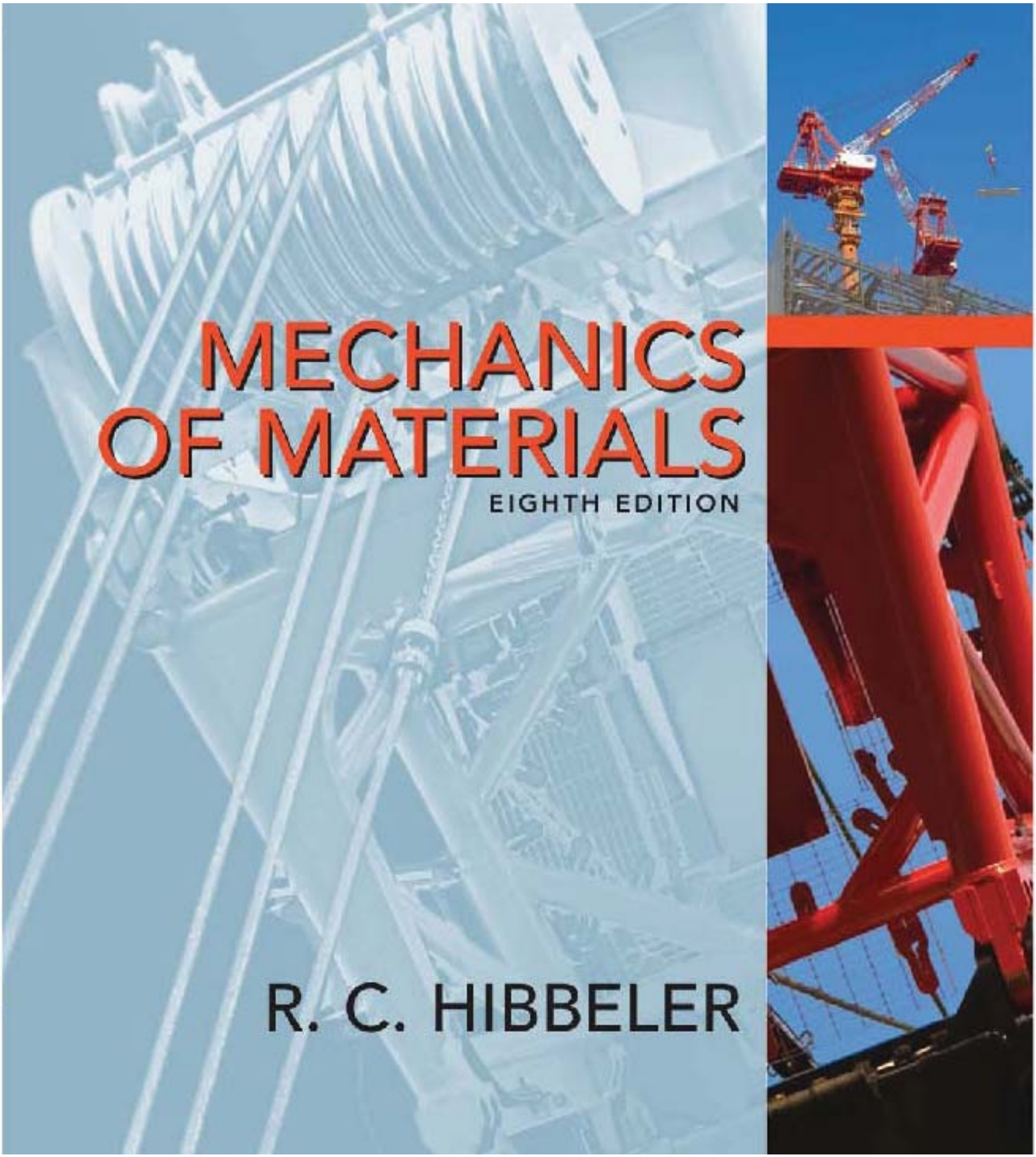


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MECHANICS OF MATERIALS

MECHANICS OF MATERIALS

EIGHTH EDITION

R. C. HIBBELER

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To the Student

With the hope that this work will stimulate
an interest in Engineering Mechanics
and provide an acceptable guide to its understanding.

PREFACE

It is intended that this book provide the student with a clear and thorough presentation of the theory and application of the principles of mechanics of materials. To achieve this objective, over the years this work has been shaped by the comments and suggestions of hundreds of reviewers in the teaching profession, as well as many of the author's students. The eighth edition has been significantly enhanced from the previous edition, and it is hoped that both the instructor and student will benefit greatly from these improvements.

New to This Edition

- **Updated Content.** Some portions of the text have been rewritten in order to enhance clarity and be more succinct. In this regard, some new examples have been added and others have been modified to provide more emphasis on the application of important concepts. Also, the artwork has been improved throughout the book to support these changes.
- **New Photos.** The relevance of knowing the subject matter is reflected by the real-world applications depicted in over 44 new or updated photos placed throughout the book. These photos generally are used to explain how the relevant principles apply to real-world situations and how materials behave under load.
- **Fundamental Problems.** These problem sets are located just after each group of example problems. They offer students simple applications of the concepts covered in each section and, therefore, provide them with the chance to develop their problem-solving skills before attempting to solve any of the standard problems that follow. The fundamental problems may be considered as extended examples, since the key equations and answers are all listed in the back of the book. Additionally, when assigned, these problems offer students an excellent means of preparing for exams, and they can be used at a later time as a review when studying for the Fundamentals of Engineering Exam.
- **Conceptual Problems.** Throughout the text, usually at the end of each chapter, there is a set of problems that involve conceptual situations related to the application of the principles contained in the chapter. These analysis and design problems are intended to engage the students in thinking through a real-life situation as depicted in a photo. They can be assigned after the students have developed some expertise in the subject matter and they work well either for individual or team projects.
- **New Problems.** There are approximately 35%, or about 550, new problems added to this edition, which involve applications to many different fields of engineering. Also, this new edition now has approximately 134 more problems than in the previous edition.

• **Problems with Hints.** With the additional homework problems in this new edition, every problem indicated with a bullet (•) before the problem number includes a suggestion, key equation, or additional numerical result that is given along with the answer in the back of the book. These problems further encourage students to solve problems on their own by providing them with additional checks to the solution.

Contents

The subject matter is organized into 14 chapters. Chapter 1 begins with a review of the important concepts of statics, followed by a formal definition of both normal and shear stress, and a discussion of normal stress in axially loaded members and average shear stress caused by direct shear.

In Chapter 2 normal and shear strain are defined, and in Chapter 3 a discussion of some of the important mechanical properties of materials is given. Separate treatments of axial load, torsion, and bending are presented in Chapters 4, 5, and 6, respectively. In each of these chapters, both linear-elastic and plastic behavior of the material are considered. Also, topics related to stress concentrations and residual stress are included. Transverse shear is discussed in Chapter 7, along with a discussion of thin-walled tubes, shear flow, and the shear center. Chapter 8 includes a discussion of thin-walled pressure vessels and provides a partial review of the material covered in the previous chapters, such that the state of stress results from combined loadings. In Chapter 9 the concepts for transforming multiaxial states of stress are presented. In a similar manner, Chapter 10 discusses the methods for strain transformation, including the application of various theories of failure. Chapter 11 provides a means for a further summary and review of previous material by covering design applications of beams and shafts. In Chapter 12 various methods for computing deflections of beams and shafts are covered. Also included is a discussion for finding the reactions on these members if they are statically indeterminate. Chapter 13 provides a discussion of column buckling, and lastly, in Chapter 14 the problem of impact and the application of various energy methods for computing deflections are considered.

Sections of the book that contain more advanced material are indicated by a star (*). Time permitting, some of these topics may be included in the course. Furthermore, this material provides a suitable reference for basic principles when it is covered in other courses, and it can be used as a basis for assigning special projects.

Alternative Method of Coverage. Some instructors prefer to cover stress and strain transformations *first*, before discussing specific applications of axial load, torsion, bending, and shear. One possible method for doing this would be first to cover stress and its transformation, Chapter 1 and Chapter 9, followed by strain and its transformation, Chapter 2 and the first part of Chapter 10. The discussion and example problems in these later chapters have been

styled so that this is possible. Also, the problem sets have been subdivided so that this material can be covered without prior knowledge of the intervening chapters. Chapters 3 through 8 can then be covered with no loss in continuity.

Hallmark Elements

Organization and Approach. The contents of each chapter are organized into well-defined sections that contain an explanation of specific topics, illustrative example problems, and a set of homework problems. The topics within each section are placed into subgroups defined by titles. The purpose of this is to present a structured method for introducing each new definition or concept and to make the book convenient for later reference and review.

Chapter Contents. Each chapter begins with a full-page illustration that indicates a broad-range application of the material within the chapter. The "Chapter Objectives" are then provided to give a general overview of the material that will be covered.

Procedures for Analysis. Found after many of the sections of the book, this unique feature provides the student with a logical and orderly method to follow when applying the theory. The example problems are solved using this outlined method in order to clarify its numerical application. It is to be understood, however, that once the relevant principles have been mastered and enough confidence and judgment have been obtained, the student can then develop his or her own procedures for solving problems.

Photographs. Many photographs are used throughout the book to enhance conceptual understanding and explain how the principles of mechanics of materials apply to real-world situations.

Important Points. This feature provides a review or summary of the most important concepts in a section and highlights the most significant points that should be realized when applying the theory to solve problems.

Example Problems. All the example problems are presented in a concise manner and in a style that is easy to understand.

Homework Problems. Numerous problems in the book depict realistic situations encountered in engineering practice. It is hoped that this realism will both stimulate the student's interest in the subject and provide a means for developing the skill to reduce any such problem from its physical description to a model or a symbolic representation to which principles may be applied. Throughout the book there is an approximate balance of problems using either SI or FPS units. Furthermore, in any set, an attempt has been made to arrange the problems in order of increasing difficulty. The answers to all but every fourth problem are listed in the back of the book. To alert the user to a

• **Problems with Hints.** With the additional homework problems in this new edition, every problem indicated with a bullet (•) before the problem number includes a suggestion, key equation, or additional numerical result that is given along with the answer in the back of the book. These problems further encourage students to solve problems on their own by providing them with additional checks to the solution.

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problem without a reported answer, an asterisk(*) is placed before the problem number. Answers are reported to three significant figures, even though the data for material properties may be known with less accuracy. Although this might appear to be a poor practice, it is done simply to be consistent and to allow the student a better chance to validate his or her solution. A solid square (■) is used to identify problems that require a numerical analysis or a computer application.

Appendices. The appendices of the book provide a source for review and a listing of tabular data. Appendix A provides information on the centroid and the moment of inertia of an area. Appendices B and C list tabular data for structural shapes, and the deflection and slopes of various types of beams and shafts.

Accuracy Checking. The Eighth Edition has undergone our rigorous Triple Accuracy Checking review. In addition to the author's review of all art pieces and pages, the text was checked by the following individuals:

- Scott Hendricks, Virginia Polytechnic University
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- Kurt Norlin, Laurel Tech Integrated Publishing Services
- Kai Beng Yap, Engineering Consultant

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I would also like to thank all my students who have used the previous edition and have made comments to improve its contents.

I would greatly appreciate hearing from you if at any time you have any comments or suggestions regarding the contents of this edition.

Russell Charles Hibbeler
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Resources for Instructors

- **Instructor's Solutions Manual.** An instructor's solutions manual was prepared by the author. The manual includes homework assignment lists and was also checked as part of the accuracy checking program.
- **Presentation Resources.** All art from the text is available in PowerPoint slide and JPEG format. These files are available for download from the Instructor Resource Center at http://www.pearsonhighered.com. If you are in need of a login and password for this site, please contact your local Pearson Prentice Hall representative.
- **Video Solutions.** Developed by Professor Edward Berger, University of Virginia, video solutions located on the Companion Website offer step-by-step solution walkthroughs of representative homework problems from each section of the text. Make efficient use of class time and office hours by showing students the complete and concise problem solving approaches that they can access anytime and view at their own pace. The videos are designed to be a flexible resource to be used however each instructor and student prefers. A valuable tutorial resource, the videos are also helpful for student self-evaluation as students can pause the videos to check their understanding and work alongside the video. Access the videos at http://www.pearsonhighered.com/hibbeler and follow the links for the *Mechanics of Materials* text.

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- **Companion Website**—The Companion Website, located at http://www.pearsonhighered.com/hibbeler includes opportunities for practice and review including:
  - **Video Solutions**—Complete, step-by-step solution walkthroughs of representative homework problems from each section. Videos offer:
    - **Fully Worked Solutions**—Showing every step of representative homework problems, to help students make vital connections between concepts.
    - **Self-Paced Instruction**—Students can navigate each problem and select, play, rewind, fast-forward, stop, and jump-to-sections within each problem's solution.
    - **24/7 Access**—Help whenever students need it with over 20 hours of helpful review.

An access code for the *Mechanics of Materials*, Eighth Edition website was included with this text. To redeem the code and gain access to the site, go to http://www.pearsonhighered.com/hibbeler and follow the directions on the access code card. Access can also be purchased directly from the site.

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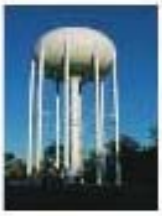
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Chapter 3, A woman stands near a collapsed bridge in one of the worst earthquake-hit areas of Yingxiu town in Wenchuan county, in China's southwestern province of Sichuan on June 2, 2008. UN Secretary of State Condoleezza Rice on June 29 met children made homeless by the devastating earthquake that hit southwest China last month and praised the country's response to the disaster. LIU JIN/Stringer\Getty Images, Inc. AFP.

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MECHANICS OF MATERIALS



The bolts used for the connections of this steel framework are subjected to stress. In this chapter we will discuss how engineers design these connections and their fasteners.

1 Stress

CHAPTER OBJECTIVES

In this chapter we will review some of the important principles of statics and show how they are used to determine the internal resultant loadings in a body. Afterwards the concepts of normal and shear stress will be introduced, and specific applications of the analysis and design of members subjected to an axial load or direct shear will be discussed.

1.1 Introduction

Mechanics of materials is a branch of mechanics that studies the internal effects of stress and strain in a solid body that is subjected to an external loading. Stress is associated with the strength of the material from which the body is made, while strain is a measure of the deformation of the body. In addition to this, mechanics of materials includes the study of the body's stability when a body such as a column is subjected to compressive loading. A thorough understanding of the fundamentals of this subject is of vital importance because many of the formulas and rules of design cited in engineering codes are based upon the principles of this subject.

1

Historical Development. The origin of mechanics of materials dates back to the beginning of the seventeenth century, when Galileo performed experiments to study the effects of loads on rods and beams made of various materials. However, at the beginning of the eighteenth century, experimental methods for testing materials were vastly improved, and at that time many experimental and theoretical studies in this subject were undertaken primarily in France, by such notables as Saint-Venant, Poisson, Lamé, and Navier.

Over the years, after many of the fundamental problems of mechanics of materials had been solved, it became necessary to use advanced mathematical and computer techniques to solve more complex problems. As a result, this subject expanded into other areas of mechanics, such as the *theory of elasticity* and the *theory of plasticity*. Research in these fields is ongoing, in order to meet the demands for solving more advanced problems in engineering.

1.2 Equilibrium of a Deformable Body

Since statics has an important role in both the development and application of mechanics of materials, it is very important to have a good grasp of its fundamentals. For this reason we will review some of the main principles of statics that will be used throughout the text.

External Loads. A body is subjected to only two types of external loads; namely, surface forces or body forces, Fig. 1–1.

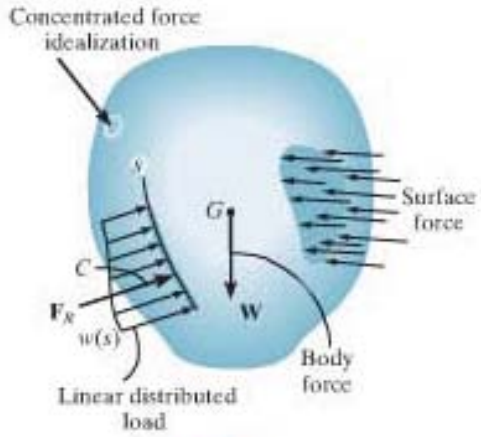


Fig. 1–1

Surface Forces. ***Surface forces*** are caused by the direct contact of one body with the surface of another. In all cases these forces are distributed over the *area* of contact between the bodies. If this area is small in comparison with the total surface area of the body, then the surface force can be *idealized* as a single ***concentrated force***, which is applied to a *point* on the body. For example, the force of the ground on the wheels of a bicycle can be considered as a concentrated force. If the surface loading is applied along a narrow strip of area, the loading can be *idealized* as a ***linear distributed load***, $w(s)$. Here the loading is measured as having an intensity of force/length along the strip and is represented graphically by a series of arrows along the line s. ***The resultant force $\mathbf{F}_R$ of $w(s)$ is equivalent to the area under the distributed loading curve, and this resultant acts through the centroid C or geometric center of this area.*** The loading along the length of a beam is a typical example of where this idealization is often applied.

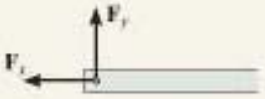
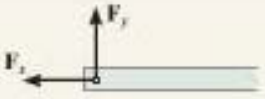
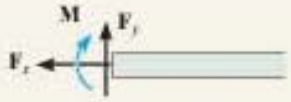
Body Forces. A ***body force*** is developed when one body exerts a force on another body without direct physical contact between the bodies. Examples include the effects caused by the earth's gravitation or its electromagnetic field. Although body forces affect each of the particles composing the body, these forces are normally represented by a single concentrated force acting on the body. In the case of gravitation, this force is called the ***weight*** of the body and acts through the body's center of gravity.

Support Reactions. The surface forces that develop at the supports or points of contact between bodies are called ***reactions***. For two-dimensional problems, i.e., bodies subjected to coplanar force systems, the supports most commonly encountered are shown in Table 1–1. Note carefully the symbol used to represent each support and the type of reactions it exerts on its contacting member. As a general rule, ***if the support prevents translation in a given direction, then a force must be developed on the member in that direction. Likewise, if rotation is prevented, a couple moment must be exerted on the member***. For example, the roller support only prevents translation perpendicular or normal to the surface. Hence, the roller exerts a normal force **F** on the member at its point of contact. Since the member can freely rotate about the roller, a couple moment cannot be developed on the member.



Many machine elements are pin connected in order to enable free rotation at their connections. These supports exert a force on a member, but no moment.

TABLE 1–1

Type of connection	Reaction	Type of connection	Reaction
 Cable	 One unknown: F	 External pin	 Two unknowns: F_x, F_y
 Roller	 One unknown: F	 Internal pin	 Two unknowns: F_x, F_y
 Smooth support	 One unknown: F	 Fixed support	 Three unknowns: F_x, F_y, M

1

Equations of Equilibrium. Equilibrium of a body requires both a ***balance of forces***, to prevent the body from translating or having accelerated motion along a straight or curved path, and a ***balance of moments***, to prevent the body from rotating. These conditions can be expressed mathematically by two vector equations

$$\begin{aligned} \Sigma \mathbf{F} &= \mathbf{0} \\ \Sigma \mathbf{M}_O &= \mathbf{0} \end{aligned} \tag{1–1}$$

Here, $\Sigma \mathbf{F}$ represents the sum of all the forces acting on the body, and $\Sigma \mathbf{M}_O$ is the sum of the moments of all the forces about any point O either on or off the body. If an x, y, z coordinate system is established with the origin at point O, the force and moment vectors can be resolved into components along each coordinate axis and the above two equations can be written in scalar form as six equations, namely,

$$\begin{aligned} \Sigma F_x &= 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0 \\ \Sigma M_x &= 0 \quad \Sigma M_y = 0 \quad \Sigma M_z = 0 \end{aligned} \tag{1–2}$$

Often in engineering practice the loading on a body can be represented as a system of *coplanar forces*. If this is the case, and the forces lie in the x–y plane, then the conditions for equilibrium of the body can be specified with only three scalar equilibrium equations; that is,

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M_O &= 0 \end{aligned} \tag{1–3}$$



In order to design the horizontal members of this building frame, it is first necessary to find the internal loadings at various points along their length.

Here all the moments are summed about point O and so they will be directed along the z axis.

Successful application of the equations of equilibrium requires complete specification of all the known and unknown forces that act *on* the body, and so ***the best way to account for all these forces is to draw the body's free-body diagram***.

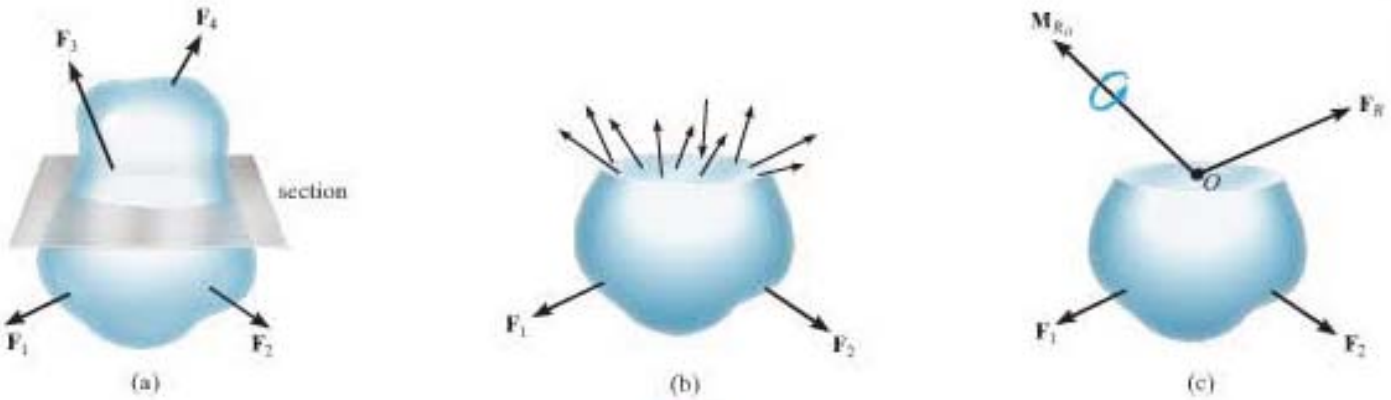


Fig. 1–2

Internal Resultant Loadings. In mechanics of materials, statics is primarily used to determine the resultant loadings that act within a body. For example, consider the body shown in Fig. 1–2*a*, which is held in equilibrium by the four external forces.* In order to obtain the internal loadings acting on a specific region within the body, it is necessary to pass an imaginary section or "cut" through the region where the internal loadings are to be determined. The two parts of the body are then separated, and a free-body diagram of one of the parts is drawn, Fig. 1–2*b*. Notice that there is actually a distribution of internal force acting on the "exposed" area of the section. These forces represent the effects of the material of the top part of the body acting on the adjacent material of the bottom part.

Although the exact distribution of this internal loading may be *unknown*, we can use the equations of equilibrium to relate the external forces on the bottom part of the body to the distribution's *resultant force and moment*, $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$, *at any specific point* O on the sectioned area, Fig. 1–2*c*. It will be shown in later portions of the text that point O is most often chosen at the *centroid* of the sectioned area, and so we will always choose this location for O, unless otherwise stated. Also, if a member is long and slender, as in the case of a rod or beam, the section to be considered is generally taken *perpendicular* to the longitudinal axis of the member. This section is referred to as the ***cross section***.

*The body's weight is not shown, since it is assumed to be quite small, and therefore negligible compared with the other loads.

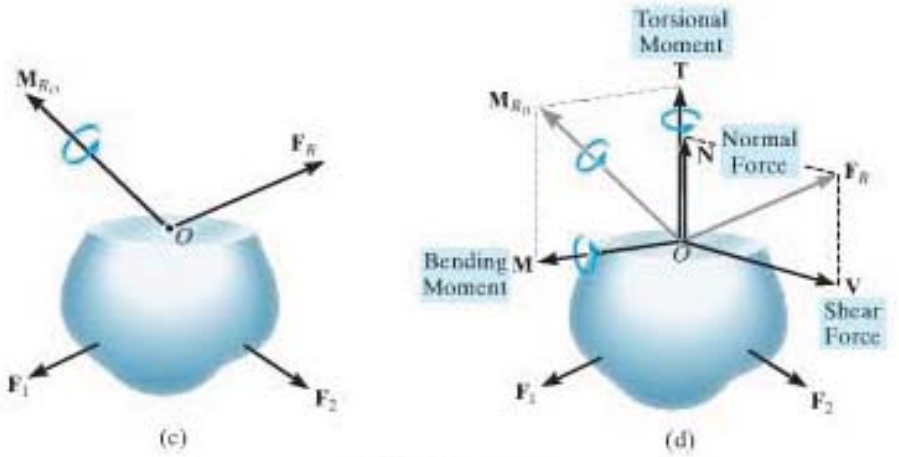


Fig. 1–2 (cont.)

Three Dimensions. Later in this text we will show how to relate the resultant loadings, $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$, to the *distribution of force* on the sectioned area, and thereby develop equations that can be used for analysis and design. To do this, however, the components of $\mathbf{F}_R$ and $\mathbf{M}_{R_O}$ acting both normal and perpendicular to the sectioned area must be considered, Fig. 1–2*d*. Four different types of resultant loadings can then be defined as follows:

Normal force, N. This force acts perpendicular to the area. It is developed whenever the external loads tend to push or pull on the two segments of the body.

Shear force, V. The shear force lies in the plane of the area and it is developed when the external loads tend to cause the two segments of the body to slide over one another.

Torsional moment or torque, T. This effect is developed when the external loads tend to twist one segment of the body with respect to the other about an axis perpendicular to the area.

Bending moment, M. The bending moment is caused by the external loads that tend to bend the body about an axis lying within the plane of the area.

In this text, note that graphical representation of a moment or torque is shown in three dimensions as a vector with an associated curl. By the *right-hand rule*, the thumb gives the arrowhead sense of this vector and the fingers or curl indicate the tendency for rotation (twisting or bending).

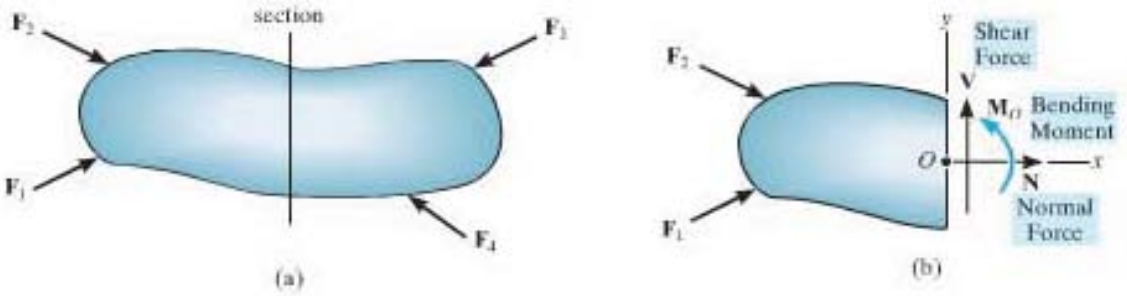


Fig. 1–3

Coplanar Loadings. If the body is subjected to a *coplanar system of forces*, Fig. 1–3*a*, then only normal-force, shear-force, and bending-moment components will exist at the section, Fig. 1–3*b*. If we use the x, y, z coordinate axes, as shown on the left segment, then $\mathbf{N}$ can be obtained by applying $\Sigma F_x = 0$, and $\mathbf{V}$ can be obtained from $\Sigma F_y = 0$. Finally, the bending moment $\mathbf{M}_O$ can be determined by summing moments about point O (the z axis), $\Sigma M_O = 0$, in order to eliminate the moments caused by the unknowns $\mathbf{N}$ and $\mathbf{V}$.

Important Points

- *Mechanics of materials* is a study of the relationship between the external loads applied to a body and the stress and strain caused by the internal loads within the body.
- External forces can be applied to a body as *distributed* or *concentrated surface loadings*, or as *body forces* that act throughout the volume of the body.
- Linear distributed loadings produce a *resultant force* having a *magnitude* equal to the *area* under the load diagram, and having a *location* that passes through the *centroid* of this area.
- A support produces a *force* in a particular direction on its attached member if it *prevents translation* of the member in that direction, and it produces a *couple moment* on the member if it *prevents rotation*.
- The equations of equilibrium $\Sigma \mathbf{F} = \mathbf{0}$ and $\Sigma \mathbf{M} = \mathbf{0}$ must be satisfied in order to prevent a body from translating with accelerated motion and from rotating.
- When applying the equations of equilibrium, it is important to first draw the free-body diagram for the body in order to account for all the terms in the equations.
- The method of sections is used to determine the internal resultant loadings acting on the surface of the sectioned body. In general, these resultants consist of a normal force, shear force, torsional moment, and bending moment.

1

Procedure for Analysis

The resultant *internal* loadings at a point located on the section of a body can be obtained using the method of sections. This requires the following steps.

Support Reactions.

- First decide which segment of the body is to be considered. If the segment has a support or connection to another body, then *before* the body is sectioned, it will be necessary to determine the reactions acting on the chosen segment. To do this draw the free-body diagram of the *entire body* and then apply the necessary equations of equilibrium to obtain these reactions.

Free-Body Diagram.

- Keep all external distributed loadings, couple moments, torques, and forces in their *exact locations*, before passing an imaginary section through the body at the point where the resultant internal loadings are to be determined.
- Draw a free-body diagram of one of the "cut" segments and indicate the unknown resultants **N**, **V**, **M**, and **T** at the section. These resultants are normally placed at the point representing the geometric center or *centroid* of the sectioned area.
- If the member is subjected to a *coplanar* system of forces, only **N**, **V**, and **M** act at the centroid.
- Establish the x, y, z coordinate axes with origin at the centroid and show the resultant internal loadings acting along the axes.

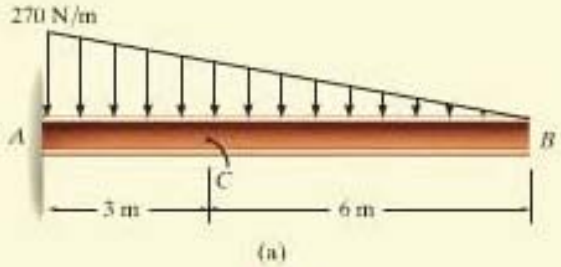
Equations of Equilibrium.

- Moments should be summed at the section, about each of the coordinate axes where the resultants act. Doing this eliminates the unknown forces **N** and **V** and allows a direct solution for **M** (and **T**).
- If the solution of the equilibrium equations yields a negative value for a resultant, the assumed *directional sense* of the resultant is *opposite* to that shown on the free-body diagram.

The following examples illustrate this procedure numerically and also provide a review of some of the important principles of statics.

EXAMPLE 1.1

Determine the resultant internal loadings acting on the cross section at C of the cantilevered beam shown in Fig. 1–4*a*.



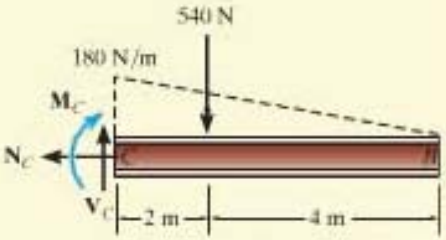
(a)

Fig. 1–4

SOLUTION

Support Reactions. The support reactions at A do not have to be determined if segment CB is considered.

Free-Body Diagram. The free-body diagram of segment CB is shown in Fig. 1–4*b*. It is important to keep the distributed loading on the segment until *after* the section is made. Only then should this loading be replaced by a single resultant force. Notice that the intensity of the distributed loading at C is found by proportion, i.e., from Fig. 1–4*a*, $w/6\text{ m} = (270\text{ N/m})/9\text{ m}$, $w = 180\text{ N/m}$. The magnitude of the resultant of the distributed load is equal to the area under the loading curve (triangle) and acts through the centroid of this area. Thus, $F = \frac{1}{2}(180\text{ N/m})(6\text{ m}) = 540\text{ N}$, which acts $\frac{1}{3}(6\text{ m}) = 2\text{ m}$ from C as shown in Fig. 1–4*b*.



(b)

Equations of Equilibrium. Applying the equations of equilibrium we have

$$\stackrel{+}{\rightarrow}\Sigma F_x = 0; \qquad -N_C = 0$$

$$N_C = 0 \qquad \textit{Ans.}$$

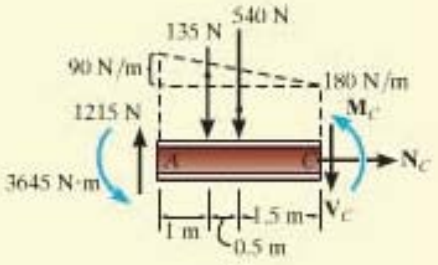
$$+\uparrow\Sigma F_y = 0; \qquad V_C - 540\text{ N} = 0$$

$$V_C = 540\text{ N} \qquad \textit{Ans.}$$

$$\circlearrowleft+\Sigma M_C = 0; \qquad -M_C - 540\text{ N}(2\text{ m}) = 0$$

$$M_C = -1080\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: The negative sign indicates that $\mathbf{M}_C$ acts in the opposite direction to that shown on the free-body diagram. Try solving this problem using segment AC, by first obtaining the support reactions at A, which are given in Fig. 1–4*c*.



(c)

EXAMPLE 1.2

Determine the resultant internal loadings acting on the cross section at C of the machine shaft shown in Fig. 1–5*a*. The shaft is supported by journal bearings at A and B, which only exert vertical forces on the shaft.

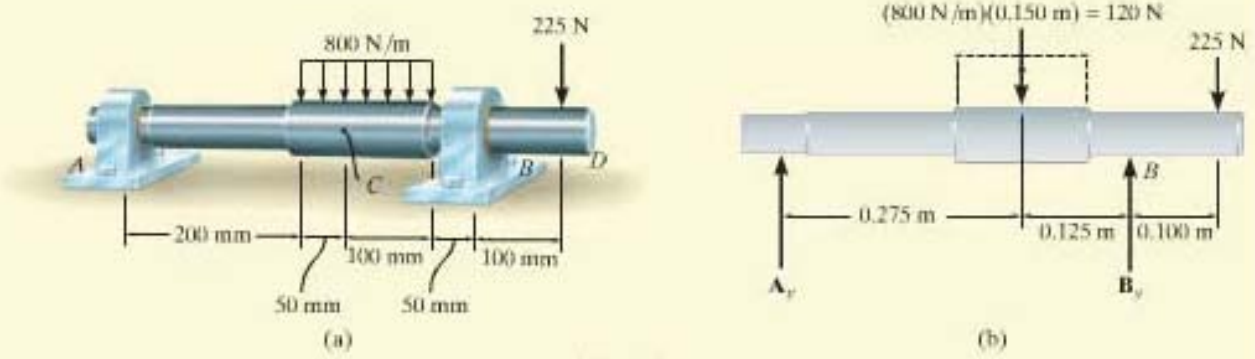


Fig. 1–5

SOLUTION

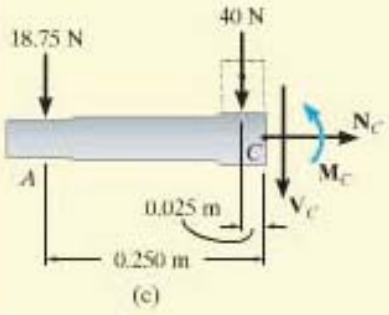
We will solve this problem using segment AC of the shaft.

Support Reactions. The free-body diagram of the entire shaft is shown in Fig. 1–5*b*. Since segment AC is to be considered, only the reaction at A has to be determined. Why?

$$\zeta + \Sigma M_B = 0;\quad -A_y(0.400\text{ m}) + 120\text{ N}(0.125\text{ m}) - 225\text{ N}(0.100\text{ m}) = 0$$

$$A_y = -18.75\text{ N}$$

The negative sign indicates that $\mathbf{A}_y$ acts in the *opposite sense* to that shown on the free-body diagram.



Free-Body Diagram. The free-body diagram of segment AC is shown in Fig. 1–5*c*.

Equations of Equilibrium.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0;\qquad N_C = 0 \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0;\qquad -18.75\text{ N} - 40\text{ N} - V_C = 0$$

$$V_C = -58.8\text{ N} \qquad \textit{Ans.}$$

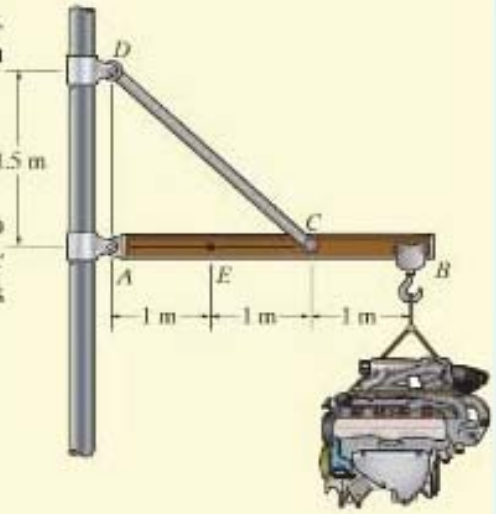
$$\zeta + \Sigma M_C = 0;\qquad M_C + 40\text{ N}(0.025\text{ m}) + 18.75\text{ N}(0.250\text{ m}) = 0$$

$$M_C = -5.69\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: The negative signs for V_C and M_C indicate they act in the opposite directions on the free-body diagram. As an exercise, calculate the reaction at B and try to obtain the same results using segment CBD of the shaft.

EXAMPLE 1.3

The 500-kg engine is suspended from the crane boom in Fig. 1–6*a*. Determine the resultant internal loadings acting on the cross section of the boom at point *E*.



(a)

SOLUTION

Support Reactions. We will consider segment *AE* of the boom so we must first determine the pin reactions at *A*. Notice that member *CD* is a two-force member. The free-body diagram of the boom is shown in Fig. 1–6*b*. Applying the equations of equilibrium,

$$\curvearrowleft + \Sigma M_A = 0; \quad F_{CD}\left(\tfrac{3}{5}\right)(2\text{ m}) - [500(9.81)\text{ N}](3\text{ m}) = 0$$

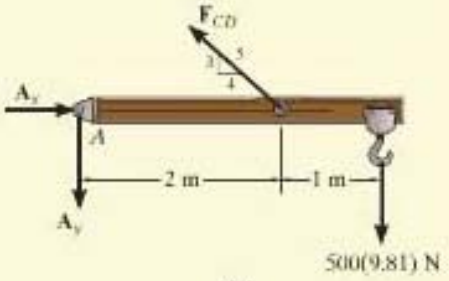
$$F_{CD} = 12\,262.5\text{ N}$$

$$\pm \Sigma F_x = 0; \quad A_x - (12\,262.5\text{ N})\left(\tfrac{4}{5}\right) = 0$$

$$A_x = 9810\text{ N}$$

$$+\uparrow \Sigma F_y = 0; \quad -A_y + (12\,262.5\text{ N})\left(\tfrac{3}{5}\right) - 500(9.81)\text{ N} = 0$$

$$A_y = 2452.5\text{ N}$$



(b)

Free-Body Diagram. The free-body diagram of segment *AE* is shown in Fig. 1–6*c*.

Equations of Equilibrium.

$$\pm \Sigma F_x = 0; \quad N_E + 9810\text{ N} = 0$$

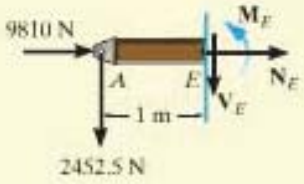
$$N_E = -9810\text{ N} = -9.81\text{ kN} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \quad -V_E - 2452.5\text{ N} = 0$$

$$V_E = -2452.5\text{ N} = -2.45\text{ kN} \qquad \textit{Ans.}$$

$$\curvearrowleft + \Sigma M_E = 0; \quad M_E + (2452.5\text{ N})(1\text{ m}) = 0$$

$$M_E = -2452.5\text{ N}\cdot\text{m} = -2.45\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



(c)

Fig. 1–6

EXAMPLE 1.4

Determine the resultant internal loadings acting on the cross section at G of the beam shown in Fig. 1–7*a*. Each joint is pin connected.

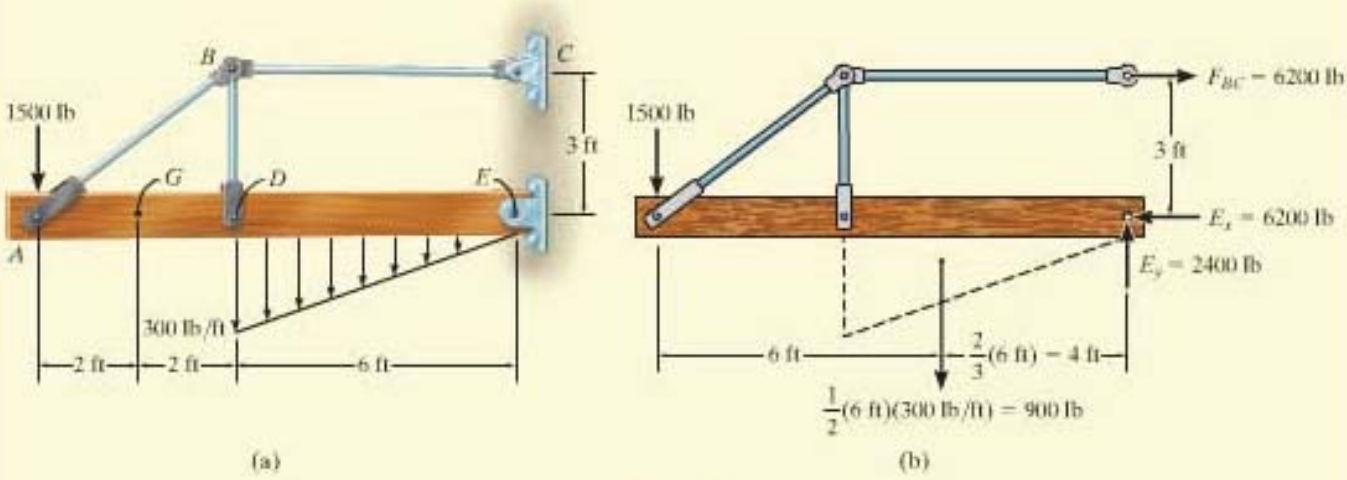
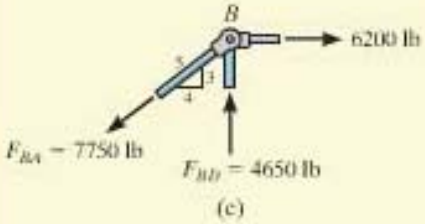


Fig. 1–7

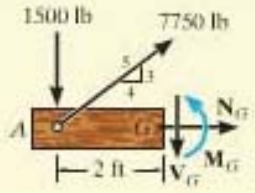
SOLUTION

Support Reactions. Here we will consider segment AG. The free-body diagram of the *entire* structure is shown in Fig. 1–7*b*. Verify the calculated reactions at E and C. In particular, note that BC is a *two-force member* since only two forces act on it. For this reason the force at C must act along BC, which is horizontal as shown.

Since BA and BD are also two-force members, the free-body diagram of joint B is shown in Fig. 1–7*c*. Again, verify the magnitudes of forces $\mathbf{F}_{BA}$ and $\mathbf{F}_{BD}$.



Free-Body Diagram. Using the result for $\mathbf{F}_{BA}$, the free-body diagram of segment AG is shown in Fig. 1–7*d*.



Equations of Equilibrium.

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 7750\text{ lb}\left(\tfrac{4}{5}\right) + N_G = 0 \qquad N_G = -6200\text{ lb} \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \quad -1500\text{ lb} + 7750\text{ lb}\left(\tfrac{3}{5}\right) - V_G = 0$$

$$V_G = 3150\text{ lb} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_G = 0; \quad M_G - (7750\text{ lb})\left(\tfrac{3}{5}\right)(2\text{ ft}) + 1500\text{ lb}(2\text{ ft}) = 0$$

$$M_G = 6300\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

EXAMPLE 1.5

Determine the resultant internal loadings acting on the cross section at B of the pipe shown in Fig. 1–8*a*. The pipe has a mass of 2 kg/m and is subjected to both a vertical force of 50 N and a couple moment of 70 N · m at its end A. It is fixed to the wall at C.

SOLUTION

The problem can be solved by considering segment AB, so we do not need to calculate the support reactions at C.

Free-Body Diagram. The x, y, z axes are established at B and the free-body diagram of segment AB is shown in Fig. 1–8*b*. The resultant force and moment components at the section are assumed to act in the positive coordinate directions and to pass through the *centroid* of the cross-sectional area at B. The weight of each segment of pipe is calculated as follows:

$$W_{BD} = (2\ \text{kg/m})(0.5\ \text{m})(9.81\ \text{N/kg}) = 9.81\ \text{N}$$

$$W_{AD} = (2\ \text{kg/m})(1.25\ \text{m})(9.81\ \text{N/kg}) = 24.525\ \text{N}$$

These forces act through the center of gravity of each segment.

Equations of Equilibrium. Applying the six scalar equations of equilibrium, we have*

$$\Sigma F_x = 0; \qquad (F_B)_x = 0 \qquad \textit{Ans.}$$

$$\Sigma F_y = 0; \qquad (F_B)_y = 0 \qquad \textit{Ans.}$$

$$\Sigma F_z = 0; \qquad (F_B)_z - 9.81\ \text{N} - 24.525\ \text{N} - 50\ \text{N} = 0$$

$$(F_B)_z = 84.3\ \text{N} \qquad \textit{Ans.}$$

$$\Sigma (M_B)_x = 0; \qquad (M_B)_x + 70\ \text{N} \cdot \text{m} - 50\ \text{N}\,(0.5\ \text{m}) - 24.525\ \text{N}\,(0.5\ \text{m}) - 9.81\ \text{N}\,(0.25\ \text{m}) = 0$$

$$(M_B)_x = -30.3\ \text{N} \cdot \text{m} \qquad \textit{Ans.}$$

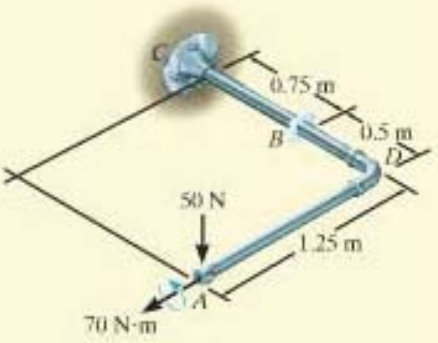
$$\Sigma (M_B)_y = 0; \quad (M_B)_y + 24.525\ \text{N}\,(0.625\ \text{m}) + 50\ \text{N}\,(1.25\ \text{m}) = 0$$

$$(M_B)_y = -77.8\ \text{N} \cdot \text{m} \qquad \textit{Ans.}$$

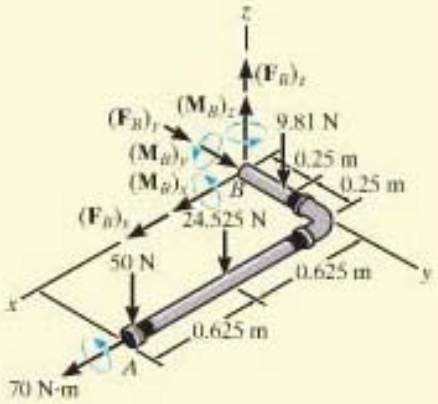
$$\Sigma (M_B)_z = 0; \qquad (M_B)_z = 0 \qquad \textit{Ans.}$$

NOTE: What do the negative signs for $(M_B)_x$ and $(M_B)_y$ indicate? Note that the normal force $N_B = (F_B)_y = 0$, whereas the shear force is $V_B = \sqrt{(0)^2 + (84.3)^2} = 84.3\ \text{N}$. Also, the torsional moment is $T_B = (M_B)_y = 77.8\ \text{N} \cdot \text{m}$ and the bending moment is $M_B = \sqrt{(30.3)^2 + (0)^2} = 30.3\ \text{N} \cdot \text{m}$.

*The *magnitude* of each moment about an axis is equal to the magnitude of each force times the perpendicular distance from the axis to the line of action of the force. The *direction* of each moment is determined using the right-hand rule, with positive moments (thumb) directed along the positive coordinate axes.



(a)

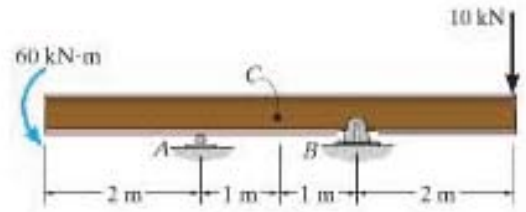


(b)

Fig. 1–8

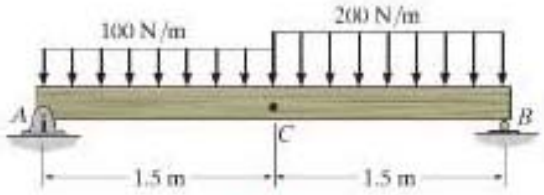
1 FUNDAMENTAL PROBLEMS

F1–1. Determine the internal normal force, shear force, and bending moment at point C in the beam.



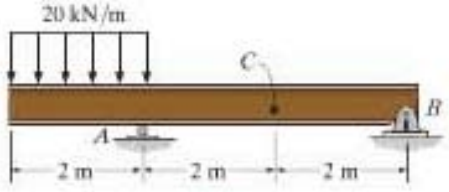
F1–1

F1–2. Determine the internal normal force, shear force, and bending moment at point C in the beam.



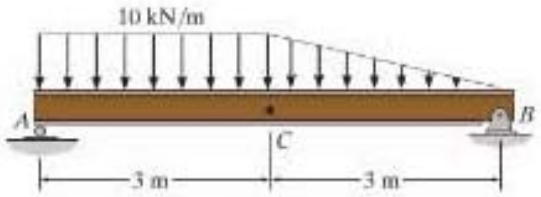
F1–2

F1–3. Determine the internal normal force, shear force, and bending moment at point C in the beam.



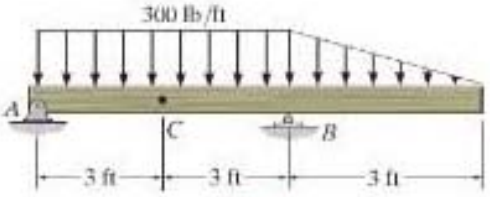
F1–3

F1–4. Determine the internal normal force, shear force, and bending moment at point C in the beam.



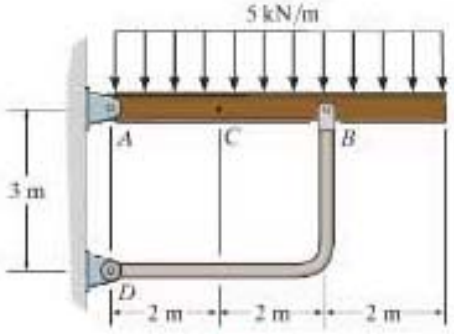
F1–4

F1–5. Determine the internal normal force, shear force, and bending moment at point C in the beam.



F1–5

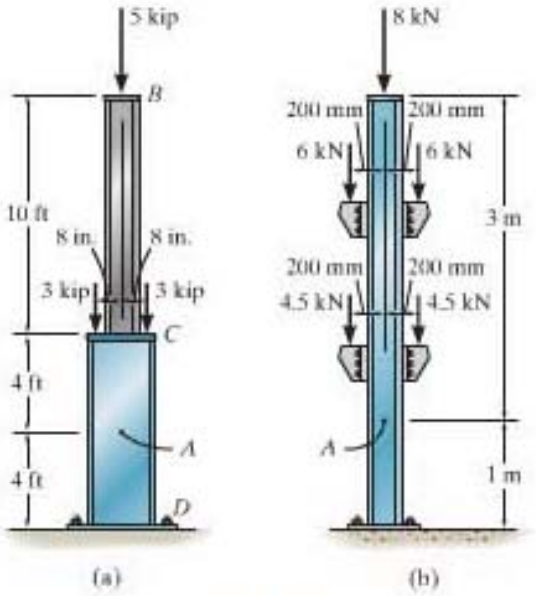
F1–6. Determine the internal normal force, shear force, and bending moment at point C in the beam.



F1–6

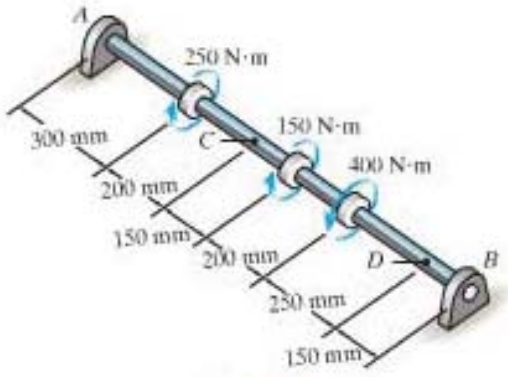
PROBLEMS

1–1. Determine the resultant internal normal force acting on the cross section through point A in each column. In (a), segment BC weighs 180 lb/ft and segment CD weighs 250 lb/ft. In (b), the column has a mass of 200 kg/m.



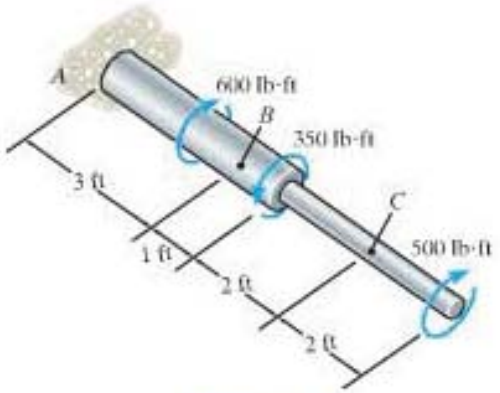
Prob. 1–1

1–2. Determine the resultant internal torque acting on the cross sections through points C and D. The support bearings at A and B allow free turning of the shaft.



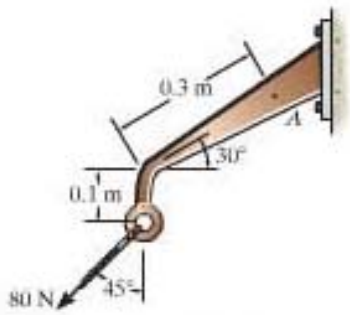
Prob. 1–2

1–3. Determine the resultant internal torque acting on the cross sections through points B and C.



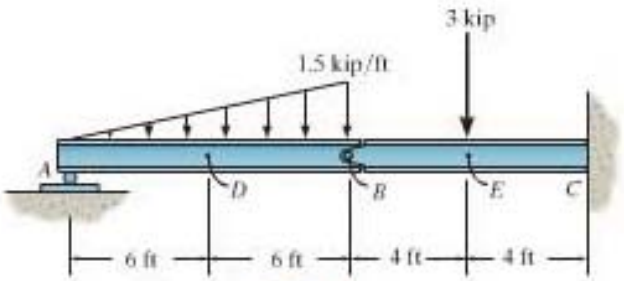
Prob. 1–3

***1–4.** A force of 80 N is supported by the bracket as shown. Determine the resultant internal loadings acting on the section through point A.



Prob. 1–4

•1–5. Determine the resultant internal loadings in the beam at cross sections through points D and E. Point E is just to the right of the 3-kip load.

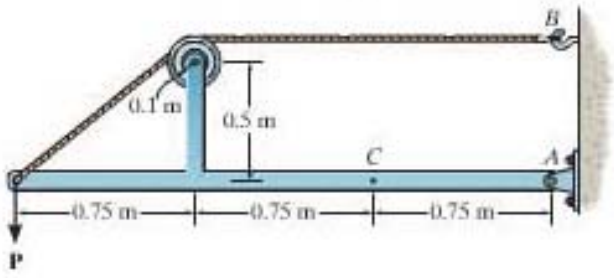


Prob. 1–5

1

1–6. Determine the normal force, shear force, and moment at a section through point C. Take $P = 8$ kN.

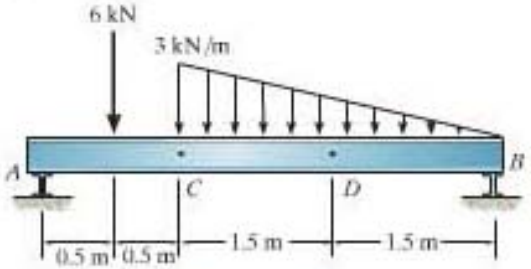
1–7. The cable will fail when subjected to a tension of 2 kN. Determine the largest vertical load P the frame will support and calculate the internal normal force, shear force, and moment at the cross section through point C for this loading.



Probs. 1–6/7

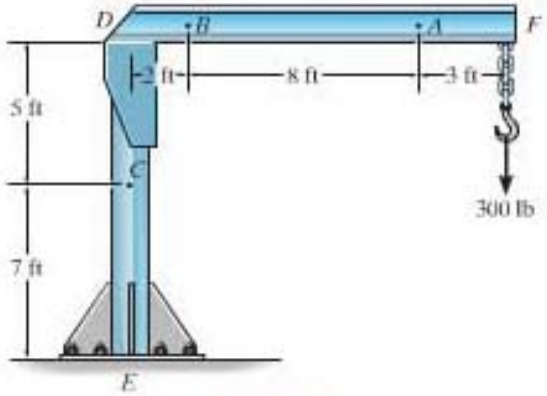
***1–8.** Determine the resultant internal loadings on the cross section through point C. Assume the reactions at the supports A and B are vertical.

•1–9. Determine the resultant internal loadings on the cross section through point D. Assume the reactions at the supports A and B are vertical.



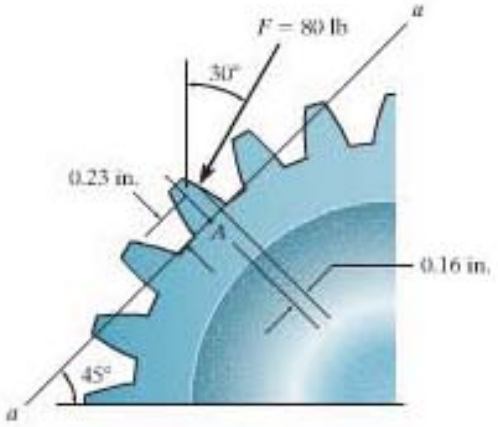
Probs. 1–8/9

1–10. The boom DF of the jib crane and the column DE have a uniform weight of 50 lb/ft. If the hoist and load weigh 300 lb, determine the resultant internal loadings in the crane on cross sections through points A, B, and C.



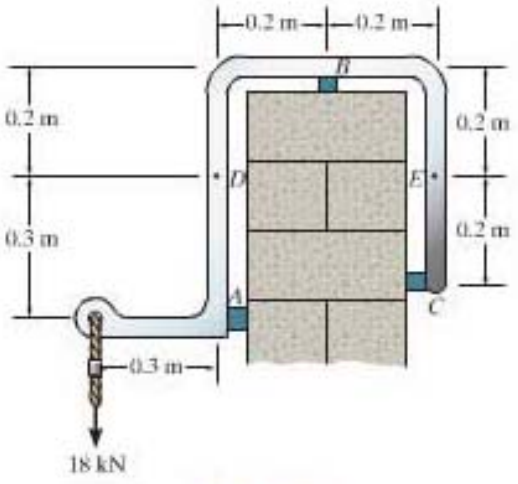
Prob. 1–10

1–11. The force $F = 80$ lb acts on the gear tooth. Determine the resultant internal loadings on the root of the tooth, i.e., at the centroid point A of section a–a.



Prob. 1–11

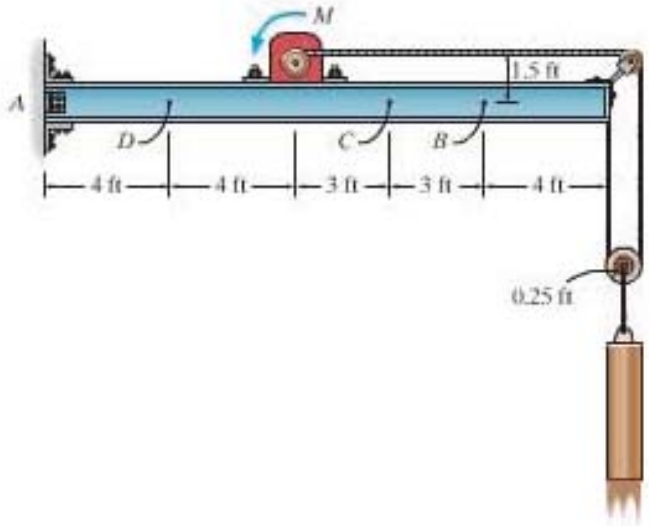
***1–12.** The sky hook is used to support the cable of a scaffold over the side of a building. If it consists of a smooth rod that contacts the parapet of a wall at points A, B, and C, determine the normal force, shear force, and moment on the cross section at points D and E.



Prob. 1–12

•1–13. The 800-lb load is being hoisted at a constant speed using the motor M, which has a weight of 90 lb. Determine the resultant internal loadings acting on the cross section through point B in the beam. The beam has a weight of 40 lb/ft and is fixed to the wall at A.

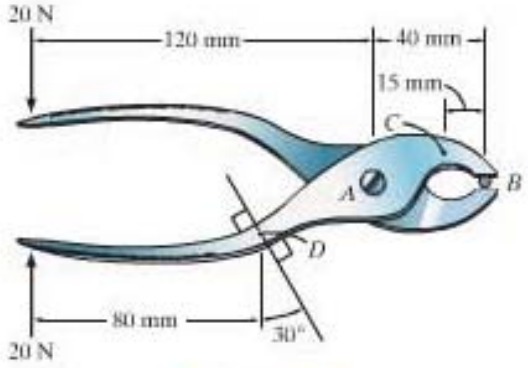
1–14. Determine the resultant internal loadings acting on the cross section through points C and D of the beam in Prob. 1–13.



Probs. 1–13/14

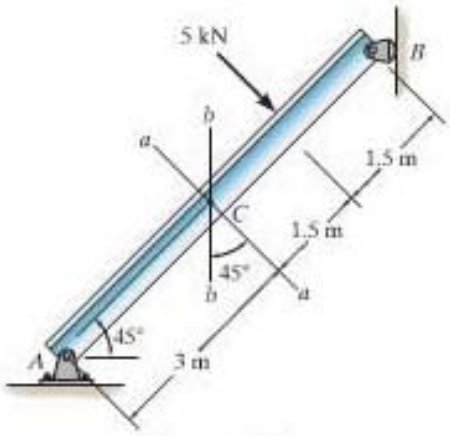
1–15. Determine the resultant internal loading on the cross section through point C of the pliers. There is a pin at A, and the jaws at B are smooth.

***1–16.** Determine the resultant internal loading on the cross section through point D of the pliers.



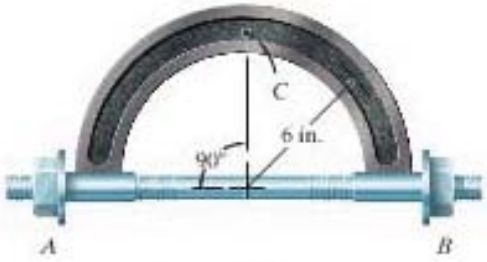
Probs. 1–15/16

•1–17. Determine resultant internal loadings acting on section a–a and section b–b. Each section passes through the centerline at point C.



Prob. 1–17

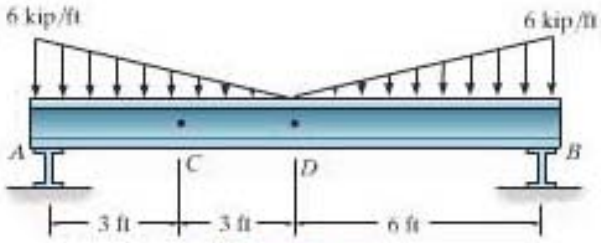
1–18. The bolt shank is subjected to a tension of 80 lb. Determine the resultant internal loadings acting on the cross section at point C.



Prob. 1–18

1–19. Determine the resultant internal loadings acting on the cross section through point C. Assume the reactions at the supports A and B are vertical.

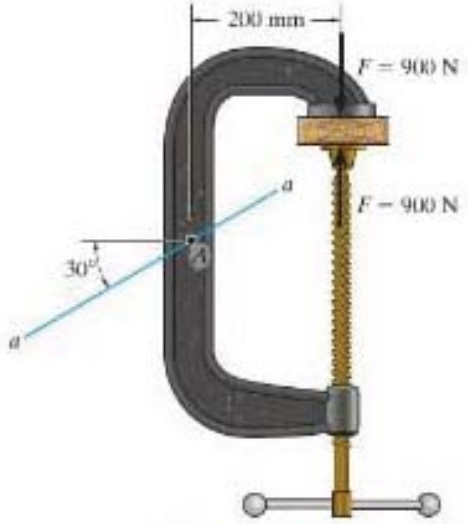
***1–20.** Determine the resultant internal loadings acting on the cross section through point D. Assume the reactions at the supports A and B are vertical.



Probs. 1–19/20

1

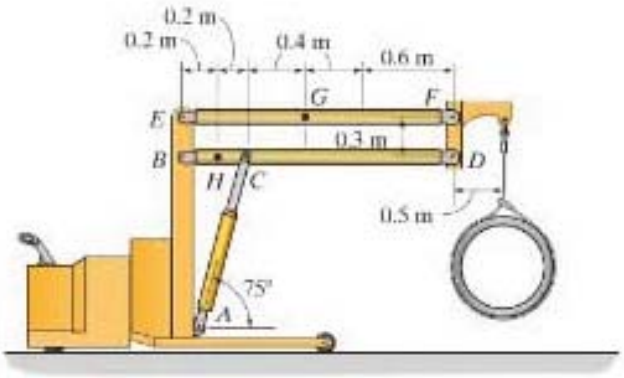
•1–21. The forged steel clamp exerts a force of $F = 900$ N on the wooden block. Determine the resultant internal loadings acting on section a–a passing through point A.



Prob. 1–21

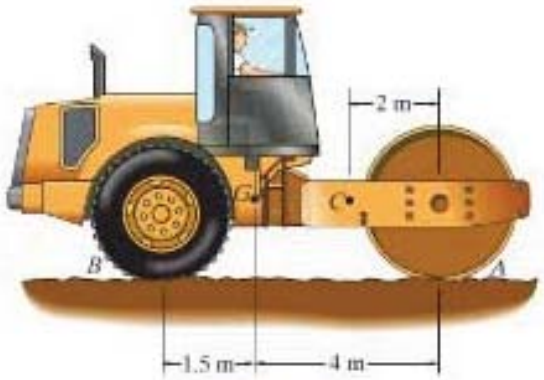
1–22. The floor crane is used to lift a 600-kg concrete pipe. Determine the resultant internal loadings acting on the cross section at G.

1–23. The floor crane is used to lift a 600-kg concrete pipe. Determine the resultant internal loadings acting on the cross section at H.



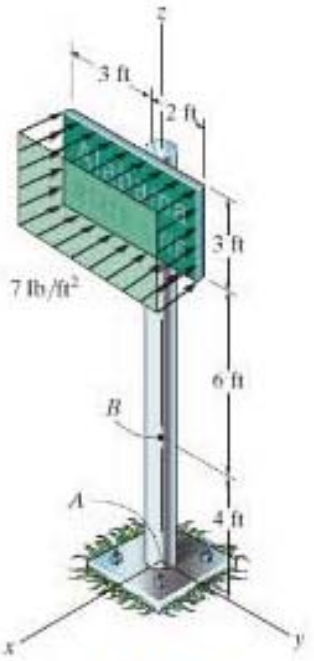
Probs. 1–22/23

***1–24.** The machine is moving with a constant velocity. It has a total mass of 20 Mg, and its center of mass is located at G, excluding the front roller. If the front roller has a mass of 5 Mg, determine the resultant internal loadings acting on point C of each of the two side members that support the roller. Neglect the mass of the side members. The front roller is free to roll.



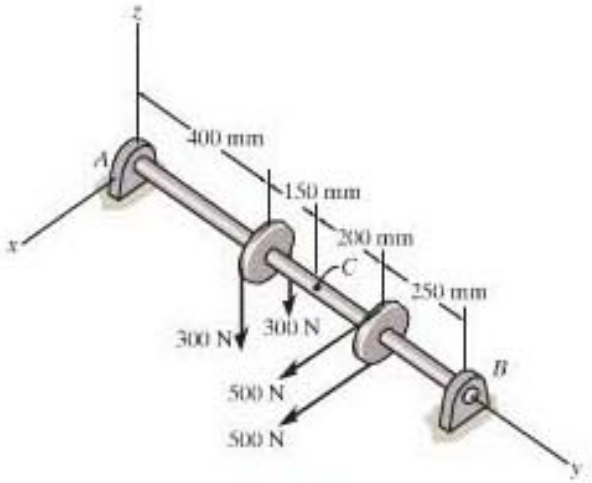
Prob. 1–24

•1–25. Determine the resultant internal loadings acting on the cross section through point B of the signpost. The post is fixed to the ground and a uniform pressure of 7 lb/ft^2 acts perpendicular to the face of the sign.



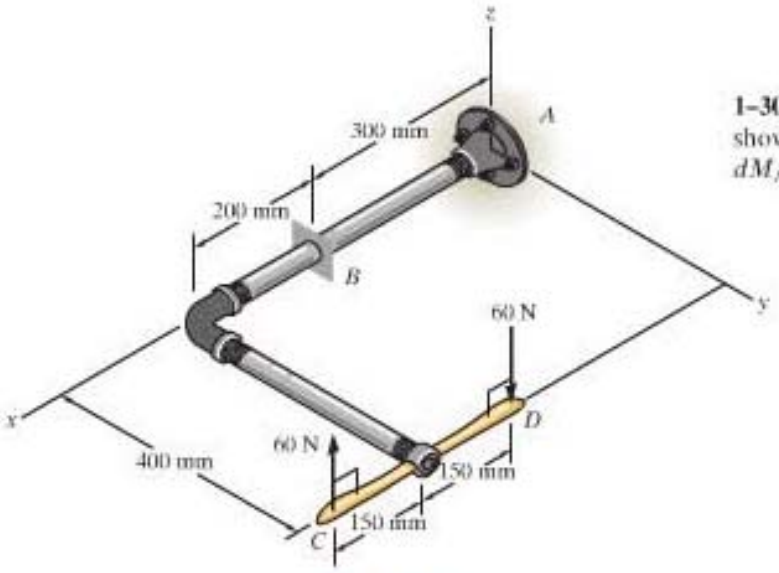
Prob. 1–25

1–26. The shaft is supported at its ends by two bearings A and B and is subjected to the forces applied to the pulleys fixed to the shaft. Determine the resultant internal loadings acting on the cross section located at point C. The 300-N forces act in the $-z$ direction and the 500-N forces act in the $+x$ direction. The journal bearings at A and B exert only x and z components of force on the shaft.



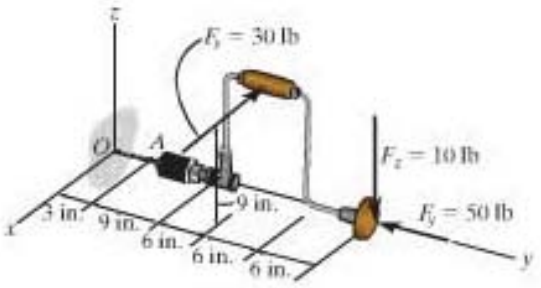
Prob. 1–26

1–27. The pipe has a mass of 12 kg/m. If it is fixed to the wall at A, determine the resultant internal loadings acting on the cross section at B. Neglect the weight of the wrench CD.



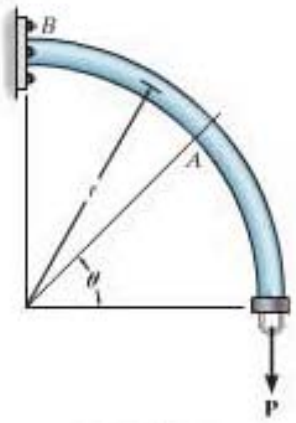
Prob. 1–27

***1–28.** The brace and drill bit is used to drill a hole at O. If the drill bit jams when the brace is subjected to the forces shown, determine the resultant internal loadings acting on the cross section of the drill bit at A.



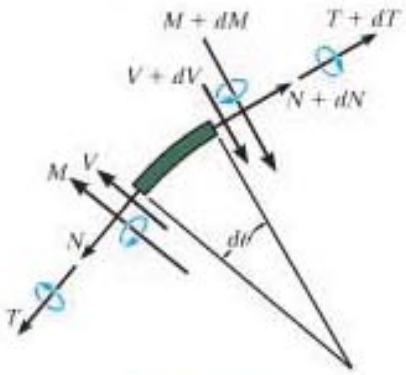
Prob. 1–28

•1–29. The curved rod has a radius r and is fixed to the wall at B. Determine the resultant internal loadings acting on the cross section through A which is located at an angle θ from the horizontal.



Prob. 1–29

1–30. A differential element taken from a curved bar is shown in the figure. Show that $dN/d\theta = V$, $dV/d\theta = -N$, $dM/d\theta = -T$, and $dT/d\theta = M$.



Prob. 1–30

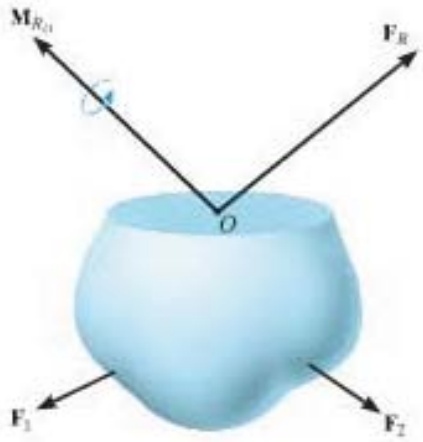


Fig. 1–9

1.3 Stress

It was stated in Section 1.2 that the force and moment acting at a specified point O on the sectioned area of the body, Fig. 1–9, represents the resultant effects of the actual *distribution of loading* acting over the sectioned area, Fig. 1–10*a*. Obtaining this *distribution* is of primary importance in mechanics of materials. To solve this problem it is necessary to establish the concept of stress.

We begin by considering the sectioned area to be subdivided into small areas, such as ΔA shown in Fig. 1–10*a*. As we reduce ΔA to a smaller and smaller size, we must make two assumptions regarding the properties of the material. We will consider the material to be ***continuous***, that is, to consist of a *continuum* or uniform distribution of matter having no voids. Also, the material must be ***cohesive***, meaning that all portions of it are connected together, without having breaks, cracks, or separations. A typical finite yet very small force $\Delta\mathbf{F}$, acting on ΔA, is shown in Fig. 1–10*a*. This force, like all the others, will have a unique direction, but for further discussion we will replace it by its *three components*, namely, $\Delta\mathbf{F}_x$, $\Delta\mathbf{F}_y$, and $\Delta\mathbf{F}_z$, which are taken tangent, tangent, and normal to the area, respectively. As ΔA approaches zero, so do $\Delta\mathbf{F}$ and its components; however, the quotient of the force and area will, in general, approach a finite limit. This quotient is called *stress*, and as noted, it describes the *intensity of the internal force* acting on a *specific plane* (area) passing through a point.

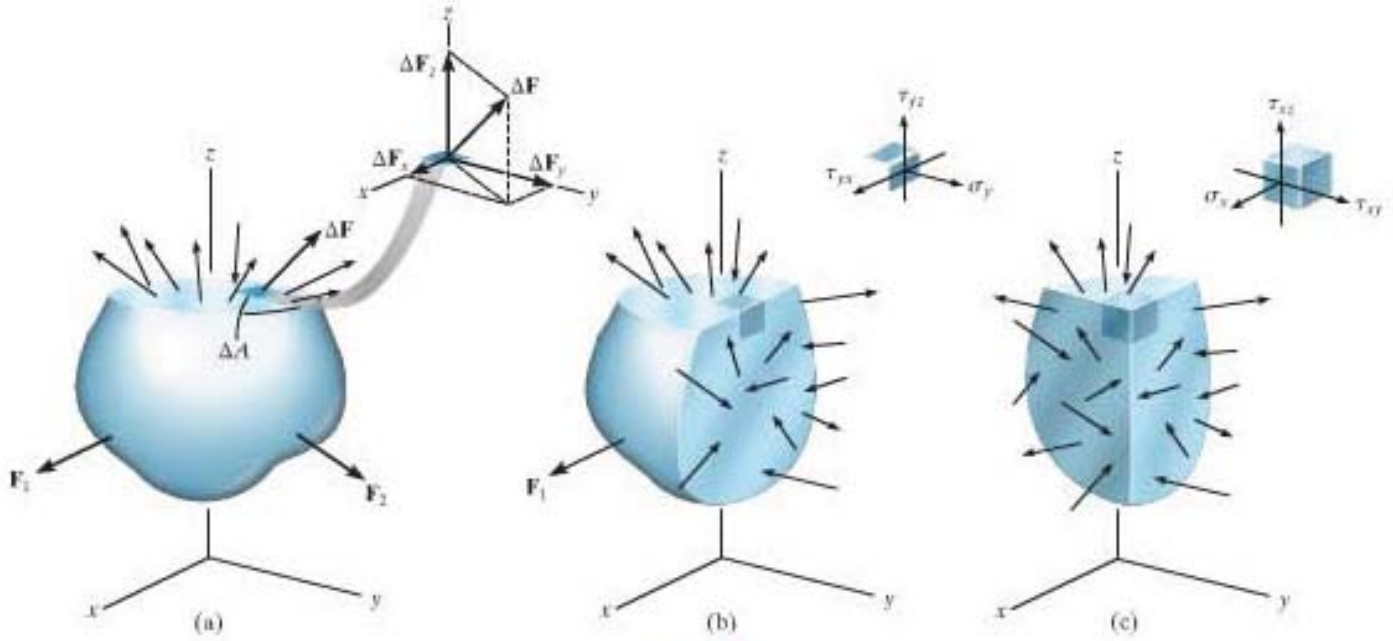


Fig. 1–10

Normal Stress. The *intensity* of the force acting normal to ΔA is defined as the ***normal stress***, σ (sigma). Since $\Delta \mathbf{F}_z$ is normal to the area then

$$\sigma_z = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \tag{1–4}$$

If the normal force or stress "pulls" on ΔA as shown in Fig. 1–10*a*, it is referred to as *tensile stress*, whereas if it "pushes" on ΔA it is called *compressive stress*.

Shear Stress. The intensity of force acting tangent to ΔA is called the ***shear stress***, τ (tau). Here we have shear stress components,

$$\tau_{zx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \qquad \tau_{zy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \tag{1–5}$$

Note that in this subscript notation z specifies the orientation of the area ΔA, Fig. 1–11, and x and y indicate the axes along which each shear stress acts.

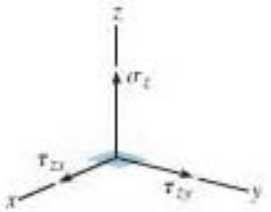


Fig. 1–11

General State of Stress. If the body is further sectioned by planes parallel to the $x-z$ plane, Fig. 1–10*b*, and the $y-z$ plane, Fig. 1–10*c*, we can then "cut out" a cubic volume element of material that represents the ***state of stress*** acting around the chosen point in the body. This state of stress is then characterized by three components acting on each face of the element, Fig. 1–12.

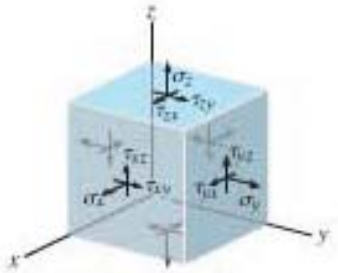


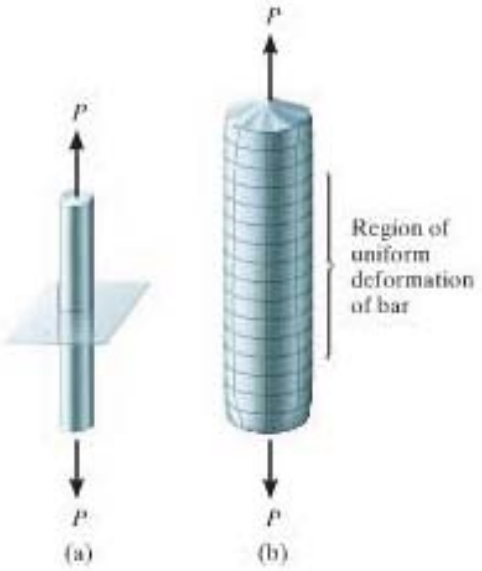
Fig. 1–12

Units. Since stress represents a force per unit area, in the International Standard or SI system, the magnitudes of both normal and shear stress are specified in the basic units of newtons per square meter (N/m^2). This unit, called a pascal ($1\ Pa = 1\ N/m^2$) is rather small, and in engineering work prefixes such as kilo- (10^3), symbolized by k, mega- (10^6), symbolized by M, or giga- (10^9), symbolized by G, are used to represent larger, more realistic values of stress.* Likewise, in the Foot-Pound-Second system of units, engineers usually express stress in pounds per square inch (psi) or kilopounds per square inch (ksi), where 1 kilopound (kip) = 1000 lb.

*Sometimes stress is expressed in units of N/mm^2, where $1\ mm = 10^{-3}\ m$. However, in the SI system, prefixes are not allowed in the denominator of a fraction and therefore it is better to use the equivalent $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$.

1

1.4 Average Normal Stress in an Axially Loaded Bar



In this section we will determine the average stress distribution acting on the cross-sectional area of an axially loaded bar such as the one shown in Fig. 1–13*a*. This bar is **prismatic** since all cross sections are the same throughout its length. When the load P is applied to the bar through the centroid of its cross-sectional area, then the bar will deform uniformly throughout the central region of its length, as shown in Fig. 1–13*b*, provided the material of the bar is both homogeneous and isotropic.

Homogeneous material has the same physical and mechanical properties throughout its volume, and ***isotropic material*** has these same properties in all directions. Many engineering materials may be approximated as being both homogeneous and isotropic as assumed here. Steel, for example, contains thousands of randomly oriented crystals in each cubic millimeter of its volume, and since most problems involving this material have a physical size that is very much larger than a single crystal, the above assumption regarding its material composition is quite realistic.

Note that anisotropic materials such as wood have different properties in different directions, and although this is the case, like wood if the anisotropy is oriented along the bar's axis, then the bar will also deform uniformly when subjected to the axial load P.

Average Normal Stress Distribution. If we pass a section through the bar, and separate it into two parts, then equilibrium requires the resultant normal force at the section to be P, Fig. 1–13*c*. Due to the *uniform* deformation of the material, it is necessary that the cross section be subjected to a *constant normal stress distribution*, Fig. 1–13*d*.

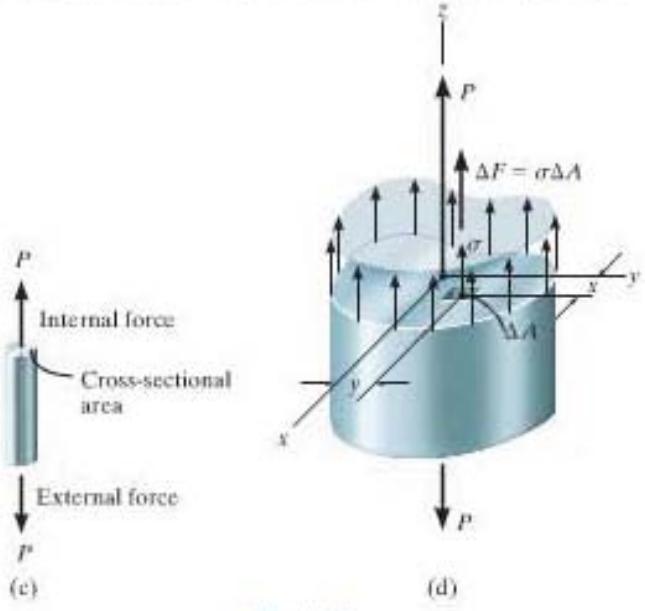


Fig. 1–13

As a result, each small area ΔA on the cross section is subjected to a force $\Delta F = \sigma\,\Delta A$, and the *sum* of these forces acting over the entire cross-sectional area must be equivalent to the internal resultant force **P** at the section. If we let $\Delta A \rightarrow dA$ and therefore $\Delta F \rightarrow dF$, then, recognizing σ is *constant*, we have

$$+\uparrow F_{R_z} = \Sigma F_z; \qquad \int dF = \int_A \sigma\, dA$$

$$P = \sigma A$$

$$\boxed{\sigma = \frac{P}{A}} \tag{1–6}$$

Here

σ = average normal stress at any point on the cross-sectional area

P = *internal resultant normal force*, which acts through the *centroid* of the cross-sectional area. P is determined using the method of sections and the equations of equilibrium

A = cross-sectional area of the bar where σ is determined

Since the internal load P passes through the centroid of the cross-section the uniform stress distribution will produce zero moments about the x and y axes passing through this point, Fig. 1–13*d*. To show this, we require the moment of P about each axis to be equal to the moment of the stress distribution about the axes, namely,

$$(M_R)_x = \Sigma M_x; \quad 0 = \int_A y\, dF = \int_A y\sigma\, dA = \sigma \int_A y\, dA$$

$$(M_R)_y = \Sigma M_y; \quad 0 = -\int_A x\, dF = -\int_A x\sigma\, dA = -\sigma \int_A x\, dA$$

These equations are indeed satisfied, since by definition of the centroid, $\int y\, dA = 0$ and $\int x\, dA = 0$. (See Appendix A.)

Equilibrium. It should be apparent that only a normal stress exists on any small volume element of material located at each point on the cross section of an axially loaded bar. If we consider vertical equilibrium of the element, Fig. 1–14, then apply the equation of force equilibrium,

$$\Sigma F_z = 0; \qquad \sigma(\Delta A) - \sigma'(\Delta A) = 0$$

$$\sigma = \sigma'$$

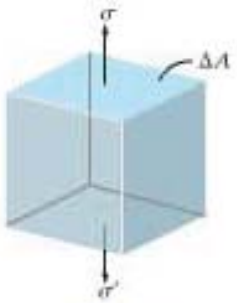


Fig. 1–14

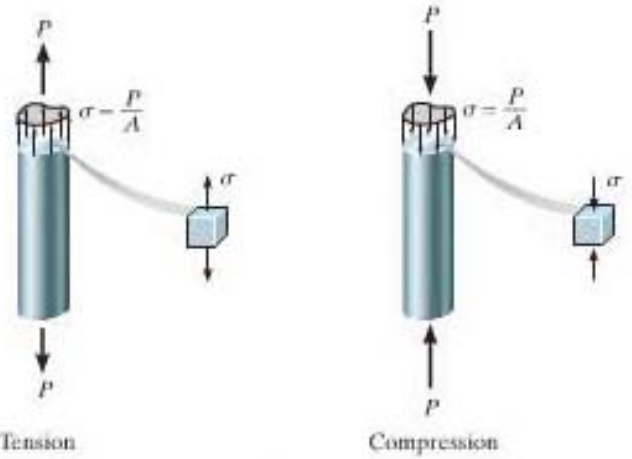


Fig. 1–15

In other words, the two normal stress components on the element must be equal in magnitude but opposite in direction. This is referred to as *uniaxial stress*.

The previous analysis applies to members subjected to either tension or compression, as shown in Fig. 1–15. As a graphical interpretation, the ***magnitude*** of the internal resultant force **P** is ***equivalent*** to the ***volume*** under the stress diagram; that is, $P = \sigma A$ (volume = height × base). Furthermore, as a consequence of the balance of moments, ***this resultant passes through the centroid of this volume***.

Although we have developed this analysis for *prismatic* bars, this assumption can be relaxed somewhat to include bars that have a *slight taper*. For example, it can be shown, using the more exact analysis of the theory of elasticity, that for a tapered bar of rectangular cross section, for which the angle between two adjacent sides is 15°, the average normal stress, as calculated by $\sigma = P/A$, is only 2.2% *less* than its value found from the theory of elasticity.



This steel tie rod is used as a hanger to suspend a portion of a staircase, and as a result it is subjected to tensile stress.

Maximum Average Normal Stress. In our analysis both the internal force P and the cross-sectional area A were *constant* along the longitudinal axis of the bar, and as a result the normal stress $\sigma = P/A$ is also *constant* throughout the bar's length. Occasionally, however, the bar may be subjected to *several* external loads along its axis, or a change in its cross-sectional area may occur. As a result, the normal stress within the bar could be different from one section to the next, and, if the *maximum* average normal stress is to be determined, then it becomes important to find the location where the ratio P/A is a *maximum*. To do this it is necessary to determine the internal force P at various sections along the bar. Here it may be helpful to show this variation by drawing an ***axial or normal force diagram***. Specifically, this diagram is a plot of the normal force P versus its position x along the bar's length. As a sign convention, P will be positive if it causes tension in the member, and negative if it causes compression. Once the internal loading throughout the bar is known, the maximum ratio of P/A can then be identified.

Important Points

- When a body subjected to external loads is sectioned, there is a distribution of force acting over the sectioned area which holds each segment of the body in equilibrium. The intensity of this internal force at a point in the body is referred to as *stress*.
- Stress is the limiting value of force per unit area, as the area approaches zero. For this definition, the material is considered to be continuous and cohesive.
- The magnitude of the stress components at a point depends upon the type of loading acting on the body, and the orientation of the element at the point.
- When a prismatic bar is made from homogeneous and isotropic material, and is subjected to an axial force acting through the centroid of the cross-sectional area, then the center region of the bar will deform uniformly. As a result, the material will be subjected *only to normal stress*. This stress is uniform or *averaged* over the cross-sectional area.

Procedure for Analysis

The equation $\sigma = P/A$ gives the *average* normal stress on the cross-sectional area of a member when the section is subjected to an internal resultant normal force **P**. For axially loaded members, application of this equation requires the following steps.

Internal Loading.

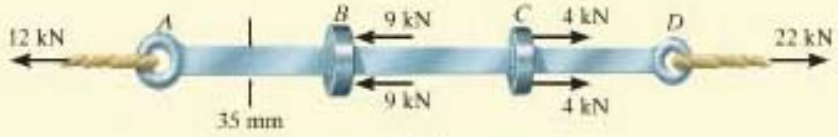
- Section the member *perpendicular* to its longitudinal axis at the point where the normal stress is to be determined and use the necessary free-body diagram and force equation of equilibrium to obtain the *internal axial force* **P** at the section.

Average Normal Stress.

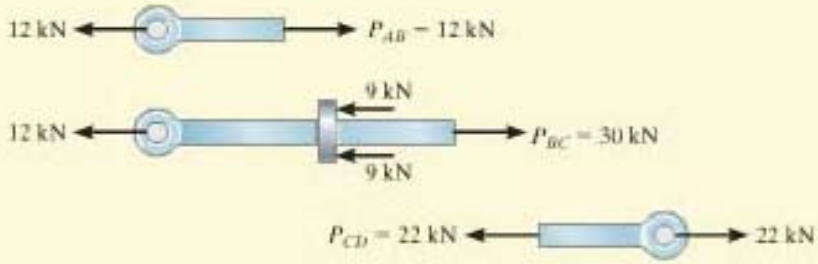
- Determine the member's cross-sectional area at the section and calculate the average normal stress $\sigma = P/A$.
- It is suggested that σ be shown acting on a small volume element of the material located at a point on the section where stress is calculated. To do this, first draw σ on the face of the element coincident with the sectioned area A. Here σ acts in the *same direction* as the internal force **P** since all the normal stresses on the cross section develop this resultant. The normal stress σ on the other face of the element acts in the opposite direction.

EXAMPLE 1.6

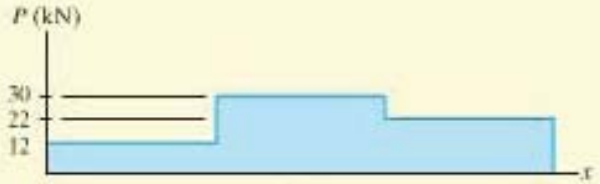
The bar in Fig. 1–16*a* has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.



(a)



(b)



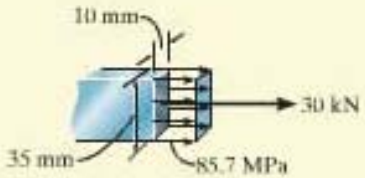
(c)

SOLUTION

Internal Loading. By inspection, the internal axial forces in regions *AB*, *BC*, and *CD* are all constant yet have different magnitudes. Using the method of sections, these loadings are determined in Fig. 1–16*b*; and the normal force diagram which represents these results graphically is shown in Fig. 1–16*c*. The largest loading is in region *BC*, where $P_{BC} = 30$ kN. Since the cross-sectional area of the bar is *constant*, the largest average normal stress also occurs within this region of the bar.

Average Normal Stress. Applying Eq. 1–6, we have

$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{30(10^3)\,\text{N}}{(0.035\,\text{m})(0.010\,\text{m})} = 85.7\,\text{MPa} \qquad \textit{Ans.}$$



(d)

Fig. 1–16

NOTE: The stress distribution acting on an arbitrary cross section of the bar within region *BC* is shown in Fig. 1–16*d*. Graphically the *volume* (or "block") represented by this distribution of stress is equivalent to the load of 30 kN; that is, 30 kN = (85.7 MPa)(35 mm)(10 mm).

EXAMPLE 1.7

The 80-kg lamp is supported by two rods AB and BC as shown in Fig. 1–17a. If AB has a diameter of 10 mm and BC has a diameter of 8 mm, determine the average normal stress in each rod.

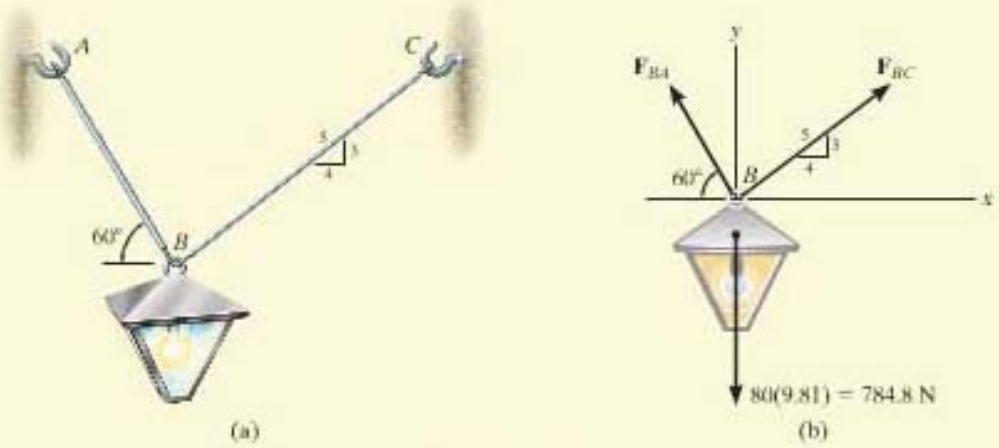


Fig. 1–17

SOLUTION

Internal Loading. We must first determine the axial force in each rod. A free-body diagram of the lamp is shown in Fig. 1–17b. Applying the equations of force equilibrium,

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad F_{BC}\left(\tfrac{4}{5}\right) - F_{BA}\cos 60° = 0$$

$$+\uparrow \Sigma F_y = 0; \quad F_{BC}\left(\tfrac{3}{5}\right) + F_{BA}\sin 60° - 784.8\ \text{N} = 0$$

$$F_{BC} = 395.2\ \text{N}, \qquad F_{BA} = 632.4\ \text{N}$$

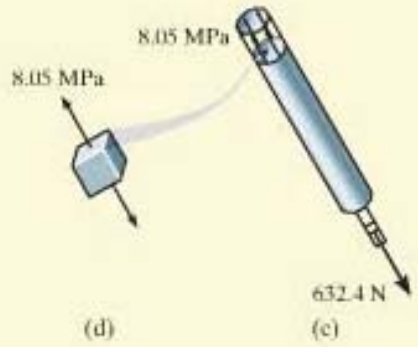
By Newton's third law of action, equal but opposite reaction, these forces subject the rods to tension throughout their length.

Average Normal Stress. Applying Eq. 1–6,

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2\ \text{N}}{\pi(0.004\ \text{m})^2} = 7.86\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.4\ \text{N}}{\pi(0.005\ \text{m})^2} = 8.05\ \text{MPa} \qquad \textit{Ans.}$$

NOTE: The average normal stress distribution acting over a cross section of rod AB is shown in Fig. 1–17c, and at a point on this cross section, an element of material is stressed as shown in Fig. 1–17d.



EXAMPLE 1.8

The casting shown in Fig. 1–18*a* is made of steel having a specific weight of $\gamma_{st} = 490 \text{ lb/ft}^3$. Determine the average compressive stress acting at points A and B.

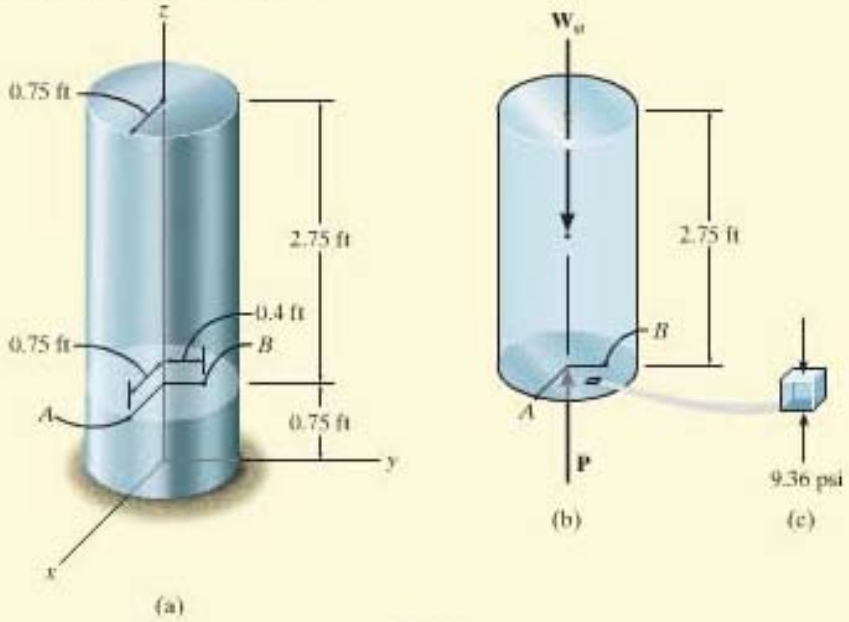


Fig. 1–18

SOLUTION

Internal Loading. A free-body diagram of the top segment of the casting where the section passes through points A and B is shown in Fig. 1–18*b*. The weight of this segment is determined from $W_{st} = \gamma_{st} V_{st}$. Thus the internal axial force P at the section is

$$+\uparrow \Sigma F_z = 0; \qquad P - W_{st} = 0$$

$$P - (490 \text{ lb/ft}^3)(2.75 \text{ ft})[\pi(0.75 \text{ ft})^2] = 0$$

$$P = 2381 \text{ lb}$$

Average Compressive Stress. The cross-sectional area at the section is $A = \pi(0.75 \text{ ft})^2$, and so the average compressive stress becomes

$$\sigma = \frac{P}{A} = \frac{2381 \text{ lb}}{\pi(0.75 \text{ ft})^2} = 1347.5 \text{ lb/ft}^2$$

$$\sigma = 1347.5 \text{ lb/ft}^2 \, (1 \text{ ft}^2/144 \text{ in}^2) = 9.36 \text{ psi} \qquad \textit{Ans.}$$

NOTE: The stress shown on the volume element of material in Fig. 1–18*c* is representative of the conditions at either point A or B. Notice that this stress acts *upward* on the bottom or shaded face of the element since this face forms part of the bottom surface area of the section, and on this surface, the resultant internal force **P** is pushing upward.

EXAMPLE 1.9

Member AC shown in Fig. 1–19*a* is subjected to a vertical force of 3 kN. Determine the position x of this force so that the average compressive stress at the smooth support C is equal to the average tensile stress in the tie rod AB. The rod has a cross-sectional area of 400 mm^2 and the contact area at C is 650 mm^2.

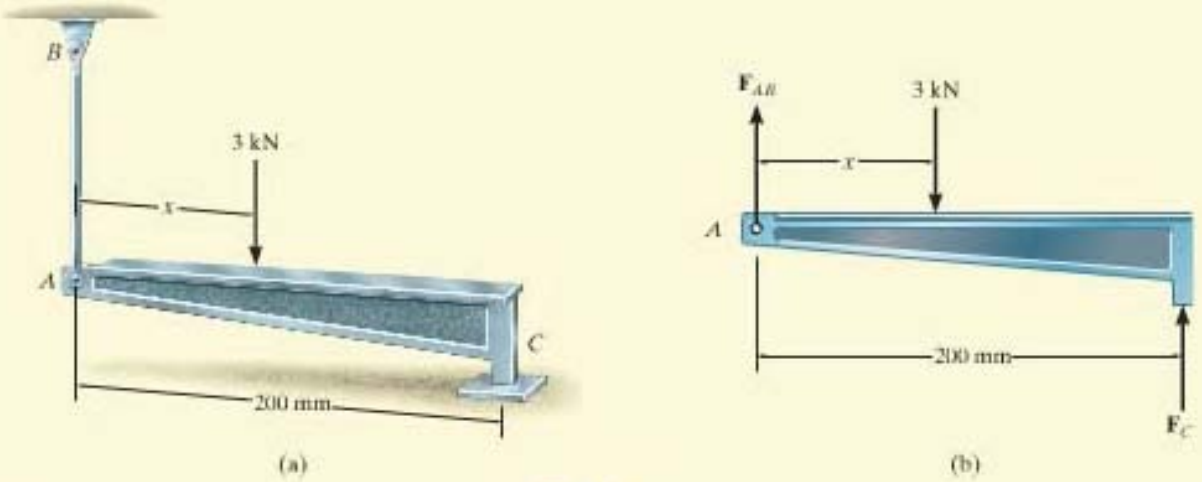


Fig. 1–19

SOLUTION

Internal Loading. The forces at A and C can be related by considering the free-body diagram for member AC, Fig. 1–19*b*. There are three unknowns, namely, F_{AB}, F_C, and x. To solve this problem we will work in units of newtons and millimeters.

$$+\uparrow \Sigma F_y = 0; \qquad F_{AB} + F_C - 3000\ \text{N} = 0 \qquad (1)$$

$$\zeta + \Sigma M_A = 0; \qquad -3000\ \text{N}(x) + F_C(200\ \text{mm}) = 0 \qquad (2)$$

Average Normal Stress. A necessary third equation can be written that requires the tensile stress in the bar AB and the compressive stress at C to be equivalent, i.e.,

$$\sigma = \frac{F_{AB}}{400\ \text{mm}^2} = \frac{F_C}{650\ \text{mm}^2}$$

$$F_C = 1.625 F_{AB}$$

Substituting this into Eq. 1, solving for F_{AB}, then solving for F_C, we obtain

$$F_{AB} = 1143\ \text{N}$$

$$F_C = 1857\ \text{N}$$

The position of the applied load is determined from Eq. 2,

$$x = 124\ \text{mm} \qquad \textit{Ans.}$$

NOTE: $0 < x < 200$ mm, as required.

