are separated by a small distance Δx. On the moment diagram, $M_1 < M_2$. Since bending stress is proportional to bending moment, $\sigma_1 < \sigma_2$. Pick a horizontal plane within the beam where you want to calculate the shear stress. In this example, the shear plane is above the neutral axis. The cross-sectional area between the shear plane and the top of the beam is A'.

Draw a free-body diagram of the beam above the shear plane. The horizontal force acting on this area at point 1 is $P_1 = \sigma_1 A'$. Likewise, the horizontal force acting on this area at point 2 is $P_2 = \sigma_2 A'$. Since $\sigma_1 < \sigma_2$, $P_1 < P_2$. The sum of the forces in the x direction equals zero, so the difference between these forces is a horizontal shear force, $V_H = P_2 - P_1$.

The normal stress acting on area A' at point 1 varies with depth: it is highest on the top surface of the beam, and lower at the shear plane. We can break up area A' into many tiny areas, where each tiny area a is a distance y from the neutral axis. The average normal stress acting on area A' at point 1 is the sum of the normal stresses acting on all of the tiny areas, so $\sigma_{1avg} = \sum_{\bar{y}}^{c} \frac{M_1 y}{I}$. The normal force acting on area A' at point 1 is the average stress times the sum of these tiny areas, so $P_1 = \sum_{\bar{y}}^{c} \frac{M_1 y a}{I} = \frac{M_1}{I} \sum_{\bar{y}}^{c} y a$. Since $\sum_{\bar{y}}^{c} y a = \bar{y} A'$, $P_1 = \frac{M_1 \bar{y} A'}{I}$. Similarly, the normal force acting on area A' at point 2 is $P_2 = \frac{M_2 \bar{y} A'}{I}$. The shear force $V_H = P_2 - P_1 = \frac{(M_2 - M_1) \bar{y} A'}{I}$.

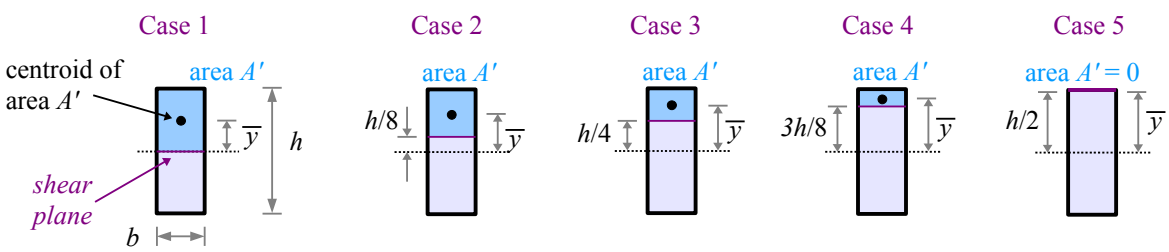
The value of the bending moment at any point along a beam equals the area of the shear diagram up to that point, so $M_1 = V x$ and $M_2 = V(x + \Delta x)$. The difference of the two moments is $M_2 - M_1 = V(x + \Delta x) - V x = V \cdot \Delta x$. Substitute

this result into the shear force equation, and we have $V_H = \frac{V \Delta x \bar{y} A'}{I}$.

The shear stress acting in the shear plane equals the shear force V_H divided by the area it acts on: the thickness of the beam in the shear plane times the length of beam between points 1 and 2: $\tau = \frac{V_H}{t \Delta x} = \frac{V \Delta x \bar{y} A'}{I t \Delta x} = \frac{V \bar{y} A'}{I t}$. The product $\bar{y} A'$ is the first moment of area A' about the neutral axis, defined as the variable $Q = \bar{y} A'$. Substituting, we have the *General Shear Formula* for beams: $\tau = \frac{V Q}{I t}$.

Example #4

Calculate the shear stress in a rectangular beam at five locations: at the neutral axis, ¼ of the way between the neutral axis and the top of the beam, ½ way, ¾ of the way, and at the top of the beam.



Case 1 When the shear plane is at the neutral axis, $\bar{y} = \frac{h}{4}$ and $A' = \frac{bh}{2}$, so $Q = \bar{y} A' = \frac{h}{4}\frac{bh}{2} = \frac{bh^2}{8}$. The moment of inertia of a rectangle is $I = \frac{bh^3}{12}$ and the thickness of the beam at the shear plane is $t = b$. The shear stress is

$$\tau = \frac{VQ}{I t} = \frac{V}{}\frac{bh^2}{8}\frac{12}{bh^3}\frac{}{b} = \frac{3V}{2bh}$$

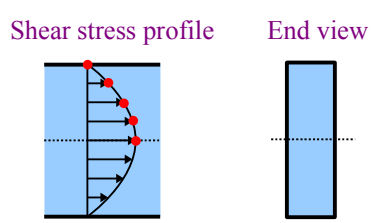
Case 2 $\bar{y} = \frac{5}{16}h$ and $A' = \frac{3}{8}bh$, so $Q = \bar{y} A' = \frac{5h}{16}\frac{3bh}{8} = \frac{15bh^2}{128}$. The other two variables in the General Shear Equation, I and t, are the same as before, so the shear stress is $\tau = \frac{VQ}{I t} = \frac{V}{}\frac{15bh^2}{128}\frac{12}{bh^3}\frac{}{b} = \frac{45V}{32bh}$

Case 3 $\bar{y} = \frac{3}{8}h$ and $A' = \frac{1}{4}bh$, so $Q = \bar{y} A' = \frac{3h}{8}\frac{bh}{4} = \frac{3bh^2}{32}$ and $\tau = \frac{VQ}{I t} = \frac{V}{}\frac{3bh^2}{32}\frac{12}{bh^3}\frac{}{b} = \frac{9V}{8bh}$

Case 4 $\bar{y} = \frac{7}{16}h$ and $A' = \frac{1}{8}bh$, so $Q = \bar{y} A' = \frac{7h}{16}\frac{bh}{8} = \frac{7bh^2}{128}$ and $\tau = \frac{VQ}{I t} = \frac{V}{}\frac{7bh^2}{128}\frac{12}{bh^3}\frac{}{b} = \frac{21V}{32bh}$

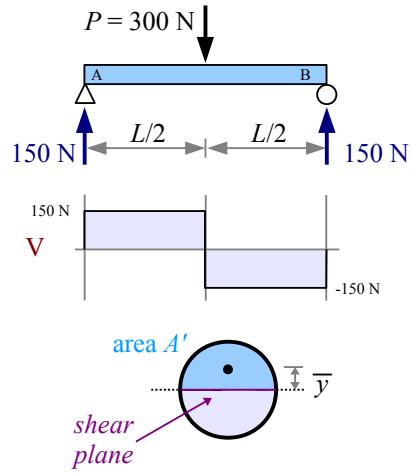
Case 5 $\bar{y} = \frac{1}{2}h$ and $A' = 0$, so $Q = \bar{y} A' = 0$ and $\tau = \frac{VQ}{I t} = 0$

We can plot the results from this Example problem as a function of depth (red dots at the right). The shear stress profile for a rectangular beam looks like a parabola. The shear stress is a maximum at the neutral axis, diminishing to zero at the top and bottom surfaces. The profile looks different for wide-flange beams because the area is distributed differently, but the maximum shear stress will still be at the core and the surface will have zero shear stress.



Example #5

A round 1.5 cm diameter rod is loaded in bending with a point load of 300 N at the midspan. Calculate the shear stress at the neutral axis. Report the result in MPa.



Solution The loading is symmetrical, so the reaction forces equal half the applied load: $R_A = R_B = \frac{300\text{ N}}{2} = 150\text{ N}$. Draw the shear diagram to find the maximum shear load, $V_{max} = R_A = 150\text{ N}$.

From Appendix C, the centroid of area A' is at $\bar{y} = \frac{2d}{3\pi}$.

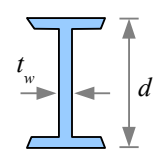
Area A' is half the area of the circle: $A' = \frac{1}{2}\frac{\pi d^2}{4} = \frac{\pi d^2}{8}$.

Calculate $Q = \bar{y}A' = \frac{2d}{3\pi}\frac{\pi d^2}{8} = \frac{d^3}{12}$. The moment of inertia of a circle is $I = \frac{\pi d^4}{64}$, and the thickness of the beam at the shear plane is d.

Combine terms to find shear stress, $\tau = \frac{VQ}{It} = \frac{V}{12}\frac{d^3}{\pi d^4}\frac{64}{d} = \frac{16V}{3\pi d^2} = \frac{16 \cdot 150\text{ N}}{3\pi(1.5\text{ cm})^2}\left|\frac{\text{MPa m}^2}{10^6\text{ N}}\right|\frac{(100\text{ cm})^2}{\text{m}^2} = 1.13\text{ MPa}$

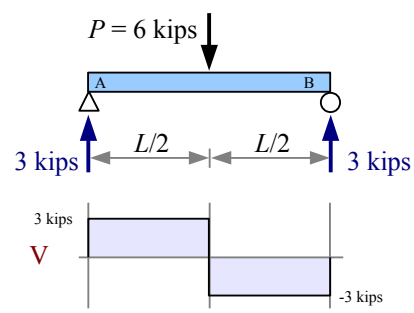
Another way to solve this problem is to calculate numerical answers for $\bar{y}$, A', Q, and I, with appropriate unit conversions, then calculate τ. Try it both ways, and you will find that the algebraic solution is shorter and requires fewer keystrokes on the calculator, which means fewer opportunities for error.

The General Shear Formula works well for simple shapes where $\bar{y}$ and A' are easy to calculate. Steel wide-flange beams have a more complicated shape, so we can approximate the shear stress at the neutral axis using the *Average Web Shear Approach:* $\tau = \frac{V}{d\,t_w}$ where d is the depth of the beam and t_w is the thickness of the web.



Example #6

A W4×13 steel wide-flange beam is loaded in bending with a point load of 6 kips at the midspan. Calculate the shear stress at the neutral axis. Report the result in ksi.



Solution The loading is symmetrical, so the reaction forces equal half the applied load: $R_A = R_B = \frac{6\text{ kips}}{2} = 3\text{ kips}$. From the shear diagram, the maximum shear load $V_{max} = R_A = 3\text{ kips}$. Use the Average Web Shear Approach to find the shear stress at the neutral axis:

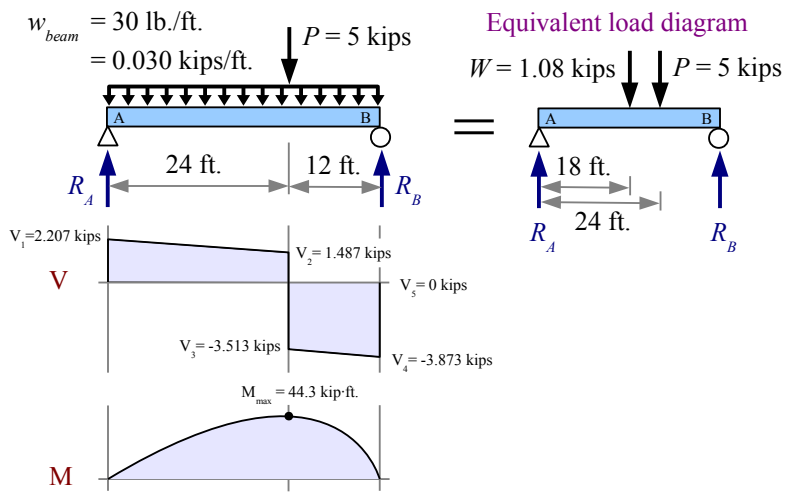
$$\tau = \frac{V}{d\,t_w} = \frac{3\text{ kips}}{4.16\text{ in.} \times 0.280\text{ in.}} = 2.58\text{ ksi}.$$

How does the calculated shear stress compare with the actual shear strength of the material? While yield and ultimate strengths are easy to find in reference books and on the internet, shear strengths are rarely reported. A good estimate for steels is $\tau_{allowable} = 0.4\,\sigma_{YS}$. If the beam in the previous Example is made of A36 steel with a yield strength of 36 ksi, the allowable shear stress is $\tau_{allowable} = 0.4\,(36\,\text{ksi}) = 14.4\,\text{ksi} > \tau_{applied}$, therefore the beam will not fail in shear.

In most of the problems so far, we have assumed that the beams are weightless. Real beams have real weight, so we need to include the weight as a uniform distributed load along the length of the beam.

Example #7

A 36 foot long W12×30 steel wide-flange beam is loaded in bending with a point load of 5 kips located 24 feet from the left end. The beam also has weight. Calculate the allowable bending moment in the beam. Calculate the maximum bending stress and maximum shear stress, and report the result in ksi. Report the locations of these maximum points.



Solution The loading is not symmetrical, so use an equivalent load diagram, sum of the moments and sum of the forces to find the reaction forces. A W12×30 beam has a weight per unit length $w = 30\,\text{lb./ft.}$, so the equivalent point load

$$W = \frac{30\,\text{lb.}}{\text{ft.}}\frac{36\,\text{ft.}}{}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\right. = 1.08\,\text{kips} \text{ at the midspan.}$$

The moment about point A is $\circlearrowleft + \sum M_A = 0 = -1.08\,\text{kips}\cdot 18\,\text{ft.} - 5\,\text{kips}\cdot 24\,\text{ft.} + R_B\cdot 36\,\text{ft.}$

Rewrite the equation to find the reaction force $R_B = \dfrac{1.08\,\text{kips}\cdot 18\,\text{ft.} + 5\,\text{kips}\cdot 24\,\text{ft.}}{36\,\text{ft.}} = 3.873\,\text{kips}$

Sum of the forces $\uparrow + \sum F_y = 0 = R_A - 1.08\,\text{kips} - 5\,\text{kips} + 3.873\,\text{kips}$. Solve for the reaction force $R_A = 1.08\,\text{kips} + 5\,\text{kips} - 3.873\,\text{kips} = 2.207\,\text{kips}$. Calculate the inflection points on the shear diagram.

$$V_1 = R_A = 2.207\,\text{kips}\,,\quad V_2 = V_1 - \frac{0.030\,\text{kips}}{\text{ft.}}\frac{24\,\text{ft.}}{} = 1.487\,\text{kips}\,,\quad V_3 = V_2 - 5\,\text{kips} = -3.5133\,\text{kips}\,,$$

$$V_4 = V_3 - \frac{0.030\,\text{kips}}{\text{ft.}}\frac{12\,\text{ft.}}{} = -3.873\,\text{kips}\,,\quad V_5 = V_4 + R_B = 0\,\text{kips}$$

The maximum moment is the area of the left-hand trapezoid in the shear diagram:

$$M_{max} = \frac{2.207\,\text{kips} + 1.487\,\text{kips}}{2}\frac{24\,\text{ft.}}{} = 44.3\,\text{kip}\cdot\text{ft.}$$

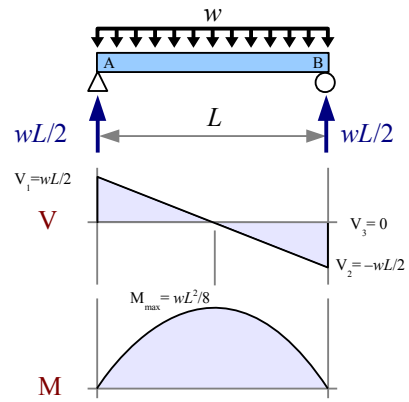
The allowable bending moment $M_{allowable} = 0.6\,\sigma_{YS}\,Z = \dfrac{0.6\cdot 50\,\text{kips}\cdot 43.1\,\text{in.}^3}{\text{in.}^2}\left|\dfrac{\text{ft.}}{12\,\text{in.}}\right. = 107.8\,\text{kip}\cdot\text{ft.}$

Since $M_{allowable} > M_{applied}$, the beam is strong enough to support the existing loads in bending.

The maximum bending stress in the beam is $\sigma = \dfrac{M}{Z} = \dfrac{44.3\,\text{kip}\cdot\text{ft.}}{43.1\,\text{in.}^3}\left|\dfrac{12\,\text{in.}}{\text{ft.}}\right. = 12.34\,\text{ksi}$, and it occurs at the point load.

From the shear diagram, the maximum shear load $|V_{max}| = 3.873\,\text{kips}$. Use the Average Web Shear Approach to find the maximum shear stress $\tau = \dfrac{V}{d\,t_w} = \dfrac{3.873\,\text{kips}}{12.34\,\text{in.}\times 0.260\,\text{in.}} = 1.21\,\text{ksi}$, which occurs at point B.

Allowable Load



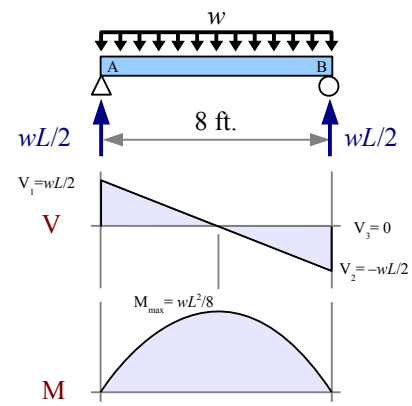
Design engineers select beams that will support known loads. However, in many cases the structure already exists, and will be used with a different loading condition than it was originally intended. A manufacturing plant may bring in a larger piece of equipment, so the engineer calculates whether the existing floor will support the machine.

A simply-supported timber beam carries a uniform distributed load; how large a load can the beam support? Since the loading is symmetrical, the reaction forces equal half the applied load: $R_A = R_B = \frac{w\,L}{2}$. The maximum shear load is $V_1 = R_A = \frac{w\,L}{2}$. The maximum moment equals the area of the left-hand triangle in the shear diagram, or one half the base times the height: $M_{max} = \frac{1}{2}\frac{L}{2}\frac{w\,L}{2} = \frac{w\,L^2}{8}$. If we look at bending stress, then $\sigma = \frac{M}{S} = \frac{w\,L^2}{8\,S}$.

Rewrite the equation to solve for the load, $w = \frac{8\,\sigma_{allowable}\,S}{L^2}$. If we look at shear stress, $\tau = \frac{V\,Q}{I\,t} = \frac{w\,L\,Q}{2\,I\,t}$. Rewrite the equation to solve for the load, $w = \frac{2\,\tau_{allowable}\,I\,t}{L\,Q}$.

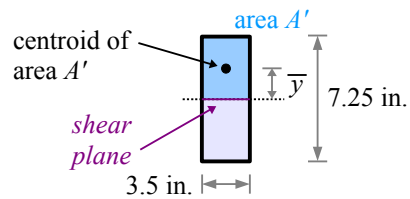
Example #8

A simply-supported, 8 foot long, 4×8 southern yellow pine timber supports a uniform distributed load. Calculate the maximum load the beam can support. Report the result in lb./ft.



Solution From the previous discussion, based on bending stress the beam can support a total load of $w_{total}=\frac{8\,\sigma_{allowable}\,S}{L^2}$. From the Appendix, the allowable bending stress of southern yellow pine is 1400 psi and the section modulus $S_x=30.7\,\text{in.}^3$.

The total load is $w_{total}=\frac{8}{(8\,\text{ft.})^2}\frac{1400\,\text{lb.}}{\text{in.}^2}\frac{30.7\,\text{in.}^3}{}\left|\frac{\text{ft.}}{12\,\text{in.}}\right.=448\,\text{lb./ft.}$. The total load includes the beam's weight per unit length: $w_{total}=w_{applied}+w_{beam}$. From the Appendix, a 4×8 timber has a weight of 7.05 lb./ft., so $w_{applied}=w_{total}-w_{beam}=448\,\text{lb./ft.}-7.05\,\text{lb./ft.}=441\,\text{lb./ft.}$



Also from the previous discussion, based on shear stress the beam can support a total load of $w_{total}=\frac{2\,\tau_{allowable}\,I\,t}{L\,Q}$. From the Appendix, $I_x=111\,\text{in.}^4$ and $\tau_{allowable}=175\,\text{psi}$.

Based on the dimensions, $\bar{y}=\frac{7.25\,\text{in.}}{4}=1.8125\,\text{in.}$, $t=3.5\,\text{in.}$, and $A'=\frac{3.5\,\text{in.}\cdot 7.25\,\text{in.}}{2}=12.6875\,\text{in.}^2$, so $Q=A'\bar{y}=12.6875\,\text{in.}^2\cdot 1.8125\,\text{in.}=22.9961\,\text{in.}^3$.

The total load is $w_{total}=\frac{2\,\tau_{allowable}\,I\,t}{L\,Q}=\frac{2}{8\,\text{ft.}}\frac{175\,\text{lb.}}{\text{in.}^2}\frac{111\,\text{in.}^4}{22.9961\,\text{in.}^3}\frac{3.5\,\text{in.}}{}=739\,\text{lb./ft.}$.

The applied load is $w_{applied}=739\,\text{lb./ft.}-7.05\,\text{lb./ft.}=732\,\text{lb./ft.}$ The beam can support an applied load of 441 lb./ft. before it fails in bending, and a load of 732 lb./ft. before it fails in shear. Pick the lower of the two numbers because this is where failure will occur first.

Key Equations

Use the Flexure Formula to find the bending stress in any beam other than a steel W-beam: $\sigma=\frac{Mc}{I_x}$ or $\sigma=\frac{M}{S_x}$

The allowable moment of a beam is $M_{allowable}=\frac{\sigma_{allowable}\,I_x}{c}$ or $M_{allowable}=\sigma_{allowable}\,S_x$

For wide-flange steel beams *only*, the allowable moment is $M_{allowable}=0.6\,\sigma_{YS}\,Z_x$ and the bending stress is $\sigma=\frac{M}{Z_x}$

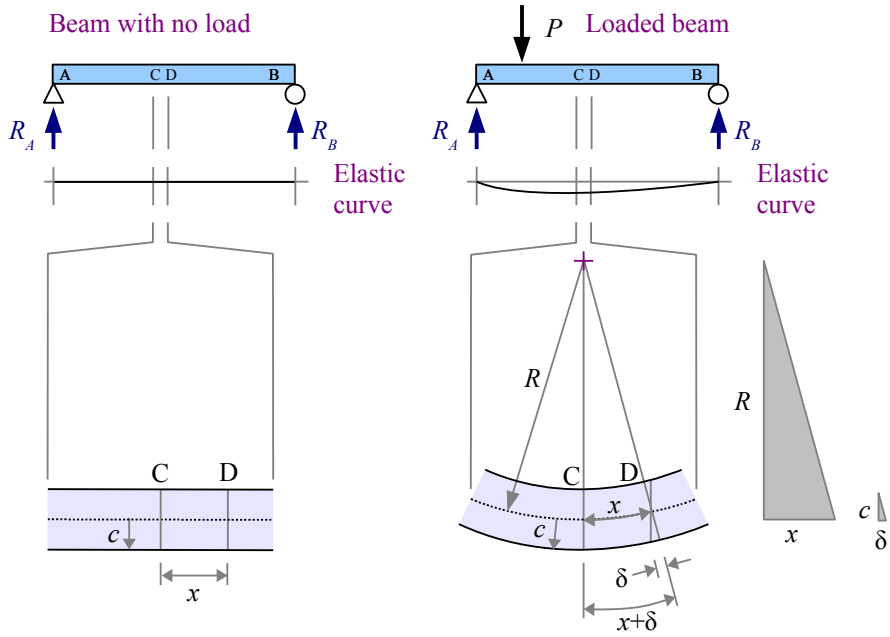
Use the General Shear Formula to find the maximum shear stress in a beam: $\tau=\frac{V\,Q}{I\,t}$ where $Q=\bar{y}\,A'$

Use the Average Web Shear Approach to find the maximum shear stress in a wide-flange steel beam: $\tau=\frac{V}{d\,t_w}$

For steels, use $\tau_{allowable}=0.4\,\sigma_{YS}$

Chapter 10: Beam Deflection

Radius of Curvature

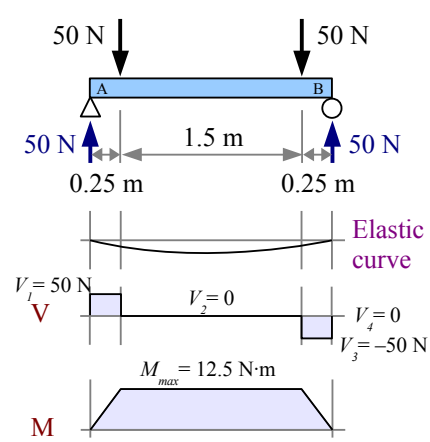


A weightless beam with no loads has a horizontal elastic curve. Pick two points C and D that are a small distance x apart. If we add a load, the beam deflects, and the elastic curve bows downward. Points C and D now lie on an arc with a *radius of curvature*, labeled R on the diagram. At the bottom of the beam, points C and D are now $x+\delta$ apart, because the bottom of the beam has stretched a little. Draw a vertical line through point D, and we have a little triangle with height c and base δ. We can use the principle of similar triangles to compare the little triangle with the big triangle having a height R and base x: $\frac{\delta}{c}=\frac{x}{R}$. Rewrite the equation to solve for $\frac{\delta}{x}=\frac{c}{R}$. Along the bottom surface of the beam, the strain is $\varepsilon=\frac{\delta}{x}$, so $\varepsilon=\frac{c}{R}$. As long as the beam is elastic, $E=\frac{\sigma}{\varepsilon}$, which we can rewrite as $\varepsilon=\frac{\sigma}{E}$. Set the strain equations equal to each other, and $\frac{\sigma}{E}=\frac{c}{R}$. Solving for stress, $\sigma=\frac{E\,c}{R}$. We know another equation for bending stress in a beam: $\sigma=\frac{M\,c}{I}$. Set the stress equations equal to each other, and $\frac{E\,c}{R}=\frac{M\,c}{I}\rightarrow\frac{E}{R}=\frac{M}{I}$. Now solve for radius of curvature, $R=\frac{E\,I}{M}$.

This equation helps us calculate the radius of curvature at a given point along the length of a beam. There are some special cases where the moment M is constant over a long length, so the elastic curve is a circular arc along this length.

Example #1

A 2 m long titanium beam is made of 1 cm square barstock. Two 50 N point loads lie 0.25 meters from each end. What is the radius of curvature of the beam between the two point loads? Report the answer in meters.



Solution Draw the elastic curve, shear diagram, and moment diagram. The loading is symmetrical, so the reaction forces $R_A=R_B=50\text{ N}$. The value of the moment diagram between the point loads equals the area of the left-hand rectangle in the shear diagram: $M=0.25\text{ m}\cdot 50\text{ N}=12.5\text{ N}\cdot\text{m}$.

The moment of inertia of a square $I=\frac{b\,h^3}{12}=\frac{b^4}{12}$.

The radius of curvature of the beam between the two point loads is

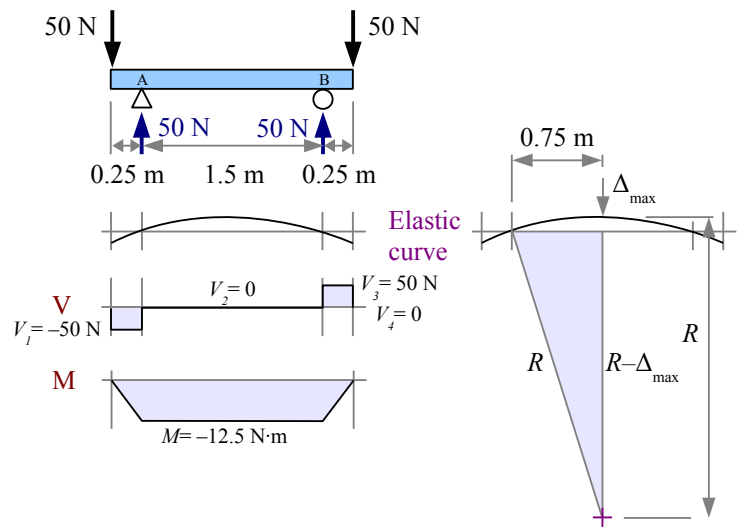
$$R=\frac{E\,I}{M}=\frac{E\,b^4}{12\,M}=\frac{114\text{ GPa}\,(1\text{ cm})^4}{12\cdot 12.5\text{ N}\cdot\text{m}}\left|\frac{10^9\text{ N}}{\text{GPa m}^2}\right|\frac{\text{m}^4}{(100\text{cm})^4}=7.6\text{ m}$$

The blade on a bandsaw obeys the radius of curvature equation because the blade flexes elastically, and forms a semicircular arc at each bandsaw wheel.

In a few rare cases, we can calculate the maximum deflection of a beam using the radius of curvature, as in the following example.

Example #2

A 2 m long titanium beam is made of 1 cm square barstock. Two 50 N point loads lie at the ends, and the supports lie 0.25 meters from each end. What is the maximum deflection at the midspan? Report the answer in cm.



Solution This problem is like Example #1 but flipped upside down. Draw the elastic curve with its radius of curvature and the center of curvature (cross at the bottom of the figure). The distance from the center of curvature to the beam is R; the beam deflects a distance Δ_{max} at the midspan; and the distance from the center of curvature to the undeflected beam is $R-\Delta_{max}$. The base of the shaded triangle is half the distance between the supports, 0.75 m. Use the Pythagorean Theorem to find the height of the triangle, $R-\Delta_{max}=\sqrt{R^2-(0.75\text{ m})^2}=\sqrt{(7.6\text{ m})^2-(0.75\text{ m})^2}=7.563\text{ m}$

Deflection at the midspan $\Delta_{max}=R-(R-\Delta_{max})=\frac{7.600\text{ m}-7.563\text{ m}}{}\left|\frac{100\text{ cm}}{\text{m}}\right.=3.71\text{ cm}$

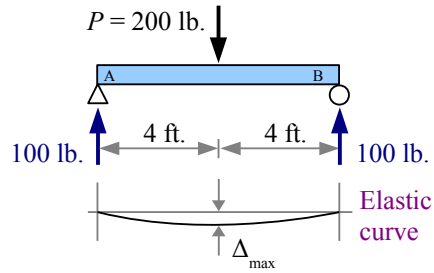
The Formula Method for Simple Cases

We can use the Radius of Curvature Method to calculate deflection as long as the moment is constant. From the examples in previous chapters, we know that the moment usually changes along the length of a beam. There are a number of geometrical and calculus-based techniques for calculating deflection in these cases. Engineers have compiled these solutions in handbooks, so future engineers can use a worked-out algebraic solution, instead of deriving the solution from scratch. The most useful compilation is *Roark's Formulas for Stress and Strain*, which contains many hundreds of formulas for beams, shells, membranes, pressure vessels, and columns.

Look in the Appendix for Formula Method solutions for various beam problems. Each case includes a load diagram, shear diagram, moment diagram, reaction forces and moments, and formulas for shear, moment, the slope at the ends, and deflection at various positions along the beam. Use the Appendix to find values for E and I, then solve.

Example #3

Use the Formula Method to calculate the maximum deflection of a simply-supported 8-foot 4×4 Southern yellow pine timber loaded with a 200 lb. point load at the midspan. Report the result in inches.



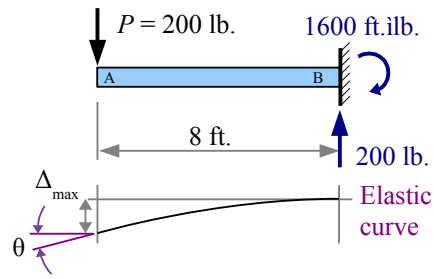
Solution From Appendix F, the deflection at the midspan $\Delta_{max} = \frac{P L^3}{48 E I}$, where Young's modulus is 1700 ksi and the moment of inertia is 12.5 in.^{4}.

$$\Delta_{max} = \frac{200\,\text{lb.}(8\,\text{ft.})^3}{48 \cdot 12.5\,\text{in.}^4} \frac{\text{in.}^2}{1700\,\text{kip}} \left| \frac{(12\,\text{in.})^3}{\text{ft.}^3} \right| \frac{\text{kip}}{10^3\,\text{lb.}} = 0.17\,\text{in.}$$

Example #4

Use the Formula Method to calculate the maximum deflection of a cantilever 8-foot 4×4 Southern yellow pine timber loaded with a 200 lb. point load at the end. Report the result in inches.

Calculate the slope at the end of the beam; report the result in degrees.



Solution From the Appendix, the deflection at the free end $\Delta_{max} = \frac{P L^3}{3 E I}$.

The moment of inertia and Young's modulus are the same as in the previous Example.

$$\Delta_{max} = \frac{200\,\text{lb.}(8\,\text{ft.})^3}{3 \cdot 12.5\,\text{in.}^4} \frac{\text{in.}^2}{1700\,\text{kip}} \left| \frac{(12\,\text{in.})^3}{\text{ft.}^3} \right| \frac{\text{kip}}{10^3\,\text{lb.}} = 2.78\,\text{in.}$$

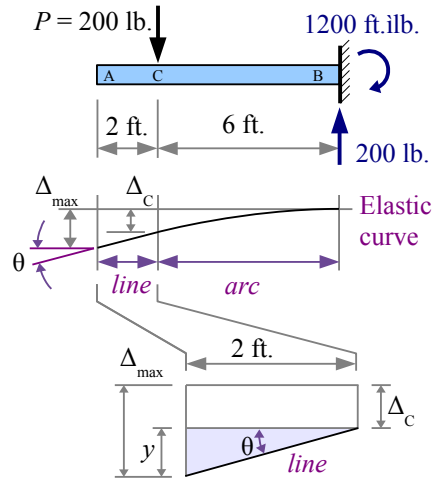
The equation for the slope at the free end is in radians, so include a conversion factor for degrees:

$$\theta = \frac{P L^2}{2 E I} = \frac{200\,\text{lb.}(8\,\text{ft.})^2}{2 \cdot 12.5\,\text{in.}^4} \frac{\text{in.}^2}{1700\,\text{kip}} \left| \frac{(12\,\text{in.})^2}{\text{ft.}^2} \right| \frac{\text{kip}}{10^3\,\text{lb.}} \left| \frac{180\,°}{\text{rad}} \right| = 7.81\,°$$

When the point load on a cantilever beam is at the end of the beam, the elastic curve is an arc. If the point load is located away from the end, then the elastic curve is an arc from the wall to the point load, and a straight line between the point load and the end of the beam. This portion of the beam is sloped but straight.

Example #5

A cantilever 8-foot 4×4 Southern yellow pine timber loaded with a 200 lb. point load 2 feet from the end. Use the Formula Method to calculate the deflection at the point load. Calculate the deflection at the point load and the maximum deflection at the end. Report the results in inches.



Solution From Appendix F, the deflection at point C is $\Delta_C = \frac{P b^3}{3 E I}$, where b is the distance from the point load to the wall. The moment of inertia and Young's modulus are the same as in the previous two Examples.

$$\Delta_C = \frac{200\,\text{lb.}(6\,\text{ft.})^3}{3 \cdot 12.5\,\text{in.}^4} \frac{\text{in.}^2}{1700\,\text{kip}} \left| \frac{(12\,\text{in.})^3}{\text{ft.}^3} \frac{\text{kip}}{10^3\,\text{lb.}} \right. = 1.17\,\text{in.}$$

The slope of the straight portion of the beam from point C to the left end is

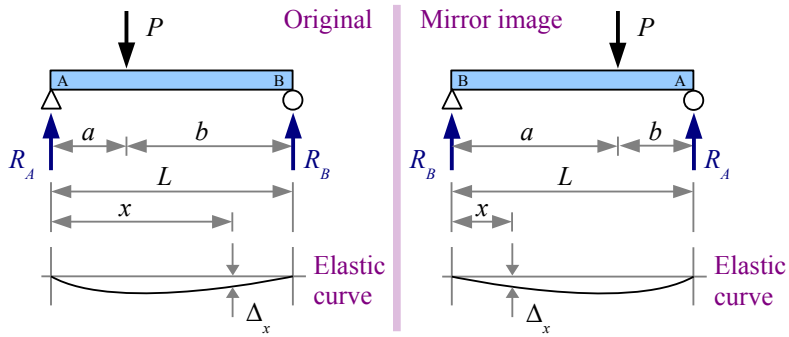
$$\theta = \frac{P b^2}{2 E I} = \frac{200\,\text{lb.}(6\,\text{ft.})^2}{2 \cdot 12.5\,\text{in.}^4} \frac{\text{in.}^2}{1700\,\text{kip}} \left| \frac{(12\,\text{in.})^2}{\text{ft.}^2} \right| \frac{\text{kip}}{10^3\,\text{lb.}} \left| \frac{180\,°}{\text{rad}} \right. = 4.39\,° .$$

The portion of the beam to the left of the point load is a straight line. Use trigonometry to find the maximum deflection: $\tan\theta = \frac{y}{2\,\text{ft.}}$ so $y = \frac{2\,\text{ft.}\ \tan(4.39\,°)}{} \left| \frac{12\,\text{in.}}{\text{ft.}} \right. = 1.84\,\text{in.}$ and

$$\Delta_{max} = y + \Delta_C = 1.84\,\text{in.} + 1.1\,\text{in.} = 3.01\,\text{in.}$$

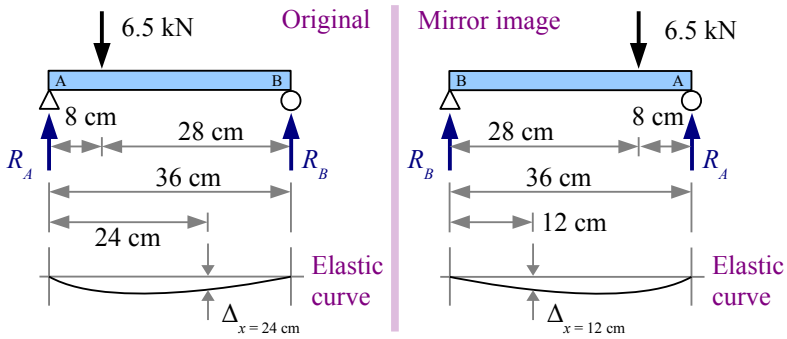
Formula Method Hints

Some Formula Method equations have conditions. Consider a simply-supported beam with a single point load located at a distance a from the left end. The deflection at a distance x from the left end is $\Delta_x = \frac{P\,b\,x}{6\,E\,I\,L}\left(L^2 - b^2 - x^2\right)$ when $x < a$, so this equation is restricted to the portion of the beam to the left of the point load. What if we want the deflection at a location to the right of the point load? Redraw the beam as a left-to-right mirror image, swapping dimensions a and b, and solve for the deflection.



Example #6

A simply-supported 36 cm long steel beam is loaded with a 6.5 kN point load 8 cm from the left end. The beam is made of 1.2 cm square barstock. Use the Formula Method to calculate the deflection 24 cm from the left end. Report the results in mm.



Solution From Appendix F, the deflection at a distance x from the left end is $\Delta_x = \frac{P\,b\,x}{6\,E\,I\,L}\left(L^2 - b^2 - x^2\right)$ when $x < a$.

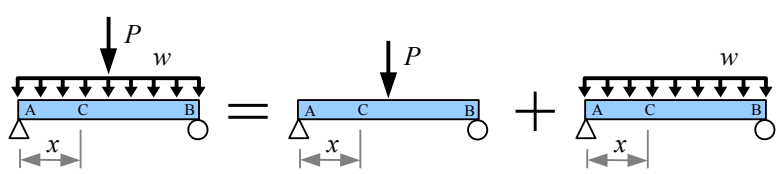
Since $x > a$, we have to draw a mirror image of the loading case.