1

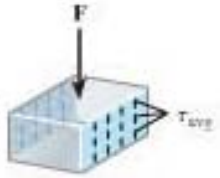
1.5 Average Shear Stress



(a)



(b)



(c)

Fig. 1–20

Shear stress has been defined in Section 1.3 as the stress component that acts *in the plane* of the sectioned area. To show how this stress can develop, consider the effect of applying a force **F** to the bar in Fig. 1–20*a*. If the supports are considered rigid, and **F** is large enough, it will cause the material of the bar to deform and fail along the planes identified by AB and CD. A free-body diagram of the unsupported center segment of the bar, Fig. 1–20*b*, indicates that the shear force $V = F/2$ must be applied at each section to hold the segment in equilibrium. The ***average shear stress*** distributed over each sectioned area that develops this shear force is defined by

$$\tau_{avg} = \frac{V}{A} \tag{1–7}$$

Here

τ_{avg} = average shear stress at the section, which is assumed to be the *same* at each point located on the section

V = internal resultant shear force on the section determined from the equations of equilibrium

A = area at the section

The distribution of average shear stress acting over the sections is shown in Fig. 1–20*c*. Notice that τ_{avg} is in the *same direction* as **V**, since the shear stress must create associated forces all of which contribute to the internal resultant force **V** at the section.

The loading case discussed here is an example of ***simple or direct shear***, since the shear is caused by the *direct action* of the applied load **F**. This type of shear often occurs in various types of simple connections that use bolts, pins, welding material, etc. In all these cases, however, application of Eq. 1–7 is *only approximate*. A more precise investigation of the shear-stress distribution over the section often reveals that much larger shear stresses occur in the material than those predicted by this equation. Although this may be the case, application of Eq. 1–7 is generally acceptable for many problems in engineering design and analysis. For example, engineering codes allow its use when considering design sizes for fasteners such as bolts and for obtaining the bonding strength of glued joints subjected to shear loadings.

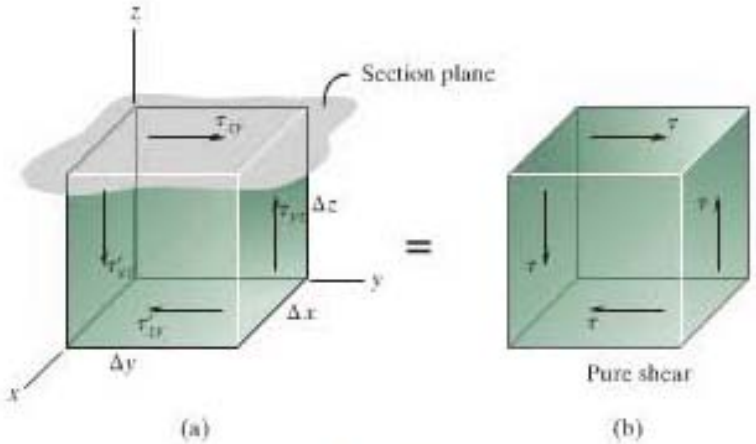


Fig. 1–21

Shear Stress Equilibrium. Figure 1–21*a* shows a volume element of material taken at a point located on the surface of a sectioned area which is subjected to a shear stress τ_{zy}. Force and moment equilibrium requires the shear stress acting on this face of the element to be accompanied by shear stress acting on three other faces. To show this we will first consider force equilibrium in the y direction. Then

$$\Sigma F_y = 0; \qquad \tau_{zy}(\Delta x\,\Delta y) - \tau'_{zy}\,\Delta x\,\Delta y = 0$$

(force; stress area)

$$\tau_{zy} = \tau'_{zy}$$

In a similar manner, force equilibrium in the z direction yields $\tau_{yz} = \tau'_{yz}$. Finally, taking moments about the x axis,

$$\Sigma M_x = 0; \qquad -\tau_{zy}(\Delta x\,\Delta y)\,\Delta z + \tau_{yz}(\Delta x\,\Delta z)\,\Delta y = 0$$

(moment; force; arm; stress area)

$$\tau_{zy} = \tau_{yz}$$

so that

$$\tau_{zy} = \tau'_{zy} = \tau_{yz} = \tau'_{yz} = \tau$$

In other words, ***all four shear stresses must have equal magnitude and be directed either toward or away from each other at opposite edges of the element***, Fig. 1–21*b*. This is referred to as the *complementary property of shear*, and under the conditions shown in Fig. 1–21, the material is subjected to *pure shear*.

1

Important Points

- If two parts are *thin or small* when joined together, the applied loads may cause shearing of the material with negligible bending. If this is the case, it is generally assumed that an *average shear stress* acts over the cross-sectional area.
- When shear stress τ acts on a plane, then equilibrium of a volume element of material at a point on the plane requires associated shear stress of the same magnitude act on three adjacent sides of the element.

Procedure for Analysis

The equation $\tau_{avg} = V/A$ is used to determine the *average shear stress* in the material. Application requires the following steps.

Internal Shear.

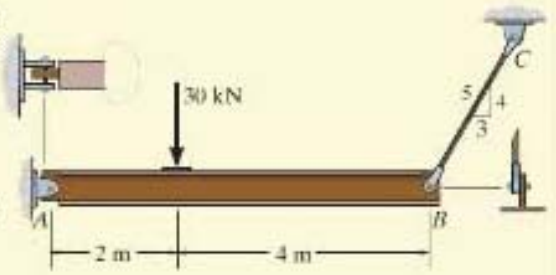
- Section the member at the point where the average shear stress is to be determined.
- Draw the necessary free-body diagram, and calculate the internal shear force **V** acting at the section that is necessary to hold the part in equilibrium.

Average Shear Stress.

- Determine the sectioned area A, and determine the average shear stress $\tau_{avg} = V/A$.
- It is suggested that τ_{avg} be shown on a small volume element of material located at a point on the section where it is determined. To do this, first draw τ_{avg} on the face of the element, coincident with the sectioned area A. This stress acts in the *same direction* as **V**. The shear stresses acting on the three adjacent planes can then be drawn in their appropriate directions following the scheme shown in Fig. 1–21.

EXAMPLE 1.10

Determine the average shear stress in the 20-mm-diameter pin at A and the 30-mm-diameter pin at B that support the beam in Fig. 1–22*a*.



(a)

SOLUTION

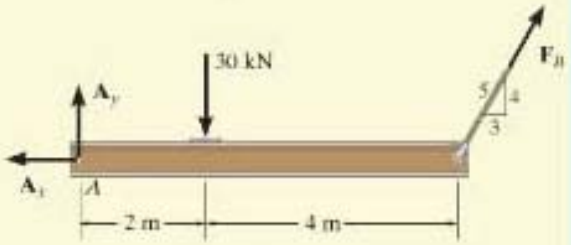
Internal Loadings. The forces on the pins can be obtained by considering the equilibrium of the beam, Fig. 1–22*b*.

$$\curvearrowleft +\Sigma M_A = 0;\quad F_B\left(\frac{4}{5}\right)(6\text{ m}) - 30\text{ kN}(2\text{ m}) = 0 \quad F_B = 12.5\text{ kN}$$

$$\pm \Sigma F_x = 0;\quad (12.5\text{ kN})\left(\frac{3}{5}\right) - A_x = 0 \quad A_x = 7.50\text{ kN}$$

$$+\uparrow \Sigma F_y = 0;\quad A_y + (12.5\text{ kN})\left(\frac{4}{5}\right) - 30\text{ kN} = 0$$

$$A_y = 20\text{ kN}$$



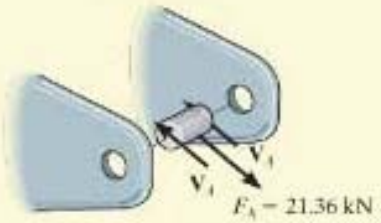
(b)

Thus, the resultant force acting on pin A is

$$F_A = \sqrt{A_x^2 + A_y^2} = \sqrt{(7.50\text{ kN})^2 + (20\text{ kN})^2} = 21.36\text{ kN}$$

The pin at A is supported by two fixed "leaves" and so the free-body diagram of the center segment of the pin shown in Fig. 1–22*c* has *two* shearing surfaces between the beam and each leaf. The force of the beam (21.36 kN) acting on the pin is therefore supported by shear force on each of these surfaces. This case is called *double shear*. Thus,

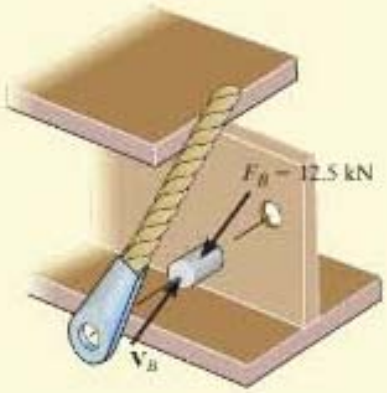
$$V_A = \frac{F_A}{2} = \frac{21.36\text{ kN}}{2} = 10.68\text{ kN}$$



(c)

In Fig. 1–22*a*, note that pin B is subjected to *single shear*, which occurs on the section between the cable and beam, Fig. 1–22*d*. For this pin segment,

$$V_B = F_B = 12.5\text{ kN}$$



(d)

Fig. 1–22

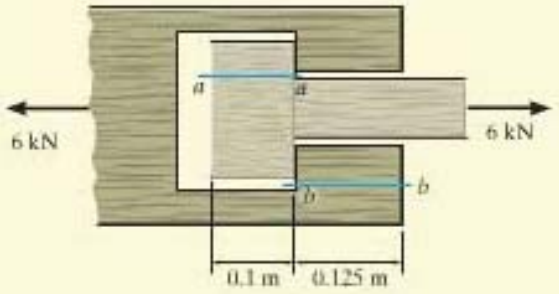
Average Shear Stress.

$$(\tau_A)_{avg} = \frac{V_A}{A_A} = \frac{10.68(10^3)\text{ N}}{\frac{\pi}{4}(0.02\text{ m})^2} = 34.0\text{ MPa} \qquad \textit{Ans}$$

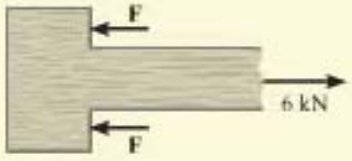
$$(\tau_B)_{avg} = \frac{V_B}{A_B} = \frac{12.5(10^3)\text{ N}}{\frac{\pi}{4}(0.03\text{ m})^2} = 17.7\text{ MPa} \qquad \textit{Ans}$$

EXAMPLE 1.11

If the wood joint in Fig. 1–23*a* has a width of 150 mm, determine the average shear stress developed along shear planes *a*–*a* and *b*–*b*. For each plane, represent the state of stress on an element of the material.



(a)



(b)

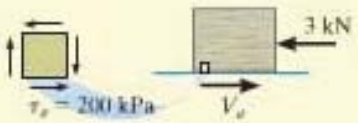
Fig. 1–23

SOLUTION

Internal Loadings. Referring to the free-body diagram of the member, Fig. 1–23*b*,

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 6\ \text{kN} - F - F = 0 \qquad F = 3\ \text{kN}$$

Now consider the equilibrium of segments cut across shear planes *a*–*a* and *b*–*b*, shown in Figs. 1–23*c* and 1–23*d*.



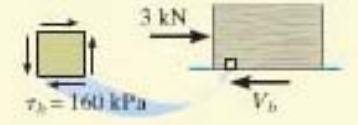
(c)

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad V_a - 3\ \text{kN} = 0 \qquad V_a = 3\ \text{kN}$$

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad 3\ \text{kN} - V_b = 0 \qquad V_b = 3\ \text{kN}$$

Average Shear Stress.

$$(\tau_a)_{\text{avg}} = \frac{V_a}{A_a} = \frac{3(10^3)\ \text{N}}{(0.1\ \text{m})(0.15\ \text{m})} = 200\ \text{kPa} \qquad \textit{Ans.}$$



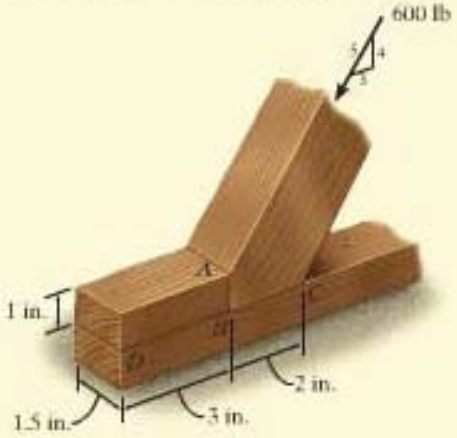
(d)

$$(\tau_b)_{\text{avg}} = \frac{V_b}{A_b} = \frac{3(10^3)\ \text{N}}{(0.125\ \text{m})(0.15\ \text{m})} = 160\ \text{kPa} \qquad \textit{Ans.}$$

The state of stress on elements located on sections *a*–*a* and *b*–*b* is shown in Figs. 1–23*c* and 1–23*d*, respectively.

EXAMPLE 1.12

The inclined member in Fig. 1–24*a* is subjected to a compressive force of 600 lb. Determine the average compressive stress along the smooth areas of contact defined by AB and BC, and the average shear stress along the horizontal plane defined by DB.



(a)

Fig. 1–24

SOLUTION

Internal Loadings. The free-body diagram of the inclined member is shown in Fig. 1–24*b*. The compressive forces acting on the areas of contact are

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad F_{AB} - 600\text{ lb}\left(\tfrac{3}{5}\right) = 0 \qquad F_{AB} = 360\text{ lb}$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{BC} - 600\text{ lb}\left(\tfrac{4}{5}\right) = 0 \qquad F_{BC} = 480\text{ lb}$$

Also, from the free-body diagram of the top segment ABD of the bottom member, Fig. 1–24*c*, the shear force acting on the sectioned horizontal plane DB is

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad V = 360\text{ lb}$$

Average Stress. The average compressive stresses along the horizontal and vertical planes of the inclined member are

$$\sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{360\text{ lb}}{(1\text{ in.})(1.5\text{ in.})} = 240\text{ psi} \qquad \textit{Ans.}$$

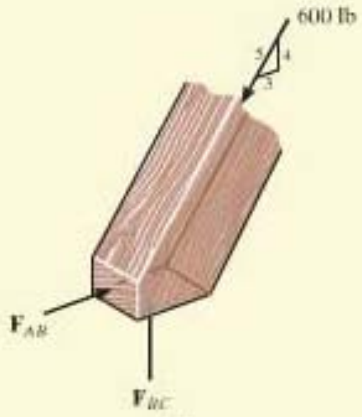
$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{480\text{ lb}}{(2\text{ in.})(1.5\text{ in.})} = 160\text{ psi} \qquad \textit{Ans.}$$

These stress distributions are shown in Fig. 1–24*d*.

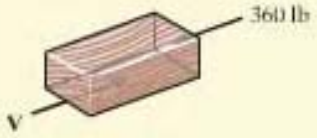
The average shear stress acting on the horizontal plane defined by DB is

$$\tau_{avg} = \frac{360\text{ lb}}{(3\text{ in.})(1.5\text{ in.})} = 80\text{ psi} \qquad \textit{Ans.}$$

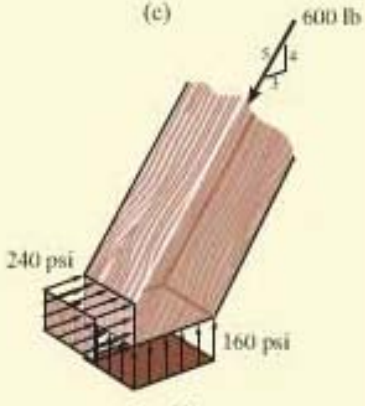
This stress is shown uniformly distributed over the sectioned area in Fig. 1–24*e*.



(b)



(c)



(d)

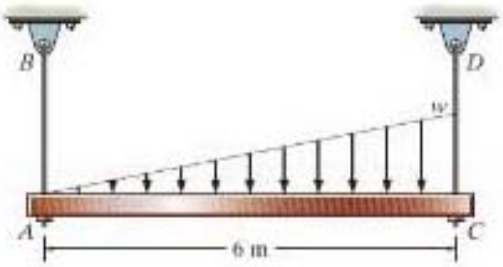


(e)

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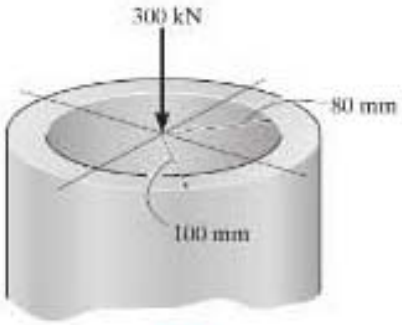
FUNDAMENTAL PROBLEMS

F1–7. The uniform beam is supported by two rods AB and CD that have cross-sectional areas of 10 mm^2 and 15 mm^2, respectively. Determine the intensity w of the distributed load so that the average normal stress in each rod does not exceed 300 kPa.



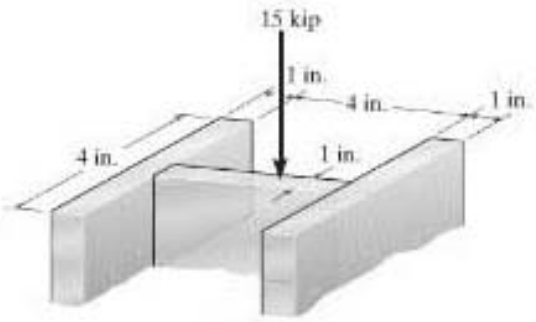
F1–7

F1–8. Determine the average normal stress developed on the cross section. Sketch the normal stress distribution over the cross section.



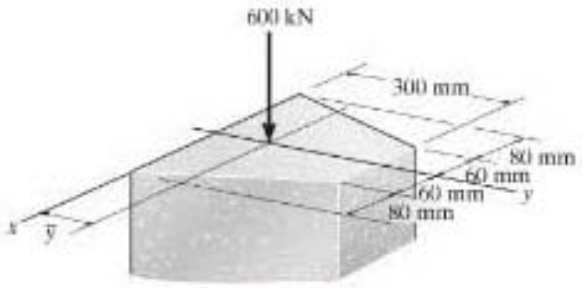
F1–8

F1–9. Determine the average normal stress developed on the cross section. Sketch the normal stress distribution over the cross section.



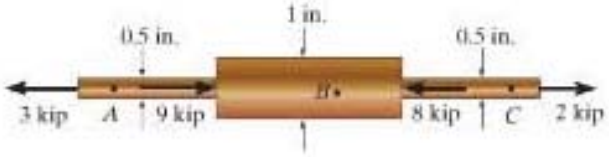
F1–9

F1–10. If the 600-kN force acts through the centroid of the cross section, determine the location $\bar{y}$ of the centroid and the average normal stress developed on the cross section. Also, sketch the normal stress distribution over the cross section.



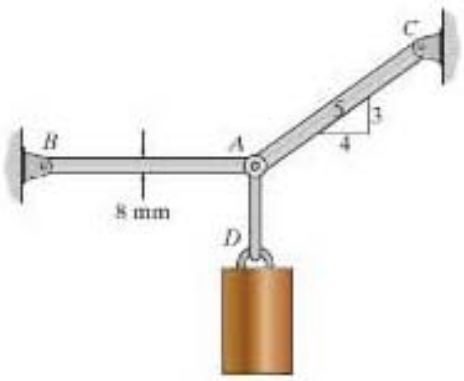
F1–10

F1–11. Determine the average normal stress developed at points A, B, and C. The diameter of each segment is indicated in the figure.



F1–11

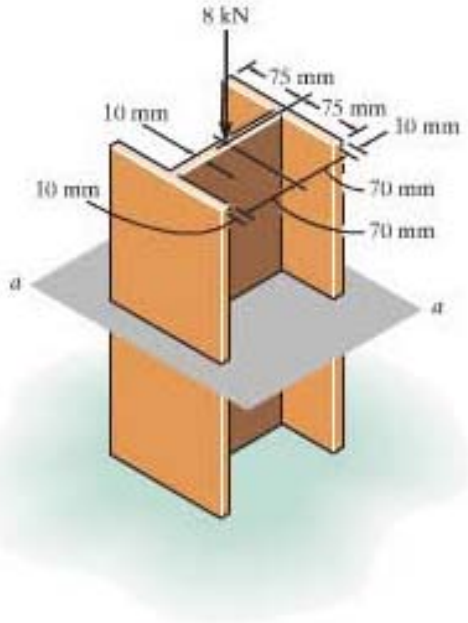
F1–12. Determine the average normal stress developed in rod AB if the load has a mass of 50 kg. The diameter of rod AB is 8 mm.



F1–12

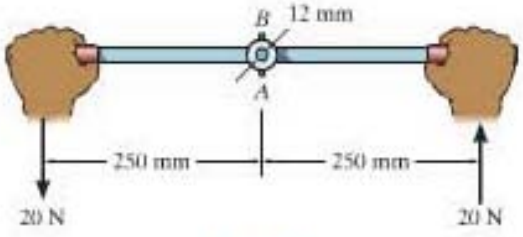
PROBLEMS

1–31. The column is subjected to an axial force of 8 kN, which is applied through the centroid of the cross-sectional area. Determine the average normal stress acting at section *a–a*. Show this distribution of stress acting over the area's cross section.



Prob. 1–31

***1–32.** The lever is held to the fixed shaft using a tapered pin *AB*, which has a mean diameter of 6 mm. If a couple is applied to the lever, determine the average shear stress in the pin between the pin and lever.



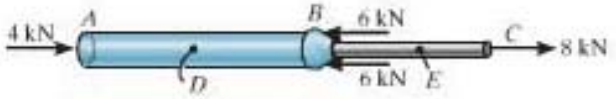
Prob. 1–32

•1–33. The bar has a cross-sectional area A and is subjected to the axial load P. Determine the average normal and average shear stresses acting over the shaded section, which is oriented at θ from the horizontal. Plot the variation of these stresses as a function of θ ($0 \le \theta \le 90°$).



Prob. 1–33

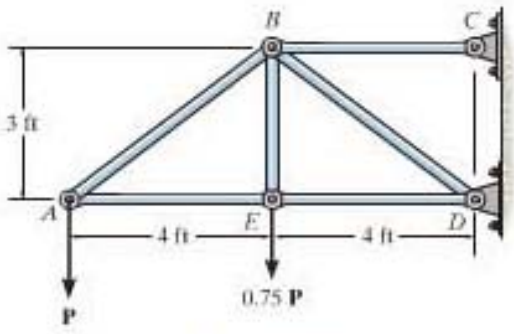
1–34. The built-up shaft consists of a pipe *AB* and solid rod *BC*. The pipe has an inner diameter of 20 mm and outer diameter of 28 mm. The rod has a diameter of 12 mm. Determine the average normal stress at points *D* and *E* and represent the stress on a volume element located at each of these points.



Prob. 1–34

1–35. The bars of the truss each have a cross-sectional area of 1.25 in^2. Determine the average normal stress in each member due to the loading $P = 8$ kip. State whether the stress is tensile or compressive.

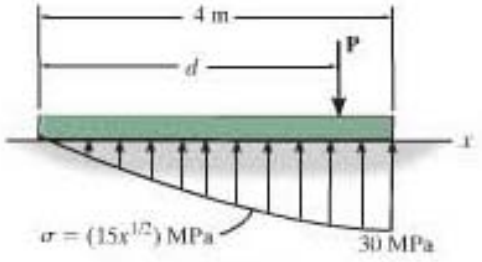
***1–36.** The bars of the truss each have a cross-sectional area of 1.25 in^2. If the maximum average normal stress in any bar is not to exceed 20 ksi, determine the maximum magnitude P of the loads that can be applied to the truss.



Probs. 1–35/36

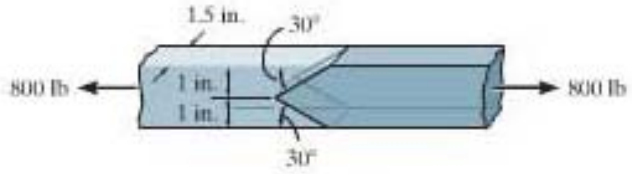
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•**1–37.** The plate has a width of 0.5 m. If the stress distribution at the support varies as shown, determine the force **P** applied to the plate and the distance d to where it is applied.



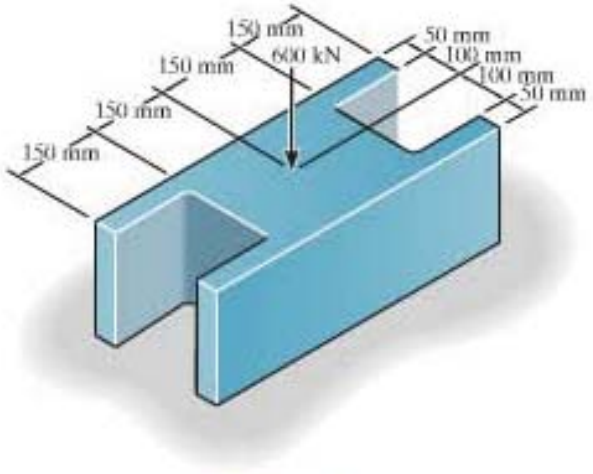
Prob. 1–37

1–38. The two members used in the construction of an aircraft fuselage are joined together using a 30° fish-mouth weld. Determine the average normal and average shear stress on the plane of each weld. Assume each inclined plane supports a horizontal force of 400 lb.



Prob. 1–38

1–39. If the block is subjected to the centrally applied force of 600 kN, determine the average normal stress in the material. Show the stress acting on a differential volume element of the material.



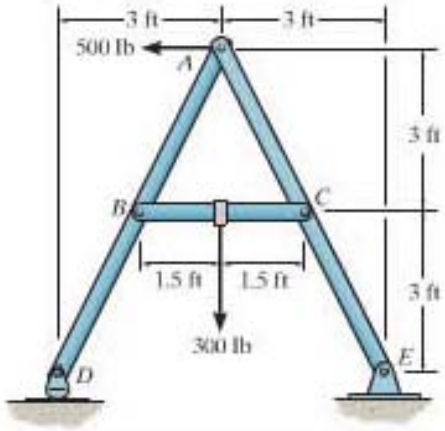
Prob. 1–39

*•**1–40.** The pins on the frame at B and C each have a diameter of 0.25 in. If these pins are subjected to *double shear*, determine the average shear stress in each pin.

•**1–41.** Solve Prob. 1–40 assuming that pins B and C are subjected to *single shear*.

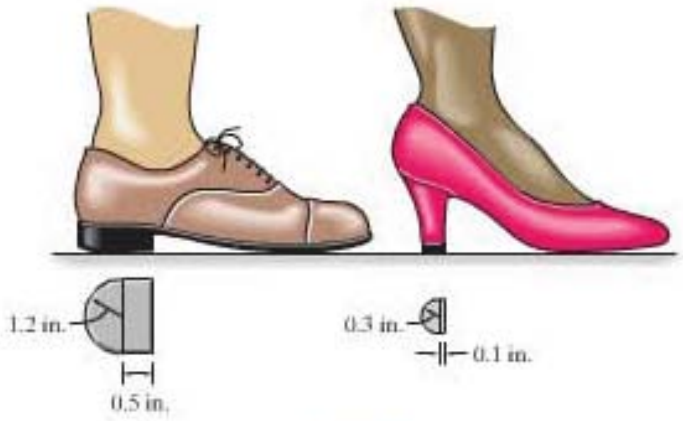
1–42. The pins on the frame at D and E each have a diameter of 0.25 in. If these pins are subjected to *double shear*, determine the average shear stress in each pin.

1–43. Solve Prob. 1–42 assuming that pins D and E are subjected to *single shear*.



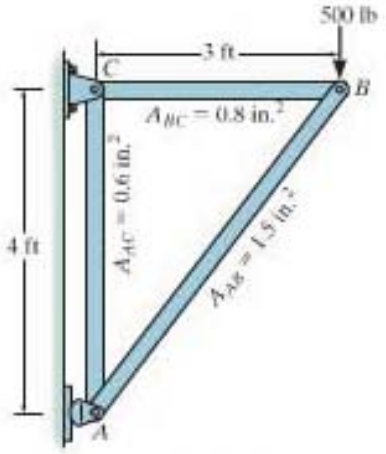
Probs. 1–40/41/42/43

*•**1–44.** A 175-lb woman stands on a vinyl floor wearing stiletto high-heel shoes. If the heel has the dimensions shown, determine the average normal stress she exerts on the floor and compare it with the average normal stress developed when a man having the same weight is wearing flat-heeled shoes. Assume the load is applied slowly, so that dynamic effects can be ignored. Also, assume the entire weight is supported only by the heel of one shoe.



Prob. 1–44

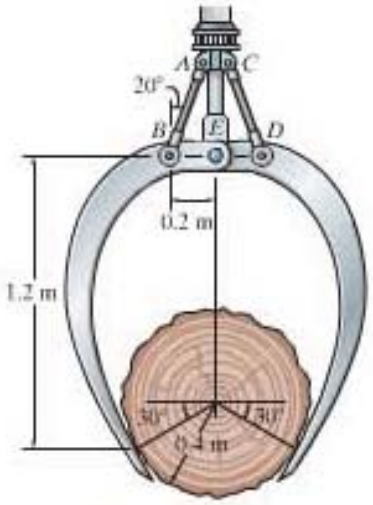
•**1–45.** The truss is made from three pin-connected members having the cross-sectional areas shown in the figure. Determine the average normal stress developed in each member when the truss is subjected to the load shown. State whether the stress is tensile or compressive.



Prob. 1–45

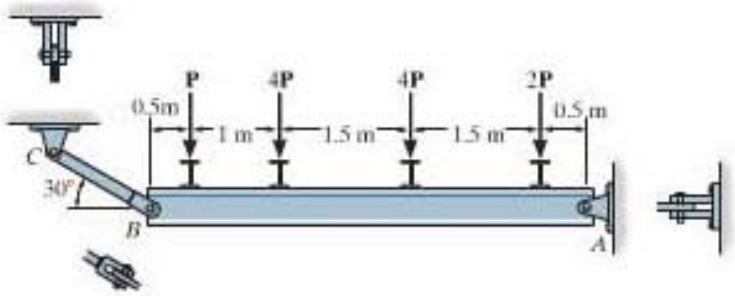
1–46. Determine the average normal stress developed in links *AB* and *CD* of the smooth two-tine grapple that supports the log having a mass of 3 Mg. The cross-sectional area of each link is 400 mm^2.

1–47. Determine the average shear stress developed in pins *A* and *B* of the smooth two-tine grapple that supports the log having a mass of 3 Mg. Each pin has a diameter of 25 mm and is subjected to double shear.



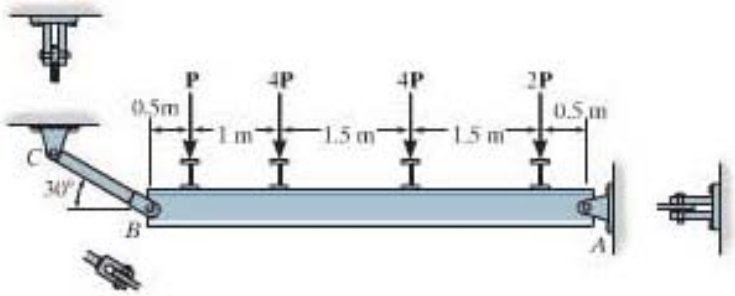
Probs. 1–46/47

*1–48. The beam is supported by a pin at *A* and a short link *BC*. If $P = 15$ kN, determine the average shear stress developed in the pins at *A*, *B*, and *C*. All pins are in double shear as shown, and each has a diameter of 18 mm.



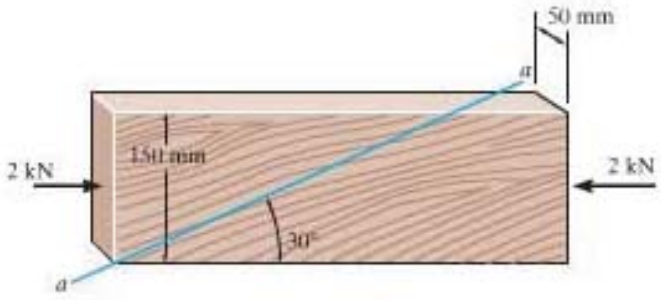
Prob. 1–48

•**1–49.** The beam is supported by a pin at *A* and a short link *BC*. Determine the maximum magnitude *P* of the loads the beam will support if the average shear stress in each pin is not to exceed 80 MPa. All pins are in double shear as shown, and each has a diameter of 18 mm.



Prob. 1–49

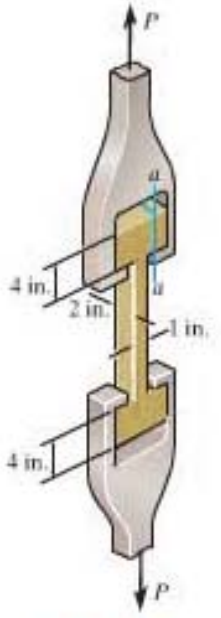
1–50. The block is subjected to a compressive force of 2 kN. Determine the average normal and average shear stress developed in the wood fibers that are oriented along section *a–a* at 30° with the axis of the block.



Prob. 1–50

1

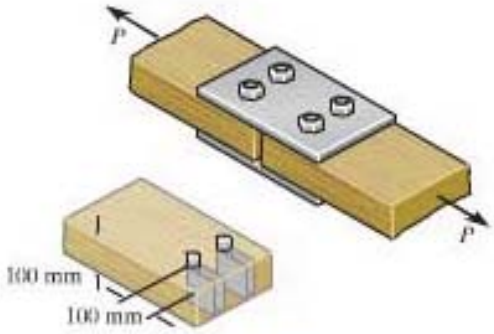
1–51. During the tension test, the wooden specimen is subjected to an average normal stress of 2 ksi. Determine the axial force **P** applied to the specimen. Also, find the average shear stress developed along section *a–a* of the specimen.



Prob. 1–51

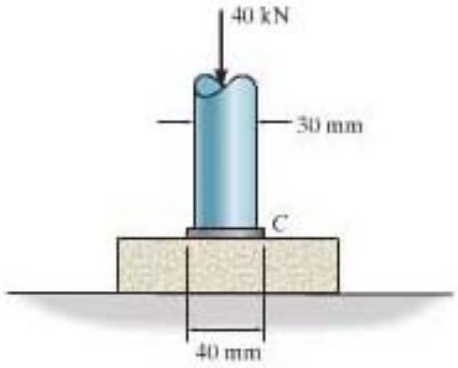
*1–52. If the joint is subjected to an axial force of $P = 9$ kN, determine the average shear stress developed in each of the 6-mm diameter bolts between the plates and the members and along each of the four shaded shear planes.

•1–53. The average shear stress in each of the 6-mm diameter bolts and along each of the four shaded shear planes is not allowed to exceed 80 MPa and 500 kPa, respectively. Determine the maximum axial force **P** that can be applied to the joint.



Probs. 1–52/53

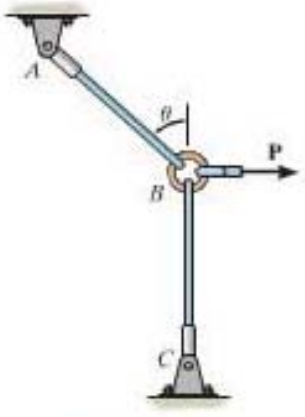
1–54. The shaft is subjected to the axial force of 40 kN. Determine the average bearing stress acting on the collar *C* and the normal stress in the shaft.



Prob. 1–54

1–55. Rods *AB* and *BC* each have a diameter of 5 mm. If the load of $P = 2$ kN is applied to the ring, determine the average normal stress in each rod if $\theta = 60°$.

*1–56. Rods *AB* and *BC* each have a diameter of 5 mm. Determine the angle θ of rod *BC* so that the average normal stress in rod *AB* is 1.5 times that in rod *BC*. What is the load **P** that will cause this to happen if the average normal stress in each rod is not allowed to exceed 100 MPa?



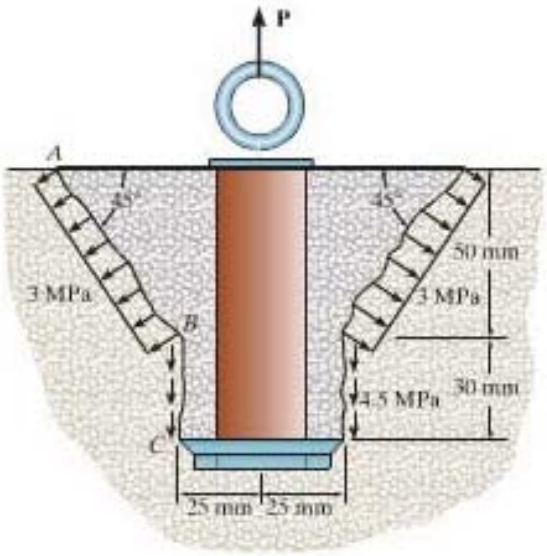
Probs. 1–55/56

•**1–57.** The specimen failed in a tension test at an angle of 52° when the axial load was 19.80 kip. If the diameter of the specimen is 0.5 in., determine the average normal and average shear stress acting on the area of the inclined failure plane. Also, what is the average normal stress acting on the *cross section* when failure occurs?



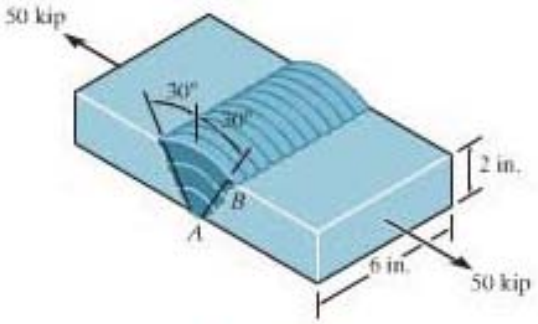
Prob. 1–57

1–58. The anchor bolt was pulled out of the concrete wall and the failure surface formed part of a frustum and cylinder. This indicates a shear failure occurred along the cylinder BC and tension failure along the frustum AB. If the shear and normal stresses along these surfaces have the magnitudes shown, determine the force **P** that must have been applied to the bolt.



Prob. 1–58

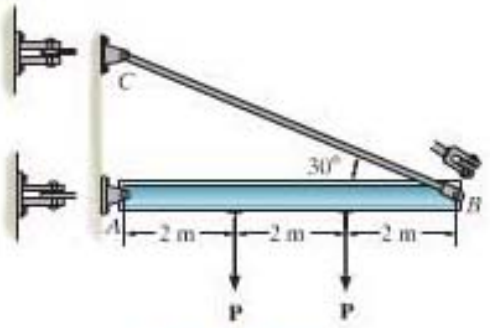
1–59. The open square butt joint is used to transmit a force of 50 kip from one plate to the other. Determine the average normal and average shear stress components that this loading creates on the face of the weld, section AB.



Prob. 1–59

*•**1–60.** If $P = 20$ kN, determine the average shear stress developed in the pins at A and C. The pins are subjected to double shear as shown, and each has a diameter of 18 mm.

•**1–61.** Determine the maximum magnitude P of the load the beam will support if the average shear stress in each pin is not to allowed to exceed 60 MPa. All pins are subjected to double shear as shown, and each has a diameter of 18 mm.

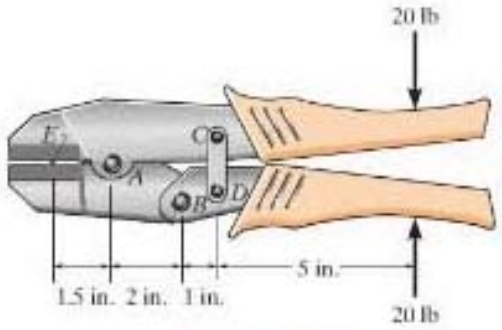


Probs. 1–60/61

1

1–62. The crimping tool is used to crimp the end of the wire E. If a force of 20 lb is applied to the handles, determine the average shear stress in the pin at A. The pin is subjected to double shear and has a diameter of 0.2 in. Only a vertical force is exerted on the wire.

1–63. Solve Prob. 1–62 for pin B. The pin is subjected to double shear and has a diameter of 0.2 in.



Probs. 1–62/63

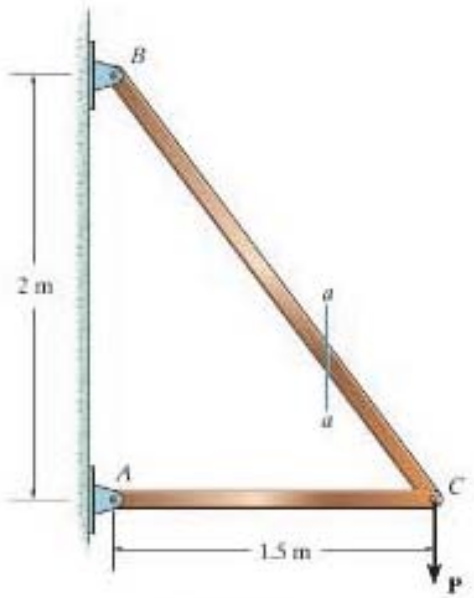
*1–64. The triangular blocks are glued along each side of the joint. A C-clamp placed between two of the blocks is used to draw the joint tight. If the glue can withstand a maximum average shear stress of 800 kPa, determine the maximum allowable clamping force **F**.

•1–65. The triangular blocks are glued along each side of the joint. A C-clamp placed between two of the blocks is used to draw the joint tight. If the clamping force is $F = 900$ N, determine the average shear stress developed in the glued shear plane.



Probs. 1–64/65

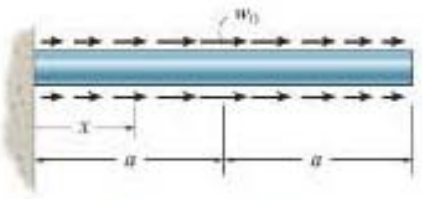
1–66. Determine the largest load **P** that can be a applied to the frame without causing either the average normal stress or the average shear stress at section a–a to exceed $\sigma = 150$ MPa and $\tau = 60$ MPa, respectively. Member CB has a square cross section of 25 mm on each side.



Prob. 1–66

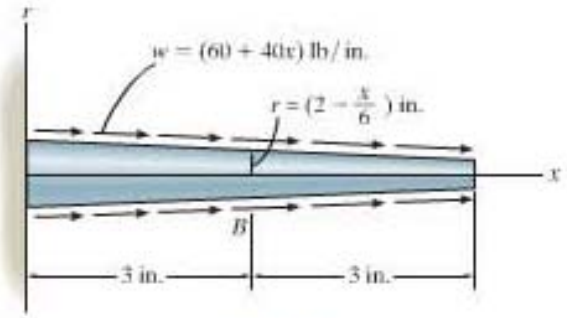
1–67. The prismatic bar has a cross-sectional area A. If it is subjected to a distributed axial loading that increases linearly from $w = 0$ at $x = 0$ to $w = w_0$ at $x = a$, and then decreases linearly to $w = 0$ at $x = 2a$, determine the average normal stress in the bar as a function of x for $0 \leq x < a$.

*1–68. The prismatic bar has a cross-sectional area A. If it is subjected to a distributed axial loading that increases linearly from $w = 0$ at $x = 0$ to $w = w_0$ at $x = a$, and then decreases linearly to $w = 0$ at $x = 2a$, determine the average normal stress in the bar as a function of x for $a < x \leq 2a$.



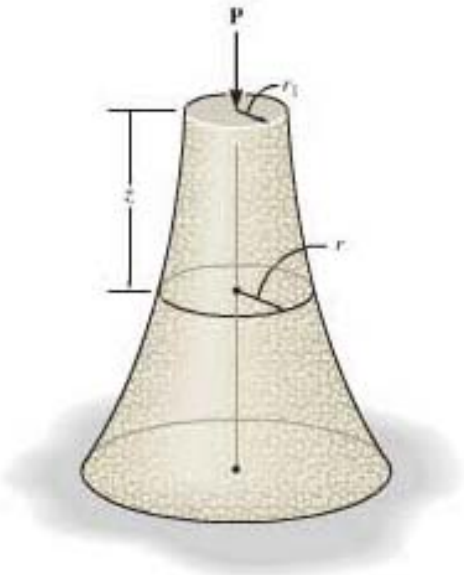
Probs. 1–67/68

•1–69. The tapered rod has a radius of $r = (2 - x/6)$ in. and is subjected to the distributed loading of $w = (60 + 40x)$ lb/in. Determine the average normal stress at the center of the rod, B.



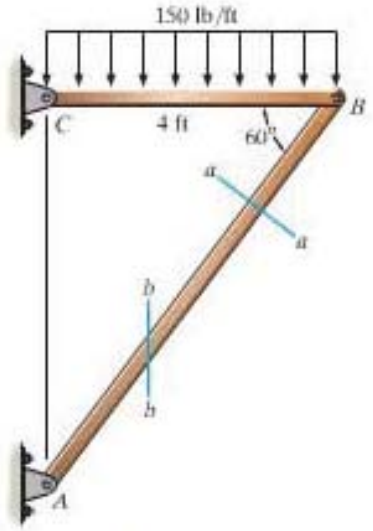
Prob. 1–69

1–70. The pedestal supports a load **P** at its center. If the material has a mass density ρ, determine the radial dimension r as a function of z so that the average normal stress in the pedestal remains constant. The cross section is circular.



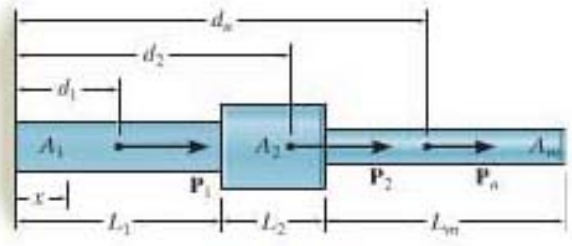
Prob. 1–70

1–71. Determine the average normal stress at section a–a and the average shear stress at section b–b in member AB. The cross section is square, 0.5 in. on each side.



Prob. 1–71

***1–72.** Consider the general problem of a bar made from m segments, each having a constant cross-sectional area A_m and length L_m. If there are n loads on the bar as shown, write a computer program that can be used to determine the average normal stress at any specified location x. Show an application of the program using the values $L_1 = 4$ ft, $d_1 = 2$ ft, $P_1 = 400$ lb, $A_1 = 3$ in^2, $L_2 = 2$ ft, $d_2 = 6$ ft, $P_2 = -300$ lb, $A_2 = 1$ in^2.



Prob. 1–72

1

1.6 Allowable Stress

To properly *design* a structural member or mechanical element it is necessary to restrict the stress in the material to a level that will be safe. To ensure this safety, it is therefore necessary to choose an allowable stress that restricts the applied load to one that is *less* than the load the member can fully support. There are many reasons for doing this. For example, the load for which the member is designed may be different from actual loadings placed on it. The intended measurements of a structure or machine may not be exact, due to errors in fabrication or in the assembly of its component parts. Unknown vibrations, impact, or accidental loadings can occur that may not be accounted for in the design. Atmospheric corrosion, decay, or weathering tend to cause materials to deteriorate during service. And lastly, some materials, such as wood, concrete, or fiber-reinforced composites, can show high variability in mechanical properties.

One method of specifying the allowable load for a member is to use a number called the factor of safety. The ***factor of safety*** (F.S.) is a ratio of the failure load F_{fail} to the allowable load F_{allow}. Here F_{fail} is found from experimental testing of the material, and the factor of safety is selected based on experience so that the above mentioned uncertainties are accounted for when the member is used under similar conditions of loading and geometry. Stated mathematically,

$$\text{F.S.} = \frac{F_{fail}}{F_{allow}} \tag{1–8}$$

If the load applied to the member is *linearly related* to the stress developed within the member, as in the case of using $\sigma = P/A$ and $\tau_{avg} = V/A$, then we can also express the factor of safety as a ratio of the failure stress σ_{fail} (or τ_{fail}) to the allowable stress σ_{allow} (or τ_{allow});* that is,

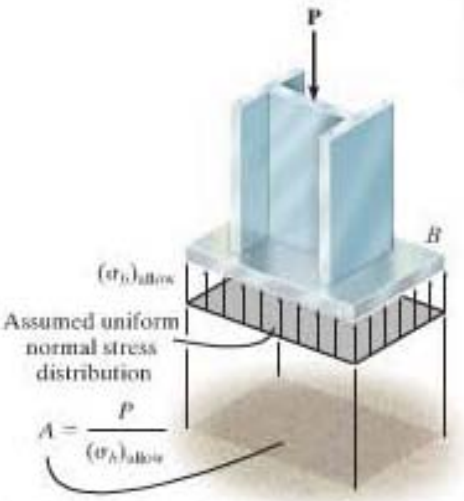
$$\text{F.S.} = \frac{\sigma_{fail}}{\sigma_{allow}} \tag{1–9}$$

or

$$\text{F.S.} = \frac{\tau_{fail}}{\tau_{allow}} \tag{1–10}$$

*In some cases, such as columns, the applied load is not linearly related to stress and therefore only Eq. 1–8 can be used to determine the factor of safety. See Chapter 13.

In any of these equations, the factor of safety must be *greater* than 1 in order to avoid the potential for failure. Specific values depend on the types of materials to be used and the intended purpose of the structure or machine. For example, the F.S. used in the design of aircraft or space-vehicle components may be close to 1 in order to reduce the weight of the vehicle. Or, in the case of a nuclear power plant, the factor of safety for some of its components may be as high as 3 due to uncertainties in loading or material behavior. In many cases, the factor of safety for a specific case can be found in design codes and engineering handbooks. These values are intended to form a balance of ensuring public and environmental safety and providing a reasonable economic solution to design.



The area of the column base plate B is determined from the allowable bearing stress for the concrete.

1.7 Design of Simple Connections

By making simplifying assumptions regarding the behavior of the material, the equations $\sigma = P/A$ and $\tau_{avg} = V/A$ can often be used to analyze or design a simple connection or mechanical element. In particular, if a member is subjected to *normal force* at a section, its required area at the section is determined from

$$A = \frac{P}{\sigma_{allow}} \tag{1–11}$$

On the other hand, if the section is subjected to an average *shear force*, then the required area at the section is

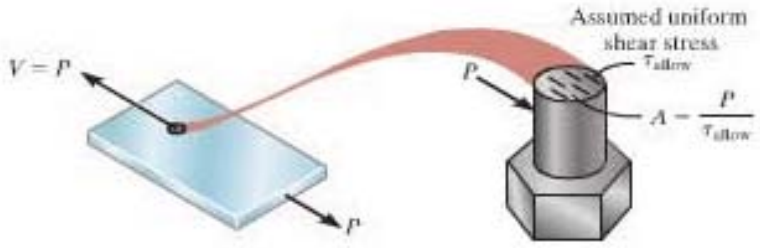
$$A = \frac{V}{\tau_{allow}} \tag{1–12}$$

As discussed in Sec. 1.6, the allowable stress used in each of these equations is determined either by applying a factor of safety to the material's normal or shear failure stress or by finding these stresses directly from an appropriate design code.

Three examples of where the above equations apply are shown in Fig. 1–25.



The embedded length l of this rod in concrete can be determined using the allowable shear stress of the bonding glue.





The area of the bolt for this lap joint is determined from the shear stress, which is largest between the plates.

Fig. 1–25

1

Important Point

- Design of a member for strength is based on selecting an allowable stress that will enable it to safely support its intended load. Since there are many unknown factors that can influence the actual stress in a member, then depending upon the intended use of the member, a *factor of safety* is applied to obtain the allowable load the member can support.

Procedure for Analysis

When solving problems using the average normal and shear stress equations, a careful consideration should first be made as to choose the section over which the critical stress is acting. Once this section is determined, the member must then be designed to have a sufficient area at the section to resist the stress that acts on it. This area is determined using the following steps.

Internal Loading.

- Section the member through the area and draw a free-body diagram of a segment of the member. The internal resultant force at the section is then determined using the equations of equilibrium.

Required Area.

- Provided the allowable stress is known or can be determined, the required area needed to sustain the load at the section is then determined from $A = P/\sigma_{allow}$ or $A = V/\tau_{allow}$.



Appropriate factors of safety must be considered when designing cranes and cables used to transfer heavy loads.

EXAMPLE 1.13

The control arm is subjected to the loading shown in Fig. 1–26*a*. Determine to the nearest $\frac{1}{4}$ in. the required diameter of the steel pin at C if the allowable shear stress for the steel is $\tau_{allow} = 8$ ksi.

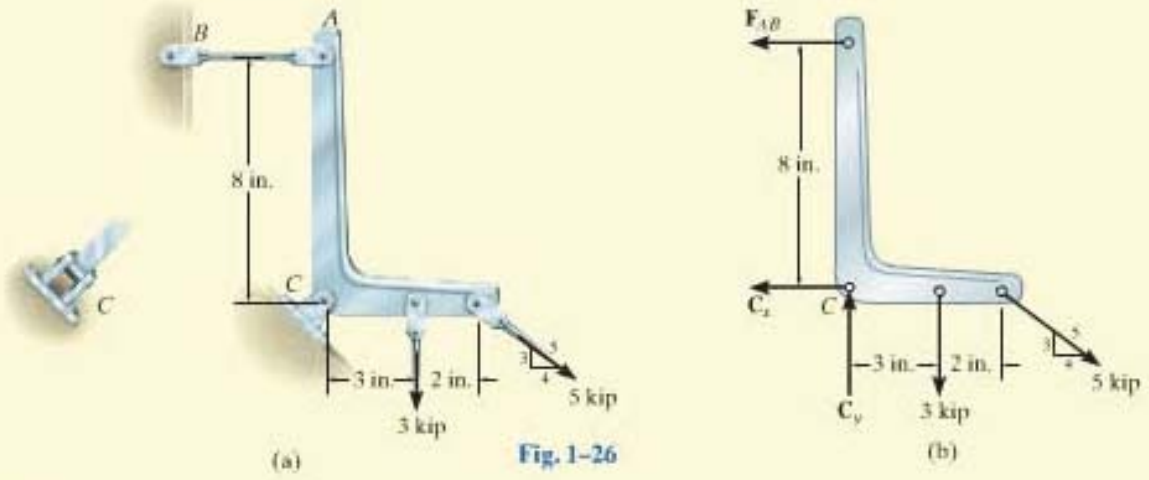


Fig. 1–26

SOLUTION

Internal Shear Force. A free-body diagram of the arm is shown in Fig. 1–26*b*. For equilibrium we have

$$\curvearrowleft + \Sigma M_C = 0; \qquad F_{AB}(8\text{ in.}) - 3\text{ kip}\,(3\text{ in.}) - 5\text{ kip}\left(\tfrac{3}{5}\right)(5\text{ in.}) = 0$$

$$F_{AB} = 3\text{ kip}$$

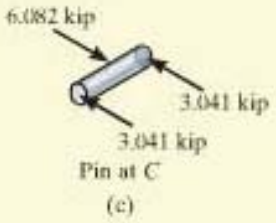
$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad -3\text{ kip} - C_x + 5\text{ kip}\left(\tfrac{4}{5}\right) = 0 \qquad C_x = 1\text{ kip}$$

$$+\uparrow \Sigma F_y = 0; \qquad C_y - 3\text{ kip} - 5\text{ kip}\left(\tfrac{3}{5}\right) = 0 \qquad C_y = 6\text{ kip}$$

The pin at C resists the resultant force at C, which is

$$F_C = \sqrt{(1\text{ kip})^2 + (6\text{ kip})^2} = 6.082\text{ kip}$$

Since the pin is subjected to double shear, a shear force of 3.041 kip acts over its cross-sectional area *between* the arm and each supporting leaf for the pin, Fig. 1–26*c*.



Required Area. We have

$$A = \frac{V}{\tau_{allow}} = \frac{3.041\text{ kip}}{8\text{ kip/in}^2} = 0.3802\text{ in}^2$$

$$\pi\left(\frac{d}{2}\right)^2 = 0.3802\text{ in}^2$$

$$d = 0.696\text{ in.}$$

Use a pin having a diameter of

$$d = \tfrac{3}{4}\text{ in.} = 0.750\text{ in.} \qquad \textit{Ans.}$$

EXAMPLE 1.14

The suspender rod is supported at its end by a fixed-connected circular disk as shown in Fig. 1–27*a*. If the rod passes through a 40-mm-diameter hole, determine the minimum required diameter of the rod and the minimum thickness of the disk needed to support the 20-kN load. The allowable normal stress for the rod is $\sigma_{allow} = 60$ MPa, and the allowable shear stress for the disk is $\tau_{allow} = 35$ MPa.

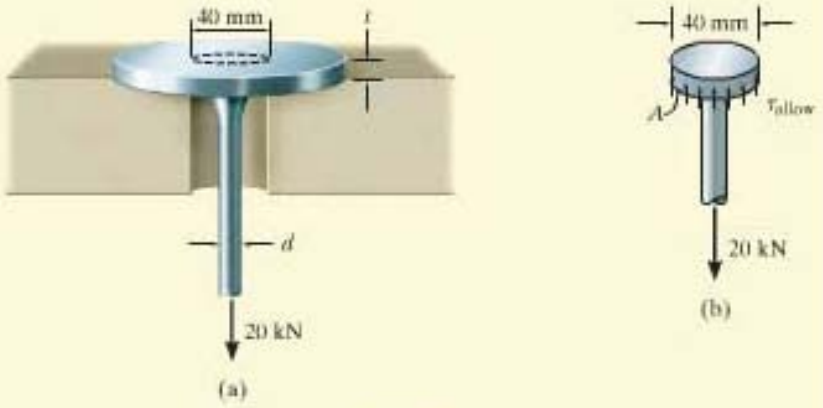


Fig. 1–27

SOLUTION

Diameter of Rod. By inspection, the axial force in the rod is 20 kN. Thus the required cross-sectional area of the rod is

$$A = \frac{P}{\sigma_{allow}}; \qquad \frac{\pi}{4}d^2 = \frac{20(10^3)\ \text{N}}{60(10^6)\ \text{N/m}^2}$$

so that

$$d = 0.0206\ \text{m} = 20.6\ \text{mm} \qquad \textit{Ans.}$$

Thickness of Disk. As shown on the free-body diagram in Fig. 1–27*b*, the material at the sectioned area of the disk must resist *shear stress* to prevent movement of the disk through the hole. If this shear stress is *assumed* to be uniformly distributed over the sectioned area, then, since $V = 20$ kN, we have

$$A = \frac{V}{\tau_{allow}}; \qquad 2\pi(0.02\ \text{m})(t) = \frac{20(10^3)\ \text{N}}{35(10^6)\ \text{N/m}^2}$$

$$t = 4.55(10^{-3})\ \text{m} = 4.55\ \text{mm} \qquad \textit{Ans.}$$

EXAMPLE 1.15

The shaft shown in Fig. 1–28*a* is supported by the collar at C, which is attached to the shaft and located on the right side of the bearing at B. Determine the largest value of P for the axial forces at E and F so that the bearing stress on the collar does not exceed an allowable stress of $(\sigma_b)_{\text{allow}} = 75$ MPa and the average normal stress in the shaft does not exceed an allowable stress of $(\sigma_t)_{\text{allow}} = 55$ MPa.

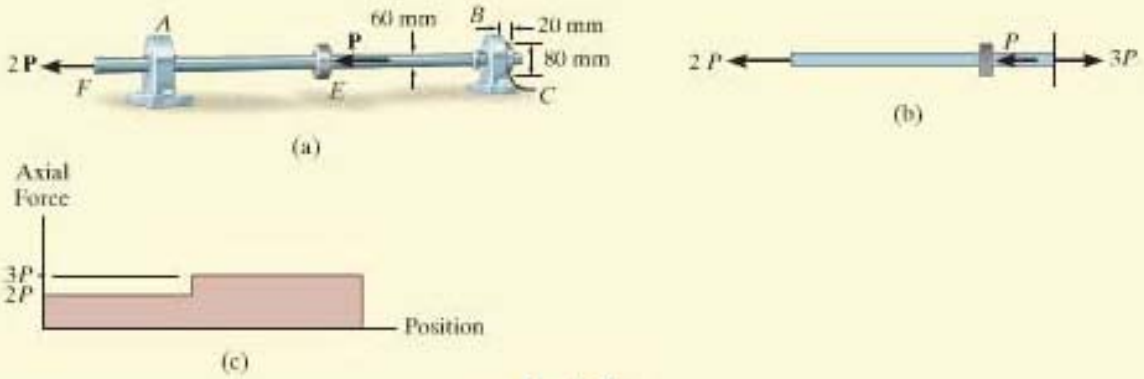


Fig. 1–28

SOLUTION

To solve the problem we will determine P for each possible failure condition. Then we will choose the *smallest* value. Why?

Normal Stress. Using the method of sections, the axial load within region FE of the shaft is $2P$, whereas the *largest* axial force, $3P$, occurs within region EC, Fig. 1–28*b*. The variation of the internal loading is clearly shown on the normal-force diagram, Fig. 1–28*c*. Since the cross-sectional area of the entire shaft is constant, region EC is subjected to the maximum average normal stress. Applying Eq. 1–11, we have

$$A = \frac{P}{\sigma_{\text{allow}}}; \qquad \pi(0.03\ \text{m})^2 = \frac{3P}{55(10^6)\ \text{N/m}^2}$$

$$P = 51.8\ \text{kN} \qquad \textit{Ans}$$

Bearing Stress. As shown on the free-body diagram in Fig. 1–28*d*, the collar at C must resist the load of $3P$, which acts over a bearing area of $A_b = [\pi(0.04\ \text{m})^2 - \pi(0.03\ \text{m})^2] = 2.199(10^{-3})\ \text{m}^2$. Thus,



(d)

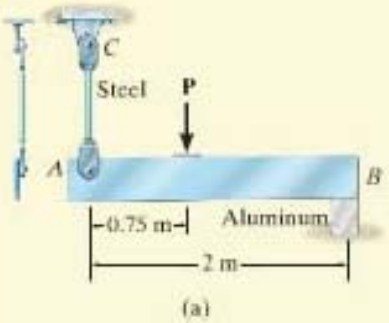
$$A = \frac{P}{\sigma_{\text{allow}}}; \qquad 2.199(10^{-3})\ \text{m}^2 = \frac{3P}{75(10^6)\ \text{N/m}^2}$$

$$P = 55.0\ \text{kN}$$

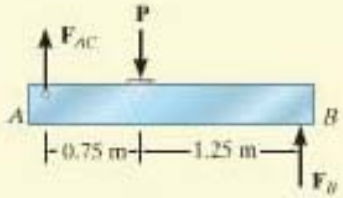
By comparison, the largest load that can be applied to the shaft is $P = 51.8$ kN, since any load larger than this will cause the allowable normal stress in the shaft to be exceeded.

NOTE: Here we have not considered a possible shear failure of the collar as in Example 1.14.

EXAMPLE 1.16



(a)



(b)

Fig. 1–29

The rigid bar AB shown in Fig. 1–29*a* is supported by a steel rod AC having a diameter of 20 mm and an aluminum block having a cross-sectional area of 1800 mm^2. The 18-mm-diameter pins at A and C are subjected to *single shear*. If the failure stress for the steel and aluminum is $(\sigma_{st})_{fail} = 680$ MPa and $(\sigma_{al})_{fail} = 70$ MPa, respectively, and the failure shear stress for each pin is $\tau_{fail} = 900$ MPa, determine the largest load P that can be applied to the bar. Apply a factor of safety of F.S. = 2.

SOLUTION

Using Eqs. 1–9 and 1–10, the allowable stresses are

$$(\sigma_{st})_{allow} = \frac{(\sigma_{st})_{fail}}{\text{F.S.}} = \frac{680\text{ MPa}}{2} = 340\text{ MPa}$$

$$(\sigma_{al})_{allow} = \frac{(\sigma_{al})_{fail}}{\text{F.S.}} = \frac{70\text{ MPa}}{2} = 35\text{ MPa}$$

$$\tau_{allow} = \frac{\tau_{fail}}{\text{F.S.}} = \frac{900\text{ MPa}}{2} = 450\text{ MPa}$$

The free-body diagram of the bar is shown in Fig. 1–29*b*. There are three unknowns. Here we will apply the moment equations of equilibrium in order to express F_{AC} and F_B in terms of the applied load P. We have

$$\circlearrowleft + \Sigma M_B = 0; \qquad P(1.25\text{ m}) - F_{AC}(2\text{ m}) = 0 \qquad (1)$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad F_B(2\text{ m}) - P(0.75\text{ m}) = 0 \qquad (2)$$

We will now determine each value of P that creates the allowable stress in the rod, block, and pins, respectively.

Rod AC. This requires

$$F_{AC} = (\sigma_{st})_{allow}(A_{AC}) = 340(10^6)\text{ N/m}^2\,[\pi(0.01\text{ m})^2] = 106.8\text{ kN}$$

Using Eq. 1,

$$P = \frac{(106.8\text{ kN})(2\text{ m})}{1.25\text{ m}} = 171\text{ kN}$$

Block B. In this case,

$$F_B = (\sigma_{al})_{allow} A_B = 35(10^6)\text{ N/m}^2\,[1800\text{ mm}^2\,(10^{-6})\text{ m}^2/\text{mm}^2] = 63.0\text{ kN}$$

Using Eq. 2,

$$P = \frac{(63.0\text{ kN})(2\text{ m})}{0.75\text{ m}} = 168\text{ kN}$$

Pin A or C. Due to single shear,

$$F_{AC} = V = \tau_{allow} A = 450(10^6)\text{ N/m}^2\,[\pi(0.009\text{ m})^2] = 114.5\text{ kN}$$

From Eq. 1,

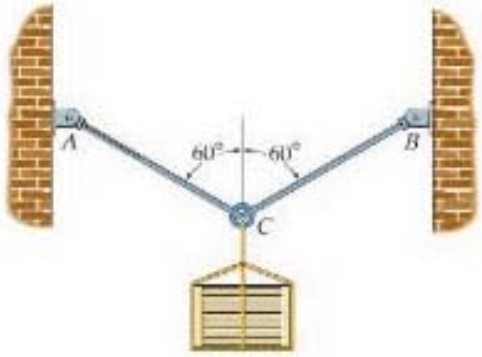
$$P = \frac{114.5\text{ kN}\,(2\text{ m})}{1.25\text{ m}} = 183\text{ kN}$$

By comparison, as P reaches its *smallest value* (168 kN), the allowable normal stress will first be developed in the aluminum block. Hence,

$$P = 168\text{ kN} \qquad \textit{Ans.}$$

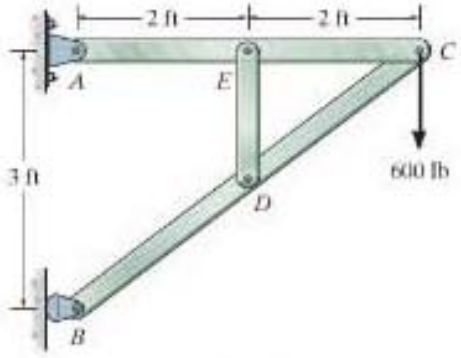
FUNDAMENTAL PROBLEMS

F1–13. Rods AC and BC are used to suspend the 200-kg mass. If each rod is made of a material for which the average normal stress can not exceed 150 MPa, determine the minimum required diameter of each rod to the nearest mm.



F1–13

F1–14. The frame supports the loading shown. The pin at A has a diameter of 0.25 in. If it is subjected to double shear, determine the average shear stress in the pin.



F1–14

F1–15. Determine the maximum average shear stress developed in each 3/4-in.-diameter bolt.



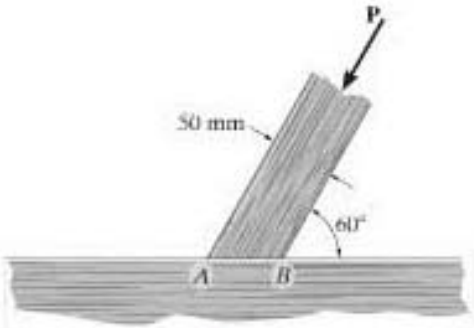
F1–15

F1–16. If each of the three nails has a diameter of 4 mm and can withstand an average shear stress of 60 MPa, determine the maximum allowable force **P** that can be applied to the board.



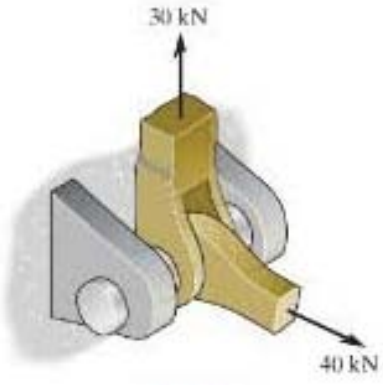
F1–16

F1–17. The strut is glued to the horizontal member at surface AB. If the strut has a thickness of 25 mm and the glue can withstand an average shear stress of 600 kPa, determine the maximum force **P** that can be applied to the strut.



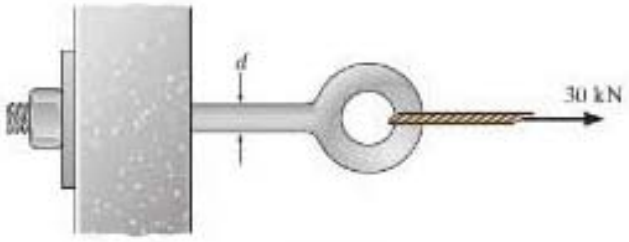
F1–17

F1–18. Determine the maximum average shear stress developed in the 30-mm-diameter pin.



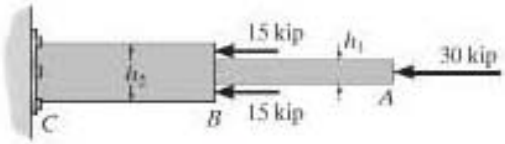
F1–18

F1–19. If the eyebolt is made of a material having a yield stress of $\sigma_Y = 250$ MPa, determine the minimum required diameter d of its shank. Apply a factor of safety F.S. = 1.5 against yielding.



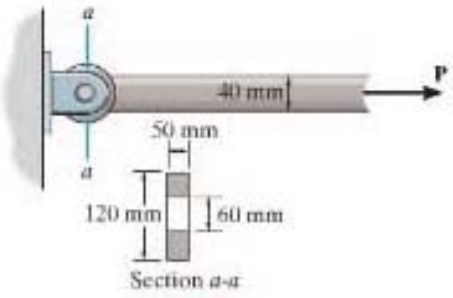
F1–19

F1–20. If the bar assembly is made of a material having a yield stress of $\sigma_Y = 50$ ksi, determine the minimum required dimensions h_1 and h_2 to the nearest 1/8 in. Apply a factor of safety F.S. = 1.5 against yielding. Each bar has a thickness of 0.5 in.



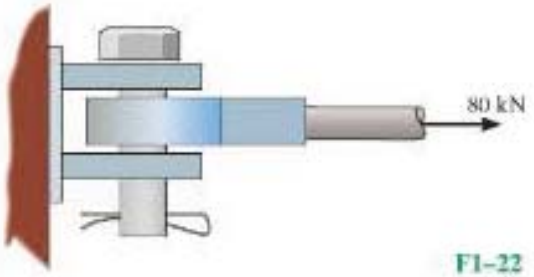
F1–20

F1–21. Determine the maximum force **P** that can be applied to the rod if it is made of material having a yield stress of $\sigma_Y = 250$ MPa. Consider the possibility that failure occurs in the rod and at section a–a. Apply a factor of safety of F.S. = 2 against yielding.



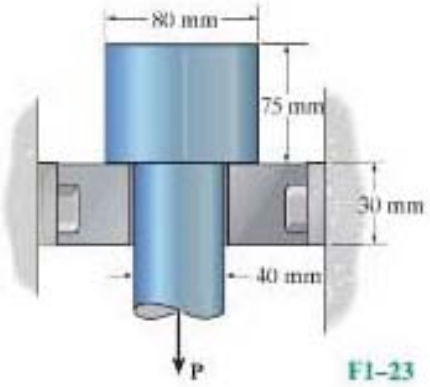
F1–21

F1–22. The pin is made of a material having a failure shear stress of $\tau_{fail} = 100$ MPa. Determine the minimum required diameter of the pin to the nearest mm. Apply a factor of safety of F.S. = 2.5 against shear failure.



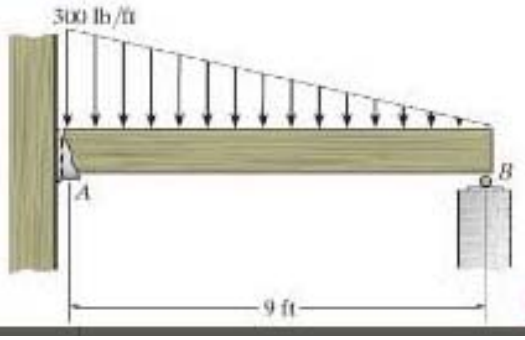
F1–22

F1–23. If the bolt head and the supporting bracket are made of the same material having a failure shear stress of $\tau_{fail} = 120$ MPa, determine the maximum allowable force **P** that can be applied to the bolt so that it does not pull through the plate. Apply a factor of safety of F.S. = 2.5 against shear failure.



F1–23

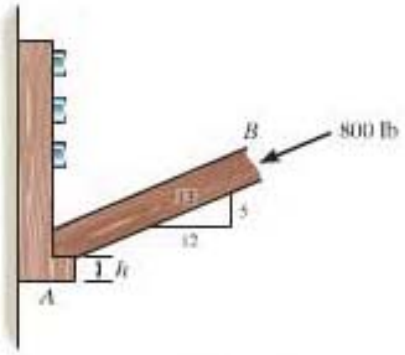
F1–24. Six nails are used to hold the hanger at A against the column. Determine the minimum required diameter of each nail to the nearest 1/16 in. if it is made of material having $\tau_{fail} = 16$ ksi. Apply a factor of safety of F.S. = 2 against shear failure.



F1–24

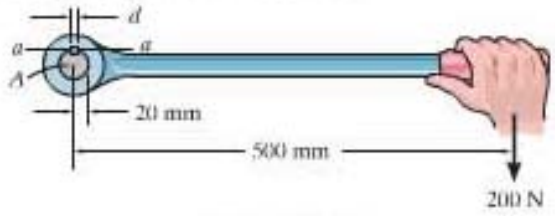
PROBLEMS

•**1–73.** Member B is subjected to a compressive force of 800 lb. If A and B are both made of wood and are $\frac{3}{8}$ in. thick, determine to the nearest $\frac{1}{4}$ in. the smallest dimension h of the horizontal segment so that it does not fail in shear. The average shear stress for the segment is $\tau_{allow} = 300$ psi.



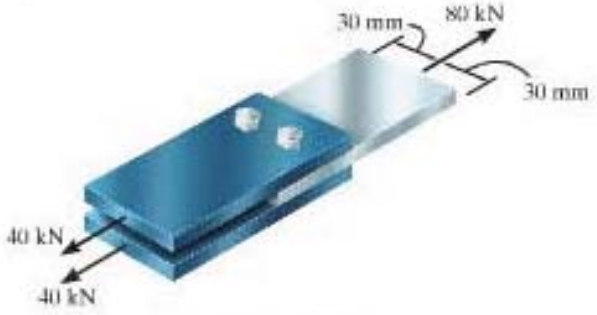
Prob. 1–73

1–74. The lever is attached to the shaft A using a key that has a width d and length of 25 mm. If the shaft is fixed and a vertical force of 200 N is applied perpendicular to the handle, determine the dimension d if the allowable shear stress for the key is $\tau_{allow} = 35$ MPa.



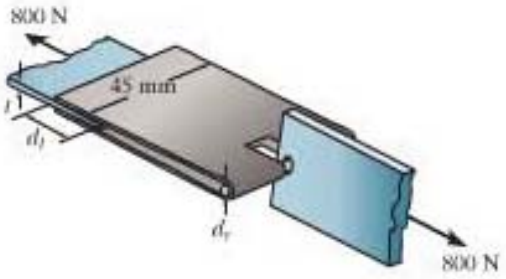
Prob. 1–74

1–75. The joint is fastened together using two bolts. Determine the required diameter of the bolts if the failure shear stress for the bolts is $\tau_{fail} = 350$ MPa. Use a factor of safety for shear of F.S. = 2.5.



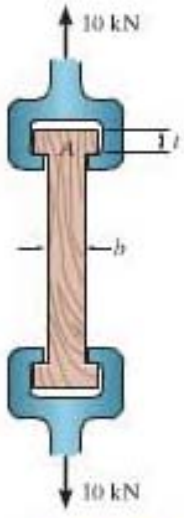
Prob. 1–75

*1–76. The lapbelt assembly is to be subjected to a force of 800 N. Determine (a) the required thickness t of the belt if the allowable tensile stress for the material is $(\sigma_t)_{allow} = 10$ MPa, (b) the required lap length d_l if the glue can sustain an allowable shear stress of $(\tau_{allow})_g = 0.75$ MPa, and (c) the required diameter d_r of the pin if the allowable shear stress for the pin is $(\tau_{allow})_p = 30$ MPa.



Prob. 1–76

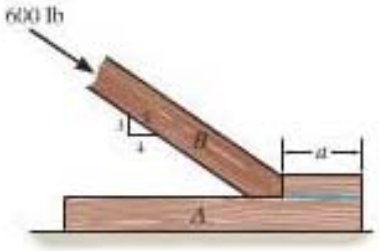
•**1–77.** The wood specimen is subjected to the pull of 10 kN in a tension testing machine. If the allowable normal stress for the wood is $(\sigma_t)_{allow} = 12$ MPa and the allowable shear stress is $\tau_{allow} = 1.2$ MPa, determine the required dimensions b and t so that the specimen reaches these stresses simultaneously. The specimen has a width of 25 mm.



Prob. 1–77

1

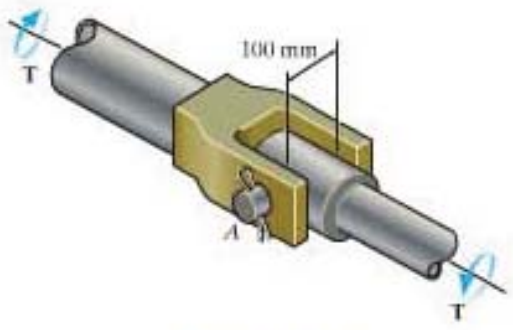
1–78. Member B is subjected to a compressive force of 600 lb. If A and B are both made of wood and are 1.5 in. thick, determine to the nearest 1/8 in. the smallest dimension a of the support so that the average shear stress along the blue line does not exceed $\tau_{allow} = 50$ psi. Neglect friction.



Prob. 1–78

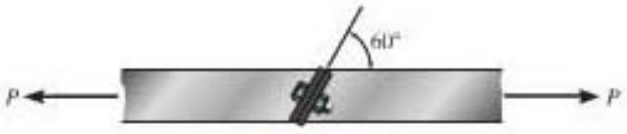
1–79. The joint is used to transmit a torque of $T = 3$ kN · m. Determine the required minimum diameter of the shear pin A if it is made from a material having a shear failure stress of $\tau_{fail} = 150$ MPa. Apply a factor of safety of 3 against failure.

***1–80.** Determine the maximum allowable torque **T** that can be transmitted by the joint. The shear pin A has a diameter of 25 mm, and it is made from a material having a failure shear stress of $\tau_{fail} = 150$ MPa. Apply a factor of safety of 3 against failure.



Probs. 1–79/80

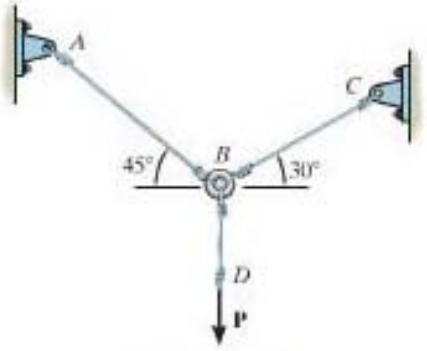
•1–81. The tension member is fastened together using *two* bolts, one on each side of the member as shown. Each bolt has a diameter of 0.3 in. Determine the maximum load P that can be applied to the member if the allowable shear stress for the bolts is $\tau_{allow} = 12$ ksi and the allowable average normal stress is $\sigma_{allow} = 20$ ksi.



Prob. 1–81

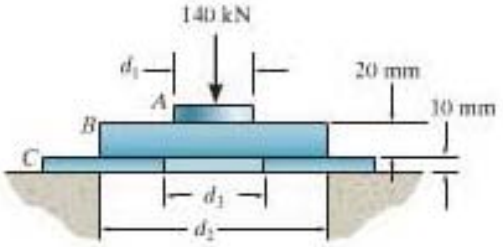
1–82. The three steel wires are used to support the load. If the wires have an allowable tensile stress of $\sigma_{allow} = 165$ MPa, determine the required diameter of each wire if the applied load is $P = 6$ kN.

1–83. The three steel wires are used to support the load. If the wires have an allowable tensile stress of $\sigma_{allow} = 165$ MPa, and wire AB has a diameter of 6 mm, BC has a diameter of 5 mm, and BD has a diameter of 7 mm, determine the greatest force P that can be applied before one of the wires fails.



Probs. 1–82/83

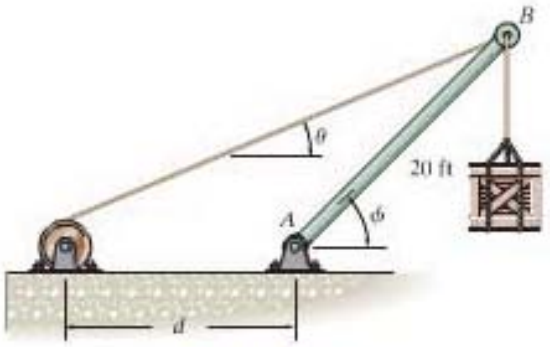
*1–84. The assembly consists of three disks A, B, and C that are used to support the load of 140 kN. Determine the smallest diameter d_1 of the top disk, the diameter d_2 within the support space, and the diameter d_3 of the hole in the bottom disk. The allowable bearing stress for the material is $(\sigma_{allow})_b = 350$ MPa and allowable shear stress is $\tau_{allow} = 125$ MPa.



Prob. 1–84

•1–85. The boom is supported by the winch cable that has a diameter of 0.25 in. and an allowable normal stress of $\sigma_{allow} = 24$ ksi. Determine the greatest load that can be supported without causing the cable to fail when $\theta = 30°$ and $\phi = 45°$. Neglect the size of the winch.

1–86. The boom is supported by the winch cable that has an allowable normal stress of $\sigma_{allow} = 24$ ksi. If it is required that it be able to slowly lift 5000 lb, from $\theta = 20°$ to $\theta = 50°$, determine the smallest diameter of the cable to the nearest $\frac{1}{16}$ in. The boom AB has a length of 20 ft. Neglect the size of the winch. Set $d = 12$ ft.



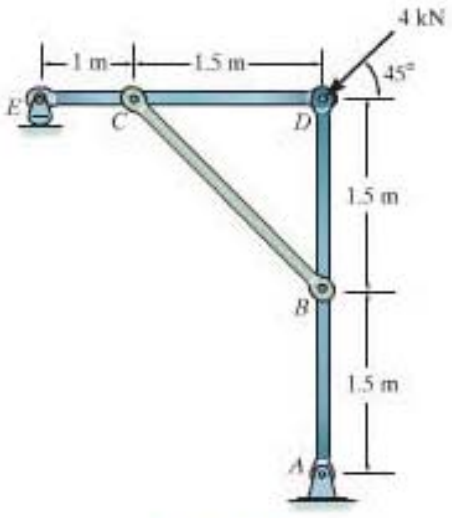
Probs. 1–85/86

1–87. The 60 mm × 60 mm oak post is supported on the pine block. If the allowable bearing stresses for these materials are $\sigma_{oak} = 43$ MPa and $\sigma_{pine} = 25$ MPa, determine the greatest load P that can be supported. If a rigid bearing plate is used between these materials, determine its required area so that the maximum load P can be supported. What is this load?



Prob. 1–87

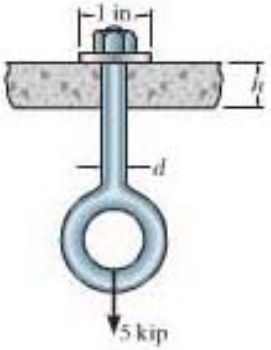
*1–88. The frame is subjected to the load of 4 kN which acts on member ABD at D. Determine the required diameter of the pins at D and C if the allowable shear stress for the material is $\tau_{allow} = 40$ MPa. Pin C is subjected to double shear, whereas pin D is subjected to single shear.



Prob. 1–88

1

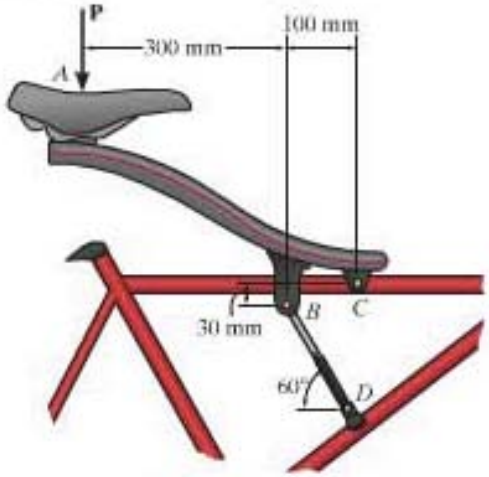
•1–89. The eye bolt is used to support the load of 5 kip. Determine its diameter d to the nearest $\frac{1}{8}$ in. and the required thickness h to the nearest $\frac{1}{8}$ in. of the support so that the washer will not penetrate or shear through it. The allowable normal stress for the bolt is $\sigma_{allow} = 21$ ksi and the allowable shear stress for the supporting material is $\tau_{allow} = 5$ ksi.



Prob. 1–89

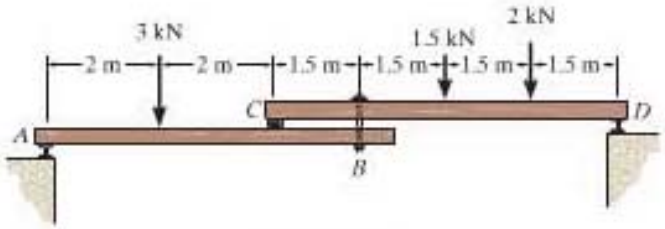
1–90. The soft-ride suspension system of the mountain bike is pinned at C and supported by the shock absorber BD. If it is designed to support a load $P = 1500$ N, determine the required minimum diameter of pins B and C. Use a factor of safety of 2 against failure. The pins are made of material having a failure shear stress of $\tau_{fail} = 150$ MPa, and each pin is subjected to double shear.

1–91. The soft-ride suspension system of the mountain bike is pinned at C and supported by the shock absorber BD. If it is designed to support a load of $P = 1500$ N, determine the factor of safety of pins B and C against failure if they are made of a material having a shear failure stress of $\tau_{fail} = 150$ MPa. Pin B has a diameter of 7.5 mm, and pin C has a diameter of 6.5 mm. Both pins are subjected to double shear.



Probs. 1–90/91

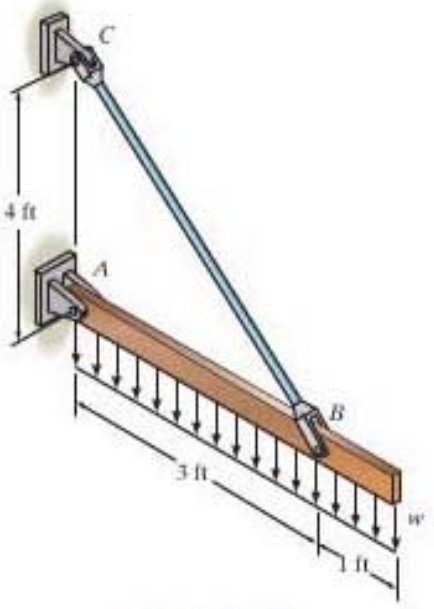
***1–92.** The compound wooden beam is connected together by a bolt at B. Assuming that the connections at A, B, C, and D exert only vertical forces on the beam, determine the required diameter of the bolt at B and the required outer diameter of its washers if the allowable tensile stress for the bolt is $(\sigma_t)_{allow} = 150$ MPa and the allowable bearing stress for the wood is $(\sigma_b)_{allow} = 28$ MPa. Assume that the hole in the washers has the same diameter as the bolt.



Prob. 1–92

•1–93. The assembly is used to support the distributed loading of $w = 500$ lb/ft. Determine the factor of safety with respect to yielding for the steel rod BC and the pins at B and C if the yield stress for the steel in tension is $\sigma_y = 36$ ksi and in shear $\tau_y = 18$ ksi. The rod has a diameter of 0.40 in., and the pins each have a diameter of 0.30 in.

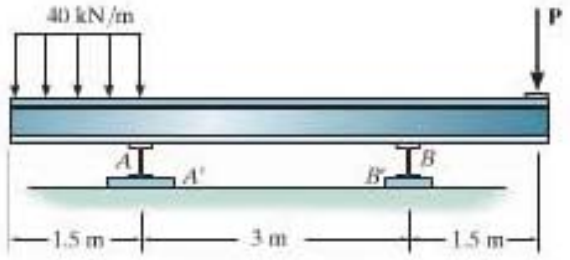
1–94. If the allowable shear stress for each of the 0.30-in.-diameter steel pins at A, B, and C is $\tau_{allow} = 12.5$ ksi, and the allowable normal stress for the 0.40-in.-diameter rod is $\sigma_{allow} = 22$ ksi, determine the largest intensity w of the uniform distributed load that can be suspended from the beam.



Probs. 1–93/94

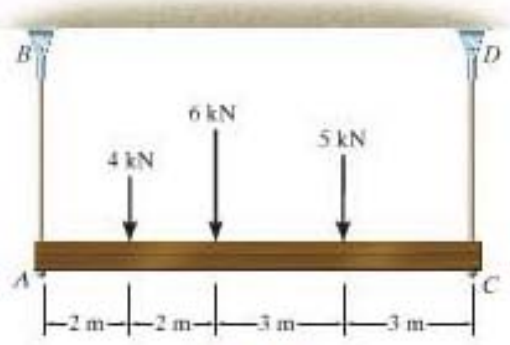
1–95. If the allowable bearing stress for the material under the supports at A and B is $(\sigma_b)_{allow} = 1.5$ MPa, determine the size of *square* bearing plates A' and B' required to support the load. Dimension the plates to the nearest mm. The reactions at the supports are vertical. Take $P = 100$ kN.

***1–96.** If the allowable bearing stress for the material under the supports at A and B is $(\sigma_b)_{allow} = 1.5$ MPa, determine the maximum load P that can be applied to the beam. The bearing plates A' and B' have square cross sections of 150 mm × 150 mm and 250 mm × 250 mm, respectively.



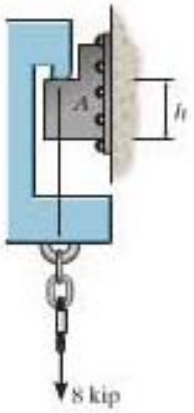
Probs. 1–95/96

•1–97. The rods AB and CD are made of steel having a failure tensile stress of $\sigma_{fail} = 510$ MPa. Using a factor of safety of F.S. = 1.75 for tension, determine their smallest diameter so that they can support the load shown. The beam is assumed to be pin connected at A and C.



Prob. 1–97

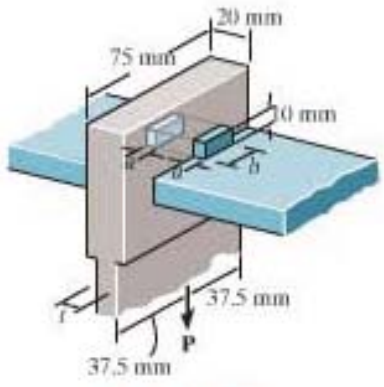
1–98. The aluminum bracket A is used to support the centrally applied load of 8 kip. If it has a constant thickness of 0.5 in., determine the smallest height h in order to prevent a shear failure. The failure shear stress is $\tau_{fail} = 23$ ksi. Use a factor of safety for shear of F.S. = 2.5.



Prob. 1–98

1–99. The hanger is supported using the rectangular pin. Determine the magnitude of the allowable suspended load **P** if the allowable bearing stress is $(\sigma_b)_{allow} = 220$ MPa, the allowable tensile stress is $(\sigma_t)_{allow} = 150$ MPa, and the allowable shear stress is $\tau_{allow} = 130$ MPa. Take $t = 6$ mm, $a = 5$ mm, and $b = 25$ mm.

***1–100.** The hanger is supported using the rectangular pin. Determine the required thickness t of the hanger, and dimensions a and b if the suspended load is $P = 60$ kN. The allowable tensile stress is $(\sigma_t)_{allow} = 150$ MPa, the allowable bearing stress is $(\sigma_b)_{allow} = 290$ MPa, and the allowable shear stress is $\tau_{allow} = 125$ MPa.

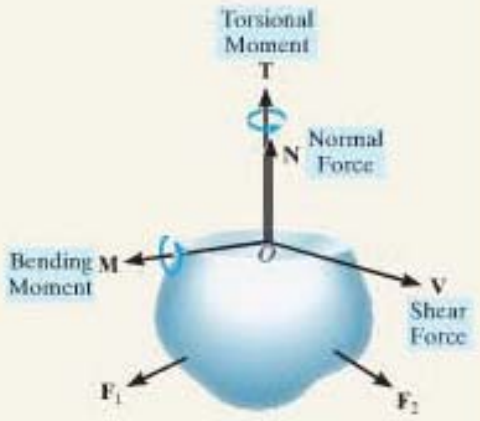


Probs. 1–99/100

CHAPTER REVIEW

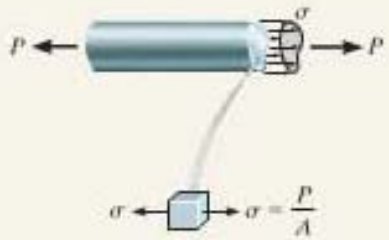
The internal loadings in a body consist of a normal force, shear force, bending moment, and torsional moment. They represent the resultants of both a normal and shear stress distribution that acts over the cross section. To obtain these resultants, use the method of sections and the equations of equilibrium.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_z = 0$$
$$\Sigma M_x = 0$$
$$\Sigma M_y = 0$$
$$\Sigma M_z = 0$$



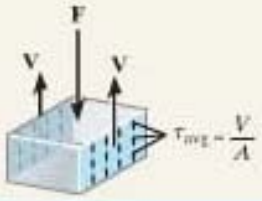
If a bar is made from homogeneous isotropic material and it is subjected to a series of external axial loads that pass through the centroid of the cross section, then a uniform normal stress distribution will act over the cross section. This average normal stress can be determined from $\sigma = P/A$, where P is the internal axial load at the section.

$$\sigma = \frac{P}{A}$$



The average shear stress can be determined using $\tau_{avg} = V/A$, where V is the shear force acting on the cross-sectional area A. This formula is often used to find the average shear stress in fasteners or in parts used for connections.

$$\tau_{avg} = \frac{V}{A}$$



The design of any simple connection requires that the average stress along any cross section not exceed an allowable stress of σ_{allow} or τ_{allow}. These values are reported in codes and are considered safe on the basis of experiments or through experience. Sometimes a factor of safety is reported provided the ultimate stress is known.

$$\text{F.S.} = \frac{\sigma_{fail}}{\sigma_{allow}} = \frac{\tau_{fail}}{\tau_{allow}}$$

CONCEPTUAL PROBLEMS



P1–1

P1–1. Here hurricane winds caused the failure of this highway sign. Assuming the wind creates a uniform pressure on the sign of 2 kPa, use reasonable dimensions for the sign and determine the resultant shear and moment at the two connections where the failure occurred.



P1–2

P1–2. The two structural tubes are connected by the pin which passes through them. If the vertical load being supported is 100 kN, draw a free-body diagram of the pin and then use the method of sections to find the maximum average shear force in the pin. If the pin has a diameter of 50 mm, what is the maximum average shear stress in the pin?



P1–3

P1–3. The hydraulic cylinder H applies a horizontal force F on the pin at A. Draw the free-body diagram of the pin and show the forces acting on it. Using the method of sections, explain why the average shear stress in the pin is largest at sections through the gaps D and E and not at some intermediate section.

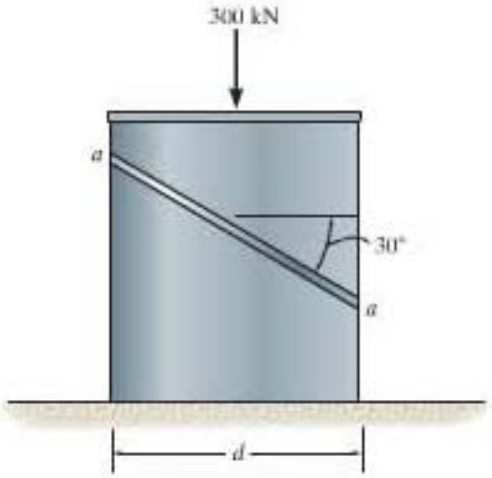


P1–4

P1–4. The vertical load on the hook is 1000 lb. Draw the appropriate free-body diagrams and determine the maximum average shear force on the pins at A, B, and C. Note that due to symmetry four wheels are used to support the loading on the railing.

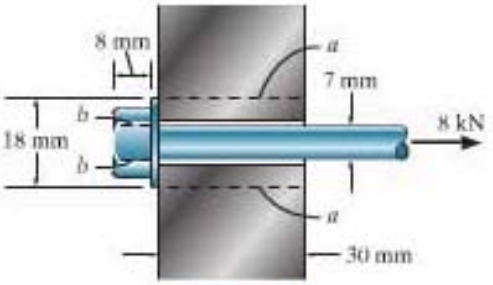
1 REVIEW PROBLEMS

•1–101. The 200-mm-diameter aluminum cylinder supports a compressive load of 300 kN. Determine the average normal and shear stress acting on section *a–a*. Show the results on a differential element located on the section.



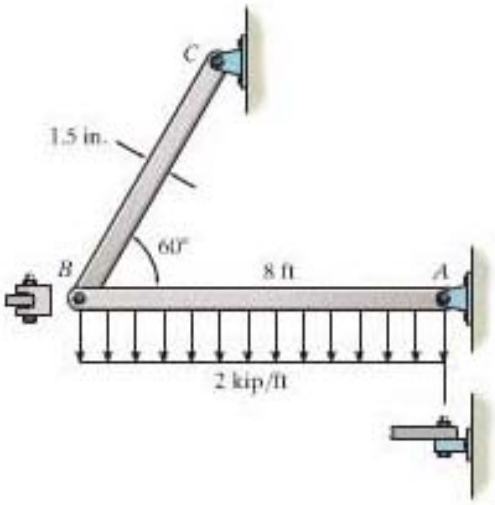
Prob. 1–101

1–102. The long bolt passes through the 30-mm-thick plate. If the force in the bolt shank is 8 kN, determine the average normal stress in the shank, the average shear stress along the cylindrical area of the plate defined by the section lines *a–a*, and the average shear stress in the bolt head along the cylindrical area defined by the section lines *b–b*.



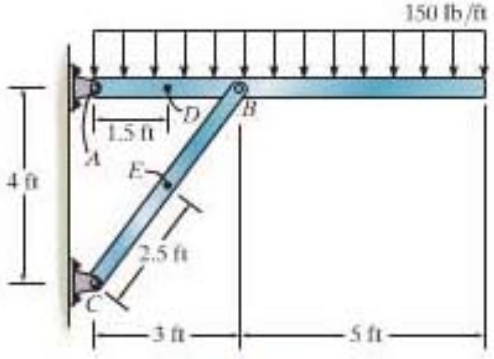
Prob. 1–102

1–103. Determine the required thickness of member *BC* and the diameter of the pins at *A* and *B* if the allowable normal stress for member *BC* is $\sigma_{allow} = 29$ ksi and the allowable shear stress for the pins is $\tau_{allow} = 10$ ksi.



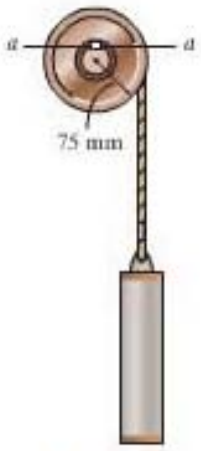
Prob. 1–103

***1–104.** Determine the resultant internal loadings acting on the cross sections located through points *D* and *E* of the frame.



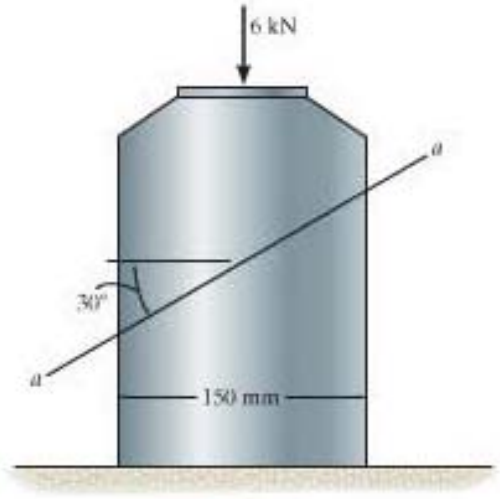
Prob. 1–104

•1–105. The pulley is held fixed to the 20-mm-diameter shaft using a key that fits within a groove cut into the pulley and shaft. If the suspended load has a mass of 50 kg, determine the average shear stress in the key along section *a–a*. The key is 5 mm by 5 mm square and 12 mm long.



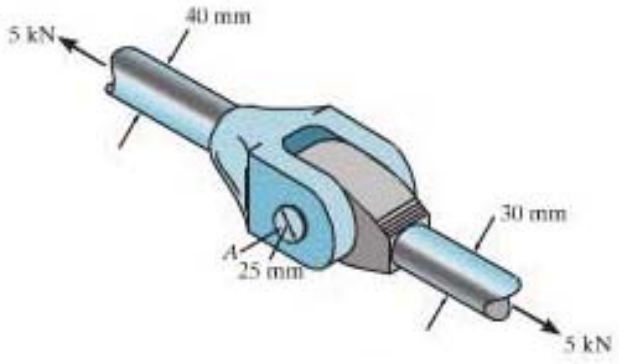
Prob. 1–105

1–106. The bearing pad consists of a 150 mm by 150 mm block of aluminum that supports a compressive load of 6 kN. Determine the average normal and shear stress acting on the plane through section *a–a*. Show the results on a differential volume element located on the plane.



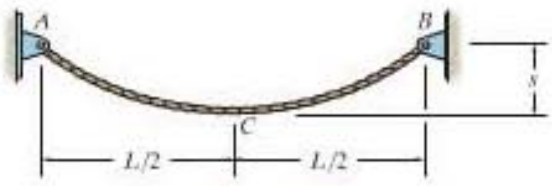
Prob. 1–106

1–107. The yoke-and-rod connection is subjected to a tensile force of 5 kN. Determine the average normal stress in each rod and the average shear stress in the pin *A* between the members.

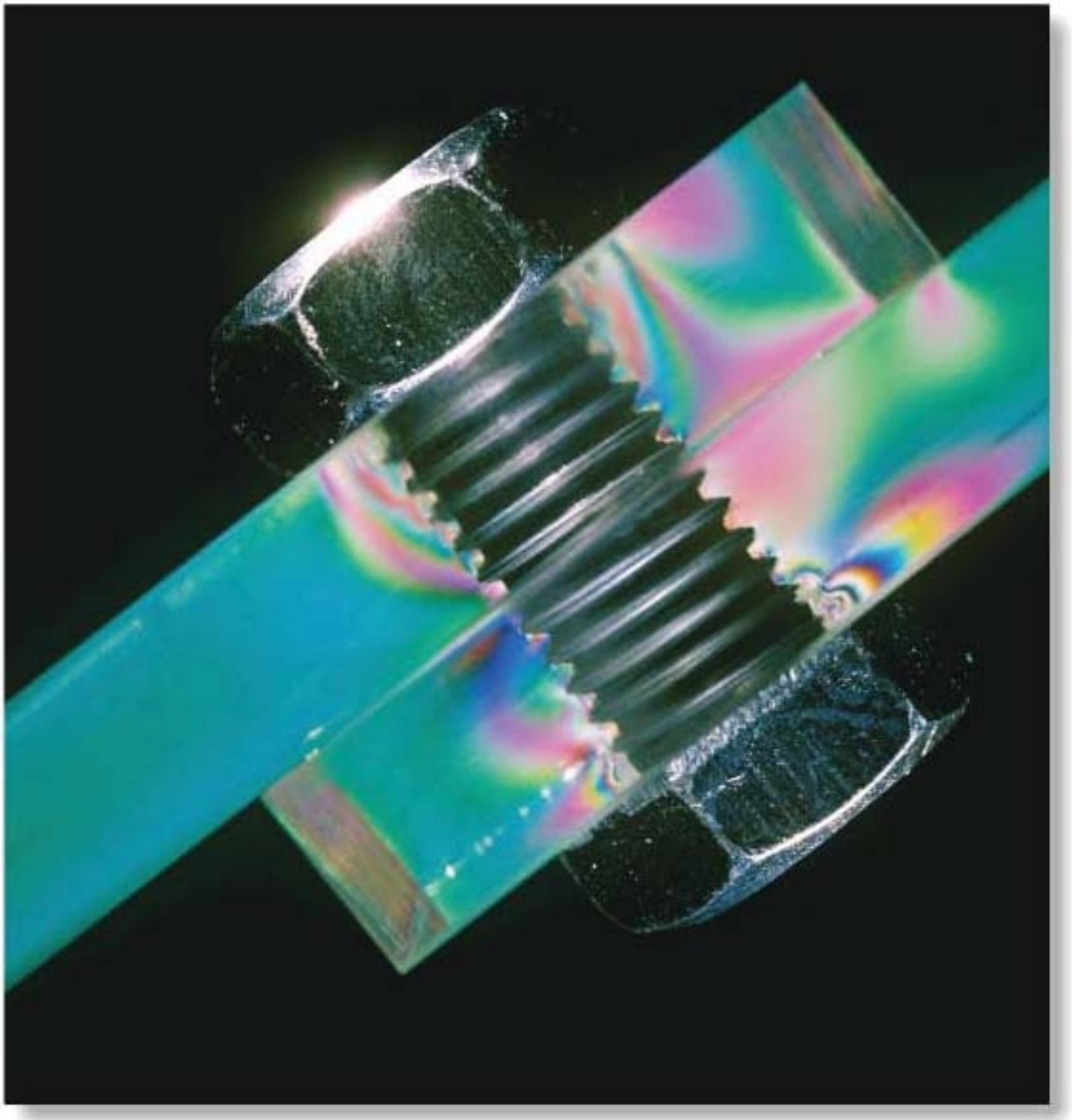


Prob. 1–107

***1–108.** The cable has a specific weight γ (weight/volume) and cross-sectional area A. If the sag s is small, so that its length is approximately L and its weight can be distributed uniformly along the horizontal axis, determine the average normal stress in the cable at its lowest point C.



Prob. 1–108



When the bolt causes compression of these two transparent plates it produces strains in the material that shows up as a spectrum of colors when displayed under polarized light. These strains can be related to the stress in the material.

Strain

2

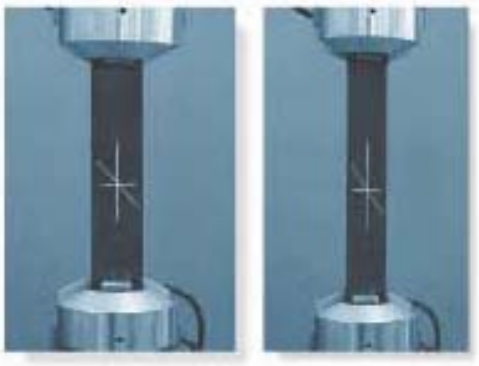
CHAPTER OBJECTIVES

In engineering the deformation of a body is specified using the concepts of normal and shear strain. In this chapter we will define these quantities and show how they can be determined for various types of problems.

2.1 Deformation

Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes are referred to as ***deformation***, and they may be either highly visible or practically unnoticeable. For example, a rubber band will undergo a very large deformation when stretched, whereas only slight deformations of structural members occur when a building is occupied by people walking about. Deformation of a body can also occur when the temperature of the body is changed. A typical example is the thermal expansion or contraction of a roof caused by the weather.

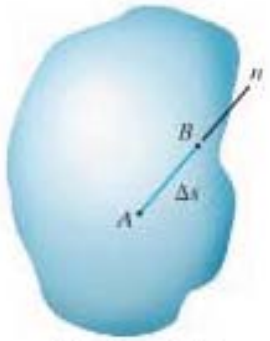
In a general sense, the deformation of a body will not be uniform throughout its volume, and so the change in geometry of any line segment within the body may vary substantially along its length. Hence, to study deformational changes in a more uniform manner, we will consider line segments that are very short and located in the neighborhood of a point. Realize, however, that these changes will also depend on the orientation of the line segment at the point. For example, a line segment may elongate if it is oriented in one direction, whereas it may contract if it is oriented in another direction.



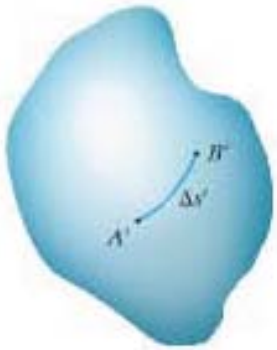
Note the before and after positions of three different line segments on this rubber membrane which is subjected to tension. The vertical line is lengthened, the horizontal line is shortened, and the inclined line changes its length and rotates.

2.2 Strain

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain. Strain is actually measured by experiments, and once the strain is obtained, it will be shown in the next chapter how it can be related to the stress acting within the body.



Undeformed body
(a)



Deformed body
(b)

Fig. 2–1

Normal Strain. If we define the normal strain as the change in length of a line per unit length, then we will not have to specify the *actual length* of any particular line segment. Consider, for example, the line AB, which is contained within the undeformed body shown in Fig. 2–1*a*. This line lies along the n axis and has an original length of Δs. After deformation, points A and B are displaced to A' and B', and the line becomes a curve having a length of $\Delta s'$, Fig. 2–1*b*. The change in length of the line is therefore $\Delta s' - \Delta s$. If we define the *average normal strain* using the symbol ϵ_{avg} (epsilon), then

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} \tag{2–1}$$

As point B is chosen closer and closer to point A, the length of the line will become shorter and shorter, such that $\Delta s \to 0$. Also, this causes B' to approach A', such that $\Delta s' \to 0$. Consequently, in the limit the normal strain at point A and in the direction of n is

$$\epsilon = \lim_{B \to A \text{ along } n} \frac{\Delta s' - \Delta s}{\Delta s} \tag{2–2}$$

Hence, when ϵ (or ϵ_{avg}) is positive the initial line will elongate, whereas if ϵ is negative the line contracts.

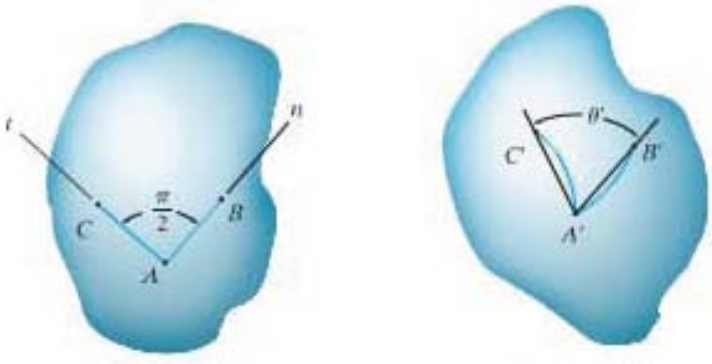
Note that normal strain is a *dimensionless quantity*, since it is a ratio of two lengths. Although this is the case, it is sometimes stated in terms of a ratio of length units. If the SI system is used, then the basic unit for length is the meter (m). Ordinarily, for most engineering applications ϵ will be very small, so measurements of strain are in micrometers per meter (μm/m), where $1\ \mu\text{m} = 10^{-6}$ m. In the Foot-Pound-Second system, strain is often stated in units of inches per inch (in./in.). Sometimes

for experimental work, strain is expressed as a percent, e.g., 0.001 m/m = 0.1%. As an example, a normal strain of $480(10^{-6})$ can be reported as $480(10^{-6})$ in./in., 480 μm/m, or 0.0480%. Also, one can state this answer as simply 480 μ (480 "micros").

Shear Strain. Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between these two line segments is referred to as ***shear strain***. This angle is denoted by γ (gamma) and is always measured in radians (rad), which are dimensionless. For example, consider the line segments AB and AC originating from the same point A in a body, and directed along the perpendicular n and t axes, Fig. 2–2*a*. After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at A is θ', Fig. 2–2*b*. Hence the shear strain at point A associated with the n and t axes becomes

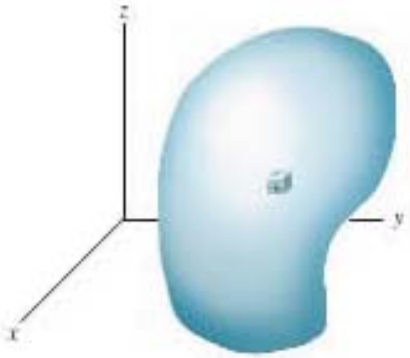
$$\gamma_{nt} = \frac{\pi}{2} - \lim_{\substack{B \to A \text{ along } n \\ C \to A \text{ along } t}} \theta' \tag{2–3}$$

Notice that if θ' is smaller than $\pi/2$ the shear strain is positive, whereas if θ' is larger than $\pi/2$ the shear strain is negative.



Undeformed body (a) Deformed body (b)

Fig. 2–2



(a)

Cartesian Strain Components. Using the definitions of normal and shear strain, we will now show how they can be used to describe the deformation of the body in Fig. 2–3*a*. To do so, imagine the body is subdivided into small elements such as the one shown in Fig. 2–3*b*. This element is rectangular, has undeformed dimensions Δx, Δy, and Δz, and is located in the neighborhood of a point in the body, Fig. 2–3*a*. If the element's dimensions are very small, then its deformed shape will be a parallelepiped, Fig. 2–3*c*, since very small line segments will remain approximately straight after the body is deformed. In order to achieve this deformed shape, we will first consider how the normal strain changes the lengths of the sides of the rectangular element, and then how the shear strain changes the angles of each side. For example, Δx elongates $\epsilon_x \Delta x$, so its new length is $\Delta x + \epsilon_x \Delta x$. Therefore, the approximate lengths of the three sides of the parallelepiped are

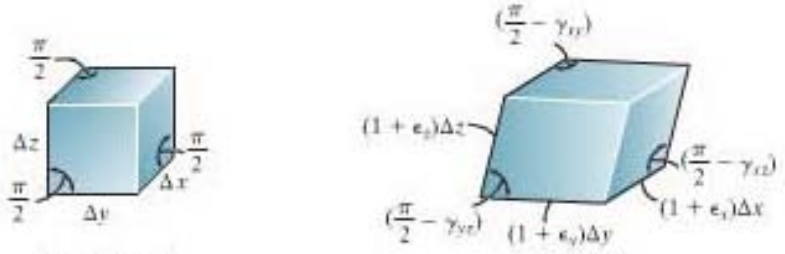
$$(1 + \epsilon_x)\,\Delta x \qquad (1 + \epsilon_y)\,\Delta y \qquad (1 + \epsilon_z)\,\Delta z$$

And the approximate angles between these sides are

$$\frac{\pi}{2} - \gamma_{xy} \qquad \frac{\pi}{2} - \gamma_{yz} \qquad \frac{\pi}{2} - \gamma_{xz}$$

Notice that the ***normal strains cause a change in volume*** of the element, whereas the ***shear strains cause a change in its shape***. Of course, both of these effects occur simultaneously during the deformation.

In summary, then, the *state of strain* at a point in a body requires specifying three normal strains, ϵ_x, ϵ_y, ϵ_z, and three shear strains, γ_{xy}, γ_{yz}, γ_{xz}. These strains completely describe the deformation of a rectangular volume element of material located at the point and oriented so that its sides are originally parallel to the x, y, z axes. Provided these strains are defined at all points in the body, then the deformed shape of the body can be determined.



Undeformed element
(b)

Deformed element
(c)

Fig. 2–3

Small Strain Analysis. Most engineering design involves applications for which only *small deformations* are allowed. In this text, therefore, we will assume that the deformations that take place within a body are almost infinitesimal. In particular, the *normal strains* occurring within the material are *very small* compared to 1, so that $\epsilon \ll 1$. This assumption has wide practical application in engineering, and it is often referred to as a *small strain analysis*. It can be used, for example, to approximate $\sin\theta = \theta$, $\cos\theta = 1$, and $\tan\theta = \theta$, provided θ is very small.



The rubber bearing support under this concrete bridge girder is subjected to both normal and shear strain. The normal strain is caused by the weight and bridge loads on the girder, and the shear strain is caused by the horizontal movement of the girder due to temperature changes.

Important Points

- Loads will cause all material bodies to deform and, as a result, points in a body will undergo *displacements or changes in position.*
- *Normal strain* is a measure per unit length of the elongation or contraction of a small line segment in the body, whereas *shear strain* is a measure of the change in angle that occurs between two small line segments that are originally perpendicular to one another.
- The state of strain at a point is characterized by six strain components: three normal strains ϵ_x, ϵ_y, ϵ_z and three shear strains γ_{xy}, γ_{yz}, γ_{xz}. These components depend upon the original orientation of the line segments and their location in the body.
- Strain is the geometrical quantity that is measured using experimental techniques. Once obtained, the stress in the body can then be determined from material property relations, as discussed in the next chapter.
- Most engineering materials undergo very small deformations, and so the normal strain $\epsilon \ll 1$. This assumption of "small strain analysis" allows the calculations for normal strain to be simplified, since first-order approximations can be made about their size.

EXAMPLE 2.1

The slender rod shown in Fig. 2–4 is subjected to an increase of temperature along its axis, which creates a normal strain in the rod of $\epsilon_z = 40(10^{-3})z^{1/2}$, where z is measured in meters. Determine (a) the displacement of the end B of the rod due to the temperature increase, and (b) the average normal strain in the rod.

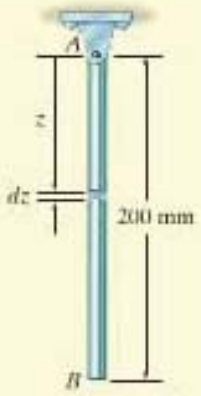


Fig. 2–4

SOLUTION

Part (a). Since the normal strain is reported at each point along the rod, a differential segment dz, located at position z, Fig. 2–4, has a deformed length that can be determined from Eq. 2–1; that is,

$$dz' = dz + \epsilon_z\, dz$$

$$dz' = \left[1 + 40(10^{-3})z^{1/2}\right] dz$$

The sum of these segments along the axis yields the *deformed length* of the rod, i.e.,

$$z' = \int_0^{0.2\text{ m}} \left[1 + 40(10^{-3})z^{1/2}\right] dz$$

$$= \left[z + 40(10^{-3})\tfrac{2}{3}z^{3/2}\right]\Big|_0^{0.2\text{ m}}$$

$$= 0.20239\text{ m}$$

The displacement of the end of the rod is therefore

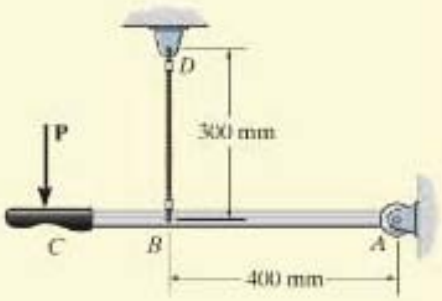
$$\Delta_B = 0.20239\text{ m} - 0.2\text{ m} = 0.00239\text{ m} = 2.39\text{ mm} \downarrow \quad \textit{Ans.}$$

Part (b). The average normal strain in the rod is determined from Eq. 2–1, which assumes that the rod or "line segment" has an original length of 200 mm and a change in length of 2.39 mm. Hence,

$$\epsilon_{avg} = \frac{\Delta s' - \Delta s}{\Delta s} = \frac{2.39\text{ mm}}{200\text{ mm}} = 0.0119\text{ mm/mm} \quad \textit{Ans.}$$

EXAMPLE 2.2

When force **P** is applied to the rigid lever arm ABC in Fig. 2–5*a*, the arm rotates counterclockwise about pin A through an angle of 0.05°. Determine the normal strain developed in wire BD.



(a)

SOLUTION I

Geometry. The orientation of the lever arm after it rotates about point A is shown in Fig. 2–5*b*. From the geometry of this figure,

$$\alpha = \tan^{-1}\left(\frac{400\text{ mm}}{300\text{ mm}}\right) = 53.1301°$$

Then

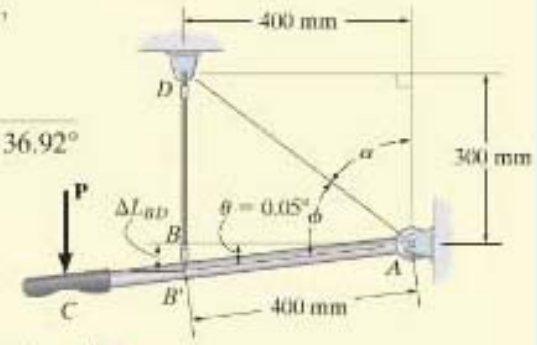
$$\phi = 90° - \alpha + 0.05° = 90° - 53.1301° + 0.05° = 36.92°$$

For triangle ABD the Pythagorean theorem gives

$$L_{AD} = \sqrt{(300\text{ mm})^2 + (400\text{ mm})^2} = 500\text{ mm}$$

Using this result and applying the law of cosines to triangle $AB'D$,

$$\begin{aligned} L_{B'D} &= \sqrt{L_{AD}^2 + L_{AB'}^2 - 2(L_{AD})(L_{AB'})\cos\phi} \\ &= \sqrt{(500\text{ mm})^2 + (400\text{ mm})^2 - 2(500\text{ mm})(400\text{ mm})\cos 36.92°} \\ &= 300.3491\text{ mm} \end{aligned}$$



(b)

Fig. 2–5

Normal Strain.

$$\epsilon_{BD} = \frac{L_{B'D} - L_{BD}}{L_{BD}} = \frac{300.3491\text{ mm} - 300\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \quad \textit{Ans.}$$

SOLUTION II

Since the strain is small, this same result can be obtained by approximating the elongation of wire BD as ΔL_{BD}, shown in Fig. 2–5*b*. Here,

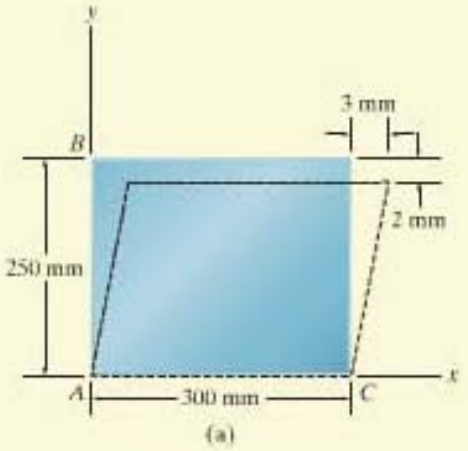
$$\Delta L_{BD} = \theta L_{AB} = \left[\left(\frac{0.05°}{180°}\right)(\pi\text{ rad})\right](400\text{ mm}) = 0.3491\text{ mm}$$

Therefore,

$$\epsilon_{BD} = \frac{\Delta L_{BD}}{L_{BD}} = \frac{0.3491\text{ mm}}{300\text{ mm}} = 0.00116\text{ mm/mm} \quad \textit{Ans.}$$

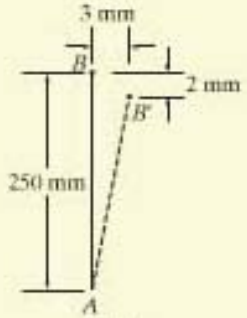
EXAMPLE 2.3

Due to a loading, the plate is deformed into the dashed shape shown in Fig. 2–6*a*. Determine (a) the average normal strain along the side AB, and (b) the average shear strain in the plate at A relative to the x and y axes.

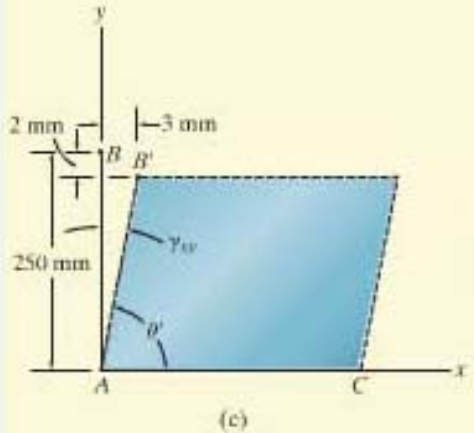


(a)

Fig. 2–6



(b)



(c)

SOLUTION

Part (a). Line AB, coincident with the y axis, becomes line AB' after deformation, as shown in Fig. 2–6*b*. The length of AB' is

$$AB' = \sqrt{(250\text{ mm} - 2\text{ mm})^2 + (3\text{ mm})^2} = 248.018\text{ mm}$$

The average normal strain for AB is therefore

$$(\epsilon_{AB})_{avg} = \frac{AB' - AB}{AB} = \frac{248.018\text{ mm} - 250\text{ mm}}{250\text{ mm}}$$

$$= -7.93(10^{-3})\text{ mm/mm} \qquad \textit{Ans.}$$

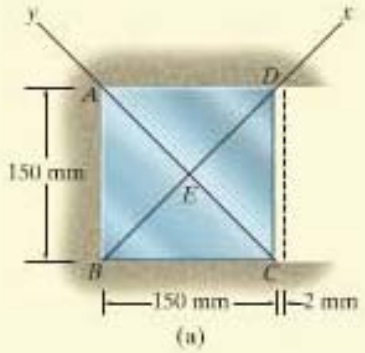
The negative sign indicates the strain causes a contraction of AB.

Part (b). As noted in Fig. 2–6*c*, the once 90° angle BAC between the sides of the plate at A changes to θ' due to the displacement of B to B'. Since $\gamma_{xy} = \pi/2 - \theta'$, then γ_{xy} is the angle shown in the figure. Thus,

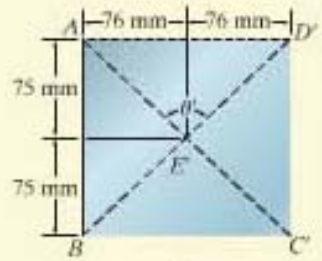
$$\gamma_{xy} = \tan^{-1}\left(\frac{3\text{ mm}}{250\text{ mm} - 2\text{ mm}}\right) = 0.0121\text{ rad} \qquad \textit{Ans.}$$

EXAMPLE 2.4

The plate shown in Fig. 2–7*a* is fixed connected along AB and held in the horizontal guides at its top and bottom, AD and BC. If its right side CD is given a uniform horizontal displacement of 2 mm, determine (a) the average normal strain along the diagonal AC, and (b) the shear strain at E relative to the x, y axes.



(a)



(b)

Fig. 2–7

SOLUTION

Part (a). When the plate is deformed, the diagonal AC becomes AC', Fig. 2–7*b*. The length of diagonals AC and AC' can be found from the Pythagorean theorem. We have

$$AC = \sqrt{(0.150\text{ m})^2 + (0.150\text{ m})^2} = 0.21213\text{ m}$$

$$AC' = \sqrt{(0.150\text{ m})^2 + (0.152\text{ m})^2} = 0.21355\text{ m}$$

Therefore the average normal strain along the diagonal is

$$(\epsilon_{AC})_{avg} = \frac{AC' - AC}{AC} = \frac{0.21355\text{ m} - 0.21213\text{ m}}{0.21213\text{ m}}$$

$$= 0.00669\text{ mm/mm} \qquad \textit{Ans.}$$

Part (b). To find the shear strain at E relative to the x and y axes, it is first necessary to find the angle θ' after deformation, Fig. 2–7*b*. We have

$$\tan\left(\frac{\theta'}{2}\right) = \frac{76\text{ mm}}{75\text{ mm}}$$

$$\theta' = 90.759^\circ = \left(\frac{\pi}{180^\circ}\right)(90.759^\circ) = 1.58404\text{ rad}$$

Applying Eq. 2–3, the shear strain at E is therefore

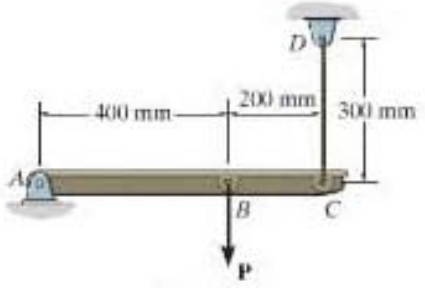
$$\gamma_{xy} = \frac{\pi}{2} - 1.58404\text{ rad} = -0.0132\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle θ' is *greater than* 90°.

NOTE: If the x and y axes were horizontal and vertical at point E, then the 90° angle between these axes would not change due to the deformation, and so $\gamma_{xy} = 0$ at point E.

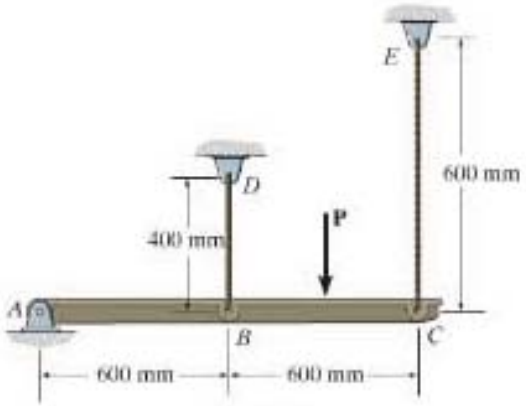
FUNDAMENTAL PROBLEMS

F2–1. When force **P** is applied to the rigid arm ABC, point B displaces vertically downward through a distance of 0.2 mm. Determine the normal strain developed in wire CD.



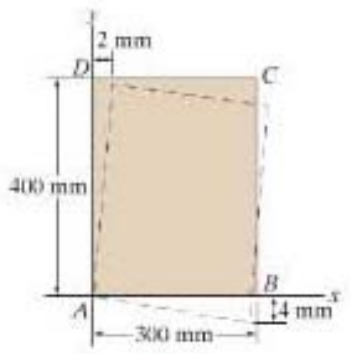
F2–1

F2–2. If the applied force **P** causes the rigid arm ABC to rotate clockwise about pin A through an angle of 0.02°, determine the normal strain developed in wires BD and CE.



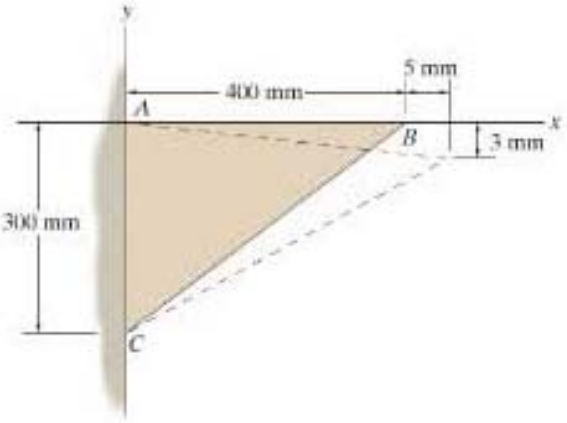
F2–2

F2–3. The rectangular plate is deformed into the shape of a rhombus shown by the dashed line. Determine the average shear strain at corner A with respect to the x and y axes.



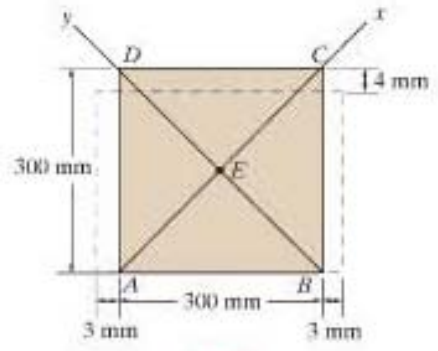
F2–3

F2–4. The triangular plate is deformed into the shape shown by the dashed line. Determine the normal strain developed along edge BC and the average shear strain at corner A with respect to the x and y axes.



F2–4

F2–5. The square plate is deformed into the shape shown by the dashed line. Determine the average normal strain along diagonal AC and the shear strain of point E with respect to the x and y axes.



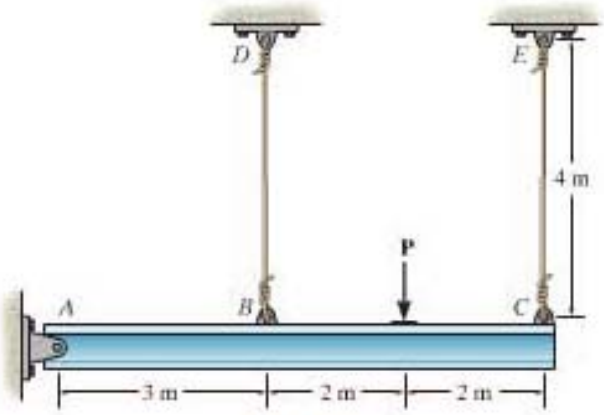
F2–5

PROBLEMS

2–1. An air-filled rubber ball has a diameter of 6 in. If the air pressure within it is increased until the ball's diameter becomes 7 in., determine the average normal strain in the rubber.

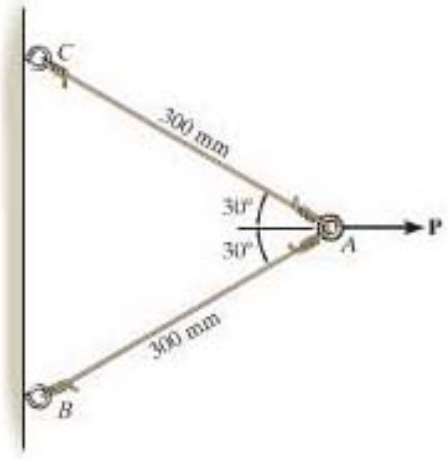
2–2. A thin strip of rubber has an unstretched length of 15 in. If it is stretched around a pipe having an outer diameter of 5 in., determine the average normal strain in the strip.

2–3. The rigid beam is supported by a pin at A and wires BD and CE. If the load **P** on the beam causes the end C to be displaced 10 mm downward, determine the normal strain developed in wires CE and BD.



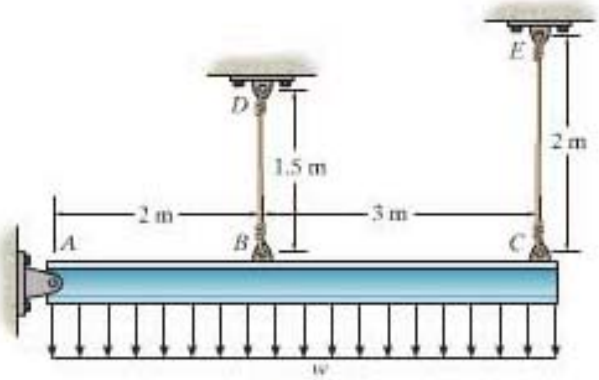
Prob. 2–3

*2–4. The two wires are connected together at A. If the force **P** causes point A to be displaced horizontally 2 mm, determine the normal strain developed in each wire.



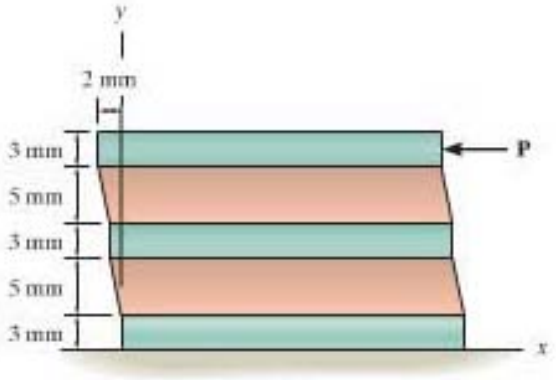
Prob. 2–4

•2–5. The rigid beam is supported by a pin at A and wires BD and CE. If the distributed load causes the end C to be displaced 10 mm downward, determine the normal strain developed in wires CE and BD.



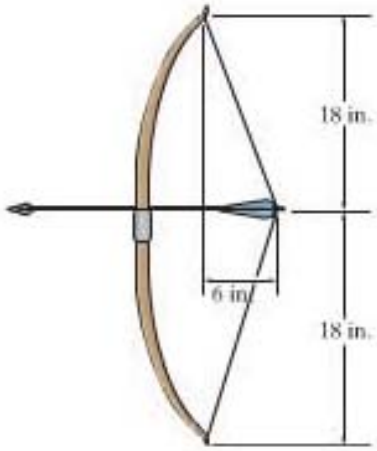
Prob. 2–5

2–6. Nylon strips are fused to glass plates. When moderately heated the nylon will become soft while the glass stays approximately rigid. Determine the average shear strain in the nylon due to the load **P** when the assembly deforms as indicated.



Prob. 2–6

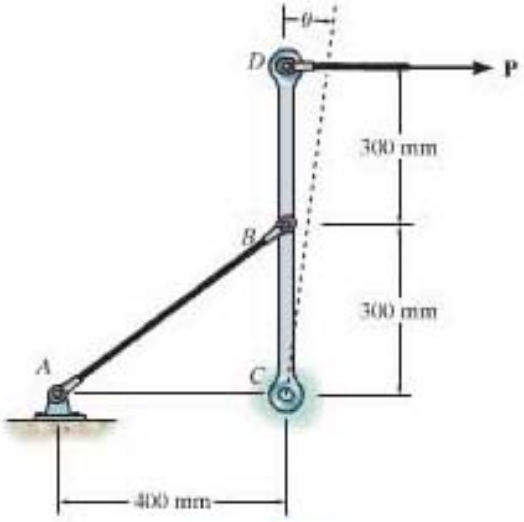
2–7. If the unstretched length of the bowstring is 35.5 in., determine the average normal strain in the string when it is stretched to the position shown.



Prob. 2–7

***2–8.** Part of a control linkage for an airplane consists of a rigid member CBD and a flexible cable AB. If a force is applied to the end D of the member and causes it to rotate by $\theta = 0.3°$, determine the normal strain in the cable. Originally the cable is unstretched.

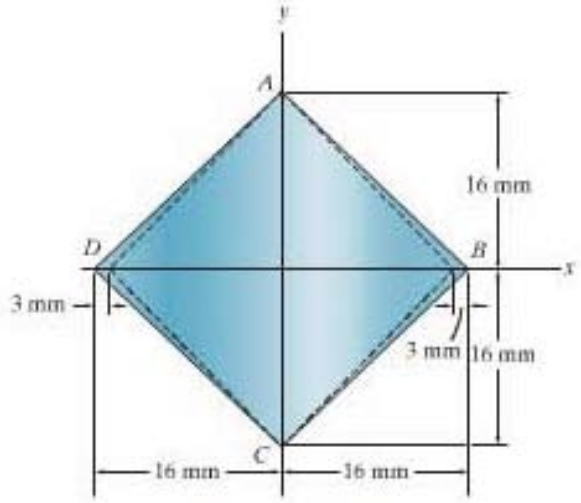
•2–9. Part of a control linkage for an airplane consists of a rigid member CBD and a flexible cable AB. If a force is applied to the end D of the member and causes a normal strain in the cable of 0.0035 mm/mm, determine the displacement of point D. Originally the cable is unstretched.



Probs. 2–8/9

2–10. The corners B and D of the square plate are given the displacements indicated. Determine the shear strains at A and B.

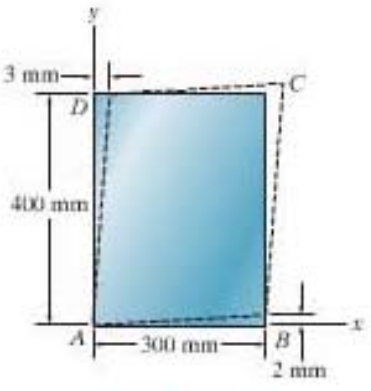
2–11. The corners B and D of the square plate are given the displacements indicated. Determine the average normal strains along side AB and diagonal DB.



Probs. 2–10/11

***2–12.** The piece of rubber is originally rectangular. Determine the average shear strain γ_{xy} at A if the corners B and D are subjected to the displacements that cause the rubber to distort as shown by the dashed lines.

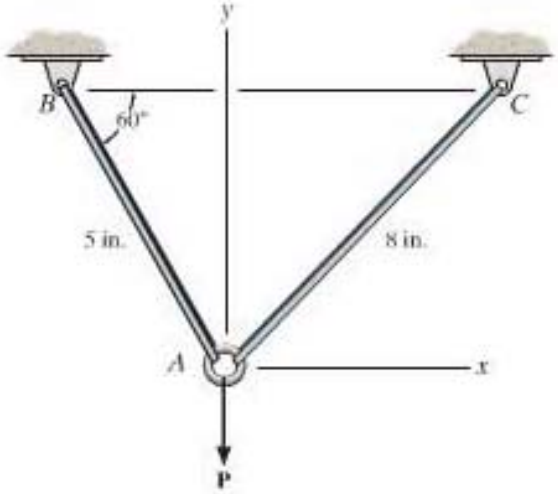
•2–13. The piece of rubber is originally rectangular and subjected to the deformation shown by the dashed lines. Determine the average normal strain along the diagonal DB and side AD.



Probs. 2–12/13

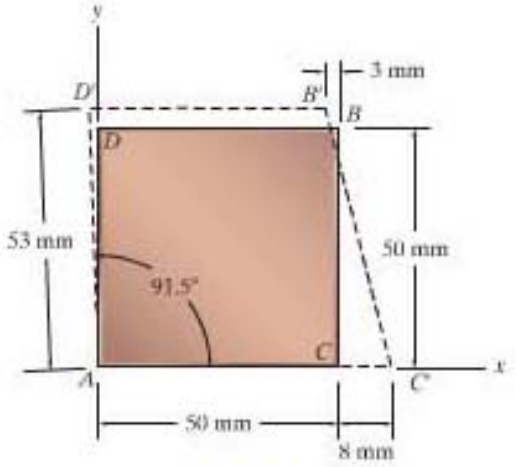
2–14. Two bars are used to support a load. When unloaded, AB is 5 in. long, AC is 8 in. long, and the ring at A has coordinates (0, 0). If a load **P** acts on the ring at A, the normal strain in AB becomes $\epsilon_{AB} = 0.02$ in./in., and the normal strain in AC becomes $\epsilon_{AC} = 0.035$ in./in. Determine the coordinate position of the ring due to the load.

2–15. Two bars are used to support a load **P**. When unloaded, AB is 5 in. long, AC is 8 in. long, and the ring at A has coordinates (0, 0). If a load is applied to the ring at A, so that it moves it to the coordinate position (0.25 in., −0.73 in.), determine the normal strain in each bar.



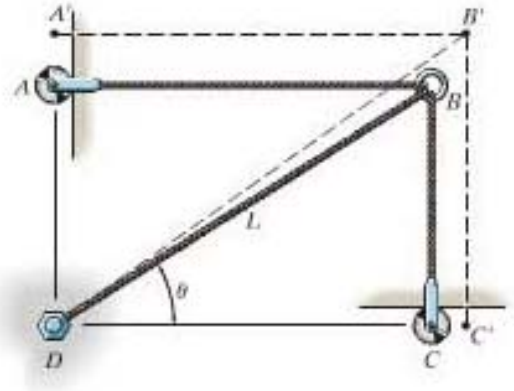
Probs. 2–14/15

*2–16. The square deforms into the position shown by the dashed lines. Determine the average normal strain along each diagonal, AB and CD. Side $D'B'$ remains horizontal.



Prob. 2–16

•2–17. The three cords are attached to the ring at B. When a force is applied to the ring it moves it to point B', such that the normal strain in AB is ϵ_{AB} and the normal strain in CB is ϵ_{CB}. Provided these strains are small, determine the normal strain in DB. Note that AB and CB remain horizontal and vertical, respectively, due to the roller guides at A and C.

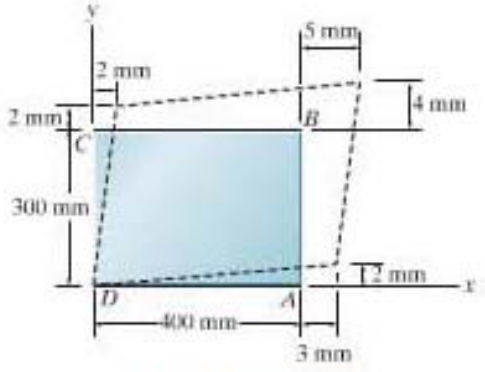


Prob. 2–17

2–18. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners A and B if the plastic distorts as shown by the dashed lines.

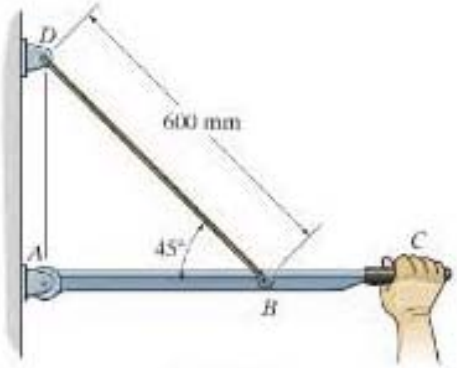
2–19. The piece of plastic is originally rectangular. Determine the shear strain γ_{xy} at corners D and C if the plastic distorts as shown by the dashed lines.

*2–20. The piece of plastic is originally rectangular. Determine the average normal strain that occurs along the diagonals AC and DB.



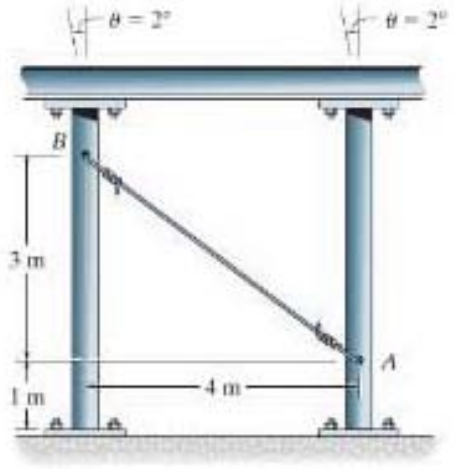
Probs. 2–18/19/20

•2–21. The force applied to the handle of the rigid lever arm causes the arm to rotate clockwise through an angle of 3° about pin A. Determine the average normal strain developed in the wire. Originally, the wire is unstretched.



Prob. 2–21

•2–25. The guy wire AB of a building frame is originally unstretched. Due to an earthquake, the two columns of the frame tilt $\theta = 2°$. Determine the approximate normal strain in the wire when the frame is in this position. Assume the columns are rigid and rotate about their lower supports.

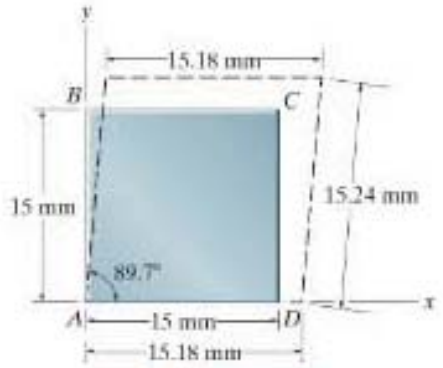


Prob. 2–25

2–22. A square piece of material is deformed into the dashed position. Determine the shear strain γ_{xy} at A.

2–23. A square piece of material is deformed into the dashed parallelogram. Determine the average normal strain that occurs along the diagonals AC and BD.

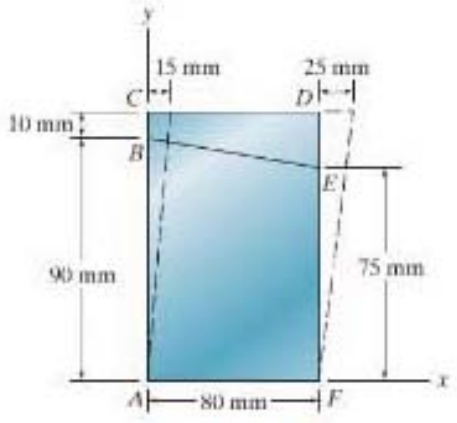
***2–24.** A square piece of material is deformed into the dashed position. Determine the shear strain γ_{xy} at C.



Probs. 2–22/23/24

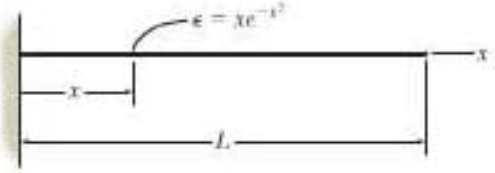
2–26. The material distorts into the dashed position shown. Determine (a) the average normal strains along sides AC and CD and the shear strain γ_{xy} at F, and (b) the average normal strain along line BE.

2–27. The material distorts into the dashed position shown. Determine the average normal strain that occurs along the diagonals AD and CF.



Probs. 2–26/27

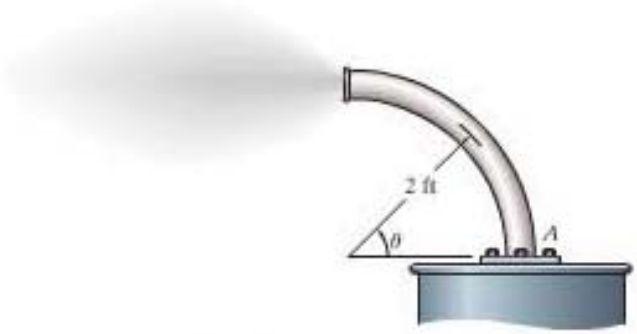
*2-28. The wire is subjected to a normal strain that is defined by $\epsilon = xe^{-x^2}$, where x is in millimeters. If the wire has an initial length L, determine the increase in its length.



Prob. 2-28

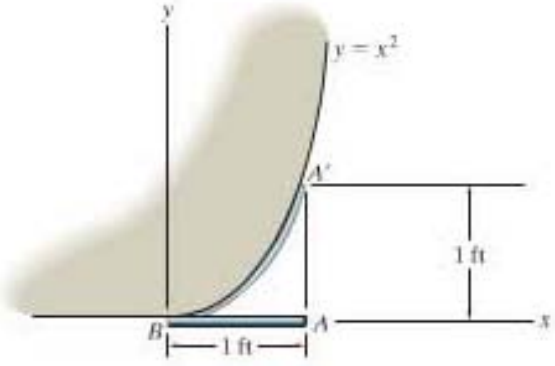
•2-29. The curved pipe has an original radius of 2 ft. If it is heated nonuniformly, so that the normal strain along its length is $\epsilon = 0.05 \cos \theta$, determine the increase in length of the pipe.

2-30. Solve Prob. 2-29 if $\epsilon = 0.08 \sin \theta$.



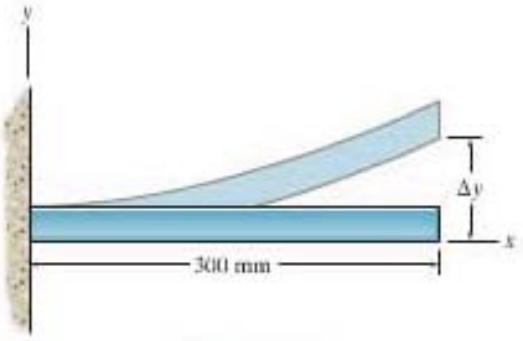
Probs. 2-29/30

2-31. The rubber band AB has an unstretched length of 1 ft. If it is fixed at B and attached to the surface at point A', determine the average normal strain in the band. The surface is defined by the function $y = (x^2)$ ft, where x is in feet.



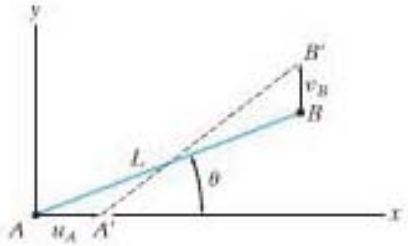
Prob. 2-31

*2-32. The bar is originally 300 mm long when it is flat. If it is subjected to a shear strain defined by $\gamma_{xy} = 0.02x$, where x is in meters, determine the displacement Δy at the end of its bottom edge. It is distorted into the shape shown, where no elongation of the bar occurs in the x direction.



Prob. 2-32

•2-33. The fiber AB has a length L and orientation θ. If its ends A and B undergo very small displacements u_A and v_B, respectively, determine the normal strain in the fiber when it is in position $A'B'$.

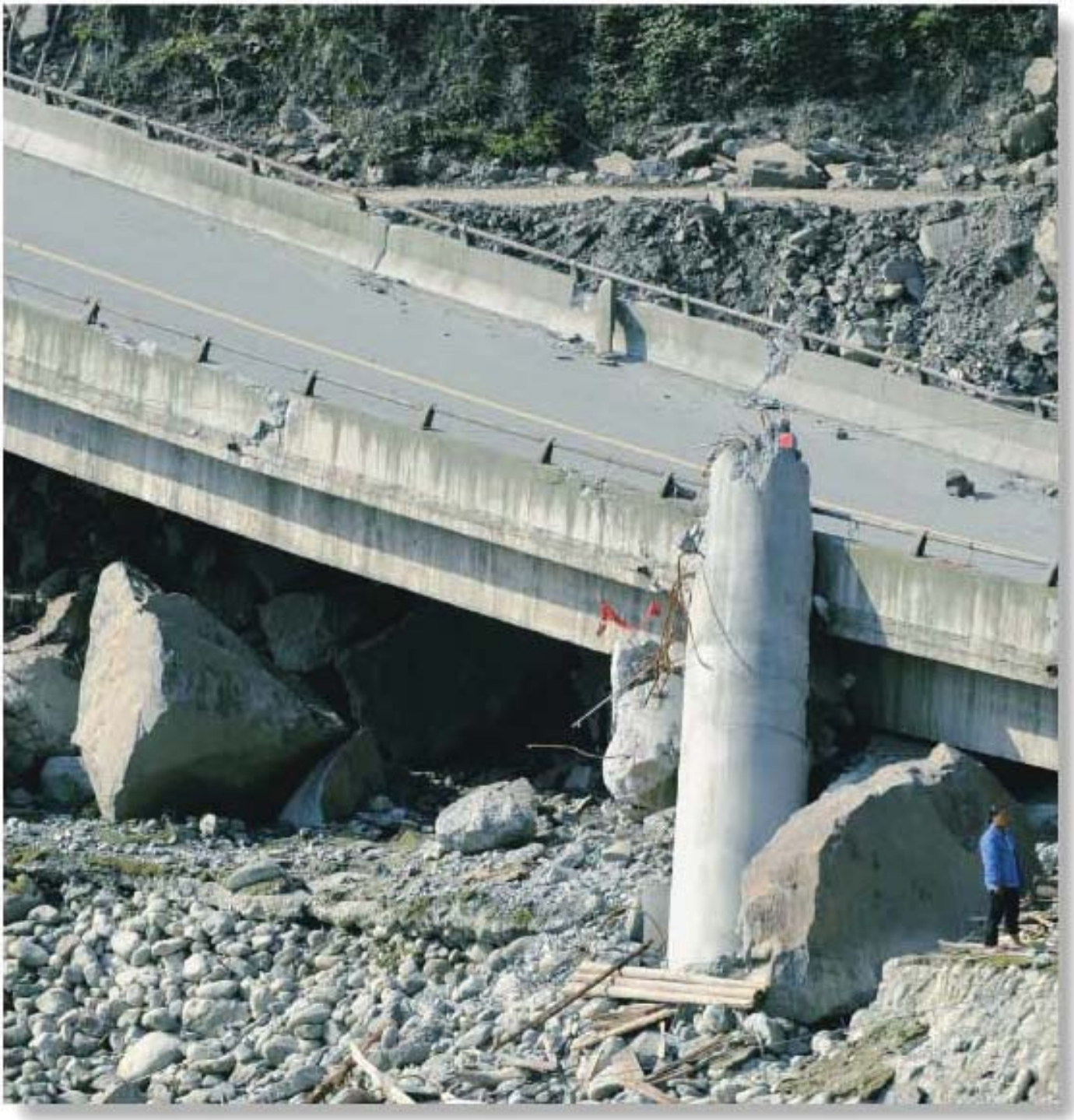


Prob. 2-33

2-34. If the normal strain is defined in reference to the final length, that is,

$$\epsilon'_n = \lim_{P' \to P} \left(\frac{\Delta s' - \Delta s}{\Delta s'} \right)$$

instead of in reference to the original length, Eq. 2-2, show that the difference in these strains is represented as a second-order term, namely, $\epsilon_n - \epsilon'_n = \epsilon_n \epsilon'_n$.



Horizontal ground displacements caused by an earthquake produced excessive strains in these bridge piers until they fractured. The material properties of the concrete and steel reinforcement must be known so that engineers can properly design this structure and thereby avoid such failures.

Mechanical Properties of Materials

3

CHAPTER OBJECTIVES

Having discussed the basic concepts of stress and strain, we will in this chapter show how stress can be related to strain by using experimental methods to determine the stress–strain diagram for a specific material. The behavior described by this diagram will then be discussed for materials that are commonly used in engineering. Also, mechanical properties and other tests that are related to the development of mechanics of materials will be discussed.

3.1 The Tension and Compression Test

The strength of a material depends on its ability to sustain a load without undue deformation or failure. This property is inherent in the material itself and must be determined by *experiment*. One of the most important tests to perform in this regard is the ***tension or compression test***. Although several important mechanical properties of a material can be determined from this test, it is used primarily to determine the relationship between the average normal stress and average normal strain in many engineering materials such as metals, ceramics, polymers, and composites.

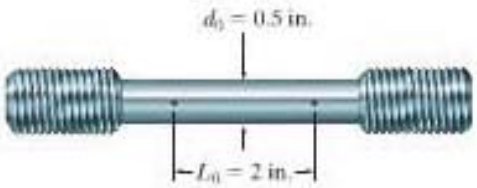


Fig. 3–1



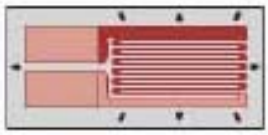
Typical steel specimen with attached strain gauge.

To perform a tension or compression test a specimen of the material is made into a "standard" shape and size. It has a constant circular cross section with enlarged ends, so that failure will not occur at the grips. Before testing, two small punch marks are placed along the specimen's uniform length. Measurements are taken of both the specimen's initial cross-sectional area, A_0, and the ***gauge-length*** distance L_0 between the punch marks. For example, when a metal specimen is used in a tension test it generally has an initial diameter of $d_0 = 0.5$ in. (13 mm) and a gauge length of $L_0 = 2$ in. (50 mm), Fig. 3–1. In order to apply an axial load with no bending of the specimen, the ends are usually seated into ball-and-socket joints. A testing machine like the one shown in Fig. 3–2 is then used to stretch the specimen at a very slow, constant rate until it fails. The machine is designed to read the load required to maintain this uniform stretching.

At frequent intervals during the test, data is recorded of the applied load P, as read on the dial of the machine or taken from a digital readout. Also, the elongation $\delta = L - L_0$ between the punch marks on the specimen may be measured using either a caliper or a mechanical or optical device called an ***extensometer***. This value of δ (delta) is then used to calculate the average normal strain in the specimen. Sometimes, however, this measurement is not taken, since it is also possible to read the strain *directly* by using an ***electrical-resistance strain gauge***, which looks like the one shown in Fig. 3–3. The operation of this gauge is based on the change in electrical resistance of a very thin wire or piece of metal foil under strain. Essentially the gauge is cemented to the specimen along its length. If the cement is very strong in comparison to the gauge, then the gauge is in effect an integral part of the specimen, so that when the specimen is strained in the direction of the gauge, the wire and specimen will experience the same strain. By measuring the electrical resistance of the wire, the gauge may be calibrated to read values of normal strain directly.



Fig. 3–2



Electrical–resistance strain gauge



Fig. 3–3

3.2 The Stress–Strain Diagram

It is not feasible to prepare a test specimen to match the size, A_0 and L_0, of each structural member. Rather, the test results must be reported so they apply to a member of *any size*. To achieve this, the load and corresponding deformation data are used to calculate various values of the stress and corresponding strain in the specimen. A plot of the results produces a curve called the ***stress–strain diagram***. There are two ways in which it is normally described.

Conventional Stress–Strain Diagram. We can determine the ***nominal*** or ***engineering stress*** by dividing the applied load P by the specimen's *original* cross-sectional area A_0. This calculation assumes that the stress is *constant* over the cross section and throughout the gauge length. We have

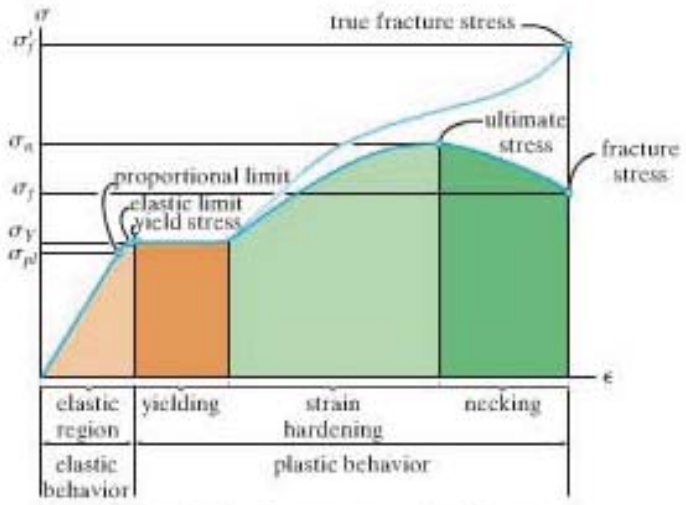
$$\sigma = \frac{P}{A_0} \tag{3–1}$$

Likewise, the ***nominal*** or ***engineering strain*** is found directly from the strain gauge reading, or by dividing the change in the specimen's gauge length, δ, by the specimen's original gauge length L_0. Here the strain is assumed to be constant throughout the region between the gauge points. Thus,

$$\epsilon = \frac{\delta}{L_0} \tag{3–2}$$

If the corresponding values of σ and ϵ are plotted so that the vertical axis is the stress and the horizontal axis is the strain, the resulting curve is called a ***conventional stress–strain diagram***. Realize, however, that two stress–strain diagrams for a particular material will be quite similar, but will never be exactly the same. This is because the results actually depend on variables such as the material's composition, microscopic imperfections, the way it is manufactured, the rate of loading, and the temperature during the time of the test.

We will now discuss the characteristics of the conventional stress–strain curve as it pertains to *steel*, a commonly used material for fabricating both structural members and mechanical elements. Using the method described above, the characteristic stress–strain diagram for a steel specimen is shown in Fig. 3–4. From this curve we can identify four different ways in which the material behaves, depending on the amount of strain induced in the material.



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

Fig. 3–4

Elastic Behavior. Elastic behavior of the material occurs when the strains in the specimen are within the light orange region shown in Fig. 3–4. Here the curve is actually a *straight line* throughout most of this region, so that the stress is *proportional* to the strain. The material in this region is said to be *linear elastic*. The upper stress limit to this linear relationship is called the ***proportional limit***, σ_{pl}. If the stress slightly exceeds the proportional limit, the curve tends to bend and flatten out as shown. This continues until the stress reaches the ***elastic limit***. Upon reaching this point, if the load is removed the specimen will still return back to its original shape. Normally for steel, however, the elastic limit is seldom determined, since it is very close to the proportional limit and therefore rather difficult to detect.

Yielding. A slight increase in stress above the elastic limit will result in a breakdown of the material and cause it to *deform permanently*. This behavior is called ***yielding***, and it is indicated by the rectangular dark orange region of the curve. The stress that causes yielding is called the ***yield stress*** or ***yield point***, σ_Y, and the deformation that occurs is called ***plastic deformation***. Although not shown in Fig. 3–4, for low-carbon steels or those that are hot rolled, the yield point is often distinguished by two values. The ***upper yield point*** occurs first, followed by a sudden decrease in load-carrying capacity to a ***lower yield point***. Notice that once the yield point is reached, then as shown in Fig. 3–4, the specimen will continue to elongate (strain) *without any increase in load*. When the material is in this state, it is often referred to as being ***perfectly plastic***.

Strain Hardening. When yielding has ended, an increase in load can be supported by the specimen, resulting in a curve that rises continuously but becomes flatter until it reaches a maximum stress referred to as the ***ultimate stress***, σ_u. The rise in the curve in this manner is called ***strain hardening***, and it is identified in Fig. 3–4 as the region in light green.

Necking. Up to the ultimate stress, as the specimen elongates, its cross-sectional area will decrease. This decrease is fairly *uniform* over the specimen's entire gauge length; however, just after, at the ultimate stress, the cross-sectional area will begin to decrease in a *localized* region of the specimen. As a result, a constriction or "neck" tends to form in this region as the specimen elongates further, Fig. 3–5*a*. This region of the curve due to necking is indicated in dark green in Fig. 3–4. Here the stress–strain diagram tends to curve downward until the specimen breaks at the ***fracture stress***, σ_f, Fig. 3–5*b*.

True Stress–Strain Diagram. Instead of always using the *original* cross-sectional area and specimen length to calculate the (engineering) stress and strain, we could have used the *actual* cross-sectional area and specimen length at the *instant* the load is measured. The values of stress and strain found from these measurements are called *true stress* and *true strain*, and a plot of their values is called the ***true stress–strain diagram***. When this diagram is plotted it has a form shown by the light-blue curve in Fig. 3–4. Note that the conventional and true $\sigma-\epsilon$ diagrams are practically coincident when the strain is small. The differences between the diagrams begin to appear in the strain-hardening range, where the magnitude of strain becomes more significant. In particular, there is a large divergence within the necking region. Here it can be seen from the conventional $\sigma-\epsilon$ diagram that the specimen *actually* supports a *decreasing load*, since A_0 is constant when calculating engineering stress, $\sigma = P/A_0$. However, from the true $\sigma-\epsilon$ diagram, the actual area A within the necking region is always decreasing until fracture, σ_f', and so the material actually sustains *increasing stress*, since $\sigma = P/A$.



Typical necking pattern which has occurred on this steel specimen just before fracture.



This steel specimen clearly shows the necking that occurred just before the specimen failed. This resulted in the formation of a "cup-cone" shape at the fracture location, which is characteristic of ductile materials.



Necking

(a)

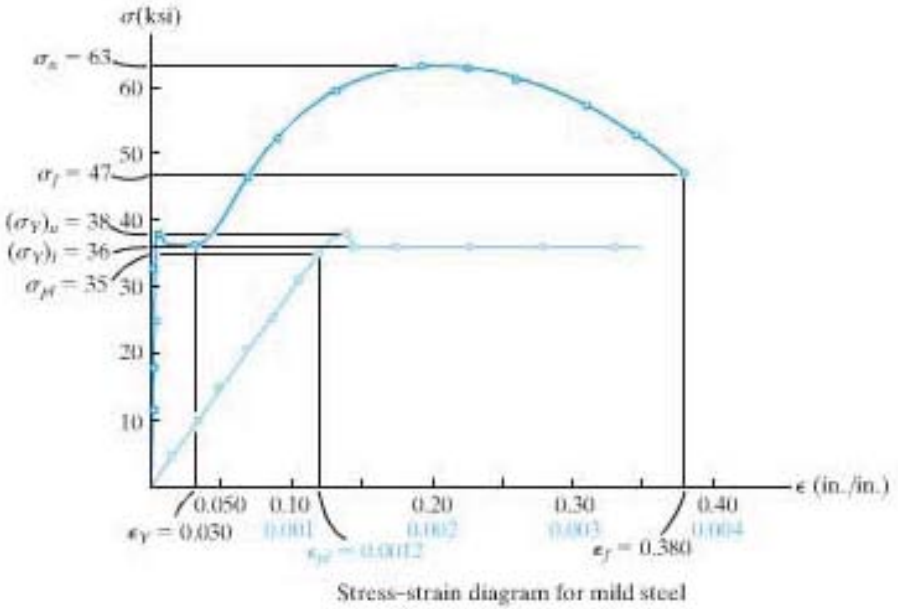
Failure of a ductile material

(b)

Fig. 3–5

Although the true and conventional stress–strain diagrams are different, most engineering design is done so that the material supports a stress within the elastic range. This is because the deformation of the material is generally not severe and the material will restore itself when the load is removed. The true strain up to the elastic limit will remain small enough so that the error in using the engineering values of σ and ϵ is very small (about 0.1%) compared with their true values. This is one of the primary reasons for using conventional stress–strain diagrams.

The above concepts can be summarized with reference to Fig. 3–6, which shows an actual conventional stress–strain diagram for a mild steel specimen. In order to enhance the details, the elastic region of the curve has been shown in light blue color using an exaggerated strain scale, also shown in light blue. Tracing the behavior, the proportional limit is reached at $\sigma_{pl} = 35$ ksi (241 MPa), where $\epsilon_{pl} = 0.0012$ in./in. This is followed by an upper yield point of $(\sigma_Y)_u = 38$ ksi (262 MPa), then suddenly a lower yield point of $(\sigma_Y)_l = 36$ ksi (248 MPa). The end of yielding occurs at a strain of $\epsilon_Y = 0.030$ in./in., which is 25 times greater than the strain at the proportional limit! Continuing, the specimen undergoes strain hardening until it reaches the ultimate stress of $\sigma_u = 63$ ksi (434 MPa), then it begins to neck down until a fracture occurs, $\sigma_f = 47$ ksi (324 MPa). By comparison, the strain at failure, $\epsilon_f = 0.380$ in./in., is 317 times greater than ϵ_{pl}!



Stress–strain diagram for mild steel

Fig. 3–6

3.3 Stress–Strain Behavior of Ductile and Brittle Materials

Materials can be classified as either being ductile or brittle, depending on their stress–strain characteristics.

Ductile Materials. Any material that can be subjected to large strains before it fractures is called a ***ductile material***. Mild steel, as discussed previously, is a typical example. Engineers often choose ductile materials for design because these materials are capable of absorbing shock or energy, and if they become overloaded, they will usually exhibit large deformation before failing.

One way to specify the ductility of a material is to report its percent elongation or percent reduction in area at the time of fracture. The ***percent elongation*** is the specimen's fracture strain expressed as a percent. Thus, if the specimen's original gauge length is L_0 and its length at fracture is L_f, then

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%) \tag{3–3}$$

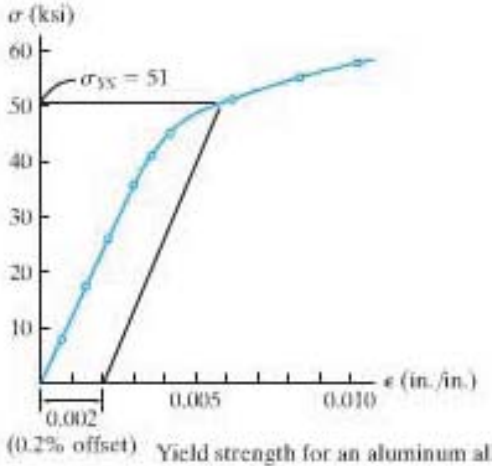
As seen in Fig. 3–6, since $\epsilon_f = 0.380$, this value would be 38% for a mild steel specimen.

The ***percent reduction in area*** is another way to specify ductility. It is defined within the region of necking as follows:

$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%) \tag{3–4}$$

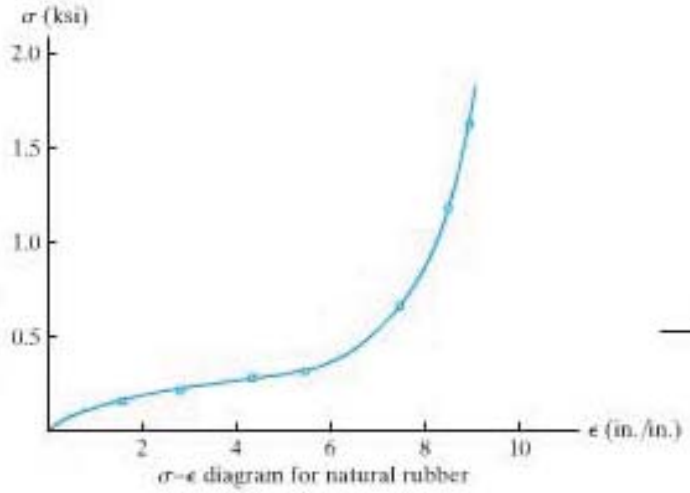
Here A_0 is the specimen's original cross-sectional area and A_f is the area of the neck at fracture. Mild steel has a typical value of 60%.

Besides steel, other metals such as brass, molybdenum, and zinc may also exhibit ductile stress–strain characteristics similar to steel, whereby they undergo elastic stress–strain behavior, yielding at constant stress, strain hardening, and finally necking until fracture. In most metals, however, constant yielding will *not occur* beyond the elastic range. One metal for which this is the case is aluminum. Actually, this metal often does not have a well-defined *yield point*, and consequently it is standard practice to define a ***yield strength*** using a graphical procedure called the ***offset method***. Normally a 0.2% strain (0.002 in./in.) is chosen, and from this point on the ϵ axis, a line parallel to the initial straight-line portion of the stress–strain diagram is drawn. The point where this line intersects the curve defines the yield strength. An example of the construction for determining the yield strength for an aluminum alloy is shown in Fig. 3–7. From the graph, the yield strength is $\sigma_{YS} = 51$ ksi (352 MPa).



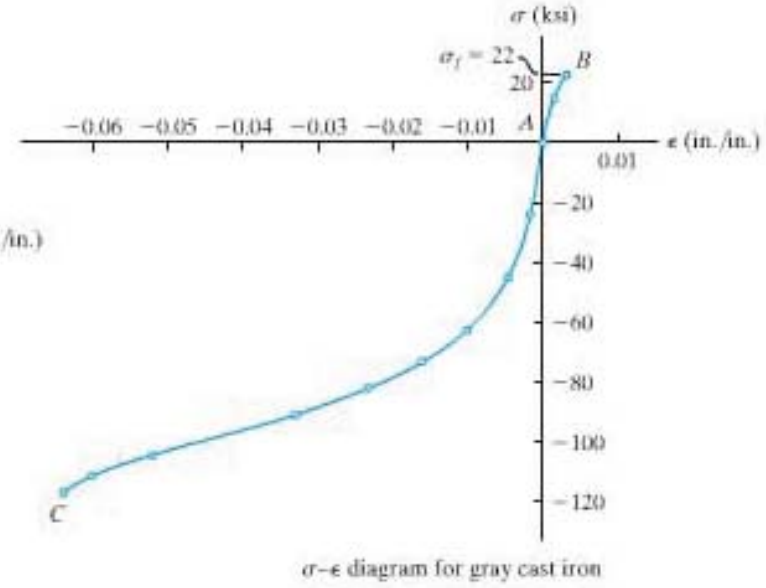
Yield strength for an aluminum alloy

Fig. 3–7



σ–ϵ diagram for natural rubber

Fig. 3–8



σ–ϵ diagram for gray cast iron

Fig. 3–9



Concrete used for structural purposes must be routinely tested in compression to be sure it provides the necessary design strength for this bridge deck. The concrete cylinders shown are compression tested for ultimate stress after curing for 30 days.

Realize that the yield strength is not a physical property of the material, since it is a stress that causes a *specified* permanent strain in the material. In this text, however, we will assume that the yield strength, yield point, elastic limit, and proportional limit all *coincide* unless otherwise stated. An exception would be natural rubber, which in fact does not even have a proportional limit, since stress and strain are *not* linearly related. Instead, as shown in Fig. 3–8, this material, which is known as a polymer, exhibits *nonlinear elastic behavior*.

Wood is a material that is often moderately ductile, and as a result it is usually designed to respond only to elastic loadings. The strength characteristics of wood vary greatly from one species to another, and for each species they depend on the moisture content, age, and the size and arrangement of knots in the wood. Since wood is a fibrous material, its tensile or compressive characteristics will differ greatly when it is loaded either parallel or perpendicular to its grain. Specifically, wood splits easily when it is loaded in tension perpendicular to its grain, and consequently tensile loads are almost always intended to be applied parallel to the grain of wood members.

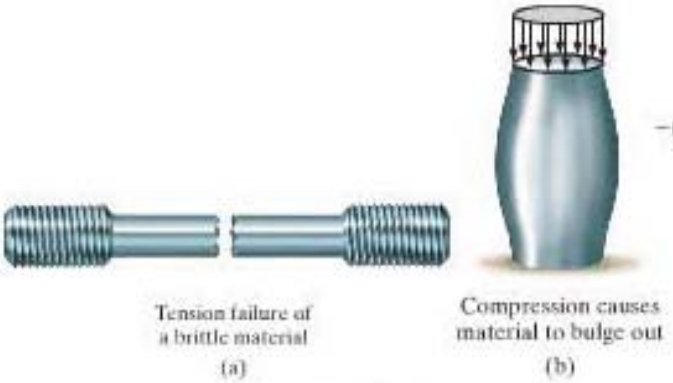
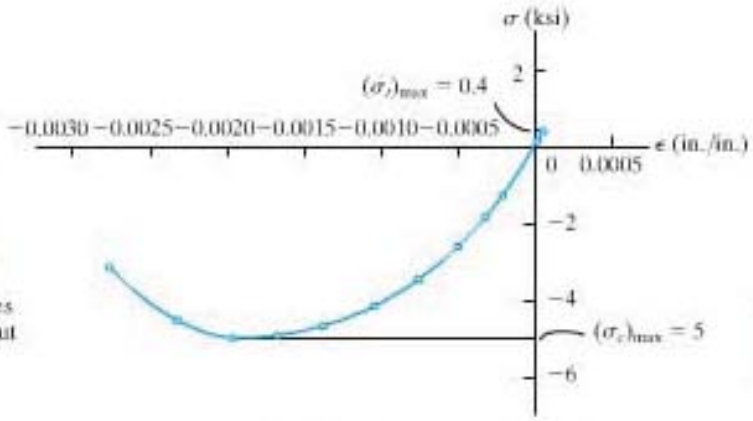


Fig. 3–10



σ–ε diagram for typical concrete mix

Fig. 3–11

Brittle Materials. Materials that exhibit little or no yielding before failure are referred to as ***brittle materials***. Gray cast iron is an example, having a stress–strain diagram in tension as shown by portion AB of the curve in Fig. 3–9. Here fracture at $\sigma_f = 22$ ksi (152 MPa) took place initially at an imperfection or microscopic crack and then spread rapidly across the specimen, causing complete fracture. Since the appearance of initial cracks in a specimen is quite random, brittle materials do not have a well-defined tensile fracture stress. Instead the *average* fracture stress from a set of observed tests is generally reported. A typical failed specimen is shown in Fig. 3–10*a*.

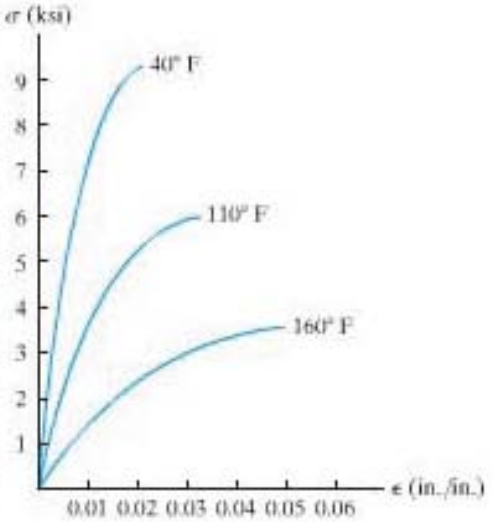
Compared with their behavior in tension, brittle materials, such as gray cast iron, exhibit a much higher resistance to axial compression, as evidenced by portion AC of the curve in Fig. 3–9. For this case any cracks or imperfections in the specimen tend to close up, and as the load increases the material will generally bulge or become barrel shaped as the strains become larger, Fig. 3–10*b*.

Like gray cast iron, concrete is classified as a brittle material, and it also has a low strength capacity in tension. The characteristics of its stress–strain diagram depend primarily on the mix of concrete (water, sand, gravel, and cement) and the time and temperature of curing. A typical example of a "complete" stress–strain diagram for concrete is given in Fig. 3–11. By inspection, its maximum compressive strength is almost 12.5 times greater than its tensile strength, $(\sigma_c)_{max} = 5$ ksi (34.5 MPa) versus $(\sigma_t)_{max} = 0.40$ ksi (2.76 MPa). For this reason, concrete is almost always reinforced with steel bars or rods whenever it is designed to support tensile loads.

It can generally be stated that most materials exhibit both ductile and brittle behavior. For example, steel has brittle behavior when it contains a high carbon content, and it is ductile when the carbon content is reduced. Also, at low temperatures materials become harder and more brittle, whereas when the temperature rises they become softer and more ductile. This effect is shown in Fig. 3–12 for a methacrylate plastic.



Steel rapidly loses its strength when heated. For this reason engineers often require main structural members to be insulated in case of fire.



σ–ε diagrams for a methacrylate plastic

Fig. 3–12

3.4 Hooke's Law

As noted in the previous section, the stress–strain diagrams for most engineering materials exhibit a *linear relationship* between stress and strain within the elastic region. Consequently, an increase in stress causes a proportionate increase in strain. This fact was discovered by Robert Hooke in 1676 using springs and is known as *Hooke's law*. It may be expressed mathematically as

$$\sigma = E\epsilon \tag{3–5}$$

Here E represents the constant of proportionality, which is called the ***modulus of elasticity*** or ***Young's modulus***, named after Thomas Young, who published an account of it in 1807.

Equation 3–5 actually represents the equation of the *initial straight-lined portion* of the stress–strain diagram up to the proportional limit. Furthermore, the modulus of elasticity represents the *slope* of this line. Since strain is dimensionless, from Eq. 3–5, E will have the same units as stress, such as psi, ksi, or pascals. As an example of its calculation, consider the stress–strain diagram for steel shown in Fig. 3–6. Here $\sigma_{pl} = 35$ ksi and $\epsilon_{pl} = 0.0012$ in./in., so that

$$E = \frac{\sigma_{pl}}{\epsilon_{pl}} = \frac{35 \text{ ksi}}{0.0012 \text{ in./in.}} = 29(10^3) \text{ ksi}$$

As shown in Fig. 3–13, the proportional limit for a particular type of steel alloy depends on its carbon content; however, most grades of steel, from the softest rolled steel to the hardest tool steel, have about the

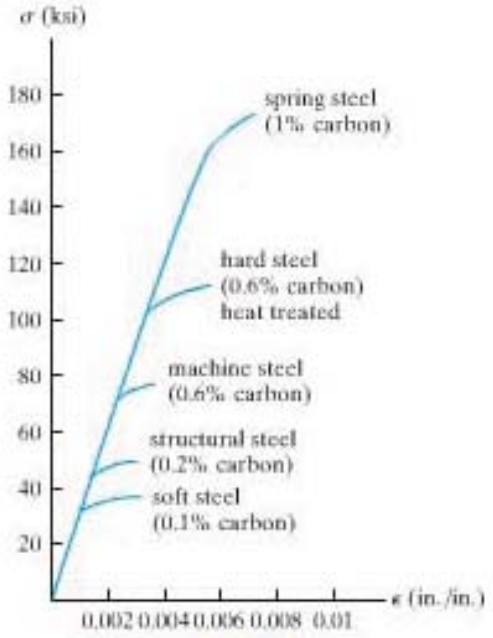


Fig. 3–13

same modulus of elasticity, generally accepted to be $E_{st} = 29(10^3)$ ksi or 200 GPa. Values of E for other commonly used engineering materials are often tabulated in engineering codes and reference books. Representative values are also listed on the inside back cover of this book. It should be noted that the modulus of elasticity is a mechanical property that indicates the *stiffness* of a material. Materials that are very stiff, such as steel, have large values of E [$E_{st} = 29(10^3)$ ksi or 200 GPa], whereas spongy materials such as vulcanized rubber may have low values [$E_r = 0.10$ ksi or 0.70 MPa].

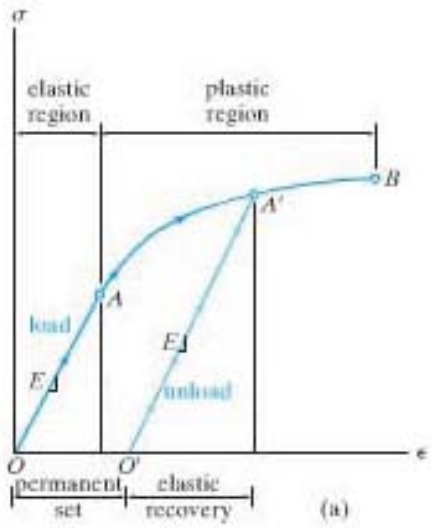
The modulus of elasticity is one of the most important mechanical properties used in the development of equations presented in this text. It must always be remembered, though, that E can be used only if a material has *linear elastic behavior*. Also, if the stress in the material is *greater* than the proportional limit, the stress–strain diagram ceases to be a straight line and so Eq. 3–5 is no longer valid.

Strain Hardening. If a specimen of ductile material, such as steel, is loaded into the *plastic region* and then unloaded, *elastic strain is recovered* as the material returns to its equilibrium state. The *plastic strain remains*, however, and as a result the material is subjected to a ***permanent set***. For example, a wire when bent (plastically) will spring back a little (elastically) when the load is removed; however, it will not fully return to its original position. This behavior can be illustrated on the stress–strain diagram shown in Fig. 3–14*a*. Here the specimen is first loaded beyond its yield point A to point A'. Since interatomic forces have to be overcome to elongate the specimen *elastically*, then these same forces pull the atoms back together when the load is removed, Fig. 3–14*a*. Consequently, the modulus of elasticity, E, is the same, and therefore the slope of line $O'A'$ is the same as line OA.

If the load is reapplied, the atoms in the material will again be displaced until yielding occurs at or near the stress A', and the stress–strain diagram continues along the same path as before, Fig. 3–14*b*. It should be noted, however, that this new stress–strain diagram, defined by $O'A'B$, now has a *higher* yield point (A'), a consequence of strain-hardening. In other words, the material now has a *greater elastic region*; however, it has *less ductility*, a smaller plastic region, than when it was in its original state.



This pin was made from a hardened steel alloy, that is, one having a high carbon content. It failed due to brittle fracture.



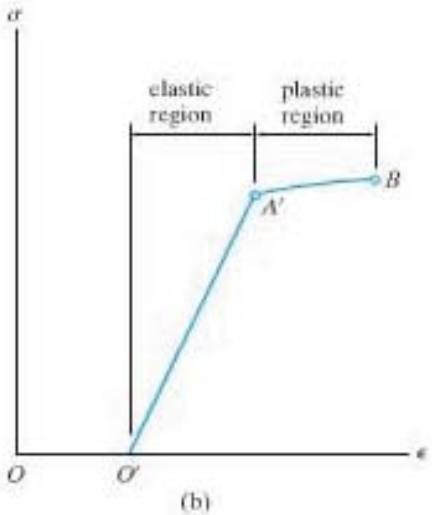


Fig. 3–14

3.5 Strain Energy

As a material is deformed by an external loading, it tends to store energy *internally* throughout its volume. Since this energy is related to the strains in the material, it is referred to as ***strain energy***. To obtain this strain energy consider a volume element of material from a tension test specimen. It is subjected to uniaxial stress as shown in Fig. 3–15. This stress develops a force $\Delta F = \sigma\,\Delta A = \sigma(\Delta x\,\Delta y)$ on the top and bottom faces of the element *after* the element of length Δz undergoes a vertical displacement $\epsilon\,\Delta z$. By definition, *work* is determined by the product of the force and displacement in the direction of the force. Since the force is increased uniformly from zero to its final magnitude ΔF when the displacement $\epsilon\,\Delta z$ is attained, the work done on the element by the force is equal to the *average* force magnitude $(\Delta F/2)$ times the displacement $\epsilon\,\Delta z$. This "external work" on the element is equivalent to the "internal work" or strain energy stored in the element—assuming that no energy is lost in the form of heat. Consequently, the strain energy ΔU is $\Delta U = (\frac{1}{2}\Delta F)\,\epsilon\,\Delta z = (\frac{1}{2}\sigma\,\Delta x\,\Delta y)\,\epsilon\,\Delta z$. Since the volume of the element is $\Delta V = \Delta x\,\Delta y\,\Delta z$, then $\Delta U = \frac{1}{2}\sigma\epsilon\,\Delta V$.

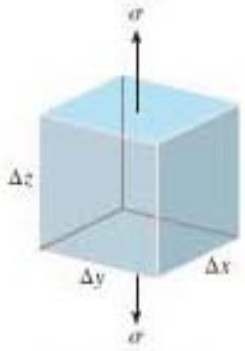


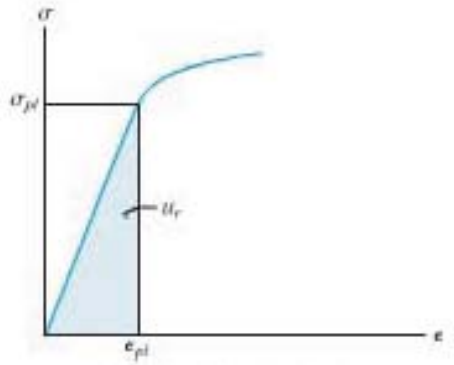
Fig. 3–15

For applications, it is sometimes convenient to specify the strain energy per unit volume of material. This is called the ***strain-energy density***, and it can be expressed as

$$u = \frac{\Delta U}{\Delta V} = \frac{1}{2}\sigma\epsilon \tag{3–6}$$

If the material behavior is *linear elastic*, then Hooke's law applies, $\sigma = E\epsilon$, and therefore we can express the elastic strain-energy density in terms of the uniaxial stress as

$$u = \frac{1}{2}\frac{\sigma^2}{E} \tag{3–7}$$



Modulus of resilience u_r

(a)

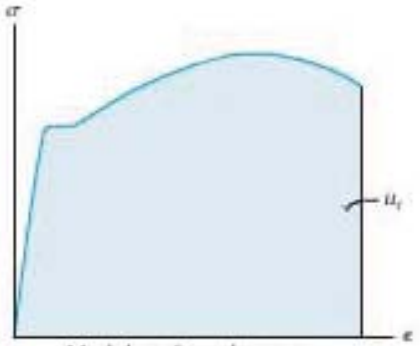
Fig. 3–16

Modulus of Resilience. In particular, when the stress σ reaches the proportional limit, the strain-energy density, as calculated by Eq. 3–6 or 3–7, is referred to as the ***modulus of resilience***, i.e.,

$$u_r = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}\frac{\sigma_{pl}^2}{E} \tag{3–8}$$

From the elastic region of the stress–strain diagram, Fig. 3–16*a*, notice that u_r is equivalent to the shaded *triangular area* under the diagram. Physically a material's resilience represents the ability of the material to absorb energy without any permanent damage to the material.

Modulus of Toughness. Another important property of a material is the ***modulus of toughness***, u_t. This quantity represents the *entire area* under the stress–strain diagram, Fig. 3–16*b*, and therefore it indicates the strain-energy density of the material just before it fractures. This property becomes important when designing members that may be accidentally overloaded. Alloying metals can also change their resilience and toughness. For example, by changing the percentage of carbon in steel, the resulting stress–strain diagrams in Fig. 3–17 show how the degrees of resilience and toughness can be changed.



Modulus of toughness u_t

(b)

Fig. 3–16 (cont.)

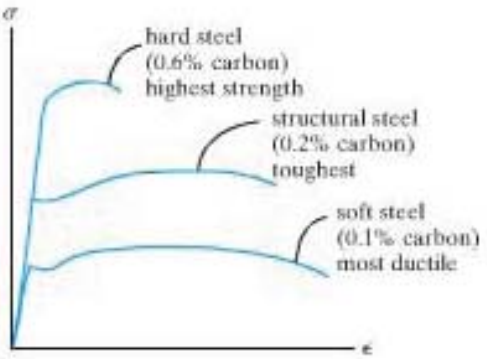


Fig. 3–17

Important Points

- A *conventional stress–strain diagram* is important in engineering since it provides a means for obtaining data about a material's tensile or compressive strength without regard for the material's physical size or shape.
- *Engineering stress and strain* are calculated using the *original* cross-sectional area and gauge length of the specimen.
- A *ductile material*, such as mild steel, has four distinct behaviors as it is loaded. They are *elastic behavior, yielding, strain hardening*, and *necking*.
- A material is *linear elastic* if the stress is proportional to the strain within the elastic region. This behavior is described by *Hooke's law*, $\sigma = E\epsilon$, where the *modulus of elasticity* E is the slope of the line.
- Important points on the stress–strain diagram are the *proportional limit, elastic limit, yield stress, ultimate stress*, and *fracture stress*.
- The *ductility* of a material can be specified by the specimen's *percent elongation* or the *percent reduction in area*.
- If a material does not have a distinct yield point, a *yield strength* can be specified using a graphical procedure such as the *offset method*.
- *Brittle materials*, such as gray cast iron, have very little or no yielding and so they can fracture suddenly.
- *Strain hardening* is used to establish a higher yield point for a material. This is done by straining the material beyond the elastic limit, then releasing the load. The modulus of elasticity remains the same; however, the material's ductility *decreases*.
- *Strain energy* is energy stored in a material due to its deformation. This energy per unit volume is called *strain-energy density*. If it is measured up to the proportional limit, it is referred to as the *modulus of resilience*, and if it is measured up to the point of fracture, it is called the *modulus of toughness*. It can be determined from the area under the $\sigma-\epsilon$ diagram.



This nylon specimen exhibits a high degree of toughness as noted by the large amount of necking that has occurred just before fracture.

EXAMPLE 3.1



A tension test for a steel alloy results in the stress–strain diagram shown in Fig. 3–18. Calculate the modulus of elasticity and the yield strength based on a 0.2% offset. Identify on the graph the ultimate stress and the fracture stress.

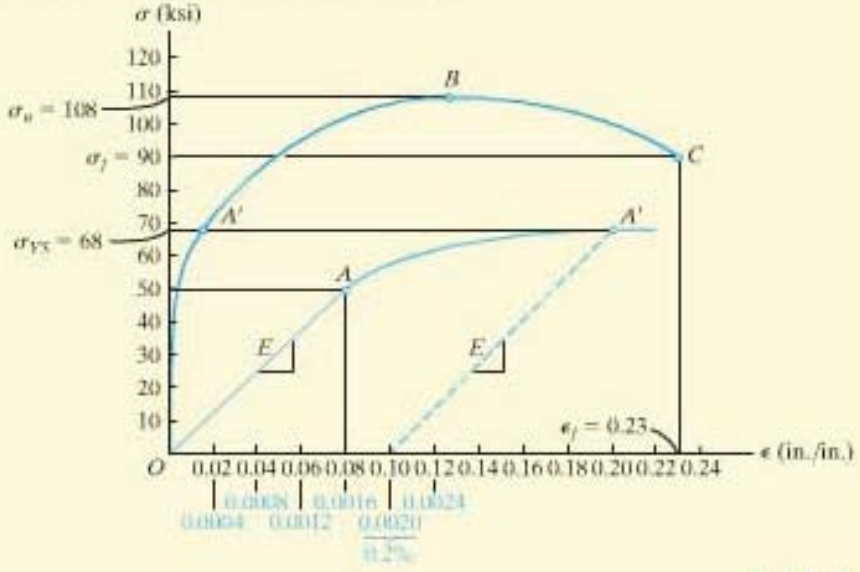


Fig. 3–18

SOLUTION

Modulus of Elasticity. We must calculate the *slope* of the initial straight-line portion of the graph. Using the magnified curve and scale shown in blue, this line extends from point O to an estimated point A, which has coordinates of approximately (0.0016 in./in., 50 ksi). Therefore,

$$E = \frac{50 \text{ ksi}}{0.0016 \text{ in./in.}} = 31.2(10^3) \text{ ksi} \qquad \textit{Ans.}$$

Note that the equation of line OA is thus $\sigma = 31.2(10^3)\epsilon$.

Yield Strength. For a 0.2% offset, we begin at a strain of 0.2% or 0.0020 in./in. and graphically extend a (dashed) line parallel to OA until it intersects the σ–ϵ curve at A'. The yield strength is approximately

$$\sigma_{YS} = 68 \text{ ksi} \qquad \textit{Ans.}$$

Ultimate Stress. This is defined by the peak of the σ–ϵ graph, point B in Fig. 3–18.

$$\sigma_u = 108 \text{ ksi} \qquad \textit{Ans.}$$

Fracture Stress. When the specimen is strained to its maximum of $\epsilon_f = 0.23$ in./in., it fractures at point C. Thus,

$$\sigma_f = 90 \text{ ksi} \qquad \textit{Ans.}$$

EXAMPLE 3.2

The stress–strain diagram for an aluminum alloy that is used for making aircraft parts is shown in Fig. 3–19. If a specimen of this material is stressed to 600 MPa, determine the permanent strain that remains in the specimen when the load is released. Also, find the modulus of resilience both before and after the load application.

SOLUTION

Permanent Strain. When the specimen is subjected to the load, it strain-hardens until point B is reached on the σ–ϵ diagram. The strain at this point is approximately 0.023 mm/mm. When the load is released, the material behaves by following the straight line BC, which is parallel to line OA. Since both lines have the same slope, the strain at point C can be determined analytically. The slope of line OA is the modulus of elasticity, i.e.,

$$E = \frac{450\text{ MPa}}{0.006\text{ mm/mm}} = 75.0\text{ GPa}$$

From triangle CBD, we require

$$E = \frac{BD}{CD}; \qquad 75.0(10^9)\text{ Pa} = \frac{600(10^6)\text{ Pa}}{CD}$$

$$CD = 0.008\text{ mm/mm}$$

This strain represents the amount of *recovered elastic strain*. The permanent strain, ϵ_{OC}, is thus

$$\epsilon_{OC} = 0.023\text{ mm/mm} - 0.008\text{ mm/mm}$$

$$= 0.0150\text{ mm/mm} \qquad \textit{Ans.}$$

Note: If gauge marks on the specimen were originally 50 mm apart, then after the load is *released* these marks will be 50 mm + (0.0150)(50 mm) = 50.75 mm apart.

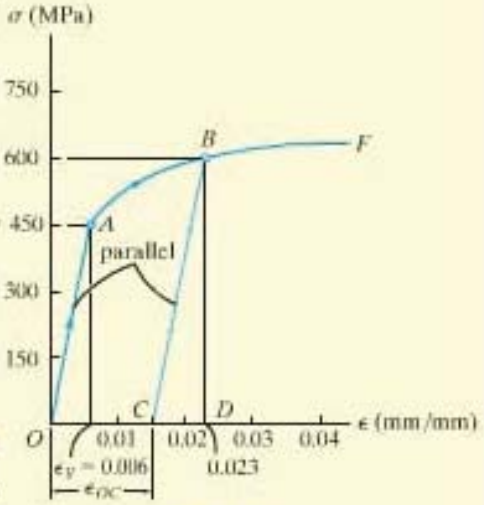


Fig. 3–19

Modulus of Resilience. Applying Eq. 3–8, we have*

$$(u_r)_{\text{initial}} = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}(450\text{ MPa})(0.006\text{ mm/mm})$$

$$= 1.35\text{ MJ/m}^3 \qquad \textit{Ans.}$$

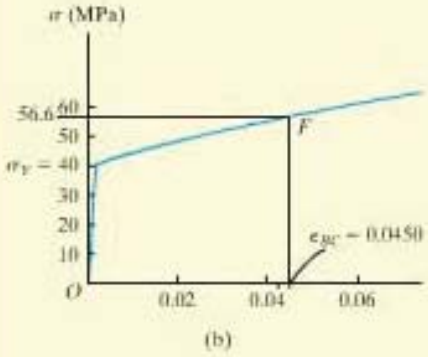
$$(u_r)_{\text{final}} = \frac{1}{2}\sigma_{pl}\epsilon_{pl} = \frac{1}{2}(600\text{ MPa})(0.008\text{ mm/mm})$$

$$= 2.40\text{ MJ/m}^3 \qquad \textit{Ans.}$$

NOTE: By comparison, the effect of strain-hardening the material has caused an increase in the modulus of resilience; however, note that the modulus of toughness for the material has decreased since the area under the original curve, $OABF$, is larger than the area under curve CBF.

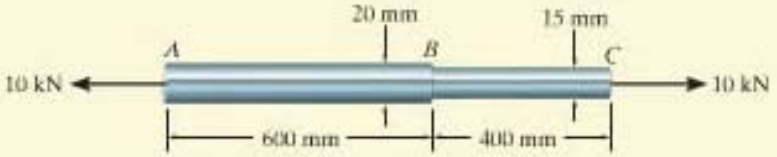
* Work in the SI system of units is measured in joules, where 1 J = 1 N · m.

EXAMPLE 3.3



(b)

An aluminum rod shown in Fig. 3–20*a* has a circular cross section and is subjected to an axial load of 10 kN. If a portion of the stress–strain diagram is shown in Fig. 3–20*b*, determine the approximate elongation of the rod when the load is applied. Take $E_{al} = 70$ GPa.



(a)

Fig. 3–20

SOLUTION

For the analysis we will neglect the *localized deformations* at the point of load application and where the rod's cross-sectional area suddenly changes. (These effects will be discussed in Sections 4.1 and 4.7.) Throughout the midsection of each segment the normal stress and deformation are uniform.

In order to find the elongation of the rod, we must first obtain the strain. This is done by calculating the stress, then using the stress–strain diagram. The normal stress within each segment is

$$\sigma_{AB} = \frac{P}{A} = \frac{10(10^3)\text{ N}}{\pi(0.01\text{ m})^2} = 31.83\text{ MPa}$$

$$\sigma_{BC} = \frac{P}{A} = \frac{10(10^3)\text{ N}}{\pi(0.0075\text{ m})^2} = 56.59\text{ MPa}$$

From the stress–strain diagram, the material in segment AB is strained *elastically* since $\sigma_{AB} < \sigma_Y = 40$ MPa. Using Hooke's law,

$$\epsilon_{AB} = \frac{\sigma_{AB}}{E_{al}} = \frac{31.83(10^6)\text{ Pa}}{70(10^9)\text{ Pa}} = 0.0004547\text{ mm/mm}$$

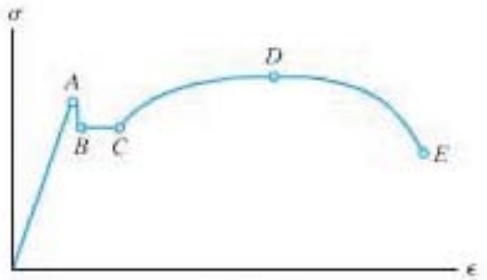
The material within segment BC is strained plastically, since $\sigma_{BC} > \sigma_Y = 40$ MPa. From the graph, for $\sigma_{BC} = 56.59$ MPa, $\epsilon_{BC} \approx 0.045$ mm/mm. The approximate elongation of the rod is therefore

$$\delta = \Sigma\epsilon L = 0.0004547(600\text{ mm}) + 0.0450(400\text{ mm})$$
$$= 18.3\text{ mm} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

F3–1. Define homogeneous material.

F3–2. Indicate the points on the stress-strain diagram which represent the proportional limit and the ultimate stress.



F3–2

F3–3. Define the modulus of elasticity E.

F3–4. At room temperature, mild steel is a ductile material. True or false?

F3–5. Engineering stress and strain are calculated using the *actual* cross-sectional area and length of the specimen. True or false?

F3–6. As the temperature increases the modulus of elasticity will increase. True or false?

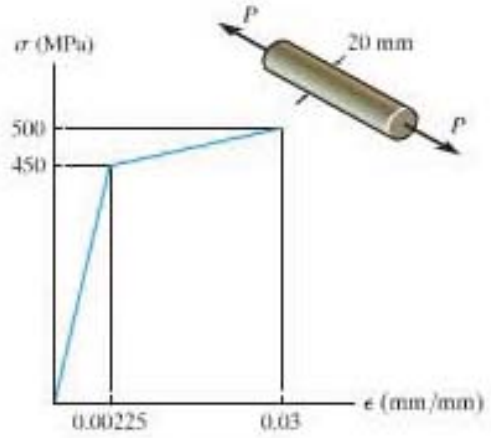
F3–7. A 100-mm long rod has a diameter of 15 mm. If an axial tensile load of 100 kN is applied, determine its change is length. $E = 200$ GPa.

F3–8. A bar has a length of 8 in. and cross-sectional area of 12 in^2. Determine the modulus of elasticity of the material if it is subjected to an axial tensile load of 10 kip and stretches 0.003 in. The material has linear-elastic behavior.

F3–9. A 10-mm-diameter brass rod has a modulus of elasticity of $E = 100$ GPa. If it is 4 m long and subjected to an axial tensile load of 6 kN, determine its elongation.

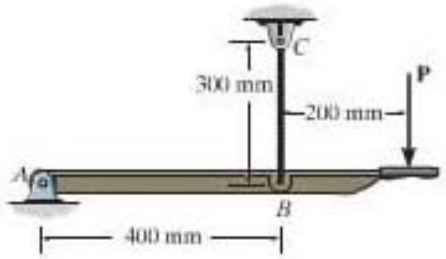
F3–10. The material for the 50-mm-long specimen has the stress–strain diagram shown. If $P = 100$ kN, determine the elongation of the specimen.

F3–11. The material for the 50-mm-long specimen has the stress–strain diagram shown. If $P = 150$ kN is applied and then released, determine the permanent elongation of the specimen.



F3–10/11

F3–12. If the elongation of wire BC is 0.2 mm after the force **P** is applied, determine the magnitude of **P**. The wire is A-36 steel and has a diameter of 3 mm.



F3–12

PROBLEMS

•**3–1.** A concrete cylinder having a diameter of 6.00 in. and gauge length of 12 in. is tested in compression. The results of the test are reported in the table as load versus contraction. Draw the stress–strain diagram using scales of 1 in. = 0.5 ksi and 1 in. = $0.2(10^{-3})$ in./in. From the diagram, determine approximately the modulus of elasticity.