The moment of inertia is $I = \frac{bh^3}{12} = \frac{b^4}{12} = \frac{(1.2\,\text{cm})^4}{12} = 0.1728\,\text{cm}^4$

$$\Delta_x = \frac{6.5\,\text{kN} \cdot 8\,\text{cm} \cdot 12\,\text{cm}}{6 \cdot 207\,\text{GPa} \cdot 0.1728\,\text{cm}^4 \cdot 36\,\text{cm}} \left| \frac{\text{GPa}\ \ \text{m}^2}{10^6\,\text{kN}} \right| \frac{(100\,\text{cm})^2}{\text{m}^2} \times \frac{\left((36\,\text{cm})^2 - (8\,\text{cm})^2 - (12\,\text{cm})^2\right)}{} \left| \frac{10\,\text{mm}}{\text{cm}} \right| = 8.79\,\text{mm}$$

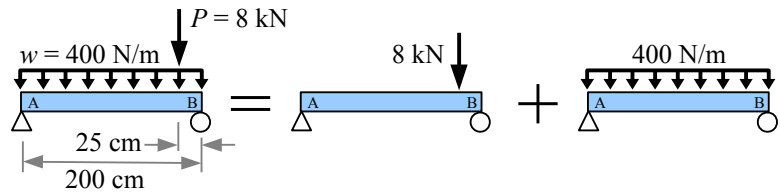
The Formula Method for Complex Cases: Superposition

The Formula Method equations in the Appendix apply to simple loading cases. Many loading conditions are more complex, but you can separate a complex loading condition into its parts, solve for V, M, or Δ at any location along the beam, then add the results together to find the total V, M, or Δ at that location: $\Delta_{total} = \Delta_1 + \Delta_2 + \Delta_3 + \ldots \Delta_n$ for n cases.



Example #7

A simply-supported 200 cm long, 51 mm standard steel pipe is loaded with a uniform distributed load of 400 N/m and a point load of 8 kN located 25 cm from the right end. Use the Formula Method and Superposition to calculate the deflection at the midspan (at $x = 1\,\text{m}$). Report the results in mm.



Solution From Appendix F, the deflection due to the point load at a distance x from the left end is $\Delta_1 = \frac{P b x}{6 E I L}(L^2 - b^2 - x^2)$. At the midspan, $x = \frac{L}{2}$, so $\Delta_1 = \frac{P b L}{6 E I L 2}\left(L^2 - b^2 - \left(\frac{L}{2}\right)^2\right) = \frac{P b}{12 E I}\left(\frac{3}{4}L^2 - b^2\right)$. In this equation, b is 25 cm. Also from the Appendix, a 51 mm standard steel pipe has a moment of inertia $I = 0.276 \times 10^6\,\text{mm}^4$

$$\Delta_1 = \frac{8\,\text{kN} \cdot 25\,\text{cm}}{12 \cdot 207\,\text{GPa} \cdot 0.276 \times 10^6\,\text{mm}^4}\left|\frac{\text{GPa}\ \ \text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^2}{\text{m}^2}\left|\frac{10\,\text{mm}}{\text{cm}} \times \frac{\left(0.75(200\,\text{cm})^2 - (25\,\text{cm})^2\right)}{}\right|\frac{(10\,\text{mm})^2}{\text{cm}^2} = 8.57\,\text{mm}$$

The deflection due to the distributed load at the midspan is $\Delta_2 = \frac{5 w L^4}{384 E I}$

$$\Delta_2 = \frac{5}{384}\frac{400\,\text{N}}{\text{m}}\frac{(200\,\text{cm})^4}{207\,\text{GPa} \cdot 0.276 \times 10^6\,\text{mm}^4}\left|\frac{\text{GPa}\ \ \text{m}^2}{10^9\,\text{N}}\right|\frac{10^3\,\text{mm}}{\text{m}}\left|\frac{(10\,\text{mm})^4}{\text{cm}^4}\right. = 1.46\,\text{mm}$$

$$\Delta_{total} = \Delta_1 + \Delta_2 = 8.57\,\text{mm} + 1.46\,\text{mm} = 10.03\,\text{mm}$$

In the previous Example, the maximum deflection due to the uniform distributed load is at the midspan (x = 1 m), but the maximum deflection due to the point load is located at $x = \sqrt{\frac{a(a+2b)}{3}} = \sqrt{\frac{175\,\text{cm}(175\,\text{cm} + 2(25\,\text{cm}))}{3}} = 114.5\,\text{cm}$, to the right of the midspan. Using the Method of Superposition, you can add the shear, moment, or deflection *at a given point along the beam*, but you cannot add the maximum shear, moment, or deflection, unless they occur at the same point along the beam. One example is a simply-supported beam with a point load at the midspan and a uniform distributed load; both loading conditions have the maximum deflection and maximum moment at the midspan.

Example #7 does not consider the weight of the pipe. From Appendix D, the mass per unit length is 5.44 kg/m. The SI unit of weight is the newton, equal to a kilogram meter per second squared. Multiply mass by gravity to get weight.

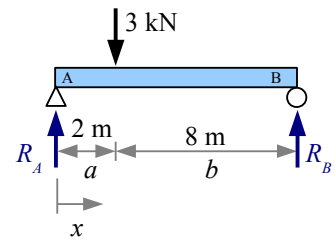
$$\frac{W}{L} = \frac{m g}{L} = \frac{5.44\,\text{kg}}{\text{m}}\frac{9.81\,\text{m}}{\text{s}^2}\left|\frac{\text{N}\ \ \text{s}^2}{\text{kg m}}\right. = 53.4\,\text{N/m}$$

Visualizing the Deflection Curve

We can use the Formula Method for calculating the deflection of a beam at many points along its length, then graphing the result to see the deflection curve.

Example #8

A 150×360 Douglas fir timber is loaded as shown. Use the Formula Method to calculate the deflection at 1 meter increments from left to right. Graph the deflection vs. position along the beam. Ignore the weight of the beam.



Solution From Appendix D, $E = 11.7\,\text{GPa}$ and $I = 471\times10^6\,\text{mm}^4$.

Since point A is pinned, $\Delta_{0\text{ft.}} = 0$.

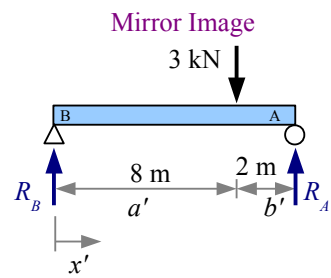
From Appendix F, the deflection equation for this loading condition is . $\Delta_x = \frac{P\,b\,x}{6\,E\,I\,L}\left[L^2 - b^2 - x^2\right]$ for $x < a$ where $a = 2\,\text{m}$ and $b = 8\,\text{m}$. We can use this equation for $x = 1\,\text{m}$.

At $x = 1\,\text{m}$, $\Delta_{1\,\text{m}} = \frac{3\,\text{kN}(8\,\text{m})(1\,\text{m})\left[(10\,\text{m})^2 - (8\,\text{m})^2 - (1\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 2.54\,\text{mm}$

At $x = 2\,\text{m}$ (the point load), it is easier to use $\Delta_{2\,\text{m}} = \frac{P\,a^2\,b^2}{3\,E\,I\,L}$

$$\Delta_{2\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})^2(8\,\text{m})^2}{3\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 4.65\,\text{mm}$$

To the right of the point load, $x > a$, so we need to flip the diagram. Let's call the new dimensions a' and b': $a' = 8\,\text{m}$ and $b' = 2\,\text{m}$. We also need new values of x, starting at point B and moving right in the mirror image...let's call it x'.



The value of x' is calculated as $x' = L - x$, so $\Delta_x = \frac{P\,b'\,x'}{6\,E\,I\,L}\left[L^2 - b'^2 - x'^2\right]$

At $x = 3\,\text{m},\ x' = 7\,\text{m}$, $\Delta_{3\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})(7\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (7\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 5.97\,\text{mm}$

At $x = 4\,\text{m},\ x' = 6\,\text{m}$, $\Delta_{4\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})(6\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (6\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 6.53\,\text{mm}$

At $x = 5\,\text{m},\ x' = 5\,\text{m}$, $\Delta_{5\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})(5\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (5\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 6.44\,\text{mm}$

At $x = 6\,\text{m},\ x' = 4\,\text{m}$, $\Delta_{6\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})(4\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (4\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 5.81\,\text{mm}$

At $x = 7\,\text{m},\ x' = 3\,\text{m}$, $\Delta_{7\,\text{m}} = \frac{3\,\text{kN}(2\,\text{m})(3\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (3\,\text{m})^2\right]}{6\;\;11.7\,\text{GPa}\;\;471\times10^6\,\text{mm}^4\;\;10\,\text{m}}\left|\frac{\text{GPa}\;\;\text{m}^2}{10^6\,\text{kN}}\right|\frac{(10^3\,\text{mm})^5}{\text{m}^5} = 4.74\,\text{mm}$

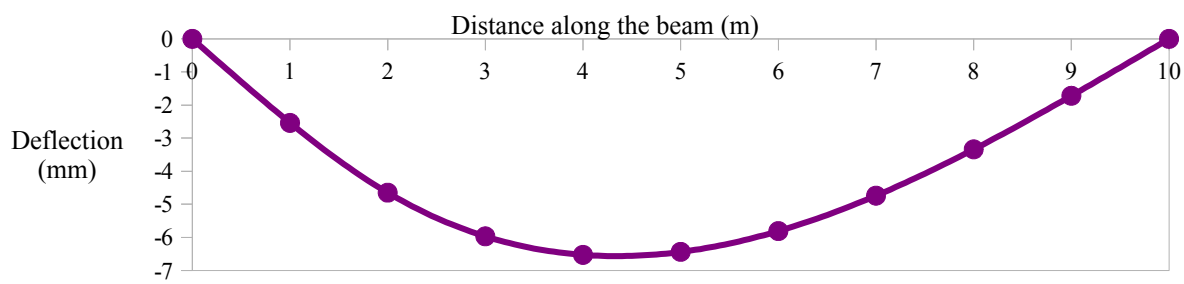
Example #8, continued

At $x = 8\,\text{m},\ x' = 2\,\text{m}$, $\Delta_{8\,\text{m}} = \dfrac{3\,\text{kN}(2\,\text{m})(2\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (2\,\text{m})^2\right]}{6\ \ 11.7\,\text{GPa}\ \ 471\times10^6\,\text{mm}^4\ \ 10\,\text{m}}\left|\dfrac{\text{GPa}\ \ \text{m}^2}{10^6\,\text{kN}}\right|\dfrac{(10^3\,\text{mm})^5}{\text{m}^5} = 3.34\,\text{mm}$

At $x = 9\,\text{m},\ x' = 1\,\text{m}$, $\Delta_{9\,\text{m}} = \dfrac{3\,\text{kN}(2\,\text{m})(1\,\text{m})\left[(10\,\text{m})^2 - (2\,\text{m})^2 - (1\,\text{m})^2\right]}{6\ \ 11.7\,\text{GPa}\ \ 471\times10^6\,\text{mm}^4\ \ 10\,\text{m}}\left|\dfrac{\text{GPa}\ \ \text{m}^2}{10^6\,\text{kN}}\right|\dfrac{(10^3\,\text{mm})^5}{\text{m}^5} = 1.72\,\text{mm}$

Since point B is pinned, $\Delta_{10\,\text{ft.}} = 0$

Deflection is plotted as negative numbers to show the beam deflecting downward under the applied load. Although the point load is 2 meters from the left end, the maximum deflection is at 4 meters.



Key Equations

If a beam segment has a constant moment, then the radius of curvature of that segment of beam is $R = \dfrac{E\,I}{M}$. The bending stress in this segment of the beam is $\sigma = \dfrac{E\,c}{R}$.

Chapter 11: Beam Design

Wide-Flange Steel Beam Design in Six Easy Steps

In the previous three chapters, we learned how to calculate reaction forces and moments, how to draw shear and moment diagrams, how to calculate deflection, and how to calculate bending and shear stresses for a given beam size and shape. When you design a beam, you start with the applied loads, beam length, allowable stresses (and sometimes allowable deflection), and you must figure out what size beam to use. Use these six steps, and you can design a wide-flange steel beam.

Step 1 Identify all applied loads and design constraints (beam length, yield strength, and maximum deflection if specified).

Step 2 Draw the load diagram and calculate the reactions. Draw an equivalent load diagram if necessary.

Step 3 Draw shear and moment diagrams, and calculate V_{max} and M_{max}.

Step 4 Select the lightest beam from Appendix D that will support M_{max}, or has enough stiffness to limit Δ_{max}. The allowable moment in a steel beam $M_{allowable} = 0.6\,\sigma_{YS}\,Z_x$, so the required plastic section modulus is $Z_{required} = \frac{M}{0.6\,\sigma_{YS}} = \frac{1.67\,M}{\sigma_{YS}}$. Solve for $Z_{required}$, then find the lightest W-beam that has a slightly larger value of Z.

U.S. Customary beams are identified by nominal depth and weight per unit length. For example, W12×40 is about 12 in. deep, and weighs 40 lb./ft.

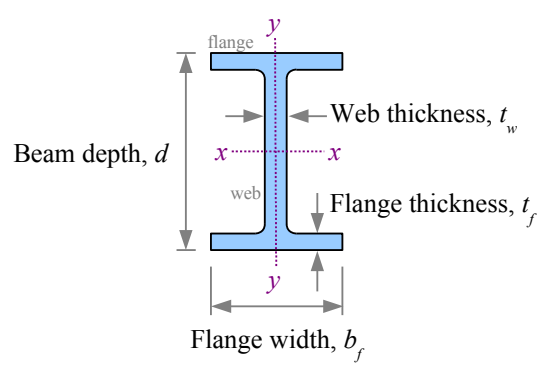
Metric beams are identified by nominal depth and mass per unit length. For example, W360×110 is about 360 mm deep, and has a mass of 110 kg/m; convert to weight per unit length by multiplying by gravity (9.81 m/s^2), using the definition of a newton, $N = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$, to convert units: $w = \frac{m}{L}\,g = \frac{110\text{ kg}}{\text{m}}\,\frac{9.81\text{ m}}{\text{s}^2}\left|\frac{\text{N s}^2}{\text{kg m}}\right|\frac{\text{kN}}{10^3\text{ N}} = 1.08\text{ kN/m}$

If the maximum deflection is specified, write the equation for Δ_{max}, solve it for $I_{required}$, then find the lightest W-beam that has a slightly larger value of I.

Step 5 Add the beam weight to the load diagram, and recheck moment and deflection. You may have to draw completely new shear and moment diagrams. If $Z_{beam} > Z_{required}$, the beam is OK in bending, and if $I_{beam} > I_{required}$, the beam is OK in deflection. If either of these conditions is not met, pick a larger beam and recalculate.

Step 6 Check shear. For steel W-beams, $V_{applied} \le 0.4\,\sigma_{YS}\,d\,t_w$ where d is the beam depth and t_w is the thickness of the web. As long as this inequality is true, the beam is OK in shear.

The rule of thumb in these calculations is to use a beam that has more than you need (more I, more Z, more shear strength). Just like in real life, having more than you need is usually better than having less than you need (money, food, education, happiness, etc.)



Example #1

Select the lightest W-beam that will support a uniform distributed load of 3 kip/ft. on a span of 20 ft.

Step 1 Identify loads and design constraints:

$w = 3\,\text{kip/ft.}$
$L = 20\,\text{ft.}$
$\sigma_{YS} = 50\,\text{ksi}$
Δ_{max} is not constrained

Step 2 The loading is symmetrical:

$$R_A = R_B = \frac{w\,L}{2} = \frac{3\,\text{kips}}{\text{ft.}}\,\frac{20\,\text{ft.}}{2} = 30\,\text{kips}$$

Step 3 $V_{max} = 30\,\text{kips}$; $M_{max} = \dfrac{w\,L^2}{8} = \dfrac{3\,\text{kips}}{\text{ft.}}\,\dfrac{(20\,\text{ft.})^2}{8} = 150\,\text{kip}\cdot\text{ft.}$

Step 4

$$Z_{required} = \frac{1.67\,M}{\sigma_{YS}} = \frac{1.67\cdot 150\,\text{kip}\cdot\text{ft.}}{}\,\frac{\text{in.}^2}{50\,\text{kips}}\left|\frac{12\,\text{in.}}{\text{ft.}}\right. = 60.1\,\text{in.}^3$$

Look in Appendix D for the lightest W-beam for which $Z > Z_{required}$. W18×35 has a plastic section modulus of 66.5 in.^{3} and is 35 lb./ft.; all other beams that meet the Z constraint are heavier.

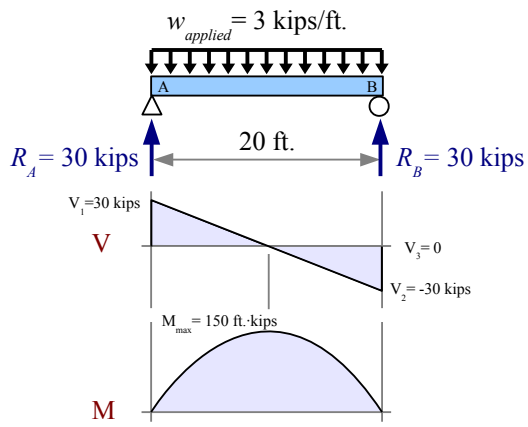
Step 5 Add the beam weight to the shear and moment diagrams, and recalculate the maximum values. Since the shape of these diagrams has not changed, we can use the same equations as before, adding the beam weight to the applied weight.

$$R_A = R_B = V_{max} = \frac{(w_{applied} + w_{beam})\,L}{2} = \frac{3.035\,\text{kips}}{\text{ft.}}\,\frac{20\,\text{ft.}}{2} = 30.35\,\text{kips}$$

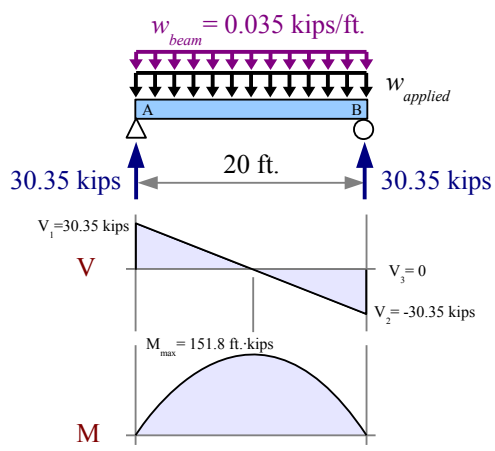
$$M_{max} = \frac{(w_{applied} + w_{beam})\,L^2}{8} = \frac{3.035\,\text{kips}}{\text{ft.}}\,\frac{(20\,\text{ft.})^2}{8} = 151.8\,\text{kip}\cdot\text{ft.}$$

$$Z_{required} = \frac{1.67\,M}{\sigma_{YS}} = \frac{1.67\cdot 151.8\,\text{kip}\cdot\text{ft.}}{}\,\frac{\text{in.}^2}{50\,\text{kips}}\left|\frac{12\,\text{in.}}{\text{ft.}}\right. = 60.8\,\text{in.}^3$$

Applied load only



Applied load + beam weight



We have $Z_{beam} = 66.5\,\text{in.}^3$ Since $Z_{beam} > Z_{required}$, the beam is OK in bending.

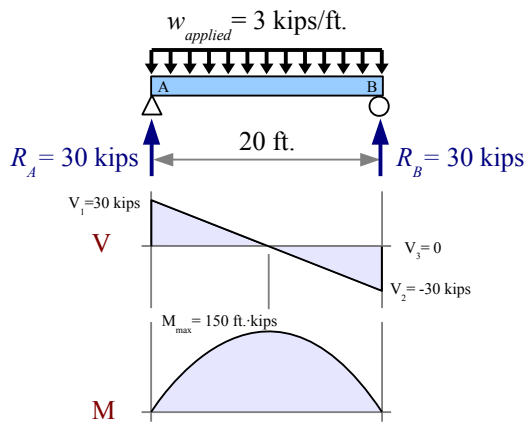
Step 6 The beam will support a shear load of $0.4\,\sigma_{YS}\,d\,t_w = 0.4\cdot 50\,\dfrac{\text{kips}}{\text{in.}^2}\cdot 17.7\,\text{in.}\cdot 0.300\,\text{in.} = 106\,\text{kips}$ Since we are applying only 30.35 kips in shear, the beam is OK in shear.

Real engineering problems have multiple constraints. In the previous problem, we designed a beam that was strong enough to support the applied load and its own weight, without failing in bending or shear. In the next problem, we will add another constraint: maximum deflection. In many structures, deflection governs design; for example, most people feel uncomfortable walking across a strong floor that moves too much.

Example #2

Select the lightest W-beam that will support a uniform distributed load of 3 kip/ft. on a span of 20 ft., *and* deflects no more than 0.6 in.

Applied load only



Step 1 Identify loads and design constraints:

$w = 3\,\text{kip/ft.}$ $\quad \sigma_{YS} = 50\,\text{ksi}$
$L = 20\,\text{ft.}$ $\quad \Delta_{max} = 0.6\,\text{in.}$

Steps 2 & 3 These steps are identical to Example #1:
$R_A = R_B = 30\,\text{kips}$, $V_{max} = 30\,\text{kips}$, and $M_{max} = 150\,\text{kip}\cdot\text{ft.}$

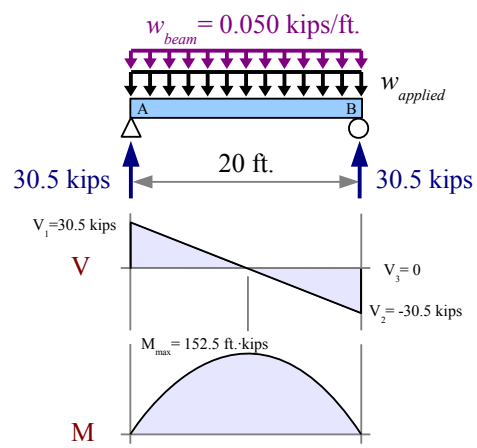
Step 4 $Z_{required} = 60.1\,\text{in.}^3$ as in Example #1, therefore we should try W18×35 which has a plastic section modulus of 66.5 in.^{3}

For this loading case, $\Delta_{max} = \dfrac{5\,w\,L^4}{384\,E\,I}$. The moment of inertia required to support the deflection constraint is

$$I_{required} = \frac{5\,w\,L^4}{384\,E\,\Delta_{max}} = \frac{5}{384}\,\frac{3\,\text{kips}}{\text{ft.}}\,\frac{(20\,\text{ft.})^4}{}\,\frac{\text{in.}^2}{30\times10^3\,\text{kip}}\,\frac{}{0.6\,\text{in.}}\left|\frac{(12\,\text{in.})^3}{\text{ft.}^3}\right. = 600\,\text{in.}^4 .$$

The moment of inertia of a W15×35 beam is 510 in.^{4}... not enough. The lightest W-beam with $I > 600\,\text{in.}^4$ is W21×50, which has $I_x = 984\,\text{in.}^4$ and $Z_x = 110\,\text{in.}^3$

Applied load + beam weight



Step 5 Add beam weight to shear and moment diagrams; recalculate the max. values. The shape of the diagrams is unchanged; use the same equations as before, adding beam weight to applied weight.

$$R_A = R_B = V_{max} = \frac{(w_{applied} + w_{beam})\,L}{2} = \frac{3.05\,\text{kips}}{\text{ft.}}\,\frac{20\,\text{ft.}}{2} = 30.5\,\text{kips}$$

$$M_{max} = \frac{(w_{applied} + w_{beam})\,L^2}{8} = \frac{3.05\,\text{kips}}{\text{ft.}}\,\frac{(20\,\text{ft.})^2}{8} = 152.5\,\text{kip}\cdot\text{ft.}$$

$$Z_{required} = \frac{1.67\,M}{\sigma_{YS}} = \frac{1.67\cdot152.5\,\text{kip}\cdot\text{ft.}}{}\,\frac{\text{in.}^2}{50\,\text{kips}}\left|\frac{12\,\text{in.}}{\text{ft.}}\right. = 61.1\,\text{in.}^3$$

We have $Z_{beam} = 110\,\text{in.}^3$ Since $Z_{beam} > Z_{required}$, the beam is OK in bending.

$$I_{required} = \frac{5(w_{applied} + w_{beam})\,L^4}{384\,E\,\Delta_{max}} = \frac{5}{384}\,\frac{3.05\,\text{kips}}{\text{ft.}}\,\frac{(20\,\text{ft.})^4}{}\,\frac{\text{in.}^2}{30\times10^3\,\text{kip}}\,\frac{}{0.6\,\text{in.}}\left|\frac{(12\,\text{in.})^3}{\text{ft.}^3}\right. = 610\,\text{in.}^4$$

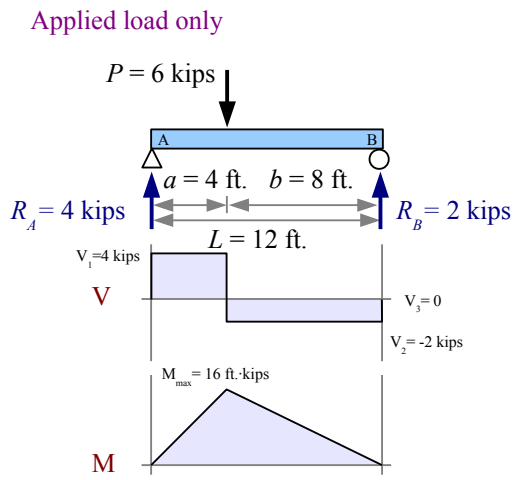
We have $I_{beam} = 984\,\text{in.}^4$. Since $I_{beam} > I_{required}$, the beam is OK in deflection.

Step 6 The beam will support a shear load of $0.4\,\sigma_{YS}\,d\,t_w = 0.4\cdot50\,\dfrac{\text{kips}}{\text{in.}^2}\cdot20.8\,\text{in.}\cdot0.380\,\text{in.} = 158\,\text{kips}$ Since we are applying only 30.5 kips in shear, the beam is OK in shear.

In these two Examples, the shear and moment diagrams retained their shapes when we added the weight of the beam. In most loading cases, we need to draw new shear and moment diagrams, as in Example #3.

Example #3

Select the lightest W-beam that will support a point load of 6 kips located 4 feet from the left end of a 12 ft. span.



Step 1 Identify loads and design constraints:

$P = 6\,\text{kips}$ $\quad \sigma_{YS} = 50\,\text{ksi}$

$L = 12\,\text{ft.}$ $\quad \Delta_{max}$ is not constrained

Step 2 For this loading case, $R_A = \dfrac{P\,b}{L} = \dfrac{6\,\text{kips}\cdot 8\,\text{ft.}}{12\,\text{ft.}} = 4\,\text{kips}$ and

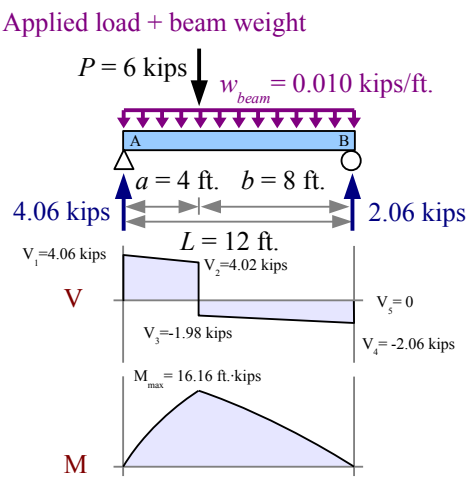
$R_B = \dfrac{P\,a}{L} = \dfrac{6\,\text{kips}\cdot 4\,\text{ft.}}{12\,\text{ft.}} = 2\,\text{kips}$

Step 3 From the shear and moment diagrams, $V_{max} = 4\,\text{kips}$ and $M_{max} = 4\,\text{kips}\cdot 4\,\text{ft.} = 16\,\text{kip}\cdot\text{ft.}$

Step 4 $Z_{required} = \dfrac{1.67\,M}{\sigma_{YS}} = \dfrac{1.67\cdot 16\,\text{kip}\cdot\text{ft.}}{50\,\text{kips}}\dfrac{\text{in.}^2}{}\Big|\dfrac{12\,\text{in.}}{\text{ft.}} = 6.41\,\text{in.}^3$

From Appendix D, the best choice is W8×10 which has a $Z_x = 8.87\,\text{in.}^3$ and a weight per unit length of 10 lb./ft.

Step 5 Add beam weight to the load diagram; redraw shear & moment diagrams. Use the Method of Superposition to add the effects of the point load and the beam weight to find the reactions.



$R_A = \dfrac{P\,b}{L} + \dfrac{w_{beam}\,L}{2} = \dfrac{6\,\text{kips}\cdot 8\,\text{ft.}}{12\,\text{ft.}} + \dfrac{0.010\,\text{kips}}{\text{ft.}}\dfrac{12\,\text{ft.}}{2} = 4.06\,\text{kips}$

$R_B = \dfrac{P\,a}{L} + \dfrac{w_{beam}\,L}{2} = \dfrac{6\,\text{kips}\cdot 4\,\text{ft.}}{12\,\text{ft.}} + \dfrac{0.010\,\text{kips}}{\text{ft.}}\dfrac{12\,\text{ft.}}{2} = 2.06\,\text{kips}$

$V_1 = R_A = 4.06\,\text{kips}$

$V_2 = V_1 - w_{beam}\,4\,\text{ft.} = 4.02\,\text{kips}$

$V_3 = V_2 - 6\,\text{kips} = -1.98\,\text{kips}$

$V_4 = V_3 - w_{beam}\,8\,\text{ft.} = -2.06\,\text{kips}$

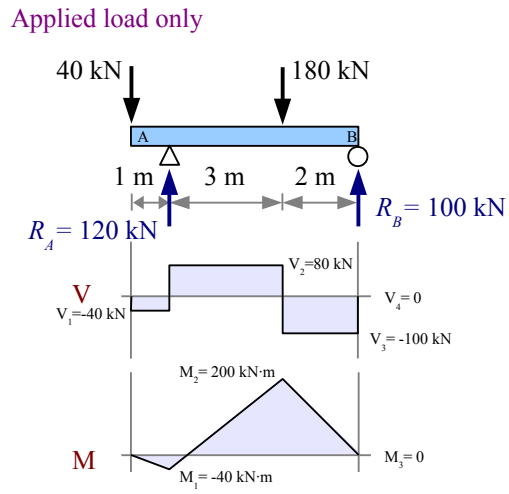
$V_5 = V_4 + R_B = 0\,\text{kips}$ Since the maximum moment for the point load occurs at the point load, and the maximum moment for the beam weight occurs at the midspan, we cannot add the M_{max} equations for each case. Instead, use the area of the first trapezoid in the shear diagram to find

$M_{max} = \left(\dfrac{4.06\,\text{kips} + 4.02\,\text{kips}}{2}\right)\cdot 4\,\text{ft.} = 16.16\,\text{kip}\cdot\text{ft.}$ $\quad Z_{required} = \dfrac{1.67\,M}{\sigma_{YS}} = \dfrac{1.67\cdot 16.16\,\text{kip}\cdot\text{ft.}}{50\,\text{kips}}\dfrac{\text{in.}^2}{}\Big|\dfrac{12\,\text{in.}}{\text{ft.}} = 6.48\,\text{in.}^3$ We have $Z_{beam} = 8.87\,\text{in.}^3$ Since $Z_{beam} > Z_{required}$, the beam is OK in bending.

Step 6 The beam will support a shear load of $0.4\,\sigma_{YS}\,d\,t_w = 0.4\cdot 50\dfrac{\text{kips}}{\text{in.}^2}\cdot 7.89\,\text{in.}\cdot 0.17\,\text{in.} = 26.8\,\text{kips}$ Since we are applying only 4.06 kips in shear, the beam is OK in shear.

Example #4

Select the lightest W-beam that will support the point loads shown.



Step 1 Identify loads and design constraints.

$\sigma_{YS} = 345\,\text{MPa}$
Δ_{max} is not constrained

Step 2 Use Sum of the Moments and Sum of the Forces to find the reaction forces.

$$R_B = \frac{-40\,\text{kN}\cdot 1\,\text{m} + 180\,\text{kN}\cdot 3\,\text{m}}{5\,\text{m}} = 100\,\text{kN}$$

$$R_A = 40\,\text{kN} + 180\,\text{kN} - 100\,\text{kN} = 120\,\text{kN}$$

Step 3 Calculate the values for shear and moment diagrams.

$$V_1 = -40\,\text{kN}\ ,\quad V_2 = V_1 + R_A = 80\,\text{kN}$$

$$V_3 = V_2 - 180\,\text{kN} = -100\,\text{kN}\ ,\quad V_4 = V_3 + R_B = 0$$

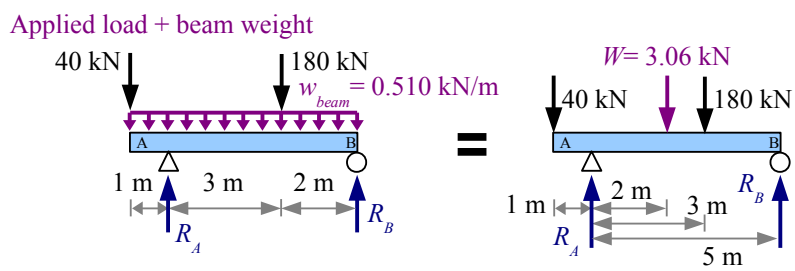
$$M_1 = -40\,\text{kN}\cdot 1\,\text{m} = -40\,\text{kN}\cdot\text{m}\ ,\quad M_2 = M_1 + 80\,\text{kN}\cdot 3\,\text{m} = 200\,\text{kN}\cdot\text{m} = M_{max}\ ,\quad M_3 = M_2 - 100\,\text{kN}\cdot 2\,\text{m} = 0$$

Step 4 $Z_{required} = \dfrac{1.67\,M}{\sigma_{YS}} = \dfrac{1.67\cdot 200\,\text{kN}\cdot\text{m}}{345\,\text{MPa}}\left|\dfrac{\text{MPa m}^2}{10^3\,\text{kN}}\right|\dfrac{(10^3\,\text{mm})^3}{\text{m}^3} = 968\times 10^3\,\text{mm}^3$ From Appendix D, the best choice is W460×52 which has a plastic section modulus of 1090×10^3 mm³ and a mass per unit length of 52 kg/m. Multiply by gravity to get the weight per unit length, $w = \dfrac{52\,\text{kg}}{\text{m}}\dfrac{9.81\,\text{m}}{\text{s}^2}\left|\dfrac{\text{N s}^2}{\text{kg m}}\right|\dfrac{\text{kN}}{10^3\,\text{N}} = 0.510\dfrac{\text{kN}}{\text{m}}$

Step 5 Add beam weight to the load diagram, then draw an Equivalent Load Diagram to find the reactions.

The equivalent point load for the beam weight per unit length is

$$W = w\cdot L = \frac{0.510\,\text{kN}}{\text{m}}\frac{6\,\text{m}}{} = 3.06\,\text{kN}$$



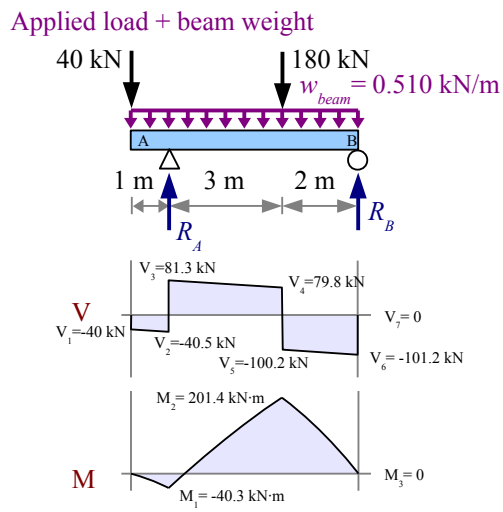
It acts at the center of the beam, 2 m to the right of reaction R_A. Use Sum of the Moments and Sum of the Forces to find the reaction forces.

$$R_B = \frac{-40\,\text{kN}\cdot 1\,\text{m} + 3.06\,\text{kN}\cdot 2\,\text{m} + 180\,\text{kN}\cdot 3\,\text{m}}{5\,\text{m}} = 101.2\,\text{kN}$$

$$R_A = 40\,\text{kN} + 3.06\,\text{kN} + 180\,\text{kN} - 101.2\,\text{kN} = 121.8\,\text{kN}$$

Example #4, continued

Now draw the shear and moment diagrams



$$V_1 = -40\,\text{kN}$$

$$V_2 = V_1 - 0.51\,\frac{\text{kN}}{\text{m}}\cdot 1\,\text{m} = -40.5\,\text{kN}$$

$$V_3 = V_2 + R_A = 81.3\,\text{kN}$$

$$V_4 = V_3 - 0.51\,\frac{\text{kN}}{\text{m}}\cdot 3\,\text{m} = 79.8\,\text{kN}$$

$$V_6 = V_5 - 0.51\,\frac{\text{kN}}{\text{m}}\cdot 2\,\text{m} = -101.2\,\text{kN}$$

$$V_5 = V_4 - 180\,\text{kN} = -100.2\,\text{kN}$$

$$V_7 = V_6 + R_B = 0$$

$$M_1 = -\left(\frac{40\,\text{kN} + 40.5\,\text{kN}}{2}\right)\cdot 1\,\text{m} = -40.3\,\text{kN}\cdot\text{m}$$

$$M_2 = M_1 + \left(\frac{81.3\,\text{kN} + 79.8\,\text{kN}}{2}\right)\cdot 3\,\text{m} = 201.4\,\text{kN}\cdot\text{m}$$

$$M_3 = M_2 - \left(\frac{100.2\,\text{kN} + 101.2\,\text{kN}}{2}\right)\cdot 2\,\text{m} = 0$$

$$Z_{required} = \frac{1.67\,M}{\sigma_{YS}} = \frac{1.67\cdot 201.4\,\text{kN}\cdot\text{m}}{345\,\text{MPa}}\left|\frac{\text{MPa m}^2}{10^3\,\text{kN}}\right|\frac{(10^3\,\text{mm})^3}{\text{m}^3} = 975\times 10^3\,\text{mm}^3$$

We have $Z_{beam} = 1090\times 10^3\,\text{mm}^3$ Since $Z_{beam} > Z_{required}$, the beam is OK in bending.

Step 6 The beam will support a shear load of

$$0.4\,\sigma_{YS}\,d\,t_w = \frac{0.4\quad 345\,\text{MPa}\quad 450\,\text{mm}\quad 7.62\,\text{mm}}{}\left|\frac{10^3\,\text{kN}}{\text{MPa m}^2}\right|\frac{\text{m}^2}{(10^3\,\text{mm})^2} = 473\,\text{kN}$$

Since we are applying only 101.2 kN in shear, the beam is OK in shear.