Load (kip)	Contraction (in.)
0	0
5.0	0.0006
9.5	0.0012
16.5	0.0020
20.5	0.0026
25.5	0.0034
30.0	0.0040
34.5	0.0045
38.5	0.0050
46.5	0.0062
50.0	0.0070
53.0	0.0075

Prob. 3–1

3–2. Data taken from a stress–strain test for a ceramic are given in the table. The curve is linear between the origin and the first point. Plot the diagram, and determine the modulus of elasticity and the modulus of resilience.

3–3. Data taken from a stress–strain test for a ceramic are given in the table. The curve is linear between the origin and the first point. Plot the diagram, and determine approximately the modulus of toughness. The rupture stress is $\sigma_r = 53.4$ ksi.

σ (ksi)	ϵ (in./in.)
0	0
33.2	0.0006
45.5	0.0010
49.4	0.0014
51.5	0.0018
53.4	0.0022

Probs. 3–2/3

*3–4. A tension test was performed on a specimen having an original diameter of 12.5 mm and a gauge length of 50 mm. The data are listed in the table. Plot the stress–strain diagram, and determine approximately the modulus of elasticity, the ultimate stress, and the fracture stress. Use a scale of 20 mm = 50 MPa and 20 mm = 0.05 mm/mm. Redraw the linear-elastic region, using the same stress scale but a strain scale of 20 mm = 0.001 mm/mm.

3–5. A tension test was performed on a steel specimen having an original diameter of 12.5 mm and gauge length of 50 mm. Using the data listed in the table, plot the stress–strain diagram, and determine approximately the modulus of toughness. Use a scale of 20 mm = 50 MPa and 20 mm = 0.05 mm/mm.

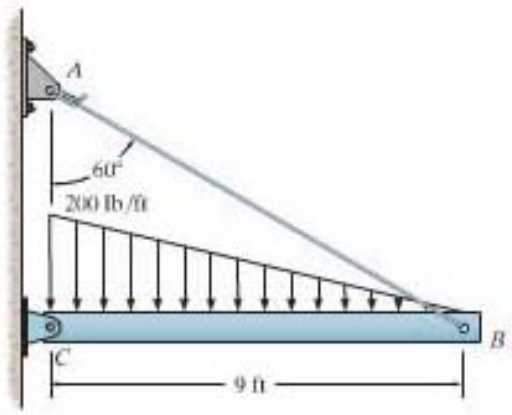
Load (kN)	Elongation (mm)
0	0
11.1	0.0175
31.9	0.0600
37.8	0.1020
40.9	0.1650
43.6	0.2490
53.4	1.0160
62.3	3.0480
64.5	6.3500
62.3	8.8900
58.8	11.9380

Probs. 3–4/5

3–6. A specimen is originally 1 ft long, has a diameter of 0.5 in., and is subjected to a force of 500 lb. When the force is increased from 500 lb to 1800 lb, the specimen elongates 0.009 in. Determine the modulus of elasticity for the material if it remains linear elastic.

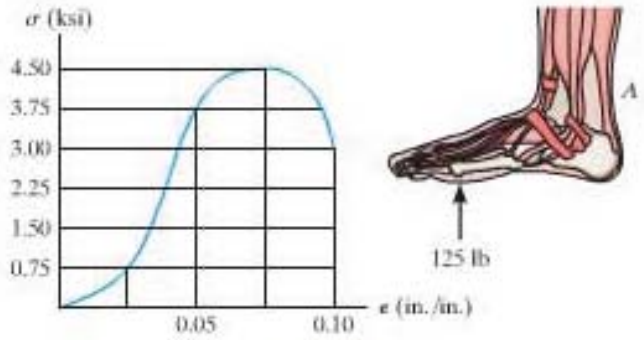
3–7. A structural member in a nuclear reactor is made of a zirconium alloy. If an axial load of 4 kip is to be supported by the member, determine its required cross-sectional area. Use a factor of safety of 3 relative to yielding. What is the load on the member if it is 3 ft long and its elongation is 0.02 in.? $E_{zr} = 14(10^3)$ ksi, $\sigma_Y = 57.5$ ksi. The material has elastic behavior.

*3–8. The strut is supported by a pin at C and an A-36 steel guy wire AB. If the wire has a diameter of 0.2 in., determine how much it stretches when the distributed load acts on the strut.



Prob. 3–8

•3–9. The $\sigma-\epsilon$ diagram for a collagen fiber bundle from which a human tendon is composed is shown. If a segment of the Achilles tendon at A has a length of 6.5 in. and an approximate cross-sectional area of 0.229 in^2, determine its elongation if the foot supports a load of 125 lb, which causes a tension in the tendon of 343.75 lb.

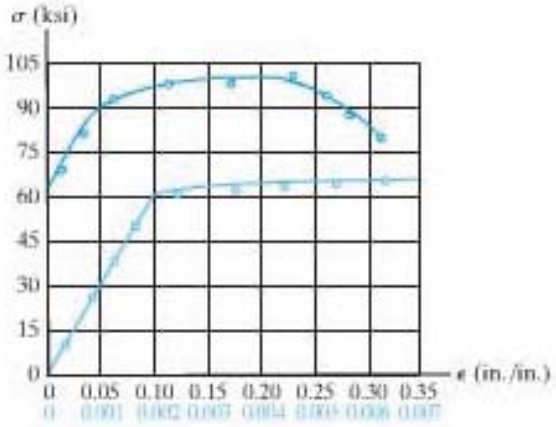


Prob. 3–9

3–10. The stress–strain diagram for a metal alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. Determine approximately the modulus of elasticity for the material, the load on the specimen that causes yielding, and the ultimate load the specimen will support.

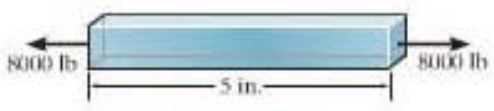
3–11. The stress–strain diagram for a steel alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. If the specimen is loaded until it is stressed to 90 ksi, determine the approximate amount of elastic recovery and the increase in the gauge length after it is unloaded.

*3–12. The stress–strain diagram for a steel alloy having an original diameter of 0.5 in. and a gauge length of 2 in. is given in the figure. Determine approximately the modulus of resilience and the modulus of toughness for the material.



Probs. 3–10/11/12

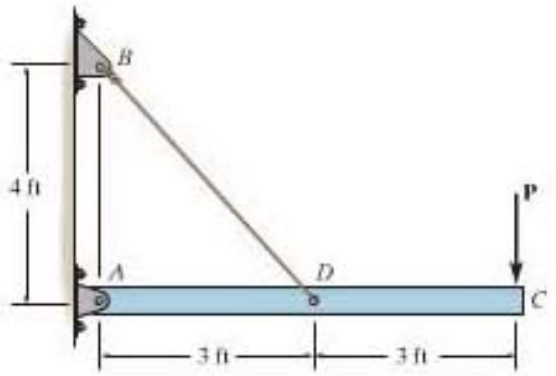
•3–13. A bar having a length of 5 in. and cross-sectional area of 0.7 in^2 is subjected to an axial force of 8000 lb. If the bar stretches 0.002 in., determine the modulus of elasticity of the material. The material has linear-elastic behavior.



Prob. 3–13

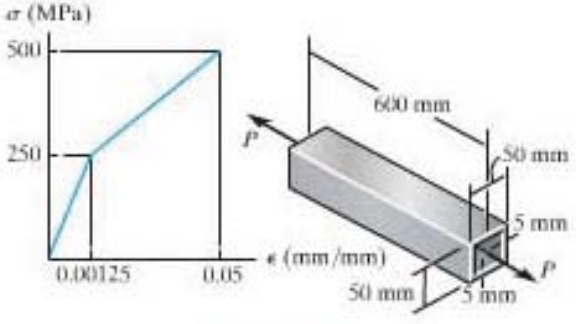
3–14. The rigid pipe is supported by a pin at A and an A-36 steel guy wire BD. If the wire has a diameter of 0.25 in., determine how much it stretches when a load of $P = 600$ lb acts on the pipe.

3–15. The rigid pipe is supported by a pin at A and an A-36 guy wire BD. If the wire has a diameter of 0.25 in., determine the load P if the end C is displaced 0.075 in. downward.



Probs. 3–14/15

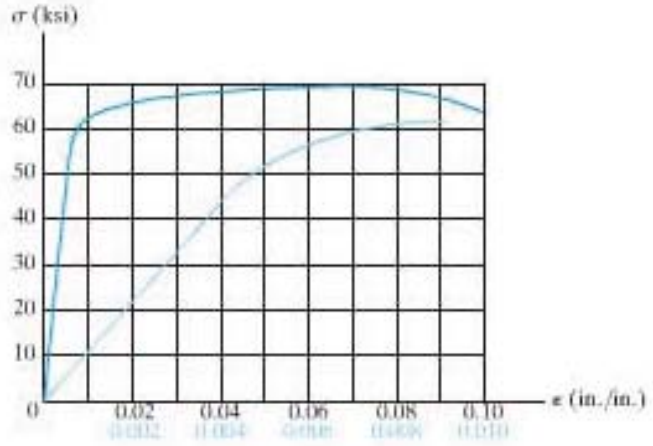
***3–16.** Determine the elongation of the square hollow bar when it is subjected to the axial force $P = 100$ kN. If this axial force is increased to $P = 360$ kN and released, find the permanent elongation of the bar. The bar is made of a metal alloy having a stress–strain diagram which can be approximated as shown.



Prob. 3–16

3–17. A tension test was performed on an aluminum 2014-T6 alloy specimen. The resulting stress–strain diagram is shown in the figure. Estimate (a) the proportional limit, (b) the modulus of elasticity, and (c) the yield strength based on a 0.2% strain offset method.

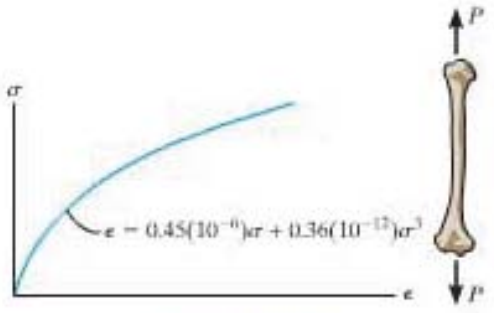
3–18. A tension test was performed on an aluminum 2014-T6 alloy specimen. The resulting stress–strain diagram is shown in the figure. Estimate (a) the modulus of resilience; and (b) modulus of toughness.



Probs. 3–17/18

3–19. The stress–strain diagram for a bone is shown, and can be described by the equation $\epsilon = 0.45(10^{-6})\sigma + 0.36(10^{-12})\sigma^3$, where σ is in kPa. Determine the yield strength assuming a 0.3% offset.

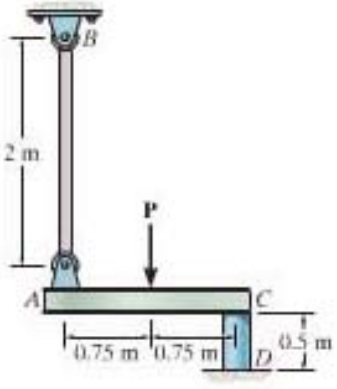
***3–20.** The stress–strain diagram for a bone is shown and can be described by the equation $\epsilon = 0.45(10^{-6})\sigma + 0.36(10^{-12})\sigma^3$, where σ is in kPa. Determine the modulus of toughness and the amount of elongation of a 200-mm-long region just before it fractures if failure occurs at $\epsilon = 0.12$ mm/mm.

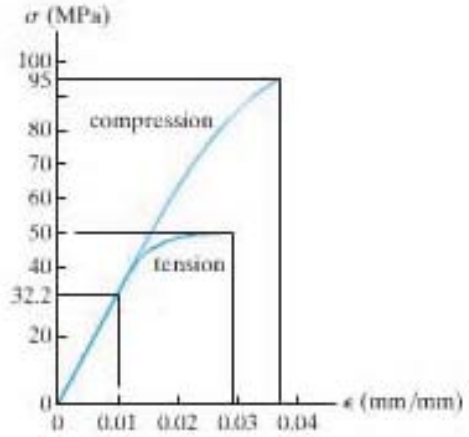


Probs. 3–19/20

•3–21. The stress–strain diagram for a polyester resin is given in the figure. If the rigid beam is supported by a strut AB and post CD, both made from this material, and subjected to a load of $P = 80$ kN, determine the angle of tilt of the beam when the load is applied. The diameter of the strut is 40 mm and the diameter of the post is 80 mm.

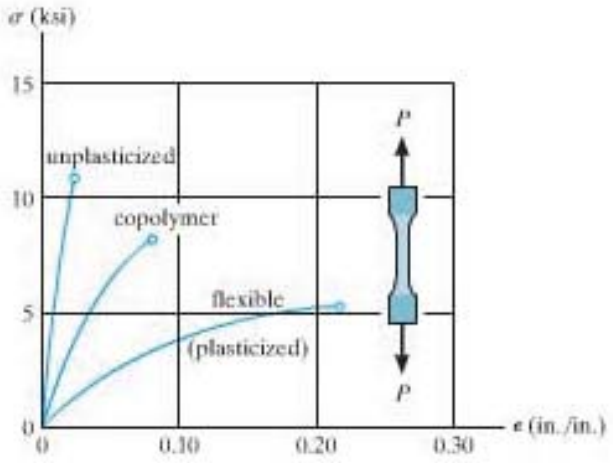
3–22. The stress–strain diagram for a polyester resin is given in the figure. If the rigid beam is supported by a strut AB and post CD made from this material, determine the largest load P that can be applied to the beam before it ruptures. The diameter of the strut is 12 mm and the diameter of the post is 40 mm.





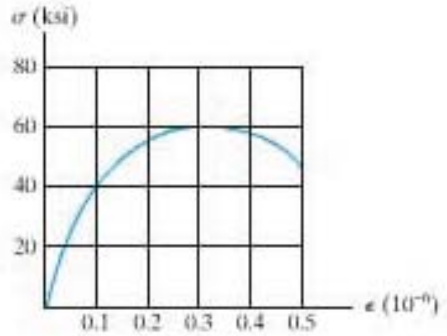
Probs. 3–21/22

3–23. By adding plasticizers to polyvinyl chloride, it is possible to reduce its stiffness. The stress–strain diagrams for three types of this material showing this effect are given below. Specify the type that should be used in the manufacture of a rod having a length of 5 in. and a diameter of 2 in., that is required to support at least an axial load of 20 kip and also be able to stretch at most $\frac{1}{4}$ in.

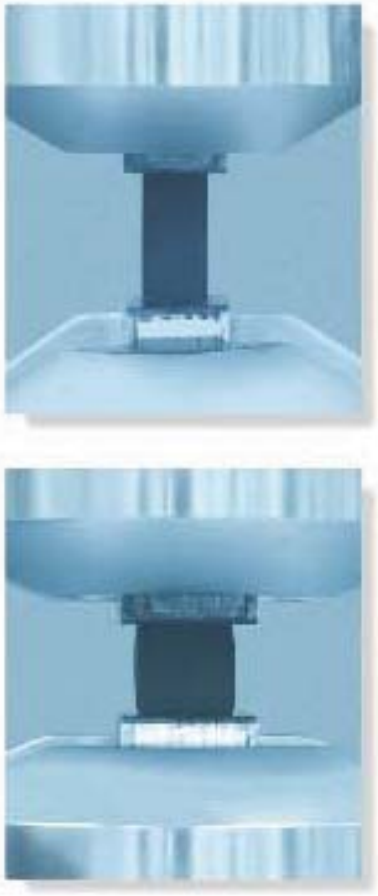


Prob. 3–23

***3–24.** The stress–strain diagram for many metal alloys can be described analytically using the Ramberg-Osgood three parameter equation $\epsilon = \sigma/E + k\sigma^n$, where E, k, and n are determined from measurements taken from the diagram. Using the stress–strain diagram shown in the figure, take $E = 30(10^3)$ ksi and determine the other two parameters k and n and thereby obtain an analytical expression for the curve.



Prob. 3–24



When the rubber block is compressed (negative strain) its sides will expand (positive strain). The ratio of these strains remains constant.

3.6 Poisson's Ratio

When a deformable body is subjected to an axial tensile force, not only does it elongate but it also contracts laterally. For example, if a rubber band is stretched, it can be noted that both the thickness and width of the band are decreased. Likewise, a compressive force acting on a body causes it to contract in the direction of the force and yet its sides expand laterally.

Consider a bar having an original radius r and length L and subjected to the tensile force P in Fig. 3–21. This force elongates the bar by an amount δ, and its radius contracts by an amount δ'. Strains in the longitudinal or axial direction and in the lateral or radial direction are, respectively,

$$\epsilon_{\text{long}} = \frac{\delta}{L} \quad \text{and} \quad \epsilon_{\text{lat}} = \frac{\delta'}{r}$$

In the early 1800s, the French scientist S. D. Poisson realized that within the *elastic range* the *ratio* of these strains is a *constant*, since the deformations δ and δ' are proportional. This constant is referred to as ***Poisson's ratio***, ν (nu), and it has a numerical value that is unique for a particular material that is both *homogeneous and isotropic*. Stated mathematically it is

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}} \tag{3–9}$$

The negative sign is included here since *longitudinal elongation* (positive strain) causes *lateral contraction* (negative strain), and vice versa. Notice that these strains are caused only by the axial or longitudinal force P; i.e., no force or stress acts in a lateral direction in order to strain the material in this direction.

Poisson's ratio is a *dimensionless* quantity, and for most nonporous solids it has a value that is generally between $\frac{1}{4}$ and $\frac{1}{3}$. Typical values of ν for common engineering materials are listed on the inside back cover. For an "ideal material" having no lateral deformation when it is stretched or compressed Poisson's ratio will be 0. Furthermore, it will be shown in Sec. 10.6 that the *maximum* possible value for Poisson's ratio is 0.5. Therefore $0 \leq \nu \leq 0.5$.

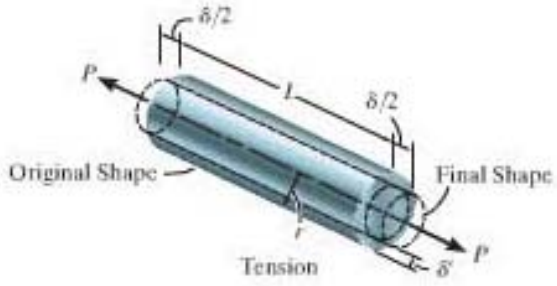


Fig. 3–21

EXAMPLE 3.4

A bar made of A-36 steel has the dimensions shown in Fig. 3–22. If an axial force of $P = 80$ kN is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.

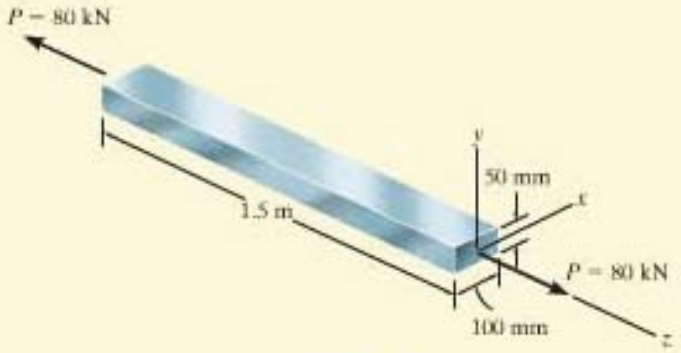


Fig. 3–22

SOLUTION

The normal stress in the bar is

$$\sigma_z = \frac{P}{A} = \frac{80(10^3)\text{ N}}{(0.1\text{ m})(0.05\text{ m})} = 16.0(10^6)\text{ Pa}$$

From the table on the inside back cover for A-36 steel $E_{st} = 200$ GPa, and so the strain in the z direction is

$$\epsilon_z = \frac{\sigma_z}{E_{st}} = \frac{16.0(10^6)\text{ Pa}}{200(10^9)\text{ Pa}} = 80(10^{-6})\text{ mm/mm}$$

The axial elongation of the bar is therefore

$$\delta_z = \epsilon_z L_z = [80(10^{-6})](1.5\text{ m}) = 120\,\mu\text{m} \qquad \textit{Ans.}$$

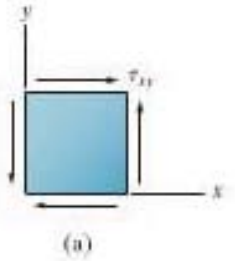
Using Eq. 3–9, where $\nu_{st} = 0.32$ as found from the inside back cover, the lateral contraction strains in *both* the x and y directions are

$$\epsilon_x = \epsilon_y = -\nu_{st}\epsilon_z = -0.32[80(10^{-6})] = -25.6\,\mu\text{m/m}$$

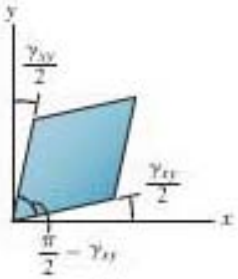
Thus the changes in the dimensions of the cross section are

$$\delta_x = \epsilon_x L_x = -[25.6(10^{-6})](0.1\text{ m}) = -2.56\,\mu\text{m} \qquad \textit{Ans.}$$

$$\delta_y = \epsilon_y L_y = -[25.6(10^{-6})](0.05\text{ m}) = -1.28\,\mu\text{m} \qquad \textit{Ans.}$$



(a)



(b)

Fig. 3–23

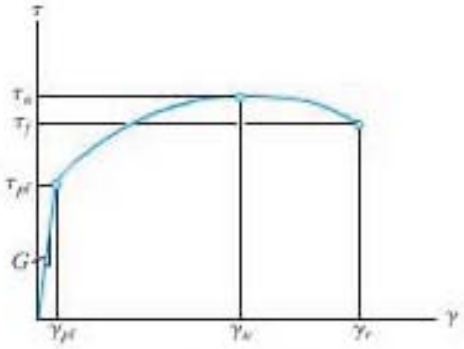


Fig. 3–24

3.7 The Shear Stress–Strain Diagram

In Sec. 1.5 it was shown that when a small element of material is subjected to *pure shear*, equilibrium requires that equal shear stresses must be developed on four faces of the element. These stresses τ_{xy} must be directed toward or away from diagonally opposite corners of the element, as shown in Fig. 3–23*a*. Furthermore, if the material is *homogeneous* and *isotropic*, then this shear stress will distort the element uniformly, Fig. 3–23*b*. As mentioned in Sec. 2.2, the shear strain γ_{xy} measures the angular distortion of the element relative to the sides originally along the x and y axes.

The behavior of a material subjected to pure shear can be studied in a laboratory using specimens in the shape of thin tubes and subjecting them to a torsional loading. If measurements are made of the applied torque and the resulting angle of twist, then by the methods to be explained in Chapter 5, the data can be used to determine the shear stress and shear strain, and a shear stress–strain diagram plotted. An example of such a diagram for a ductile material is shown in Fig. 3–24. Like the tension test, this material when subjected to shear will exhibit linear-elastic behavior and it will have a defined *proportional limit* τ_{pl}. Also, strain hardening will occur until an *ultimate shear stress* τ_u is reached. And finally, the material will begin to lose its shear strength until it reaches a point where it fractures, τ_f.

For most engineering materials, like the one just described, the elastic behavior is *linear*, and so Hooke's law for shear can be written as

$$\tau = G\gamma \tag{3–10}$$

Here G is called the ***shear modulus of elasticity*** or the ***modulus of rigidity***. Its value represents the slope of the line on the $\tau-\gamma$ diagram, that is, $G = \tau_{pl}/\gamma_{pl}$. Typical values for common engineering materials are listed on the inside back cover. Notice that the units of measurement for G will be the *same* as those for τ (Pa or psi), since γ is measured in radians, a dimensionless quantity.

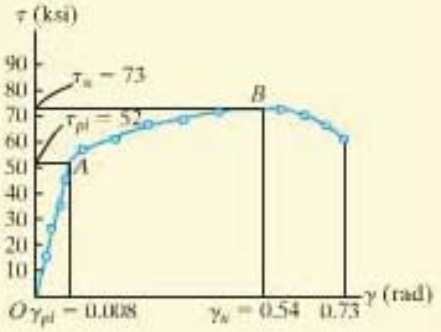
It will be shown in Sec. 10.6 that the three material constants, E, ν, and G are actually *related* by the equation

$$G = \frac{E}{2(1+\nu)} \tag{3–11}$$

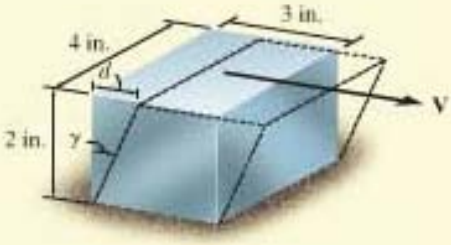
Provided E and G are known, the value of ν can then be determined from this equation rather than through experimental measurement. For example, in the case of A-36 steel, $E_{st} = 29(10^3)$ ksi and $G_{st} = 11.0(10^3)$ ksi, so that, from Eq. 3–11, $\nu_{st} = 0.32$.

EXAMPLE 3.5

A specimen of titanium alloy is tested in torsion and the shear stress–strain diagram is shown in Fig. 3–25*a*. Determine the shear modulus G, the proportional limit, and the ultimate shear stress. Also, determine the maximum distance d that the top of a block of this material, shown in Fig. 3–25*b*, could be displaced horizontally if the material behaves elastically when acted upon by a shear force **V**. What is the magnitude of **V** necessary to cause this displacement?



(a)



(b)

Fig. 3–25

SOLUTION

Shear Modulus. This value represents the slope of the straight-line portion OA of the $\tau-\gamma$ diagram. The coordinates of point A are (0.008 rad, 52 ksi). Thus,

$$G = \frac{52 \text{ ksi}}{0.008 \text{ rad}} = 6500 \text{ ksi} \qquad \textit{Ans.}$$

The equation of line OA is therefore $\tau = G\gamma = 6500\gamma$, which is Hooke's law for shear.

Proportional Limit. By inspection, the graph ceases to be linear at point A. Thus,

$$\tau_{pl} = 52 \text{ ksi} \qquad \textit{Ans.}$$

Ultimate Stress. This value represents the maximum shear stress, point B. From the graph,

$$\tau_u = 73 \text{ ksi} \qquad \textit{Ans.}$$

Maximum Elastic Displacement and Shear Force. Since the maximum elastic shear strain is 0.008 rad, a very small angle, the top of the block in Fig. 3–25*b* will be displaced horizontally:

$$\tan(0.008 \text{ rad}) \approx 0.008 \text{ rad} = \frac{d}{2 \text{ in.}}$$

$$d = 0.016 \text{ in.} \qquad \textit{Ans.}$$

The corresponding *average* shear stress in the block is $\tau_{pl} = 52$ ksi. Thus, the shear force V needed to cause the displacement is

$$\tau_{avg} = \frac{V}{A}; \qquad 52 \text{ ksi} = \frac{V}{(3 \text{ in.})(4 \text{ in.})}$$

$$V = 624 \text{ kip} \qquad \textit{Ans.}$$

EXAMPLE 3.6

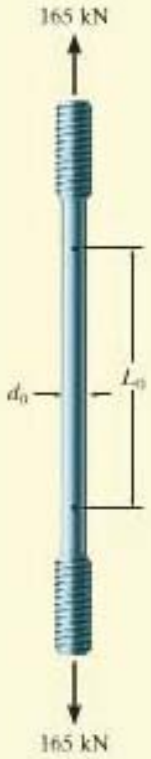


Fig. 3–26

An aluminum specimen shown in Fig. 3–26 has a diameter of $d_0 = 25$ mm and a gauge length of $L_0 = 250$ mm. If a force of 165 kN elongates the gauge length 1.20 mm, determine the modulus of elasticity. Also, determine by how much the force causes the diameter of the specimen to contract. Take $G_{al} = 26$ GPa and $\sigma_Y = 440$ MPa.

SOLUTION

Modulus of Elasticity. The average normal stress in the specimen is

$$\sigma = \frac{P}{A} = \frac{165(10^3)\text{ N}}{(\pi/4)(0.025\text{ m})^2} = 336.1\text{ MPa}$$

and the average normal strain is

$$\epsilon = \frac{\delta}{L} = \frac{1.20\text{ mm}}{250\text{ mm}} = 0.00480\text{ mm/mm}$$

Since $\sigma < \sigma_Y = 440$ MPa, the material behaves elastically. The modulus of elasticity is therefore

$$E_{al} = \frac{\sigma}{\epsilon} = \frac{336.1(10^6)\text{ Pa}}{0.00480} = 70.0\text{ GPa} \qquad \textit{Ans.}$$

Contraction of Diameter. First we will determine Poisson's ratio for the material using Eq. 3–11.

$$G = \frac{E}{2(1+\nu)}$$

$$26\text{ GPa} = \frac{70.0\text{ GPa}}{2(1+\nu)}$$

$$\nu = 0.347$$

Since $\epsilon_{long} = 0.00480$ mm/mm, then by Eq. 3–9,

$$\nu = -\frac{\epsilon_{lat}}{\epsilon_{long}}$$

$$0.347 = -\frac{\epsilon_{lat}}{0.00480\text{ mm/mm}}$$

$$\epsilon_{lat} = -0.00166\text{ mm/mm}$$

The contraction of the diameter is therefore

$$\delta' = (0.00166)(25\text{ mm})$$
$$= 0.0416\text{ mm} \qquad \textit{Ans.}$$

*3.8 Failure of Materials Due to Creep and Fatigue

The mechanical properties of a material have up to this point been discussed only for a static or slowly applied load at constant temperature. In some cases, however, a member may have to be used in an environment for which loadings must be sustained over long periods of time at elevated temperatures, or in other cases, the loading may be repeated or cycled. We will not consider these effects in this book, although we will briefly mention how one determines a material's strength for these conditions, since they are given special treatment in design.

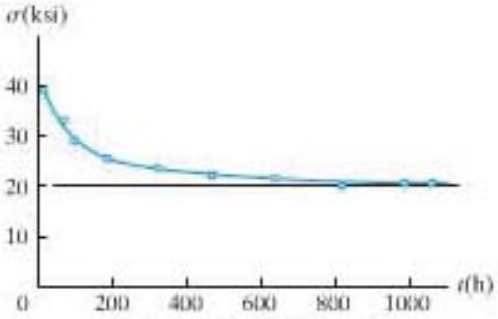
Creep. When a material has to support a load for a very long period of time, it may continue to deform until a sudden fracture occurs or its usefulness is impaired. This time-dependent permanent deformation is known as ***creep***. Normally creep is considered when metals and ceramics are used for structural members or mechanical parts that are subjected to high temperatures. For some materials, however, such as polymers and composite materials—including wood or concrete—temperature is *not* an important factor, and yet creep can occur strictly from long-term load application. As a typical example, consider the fact that a rubber band will not return to its original shape after being released from a stretched position in which it was held for a very long period of time. In the general sense, therefore, both *stress and/or temperature* play a significant role in the *rate* of creep.



The long-term application of the cable loading on this pole has caused the pole to deform due to creep.

For practical purposes, when creep becomes important, a member is usually designed to resist a specified creep strain for a given period of time. An important mechanical property that is used in this regard is called the *creep strength*. This value represents the highest stress the material can withstand during a specified time without exceeding an allowable creep strain. The creep strength will vary with temperature, and for design, a given temperature, duration of loading, and allowable creep strain must all be specified. For example, a creep strain of 0.1% per year has been suggested for steel in bolts and piping.

Several methods exist for determining an allowable creep strength for a particular material. One of the simplest involves testing several specimens simultaneously at a constant temperature, but with each subjected to a *different axial stress*. By measuring the length of time needed to produce either an allowable strain or the fracture strain for each specimen, a curve of stress versus time can be established. Normally these tests are run to a maximum of 1000 hours. An example of the results for stainless steel at a temperature of 1200°F and prescribed creep strain of 1% is shown in Fig. 3–27. As noted, this material has a yield strength of 40 ksi (276 MPa) at room temperature (0.2% offset) and the creep strength at 1000 h is found to be approximately $\sigma_c = 20$ ksi (138 MPa).



σ–t diagram for stainless steel at 1200°F and creep strain at 1%

Fig. 3–27

In general, the creep strength will *decrease* for *higher temperatures* or for *higher applied stresses*. For longer periods of time, extrapolations from the curves must be made. To do this usually requires a certain amount of experience with creep behavior, and some supplementary knowledge about the creep properties of the material. Once the material's creep strength has been determined, however, a factor of safety is applied to obtain an appropriate allowable stress for design.



The design of members used for amusement park rides requires careful consideration of cyclic loadings that can cause fatigue.

Fatigue. When a metal is subjected to repeated *cycles* of stress or strain, it causes its structure to break down, ultimately leading to fracture. This behavior is called ***fatigue***, and it is usually responsible for a large percentage of failures in connecting rods and crankshafts of engines; steam or gas turbine blades; connections or supports for bridges, railroad wheels, and axles; and other parts subjected to cyclic loading. In all these cases, fracture will occur at a stress that is *less* than the material's yield stress.

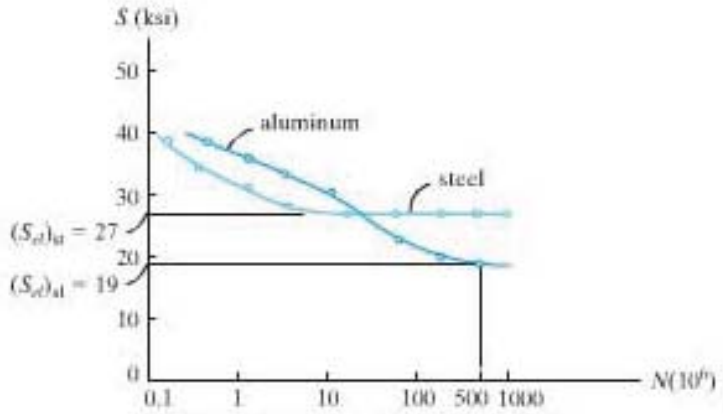
The nature of this failure apparently results from the fact that there are microscopic imperfections, usually on the surface of the member, where the localized stress becomes *much greater* than the average stress acting over the cross section. As this higher stress is cycled, it leads to the formation of minute cracks. Occurrence of these cracks causes a further increase of stress at their tips or boundaries, which in turn causes a further extension of the cracks into the material as the stress continues to be cycled. Eventually the cross-sectional area of the member is reduced to the point where the load can no longer be sustained, and as a result sudden fracture occurs. The material, even though known to be ductile, behaves as if it were *brittle*.

In order to specify a safe strength for a metallic material under repeated loading, it is necessary to determine a limit below which no evidence of failure can be detected after applying a load for a specified number of cycles. This limiting stress is called the *endurance* or *fatigue limit*. Using a testing machine for this purpose, a series of specimens are each subjected to a specified stress and cycled to failure. The results are plotted as a graph representing the stress S (or σ) on the vertical axis and the number of cycles-to-failure N on the horizontal axis. This graph is called an *S–N diagram* or *stress–cycle diagram*, and most often the values of N are plotted on a logarithmic scale since they are generally quite large.



Engineers must account for possible fatigue failure of the moving parts of this oil-pumping rig.

Examples of S–N diagrams for two common engineering metals are shown in Fig. 3–28. The endurance limit is usually identified as the stress for which the S–N graph becomes horizontal or asymptotic. As noted, it has a well-defined value of $(S_{el})_{st} = 27$ ksi (186 MPa) for steel. For aluminum, however, the endurance limit is not well defined, and so it is normally specified as the stress having a limit of 500 million cycles, $(S_{el})_{al} = 19$ ksi (131 MPa). Once a particular value is obtained, it is often assumed that for any stress below this value the fatigue life is infinite, and therefore the number of cycles to failure is no longer given consideration.



S–N diagram for steel and aluminum alloys (N axis has a logarithmic scale)

Fig. 3–28

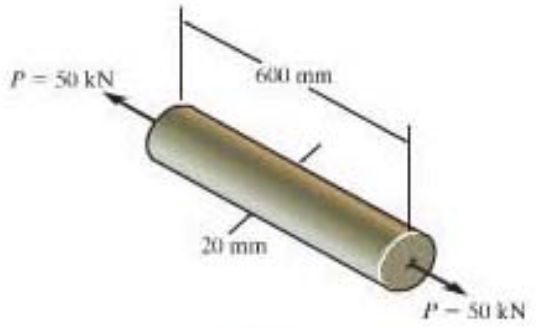
Important Points

- *Poisson's ratio*, ν, is a ratio of the lateral strain of a homogeneous and isotropic material to its longitudinal strain. Generally these strains are of opposite signs, that is, if one is an elongation, the other will be a contraction.
- The *shear stress–strain diagram* is a plot of the shear stress versus the shear strain. If the material is homogeneous and isotropic, and is also linear elastic, the slope of the straight line within the elastic region is called the modulus of rigidity or the shear modulus, G.
- There is a mathematical relationship between G, E, and ν.
- *Creep* is the time-dependent deformation of a material for which stress and/or temperature play an important role. Members are designed to resist the effects of creep based on their material creep strength, which is the largest initial stress a material can withstand during a specified time without exceeding a specified creep strain.
- *Fatigue* occurs in metals when the stress or strain is cycled. This phenomenon causes brittle fracture of the material. Members are designed to resist fatigue by ensuring that the stress in the member does not exceed its *endurance* or *fatigue limit*. This value is determined from an S–N diagram as the maximum stress the material can resist when subjected to a specified number of cycles of loading.

FUNDAMENTAL PROBLEMS

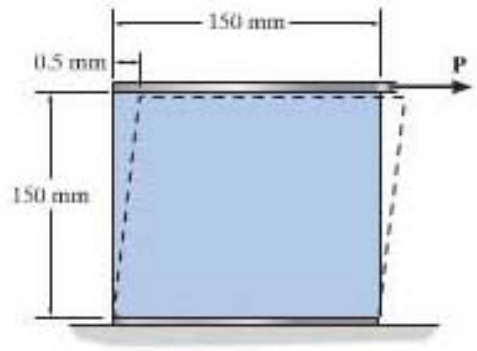
F3–13. A 100-mm long rod has a diameter of 15 mm. If an axial tensile load of 10 kN is applied to it, determine the change in its diameter. $E = 70$ GPa, $\nu = 0.35$.

F3–14. A solid circular rod that is 600 mm long and 20 mm in diameter is subjected to an axial force of $P = 50$ kN. The elongation of the rod is $\delta = 1.40$ mm, and its diameter becomes $d' = 19.9837$ mm. Determine the modulus of elasticity and the modulus of rigidity of the material. Assume that the material does not yield.



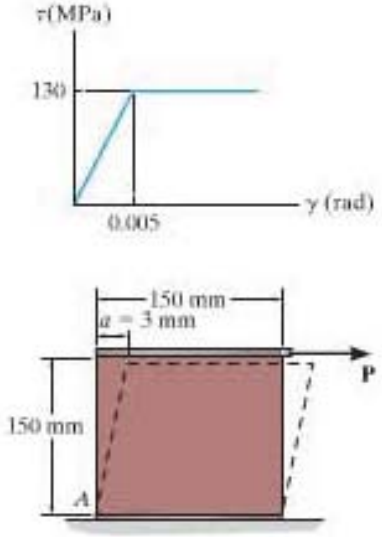
F3–14

F3–15. A 20-mm-wide block is firmly bonded to rigid plates at its top and bottom. When the force **P** is applied the block deforms into the shape shown by the dashed line. Determine the magnitude of **P**. The block's material has a modulus of rigidity of $G = 26$ GPa. Assume that the material does not yield and use small angle analysis.



F3–15

F3–16. A 20-mm-wide block is bonded to rigid plates at its top and bottom. When the force **P** is applied the block deforms into the shape shown by the dashed line. If $a = 3$ mm and **P** is released, determine the permanent shear strain in the block.



F3–16

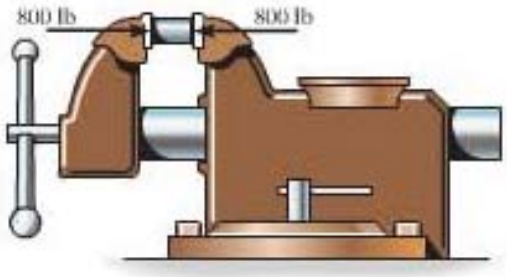
PROBLEMS

•3–25. The acrylic plastic rod is 200 mm long and 15 mm in diameter. If an axial load of 300 N is applied to it, determine the change in its length and the change in its diameter. $E_p = 2.70$ GPa, $\nu_p = 0.4$.



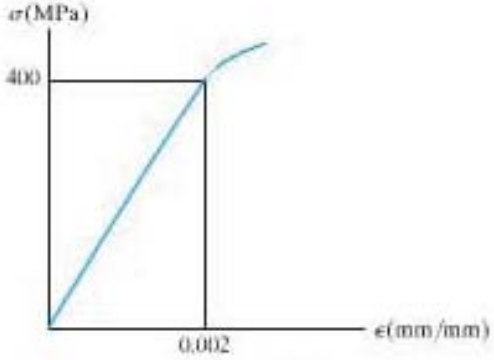
Prob. 3–25

3–26. The short cylindrical block of 2014-T6 aluminum, having an original diameter of 0.5 in. and a length of 1.5 in., is placed in the smooth jaws of a vise and squeezed until the axial load applied is 800 lb. Determine (a) the decrease in its length and (b) its new diameter.



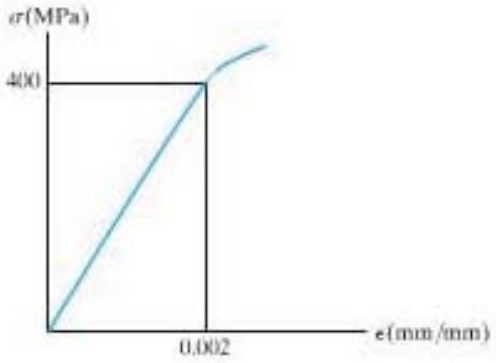
Prob. 3–26

3–27. The elastic portion of the stress–strain diagram for a steel alloy is shown in the figure. The specimen from which it was obtained had an original diameter of 13 mm and a gauge length of 50 mm. When the applied load on the specimen is 50 kN, the diameter is 12.99265 mm. Determine Poisson's ratio for the material.



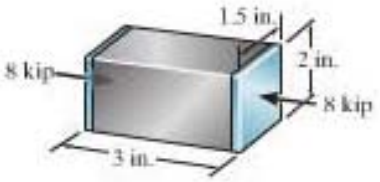
Prob. 3–27

***3–28.** The elastic portion of the stress–strain diagram for a steel alloy is shown in the figure. The specimen from which it was obtained had an original diameter of 13 mm and a gauge length of 50 mm. If a load of $P = 20$ kN is applied to the specimen, determine its diameter and gauge length. Take $\nu = 0.4$.



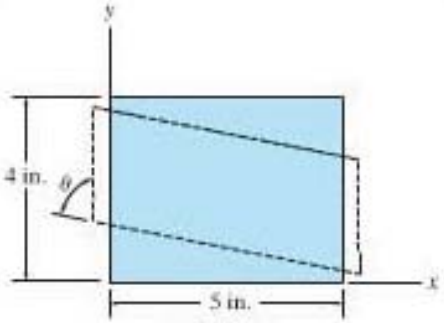
Prob. 3–28

•3–29. The aluminum block has a rectangular cross section and is subjected to an axial compressive force of 8 kip. If the 1.5-in. side changed its length to 1.500132 in., determine Poisson's ratio and the new length of the 2-in. side. $E_{al} = 10(10^3)$ ksi.



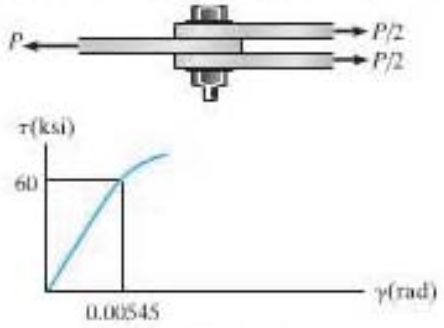
Prob. 3–29

3–30. The block is made of titanium Ti-6A1-4V and is subjected to a compression of 0.06 in. along the y axis, and its shape is given a tilt of $\theta = 89.7°$. Determine ϵ_x, ϵ_y, and γ_{xy}.



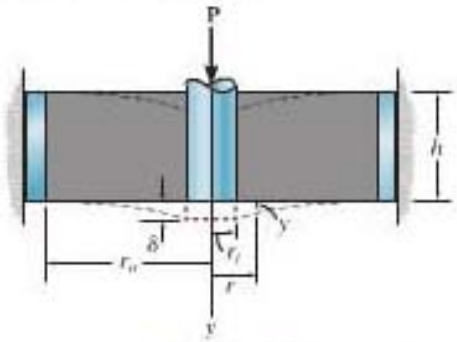
Prob. 3–30

3–31. The shear stress–strain diagram for a steel alloy is shown in the figure. If a bolt having a diameter of 0.75 in. is made of this material and used in the double lap joint, determine the modulus of elasticity E and the force P required to cause the material to yield. Take $\nu = 0.3$.



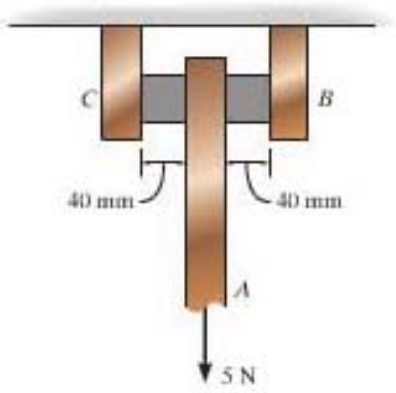
Prob. 3–31

*3–32. A shear spring is made by bonding the rubber annulus to a rigid fixed ring and a plug. When an axial load **P** is placed on the plug, show that the slope at point y in the rubber is $dy/dr = -\tan\gamma = -\tan(P/(2\pi hGr))$. For small angles we can write $dy/dr = -P/(2\pi hGr)$. Integrate this expression and evaluate the constant of integration using the condition that $y = 0$ at $r = r_o$. From the result compute the deflection $y = \delta$ of the plug.



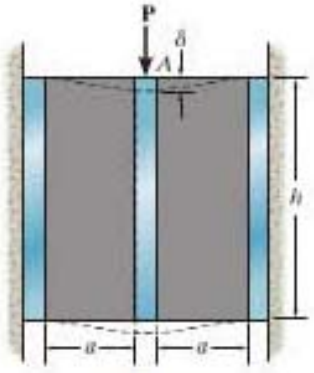
Prob. 3–32

•3–33. The support consists of three rigid plates, which are connected together using two symmetrically placed rubber pads. If a vertical force of 5 N is applied to plate A, determine the approximate vertical displacement of this plate due to shear strains in the rubber. Each pad has cross-sectional dimensions of 30 mm and 20 mm. $G_r = 0.20$ MPa.



Prob. 3–33

3–34. A shear spring is made from two blocks of rubber, each having a height h, width b, and thickness a. The blocks are bonded to three plates as shown. If the plates are rigid and the shear modulus of the rubber is G, determine the displacement of plate A if a vertical load **P** is applied to this plate. Assume that the displacement is small so that $\delta = a\tan\gamma \approx a\gamma$.



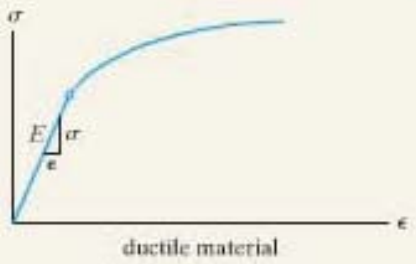
Prob. 3–34

CHAPTER REVIEW

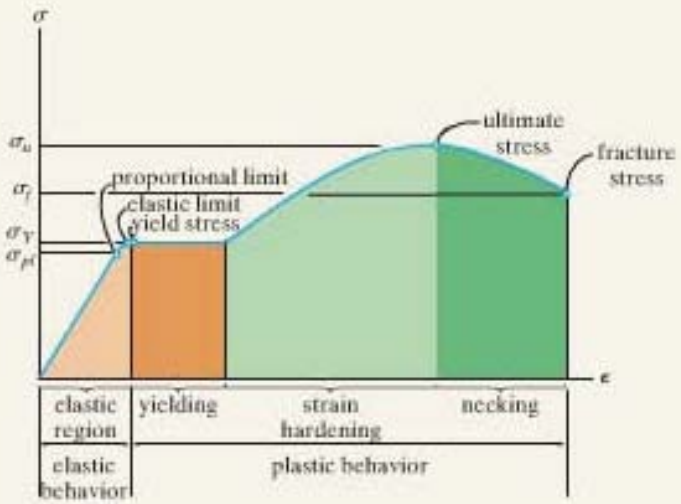
One of the most important tests for material strength is the tension test. The results, found from stretching a specimen of known size, are plotted as normal stress on the vertical axis and normal strain on the horizontal axis.

Many engineering materials exhibit initial linear elastic behavior, whereby stress is proportional to strain, defined by Hooke's law, $\sigma = E\epsilon$. Here E, called the modulus of elasticity, is the slope of this straight line on the stress–strain diagram.

$$\sigma = E\epsilon$$



When the material is stressed beyond the yield point, permanent deformation will occur. In particular, steel has a region of yielding, whereby the material will exhibit an increase in strain with no increase in stress. The region of strain hardening causes further yielding of the material with a corresponding increase in stress. Finally, at the ultimate stress, a localized region on the specimen will begin to constrict, forming a neck. It is after this that the fracture occurs.



Ductile materials, such as most metals, exhibit both elastic and plastic behavior. Wood is moderately ductile. Ductility is usually specified by the permanent elongation to failure or by the percent reduction in the cross-sectional area.

$$\text{Percent elongation} = \frac{L_f - L_0}{L_0}(100\%)$$

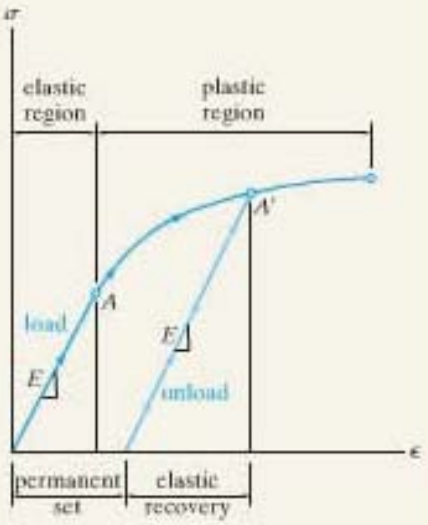
$$\text{Percent reduction of area} = \frac{A_0 - A_f}{A_0}(100\%)$$

Brittle materials exhibit little or no yielding before failure. Cast iron, concrete, and glass are typical examples.

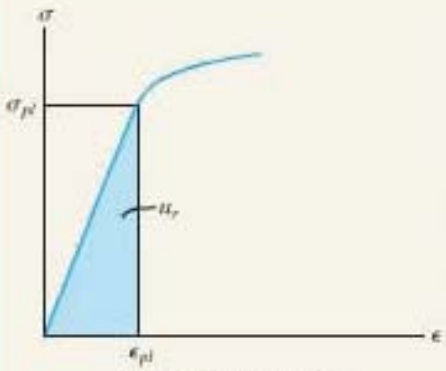


brittle material

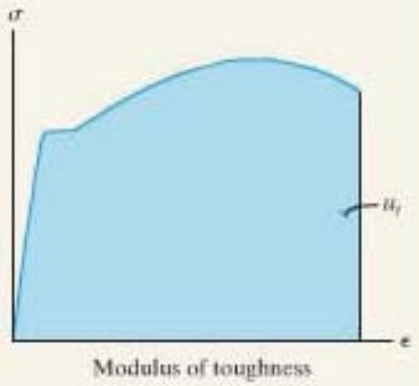
The yield point of a material at A can be increased by strain hardening. This is accomplished by applying a load that causes the stress to be greater than the yield stress, then releasing the load. The larger stress A' becomes the new yield point for the material.



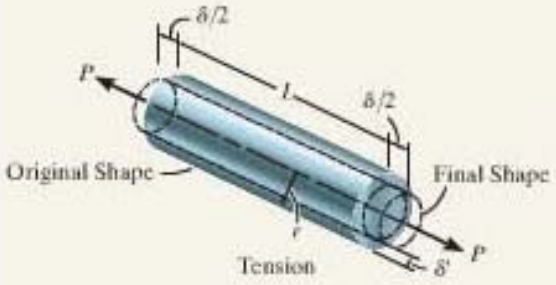
When a load is applied to a member, the deformations cause strain energy to be stored in the material. The strain energy per unit volume or strain energy density is equivalent to the area under the stress–strain curve. This area up to the yield point is called the modulus of resilience. The entire area under the stress–strain diagram is called the modulus of toughness.



Modulus of resilience



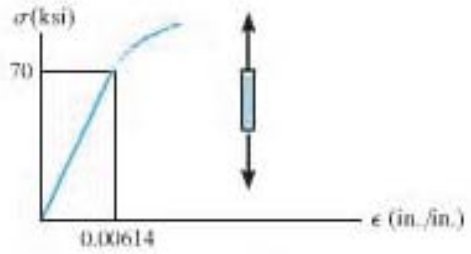
Modulus of toughness

Poisson's ratio ν is a dimensionless material property that relates the lateral strain to the longitudinal strain. Its range of values is $0 \leq \nu \leq 0.5$.	$\nu = -\frac{\epsilon_{lat}}{\epsilon_{long}}$	
Shear stress versus shear strain diagrams can also be established for a material. Within the elastic region, $\tau = G\gamma$, where G is the shear modulus, found from the slope of the line. The value of ν can be obtained from the relationship that exists between G, E and ν.	$G = \frac{E}{2(1 + \nu)}$	
When materials are in service for long periods of time, considerations of creep become important. Creep is the time rate of deformation, which occurs at high stress and/or high temperature. Design requires that the stress in the material not exceed an allowable stress which is based on the material's creep strength. Fatigue can occur when the material undergoes a large number of cycles of loading. This effect will cause microscopic cracks to form, leading to a brittle failure. To prevent fatigue, the stress in the material must not exceed a specified endurance or fatigue limit.		

REVIEW PROBLEMS

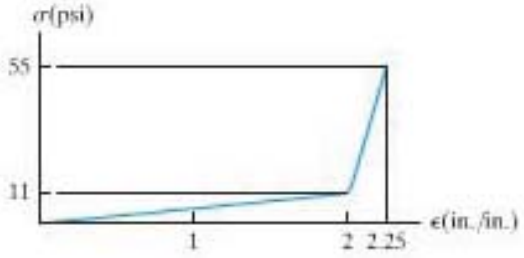
3–35. The elastic portion of the tension stress-strain diagram for an aluminum alloy is shown in the figure. The specimen used for the test has a gauge length of 2 in. and a diameter of 0.5 in. When the applied load is 9 kip, the new diameter of the specimen is 0.49935 in. Compute the shear modulus G_{al} for the aluminum.

***3–36.** The elastic portion of the tension stress-strain diagram for an aluminum alloy is shown in the figure. The specimen used for the test has a gauge length of 2 in. and a diameter of 0.5 in. If the applied load is 10 kip, determine the new diameter of the specimen. The shear modulus is $G_{al} = 3.8(10^3)$ ksi.



Probs. 3–35/36

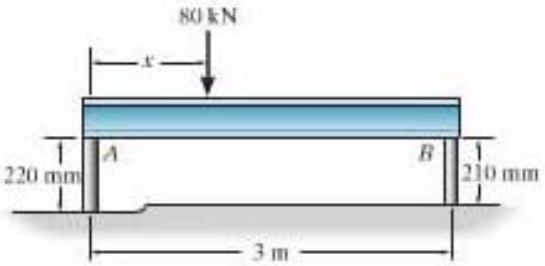
3–37. The $\sigma-\epsilon$ diagram for elastic fibers that make up human skin and muscle is shown. Determine the modulus of elasticity of the fibers and estimate their modulus of toughness and modulus of resilience.



Prob. 3–37

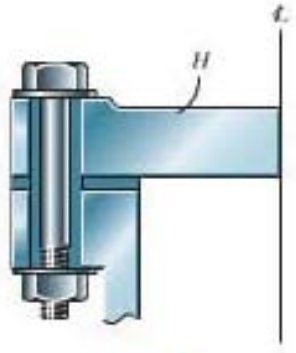
3–38. A short cylindrical block of 6061-T6 aluminum, having an original diameter of 20 mm and a length of 75 mm, is placed in a compression machine and squeezed until the axial load applied is 5 kN. Determine (a) the decrease in its length and (b) its new diameter.

3–39. The rigid beam rests in the horizontal position on two 2014-T6 aluminum cylinders having the *unloaded* lengths shown. If each cylinder has a diameter of 30 mm, determine the placement x of the applied 80-kN load so that the beam remains horizontal. What is the new diameter of cylinder A after the load is applied? $\nu_{al} = 0.35$.



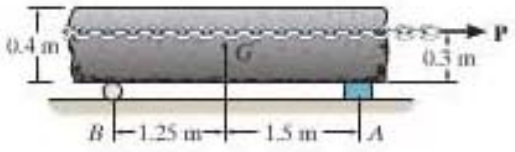
Prob. 3–39

***3–40.** The head H is connected to the cylinder of a compressor using six steel bolts. If the clamping force in each bolt is 800 lb, determine the normal strain in the bolts. Each bolt has a diameter of $\frac{3}{16}$ in. If $\sigma_Y = 40$ ksi and $E_{st} = 29(10^3)$ ksi, what is the strain in each bolt when the nut is unscrewed so that the clamping force is released?



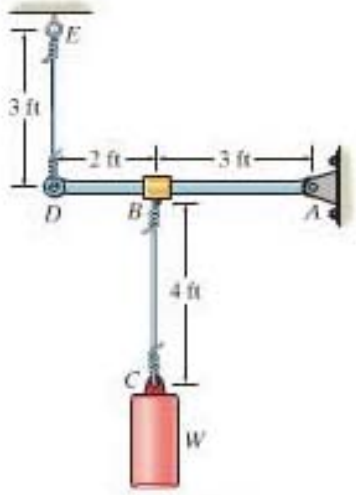
Prob. 3–40

•3–41. The stone has a mass of 800 kg and center of gravity at G. It rests on a pad at A and a roller at B. The pad is fixed to the ground and has a compressed height of 30 mm, a width of 140 mm, and a length of 150 mm. If the coefficient of static friction between the pad and the stone is $\mu_s = 0.8$, determine the approximate horizontal displacement of the stone, caused by the shear strains in the pad, before the stone begins to slip. Assume the normal force at A acts 1.5 m from G as shown. The pad is made from a material having $E = 4$ MPa and $\nu = 0.35$.



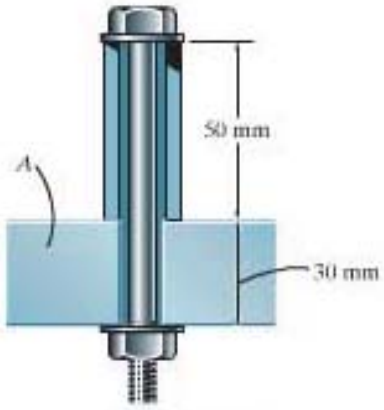
Prob. 3–41

3–42. The bar DA is rigid and is originally held in the horizontal position when the weight W is supported from C. If the weight causes B to be displaced downward 0.025 in., determine the strain in wires DE and BC. Also, if the wires are made of A-36 steel and have a cross-sectional area of 0.002 in^2, determine the weight W.



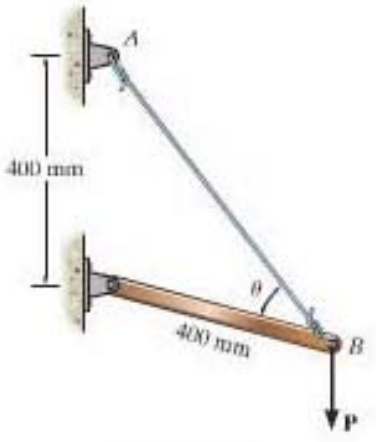
Prob. 3–42

3–43. The 8-mm-diameter bolt is made of an aluminum alloy. It fits through a magnesium sleeve that has an inner diameter of 12 mm and an outer diameter of 20 mm. If the original lengths of the bolt and sleeve are 80 mm and 50 mm, respectively, determine the strains in the sleeve and the bolt if the nut on the bolt is tightened so that the tension in the bolt is 8 kN. Assume the material at A is rigid. $E_{al} = 70$ GPa, $E_{mg} = 45$ GPa.



Prob. 3–43

***3–44.** The A-36 steel wire AB has a cross-sectional area of 10 mm^2 and is unstretched when $\theta = 45.0°$. Determine the applied load P needed to cause $\theta = 44.9°$.



Prob. 3–44



The string of drill pipe suspended from this traveling block on an oil rig is subjected to extremely large loadings and axial deformations.

Axial Load

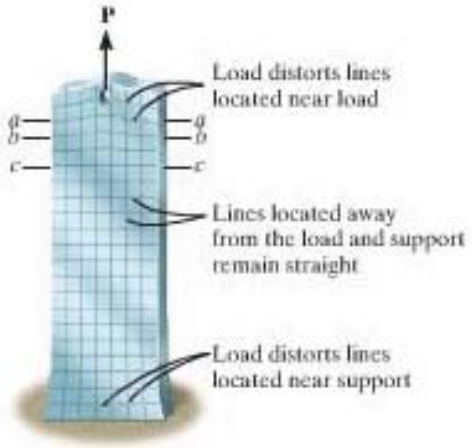
4

CHAPTER OBJECTIVES

In Chapter 1, we developed the method for finding the normal stress in axially loaded members. In this chapter we will discuss how to determine the deformation of these members, and we will also develop a method for finding the support reactions when these reactions cannot be determined strictly from the equations of equilibrium. An analysis of the effects of thermal stress, stress concentrations, inelastic deformations, and residual stress will also be discussed.

4.1 Saint-Venant's Principle

In the previous chapters, we have developed the concept of *stress* as a means of measuring the force distribution within a body and *strain* as a means of measuring a body's deformation. We have also shown that the mathematical relationship between stress and strain depends on the type of material from which the body is made. In particular, if the material behaves in a linear elastic manner, then Hooke's law applies, and there is a proportional relationship between stress and strain.



(a)

Fig. 4–1

Using this idea, consider the manner in which a rectangular bar will deform elastically when the bar is subjected to a force **P** applied along its centroidal axis, Fig. 4–1*a*. Here the bar is fixed connected at one end, with the force applied through a hole at its other end. Due to the loading, the bar deforms as indicated by the once horizontal and vertical grid lines drawn on the bar. Notice how the *localized deformation* that occurs at each end tends to even out and become uniform throughout the midsection of the bar.

If the material remains elastic then the strains caused by this deformation are directly related to the stress in the bar. As a result, the stress will be distributed more uniformly throughout the cross-sectional area when the section is taken farther and farther from the point where any external load is applied. For example, consider a profile of the variation of the stress distribution acting at sections *a–a*, *b–b*, and *c–c*, each of which is shown in Fig. 4–1*b*. By comparison, the stress tends to reach a uniform value at section *c–c*, which is sufficiently removed from the end since the localized deformation caused by **P** *vanishes*. The minimum distance from the bar's end where this occurs can be determined using a mathematical analysis based on the theory of elasticity.

It has been found that this distance should at least be equal to the *largest dimension* of the loaded cross section. Hence, section *c–c* should be located at a distance at least equal to the width (not the thickness) of the bar.*

*When section *c–c* is so located, the theory of elasticity predicts the maximum stress to be $\sigma_{max} = 1.02\sigma_{avg}$.

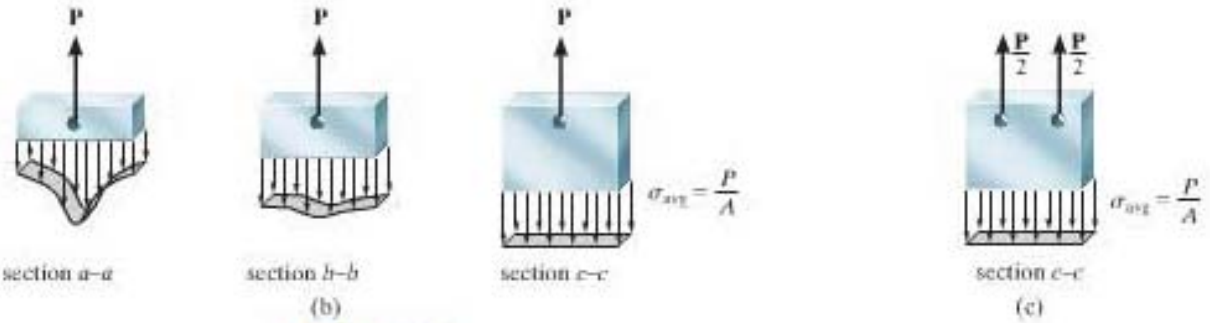


Fig. 4–1 (cont.)

In the same way, the stress distribution at the support will also even out and become uniform over the cross section located the same distance away from the support.

The fact that stress and deformation behave in this manner is referred to as *Saint-Venant's principle*, since it was first noticed by the French scientist Barré de Saint-Venant in 1855. Essentially it states that the *stress and strain produced at points in a body sufficiently removed from the region of load application will be the same as the stress and strain produced by any applied loadings that have the same statically equivalent resultant, and are applied to the body within the same region.* For example, if two symmetrically applied forces $P/2$ act on the bar, Fig. 4–1*c*, the stress distribution at section c–c will be uniform and therefore equivalent to $\sigma_{avg} = P/A$ as in Fig. 4–1*b*.



Notice how the lines on this rubber membrane distort after it is stretched. The localized distortions at the grips smooth out as stated by Saint-Venant's principle.

4.2 Elastic Deformation of an Axially Loaded Member

Using Hooke's law and the definitions of stress and strain, we will now develop an equation that can be used to determine the *elastic* displacement of a member subjected to axial loads. To generalize the development, consider the bar shown in Fig. 4–2*a*, which has a cross-sectional area that *gradually* varies along its length L. The bar is subjected to concentrated loads at its ends and a variable external load distributed along its length. This distributed load could, for example, represent the weight of the bar if it does not remain horizontal, or friction forces acting on the bar's surface. Here we wish to find the ***relative displacement*** δ (delta) of one end of the bar with respect to the other end as caused by this loading. We will *neglect* the localized deformations that occur at points of concentrated loading and where the cross section suddenly changes. From Saint-Venant's principle, these effects occur within small regions of the bar's length and will therefore have only a slight effect on the final result. For the most part, the bar will deform uniformly, so the normal stress will be uniformly distributed over the cross section.

Using the method of sections, a differential element (or wafer) of length dx and cross-sectional area $A(x)$ is isolated from the bar at the arbitrary position x. The free-body diagram of this element is shown in Fig. 4–2*b*. The resultant internal axial force will be a function of x since the external distributed loading will cause it to vary along the length of the bar. This load, $P(x)$, will deform the element into the shape indicated by the dashed outline, and therefore the displacement of one end of the element with respect to the other end is $d\delta$. The stress and strain in the element are

$$\sigma = \frac{P(x)}{A(x)} \qquad \text{and} \qquad \epsilon = \frac{d\delta}{dx}$$

Provided the stress does not exceed the proportional limit, we can apply Hooke's law; i.e.,

$$\sigma = E\epsilon$$

$$\frac{P(x)}{A(x)} = E\left(\frac{d\delta}{dx}\right)$$

$$d\delta = \frac{P(x)\,dx}{A(x)E}$$

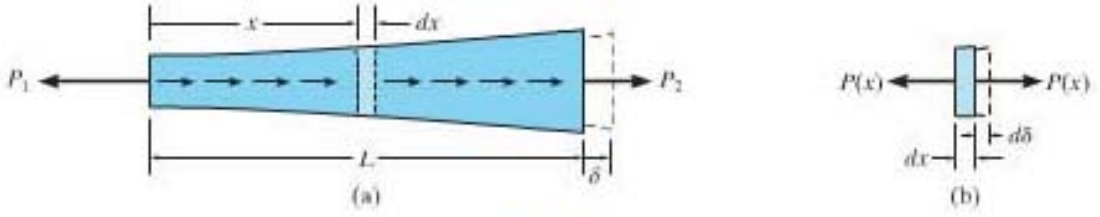


Fig. 4–2

For the entire length L of the bar, we must integrate this expression to find δ. This yields

$$\delta = \int_0^L \frac{P(x)\,dx}{A(x)E} \tag{4–1}$$

where

δ = displacement of one point on the bar relative to the other point
L = original length of bar
$P(x)$ = internal axial force at the section, located a distance x from one end
$A(x)$ = cross-sectional area of the bar, expressed as a function of x
E = modulus of elasticity for the material

Constant Load and Cross-Sectional Area. In many cases the bar will have a constant cross-sectional area A; and the material will be homogeneous, so E is constant. Furthermore, if a constant external force is applied at each end, Fig. 4–3, then the internal force P throughout the length of the bar is also constant. As a result, Eq. 4–1 can be integrated to yield

$$\delta = \frac{PL}{AE} \tag{4–2}$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each *segment* of the bar where these quantities remain *constant*. The displacement of one end of the bar with respect to the other is then found from the *algebraic addition* of the relative displacements of the ends of each segment. For this general case,

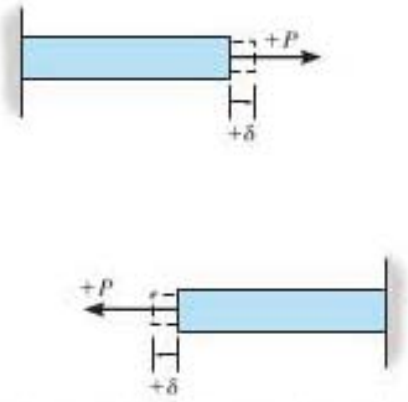
$$\delta = \Sigma \frac{PL}{AE} \tag{4–3}$$



Fig. 4–3



The vertical displacement at the top of these building columns depends upon the loading applied on the roof and to the floor attached to their midpoint.



Positive sign convention for P and δ

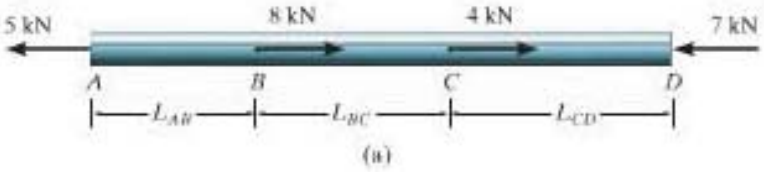
Fig. 4–4

Sign Convention. In order to apply Eq. 4–3, we must develop a sign convention for the internal axial force and the displacement of one end of the bar with respect to the other end. To do so, we will consider both the force and displacement to be *positive* if they cause *tension and elongation*, respectively, Fig. 4–4; whereas a *negative* force and displacement will cause *compression* and *contraction*, respectively.

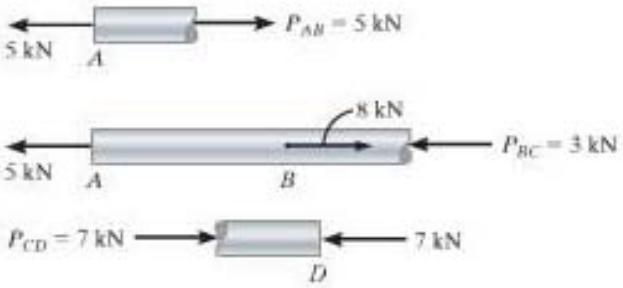
For example, consider the bar shown in Fig. 4–5*a*. The *internal axial forces* "P" are determined by the method of sections for each segment, Fig. 4–5*b*. They are $P_{AB} = +5\text{ kN}$, $P_{BC} = -3\text{ kN}$, $P_{CD} = -7\text{ kN}$. This variation in axial load is shown on the axial or *normal force diagram* for the bar, Fig. 4–5*c*. Since we now know how the *internal* force varies throughout the bar's length, the displacement of end A relative to end D is determined from

$$\delta_{A/D} = \sum \frac{PL}{AE} = \frac{(5\text{ kN})L_{AB}}{AE} + \frac{(-3\text{ kN})L_{BC}}{AE} + \frac{(-7\text{ kN})L_{CD}}{AE}$$

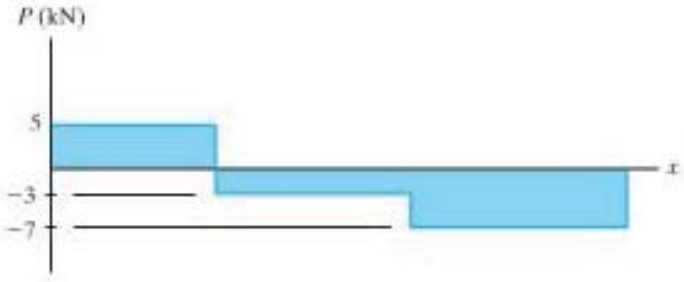
If the other data are substituted and a positive answer is calculated, it means that end A will move away from end D (the bar elongates), whereas a negative result would indicate that end A moves toward end D (the bar shortens). The double subscript notation is used to indicate this relative displacement ($\delta_{A/D}$); however, if the displacement is to be determined relative to a *fixed point*, then only a single subscript will be used. For example, if D is located at a *fixed* support, then the displacement will be denoted as simply δ_A.



(a)



(b)



(c)

Fig. 4–5

Important Points

- *Saint-Venant's principle* states that both the localized deformation and stress which occur within the regions of load application or at the supports tend to "even out" at a distance sufficiently removed from these regions.
- The displacement of one end of an axially loaded member relative to the other end is determined by relating the applied *internal* load to the stress using $\sigma = P/A$ and relating the displacement to the strain using $\epsilon = d\delta/dx$. Finally these two equations are combined using Hooke's law, $\sigma = E\epsilon$, which yields Eq. 4–1.
- Since Hooke's law has been used in the development of the displacement equation, it is important that no internal load causes yielding of the material, and that the material is homogeneous and behaves in a linear elastic manner.

Procedure for Analysis

The relative displacement between any two points A and B on an axially loaded member can be determined by applying Eq. 4–1 (or Eq. 4–2). Application requires the following steps.

Internal Force.

- Use the method of sections to determine the internal axial force P within the member.
- If this force varies along the member's length due to an *external distributed loading*, a section should be made at the arbitrary location x from one end of the member and the force represented as a function of x, i.e., $P(x)$.
- If several *constant external forces* act on the member, the internal force in each *segment* of the member, between any two external forces, must be determined.
- For any segment, an internal *tensile force* is *positive* and an internal *compressive force* is *negative*. For convenience, the results of the internal loading can be shown graphically by constructing the normal-force diagram.

Displacement.

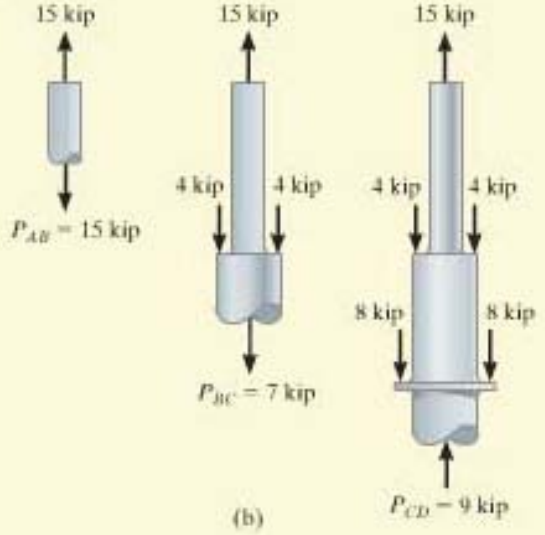
- When the member's cross-sectional area *varies* along its length, the area must be expressed as a function of its position x, i.e., $A(x)$.
- If the cross-sectional area, the modulus of elasticity, or the internal loading *suddenly changes*, then Eq. 4–2 should be applied to each segment for which these quantities are constant.
- When substituting the data into Eqs. 4–1 through 4–3, be sure to account for the proper sign for the internal force **P**. Tensile loadings are positive and compressive loadings are negative. Also, use a consistent set of units. For any segment, if the result is a *positive* numerical quantity, it indicates *elongation*; if it is *negative*, it indicates a *contraction*.

EXAMPLE 4.1

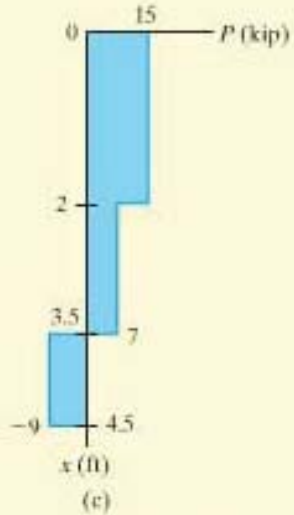
The A-36 steel bar shown in Fig. 4–6*a* is made from two segments having cross-sectional areas of $A_{AB} = 1\ \text{in}^2$ and $A_{BD} = 2\ \text{in}^2$. Determine the vertical displacement of end A and the displacement of B relative to C.



(a)



(b)



(c)

Fig. 4–6

SOLUTION

Internal Force. Due to the application of the external loadings, the internal axial forces in regions AB, BC, and CD will all be different. These forces are obtained by applying the method of sections and the equation of vertical force equilibrium as shown in Fig. 4–6*b*. This variation is plotted in Fig. 4–6*c*.

Displacement. From the inside back cover, $E_{st} = 29(10^3)$ ksi. Using the sign convention, i.e., internal tensile forces are positive and compressive forces are negative, the vertical displacement of A relative to the *fixed* support D is

$$\delta_A = \sum \frac{PL}{AE} = \frac{[+15\ \text{kip}](2\ \text{ft})(12\ \text{in./ft})}{(1\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} + \frac{[+7\ \text{kip}](1.5\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} + \frac{[-9\ \text{kip}](1\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]}$$

$$= +0.0127\ \text{in.} \qquad \textit{Ans.}$$

Since the result is *positive*, the bar *elongates* and so the displacement at A is upward.

Applying Eq. 4–2 between points B and C, we obtain,

$$\delta_{B/C} = \frac{P_{BC}L_{BC}}{A_{BC}E} = \frac{[+7\ \text{kip}](1.5\ \text{ft})(12\ \text{in./ft})}{(2\ \text{in}^2)[29(10^3)\ \text{kip/in}^2]} = +0.00217\ \text{in.} \qquad \textit{Ans.}$$

Here B moves away from C, since the segment elongates.

EXAMPLE 4.2

The assembly shown in Fig. 4–7*a* consists of an aluminum tube *AB* having a cross-sectional area of 400 mm^2. A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. Take $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.

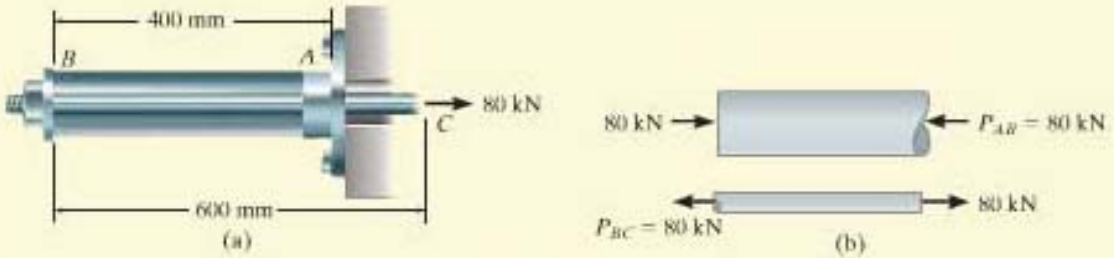


Fig. 4–7

SOLUTION

Internal Force. The free-body diagram of the tube and rod segments in Fig. 4–7*b*, shows that the rod is subjected to a tension of 80 kN and the tube is subjected to a compression of 80 kN.

Displacement. We will first determine the displacement of end *C* with respect to end *B*. Working in units of newtons and meters, we have

$$\delta_{C/B} = \frac{PL}{AE} = \frac{[+80(10^3)\text{ N}](0.6\text{ m})}{\pi(0.005\text{ m})^2[200(10^9)\text{ N/m}^2]} = +0.003056\text{ m} \rightarrow$$

The positive sign indicates that end *C* moves *to the right* relative to end *B*, since the bar elongates.

The displacement of end *B* with respect to the *fixed* end *A* is

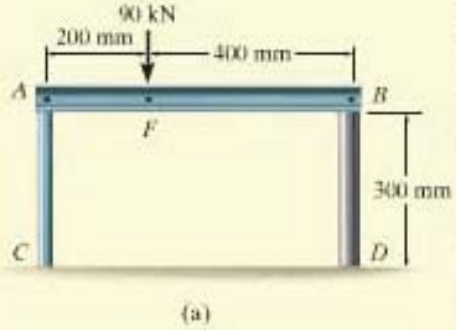
$$\delta_B = \frac{PL}{AE} = \frac{[-80(10^3)\text{ N}](0.4\text{ m})}{[400\text{ mm}^2(10^{-6})\text{ m}^2/\text{mm}^2][70(10^9)\text{ N/m}^2]}$$
$$= -0.001143\text{ m} = 0.001143\text{ m} \rightarrow$$

Here the negative sign indicates that the tube shortens, and so *B* moves to the *right* relative to *A*.

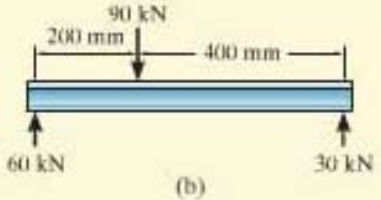
Since both displacements are to the right, the displacement of *C* relative to the fixed end *A* is therefore

$$(\stackrel{+}{\rightarrow}) \qquad \delta_C = \delta_B + \delta_{C/B} = 0.001143\text{ m} + 0.003056\text{ m}$$
$$= 0.00420\text{ m} = 4.20\text{ mm} \rightarrow \qquad \textit{Ans.}$$

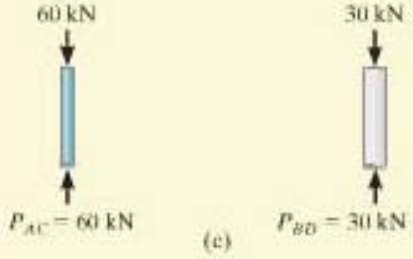
EXAMPLE 4.3



(a)



(b)



(c)

Rigid beam AB rests on the two short posts shown in Fig. 4–8*a*. AC is made of steel and has a diameter of 20 mm, and BD is made of aluminum and has a diameter of 40 mm. Determine the displacement of point F on AB if a vertical load of 90 kN is applied over this point. Take $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.

SOLUTION

Internal Force. The compressive forces acting at the top of each post are determined from the equilibrium of member AB, Fig. 4–8*b*. These forces are equal to the internal forces in each post, Fig. 4–8*c*.

Displacement. The displacement of the top of each post is

Post *AC*:

$$\delta_A = \frac{P_{AC}L_{AC}}{A_{AC}E_{st}} = \frac{[-60(10^3)\text{ N}](0.300\text{ m})}{\pi(0.010\text{ m})^2[200(10^9)\text{ N/m}^2]} = -286(10^{-6})\text{ m}$$

$$= 0.286\text{ mm} \downarrow$$

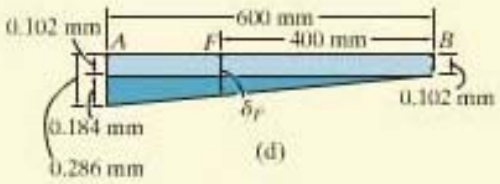
Post *BD*:

$$\delta_B = \frac{P_{BD}L_{BD}}{A_{BD}E_{al}} = \frac{[-30(10^3)\text{ N}](0.300\text{ m})}{\pi(0.020\text{ m})^2[70(10^9)\text{ N/m}^2]} = -102(10^{-6})\text{ m}$$

$$= 0.102\text{ mm} \downarrow$$

A diagram showing the centerline displacements at A, B, and F on the beam is shown in Fig. 4–8*d*. By proportion of the blue shaded triangle, the displacement of point F is therefore

$$\delta_F = 0.102\text{ mm} + (0.184\text{ mm})\left(\frac{400\text{ mm}}{600\text{ mm}}\right) = 0.225\text{ mm} \downarrow \quad \textit{Ans.}$$



(d)

Fig. 4–8

EXAMPLE 4.4

A member is made from a material that has a specific weight γ and modulus of elasticity E. If it is in the form of a *cone* having the dimensions shown in Fig. 4–9*a*, determine how far its end is displaced due to gravity when it is suspended in the vertical position.



(a)

SOLUTION

Internal Force. The internal axial force varies along the member since it is dependent on the weight $W(y)$ of a segment of the member below any section, Fig. 4–9*b*. Hence, to calculate the displacement, we must use Eq. 4–1. At the section located a distance y from its free end, the radius x of the cone as a function of y is determined by proportion; i.e.,

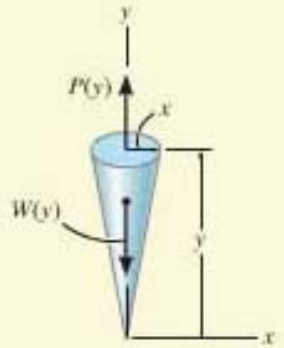
$$\frac{x}{y} = \frac{r_0}{L}; \qquad x = \frac{r_0}{L}y$$

The volume of a cone having a base of radius x and height y is

$$V = \frac{1}{3}\pi y x^2 = \frac{\pi r_0^2}{3L^2}y^3$$

Since $W = \gamma V$, the internal force at the section becomes

$$+\uparrow \Sigma F_y = 0; \qquad P(y) = \frac{\gamma \pi r_0^2}{3L^2}y^3$$



(b)

Fig. 4–9

Displacement. The area of the cross section is also a function of position y, Fig. 4–9*b*. We have

$$A(y) = \pi x^2 = \frac{\pi r_0^2}{L^2}y^2$$

Applying Eq. 4–1 between the limits of $y = 0$ and $y = L$ yields

$$\delta = \int_0^L \frac{P(y)\,dy}{A(y)E} = \int_0^L \frac{\left[(\gamma \pi r_0^2/3L^2)\,y^3\right]dy}{\left[(\pi r_0^2/L^2)\,y^2\right]E}$$

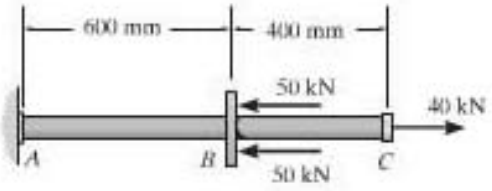
$$= \frac{\gamma}{3E}\int_0^L y\,dy$$

$$= \frac{\gamma L^2}{6E} \qquad \textit{Ans.}$$

NOTE: As a partial check of this result, notice how the units of the terms, when canceled, give the displacement in units of length as expected.

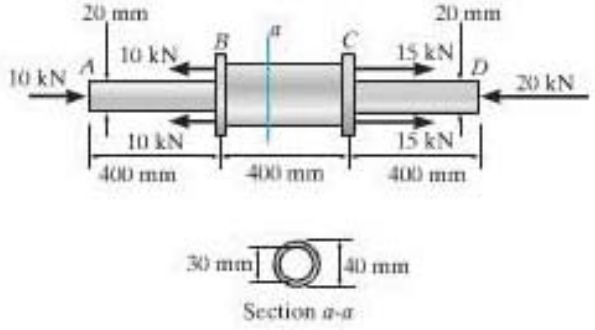
FUNDAMENTAL PROBLEMS

F4–1. The 20-mm-diameter A-36 steel rod is subjected to the axial forces shown. Determine the displacement of end C with respect to the fixed support at A.



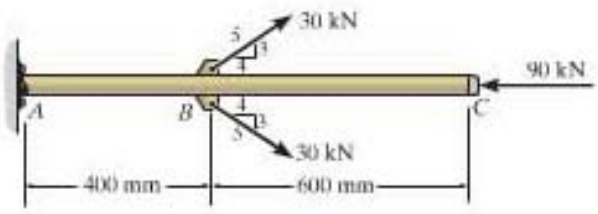
F4–1

F4–2. Segments AB and CD of the assembly are solid circular rods, and segment BC is a tube. If the assembly is made of 6061-T6 aluminum, determine the displacement of end D with respect to end A.



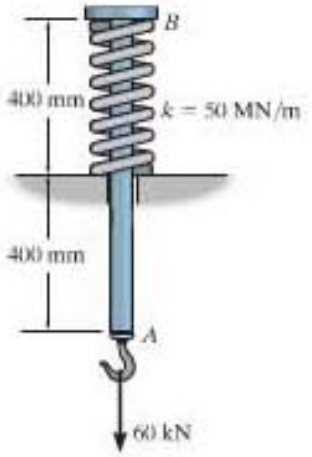
F4–2

F4–3. The 30-mm-diameter A-36 steel rod is subjected to the loading shown. Determine the displacement of end A with respect to end C.



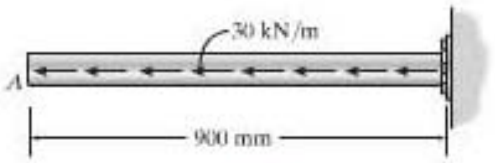
F4–3

F4–4. If the 20-mm-diameter rod is made of A-36 steel and the stiffness of the spring is $k = 50$ MN/m, determine the displacement of end A when the 60-kN force is applied.



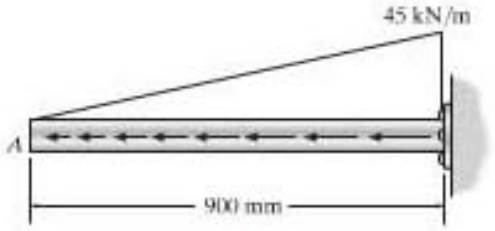
F4–4

F4–5. The 20-mm-diameter 2014-T6 aluminum rod is subjected to the uniform distributed axial load. Determine the displacement of end A.



F4–5

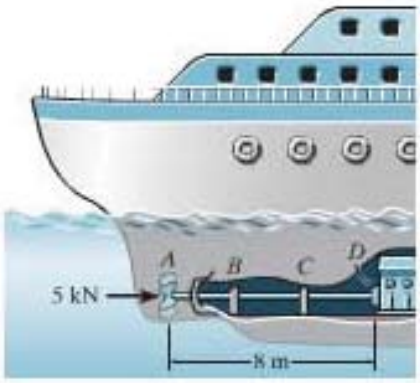
F4–6. The 20-mm-diameter 2014-T6 aluminum rod is subjected to the triangular distributed axial load. Determine the displacement of end A.



F4–6

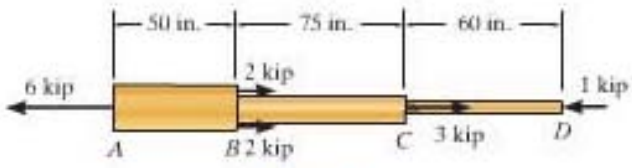
PROBLEMS

•4–1. The ship is pushed through the water using an A-36 steel propeller shaft that is 8 m long, measured from the propeller to the thrust bearing D at the engine. If it has an outer diameter of 400 mm and a wall thickness of 50 mm, determine the amount of axial contraction of the shaft when the propeller exerts a force on the shaft of 5 kN. The bearings at B and C are journal bearings.



Prob. 4–1

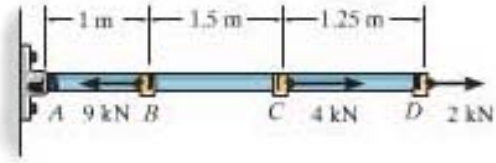
4–2. The copper shaft is subjected to the axial loads shown. Determine the displacement of end A with respect to end D. The diameters of each segment are $d_{AB} = 3$ in., $d_{BC} = 2$ in., and $d_{CD} = 1$ in. Take $E_{cu} = 18(10^3)$ ksi.



Prob. 4–2

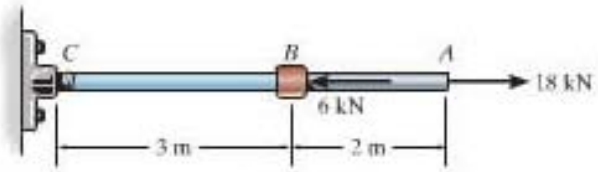
4–3. The A-36 steel rod is subjected to the loading shown. If the cross-sectional area of the rod is 50 mm^2, determine the displacement of its end D. Neglect the size of the couplings at B, C, and D.

***4–4.** The A-36 steel rod is subjected to the loading shown. If the cross-sectional area of the rod is 50 mm^2, determine the displacement of C. Neglect the size of the couplings at B, C, and D.



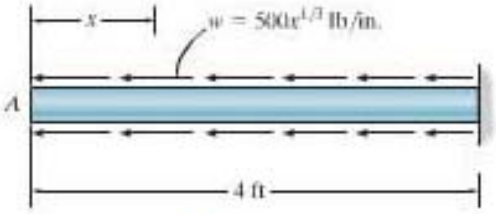
Probs. 4–3/4

4–5. The assembly consists of a steel rod CB and an aluminum rod BA, each having a diameter of 12 mm. If the rod is subjected to the axial loadings at A and at the coupling B, determine the displacement of the coupling B and the end A. The unstretched length of each segment is shown in the figure. Neglect the size of the connections at B and C, and assume that they are rigid. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.



Prob. 4–5

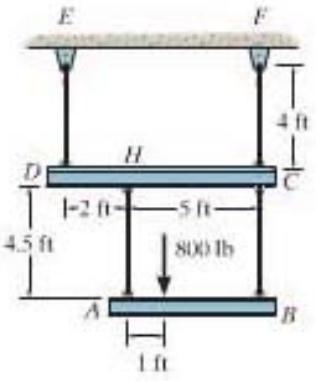
4–6. The bar has a cross-sectional area of 3 in^2, and $E = 35(10^3)$ ksi. Determine the displacement of its end A when it is subjected to the distributed loading.



Prob. 4–6

4–7. The load of 800 lb is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the vertical displacement of the load if the members were horizontal before the load was applied. Each wire has a cross-sectional area of 0.05 in^2.

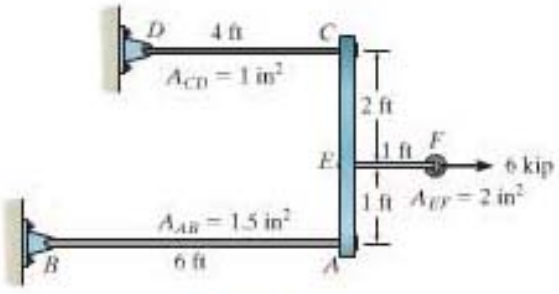
***4–8.** The load of 800 lb is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the angle of tilt of each member after the load is applied. The members were originally horizontal, and each wire has a cross-sectional area of 0.05 in^2.



Probs. 4–7/8

***4–9.** The assembly consists of three titanium (Ti-6A1-4V) rods and a rigid bar AC. The cross-sectional area of each rod is given in the figure. If a force of 6 kip is applied to the ring F, determine the horizontal displacement of point F.

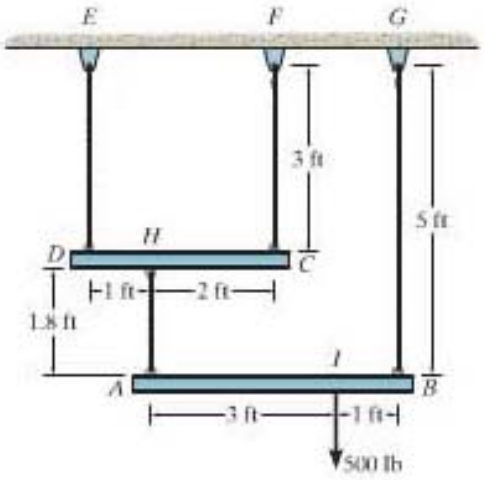
4–10. The assembly consists of three titanium (Ti-6A1-4V) rods and a rigid bar AC. The cross-sectional area of each rod is given in the figure. If a force of 6 kip is applied to the ring F, determine the angle of tilt of bar AC.



Probs. 4–9/10

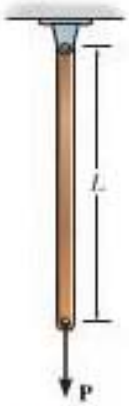
4–11. The load is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the vertical displacement of the 500-lb load if the members were originally horizontal when the load was applied. Each wire has a cross-sectional area of 0.025 in^2.

***4–12.** The load is supported by the four 304 stainless steel wires that are connected to the rigid members AB and DC. Determine the angle of tilt of each member after the 500-lb load is applied. The members were originally horizontal, and each wire has a cross-sectional area of 0.025 in^2.



Probs. 4–11/12

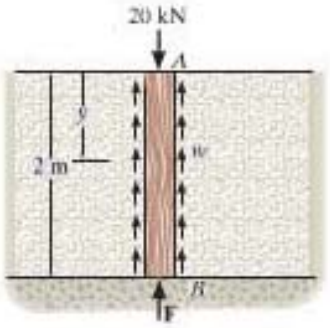
***4–13.** The bar has a length L and cross-sectional area A. Determine its elongation due to the force **P** and its own weight. The material has a specific weight γ (weight/volume) and a modulus of elasticity E.



Prob. 4–13

4–14. The post is made of Douglas fir and has a diameter of 60 mm. If it is subjected to the load of 20 kN and the soil provides a frictional resistance that is uniformly distributed along its sides of $w = 4$ kN/m, determine the force **F** at its bottom needed for equilibrium. Also, what is the displacement of the top of the post A with respect to its bottom B? Neglect the weight of the post.

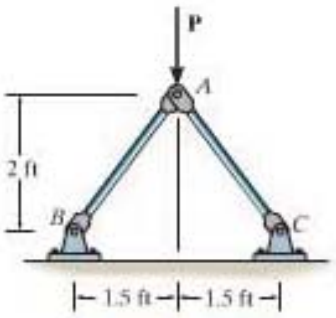
4–15. The post is made of Douglas fir and has a diameter of 60 mm. If it is subjected to the load of 20 kN and the soil provides a frictional resistance that is distributed along its length and varies linearly from $w = 0$ at $y = 0$ to $w = 3$ kN/m at $y = 2$ m, determine the force **F** at its bottom needed for equilibrium. Also, what is the displacement of the top of the post A with respect to its bottom B? Neglect the weight of the post.



Probs. 4–14/15

*4–16. The linkage is made of two pin-connected A-36 steel members, each having a cross-sectional area of 1.5 in^2. If a vertical force of $P = 50$ kip is applied to point A, determine its vertical displacement at A.

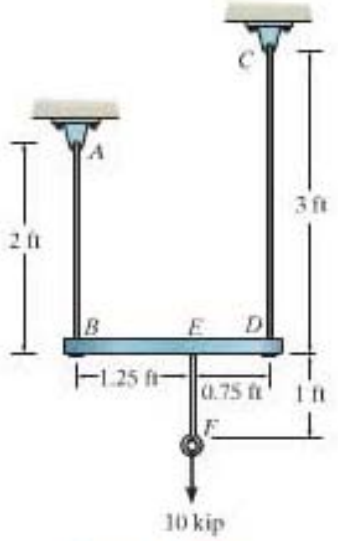
•4–17. The linkage is made of two pin-connected A-36 steel members, each having a cross-sectional area of 1.5 in^2. Determine the magnitude of the force **P** needed to displace point A 0.025 in. downward.



Probs. 4–16/17

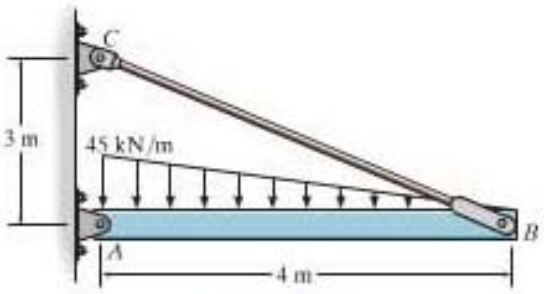
4–18. The assembly consists of two A-36 steel rods and a rigid bar BD. Each rod has a diameter of 0.75 in. If a force of 10 kip is applied to the bar as shown, determine the vertical displacement of the load.

4–19. The assembly consists of two A-36 steel rods and a rigid bar BD. Each rod has a diameter of 0.75 in. If a force of 10 kip is applied to the bar, determine the angle of tilt of the bar.



Probs. 4–18/19

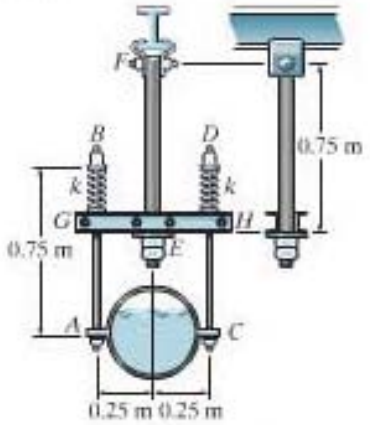
*4–20. The rigid bar is supported by the pin-connected rod CB that has a cross-sectional area of 500 mm^2 and is made of A-36 steel. Determine the vertical displacement of the bar at B when the load is applied.



Prob. 4–20

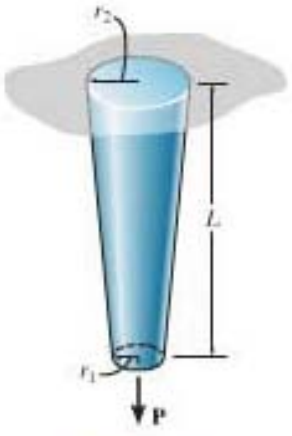
•**4–21.** A spring-supported pipe hanger consists of two springs which are originally unstretched and have a stiffness of $k = 60$ kN/m, three 304 stainless steel rods, AB and CD, which have a diameter of 5 mm, and EF, which has a diameter of 12 mm, and a rigid beam GH. If the pipe and the fluid it carries have a total weight of 4 kN, determine the displacement of the pipe when it is attached to the support.

4–22. A spring-supported pipe hanger consists of two springs, which are originally unstretched and have a stiffness of $k = 60$ kN/m, three 304 stainless steel rods, AB and CD, which have a diameter of 5 mm, and EF, which has a diameter of 12 mm, and a rigid beam GH. If the pipe is displaced 82 mm when it is filled with fluid, determine the weight of the fluid.



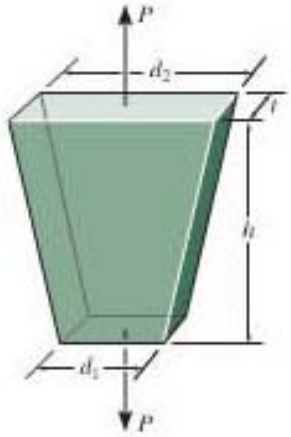
Probs. 4–21/22

4–23. The rod has a slight taper and length L. It is suspended from the ceiling and supports a load **P** at its end. Show that the displacement of its end due to this load is $\delta = PL/(\pi E r_2 r_1)$. Neglect the weight of the material. The modulus of elasticity is E.



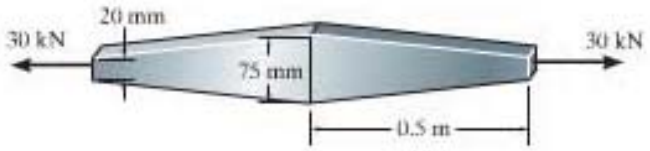
Prob. 4–23

*4–24. Determine the relative displacement of one end of the tapered plate with respect to the other end when it is subjected to an axial load P.



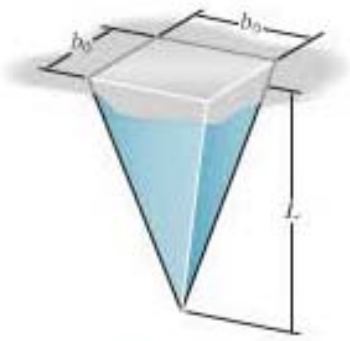
Prob. 4–24

4–25. Determine the elongation of the A-36 steel member when it is subjected to an axial force of 30 kN. The member is 10 mm thick. Use the result of Prob. 4–24.



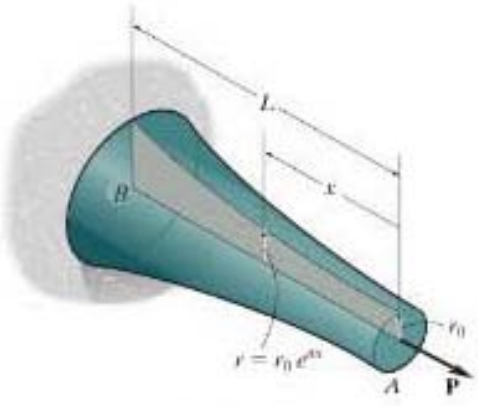
Prob. 4–25

4–26. The casting is made of a material that has a specific weight γ and modulus of elasticity E. If it is formed into a pyramid having the dimensions shown, determine how far its end is displaced due to gravity when it is suspended in the vertical position.



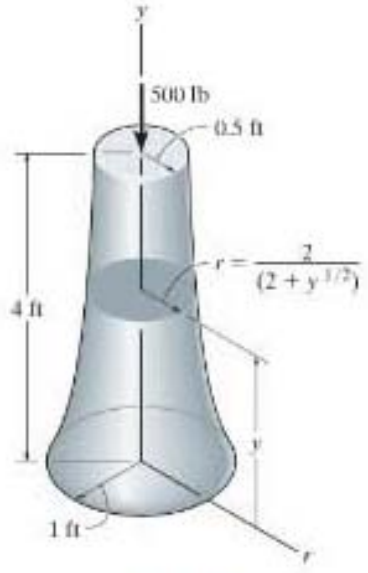
Prob. 4–26

4–27. The circular bar has a variable radius of $r = r_0 e^{ax}$ and is made of a material having a modulus of elasticity of E. Determine the displacement of end A when it is subjected to the axial force **P**.



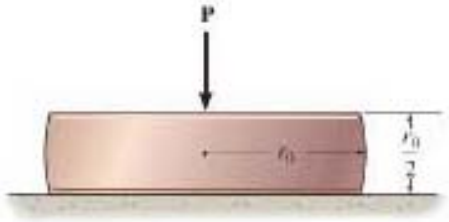
Prob. 4–27

*4–28. The pedestal is made in a shape that has a radius defined by the function $r = 2/(2 + y^{1/2})$ ft, where y is in feet. If the modulus of elasticity for the material is $E = 14(10^3)$ psi, determine the displacement of its top when it supports the 500-lb load.



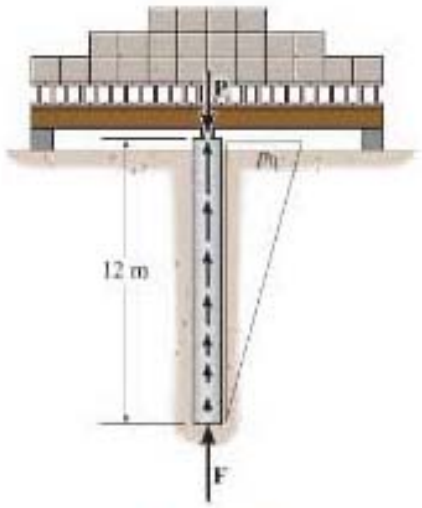
Prob. 4–28

•4–29. The support is made by cutting off the two opposite sides of a sphere that has a radius r_0. If the original height of the support is $r_0/2$, determine how far it shortens when it supports a load **P**. The modulus of elasticity is E.



Prob. 4–29

4–30. The weight of the kentledge exerts an axial force of $P = 1500$ kN on the 300-mm diameter high strength concrete bore pile. If the distribution of the resisting skin friction developed from the interaction between the soil and the surface of the pile is approximated as shown, and the resisting bearing force **F** is required to be zero, determine the maximum intensity p_0 kN/m for equilibrium. Also, find the corresponding elastic shortening of the pile. Neglect the weight of the pile.



Prob. 4–30

4.3 Principle of Superposition

The principle of superposition is often used to determine the stress or displacement at a point in a member when the member is subjected to a complicated loading. By subdividing the loading into components, the ***principle of superposition*** states that the resultant stress or displacement at the point can be determined by algebraically summing the stress or displacement caused by each load component applied separately to the member.

The following two conditions must be satisfied if the principle of superposition is to be applied.

1. ***The loading must be linearly related to the stress or displacement that is to be determined.*** For example, the equations $\sigma = P/A$ and $\delta = PL/AE$ involve a linear relationship between P and σ or δ.

2. ***The loading must not significantly change the original geometry or configuration of the member.*** If significant changes do occur, the direction and location of the applied forces and their moment arms will change. For example, consider the slender rod shown in Fig. 4–10*a*, which is subjected to the load **P**. In Fig. 4–10*b*, **P** is replaced by two of its components, $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. If **P** causes the rod to deflect a large amount, as shown, the moment of the load about its support, Pd, will not equal the sum of the moments of its component loads, $Pd \neq P_1d_1 + P_2d_2$, because $d_1 \neq d_2 \neq d$.

This principle will be used throughout this text whenever we assume Hooke's law applies and also, the bodies that are considered will be such that the loading will produce deformations that are so small that the change in position and direction of the loading will be insignificant and can be neglected.

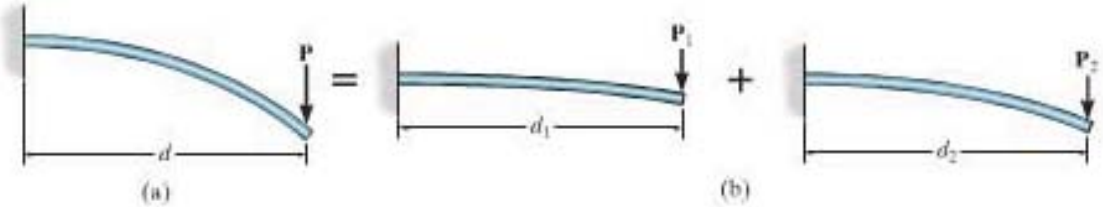


Fig. 4–10

4.4 Statically Indeterminate Axially Loaded Member

Consider the bar shown in Fig. 4–11*a* which is fixed supported at both of its ends. From the free-body diagram, Fig. 4–11*b*, equilibrium requires

$$+\uparrow \Sigma F = 0; \qquad F_B + F_A - P = 0$$

This type of problem is called ***statically indeterminate***, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic condition***. In this case, a suitable compatibility condition would require the displacement of one end of the bar with respect to the other end to be equal to zero, since the end supports are fixed. Hence, the compatibility condition becomes

$$\delta_{A/B} = 0$$

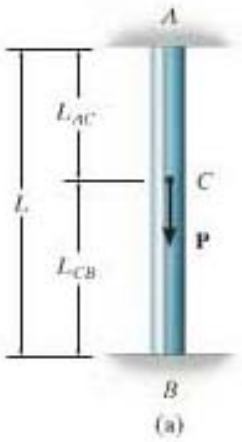
This equation can be expressed in terms of the applied loads by using a *load–displacement relationship*, which depends on the material behavior. For example, if linear-elastic behavior occurs, $\delta = PL/AE$ can be used. Realizing that the internal force in segment AC is $+F_A$, and in segment CB the internal force is $-F_B$, Fig. 4–11*c*, the above equation can be written as

$$\frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE} = 0$$

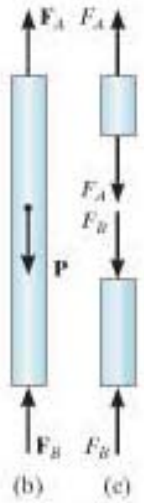
Assuming that AE is constant, then $F_A = F_B(L_{CB}/L_{AC})$, so that using the equilibrium equation, the equations for the reactions become

$$F_A = P\left(\frac{L_{CB}}{L}\right) \quad \text{and} \quad F_B = P\left(\frac{L_{AC}}{L}\right)$$

Since both of these results are positive, the direction of the reactions is shown correctly on the free-body diagram.



(a)



(b) (c)

Fig. 4–11

Important Points

- The *principle of superposition* is sometimes used to simplify stress and displacement problems having complicated loadings. This is done by subdividing the loading into components, then algebracially adding the results.
- Superposition requires that the loading be linearly related to the stress or displacement, and the loading does not significantly change the original geometry of the member.
- A problem is *statically indeterminate* if the equations of equilibrium are not sufficient to determine all the reactions on a member.
- *Compatibility conditions* specify the displacement constraints that occur at the supports or other points on a member.



Most concrete columns are reinforced with steel rods; and since these two materials work together in supporting the applied load, the forces in each material become statically indeterminate.

Procedure for Analysis

The support reactions for statically indeterminate problems are determined by satisfying equilibrium, compatibility, and force-displacement requirements for the member.

Equilibrium.

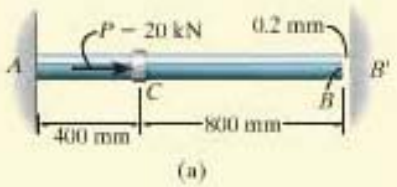
- Draw a free-body diagram of the member in order to identify all the forces that act on it.
- The problem can be classified as statically indeterminate if the number of unknown reactions on the free-body diagram is greater than the number of available equations of equilibrium.
- Write the equations of equilibrium for the member.

Compatibility.

- Consider drawing a displacement diagram in order to investigate the way the member will elongate or contract when subjected to the external loads.
- Express the compatibility conditions in terms of the displacements caused by the loading.
- Use a load–displacement relation, such as $\delta = PL/AE$, to relate the unknown displacements to the reactions.
- Solve the equilibrium and compatibility equations for the reactions. If any of the results has a negative numerical value, it indicates that this force acts in the opposite sense of direction to that indicated on the free-body diagram.

EXAMPLE 4.5

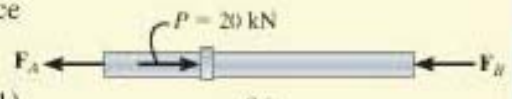
The steel rod shown in Fig. 4–12*a* has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded, there is a gap of 0.2 mm between the wall at B' and the rod. Determine the reactions at A and B' if the rod is subjected to an axial force of $P = 20$ kN as shown. Neglect the size of the collar at C. Take $E_{st} = 200$ GPa.



(a)

SOLUTION

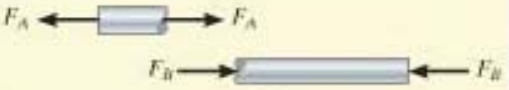
Equilibrium. As shown on the free-body diagram, Fig. 4–12*b*, we will *assume* that force P is large enough to cause the rod's end B to contact the wall at B'. The problem is statically indeterminate since there are two unknowns and only one equation of equilibrium.



(b)

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \qquad -F_A - F_B + 20(10^3)\,\text{N} = 0 \qquad (1)$$

Compatibility. The force P causes point B to move to B', with no further displacement. Therefore the compatibility condition for the rod is



(c)

Fig. 4–12

$$\delta_{B/A} = 0.0002\ \text{m}$$

This displacement can be expressed in terms of the unknown reactions using the load–displacement relationship, Eq. 4–2, applied to segments AC and CB, Fig. 4–12*c*. Working in units of newtons and meters, we have

$$\delta_{B/A} = 0.0002\ \text{m} = \frac{F_A L_{AC}}{AE} - \frac{F_B L_{CB}}{AE}$$

$$0.0002\ \text{m} = \frac{F_A(0.4\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} - \frac{F_B(0.8\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]}$$

or

$$F_A(0.4\ \text{m}) - F_B(0.8\ \text{m}) = 3141.59\ \text{N}\cdot\text{m} \qquad (2)$$

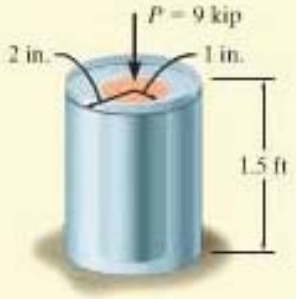
Solving Eqs. 1 and 2 yields

$$F_A = 16.0\ \text{kN} \qquad F_B = 4.05\ \text{kN} \qquad \textit{Ans}$$

Since the answer for F_B is *positive*, indeed end B contacts the wall at B' as originally assumed.

NOTE: If F_B were a negative quantity, the problem would be statically determinate, so that $F_B = 0$ and $F_A = 20$ kN.

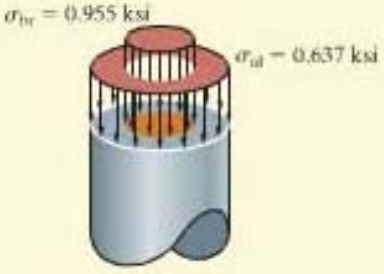
EXAMPLE 4.6



(a)



(b)



(c)

Fig. 4–13

The aluminum post shown in Fig. 4–13*a* is reinforced with a brass core. If this assembly supports an axial compressive load of $P = 9$ kip, applied to the rigid cap, determine the average normal stress in the aluminum and the brass. Take $E_{al} = 10(10^3)$ ksi and $E_{br} = 15(10^3)$ ksi.

SOLUTION

Equilibrium. The free-body diagram of the post is shown in Fig. 4–13*b*. Here the resultant axial force at the base is represented by the unknown components carried by the aluminum, $\mathbf{F}_{al}$, and brass, $\mathbf{F}_{br}$. The problem is statically indeterminate. Why?

Vertical force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad -9\text{ kip} + F_{al} + F_{br} = 0 \tag{1}$$

Compatibility. The rigid cap at the top of the post causes both the aluminum and brass to displace the same amount. Therefore,

$$\delta_{al} = \delta_{br}$$

Using the load–displacement relationships,

$$\frac{F_{al}L}{A_{al}E_{al}} = \frac{F_{br}L}{A_{br}E_{br}}$$

$$F_{al} = F_{br}\left(\frac{A_{al}}{A_{br}}\right)\left(\frac{E_{al}}{E_{br}}\right)$$

$$F_{al} = F_{br}\left[\frac{\pi[(2\text{ in.})^2 - (1\text{ in.})^2]}{\pi(1\text{ in.})^2}\right]\left[\frac{10(10^3)\text{ ksi}}{15(10^3)\text{ ksi}}\right]$$

$$F_{al} = 2F_{br} \tag{2}$$

Solving Eqs. 1 and 2 simultaneously yields

$$F_{al} = 6\text{ kip} \qquad F_{br} = 3\text{ kip}$$

Since the results are positive, indeed the stress will be compressive.

The average normal stress in the aluminum and brass is therefore

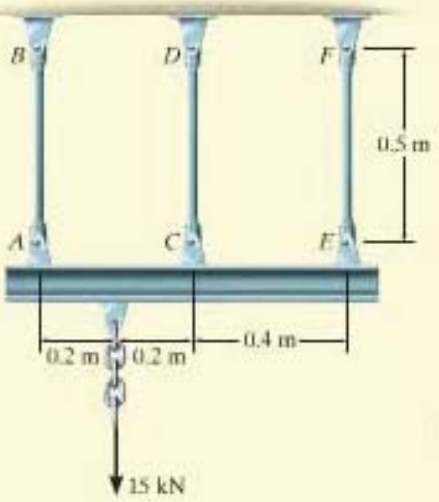
$$\sigma_{al} = \frac{6\text{ kip}}{\pi[(2\text{ in.})^2 - (1\text{ in.})^2]} = 0.637\text{ ksi} \qquad \textit{Ans.}$$

$$\sigma_{br} = \frac{3\text{ kip}}{\pi(1\text{ in.})^2} = 0.955\text{ ksi} \qquad \textit{Ans.}$$

NOTE: Using these results, the stress distributions are shown in Fig. 4–13*c*.

EXAMPLE 4.7

The three A-36 steel bars shown in Fig. 4–14*a* are pin connected to a *rigid* member. If the applied load on the member is 15 kN, determine the force developed in each bar. Bars AB and EF each have a cross-sectional area of 50 mm^2, and bar CD has a cross-sectional area of 30 mm^2.



(a)

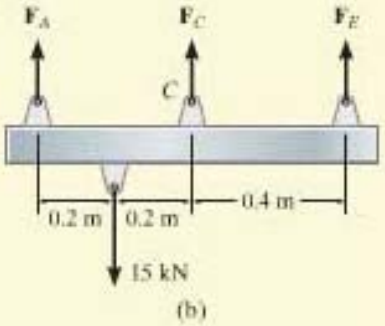
SOLUTION

Equilibrium. The free-body diagram of the rigid member is shown in Fig. 4–14*b*. This problem is statically indeterminate since there are three unknowns and only two available equilibrium equations.

$$+\uparrow \Sigma F_y = 0; \qquad F_A + F_C + F_E - 15\ \text{kN} = 0 \qquad (1)$$

$$\circlearrowleft + \Sigma M_C = 0; \quad -F_A(0.4\ \text{m}) + 15\ \text{kN}(0.2\ \text{m}) + F_E(0.4\ \text{m}) = 0 \qquad (2)$$

Compatibility. The applied load will cause the horizontal line ACE shown in Fig. 4–14*c* to move to the inclined line $A'C'E'$. The displacements of points A, C, and E can be related by similar triangles. Thus the compatibility equation that relates these displacements is



(b)

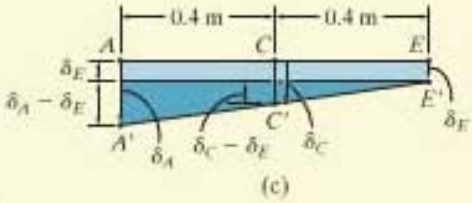
$$\frac{\delta_A - \delta_E}{0.8\ \text{m}} = \frac{\delta_C - \delta_E}{0.4\ \text{m}}$$

$$\delta_C = \frac{1}{2}\delta_A + \frac{1}{2}\delta_E$$

Using the load–displacement relationship, Eq. 4–2, we have

$$\frac{F_C L}{(30\ \text{mm}^2)E_{st}} = \frac{1}{2}\left[\frac{F_A L}{(50\ \text{mm}^2)E_{st}}\right] + \frac{1}{2}\left[\frac{F_E L}{(50\ \text{mm}^2)E_{st}}\right]$$

$$F_C = 0.3F_A + 0.3F_E \qquad (3)$$



(c)

Fig. 4–14

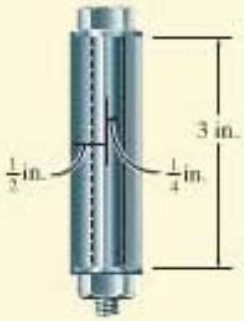
Solving Eqs. 1–3 simultaneously yields

$$F_A = 9.52\ \text{kN} \quad \textit{Ans.}$$

$$F_C = 3.46\ \text{kN} \quad \textit{Ans.}$$

$$F_E = 2.02\ \text{kN} \quad \textit{Ans.}$$

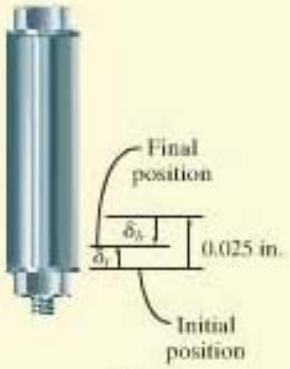
EXAMPLE 4.8



(a)



(b)



(c)

Fig. 4–15

The bolt shown in Fig. 4–15*a* is made of 2014-T6 aluminum alloy and is tightened so it compresses a cylindrical tube made of Am 1004-T61 magnesium alloy. The tube has an outer radius of $\frac{1}{2}$ in., and it is assumed that both the inner radius of the tube and the radius of the bolt are $\frac{1}{4}$ in. The washers at the top and bottom of the tube are considered to be rigid and have a negligible thickness. Initially the nut is hand tightened snugly; then, using a wrench, the nut is further tightened one-half turn. If the bolt has 20 threads per inch, determine the stress in the bolt.

SOLUTION

Equilibrium. The free-body diagram of a section of the bolt and the tube, Fig. 4–15*b*, is considered in order to relate the force in the bolt F_b to that in the tube, F_t. Equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad F_b - F_t = 0 \tag{1}$$

Compatibility. When the nut is tightened on the bolt, the tube will shorten δ_t, and the bolt will *elongate* δ_b, Fig. 4–15*c*. Since the nut undergoes one-half turn, it advances a distance of $(\frac{1}{2})(\frac{1}{20}\text{ in.}) = 0.025$ in. along the bolt. Thus, the compatibility of these displacements requires

$$(+\uparrow) \qquad \delta_t = 0.025 \text{ in.} - \delta_b$$

Taking the moduli of elasticity from the table on the inside back cover, and applying Eq. 4–2, yields

$$\frac{F_t(3 \text{ in.})}{\pi[(0.5 \text{ in.})^2 - (0.25 \text{ in.})^2][6.48(10^3) \text{ ksi}]} = 0.025 \text{ in.} - \frac{F_b(3 \text{ in.})}{\pi(0.25 \text{ in.})^2[10.6(10^3) \text{ ksi}]}$$

$$0.78595F_t = 25 - 1.4414F_b \tag{2}$$

Solving Eqs. 1 and 2 simultaneously, we get

$$F_b = F_t = 11.22 \text{ kip}$$

The stresses in the bolt and tube are therefore

$$\sigma_b = \frac{F_b}{A_b} = \frac{11.22 \text{ kip}}{\pi(0.25 \text{ in.})^2} = 57.2 \text{ ksi} \qquad \textit{Ans.}$$

$$\sigma_t = \frac{F_t}{A_t} = \frac{11.22 \text{ kip}}{\pi[(0.5 \text{ in.})^2 - (0.25 \text{ in.})^2]} = 19.1 \text{ ksi}$$

These stresses are less than the reported yield stress for each material, $(\sigma_Y)_{al} = 60$ ksi and $(\sigma_Y)_{mg} = 22$ ksi (see the inside back cover), and therefore this "elastic" analysis is valid.

4.5 The Force Method of Analysis for Axially Loaded Members

It is also possible to solve statically indeterminate problems by writing the compatibility equation using the principle of superposition. This method of solution is often referred to as the ***flexibility or force method of analysis***. To show how it is applied, consider again the bar in Fig. 4–16*a*. If we choose the support at B as "redundant" and *temporarily* remove its effect on the bar, then the bar will become statically determinate as in Fig. 4–16*b*. By using the principle of superposition, we must add back the unknown redundant load $\mathbf{F}_B$, as shown in Fig. 4–16*c*.

If load $\mathbf{P}$ causes B to be displaced *downward* by an amount δ_P, the reaction $\mathbf{F}_B$ must displace end B of the bar *upward* by an amount δ_B, such that no displacement occurs at B when the two loadings are superimposed. Thus,

$$(+\downarrow) \qquad 0 = \delta_P - \delta_B$$

This equation represents the compatibility equation for displacements at point B, for which we have assumed that displacements are positive downward.

Applying the load–displacement relationship to each case, we have $\delta_P = PL_{AC}/AE$ and $\delta_B = F_B L/AE$. Consequently,

$$0 = \frac{PL_{AC}}{AE} - \frac{F_B L}{AE}$$

$$F_B = P\left(\frac{L_{AC}}{L}\right)$$

From the free-body diagram of the bar, Fig. 4–11*b*, the reaction at A can now be determined from the equation of equilibrium,

$$+\uparrow \Sigma F_y = 0; \qquad P\left(\frac{L_{AC}}{L}\right) + F_A - P = 0$$

Since $L_{CB} = L - L_{AC}$, then

$$F_A = P\left(\frac{L_{CB}}{L}\right)$$

These results are the same as those obtained in Sec. 4.4, except that here we have applied the condition of compatibility to obtain one reaction *and then* the equilibrium condition to obtain the other.

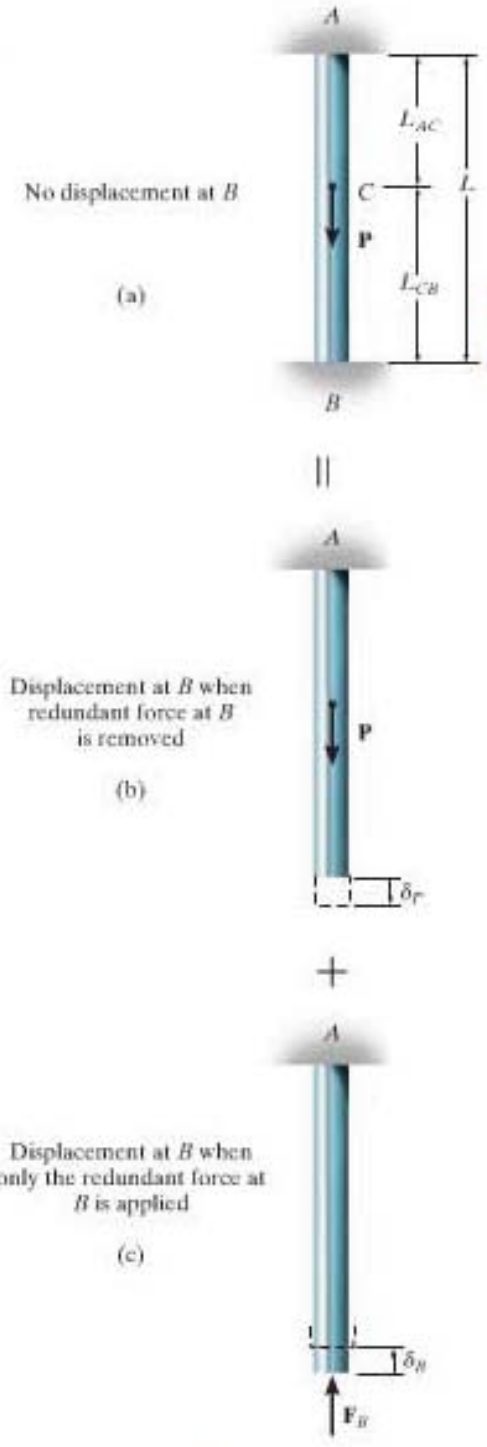


Fig. 4–16

Procedure for Analysis

The force method of analysis requires the following steps.

Compatibility.

- Choose one of the supports as redundant and write the equation of compatibility. To do this, the known displacement at the redundant support, which is usually zero, is equated to the displacement at the support caused *only* by the external loads acting on the member *plus* (vectorially) the displacement at this support caused *only* by the redundant reaction acting on the member.
- Express the external load and redundant displacements in terms of the loadings by using a load–displacement relationship, such as $\delta = PL/AE$.
- Once established, the compatibility equation can then be solved for the magnitude of the redundant force.

Equilibrium.

- Draw a free-body diagram and write the appropriate equations of equilibrium for the member using the calculated result for the redundant. Solve these equations for any other reactions.

EXAMPLE 4.9

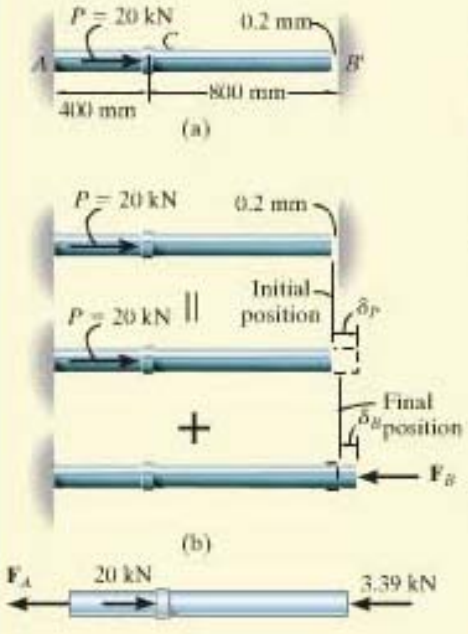


Fig. 4–17

The A-36 steel rod shown in Fig. 4–17*a* has a diameter of 10 mm. It is fixed to the wall at A, and before it is loaded there is a gap between the wall at B' and the rod of 0.2 mm. Determine the reactions at A and B'. Neglect the size of the collar at C. Take $E_{st} = 200$ GPa.

SOLUTION

Compatibility. Here we will consider the support at B' as redundant. Using the principle of superposition, Fig. 4–17*b*, we have

$$(\stackrel{+}{\rightarrow}) \qquad 0.0002\ \text{m} = \delta_P - \delta_B \qquad (1)$$

The deflections δ_P and δ_B are determined from Eq. 4–2.

$$\delta_P = \frac{PL_{AC}}{AE} = \frac{[20(10^3)\ \text{N}](0.4\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} = 0.5093(10^{-3})\ \text{m}$$

$$\delta_B = \frac{F_B L_{AB}}{AE} = \frac{F_B(1.20\ \text{m})}{\pi(0.005\ \text{m})^2[200(10^9)\ \text{N/m}^2]} = 76.3944(10^{-9})F_B$$

Substituting into Eq. 1, we get

$$0.0002\ \text{m} = 0.5093(10^{-3})\ \text{m} - 76.3944(10^{-9})F_B$$

$$F_B = 4.05(10^3)\ \text{N} = 4.05\ \text{kN} \qquad \textit{Ans.}$$

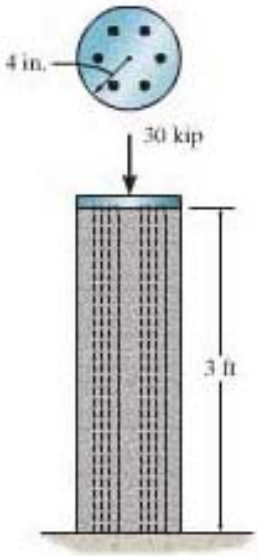
Equilibrium. From the free-body diagram, Fig. 4–17*c*,

$$\stackrel{+}{\rightarrow} \Sigma F_x = 0; \quad -F_A + 20\ \text{kN} - 4.05\ \text{kN} = 0 \quad F_A = 16.0\ \text{kN} \qquad \textit{Ans.}$$

PROBLEMS

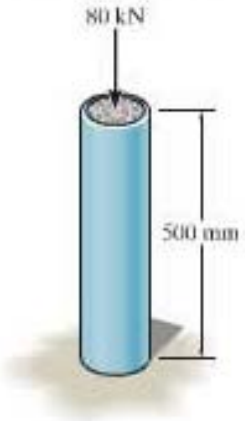
4–31. The column is constructed from high-strength concrete and six A-36 steel reinforcing rods. If it is subjected to an axial force of 30 kip, determine the average normal stress in the concrete and in each rod. Each rod has a diameter of 0.75 in.

***4–32.** The column is constructed from high-strength concrete and six A-36 steel reinforcing rods. If it is subjected to an axial force of 30 kip, determine the required diameter of each rod so that one-fourth of the load is carried by the concrete and three-fourths by the steel.



Probs. 4–31/32

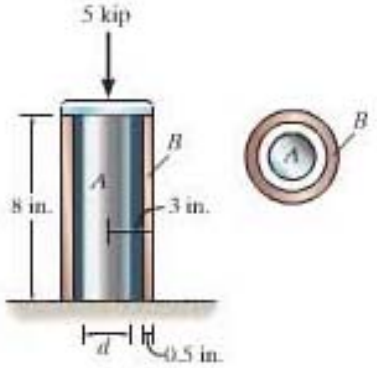
•4–33. The steel pipe is filled with concrete and subjected to a compressive force of 80 kN. Determine the average normal stress in the concrete and the steel due to this loading. The pipe has an outer diameter of 80 mm and an inner diameter of 70 mm. $E_{st} = 200$ GPa, $E_c = 24$ GPa.



Prob. 4–33

4–34. The 304 stainless steel post A has a diameter of $d = 2$ in. and is surrounded by a red brass C83400 tube B. Both rest on the rigid surface. If a force of 5 kip is applied to the rigid cap, determine the average normal stress developed in the post and the tube.

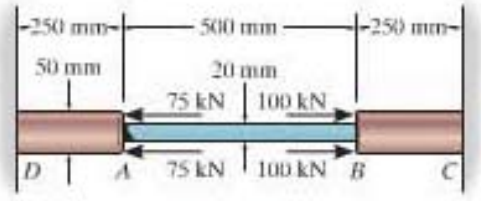
4–35. The 304 stainless steel post A is surrounded by a red brass C83400 tube B. Both rest on the rigid surface. If a force of 5 kip is applied to the rigid cap, determine the required diameter d of the steel post so that the load is shared equally between the post and tube.



Probs. 4–34/35

***4–36.** The composite bar consists of a 20-mm-diameter A-36 steel segment AB and 50-mm-diameter red brass C83400 end segments DA and CB. Determine the average normal stress in each segment due to the applied load.

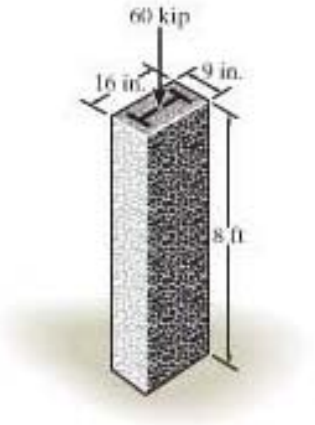
•4–37. The composite bar consists of a 20-mm-diameter A-36 steel segment AB and 50-mm-diameter red brass C83400 end segments DA and CB. Determine the displacement of A with respect to B due to the applied load.



Probs. 4–36/37

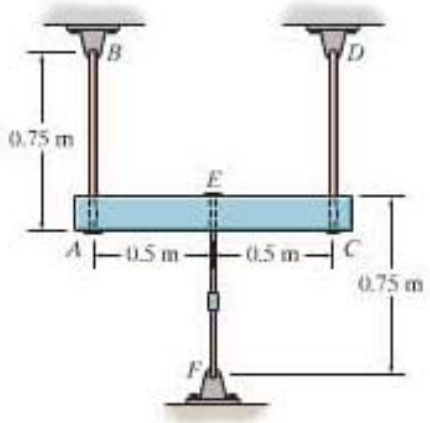
4–38. The A-36 steel column, having a cross-sectional area of 18 in^2, is encased in high-strength concrete as shown. If an axial force of 60 kip is applied to the column, determine the average compressive stress in the concrete and in the steel. How far does the column shorten? It has an original length of 8 ft.

4–39. The A-36 steel column is encased in high-strength concrete as shown. If an axial force of 60 kip is applied to the column, determine the required area of the steel so that the force is shared equally between the steel and concrete. How far does the column shorten? It has an original length of 8 ft.



Probs. 4–38/39

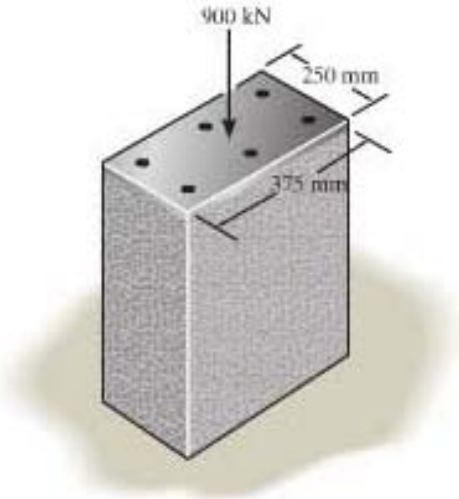
*4–40. The rigid member is held in the position shown by three A-36 steel tie rods. Each rod has an unstretched length of 0.75 m and a cross-sectional area of 125 mm^2. Determine the forces in the rods if a turnbuckle on rod *EF* undergoes one full turn. The lead of the screw is 1.5 mm. Neglect the size of the turnbuckle and assume that it is rigid. *Note:* The lead would cause the rod, when *unloaded*, to shorten 1.5 mm when the turnbuckle is rotated one revolution.



Prob. 4–40

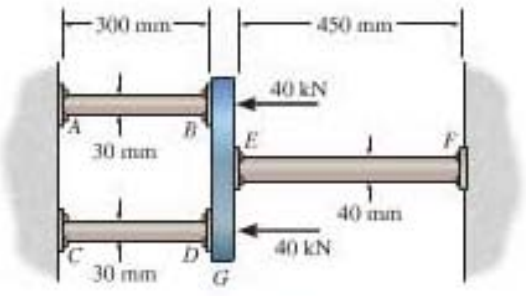
•4–41. The concrete post is reinforced using six steel reinforcing rods, each having a diameter of 20 mm. Determine the stress in the concrete and the steel if the post is subjected to an axial load of 900 kN. E_{st} = 200 GPa, E_c = 25 GPa.

4–42. The post is constructed from concrete and six A-36 steel reinforcing rods. If it is subjected to an axial force of 900 kN, determine the required diameter of each rod so that one-fifth of the load is carried by the steel and four-fifths by the concrete. E_{st} = 200 GPa, E_c = 25 GPa.



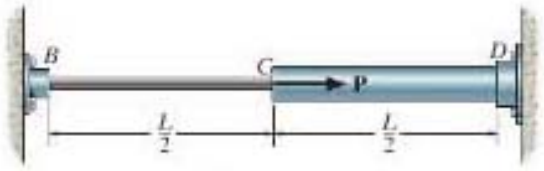
Probs. 4–41/42

4–43. The assembly consists of two red brass C83400 copper alloy rods *AB* and *CD* of diameter 30 mm, a stainless 304 steel alloy rod *EF* of diameter 40 mm, and a rigid cap *G*. If the supports at *A*, *C* and *F* are rigid, determine the average normal stress developed in rods *AB*, *CD* and *EF*.



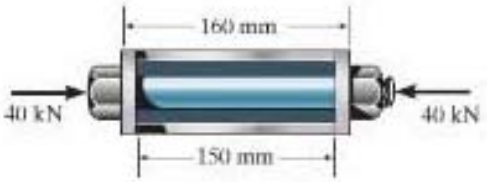
Prob. 4–43

*4–44. The two pipes are made of the same material and are connected as shown. If the cross-sectional area of BC is A and that of CD is $2A$, determine the reactions at B and D when a force $\mathbf{P}$ is applied at the junction C.



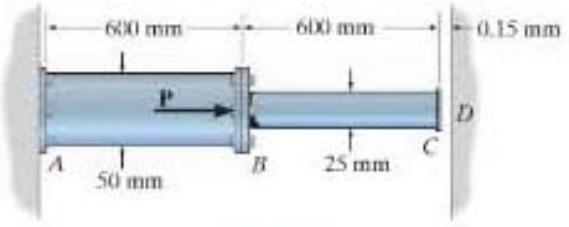
Prob. 4–44

•4–45. The bolt has a diameter of 20 mm and passes through a tube that has an inner diameter of 50 mm and an outer diameter of 60 mm. If the bolt and tube are made of A-36 steel, determine the normal stress in the tube and bolt when a force of 40 kN is applied to the bolt. Assume the end caps are rigid.



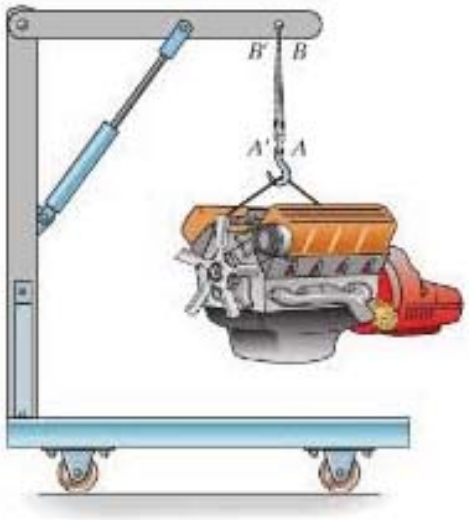
Prob. 4–45

4–46. If the gap between C and the rigid wall at D is initially 0.15 mm, determine the support reactions at A and D when the force $\mathbf{P} = 200$ kN is applied. The assembly is made of A36 steel.



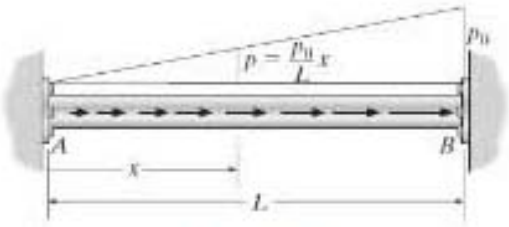
Prob. 4–46

4–47. Two A-36 steel wires are used to support the 650-lb engine. Originally, AB is 32 in. long and $A'B'$ is 32.008 in. long. Determine the force supported by each wire when the engine is suspended from them. Each wire has a cross-sectional area of 0.01 in^2.



Prob. 4–47

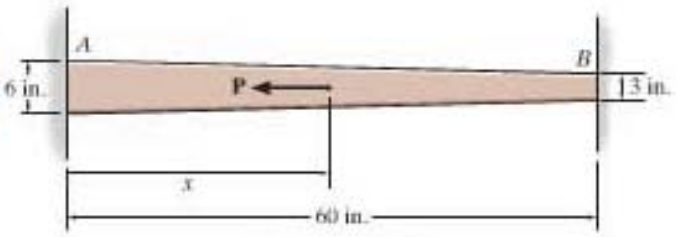
*4–48. Rod AB has a diameter d and fits snugly between the rigid supports at A and B when it is unloaded. The modulus of elasticity is E. Determine the support reactions at A and B if the rod is subjected to the linearly distributed axial load.



Prob. 4–48

•**4–49.** The tapered member is fixed connected at its ends A and B and is subjected to a load $P = 7$ kip at $x = 30$ in. Determine the reactions at the supports. The material is 2 in. thick and is made from 2014-T6 aluminum.

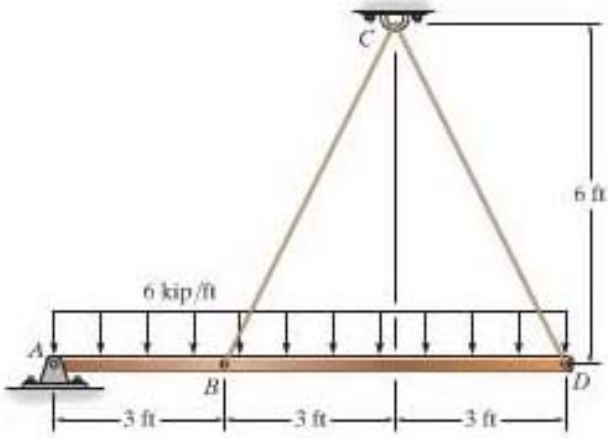
4–50. The tapered member is fixed connected at its ends A and B and is subjected to a load **P**. Determine the location x of the load and its greatest magnitude so that the average normal stress in the bar does not exceed $\sigma_{allow} = 4$ ksi. The member is 2 in. thick.



Probs. 4–49/50

4–51. The rigid bar supports the uniform distributed load of 6 kip/ft. Determine the force in each cable if each cable has a cross-sectional area of 0.05 in^2, and $E = 31(10^3)$ ksi.

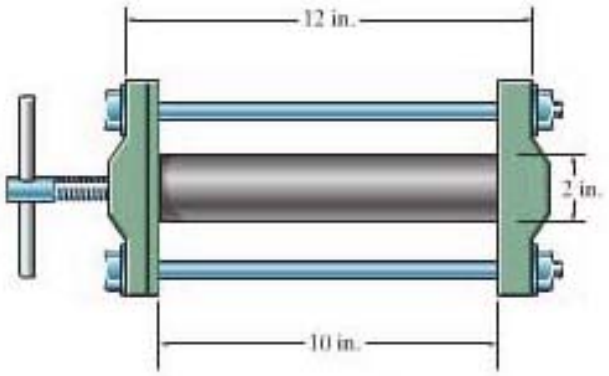
*4–52. The rigid bar is originally horizontal and is supported by two cables each having a cross-sectional area of 0.05 in^2, and $E = 31(10^3)$ ksi. Determine the slight rotation of the bar when the uniform load is applied.



Probs. 4–51/52

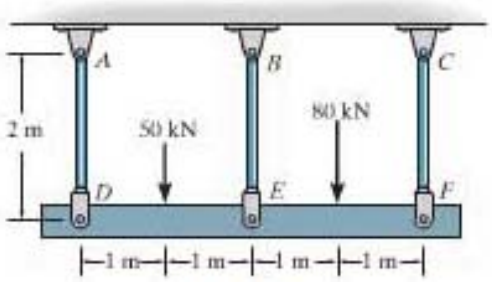
•**4–53.** The press consists of two rigid heads that are held together by the two A-36 steel $\frac{1}{2}$-in.-diameter rods. A 6061-T6-solid-aluminum cylinder is placed in the press and the screw is adjusted so that it just presses up against the cylinder. If it is then tightened one-half turn, determine the average normal stress in the rods and in the cylinder. The single-threaded screw on the bolt has a lead of 0.01 in. *Note*: The lead represents the distance the screw advances along its axis for one complete turn of the screw.

4–54. The press consists of two rigid heads that are held together by the two A-36 steel $\frac{1}{2}$-in.-diameter rods. A 6061-T6-solid-aluminum cylinder is placed in the press and the screw is adjusted so that it just presses up against the cylinder. Determine the angle through which the screw can be turned before the rods or the specimen begin to yield. The single-threaded screw on the bolt has a lead of 0.01 in. *Note*: The lead represents the distance the screw advances along its axis for one complete turn of the screw.



Probs. 4–53/54

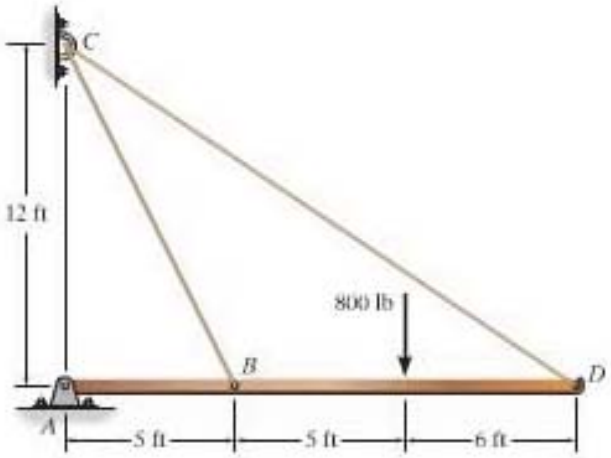
4–55. The three suspender bars are made of A-36 steel and have equal cross-sectional areas of 450 mm^2. Determine the average normal stress in each bar if the rigid beam is subjected to the loading shown.



Prob. 4–55

*4–56. The rigid bar supports the 800-lb load. Determine the normal stress in each A-36 steel cable if each cable has a cross-sectional area of 0.04 in^2.

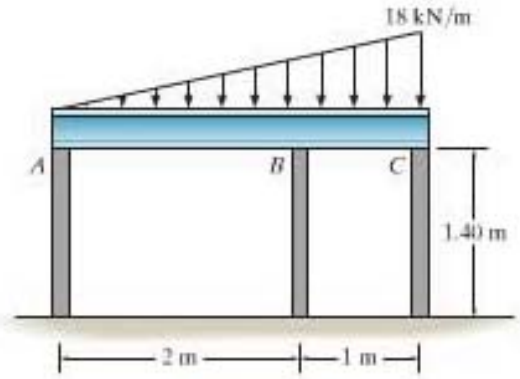
•4–57. The rigid bar is originally horizontal and is supported by two A-36 steel cables each having a cross-sectional area of 0.04 in^2. Determine the rotation of the bar when the 800-lb load is applied.



Probs. 4–56/57

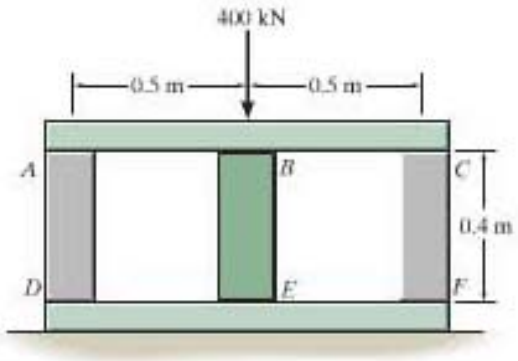
4–58. The horizontal beam is assumed to be rigid and supports the distributed load shown. Determine the vertical reactions at the supports. Each support consists of a wooden post having a diameter of 120 mm and an unloaded (original) length of 1.40 m. Take $E_w = 12$ GPa.

4–59. The horizontal beam is assumed to be rigid and supports the distributed load shown. Determine the angle of tilt of the beam after the load is applied. Each support consists of a wooden post having a diameter of 120 mm and an unloaded (original) length of 1.40 m. Take $E_w = 12$ GPa.



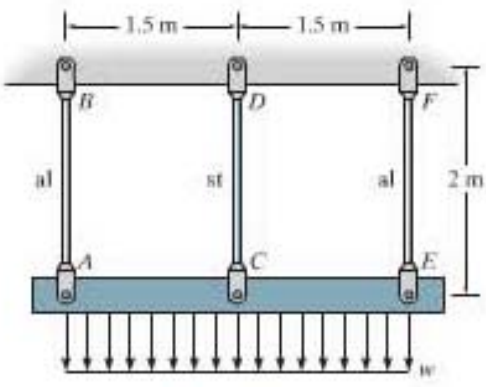
Probs. 4–58/59

*4–60. The assembly consists of two posts AD and CF made of A-36 steel and having a cross-sectional area of 1000 mm^2, and a 2014-T6 aluminum post BE having a cross-sectional area of 1500 mm^2. If a central load of 400 kN is applied to the rigid cap, determine the normal stress in each post. There is a small gap of 0.1 mm between the post BE and the rigid member ABC.



Prob. 4–60

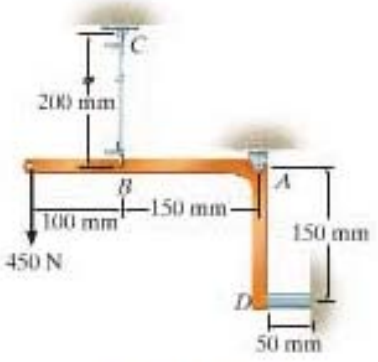
•4–61. The distributed loading is supported by the three suspender bars. AB and EF are made of aluminum and CD is made of steel. If each bar has a cross-sectional area of 450 mm^2, determine the maximum intensity w of the distributed loading so that an allowable stress of $(\sigma_{allow})_{st} = 180$ MPa in the steel and $(\sigma_{allow})_{al} = 94$ MPa in the aluminum is not exceeded. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa. Assume ACE is rigid.



Prob. 4–61

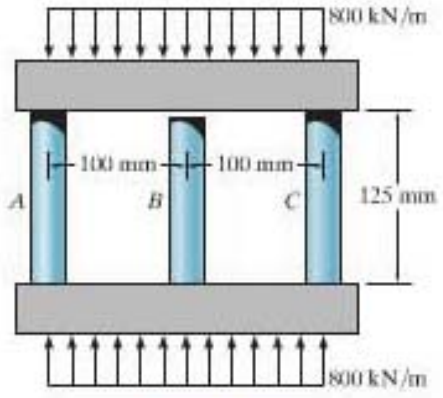
4–62. The rigid link is supported by a pin at A, a steel wire BC having an unstretched length of 200 mm and cross-sectional area of 22.5 mm^2, and a short aluminum block having an unloaded length of 50 mm and cross-sectional area of 40 mm^2. If the link is subjected to the vertical load shown, determine the average normal stress in the wire and the block. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.

4–63. The rigid link is supported by a pin at A, a steel wire BC having an unstretched length of 200 mm and cross-sectional area of 22.5 mm^2, and a short aluminum block having an unloaded length of 50 mm and cross-sectional area of 40 mm^2. If the link is subjected to the vertical load shown, determine the rotation of the link about the pin A. Report the answer in radians. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa.



Probs. 4–62/63

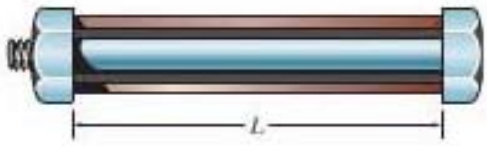
*4–64. The center post B of the assembly has an original length of 124.7 mm, whereas posts A and C have a length of 125 mm. If the caps on the top and bottom can be considered rigid, determine the average normal stress in each post. The posts are made of aluminum and have a cross-sectional area of 400 mm^2. $E_{al} = 70$ GPa.



Prob. 4–64

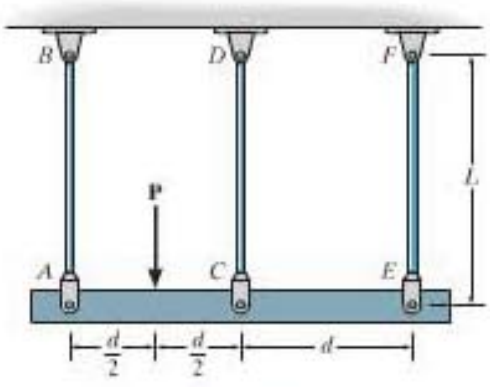
•4–65. The assembly consists of an A-36 steel bolt and a C83400 red brass tube. If the nut is drawn up snug against the tube so that $L = 75$ mm, then turned an additional amount so that it advances 0.02 mm on the bolt, determine the force in the bolt and the tube. The bolt has a diameter of 7 mm and the tube has a cross-sectional area of 100 mm^2.

4–66. The assembly consists of an A-36 steel bolt and a C83400 red brass tube. The nut is drawn up snug against the tube so that $L = 75$ mm. Determine the maximum additional amount of advance of the nut on the bolt so that none of the material will yield. The bolt has a diameter of 7 mm and the tube has a cross-sectional area of 100 mm^2.



Probs. 4–65/66

4–67. The three suspender bars are made of the same material and have equal cross-sectional areas A. Determine the average normal stress in each bar if the rigid beam ACE is subjected to the force **P**.



Prob. 4–67

4.6 Thermal Stress

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. Ordinarily this expansion or contraction is *linearly* related to the temperature increase or decrease that occurs. If this is the case, and the material is homogeneous and isotropic, it has been found from experiment that the displacement of a member having a length L can be calculated using the formula

$$\delta_T = \alpha \Delta T L \tag{4–4}$$



Most traffic bridges are designed with expansion joints to accommodate the thermal movement of the deck and thus avoid any thermal stress.

where

α = a property of the material, referred to as the ***linear coefficient of thermal expansion***. The units measure strain per degree of temperature. They are 1/°F (Fahrenheit) in the FPS system, and 1/°C (Celsius) or 1/K (Kelvin) in the SI system. Typical values are given on the inside back cover

ΔT = the algebraic change in temperature of the member

L = the original length of the member

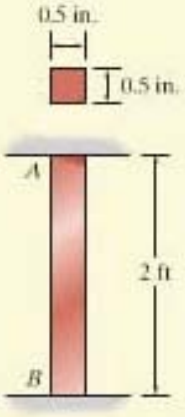
δ_T = the algebraic change in the length of the member

The change in length of a *statically determinate* member can easily be calculated using Eq. 4–4, since the member is free to expand or contract when it undergoes a temperature change. However, in a *statically indeterminate* member, these thermal displacements will be constrained by the supports, thereby producing ***thermal stresses*** that must be considered in design. Determining these thermal stresses is possible using the methods outlined in the previous sections. The following examples illustrate some applications.



Long extensions of ducts and pipes that carry fluids are subjected to variations in climate that will cause them to expand and contract. Expansion joints, such as the one shown, are used to mitigate thermal stress in the material.

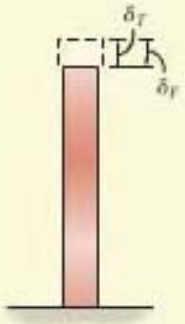
EXAMPLE 4.10



(a)



(b)



(c)

Fig. 4–18

The A-36 steel bar shown in Fig. 4–18*a* is constrained to just fit between two fixed supports when $T_1 = 60°F$. If the temperature is raised to $T_2 = 120°F$, determine the average normal thermal stress developed in the bar.

SOLUTION

Equilibrium. The free-body diagram of the bar is shown in Fig. 4–18*b*. Since there is no external load, the force at A is equal but opposite to the force at B; that is,

$$+\uparrow \Sigma F_y = 0; \qquad F_A = F_B = F$$

The problem is statically indeterminate since this force cannot be determined from equilibrium.

Compatibility. Since $\delta_{A/B} = 0$, the thermal displacement δ_T at A that occurs, Fig. 4–18*c*, is counteracted by the force **F** that is required to push the bar δ_F back to its original position. The compatibility condition at A becomes

$$(+\uparrow) \qquad \delta_{A/B} = 0 = \delta_T - \delta_F$$

Applying the thermal and load–displacement relationships, we have

$$0 = \alpha \Delta T L - \frac{FL}{AE}$$

Thus, from the data on the inside back cover,

$$\begin{aligned} F &= \alpha \Delta T AE \\ &= [6.60(10^{-6})/°F](120°F - 60°F)(0.5\text{ in.})^2[29(10^3)\text{ kip/in}^2] \\ &= 2.871\text{ kip} \end{aligned}$$

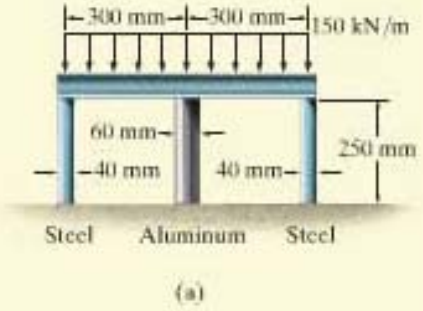
Since **F** also represents the internal axial force within the bar, the average normal compressive stress is thus

$$\sigma = \frac{F}{A} = \frac{2.871\text{ kip}}{(0.5\text{ in.})^2} = 11.5\text{ ksi} \qquad \textit{Ans.}$$

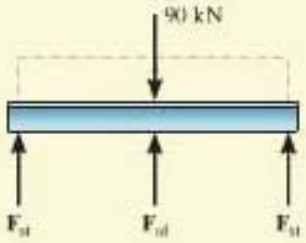
NOTE: From the magnitude of **F**, it should be apparent that changes in temperature can cause large reaction forces in statically indeterminate members.

EXAMPLE 4.11

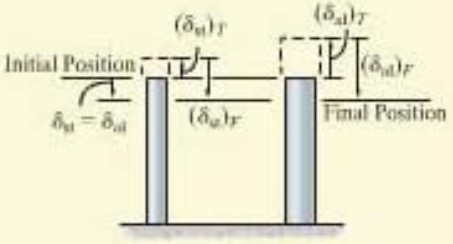
The rigid beam shown in Fig. 4–19*a* is fixed to the top of the three posts made of A-36 steel and 2014-T6 aluminum. The posts each have a length of 250 mm when no load is applied to the beam, and the temperature is $T_1 = 20°C$. Determine the force supported by each post if the bar is subjected to a uniform distributed load of 150 kN/m and the temperature is raised to $T_2 = 80°C$.



(a)



(b)



(c)

Fig. 4–19

SOLUTION

Equilibrium. The free-body diagram of the beam is shown in Fig. 4–19*b*. Moment equilibrium about the beam's center requires the forces in the steel posts to be equal. Summing forces on the free-body diagram, we have

$$+\uparrow \Sigma F_y = 0; \qquad 2F_{st} + F_{al} - 90(10^3)\,\text{N} = 0 \tag{1}$$

Compatibility. Due to load, geometry, and material symmetry, the top of each post is displaced by an equal amount. Hence,

$$(+\downarrow) \qquad \delta_{st} = \delta_{al} \tag{2}$$

The final position of the top of each post is equal to its displacement caused by the temperature increase, plus its displacement caused by the internal axial compressive force, Fig. 4–19*c*. Thus, for the steel and aluminum post, we have

$$(+\downarrow) \qquad \delta_{st} = -(\delta_{st})_T + (\delta_{st})_F$$

$$(+\downarrow) \qquad \delta_{al} = -(\delta_{al})_T + (\delta_{al})_F$$

Applying Eq. 2 gives

$$-(\delta_{st})_T + (\delta_{st})_F = -(\delta_{al})_T + (\delta_{al})_F$$

Using Eqs. 4–2 and 4–4 and the material properties on the inside back cover, we get

$$-[12(10^{-6})/°C](80°C - 20°C)(0.250\,\text{m}) + \frac{F_{st}(0.250\,\text{m})}{\pi(0.020\,\text{m})^2[200(10^9)\,\text{N/m}^2]}$$

$$= -[23(10^{-6})/°C](80°C - 20°C)(0.250\,\text{m}) + \frac{F_{al}(0.250\,\text{m})}{\pi(0.030\,\text{m})^2[73.1(10^9)\,\text{N/m}^2]}$$

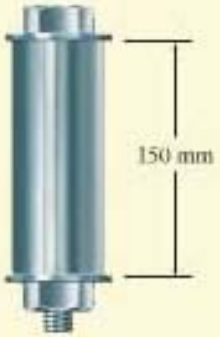
$$F_{st} = 1.216F_{al} - 165.9(10^3) \tag{3}$$

To be *consistent*, all numerical data has been expressed in terms of newtons, meters, and degrees Celsius. Solving Eqs. 1 and 3 simultaneously yields

$$F_{st} = -16.4\,\text{kN} \quad F_{al} = 123\,\text{kN} \qquad \textit{Ans.}$$

The negative value for F_{st} indicates that this force acts opposite to that shown in Fig. 4–19*b*. In other words, the steel posts are in tension and the aluminum post is in compression.

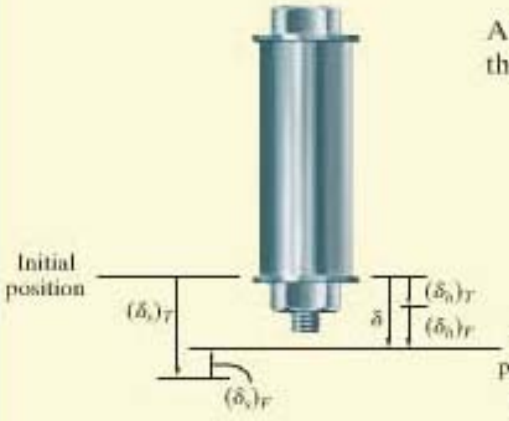
EXAMPLE 4.12



(a)



(b)



(c)

Fig. 4–20

A 2014-T6 aluminum tube having a cross-sectional area of 600 mm^2 is used as a sleeve for an A-36 steel bolt having a cross-sectional area of 400 mm^2, Fig. 4–20*a*. When the temperature is $T_1 = 15°C$, the nut holds the assembly in a snug position such that the axial force in the bolt is negligible. If the temperature increases to $T_2 = 80°C$, determine the force in the bolt and sleeve.

SOLUTION

Equilibrium. The free-body diagram of a top segment of the assembly is shown in Fig. 4–20*b*. The forces F_b and F_s are produced since the sleeve has a higher coefficient of thermal expansion than the bolt, and therefore the sleeve will expand more when the temperature is increased. It is required that

$$+\uparrow \Sigma F_y = 0; \qquad F_s = F_b \tag{1}$$

Compatibility. The temperature increase causes the sleeve and bolt to expand $(\delta_s)_T$ and $(\delta_b)_T$, Fig. 4–20*c*. However, the redundant forces F_b and F_s elongate the bolt and shorten the sleeve. Consequently, the end of the assembly reaches a final position, which is not the same as its initial position. Hence, the compatibility condition becomes

$$(+\downarrow) \qquad \delta = (\delta_b)_T + (\delta_b)_F = (\delta_s)_T - (\delta_s)_F$$

Applying Eqs. 4–2 and 4–4, and using the mechanical properties from the table on the inside back cover, we have

$$[12(10^{-6})/°C](80°C - 15°C)(0.150\ m) + \frac{F_b(0.150\ m)}{(400\ mm^2)(10^{-6}\ m^2/mm^2)[200(10^9)\ N/m^2]} = [23(10^{-6})/°C](80°C - 15°C)(0.150\ m) - \frac{F_s(0.150\ m)}{(600\ mm^2)(10^{-6}\ m^2/mm^2)[73.1(10^9)\ N/m^2]}$$

Using Eq. 1 and solving gives

$$F_s = F_b = 20.3\ kN \qquad \textit{Ans.}$$

NOTE: Since linear elastic material behavior was assumed in this analysis, the average normal stresses should be checked to make sure that they do not exceed the proportional limits for the material.

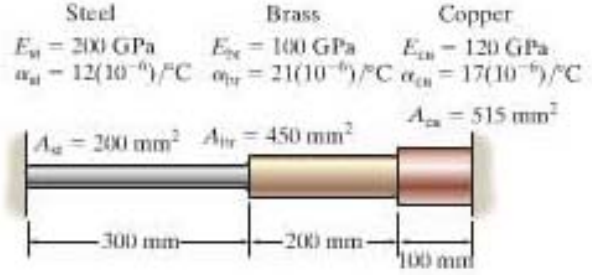
PROBLEMS

*4–68. A steel surveyor's tape is to be used to measure the length of a line. The tape has a rectangular cross section of 0.05 in. by 0.2 in. and a length of 100 ft when $T_1 = 60°F$ and the tension or pull on the tape is 20 lb. Determine the true length of the line if the tape shows the reading to be 463.25 ft when used with a pull of 35 lb at $T_2 = 90°F$. The ground on which it is placed is flat. $\alpha_{st} = 9.60(10^{-6})/°F$, $E_{st} = 29(10^3)$ ksi.



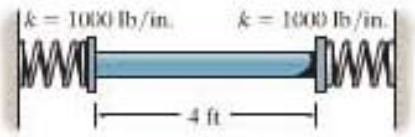
Prob. 4–68

•4–69. Three bars each made of different materials are connected together and placed between two walls when the temperature is $T_1 = 12°C$. Determine the force exerted on the (rigid) supports when the temperature becomes $T_2 = 18°C$. The material properties and cross-sectional area of each bar are given in the figure.



Prob. 4–69

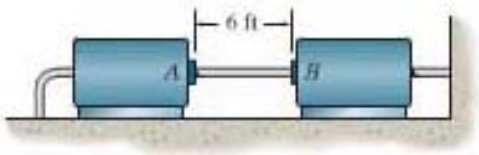
4–70. The rod is made of A-36 steel and has a diameter of 0.25 in. If the rod is 4 ft long when the springs are compressed 0.5 in. and the temperature of the rod is $T = 40°F$, determine the force in the rod when its temperature is $T = 160°F$.



Prob. 4–70

4–71. A 6-ft-long steam pipe is made of A-36 steel with $\sigma_Y = 40$ ksi. It is connected directly to two turbines A and B as shown. The pipe has an outer diameter of 4 in. and a wall thickness of 0.25 in. The connection was made at $T_1 = 70°F$. If the turbines' points of attachment are assumed rigid, determine the force the pipe exerts on the turbines when the steam and thus the pipe reach a temperature of $T_2 = 275°F$.

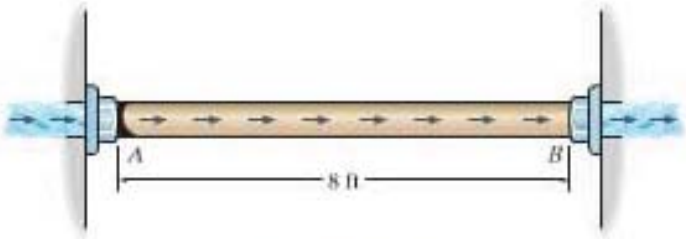
*4–72. A 6-ft-long steam pipe is made of A-36 steel with $\sigma_Y = 40$ ksi. It is connected directly to two turbines A and B as shown. The pipe has an outer diameter of 4 in. and a wall thickness of 0.25 in. The connection was made at $T_1 = 70°F$. If the turbines' points of attachment are assumed to have a stiffness of $k = 80(10^3)$ kip/in., determine the force the pipe exerts on the turbines when the steam and thus the pipe reach a temperature of $T_2 = 275°F$.



Probs. 4–71/72

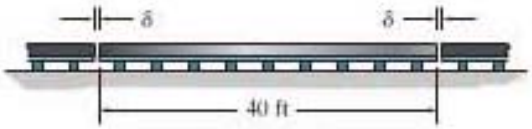
•4–73. The pipe is made of A-36 steel and is connected to the collars at A and B. When the temperature is 60° F, there is no axial load in the pipe. If hot gas traveling through the pipe causes its temperature to rise by $\Delta T = (40 + 15x)°F$, where x is in feet, determine the average normal stress in the pipe. The inner diameter is 2 in., the wall thickness is 0.15 in.

4–74. The bronze C86100 pipe has an inner radius of 0.5 in. and a wall thickness of 0.2 in. If the gas flowing through it changes the temperature of the pipe uniformly from $T_A = 200°F$ at A to $T_B = 60°F$ at B, determine the axial force it exerts on the walls. The pipe was fitted between the walls when $T = 60°F$.



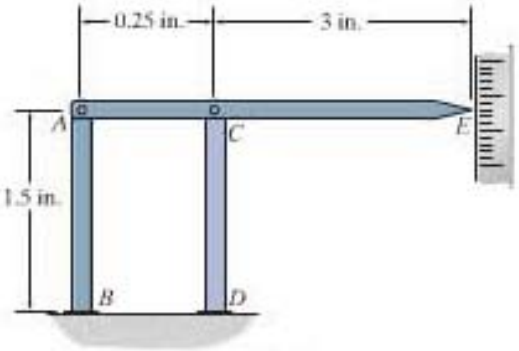
Probs. 4–73/74

4–75. The 40-ft-long A-36 steel rails on a train track are laid with a small gap between them to allow for thermal expansion. Determine the required gap δ so that the rails just touch one another when the temperature is increased from $T_1 = -20°F$ to $T_2 = 90°F$. Using this gap, what would be the axial force in the rails if the temperature were to rise to $T_3 = 110°F$? The cross-sectional area of each rail is 5.10 in^2.



Prob. 4–75

***4–76.** The device is used to measure a change in temperature. Bars AB and CD are made of A-36 steel and 2014-T6 aluminum alloy respectively. When the temperature is at 75°F, ACE is in the horizontal position. Determine the vertical displacement of the pointer at E when the temperature rises to 150°F.



Prob. 4–76

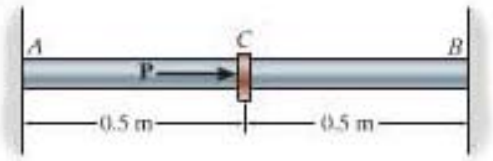
•4–77. The bar has a cross-sectional area A, length L, modulus of elasticity E, and coefficient of thermal expansion α. The temperature of the bar changes uniformly along its length from T_A at A to T_B at B so that at any point x along the bar $T = T_A + x(T_B - T_A)/L$. Determine the force the bar exerts on the rigid walls. Initially no axial force is in the bar and the bar has a temperature of T_A.



Prob. 4–77

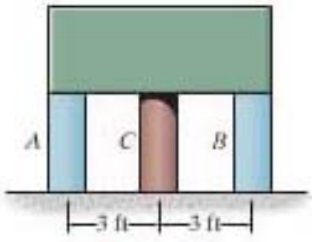
4–78. The A-36 steel rod has a diameter of 50 mm and is lightly attached to the rigid supports at A and B when $T_1 = 80°C$. If the temperature becomes $T_2 = 20°C$ and an axial force of $P = 200$ kN is applied to its center, determine the reactions at A and B.

4–79. The A-36 steel rod has a diameter of 50 mm and is lightly attached to the rigid supports at A and B when $T_1 = 50°C$. Determine the force P that must be applied to the collar at its midpoint so that, when $T_2 = 30°C$, the reaction at B is zero.



Probs. 4–78/79

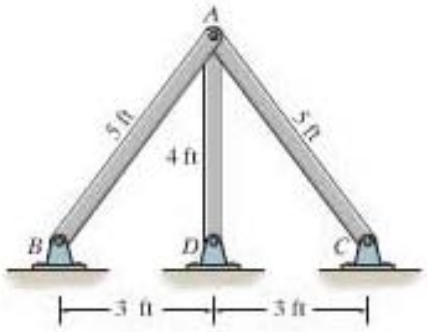
***4–80.** The rigid block has a weight of 80 kip and is to be supported by posts A and B, which are made of A-36 steel, and the post C, which is made of C83400 red brass. If all the posts have the same original length before they are loaded, determine the average normal stress developed in each post when post C is heated so that its temperature is increased by 20°F. Each post has a cross-sectional area of 8 in^2.



Prob. 4–80

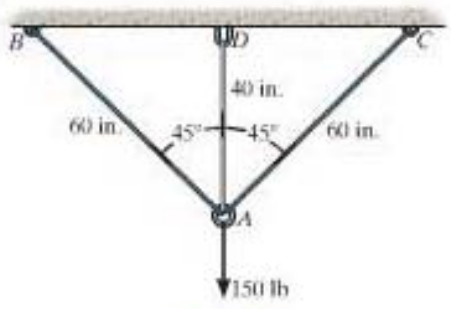
•4–81. The three bars are made of A-36 steel and form a pin-connected truss. If the truss is constructed when $T_1 = 50°F$, determine the force in each bar when $T_2 = 110°F$. Each bar has a cross-sectional area of 2 in^2.

4–82. The three bars are made of A-36 steel and form a pin-connected truss. If the truss is constructed when $T_1 = 50°F$, determine the vertical displacement of joint A when $T_2 = 150°F$. Each bar has a cross-sectional area of 2 in^2.



Probs. 4–81/82

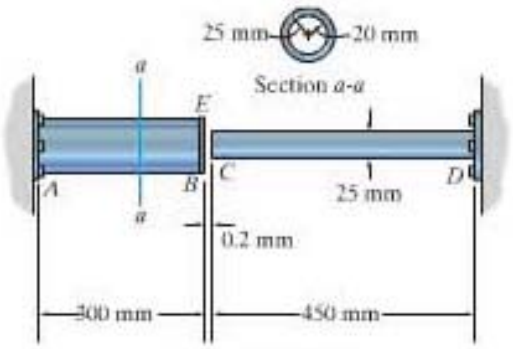
4–83. The wires AB and AC are made of steel, and wire AD is made of copper. Before the 150-lb force is applied, AB and AC are each 60 in. long and AD is 40 in. long. If the temperature is increased by 80°F, determine the force in each wire needed to support the load. Take $E_{st} = 29(10^3)$ ksi, $E_{cu} = 17(10^3)$ ksi, $\alpha_{st} = 8(10^{-6})/°F$, $\alpha_{cu} = 9.60(10^{-6})/°F$. Each wire has a cross-sectional area of 0.0123 in^2.



Prob. 4–83

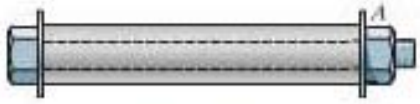
***4–84.** The AM1004-T61 magnesium alloy tube AB is capped with a rigid plate E. The gap between E and end C of the 6061-T6 aluminum alloy solid circular rod CD is 0.2 mm when the temperature is at 30° C. Determine the normal stress developed in the tube and the rod if the temperature rises to 80° C. Neglect the thickness of the rigid cap.

•4–85. The AM1004-T61 magnesium alloy tube AB is capped with a rigid plate. The gap between E and end C of the 6061-T6 aluminum alloy solid circular rod CD is 0.2 mm when the temperature is at 30° C. Determine the highest temperature to which it can be raised without causing yielding either in the tube or the rod. Neglect the thickness of the rigid cap.



Probs. 4–84/85

4–86. The steel bolt has a diameter of 7 mm and fits through an aluminum sleeve as shown. The sleeve has an inner diameter of 8 mm and an outer diameter of 10 mm. The nut at A is adjusted so that it just presses up against the sleeve. If the assembly is originally at a temperature of $T_1 = 20°C$ and then is heated to a temperature of $T_2 = 100°C$, determine the average normal stress in the bolt and the sleeve. $E_{st} = 200$ GPa, $E_{al} = 70$ GPa, $\alpha_{st} = 14(10^{-6})/°C$, $\alpha_{al} = 23(10^{-6})/°C$.



Prob. 4–86



This saw blade has grooves cut into it in order to relieve both the dynamic stress that develops within it as it rotates and the thermal stress that develops as it heats up. Note the small circles at the end of each groove. These serve to reduce the stress concentrations that develop at the end of each groove.

4.7 Stress Concentrations

In Sec. 4.1, it was pointed out that when an axial force is applied to a member, it creates a complex stress distribution within the localized region of the point of load application. Not only do complex stress distributions arise just under a concentrated loading, they can also arise at sections where the member's cross-sectional area changes. For example, consider the bar in Fig. 4–21*a*, which is subjected to an axial force P. Here the once horizontal and vertical grid lines deflect into an irregular pattern around the hole centered in the bar. The maximum normal stress in the bar occurs on section *a–a*, which is taken through the bar's *smallest* cross-sectional area. Provided the material behaves in a linear-elastic manner, the stress distribution acting on this section can be determined either from a mathematical analysis, using the theory of elasticity, or experimentally by measuring the strain normal to section *a–a* and then calculating the stress using Hooke's law, $\sigma = E\epsilon$. Regardless of the method used, the general shape of the stress distribution will be like that shown in Fig. 4–21*b*. In a similar manner, if the bar has a reduction in its cross section, achieved using shoulder fillets as in Fig. 4–22*a*, then again the maximum normal stress in the bar will occur at the *smallest* cross-sectional area, section *a–a*, and the stress distribution will look like that shown in Fig. 4–22*b*.

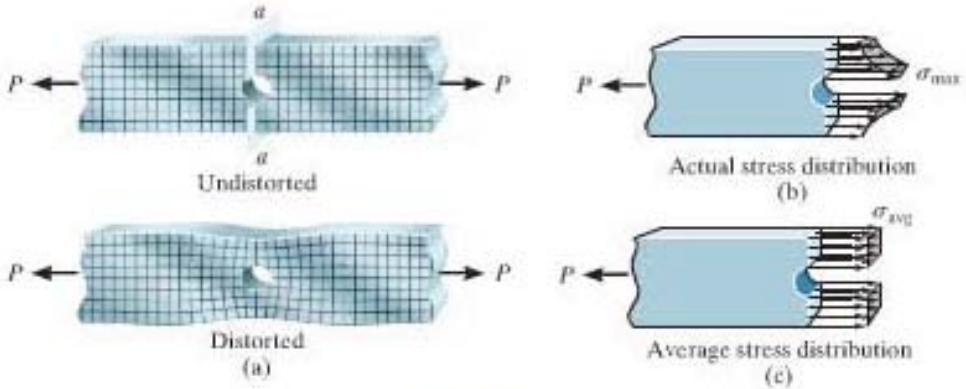


Fig. 4–21

In both of these cases, *force equilibrium* requires the magnitude of the *resultant force* developed by the stress distribution to be equal to P. In other words,

$$P = \int_A \sigma \, dA \tag{4–5}$$

This integral *graphically* represents the total *volume* under each of the stress-distribution diagrams shown in Fig. 4–21*b* or Fig. 4–22*b*. The resultant **P** must act through the *centroid* of each *volume*.

In engineering practice, the actual stress distributions in Fig. 4–21*b* and Fig. 4–22*b* do *not* have to be determined. Instead, only the *maximum stress* at these sections must be known, and the member is then designed to resist this stress when the axial load **P** is applied. Specific values of this maximum normal stress can be determined by experimental methods or by advanced mathematical techniques using the theory of elasticity. The results of these investigations are usually reported in graphical form using a ***stress-concentration factor*** K. We define K as a ratio of the maximum stress to the average normal stress acting at the cross section; i.e.,

$$K = \frac{\sigma_{max}}{\sigma_{avg}} \tag{4–6}$$

Provided K is known, and the average normal stress has been calculated from $\sigma_{avg} = P/A$, where A is the *smallest* cross-sectional area, Figs. 4–21*c* and 4–22*c*, then the maximum normal stress at the cross section is $\sigma_{max} = K(P/A)$.



Stress concentrations often arise at sharp corners on heavy machinery. Engineers can mitigate this effect by using stiffeners welded to the corners.

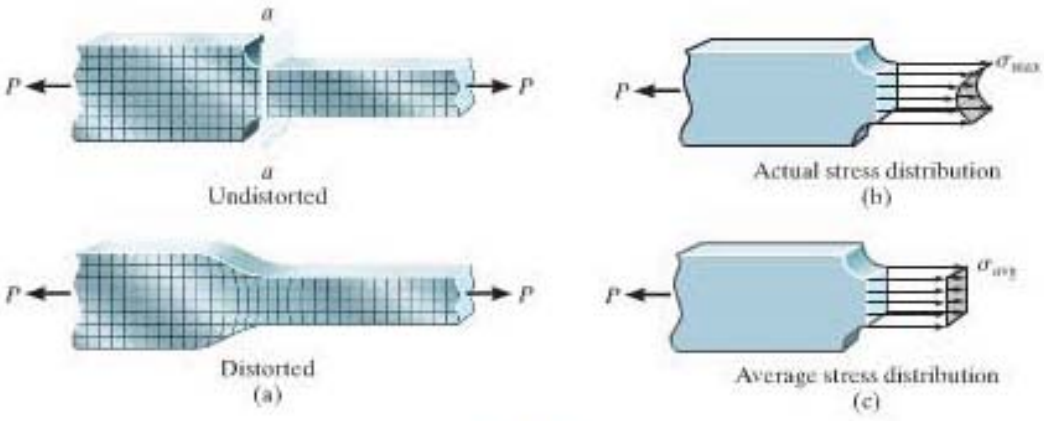


Fig. 4–22

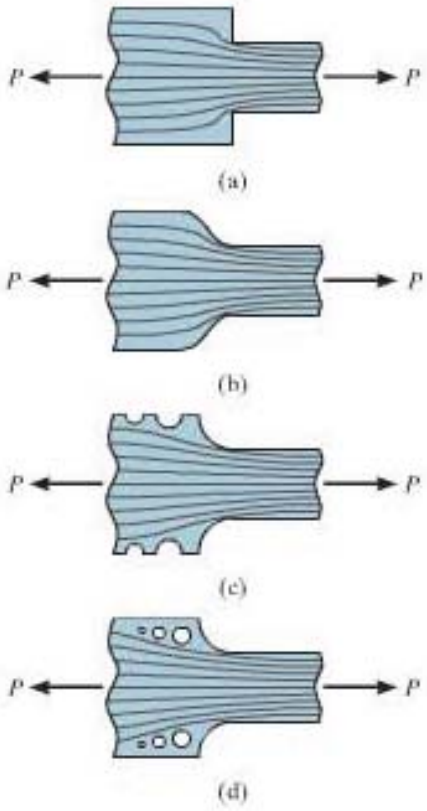


Fig. 4–23

Specific values of K are generally reported in handbooks related to stress analysis.* Examples are given in Figs. 4–24 and 4–25. Note that K is independent of the bar's material properties; rather, it *depends only* on the bar's *geometry* and the type of discontinuity. As the size r of the discontinuity is *decreased*, the stress concentration is increased. For example, if a bar requires a change in cross section, it has been determined that a sharp corner, Fig. 4–23*a*, produces a stress-concentration factor greater than 3. In other words, the maximum normal stress will be three times greater than the average normal stress on the smallest cross section. However, this can be reduced to, say, 1.5 by introducing a fillet, Fig. 4–23*b*. A further reduction can be made by means of small grooves or holes placed at the transition, Fig. 4–23*c* and 4–23*d*. In all of these cases these designs help to reduce the rigidity of the material surrounding the corners, so that both the strain and the stress are more evenly spread throughout the bar.

The stress-concentration factors given in Figs. 4–24 and 4–25 were determined on the basis of a static loading, with the assumption that the stress in the material does not exceed the proportional limit. If the material is *very brittle*, the proportional limit may be at the fracture stress, and so for this material, failure will begin *at* the point of stress concentration. Essentially a crack begins to form at this point, and a higher stress concentration will develop at the *tip* of this crack. This, in turn, causes the crack to propagate over the cross section, resulting in sudden fracture. For this reason, it is very important to use stress-concentration factors in design when using brittle materials. On the other hand, if the material is ductile and subjected to a static load, it is often not necessary to use stress-concentration factors since any stress that exceeds the proportional limit will not result in a crack. Instead, the material will have reserve strength due to yielding and strain-hardening. In the next section we will discuss the effects caused by this phenomenon.

Stress concentrations are also responsible for many failures of structural members or mechanical elements subjected to *fatigue loadings*. For these cases, a stress concentration will cause the material to crack if the stress exceeds the material's endurance limit, whether or not the material is ductile or brittle. Here, the material *localized* at the tip of the crack remains in a *brittle state*, and so the crack continues to grow, leading to a progressive fracture. As a result, one must seek ways to limit the amount of damage that can be caused by fatigue.

*See Lipson, C. and R. C. Juvinall, *Handbook of Stress and Strength*, Macmillan.

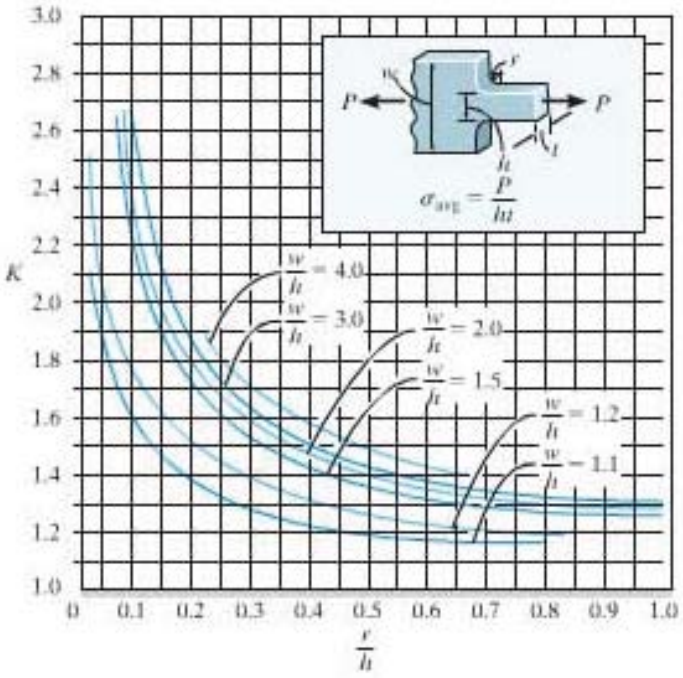


Fig. 4–24

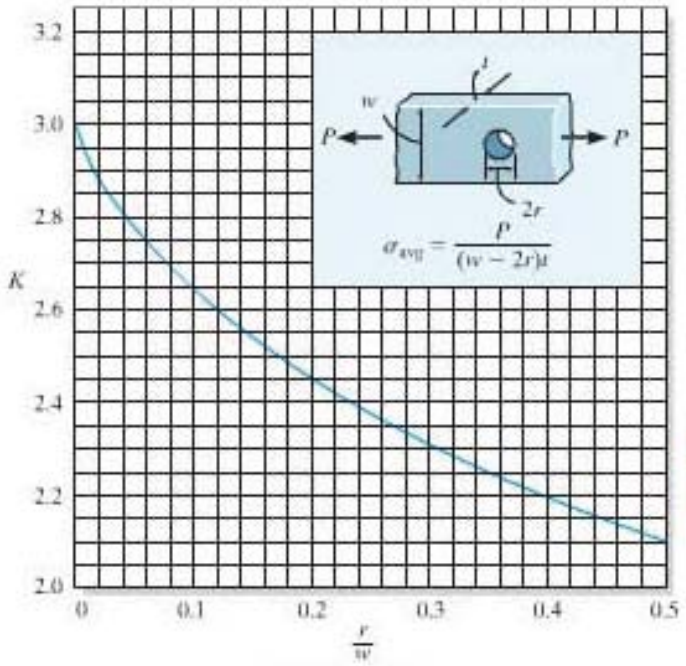


Fig. 4–25

Important Points

- *Stress concentrations* occur at sections where the cross-sectional area suddenly changes. The more severe the change, the larger the stress concentration.
- For design or analysis, it is only necessary to determine the maximum stress acting on the smallest cross-sectional area. This is done using a *stress concentration factor*, K, that has been determined through experiment and is only a function of the geometry of the specimen.
- Normally the stress concentration in a ductile specimen that is subjected to a static loading will *not* have to be considered in design; however, if the material is *brittle*, or subjected to *fatigue* loadings, then stress concentrations become important.



Failure of this steel pipe in tension occurred at its smallest cross-sectional area, which is through the hole. Notice how the material yielded around the fractured surface.

*4.8 Inelastic Axial Deformation

Up to this point we have only considered loadings that cause the material of a member to behave elastically. Sometimes, however, a member may be designed so that the loading causes the material to yield and thereby permanently deform. Such members are often made from a highly ductile metal such as annealed low-carbon steel having a stress–strain diagram that is similar to that of Fig. 3–6 and for simplicity can be *modeled* as shown in Fig. 4–26*b*. A material that exhibits this behavior is referred to as being ***elastic perfectly plastic*** or ***elastoplastic.***

To illustrate physically how such a material behaves, consider the bar in Fig. 4–26*a*, which is subjected to the axial load **P**. If the load causes an *elastic stress* $\sigma = \sigma_1$ to be developed in the bar, then applying Eq. 4–5, *equilibrium* requires $P = \int \sigma_1 \, dA = \sigma_1 A$. Furthermore, the stress σ_1 causes the bar to strain ϵ_1 as indicated on the stress–strain diagram, Fig. 4–26*b*. If P is now increased to P_p such that it causes yielding of the material, that is, $\sigma = \sigma_Y$, then again $P_p = \int \sigma_Y \, dA = \sigma_Y A$. The load P_p is called the *plastic load* since it represents the maximum load that can be supported by an elastoplastic material. For this case, the strains are *not* uniquely defined. Instead, at the instant σ_Y is attained, the bar is *first* subjected to the yield strain ϵ_Y, Fig. 4–26*b*, after which the bar will *continue to yield* (or elongate) such that the strains ϵ_2, then ϵ_3, etc., are generated. Since our "model" of the material exhibits perfectly plastic material behavior, this elongation will continue indefinitely with no increase in load. In reality, however, the material will, after some yielding, actually begin to strain-harden so that the extra strength it attains will *stop* any further straining. As a result, any design based on this behavior will be safe, since strain-hardening provides the potential for the material to support an *additional* load if necessary.

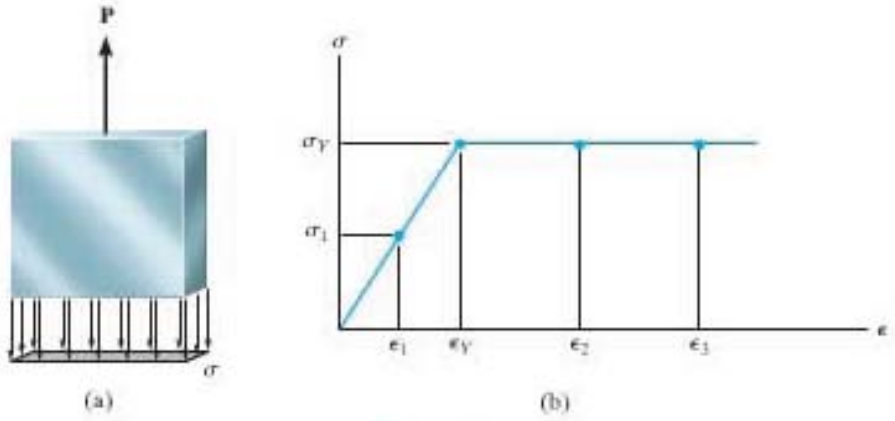
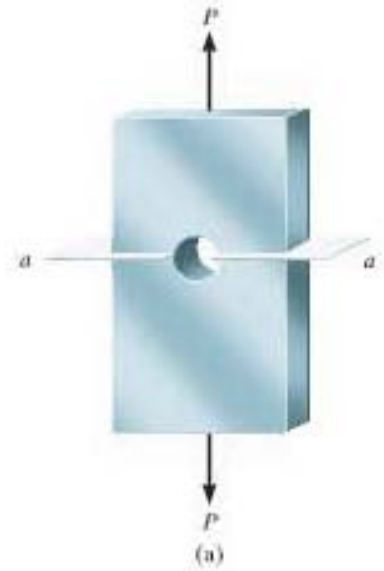


Fig. 4–26

Consider now the case of a bar having a hole through it as shown in Fig. 4–27*a*. As the magnitude of **P** is increased, a stress concentration occurs in the material at the edge of the hole, on section *a–a*. The stress here will reach a maximum value of, say, $\sigma_{max} = \sigma_1$ and have a corresponding *elastic strain* of ϵ_1, Fig. 4–27*b*. The stresses and corresponding strains at other points on the cross section will be smaller, as indicated by the stress distribution shown in Fig. 4–27*c*. Equilibrium requires $P = \int \sigma \, dA$. In other words, P is geometrically equivalent to the "volume" contained within the stress distribution. If the load is now increased to P', so that $\sigma_{max} = \sigma_Y$, then the material will begin to yield outward from the hole, until the equilibrium condition $P' = \int \sigma \, dA$ is satisfied, Fig. 4–27*d*. As shown, this produces a stress distribution that has a geometrically *greater* "volume" than that shown in Fig. 4–27*c*. A further increase in load will cause the material over the *entire cross section* to yield eventually. When this happens, *no greater load* can be sustained by the bar. This *plastic load* P_p is shown in Fig. 4–27*e*. It can be calculated from the equilibrium condition

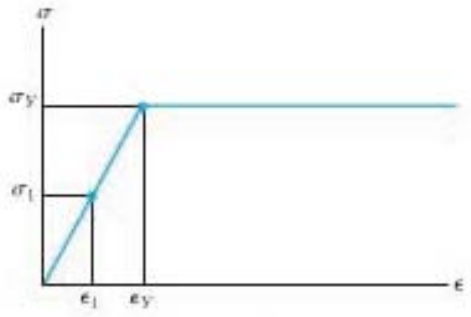


(a)

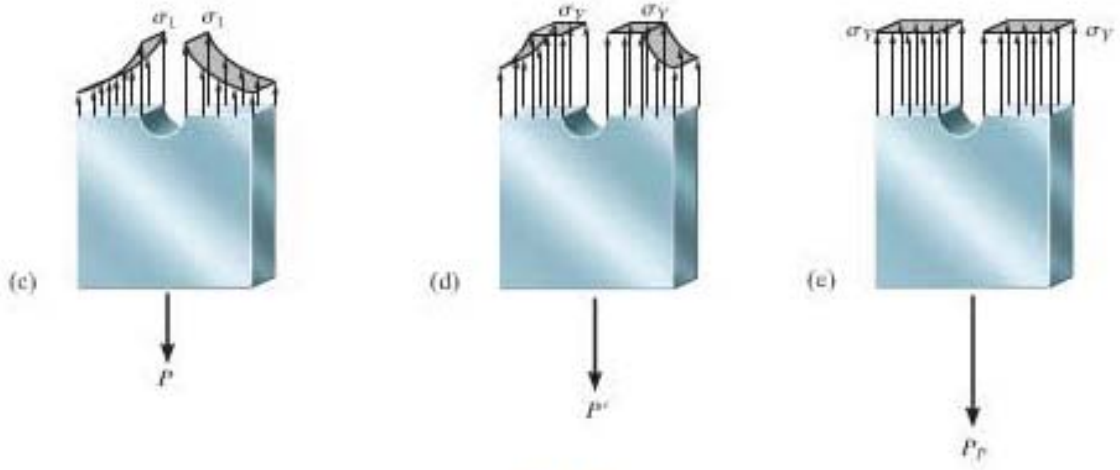
$$P_p = \int_A \sigma_Y \, dA = \sigma_Y A$$

where A is the bar's cross-sectional area at section *a–a*.

The following examples illustrate numerically how these concepts apply to other types of problems for which the material has elastoplastic behavior.



(b)



(c) (d) (e)

Fig. 4–27

*4.9 Residual Stress

If an axially loaded member or group of such members forms a statically indeterminate system that can support both tensile and compressive loads, then excessive external loadings, which cause yielding of the material, will create *residual stresses* in the members when the loads are removed. The reason for this has to do with the elastic recovery of the material that occurs during unloading. To show this, consider a prismatic member made from an elastoplastic material having the stress–strain diagram shown in Fig. 4–28. If an axial load produces a stress σ_Y in the material and a corresponding plastic strain ϵ_C, then when the load is *removed*, the material will respond elastically and follow the line CD in order to recover some of the plastic strain. A recovery to zero stress at point O' will be possible only if the member is statically determinate, since the support reactions for the member must be zero when the load is removed. Under these circumstances the member will be permanently deformed so that the permanent set or strain in the member is $\epsilon_{O'}$.

If the member is *statically indeterminate*, however, removal of the external load will cause the support forces to respond to the elastic recovery CD. Since these forces will constrain the member from full recovery, they will induce ***residual stresses*** in the member. To solve a problem of this kind, the complete cycle of loading and then unloading of the member can be considered as the *superposition* of a positive load (loading) on a negative load (unloading). The loading, O to C, results in a plastic stress distribution, whereas the unloading, along CD, results only in an elastic stress distribution. Superposition requires the loads to cancel; however, the stress distributions will not cancel, and so residual stresses will remain.

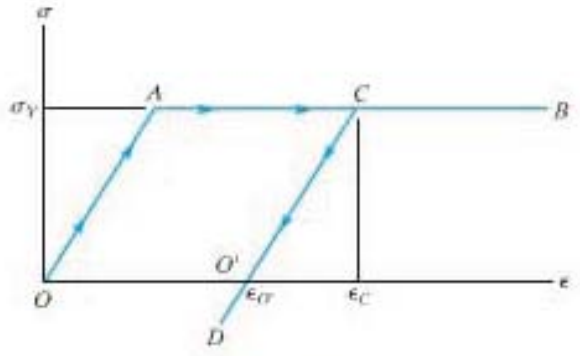


Fig. 4–28

EXAMPLE 4.13

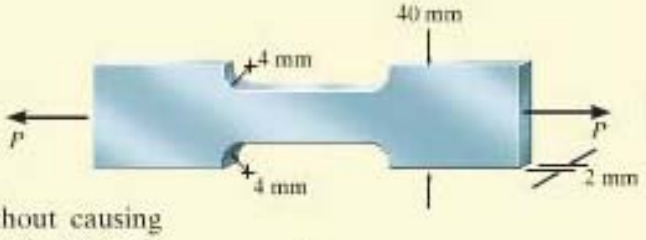
The bar in Fig. 4–29*a* is made of steel that is assumed to be elastic perfectly plastic, with $\sigma_Y = 250$ MPa. Determine (a) the maximum value of the applied load P that can be applied without causing the steel to yield and (b) the maximum value of P that the bar can support. Sketch the stress distribution at the critical section for each case.

SOLUTION

Part (a). When the material behaves elastically, we must use a stress-concentration factor determined from Fig. 4–24 that is unique for the bar's geometry. Here

$$\frac{r}{h} = \frac{4\text{ mm}}{(40\text{ mm} - 8\text{ mm})} = 0.125$$

$$\frac{w}{h} = \frac{40\text{ mm}}{(40\text{ mm} - 8\text{ mm})} = 1.25$$



(a)

From the figure $K \approx 1.75$. The maximum load, without causing yielding, occurs when $\sigma_{max} = \sigma_Y$. The average normal stress is $\sigma_{avg} = P/A$. Using Eq. 4–6, we have

$$\sigma_{max} = K\sigma_{avg}; \qquad \sigma_Y = K\left(\frac{P_Y}{A}\right)$$

$$250(10^6)\text{ Pa} = 1.75\left[\frac{P_Y}{(0.002\text{ m})(0.032\text{ m})}\right]$$

$$P_Y = 9.14\text{ kN} \qquad \textit{Ans.}$$



(b)

This load has been calculated using the *smallest* cross section. The resulting stress distribution is shown in Fig. 4–29*b*. For equilibrium, the "volume" contained within this distribution must equal 9.14 kN.

Part (b). The maximum load sustained by the bar will cause *all the material* at the smallest cross section to yield. Therefore, as P is increased to the *plastic load* P_p, it gradually changes the stress distribution from the elastic state shown in Fig. 4–29*b* to the plastic state shown in Fig. 4–29*c*. We require



(c)

Fig. 4–29

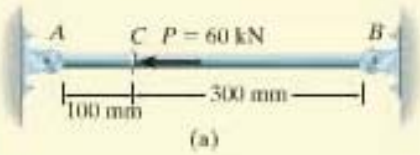
$$\sigma_Y = \frac{P_p}{A}$$

$$250(10^6)\text{ Pa} = \frac{P_p}{(0.002\text{ m})(0.032\text{ m})}$$

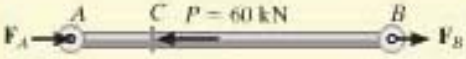
$$P_p = 16.0\text{ kN} \qquad \textit{Ans.}$$

Here P_p equals the "volume" contained within the stress distribution, which in this case is $P_p = \sigma_Y A$.

EXAMPLE 4.14



(a)



(b)

Fig. 4–30

The rod shown in Fig. 4–30*a* has a radius of 5 mm and is made of an elastic perfectly plastic material for which $\sigma_Y = 420$ MPa, $E = 70$ GPa, Fig. 4–30*c*. If a force of $P = 60$ kN is applied to the rod and then removed, determine the residual stress in the rod.

SOLUTION

The free-body diagram of the rod is shown in Fig. 4–30*b*. Application of the load **P** will cause one of three possibilities, namely, both segments *AC* and *CB* remain elastic, *AC* is plastic while *CB* is elastic, or both *AC* and *CB* are plastic.*

An *elastic analysis*, similar to that discussed in Sec. 4.4, will produce $F_A = 45$ kN and $F_B = 15$ kN at the supports. However, this results in a stress of

$$\sigma_{AC} = \frac{45\text{ kN}}{\pi(0.005\text{ m})^2} = 573\text{ MPa (compression)} > \sigma_Y = 420\text{ MPa}$$

$$\sigma_{CB} = \frac{15\text{ kN}}{\pi(0.005\text{ m})^2} = 191\text{ MPa (tension)}$$

Since the material in segment *AC* will yield, we will assume that *AC* becomes plastic, while *CB* remains elastic.