Timber Beam Design in Six Easy Steps

Rectangular cross-section timber beam design is similar to wide-flange steel beam design, except we use section modulus S instead of plastic section modulus Z. Otherwise, the steps are the same:

Step 1 Identify all applied loads and design constraints.

Step 2 Draw the load diagram and calculate the reactions. Draw an equivalent load diagram if necessary.

Step 3 Draw shear and moment diagrams, and calculate V_{max} and M_{max}.

Step 4 Select the lightest beam from Appendix E that will support M_{max}, or has enough stiffness to limit Δ_{max}. The allowable moment in a timber beam $M_{allowable} = \sigma S_x$, so the required section modulus is $S_{required} = \frac{M}{\sigma_{allowable\,bending}}$. Solve for $S_{required}$, then find the lightest timber beam that has a slightly larger value of S.

If the maximum deflection is specified, write the equation for Δ_{max}, solve it for $I_{required}$, then find the lightest timber beam that has a slightly larger value of I, just like we did for steel beams previously.

Step 5 Add the beam weight to the load diagram, and recheck moment and deflection. You may have to draw completely new shear and moment diagrams. If $S_{beam} > S_{required}$, the beam is OK in bending, and if $I_{beam} > I_{required}$, the beam is OK in deflection. If either of these conditions is not met, pick a larger beam and recalculate.

Step 6 Check shear. For rectangular cross-section timbers, $\tau = \frac{3V}{2A}$. If $V_{applied} < \frac{2A\tau_{allowable}}{3}$, the beam is OK in shear.

Interpolation

The weight per unit length of a steel beam is the specific weight of steel (0.284 lb./in.^{3}) times the cross-sectional area of the beam. For metric sizes, the mass per unit length of a steel beam is the density of steel (7.85 g/cm^3) times the cross-sectional area of the beam. These values are listed in Appendix D.

With timber beams, the weight per unit length for a given timber size depends on the type of wood, because different wood species have different specific weights. Appendix E lists the weight per unit length for three different specific weights: 20, 30, and 40 lb./ft.^{3}. For metric sizes, the mass per unit length is listed for three densities: 320, 480, and 640 kg/m^3. Douglas fir has a specific weight of 30 lb./ft.^{3}, therefore you can find the weight per unit length of a 2×6 Douglas fir timber in Table E3: it is listed as 1.72 lb./ft. in the table.

What if your 2×6 is made of Southern yellow pine? Specific weight $\gamma = 34\,\text{lb./ft.}^3$, which is not in the table...so you need to interpolate to find the weight per unit length w of this beam.

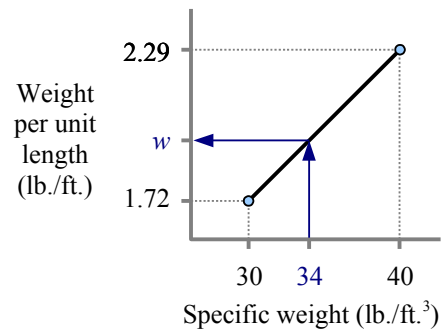
Here are the first few rows of Table E3:

Designation	Width	Depth	Area	Moment of inertia	Section modulus	Weight per unit length γ = 20 lb/ft.^{3}	Weight per unit length γ = 30 lb/ft.^{3}	Weight per unit length γ = 40 lb/ft.^{3}
	b (in.)	d (in.)	A (in.^{2})	I_x (in.^{4})	S_x (in.^{3})	w (lb./ft.)	w (lb./ft.)	w (lb./ft.)
2×2	1.5	1.5	2.25	0.422	0.563	0.313	0.469	0.625
2×3	1.5	2.5	3.75	1.95	1.56	0.521	0.781	1.04
2×4	1.5	3.5	5.25	5.36	3.06	0.729	1.09	1.46
2×6	1.5	5.5	8.25	20.8	7.56	1.15	1.72	2.29

From the 2×6 row, $w = 1.72\,\text{lb./ft.}$ for $\gamma = 30\,\text{lb./ft.}^3$ and $w = 2.29\,\text{lb./ft.}$ for $\gamma = 40\,\text{lb./ft.}^3$. If you graph these values, you can see that the ratio of $\frac{34\frac{\text{lb.}}{\text{ft.}^3} - 30\frac{\text{lb.}}{\text{ft.}^3}}{40\frac{\text{lb.}}{\text{ft.}^3} - 30\frac{\text{lb.}}{\text{ft.}^3}}$ is equal to $\frac{w - 1.72\frac{\text{lb.}}{\text{ft.}}}{2.29\frac{\text{lb.}}{\text{ft.}} - 1.72\frac{\text{lb.}}{\text{ft.}}}$. The math is essentially the same as a similar triangles calculation.

Rewrite the equation to solve for the weight per unit length:

$$w = \left[\frac{34\frac{\text{lb.}}{\text{ft.}^3} - 30\frac{\text{lb.}}{\text{ft.}^3}}{40\frac{\text{lb.}}{\text{ft.}^3} - 30\frac{\text{lb.}}{\text{ft.}^3}}\right]\left(2.29\frac{\text{lb.}}{\text{ft.}} - 1.72\frac{\text{lb.}}{\text{ft.}}\right) + 1.72\frac{\text{lb.}}{\text{ft.}} = 1.95\frac{\text{lb.}}{\text{ft.}}$$



Use the same method to find the mass per unit length for SI timbers, and be sure to multiply by gravity to find the weight per unit length.

Interpolation is a good skill to develop because it is used in many other technical fields, including physics, chemistry, heat transfer, fluid mechanics, and thermodynamics.

Example #5

Select the lightest Southern yellow pine beam for this load and deflects no more than 0.5 in. at the free end.

Step 1 Identify loads and design constraints.

P	=	800 lb.
L	=	14 ft.
b	=	12 ft.
Δ_{max}	$\leq$	0.5 in.
$\sigma_{allowable\ bending}$	=	900 psi (from Appendix E)
$\tau_{allowable}$	=	180 psi (from Appendix E)
E	=	1700 ksi (from Appendix E)
γ	=	34 lb./ft.^{3}

Step 2 Sove for reactions:

$R_B = P = 800\text{ lb.}$ and $M_{max} = P\,b = 800\text{ lb.} \times 12\text{ ft.} = 9600\text{ ft.}\cdot\text{lb.}$

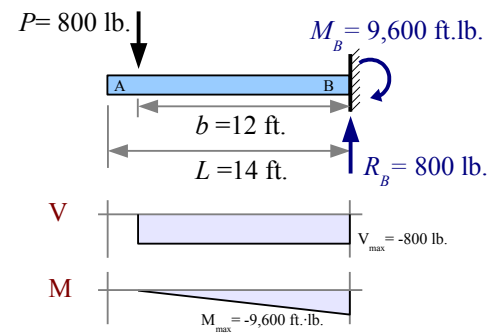
Step 3 Draw shear and moment diagrams, and identify maximum values:

$V_{max} = 800\text{ lb.}$, $M_{max} = 9600\text{ ft.}\cdot\text{lb.}$

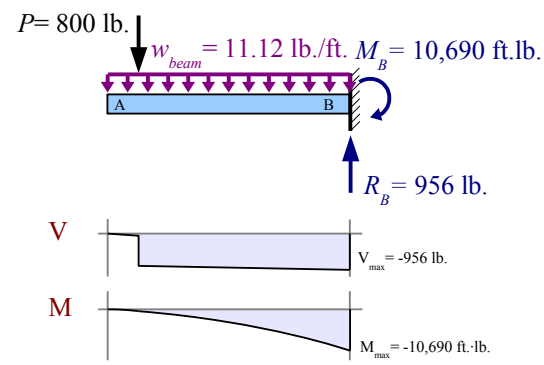
Step 4 $S_{required} = \dfrac{M}{\sigma_{all.\ bending}} = \dfrac{9{,}600\text{ ft.}\cdot\text{lb.}}{900\text{ lb.}}\left|\dfrac{\text{in.}^2}{}\right|\dfrac{12\text{ in.}}{\text{ft.}} = 128\text{ in.}^3$

From Appendix F, $\Delta_{max} = \dfrac{P\,b^2}{6\,E\,I}(3\,L - b)$

Applied load only



Applied load + beam weight, 4×16 timber



Rewrite to solve for $I_{required} = \dfrac{P\,b^2}{6\,E\,\Delta_{max}}(3\,L - b) = \dfrac{800\text{ lb.}(12\text{ ft.})^2\text{in.}^2}{6\ \ 1700\text{ kip}\ \ 0.5\text{ in.}}\left[3(14\text{ ft.}) - 12\text{ ft.}\right]\left|\dfrac{\text{kip}}{10^3\text{lb.}}\right|\dfrac{(12\text{ in.})^3}{\text{ft.}^3} = 1171\text{ in.}^4$

Look in Appendix E for the lightest Douglas fir beam for which $S > S_{required}$ and $I > I_{required}$. The lightest beam that meets the section modulus criterion is a 4×16 beam, with $S = 136\text{ in.}^3$. However, this beam does not meet the moment of inertia criterion. The lightest beam that meets both criteria is a 6×16 beam, with $S = 220\text{ in.}^3$ and $I = 1710\text{ in.}^4$.

Example #5 (continued)

Step 5 Add the beam weight to the load diagram, and redraw the shear and moment diagrams. Since the specific weight of Southern yellow pine is $34\ \text{lb.}/\text{ft.}^3$, interpolation is necessary.

$$w = \left[\frac{34\frac{\text{lb.}}{\text{ft.}^3}-30\frac{\text{lb.}}{\text{ft.}^3}}{40\frac{\text{lb.}}{\text{ft.}^3}-30\frac{\text{lb.}}{\text{ft.}^3}}\right]\left(23.7\frac{\text{lb.}}{\text{ft.}}-17.8\frac{\text{lb.}}{\text{ft.}}\right) + 17.8\frac{\text{lb.}}{\text{ft.}} = 20.2\frac{\text{lb.}}{\text{ft.}}$$

Using the Method of Superposition, we can add the effects of the point load and the distributed load to find the reaction force and reaction moment.

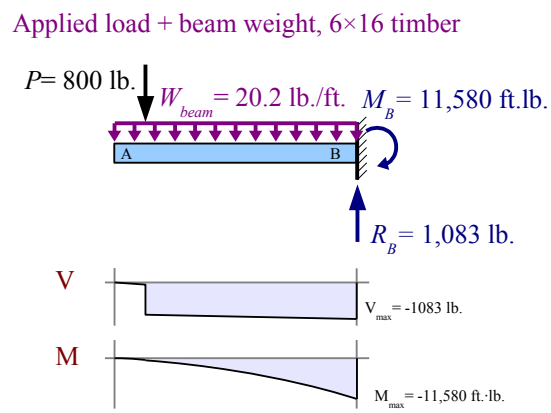
$$R_B = V_{max} = P + w\,L = 800\ \text{lb.} + 20.2\ \text{lb./ft.} \times 14\ \text{ft.} = 1083\ \text{lb.}$$

$$M_B = M_{max} = P\,b + \frac{w_{beam}\,L^2}{2}$$

$$= 800\ \text{lb.} \times 12\ \text{ft.} + \frac{20.2\ \text{lb.}}{\text{ft.}}\frac{(14\ \text{ft.})^2}{2} = 11{,}580\ \text{ft.}\cdot\text{lb.}$$

$$S_{required} = \frac{M}{\sigma_{all.\ bending}} = \frac{10{,}690\ \text{ft.}\cdot\text{lb.}}{}\frac{\text{in.}^2}{900\ \text{lb.}}\left|\frac{12\ \text{in.}}{\text{ft.}}\right. = 143\ \text{in.}^3$$

We have $S_{beam} = 220\ \text{in.}^3$ Since $S_{beam} > S_{required}$, the beam is OK in bending.



Using the Method of Superposition, we can add the effect of the beam weight to solve for $I_{required}$. From Appendix F, the maximum deflection due to a uniform distributed load on a cantilever beam is $\Delta_{max} = \frac{w\,L^4}{8\,E\,I}$, therefore

$$I_{required} = \frac{P\,b^2}{6\,E\,\Delta_{max}}(3\,L - b) + \frac{w\,L^4}{8\,E\,\Delta_{max}} = 1171\ \text{in.}^4 + \frac{20.2\ \text{lb.}(14\ \text{ft.})^4\ \text{in.}^2}{8\ \ \text{ft.}\ \ 1700\ \text{kip}\ \ 0.5\ \text{in.}}\left|\frac{\text{kip}}{10^3\ \text{lb.}}\right|\frac{(12\ \text{in.})^3}{\text{ft.}^3} = 1368\ \text{in.}^4$$

We have $I_{beam} = 1710\ \text{in.}^4$ Since $I_{beam} > I_{required}$, the beam is OK in deflection.

Step 6 The beam supports a shear load of $\frac{2\,A\,\tau_{allowable}}{3} = \frac{2 \cdot 85.3\ \text{in.}^2}{3}\frac{175\ \text{lb.}}{\text{in.}^2} = 9{,}952\ \text{lb.}$ Since we are applying only 1,083 lb. in shear, the beam is OK in shear.

All Other Beams

Beams can be made of other shapes and materials: aluminum extrusions, laminated wood with a nonrectangular cross-section, steel pipe, and so forth. If your beam is neither a wide-flange steel beam nor a rectangular cross-section timber beam, follow these 6 steps (essentially the same as the Timber Beam process, except for Step 6).

Step 1 Identify all applied loads and design constraints.

Step 2 Draw the load diagram and calculate the reactions. Draw an equivalent load diagram if necessary.

Step 3 Draw shear and moment diagrams, and calculate V_{max} and M_{max}.

Step 4 Select the lightest beam that will support M_{max}, or has enough stiffness to limit Δ_{max}. Solve for $S_{required} = \frac{M}{\sigma_{allowable}}$, then find the lightest beam that has a slightly larger value of S.

If the maximum deflection is specified, write the equation for Δ_{max}, solve it for $I_{required}$, then find the lightest beam that has a slightly larger value of I.

Step 5 Add the beam weight to the load diagram, and recheck moment and deflection. If $S_{beam} > S_{required}$, the beam is OK in bending, and if $I_{beam} > I_{required}$, the beam is OK in deflection. If either of these conditions is not met, pick a larger beam and recalculate.

Step 6 Check shear. For rectangular cross-sections, use $V_{applied} < \frac{2 A \tau_{allowable}}{3}$ as we did with timber beams. For all other shapes, rewrite the General Shear Formula $\tau = \frac{V Q}{I t}$ and use $V_{applied} < \frac{\tau_{allowable} I Q}{t}$. The allowable shear stress for timber is given in Appendix E. Allowable shear stress for steel is $0.4 \sigma_{YS}$.

Key Equations

Steel wide-flange beam design

Follow the 6-step method for steel W-beams, using $Z_{required} = \frac{1.67 M}{\sigma_{YS}}$ and $V_{applied} \leq 0.4 \sigma_{YS} d t_w$.

Timber beam design

Follow the 6-step method for timber beams, using $S_{required} = \frac{M}{\sigma_{allowable\,bending}}$ and $V_{applied} < \frac{2 A \tau_{allowable}}{3}$.

All other beams

Follow the 6-step method for timber beams, using $S_{required} = \frac{M}{\sigma_{allowable}}$ and $V_{applied} < \frac{\tau_{allowable} I Q}{t}$.

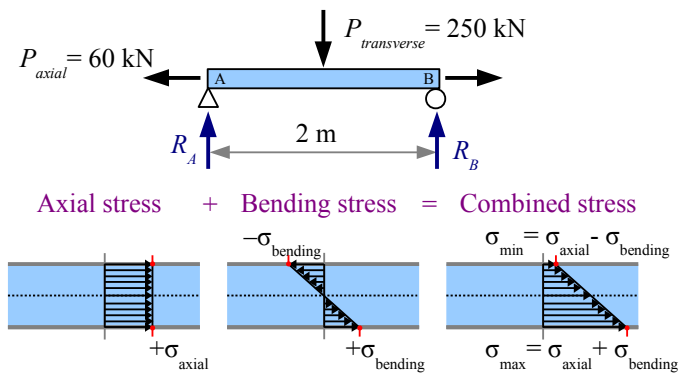
Chapter 12: Combined Stresses

Tension + Bending

Previous chapters discussed how to solve bending, torsion, tension, thermal, and pressure problems independently. Many real problems combine states of stress: a steam pipe restrained at both ends (pressure + thermal stress), horizontal beams in a tall building (vertical bending due to gravity + horizontal bending due to wind loads), or a bolt as it is tightened (torsion + tension). We can use the *Method of Superposition* to solve these problems. Calculate the stresses at a given point due to each loading condition, then add the stresses. We use the same method to calculate deflections.

Example #1

A 2 meter long, 51 mm diameter standard pipe is loaded axially in tension with a load of 60 kN. The bar is also a simply-supported beam, loaded transversely at the midspan with a load of 250 kN. Calculate the maximum stress in the bar. Report the results in MPa.



Solution Using the Method of Superposition, add the maximum tensile stresses resulting from the axial load and the bending load. The tensile stress in the bar due to the axial load is

$$\sigma_{axial} = \frac{P_{axial}}{A} = \frac{60\,\text{kN}}{658\,\text{mm}^2}\left|\frac{10^3\,\text{N}}{\text{kN}}\right|\frac{\text{MPa mm}^2}{\text{N}} = 91.2\,\text{MPa}$$

From the Formula Method (Appendix F), the maximum bending stress due to the transverse load at the midspan is

$$\sigma_{bending} = \frac{P_{transverse}\,L}{4\;\;S} = \frac{250\,\text{kN}\cdot 2\,\text{m}}{4\cdot 8650\,\text{mm}^3}\left|\frac{10^3\,\text{N}}{\text{kN}}\right|\frac{\text{MPa mm}^2}{\text{N}} = 14.5\,\text{MPa}\ .$$

The combined maximum stress is $\sigma_{combined} = \sigma_{axial} + \sigma_{bending} = 91.2\,\text{MPa} + 14.5\,\text{MPa} = 105.6\,\text{MPa}$. This maximum occurs on the bottom side of the pipe at the midspan. (On the top side of the pipe at the midspan, the combined stress is $\sigma_{combined} = \sigma_{axial} - \sigma_{bending} = 91.2\,\text{MPa} - 14.5\,\text{MPa} = 76.7\,\text{MPa}$.)

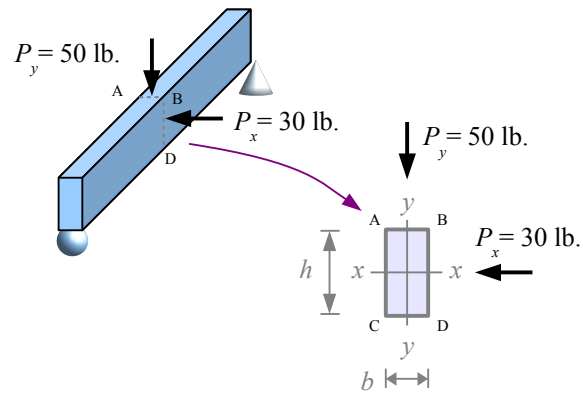
Bending in Two Directions

Consider a square cross-section beam loaded vertically at the midspan: the maximum bending stress occurs on the top and bottom surfaces. If the beam is loaded horizontally at the midspan, the maximum bending stress occurs on the side surfaces. We can add the stresses at the corners to find the extremes.

The beam deflects down and to the left; we can calculate these deflections, then combine them using trigonometry.

Example #2

A simply supported Douglas fir 2×4 timber is 84 in. long. It has a vertical load of 50 lb. and a horizontal load of 30 lb. at the midspan. Calculate the combined stresses at the edges of the beam, at the midspan (in psi). Also calculate vertical, horizontal, and total deflections (in inches).



Stress Solution From the Formula Method, the bending stress at the midspan due to the vertical load is $\sigma_y = \frac{M_y}{S_x} = \frac{P_y}{4}\frac{L}{S_x}$ where S_x is the section modulus about the x-x axis (listed in Appendix E). This stress is positive (tension) at points C and D, because they are on the far side of the beam from the applied load.

The stress is negative (compression) at points A and B, because they are on the same side of the beam as the applied load.

$\sigma_y = \frac{P_y}{4}\frac{L}{S_x} = \frac{50\,\text{lb.}\cdot 84\,\text{in.}}{4\cdot 3.06\,\text{in.}^3} = 343\text{ psi}$, therefore $\sigma_{y(C)} = \sigma_{y(D)} = +343\text{ psi}$ and $\sigma_{y(A)} = \sigma_{y(B)} = -343\text{ psi}$.

The bending stress at the midspan due to the horizontal load is $\sigma_x = \frac{M_x}{S_y} = \frac{P_x}{4}\frac{L}{S_y}$. This stress is positive (tension) at points A and C, negative (compression) at points B and D. Appendix E does not list S_y for timber beams, so we have to calculate it. $I_y = \frac{hb^3}{12} = \frac{3.5\,\text{in.}(1.5\,\text{in.})^3}{12} = 0.984\,\text{in.}^4$, therefore $S_y = \frac{I_y}{c} = \frac{I_y}{(b/2)} = \frac{0.984\,\text{in.}^4}{(1.5\,\text{in.}/2)} = 1.313\,\text{in.}^3$

$\sigma_x = \frac{P_x}{4}\frac{L}{S_y} = \frac{30\,\text{lb.}\cdot 84\,\text{in.}}{4\cdot 1.313\,\text{in.}^3} = 480\text{ psi}$, therefore $\sigma_{x(A)} = \sigma_{x(C)} = +480\text{ psi}$ and $\sigma_{x(B)} = \sigma_{x(D)} = -480\text{ psi}$.

The final step is to add the stresses at each location. The maximum tensile stress is 823 psi at point C; the maximum compressive stress is -823 psi at point B.

Point	σ_y (psi)	σ_x (psi)	σ_{total} (psi)
A	−343	480	137
B	−343	−480	−823
C	343	480	823
D	343	−480	−137

Deflection Solution From the Appendix, deflection due to a point load at the midspan is $\Delta_{max} = \frac{PL^3}{48EI}$.

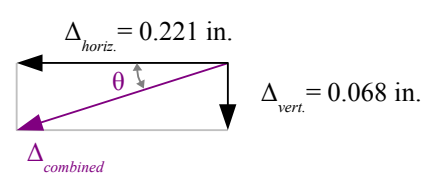
Vertical deflection is $\Delta_{vert.} = \frac{P_y L^3}{48 E I_x} = \frac{50\,\text{lb.}(84\,\text{in.})^3}{48}\frac{\text{in.}^2}{1700\text{ kips}}\frac{}{5.36\,\text{in.}^4}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\right. = 0.068\,\text{in.}$ downward.

Horizontal deflection is $\Delta_{horiz.} = \frac{P_x L^3}{48 E I_y} = \frac{30\,\text{lb.}(84\,\text{in.})^3}{48}\frac{\text{in.}^2}{1700\text{ kips}}\frac{}{0.984\,\text{in.}^4}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\right. = 0.221\,\text{in.}$ to the left.

Use trigonometry to find the combined deflection and its direction.

$\Delta_{combined} = \sqrt{\Delta_{horiz.}^2 + \Delta_{vert.}^2} = \sqrt{(0.221\,\text{in.})^2 + (0.068\,\text{in.})^2} = 0.231\,\text{in.}$

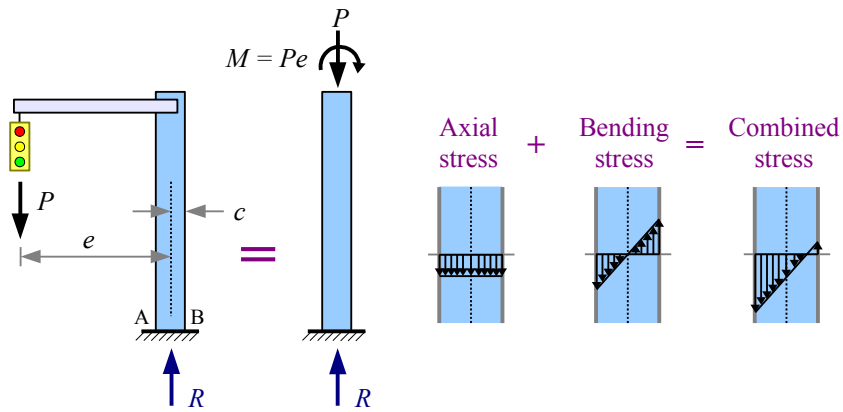
$\theta = \tan^{-1}\frac{\Delta_{vert.}}{\Delta_{horiz.}} = \tan^{-1}\frac{0.068\,\text{in.}}{0.221\,\text{in.}} = 17.1°$



Eccentric Loading

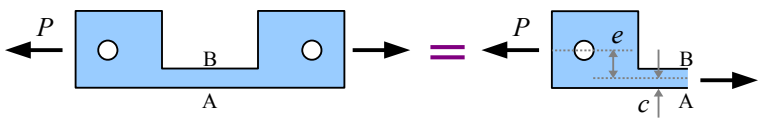
Consider an axial member loaded in tension or compression. If the load path is not along the neutral axis, then a bending moment develops in addition to the axial load.

The weight of a traffic light hanging from a cantilever arm creates an axial stress $\sigma_{axial} = -P/A$ in the vertical support pole (negative because the load is compressive), while the weight times the moment arm creates a bending moment in the support pole, causing a bending stress.



Moment is force times perpendicular distance; if we define this eccentric distance as e, the moment $M = Pe$. The bending stress at points A and B is $\sigma_{bending} = \pm\frac{Mc}{I} = \pm\frac{Pec}{I}$. The combined stress in the pole is $\sigma_{combined} = \sigma_{axial} + \sigma_{bending} = \frac{-P}{A} \pm \frac{Pec}{I}$. At point A, the bending stress is compressive, so $\sigma_A = \frac{-P}{A} - \frac{Pec}{I}$. At point B, the bending stress is tensile, so $\sigma_B = \frac{-P}{A} + \frac{Pec}{I}$.

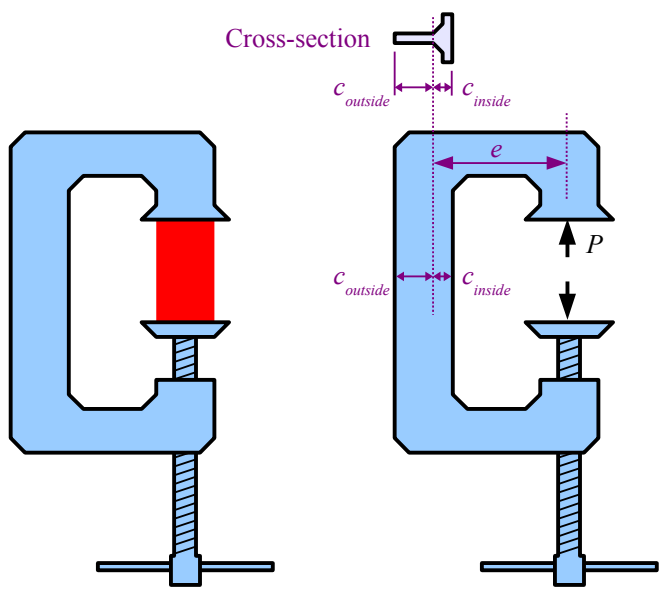
Some problems that look like axial loading problems are actually eccentric loading problems. Consider this link in a conveyer belt chain. To find the stresses at points A and B, cut the link as shown. The total stress at these points is



$\sigma_{combined} = \sigma_{axial} + \sigma_{bending} = \frac{P}{A} \pm \frac{Pec}{I}$ where I is the moment of inertia in the link at the cut section.

A C-clamp or an arbor press is an eccentric load problem: the frame is eccentrically loaded with respect to the workpiece clamped between the anvils.

The cross-section of the body of a cast-iron C-clamp or arbor press is not a rectangle, because cast iron is weaker in tension than in compression. The inside surface of the clamp is loaded in tension, therefore it needs more material than the outside surface of the clamp. Calculate the moment of inertia of a compound shape, find the neutral axis, and determine two different values of c – one for the inside surface, and the other for the outside surface. Eccentricity e is the distance from the centerline of the clamp screw to the neutral axis of the clamp body.



Another type of eccentric load involves a block of stone or concrete...a so-called short-block problem. The block is short in comparison with a masonry column.

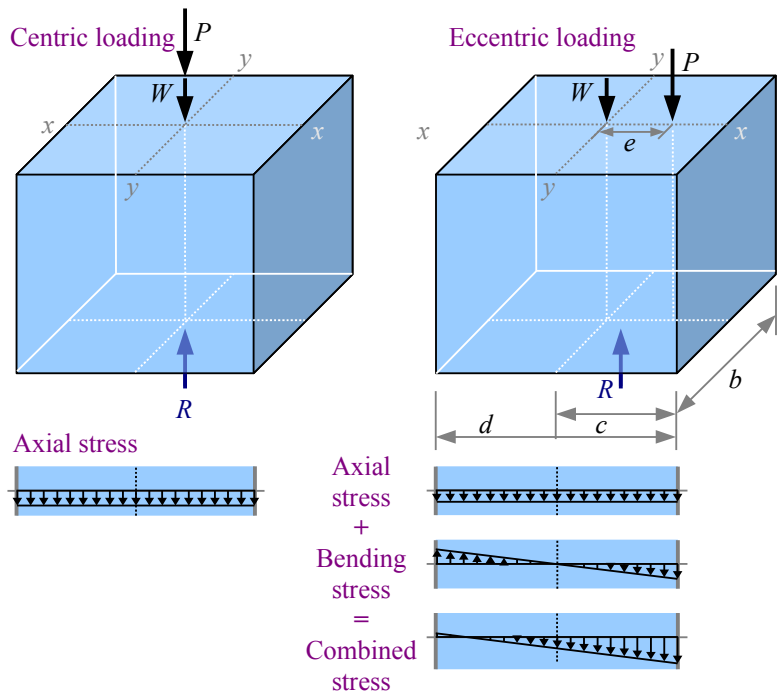
Consider a short masonry block with a vertical point load. If the load is centric, then the stress at the base of the block is due to the point load and the weight of the block:

$\sigma = \frac{-W}{A} - \frac{P}{A}$. Both terms are negative because they are both compressive loads.

If the load is offset from the center, we have an eccentric load problem. The combined stress at the base of the block is the axial term plus the bending moment term. The weight (a centric load) adds to the axial term, but not to the bending moment term.

The combined stress at the base of an eccentrically loaded short block is

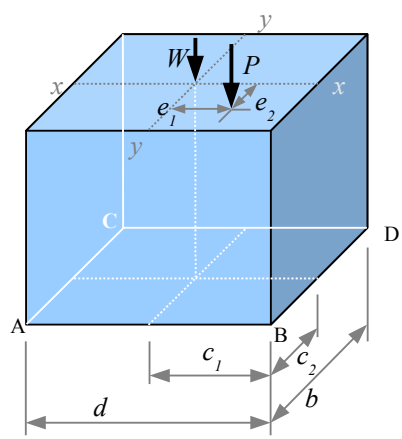
$\sigma = \frac{-W}{A} - \frac{P}{A} \pm \frac{P e c}{I_y}$ where $I_y = \frac{b d^3}{12}$.



See Example #3 for a worked-out problem with numbers and units.

We can modify this short block problem by shifting the point load so it is eccentric with respect to both the *x-x* and *y-y* neutral axes. Use the same technique as before, but add another term for the new eccentricity:

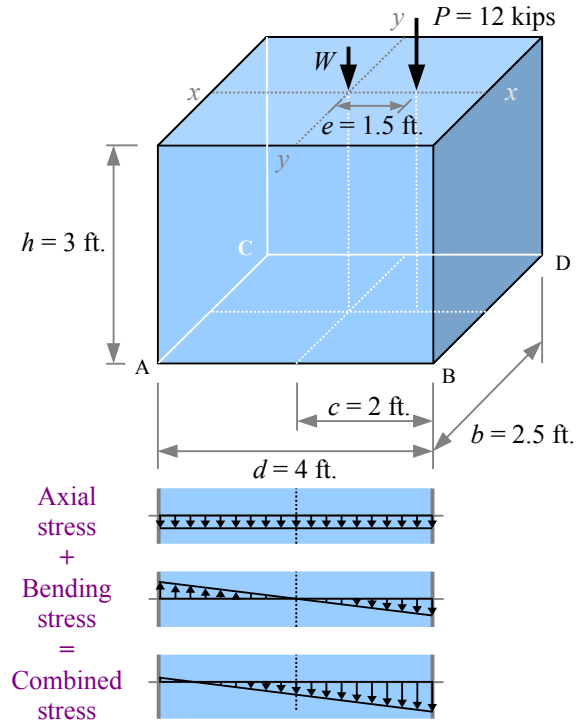
$\sigma = \frac{-W}{A} - \frac{P}{A} \pm \frac{P e_1 c_1}{I_y} \pm \frac{P e_2 c_2}{I_x}$ where $I_y = \frac{b d^3}{12}$ and $I_x = \frac{d b^3}{12}$. Now the short block problem starts to look like an Accounting problem. The way this block is drawn, eccentricity e_1 will create a positive stress as A and C, a negative stress at B and D. Eccentricity e_2 will create a positive stress as C and D, a negative stress at A and B. The maximum compressive stress occurs at point B because the bending stresses due to both eccentricities are negative at that corner; the maximum tensile stress (or smallest compressive stress, if we don't achieve tension) is at point C, because the bending stresses due to both eccentricities are positive at that corner.



See Example #4 for a worked-out problem with numbers and units.

Example #3

A stone block measuring 2.5 ft. × 4 ft. × 3 ft. has an eccentric load of 12 kips placed 1.5 ft. to the right of center as shown. The specific gravity of the stone is 200 lb./ft.^{3}. Calculate the stresses at corners A, B, C, and D. Report the results in psi.



Solution Weight is specific weight times volume:

$$W = \gamma V = \frac{200\,\text{lb.}}{\text{ft.}^3}\,\frac{2.5\,\text{ft.}\times 4\,\text{ft.}\times 3\,\text{ft.}}{}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\right. = 6\,\text{kips}\ .$$

Axial stress due to the weight is

$$\sigma_W = \frac{-W}{A} = \frac{-6\,\text{kips}}{2.5\,\text{ft.}\times 4\,\text{ft.}}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right|\frac{\text{ft.}^2}{(12\,\text{in.})^2} = -4.17\,\text{psi}$$

Axial stress due to the point load is

$$\sigma_P = \frac{-P}{A} = \frac{-12\,\text{kips}}{2.5\,\text{ft.}\times 4\,\text{ft.}}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right|\frac{\text{ft.}^2}{(12\,\text{in.})^2} = -8.33\,\text{psi}$$

The moment of inertia of the plan-view of the block with respect to the y-y axis is $I_y = \frac{b\,d^3}{12} = \frac{2.5\,\text{ft.}\,(4\,\text{ft.})^3}{12} = 13.3\,\text{ft.}^4$

Bending stress due to the eccentric load is $\sigma_{bending} = \pm\frac{P\,e\,c}{I_y} = \pm\frac{12\,\text{kips}\cdot 1.5\,\text{ft.}\cdot 2\,\text{ft.}}{13.3\,\text{ft.}^4}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right|\frac{\text{ft.}^2}{(12\,\text{in.})^2} = \pm 18.75\,\text{psi}$

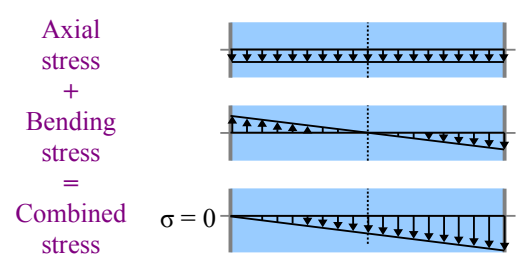
This stress is positive (tension) at points A and C, negative (compression) at points B and D.

Create a table to add up the stresses. Along the left edge of the block, the stress is 6.25 psi in tension; along the right edge, the stress is 31.25 psi in compression.

Point	σ_W (psi)	σ_P (psi)	$\sigma_{bending}$ (psi)	σ_{total} (psi)
A	−4.17	−8.33	18.75	6.25
B	−4.17	−8.33	−18.75	−31.25
C	−4.17	−8.33	18.75	6.25
D	−4.17	−8.33	−18.75	−31.25

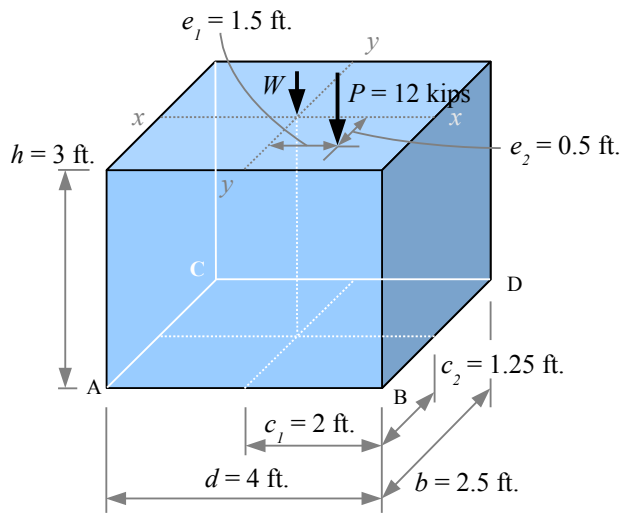
In a masonry structure, tensile stress leads to cracks because stone and unreinforced concrete tend to be many times stronger in compression than in tension. One type of short block problem is to find the optimum location of P such that no tensile stress develops. We want the stress at the left edge of this diagram to equal zero, so $\sigma = 0 = -\frac{W}{A} - \frac{P}{A} + \frac{P\,e\,c}{I_y}$.

Rewrite the equation to solve for e, then calculate the answer.



Example #4

The stone block from Example #3 has an eccentric load of 12 kips placed 1.5 ft. to the right of center and 0.5 ft. to the front of center. Calculate the corner stresses in psi.



Solution Axial stress due to the weight and the point load are the same as before: $\sigma_W = -4.17\,\text{psi}$ and $\sigma_P = -8.33\,\text{psi}$. The moment of inertia with respect to the y-y axis is the same as before: $I_y = \frac{b\,d^3}{12} = 13.3\,\text{ft.}^4$ The moment of inertia with respect to the x-x axis is $I_x = \frac{d\,b^3}{12} = \frac{4\,\text{ft.}(2.5\,\text{ft.})^3}{12} = 5.21\,\text{ft.}^4$.

Bending stress due to eccentricity e_1 is the same as before:

$$\sigma_{bending\,1} = \pm\frac{P\,e_1\,c_1}{I_y} = \pm\frac{12\,\text{kips}\cdot 1.5\,\text{ft.}\cdot 2\,\text{ft.}}{13.3\,\text{ft.}^4}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right|\frac{\text{ft.}^2}{(12\,\text{in.})^2} = \pm 18.75\,\text{psi}$$

. Bending stress due to eccentricity e_2 is

$$\sigma_{bending\,2} = \pm\frac{P\,e_2\,c_2}{I_x} = \pm\frac{12\,\text{kips}\cdot 0.5\,\text{ft.}\cdot 1.25\,\text{ft.}}{5.21\,\text{ft.}^4}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right|\frac{\text{ft.}^2}{(12\,\text{in.})^2} = \pm 10\,\text{psi}.$$

Create a table to add up the stresses. The maximum compressive stress is at the front right corner; the rear left is the only corner with a tensile stress.