For this case, the maximum possible force developed in *AC* is

$$(F_A)_Y = \sigma_Y A = 420(10^3)\text{ kN/m}^2[\pi(0.005\text{ m})^2] = 33.0\text{ kN}$$

and from the equilibrium of the rod, Fig. 4–31*b*,

$$F_B = 60\text{ kN} - 33.0\text{ kN} = 27.0\text{ kN}$$

The stress in each segment of the rod is therefore

$$\sigma_{AC} = \sigma_Y = 420\text{ MPa (compression)}$$

$$\sigma_{CB} = \frac{27.0\text{ kN}}{\pi(0.005\text{ m})^2} = 344\text{ MPa (tension)} < 420\text{ MPa (OK)}$$

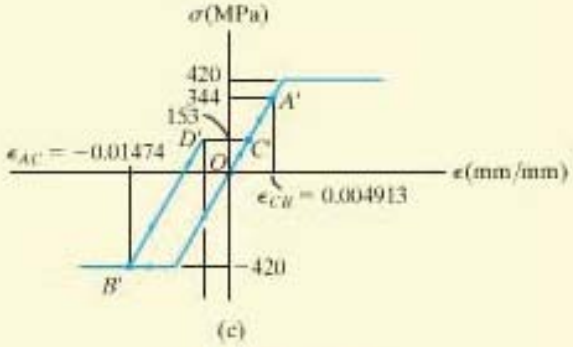
*The possibility of *CB* becoming plastic before *AC* will not occur because when point *C* moves, the *strain* in *AC* (since it is shorter) will always be larger than the strain in *CB*.

Residual Stress. In order to obtain the residual stress, it is also necessary to know the strain in each segment due to the loading. Since CB responds elastically,

$$\delta_C = \frac{F_B L_{CB}}{AE} = \frac{(27.0\text{ kN})(0.300\text{ m})}{\pi(0.005\text{ m})^2[70(10^6)\text{ kN/m}^2]} = 0.001474\text{ m}$$

$$\epsilon_{CB} = \frac{\delta_C}{L_{CB}} = \frac{0.001474\text{ m}}{0.300\text{ m}} = +0.004913$$

$$\epsilon_{AC} = \frac{\delta_C}{L_{AC}} = -\frac{0.001474\text{ m}}{0.100\text{ m}} = -0.01474$$



(c)

Fig. 4–30 (cont.)

Here the yield strain is

$$\epsilon_Y = \frac{\sigma_Y}{E} = \frac{420(10^6)\text{ N/m}^2}{70(10^9)\text{ N/m}^2} = 0.006$$

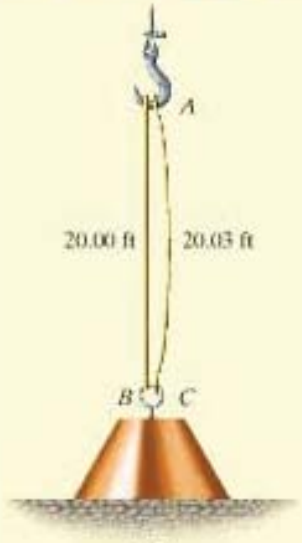
Therefore, when **P** is *applied*, the stress–strain behavior for the material in segment CB moves from O to A', Fig. 4–30*c*, and the stress–strain behavior for the material in segment AC moves from O to B'. If the load **P** is applied in the *reverse direction*, in other words, the load is removed, then an elastic response occurs and a reverse force of $F_A = 45$ kN and $F_B = 15$ kN must be applied to each segment. As calculated previously, these forces now produce stresses $\sigma_{AC} = 573$ MPa (tension) and $\sigma_{CB} = 191$ MPa (compression), and as a result the residual stress in each member is

$$(\sigma_{AC})_r = -420\text{ MPa} + 573\text{ MPa} = 153\text{ MPa} \qquad \textit{Ans}$$

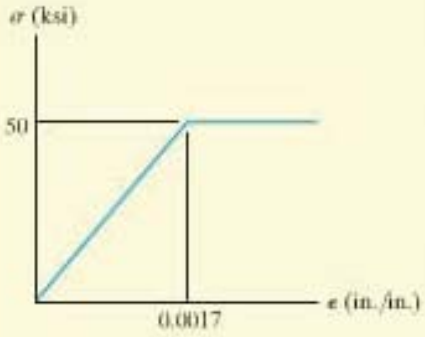
$$(\sigma_{CB})_r = 344\text{ MPa} - 191\text{ MPa} = 153\text{ MPa} \qquad \textit{Ans}$$

This residual stress is the *same* for both segments, which is to be expected. Also note that the stress–strain behavior for segment AC moves from B' to D' in Fig. 4–30*c*, while the stress–strain behavior for the material in segment CB moves from A' to C' when the load is removed.

EXAMPLE 4.15



(a)

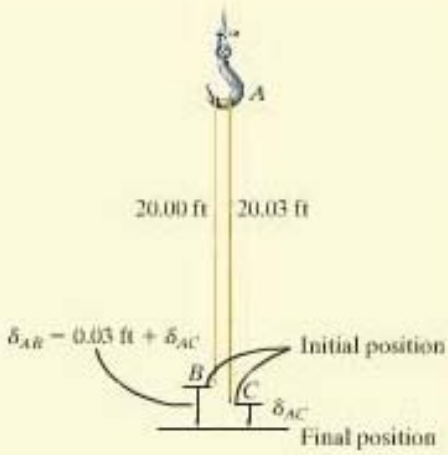


(b)



(c)

Fig. 4–31



(d)

Two steel wires are used to lift the weight of 3 kip, Fig. 4–31*a*. Wire AB has an unstretched length of 20.00 ft and wire AC has an unstretched length of 20.03 ft. If each wire has a cross-sectional area of 0.05 in^2, and the steel can be considered elastic perfectly plastic as shown by the $\sigma-\epsilon$ graph in Fig. 4–31*b*, determine the force in each wire and its elongation.

SOLUTION

Once the weight is supported by both wires, then the stress in the wires depends on the corresponding strain. There are three possibilities, namely, the strains in both wires are elastic, wire AB is plastically strained while wire AC is elastically strained, or both wires are plastically strained. We will assume that AC remains *elastic* and AB is plastically strained.

Investigation of the free-body diagram of the suspended weight, Fig. 4–31*c*, indicates that the problem is statically indeterminate. The equation of equilibrium is

$$+\uparrow \Sigma F_y = 0; \qquad T_{AB} + T_{AC} - 3\ \text{kip} = 0 \tag{1}$$

Since AB becomes plastically strained then it must support its maximum load.

$$T_{AB} = \sigma_Y A_{AB} = 50\ \text{ksi}\ (0.05\ \text{in}^2) = 2.50\ \text{kip} \qquad \textit{Ans.}$$

Therefore, from Eq. 1,

$$T_{AC} = 0.500\ \text{kip} \qquad \textit{Ans.}$$

Note that wire AC remains elastic as assumed since the stress in the wire is $\sigma_{AC} = 0.500\ \text{kip}/0.05\ \text{in}^2 = 10\ \text{ksi} < 50\ \text{ksi}$. The corresponding elastic strain is determined by proportion, Fig. 4–31*b*; i.e.,

$$\frac{\epsilon_{AC}}{10\ \text{ksi}} = \frac{0.0017}{50\ \text{ksi}}$$

$$\epsilon_{AC} = 0.000340$$

The elongation of AC is thus

$$\delta_{AC} = (0.000340)(20.03\ \text{ft}) = 0.00681\ \text{ft} \qquad \textit{Ans.}$$

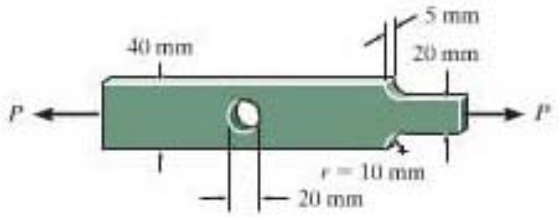
And from Fig. 4–31*d*, the elongation of AB is then

$$\delta_{AB} = 0.03\ \text{ft} + 0.00681\ \text{ft} = 0.0368\ \text{ft} \qquad \textit{Ans.}$$

PROBLEMS

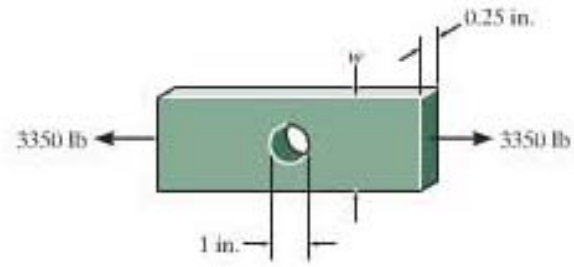
4–87. Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 8$ kN.

***4–88.** If the allowable normal stress for the bar is $\sigma_{allow} = 120$ MPa, determine the maximum axial force P that can be applied to the bar.



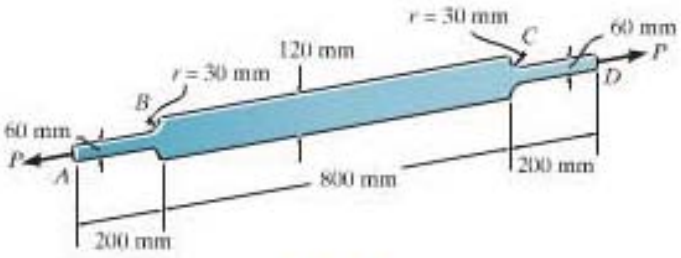
Probs. 4–87/88

•4–89. The member is to be made from a steel plate that is 0.25 in. thick. If a 1-in. hole is drilled through its center, determine the approximate width w of the plate so that it can support an axial force of 3350 lb. The allowable stress is $\sigma_{allow} = 22$ ksi.



Prob. 4–89

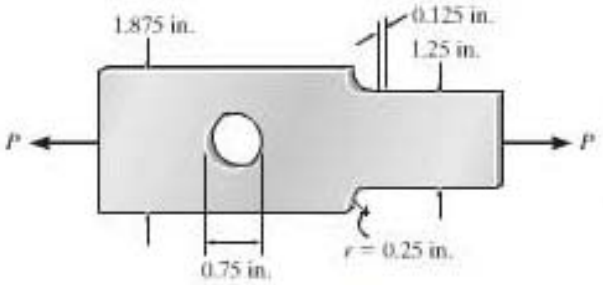
4–90. The A-36 steel plate has a thickness of 12 mm. If there are shoulder fillets at B and C, and $\sigma_{allow} = 150$ MPa, determine the maximum axial load P that it can support. Calculate its elongation, neglecting the effect of the fillets.



Prob. 4–90

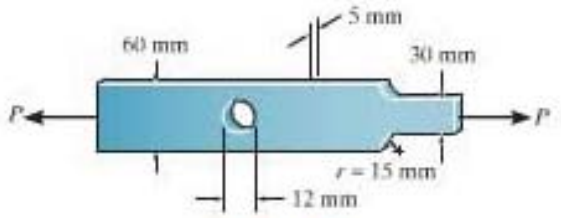
4–91. Determine the maximum axial force P that can be applied to the bar. The bar is made from steel and has an allowable stress of $\sigma_{allow} = 21$ ksi.

***4–92.** Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 2$ kip.



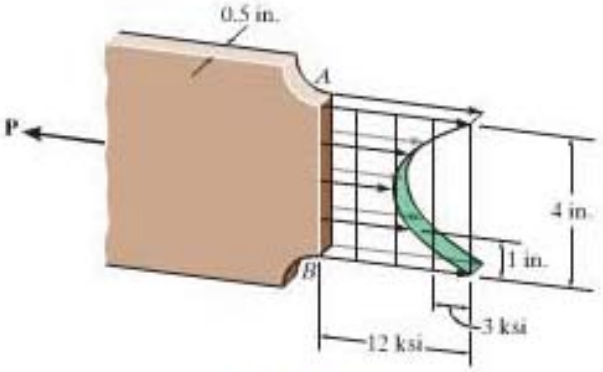
Probs. 4–91/92

•4–93. Determine the maximum normal stress developed in the bar when it is subjected to a tension of $P = 8$ kN.



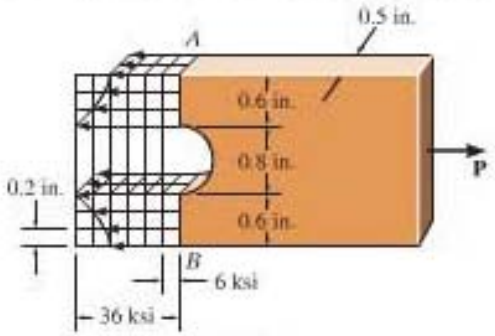
Prob. 4–93

4–94. The resulting stress distribution along section AB for the bar is shown. From this distribution, determine the approximate resultant axial force P applied to the bar. Also, what is the stress-concentration factor for this geometry?



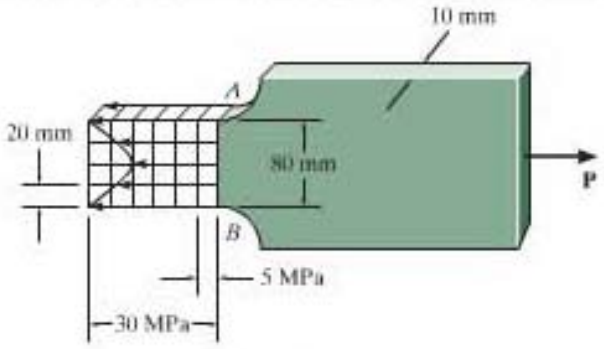
Prob. 4–94

4–95. The resulting stress distribution along section AB for the bar is shown. From this distribution, determine the approximate resultant axial force P applied to the bar. Also, what is the stress-concentration factor for this geometry?



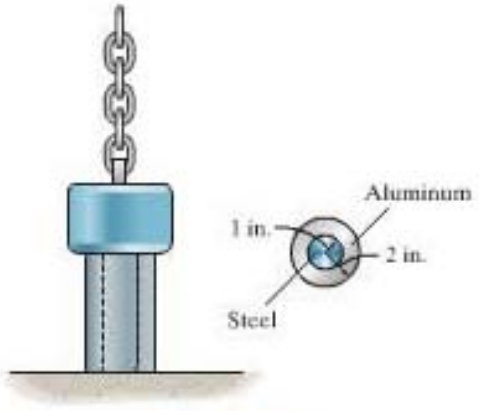
Prob. 4–95

***4–96.** The resulting stress distribution along section AB for the bar is shown. From this distribution, determine the approximate resultant axial force P applied to the bar. Also, what is the stress-concentration factor for this geometry?



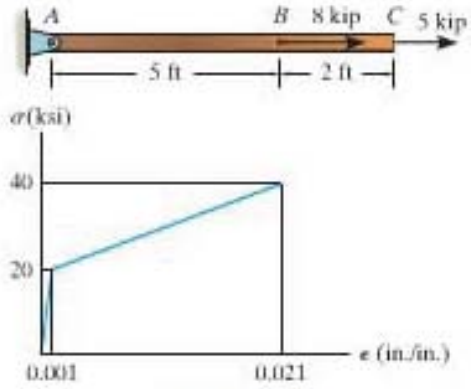
Prob. 4–96

•4–97. The 300-kip weight is slowly set on the top of a post made of 2014-T6 aluminum with an A-36 steel core. If both materials can be considered elastic perfectly plastic, determine the stress in each material.



Prob. 4–97

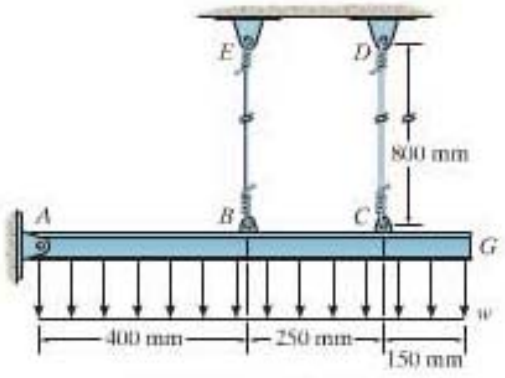
4–98. The bar has a cross-sectional area of 0.5 in^2 and is made of a material that has a stress–strain diagram that can be approximated by the two line segments shown. Determine the elongation of the bar due to the applied loading.



Prob. 4–98

4–99. The rigid bar is supported by a pin at A and two steel wires, each having a diameter of 4 mm. If the yield stress for the wires is $\sigma_Y = 530$ MPa, and $E_{st} = 200$ GPa, determine the intensity of the distributed load w that can be placed on the beam and will just cause wire EB to yield. What is the displacement of point G for this case? For the calculation, assume that the steel is elastic perfectly plastic.

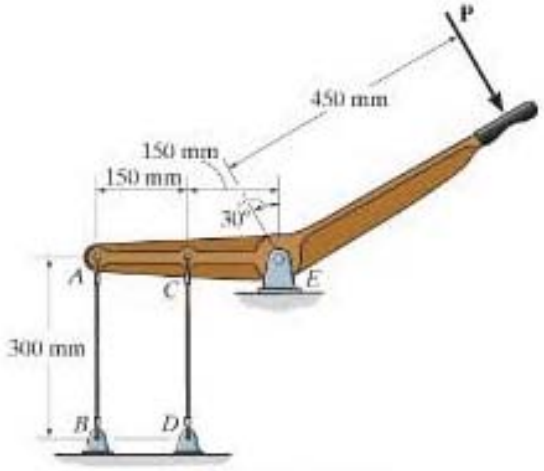
***4–100.** The rigid bar is supported by a pin at A and two steel wires, each having a diameter of 4 mm. If the yield stress for the wires is $\sigma_Y = 530$ MPa, and $E_{st} = 200$ GPa, determine (a) the intensity of the distributed load w that can be placed on the beam that will cause only one of the wires to start to yield and (b) the smallest intensity of the distributed load that will cause both wires to yield. For the calculation, assume that the steel is elastic perfectly plastic.



Probs. 4–99/100

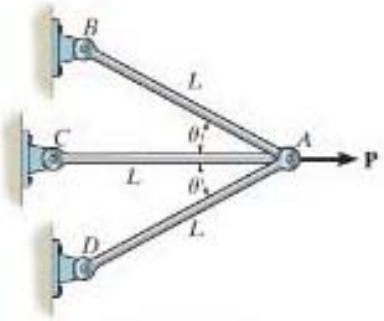
•4–101. The rigid lever arm is supported by two A-36 steel wires having the same diameter of 4 mm. If a force of $P = 3$ kN is applied to the handle, determine the force developed in both wires and their corresponding elongations. Consider A-36 steel as an elastic-perfectly plastic material.

4–102. The rigid lever arm is supported by two A-36 steel wires having the same diameter of 4 mm. Determine the smallest force **P** that will cause (a) only one of the wires to yield; (b) both wires to yield. Consider A-36 steel as an elastic-perfectly plastic material.



Probs. 4–101/102

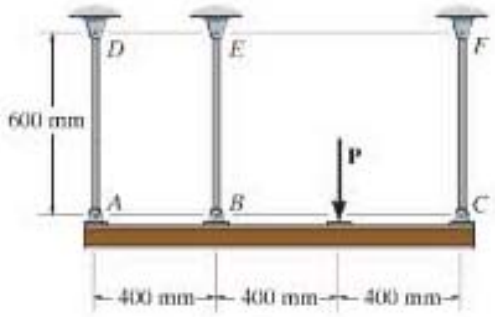
4–103. The three bars are pinned together and subjected to the load **P**. If each bar has a cross-sectional area A, length L, and is made from an elastic perfectly plastic material, for which the yield stress is σ_Y, determine the largest load (ultimate load) that can be supported by the bars, i.e., the load P that causes all the bars to yield. Also, what is the horizontal displacement of point A when the load reaches its ultimate value? The modulus of elasticity is E.



Prob. 4–103

***4–104.** The rigid beam is supported by three 25-mm diameter A-36 steel rods. If the beam supports the force of $P = 230$ kN, determine the force developed in each rod. Consider the steel to be an elastic-perfectly-plastic material.

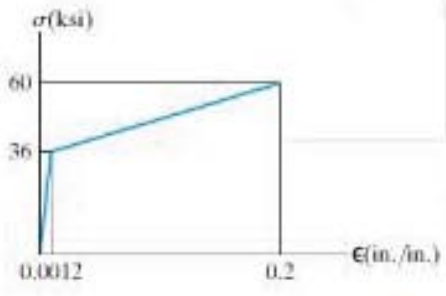
•4–105. The rigid beam is supported by three 25-mm diameter A-36 steel rods. If the force of $P = 230$ kN is applied on the beam and removed, determine the residual stresses in each rod. Consider the steel to be an elastic perfectly-plastic material.

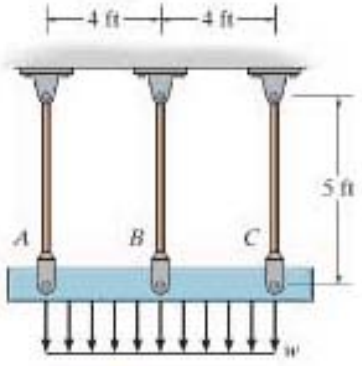


Probs. 4–104/105

4–106. The distributed loading is applied to the rigid beam, which is supported by the three bars. Each bar has a cross-sectional area of 1.25 in^2 and is made from a material having a stress–strain diagram that can be approximated by the two line segments shown. If a load of $w = 25$ kip/ft is applied to the beam, determine the stress in each bar and the vertical displacement of the beam.

4–107. The distributed loading is applied to the rigid beam, which is supported by the three bars. Each bar has a cross-sectional area of 0.75 in^2 and is made from a material having a stress–strain diagram that can be approximated by the two line segments shown. Determine the intensity of the distributed loading w needed to cause the beam to be displaced downward 1.5 in.

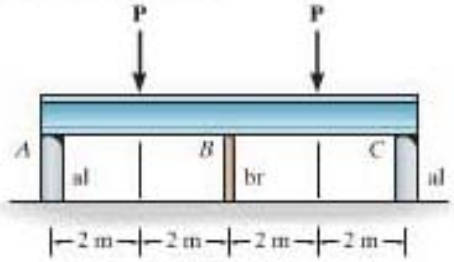




Probs. 4–106/107

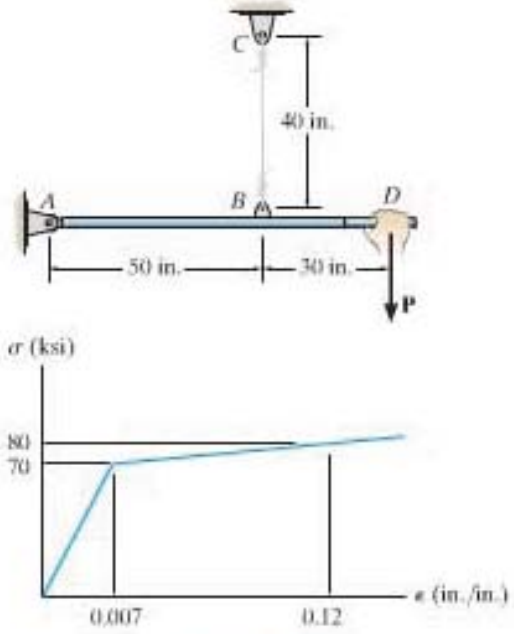
*4–108. The rigid beam is supported by the three posts A, B, and C of equal length. Posts A and C have a diameter of 75 mm and are made of aluminum, for which $E_{al} = 70$ GPa and $(\sigma_Y)_{al} = 20$ MPa. Post B has a diameter of 20 mm and is made of brass, for which $E_{br} = 100$ GPa and $(\sigma_Y)_{br} = 590$ MPa. Determine the smallest magnitude of **P** so that (a) only rods A and C yield and (b) all the posts yield.

•4–109. The rigid beam is supported by the three posts A, B, and C. Posts A and C have a diameter of 60 mm and are made of aluminum, for which $E_{al} = 70$ GPa and $(\sigma_Y)_{al} = 20$ MPa. Post B is made of brass, for which $E_{br} = 100$ GPa and $(\sigma_Y)_{br} = 590$ MPa. If $P = 130$ kN, determine the largest diameter of post B so that all the posts yield at the same time.



Probs. 4–108/109

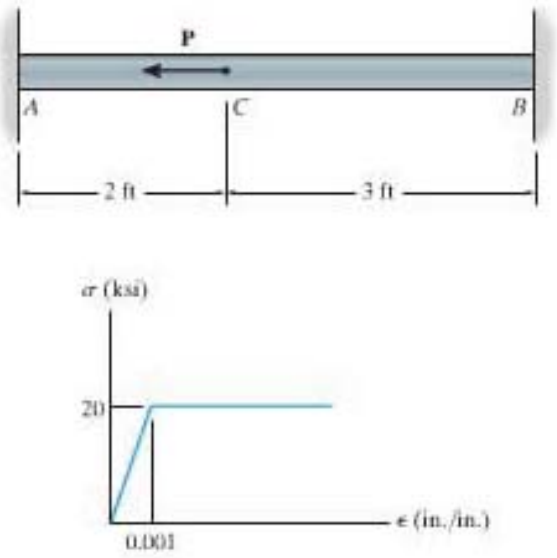
4–110. The wire BC has a diameter of 0.125 in. and the material has the stress–strain characteristics shown in the figure. Determine the vertical displacement of the handle at D if the pull at the grip is slowly increased and reaches a magnitude of (a) $P = 450$ lb, (b) $P = 600$ lb.



Prob. 4–110

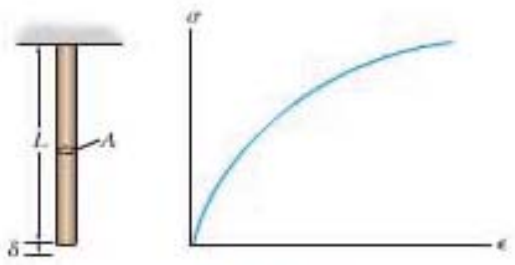
4–111. The bar having a diameter of 2 in. is fixed connected at its ends and supports the axial load **P**. If the material is elastic perfectly plastic as shown by the stress–strain diagram, determine the smallest load P needed to cause segment CB to yield. If this load is released, determine the permanent displacement of point C.

*4–112. Determine the elongation of the bar in Prob. 4–111 when both the load **P** and the supports are removed.



Probs. 4–111/112

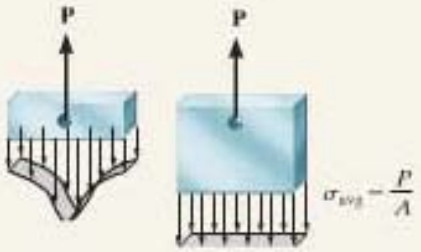
•4–113. A material has a stress–strain diagram that can be described by the curve $\sigma = c\epsilon^{1/2}$. Determine the deflection δ of the end of a rod made from this material if it has a length L, cross-sectional area A, and a specific weight γ.



Prob. 4–113

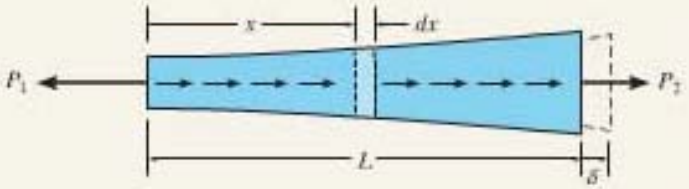
CHAPTER REVIEW

When a loading is applied at a point on a body, it tends to create a stress distribution within the body that becomes more uniformly distributed at regions removed from the point of application of the load. This is called Saint-Venant's principle.



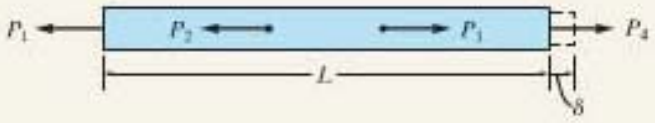
The relative displacement at the end of an axially loaded member relative to the other end is determined from

$$\delta = \int_0^L \frac{P(x)\,dx}{AE}$$



If a series of concentrated external axial forces are applied to a member and AE is also constant for the member, then

$$\delta = \Sigma \frac{PL}{AE}$$



For application, it is necessary to use a sign convention for the internal load P and displacement δ. We considered tension and elongation as positive values. Also, the material must not yield, but rather it must remain linear elastic.

Superposition of load and displacement is possible provided the material remains linear elastic and no significant changes in the geometry of the member occur after loading.

The reactions on a statically indeterminate bar can be determined using the equilibrium equations and compatibility conditions that specify the displacement at the supports. These displacements are related to the loads using a load–displacement relationship such as $\delta = PL/AE$.

A change in temperature can cause a member made of homogeneous isotropic material to change its length by

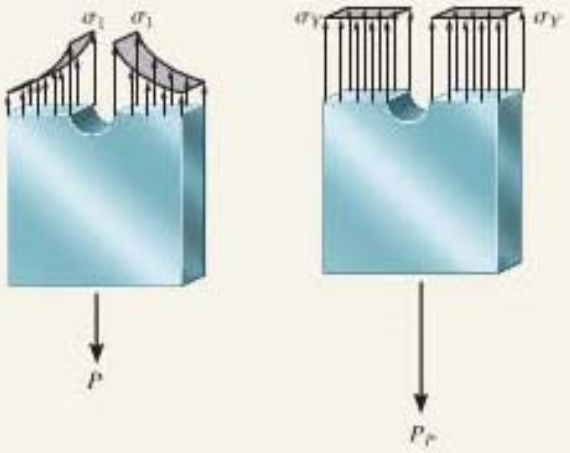
$$\delta = \alpha \Delta T L$$

If the member is confined, this change will produce thermal stress in the member.

Holes and sharp transitions at a cross section will create stress concentrations. For the design of a member made of brittle material one obtains the stress concentration factor K from a graph, which has been determined from experiment. This value is then multiplied by the average stress to obtain the maximum stress at the cross section.

$$\sigma_{max} = K\sigma_{avg}$$

If the loading in a bar made of ductile material causes the material to yield, then the stress distribution that is produced can be determined from the strain distribution and the stress–strain diagram. Assuming the material is perfectly plastic, yielding will cause the stress distribution at the cross section of a hole or transition to even out and become uniform.



If a member is constrained and an external loading causes yielding, then when the load is released, it will cause residual stress in the member.

CONCEPTUAL PROBLEMS



P4–1



P4–2

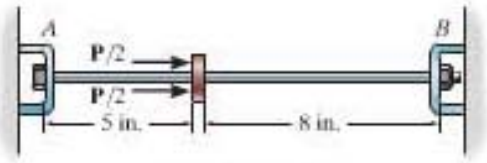
P4–1. The concrete footing *A* was poured when this column was put in place. Later the rest of the foundation slab was poured. Can you explain why the 45° cracks occurred at each corner? Can you think of a better design that would avoid such cracks?

P4–2. The row of bricks, along with mortar and an internal steel reinforcing rod, was intended to serve as a lintel beam to support the bricks above this ventilation opening on an exterior wall of a building. Explain what may have caused the bricks to fail in the manner shown.

REVIEW PROBLEMS

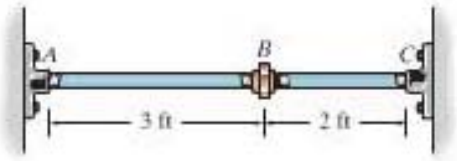
4–114. The 2014-T6 aluminum rod has a diameter of 0.5 in. and is lightly attached to the rigid supports at A and B when $T_1 = 70°F$. If the temperature becomes $T_2 = -10°F$, and an axial force of $P = 16$ lb is applied to the rigid collar as shown, determine the reactions at A and B.

4–115. The 2014-T6 aluminum rod has a diameter of 0.5 in. and is lightly attached to the rigid supports at A and B when $T_1 = 70°F$. Determine the force P that must be applied to the collar so that, when $T = 0°F$, the reaction at B is zero.



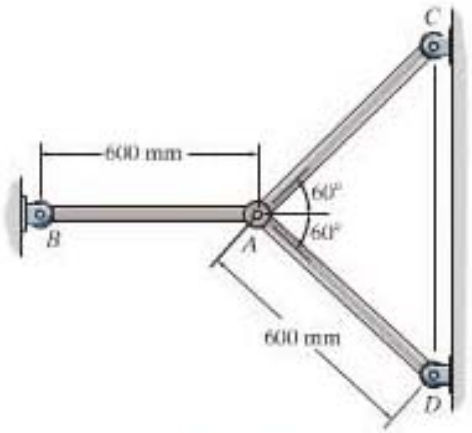
Probs. 114/115

•4–117. Two A-36 steel pipes, each having a cross-sectional area of 0.32 in^2, are screwed together using a union at B as shown. Originally the assembly is adjusted so that no load is on the pipe. If the union is then tightened so that its screw, having a lead of 0.15 in., undergoes two full turns, determine the average normal stress developed in the pipe. Assume that the union at B and couplings at A and C are rigid. Neglect the size of the union. *Note:* The lead would cause the pipe, when *unloaded*, to shorten 0.15 in. when the union is rotated one revolution.



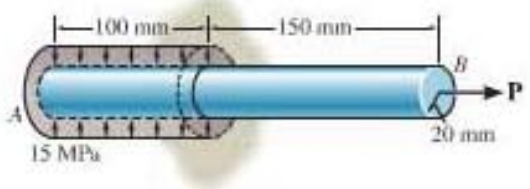
Prob. 4–117

***4–116.** The rods each have the same 25-mm diameter and 600-mm length. If they are made of A-36 steel, determine the forces developed in each rod when the temperature increases to 50° C.



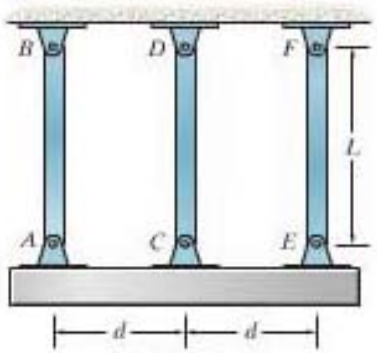
Prob. 4–116

4–118. The brass plug is force-fitted into the rigid casting. The uniform normal bearing pressure on the plug is estimated to be 15 MPa. If the coefficient of static friction between the plug and casting is $\mu_s = 0.3$, determine the axial force P needed to pull the plug out. Also, calculate the displacement of end B relative to end A just before the plug starts to slip out. $E_{br} = 98$ GPa.



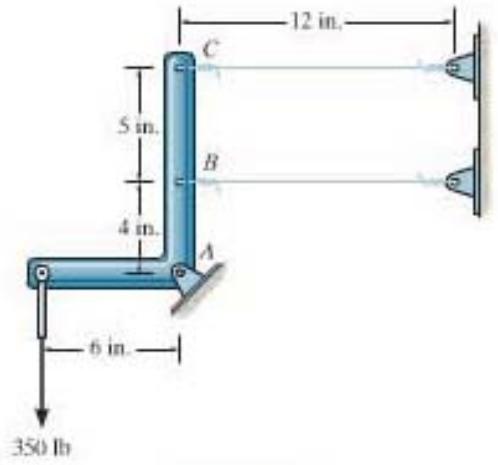
Prob. 4–118

4–119. The assembly consists of two bars AB and CD of the same material having a modulus of elasticity E_1 and coefficient of thermal expansion α_1, and a bar EF having a modulus of elasticity E_2 and coefficient of thermal expansion α_2. All the bars have the same length L and cross-sectional area A. If the rigid beam is originally horizontal at temperature T_1, determine the angle it makes with the horizontal when the temperature is increased to T_2.

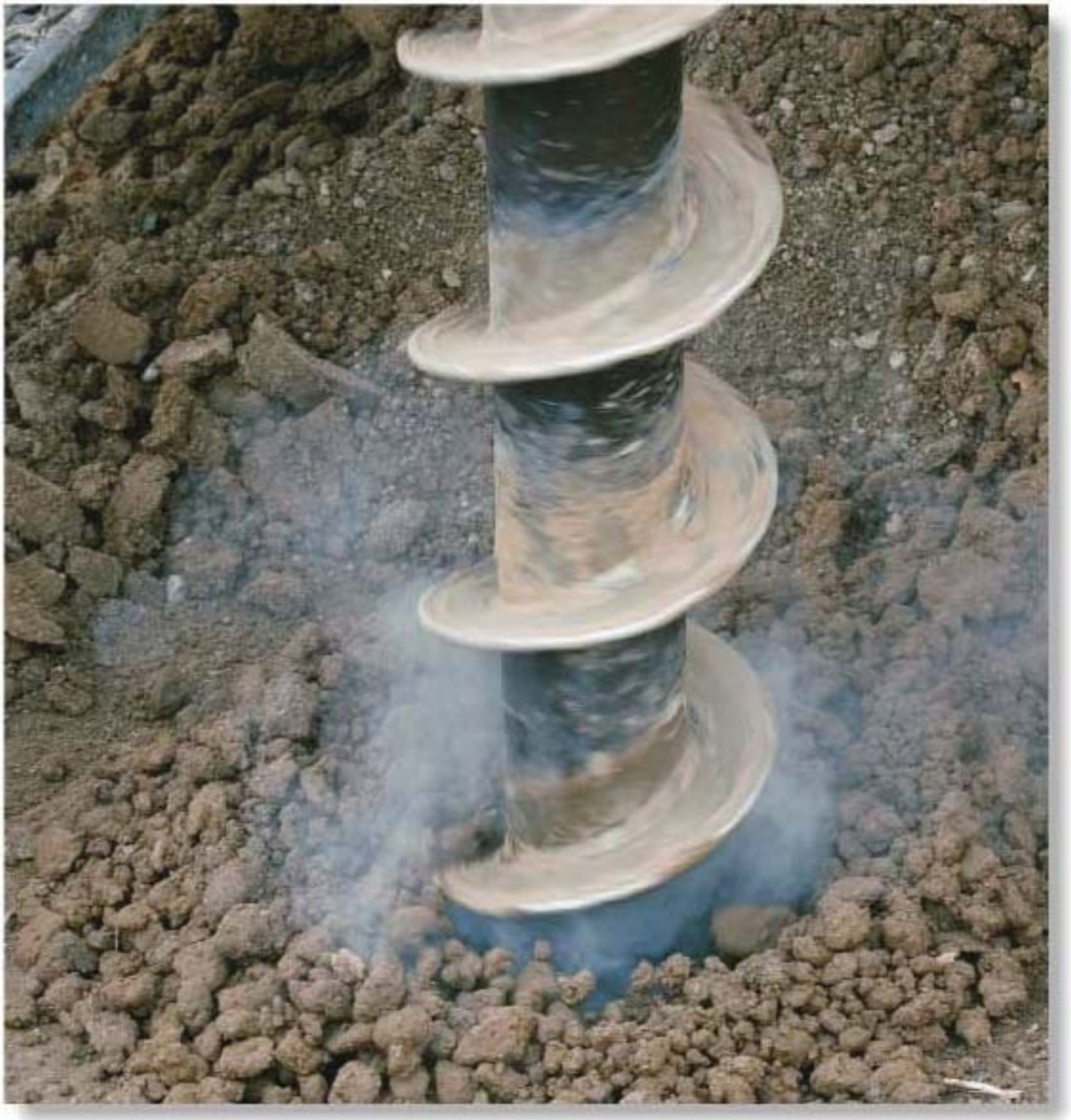


Prob. 4–119

*4–120. The rigid link is supported by a pin at A and two A-36 steel wires, each having an unstretched length of 12 in. and cross-sectional area of 0.0125 in^2. Determine the force developed in the wires when the link supports the vertical load of 350 lb.



Prob. 4–120



The torsional stress and angle of twist of this soil auger depend upon the output of the machine turning the bit as well as the resistance of the soil in contact with the shaft.

Torsion

5

CHAPTER OBJECTIVES

In this chapter we will discuss the effects of applying a torsional loading to a long straight member such as a shaft or tube. Initially we will consider the member to have a circular cross section. We will show how to determine both the stress distribution within the member and the angle of twist when the material behaves in a linear elastic manner and also when it is inelastic. Statically indeterminate analysis of shafts and tubes will also be discussed, along with special topics that include those members having noncircular cross sections. Lastly, stress concentrations and residual stress caused by torsional loadings will be given special consideration.

5.1 Torsional Deformation of a Circular Shaft

Torque is a moment that tends to twist a member about its longitudinal axis. Its effect is of primary concern in the design of axles or drive shafts used in vehicles and machinery. We can illustrate physically what happens when a torque is applied to a circular shaft by considering the shaft to be made of a highly deformable material such as rubber, Fig. 5–1*a*. When the torque is applied, the circles and longitudinal grid lines originally marked on the shaft tend to distort into the pattern shown in Fig. 5–1*b*. Note that twisting causes the circles to *remain circles*, and each longitudinal grid line deforms into a helix that intersects the circles at equal angles. Also, the cross sections from the ends along the shaft will remain *flat*—that is, they do not warp or bulge in or out—and radial lines *remain straight* during the deformation, Fig. 5–1*b*. From these observations we can assume that if the angle of twist is *small*, the *length of the shaft* and its *radius* will *remain unchanged*.

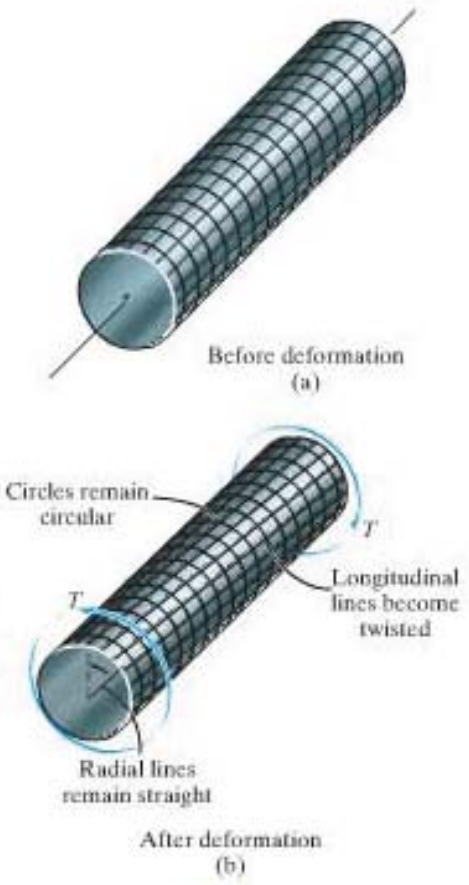
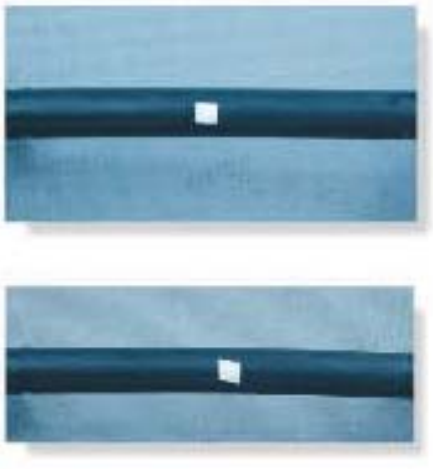


Fig. 5–1

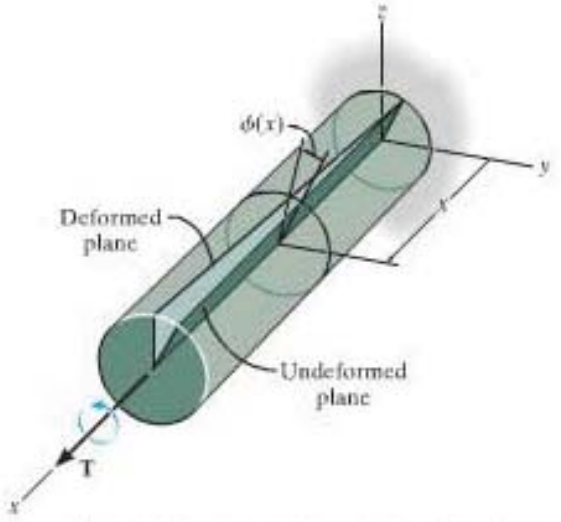
If the shaft is fixed at one end and a torque is applied to its other end, the dark green shaded plane in Fig. 5–2 will distort into a skewed form as shown. Here a radial line located on the cross section at a distance x from the fixed end of the shaft will rotate through an angle $\phi(x)$. The angle $\phi(x)$, so defined, is called the *angle of twist*. It depends on the position x and will vary along the shaft as shown.

In order to understand how this distortion strains the material, we will now isolate a small element located at a radial distance ρ (rho) from the axis of the shaft, Fig. 5–3. Due to the deformation as noted in Fig. 5–2, the front and rear faces of the element will undergo a rotation—the back face by $\phi(x)$, and the front face by $\phi(x) + \Delta\phi$. As a result, the *difference* in these rotations, $\Delta\phi$, causes the element to be subjected to a *shear strain*. To calculate this strain, note that before deformation the angle between the edges AB and AC is 90°; after deformation, however, the edges of the element are AD and AC and the angle between them is θ'. From the definition of shear strain, Eq. 2–4, we have

$$\gamma = \frac{\pi}{2} - \theta'$$



Notice the deformation of the rectangular element when this rubber bar is subjected to a torque.



The angle of twist $\phi(x)$ increases as x increases.

Fig. 5–2

This angle, γ, is indicated on the element. It can be related to the length Δx of the element and the angle $\Delta\phi$ between the shaded planes by considering the length of arc BD, that is

$$BD = \rho\Delta\phi = \Delta x\,\gamma$$

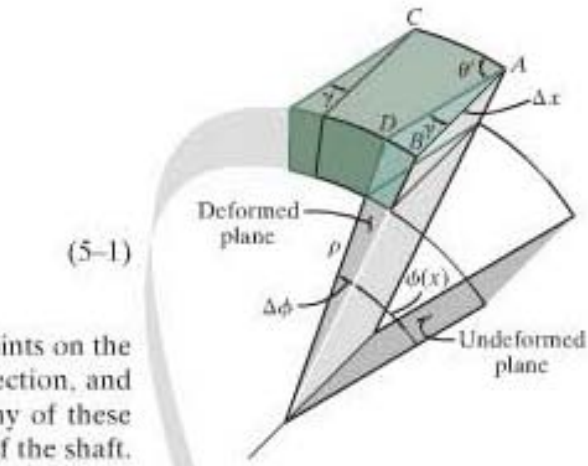
Therefore, if we let $\Delta x \rightarrow dx$ and $\Delta\phi \rightarrow d\phi$,

$$\gamma = \rho\frac{d\phi}{dx} \tag{5–1}$$

Since dx and $d\phi$ are the *same* for *all elements* located at points on the cross section at x, then $d\phi/dx$ is constant over the cross section, and Eq. 5–1 states that the magnitude of the shear strain for any of these elements varies only with its radial distance ρ from the axis of the shaft. In other words, the shear strain within the shaft varies linearly along any radial line, from zero at the axis of the shaft to a maximum γ_{max} at its outer boundary, Fig. 5–4. Since $d\phi/dx = \gamma/\rho = \gamma_{max}/c$, then

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{max} \tag{5–2}$$

The results obtained here are also valid for circular tubes. They depend only on the assumptions regarding the deformations mentioned above.



Shear strain of element

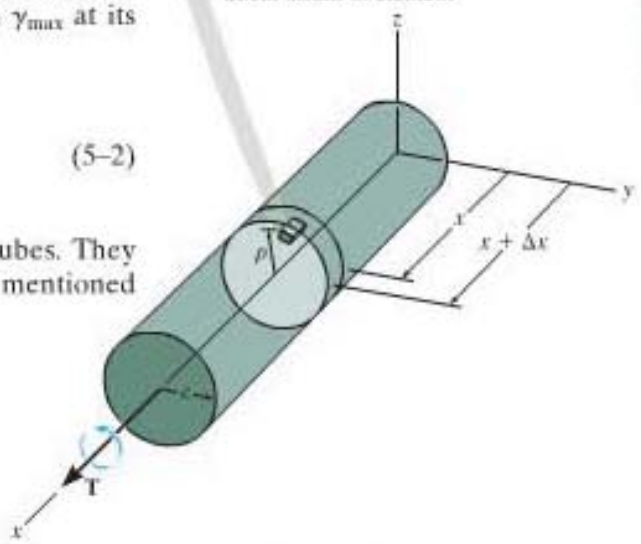
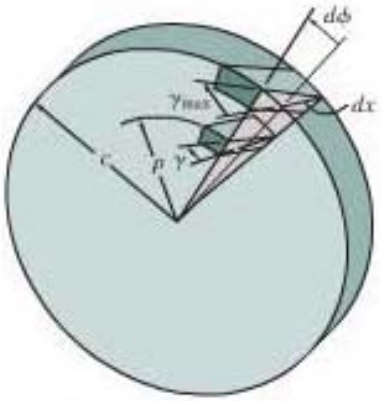


Fig. 5–3



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

Fig. 5–4

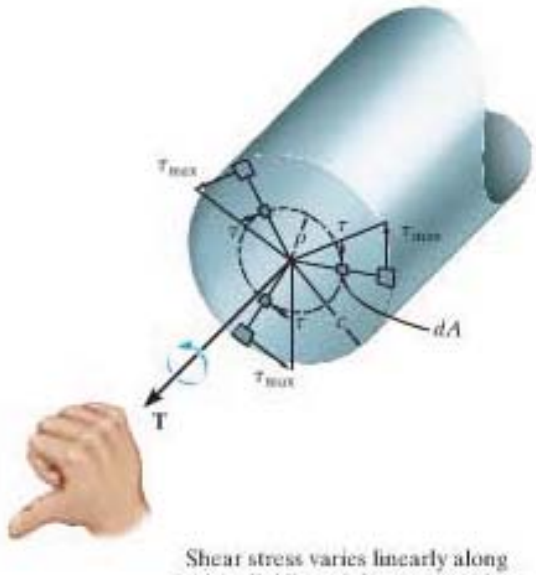
5.2 The Torsion Formula

When an external torque is applied to a shaft it creates a corresponding internal torque within the shaft. In this section, we will develop an equation that relates this internal torque to the shear stress distribution on the cross section of a circular shaft or tube.

If the material is linear-elastic, then Hooke's law applies, $\tau = G\gamma$, and consequently a ***linear variation in shear strain***, as noted in the previous section, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section. Hence, τ will vary from zero at the shaft's longitudinal axis to a maximum value, τ_{max}, at its outer surface. This variation is shown in Fig. 5–5 on the front faces of a selected number of elements, located at an intermediate radial position ρ and at the outer radius c. Due to the proportionality of triangles, we can write

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max} \tag{5–3}$$

This equation expresses the shear-stress distribution over the cross section in terms of the radial position ρ of the element. Using it, we can now apply the condition that requires the torque produced by the stress distribution over the entire cross section to be equivalent to the resultant internal torque T at the section, which holds the shaft in equilibrium, Fig. 5–5.



Shear stress varies linearly along each radial line of the cross section.

Fig. 5–5

Specifically, each element of area dA, located at ρ, is subjected to a force of $dF = \tau\, dA$. The torque produced by this force is $dT = \rho(\tau\, dA)$. We therefore have for the entire cross section

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\, dA \tag{5–4}$$

Since $\tau_{\max}/c$ is constant,

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\, dA \tag{5–5}$$

The integral depends only on the geometry of the shaft. It represents the ***polar moment of inertia*** of the shaft's cross-sectional area about the shaft's longitudinal axis. We will symbolize its value as J, and therefore the above equation can be rearranged and written in a more compact form, namely,

$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{5–6}$$

Here

$\tau_{\max}$ = the maximum shear stress in the shaft, which occurs at the outer surface

T = the resultant *internal torque* acting at the cross section. Its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis

J = the polar moment of inertia of the cross-sectional area

c = the outer radius of the shaft

Combining Eqs. 5–3 and 5–6, the shear stress at the intermediate distance ρ can be determined from

$$\boxed{\tau = \frac{T\rho}{J}} \tag{5–7}$$

Either of the above two equations is often referred to as the ***torsion formula***. Recall that it is used only if the shaft is circular and the material is homogeneous and behaves in a linear elastic manner, since the derivation is based on Hooke's law.

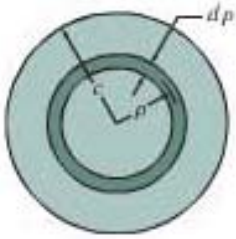


Fig. 5–6

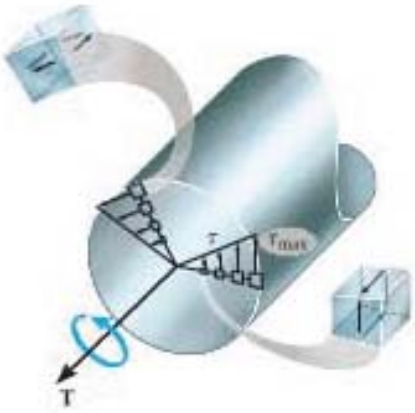
Solid Shaft. If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using an area element in the form of a *differential ring* or annulus having a thickness $d\rho$ and circumference $2\pi\rho$, Fig. 5–6. For this ring, $dA = 2\pi\rho\, d\rho$, and so

$$J = \int_A \rho^2\, dA = \int_0^c \rho^2(2\pi\rho\, d\rho) = 2\pi\int_0^c \rho^3\, d\rho = 2\pi\left(\frac{1}{4}\right)\rho^4\Big|_0^c$$

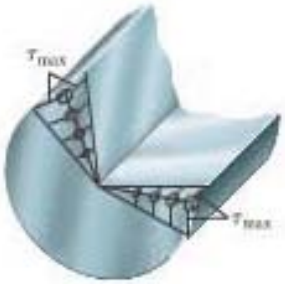
$$\boxed{J = \frac{\pi}{2}c^4} \tag{5–8}$$

Note that J is a *geometric property* of the circular area and is always positive. Common units used for its measurement are mm^4 or in^4.

The shear stress has been shown to vary linearly along each radial line of the cross section of the shaft. However, if an element of material on the cross section is isolated, then due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces as shown in Fig. 5–7*a*. Hence, ***not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane***, Fig. 5–7*b*. It is interesting to note that because of this axial distribution of shear stress, shafts made from wood tend to *split* along the axial plane when subjected to excessive torque, Fig. 5–8. This is because wood is an anisotropic material. Its shear resistance parallel to its grains or fibers, directed along the axis of the shaft, is much less than its resistance perpendicular to the fibers, directed in the plane of the cross section.



(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

Fig. 5–7





Failure of a wooden shaft due to torsion.

Fig. 5–8

Tubular Shaft. If a shaft has a tubular cross section, with inner radius c_i and outer radius c_o, then from Eq. 5–8 we can determine its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o. The result is

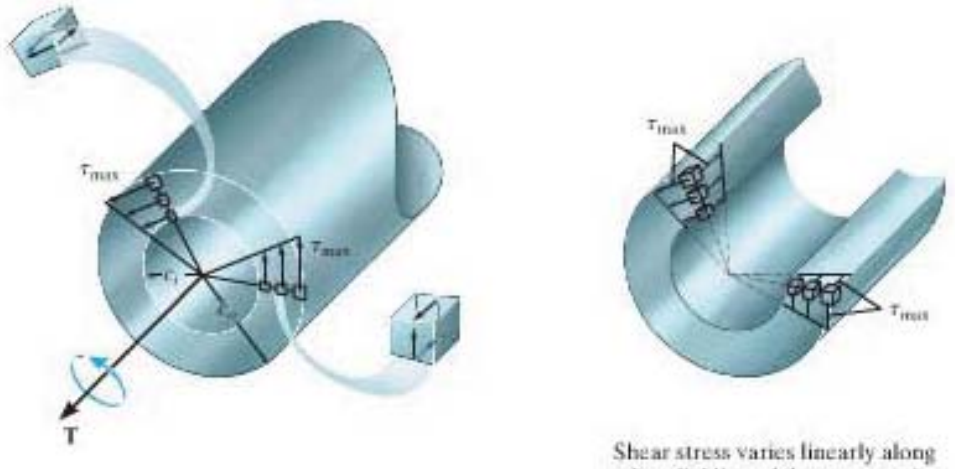
$$J = \frac{\pi}{2}(c_o^4 - c_i^4) \tag{5–9}$$

Like the solid shaft, the shear stress distributed over the tube's cross-sectional area varies linearly along any radial line, Fig. 5–9*a*. Furthermore, the shear stress varies along an axial plane in this same manner, Fig. 5–9*b*.



This tubular drive shaft for a truck was subjected to an excessive torque, resulting in failure caused by yielding of the material.

Absolute Maximum Torsional Stress. If the absolute maximum torsional stress is to be determined, then it becomes important to find the location where the ratio Tc/J is a maximum. In this regard, it may be helpful to show the variation of the internal torque T at each section along the axis of the shaft by drawing a ***torque diagram***, which is a plot of the internal torque T versus its position x along the shaft's length. As a sign convention, T will be positive if by the right-hand rule the thumb is directed outward from the shaft when the fingers curl in the direction of twist as caused by the torque, Fig. 5–5. Once the internal torque throughout the shaft is determined, the maximum ratio of Tc/J can then be identified.



(a)

Shear stress varies linearly along each radial line of the cross section.
(b)

Fig. 5–9

Important Points

- When a shaft having a *circular cross section* is subjected to a torque, the cross section *remains plane* while radial lines rotate. This causes a *shear strain* within the material that *varies linearly* along any radial line, from zero at the axis of the shaft to a maximum at its outer boundary.
- For linear elastic homogeneous material the *shear stress* along any radial line of the shaft also *varies linearly*, from zero at its axis to a maximum at its outer boundary. This maximum shear stress *must not* exceed the proportional limit.
- Due to the complementary property of shear, the linear shear stress distribution within the plane of the cross section is also distributed along an adjacent axial plane of the shaft.
- The torsion formula is based on the requirement that the resultant torque on the cross section is equal to the torque produced by the shear stress distribution about the longitudinal axis of the shaft. It is required that the shaft or tube have a *circular* cross section and that it is made of *homogeneous* material which has *linear-elastic* behavior.

5

Procedure for Analysis

The torsion formula can be applied using the following procedure.

Internal Loading.

- Section the shaft perpendicular to its axis at the point where the shear stress is to be determined, and use the necessary free-body diagram and equations of equilibrium to obtain the internal torque at the section.

Section Property.

- Calculate the polar moment of inertia of the cross-sectional area. For a solid section of radius c, $J = \pi c^4/2$, and for a tube of outer radius c_o and inner radius c_i, $J = \pi(c_o^4 - c_i^4)/2$.

Shear Stress.

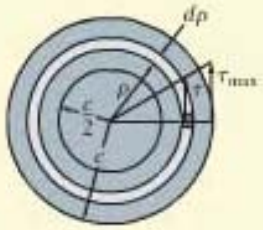
- Specify the radial distance ρ, measured from the center of the cross section to the point where the shear stress is to be found. Then apply the torsion formula $\tau = T\rho/J$, or if the maximum shear stress is to be determined use $\tau_{max} = Tc/J$. When substituting the data, make sure to use a consistent set of units.
- The shear stress acts on the cross section in a direction that is always perpendicular to ρ. The force it creates must contribute a torque about the axis of the shaft that is in the *same direction* as the internal resultant torque **T** acting on the section. Once this direction is established, a volume element located at the point where τ is determined can be isolated, and the direction of τ acting on the remaining three adjacent faces of the element can be shown.

EXAMPLE 5.1

The *solid* shaft of radius c is subjected to a torque $\mathbf{T}$, Fig. 5–10*a*. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius of $c/2$ and outer radius c.



(a)



(b)

Fig. 5–10

SOLUTION

The stress in the shaft varies linearly, such that $\tau = (\rho/c)\tau_{max}$, Eq. 5–3. Therefore, the torque dT' on the ring (area) located within the lighter-shaded region, Fig. 5–10*b*, is

$$dT' = \rho(\tau\, dA) = \rho(\rho/c)\tau_{max}(2\pi\rho\, d\rho)$$

For the entire lighter-shaded area the torque is

$$T' = \frac{2\pi\tau_{max}}{c}\int_{c/2}^{c}\rho^3\, d\rho$$

$$= \frac{2\pi\tau_{max}}{c}\frac{1}{4}\rho^4\Big|_{c/2}^{c}$$

So that

$$T' = \frac{15\pi}{32}\tau_{max}c^3 \qquad (1)$$

This torque T' can be expressed in terms of the applied torque T by first using the torsion formula to determine the maximum stress in the shaft. We have

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{(\pi/2)c^4}$$

or

$$\tau_{max} = \frac{2T}{\pi c^3}$$

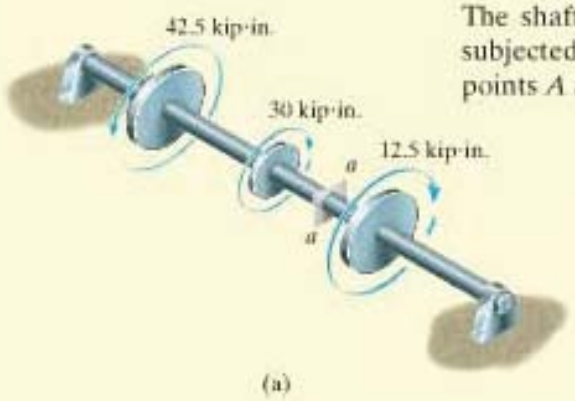
Substituting this into Eq. 1 yields

$$T' = \frac{15}{16}T \qquad \textit{Ans.}$$

NOTE: Here, approximately 94% of the torque is resisted by the lighter-shaded region, and the remaining 6% (or $\frac{1}{16}$) of T is resisted by the inner "core" of the shaft, $\rho = 0$ to $\rho = c/2$. As a result, the material located at the *outer region* of the shaft is highly effective in resisting torque, which justifies the use of tubular shafts as an efficient means for transmitting torque, and thereby saving material.

EXAMPLE 5.2

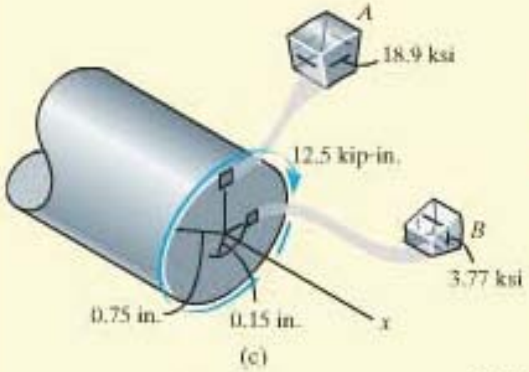
The shaft shown in Fig. 5–11*a* is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points *A* and *B*, located at section *a–a* of the shaft, Fig. 5–11*c*.



(a)



(b)



(c)

Fig. 5–11

SOLUTION

Internal Torque. The bearing reactions on the shaft are zero, provided the shaft's weight is neglected. Furthermore, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section *a–a* will be determined from the free-body diagram of the left segment, Fig. 5–11*b*. We have

$$\Sigma M_x = 0; \quad 42.5 \text{ kip} \cdot \text{in.} - 30 \text{ kip} \cdot \text{in.} - T = 0 \qquad T = 12.5 \text{ kip} \cdot \text{in.}$$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.75 \text{ in.})^4 = 0.497 \text{ in}^4$$

Shear Stress. Since point *A* is at $\rho = c = 0.75$ in.,

$$\tau_A = \frac{Tc}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{(0.497 \text{ in}^4)} = 18.9 \text{ ksi} \qquad \textit{Ans.}$$

Likewise for point *B*, at $\rho = 0.15$ in., we have

$$\tau_B = \frac{T\rho}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.15 \text{ in.})}{(0.497 \text{ in}^4)} = 3.77 \text{ ksi} \qquad \textit{Ans.}$$

NOTE: The directions of these stresses on each element at *A* and *B*, Fig. 5–11*c*, are established from the direction of the resultant internal torque **T**, shown in Fig. 5–11*b*. Note carefully how the shear stress acts on the planes of each of these elements.

EXAMPLE 5.3

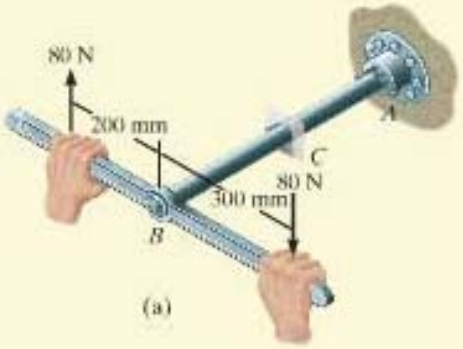
The pipe shown in Fig. 5–12*a* has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

SOLUTION

Internal Torque. A section is taken at an intermediate location *C* along the pipe's axis, Fig. 5–12*b*. The only unknown at the section is the internal torque **T**. We require

$$\Sigma M_y = 0; \quad 80\ \text{N}\,(0.3\ \text{m}) + 80\ \text{N}\,(0.2\ \text{m}) - T = 0$$

$$T = 40\ \text{N}\cdot\text{m}$$



(a)

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

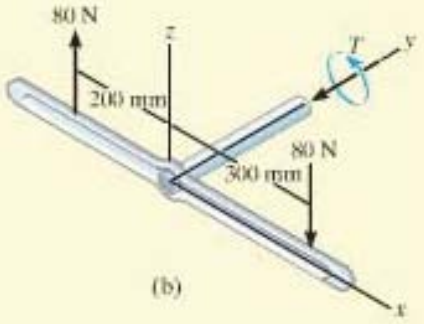
$$J = \frac{\pi}{2}[(0.05\ \text{m})^4 - (0.04\ \text{m})^4] = 5.796(10^{-6})\ \text{m}^4$$

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05$ m, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40\ \text{N}\cdot\text{m}\,(0.05\ \text{m})}{5.796(10^{-6})\ \text{m}^4} = 0.345\ \text{MPa} \qquad \textit{Ans.}$$

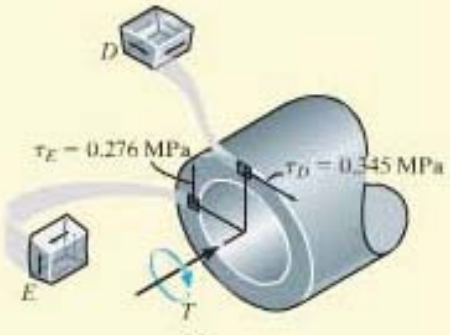
And for any point located on the inside surface, $\rho = c_i = 0.04$ m, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40\ \text{N}\cdot\text{m}\,(0.04\ \text{m})}{5.796(10^{-6})\ \text{m}^4} = 0.276\ \text{MPa} \qquad \textit{Ans.}$$



(b)

NOTE: To show how these stresses act at representative points *D* and *E* on the cross-section, we will first view the cross section from the front of segment *CA* of the pipe, Fig. 5–12*a*. On this section, Fig. 5–12*c*, the resultant internal torque is equal but opposite to that shown in Fig. 5–12*b*. The shear stresses at *D* and *E* contribute to this torque and therefore act on the shaded faces of the elements in the directions shown. As a consequence, notice how the shear-stress components act on the other three faces. Furthermore, since the top face of *D* and the inner face of *E* are in stress-free regions taken from the pipe's outer and inner walls, no shear stress can exist on these faces or on the other corresponding faces of the elements.



(c)

Fig. 5–12



The chain drive transmits the torque developed by the electric motor to the shaft. The stress developed in the shaft depends upon the power transmitted by the motor and the rate of rotation of the connecting shaft. $P = T\omega$.

5.3 Power Transmission

Shafts and tubes having circular cross sections are often used to transmit power developed by a machine. When used for this purpose, they are subjected to a torque that depends on the power generated by the machine and the angular speed of the shaft. ***Power*** is defined as the work performed per unit of time. Also, the work transmitted by a rotating shaft equals the torque applied times the angle of rotation. Therefore, if during an instant of time dt an applied torque $\mathbf{T}$ causes the shaft to rotate $d\theta$, then the instantaneous power is

$$P = \frac{T\,d\theta}{dt}$$

Since the shaft's angular velocity is $\omega = d\theta/dt$, we can express the power as

$$\boxed{P = T\omega} \tag{5–10}$$

In the SI system, power is expressed in *watts* when torque is measured in newton-meters (N · m) and ω is in radians per second (rad/s) (1 W = 1 N · m/s). In the FPS system, the basic units of power are foot-pounds per second (ft · lb/s); however, horsepower (hp) is often used in engineering practice, where

$$1\text{ hp} = 550\text{ ft} \cdot \text{lb/s}$$

For machinery, the *frequency* of a shaft's rotation, f, is often reported. This is a measure of the number of revolutions or cycles the shaft makes per second and is expressed in hertz (1 Hz = 1 cycle/s). Since 1 cycle = 2π rad, then $\omega = 2\pi f$, and so the above equation for power becomes

$$\boxed{P = 2\pi f T} \tag{5–11}$$

Shaft Design. When the power transmitted by a shaft and its frequency of rotation are known, the torque developed in the shaft can be determined from Eq. 5–11, that is, $T = P/2\pi f$. Knowing T and the allowable shear stress for the material, τ_{allow}, we can determine the size of the shaft's cross section using the torsion formula, provided the material behavior is linear elastic. Specifically, the design or geometric parameter J/c becomes

$$\frac{J}{c} = \frac{T}{\tau_{allow}} \tag{5–12}$$

For a *solid shaft*, $J = (\pi/2)c^4$, and thus, upon substitution, a *unique value* for the shaft's radius c can be determined. If the shaft is *tubular*, so that $J = (\pi/2)(c_o^4 - c_i^4)$, design permits a wide range of possibilities for the solution. This is because an *arbitrary choice* can be made for either c_o or c_i and the other radius can then be determined from Eq. 5–12.

EXAMPLE 5.4

A solid steel shaft AB shown in Fig. 5–13 is to be used to transmit 5 hp from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{allow} = 14.5$ ksi, determine the required diameter of the shaft to the nearest $\frac{1}{8}$ in.

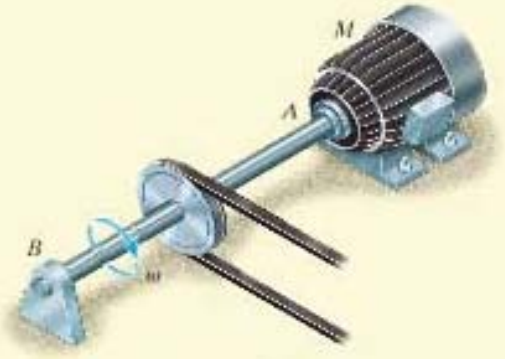


Fig. 5–13

SOLUTION

The torque on the shaft is determined from Eq. 5–10, that is, $P = T\omega$. Expressing P in foot-pounds per second and ω in radians/second, we have

$$P = 5 \text{ hp}\left(\frac{550 \text{ ft} \cdot \text{lb/s}}{1 \text{ hp}}\right) = 2750 \text{ ft} \cdot \text{lb/s}$$

$$\omega = \frac{175 \text{ rev}}{\text{min}}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 18.33 \text{ rad/s}$$

Thus,

$$P = T\omega; \qquad 2750 \text{ ft} \cdot \text{lb/s} = T(18.33 \text{ rad/s})$$

$$T = 150.1 \text{ ft} \cdot \text{lb}$$

Applying Eq. 5–12 yields

$$\frac{J}{c} = \frac{\pi}{2}\frac{c^4}{c} = \frac{T}{\tau_{allow}}$$

$$c = \left(\frac{2T}{\pi\tau_{allow}}\right)^{1/3} = \left(\frac{2(150.1 \text{ ft} \cdot \text{lb})(12 \text{ in./ft})}{\pi(14\,500 \text{ lb/in}^2)}\right)^{1/3}$$

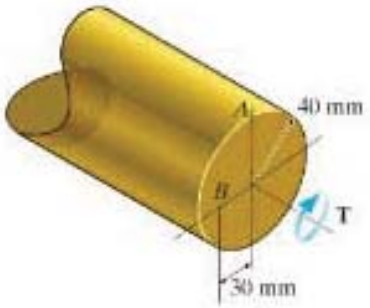
$$c = 0.429 \text{ in.}$$

Since $2c = 0.858$ in., select a shaft having a diameter of

$$d = \frac{7}{8} \text{ in.} = 0.875 \text{ in.} \qquad \textit{Ans}$$

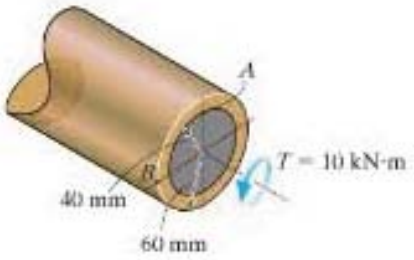
FUNDAMENTAL PROBLEMS

F5–1. The solid circular shaft is subjected to an internal torque of $T = 5$ kN·m. Determine the shear stress developed at points A and B. Represent each state of stress on a volume element.



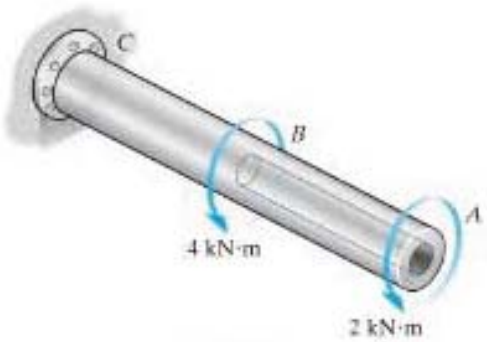
F5–1

F5–2. The hollow circular shaft is subjected to an internal torque of $T = 10$ kN·m. Determine the shear stress developed at points A and B. Represent each state of stress on a volume element.



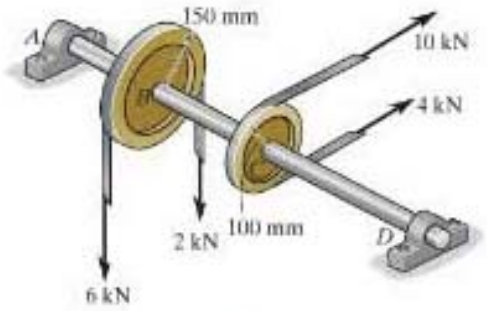
F5–2

F5–3. The shaft is hollow from A to B and solid from B to C. Determine the maximum shear stress developed in the shaft. The shaft has an outer diameter of 80 mm, and the thickness of the wall of the hollow segment is 10 mm.



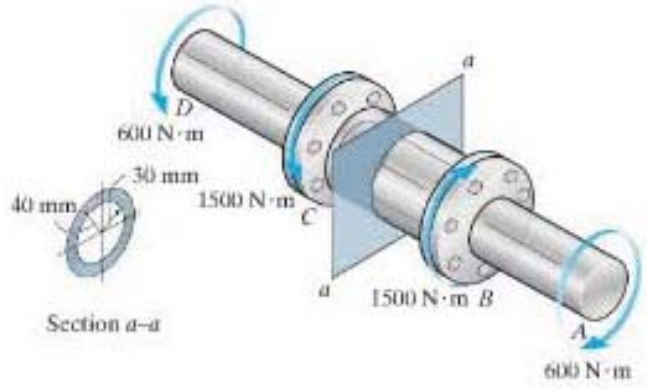
F5–3

F5–4. Determine the maximum shear stress developed in the 40-mm diameter shaft.



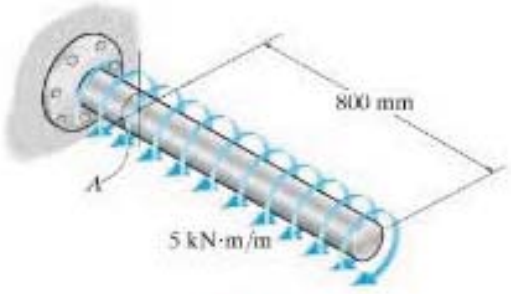
F5–4

F5–5. Determine the maximum shear stress developed in the shaft at section a–a.



F5–5

F5–6. Determine the shear stress developed at point A on the surface of the shaft. Represent the state of stress on a volume element at this point. The shaft has a radius of 40 mm.



F5–6

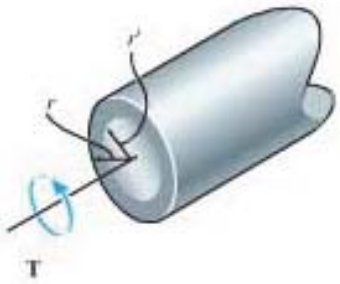
PROBLEMS

•5–1. A shaft is made of a steel alloy having an allowable shear stress of $\tau_{allow} = 12$ ksi. If the diameter of the shaft is 1.5 in., determine the maximum torque **T** that can be transmitted. What would be the maximum torque **T'** if a 1-in.-diameter hole is bored through the shaft? Sketch the shear-stress distribution along a radial line in each case.



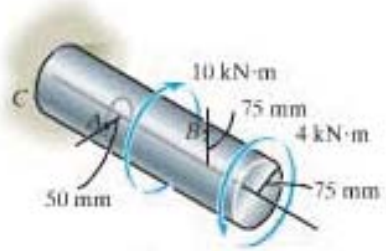
Prob. 5–1

5–2. The solid shaft of radius r is subjected to a torque **T**. Determine the radius r' of the inner core of the shaft that resists one-half of the applied torque $(T/2)$. Solve the problem two ways: (a) by using the torsion formula, (b) by finding the resultant of the shear-stress distribution.



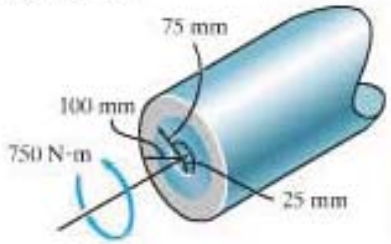
Prob. 5–2

5–3. The solid shaft is fixed to the support at C and subjected to the torsional loadings shown. Determine the shear stress at points A and B and sketch the shear stress on volume elements located at these points.



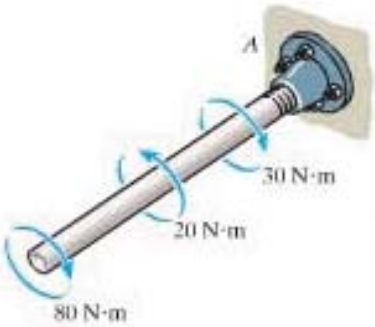
Prob. 5–3

***5–4.** The tube is subjected to a torque of 750 N · m. Determine the amount of this torque that is resisted by the gray shaded section. Solve the problem two ways: (a) by using the torsion formula, (b) by finding the resultant of the shear-stress distribution.



Prob. 5–4

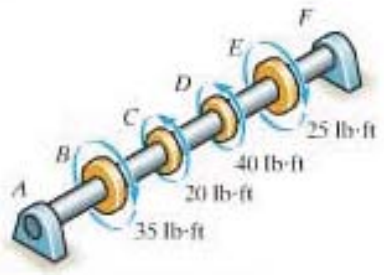
5–5. The copper pipe has an outer diameter of 40 mm and an inner diameter of 37 mm. If it is tightly secured to the wall at A and three torques are applied to it as shown, determine the absolute maximum shear stress developed in the pipe.



Prob. 5–5

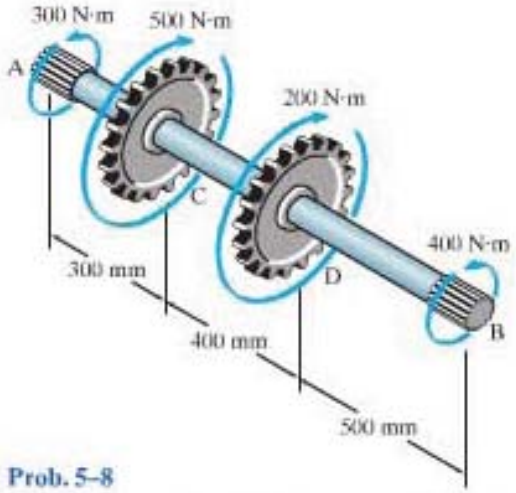
5–6. The solid shaft has a diameter of 0.75 in. If it is subjected to the torques shown, determine the maximum shear stress developed in regions BC and DE of the shaft. The bearings at A and F allow free rotation of the shaft.

5–7. The solid shaft has a diameter of 0.75 in. If it is subjected to the torques shown, determine the maximum shear stress developed in regions CD and EF of the shaft. The bearings at A and F allow free rotation of the shaft.



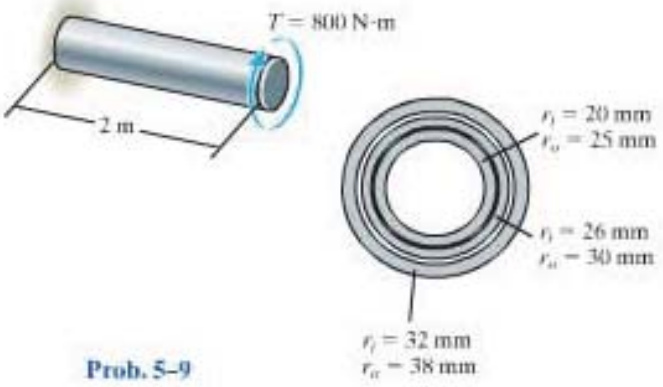
Probs. 5–6/7

*5–8. The solid 30-mm-diameter shaft is used to transmit the torques applied to the gears. Determine the absolute maximum shear stress on the shaft.



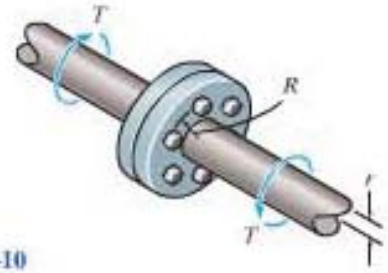
Prob. 5–8

•5–9. The shaft consists of three concentric tubes, each made from the same material and having the inner and outer radii shown. If a torque of $T = 800\ \text{N} \cdot \text{m}$ is applied to the rigid disk fixed to its end, determine the maximum shear stress in the shaft.



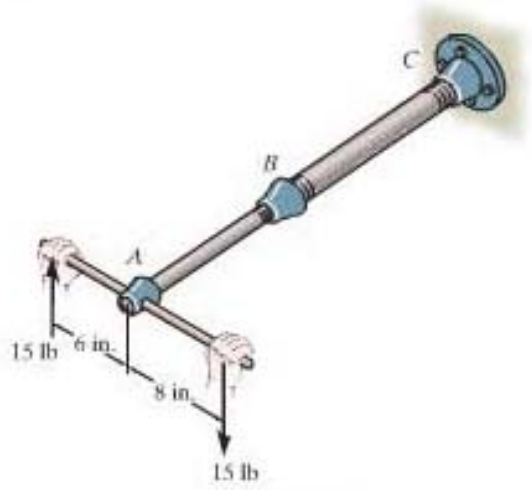
Prob. 5–9

5–10. The coupling is used to connect the two shafts together. Assuming that the shear stress in the bolts is *uniform*, determine the number of bolts necessary to make the maximum shear stress in the shaft equal to the shear stress in the bolts. Each bolt has a diameter d.



Prob. 5–10

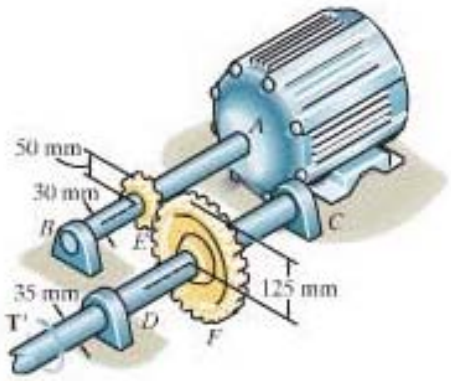
5–11. The assembly consists of two sections of galvanized steel pipe connected together using a reducing coupling at B. The smaller pipe has an outer diameter of 0.75 in. and an inner diameter of 0.68 in., whereas the larger pipe has an outer diameter of 1 in. and an inner diameter of 0.86 in. If the pipe is tightly secured to the wall at C, determine the maximum shear stress developed in each section of the pipe when the couple shown is applied to the handles of the wrench.



Prob. 5–11

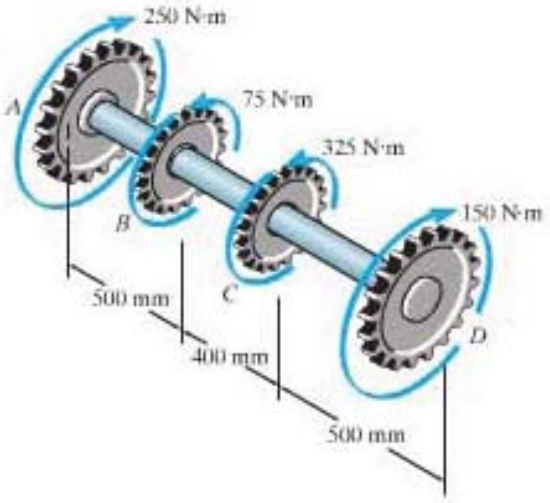
*5–12. The motor delivers a torque of $50\ \text{N} \cdot \text{m}$ to the shaft AB. This torque is transmitted to shaft CD using the gears at E and F. Determine the equilibrium torque $\mathbf{T}'$ on shaft CD and the maximum shear stress in each shaft. The bearings B, C, and D allow free rotation of the shafts.

•5–13. If the applied torque on shaft CD is $T' = 75\ \text{N} \cdot \text{m}$, determine the absolute maximum shear stress in each shaft. The bearings B, C, and D allow free rotation of the shafts, and the motor holds the shafts fixed from rotating.



Probs. 5–12/13

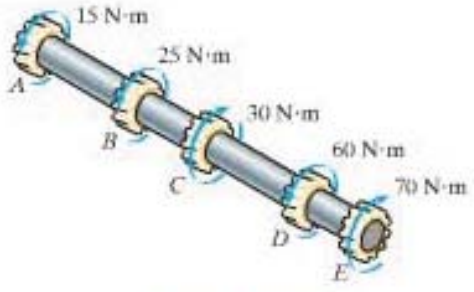
5–14. The solid 50-mm-diameter shaft is used to transmit the torques applied to the gears. Determine the absolute maximum shear stress in the shaft.



Prob. 5–14

5–15. The solid shaft is made of material that has an allowable shear stress of $\tau_{allow} = 10$ MPa. Determine the required diameter of the shaft to the nearest mm.

***5–16.** The solid shaft has a diameter of 40 mm. Determine the absolute maximum shear stress in the shaft and sketch the shear-stress distribution along a radial line of the shaft where the shear stress is maximum.



Probs. 5–15/16

•**5–17.** The rod has a diameter of 1 in. and a weight of 10 lb/ft. Determine the maximum torsional stress in the rod at a section located at A due to the rod's weight.

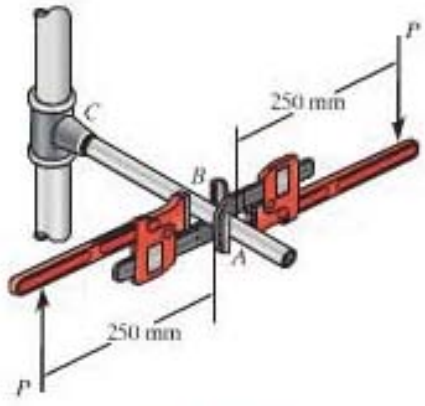
5–18. The rod has a diameter of 1 in. and a weight of 15 lb/ft. Determine the maximum torsional stress in the rod at a section located at B due to the rod's weight.



Probs. 5–17/18

5–19. Two wrenches are used to tighten the pipe. If $P = 300$ N is applied to each wrench, determine the maximum torsional shear stress developed within regions AB and BC. The pipe has an outer diameter of 25 mm and inner diameter of 20 mm. Sketch the shear stress distribution for both cases.

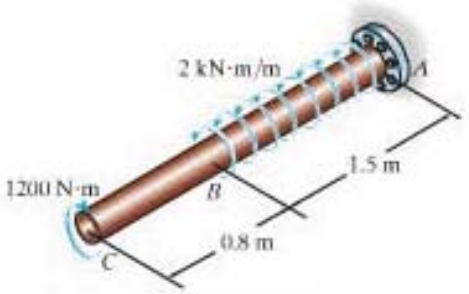
***5–20.** Two wrenches are used to tighten the pipe. If the pipe is made from a material having an allowable shear stress of $\tau_{allow} = 85$ MPa, determine the allowable maximum force **P** that can be applied to each wrench. The pipe has an outer diameter of 25 mm and inner diameter of 20 mm.



Probs. 5–19/20

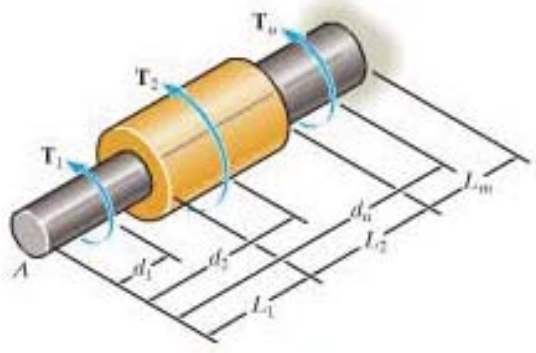
•5–21. The 60-mm-diameter solid shaft is subjected to the distributed and concentrated torsional loadings shown. Determine the absolute maximum and minimum shear stresses on the outer surface of the shaft and specify their locations, measured from the fixed end A.

5–22. The solid shaft is subjected to the distributed and concentrated torsional loadings shown. Determine the required diameter d of the shaft to the nearest mm if the allowable shear stress for the material is $\tau_{allow} = 50$ MPa.



Probs. 5–21/22

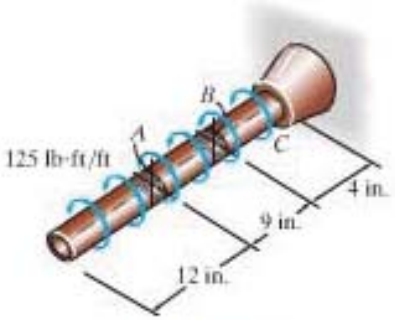
5–23. Consider the general problem of a circular shaft made from m segments each having a radius of c_m. If there are n torques on the shaft as shown, write a computer program that can be used to determine the maximum shear stress at any specified location x along the shaft. Show an application of the program using the values $L_1 = 2$ ft, $c_1 = 2$ in., $L_2 = 4$ ft, $c_2 = 1$ in., $T_1 = 800$ lb · ft, $d_1 = 0$, $T_2 = -600$ lb · ft, $d_2 = 5$ ft.



Prob. 5–23

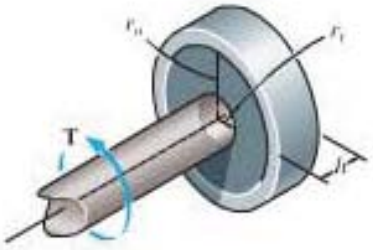
*5–24. The copper pipe has an outer diameter of 2.50 in. and an inner diameter of 2.30 in. If it is tightly secured to the wall at C and a uniformly distributed torque is applied to it as shown, determine the shear stress developed at points A and B. These points lie on the pipe's outer surface. Sketch the shear stress on volume elements located at A and B.

•5–25. The copper pipe has an outer diameter of 2.50 in. and an inner diameter of 2.30 in. If it is tightly secured to the wall at C and it is subjected to the uniformly distributed torque along its entire length, determine the absolute maximum shear stress in the pipe. Discuss the validity of this result.



Probs. 5–24/25

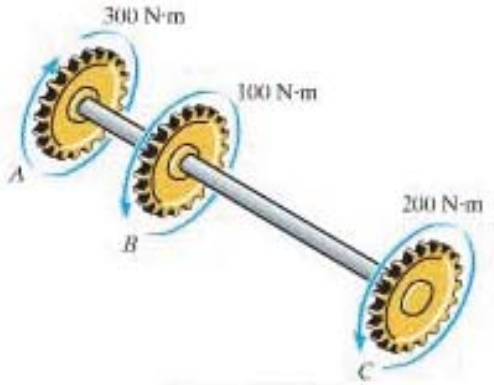
5–26. A cylindrical spring consists of a rubber annulus bonded to a rigid ring and shaft. If the ring is held fixed and a torque **T** is applied to the shaft, determine the maximum shear stress in the rubber.



Prob. 5–26

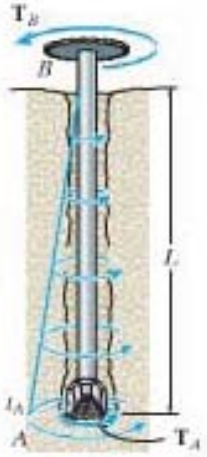
5–27. The A-36 steel shaft is supported on smooth bearings that allow it to rotate freely. If the gears are subjected to the torques shown, determine the maximum shear stress developed in the segments AB and BC. The shaft has a diameter of 40 mm.

***5–28.** The A-36 steel shaft is supported on smooth bearings that allow it to rotate freely. If the gears are subjected to the torques shown, determine the required diameter of the shaft to the nearest mm if $\tau_{allow} = 60$ MPa.



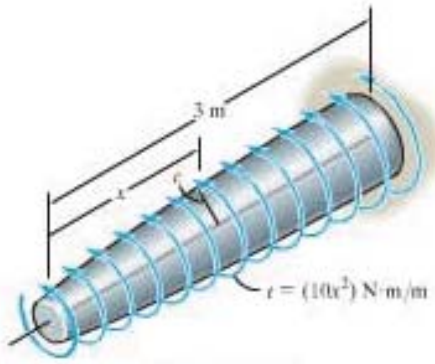
Probs. 5–27/28

•**5–29.** When drilling a well at constant angular velocity, the bottom end of the drill pipe encounters a torsional resistance T_A. Also, soil along the sides of the pipe creates a distributed frictional torque along its length, varying uniformly from zero at the surface B to t_A at A. Determine the minimum torque T_B that must be supplied by the drive unit to overcome the resisting torques, and compute the maximum shear stress in the pipe. The pipe has an outer radius r_o and an inner radius r_i.



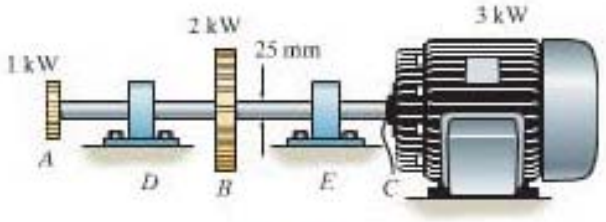
Prob. 5–29

5–30. The shaft is subjected to a distributed torque along its length of $t = (10x^2)$ N·m/m, where x is in meters. If the maximum stress in the shaft is to remain constant at 80 MPa, determine the required variation of the radius c of the shaft for $0 \le x \le 3$ m.



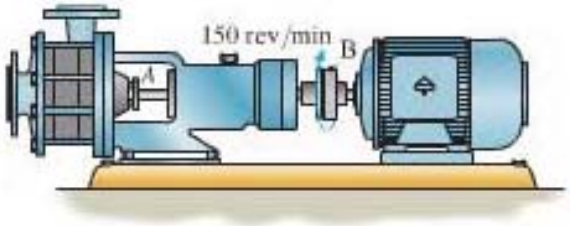
Prob. 5–30

5–31. The solid steel shaft AC has a diameter of 25 mm and is supported by smooth bearings at D and E. It is coupled to a motor at C, which delivers 3 kW of power to the shaft while it is turning at 50 rev/s. If gears A and B remove 1 kW and 2 kW, respectively, determine the maximum shear stress developed in the shaft within regions AB and BC. The shaft is free to turn in its support bearings D and E.



Prob. 5–31

***5–32.** The pump operates using the motor that has a power of 85 W. If the impeller at B is turning at 150 rev/min, determine the maximum shear stress developed in the 20-mm-diameter transmission shaft at A.



Prob. 5–32

•5–33. The gear motor can develop 2 hp when it turns at 450 rev/min. If the shaft has a diameter of 1 in., determine the maximum shear stress developed in the shaft.

5–34. The gear motor can develop 3 hp when it turns at 150 rev/min. If the allowable shear stress for the shaft is $\tau_{allow} = 12$ ksi, determine the smallest diameter of the shaft to the nearest $\frac{1}{8}$ in. that can be used.



Probs. 5–33/34

5–35. The 25-mm-diameter shaft on the motor is made of a material having an allowable shear stress of $\tau_{allow} = 75$ MPa. If the motor is operating at its maximum power of 5 kW, determine the minimum allowable rotation of the shaft.

*5–36. The drive shaft of the motor is made of a material having an allowable shear stress of $\tau_{allow} = 75$ MPa. If the outer diameter of the tubular shaft is 20 mm and the wall thickness is 2.5 mm, determine the maximum allowable power that can be supplied to the motor when the shaft is operating at an angular velocity of 1500 rev/min.



Probs. 5–35/36

•5–37. A ship has a propeller drive shaft that is turning at 1500 rev/min while developing 1800 hp. If it is 8 ft long and has a diameter of 4 in., determine the maximum shear stress in the shaft caused by torsion.

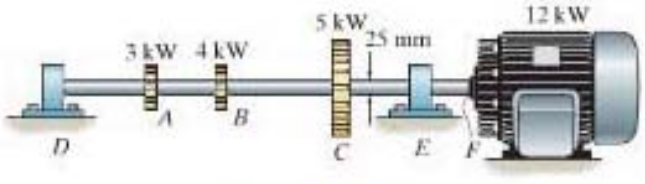
5–38. The motor A develops a power of 300 W and turns its connected pulley at 90 rev/min. Determine the required diameters of the steel shafts on the pulleys at A and B if the allowable shear stress is $\tau_{allow} = 85$ MPa.



Prob. 5–38

5–39. The solid steel shaft DF has a diameter of 25 mm and is supported by smooth bearings at D and E. It is coupled to a motor at F, which delivers 12 kW of power to the shaft while it is turning at 50 rev/s. If gears A, B, and C remove 3 kW, 4 kW, and 5 kW respectively, determine the maximum shear stress developed in the shaft within regions CF and BC. The shaft is free to turn in its support bearings D and E.

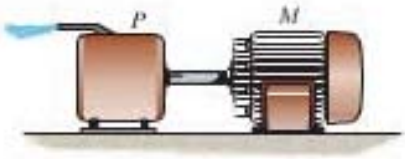
*5–40. Determine the absolute maximum shear stress developed in the shaft in Prob. 5–39.



Probs. 5–39/40

•5–41. The A-36 steel tubular shaft is 2 m long and has an outer diameter of 50 mm. When it is rotating at 40 rad/s, it transmits 25 kW of power from the motor M to the pump P. Determine the smallest thickness of the tube if the allowable shear stress is $\tau_{allow} = 80$ MPa.

5–42. The A-36 solid tubular steel shaft is 2 m long and has an outer diameter of 60 mm. It is required to transmit 60 kW of power from the motor M to the pump P. Determine the smallest angular velocity the shaft can have if the allowable shear stress is $\tau_{allow} = 80$ MPa.



Probs. 5–41/42

5–43. A steel tube having an outer diameter of 2.5 in. is used to transmit 35 hp when turning at 2700 rev/min. Determine the inner diameter d of the tube to the nearest $\frac{1}{8}$ in. if the allowable shear stress is $\tau_{allow} = 10$ ksi.



Prob. 5–43

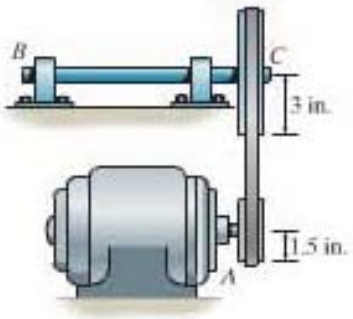
***5–44.** The drive shaft AB of an automobile is made of a steel having an allowable shear stress of $\tau_{allow} = 8$ ksi. If the outer diameter of the shaft is 2.5 in. and the engine delivers 200 hp to the shaft when it is turning at 1140 rev/min, determine the minimum required thickness of the shaft's wall.

•5–45. The drive shaft AB of an automobile is to be designed as a thin-walled tube. The engine delivers 150 hp when the shaft is turning at 1500 rev/min. Determine the minimum thickness of the shaft's wall if the shaft's outer diameter is 2.5 in. The material has an allowable shear stress of $\tau_{allow} = 7$ ksi.



Probs. 5–44/45

5–46. The motor delivers 15 hp to the pulley at A while turning at a constant rate of 1800 rpm. Determine to the nearest $\frac{1}{8}$ in. the smallest diameter of shaft BC if the allowable shear stress for steel is $\tau_{allow} = 12$ ksi. The belt does not slip on the pulley.



Prob. 5–46

5.4 Angle of Twist



Oil wells are commonly drilled to depths exceeding a thousand meters. As a result, the total angle of twist of a string of drill pipe can be substantial and must be determined.

Occasionally the design of a shaft depends on restricting the amount of rotation or twist that may occur when the shaft is subjected to a torque. Furthermore, being able to compute the angle of twist for a shaft is important when analyzing the reactions on statically indeterminate shafts.

In this section we will develop a formula for determining the ***angle of twist*** ϕ (phi) of one end of a shaft with respect to its other end. The shaft is assumed to have a circular cross section that can gradually vary along its length, Fig. 5–14*a*. Also, the material is assumed to be homogeneous and to behave in a linear-elastic manner when the torque is applied. Like the case of an axially loaded bar, we will neglect the localized deformations that occur at points of application of the torques and where the cross section changes abruptly. By Saint-Venant's principle, these effects occur within small regions of the shaft's length and generally they will have only a slight effect on the final result.

Using the method of sections, a differential disk of thickness dx, located at position x, is isolated from the shaft, Fig. 5–14*b*. The internal resultant torque is $T(x)$, since the external loading may cause it to vary along the axis of the shaft. Due to $T(x)$, the disk will twist, such that the *relative rotation* of one of its faces with respect to the other face is $d\phi$, Fig. 5–14*b*. As a result an element of material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The values of γ and $d\phi$ are related by Eq. 5–1, namely,

$$d\phi = \gamma \frac{dx}{\rho} \tag{5–13}$$

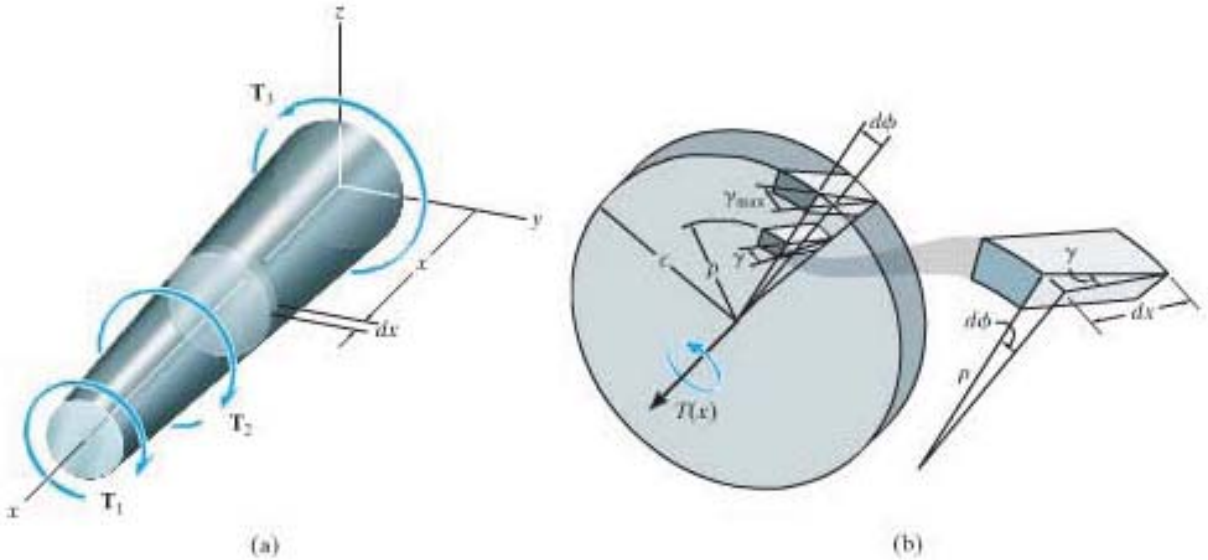


Fig. 5–14

Since Hooke's law, $\gamma = \tau/G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho/J(x)$, then $\gamma = T(x)\rho/J(x)G$. Substituting this into Eq. 5–13, the angle of twist for the disk is

$$d\phi = \frac{T(x)}{J(x)G}\,dx$$

Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

$$\boxed{\phi = \int_0^L \frac{T(x)\,dx}{J(x)G}} \tag{5–14}$$

Here

ϕ = the angle of twist of one end of the shaft with respect to the other end, measured in radians

$T(x)$ = the *internal torque* at the arbitrary position x, found from the method of sections and the equation of moment equilibrium applied about the shaft's axis

$J(x)$ = the shaft's polar moment of inertia expressed as a function of position x

G = the shear modulus of elasticity for the material

Constant Torque and Cross-Sectional Area. Usually in engineering practice the material is homogeneous so that G is constant. Also, the shaft's cross-sectional area and the external torque are constant along the length of the shaft, Fig. 5–15. If this is the case, the internal torque $T(x) = T$, the polar moment of inertia $J(x) = J$, and Eq. 5–14 can be integrated, which gives

$$\boxed{\phi = \frac{TL}{JG}} \tag{5–15}$$

The similarities between the above two equations and those for an axially loaded bar ($\delta = \int P(x)\,dx/A(x)E$ and $\delta = PL/AE$) should be noted.

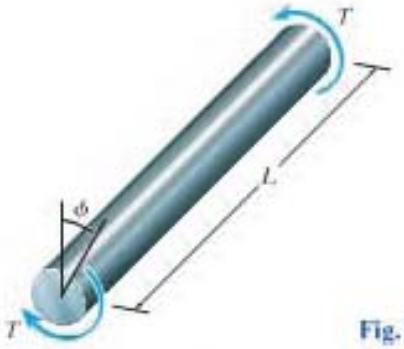


Fig. 5–15



When computing both the stress and the angle of twist of this soil auger, it is necessary to consider the variable torsional loading which acts along its length.

5

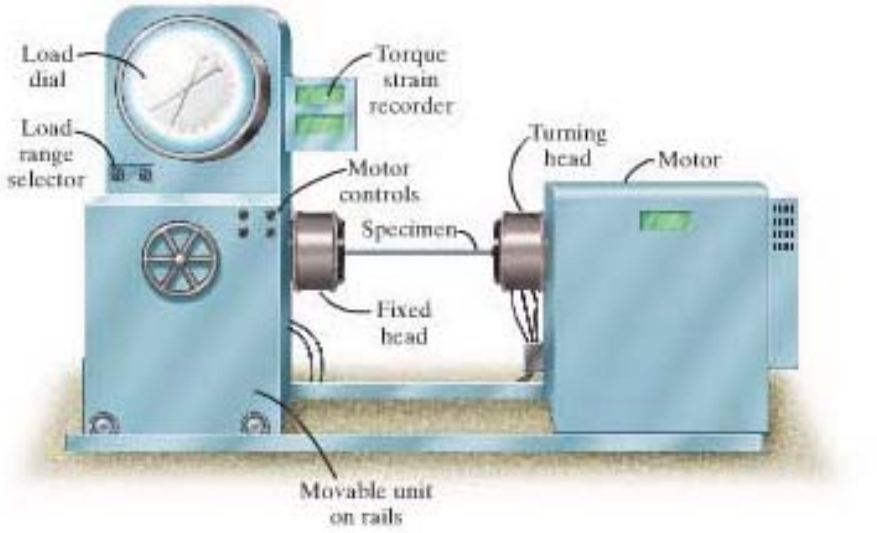


Fig. 5–16

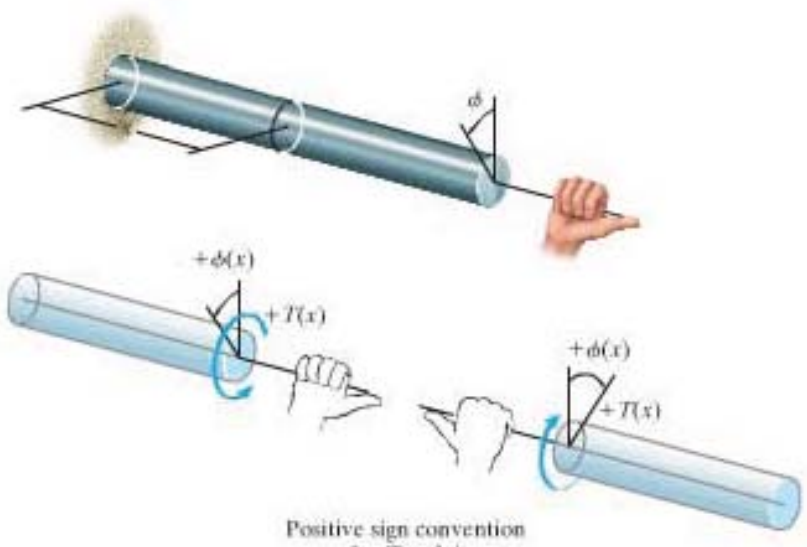
Equation 5–15 is often used to determine the shear modulus of elasticity G of a material. To do so, a specimen of known length and diameter is placed in a torsion testing machine like the one shown in Fig. 5–16. The applied torque T and angle of twist ϕ are then measured along the length L. Using Eq. 5–15, $G = TL/J\phi$. Usually, to obtain a more reliable value of G, several of these tests are performed and the average value is used.

Multiple Torques. If the shaft is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next, Eq. 5–15 can be applied to each segment of the shaft where these quantities are all constant. The angle of twist of one end of the shaft with respect to the other is then found from the vector addition of the angles of twist of each segment. For this case,

$$\phi = \sum \frac{TL}{JG} \tag{5–16}$$

Sign Convention. In order to apply this equation, we must develop a sign convention for both the internal torque and the angle of twist of one end of the shaft with respect to the other end. To do this, we will use the right-hand rule, whereby both the torque and angle will be *positive*, provided the *thumb* is directed *outward* from the shaft when the fingers curl to give the tendency for rotation, Fig. 5–17.

To illustrate the use of this sign convention, consider the shaft shown in Fig. 5–18*a*. The angle of twist of end A with respect to end D is to be determined. Three segments of the shaft must be considered, since the



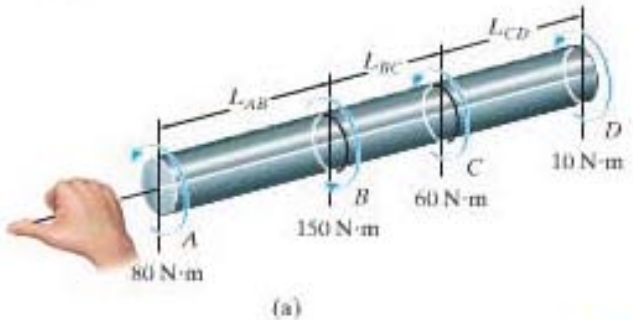
Positive sign convention for T and ϕ.

Fig. 5–17

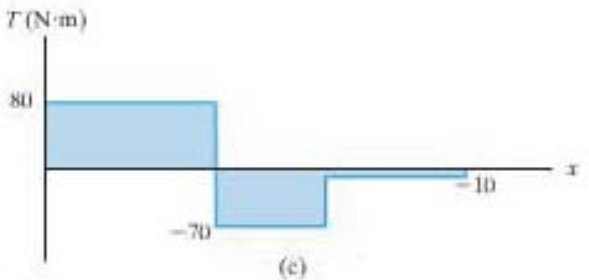
internal torque will change at B and at C. Using the method of sections, the *internal torques* are found for each segment, Fig. 5–18*b*. By the right-hand rule, with positive torques directed away from the *sectioned end* of the shaft, we have $T_{AB} = +80\ \text{N}\cdot\text{m}$, $T_{BC} = -70\ \text{N}\cdot\text{m}$, and $T_{CD} = -10\ \text{N}\cdot\text{m}$. These results are also shown on the *torque diagram* for the shaft, Fig. 5–18*c*. Applying Eq. 5–16, we have

$$\phi_{A/D} = \frac{(+80\ \text{N}\cdot\text{m})\,L_{AB}}{JG} + \frac{(-70\ \text{N}\cdot\text{m})\,L_{BC}}{JG} + \frac{(-10\ \text{N}\cdot\text{m})\,L_{CD}}{JG}$$

If the other data is substituted and the answer is found as a *positive* quantity, it means that end A will rotate as indicated by the curl of the right-hand fingers when the thumb is directed *away* from the shaft, Fig. 5–18*a*. The double subscript notation is used to indicate this relative angle of twist ($\phi_{A/D}$); however, if the angle of twist is to be determined relative to a *fixed support*, then only a single subscript will be used. For example, if D is a fixed support, then the angle of twist will be denoted as ϕ_A.

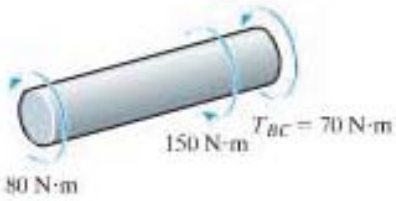


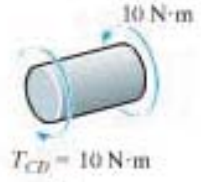
(a)



(c)







(b)

Fig. 5–18

Important Point

- When applying Eq. 5–14 to determine the angle of twist, it is important that the applied torques do not cause yielding of the material and that the material is homogeneous and behaves in a linear elastic manner.

Procedure for Analysis

The angle of twist of one end of a shaft or tube with respect to the other end can be determined using the following procedure.

Internal Torque.

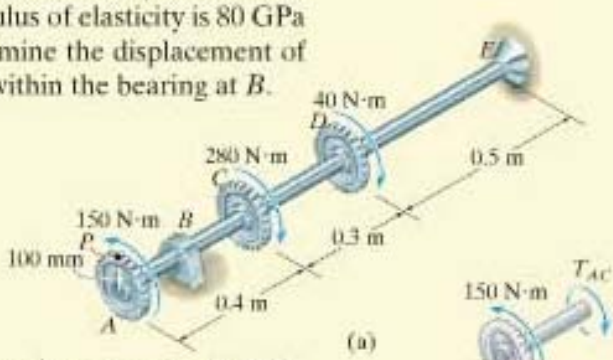
- The *internal torque* is found at a point on the axis of the shaft by using the method of sections and the equation of moment equilibrium, applied along the shaft's axis.
- If the torque varies along the shaft's length, a section should be made at the arbitrary position x along the shaft and the *internal torque* represented as a function of x, i.e., $T(x)$.
- If several constant external torques act on the shaft between its ends, the internal torque in each *segment* of the shaft, between any two external torques, must be determined. The results can be represented graphically as a torque diagram.

Angle of Twist.

- When the circular cross-sectional area of the shaft varies along the shaft's axis, the polar moment of inertia must be expressed as a function of its position x along the axis, $J(x)$.
- If the polar moment of inertia or the internal torque *suddenly changes* between the ends of the shaft, then $\phi = \int (T(x)/J(x)G)\, dx$ or $\phi = TL/JG$ must be applied to *each segment* for which J, G, and T are continuous or constant.
- When the internal torque in each segment is determined, be sure to use a consistent sign convention for the shaft, such as the one discussed in Fig. 5–17. Also make sure that a consistent set of units is used when substituting numerical data into the equations.

EXAMPLE 5.5

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–19*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. The shaft turns freely within the bearing at B.



(a)

SOLUTION

Internal Torque. By inspection, the torques in segments AC, CD, and DE are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 5–19*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150\,\text{N}\cdot\text{m} \qquad T_{CD} = -130\,\text{N}\cdot\text{m} \qquad T_{DE} = -170\,\text{N}\cdot\text{m}$$

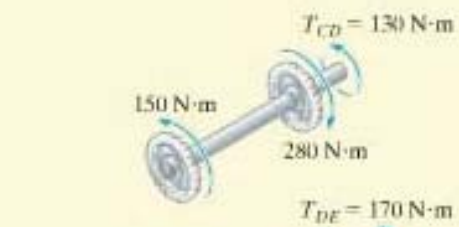
These results are also shown on the torque diagram, Fig. 5–19*c*.

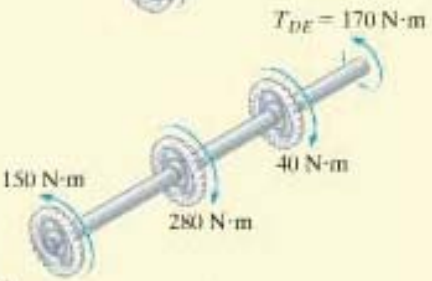
Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\,\text{m})^4 = 3.771(10^{-9})\,\text{m}^4$$

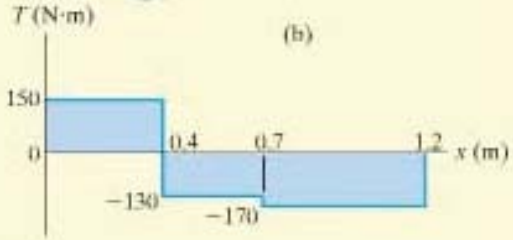
Applying Eq. 5–16 to each segment and adding the results algebraically, we have

$$\phi_A = \sum\frac{TL}{JG} = \frac{(+150\,\text{N}\cdot\text{m})(0.4\,\text{m})}{3.771(10^{-9})\,\text{m}^4\,[80(10^9)\,\text{N/m}^2]} + \frac{(-130\,\text{N}\cdot\text{m})(0.3\,\text{m})}{3.771(10^{-9})\,\text{m}^4\,[80(10^9)\,\text{N/m}^2)]} + \frac{(-170\,\text{N}\cdot\text{m})(0.5\,\text{m})}{3.771(10^{-9})\,\text{m}^4\,[80(10^9)\,\text{N/m}^2)]} = -0.2121\,\text{rad}$$





(b)

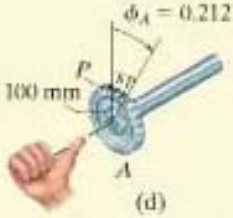


(c)

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end E of the shaft, and therefore gear A will rotate as shown in Fig. 5–19*d*.

The displacement of tooth P on gear A is

$$s_P = \phi_A r = (0.2121\,\text{rad})(100\,\text{mm}) = 21.2\,\text{mm} \qquad \textit{Ans.}$$



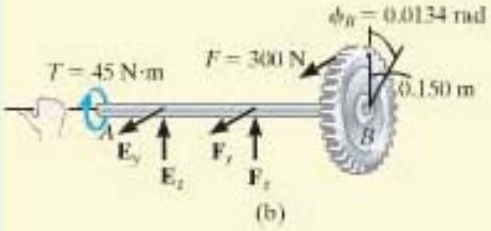
(d)

NOTE: Remember that this analysis is valid only if the shear stress does not exceed the proportional limit of the material.

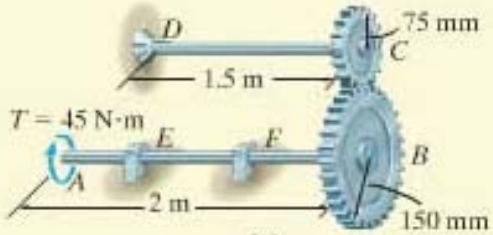
Fig. 5–19

EXAMPLE 5.6

The two solid steel shafts shown in Fig. 5–20*a* are coupled together using the meshed gears. Determine the angle of twist of end *A* of shaft *AB* when the torque $T = 45\ \text{N} \cdot \text{m}$ is applied. Take $G = 80\ \text{GPa}$. Shaft *AB* is free to rotate within bearings *E* and *F*, whereas shaft *DC* is fixed at *D*. Each shaft has a diameter of 20 mm.

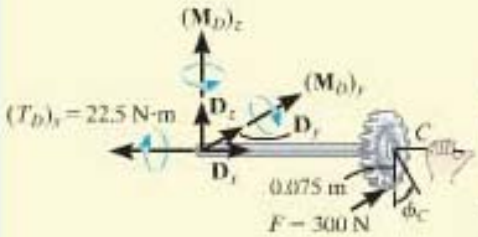


(b)



(a)

Fig. 5–20



(c)

SOLUTION

Internal Torque. Free-body diagrams for each shaft are shown in Fig. 5–20*b* and 5–20*c*. Summing moments along the *x* axis of shaft *AB* yields the tangential reaction between the gears of $F = 45\ \text{N} \cdot \text{m}/0.15\ \text{m} = 300\ \text{N}$. Summing moments about the *x* axis of shaft *DC*, this force then creates a torque of $(T_D)_x = 300\ \text{N}\ (0.075\ \text{m}) = 22.5\ \text{N} \cdot \text{m}$ on shaft *DC*.

Angle of Twist. To solve the problem, we will first calculate the rotation of gear *C* due to the torque of $22.5\ \text{N} \cdot \text{m}$ in shaft *DC*, Fig. 5–20*c*. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5\ \text{N} \cdot \text{m})(1.5\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0269\ \text{rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear *C* causes gear *B* to rotate ϕ_B, Fig. 5–20*b*, where

$$\phi_B(0.15\ \text{m}) = (0.0269\ \text{rad})(0.075\ \text{m})$$
$$\phi_B = 0.0134\ \text{rad}$$

We will now determine the angle of twist of end *A* with respect to end *B* of shaft *AB* caused by the $45\ \text{N} \cdot \text{m}$ torque, Fig. 5–20*b*. We have

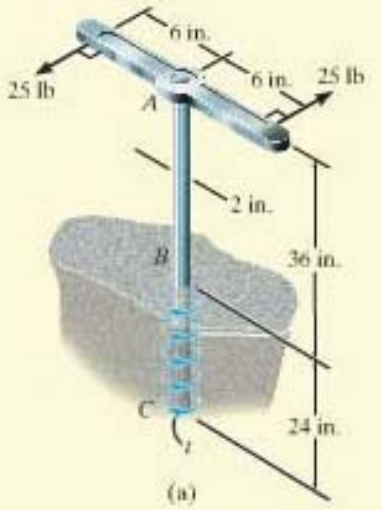
$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45\ \text{N} \cdot \text{m})(2\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0716\ \text{rad}$$

The rotation of end *A* is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the *same direction*, Fig. 5–20*b*. We have

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134\ \text{rad} + 0.0716\ \text{rad} = +0.0850\ \text{rad} \quad \textit{Ans.}$$

EXAMPLE 5.7

The 2-in.-diameter solid cast-iron post shown in Fig. 5–21*a* is buried 24 in. in soil. If a torque is applied to its top using a rigid wrench, determine the maximum shear stress in the post and the angle of twist at its top. Assume that the torque is about to turn the post, and the soil exerts a uniform torsional resistance of t lb · in./in. along its 24-in. buried length. $G = 5.5(10^3)$ ksi.



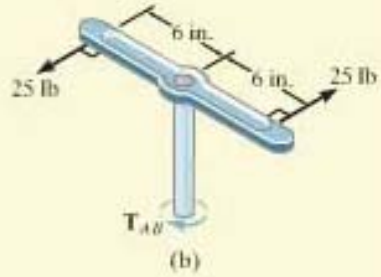
SOLUTION

Internal Torque. The internal torque in segment AB of the post is constant. From the free-body diagram, Fig. 5–21*b*, we have

$$\Sigma M_z = 0; \qquad T_{AB} = 25 \text{ lb}\,(12 \text{ in.}) = 300 \text{ lb} \cdot \text{in.}$$

The magnitude of the uniform distribution of torque along the buried segment BC can be determined from equilibrium of the entire post, Fig. 5–21*c*. Here

$$\Sigma M_z = 0 \qquad 25 \text{ lb}\,(12 \text{ in.}) - t(24 \text{ in.}) = 0$$
$$t = 12.5 \text{ lb} \cdot \text{in./in.}$$



Hence, from a free-body diagram of a section of the post located at the position x, Fig. 5–21*d*, we have

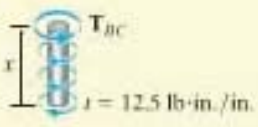
$$\Sigma M_z = 0; \qquad T_{BC} - 12.5x = 0$$
$$T_{BC} = 12.5x$$

Maximum Shear Stress. The largest shear stress occurs in region AB, since the torque is largest there and J is constant for the post. Applying the torsion formula, we have

$$\tau_{max} = \frac{T_{AB}c}{J} = \frac{(300 \text{ lb} \cdot \text{in.})(1 \text{ in.})}{(\pi/2)(1 \text{ in.})^4} = 191 \text{ psi} \qquad Ans$$

Angle of Twist. The angle of twist at the top can be determined relative to the bottom of the post, since it is fixed and yet is about to turn. Both segments AB and BC twist, and so in this case we have

$$\phi_A = \frac{T_{AB}L_{AB}}{JG} + \int_0^{L_{BC}} \frac{T_{BC}\,dx}{JG}$$
$$= \frac{(300 \text{ lb} \cdot \text{in.})36 \text{ in.}}{JG} + \int_0^{24 \text{ in.}} \frac{12.5x\,dx}{JG}$$
$$= \frac{10\,800 \text{ lb} \cdot \text{in}^2}{JG} + \frac{12.5[(24)^2/2] \text{ lb} \cdot \text{in}^2}{JG}$$
$$= \frac{14\,400 \text{ lb} \cdot \text{in}^2}{(\pi/2)(1 \text{ in.})^4 5500(10^3) \text{ lb/in}^2} = 0.00167 \text{ rad} \qquad Ans$$



(d)

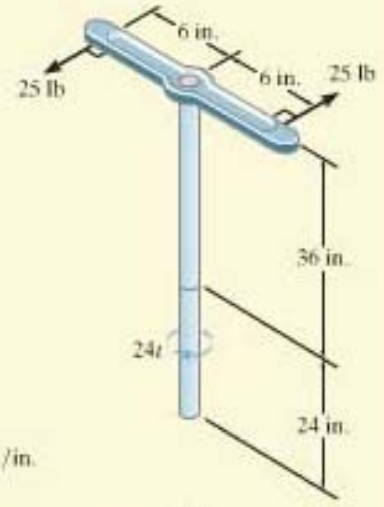
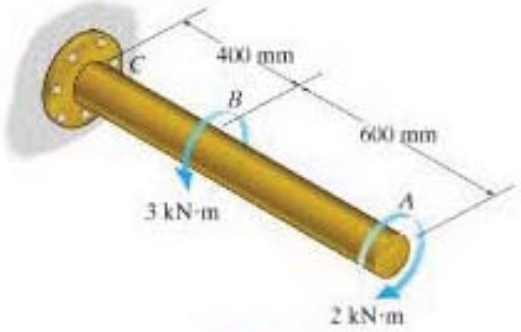


Fig. 5–21

FUNDAMENTAL PROBLEMS

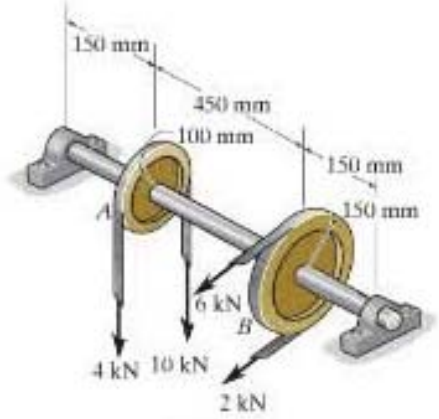
F5–7. The 60-mm-diameter A-36 steel shaft is subjected to the torques shown. Determine the angle of twist of end A with respect to C.



F5–7

5

F5–8. Determine the angle of twist of wheel B with respect to wheel A. The shaft has a diameter of 40 mm and is made of A-36 steel.



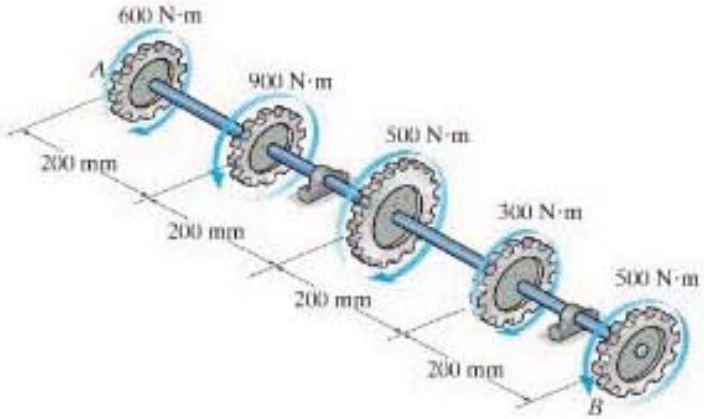
F5–8

F5–9. The hollow 6061-T6 aluminum shaft has an outer and inner radius of $c_o = 40$ mm and $c_i = 30$ mm, respectively. Determine the angle of twist of end A. The flexible support at B has a torsional stiffness of $k = 90$ kN · m/rad.



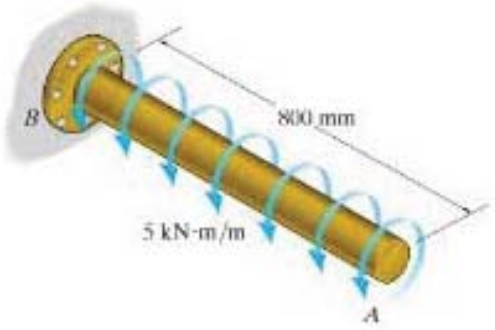
F5–9

F5–10. A series of gears are mounted on the 40-mm-diameter A-36 steel shaft. Determine the angle of twist of gear B relative to gear A.



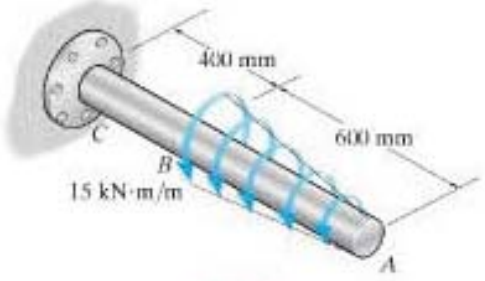
F5–10

F5–11. The 80-mm-diameter shaft is made of A-36 steel. If it is subjected to the uniform distributed torque, determine the angle of twist of end A with respect to B.



F5–11

F5–12. The 80-mm-diameter shaft is made of A-36 steel. If it is subjected to the triangular distributed load, determine the angle of twist of end A with respect to C.



F5–12

PROBLEMS

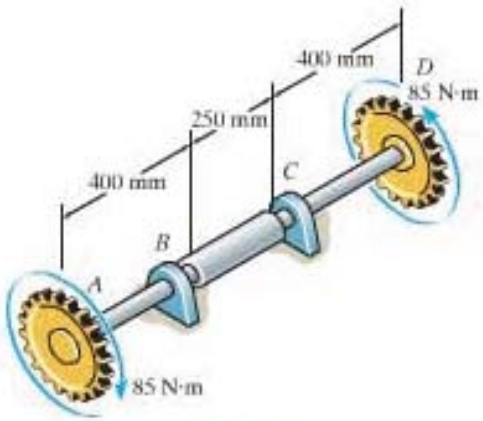
5–47. The propellers of a ship are connected to a A-36 steel shaft that is 60 m long and has an outer diameter of 340 mm and inner diameter of 260 mm. If the power output is 4.5 MW when the shaft rotates at 20 rad/s, determine the maximum torsional stress in the shaft and its angle of twist.

***5–48.** A shaft is subjected to a torque **T**. Compare the effectiveness of using the tube shown in the figure with that of a solid section of radius c. To do this, compute the percent increase in torsional stress and angle of twist per unit length for the tube versus the solid section.



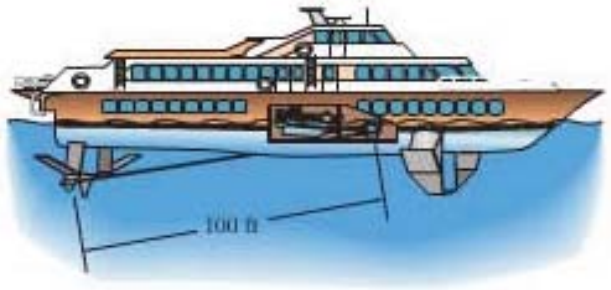
Prob. 5–48

•5–49. The A-36 steel axle is made from tubes AB and CD and a solid section BC. It is supported on smooth bearings that allow it to rotate freely. If the gears, fixed to its ends, are subjected to 85-N · m torques, determine the angle of twist of gear A relative to gear D. The tubes have an outer diameter of 30 mm and an inner diameter of 20 mm. The solid section has a diameter of 40 mm.



Prob. 5–49

5–50. The hydrofoil boat has an A-36 steel propeller shaft that is 100 ft long. It is connected to an in-line diesel engine that delivers a maximum power of 2500 hp and causes the shaft to rotate at 1700 rpm. If the outer diameter of the shaft is 8 in. and the wall thickness is $\frac{1}{8}$ in., determine the maximum shear stress developed in the shaft. Also, what is the "wind up," or angle of twist in the shaft at full power?



Prob. 5–50

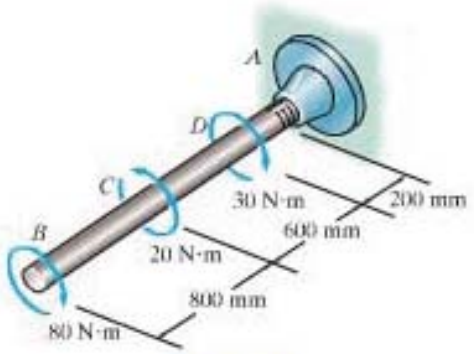
5–51. The engine of the helicopter is delivering 600 hp to the rotor shaft AB when the blade is rotating at 1200 rev/min. Determine to the nearest $\frac{1}{8}$ in. the diameter of the shaft AB if the allowable shear stress is $\tau_{allow} = 8$ ksi and the vibrations limit the angle of twist of the shaft to 0.05 rad. The shaft is 2 ft long and made from L2 steel.

***5–52.** The engine of the helicopter is delivering 600 hp to the rotor shaft AB when the blade is rotating at 1200 rev/min. Determine to the nearest $\frac{1}{8}$ in. the diameter of the shaft AB if the allowable shear stress is $\tau_{allow} = 10.5$ ksi and the vibrations limit the angle of twist of the shaft to 0.05 rad. The shaft is 2 ft long and made from L2 steel.



Probs. 5–51/52

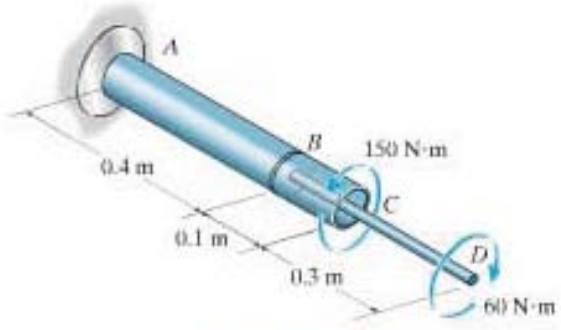
•5–53. The 20-mm-diameter A-36 steel shaft is subjected to the torques shown. Determine the angle of twist of the end B.



Prob. 5–53

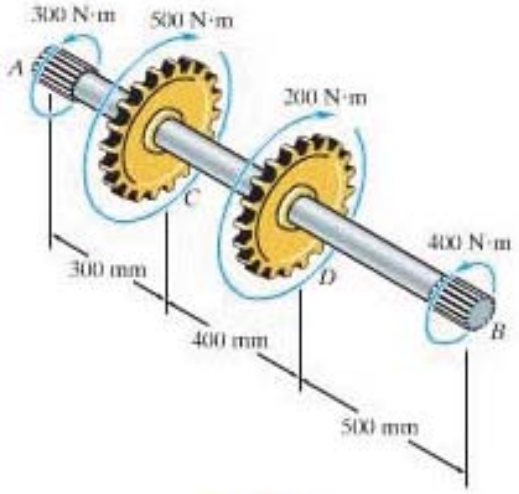
5–54. The assembly is made of A-36 steel and consists of a solid rod 20 mm in diameter fixed to the inside of a tube using a rigid disk at B. Determine the angle of twist at D. The tube has an outer diameter of 40 mm and wall thickness of 5 mm.

5–55. The assembly is made of A-36 steel and consists of a solid rod 20 mm in diameter fixed to the inside of a tube using a rigid disk at B. Determine the angle of twist at C. The tube has an outer diameter of 40 mm and wall thickness of 5 mm.



Probs. 5–54/55

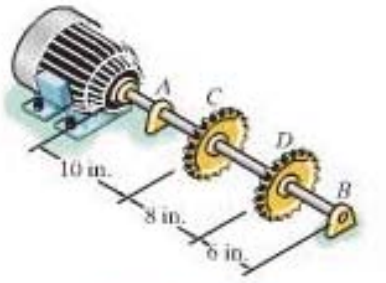
*5–56. The splined ends and gears attached to the A-36 steel shaft are subjected to the torques shown. Determine the angle of twist of end B with respect to end A. The shaft has a diameter of 40 mm.



Prob. 5–56

•5–57. The motor delivers 40 hp to the 304 stainless steel shaft while it rotates at 20 Hz. The shaft is supported on smooth bearings at A and B, which allow free rotation of the shaft. The gears C and D fixed to the shaft remove 25 hp and 15 hp, respectively. Determine the diameter of the shaft to the nearest $\frac{1}{8}$ in. if the allowable shear stress is $\tau_{allow} = 8$ ksi and the allowable angle of twist of C with respect to D is 0.20°.

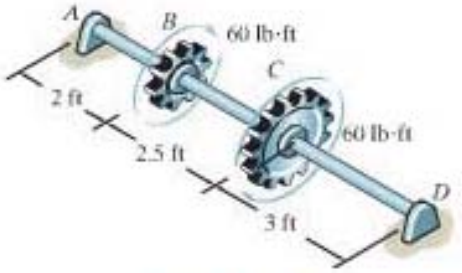
5–58. The motor delivers 40 hp to the 304 stainless steel solid shaft while it rotates at 20 Hz. The shaft has a diameter of 1.5 in. and is supported on smooth bearings at A and B, which allow free rotation of the shaft. The gears C and D fixed to the shaft remove 25 hp and 15 hp, respectively. Determine the absolute maximum stress in the shaft and the angle of twist of gear C with respect to gear D.



Probs. 5–57/58

5–59. The shaft is made of A-36 steel. It has a diameter of 1 in. and is supported by bearings at A and D, which allow free rotation. Determine the angle of twist of B with respect to D.

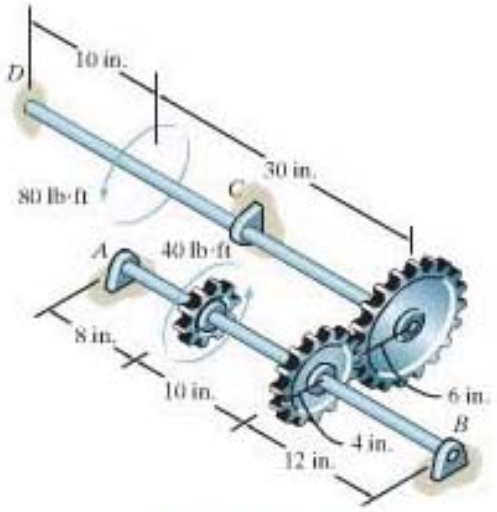
***5–60.** The shaft is made of A-36 steel. It has a diameter of 1 in. and is supported by bearings at A and D, which allow free rotation. Determine the angle of twist of gear C with respect to B.



Probs. 5–59/60

•5–61. The two shafts are made of A-36 steel. Each has a diameter of 1 in., and they are supported by bearings at A, B, and C, which allow free rotation. If the support at D is fixed, determine the angle of twist of end B when the torques are applied to the assembly as shown.

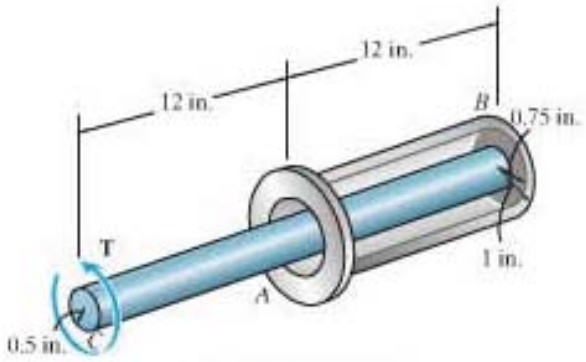
5–62. The two shafts are made of A-36 steel. Each has a diameter of 1 in., and they are supported by bearings at A, B, and C, which allow free rotation. If the support at D is fixed, determine the angle of twist of end A when the torques are applied to the assembly as shown.



Probs. 5–61/62

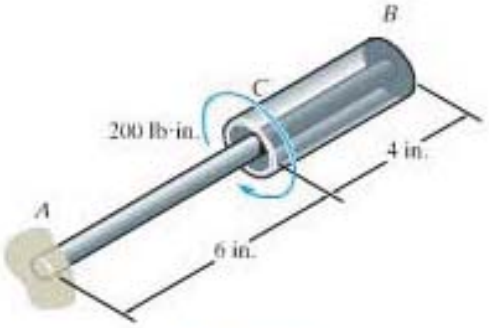
5–63. The device serves as a compact torsional spring. It is made of A-36 steel and consists of a solid inner shaft CB which is surrounded by and attached to a tube AB using a rigid ring at B. The ring at A can also be assumed rigid and is fixed from rotating. If a torque of $T = 2$ kip · in. is applied to the shaft, determine the angle of twist at the end C and the maximum shear stress in the tube and shaft.

***5–64.** The device serves as a compact torsion spring. It is made of A-36 steel and consists of a solid inner shaft CB which is surrounded by and attached to a tube AB using a rigid ring at B. The ring at A can also be assumed rigid and is fixed from rotating. If the allowable shear stress for the material is $\tau_{allow} = 12$ ksi and the angle of twist at C is limited to $\phi_{allow} = 3°$, determine the maximum torque T that can be applied at the end C.



Probs. 5–63/64

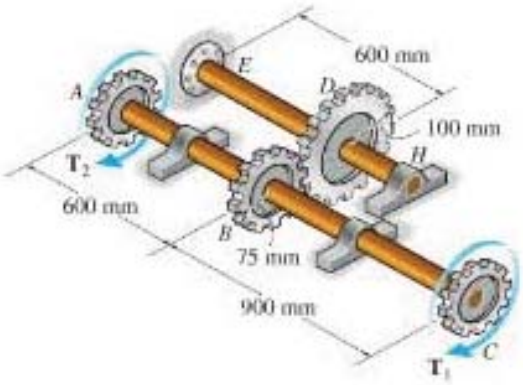
•5–65. The A-36 steel assembly consists of a tube having an outer radius of 1 in. and a wall thickness of 0.125 in. Using a rigid plate at B, it is connected to the solid 1-in-diameter shaft AB. Determine the rotation of the tube's end C if a torque of 200 lb · in. is applied to the tube at this end. The end A of the shaft is fixed supported.



Prob. 5–65

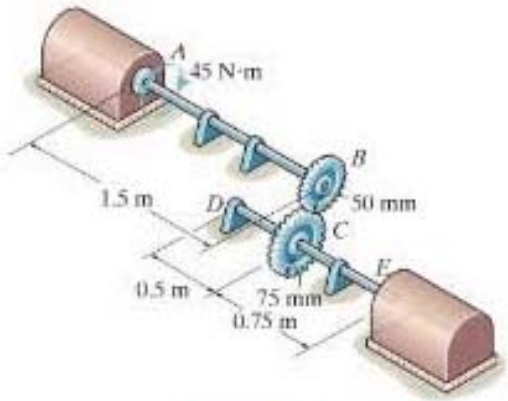
5–66. The 60-mm diameter shaft ABC is supported by two journal bearings, while the 80-mm diameter shaft EH is fixed at E and supported by a journal bearing at H. If $T_1 = 2$ kN·m and $T_2 = 4$ kN·m, determine the angle of twist of gears A and C. The shafts are made of A-36 steel.

5–67. The 60-mm diameter shaft ABC is supported by two journal bearings, while the 80-mm diameter shaft EH is fixed at E and supported by a journal bearing at H. If the angle of twist at gears A and C is required to be 0.04 rad, determine the magnitudes of the torques T_1 and T_2. The shafts are made of A-36 steel.



Probs. 5–66/67

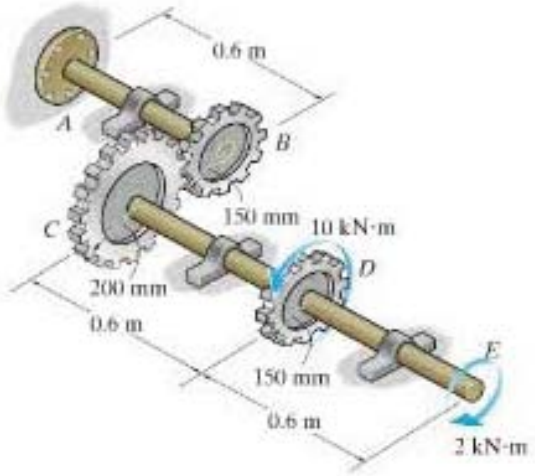
***5–68.** The 30-mm-diameter shafts are made of L2 tool steel and are supported on journal bearings that allow the shaft to rotate freely. If the motor at A develops a torque of $T = 45$ N·m on the shaft AB, while the turbine at E is fixed from turning, determine the amount of rotation of gears B and C.



Prob. 5–68

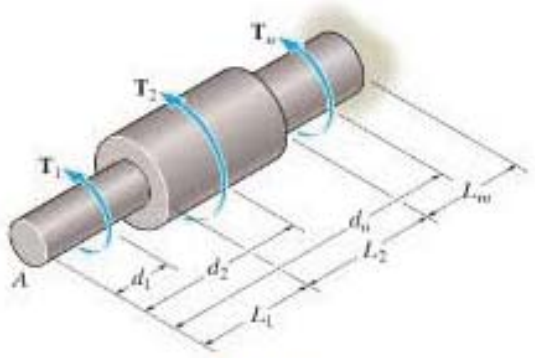
•5–69. The shafts are made of A-36 steel and each has a diameter of 80 mm. Determine the angle of twist at end E.

5–70. The shafts are made of A-36 steel and each has a diameter of 80 mm. Determine the angle of twist of gear D.



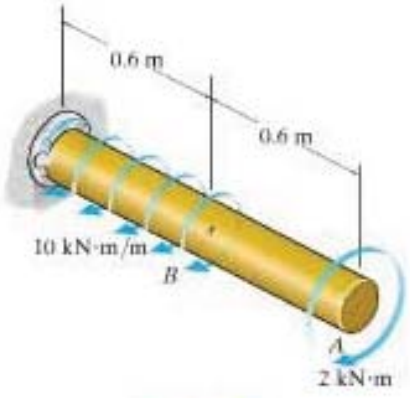
Probs. 5–69/70

5–71. Consider the general problem of a circular shaft made from m segments, each having a radius of c_m and shear modulus G_m. If there are n torques on the shaft as shown, write a computer program that can be used to determine the angle of twist of its end A. Show an application of the program using the values $L_1 = 0.5$ m, $c_1 = 0.02$ m, $G_1 = 30$ GPa, $L_2 = 1.5$ m, $c_2 = 0.05$ m, $G_2 = 15$ GPa, $T_1 = -450$ N·m, $d_1 = 0.25$ m, $T_2 = 600$ N·m, $d_2 = 0.8$ m.



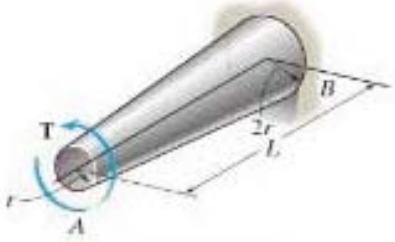
Prob. 5–71

***5–72.** The 80-mm diameter shaft is made of 6061-T6 aluminum alloy and subjected to the torsional loading shown. Determine the angle of twist at end A.



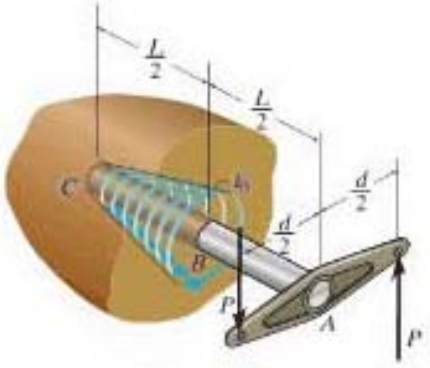
Prob. 5–72

•5–73. The tapered shaft has a length L and a radius r at end A and $2r$ at end B. If it is fixed at end B and is subjected to a torque T, determine the angle of twist of end A. The shear modulus is G.



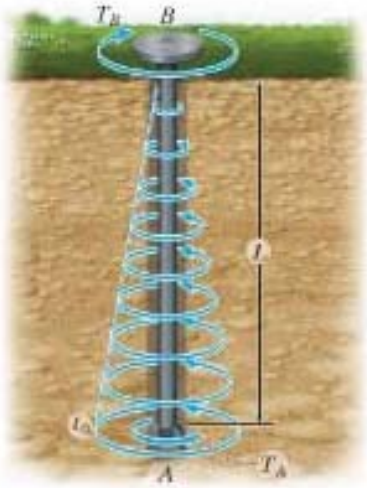
Prob. 5–73

5–74. The rod ABC of radius c is embedded into a medium where the distributed torque reaction varies linearly from zero at C to t_0 at B. If couple forces P are applied to the lever arm, determine the value of t_0 for equilibrium. Also, find the angle of twist of end A. The rod is made from material having a shear modulus of G.



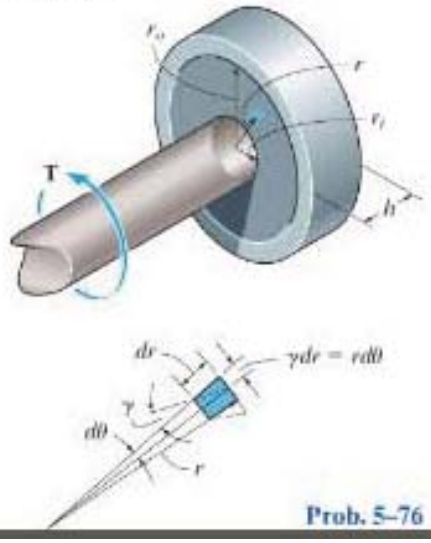
Prob. 5–74

5–75. When drilling a well, the deep end of the drill pipe is assumed to encounter a torsional resistance T_A. Furthermore, soil friction along the sides of the pipe creates a linear distribution of torque per unit length, varying from zero at the surface B to t_0 at A. Determine the necessary torque T_B that must be supplied by the drive unit to turn the pipe. Also, what is the relative angle of twist of one end of the pipe with respect to the other end at the instant the pipe is about to turn? The pipe has an outer radius r_o and an inner radius r_i. The shear modulus is G.



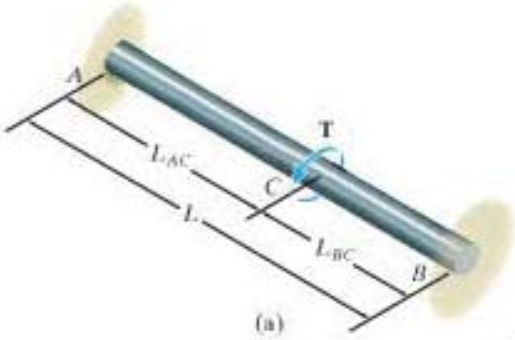
Prob. 5–75

***5–76.** A cylindrical spring consists of a rubber annulus bonded to a rigid ring and shaft. If the ring is held fixed and a torque T is applied to the rigid shaft, determine the angle of twist of the shaft. The shear modulus of the rubber is G. *Hint:* As shown in the figure, the deformation of the element at radius r can be determined from $r d\theta = dr\gamma$. Use this expression along with $\tau = T/(2\pi r^2 h)$ from Prob. 5–26, to obtain the result.



Prob. 5–76

5.5 Statically Indeterminate Torque-Loaded Members



A torsionally loaded shaft may be classified as statically indeterminate if the moment equation of equilibrium, applied about the axis of the shaft, is not adequate to determine the unknown torques acting on the shaft. An example of this situation is shown in Fig. 5–22*a*. As shown on the free-body diagram, Fig. 5–22*b*, the reactive torques at the supports A and B are unknown. We require that

$$\Sigma M_x = 0; \qquad T - T_A - T_B = 0$$

In order to obtain a solution, we will use the method of analysis discussed in Sec. 4.4. The necessary condition of compatibility, or the kinematic condition, requires the angle of twist of one end of the shaft with respect to the other end to be equal to zero, since the end supports are fixed. Therefore,

$$\phi_{A/B} = 0$$

Provided the material is linear elastic, we can apply the load–displacement relation $\phi = TL/JG$ to express the compatibility condition in terms of the unknown torques. Realizing that the internal torque in segment AC is $+T_A$ and in segment CB it is $-T_B$, Fig. 5–22*c*, we have

$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{BC}}{JG} = 0$$

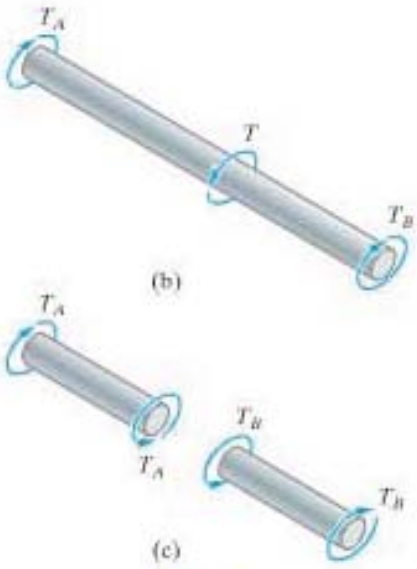


Fig. 5–22

Solving the above two equations for the reactions, realizing that $L = L_{AC} + L_{BC}$, we get

$$T_A = T\left(\frac{L_{BC}}{L}\right) \quad \text{and} \quad T_B = T\left(\frac{L_{AC}}{L}\right)$$

Procedure for Analysis

The unknown torques in statically indeterminate shafts are determined by satisfying equilibrium, compatibility, and torque-displacement requirements for the shaft.

Equilibrium.

- Draw a free-body diagram of the shaft in order to identify all the external torques that act on it. Then write the equation of moment equilibrium about the axis of the shaft.

Compatibility.

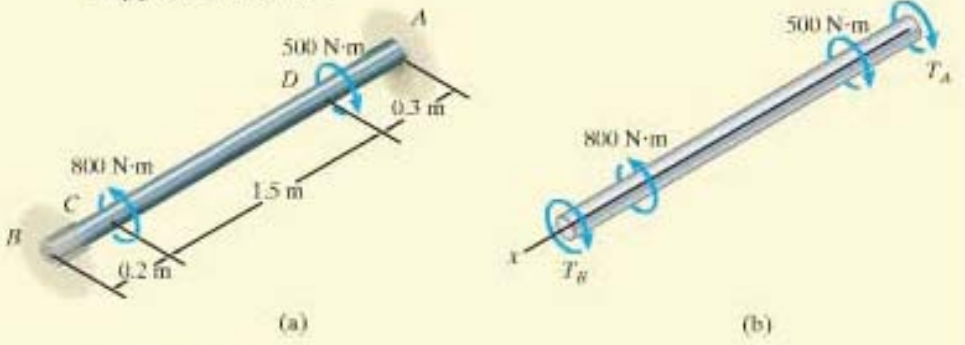
- Write the compatibility equation between two points along the shaft. Give consideration as to how the supports constrain the shaft when it is twisted.
- Express the angles of twist in the compatibility condition in terms of the torques, using a torque-displacement relation, such as $\phi = TL/JG$.
- Solve the equilibrium and compatibility equations for the unknown reactive torques. If any of the magnitudes have a negative numerical value, it indicates that this torque acts in the opposite sense of direction to that shown on the free-body diagram.



The shaft of this cutting machine is fixed at its ends and subjected to a torque at its center, allowing it to act as a torsional spring.

EXAMPLE 5.8

The solid steel shaft shown in Fig. 5–23*a* has a diameter of 20 mm. If it is subjected to the two torques, determine the reactions at the fixed supports A and B.



SOLUTION

Equilibrium. By inspection of the free-body diagram, Fig. 5–23*b*, it is seen that the problem is statically indeterminate since there is only *one* available equation of equilibrium and there are two unknowns. We require

$$\Sigma M_x = 0; \qquad -T_B + 800\ \text{N}\cdot\text{m} - 500\ \text{N}\cdot\text{m} - T_A = 0 \tag{1}$$

Compatibility. Since the ends of the shaft are fixed, the angle of twist of one end of the shaft with respect to the other must be zero. Hence, the compatibility equation becomes

$$\phi_{A/B} = 0$$

This condition can be expressed in terms of the unknown torques by using the load–displacement relationship, $\phi = TL/JG$. Here there are three regions of the shaft where the internal torque is constant. On the free-body diagrams in Fig. 5–23*c* we have shown the internal torques acting on the left segments of the shaft which are sectioned in each of these regions. This way the internal torque is only a function of T_B. Using the sign convention established in Sec. 5.4, we have

$$\frac{-T_B(0.2\ \text{m})}{JG} + \frac{(800 - T_B)(1.5\ \text{m})}{JG} + \frac{(300 - T_B)(0.3\ \text{m})}{JG} = 0$$

so that

$$T_B = 645\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Using Eq. 1,

$$T_A = -345\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

The negative sign indicates that $\mathbf{T}_A$ acts in the opposite direction of that shown in Fig. 5–23*b*.

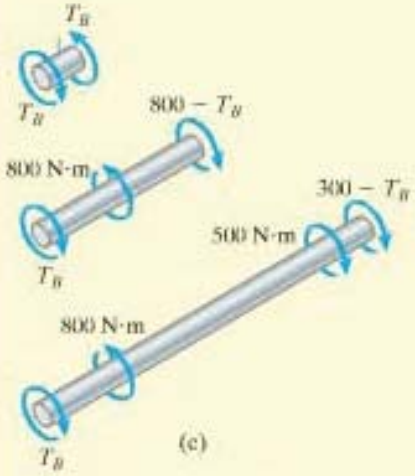


Fig. 5–23

EXAMPLE 5.9

The shaft shown in Fig. 5–24*a* is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\ \text{lb}\cdot\text{ft}$ is applied at its end, plot the shear-stress distribution along a radial line of its cross-sectional area. Take $G_{st} = 11.4(10^3)$ ksi, $G_{br} = 5.20(10^3)$ ksi.

SOLUTION

Equilibrium. A free-body diagram of the shaft is shown in Fig. 5–24*b*. The reaction at the wall has been represented by the unknown amount of torque resisted by the steel, T_{st}, and by the brass, T_{br}. Working in units of pounds and inches, equilibrium requires

$$-T_{st} - T_{br} + (250\ \text{lb}\cdot\text{ft})(12\ \text{in./ft}) = 0 \qquad (1)$$

Compatibility. We require the angle of twist of end A to be the same for both the steel and brass since they are bonded together. Thus,

$$\phi = \phi_{st} = \phi_{br}$$

Applying the load–displacement relationship, $\phi = TL/JG$,

$$\frac{T_{st}L}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]11.4(10^3)\ \text{kip/in}^2} = \frac{T_{br}L}{(\pi/2)(0.5\ \text{in.})^4 5.20(10^3)\ \text{kip/in}^2}$$

$$T_{st} = 32.88T_{br} \qquad (2)$$

Solving Eqs. 1 and 2, we get

$$T_{st} = 2911.5\ \text{lb}\cdot\text{in.} = 242.6\ \text{lb}\cdot\text{ft}$$
$$T_{br} = 88.5\ \text{lb}\cdot\text{in.} = 7.38\ \text{lb}\cdot\text{ft}$$

The shear stress in the brass core varies from zero at its center to a maximum at the interface where it contacts the steel tube. Using the torsion formula,

$$(\tau_{br})_{max} = \frac{(88.5\ \text{lb}\cdot\text{in.})(0.5\ \text{in.})}{(\pi/2)(0.5\ \text{in.})^4} = 451\ \text{psi}$$

For the steel, the minimum and maximum shear stresses are

$$(\tau_{st})_{min} = \frac{(2911.5\ \text{lb}\cdot\text{in.})(0.5\ \text{in.})}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]} = 989\ \text{psi}$$

$$(\tau_{st})_{max} = \frac{(2911.5\ \text{lb}\cdot\text{in.})(1\ \text{in.})}{(\pi/2)[(1\ \text{in.})^4 - (0.5\ \text{in.})^4]} = 1977\ \text{psi}$$

The results are plotted in Fig. 5–24*c*. Note the discontinuity of *shear stress* at the brass and steel interface. This is to be expected, since the materials have different moduli of rigidity; i.e., steel is stiffer than brass ($G_{st} > G_{br}$) and thus it carries more shear stress at the interface. Although the shear stress is discontinuous here, the *shear strain* is not. Rather, the shear strain is the *same* for both the brass and the steel.

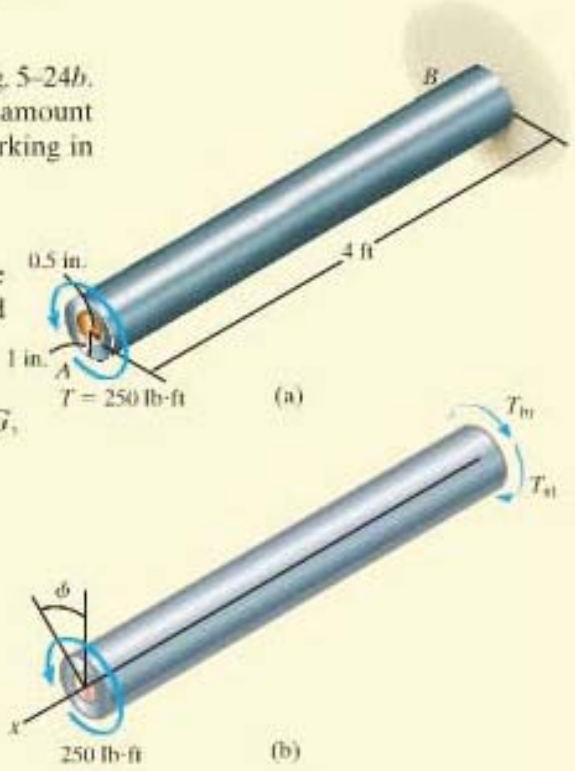
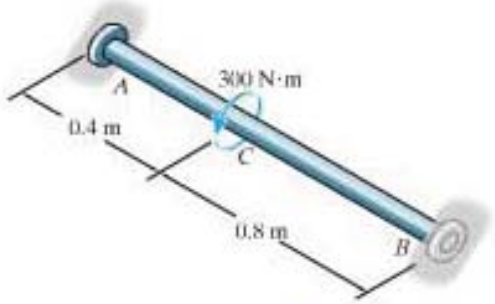


Fig. 5–24

PROBLEMS

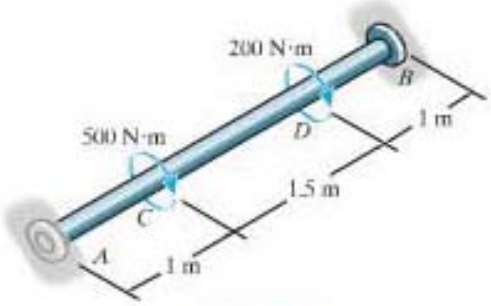
•5–77. The A-36 steel shaft has a diameter of 50 mm and is fixed at its ends A and B. If it is subjected to the torque, determine the maximum shear stress in regions AC and CB of the shaft.



Prob. 5–77

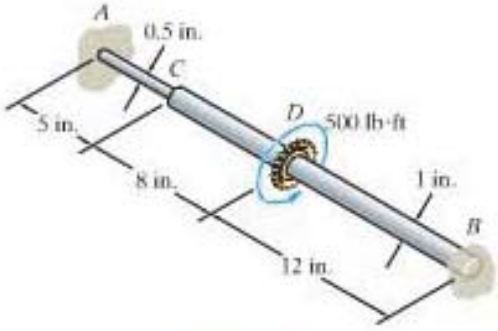
5

5–78. The A-36 steel shaft has a diameter of 60 mm and is fixed at its ends A and B. If it is subjected to the torques shown, determine the absolute maximum shear stress in the shaft.



Prob. 5–78

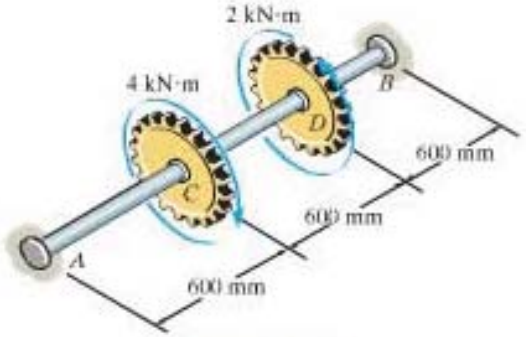
5–79. The steel shaft is made from two segments: AC has a diameter of 0.5 in. and CB has a diameter of 1 in. If it is fixed at its ends A and B and subjected to a torque of 500 lb · ft, determine the maximum shear stress in the shaft. $G_{st} = 10.8(10^3)$ ksi.



Prob. 5–79

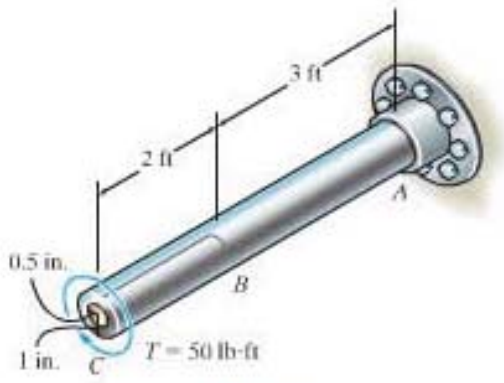
***5–80.** The shaft is made of A-36 steel, has a diameter of 80 mm, and is fixed at B while A is loose and can rotate 0.005 rad before becoming fixed. When the torques are applied to C and D, determine the maximum shear stress in regions AC and CD of the shaft.

•5–81. The shaft is made of A-36 steel and has a diameter of 80 mm. It is fixed at B and the support at A has a torsional stiffness of $k = 0.5$ MN · m/rad. If it is subjected to the gear torques shown, determine the absolute maximum shear stress in the shaft.



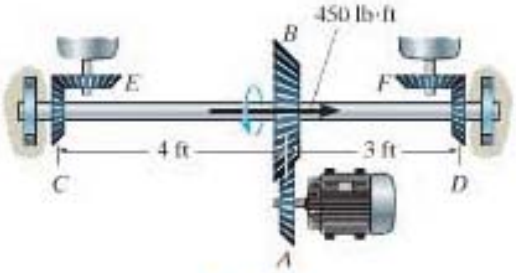
Probs. 5–80/81

5–82. The shaft is made from a solid steel section AB and a tubular portion made of steel and having a brass core. If it is fixed to a rigid support at A, and a torque of $T = 50$ lb · ft is applied to it at C, determine the angle of twist that occurs at C and compute the maximum shear stress and maximum shear strain in the brass and steel. Take $G_{st} = 11.5(10^3)$ ksi, $G_{bc} = 5.6(10^3)$ ksi.



Prob. 5–82

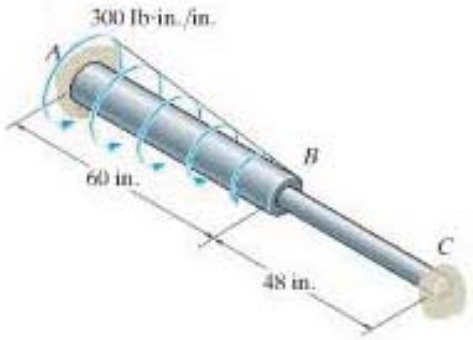
5–83. The motor A develops a torque at gear B of 450 lb · ft, which is applied along the axis of the 2-in.-diameter steel shaft CD. This torque is to be transmitted to the pinion gears at E and F. If these gears are temporarily fixed, determine the maximum shear stress in segments CB and BD of the shaft. Also, what is the angle of twist of each of these segments? The bearings at C and D only exert force reactions on the shaft and do not resist torque. $G_{st} = 12(10^3)$ ksi.



Prob. 5–83

*5–84. A portion of the A-36 steel shaft is subjected to a linearly distributed torsional loading. If the shaft has the dimensions shown, determine the reactions at the fixed supports A and C. Segment AB has a diameter of 1.5 in. and segment BC has a diameter of 0.75 in.

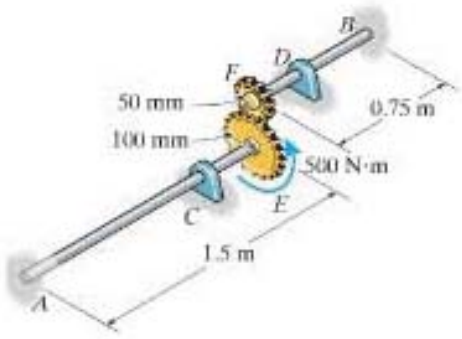
•5–85. Determine the rotation of joint B and the absolute maximum shear stress in the shaft in Prob. 5–84.



Probs. 5–84/85

5–86. The two shafts are made of A-36 steel. Each has a diameter of 25 mm and they are connected using the gears fixed to their ends. Their other ends are attached to fixed supports at A and B. They are also supported by journal bearings at C and D, which allow free rotation of the shafts along their axes. If a torque of 500 N · m is applied to the gear at E as shown, determine the reactions at A and B.

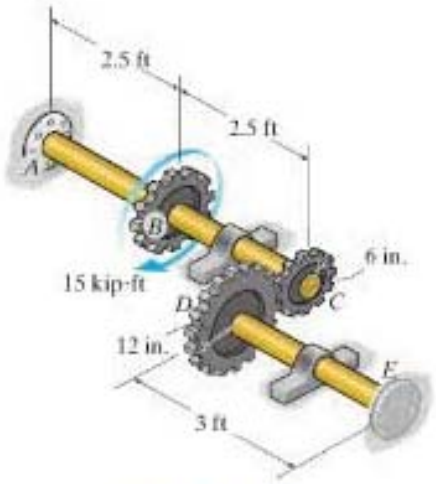
5–87. Determine the rotation of the gear at E in Prob. 5–86.



Probs. 5–86/87

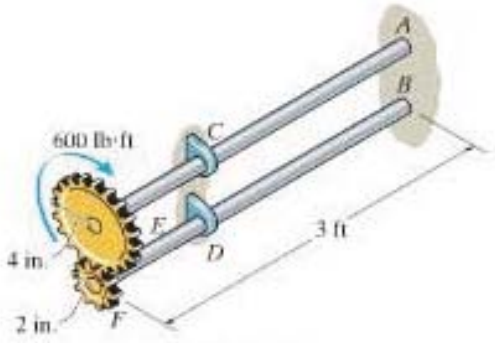
*5–88. The shafts are made of A-36 steel and have the same diameter of 4 in. If a torque of 15 kip · ft is applied to gear B, determine the absolute maximum shear stress developed in the shaft.

•5–89. The shafts are made of A-36 steel and have the same diameter of 4 in. If a torque of 15 kip · ft is applied to gear B, determine the angle of twist of gear B.



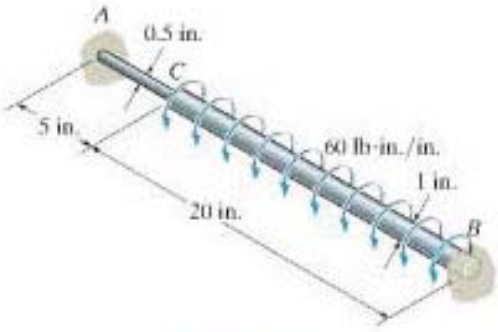
Probs. 5–88/89

5–90. The two 3-ft-long shafts are made of 2014-T6 aluminum. Each has a diameter of 1.5 in. and they are connected using the gears fixed to their ends. Their other ends are attached to fixed supports at A and B. They are also supported by bearings at C and D, which allow free rotation of the shafts along their axes. If a torque of 600 lb · ft is applied to the top gear as shown, determine the maximum shear stress in each shaft.



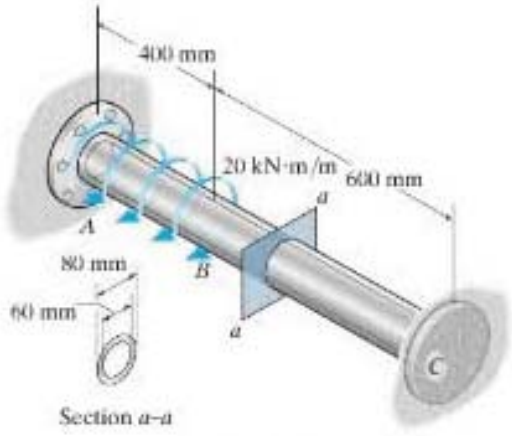
Prob. 5–90

5–91. The A-36 steel shaft is made from two segments: AC has a diameter of 0.5 in. and CB has a diameter of 1 in. If the shaft is fixed at its ends A and B and subjected to a uniform distributed torque of 60 lb · in./in. along segment CB, determine the absolute maximum shear stress in the shaft.



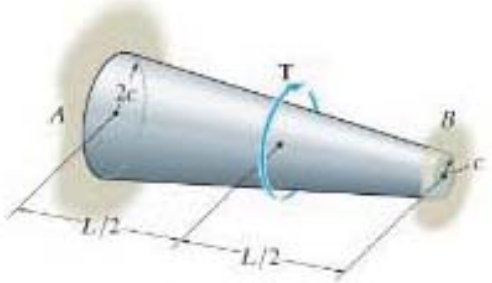
Prob. 5–91

***5–92.** If the shaft is subjected to a uniform distributed torque of $t = 20$ kN · m/m, determine the maximum shear stress developed in the shaft. The shaft is made of 2014-T6 aluminum alloy and is fixed at A and C.



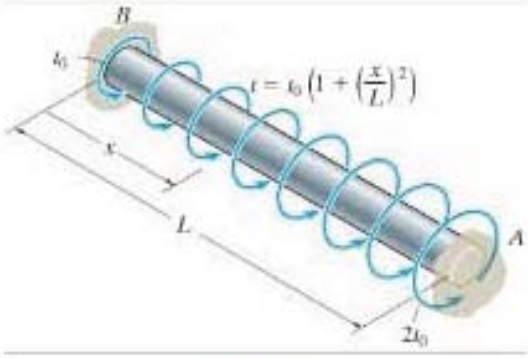
Prob. 5–92

•**5–93.** The tapered shaft is confined by the fixed supports at A and B. If a torque $\mathbf{T}$ is applied at its mid-point, determine the reactions at the supports.



Prob. 5–93

5–94. The shaft of radius c is subjected to a distributed torque t, measured as torque/length of shaft. Determine the reactions at the fixed supports A and B.



Prob. 5–94

*5.6 Solid Noncircular Shafts

It was demonstrated in Sec. 5.1 that when a torque is applied to a shaft having a circular cross section—that is, one that is axisymmetric—the shear strains vary linearly from zero at its center to a maximum at its outer surface. Furthermore, due to the uniformity of the shear strain at all points on the same radius, the cross sections do not deform, but rather remain plane after the shaft has twisted. Shafts that have a noncircular cross section, however, are *not* axisymmetric, and so their cross sections will ***bulge*** or ***warp*** when the shaft is twisted. Evidence of this can be seen from the way grid lines deform on a shaft having a square cross section when the shaft is twisted, Fig. 5–25. As a consequence of this deformation the torsional analysis of *noncircular* shafts becomes considerably more complicated and will not be considered in this text.

Using a mathematical analysis based on the theory of elasticity, however, it is possible to determine the shear-stress distribution within a shaft of square cross section. Examples of how this shear stress varies along two radial lines of the shaft are shown in Fig. 5–26*a*. Because these shear-stress distributions vary in a complex manner, the shear strains they create will *warp* the cross section as shown in Fig. 5–26*b*. In particular notice that the corner points of the shaft must be subjected to zero shear stress and therefore zero shear strain. The reason for this can be shown by considering an element of material located at one of these points, Fig. 5–26*c*. One would expect the top face of this element to be subjected to a shear stress in order to aid in resisting the applied torque **T**. This, however, cannot occur since the complementary shear stresses τ and τ', acting on the *outer surface* of the shaft, must be zero,

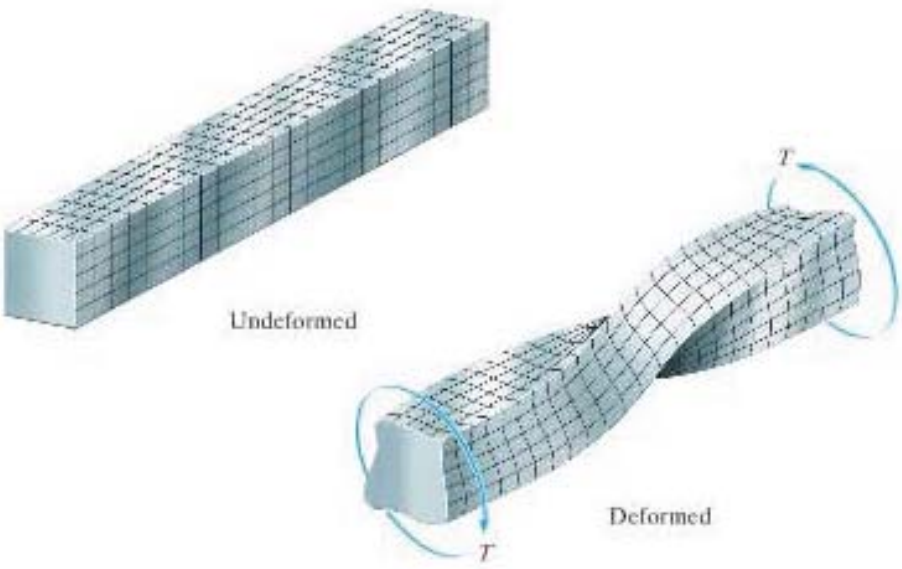
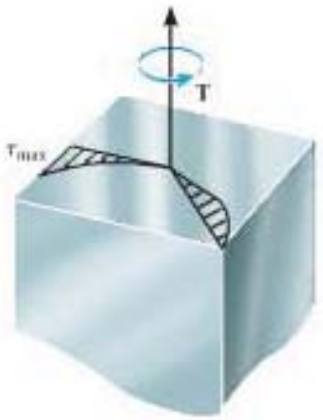


Fig. 5–25



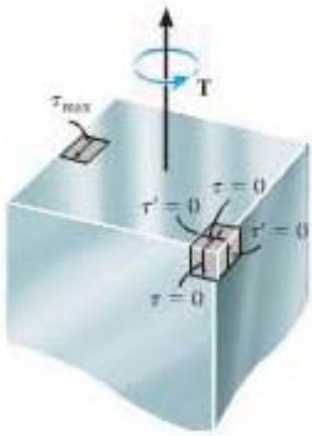
Shear stress distribution along two radial lines

(a)



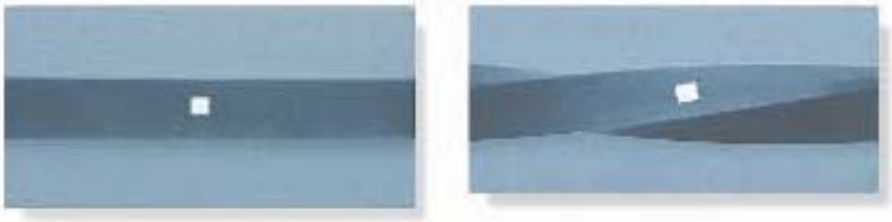
Warping of cross-sectional area

(b)



(c)

Fig. 5–26



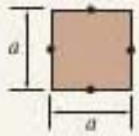
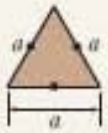
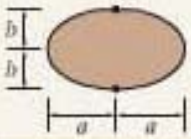
Notice the deformation of the square element when this rubber bar is subjected to a torque.

The results of the analysis for square cross sections, along with other results from the theory of elasticity, for shafts having triangular and elliptical cross sections, are reported in Table 5–1. In all cases the *maximum shear stress* occurs at a point on the edge of the cross section that is *closest to* the center axis of the shaft. In Table 5–1 these points are indicated as "dots" on the cross sections. Also given are formulas for the angle of twist of each shaft. By extending these results to a shaft having an *arbitrary* cross section, it can also be shown that a shaft having a *circular* cross section is most efficient, since it is subjected to both a *smaller* maximum shear stress and a *smaller* angle of twist than a corresponding shaft having a noncircular cross section and subjected to the same torque.



The drill shaft is connected to the soil auger using a shaft having a square cross section.

TABLE 5–1

Shape of cross section	τ_{max}	ϕ
Square 	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4G}$
Equilateral triangle 	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4G}$
Ellipse 	$\frac{2\,T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

EXAMPLE 5.10

The 6061-T6 aluminum shaft shown in Fig. 5–27 has a cross-sectional area in the shape of an equilateral triangle. Determine the largest torque **T** that can be applied to the end of the shaft if the allowable shear stress is $\tau_{allow} = 8$ ksi and the angle of twist at its end is restricted to $\phi_{allow} = 0.02$ rad. How much torque can be applied to a shaft of circular cross section made from the same amount of material?

SOLUTION

By inspection, the resultant internal torque at any cross section along the shaft's axis is also **T**. Using the formulas for τ_{max} and ϕ in Table 5–1, we require

$$\tau_{allow} = \frac{20T}{a^3}; \qquad 8(10^3)\ \text{lb/in}^2 = \frac{20T}{(1.5\ \text{in.})^3}$$

$$T = 1350\ \text{lb} \cdot \text{in.}$$

Also,

$$\phi_{allow} = \frac{46TL}{a^4 G_{al}}; \quad 0.02\ \text{rad} = \frac{46T(4\ \text{ft})(12\ \text{in./ft})}{(1.5\ \text{in.})^4[3.7(10^6)\ \text{lb/in}^2]}$$

$$T = 170\ \text{lb} \cdot \text{in.} \qquad \textit{Ans}$$

By comparison, the torque is limited due to the angle of twist.

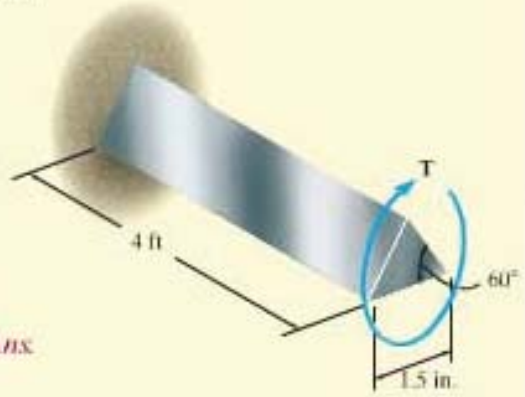


Fig. 5–27

Circular Cross Section. If the same amount of aluminum is to be used in making the same length of shaft having a circular cross section, then the radius of the cross section can be calculated. We have

$$A_{circle} = A_{triangle}; \qquad \pi c^2 = \frac{1}{2}(1.5\ \text{in.})(1.5 \sin 60°)$$

$$c = 0.557\ \text{in.}$$

The limitations of stress and angle of twist then require

$$\tau_{allow} = \frac{Tc}{J}; \qquad 8(10^3)\ \text{lb/in}^2 = \frac{T(0.557\ \text{in.})}{(\pi/2)(0.557\ \text{in.})^4}$$

$$T = 2170\ \text{lb} \cdot \text{in.}$$

$$\phi_{allow} = \frac{TL}{JG_{al}}; \quad 0.02\ \text{rad} = \frac{T(4\ \text{ft})(12\ \text{in./ft})}{(\pi/2)(0.557\ \text{in.})^4[3.7(10^6)\ \text{lb/in}^2]}$$

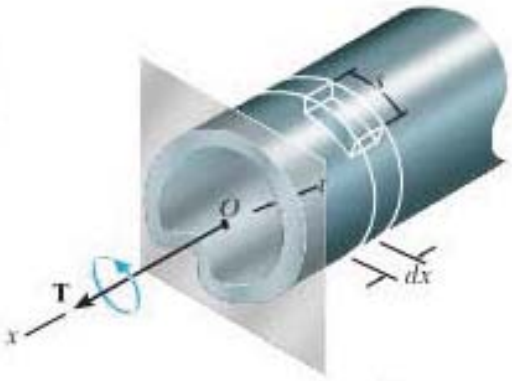
$$T = 233\ \text{lb} \cdot \text{in.} \qquad \textit{Ans}$$

Again, the angle of twist limits the applied torque.

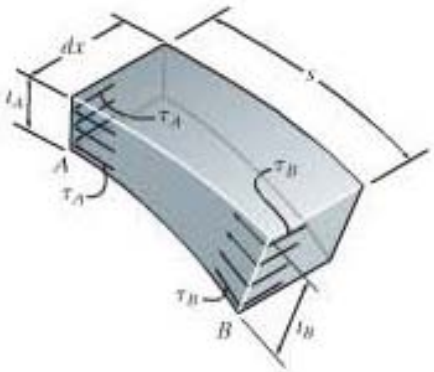
NOTE: Comparing this result (233 lb · in.) with that given above (170 lb · in.), it is seen that a shaft of circular cross section can support 37% more torque than the one having a triangular cross section.

*5.7 Thin-Walled Tubes Having Closed Cross Sections

Thin-walled tubes of noncircular cross section are often used to construct light-weight frameworks such as those used in aircraft. In some applications, they may be subjected to a torsional loading. In this section we will analyze the effects of applying a torque to a thin-walled tube having a *closed* cross section, that is, a tube that does not have any breaks or slits along its length. Such a tube, having a constant yet arbitrary cross-sectional shape, and variable thickness t, is shown in Fig. 5–28*a*. Since the walls are thin, we will obtain the average shear stress by assuming that this stress is *uniformly distributed* across the thickness of the tube at any given point. Before we do this, however, we will first discuss some preliminary concepts regarding the action of shear stress over the cross section.



(a)



(b)

Fig. 5–28

Shear Flow. Shown in Figs. 5–28*a* and 5–28*b* is a small element of the tube having a finite length s and differential width dx. At one end the element has a thickness t_A, and at the other end the thickness is t_B. Due to the internal torque **T**, shear stress is developed on the front face of the element. Specifically, at end A the shear stress is τ_A, and at end B it is τ_B. These stresses can be related by noting that equivalent shear stresses τ_A and τ_B must also act on the longitudinal sides of the element. Since these sides have a *constant* width dx, the forces acting on them are $dF_A = \tau_A(t_A\,dx)$ and $dF_B = \tau_B(t_B\,dx)$. Equilibrium requires these forces to be of equal magnitude but opposite direction, so that

$$\tau_A t_A = \tau_B t_B$$

This important result states that ***the product of the average shear stress times the thickness of the tube is the same at each point on the tube's cross-sectional area***. This product is called ***shear flow***,* q, and in general terms we can express it as

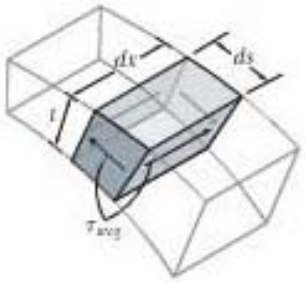
$$q = \tau_{avg} t \tag{5–17}$$

Since q is constant over the cross section, the *largest* average shear stress must occur where the tube's thickness is the *smallest*.

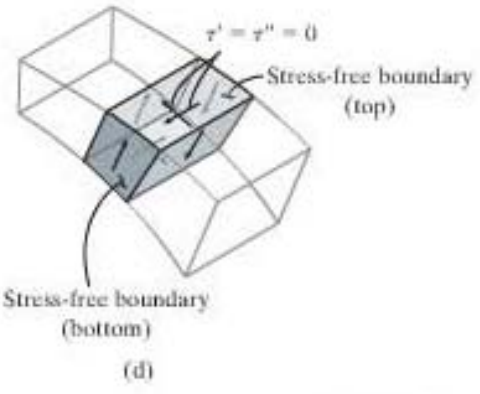
*The terminology "flow" is used since q is analogous to water flowing through a tube of rectangular cross section having a constant depth and variable width w. Although the water's velocity v at each point along the tube will be different (like τ_{avg}), the flow $q = vw$ will be constant.

Now if a differential element having a thickness t, length ds, and width dx is isolated from the tube, Fig. 5–28*c*, it is seen that the front face over which the average shear stress acts is $dA = t\,ds$. Hence, $dF = \tau_{avg}(t\,ds) = q\,ds$, or $q = dF/ds$. In other words, *the shear flow measures the force per unit length along the tube's cross-sectional area.*

It is important to realize that the shear-stress components shown in Fig. 5–28*c* are the only ones acting on the tube. Components acting in the other direction, as shown in Fig. 5–28*d*, cannot exist. This is because the top and bottom faces of the element are at the inner and outer walls of the tube, and these boundaries must be free of stress. Instead, as noted above, the applied torque causes *the shear flow and the average stress to always be directed tangent to the wall of the tube, such that it contributes to the resultant internal torque* **T**.



(c)



(d)

Average Shear Stress. The average shear stress can be related to the torque T by considering the torque produced by this shear stress about a selected point O within the tube's boundary, Fig. 5–28*e*. As shown, the shear stress develops a force $dF = \tau_{avg}\,dA = \tau_{avg}(t\,ds)$ on an element of the tube. This force acts tangent to the centerline of the tube's wall, and if the moment arm is h, the torque is

$$dT = h(dF) = h(\tau_{avg}\,t\,ds)$$

For the entire cross section, we require

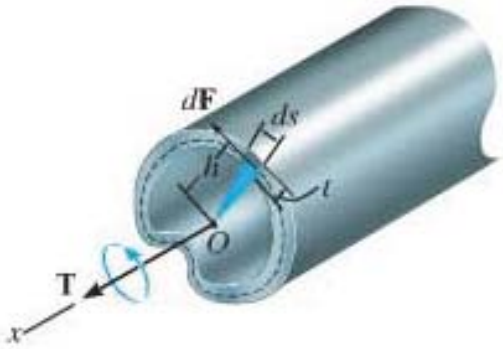
$$T = \oint h\tau_{avg}\,t\,ds$$

Here the "line integral" indicates that integration must be performed *around* the entire boundary of the area. Since the shear flow $q = \tau_{avg}t$ is *constant*, it can be factored out of the integral, so that

$$T = \tau_{avg}\,t \oint h\,ds$$

A graphical simplification can be made for evaluating the integral by noting that the *mean area*, shown by the blue colored triangle in Fig. 5–28*e*, is $dA_m = (1/2)h\,ds$. Thus,

$$T = 2\tau_{avg}\,t \int dA_m = 2\tau_{avg}\,tA_m$$



(e)



(f)

Fig. 5–28 (cont.)

Solving for τ_{avg}, we have

$$\tau_{avg} = \frac{T}{2tA_m} \tag{5–18}$$

Here

τ_{avg} = the average shear stress acting over a particular thickness of the tube

T = the resultant internal torque at the cross section

t = the thickness of the tube where τ_{avg} is to be determined

A_m = the mean area enclosed within the boundary of the *centerline* of the tube's thickness. A_m is shown shaded in Fig. 5–28*f*

Since $q = \tau_{avg} t$, then the shear flow throughout the cross section becomes

$$q = \frac{T}{2A_m} \tag{5–19}$$

Angle of Twist. The angle of twist of a thin-walled tube of length L can be determined using energy methods, and the development of the necessary equation is given as a problem later in the text.* If the material behaves in a linear elastic manner and G is the shear modulus, then this angle ϕ, given in radians, can be expressed as

$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t} \tag{5–20}$$

Here again the integration must be performed around the entire boundary of the tube's cross-sectional area.

Important Points

- Shear flow q is the product of the tube's thickness and the average shear stress. This value is the same at all points along the tube's cross section. As a result, the *largest* average shear stress on the cross section occurs where the thickness is *smallest*.
- Both shear flow and the average shear stress act *tangent* to the wall of the tube at all points and in a direction so as to contribute to the resultant internal torque.

*See Prob. 14–12.

EXAMPLE 5.11

Calculate the average shear stress in a thin-walled tube having a circular cross section of mean radius r_m and thickness t, which is subjected to a torque T, Fig. 5–29*a*. Also, what is the relative angle of twist if the tube has a length L?



(a)

SOLUTION

Average Shear Stress. The mean area for the tube is $A_m = \pi r_m^2$. Applying Eq. 5–18 gives

$$\tau_{avg} = \frac{T}{2tA_m} = \frac{T}{2\pi t r_m^2} \qquad \textit{Ans.}$$

We can check the validity of this result by applying the torsion formula. In this case, using Eq. 5–9, we have

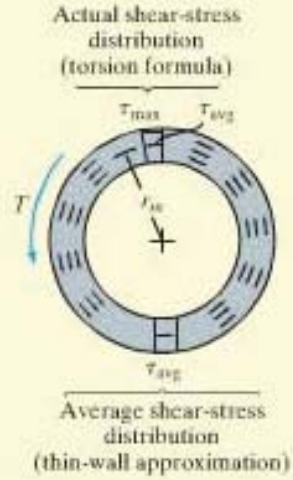
$$\begin{aligned} J &= \frac{\pi}{2}(r_o^4 - r_i^4) \\ &= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o^2 - r_i^2) \\ &= \frac{\pi}{2}(r_o^2 + r_i^2)(r_o + r_i)(r_o - r_i) \end{aligned}$$

Since $r_m \approx r_o \approx r_i$ and $t = r_o - r_i$, $J = \frac{\pi}{2}(2r_m^2)(2r_m)t = 2\pi r_m^3 t$

so that $$\tau_{avg} = \frac{Tr_m}{J} = \frac{Tr_m}{2\pi r_m^3 t} = \frac{T}{2\pi t r_m^2} \qquad \textit{Ans.}$$

which agrees with the previous result.

The average shear-stress distribution acting throughout the tube's cross section is shown in Fig. 5–29*b*. Also shown is the shear-stress distribution acting on a radial line as calculated using the torsion formula. Notice how each τ_{avg} acts in a direction such that it contributes to the resultant torque **T** at the section. As the tube's thickness decreases, the shear stress throughout the tube becomes more uniform.



(b)

Fig. 5–29

Angle of Twist. Applying Eq. 5–20, we have

$$\phi = \frac{TL}{4A_m^2 G}\oint \frac{ds}{t} = \frac{TL}{4(\pi r_m^2)^2 Gt}\oint ds$$

The integral represents the length around the centerline boundary, which is $2\pi r_m$. Substituting, the final result is

$$\phi = \frac{TL}{2\pi r_m^3 Gt} \qquad \textit{Ans.}$$

Show that one obtains this same result using Eq. 5–15.

EXAMPLE 5.12

The tube is made of C86100 bronze and has a rectangular cross section as shown in Fig. 5–30*a*. If it is subjected to the two torques, determine the average shear stress in the tube at points *A* and *B*. Also, what is the angle of twist of end *C*? The tube is fixed at *E*.

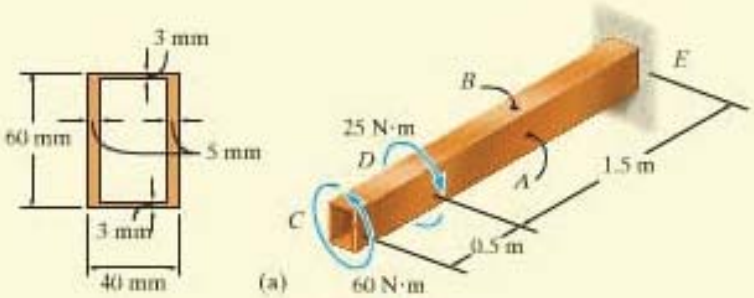


Fig. 5–30

SOLUTION

Average Shear Stress. If the tube is sectioned through points *A* and *B*, the resulting free-body diagram is shown in Fig. 5–30*b*. The internal torque is 35 N · m. As shown in Fig. 5–30*d*, the mean area is

$$A_m = (0.035\ \text{m})(0.057\ \text{m}) = 0.00200\ \text{m}^2$$

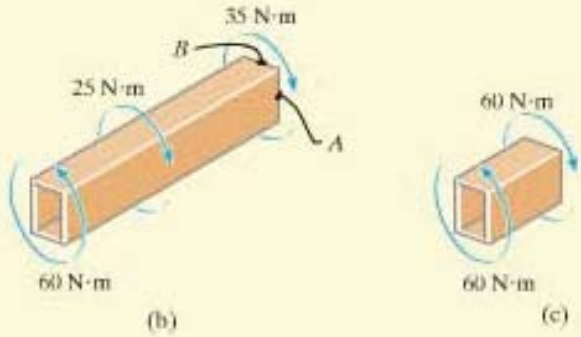
Applying Eq. 5–18 for point *A*, $t_A = 5$ mm, so that

$$\tau_A = \frac{T}{2tA_m} = \frac{35\ \text{N} \cdot \text{m}}{2(0.005\ \text{m})(0.00200\ \text{m}^2)} = 1.75\ \text{MPa} \quad \textit{Ans.}$$

And for point *B*, $t_B = 3$ mm, and therefore

$$\tau_B = \frac{T}{2tA_m} = \frac{35\ \text{N} \cdot \text{m}}{2(0.003\ \text{m})(0.00200\ \text{m}^2)} = 2.92\ \text{MPa} \quad \textit{Ans.}$$

These results are shown on elements of material located at points *A* and *B*, Fig. 5–30*e*. Note carefully how the 35-N · m torque in Fig. 5–30*b* creates these stresses on the back sides of each element.



Angle of Twist. From the free-body diagrams in Fig. 5–30*b* and 5–30*c*, the internal torques in regions *DE* and *CD* are 35 N · m and 60 N · m, respectively. Following the sign convention outlined in Sec. 5.4, these torques are both positive. Thus, Eq. 5–20 becomes

$$\phi = \sum \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

$$= \frac{60\ \text{N}\cdot\text{m}\ (0.5\ \text{m})}{4(0.00200\ \text{m}^2)^2(38(10^9)\ \text{N/m}^2)}\left[2\left(\frac{57\ \text{mm}}{5\ \text{mm}}\right) + 2\left(\frac{35\ \text{mm}}{3\ \text{mm}}\right)\right]$$

$$+ \frac{35\ \text{N}\cdot\text{m}\ (1.5\ \text{m})}{4(0.00200\ \text{m}^2)^2(38(10^9)\ \text{N/m}^2)}\left[2\left(\frac{57\ \text{mm}}{5\ \text{mm}}\right) + 2\left(\frac{35\ \text{mm}}{3\ \text{mm}}\right)\right]$$

$$= 6.29(10^{-3})\ \text{rad} \qquad \textit{Ans}$$

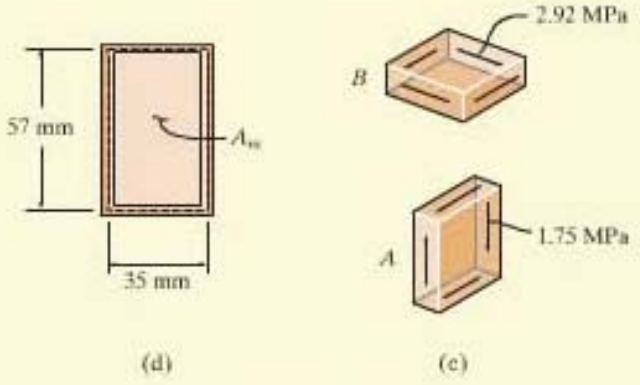
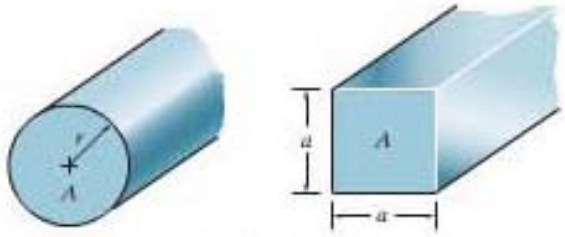


Fig. 5–30

PROBLEMS

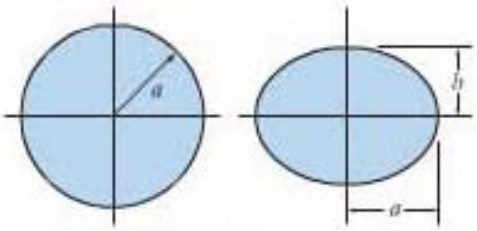
5–95. Compare the values of the maximum elastic shear stress and the angle of twist developed in 304 stainless steel shafts having circular and square cross sections. Each shaft has the same cross-sectional area of 9 in^2, length of 36 in., and is subjected to a torque of 4000 lb · in.



Prob. 5–95

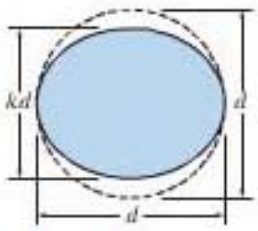
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***5–96.** If $a = 25$ mm and $b = 15$ mm, determine the maximum shear stress in the circular and elliptical shafts when the applied torque is $T = 80$ N · m. By what percentage is the shaft of circular cross section more efficient at withstanding the torque than the shaft of elliptical cross section?



Prob. 5–96

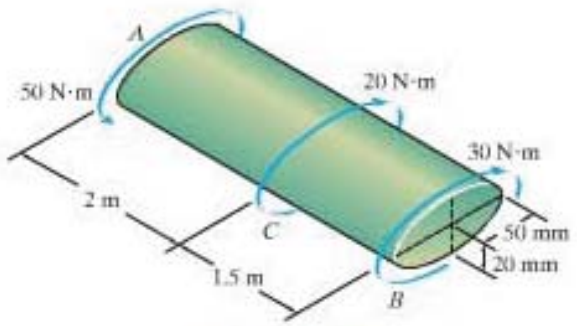
•5–97. It is intended to manufacture a circular bar to resist torque; however, the bar is made elliptical in the process of manufacturing, with one dimension smaller than the other by a factor k as shown. Determine the factor by which the maximum shear stress is increased.



Prob. 5–97

5–98. The shaft is made of red brass C83400 and has an elliptical cross section. If it is subjected to the torsional loading shown, determine the maximum shear stress within regions AC and BC, and the angle of twist ϕ of end B relative to end A.

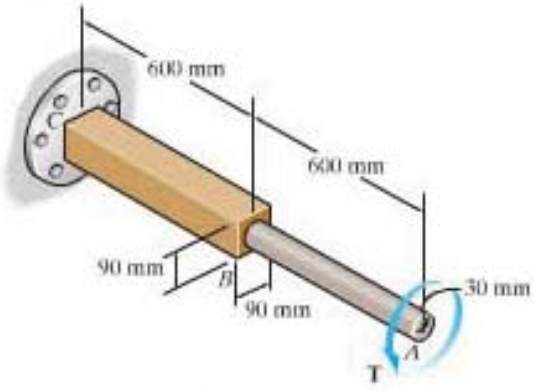
5–99. Solve Prob. 5–98 for the maximum shear stress within regions AC and BC, and the angle of twist ϕ of end B relative to C.



Probs. 5–98/99

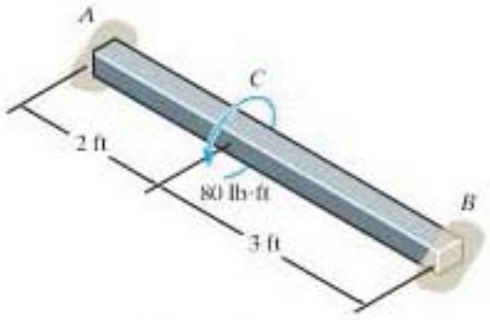
***5–100.** Segments AB and BC of the shaft have circular and square cross sections, respectively. If end A is subjected to a torque of $T = 2$ kN · m, determine the absolute maximum shear stress developed in the shaft and the angle of twist of end A. The shaft is made from A-36 steel and is fixed at C.

•5–101. Segments AB and BC of the shaft have circular and square cross sections, respectively. The shaft is made from A-36 steel with an allowable shear stress of $\tau_{allow} = 75$ MPa, and an angle of twist at end A which is not allowed to exceed 0.02 rad. Determine the maximum allowable torque **T** that can be applied at end A. The shaft is fixed at C.



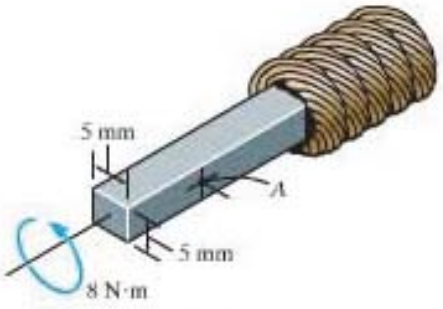
Probs. 5–100/101

5–102. The aluminum strut is fixed between the two walls at A and B. If it has a 2 in. by 2 in. square cross section, and it is subjected to the torque of 80 lb · ft at C, determine the reactions at the fixed supports. Also, what is the angle of twist at C? $G_{al} = 3.8(10^3)$ ksi.



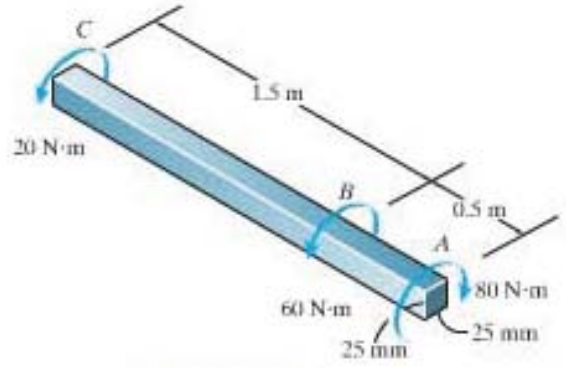
Prob. 5–102

5–103. The square shaft is used at the end of a drive cable in order to register the rotation of the cable on a gauge. If it has the dimensions shown and is subjected to a torque of 8 N · m, determine the shear stress in the shaft at point A. Sketch the shear stress on a volume element located at this point.



Prob. 5–103

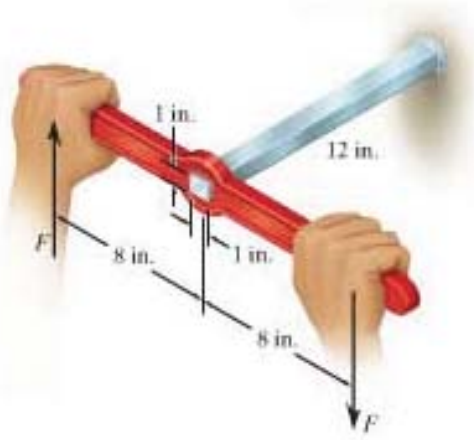
*5–104. The 6061-T6 aluminum bar has a square cross section of 25 mm by 25 mm. If it is 2 m long, determine the maximum shear stress in the bar and the rotation of one end relative to the other end.



Prob. 5–104

•5–105. The steel shaft is 12 in. long and is screwed into the wall using a wrench. Determine the largest couple forces F that can be applied to the shaft without causing the steel to yield. $\tau_Y = 8$ ksi.