Point	σ_W (psi)	σ_P (psi)	$\sigma_{bending\,1}$ (psi)	$\sigma_{bending\,2}$ (psi)	σ_{total} (psi)
A	−4.17	−8.33	+18.75	−10.00	−3.75
B	−4.17	−8.33	−18.75	−10.00	−41.25
C	−4.17	−8.33	+18.75	+10.00	+16.25
D	−4.17	−8.33	−18.75	+10.00	−21.25

Solve a short-block problem in SI units the same way, except you will have to convert mass to weight: $W = mg$. For example, if a granite block measures 1 m by 0.8 m by 0.6 m, the weight is

$$W = mg = \rho\,V\,g = \frac{2.69\,\text{g}}{\text{cm}^3}\,\frac{1\,\text{m}\times 0.8\,\text{m}\times 0.6\,\text{m}}{}\,\frac{9.81\,\text{m}}{\text{s}^2}\left|\frac{\text{N s}^2}{\text{kg m}}\right|\frac{\text{kN}}{10^3\,\text{N}}\left|\frac{(100\,\text{cm})^3}{\text{m}^3}\right|\frac{\text{kg}}{10^3\,\text{g}} = 12.7\,\text{kN}$$

Key Equations

Combine stresses due to multiple loads by adding the individual stresses.

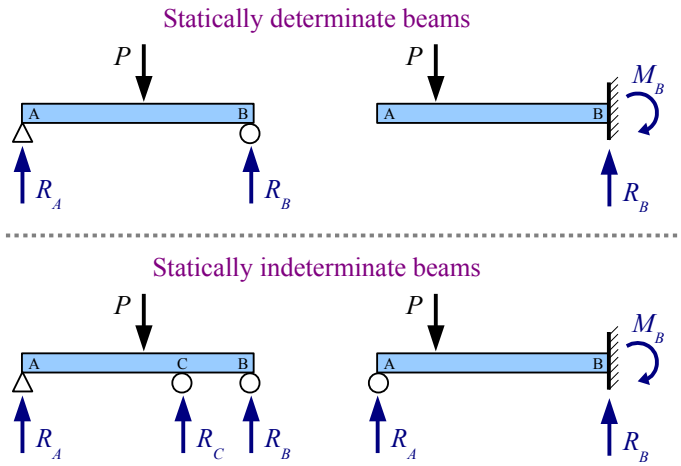
Solve short block problems with $\sigma = \frac{-W}{A} - \frac{P}{A} \pm \frac{P\,e_1\,c_1}{I_y} \pm \frac{P\,e_2\,c_2}{I_x}$ where $I_x = \frac{d\,b^3}{12}$ and $I_y = \frac{b\,d^3}{12}$

Chapter 13: Statically Indeterminate Beams

Defining Determinate and Indeterminate Beams

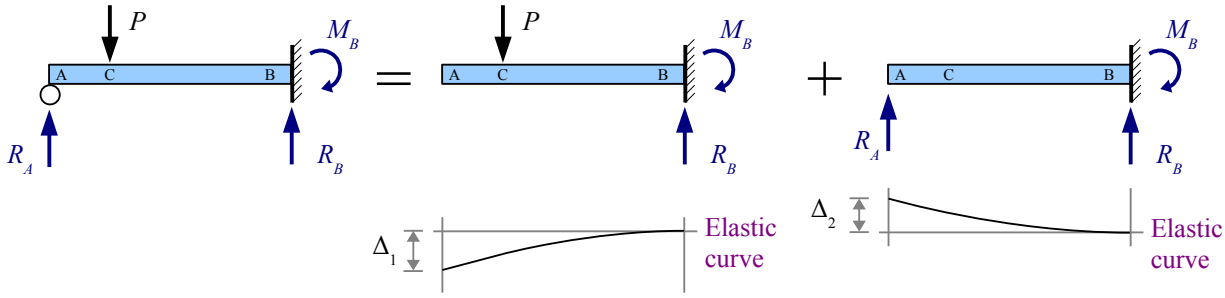
We know how to calculate deflection, reactions, shear stresses, and bending stresses in beams that have two reactions (either two reaction forces, or a reaction force and a reaction moment). These problems have two unknowns and two equations (sum of the moments, sum of the forces). Since the number of unknowns equals the number of equations, we can solve the problems directly with simple algebra, and we call these problems *statically determinate*.

From a design perspective, a statically determinate structure is risky; if one of the supports fails, the entire structure collapses. If you are building a deck, adding extra supports will preserve the deck even if one support is damaged. However, each additional support adds an unknown without adding an equation, so we cannot solve the problem directly; it is a *statically indeterminate* beam.



Method of Superposition

We can split a statically *indeterminate* problem into two statically *determinate* problems. Consider a propped cantilever beam: if we remove the prop, the end of the beam deflects downward by an amount Δ_1. If instead we remove the applied load and leave the prop in place, reaction force R_A causes the end of the beam to deflect upward by an amount Δ_2. Since the end of the propped cantilever beam is a simple support and does not deflect, $\Delta_1 = \Delta_2$. Use the Formula Method to find expressions for deflection, set them equal, and solve for reaction force R_A.



Procedure

Step 1 Identify the location of the extra support, having a force reaction only (no moment reaction).

Step 2 Remove the extra support from the original load diagram, and find the deflection formula in Appendix F for the location you selected in Step 1.

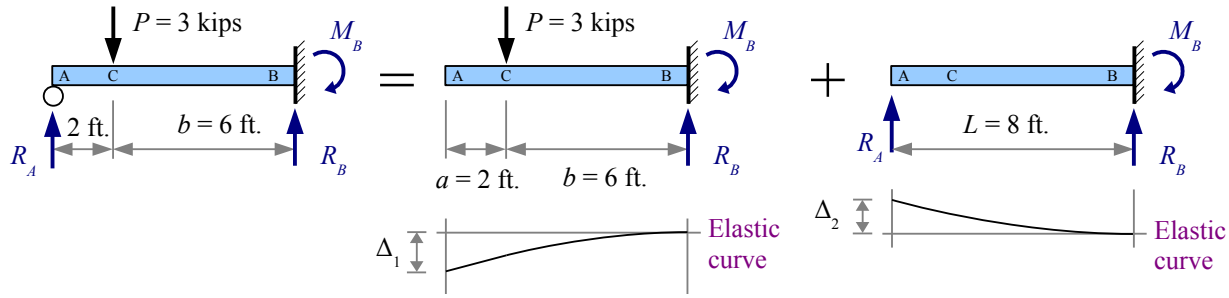
Step 3 Remove the applied loads from the original load diagram and find the deflection formula in Appendix F for the same location used in Step 2.

Step 4 Set the deflection formulas equal to each other, and solve for the unknown reaction force.

Step 5 Use sum of the forces and sum of the moments to find the remaining beam reactions.

Example #1

A propped 8 foot long cantilever beam has an applied load of 3 kips located 2 feet from the prop. Calculate the reaction forces and reaction moment. Report the results in kips and kip·ft., respectively.



Step 1 Select the free end of the beam (Point **A**) for the deflection location.

Step 2 Remove support R_A. From Appendix F, the deflection at Point **A** due to the applied load P is $\Delta_1 = \frac{P b^2}{6 E I}(3L - b)$.

Step 3 Restore support R_A and remove the applied load P. From Appendix F, the deflection at Point **A** due to a point load at the free end is $\Delta_2 = \frac{P L^3}{3 E I}$. However, the point load P in this equation is actually R_A, so $\Delta_2 = \frac{R_A L^3}{3 E I}$.

Step 4 Set the deflection equations equal to each other: $\Delta_1 = \Delta_2 = \frac{P b^2}{6 E I}(3L - b) = \frac{R_A L^3}{3 E I}$. Young's modulus and moment of inertia cancel, so the solution is independent of the cross-sectional shape and material of the beam:

$\frac{P b^2}{6}(3L - b) = \frac{R_A L^3}{3}$. Divide both sides by 3 to get $\frac{P b^2}{2}(3L - b) = R_A L^3$. Solve for the reaction force,

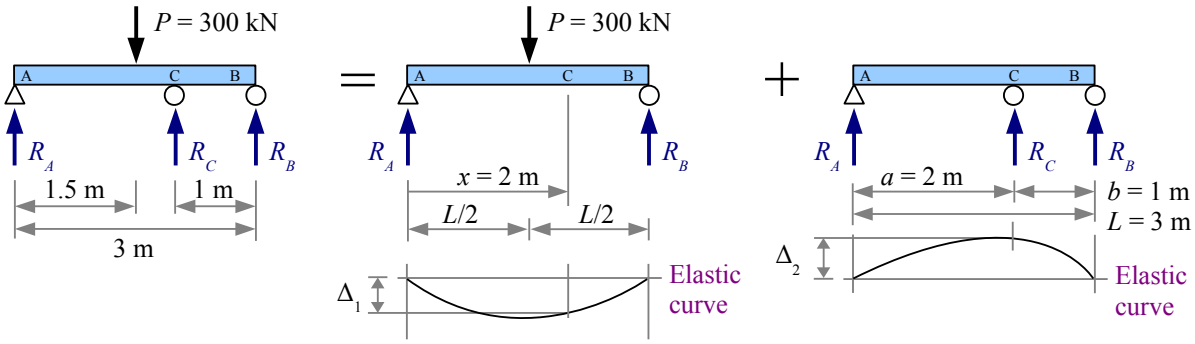
$$R_A = \frac{P b^2}{2 L^3}(3L - b) = \frac{3\,\text{kips}(6\,\text{ft.})^2}{2(8\,\text{ft.})^3}(3(8\,\text{ft.}) - 6\,\text{ft.}) = 1.90\,\text{kips}$$

Step 5 Sum of the Forces gives us $R_B = P - R_A = 3.00\,\text{kips} - 1.90\,\text{kips} = 1.10\,\text{kips}$

Sum of the Moments gives us $M_B = P \cdot b - R_A \cdot L = 3.00\,\text{kips} \cdot 6\,\text{ft.} - 1.90\,\text{kips} \cdot 8\,\text{ft.} = 2.80\,\text{kip} \cdot \text{ft.}$

Example #2

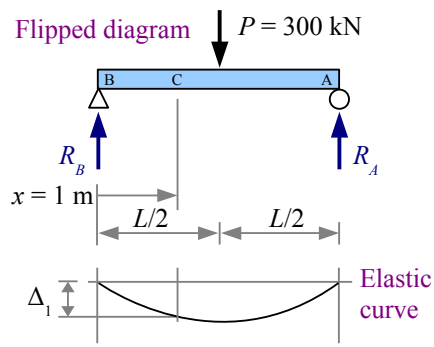
A 3 meter long beam has an applied load of 300 kN located at the midspan. The beam is supported at the ends and with a third support 1 meter from the right end. Calculate the three reaction forces. Report the results in kN.



Step 1 Select location **C** for the deflection location.

Step 2 Remove support R_C. From Appendix F, the deflection at Point **C** due to the applied load P at the midspan is $\Delta_1 = \frac{P\,x}{48\,E\,I}\left[3\,L^2 - 4\,x^2\right]$ for $x < \frac{L}{2}$.

Since $x = 2\,\text{m} > \frac{L}{2}$, we have to flip the diagram, so $x = 1\,\text{m}$, measured from the other end.



Step 3 Restore support R_C and remove the applied load P. From Appendix F, the deflection at Point **C** due to a point load at Point **C** is $\Delta_2 = \frac{P\,a^2\,b^2}{3\,E\,I\,L}$.

However, the point load P in this equation is actually R_C, so $\Delta_2 = \frac{R_C\,a^2\,b^2}{3\,E\,I\,L}$.

Step 4 Set the deflection equations equal to each other: $\Delta_1 = \Delta_2 = \frac{P\,x}{48\,E\,I}\left[3\,L^2 - 4\,x^2\right] = \frac{R_C\,a^2\,b^2}{3\,E\,I\,L}$. Once again, E and I cancel. Solving, we get $R_C = \frac{P\,x\,L}{16\,a^2\,b^2}\left[3\,L^2 - 4\,x^2\right] = \frac{300\,\text{kN}\cdot 1\,\text{m}\cdot 3\,\text{m}}{16\,(2\,\text{m})^2\,(1\,\text{m})^2}\left[3\,(3\,\text{m})^2 - 4\,(1\,\text{m})^2\right] = 323.4\,\text{kN}$

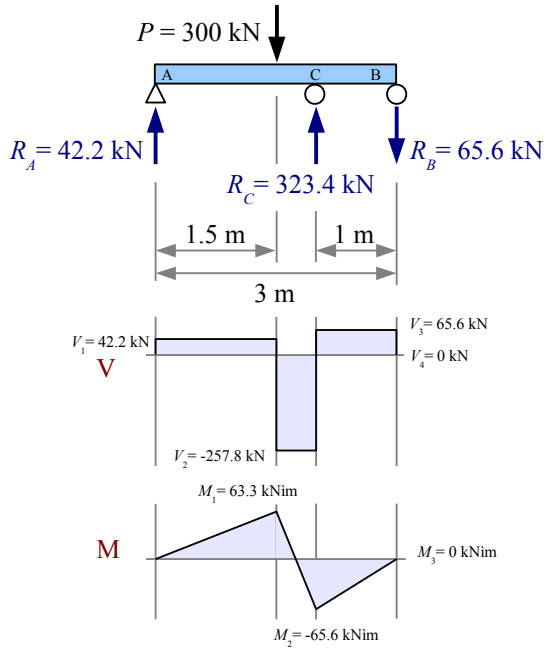
Step 5 Sum of the Moments about point **A** gives us $M_B = 0 = -300\,\text{kN}\cdot 1.5\,\text{m} + 323.4\,\text{kN}\cdot 2\,\text{m} + R_B\cdot 3\,\text{m}$. Solve for $R_B = \frac{300\,\text{kN}\cdot 1.5\,\text{m} - 323.4\,\text{kN}\cdot 2\,\text{m}}{3\,\text{m}} = -65.6\,\text{kN}$. The negative sign means R_B acts downward.

Sum of the Forces gives us $R_A = P - R_B - R_C = 300\,\text{kN} - (-65.6\,\text{kN}) - 323.4\,\text{kN} = 42.2\,\text{kN}$, positive, therefore acting upward.

Once you have solved for the reactions, you can draw shear and moment diagrams to find maximum bending and shear stresses.

Example #3

Calculate the maximum bending and shear stresses for Example #2, assuming a Southern yellow pine 250×560 timber. Ignore the weight of the beam. Report the results in MPa.



Solution Calculate the values on the shear diagram values as follows:

$$V_1 = R_A = 42.2\,\text{kN}$$

$$V_2 = V_1 - 300\,\text{kN} = -257.8\,\text{kN}$$

$$V_3 = V_2 + 323.4\,\text{kN} = 65.6\,\text{kN}$$

$$V_4 = V_3 - 65.6\,\text{kN} = 0\,\text{kN}$$

Calculate the values on the moment diagram as follows:

$$M_1 = 42.2\,\text{kN} \times 1.5\,\text{m} = 63.3\,\text{kN}\cdot\text{m}$$

$$M_2 = M_1 - 257.8\,\text{kN} \times 0.5\,\text{m} = -65.6\,\text{kN}\cdot\text{m}$$

$$M_3 = M_2 + 65.6\,\text{kN} \times 1\,\text{m} = 0\,\text{kN}\cdot\text{m}$$

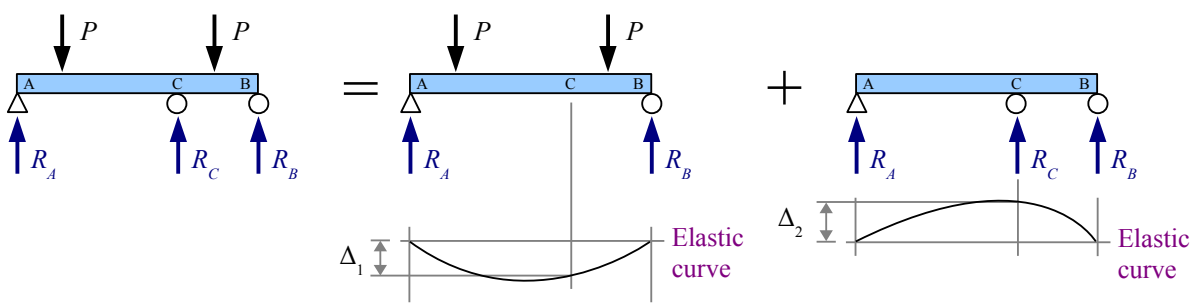
The maximum moment, 65.6 kNìm, is at point **C**. The maximum bending stress is

$$\sigma = \frac{M}{S} = \frac{65.6\,\text{kN}\cdot\text{m}}{12.0\times10^6\,\text{mm}^3}\left|\frac{(10^3\,\text{mm})^3}{\text{m}^3}\right|\frac{\text{MPa}\quad\text{m}^2}{10^3\,\text{kN}} = 5.47\,\text{MPa}\ ,$$

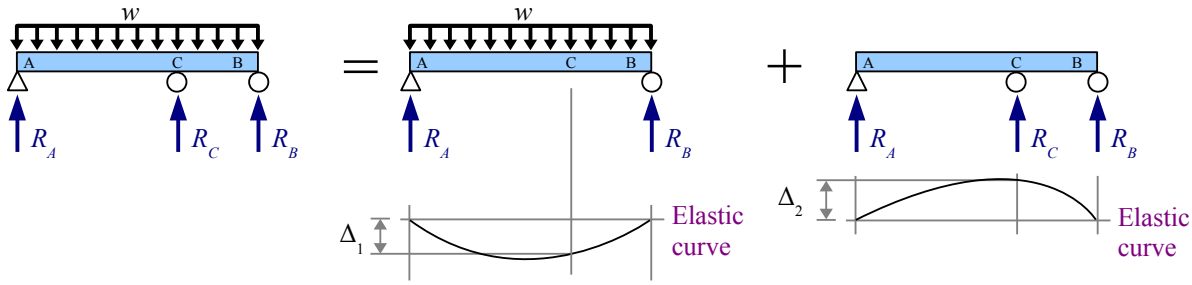
which is less than the allowable 9.65 MPa for Southern yellow pine, therefore the beam is OK in bending.

The maximum shear load is 257.8 kN, just to the right of the point load. The maximum shear stress for a rectangular cross-section is $\tau = \frac{3V}{2A} = \frac{3\cdot 257.8\,\text{kN}}{2\cdot 132\times10^3\,\text{mm}^2}\left|\frac{(10^3\,\text{mm})^2}{\text{m}^2}\right|\frac{\text{MPa}\quad\text{m}^2}{10^3\,\text{kN}} = 2.93\,\text{MPa}$, which is more than the 1.21 MPa allowable for Southern yellow pine, therefore the beam will fail in shear.

Each of these examples shows a single point load. If there is more than one point load, set up the problem by removing a support, then by removing the applied loads. Otherwise, the solution is the same as before.

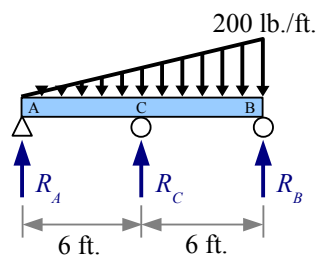


Distributed load problems are solved the same way.



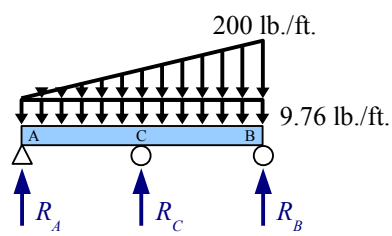
Example #4

A 12 foot long 8×8 hem-fir timber beam has a nonuniform applied load as shown. The beam is supported at the ends and with a third support at the midspan. Calculate the three reaction forces due to the applied load and the weight of the beam. Report the results in pounds.

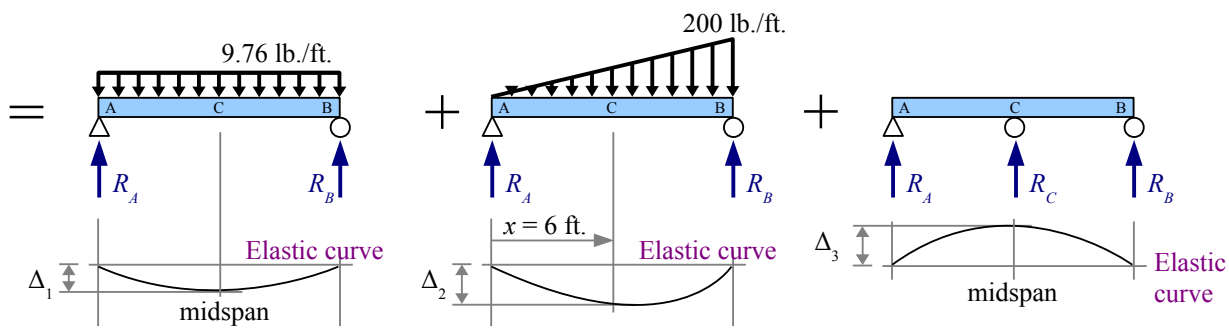


Solution Calculate the weight per unit length of the beam. Since the specific weight of hem-fir is $25\,\text{lb./ft.}^3$, interpolation is necessary.

$$w = \left[\frac{25\frac{\text{lb.}}{\text{ft.}^3} - 20\frac{\text{lb.}}{\text{ft.}^3}}{30\frac{\text{lb.}}{\text{ft.}^3} - 20\frac{\text{lb.}}{\text{ft.}^3}}\right]\left(11.7\frac{\text{lb.}}{\text{ft.}} - 7.81\frac{\text{lb.}}{\text{ft.}}\right) + 7.81\frac{\text{lb.}}{\text{ft.}} = 9.76\frac{\text{lb.}}{\text{ft.}}$$



Now the beam has two loads: the applied nonuniform distributed load plus the beam weight. Each load will create its own elastic curve.



Example #4, continued

Step 1 Select location **C** for the deflection location.

Step 2 Remove support R_C. From Appendix F, the deflection at the midspan due to the weight of the beam is $\Delta_1 = \frac{5 w_{beam} L^4}{384 E I}$, and the deflection at any location x due to the nonuniform distributed load is $\Delta_2 = \frac{w_{max} x}{360 E I L}(3x^4 - 10 L^2 x^2 + 7 L^4)$, so the total deflection due to the loads is $\Delta_{1+2} = \frac{5 w_{beam} L^4}{384 E I} + \frac{w_{max} x}{360 E I L}(3x^4 - 10 L^2 x^2 + 7 L^4)$. At the midspan, $x = \frac{L}{2}$, so the deflection equation simplifies to

$$\Delta_{1+2} = \frac{5 w_{beam} L^4}{384 E I} + \frac{w_{max} L}{360 E I L 2}\left(3\left(\frac{L}{2}\right)^4 - 10 L^2\left(\frac{L}{2}\right)^2 + 7 L^4\right) = \frac{5 w_{beam} L^4}{384 E I} + \frac{15 w_{max} L^4}{2304 E I}.$$

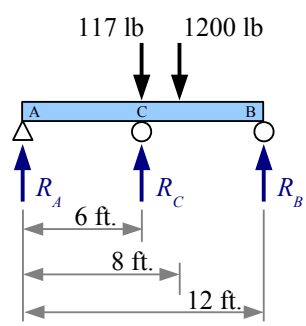
Step 3 Restore support R_C and remove the applied nonuniform distributed load. From Appendix F, the deflection is $\Delta_3 = \frac{P L^3}{48 E I}$ where P is R_C.

Step 4 Set the deflection equations equal to each other: $\frac{R_C L^3}{48 E I} = \frac{5 w_{beam} L^4}{384 E I} + \frac{15 w_{max} L^4}{2304 E I}$. Terms L^3, E and I cancel. Solving for the reaction, we get $R_C = 48\left[\frac{5 w_{beam} L}{384} + \frac{15 w_{max} L}{2304}\right] = 48\left[\frac{5 \;\; 9.76 \text{ lb.} 12 \text{ ft.}}{384 \text{ ft.}} + \frac{15 \;\; 200 \text{ lb.} 12 \text{ ft.}}{2304 \text{ ft.}}\right] = 823 \text{ lb.}$

Use an Equivalent Load Diagram and Sum of the Moments to find the reaction R_B; use Sum of the Forces to find reaction R_A.

$$R_B = \frac{117 \text{ lb.} \times 6 \text{ ft.} - 823 \text{ lb.} \times 6 \text{ ft.} + 1200 \text{ lb.} \times 8 \text{ ft.}}{12 \text{ ft.}} = 447 \text{ lb.}$$

$$R_A = (117 + 1200 - 823 - 447) \text{ lb.} = 47 \text{ lb.}$$

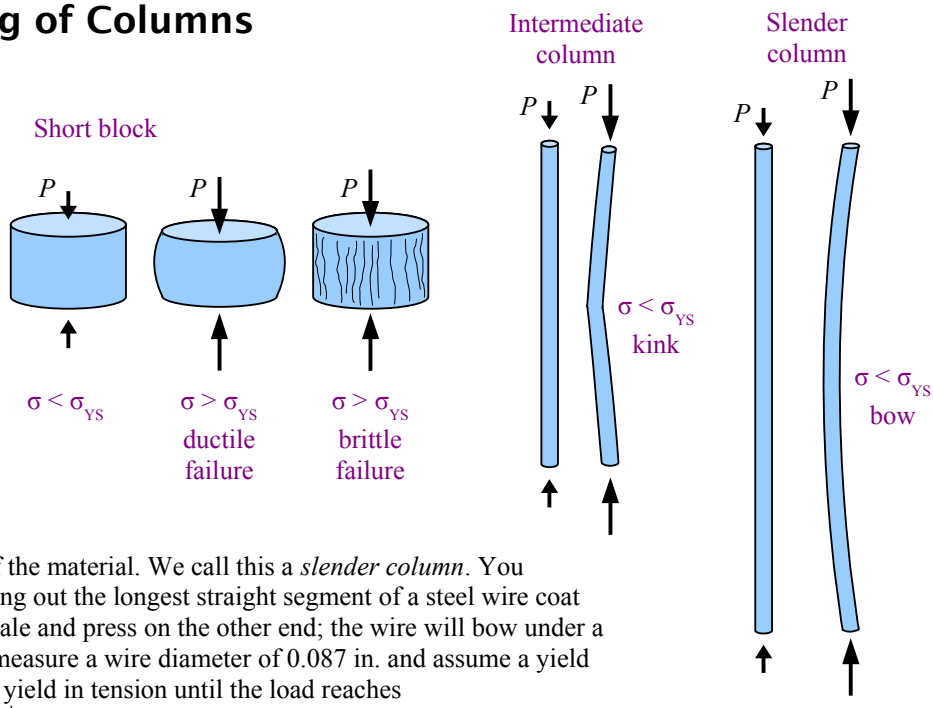


Key Equations

Use the Method of Superposition to solve statically indeterminate beam problems.

Chapter 14: Buckling of Columns

Types of Columns



Squeeze a short cylindrical block elastically, and the normal stress is $\sigma = P/A$. Squeeze hard enough to exceed the yield strength, and the block will either fail by distorting (if it is ductile) or by fracturing (if it is brittle). A ductile short block will yield at the same load, whether it is in tension or in compression.

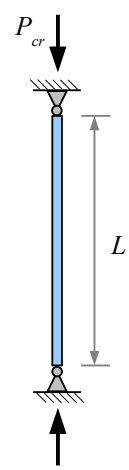
Change the dimensions of the cylinder so it is very long and thin, and push on the ends: it will form a gentle bow at a normal stress less than the yield strength of the material. We call this a *slender column*. You can make a slender column by cutting out the longest straight segment of a steel wire coat hanger. Put one end on a kitchen scale and press on the other end; the wire will bow under a force of a couple of pounds. If we measure a wire diameter of 0.087 in. and assume a yield strength of 30 ksi, the wire will not yield in tension until the load reaches

$P_{tension} = \sigma A = \frac{30\,\text{kips}}{\text{in.}^2}\frac{\pi(0.087\,\text{in.})^2}{4}\left|\frac{10^3\,\text{lb.}}{\text{kip}}\right. = 178\,\text{lb.}$ In tension, it takes almost 200 lb. to permanently deform the wire, while in compression it takes about 2 lb. – a difference of two orders of magnitude! The extreme example of a slender column is a rope or cable: high tensile strength; no compressive strength.

Between the short block and the slender column is the *intermediate column*, not as skinny as the slender column, but longer and thinner than the short block. Push on the ends, and an intermediate column forms a kink about halfway up, at a normal stress less than the yield strength of the material. You can make an intermediate column with a thick piece of solder about 10 in. long.

Ideal Slender Columns

Euler Critical Load

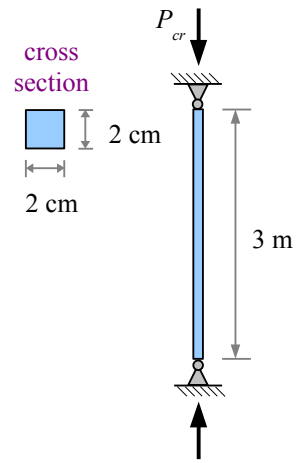


Leonhard Euler, the Swiss mathematician who gave us the e^x button on the calculator, studied the mathematics of columns. He derived an equation for the critical load that causes buckling in an ideal slender column that is pinned at both ends. The Euler critical load $P_{cr} = \frac{\pi^2 E I}{L^2}$ where I is the smallest moment of inertia for the cross-sectional area of the column. For example, if the column is a wide-flange beam, $I_y < I_x$, so use I_y. Euler's equation tells us the critical load for a 16.5 in. long coat hanger wire is $P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 E}{L^2}\frac{\pi d^4}{64} = \frac{\pi^2}{(16.5\,\text{in.})^2}\frac{30\times 10^6\,\text{lb.}}{\text{in.}^2}\frac{\pi(0.087\,\text{in.})^4}{64} = 3.1\,\text{lb.}$, which is pretty close to, but not the same as, the 2 lb. measured in the kitchen. Why are the numbers different? Euler's equation assumes the wire is perfectly straight, the loads are applied in perfect alignment with the axis of the column, there are no materials defects in the steel, the cross-section is perfect and uniform along the length of the wire, there are no scratches on the wire's surface, and the ends can rotate freely with no friction.

What if you know the applied load, and have to select a solid round rod to support that load? Rewrite the Euler critical load equation, solving for diameter: $P_{cr}=\frac{\pi^2 E I}{L^2}=\frac{\pi^2 E}{L^2}\frac{\pi d^4}{64} \rightarrow d=\sqrt[4]{\frac{64 P L^2}{\pi^3 E}}$.

Example #1

An aluminum column made of 2 cm square barstock is loaded as a pin-connected column. Calculate the Euler critical load for this column if the length is 3 m. Report the result in kN.



Solution The moment of inertia of a solid square is $I=\frac{bh^3}{12}=\frac{b^4}{12}$.

From Appendix B, $E = 70$ GPa.

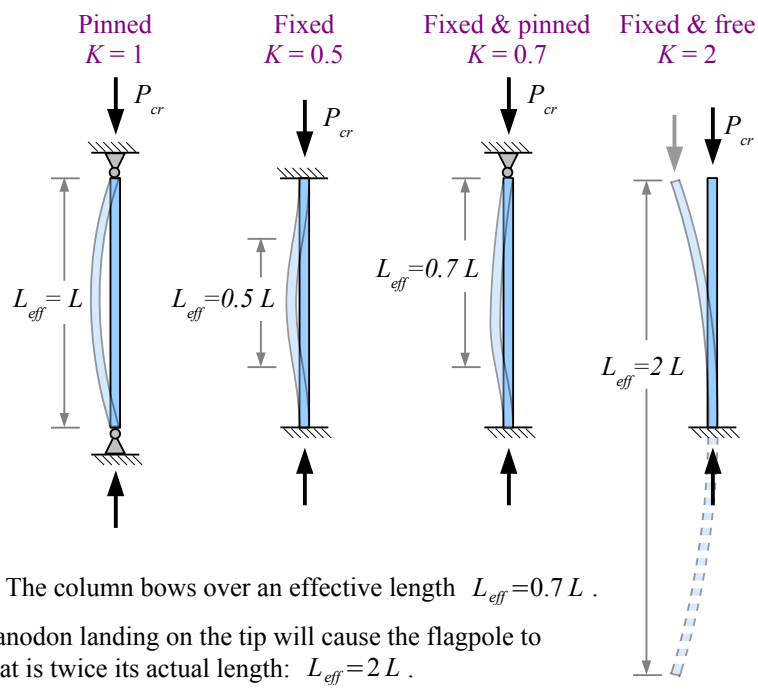
$$P_{cr}=\frac{\pi^2 E I}{L^2}=\frac{\pi^2 b^4 E}{12 L^2}=\frac{\pi^2 (2\,\text{cm})^4\, 70\,\text{GPa}}{12\,(3\,\text{m})^2}\left|\frac{10^6\,\text{kN}}{\text{GPa m}^2}\right|\frac{\text{m}^4}{(100\,\text{cm})^4}=1.02\,\text{kN}$$

Euler Stress

Recall the definition for radius of gyration of an area: $r_G=\sqrt{I/A}$. Square both sides to get $r_G^2=I/A$. The Euler critical load divided by the cross-sectional area of the column gives us the stress in the column: $\frac{P_{cr}}{A}=\sigma_{cr}=\frac{\pi^2 E I}{L^2 A}=\frac{\pi^2 E r_G^2}{L^2}=\frac{\pi^2 E}{(L/r_G)^2}$. The quantity L/r_G is called the *slenderness ratio* of the column; the larger this value, the more slender the column.

Support Conditions

The end of a column can have one of three support conditions. It can be pinned: free to rotate, but prevented from sliding sideways. It can be fixed, like the wall-end of a cantilever beam: prevented from rotating or sliding. It can be free to rotate and slide, like the free end of a cantilever beam.



If both ends are pinned, the column bows in an arc equal to the length of the column. The *effective length* of the column is its own length.

If both ends are fixed, then the column cannot rotate at its ends. It bows in the center relative to two inflection points located $L/4$ from each end, so the effective length of the bow is half the column length, and $L_{eff}=0.5\,L$.

If one end is fixed while the other is pinned, then the column can rotate about the pinned connection. The column bows over an effective length $L_{eff}=0.7\,L$.

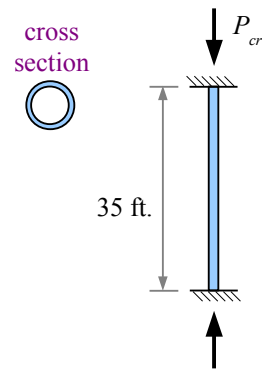
A fixed & free column looks like a flagpole; a pteranodon landing on the tip will cause the flagpole to sway. The column bows over an effective length that is twice its actual length: $L_{eff}=2\,L$.

We can modify Euler's critical load equation to include these support conditions: $P_{cr}=\dfrac{\pi^2 E I}{(KL)^2}$ where K is the effective length factor 1, 0.5, 0.7, or 2, as appropriate. Likewise, the Euler stress is $\sigma_{cr}=\dfrac{\pi^2 E}{(KL/r_G)^2}$.

These equations help us find the load and stress when buckling first starts, assuming perfect loading conditions and materials. In real life, we need to include a Factor of Safety, so $P_{cr}=\dfrac{\pi^2 E I}{(KL)^2 \text{F.S.}}$ and $\sigma_{cr}=\dfrac{\pi^2 E}{(KL/r_G)^2 \text{F.S.}}$.

Example #2

A standard 4 inch pipe 35 feet long is used as an Ideal fixed column. Using a factor of safety of 2, calculate the load that the pipe can support; report the result in kips. Calculate the stress in the pipe; report the result in ksi.



Solution Since the column is fixed, $K=0.5$. From Appendix D, a standard 4 inch pipe has a moment of inertia $I=6.82\text{ in.}^4$, a radius of gyration $r_G=1.51\text{ in.}$, and a cross-sectional area of 2.96 in.^{2}.

$$P_{cr}=\frac{\pi^2 E I}{(KL)^2 \text{F.S.}}=\frac{\pi^2}{(0.5\cdot 35\text{ ft.})^2\, 2}\,\frac{30\times 10^6\text{ lb.}}{\text{in.}^2}\,\frac{6.82\text{ in.}^4}{}\left|\frac{\text{kips}}{10^3\text{ lb.}}\right|\frac{(\text{ft.})^2}{(12\text{ in.})^2}=24.3\text{ kips}$$

$$\sigma_{cr}=\frac{\pi^2 E}{\left(\dfrac{KL}{r_G}\right)^2 \text{F.S.}}=\frac{\pi^2}{\left(\dfrac{0.5\cdot 35\text{ ft.}}{1.51\text{ in.}}\left|\dfrac{12\text{ in.}}{\text{ft.}}\right.\right)^2\cdot 2}\,\frac{30\times 10^6\text{ lb.}}{\text{in.}^2}\left|\frac{\text{kips}}{10^3\text{ lb.}}\right.=7.65\text{ ksi}$$

Structural Steel Columns

W-beams are often used as columns in steel structures. AISC recommends using the following formulas for calculating the allowable load. A factor of safety is built in to the equations.

If $\dfrac{KL}{r_G}>200$ then the column is too slender for safe use; stop all calculations, and pick a shorter or deeper column.

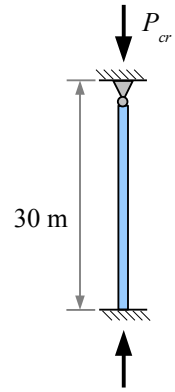
If $\dfrac{KL}{r_G}>4.71\sqrt{\dfrac{E}{\sigma_{YS}}}$ then $P_{all}=0.525\sigma_{cr}A$

If $\dfrac{KL}{r_G}<4.71\sqrt{\dfrac{E}{\sigma_{YS}}}$ then $P_{all}=\dfrac{0.658^{(\sigma_{YS}/\sigma_{cr})}\sigma_{YS}A}{1.67}$

Solve Structural Steel Column problems in four steps:

1. Calculate KL/r_G and $4.71\sqrt{E/\sigma_{YS}}$. Since $r_{Gy}<r_{Gx}$ for W-beams, use r_{Gy}
2. Calculate σ_{cr}
3. Use the appropriate formula to calculate P_{all}
4. Use $\sigma=P_{all}/A$ to find the stress in the column

Example #3



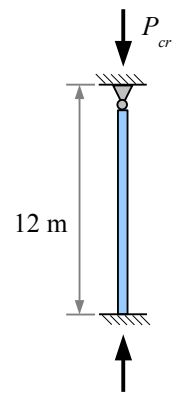
A 30 m long W200×59 wide-flange beam made from A36 steel is used as a fixed and pinned column. Calculate the allowable load; report the result in kN. Calculate the stress in the column; report the result in MPa.

Solution Since the column is fixed and pinned, $K=0.7$. From Appendix D, $r_{Gx}=89.7\,\text{mm}$ and $r_{Gy}=51.8\,\text{mm}$. Use the smaller of these numbers. From Appendix B, the yield strength of A36 steel is 250 MPa.

$$\frac{KL}{r_G}=\frac{0.7\cdot 30\,\text{m}}{51.8\,\text{mm}}\left|\frac{10^3\,\text{mm}}{\text{m}}\right.=405$$

Since $\frac{KL}{r_G}>200$, the column is too slender for safe use.

Example #4



Solve Example #3 using a 12 m long W200×59 wide-flange beam.

Solution The material and dimensional properties are the same as in Example #3.

$$\frac{KL}{r_G}=\frac{0.7\cdot 12\,\text{m}}{51.8\,\text{mm}}\left|\frac{10^3\,\text{mm}}{\text{m}}\right.=162.2$$

$$4.71\sqrt{\frac{E}{\sigma_{YS}}}=4.71\sqrt{\frac{207\,\text{GPa}}{250\,\text{MPa}}\left|\frac{10^3\,\text{MPa}}{\text{GPa}}\right.}=136$$

Since $\frac{KL}{r_G}>4.71\sqrt{\frac{E}{\sigma_{YS}}}$, use $P_{all}=0.525\sigma_{cr}A$ where $\sigma_{cr}=\frac{\pi^2 E}{(KL/r_G)^2}$. We can plug in the calculated value of KL/r_G from above.

$$\sigma_{cr}=\frac{\pi^2 207\,\text{GPa}}{(162.2)^2}\left|\frac{10^3\,\text{MPa}}{\text{GPa}}\right.=77.7\,\text{MPa}$$

$$P_{all}=0.525\sigma_{cr}A=\frac{0.525\cdot 77.7\,\text{MPa}\cdot 7550\,\text{mm}^2}{}\left|\frac{10^3\,\text{kN}}{\text{MPa}\ \ \text{m}^2}\right|\frac{\text{m}^2}{(10^3\,\text{mm})^2}=308\,\text{kN}$$

The actual stress in the column is $\sigma=\frac{P_{all}}{A}=\frac{308\,\text{kN}}{7550\,\text{mm}^2}\left|\frac{\text{MPa}\ \ \text{m}^2}{10^3\,\text{kN}}\right|\frac{(10^3\,\text{mm})^2}{\text{m}^2}=40.8\,\text{MPa}$

Steel Machine Parts

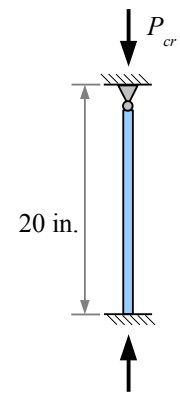
Structural columns are large – several feet to dozens of feet tall. Machines also use long, thin compression members, such as connecting rods in internal combustion engines. If $\frac{KL}{r_G}\geq\sqrt{\frac{2\pi^2 E}{\sigma_{YS}}}$ then the column is slender, and the allowable stress is $\sigma_{all}=\frac{\pi^2 E}{(KL/r_G)^2\text{F.S.}}$. This is the Euler equation for ideal columns.

On the other hand, if $\frac{KL}{r_G} < \sqrt{\frac{2\pi^2 E}{\sigma_{YS}}}$ then the column is intermediate. Based on empirical data from real machine parts, J.B. Johnson developed a formula for the allowable stress: $\sigma_{all} = \left[1 - \frac{\sigma_{YS}(KL/r_G)^2}{4\pi^2 E}\right]\frac{\sigma_{YS}}{\text{F.S.}}$.

Calculate the allowable load of a steel machine part by multiplying the allowable stress by the cross-sectional area.

Example #5

A 20 inch long , 1 inch diameter steel rod with a yield strength of 35 ksi is used as a fixed and pinned column with a Factor of Safety of 2. Calculate the allowable stress in the machine part; report the result in ksi. Calculate the the allowable load in the machine part; report the result in kips.



Solution Since the column is fixed and pinned, $K = 0.7$. The radius of gyration of a solid circle is one quarter the diameter, so $r_G = 0.25$ in.

$$\frac{KL}{r_G} = \frac{0.7 \cdot 20\,\text{in.}}{0.25\,\text{in.}} = 56$$

$$\sqrt{\frac{2\pi^2 E}{\sigma_{YS}}} = \frac{KL}{r_G} < \sqrt{\frac{2\pi^2 \cdot 30\times 10^3\,\text{ksi}}{35\,\text{ksi}}} = 130$$

Since $\frac{KL}{r_G} < \sqrt{\frac{2\pi^2 E}{\sigma_{YS}}}$, the column is intermediate.

$$\sigma_{all} = \left[1 - \frac{\sigma_{YS}(KL/r_G)^2}{4\pi^2 E}\right]\frac{\sigma_{YS}}{\text{F.S.}} = \left[1 - \frac{35\,\text{ksi}\,(56)^2}{4\pi^2 30\times 10^3\,\text{ksi}}\right]\frac{35\,\text{ksi}}{2} = 15.9\,\text{ksi} \quad \text{and} \quad P_{all} = \sigma_{all} A = \frac{15.9\,\text{kips}}{\text{in.}^2}\frac{\pi(1\,\text{in.})^2}{4} = 12.5\,\text{kips}$$

Some machine design problems start with a known load and column length; your job is to calculate the minimum diameter of a solid round rod to allow the column to survive buckling. Remember that the radius of gyration of a solid round rod is $d/4$ and the allowable stress is $\sigma_{all} = \frac{P}{A} = \frac{P}{(\pi/4)d^2}$, then follow these six steps:

Step 1 Assume the column is slender (because the math is easier): $\sigma_{all} = \frac{\pi^2 E}{(KL/r_G)^2\,\text{F.S.}} = \frac{P}{(\pi/4)d^2}$

Step 2 Rewrite the equation to solve for diameter.

Step 3 Check that the column really is slender.

Step 4 If the column is not slender, then $\sigma_{all} = \left[1 - \frac{\sigma_{YS}(KL/r_G)^2}{4\pi^2 E}\right]\frac{\sigma_{YS}}{\text{F.S.}} = \frac{P}{(\pi/4)d^2}$

Step 5 Rewrite the equation to solve for diameter. This is a little tricky because of the number of terms.

Step 6 Check that the column really is intermediate; if it isn't, you made a mistake someplace.

If the cross-sectional area of the column is a different shape (such as a rectangle), follow these steps, using the appropriate equations for area and r_G.

Key Equations

Ideal Slender Columns

The Euler critical load is $P_{cr}=\frac{\pi^2 E I}{(KL)^2 \text{F.S.}}$, and the Euler critical stress is $\sigma_{cr}=\frac{\pi^2 E}{(KL/r_G)^2 \text{F.S.}}$

Structural Steel Columns

If $\frac{KL}{r_G}>200$ then the column is too slender for safe use.

If $\frac{KL}{r_G}>4.71\sqrt{\frac{E}{\sigma_{YS}}}$ then $P_{all}=0.525\sigma_{cr} A$

If $\frac{KL}{r_G}<4.71\sqrt{\frac{E}{\sigma_{YS}}}$ then $P_{all}=\frac{0.658^{(\sigma_{YS}/\sigma_{cr})}\sigma_{YS} A}{1.67}$

Steel Machine Columns

If $\frac{KL}{r_G}\geq\sqrt{\frac{2\pi^2 E}{\sigma_{YS}}}$ then the column is slender, and the allowable stress is $\sigma_{all}=\frac{\pi^2 E}{(KL/r_G)^2 \text{F.S.}}$.

If $\frac{KL}{r_G}<\sqrt{\frac{2\pi^2 E}{\sigma_{YS}}}$ then the column is intermediate, and $\sigma_{all}=\left[1-\frac{\sigma_{YS}(KL/r_G)^2}{4\pi^2 E}\right]\frac{\sigma_{YS}}{\text{F.S.}}$.

Calculate $P_{all}=\sigma_{all} A$

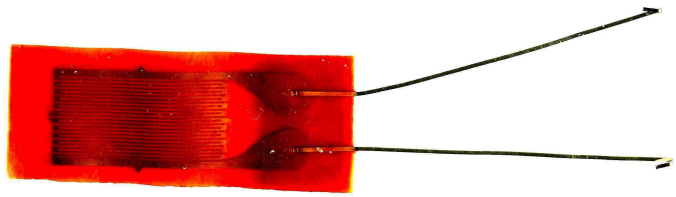
To find the smallest allowable diameter of a round solid rod, follow the 6-step procedure described above.

Chapter 15: Visualizing Stress and Strain

Measuring Stress

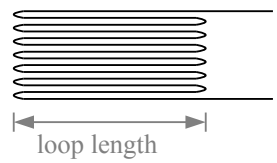
We can see deformation, if it's large enough, but we cannot see stress, and in most cases we cannot measure it directly. Instead, we can use a *strain gauge* to measure the strain on the surface of an object that is being elastically deformed, and use Young's Modulus to estimate the stress.

The earliest strain gauges in the 1930s were made of fine wire. If you glue a very thin wire along the bottom surface of a beam, then load the beam in bending, the wire will stretch and become thinner. The thinner the wire becomes, the higher its electrical resistance. By measuring this resistance you can determine strain, then calculate stress as $\sigma = \frac{E}{\varepsilon}$, provided the object is being strained elastically.



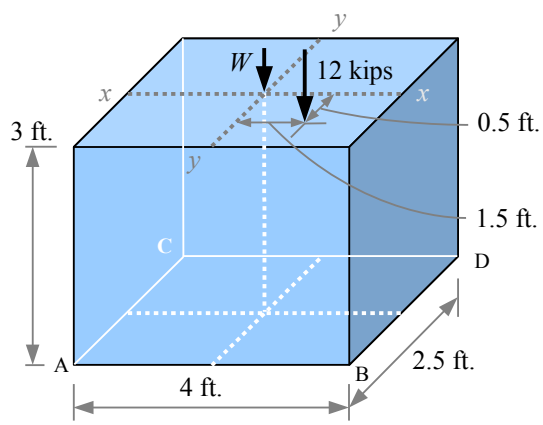
Modern strain gauges use the same principal, but are made of thin metal foil instead of wire. The foil is embedded in a pad that is easy to glue to the part being tested.

The longer the conductor, the more sensitive the gauge. With a foil strain gauge, the conductor loops back on itself many times. The total wire length is the loop length times twice the number of loops; in this cartoon, the total length is 14 times the loop length.



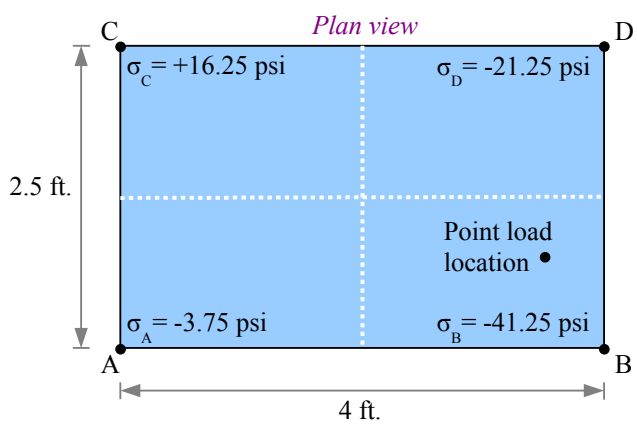
Stress at the Base of a Short Block

Consider the short block in Chapter 12, Example #4. A point load is offset from both neutral axes. The total stress at the four corners is -3.75 psi, -41.25 psi, +16.25 psi, and -21.25 psi.

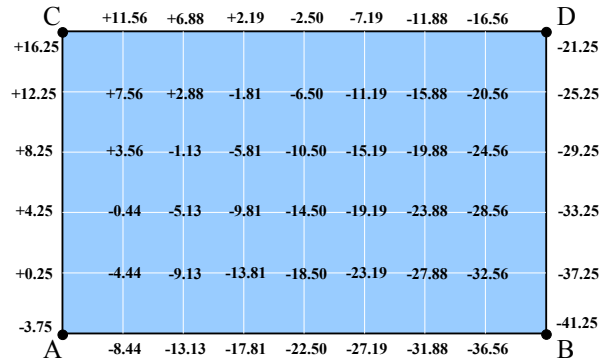


In plan view, point C has a tensile stress, while points A, B, and D have compressive stresses.

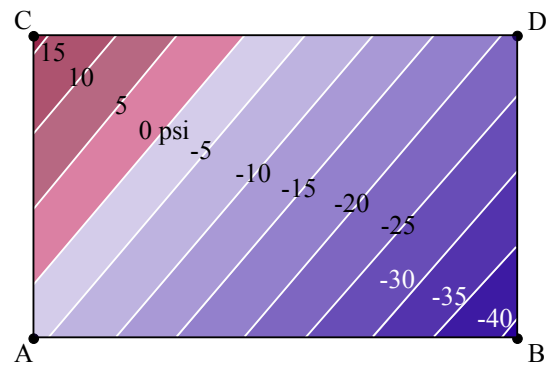
We can calculate the stress at any point along the edges by interpolating from the corner stresses, because the stress varies linearly from one corner to the next. Likewise we can calculate the stresses within the area of the block by interpolating from the edge stresses.



One way to visualize these stresses is with a grid pattern in plan view, identifying the stress at each location, or node. In the diagram, the nodes are at half-foot intervals.



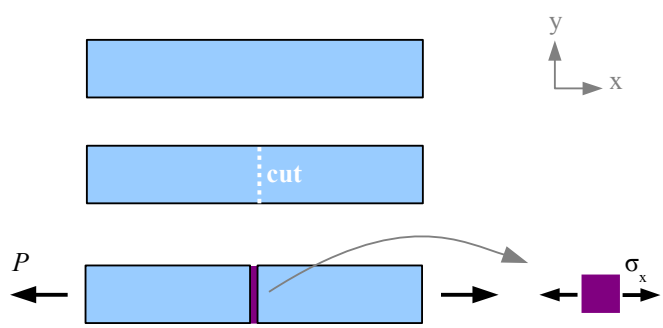
Alternatively, we can calculate contour lines of stress (a.k.a. isostress lines), coloring the spaces between the lines. The pattern shows that the pink areas are in tension, while the blue areas are in compression. The picture helps us see that the isostress lines are diagonals, and there is a zero stress line stretching from the AC edge to the CD edge.



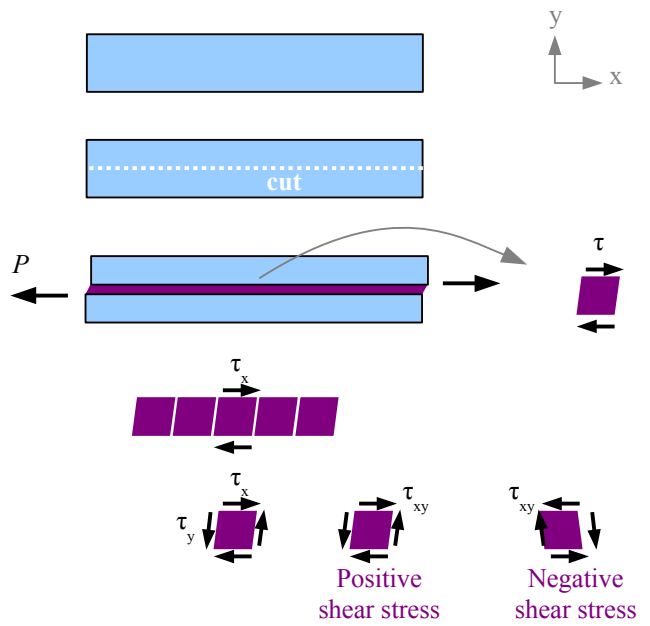
Mohr's Circle

Normal and shear stresses in uniaxial tension

Take a tensile bar, cut it in half transversely to the axis, and glue the bar back together. If you pull on the bar along the x-axis, a normal stress $\sigma_x = \frac{P}{A}$ develops in the adhesive. Area A is the cut surface area of the bar (the cross-sectional area). There is no shear stress, because the adhesive is not loaded in shear. Consider a little square piece of adhesive: it has a normal stress σ_x acting on the left and right faces, but no normal stress on the top or bottom faces, and no shear stresses.

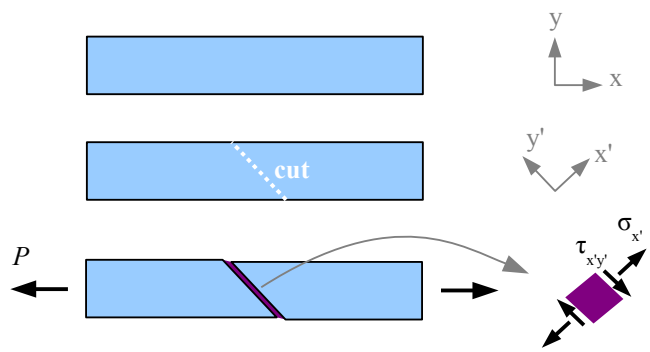


Cut a similar tensile bar in half along its x-axis, and glue the bar back together. If you load the bar as shown, a shear stress $\tau = \frac{P}{A}$ develops in the adhesive. Again, area A is the cut surface area of the bar. There is no normal stress, because the adhesive is only loaded in shear. A little square piece of adhesive has a shear stress acting on the top and bottom faces, in the x direction. The square becomes a parallelogram as it distorts.



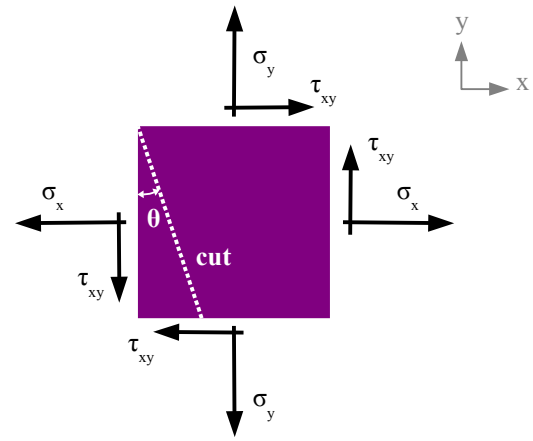
We could call this shear stress τ_x, because it acts along the x direction. Think about how the little square of adhesive distorts relative to other little squares of adhesive on its left and right: its neighbors try to prevent the piece from parallelogramming, so there are shear stresses in the y direction too. The stress required to produce the parallelogram is the same along each edge, so $\tau_x = \tau_y$; we'll use a single term $\tau_{xy} = \tau_x = \tau_y$. A positive shear stress is one that distorts the square clockwise relative to its base; a negative shear stress distorts the square counterclockwise.

Cut another bar at an angle to the axis, and glue it back together. If you pull on the bar, the adhesive is loaded in normal tension *and* in shear. The two segments of the bar want to pull away from each other *and* slide along each other. Let's define two new axes, x' and y', perpendicular and parallel to the cut surface. We have a shear stress τ acting parallel to the cut surface, and a tensile stress $\sigma_{x'}$ acting perpendicular to the cut surface, in the x' direction.

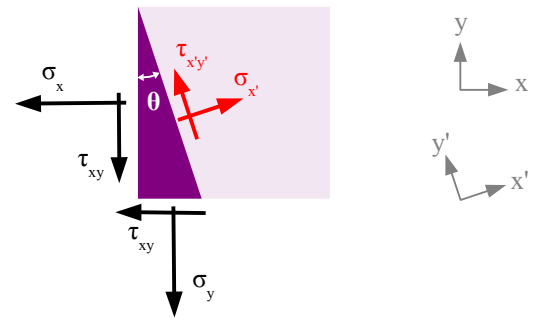


We don't have to actually cut the bar and use an adhesive; instead, we can imagine the stresses that act in various directions within the material. Draw a square that represents the piece of adhesive, and rotate it in different directions. We can calculate the stresses in the x' and y' directions by using equations from Statics: sum of the forces in any particular direction equals zero.

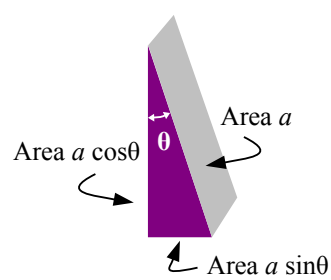
First, draw the square element. In the general case, there could be a vertical normal stress σ_y, so include this stress in the sketch.



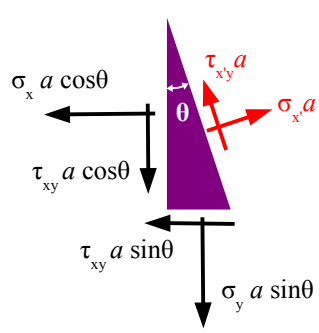
Next, cut the square along a diagonal in the y' direction, so the normal stress acting on this surface is $\sigma_{x'}$ and the shear stress acting on this surface is $\tau_{x'y'}$. The new x'-y' coordinate axes are tilted at angle θ counterclockwise from the original x-y coordinate axes.



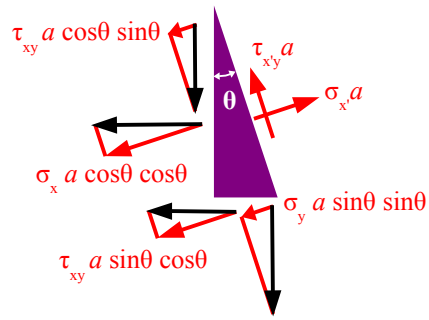
The square has a depth (the thickness of the stressed part), so each edge is actually an area in three dimensions. Let the area of the cut surface be area a. Using trigonometry, the area of the left surface is $a\cos\theta$ and the area of the bottom surface is $a\sin\theta$.



The force acting on each face is equal to the stress times the area. We have 6 forces acting on the triangular body.



Look at the components of the forces acting in the x' direction. If you add them up, the sum is zero.



$$\sigma_{x'}a = \sigma_x a\cos^2\theta + \sigma_y a\sin^2\theta + 2\tau_{xy} a\cos\theta\sin\theta$$

Notice that area a cancels, so we can write

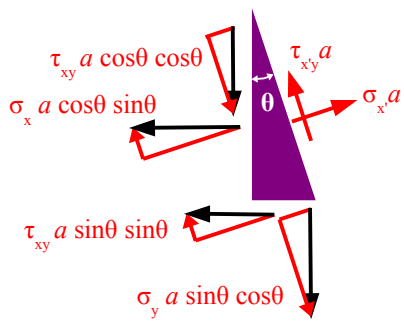
$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta$$

There are some trig identities that will help simplify the equation into something useful: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\sin 2\theta = 2\cos\theta\sin\theta$, and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Now we get

$$\begin{aligned}\sigma_{x'} &= \sigma_x\frac{1+\cos 2\theta}{2} + \sigma_y\frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta \\ &= \frac{\sigma_x + \sigma_x\cos 2\theta + \sigma_y + \sigma_y\cos 2\theta}{2} + \tau_{xy}\sin 2\theta \\ &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\end{aligned}$$

Following the same procedure, we can calculate the sum of the forces in the y' direction.

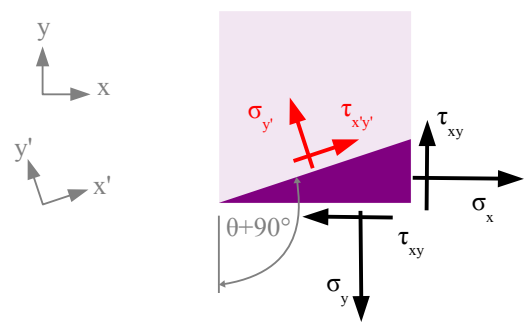


$$\begin{aligned}\tau_{x'y'}a = &-\sigma_x a\cos\theta\sin\theta + \sigma_y a\sin\theta\cos\theta \\ &-\tau_{xy} a\sin^2\theta + \tau_{xy} a\cos^2\theta\end{aligned}$$

Area a cancels. Using the trig identities, we can simplify the equation:

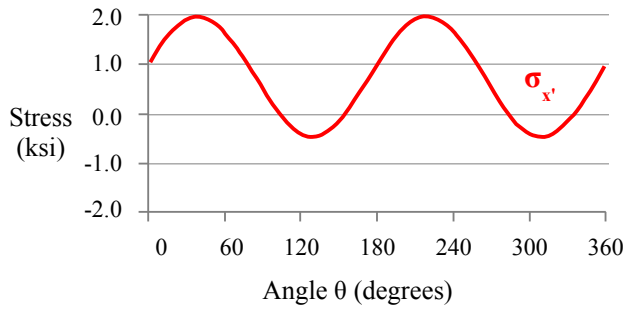
$$\begin{aligned}\tau_{x'y'} &= (-\sigma_x + \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\ &= -(\sigma_x - \sigma_y)\frac{\sin 2\theta}{2} + \tau_{xy}\cos 2\theta \\ &= \frac{-(\sigma_x-\sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}$$

We can calculate the stress in the y' direction by taking angle $\theta + 90°$. Using free-body diagrams and trig identities, we get:



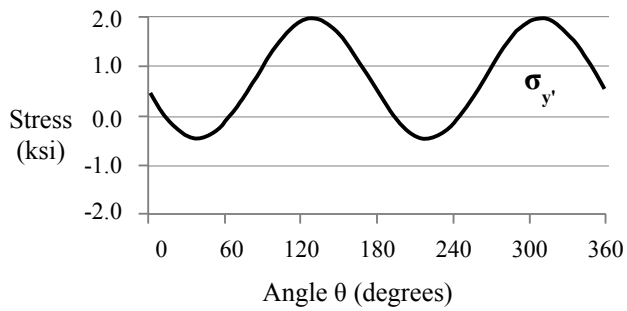
$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

Now we have three equations that are functions of σ_x, σ_y, and τ_{xy}, and θ. Let's pick values of normal and shear stresses, and see how $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ vary by plotting stress vs. angle θ. The graph at the right shows $\sigma_{x'}$ as a function of angle θ if we pick $\sigma_x = 1\text{ ksi}$, $\sigma_y = 0.5\text{ ksi}$, and $\tau_{xy} = 1.2\text{ ksi}$. The graph is a sine wave, with peaks, valleys, and zero values. Picking different values of σ_x, σ_y, and τ_{xy} will change the amplitude and vertical position of the sine wave, but not the period.



Notice that the maximum values of $\sigma_{x'}$ occur at $\theta = 40°$ and 220°, the minimum values occur at $\theta = 130°$ and 310°, and $\sigma_{x'} = 0$ at $\theta = 100°$, 155°, 283°, and 385°.

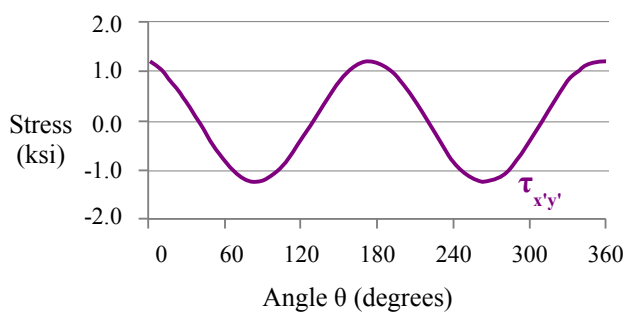
This graph shows $\sigma_{y'}$ as a function of angle θ for the same set of σ_x, σ_y, and τ_{xy}.



The maximum values of $\sigma_{y'}$ occur at $\theta = 130°$ and 310°, the minimum values occur at $\theta = 40°$ and 220°, and $\sigma_{y'} = 0$ at $\theta = 13°$, 65°, 193°, and 235°.

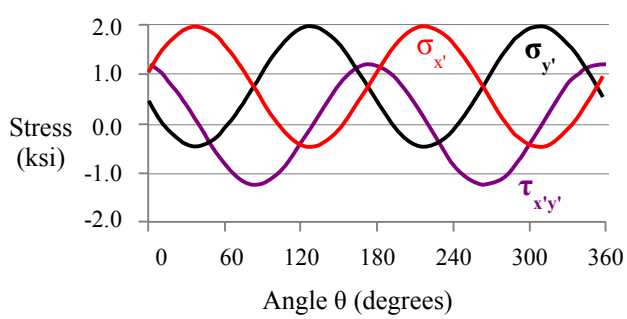
This graph of $\sigma_{y'}$ is shifted 90° to the right of the graph of $\sigma_{x'}$; as a result, you can find the angles for the maximum values of $\sigma_{y'}$ by adding 90° to the angles for the maximum values of $\sigma_{x'}$.

This graph shows $\tau_{x'y'}$ as a function of angle θ for the same set of σ_x, σ_y, and τ_{xy}.



The maximum values of $\tau_{x'y'}$ occur at $\theta = 175°$ and 355°, shifted 45° from the maximum values of σ_x. The minimum values of $\tau_{x'y'}$ occur at $\theta = 85°$ and 265°, and $\tau_{x'y'} = 0$ at $\theta = 39°$, 129°, 219°, and 309°.

We can see how the three stresses interrelate by plotting them together. There are normal stresses for which the shear stress is zero, and there are shear stresses for which the normal stress is zero. The values and angles depend on the input conditions (σ_x, σ_y, and τ_{xy}).



Let's take another look at the equations for $\sigma_{x'}$ and $\tau_{x'y'}$:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Rewrite the $\sigma_{x'}$ equation, putting two terms on the left:

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Square both sides of the $\sigma_{x'}$ and $\tau_{x'y'}$ equations:

$$\left(\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}\right)^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 2\theta+(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta+\tau_{xy}^2\sin^2 2\theta$$

$$\tau_{x'y'}^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\sin^2 2\theta-(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta+\tau_{xy}^2\cos^2 2\theta$$

Next, add these two equations together:

$$\left(\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}\right)^2+\tau_{x'y'}^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 2\theta+(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta+\tau_{xy}^2\sin^2 2\theta$$
$$+\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\sin^2 2\theta-(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta+\tau_{xy}^2\cos^2 2\theta$$

We can eliminate two terms because $(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta-(\sigma_x-\sigma_y)\tau_{xy}\sin 2\theta\cos 2\theta=0$. Rewriting the equation, we have:

$$\left(\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}\right)^2+\tau_{x'y'}^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2(\cos^2 2\theta+\sin^2 2\theta)+\tau_{xy}^2(\sin^2 2\theta+\cos^2 2\theta)$$

Using the trig identity $\cos^2 2\theta+\sin^2 2\theta=1$, the equation simplifies to $\left(\sigma_{x'}-\frac{\sigma_x+\sigma_y}{2}\right)^2+\tau_{x'y'}^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2$.

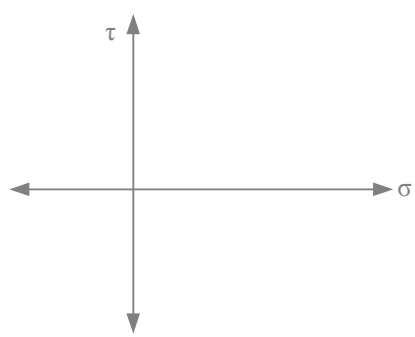
The term $\frac{\sigma_x+\sigma_y}{2}$ is an average of two stress values, so let's call it σ_{avg}. Now the equation is even simpler:

$(\sigma_{x'}-\sigma_{avg})^2+\tau_{x'y'}^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2$. If we define the right side of the equation as $R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$ then $(\sigma_{x'}-\sigma_{avg})^2+\tau_{x'y'}^2=R^2$. This is the equation of a circle with a radius R and a center at coordinates $(\sigma_{avg},0)$. In 1880, Otto Mohr developed this method for visualizing stresses in two-dimensional and three-dimensional objects. His graphical method is also used for diagramming strains. We can use Mohr's Circle to determine the value and direction of maximum stresses within a loaded part.

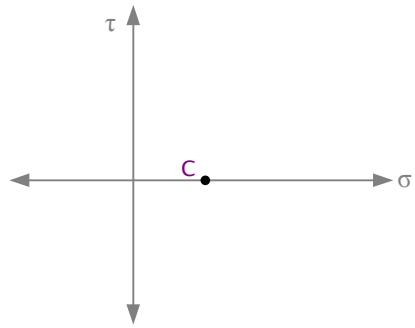
Follow this 6-step process for drawing Mohr's circle for the stresses acting at a point within a stressed object. For best results, use a compass and straightedge on graph paper.

In this example, use the same stresses as before:
$\sigma_x=1\,\text{ksi}$, $\sigma_y=0.5\,\text{ksi}$, and $\tau_{xy}=1.2\,\text{ksi}$.

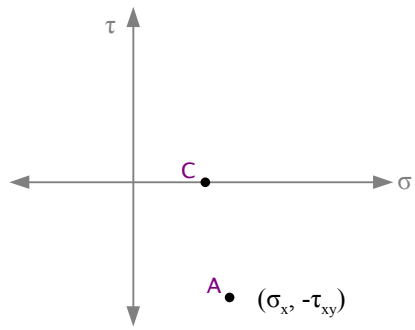
Step 1 Draw two axes. Label the horizontal axis σ and the vertical axis τ.



Step 2 Mark the center of the circle at coordinates $(\sigma_{avg}, 0)$ and label it **C** for center. In this case, $\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{1\,\text{ksi} + 0.5\,\text{ksi}}{2} = 0.75\,\text{ksi}$.



Step 3 Mark point $(\sigma_x, -\tau_{xy})$ on the graph; in this case, $(1\,\text{ksi}, -1.2\,\text{ksi})$. Label this point **A**.

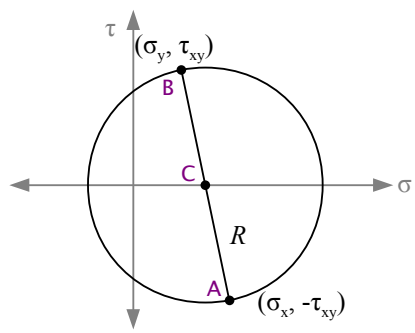


Step 4 Placing your compass point at the center of the circle, draw a circle passing through this point. The radius of Mohr's circle is:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{1\,\text{ksi} - 0.5\,\text{ksi}}{2}\right)^2 + (1.2\,\text{ksi})^2} = 1.226\,\text{ksi}$$

Notice the circle also passes through point (σ_y, τ_{xy}) , which is $(0.5\,\text{ksi}, 1.2\,\text{ksi})$. Mark this point and label it **B**. A line passing through points **A** and **B** is the diameter of Mohr's circle.



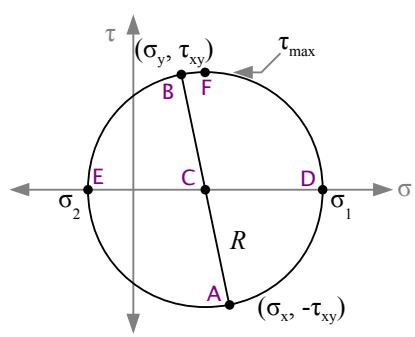
Step 5 The maximum normal stress occurs at point **D**; we call it σ_1. The minimum normal stress occurs at point **E**; we call it σ_2. The maximum shear stress occurs at point **F**; we call this τ_{max}. These three stresses are called *Principal Stresses*.

You can measure these values off the graph, or you can calculate them:

$$\sigma_1 = \sigma_{avg} + R = 0.75\,\text{ksi} + 1.226\,\text{ksi} = 1.976\,\text{ksi}$$

$$\sigma_2 = \sigma_{avg} - R = 0.75\,\text{ksi} - 1.226\,\text{ksi} = -0.476\,\text{ksi}$$

$$\tau_{max} = R = 1.226\,\text{ksi}$$



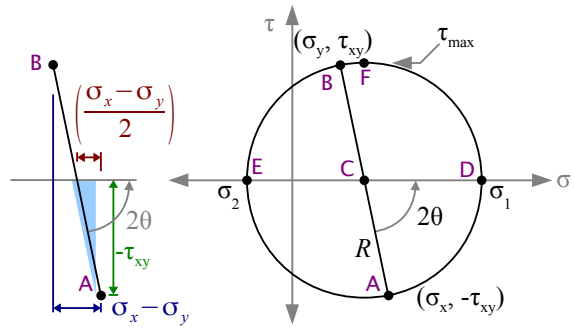
Step 6 Starting on line segment **A-C**, measure the angle counterclockwise about point **C** to line segment **C-D**. This angle is 2θ.

$$\tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(1.2\text{ ksi})}{1\text{ ksi} - 0.5\text{ ksi}} = 4.8$$

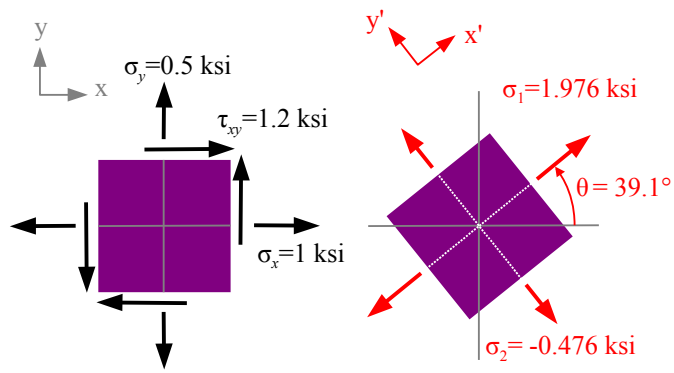
$$2\theta = \tan^{-1} 4.8 = 78.2°$$

$$\theta = \frac{78.2°}{2} = 39.1°$$

This is the angle of the principal stress σ_1. The angle of principal stress σ_2 is $\theta = 90° + 39.1° = 129.1°$.



Going back to the original stress element, we had $\sigma_x = 1\text{ ksi}$, $\sigma_y = 0.5\text{ ksi}$, and $\tau_{xy} = 1.2\text{ ksi}$. If we rotate the element counterclockwise 39.1°, in this orientation there are no shear stresses, and the principal stresses are $\sigma_1 = 1.976\text{ ksi}$ tension and $\sigma_2 = -0.476\text{ ksi}$ compression. The maximum shear stress $\tau_{max} = 1.226\text{ ksi}$ is oriented 45° from the maximum normal stress, at $\theta = 45° + 39.1° = 84.1°$.

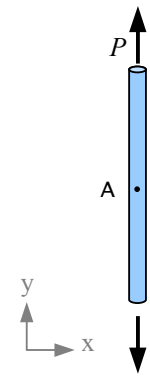


Terms and Equations for Mohr's Circle

σ_x	Applied normal stress in the x direction, which could be from tension or compression (*P*/*A*), bending (*Mc*/*I* or *M*/*Z*), hoop stress ($p\ d_i/2t$), longitudinal stress ($p\ d_i/4t$), etc.
σ_y	Applied normal stress in the y direction, which could be from tension or compression (*P*/*A*), bending (*Mc*/*I* or *M*/*Z*), hoop stress ($p\ d_i/2t$), longitudinal stress ($p\ d_i/4t$), etc.
τ_{xy}	Applied shear stress. Could be from bending (*VQ*/*It*), torsion (*Tc*/*J*), applied shear load (*P*/*A*), etc.
R	Radius of Mohr's circle, equal to maximum shear stress τ_{max} and calculated as $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
Point **A**	Point on Mohr's circle, plotted as (σ_x, $-\tau_{xy}$)
Point **B**	Point on Mohr's circle, plotted as (σ_y, $+\tau_{xy}$)
Point **C**	Point on Mohr's circle, plotted as (σ_{avg}, 0) where $\sigma_{avg} = (\sigma_x + \sigma_y)/2$
Point **D**	Principal stress σ_1 which is the maximum normal stress, calculated as $\sigma_1 = \sigma_{avg} + R$
Point **E**	Principal stress σ_2 which is the minimum normal stress, calculated as $\sigma_2 = \sigma_{avg} - R$
θ	Angle at which the principal stresses act. All angles on Mohr's circle axes are double angles. Calculate as $\tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$. One common mistake is to enter τ_{max} in place of τ_{xy}.
x', y'	Axes along which the principal stresses act.

Example #1

A brass rod with a diameter of 0.5 inches is pulled with a load of 2000 lb. as shown. Calculate the applied stresses σ_x, σ_y, and τ_{xy}. Use Mohr's circle to find the principal stresses σ_1 and σ_2, angle θ, and the maximum shear stress τ_{max} at point **A**.



Solution

The applied normal stress is in the y direction; there is no applied shear stress.

$$\sigma_x = 0 \qquad \sigma_y = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4 \cdot 2000\,\text{lb.}}{\pi (0.5\,\text{in.})^2}\left|\frac{\text{kip}}{10^3\,\text{lb.}}\right. = 10.2\,\text{ksi} \qquad \tau_{xy} = 0$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{0\,\text{ksi} + 10.2\,\text{ksi}}{2} = 5.09\,\text{ksi}$$

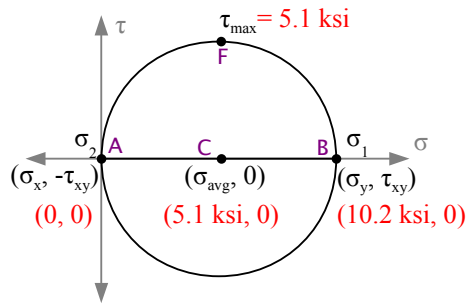
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{0\,\text{ksi} - 10.2\,\text{ksi}}{2}\right)^2 + (0\,\text{ksi})^2} = 5.09\,\text{ksi}$$

$$\sigma_1 = \sigma_{avg} + R = 5.09\,\text{ksi} + 5.09\,\text{ksi} = 10.2\,\text{ksi}$$

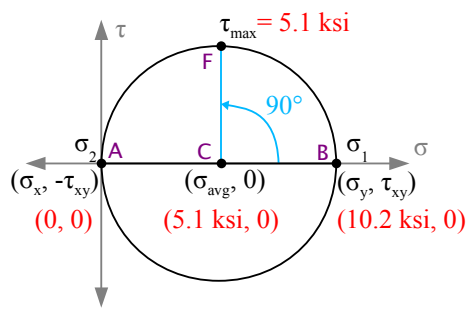
$$\sigma_2 = \sigma_{avg} - R = 5.09\,\text{ksi} - 5.09\,\text{ksi} = 0\,\text{ksi}$$

$$\tau_{max} = R = 5.09\,\text{ksi}$$



Angle $\theta = 0$, by inspection, confirming that the maximum normal stress (principal stress #1) is the normal stress in the y direction.

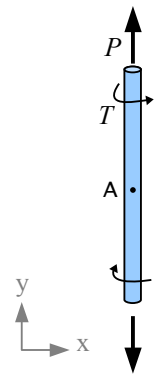
Rotate line segment **B-C** about point **C** to create line segment **C-F** 90° to the principal stress direction. Since all angles in Mohr's circle are double angles, we see that the maximum shear stress acts 45° to the principal stresses.



This photograph of a broken tensile specimen proves the existence of the maximum shear stress acting 45° to the axis of the rod. The specimen is pulled beyond its yield strength. Internally, microscopic voids form in a plane perpendicular to the axis; these voids coalesce to form a crack that grows from the center towards the surface of the rod. At some point, there is not enough material to support the load, and the rod shears at 45°, creating the classic cup-and-cone fracture surface of a ductile tensile specimen.

Example #2

The rod in Example #1 is pulled as before, and also twisted with a torque of 45 ft.ìlb.



Solution

Once again the applied normal stress is in the y direction, but the torque creates an applied shear stress.

$$\sigma_x = 0$$

$$\sigma_y = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4 \cdot 2000\,\text{lb.}}{\pi (0.5\,\text{in.})^2} \left| \frac{\text{kip}}{10^3\,\text{lb.}} \right. = 10.2\,\text{ksi} \quad \text{as before}$$

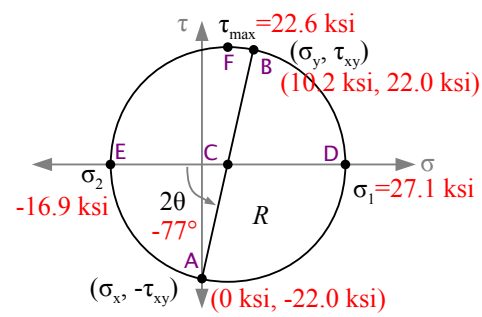
$\tau_{xy} = \dfrac{T\,c}{J}$ where $c = \dfrac{d}{2}$ and $J = \dfrac{\pi d^4}{32}$ (see Chapter 7 – torsion in a round shaft produces a shear stress at the surface)

Substituting, $\tau_{xy} = \dfrac{T}{2}\dfrac{d}{\pi d^4}\dfrac{32}{} = \dfrac{16\,T}{\pi d^3} = \dfrac{16 \quad 45\,\text{ft}\cdot\text{lb.}}{\pi (0.5\,\text{in.})^3} \left| \dfrac{12\,\text{in.}}{\text{ft.}} \right| \dfrac{\text{kip}}{10^3\,\text{lb.}} = 22.0\,\text{ksi}$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{0\,\text{ksi} + 10.2\,\text{ksi}}{2} = 5.09\,\text{ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{0\,\text{ksi} - 10.2\,\text{ksi}}{2}\right)^2 + (22.0\,\text{ksi})^2} = 22.6\,\text{ksi}$$



$$\sigma_1 = \sigma_{avg} + R = 5.09\,\text{ksi} + 22.6\,\text{ksi} = 27.1\,\text{ksi}$$

$$\sigma_2 = \sigma_{avg} - R = 5.09\,\text{ksi} - 22.6\,\text{ksi} = -16.9\,\text{ksi}$$

$$\tau_{max} = R = 22.6\,\text{ksi}$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(22.0\,\text{ksi})}{0\,\text{ksi} - 10.2\,\text{ksi}} = -4.32$$

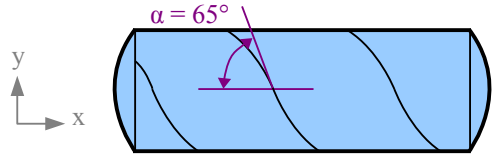
$$2\theta = \tan^{-1} -4.32 = -77.0°$$

$$\theta = \frac{-77.0°}{2} = -38.5°\,; \quad -38.5° + 90° = 51.5°$$

We can also use Mohr's circle to find out the stress in a plane at any angle α. Rotate counterclockwise through angle 2α from line segment **A-C** to a new line segment **A'-C** to find the shear and normal stresses acting in this direction.

Example #3

A 20" diameter cylindrical tank with a ¼" wall thickness is pressurized to 700 psi. If the barrel of the tank is made from a strip of steel welded in a spiral at 65°, what are the normal and shear stresses acting on the weld?



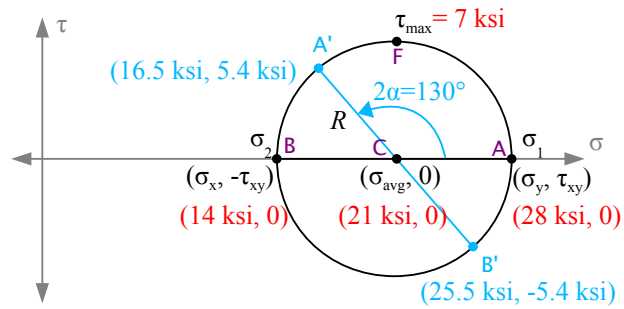
Solution

From Chapter 4,

$$\sigma_x = \sigma_{long} = \frac{p\,d_i}{4t} = \frac{700\,\text{psi}\;\;20\,\text{in.}}{4\;\;0.25\,\text{in.}}\left|\frac{\text{ksi}}{10^3\,\text{psi}}\right. = 14\,\text{ksi}$$

$$\sigma_y = \sigma_{hoop} = \frac{p\,d_i}{2t} = \frac{700\,\text{psi}\;\;20\,\text{in.}}{2\;\;0.25\,\text{in.}}\left|\frac{\text{ksi}}{10^3\,\text{psi}}\right. = 28\,\text{ksi}$$

$$\tau_{xy} = 0$$



Now solve the Mohr's circle equations.

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{14\,\text{ksi} + 28\,\text{ksi}}{2} = 21\,\text{ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{14\,\text{ksi} - 28\,\text{ksi}}{2}\right)^2 + (0\,\text{ksi})^2} = 7\,\text{ksi}$$

$$\sigma_1 = \sigma_{avg} + R = 21\,\text{ksi} + 7\,\text{ksi} = 28\,\text{ksi}$$

$$\sigma_2 = \sigma_{avg} - R = 21\,\text{ksi} - 7\,\text{ksi} = 14\,\text{ksi}$$

$$\tau_{max} = R = 7\,\text{ksi}$$

Angle $\theta = 0$, by inspection.

Rotate counterclockwise through angle $2\alpha = 2 \cdot 65° = 130°$ from line segment **A-C** to a new line segment **A'-C**. The normal stresses acting perpendicular and parallel to the weld are $\sigma_{avg} \pm R\cos 2\alpha = 21\,\text{ksi} \pm 7\,\text{ksi}\cos(130°) = 16.5\,\text{ksi}\ \&\ 25.5\,\text{ksi}$. The shear stress acting along the weld $R\sin 2\alpha = 7\,\text{ksi}\sin(130°) = 5.4\,\text{ksi}$.

Since the normal stress acting on the spiral weld is 91% of the maximum normal stress (the hoop stress), the weld efficiency can be as low as 91% before the tank will fail at a weld.

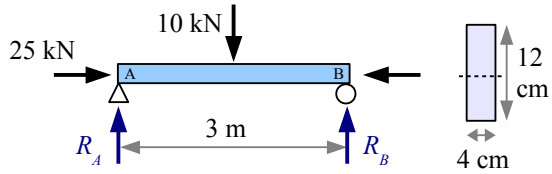
If the tank were manufactured like welded pipe, with a longitudinal weld instead of a spiral weld, then anything less than 100% weld efficiency would cause the tank to fail at the weld instead of splitting in the base metal from excessive hoop stress.

The end cap welds can have as little as 50% weld efficiency and still survive, because the longitudinal stress is 50% of the hoop stress.

Chapter 9 discussed normal and shear stresses in beams. We can use Mohr's circle to look at the normal and shear stresses at any location within a beam.

Example #4

A 4 cm × 12 cm titanium bar is loaded as a beam and as a compression member. Find the stresses in the bar 1 cm above the neutral axis, at the midspan.



Solution

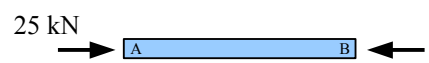
First, determine whether the bar will fail in buckling. From Chapter 14, $P_{cr}=\dfrac{\pi^2 E I}{L^2}$.

If the load is greater than P_{cr}, then the bar fails in buckling. The moment of inertia about the weak axis is $I=\dfrac{12\,\text{cm}(4\,\text{cm})^3}{12}=64\,\text{cm}^4$, therefore $P_{cr}=\dfrac{\pi^2 114\,\text{GPa}\;\;64\,\text{cm}^4}{(3\,\text{m})^2}\left|\dfrac{10^6\,\text{kN}}{\text{GPa}\;\;\text{m}^2}\right|\dfrac{\text{m}^4}{(100\,\text{cm})^4}=80\,\text{kN}$. Since the applied axial load is less than 80 kN, the bar does not buckle.

The axial stress in the bar is

$$\sigma_{x\,axial}=\frac{P}{A}=\frac{-25\,\text{kN}}{4\,\text{cm}\cdot 12\,\text{cm}}\left|\frac{\text{MPa m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=-5.2\,\text{MPa}$$

Axial load only

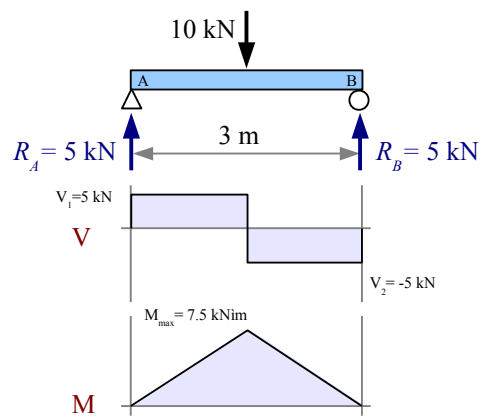


The bending stress in the bar at the top or bottom surface is $\sigma_{x\,bending}=\dfrac{M\,c}{I}$, but we need the bending stress at a location 1 cm above the neutral axis, so $\sigma_{x\,bending}=\dfrac{M\,y}{I}$ where $y=1\,\text{cm}$. The beam is being bent in the strong direction, so

$$I=\frac{4\,\text{cm}(12\,\text{cm})^3}{12}=576\,\text{cm}^4$$

$$\sigma_{x\,bending}=\frac{7.5\,\text{kN}\cdot\text{m}\;\;1\,\text{cm}}{576\,\text{cm}^4}\left|\frac{\text{MPa m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^3}{\text{m}^3}=13.0\,\text{MPa}$$

Bending load only



Add the axial and bending stresses together to find the total stress in the x direction.

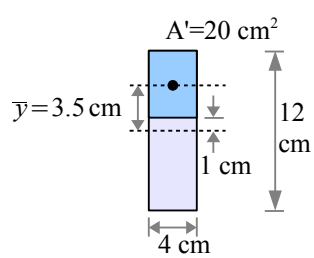
$$\sigma_{x\,total}=\sigma_{x\,axial}+\sigma_{x\,bending}=-5.2\,\text{MPa}+13.0\,\text{MPa}=7.8\,\text{MPa}$$

$$\sigma_y=0$$

Shear stress in the beam is $\tau_{xy}=\dfrac{V\,Q}{I\,t}$ where

$$Q=A'\,\bar{y}=(4\,\text{cm}\times 5\,\text{cm})3.5\,\text{cm}=70\,\text{cm}^3$$

$$\tau_{xy}=\frac{5\,\text{kN}\;\;70\,\text{cm}^3}{576\,\text{cm}^4\;\;4\,\text{cm}}\left|\frac{\text{MPa}\;\;\text{m}^2}{10^3\,\text{kN}}\right|\frac{(100\,\text{cm})^2}{\text{m}^2}=1.5\,\text{MPa}$$



Example #4, continued

Now solve the Mohr's circle equations.

$$\sigma_{avg}=\frac{\sigma_x+\sigma_y}{2}=\frac{7.8\,\text{MPa}+0\,\text{MPa}}{2}=3.9\,\text{MPa}$$

$$R=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\;=\;\sqrt{\left(\frac{7.8\,\text{MPa}-0\,\text{MPa}}{2}\right)^2+(1.5\,\text{MPa})^2}\;=\;4.2\,\text{MPa}$$

$$\sigma_1=\sigma_{avg}+R=3.9\,\text{MPa}+4.2\,\text{MPa}=8.1\,\text{MPa}$$

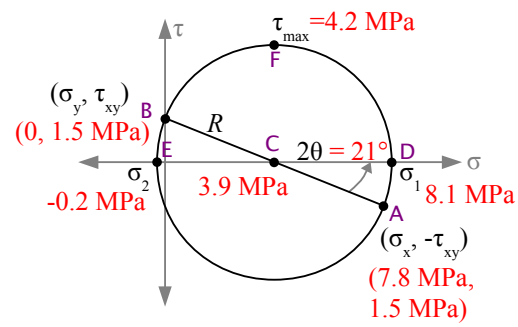
$$\sigma_2=\sigma_{avg}-R=3.9\,\text{MPa}-4.2\,\text{MPa}=-0.2\,\text{MPa}$$

$$\tau_{max}=R=4.2\,\text{MPa}$$

$$\tan 2\theta=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}=\frac{2(1.5\,\text{MPa})}{7.8\,\text{MPa}-0}=0.385$$

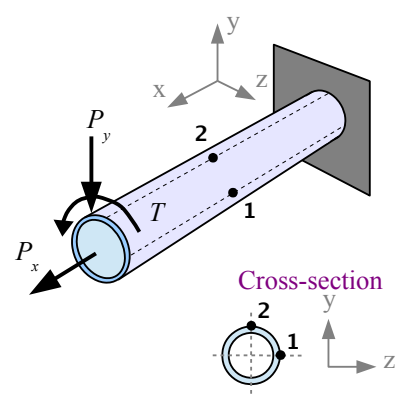
$$2\theta=\tan^{-1}(0.385)=21.0\,°$$

$$\theta=\frac{21.0\,°}{2}=10.5\,°;\;\;10.5\,°+90\,°=100.5\,°$$



Example #5

A 6 ft. long, 3 in. diameter thin-walled hollow tube has a wall thickness of 0.060 in. The tube is loaded in tension, torsion and bending, as shown, and it is pressurized to 450 psi. $P_x=600\,\text{lb.}$, $P_y=150\,\text{lb.}$ and $T=75\,\text{ft.}\cdot\text{lb.}$. Points **1** and **2** are on the surface of the tube at the midspan (3 ft. from the wall), with point **1** at the neutral axis and point **2** at the top.



The x direction is horizontal along the axis of the pipe, the y direction is vertical, and the z direction is horizontal and transverse to the axis of the pipe.

Solution Calculate the normal and shear stresses due to the applied loads, then add up the terms. Horizontal applied load P_x creates a tensile stress $\sigma_x=P_x/A$ at points **1** and **2**. Vertical applied load P_y creates a a shear stress $\tau_{xy}=VQ/It$ at point **1** and a bending stress $\sigma_x=Mc/I$ at point **2**. Torque T creates a shear stress $\tau_{xy}=Tc/J$ at point **1** and a shear stress $\tau_{xz}=Tc/J$ at point **2**. Pressure p creates a hoop stress $\sigma_y=pd_i/2t$ at point **1**, a hoop stress $\sigma_z=pd_i/2t$ at point **2**, and a longitudinal stress $\sigma_x=pd_i/4t$ at points **1** and **2**.

After combining the σ_x terms, the σ_y terms, the σ_z terms, the τ_{xy} terms, and the τ_{xz} terms. Draw Mohr's circle at for **1** using σ_x, σ_y, and τ_{xy}. Draw Mohr's circle for point **2** using σ_x, σ_z, and τ_{xz}.

Example #5, continued

First, calculate quantities that will be needed in multiple equations.

Inside diameter $d_i = d_o - 2t = 3\text{ in.} - 2\times 0.060\text{ in.} = 2.88\text{ in.}$

Dimension $c = \frac{d_o}{2} = \frac{3\text{ in.}}{2} = 1.5\text{ in.}$

Cross-sectional area $A = \frac{\pi}{4}\left[d_o^2 - d_i^2\right] = \frac{\pi}{4}\left[(3\text{ in.})^2 - (2.88\text{ in.})^2\right] = 0.5542\text{ in.}^2$

Moment of inertia about the x-x axis $I_x = \frac{\pi}{64}\left[d_o^4 - d_i^4\right] = \frac{\pi}{64}\left[(3\text{ in.})^4 - (2.88\text{ in.})^4\right] = 0.5990\text{ in.}^4$

Polar moment of inertia about the x-x axis $I_x = \frac{\pi}{32}\left[d_o^4 - d_i^4\right] = \frac{\pi}{64}\left[(3\text{ in.})^4 - (2.88\text{ in.})^4\right] = 0.2995\text{ in.}^4$

Stresses due to P_x only

At points **1** and **2**, $\sigma_{x1} = \sigma_{x2} = \frac{P_x}{A} = \frac{600\text{ lb.}}{0.5542\text{ in.}^2} = 1{,}083\text{ psi}$

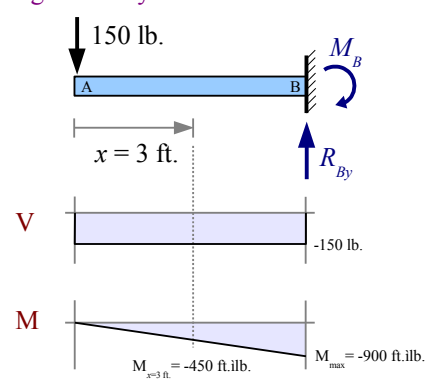
Axial load only

Stresses due to P_y only

There is no bending stress at point **1** because it is at the neutral axis.

To calculate the bending stress at point **2**, we need the bending moment at the midspan of a cantilever beam due to a point load at the free end. Use the Formula Method to find $M_{x=3\text{ ft.}} = P_y x = 150\text{ lb.} \times 3\text{ ft.} = 450\text{ ft.lb.}$

Bending load only



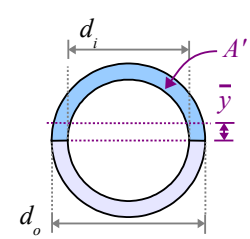
Bending stress $\sigma_{x2} = \frac{M\,c}{I} = \frac{450\text{ ft.lb. }1.5\text{ in.}}{0.5990\text{ in.}^4}\left|\frac{12\text{ in.}}{\text{ft.}}\right. = 13{,}520\text{ psi}$

Vertical load P_y creates a VQ/It shear stress due to bending at point **1** but not at point **2**, where shear stress due to bending is zero.

Point **1** is at the neutral axis, so A' is the cross-sectional area of a half pipe: $A' = \frac{\pi}{8}\left[d_o^2 - d_i^2\right] = \frac{\pi}{8}\left[(3\text{ in.})^2 - (2.88\text{ in.})^2\right] = 0.2271\text{ in.}^2$

The distance from the neutral axis to the center of gravity of a half pipe is

$$\bar{y} = \frac{2\left[d_o^3 - d_i^3\right]}{3\pi\left[d_o^2 - d_i^2\right]} = \frac{2\left[(3\text{ in.})^3 - (2.88\text{ in.})^3\right]}{3\pi\left[3\text{ in.}^2 - (2.88\text{ in.})^2\right]} = 0.9360\text{ in.}$$



$Q = A'\bar{y} = 0.2271\text{ in.}^2 \times 0.9360\text{ in.} = 0.2593\text{ in.}^3$, so $\tau_{xy1} = \frac{V\,Q}{I\,t} = \frac{150\text{ lb. }\ 0.2593\text{ in.}^3}{0.5990\text{ in.}^4\ \ 3\text{ in.}} = 21.65\text{ psi}$

Stresses due to T only

Torque produces the same value of shear stress at points **1** and **2**.

$$\tau_{xy1} = \tau_{xz2} = \frac{T\,c}{J} = \frac{75\text{ ft.lb. }\ 3\text{ in.}}{0.2995\text{ in.}^4}\left|\frac{12\text{ in.}}{\text{ft.}}\right. = 9{,}015\text{ psi}$$

Example #5, continued

Stresses due to *p* only

At both points, longitudinal stress $\sigma_{x1}=\sigma_{x2}=\dfrac{p\,d_i}{4t}=\dfrac{450\text{ lb.}}{\text{in.}^2}\dfrac{3\text{ in.}}{4\ \ 0.060\text{ in.}}=5{,}625\text{ psi}$

At both points, hoop stress $\sigma_{y1}=\sigma_{z2}=\dfrac{p\,d_i}{2t}=\dfrac{450\text{ lb.}}{\text{in.}^2}\dfrac{3\text{ in.}}{4\ \ 0.060\text{ in.}}=11{,}250\text{ psi}$

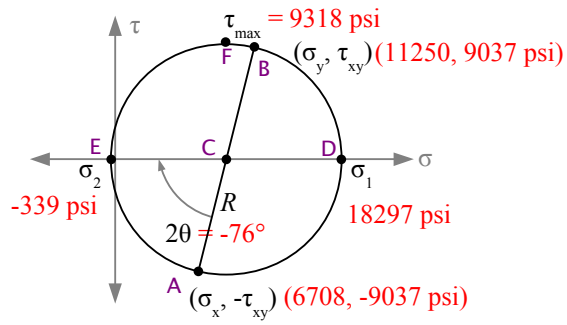
Mohr's Circle for Point 1

$\sigma_{x1}=1{,}083\text{ psi}+5{,}625\text{ psi}=6{,}708\text{ psi}$ $\qquad\sigma_{y1}=11{,}250\text{ psi}$ $\qquad\tau_{xy1}=9{,}015\text{ psi}+22\text{ psi}=9{,}037\text{ psi}$

$$\sigma_{avg1}=\frac{\sigma_{x1}+\sigma_{y1}}{2}=\frac{6{,}708\text{ psi}+11{,}250\text{ psi}}{2}=8{,}979\text{ psi}$$

$$R=\sqrt{\left(\frac{\sigma_{x1}-\sigma_{y1}}{2}\right)^2+\tau_{xy1}^2}$$

$$=\sqrt{\left(\frac{6{,}708\text{ psi}-11{,}250\text{ psi}}{2}\right)^2+(9{,}037\text{ psi})^2}=9{,}318\text{ psi}$$



$\sigma_{P1\#1}=\sigma_{avg1}+R=18{,}297\text{ psi}$ $\qquad\sigma_{P1\#2}=\sigma_{avg1}-R=-339\text{ psi}$

$$\tan 2\theta=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}=\frac{2(9{,}037\text{ psi})}{6{,}708\text{ psi}-11{,}250\text{ psi}}=-3.98$$

$$2\theta=\tan^{-1}-3.98=-76°,\quad \theta=\frac{-76.0°}{2}=-38°,\quad -38°+90°=52°$$

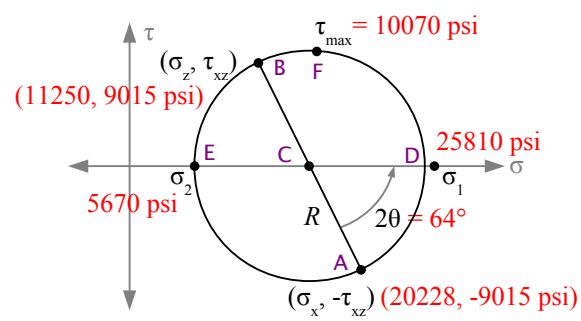
Mohr's Circle for Point 2

$\sigma_{x2}=1{,}083\text{ psi}+13{,}520\text{ psi}+5{,}625\text{ psi}=20{,}228\text{ psi}$ $\qquad\sigma_{z2}=11{,}250\text{ psi}$ $\qquad\tau_{xy2}=9{,}015\text{ psi}$

$$\sigma_{avg2}=\frac{\sigma_{x2}+\sigma_{y2}}{2}=\frac{20{,}228\text{ psi}+11{,}250\text{ psi}}{2}=15{,}740\text{ psi}$$

$$R=\sqrt{\left(\frac{\sigma_{x2}-\sigma_{z2}}{2}\right)^2+\tau_{xz2}^2}$$

$$=\sqrt{\left(\frac{20{,}228\text{ psi}-11{,}250\text{ psi}}{2}\right)^2+(9{,}015\text{ psi})^2}=10{,}070\text{ psi}$$



$\sigma_{P2\#1}=\sigma_{avg2}+R=25{,}810\text{ psi}$ $\qquad\sigma_{P2\#2}=\sigma_{avg2}-R=5{,}670\text{ psi}$

$$\tan 2\theta=\frac{2\tau_{xy}}{\sigma_x-\sigma_z}=\frac{2(9{,}015\text{ psi})}{20{,}228\text{ psi}-11{,}250\text{ psi}}=2.01$$

$$2\theta=\tan^{-1}2.01=64°,\quad \theta=\frac{64°}{2}=32°,\quad 32°+90°=122°$$

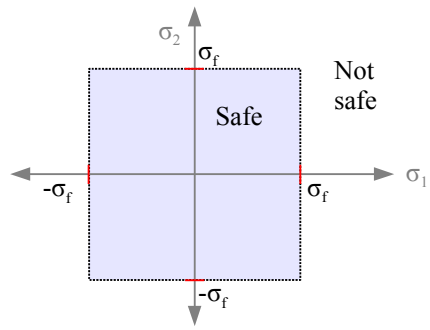
Theories of Failure

Design engineers don't prevent failure, they manage it by designing products and structures so that *when* they fail due to old age, abuse, or ordinary service, nobody is injured. How do we define failure? It depends on the product or structure. A bridge deck could be described as a failure if it deflects too far, even though it does not break. A shear pin in a snowblower

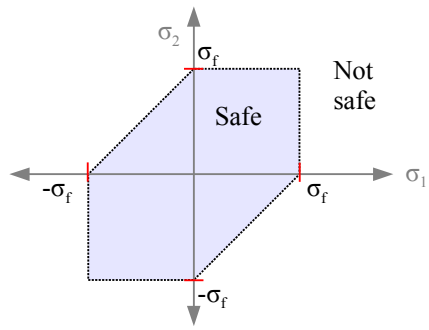
fails if it does *not* break when the blades are overloaded by ice, or *does* break under ordinary snow load.

In Strength of Materials, elastic formulas do not work in the plastic zone, and usually we do not want plastic deformation during service, therefore we can define failure as yielding for most products or structures.

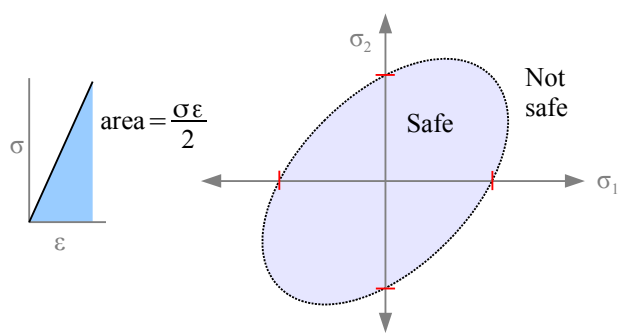
William Rankine, a 19th century Scottish engineer, developed the *Maximum Normal Stress Theory*. He considered materials having the same strength in tension and compression (such as steel, not cast iron). He proposed a failure stress criterion $\sigma_f = \sigma_{P\,max}$ where σ_f is the failure stress and σ_{Pmax} is the maximum principal stress. If we plot principal stresses σ_1 and σ_2 on x and y coordinates, the part is "safe" if the combined stress point is within the square.



Charles-Augustin de Coulomb, a 19th century French physicist, and Henri Tresca, a 19th century French engineer, developed the *Maximum Shear Stress Theory* (also known as the *Tresca Theory*), which defines failure as $\tau_f = \tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$. This theory is more conservative than the *Maximum Normal Stress Theory*, as it cuts off the upper left and lower right corners of the square "safe" zone.



In metals, we have a linear elastic curve on the stress-strain diagram. The area under the curve equals the strain energy per unit volume, $u = \frac{\sigma \varepsilon}{2}$. Richard Edler von Mises was a Jewish mathematician who escaped Austria when the Nazis rose to power, and later joined the faculty at Harvard. He developed the most widely-used failure criterion. From the energy per unit volume equation, he derived: $\sigma_f = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}$ where σ_1 and σ_2 are the principal stresses. The *von Mises* or *Maximum Distortion Energy Theory* produces an elliptical "safe zone".



Notice on all three stress graphs, the "safe" envelopes all pass through the same four points on the two axes, marked in red.

A number of other failure theories apply to special cases, such as the *Maximum Strain Theory* of Barré de Saint-Venant, and the *Internal Friction Theory*. However, *von Mises* is used widely in finite element programs because it fits more data than other methods.

Key Equations

The average normal stress (center of Mohr's circle) is $\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$, and the radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

The principal stresses are $\sigma_1 = \sigma_{avg} + R$ and $\sigma_2 = \sigma_{avg} - R$. The maximum shear stress $\tau_{max} = R$

Find the angle of principal stresses with

$$\tan 2\theta = \left[\frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}\right] \rightarrow 2\theta = \tan^{-1}\left[\frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}\right] \rightarrow \theta = \frac{[2\theta]}{2} \text{ and } 90° + \frac{[2\theta]}{2}$$

Bibliography

Textbooks

Sometimes when you are learning a new topic, it's helpful to look at other textbooks. The Engineering Technology textbooks focus on algebraic and trigonometric solutions, while Engineering textbooks also use calculus. Symbols used in these textbooks vary; authors use different symbols for force (C, F, N, P, Q, T), normal stress/strength (σ, s, f, F, S), shear stress/strength (τ, s_s, S_s), Poisson's ratio (ν, μ), section modulus (S, Z), length (L, l), temperature (T, t), pressure (p, R), radius of curvature (R, ρ), radius of gyration (r, k, q), and stress concentration (k, K, K_{conc}, K_t). Check the table of terminology before using the equations.

The basic theory of Strength of Materials has not changed in many decades. Material properties have improved – structural steel is better than it used to be – and some industry standards for factors of safety have changed. Newer textbooks have full-color diagrams and online resources, including videos and animations.

Engineering Technology Textbooks

George Limbrunner & Craig D'Allaird, *Applied Statics and Strength of Materials, 6th ed.*, Pearson Prentice Hall, 2016.

Fa-Hwa Cheng, *Statics and Strength of Materials, 2nd ed.*, Glencoe/McGraw-Hill, 1997.

Lawrence J. Wolf, *Statics and Strength of Materials: A Parallel Approach to Understanding Structures*, Merrill, 1988.

Ottmar W. Eckardt, *Strength of Materials*, Holt, Rinehart and Winston, 1969.

Alfred Jensen, *Statics and Strength of Materials,* McGraw-Hill, 1962.

Engineering Textbooks

R.C. Hibbeler, *Statics and Mechanics of Materials, 3rd ed.*, Pearson Prentice Hall, 2011.

Madhukar Vable, *Intermediate Mechanics of Materials, 2nd ed.,* Expanding Educational Horizons, 2013.

William Riley, Leroy Sturges, & Don Morris, *Mechanics of Materials, 6th ed.,* John Wiley & Sons, 2007.

Ferdinand Beer, E. Russell Johnston, Jr., John DeWolf, & David Mazurek, *Mechanics of Materials, 7th ed.*, McGraw-Hill, 2014. Beer & Johnston pioneered the design of modern mechanics textbooks, using many examples and drawings.

Joseph Edward Shigley, *Applied Mechanics of Materials,* McGraw-Hill, 1976.

Other Reading Material

A textbook is usually written for a specific course. Other books that are not course-specific are still useful in engineering practice; these include histories of technology, handbooks of design specifications, anthologies of mechanism designs, etc.

Warren Young, Richard Budynas, & Ali Sadegh, *Roark's Formulas for Stress and Strain, 8th ed.,* McGraw-Hill, 2011.

Wood Handbook – 2010 Centennial Edition, General Technical Report FPL-GTR-190. Madison, WI: U.S. Department of Agriculture, Forest Service, Forest Products Laboratory. Available online at www.fpl.fs.fed.us.

Walter Pilkey & Deborah Pilkey, *Peterson's Stress Concentration Factors, 3rd ed.*, Wiley, 2008. The original edition of 1953 contains hundreds of 7"×10" hand-drawn graphs. Most of the data was developed experimentally using photoelastic tests; some graphs are based on mathematical models. Rudolph Peterson was the manager of the Mechanics department at Westinghouse Research Laboratories.

Eugene Avallone, Theodore Baumeister III, & Ali Sadegh, *Mark's Standard Handbook for Mechanical Engineers, 11th ed.*, McGraw-Hill, 2006.

J.E. Gordon, *Structures: Or Why Things Don't Fall Down*, Da Capo Press, 2003

Stephen P. Timoshenko, *History of Strength of Materials,* Dover, 1983 (reprint from McGraw-Hill, 1953).

Appendix A: Units

SI System of Units

The SI system of units (Système International), also commonly called the metric system, uses prefixes to indicate scale. Learn the prefixes, and unit conversion will become easy. For example, 1 kN is 10^3 newtons, 1 MPa is 10^6 pascals, and 1 Pa is 1 N/m^2. If you are converting 600 kN/m^2 to MPa, you can either convert everything to the lowest level of meters and newtons:

$$\frac{600\,\text{kN}}{\text{m}^2}\left|\frac{10^3\,\text{N}}{\text{kN}}\right|\frac{\text{MPa m}^2}{10^6\,\text{N}}=0.6\,\text{MPa}$$

or you can recognize that 1 MPa is 10^3 kPa and take fewer steps:

$$\frac{600\,\text{kN}}{\text{m}^2}\left|\frac{\text{MPa m}^2}{10^3\,\text{kN}}\right.=0.6\,\text{MPa}$$

Prefix	Abbrev.		Multiplier	Literal meaning
nano-	n	=	10^{-9}	dwarf (Greek)
micro-	μ	=	10^{-6}	small (Greek)
milli-	m	=	10^{-3}	thousand (Latin)
centi-	c	=	10^{-2}	hundred (Latin)
kilo-	k	=	10^{3}	thousand (Greek)
Mega-	M	=	10^{6}	great (Greek)
Giga-	G	=	10^{9}	giant (Greek)
Tera-	T	=	10^{12}	monster (Greek)

Quantity	Unit	Symbol	Definition
Length	meter	m	
Mass	gram	g	
Force or Weight	newton	N	$\text{kg}\cdot\text{m/s}^2$
Stress or Pressure	pascal	Pa	N/m^2
Moment or Torque	newton meter	$\text{N}\cdot\text{m}$	

A very useful SI conversion factor for stress: $1\,\text{MPa}=1\,\text{N/mm}^2$

US Customary System of Units

The US Customary system of units, also commonly called the English system, generally does not use prefixes to indicate scale; the exception is "kips" for "kilopounds" (1000 lb.). Instead, we use conversion factors to go from one scale to the next. Most customary unit symbols are abbreviations with periods: "ft." not "ft", "in." not "in"; exceptions include "psi" and "ksi" which have no periods.

In the construction trades, the term "yard" is an abbreviation for "cubic yard", so 3 yards of concrete is actually 3 cubic yards, or 27 cubic feet.

Quantity	Unit	Symbol	Definition
Length	foot	ft.	
Mass	slug	S	
Force or Weight	pound	lb.	
Stress or Pressure	pounds per sq. in.	psi	$\text{lb.}/\text{in.}^2$
Moment or Torque	foot pound	$\text{ft.}\cdot\text{lb.}$	

Unit		Equivalent
1 ft.	=	12 in.
1 yd.	=	3 ft.
1 kip	=	1,000 lb.
1 ksi	=	1,000 psi
1 ton	=	2,000 lb.

Appendix B: Materials Properties

Apply a load to a material and measure the response to obtain *Mechanical Properties* such as yield strength, Young's modulus, elongation, impact resistance, and fatigue life. All other properties are called *Physical Properties*, including thermal expansion, electrical conductivity, density, color, and specific heat. Some properties vary very little, while others have large ranges. For example, Young's modulus for steels varies less than ±1½% because this property is related to bond strength at the atomic level. The yield strength of a grade of steel can vary ±15% due to variations in composition and processing. The weakest steels have a yield strength < 25 ksi, while the strongest have a yield strength > 300 ksi.

Metals, Concrete, & Stone

Table B1: Materials Properties at Room Temperature (U.S. Customary units)

Material	Specific weight	Yield strength (tension)	Ultimate strength (tension)	Ultimate strength (compression)	Ultimate strength (shear)	Young's modulus	Shear modulus	Thermal expansion coefficient	Poisson's ratio
	γ (lb./in.^{3})	σ_{YS} (ksi)	σ_{UTS} (ksi)	σ_{UCS} (ksi)	τ_{ult} (ksi)	E (10^6 psi)	G (10^6 psi)	α (10^{-6} in./(in.°F))	ν -
Steels									
ASTM A36	0.284	36	58	-	-	30	12	6.5	0.25
ASTM A992 (HSLA)	0.284	50	65	-	-	30	12	6.5	0.25
AISI 80X (HSLA)	0.284	80	100	-	-	30	11.6	6.5	0.28
AISI 1020, annealed	0.284	43	57	-	-	30	12	6.8	0.25
AISI 1020, cold rolled	0.284	48	65	-	-	30	12	6.8	0.25
AISI 1040, annealed	0.284	51	75	-	-	30	12	6.3	0.25
AISI 1040, cold rolled	0.284	60	90	-	-	30	12	6.3	0.25
AISI 4140, annealed	0.284	61	95	-	-	30	12	6.8	0.25
AISI 8620, annealed	0.284	56	78	-	-	30	12	6.2	0.25
Stainless steels									
Type 304, annealed	0.290	30	75	-	-	28	12.5	9.6	0.29
Type 304, cold rolled	0.290	110	150	-	-	28	12.5	9.6	0.29
Type 409, annealed	0.280	30	55	-	-	29	11.3	6.5	0.28
Cast irons									
Grade 4 austempered ductile iron	0.260	155	200	-	-	24.4	9.4	5.9	0.25
Class 35 gray cast iron	0.260	-	35	125	48.5	10.9	6.4	7.2	0.29
Aluminum alloys									
2024-T4	0.101	47	68	-	41	10.6	4.0	12.9	0.33
6061-T6	0.098	40	45	-	30	10.0	3.8	13.1	0.33
7075-T6	0.101	73	83	-	48	10.4	3.9	13.1	0.33
Magnesium alloys									
AZ80A, as extruded	0.065	36	49	-	22	6.5	2.4	14.0	0.35
AZ80A, T5 temper	0.065	40	55	-	24	6.5	2.4	14.0	0.35
Copper alloys									
C10800 copper, hard drawn	0.323	32	40	-	26	17.0	6.4	9.4	0.33
C26000 cartridge brass, annealed	0.308	16	48	-	34	16.0	6.0	11.1	0.38
C26000 cartridge brass, rolled	0.308	52	70	-	42	16.0	6.0	11.1	0.38
C63000 aluminum bronze, annealed	0.274	53	111	-	-	17.0	6.4	9.0	0.33
Titanium alloys									
Ti-6Al-4V, annealed	0.160	120	130	-	80	16.5	6.1	4.8	0.34
Ti-6Al-4V, heat treated	0.160	160	170	-	-	16.5	6.1	4.8	0.34
Concrete & Stone									
Unreinforced concrete (range of grades)	0.0868		$0.1\times\sigma_{UCS}$	3 to 4	$0.5\times\sigma_{UCS}$	2.5 to 4		5.5	0.1 to 0.20
Granite (typical)	0.0972			25		7.0		3.6	0.28
Limestone (typical)	0.0961			8 to 16		6.0		2.8	0.21
Sandstone (typical)	0.0903			6		2.5		5.2	0.28

Table B2: Materials Properties at Room Temperature (SI units)

Material	Density ρ (g/cm^3)	Yield strength (tension) σ_{YS} (MPa)	Ultimate strength (tension) σ_{UTS} (MPa)	Ultimate strength (compression) σ_{UCS} (MPa)	Ultimate strength (shear) τ_{ult} (MPa)	Young's modulus E (GPa)	Shear modulus G (GPa)	Thermal expansion coefficient α (10^{-6} mm/(mm°C))	Poisson's ratio ν -
Steels									
ASTM A36	7.85	250	400	-	-	207	83	11.7	0.25
ASTM A992 (HSLA)	7.85	345	450	-	-	207	83	11.7	0.25
AISI 80X (HSLA)	7.85	550	690	-	-	207	80	11.7	0.28
AISI 1020, annealed	7.85	295	395	-	-	207	83	11.7	0.25
AISI 1020, cold rolled	7.85	330	450	-	-	207	83	11.7	0.25
AISI 1040, annealed	7.85	350	520	-	-	207	83	11.3	0.25
AISI 1040, cold rolled	7.85	415	620	-	-	207	83	11.3	0.25
AISI 4140, annealed	7.85	420	655	-	-	207	83	12.3	0.25
AISI 8620, annealed	7.85	385	540	-	-	207	83	11.1	0.25
Stainless steels									
Type 304, annealed	8.00	205	515	-	-	193	86	17.2	0.29
Type 304, cold rolled	8.00	760	1035	-	-	193	86	17.2	0.29
Type 409, annealed	7.80	205	380	-	-	200	78	11.7	0.28
Cast irons									
Grade 4 austempered ductile iron	7.1	1100	1400	-	-	168	65	10.6	0.25
Class 35 gray cast iron	7.1		241	855	334	110	44	13.0	0.29
Aluminum alloys									
2024-T4	2.78	325	470	-	285	73	28.0	23.2	0.33
6061-T6	2.70	275	310	-	205	69	26.0	23.6	0.33
7075-T6	2.80	505	570	-	330	72	26.9	23.6	0.33
Magnesium alloys									
AZ80A, as extruded	1.80	250	340	-	150	45	17	26.0	0.35
AZ80A, T5 temper	1.80	275	380	-	165	45	17	26.0	0.35
Copper alloys									
C10800 copper, hard-drawn	8.94	220	275	-	180	115	44	17.0	0.33
C26000 cartridge brass, annealed	8.53	110	330	-	235	110	40	19.9	0.38
C26000 cartridge brass, rolled	8.53	360	480	-	290	110	40	19.9	0.38
C63000 aluminum bronze, annealed	7.58	370	766			115	44	16.2	0.33
Titanium alloys									
Ti-6Al-4V, annealed	4.43	830	900	-	550	114	42	8.6	0.34
Ti-6Al-4V, heat treated	4.43	1100	1170	-	-	114	42	8.6	0.34
Concrete & Stone									
Unreinforced concrete (range of grades)	2.40		$0.1\times \sigma_{UCS}$	21 to 28	$0.5\times \sigma_{UCS}$	17 to 28		9.9	0.1 to 0.20
Granite (typical)	2.69			170		48		6.5	0.28
Limestone (typical)	2.66			55 to 110		40		5.0	0.21
Sandstone (typical)	2.50			40		17		9.4	0.28

For most metals, the ultimate strength in compression is equal to the ultimate strength in tension.

Concrete, stone, and gray cast iron are significantly stronger in compression than in tension. The elastic portion of their stress-strain curves are not linear, so Young's modulus is not clearly defined for these materials.

The numbers in these tables are either typical or minimum values, for use in homework problems only. Use standards published by ASTM, SAE, et al., for materials property values before designing anything.

Table B3: Materials Properties of Bolts in Bolted Joints

Bolt Specification	τ_{all} (ksi)	τ_{all} (MPa)
ASTM A307 low-carbon steel	12.0	82.5
ASTM A325N, threads in the shear plane	24.0	165
ASTM A325X, threads excluded from the shear plane	30.0	207
ASTM A490N, threads in the shear plane	30.0	207
ASTM A490X, threads excluded from the shear plane	37.5	260

Table B4: Materials Properties of Plates in Bolted Joints or Welded Joints

Plate Material	σ_{YS} (ksi)	σ_{YS} (MPa)	σ_{UTS} (ksi)	σ_{UTS} (MPa)	$\sigma_{P\text{-}all}$ (ksi)	$\sigma_{P\text{-}all}$ (MPa)	$\sigma_{G\text{-}all}$ (ksi)	$\sigma_{G\text{-}all}$ (MPa)	$\sigma_{N\text{-}all}$ (ksi)	$\sigma_{N\text{-}all}$ (MPa)
ASTM A36 low-carbon steel	36	250	58	400	87	600	21.6	150	29	200
ASTM A992 high-strength low-alloy steel	50	345	65	448	97.5	672	30	207	32.5	224

Table B5: Unit Strengths for Fillet Welds

Weld Size (in.)	**E60 electrode** (kip/in.)	**E70 electrode** (kip/in.)
1/16	0.795	0.925
1/8	1.59	1.85
3/16	2.39	2.78
1/4	3.18	3.70
5/16	3.98	4.63
3/8	4.77	5.55
7/16	5.57	6.48
1/2	6.36	7.40
9/16	7.16	8.33
5/8	7.95	9.25
11/16	8.75	10.18
3/4	9.54	11.10
13/16	10.34	12.03
7/8	11.13	12.95

Table B6: AISC Recommended Weld Sizes

Plate Thickness (in.)	**Weld Size** (in.)
1/8	1/8
3/16	1/8 to 3/16
1/4	1/8 to 3/16
5/16	3/16 to 1/4
3/8	3/16 to 5/16
7/16	3/16 to 3/8
1/2	3/16 to 7/16
9/16	1/4 to 1/2
5/8	1/4 to 9/16
11/16	1/4 to 5/8
3/4	1/4 to 11/16
13/16	5/16 to 3/4
7/8	5/16 to 13/16

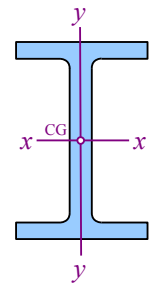
Table B7: Descriptions of Materials and their Common Uses[10]

Steels	
ASTM A36	Low-carbon, low-strength, low-cost rolled structural steel used for W-beams, I-beams, C-channels, etc.
ASTM A992 (HSLA) & AISI 80X (HSLA)	High-strength, low-alloy steels contain grain-refining elements such as boron which create very small grains, resulting in higher strength. Reheating to welding temperatures causes grains to grow, ruining the strength of the steel, therefore bolting or riveting are preferred. Used for structural steel in buildings and large trucks. By definition, few alloying elements are added, keeping costs low.
AISI 1020	Low-carbon steel containing 0.20% carbon, 0.45% manganese. Cold rolling increases strength & decreases ductility.
AISI 1040	Medium-carbon steel containing 0.40% carbon, 0.75% manganese. Added carbon makes the alloy stronger than AISI 1020. Cold rolling further increases strength and decreases ductility.
AISI 4140	Add 0.95% chromium, 0.20% molybdenum, and a little manganese to AISI 1040 steel. Cost and strength increase. Quench & temper heat treatment can raise yield strength above 230 ksi. Used for machine parts.
AISI 8620	Add 0.55% nickel, 0.50% chromium, 0.20% molybdenum, and a little manganese to AISI 1020 steel. Cost and strength increase. Carburize to make a hard, wear-resistant high-carbon surface for gears and bearings.
Stainless steels	
Type 304	Most common stainless steel, with 18% Cr, 8% Ni. Used for food handling equipment (cutlery, pots and pans), tubing, cryogenic equipment. Hardenable by cold work, but not through heat treatment. Not magnetic.
Type 409, annealed	Contains 11% Cr, ½% Ni, therefore cheaper than type 304, with less corrosion resistance. Widely used for stainless exhaust pipes on cars and trucks. Magnetic.
Cast irons	
Grade 4 austempered ductile iron	Contains 3 to 4% carbon as spherical graphite particles which reduce stress concentrations. Austempering heat treatment gives it the strength and wear resistance of steel with the low cost and castability of cast iron. Used for crankshafts and other machine parts.
Class 35 gray cast iron	Contains 2.5 to 4% carbon in the form of graphite flakes which help absorb vibration, but create stress concentrations which reduce strength. Used for engine blocks and machinery bases. The elastic stress-strain curve is not linear, so yield strength is not reported.
Aluminum alloys	
2024-T4	Contains 4.4% copper, 1.5% magnesium, 0.6% manganese. Used widely in the aircraft industry owing to its high strength to weight ratio. T4 heat treatment quadruples yield strength of the soft annealed condition.
6061-T6	Contains 1% Mg, 0.6% Si, 03% Cu, 0.2% Cr. Used in canoes, trucks, pipelines, and other applications where good strength to weight ratio, corrosion resistance, and weldability are needed. T6 heat treat raises yield strength 2½ times vs. annealed condition.
7075-T6	Contains 5.6% zinc, 2.5% magnesium, 1.6% copper, 0.23% chromium. Used widely in the aircraft industry owing to its high strength to weight ratio and corrosion resistance. T6 heat treatment quadruples yield strength.
Magnesium alloys	
AZ80A	Contains 8.5% aluminum, 0.5% zinc. Commonly used as an extrusion. T5 heat treatment increases strength by 10%.
Copper alloys	
C12200 phosphorus deoxidized copper	Contains min. 99.9% copper. Copper tube used for domestic water, medical gas, air conditioning, and refrigeration. Allowable stress depends on the amount of cold work, from 6 ksi (annealed) to 10 ksi (as-drawn) at room temp.
C26000 cartridge brass	Contains 70% Cu, 30% Zn. Cold work and heat treatment substantially improve mechanical properties. Widely used in electrical, hardware, ammunition, plumbing, and automotive industries. Susceptible to stress-corrosion cracking in the presence of ammonia.
C63000 Al bronze	Contains 82% copper, 10% aluminum, 5% nickel, 3% iron. Used for pump parts, valve seats, faucet balls, gears, and cams.
Titanium alloys	
Ti-6Al-4V	Titanium alloys are expensive to produce partly because molten titanium explodes in air. Ti-6-4 is widely used in aerospace because it has half the density of steel, but higher strength and stiffness than aluminum aerospace alloys.
Concrete & Stone	
Unreinforced concrete	10 times stronger in compression than in tension; commonly reinforced with steel to improve tensile strength. Wet curing increases strength as the concrete ages; 90% of strength reached within 28 days of pouring.
Granite	Igneous rock. Hard, strong, durable material used for building exteriors and curbstones. Resistant to acid rain.
Limestone	Sedimentary rock. Softer than granite, used for building exteriors and gravel for paving. Sensitive to acid rain.
Sandstone	Sedimentary rock. Easier to carve than other rock. Used for paving and building exteriors.

10 For more information, consult ASTM Standards, SAE Standards, the ASM Handbook series published by ASM International, the Copper Tube Handbook, and other engineering handbooks.

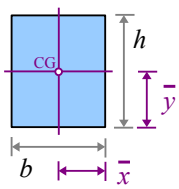
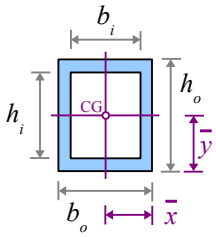
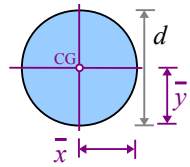
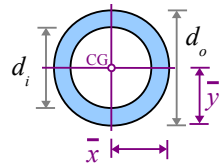
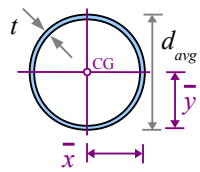
Appendix C: Properties of Areas

Center of Gravity, Area, Moment of Inertia, and Radius of Gyration



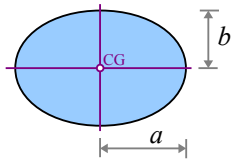
In these diagrams, the polar moment of inertia, J, is about the center of gravity (the small circle marked "CG").

The moments of inertia, I_x and I_y, and radii of gyration, r_{Gx} and r_{Gy}, are about the x-x and y-y neutral axes (horizontal and vertical magenta lines).

Rectangle	Hollow rectangle	Circle	Hollow circle	Thin-walled hollow circle
				
$\bar{x}=\frac{b}{2}$, $\bar{y}=\frac{h}{2}$ $A=b\,h$ $I_x=\frac{b\,h^3}{12}$ $I_y=\frac{h\,b^3}{12}$ $J=\frac{b\,h}{12}(h^2+b^2)$ $r_{Gx}=\frac{h}{\sqrt{12}}$ $r_{Gy}=\frac{b}{\sqrt{12}}$	$\bar{x}=\frac{b_o}{2}$, $\bar{y}=\frac{h_o}{2}$ $A=b_o h_o-b_i h_i$ $I_x=\frac{b_o h_o^3-b_i h_i^3}{12}$ $I_y=\frac{h_o b_o^3-h_i b_i^3}{12}$ $J=I_x+I_y$ $r_{Gx}=\sqrt{\frac{b_o h_o^3-b_i h_i^3}{12\,A}}$ $r_{Gy}=\sqrt{\frac{h_o b_o^3-h_i b_i^3}{12\,A}}$	$\bar{x}=\bar{y}=\frac{d}{2}$ $A=\frac{\pi d^2}{4}$ $I_x=I_y=\frac{\pi d^4}{64}$ $J=\frac{\pi d^4}{32}$ $r_{Gx}=r_{Gy}=\frac{d}{4}$	$\bar{x}=\bar{y}=\frac{d_o}{2}$ $A=\frac{\pi(d_o^2-d_i^2)}{4}$ $I_x=I_y=\frac{\pi(d_o^4-d_i^4)}{64}$ $J=\frac{\pi(d_o^4-d_i^4)}{32}$ $r_{Gx}=r_{Gy}=\frac{\sqrt{d_o^2+d_i^2}}{4}$	If a hollow circle has a thin wall of thickness t, then the area is equal to the circumference times the wall thickness. The cross-section of a copper water pipe is a good example of a thin-walled hollow circle. $d_{avg}=\frac{d_o+d_i}{2}$ $\bar{x}=\bar{y}=\frac{d_{avg}}{2}$ $A=\pi\,d_{avg}\,t$ $I_x=I_y=\frac{\pi\,d_{avg}^3\,t}{8}$ $r_{Gx}=r_{Gy}=\frac{\sqrt{d_{avg}\,t}}{2}$

Appendix C: Properties of Areas

Ellipse



$$A=\pi a b$$

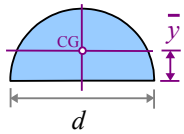
$$I_x=\frac{\pi a b^3}{4}$$

$$I_y=\frac{\pi b a^3}{4}$$

$$J=\frac{\pi a b}{4}\left(a^2+b^2\right)$$

$$r_{Gx}=\frac{b}{2}\ ,\ r_{Gy}=\frac{a}{2}$$

Half circle



$$\bar{y}=\frac{2d}{3\pi}$$

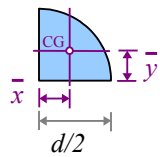
$$A=\frac{\pi d^2}{8}$$

$$I_x=\frac{\pi d^4}{128}\left[1-\left(\frac{8}{3\pi}\right)^2\right]=\frac{d^4}{145.78}$$

$$I_y=\frac{\pi d^4}{128}\ ,\ J=\frac{d^4}{31.84}$$

$$r_{Gx}=\frac{d}{7.566}\ ,\ r_{Gy}=\frac{d}{4}$$

Quarter circle



$$\bar{x}=\bar{y}=\frac{2d}{3\pi}$$

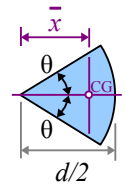
$$A=\frac{\pi d^2}{16}$$

$$I_x=I_y=\frac{d^4}{291.4}$$

$$J=\frac{d^4}{145.7}$$

$$r_{Gx}=r_{Gy}=\frac{d}{7.564}$$

Segment of a circle



$$\bar{x}=\frac{d\sin\theta}{3\theta}$$

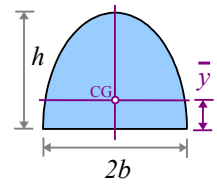
(θ in radians)

$$A=\frac{\theta d^2}{4}$$

$$I_x=\frac{d^4(2\theta-\sin 2\theta)}{128}$$

$$I_y=\frac{d^4(2\theta+\sin 2\theta)}{128}$$

2nd degree parabola



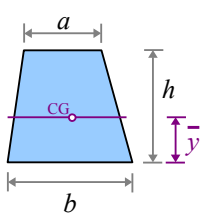
$$\bar{y}=\frac{2h}{5}$$

$$A=\frac{4bh}{3}$$

$$I_x=0.914\,b\,h^3$$

$$I_y=\frac{4hb^3}{15}$$

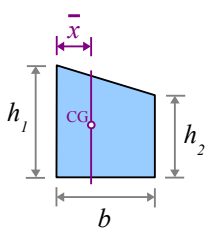
Trapezoid, parallel top and bottom edges



$$\bar{y}=\frac{h}{3}\left(\frac{2a+b}{a+b}\right)$$

$$A=h\left(\frac{a+b}{2}\right)$$

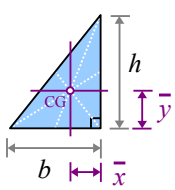
Trapezoid, parallel left and right edges



$$\bar{x}=\frac{b}{3}\left(\frac{h_1+2h_2}{h_1+h_2}\right)$$

$$A=b\left(\frac{h_1+h_2}{2}\right)$$

Right triangle



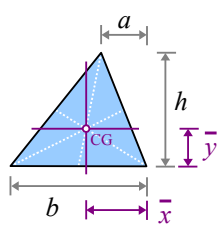
$$\bar{x}=\frac{b}{3}\ ,\ \bar{y}=\frac{h}{3}$$

$$A=\frac{bh}{2}$$

$$I_x=\frac{bh^3}{36}\ ,\ I_y=\frac{hb^3}{36}$$

$$r_{Gx}=\frac{h}{\sqrt{18}}$$

General triangle



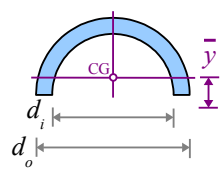
$$\bar{x}=\frac{a+b}{3}\ ,\ \bar{y}=\frac{h}{3}$$

$$A=\frac{bh}{2}$$

$$I_x=\frac{bh^3}{36}$$

$$r_{Gx}=\frac{h}{\sqrt{18}}$$

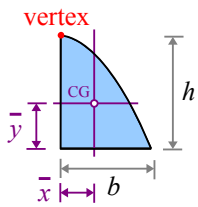
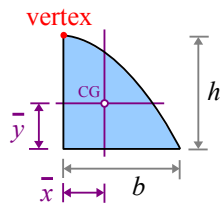
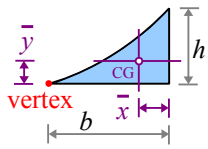
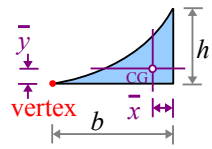
Half pipe



$$\bar{y}=\frac{2\left(d_o^3-d_i^3\right)}{3\pi\left(d_o^2-d_i^2\right)}$$

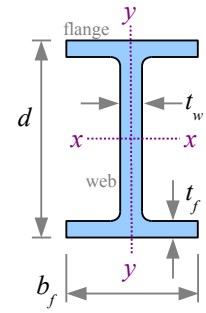
$$A=\frac{\pi}{8}\left(d_o^2-d_i^2\right)$$

$$I_x=\frac{d_o^4-d_i^4}{145.7}-\frac{d_o^2 d_i^2\left(d_o-d_i\right)}{56.5\left(d_o+d_i\right)}$$

2nd degree half parabola, vertex at the top	nth degree half parabola, vertex at the top	2nd degree half parabola, vertex at the left	nth degree half parabola, vertex at the left	
				
$\bar{x}=\frac{3}{8}b$, $\bar{y}=\frac{2}{5}h$ $A=\frac{2}{3}bh$	$\bar{x}=\frac{b(n+1)}{2(n+2)}$ $\bar{y}=\frac{hn}{2n+1}$ $A=\frac{nbh}{n+1}$	$\bar{x}=\frac{b}{4}$, $\bar{y}=\frac{3}{10}h$ $A=\frac{1}{3}bh$	$\bar{x}=\frac{b}{n+2}$ $\bar{y}=h\frac{n+1}{4n+2}$ $A=\frac{bh}{n+1}$	

Appendix D: Properties of Steel Beams and Pipes

W-beams



Steel is hot-rolled to manufacture beams in shapes that look like the letters `I` or `L`. In this diagram, the flanges are the top and bottom horizontal portions of the beam; the web is the vertical portion. `I`-shaped steel beams with wide flanges are called W-beams.

The tables below list geometric properties of hot-rolled steel beams.

U.S. Customary W-beams are designated by the nominal depth and the weight per unit length: a W18×40 beam has a depth of about 18 inches and a weight per unit length of 40 lb./ft.

Table D1: U.S. Customary W-Beams

Beam	Area	Depth	Flange width	Flange thickness	Web thickness	Properties with respect to the x-x neutral axis (moment of inertia, section modulus, radius of gyration, and plastic section modulus)				Properties with respect to the y-y neutral axis		
w (in.)×(lb./ft.)	A (in.^{2})	d (in.)	b_f (in.)	t_f (in.)	t_w (in.)	I_x (in.^{4})	S_x (in.^{3})	r_{Gx} (in.)	Z_x (in.^{3})	I_y (in.^{4})	S_y (in.^{3})	r_{Gy} (in.)
W44×335	98.5	44.0	15.9	1.77	1.03	31100	1410	17.8	1620	1200	150	3.49
W44×290	85.4	43.6	15.8	1.58	0.865	27000	1240	17.8	1410	1040	132	3.49
W44×262	77.2	43.3	15.8	1.42	0.785	24100	1110	17.7	1270	923	117	3.47
W44×230	67.8	42.9	15.8	1.22	0.710	20800	971	17.5	1100	796	101	3.43
W40×397	117	41.0	16.1	2.20	1.22	32000	1560	16.6	1800	1540	191	3.64
W40×297	87.3	39.8	15.8	1.65	0.930	23200	1170	16.3	1330	1090	138	3.54
W40×199	58.8	38.7	15.8	1.07	0.650	14900	770	16.0	869	695	88.2	3.45
W40×149	43.8	38.2	11.8	0.830	0.630	9800	513	15.0	598	229	38.8	2.29
W36×395	116	38.4	16.8	2.20	1.22	28500	1490	15.7	1710	1750	208	3.88
W36×302	89.0	37.3	16.7	1.68	0.945	21100	1130	15.4	1280	1300	156	3.82
W36×194	57.0	36.5	12.1	1.26	0.765	12100	664	14.6	767	375	61.9	2.56
W36×150	44.3	35.9	12.0	0.940	0.625	9040	504	14.3	581	270	45.1	2.47
W36×135	39.9	35.6	12.0	0.790	0.600	7800	439	14.0	509	225	37.7	2.38
W33×387	114	36.0	16.2	2.28	1.26	24300	1350	14.6	1560	1620	200	3.77
W33×201	59.1	33.7	15.7	1.15	0.715	11600	686	14.0	773	749	95.2	3.56
W33×130	38.3	33.1	11.5	0.855	0.580	6710	406	13.2	467	218	37.9	2.39
W33×118	34.7	32.9	11.5	0.740	0.550	5900	359	13.0	415	187	32.6	2.32
W30×391	115	33.2	15.6	2.44	1.36	20700	1250	13.4	1450	1550	198	3.67
W30×292	86.0	32.0	15.3	1.85	1.02	14900	930	13.2	1060	1100	144	3.58
W30×211	62.3	30.9	15.1	1.32	0.775	10300	665	12.9	751	757	100	3.49
W30×148	43.6	30.7	10.5	1.18	0.650	6680	436	12.4	500	227	43.3	2.28
W30×90	26.3	29.5	10.4	0.610	0.470	3610	245	11.7	283	115	22.1	2.09
W27×368	109	30.4	14.7	2.48	1.38	16200	1060	12.2	1240	1310	179	3.48
W27×281	83.1	29.3	14.4	1.93	1.06	11900	814	12.0	936	953	133	3.39
W27×194	57.1	28.1	14.0	1.34	0.750	7860	559	11.7	631	619	88.1	3.29
W27×146	43.2	27.4	14.0	0.975	0.605	5660	414	11.5	464	443	63.5	3.20
W27×114	33.6	27.3	10.1	0.930	0.570	4080	299	11.0	343	159	31.5	2.18
W27×84	24.7	26.7	10.0	0.640	0.460	2850	213	10.7	244	106	21.2	2.07
W24×370	109	28.0	13.7	2.72	1.52	13400	957	11.1	1130	1160	170	3.27
W24×279	81.9	26.7	13.3	2.09	1.16	9600	718	10.8	835	823	124	3.17
W24×192	56.5	25.5	13.0	1.46	0.810	6260	491	10.5	559	530	81.8	3.07
W24×104	30.7	24.1	12.8	0.750	0.500	3100	258	10.1	289	259	40.7	2.91
W24×76	22.4	23.9	8.99	0.680	0.440	2100	176	9.69	200	82.5	18.4	1.92
W24×55	16.2	23.6	7.01	0.505	0.395	1350	114	9.11	134	29.1	8.30	1.34
W21×201	59.3	23.0	12.6	1.63	0.910	5310	461	9.47	530	542	86.1	3.02
W21×166	48.8	22.5	12.4	1.36	0.750	4280	380	9.36	432	435	70.0	2.99
W21×147	43.2	22.1	12.5	1.15	0.720	3630	329	9.17	373	376	60.1	2.95
W21×83	24.4	21.4	8.36	0.835	0.515	1830	171	8.67	196	81.4	19.5	1.83
W21×62	18.3	21.0	8.24	0.615	0.400	1330	127	8.54	144	57.5	14.0	1.77
W21×50	14.7	20.8	6.53	0.535	0.380	984	94.5	8.18	110	24.9	7.64	1.30

Table D1: U.S. Customary W-Beams (continued)

Beam	Area	Depth	Flange width	Flange thickness	Web thickness	Properties with respect to the x-x neutral axis (moment of inertia, section modulus, radius of gyration, and plastic section modulus)				Properties with respect to the y-y neutral axis		
w	A	d	b_f	t_f	t_w	I_x	S_x	r_{Gx}	Z_x	I_y	S_y	r_{Gy}
(in.)×(lb./ft.)	(in.^{2})	(in.)	(in.)	(in.)	(in.)	(in.^{4})	(in.^{3})	(in.)	(in.^{3})	(in.^{4})	(in.^{3})	(in.)
W18×311	91.6	22.3	12.0	2.74	1.52	6970	624	8.72	754	795	132	2.95
W18×211	62.3	20.7	11.6	1.91	1.06	4330	419	8.35	490	493	85.3	2.82
W18×130	38.3	19.3	11.2	1.20	0.670	2460	256	8.03	290	278	49.9	2.70
W18×86	25.3	18.4	11.1	0.770	0.480	1530	166	7.77	186	175	31.6	2.63
W18×60	17.6	18.2	7.56	0.695	0.415	984	108	7.47	123	50.1	13.3	1.68
W18×35	10.3	17.7	6.00	0.425	0.300	510	57.6	7.04	66.5	15.3	5.12	1.22
W16×100	29.4	17.0	10.4	0.985	0.585	1490	175	7.10	198	186	35.7	2.51
W16×67	19.6	16.3	10.2	0.665	0.395	954	117	6.96	130	119	23.2	2.46
W16×40	11.8	16.0	7.00	0.505	0.305	518	64.7	6.63	73.0	28.9	8.25	1.57
W16×31	9.13	15.9	5.53	0.440	0.275	375	47.2	6.41	54.0	12.4	4.49	1.17
W16×26	7.68	15.7	5.50	0.345	0.250	301	38.4	6.26	44.2	9.59	3.49	1.12
W14×159	46.7	15.0	15.6	1.19	0.745	1900	254	6.38	287	748	96.2	4.00
W14×99	29.1	14.2	14.6	0.780	0.485	1110	157	6.17	173	402	55.2	3.71
W14×82	24.0	14.3	10.1	0.855	0.510	881	123	6.05	139	148	29.3	2.48
W14×38	11.2	14.1	6.77	0.515	0.310	385	54.6	5.87	61.5	26.7	7.88	1.55
W14×22	6.49	13.7	5.00	0.335	0.230	199	29.0	5.54	33.2	7.00	2.80	1.04
W12×152	44.7	13.7	12.5	1.40	0.870	1430	209	5.66	243	454	72.8	3.19
W12×106	31.2	12.9	12.2	0.990	0.610	933	145	5.47	164	301	49.3	3.11
W12×72	21.1	12.3	12.0	0.670	0.430	597	97.4	5.31	108	195	32.4	3.04
W12×50	14.6	12.2	8.08	0.640	0.370	391	64.2	5.18	71.9	56.3	13.9	1.96
W12×40	11.7	11.9	8.01	0.515	0.295	307	51.5	5.13	57.0	44.1	11.0	1.94
W12×30	8.79	12.3	6.52	0.440	0.260	238	38.6	5.21	43.1	20.3	6.24	1.52
W12×19	5.57	12.2	4.01	0.350	0.235	130	21.3	4.82	24.7	3.76	1.88	0.822
W12×14	4.16	11.9	3.97	0.225	0.200	88.6	14.9	4.62	17.4	2.36	1.19	0.753
W10×100	29.3	11.1	10.3	1.12	0.680	623	112	4.60	130	207	40.0	2.65
W10×60	17.7	10.2	10.1	0.680	0.420	341	66.7	4.39	74.6	116	23.0	2.57
W10×49	14.4	10.0	10.0	0.560	0.340	272	54.6	4.35	60.4	93.4	18.7	2.54
W10×39	11.5	9.92	7.99	0.530	0.315	209	42.1	4.27	46.8	45.0	11.3	1.98
W10×30	8.84	10.5	5.81	0.510	0.300	170	32.4	4.38	36.6	16.7	5.75	1.37
W10×19	5.62	10.2	4.02	0.395	0.250	96.3	18.8	4.14	21.6	4.29	2.14	0.874
W10×12	3.54	9.87	3.96	0.210	0.190	53.8	10.9	3.90	12.6	2.18	1.10	0.785
W8×67	19.7	9.00	8.28	0.935	0.570	272	60.4	3.72	70.1	88.6	21.4	2.12
W8×40	11.7	8.25	8.07	0.560	0.360	146	35.5	3.53	39.8	49.1	12.2	2.04
W8×31	9.13	8.00	8.00	0.435	0.285	110	27.5	3.47	30.4	37.1	9.27	2.02
W8×24	7.08	7.93	6.50	0.400	0.245	82.7	20.9	3.42	23.1	18.3	5.63	1.61
W8×10	2.96	7.89	3.94	0.205	0.170	30.8	7.81	3.22	8.87	2.09	1.06	0.841
W6×25	7.34	6.38	6.08	0.455	0.320	53.4	16.7	2.70	18.9	17.1	5.61	1.52
W6×15	4.43	5.99	5.99	0.260	0.230	29.1	9.72	2.56	10.8	9.32	3.11	1.45
W6×9	2.68	5.90	3.94	0.215	0.170	16.4	5.56	2.47	6.23	2.20	1.11	0.905
W4×13	3.83	4.16	4.06	0.345	0.280	11.3	5.46	1.72	6.28	3.86	1.90	1.00

Appendix D: Properties of Steel Beams and Pipes

W-beams in SI units are designated by the nominal depth and the *mass* per unit length. See the first part of Chapter 8 for the method to calculate *w*, the *weight* per unit length.

Table D2: SI W-Beams

Beam	Area	Depth	Flange width	Flange thickness	Web thickness	Properties with respect to the x-x neutral axis (moment of inertia, section modulus, radius of gyration, and plastic section modulus)				Properties with respect to the y-y neutral axis		
	A	d	b_f	t_f	t_w	I_x	S_x	r_{Gx}	Z_x	I_y	S_y	r_{Gy}
(mm)×(kg/m)	(mm^2)	(mm)	(mm)	(mm)	(mm)	(10^6 mm^4)	(10^3 mm^3)	(mm)	(10^3 mm^3)	(10^6 mm^4)	(10^3 mm^3)	(mm)
W1100×499	63500	1120	404	45.0	26.2	12900	23100	452	26500	499	2460	88.6
W1100×433	55100	1110	401	40.1	22.0	11200	20300	452	23100	433	2160	88.6
W1100×390	49800	1100	401	36.1	19.9	10000	18200	450	20800	384	1920	88.1
W1100×343	43700	1090	401	31.0	18.0	8660	15900	445	18000	331	1660	87.1
W1000×591	75500	1040	409	55.9	31.0	13300	25600	422	29500	641	3130	92.5
W1000×443	56300	1010	401	41.9	23.6	9660	19200	414	21800	454	2260	89.9
W1000×296	37900	983	401	27.2	16.5	6200	12600	406	14200	289	1450	87.6
W1000×222	28300	970	300	21.1	16.0	4080	8410	381	9800	95.3	636	58.2
W920×588	74800	975	427	55.9	31.0	11900	24400	399	28000	728	3410	98.6
W920×449	57400	947	424	42.7	24.0	8780	18500	391	21000	541	2560	97.0
W920×289	36800	927	307	32.0	19.4	5040	10900	371	12600	156	1010	65.0
W920×223	28600	912	305	23.9	15.9	3760	8260	363	9520	112	739	62.7
W920×201	25700	904	305	20.1	15.2	3250	7190	356	8340	93.7	618	60.5
W840×576	73500	914	411	57.9	32.0	10100	22100	371	25600	674	3280	95.8
W840×299	38100	856	399	29.2	18.2	4830	11200	356	12700	312	1560	90.4
W840×193	24700	841	292	21.7	14.7	2790	6650	335	7650	90.7	621	60.7
W840×176	22400	836	292	18.8	14.0	2460	5880	330	6800	77.8	534	58.9
W760×582	74200	843	396	62.0	34.5	8620	20500	340	23800	645	3240	93.2
W760×434	55500	813	389	47.0	25.9	6200	15200	335	17400	458	2360	90.9
W760×314	40200	785	384	33.5	19.7	4290	10900	328	12300	315	1640	88.6
W760×220	28100	780	267	30.0	16.5	2780	7140	315	8190	94.5	710	57.9
W760×134	17000	749	264	15.5	11.9	1500	4010	297	4640	47.9	362	53.1
W690×548	70300	772	373	63.0	35.1	6740	17400	310	20300	545	2930	88.4
W690×419	53600	744	366	49.0	26.9	4950	13300	305	15300	397	2180	86.1
W690×289	36800	714	356	34.0	19.1	3270	9160	297	10300	258	1440	83.6
W690×217	27900	696	356	24.8	15.4	2360	6780	292	7600	184	1040	81.3
W690×170	21700	693	257	23.6	14.5	1700	4900	279	5620	66.2	516	55.4
W690×125	15900	678	254	16.3	11.7	1190	3490	272	4000	44.1	347	52.6
W610×551	70300	711	348	69.1	38.6	5580	15700	282	18500	483	2790	83.1
W610×415	52800	678	338	53.1	29.5	4000	11800	274	13700	343	2030	80.5
W610×285	36500	648	330	37.1	20.6	2610	8050	267	9160	221	1340	78.0
W610×155	19800	612	325	19.1	12.7	1290	4230	257	4740	108	667	73.9
W610×113	14500	607	228	17.3	11.2	874	2880	246	3280	34.3	302	48.8
W610×82	10500	599	178	12.8	10.0	562	1870	231	2200	12.1	136	34.0
W530×300	38300	584	320	41.4	23.1	2210	7550	241	8690	226	1410	76.7
W530×248	31500	572	315	34.5	19.1	1780	6230	238	7080	181	1150	75.9
W530×219	27900	561	318	29.2	18.3	1510	5390	233	6110	157	985	74.9
W530×123	15700	544	212	21.2	13.1	762	2800	220	3210	33.9	320	46.5
W530×92	11800	533	209	15.6	10.2	554	2080	217	2360	23.9	229	45.0
W530×74	9480	528	166	13.6	9.65	410	1550	208	1800	10.4	125	33.0

Table D2: SI W-Beams (continued)

Beam	Area	Depth	Flange width	Flange thickness	Web thickness	Properties with respect to the x-x neutral axis (moment of inertia, section modulus, radius of gyration, and plastic section modulus)				Properties with respect to the y-y neutral axis		
	A	d	b_f	t_f	t_w	I_x	S_x	r_{Gx}	Z_x	I_y	S_y	r_{Gy}
(mm)×(kg/m)	(mm^2)	(mm)	(mm)	(mm)	(mm)	(10^6 mm^4)	(10^3 mm^3)	(mm)	(10^3 mm^3)	(10^6 mm^4)	(10^3 mm^3)	(mm)
W460×464	59100	566	305	69.6	38.6	2900	10200	221	12400	331	2160	74.9
W460×315	40200	526	295	48.5	26.9	1800	6870	212	8030	205	1400	71.6
W460×193	24700	490	284	30.5	17.0	1020	4200	204	4750	116	818	68.6
W460×128	16300	467	282	19.6	12.2	637	2720	197	3050	72.8	518	66.8
W460×89	11400	462	192	17.7	10.5	410	1770	190	2020	20.9	218	42.7
W460×52	6650	450	152	10.8	7.62	212	944	179	1090	6.37	83.9	31.0
W410×149	19000	432	264	25.0	14.9	620	2870	180	3240	77.4	585	63.8
W410×100	12600	414	259	16.9	10.0	397	1920	177	2130	49.5	380	62.5
W410×60	7610	406	178	12.8	7.75	216	1060	168	1200	12.0	135	39.9
W410×46.1	5890	404	140	11.2	6.99	156	773	163	885	5.16	73.6	29.7
W410×38.8	4950	399	140	8.76	6.35	125	629	159	724	3.99	57.2	28.4
W360×237	30100	381	396	30.2	18.9	791	4160	162	4700	311	1580	102
W360×147	18800	361	371	19.8	12.3	462	2570	157	2830	167	905	94.2
W360×122	15500	363	257	21.7	13.0	367	2020	154	2280	61.6	480	63.0
W360×57.8	7230	358	172	13.1	7.87	160	895	149	1010	11.1	129	39.4
W360×32.9	4190	348	127	8.51	5.84	82.8	475	141	544	2.91	45.9	26.4
W310×226	28800	348	318	35.6	22.1	595	3420	144	3980	189	1190	81.0
W310×158	20100	328	310	25.1	15.5	388	2380	139	2690	125	808	79.0
W310×107	13600	312	305	17.0	10.9	248	1600	135	1770	81.2	531	77.2
W310×74	9420	310	205	16.3	9.40	163	1050	132	1180	23.4	228	49.8
W310×60	7550	302	203	13.1	7.49	128	844	130	934	18.4	180	49.3
W310×44.5	5670	312	166	11.2	6.60	99.1	633	132	706	8.45	102	38.6
W310×28.3	3590	310	102	8.89	5.97	54.1	349	122	405	1.57	30.8	20.9
W310×21	2680	302	101	5.72	5.08	36.9	244	117	285	0.982	19.5	19.1
W250×149	18900	282	262	28.4	17.3	259	1840	117	2130	86.2	655	67.3
W250×89	11400	259	257	17.3	10.7	142	1090	112	1220	48.3	377	65.3
W250×73	9290	254	254	14.2	8.64	113	895	110	990	38.9	306	64.5
W250×58	7420	252	203	13.5	8.00	87.0	690	108	767	18.7	185	50.3
W250×44.8	5700	267	148	13.0	7.62	70.8	531	111	600	6.95	94.2	34.8
W250×28.4	3630	259	102	10.0	6.35	40.1	308	105	354	1.79	35.1	22.2
W250×17.9	2280	251	101	5.33	4.83	22.4	179	99.1	206	0.907	18.0	19.9
W200×100	12700	229	210	23.7	14.5	113	990	94.5	1150	36.9	351	53.8
W200×59	7550	210	205	14.2	9.14	60.8	582	89.7	652	20.4	200	51.8
W200×46.1	5890	203	203	11.0	7.24	45.8	451	88.1	498	15.4	152	51.3
W200×35.9	4570	201	165	10.2	6.22	34.4	342	86.9	379	7.62	92.3	40.9
W200×15	1910	200	100	5.21	4.32	12.8	128	81.8	145	0.870	17.4	21.4
W150×37.1	4740	162	154	11.6	8.13	22.2	274	68.6	310	7.12	91.9	38.6
W150×22.5	2860	152	152	6.60	5.84	12.1	159	65.0	177	3.88	51.0	36.8
W150×13.5	1730	150	100	5.46	4.32	6.83	91.1	62.7	102	0.916	18.2	23.0
W100×19.3	2470	106	103	8.76	7.11	4.70	89.5	43.7	103	1.61	31.1	25.4

Steel Pipes

In the 1920s, iron and steel pipes were available in three thickness grades: Standard Weight, Extra Strong (XS), and Double Extra Strong (XXS). Subsequently, the "schedule" designation system was developed to cover other pipe wall thicknesses. Schedule 40 is the same as Standard Weight; Schedule 80 is the same as Extra-Strong. Pipes are available from Schedule 5 to Schedule 160. Regardless of the Schedule, the OD is identical between nominal sizes, so pipes of different Schedules can thread into the same fittings. The heavier Schedules have thicker walls, therefore smaller IDs.

Table D3: US Customary Steel Pipes

	Nominal diameter	Inside diameter	Outside diameter	Wall thickness	Area	Moment of inertia	Section modulus	Radius of gyration	Polar moment of inertia	Weight per unit length
		d_i	d_o	t	A	I	S	r_G	J	w
	(in.)	(in.)	(in.)	(in.)	(in.^{2})	(in.^{4})	(in.^{3})	(in.)	(in.^{4})	(lb./ft.)
Schedule 40 pipes (Standard Weight)	1	1.049	1.315	0.133	0.469	0.0830	0.126	0.423	0.166	1.68
	2	2.067	2.375	0.154	1.02	0.627	0.528	0.791	1.25	3.66
	3	3.068	3.500	0.216	2.07	2.85	1.63	1.17	5.69	7.58
	4	4.026	4.500	0.237	2.96	6.82	3.03	1.51	13.6	10.8
	5	5.047	5.563	0.258	4.01	14.3	5.14	1.88	28.6	14.6
	6	6.065	6.625	0.280	5.20	26.5	7.99	2.25	52.9	19.0
	8	7.981	8.625	0.322	7.85	68.1	15.8	2.95	136	28.6
Schedule 80 pipes (Extra-Strong)	1	0.957	1.315	0.179	0.602	0.101	0.154	0.410	0.202	2.17
	2	1.939	2.375	0.218	1.40	0.827	0.696	0.771	1.65	5.03
	3	2.900	3.500	0.300	2.83	3.70	2.11	1.14	7.40	10.3
	4	3.826	4.500	0.337	4.14	9.12	4.05	1.48	18.2	15.0
	5	4.813	5.563	0.375	5.73	19.5	7.02	1.85	39.0	20.8
	6	5.761	6.625	0.432	7.83	38.3	11.6	2.20	76.6	28.6
	8	7.625	8.625	0.500	11.9	100	23.1	2.89	199	43.4

Table D4: SI Steel Pipes

	Nominal diameter	Inside diameter	Outside diameter	Wall thickness	Area	Moment of inertia	Section modulus	Radius of gyration	Polar moment of inertia	Mass per unit length
		d_i	d_o	t	A	I	S	r_G	J	
	(mm)	(mm)	(mm)	(mm)	(mm^2)	(10^6 mm^4)	(10^3 mm^3)	(mm)	(10^3 mm.^{4})	(kg/m)
Schedule 40 pipes (Standard Weight)	25	26.7	33.5	3.38	303	0.0345	2.06	10.7	69.1	2.50
	51	52.5	60.5	3.91	658	0.261	8.65	20.1	520	5.44
	76	78.0	88.9	5.49	1340	1.19	26.7	29.7	2370	11.3
	102	102	114	6.02	1910	2.84	49.7	38.4	5660	16.1
	127	128	141	6.55	2590	5.95	84.2	47.8	11900	21.7
	152	154	168	7.11	3350	11.0	131	57.2	22000	28.3
	203	203	219	8.18	5060	28.3	259	74.9	56600	42.5
Schedule 80 pipes (Extra-Strong)	25	24.3	33.5	4.55	388	0.0420	2.52	10.4	84.1	3.23
	51	49.3	60.5	5.54	903	0.344	11.4	19.6	687	7.48
	76	73.7	88.9	7.62	1830	1.54	34.6	29.0	3080	15.3
	102	97.3	114	8.56	2670	3.80	66.4	37.6	7580	22.3
	127	122	141	9.53	3700	8.12	115	47.0	16200	30.9
	152	146	168	11.0	5050	15.9	190	55.9	31900	42.5
	203	194	219	12.7	7680	41.6	379	73.4	82800	64.5

Additional data on steel beams and pipes are freely available online from the American Institute of Steel Construction (AISC).

Copper Tubing

In the 1930s, thin-walled copper tubing was introduced throughout North America for domestic potable water and baseboard hot water heating. Three types are available in the U.S.:

Type K with the thickest wall, useable for underground applications.
Type L with an intermediate wall thickness, used in interior applications.
Type M with the thinnest wall, used in interior applications (in some areas, legal only for nonpressurized applications such as condensate drains).

Copper pipes are made with C10800 oxygen-free low-phosphorus copper. The nominal diameter is approximately equal to the inside diameter. The outside diameters are identical for all Types so they will fit the same elbows, valves, and other fittings, and are 1/8 of an inch larger than the nominal diameters. Wall thickness $t = \frac{d_o - d_i}{2}$.

Table D5: U.S. Customary Copper Tubing

Nominal diameter (in.)	Type K Inside diameter d_i (in.)	Type L Inside diameter d_i (in.)	Type M Inside diameter d_i (in.)	All Types Outside diameter d_o (in.)
3/8	0.402	0.430	0.450	0.500
1/2	0.528	0.545	0.569	0.625
5/8	0.652	0.668	0.690	0.750
3/4	0.745	0.785	0.811	0.875
1	0.995	1.025	1.055	1.125
1 1/4	1.245	1.265	1.291	1.375
1 1/2	1.481	1.505	1.527	1.625
2	1.959	1.985	2.009	2.125
2 1/2	2.435	2.465	2.495	2.625
3	2.907	2.945	2.981	3.125

Copper water tube sold in SI units comes in Types A, B, and C (similar to Types K, L, and M, above). Nominal diameter is the outside diameter.

Table D6: SI Copper Tubing

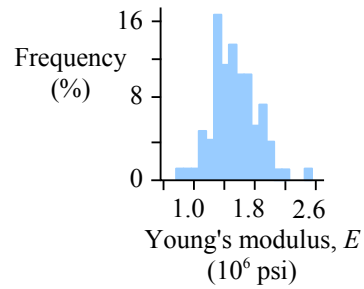
Nominal diameter (mm)	Type A Inside diameter d_i (mm)	Type B Inside diameter d_i (mm)	Type C Inside diameter d_i (mm)	All Types Outside diameter d_o (mm)
10	8.2	8.4	8.8	10
12	9.6	10.2	10.8	12
15	12.6	13.0	13.6	15
18	15.6	16.0	16.6	18
22	18.8	19.8	20.4	22
28	24.8	25.6	26.2	28
35	31.8	32.2	32.8	35
42	38.4	39.0	39.6	42
54	49.8	50.6	51.0	54
67	62.2	63.0	63.8	67

More sizes are available; see ASTM Standards B 88 and B 89 for a complete list.

Appendix E: Mechanical and Dimensional Properties of Wood

Mechanical Properties of Air-Dried Boards and Timber

Wood is a natural material which varies in mechanical properties. Factors affecting strength and stiffness include defects (knots, insect and woodpecker holes), moisture content, service temperature, age of the tree at cutting, direction within the wood (along the grain, tangential, and radial), and geographic source of the tree. For example, this graph from the Forest Products Laboratory of the U.S. Department of Agriculture* shows variation in the same visual grade of lumber (same quality & equivalent species), cut from a wide geographical area.



The Allowable Stress Design method (ASD) sets the design stress at the bottom 5th percentile of wood within a stress grade. In theory, 5% of the wood used in any structure is weaker than the required strength...so use a Factor of Safety to prevent failure.

Table E1: U.S. Customary Materials Properties of Timber

Tree species	Allowable Stresses					
	Bending	Horizontal shear	Tension parallel to grain	Compression parallel to grain	Specific weight	Young's modulus
					γ	E
	(psi)	(psi)	(psi)	(psi)	(lb./ft.^{3})	(ksi)
Eastern white pine	600	135	275	725	22	1100
Southern yellow pine	1400	175	825	1250	34	1700
Hem-fir	1000	150	500	1300	25	1400
Douglas fir	900	180	625	1050	30	1700
Redwood	1000	160	600	1050	25	1300

Table E2: SI Materials Properties of Timber

Tree species	Allowable Stresses					
	Bending	Horizontal shear	Tension parallel to grain	Compression parallel to grain	Density	Young's modulus
					ρ	E
	(MPa)	(MPa)	(MPa)	(MPa)	(kg/m^3)	(GPa)
Eastern white pine	4.1	0.93	1.9	5.0	350	7.6
Southern yellow pine	9.7	1.2	5.7	8.6	550	11.7
Hem-fir	6.9	1.0	3.4	9.0	400	9.7
Douglas fir	6.2	1.2	4.3	7.2	480	11.7
Redwood	6.9	1.1	4.1	7.2	400	9.0

Additional data on timber are freely available online from the U.S. Forest Products Laboratory of the U.S. Department of Agriculture at http://www.fpl.fs.fed.us.

* General Technical Report FPL GTR 190, Chapter 7, p. 7-6.

Softwood Lumber and Timber Sizes

Softwood is "dressed" or planed from nominal size to the sizes listed below. These sizes are *S4S* (surfaced four sides). Weight per unit length depends on the density of the wood. Dried softwoods range in specific weight from 20 to 40 lb./ft.^{3} (320 to 640 kg/m^3). If a 2×4 is made of wood with a specific weight of 30 lb./ft.^{3}, its weight per unit length is 1.09 lb./ft.

Table E3: U.S. Customary Timber Sizes

Designation	Width	Depth	Area	Moment of inertia	Section modulus	Weight per unit length γ = 20 lb/ft.^{3}	Weight per unit length γ = 30 lb/ft.^{3}	Weight per unit length γ = 40 lb/ft.^{3}
	b (in.)	d (in.)	A (in.^{2})	I_x (in.^{4})	S_x (in.^{3})	w (lb./ft.)	w (lb./ft.)	w (lb./ft.)
2×2	1.5	1.5	2.25	0.422	0.563	0.313	0.469	0.625
2×3	1.5	2.5	3.75	1.95	1.56	0.521	0.781	1.04
2×4	1.5	3.5	5.25	5.36	3.06	0.729	1.09	1.46
2×6	1.5	5.5	8.25	20.8	7.56	1.15	1.72	2.29
2×8	1.5	7.25	10.9	47.6	13.1	1.51	2.27	3.02
2×10	1.5	9.25	13.9	98.9	21.4	1.93	2.89	3.85
2×12	1.5	11.25	16.9	178	31.6	2.34	3.52	4.69
4×4	3.5	3.5	12.3	12.5	7.1	1.70	2.55	3.40
4×6	3.5	5.5	19.3	48.5	17.6	2.67	4.01	5.35
4×8	3.5	7.25	25.4	111	30.7	3.52	5.29	7.05
4×10	3.5	9.25	32.4	231	49.9	4.50	6.74	8.99
4×12	3.5	11.25	39.4	415	73.8	5.47	8.20	10.9
4×14	3.5	13.25	46.4	678	102	6.44	9.66	12.9
4×16	3.5	15.25	53.4	1034	136	7.41	11.12	14.8
6×6	5.5	5.5	30.3	76.3	27.7	4.20	6.30	8.40
6×8	5.5	7.5	41.3	193	51.6	5.73	8.59	11.5
6×10	5.5	9.5	52.3	393	82.7	7.26	10.9	14.5
6×12	5.5	11.5	63.3	697	121	8.78	13.2	17.6
6×14	5.5	13.5	74.3	1130	167	10.3	15.5	20.6
6×16	5.5	15.5	85.3	1710	220	11.8	17.8	23.7
6×18	5.5	17.5	96.3	2460	281	13.4	20.1	26.7
6×20	5.5	19.5	107	3400	349	14.9	22.3	29.8
8×8	7.5	7.5	56.3	264	70.3	7.81	11.7	15.6
8×10	7.5	9.5	71.3	536	113	9.90	14.8	19.8
8×12	7.5	11.5	86.3	951	165	12.0	18.0	24.0
8×14	7.5	13.5	101	1540	228	14.1	21.1	28.1
8×16	7.5	15.5	116	2330	300	16.1	24.2	32.3
8×18	7.5	17.5	131	3350	383	18.2	27.3	36.5
8×20	7.5	19.5	146	4630	475	20.3	30.5	40.6
10×10	9.5	9.5	90.3	679	143	12.5	18.8	25.1
10×12	9.5	11.5	109	1200	209	15.2	22.8	30.3
10×14	9.5	13.5	128	1950	289	17.8	26.7	35.6
10×16	9.5	15.5	147	2950	380	20.5	30.7	40.9
10×18	9.5	17.5	166	4240	485	23.1	34.6	46.2
10×20	9.5	19.5	185	5870	602	25.7	38.6	51.5
10×22	9.5	21.5	204	7870	732	28.4	42.6	56.7
10×24	9.5	23.5	223	10300	874	31.0	46.5	62.0
12×12	11.5	11.5	132	1460	253	18.4	27.6	36.7
12×14	11.5	13.5	155	2360	349	21.6	32.3	43.1
12×16	11.5	15.5	178	3570	460	24.8	37.1	49.5
12×18	11.5	17.5	201	5140	587	28.0	41.9	55.9
12×20	11.5	19.5	224	7110	729	31.1	46.7	62.3
12×22	11.5	21.5	247	9520	886	34.3	51.5	68.7
12×24	11.5	23.5	270	12400	1060	37.5	56.3	75.1

Appendix E: Mechanical and Dimensional Properties of Wood

Timber beams in SI units are listed by their *mass* per unit length, not by their weight per unit length. A 200×360 timber beam made of wood with a density of 320 kg/m^3 has a *mass* per unit length of 21 kg/m. Multiply by the mass per unit length by gravity to obtain weight per unit length: $\frac{21\,\text{kg}}{\text{m}}\frac{9.81\,\text{m}}{\text{s}^2}\left|\frac{\text{N}\ \text{s}^2}{\text{kg m}}\right.=206\,\text{N/m}$.

Table E4: SI Timber Sizes

Designation	Width b (mm)	Depth d (mm)	Area A (10^3 mm^2)	Moment of inertia I_x (10^6 mm^4)	Section modulus S_x (10^3 mm^3)	Mass per unit length ρ = 320 kg/m^3 m/L (kg/m)	Mass per unit length ρ = 480 kg/m^3 m/L (kg/m)	Mass per unit length ρ = 640 kg/m^3 m/L (kg/m)
50×100	38.1	88.9	3.39	2.23	50.2	1.09	1.63	2.17
50×150	38.1	140	5.33	8.71	124	1.71	2.56	3.42
50×200	38.1	184	7.01	19.8	215	2.25	3.37	4.49
50×250	38.1	235	8.95	41.2	351	2.87	4.30	5.74
50×300	38.1	286	10.9	74.3	519	3.49	5.24	6.98
50×360	38.1	337	12.8	122	721	4.12	6.17	8.23
50×410	38.1	387	14.7	184	951	4.73	7.09	9.45
100×100	88.9	88.9	7.90	5.21	117	2.53	3.80	5.07
100×150	88.9	140	12.4	20.3	290	3.99	5.98	7.98
100×200	88.9	184	16.4	46.2	502	5.24	7.86	10.5
100×250	88.9	235	20.9	96.1	818	6.70	10.0	13.4
100×300	88.9	286	25.4	173	1212	8.15	12.2	16.3
100×360	88.9	337	30.0	284	1683	9.60	14.4	19.2
100×410	88.9	388	34.5	433	2230	11.1	16.6	22.1
150×150	140	140	19.6	32.0	457	6.28	9.42	12.6
150×200	140	191	26.7	81.3	851	8.57	12.9	17.1
150×250	140	241	33.7	163	1360	10.8	16.2	21.6
150×300	140	292	40.9	290	1990	13.1	19.7	26.2
150×360	140	343	48.0	471	2750	15.4	23.1	30.8
150×410	140	394	55.2	714	3620	17.7	26.5	35.4
150×460	140	445	62.3	1030	4620	20.0	30.0	39.9
150×510	140	496	69.4	1420	5740	22.3	33.4	44.5
200×200	191	191	36.5	111	1160	11.7	17.5	23.4
200×250	191	241	46.0	223	1850	14.8	22.1	29.5
200×300	191	292	55.8	396	2710	17.9	26.8	35.7
200×360	191	343	65.5	642	3750	21.0	31.5	42.0
200×410	191	394	75.3	974	4940	24.1	36.2	48.2
200×460	191	445	85.0	1400	6300	27.2	40.9	54.5
200×510	191	496	94.7	1940	7830	30.4	45.5	60.7
250×250	241	241	58.1	281	2330	18.6	27.9	37.2
250×300	241	292	70.4	500	3420	22.6	33.8	45.1
250×360	241	343	82.7	810	4730	26.5	39.7	53.0
250×410	241	394	95.0	1230	6240	30.4	45.7	60.9
250×460	241	445	107	1770	7950	34.4	51.6	68.7
250×510	241	495	119	2440	9840	38.2	57.4	76.5
250×560	241	546	132	3270	12000	42.2	63.3	84.3
250×610	241	597	144	4270	14300	46.1	69.2	92.2
300×300	292	292	85	606	4150	27.3	41.0	54.7
300×360	292	343	100	982	5730	32.1	48.2	64.2
300×410	292	394	115	1490	7550	36.9	55.3	73.7
300×460	292	445	130	2140	9640	41.6	62.5	83.3
300×510	292	495	145	2950	11900	46.3	69.5	92.7
300×560	292	546	159	3960	14500	51.1	76.7	102
300×610	292	597	174	5180	17300	55.9	83.8	112

Appendix F: Beam Equations

Terminology Used in Beam Equations

R_A and R_B are the reaction forces at points A and B. M_B is the reaction moment at point B.

V_x, M_x and Δ_x are the shear load, bending moment, and deflection at a distance x from the left end of the beam.

M_a and Δ_a are the bending moment and deflection at a distance $x = a$ from the left end of the beam.

Every beam in this Appendix has a length L.

Point load at the midspan of a simply-supported beam

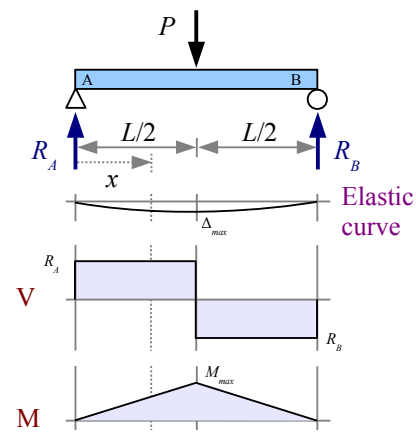
$$R_A = R_B = V_{max} = \frac{P}{2}$$

$$M_{max} = \frac{P\,L}{4} \quad \text{at the point load}$$

$$M_x = \frac{P\,x}{2} \quad \text{for} \quad x < \frac{L}{2}$$

$$\Delta_{max} = \frac{P\,L^3}{48\,E\,I} \quad \text{at the point load}$$

$$\Delta_x = \frac{P\,x}{48\,E\,I}\left(3\,L^2 - 4\,x^2\right) \quad \text{for} \quad x \le \frac{L}{2}$$



Point load at any location on a simply-supported beam

$$R_A = \frac{P\,b}{L}, \quad R_B = \frac{P\,a}{L}, \quad V_{max} \text{ is the greater of the two}$$

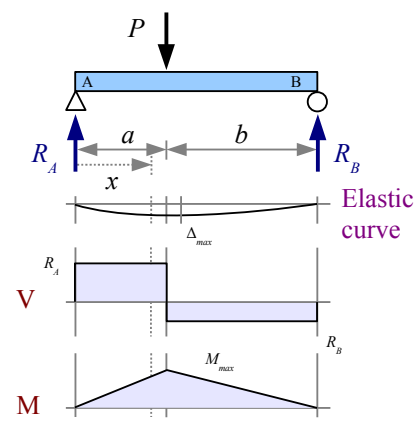
$$M_{max} = \frac{P\,a\,b}{L} \quad \text{at the point load}$$

$$M_x = \frac{P\,b\,x}{L} \quad \text{for } x < a$$

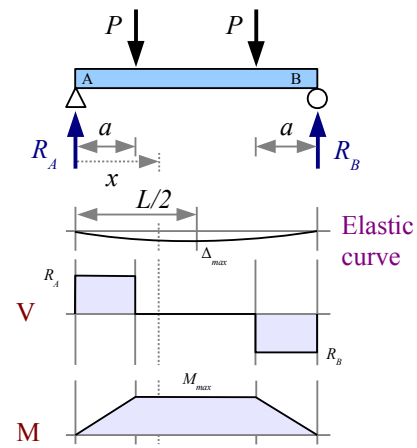
$$\Delta_{max} = \frac{P\,a\,b(a+2\text{b})\sqrt{3\,a(a+2\,b)}}{27\,E\,I\,L} \quad \text{at} \quad x = \sqrt{\frac{a(a+2\,b)}{3}} \quad \text{when } a > b$$

$$\Delta_a = \frac{P\,a^2 b^2}{3\,E\,I\,L} \quad \text{at the point load}$$

$$\Delta_x = \frac{P\,b\,x}{6\,E\,I\,L}\left(L^2 - b^2 - x^2\right) \quad \text{for } x < a$$



Two identical point loads, symmetrically placed on a simply-supported beam



$R_A = R_B = V_{max} = P$

$M_{max} = P\,a$ between the point loads

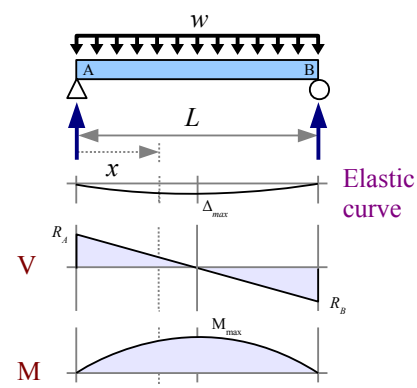
$M_x = P\,x$ for $x < a$

$\Delta_{max} = \frac{P\,a}{24\,E\,I}(3\,L^2 - 4\,a^2)$ at the midspan

$\Delta_x = \frac{P\,x}{6\,E\,I}(3\,L\,a - 3\,a^2 - x^2)$ for $x \le a$

$\Delta_x = \frac{P\,a}{6\,E\,I}(3\,L\,x - 3\,x^2 - a^2)$ between the point loads

Uniform distributed load along the entire length of a simply-supported beam



$R_A = R_B = V_{max} = \frac{w\,L}{2}$

$V_x = w\left(\frac{L}{2} - x\right)$

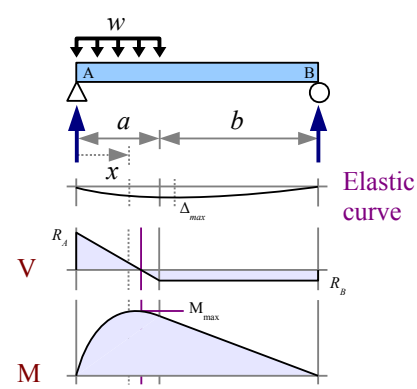
$M_{max} = \frac{w\,L^2}{8}$ at the midspan

$M_x = \frac{w\,x}{2}(L - x)$

$\Delta_{max} = \frac{5\,w\,L^4}{384\,E\,I}$ at the midspan

$\Delta_x = \frac{w\,x}{24\,E\,I}(L^3 - 2\,L\,x^2 + x^3)$

Uniform distributed load on a simply-supported beam



$R_A = V_{max} = \frac{w\,a}{2\,L}(2\,L - a)$, $R_B = \frac{w\,a^2}{2\,L}$

$V_x = R_A - w\,x$ for $x \le a$; $V_x = -R_B$ for $x \ge a$

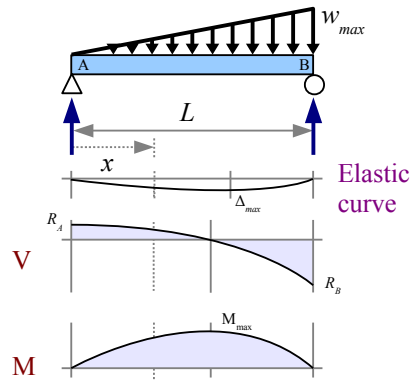
$M_{max} = \frac{R_A^2}{2\,w}$ at $x = \frac{R_A}{w}$

$M_x = R_A\,x - \frac{w\,x^2}{2}$ for $x \le a$; $M_x = R_B(L - x)$ for $x \ge a$

$\Delta_x = \frac{w\,x}{24\,E\,I\,L}\left[a^2(2\,L - a)^2 - 2\,a\,x^2(2\,L - a) + L\,x^3\right]$ for $x \le a$

$\Delta_x = \frac{w\,a^2(L - x)}{24\,E\,I\,L}(4\,x\,L - 2\,x^2 - a^2)$ for $x \ge a$

Nonuniform distributed load, tapered along the length



$$R_A = \frac{w_{max} L}{6}, \quad R_B = \frac{w_{max} L}{3}$$

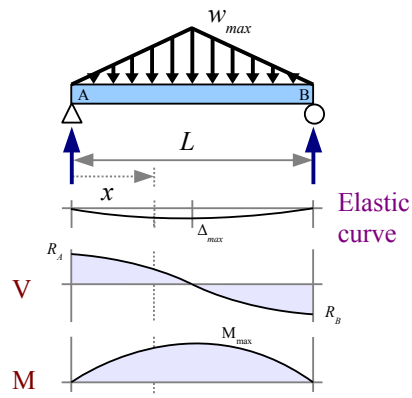
$$V_x = \frac{w_{max} L}{6} - \frac{w_{max} x^2}{2L}$$

$$M_{max} = \frac{w_{max} L^2}{9\sqrt{3}} \text{ at } x = \frac{L}{\sqrt{3}} \qquad M_x = \frac{w_{max} L x}{6L^2}(L^2 - x^2)$$

$$\Delta_{max} = 0.00652 \frac{w_{max} L^4}{E I} \text{ at } x = L\sqrt{1 - \sqrt{\frac{8}{15}}}$$

$$\Delta_x = \frac{w_{max} x}{360 E I L}(3x^4 - 10L^2x^2 + 7L^4)$$

Nonuniform distributed load on a simply-supported beam, tapering from the midspan to the ends



$$R_A = R_B = V_{max} = \frac{w_{max} L}{4}$$

$$V_x = \frac{w_{max}}{4L}(L^2 - 4x^2) \text{ for } x < \frac{L}{2}$$

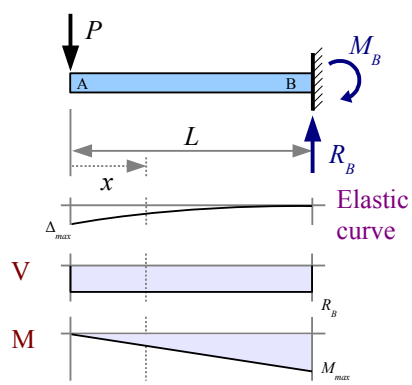
$$M_{max} = \frac{w_{max} L^2}{12} \text{ at the midspan}$$

$$M_x = \frac{w_{max} L x}{2}\left(\frac{1}{2} - \frac{2x^2}{3L^2}\right) \text{ for } x < \frac{L}{2}$$

$$\Delta_{max} = \frac{w_{max} L^4}{120 E I} \text{ at the midspan}$$

$$\Delta_x = \frac{w_{max} x}{960 E I L}(5L^2 - 4x^2)^2 \text{ for } x < \frac{L}{2}$$

Point load at the end of a cantilever beam



$$R_B = V_{max} = P$$

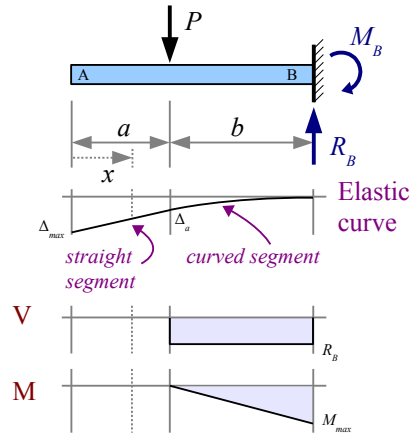
$$M_B = M_{max} = PL$$

$$M_x = Px$$

$$\Delta_{max} = \frac{PL^3}{3EI} \text{ at the free end}$$

$$\Delta_x = \frac{P}{6EI}(2L^3 - 3L^2x + x^3)$$

Point load at any location on a cantilever beam



$R_B = V_{max} = P$

$M_B = M_{max} = P\,b$

$M_x = P(x-a)$ for $x > a$

$\Delta_{max} = \frac{P\,b^2}{6\,E\,I}(3\,L-b)$ at the free end

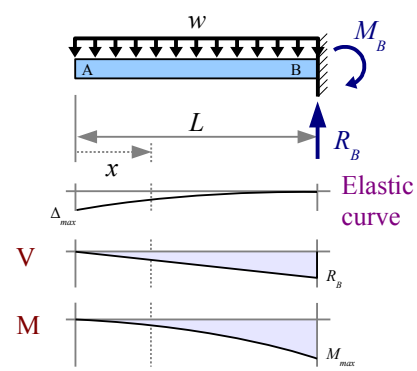
$\Delta_a = \frac{P\,b^3}{3\,E\,I}$ at the point load

$\Delta_x = \frac{P\,b^2}{6\,E\,I}(3\,L-3\,x-b)$ for $x < a$

$\Delta_x = \frac{P\,(L-x)^2}{6\,E\,I}(3\,b-L+x)$ for $x > a$

The beam bends in an arc between the support and the point load. To the left of the point load, the beam is straight, but at an angle.

Uniform distributed load along the entire length of a cantilever beam



$R_B = V_{max} = wL$

$V_x = w\,x$

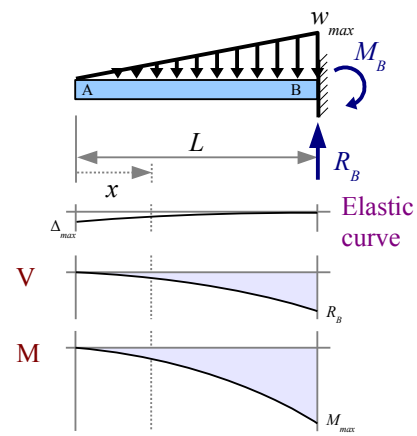
$M_B = M_{max} = \frac{w\,L^2}{2}$

$M_x = \frac{w\,x^2}{2}$

$\Delta_{max} = \frac{w\,L^4}{8\,E\,I}$ at the free end

$\Delta_x = \frac{w}{24\,E\,I}(x^4 - 4\,L^3\,x + 3\,L^4)$

Nonuniform distributed load along the entire length of a cantilever beam, tapering from the support to the free end



$$R_B = V_{max} = \frac{w_{max} L}{2}$$

$$V_x = \frac{w_{max} x^2}{2L}$$

$$M_B = M_{max} = \frac{w_{max} L^2}{6}$$

$$M_x = \frac{w_{max} x^3}{6L}$$

$$\Delta_{max} = \frac{w_{max} L^4}{30 E I} \quad \text{at the free end}$$

$$\Delta_x = \frac{w_{max}}{120 E I L}\left(x^5 - 5 L^4 x + 4 L^5\right)$$

Index

Allowable moment in a beam....83
Allowable stress in welded tanks....29
Angle of twist in a round shaft....57
Average web shear approach....89
Beam deflection....94
Beam design, steel....102
Beam design, wood....108, 110
Beam reactions....60
Bending moment diagrams....72
Bending stress in beams....82
Bending stress in steel beams....84
Bolted joints....33
Buckling of columns....124
Cantilever beams....60
Centroid 41
Centroidal axis....41
Column buckling....124
Columns, ideal slender....124
Columns, steel....126
Columns, steel machine parts....127
Columns, support conditions....125
Combined stresses....112
Deflection diagrams....82
Eccentric loading....114
Effective length of a column....125
Efficiency, bolted & welded joints....36
Elastic deformation....18
Elastic modulus....18
Equivalent load diagram....63
Euler critical load....124
Euler stress....125
Factor-label method of unit conversion....12
Failure theories....145
Flexure formula....82
Formula method of beam analysis....95
Hollow shafts....56
Hoop stress....28
Ideal slender columns....124
Interpolation....108
Isostress lines....131
Joint efficiency in bolted & welded joints....36
Load diagrams....61
Longitudinal stress in a pipe....29
Microstrain....17
Modulus of elasticity....18
Modulus, shear....55
Mohr's circle....131
Moment diagrams....72
Moment of inertia....41
Neutral axis....82
Overhanging beams....64
Pipes....28
Plastic deformation....18
Plastic moment....85
Plastic section modulus....85
Poisson's ratio....23
Polar moment of inertia....54
Pressure vessels....28
Radius of curvature of a beam....94
Radius of gyration....54
Reference axis....45
Rupture strength....18
S4S....164
Section modulus....83
Shafts in torsion....55
Shear diagrams....66
Shear stress....20
Shear stress in beams....87
Shear stress in round shafts....55
Short block....115
Simply-supported beams....60
Statically indeterminate beams....118
Steel beam design....102
Steel columns....126
Steel machine part columns....127
Strain....17
Stress concentration in tension....30
Stress concentration in torsion....58
Stress-strain curve....17
Stress, tensile....17
Stress, thermal....26
Superposition....98, 112, 118
Tensile strength....18
Terminology....6
Thermal expansion....25
Thin-walled pipe....28
Timber beam design....108, 110
Torsion in round shafts....55
Transfer formula....44
Ultimate tensile strength....18
Unit systems & conversions....149p.
Visualizing Stress and Strain....130
Welded lap joints....38
Wood beam design....108, 110
Yield strength....18
Young's modulus....18