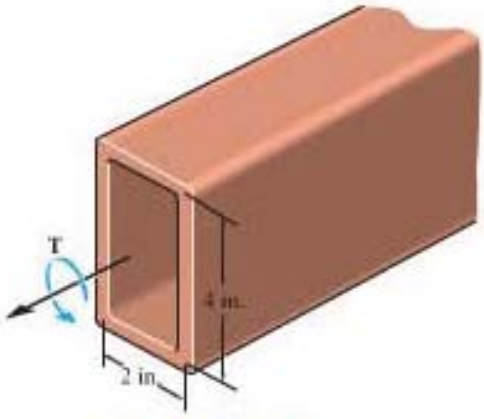
5–106. The steel shaft is 12 in. long and is screwed into the wall using a wrench. Determine the maximum shear stress in the shaft and the amount of displacement that each couple force undergoes if the couple forces have a magnitude of $F = 30$ lb, $G_{st} = 10.8(10^3)$ ksi.



Probs. 5–105/106

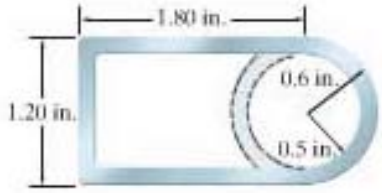
5–107. Determine the constant thickness of the rectangular tube if the average shear stress is not to exceed 12 ksi when a torque of $T = 20$ kip · in. is applied to the tube. Neglect stress concentrations at the corners. The mean dimensions of the tube are shown.

*5–108. Determine the torque T that can be applied to the rectangular tube if the average shear stress is not to exceed 12 ksi. Neglect stress concentrations at the corners. The mean dimensions of the tube are shown and the tube has a thickness of 0.125 in.



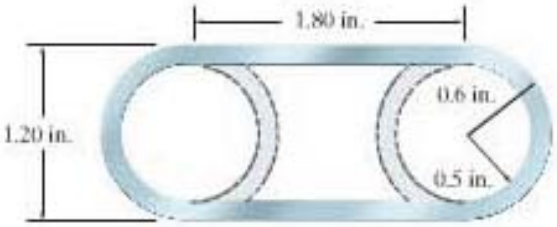
Probs. 5–107/108

•5–109. For a given maximum shear stress, determine the factor by which the torque carrying capacity is increased if the half-circular section is reversed from the dashed-line position to the section shown. The tube is 0.1 in. thick.



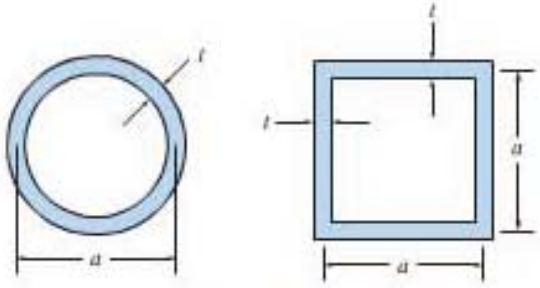
Prob. 5–109

5–110. For a given average shear stress, determine the factor by which the torque-carrying capacity is increased if the half-circular sections are reversed from the dashed-line positions to the section shown. The tube is 0.1 in. thick.



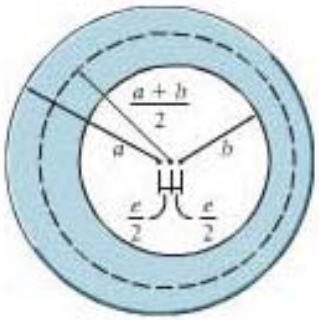
Prob. 5–110

5–111. A torque **T** is applied to two tubes having the cross sections shown. Compare the shear flow developed in each tube.



Prob. 5–111

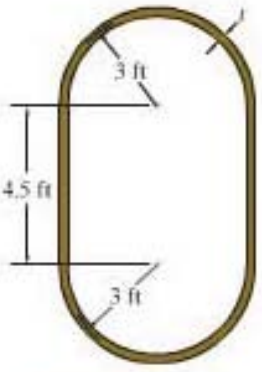
*5–112. Due to a fabrication error the inner circle of the tube is eccentric with respect to the outer circle. By what percentage is the torsional strength reduced when the eccentricity e is one-fourth of the difference in the radii?



Prob. 5–112

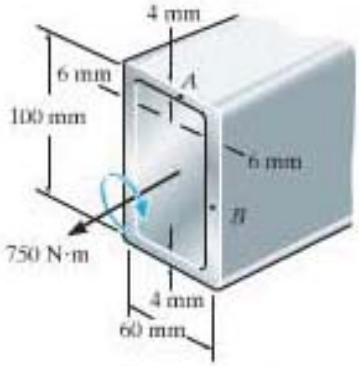
•5–113. The mean dimensions of the cross section of an airplane fuselage are shown. If the fuselage is made of 2014-T6 aluminum alloy having allowable shear stress of $\tau_{allow} = 18$ ksi, and it is subjected to a torque of 6000 kip · ft, determine the required minimum thickness t of the cross section to the nearest 1/16 in. Also, find the corresponding angle of twist per foot length of the fuselage.

5–114. The mean dimensions of the cross section of an airplane fuselage are shown. If the fuselage is made from 2014-T6 aluminum alloy having an allowable shear stress of $\tau_{allow} = 18$ ksi and the angle of twist per foot length of fuselage is not allowed to exceed 0.001 rad/ft, determine the maximum allowable torque that can be sustained by the fuselage. The thickness of the wall is $t = 0.25$ in.



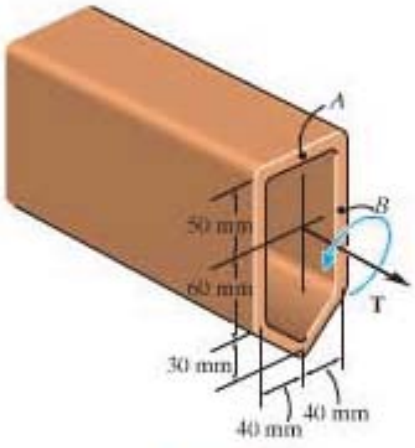
Probs. 5–113/114

5–115. The tube is subjected to a torque of 750 N · m. Determine the average shear stress in the tube at points A and B.



Prob. 5–115

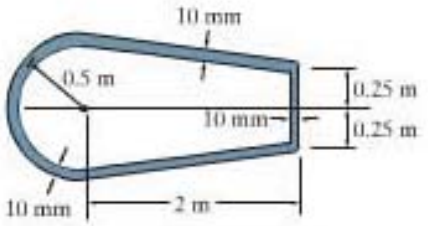
*5–116. The tube is made of plastic, is 5 mm thick, and has the mean dimensions shown. Determine the average shear stress at points A and B if it is subjected to the torque of $T = 5$ N · m. Show the shear stress on volume elements located at these points.



Prob. 5–116

•5–117. The mean dimensions of the cross section of the leading edge and torsion box of an airplane wing can be approximated as shown. If the wing is made of 2014-T6 aluminum alloy having an allowable shear stress of $\tau_{allow} = 125$ MPa and the wall thickness is 10 mm, determine the maximum allowable torque and the corresponding angle of twist per meter length of the wing.

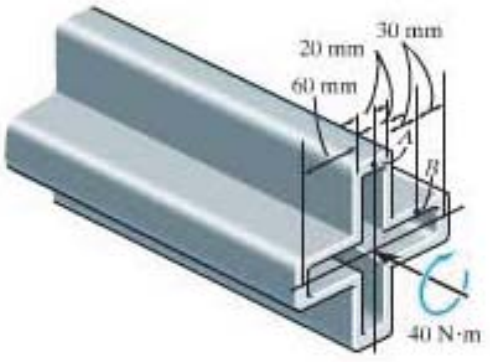
5–118. The mean dimensions of the cross section of the leading edge and torsion box of an airplane wing can be approximated as shown. If the wing is subjected to a torque of 4.5 MN · m and the wall thickness is 10 mm, determine the average shear stress developed in the wing and the angle of twist per meter length of the wing. The wing is made of 2014-T6 aluminum alloy.



Probs. 5–117/118

5

5–119. The symmetric tube is made from a high-strength steel, having the mean dimensions shown and a thickness of 5 mm. If it is subjected to a torque of $T = 40$ N · m, determine the average shear stress developed at points A and B. Indicate the shear stress on volume elements located at these points.



Prob. 5–119

5.8 Stress Concentration

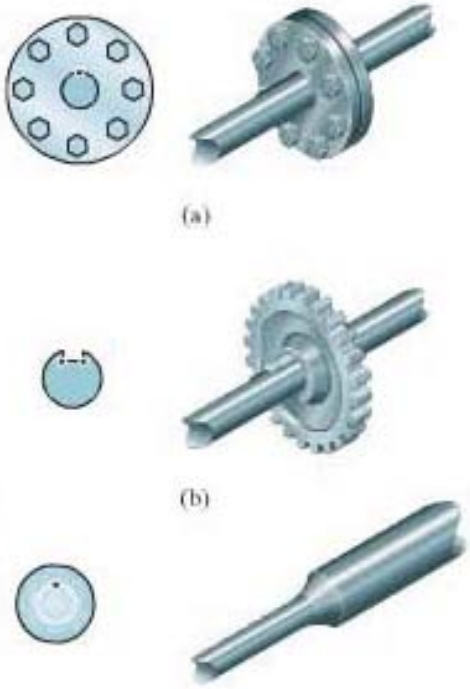


Fig. 5–31

The torsion formula, $\tau_{max} = Tc/J$, cannot be applied to regions of a shaft having a sudden change in the cross section. Here the shear-stress and shear-strain distributions in the shaft become complex and can be obtained only by using experimental methods or possibly by a mathematical analysis based on the theory of elasticity. Three common discontinuities of the cross section that occur in practice are shown in Fig. 5–31. They are at *couplings*, which are used to connect two collinear shafts together, Fig. 5–31*a*, *keyways*, used to connect gears or pulleys to a shaft, Fig. 5–31*b*, and *shoulder fillets*, used to fabricate a single collinear shaft from two shafts having different diameters, Fig. 5–31*c*. In each case the maximum shear stress will occur at the point (dot) indicated on the cross section.

The necessity to perform a complex stress analysis at a shaft discontinuity to obtain the maximum shear stress can be eliminated by using a ***torsional stress-concentration factor***, K. As in the case of axially loaded members, Sec. 4.7, K is usually taken from a graph based on experimental data. An example, for the shoulder-fillet shaft, is shown in Fig. 5–32. To use this graph, one first finds the geometric ratio D/d to define the appropriate curve, and then once the abscissa r/d is calculated, the value of K is found along the ordinate.

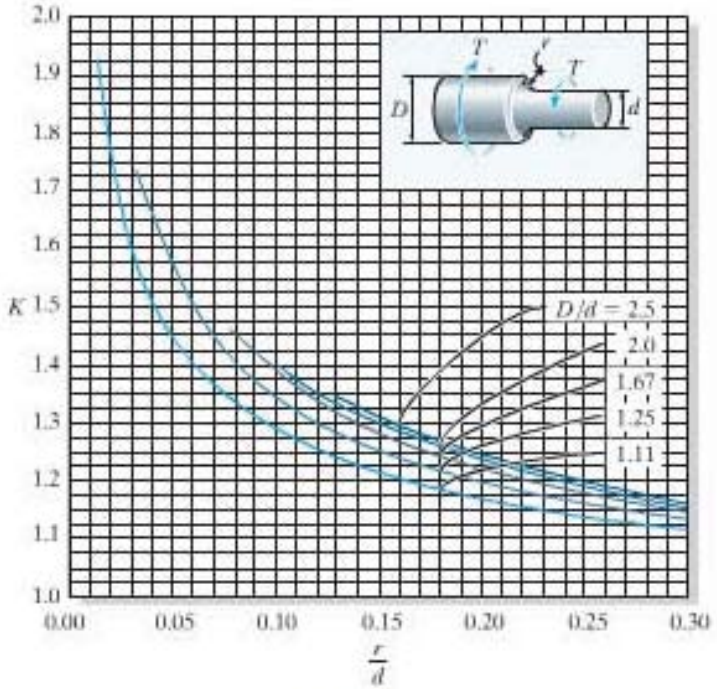


Fig. 5–32

The maximum shear stress is then determined from

$$\tau_{max} = K\frac{Tc}{J} \tag{5–21}$$

Here the torsion formula is applied to the *smaller* of the two connected shafts, since τ_{max} occurs at the base of the fillet, Fig. 5–31*c*.

Note from the graph that an *increase* in fillet radius r causes a *decrease* in K. Hence the maximum shear stress in the shaft can be *reduced* by *increasing* the fillet radius. Also, if the diameter of the larger shaft is reduced, the D/d ratio will be lower and so the value of K and therefore τ_{max} will be lower.

Like the case of axially loaded members, torsional stress concentration factors should *always* be used when designing shafts made from *brittle materials*, or when designing shafts that will be subjected to *fatigue or cyclic torsional loadings*. These conditions give rise to the formation of cracks at the stress concentration, and this can often lead to a sudden fracture. On the other hand, if large *static* torsional loadings are applied to a shaft made from *ductile material*, then *inelastic strains* will develop within the shaft. Yielding of the material will cause the stress distribution to become more *evenly distributed* throughout the shaft, so that the maximum stress will not be limited to the region of stress concentration. This phenomenon will be discussed further in the next section.



Stress concentrations can arise at the coupling of these shafts, and this must be taken into account when the shaft is designed.

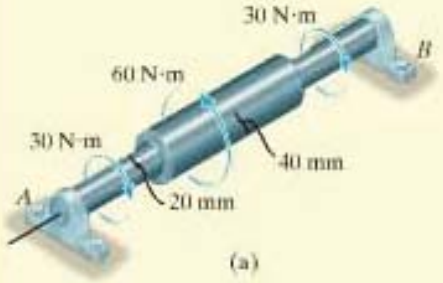
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Important Points

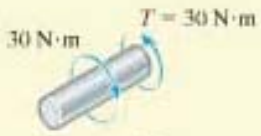
- *Stress concentrations* in shafts occur at points of sudden cross-sectional change, such as couplings, keyways, and at shoulder fillets. The more severe the change in geometry, the larger the stress concentration.
- For design or analysis, it is not necessary to know the exact shear-stress distribution on the cross section. Instead, it is possible to obtain the maximum shear stress using a stress concentration factor, K, that has been determined through experiment, and is only a function of the geometry of the shaft.
- Normally a stress concentration in a ductile shaft subjected to a static torque will *not* have to be considered in design; however, if the material is *brittle*, or subjected to *fatigue* loadings, then stress concentrations become important.

EXAMPLE 5.13

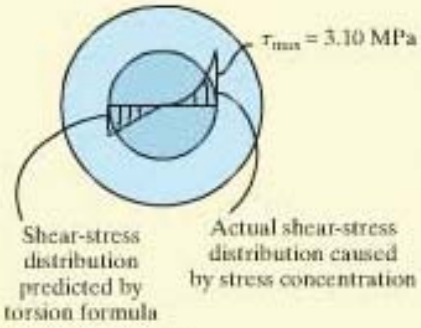
The stepped shaft shown in Fig. 5–33*a* is supported by bearings at A and B. Determine the maximum stress in the shaft due to the applied torques. The shoulder fillet at the junction of each shaft has a radius of $r = 6$ mm.



(a)



(b)



(c)

Fig. 5–33

SOLUTION

Internal Torque. By inspection, moment equilibrium about the axis of the shaft is satisfied. Since the maximum shear stress occurs at the rooted ends of the *smaller* diameter shafts, the internal torque (30 N · m) can be found there by applying the method of sections, Fig. 5–33*b*.

Maximum Shear Stress. The stress-concentration factor can be determined by using Fig. 5–32. From the shaft geometry we have

$$\frac{D}{d} = \frac{2(40\text{ mm})}{2(20\text{ mm})} = 2$$

$$\frac{r}{d} = \frac{6\text{ mm}}{2(20\text{ mm})} = 0.15$$

Thus, the value of $K = 1.3$ is obtained.

Applying Eq. 5–21, we have

$$\tau_{max} = K\frac{Tc}{J};\quad \tau_{max} = 1.3\left[\frac{30\text{ N}\cdot\text{m}\,(0.020\text{ m})}{(\pi/2)(0.020\text{ m})^4}\right] = 3.10\text{ MPa} \qquad \textit{Ans.}$$

NOTE: From experimental evidence, the actual stress distribution along a radial line of the cross section at the critical section looks similar to that shown in Fig. 5–33*c*. Notice how this compares with the linear stress distribution found from the torsion formula.

*5.9 Inelastic Torsion

If the torsional loadings applied to the shaft are excessive, then the material may yield, and, consequently, a "plastic analysis" must be used to determine the shear-stress distribution and the angle of twist. To perform this analysis, then as before, it is necessary to meet the conditions of both deformation and equilibrium for the shaft.

It was shown in Sec. 5.1 that regardless of the material behavior, the shear strains that develop in a circular shaft will vary *linearly*, from zero at the center of the shaft to a maximum at its outer boundary, Fig. 5–34*a*. Also, the resultant internal torque at the section must be equivalent to the torque caused by the entire shear-stress distribution over the cross section. This condition can be expressed mathematically by considering the shear stress τ acting on an element of area dA located a distance ρ from the center of the shaft, Fig. 5–34*b*. The force produced by this stress is $dF = \tau\, dA$, and the torque produced is $dT = \rho\, dF = \rho(\tau\, dA)$. For the entire shaft we require

$$T = \int_A \rho\tau\, dA \tag{5–22}$$

If the area dA over which τ acts can be defined as a *differential ring* having an area of $dA = 2\pi\rho\, d\rho$, Fig. 5–34*c*, then the above equation can be written as

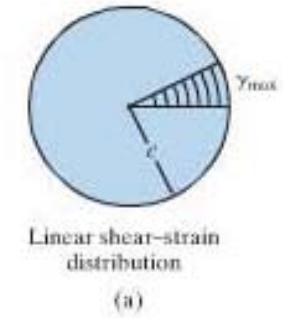
$$T = 2\pi \int_0^c \tau\rho^2\, d\rho \tag{5–23}$$

These conditions of geometry and loading will now be used to determine the shear-stress distribution in a shaft when the shaft is subjected to two types of torque.

Elastic-Plastic Torque. Let us consider the material in the shaft to exhibit an elastic-perfectly plastic behavior. As shown in Fig. 5–35*a*, this is characterized by a shear stress–strain diagram for which the material undergoes an increasing amount of shear strain when the shear stress reaches the yield point τ_Y.

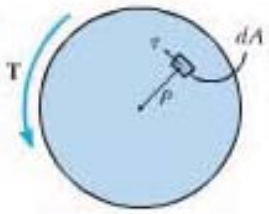


Severe twist of an aluminum specimen caused by the application of a plastic torque.

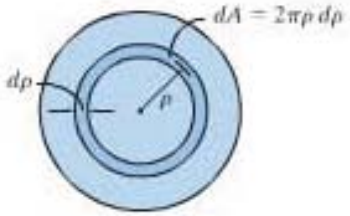


Linear shear–strain distribution

(a)

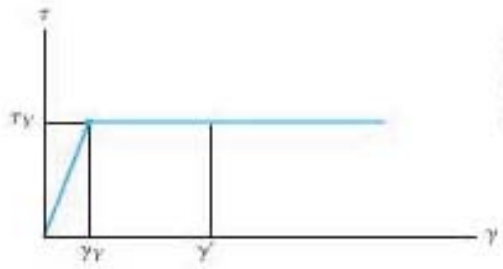


(b)

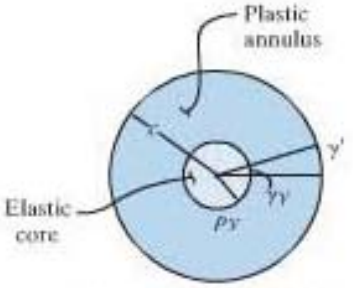


(c)

Fig. 5–34

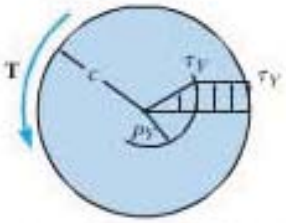


(a)



Shear–strain distribution

(b)



Shear–stress distribution

(c)

Fig. 5–35

If the internal torque produces the maximum *elastic* shear strain γ_Y, at the outer boundary of the shaft, then the maximum elastic torque T_Y that produces this distribution can be found from the torsion formula, $\tau_Y = T_Y c/[(\pi/2)c^4]$, so that

$$T_Y = \frac{\pi}{2}\tau_Y c^3 \tag{5–24}$$

Furthermore, the angle of twist can be determined from Eq. 5–13, namely,

$$d\phi = \gamma \frac{dx}{\rho} \tag{5–25}$$

If the applied torque increases in magnitude above T_Y, it will begin to cause yielding. First at the outer boundary of the shaft, $\rho = c$, and then, as the maximum shear strain increases to, say, γ' in Fig. 5–35*a*, the yielding boundary will progress inward toward the shaft's center, Fig. 5–35*b*. As shown, this produces an *elastic core*, where, by proportion, the radius of the core is $\rho_Y = (\gamma_Y/\gamma')c$. Also, the outer portion of the material forms a *plastic annulus* or ring, since the shear strains γ within this region are greater than γ_Y. The corresponding shear-stress distribution along a radial line of the shaft is shown in Fig. 5–35*c*. It is established by taking successive points on the shear-strain distribution in Fig. 5–35*b* and finding the corresponding value of shear stress from the τ–γ diagram, Fig. 5–35*a*. For example, at $\rho = c$, γ' gives τ_Y, and at $\rho = \rho_Y$, γ_Y also gives τ_Y; etc.

Since τ in Fig. 5–35*c* can now be expressed as a function of ρ, we can apply Eq. 5–23 to determine the torque. We have

$$\begin{aligned} T &= 2\pi \int_0^c \tau \rho^2 \, d\rho \\ &= 2\pi \int_0^{\rho_Y} \left(\tau_Y \frac{\rho}{\rho_Y}\right)\rho^2 \, d\rho + 2\pi \int_{\rho_Y}^c \tau_Y \rho^2 \, d\rho \\ &= \frac{2\pi}{\rho_Y}\tau_Y \int_0^{\rho_Y} \rho^3 \, d\rho + 2\pi\tau_Y \int_{\rho_Y}^c \rho^2 \, d\rho \\ &= \frac{\pi}{2\rho_Y}\tau_Y \rho_Y^4 + \frac{2\pi}{3}\tau_Y(c^3 - \rho_Y^3) \\ &= \frac{\pi\tau_Y}{6}(4c^3 - \rho_Y^3) \end{aligned} \tag{5–26}$$

Plastic Torque. Further increases in T tend to shrink the radius of the elastic core until all the material will yield, i.e., $\rho_Y \to 0$, Fig. 5–35*b*. The material of the shaft will then be subjected to *perfectly plastic behavior* and the shear-stress distribution becomes uniform, so that $\tau = \tau_Y$, Fig. 5–35*d*. We can now apply Eq. 5–23 to determine the ***plastic torque*** T_p, which represents the largest possible torque the shaft will support.

$$T_p = 2\pi \int_0^c \tau_Y \rho^2 \, d\rho$$

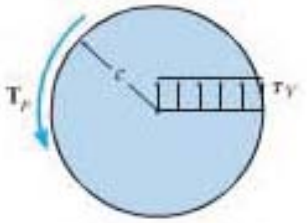
$$= \frac{2\pi}{3} \tau_Y c^3 \tag{5–27}$$

Compared with the maximum elastic torque T_Y, Eq. 5–24, it can be seen that

$$T_p = \frac{4}{3} T_Y$$

In other words, the plastic torque is 33% greater than the maximum elastic torque.

Unfortunately, the angle of twist ϕ for the shear-stress distribution *cannot* be uniquely defined. This is because $\tau = \tau_Y$ does not correspond to any unique value of shear strain $\gamma \geq \gamma_Y$. As a result, once $\mathbf{T}_p$ is applied, the shaft will continue to deform or twist with no corresponding increase in shear stress.



Fully plastic torque

(d)

Fig. 5–35 (cont.)

*5.10 Residual Stress

When a shaft is subjected to plastic shear strains caused by torsion, removal of the torque will cause some shear stress to remain in the shaft. This stress is referred to as ***residual stress***, and its distribution can be calculated using superposition and elastic recovery. (See Sec. 4.9.)

For example, if T_p causes the material at the outer boundary of the shaft to be strained to γ_1, shown as point C on the $\tau-\gamma$ curve in Fig. 5–36, the release of T_p will cause a reverse shear stress, such that the material behavior will follow the straight-lined segment CD, creating some *elastic recovery* of the shear strain γ_1. This line is parallel to the initial straight-lined portion AB of the $\tau-\gamma$ diagram, and thus both lines have a slope G as indicated.

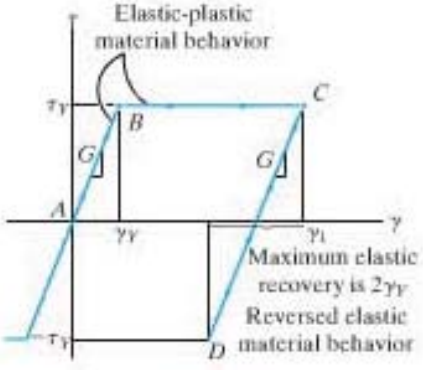
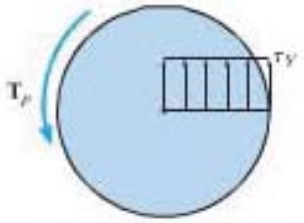
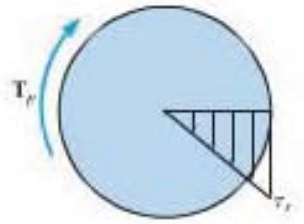


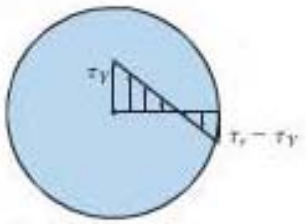
Fig. 5–36



Plastic torque applied causing plastic shear strains throughout the shaft
(a)



Plastic torque reversed causing elastic shear strains throughout the shaft
(b)



Residual shear–stress distribution in shaft
(c)

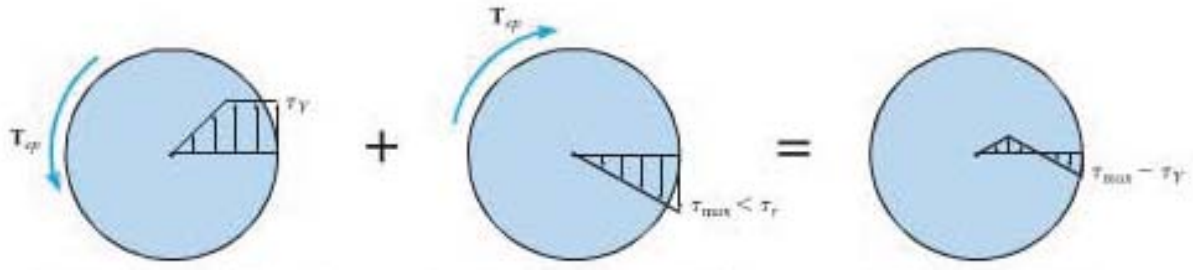
Since elastic recovery occurs, we can superimpose on the plastic torque stress distribution in Fig. 5–37*a* a *linear stress distribution* caused by applying the plastic torque $\mathbf{T}_p$ in the *opposite* direction, Fig. 5–37*b*. Here the maximum shear stress τ_r, for this stress distribution, is called the *modulus of rupture* for torsion. It is determined from the torsion formula,* which gives

$$\tau_r = \frac{T_p c}{J} = \frac{T_p c}{(\pi/2)c^4}$$

Using Eq. 5–27,

$$\tau_r = \frac{[(2/3)\pi\tau_Y c^3]c}{(\pi/2)c^4} = \frac{4}{3}\tau_Y$$

Note that reversed application of $\mathbf{T}_p$ using the linear shear-stress distribution in Fig. 5–37*b* is possible here, since the maximum recovery for the elastic shear strain is $2\gamma_Y$, as noted in Fig. 5–37. This corresponds to a maximum applied shear stress of $2\tau_Y$, which is *greater* than the maximum shear stress of $\frac{4}{3}\tau_Y$ calculated above. Hence, by superimposing the stress distributions involving applications and then removal of the plastic torque, we obtain the residual shear-stress distribution in the shaft as shown in Fig. 5–37*c*. It should be noted from this diagram that the shear stress at the center of the shaft, shown as τ_Y, must actually be *zero*, since the material along the axis of the shaft is never strained. The reason this is not zero is because we assumed that *all* the material of the shaft was strained beyond the yield point in order to determine the plastic torque, Fig. 5–37*a*. To be more realistic, an elastic–plastic torque should be considered when modeling the material behavior. Doing so leads to the superposition of the stress distribution shown in Fig. 5–37*d*.



Elastic–plastic torque applied

Elastic–plastic torque reversed

Residual shear–stress distribution in shaft

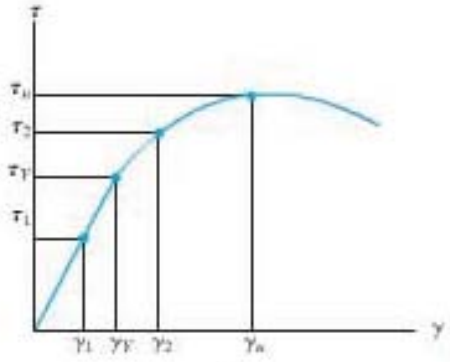
(d)

Fig. 5–37

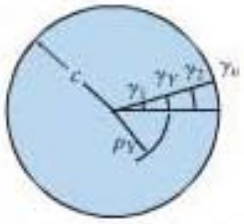
*The torsion formula is valid only when the material behaves in a linear elastic manner; however, the modulus of rupture is so named because it assumes that the material behaves elastically and then suddenly *ruptures* at the proportional limit.

Ultimate Torque. In the general case, most engineering materials will have a shear stress–strain diagram as shown in Fig. 5–38*a*. Consequently, if T is increased so that the maximum shear strain in the shaft becomes $\gamma = \gamma_u$, Fig. 5–38*b*, then, by proportion γ_Y occurs at $\rho_Y = (\gamma_Y/\gamma_u)c$. Likewise, the shear strains at, say, $\rho = \rho_1$ and $\rho = \rho_2$, can be found by proportion, i.e., $\gamma_1 = (\rho_1/c)\gamma_u$ and $\gamma_2 = (\rho_2/c)\gamma_u$. If the corresponding values of τ_1, τ_Y, τ_2, and τ_u are taken from the τ–γ diagram and plotted, we obtain the shear-stress distribution, which acts along a radial line on the cross section, Fig. 5–38*c*. The torque produced by this stress distribution is called the ***ultimate torque***, T_u.

The magnitude of $\mathbf{T}_u$ can be determined by "graphically" integrating Eq. 5–23. To do this, the cross-sectional area of the shaft is segmented into a finite number of rings, such as the one shown shaded in Fig. 5–38*d*. The area of this ring, $\Delta A = 2\pi\rho\,\Delta\rho$, is multiplied by the shear stress τ that acts on it, so that the force $\Delta F = \tau\,\Delta A$ can be determined. The torque created by this force is then $\Delta T = \rho\,\Delta F = \rho(\tau\,\Delta A)$. The addition of all the torques for the entire cross section, as determined in this manner, gives the ultimate torque T_u; that is, Eq. 5–23 becomes $T_u \approx 2\pi\Sigma\tau\rho^2\,\Delta\rho$. If, however, the stress distribution can be expressed as an analytical function, $\tau = f(\rho)$, as in the elastic and plastic torque cases, then the integration of Eq. 5–23 can be carried out directly.

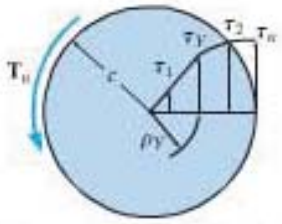


(a)



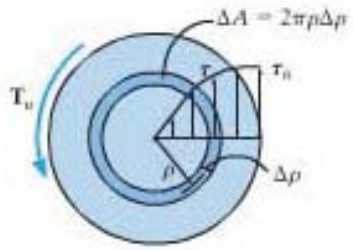
Ultimate shear-strain distribution

(b)



Ultimate shear-stress distribution

(c)



(d)

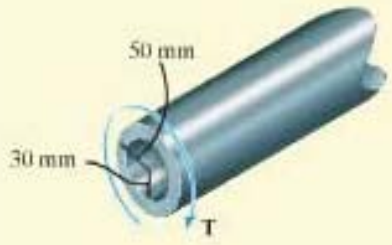
Fig. 5–38

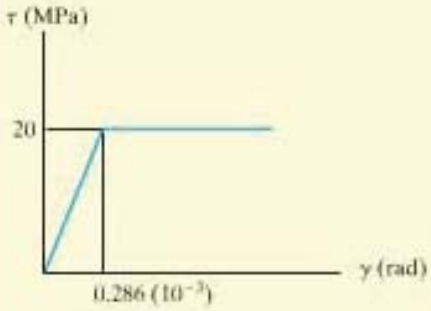
Important Points

- The *shear-strain distribution* along a radial line on the cross section of a shaft is based on geometric considerations, and it is found to *always* vary linearly along the radial line. Once it is established, the shear-stress distribution can then be determined using the shear stress–strain diagram.
- If the shear-stress distribution for the shaft is established, it produces a torque about the axis of the shaft that is equivalent to the resultant internal torque acting on the cross section.
- *Perfectly plastic behavior* assumes the shear-stress distribution is *constant*. When it occurs, the shaft will continue to twist with no increase in torque. This torque is called the *plastic torque*.

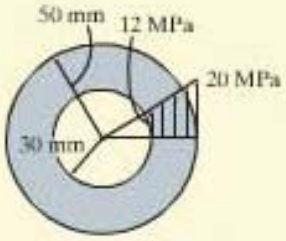
EXAMPLE 5.14

The tubular shaft in Fig. 5–39*a* is made of an aluminum alloy that is assumed to have an elastic-plastic $\tau-\gamma$ diagram as shown. Determine the maximum torque that can be applied to the shaft without causing the material to yield, and the maximum torque or plastic torque that can be applied to the shaft. Also, what should the minimum shear strain at the outer wall be in order to develop a fully plastic torque?

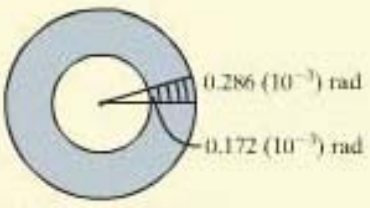




(a)



Elastic shear–stress distribution



Elastic shear–strain distribution

(b)

Fig. 5–39

SOLUTION

Maximum Elastic Torque. We require the shear stress at the outer fiber to be 20 MPa. Using the torsion formula, we have

$$\tau_Y = \frac{T_Y c}{J}; \qquad 20(10^6)\ \text{N/m}^2 = \frac{T_Y(0.05\ \text{m})}{(\pi/2)[(0.05\ \text{m})^4 - (0.03\ \text{m})^4]}$$

$$T_Y = 3.42\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

The shear-stress and shear-strain distributions for this case are shown in Fig. 5–39*b*. The values at the tube's inner wall have been obtained by proportion.

Plastic Torque. The shear-stress distribution in this case is shown in Fig. 5–39*c*. Application of Eq. 5–23 requires $\tau = \tau_Y$. We have

$$T_p = 2\pi \int_{0.03\ \text{m}}^{0.05\ \text{m}} [20(10^6)\ \text{N/m}^2]\rho^2\, d\rho = 125.66(10^6)\frac{1}{3}\rho^3 \Big|_{0.03\ \text{m}}^{0.05\ \text{m}}$$

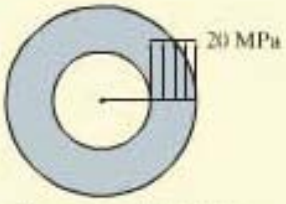
$$= 4.11\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

For this tube T_p represents a 20% increase in torque capacity compared with the elastic torque T_Y.

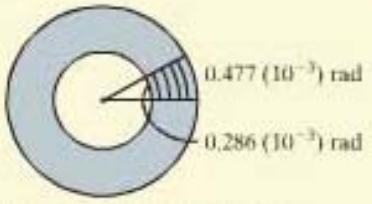
Outer Radius Shear Strain. The tube becomes fully plastic when the shear strain at the *inner wall* becomes $0.286(10^{-3})$ rad, as shown in Fig. 5–39*c*. Since the shear strain *remains linear* over the cross section, the plastic strain at the outer fibers of the tube in Fig. 5–39*c* is determined by proportion.

$$\frac{\gamma_o}{50\ \text{mm}} = \frac{0.286(10^{-3})\ \text{rad}}{30\ \text{mm}}$$

$$\gamma_o = 0.477(10^{-3})\ \text{rad} \qquad \textit{Ans.}$$



Plastic shear–stress distribution

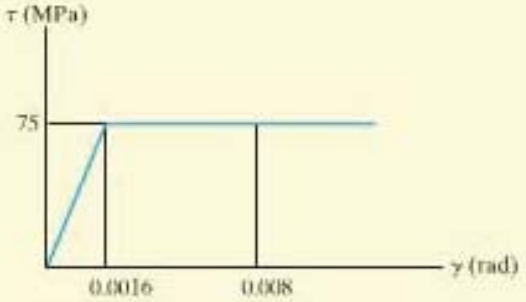


Initial plastic shear–strain distribution

(c)

EXAMPLE 5.15

A solid circular shaft has a radius of 20 mm and length of 1.5 m. The material has an elastic–plastic $\tau-\gamma$ diagram as shown in Fig. 5–40*a*. Determine the torque needed to twist the shaft $\phi = 0.6$ rad.



(a)

SOLUTION

We will first obtain the shear-strain distribution, then establish the shear-stress distribution. Once this is known, the applied torque can be determined.

The maximum shear strain occurs at the surface of the shaft, $\rho = c$. Since the angle of twist is $\phi = 0.6$ rad for the entire 1.5-m length of the shaft, then using Eq. 5–25, for the entire length we have

$$\phi = \gamma\frac{L}{\rho}; \qquad 0.6 = \frac{\gamma_{max}(1.5\text{ m})}{(0.02\text{ m})}$$

$$\gamma_{max} = 0.008\text{ rad}$$

The shear-strain distribution is shown in Fig. 5–40*b*. Note that yielding of the material occurs since $\gamma_{max} > \gamma_Y = 0.0016$ rad in Fig. 5–40*a*. The radius of the elastic core, ρ_Y, can be obtained by proportion. From Fig. 5–40*b*,

$$\frac{\rho_Y}{0.0016} = \frac{0.02\text{ m}}{0.008}$$

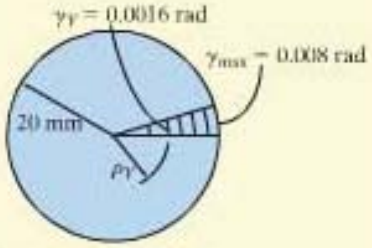
$$\rho_Y = 0.004\text{ m} = 4\text{ mm}$$

Based on the shear-strain distribution, the shear-stress distribution, plotted over a radial line segment, is shown in Fig. 5–40*c*. The torque can now be obtained using Eq. 5–26. Substituting in the numerical data yields

$$T = \frac{\pi\tau_Y}{6}(4c^3 - \rho_Y^3)$$

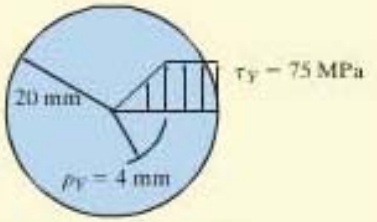
$$= \frac{\pi[75(10^6)\text{ N/m}^2]}{6}[4(0.02\text{ m})^3 - (0.004\text{ m})^3]$$

$$= 1.25\text{ kN}\cdot\text{m} \qquad \textit{Ans}$$



Shear–strain distribution

(b)



Shear–stress distribution

(c)

Fig. 5–40

EXAMPLE 5.16

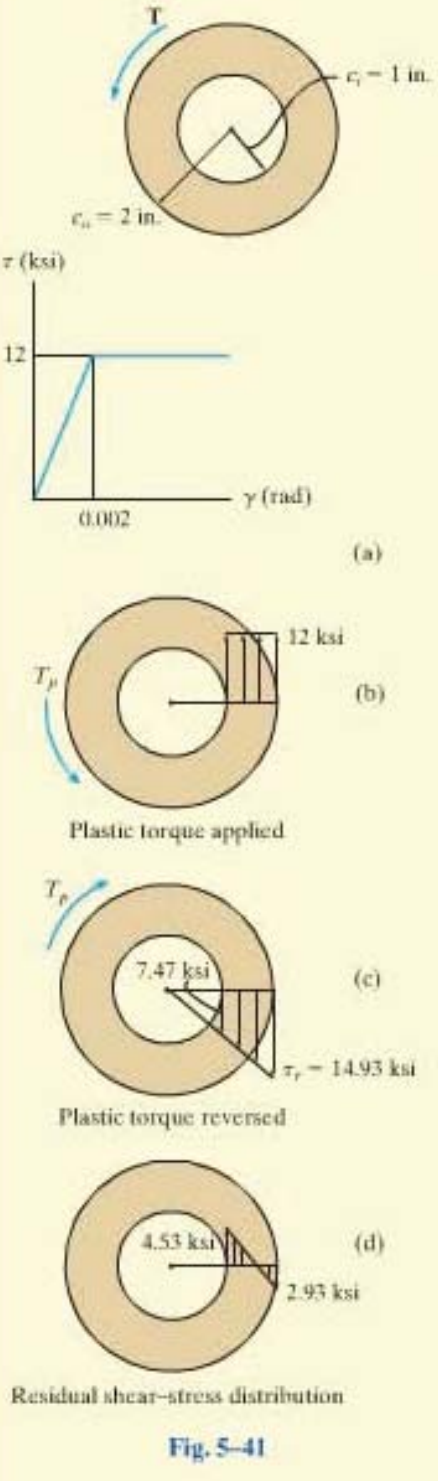


Fig. 5–41

A tube in Fig. 5–41*a* has a length of 5 ft and the material has an elastic-plastic $\tau-\gamma$ diagram, also shown in Fig. 5–41*a*. Determine the plastic torque T_p. What is the residual shear-stress distribution if $\mathbf{T}_p$ is removed *just after* the tube becomes fully plastic?

SOLUTION

Plastic Torque. The plastic torque $\mathbf{T}_p$ will strain the tube such that all the material yields. Hence the stress distribution will appear as shown in Fig. 5–41*b*. Applying Eq. 5–23, we have

$$T_p = 2\pi \int_{c_i}^{c_o} \tau_Y \rho^2 \, d\rho = \frac{2\pi}{3}\tau_Y (c_o^3 - c_i^3)$$

$$= \frac{2\pi}{3}(12(10^3)\ \text{lb/in}^2)[(2\ \text{in.})^3 - (1\ \text{in.})^3] = 175.9\ \text{kip}\cdot\text{in.} \quad \textit{Ans.}$$

When the tube just becomes fully plastic, yielding has started at the inner wall, i.e., at $c_i = 1$ in., $\gamma_Y = 0.002$ rad, Fig. 5–41*a*. The angle of twist that occurs can be determined from Eq. 5–25, which for the entire tube becomes

$$\phi_p = \gamma_Y \frac{L}{c_i} = \frac{(0.002)(5\ \text{ft})(12\ \text{in./ft})}{(1\ \text{in.})} = 0.120\ \text{rad}$$

When $\mathbf{T}_p$ is *removed*, or in effect reapplied in the opposite direction, then the "fictitious" linear shear-stress distribution shown in Fig. 5–41*c* must be superimposed on the one shown in Fig. 5–41*b*. In Fig. 5–41*c* the maximum shear stress or the modulus of rupture is found from the torsion formula

$$\tau_r = \frac{T_p c_o}{J} = \frac{(175.9\ \text{kip}\cdot\text{in.})(2\ \text{in.})}{(\pi/2)[(2\ \text{in.})^4 - (1\ \text{in.})^4]} = 14.93\ \text{ksi}$$

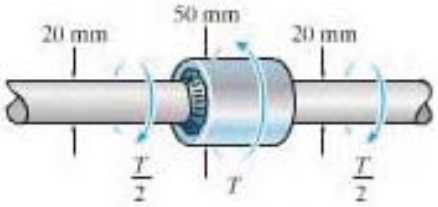
Also, at the inner wall of the tube the shear stress is

$$\tau_i = (14.93\ \text{ksi})\left(\frac{1\ \text{in.}}{2\ \text{in.}}\right) = 7.47\ \text{ksi} \quad \textit{Ans.}$$

The resultant residual shear-stress distribution is shown in Fig. 5–41*d*.

PROBLEMS

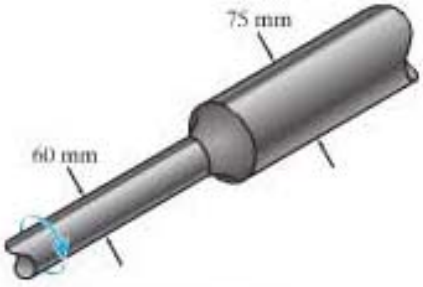
***5–120.** The steel used for the shaft has an allowable shear stress of $\tau_{allow} = 8$ MPa. If the members are connected with a fillet weld of radius $r = 4$ mm, determine the maximum torque T that can be applied.



Prob. 5–120

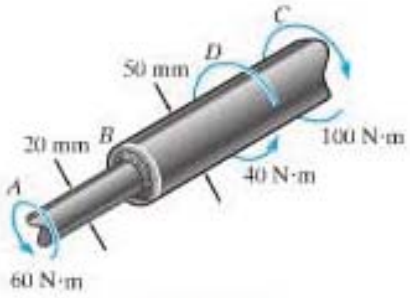
•5–121. The built-up shaft is to be designed to rotate at 720 rpm while transmitting 30 kW of power. Is this possible? The allowable shear stress is $\tau_{allow} = 12$ MPa.

5–122. The built-up shaft is designed to rotate at 540 rpm. If the radius of the fillet weld connecting the shafts is $r = 7.20$ mm, and the allowable shear stress for the material is $\tau_{allow} = 55$ MPa, determine the maximum power the shaft can transmit.



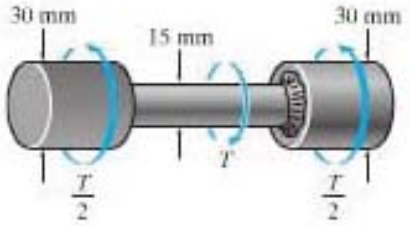
Probs. 5–121/122

5–123. The steel shaft is made from two segments: AB and BC, which are connected using a fillet weld having a radius of 2.8 mm. Determine the maximum shear stress developed in the shaft.



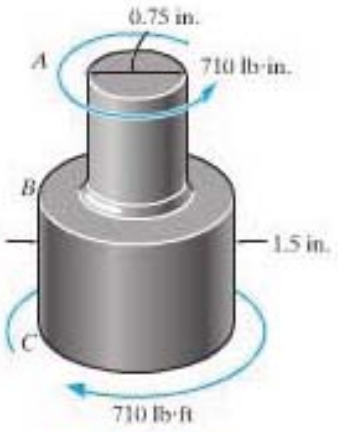
Prob. 5–123

***5–124.** The steel used for the shaft has an allowable shear stress of $\tau_{allow} = 8$ MPa. If the members are connected together with a fillet weld of radius $r = 2.25$ mm, determine the maximum torque T that can be applied.



Prob. 5–124

•5–125. The assembly is subjected to a torque of 710 lb · in. If the allowable shear stress for the material is $\tau_{allow} = 12$ ksi, determine the radius of the smallest size fillet that can be used to transmit the torque.



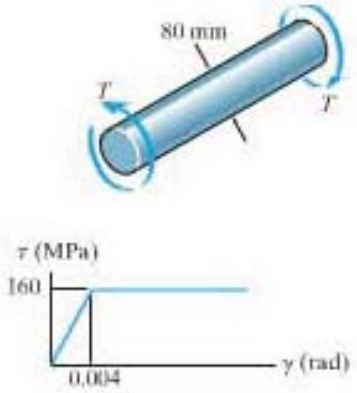
Prob. 5–125

5–126. A solid shaft is subjected to the torque T, which causes the material to yield. If the material is elastic plastic, show that the torque can be expressed in terms of the angle of twist ϕ of the shaft as $T = \frac{4}{3}T_Y(1 - \phi^3_Y/4\phi^3)$, where T_Y and ϕ_Y are the torque and angle of twist when the material begins to yield.

5–127. A solid shaft having a diameter of 2 in. is made of elastic-plastic material having a yield stress of $\tau_Y = 16$ ksi and shear modulus of $G = 12(10^3)$ ksi. Determine the torque required to develop an elastic core in the shaft having a diameter of 1 in. Also, what is the plastic torque?

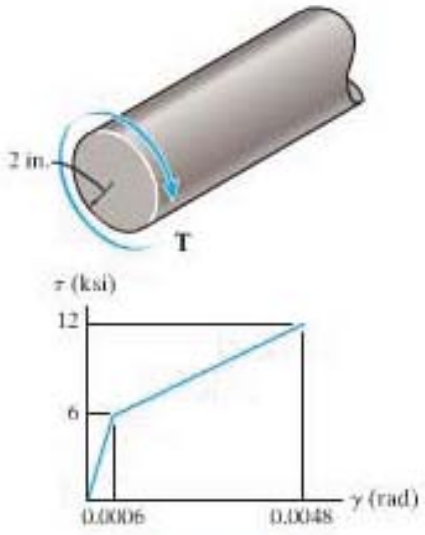
***5–128.** Determine the torque needed to twist a short 3-mm-diameter steel wire through several revolutions if it is made from steel assumed to be elastic plastic and having a yield stress of $\tau_Y = 80$ MPa. Assume that the material becomes fully plastic.

•5–129. The solid shaft is made of an elastic-perfectly plastic material as shown. Determine the torque T needed to form an elastic core in the shaft having a radius of $\rho_Y = 20$ mm. If the shaft is 3 m long, through what angle does one end of the shaft twist with respect to the other end? When the torque is removed, determine the residual stress distribution in the shaft and the permanent angle of twist.



Prob. 5–129

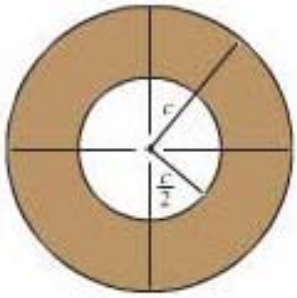
5–130. The shaft is subjected to a maximum shear strain of 0.0048 rad. Determine the torque applied to the shaft if the material has strain hardening as shown by the shear stress–strain diagram.



Prob. 5–130

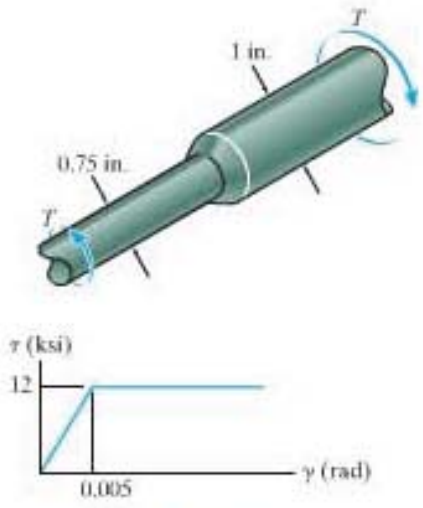
5–131. An 80-mm diameter solid circular shaft is made of an elastic-perfectly plastic material having a yield shear stress of $\tau_Y = 125\,\text{MPa}$. Determine (a) the maximum elastic torque T_Y; and (b) the plastic torque T_p.

***5–132.** The hollow shaft has the cross section shown and is made of an elastic-perfectly plastic material having a yield shear stress of τ_Y. Determine the ratio of the plastic torque T_p to the maximum elastic torque T_Y.



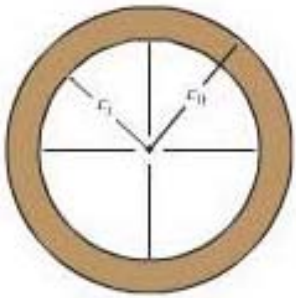
Prob. 5–132

5–133. The shaft consists of two sections that are rigidly connected. If the material is elastic plastic as shown, determine the largest torque T that can be applied to the shaft. Also, draw the shear-stress distribution over a radial line for each section. Neglect the effect of stress concentration.



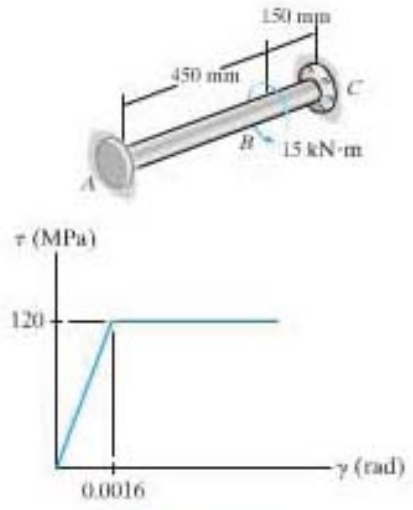
Prob. 5–133

5–134. The hollow shaft is made of an elastic-perfectly plastic material having a shear modulus of G and a yield shear stress of τ_Y. Determine the applied torque $\mathbf{T}_p$ when the material of the inner surface is about to yield (plastic torque). Also, find the corresponding angle of twist and the maximum shear strain. The shaft has a length of L.



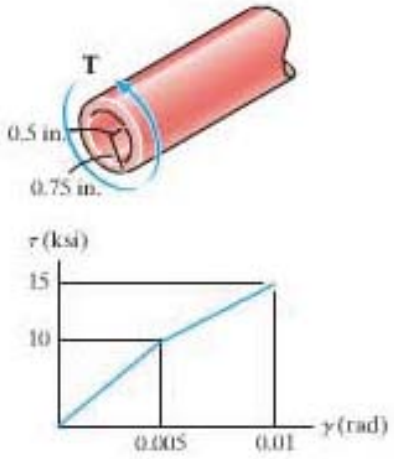
Prob. 5–134

5–135. The hollow shaft has inner and outer diameters of 60 mm and 80 mm, respectively. If it is made of an elastic-perfectly plastic material, which has the $\tau - \gamma$ diagram shown, determine the reactions at the fixed supports A and C.



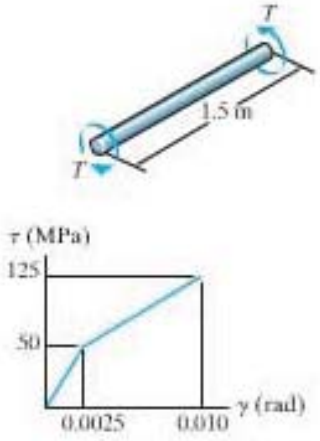
Prob. 5–135

***5–136.** The tubular shaft is made of a strain-hardening material having a $\tau-\gamma$ diagram as shown. Determine the torque T that must be applied to the shaft so that the maximum shear strain is 0.01 rad.



Prob. 5–136

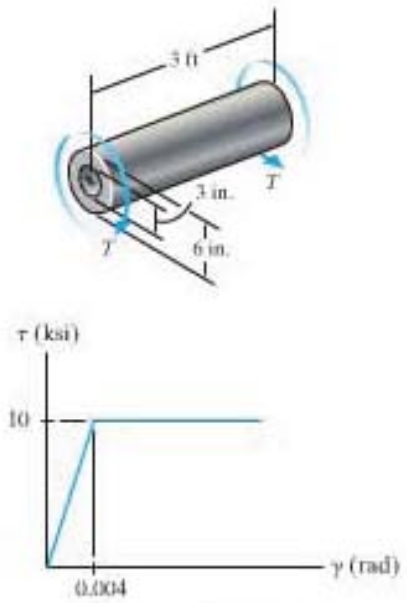
•5–137. The shear stress–strain diagram for a solid 50-mm-diameter shaft can be approximated as shown in the figure. Determine the torque T required to cause a maximum shear stress in the shaft of 125 MPa. If the shaft is 1.5 m long, what is the corresponding angle of twist?



Prob. 5–137

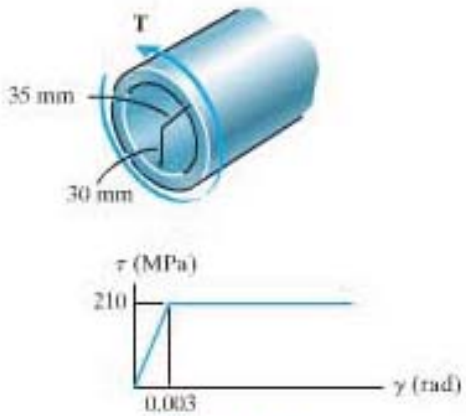
5–138. A tube is made of elastic-perfectly plastic material, which has the $\tau-\gamma$ diagram shown. If the radius of the elastic core is $\rho_Y = 2.25$ in., determine the applied torque T. Also, find the residual shear-stress distribution in the shaft and the permanent angle of twist of one end relative to the other when the torque is removed.

5–139. The tube is made of elastic-perfectly plastic material, which has the $\tau-\gamma$ diagram shown. Determine the torque T that just causes the inner surface of the shaft to yield. Also, find the residual shear-stress distribution in the shaft when the torque is removed.



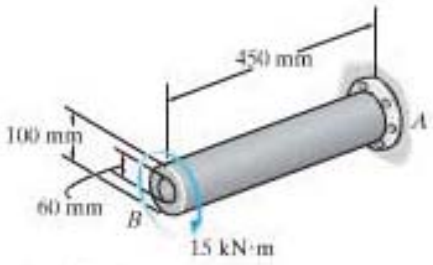
Probs. 5–138/139

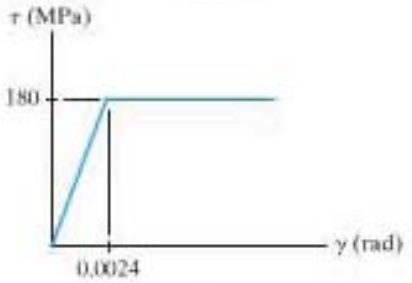
***5–140.** The 2-m-long tube is made of an elastic-perfectly plastic material as shown. Determine the applied torque T that subjects the material at the tube's outer edge to a shear strain of $\gamma_{max} = 0.006$ rad. What would be the permanent angle of twist of the tube when this torque is removed? Sketch the residual stress distribution in the tube.



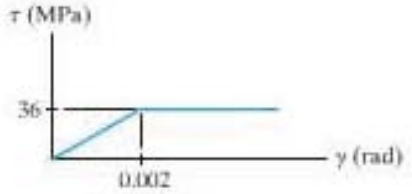
Prob. 5–140

•5–141. A steel alloy core is bonded firmly to the copper alloy tube to form the shaft shown. If the materials have the $\tau-\gamma$ diagrams shown, determine the torque resisted by the core and the tube.





Steel Alloy



Copper Alloy

Prob. 5–141

5–142. A torque is applied to the shaft of radius r. If the material has a shear stress–strain relation of $\tau = k\gamma^{1/6}$, where k is a constant, determine the maximum shear stress in the shaft.



Prob. 5–142

CHAPTER REVIEW

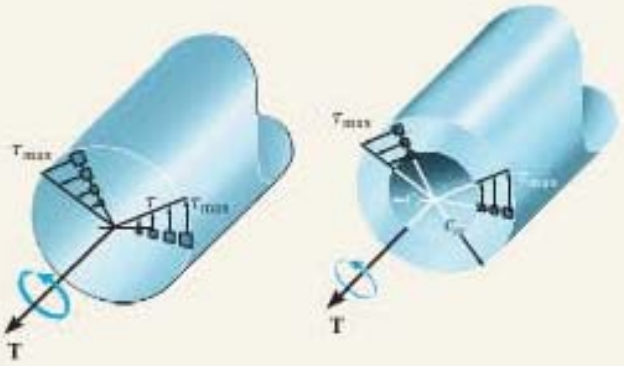
Torque causes a shaft having a circular cross section to twist, such that the shear strain in the shaft is proportional to its radial distance from the center of the shaft. Provided the material is homogeneous and linear elastic, then the shear stress is determined from the torsion formula,

$$\tau = \frac{T\rho}{J}$$

The design of a shaft requires finding the geometric parameter,

$$\frac{J}{c} = \frac{T}{\tau_{allow}}$$

Often the power P supplied to a shaft rotating at ω is reported, in which case the torque is determined from $P = T\omega$.



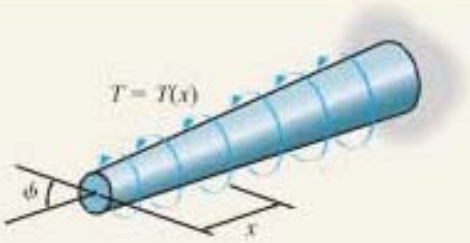
The angle of twist of a circular shaft is determined from

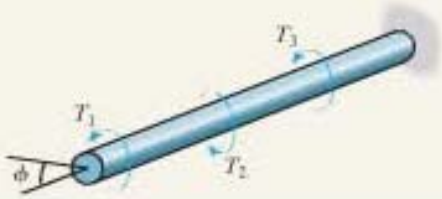
$$\phi = \int_0^L \frac{T(x)\,dx}{JG}$$

If the internal torque and JG are constant within each segment of the shaft then

$$\phi = \Sigma \frac{TL}{JG}$$

For application, it is necessary to use a sign convention for the internal torque and to be sure the material remains linear elastic.



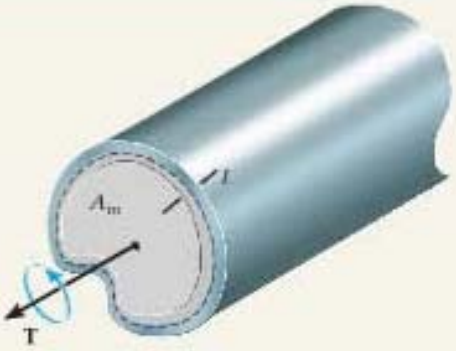


If the shaft is statically indeterminate, then the reactive torques are determined from equilibrium, compatibility of twist, and a torque-twist relationship, such as $\phi = TL/JG$.

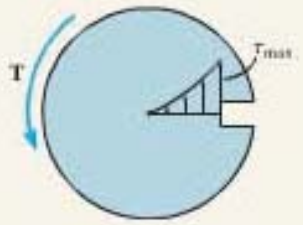
Solid non-circular shafts tend to warp out of plane when subjected to a torque. Formulas are available to determine the maximum elastic shear stress and the twist for these cases.



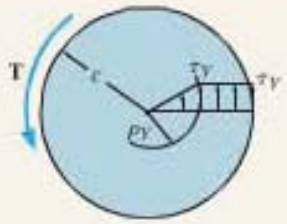
The average shear stress in thin-walled tubes is determined by assuming the shear stress across each thickness t of the tube is constant. Its value is determined from $\tau_{avg} = \frac{T}{2tA_m}$.



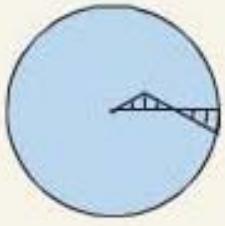
Stress concentrations occur in shafts when the cross section suddenly changes. The maximum shear stress is determined using a stress concentration factor K, which is determined from experiment and represented in graphical form. Once obtained, $\tau_{max} = K\left(\frac{Tc}{J}\right)$.



If the applied torque causes the material to exceed the elastic limit, then the stress distribution will not be proportional to the radial distance from the centerline of the shaft. Instead, the internal torque is related to the stress distribution using the shear-stress–shear-strain diagram and equilibrium.



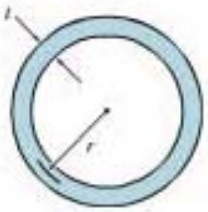
If a shaft is subjected to a plastic torque, which is then released, it will cause the material to respond elastically, thereby causing residual shear stress to be developed in the shaft.



5

REVIEW PROBLEMS

5–143. Consider a thin-walled tube of mean radius r and thickness t. Show that the maximum shear stress in the tube due to an applied torque T approaches the average shear stress computed from Eq. 5–18 as $r/t \to \infty$.



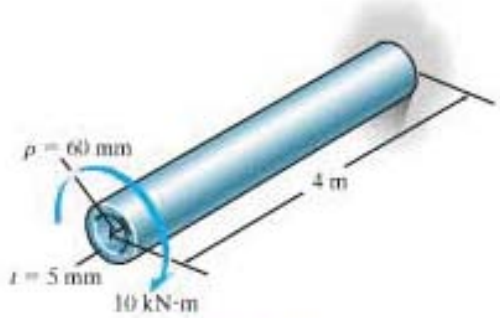
Prob. 5–143

*5–144. The 304 stainless steel shaft is 3 m long and has an outer diameter of 60 mm. When it is rotating at 60 rad/s, it transmits 30 kW of power from the engine E to the generator G. Determine the smallest thickness of the shaft if the allowable shear stress is $\tau_{allow} = 150$ MPa and the shaft is restricted not to twist more than 0.08 rad.



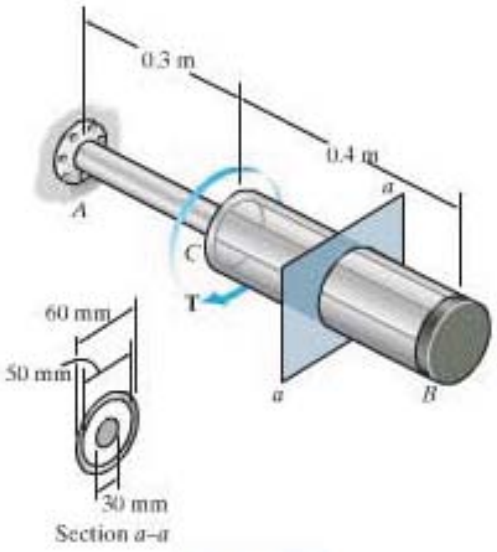
Prob. 5–144

•5–145. The A-36 steel circular tube is subjected to a torque of 10 kN · m. Determine the shear stress at the mean radius $\rho = 60$ mm and compute the angle of twist of the tube if it is 4 m long and fixed at its far end. Solve the problem using Eqs. 5–7 and 5–15 and by using Eqs. 5–18 and 5–20.



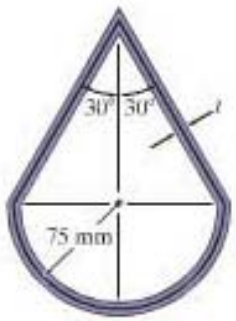
Prob. 5–145

5–146. Rod AB is made of A-36 steel with an allowable shear stress of $(\tau_{allow})_{st} = 75$ MPa, and tube BC is made of AM1004-T61 magnesium alloy with an allowable shear stress of $(\tau_{allow})_{mg} = 45$ MPa. The angle of twist of end C is not allowed to exceed 0.05 rad. Determine the maximum allowable torque **T** that can be applied to the assembly.



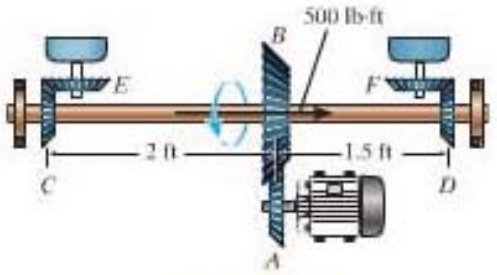
Prob. 5–146

5–147. A shaft has the cross section shown and is made of 2014-T6 aluminum alloy having an allowable shear stress of $\tau_{allow} = 125$ MPa. If the angle of twist per meter length is not allowed to exceed 0.03 rad, determine the required minimum wall thickness t to the nearest millimeter when the shaft is subjected to a torque of $T = 15$ kN · m.



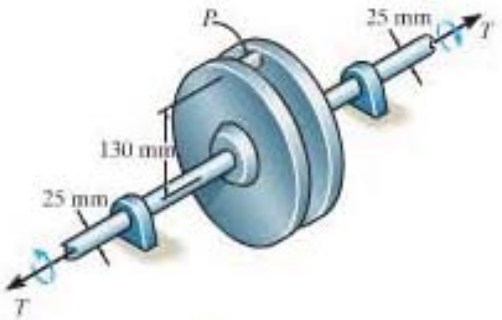
Prob. 5–147

*5–148. The motor A develops a torque at gear B of 500 lb·ft, which is applied along the axis of the 2-in.-diameter A-36 steel shaft CD. This torque is to be transmitted to the pinion gears at E and F. If these gears are temporarily fixed, determine the maximum shear stress in segments CB and BD of the shaft. Also, what is the angle of twist of each of these segments? The bearings at C and D only exert force reactions on the shaft.



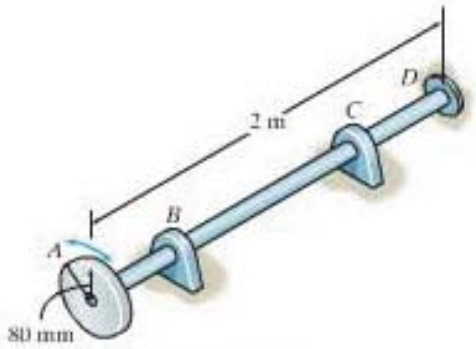
Prob. 5–148

5–149. The coupling consists of two disks fixed to separate shafts, each 25 mm in diameter. The shafts are supported on journal bearings that allow free rotation. In order to limit the torque **T** that can be transmitted, a "shear pin" P is used to connect the disks together. If this pin can sustain an *average* shear force of 550 N before it fails, determine the maximum constant torque T that can be transmitted from one shaft to the other. Also, what is the maximum shear stress in each shaft when the "shear pin" is about to fail?



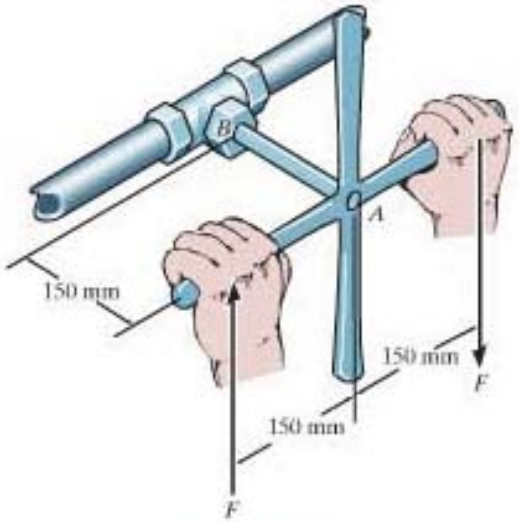
Prob. 5–149

5–150. The rotating flywheel and shaft is brought to a sudden stop at D when the bearing freezes. This causes the flywheel to oscillate clockwise–counterclockwise, so that a point A on the outer edge of the flywheel is displaced through a 10-mm arc in either direction. Determine the maximum shear stress developed in the tubular 304 stainless steel shaft due to this oscillation. The shaft has an inner diameter of 25 mm and an outer diameter of 35 mm. The journal bearings at B and C allow the shaft to rotate freely.



Prob. 5–150

5–151. If the solid shaft AB to which the valve handle is attached is made of C83400 red brass and has a diameter of 10 mm, determine the maximum couple forces F that can be applied to the handle just before the material starts to fail. Take $\tau_{allow} = 40$ MPa. What is the angle of twist of the handle? The shaft is fixed at A.



Prob. 5–151



Beams are important structural members used in building construction. Their design is often based upon their ability to resist bending stress, which forms the subject matter of this chapter.

Bending

6

CHAPTER OBJECTIVES

Beams and shafts are important structural and mechanical elements in engineering. In this chapter we will determine the stress in these members caused by bending. The chapter begins with a discussion of how to establish the shear and moment diagrams for a beam or shaft. Like the normal-force and torque diagrams, the shear and moment diagrams provide a useful means for determining the largest shear and moment in a member, and they specify where these maximums occur. Once the internal moment at a section is determined, the bending stress can then be calculated. First we will consider members that are straight, have a symmetric cross section, and are made of homogeneous linear elastic material. Afterward we will discuss special cases involving unsymmetric bending and members made of composite materials. Consideration will also be given to curved members, stress concentrations, inelastic bending, and residual stresses.

6.1 Shear and Moment Diagrams

Members that are slender and support loadings that are applied perpendicular to their longitudinal axis are called ***beams***. In general, beams are long, straight bars having a constant cross-sectional area. Often they are classified as to how they are supported. For example, a *simply supported beam* is pinned at one end and roller supported at the other, Fig. 6–1, a *cantilevered beam* is fixed at one end and free at the other, and an *overhanging beam* has one or both of its ends freely extended over the supports. Beams are considered among the most important of all structural elements. They are used to support the floor of a building, the deck of a bridge, or the wing of an aircraft. Also, the axle of an automobile, the boom of a crane, even many of the bones of the body act as beams.

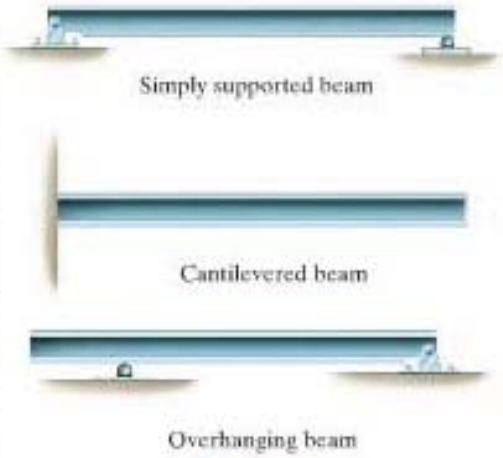


Fig. 6–1

Because of the applied loadings, beams develop an internal shear force and bending moment that, in general, vary from point to point along the axis of the beam. In order to properly design a beam it therefore becomes necessary to determine the *maximum* shear and moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis. These *shear and moment functions* can then be plotted and represented by graphs called ***shear and moment diagrams***. The maximum values of V and M can then be obtained from these graphs. Also, since the shear and moment diagrams provide detailed information about the *variation* of the shear and moment along the beam's axis, they are often used by engineers to decide where to place reinforcement materials within the beam or how to proportion the size of the beam at various points along its length.

In order to formulate V and M in terms of x we must choose the origin and the positive direction for x. Although the choice is arbitrary, most often the origin is located at the left end of the beam and the positive direction is to the right.

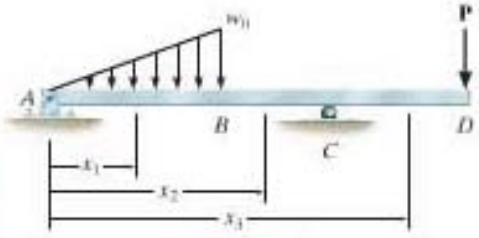


Fig. 6–2

In general, the internal shear and moment functions of x will be *discontinuous*, or their slope will be discontinuous, at points where a distributed load changes or where concentrated forces or couple moments are applied. Because of this, the shear and moment functions must be determined for *each region* of the beam *between* any two discontinuities of loading. For example, coordinates x_1, x_2, and x_3 will have to be used to describe the variation of V and M throughout the length of the beam in Fig. 6–2. These coordinates will be valid *only* within the regions from A to B for x_1, from B to C for x_2, and from C to D for x_3.

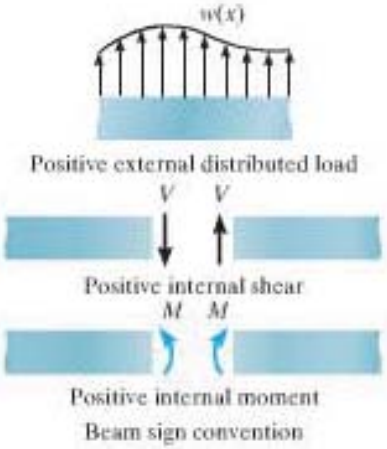


Fig. 6–3

Beam Sign Convention. Before presenting a method for determining the shear and moment as functions of x and later plotting these functions (shear and moment diagrams), it is first necessary to establish a *sign convention* so as to define "positive" and "negative" values for V and M. Although the choice of a sign convention is arbitrary, here we will use the one often used in engineering practice and shown in Fig. 6–3. The *positive directions* are as follows: the *distributed load* acts *upward* on the beam; the internal *shear force* causes a *clockwise* rotation of the beam segment on which it acts; and the internal *moment* causes *compression* in the *top fibers* of the segment such that it bends the segment so that it holds water. Loadings that are opposite to these are considered negative.

Important Points

- *Beams* are long straight members that are subjected to loads perpendicular to their longitudinal axis. They are classified according to the way they are supported, e.g., simply supported, cantilevered, or overhanging.
- In order to properly design a beam, it is important to know the *variation* of the internal shear and moment along its axis in order to find the points where these values are a maximum.
- Using an established sign convention for positive shear and moment, the shear and moment in the beam can be determined as a function of its position x on the beam, and then these functions can be plotted to form the shear and moment diagrams.

Procedure for Analysis

The shear and moment diagrams for a beam can be constructed using the following procedure.

Support Reactions.

- Determine all the reactive forces and couple moments acting on the beam, and resolve all the forces into components acting perpendicular and parallel to the beam's axis.

Shear and Moment Functions.

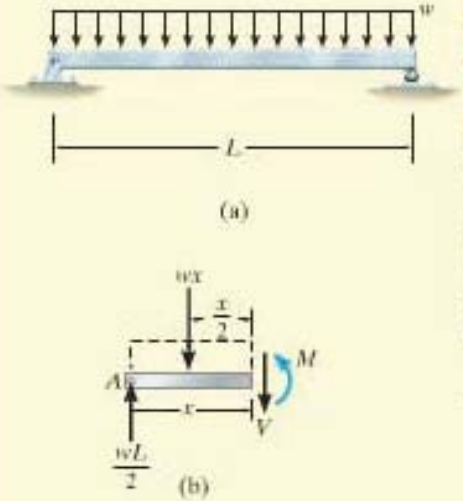
- Specify separate coordinates x having an origin at the beam's *left end* and extending to regions of the beam between concentrated forces and/or couple moments, or where there is no discontinuity of distributed loading.
- Section the beam at each distance x, and draw the free-body diagram of one of the segments. Be sure **V** and **M** are shown acting in their positive sense, in accordance with the sign convention given in Fig. 6–3.
- The shear is obtained by summing forces perpendicular to the beam's axis.
- To eliminate V, the moment is obtained directly by summing moments about the sectioned end of the segment.

Shear and Moment Diagrams.

- Plot the shear diagram (V versus x) and the moment diagram (M versus x). If numerical values of the functions describing V and M are *positive*, the values are plotted above the x axis, whereas negative values are plotted below the axis.
- Generally it is convenient to show the shear and moment diagrams below the free-body diagram of the beam.

EXAMPLE 6.1

Draw the shear and moment diagrams for the beam shown in Fig. 6–4*a*.



SOLUTION

Support Reactions. The support reactions are shown in Fig. 6–4*c*.

Shear and Moment Functions. A free-body diagram of the left segment of the beam is shown in Fig. 6–4*b*. The distributed loading on this segment, wx, is represented by its resultant force only *after* the segment is isolated as a free-body diagram. This force acts through the centroid of the area comprising the distributed loading, a distance of $x/2$ from the right end. Applying the two equations of equilibrium yields

$$+\uparrow \Sigma F_y = 0; \qquad \frac{wL}{2} - wx - V = 0$$

$$V = w\left(\frac{L}{2} - x\right) \tag{1}$$

$$\circlearrowleft +\Sigma M = 0; \qquad -\left(\frac{wL}{2}\right)x + (wx)\left(\frac{x}{2}\right) + M = 0$$

$$M = \frac{w}{2}(Lx - x^2) \tag{2}$$

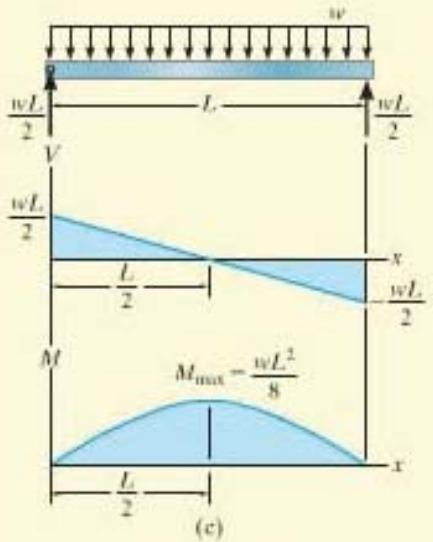


Fig. 6–4

Shear and Moment Diagrams. The shear and moment diagrams shown in Fig. 6–4*c* are obtained by plotting Eqs. 1 and 2. The point of *zero shear* can be found from Eq. 1:

$$V = w\left(\frac{L}{2} - x\right) = 0$$

$$x = \frac{L}{2}$$

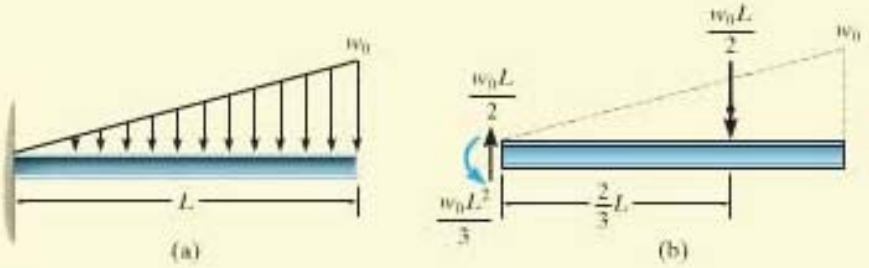
NOTE: From the moment diagram, this value of x represents the point on the beam where the *maximum moment* occurs, since by Eq. 6–2 (see Sec. 6.2) the *slope* $V = dM/dx = 0$. From Eq. 2, we have

$$M_{\max} = \frac{w}{2}\left[L\left(\frac{L}{2}\right) - \left(\frac{L}{2}\right)^2\right]$$

$$= \frac{wL^2}{8}$$

EXAMPLE 6.2

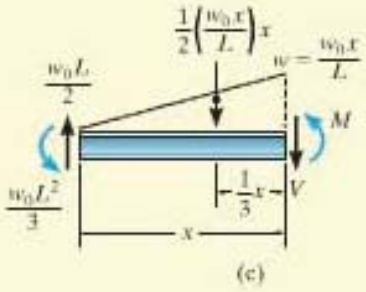
Draw the shear and moment diagrams for the beam shown in Fig. 6–5*a*.



SOLUTION

Support Reactions. The distributed load is replaced by its resultant force and the reactions have been determined as shown in Fig. 6–5*b*.

Shear and Moment Functions. A free-body diagram of a beam segment of length x is shown in Fig. 6–5*c*. Note that the intensity of the triangular load at the section is found by proportion, that is, $w/x = w_0/L$ or $w = w_0x/L$. With the load intensity known, the resultant of the distributed loading is determined from the area under the diagram. Thus,



$$+\uparrow \Sigma F_y = 0; \qquad \frac{w_0L}{2} - \frac{1}{2}\left(\frac{w_0x}{L}\right)x - V = 0$$

$$V = \frac{w_0}{2L}(L^2 - x^2) \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad \frac{w_0L^2}{3} - \frac{w_0L}{2}(x) + \frac{1}{2}\left(\frac{w_0x}{L}\right)x\left(\frac{1}{3}x\right) + M = 0$$

$$M = \frac{w_0}{6L}(-2L^3 + 3L^2x - x^3) \qquad (2)$$

These results can be checked by applying Eqs. 6–1 and 6–2 of Sec. 6.2, that is,

$$w = \frac{dV}{dx} = \frac{w_0}{2L}(0 - 2x) = -\frac{w_0x}{L} \qquad \text{OK}$$

$$V = \frac{dM}{dx} = \frac{w_0}{6L}(0 + 3L^2 - 3x^2) = \frac{w_0}{2L}(L^2 - x^2) \qquad \text{OK}$$

Shear and Moment Diagrams. The graphs of Eqs. 1 and 2 are shown in Fig. 6–5*d*.

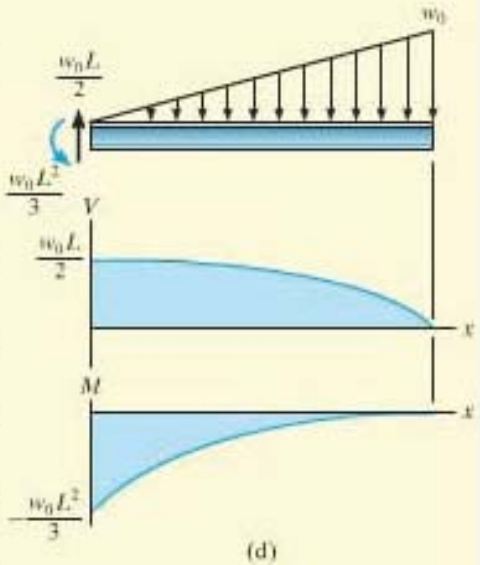
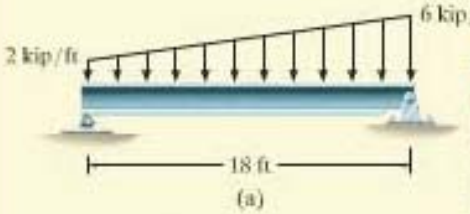


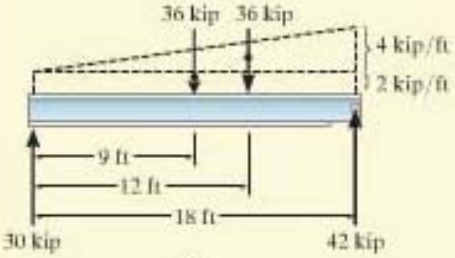
Fig. 6–5

EXAMPLE 6.3

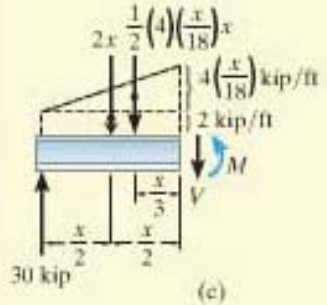
Draw the shear and moment diagrams for the beam shown in Fig. 6–6*a*.



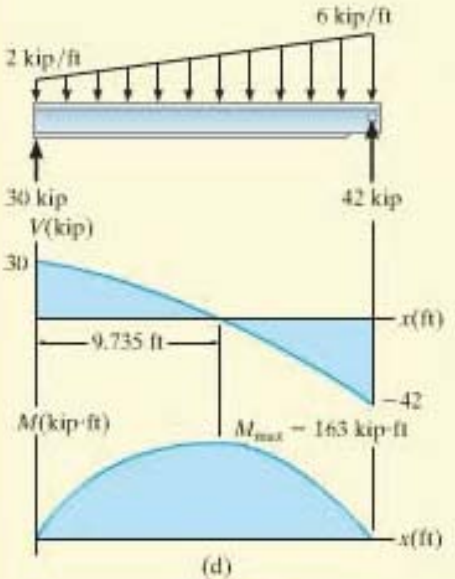
(a)



(b)



(c)



(d)

Fig. 6–6

SOLUTION

Support Reactions. The distributed load is divided into triangular and rectangular component loadings and these loadings are then replaced by their resultant forces. The reactions have been determined as shown on the beam's free-body diagram, Fig. 6–6*b*.

Shear and Moment Functions. A free-body diagram of the left segment is shown in Fig. 6–6*c*. As above, the trapezoidal loading is replaced by rectangular and triangular distributions. Note that the intensity of the triangular load at the section is found by proportion. The resultant force and the location of each distributed loading are also shown. Applying the equilibrium equations, we have

$$+\uparrow \Sigma F_y = 0;\ 30\ \text{kip} - (2\ \text{kip/ft})x - \frac{1}{2}(4\ \text{kip/ft})\left(\frac{x}{18\ \text{ft}}\right)x - V = 0$$

$$V = \left(30 - 2x - \frac{x^2}{9}\right)\ \text{kip} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0;$$

$$-30\ \text{kip}(x) + (2\ \text{kip/ft})x\left(\frac{x}{2}\right) + \frac{1}{2}(4\ \text{kip/ft})\left(\frac{x}{18\ \text{ft}}\right)x\left(\frac{x}{3}\right) + M = 0$$

$$M = \left(30x - x^2 - \frac{x^3}{27}\right)\ \text{kip}\cdot\text{ft} \qquad (2)$$

Equation 2 may be checked by noting that $dM/dx = V$, that is, Eq. 1. Also, $w = dV/dx = -2 - \frac{2}{9}x$. This equation checks, since when $x = 0$, $w = -2$ kip/ft, and when $x = 18$ ft, $w = -6$ kip/ft, Fig. 6–6*a*.

Shear and Moment Diagrams. Equations 1 and 2 are plotted in Fig. 6–6*d*. Since the point of maximum moment occurs when $dM/dx = V = 0$ (Eq. 6–2), then, from Eq. 1,

$$V = 0 = 30 - 2x - \frac{x^2}{9}$$

Choosing the positive root,

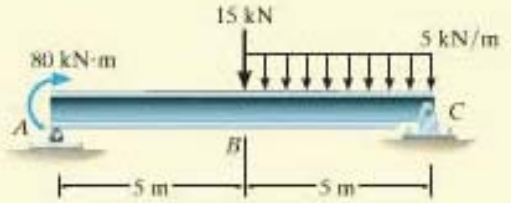
$$x = 9.735\ \text{ft}$$

Thus, from Eq. 2,

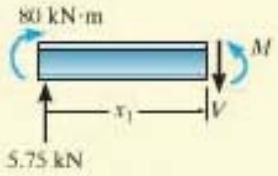
$$M_{max} = 30(9.735) - (9.735)^2 - \frac{(9.735)^3}{27}$$
$$= 163\ \text{kip}\cdot\text{ft}$$

EXAMPLE 6.4

Draw the shear and moment diagrams for the beam shown in Fig. 6–7*a*.



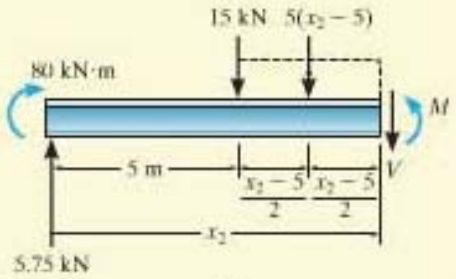
(a)



(b)

SOLUTION

Support Reactions. The reactions at the supports have been determined and are shown on the free-body diagram of the beam, Fig. 6–7*d*.



(c)

Shear and Moment Functions. Since there is a discontinuity of distributed load and also a concentrated load at the beam's center, two regions of x must be considered in order to describe the shear and moment functions for the entire beam.

$0 \le x_1 < 5$ m, Fig. 6–7*b*:

$$+\uparrow \Sigma F_y = 0; \quad 5.75\ \text{kN} - V = 0$$

$$V = 5.75\ \text{kN} \tag{1}$$

$$\circlearrowleft + \Sigma M = 0; \quad -80\ \text{kN}\cdot\text{m} - 5.75\ \text{kN}\, x_1 + M = 0$$

$$M = (5.75x_1 + 80)\ \text{kN}\cdot\text{m} \tag{2}$$

$5\ \text{m} < x_2 \le 10$ m, Fig. 6–7*c*:

$$+\uparrow \Sigma F_y = 0; \quad 5.75\ \text{kN} - 15\ \text{kN} - 5\ \text{kN/m}(x_2 - 5\ \text{m}) - V = 0$$

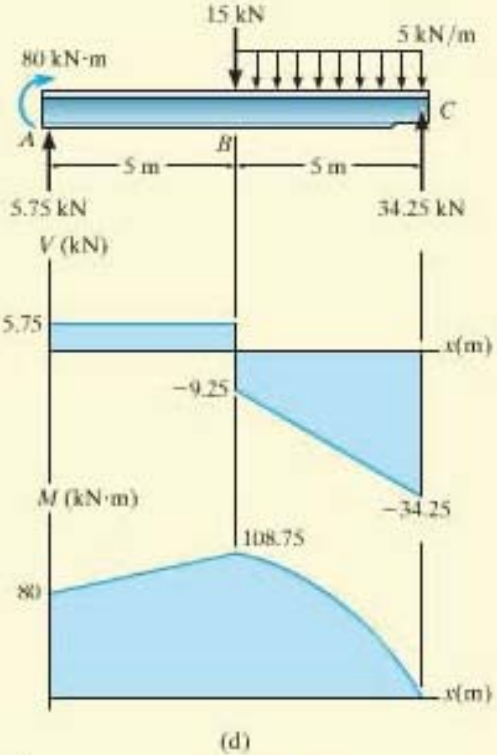
$$V = (15.75 - 5x_2)\ \text{kN} \tag{3}$$

$$\circlearrowleft + \Sigma M = 0; \quad -80\ \text{kN}\cdot\text{m} - 5.75\ \text{kN}\, x_2 + 15\ \text{kN}(x_2 - 5\ \text{m}) + 5\ \text{kN/m}(x_2 - 5\ \text{m})\left(\frac{x_2 - 5\ \text{m}}{2}\right) + M = 0$$

$$M = (-2.5x_2^2 + 15.75x_2 + 92.5)\ \text{kN}\cdot\text{m} \tag{4}$$

These results can be checked in part by noting that $w = dV/dx$ and $V = dM/dx$. Also, when $x_1 = 0$, Eqs. 1 and 2 give $V = 5.75$ kN and $M = 80$ kN·m; when $x_2 = 10$ m, Eqs. 3 and 4 give $V = -34.25$ kN and $M = 0$. These values check with the support reactions shown on the free-body diagram, Fig. 6–7*d*.

Shear and Moment Diagrams. Equations 1 through 4 are plotted in Fig. 6–7*d*.



(d)

Fig. 6–7



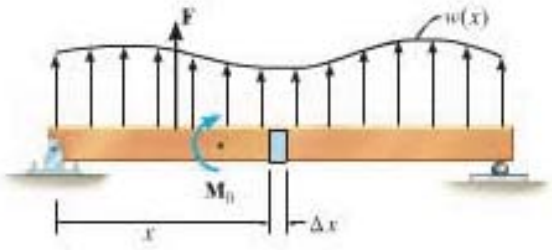
Failure of this table occurred at the brace support on its right side. If drawn, the bending moment diagram for the table loading would indicate this to be the point of maximum internal moment.

6.2 Graphical Method for Constructing Shear and Moment Diagrams

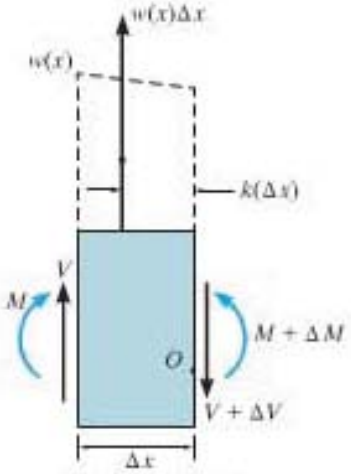
In cases where a beam is subjected to *several* different loadings, determining V and M as functions of x and then plotting these equations can become quite tedious. In this section a simpler method for constructing the shear and moment diagrams is discussed—a method based on two differential relations, one that exists between distributed load and shear, and the other between shear and moment.

Regions of Distributed Load. For purposes of generality, consider the beam shown in Fig. 6–8*a*, which is subjected to an arbitrary loading. A free-body diagram for a small segment Δx of the beam is shown in Fig. 6–8*b*. Since this segment has been chosen at a position x where there is no concentrated force or couple moment, the results to be obtained will *not* apply at these points of concentrated loading.

Notice that all the loadings shown on the segment act in their positive directions according to the established sign convention, Fig. 6–3. Also, both the internal resultant shear and moment, acting on the right face of the segment, must be changed by a small amount in order to keep the segment in equilibrium. The distributed load has been replaced by a resultant force $w(x)\,\Delta x$ that acts at a fractional distance $k(\Delta x)$ from the right side, where $0 < k < 1$ [for example, if $w(x)$ is *uniform*, $k = \frac{1}{2}$]. Applying the equations of equilibrium to the segment, we have



(a)



Free-body diagram of segment Δx

(b)

Fig. 6–8

$$+\uparrow \Sigma F_y = 0; \qquad V + w(x)\,\Delta x - (V + \Delta V) = 0$$
$$\Delta V = w(x)\,\Delta x$$

$$\circlearrowleft +\Sigma M_O = 0; \quad -V\,\Delta x - M - w(x)\,\Delta x[k(\Delta x)] + (M + \Delta M) = 0$$
$$\Delta M = V\,\Delta x + w(x)\,k(\Delta x)^2$$

Dividing by Δx and taking the limit as $\Delta x \to 0$, the above two equations become

$$\frac{dV}{dx} = w(x) \tag{6–1}$$

slope of shear diagram at each point = distributed load intensity at each point

$$\frac{dM}{dx} = V \tag{6–2}$$

slope of moment diagram at each point = shear at each point

These two equations provide a convenient means for quickly obtaining the shear and moment diagrams for a beam. Equation 6–1 states that at a point the *slope* of the shear diagram equals the intensity of the distributed loading. For example, consider the beam in Fig. 6–9*a*. The distributed loading is negative and increases from zero to w_B. Therefore, the shear diagram will be a curve that has a *negative slope*, increasing from zero to $-w_B$. Specific slopes $w_A = 0$, $-w_C$, $-w_D$, and $-w_B$ are shown in Fig. 6–9*b*.

In a similar manner, Eq. 6–2 states that at a point the *slope* of the moment diagram is equal to the shear. Notice that the shear diagram in Fig. 6–9*b* starts at $+V_A$, decreases to zero, and then becomes negative and decreases to $-V_B$. The moment diagram will then have an initial slope of $+V_A$ which decreases to zero, then the slope becomes negative and decreases to $-V_B$. Specific slopes V_A, V_C, V_D, 0, and $-V_B$ are shown in Fig. 6–9*c*.

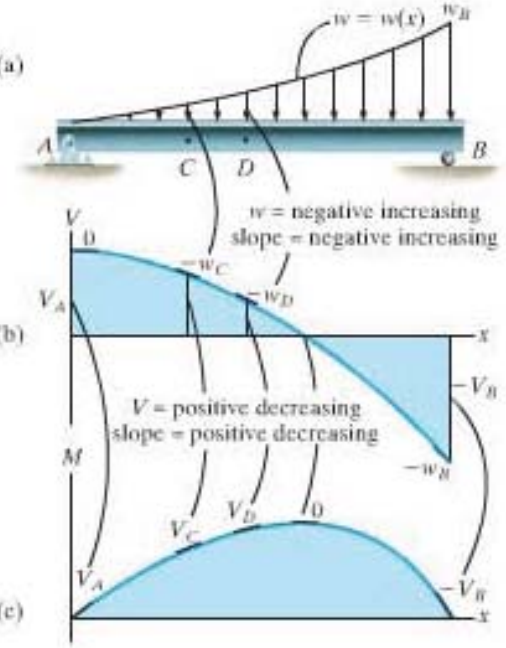


Fig. 6–9

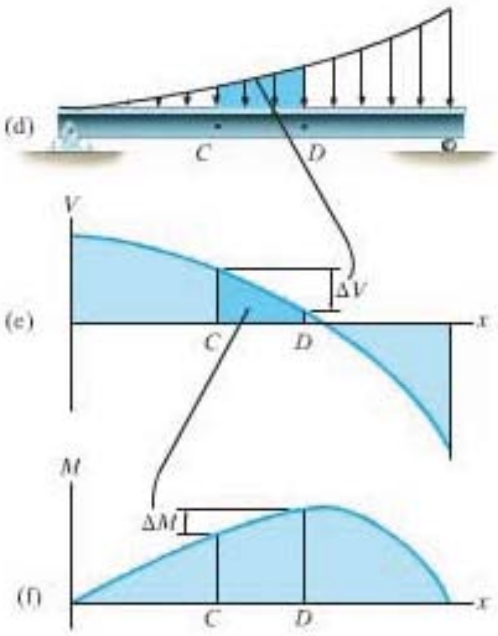


Fig. 6–9 (cont.)

Equations 6–1 and 6–2 may also be rewritten in the form $dV = w(x)\,dx$ and $dM = V\,dx$. Noting that $w(x)\,dx$ and $V\,dx$ represent differential areas under the distributed loading and shear diagram, respectively, we can integrate these areas between any two points C and D on the beam, Fig. 6–9*d*, and write

$$\Delta V = \int w(x)\,dx \qquad \text{(6–3)}$$

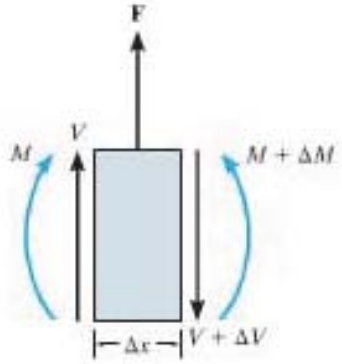
change in shear = area under distributed loading

$$\Delta M = \int V(x)\,dx \qquad \text{(6–4)}$$

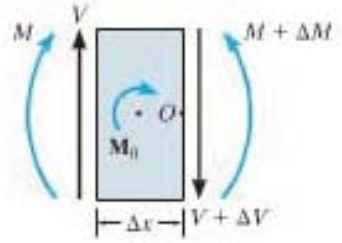
change in moment = area under shear diagram

Equation 6–3 states that the *change in shear* between C and D is equal to the *area* under the distributed-loading curve between these two points, Fig. 6–9*d*. In this case the change is negative since the distributed load acts downward. Similarly, from Eq. 6–4, the change in moment between C and D, Fig. 6–9*f*, is equal to the area under the shear diagram within the region from C to D. Here the change is positive.

Since the above equations do not apply at points where a concentrated force or couple moment acts, we will now consider each of these cases.



(a)



(b)

Fig. 6–10

Regions of Concentrated Force and Moment. A free-body diagram of a small segment of the beam in Fig. 6–10*a* taken from under the force is shown in Fig. 6–10*a*. Here it can be seen that force equilibrium requires

$$+\uparrow \Sigma F_y = 0; \qquad V + F - (V + \Delta V) = 0$$

$$\Delta V = F \qquad \text{(6–5)}$$

Thus, when $\mathbf{F}$ acts *upward* on the beam, ΔV is *positive* so the shear will "jump" *upward*. Likewise, if $\mathbf{F}$ acts *downward*, the jump (ΔV) will be *downward*.

When the beam segment includes the couple moment M_0, Fig. 6–10*b*, then moment equilibrium requires the change in moment to be

$$\downarrow + \Sigma M_O = 0; \qquad M + \Delta M - M_0 - V\,\Delta x - M = 0$$

Letting $\Delta x \to 0$, we get

$$\Delta M = M_0 \qquad \text{(6–6)}$$

In this case, if $\mathbf{M}_0$ is applied *clockwise*, ΔM is *positive* so the moment diagram will "jump" *upward*. Likewise, when $\mathbf{M}_0$ acts *counterclockwise*, the jump (ΔM) will be *downward*.

Procedure for Analysis

The following procedure provides a method for constructing the shear and moment diagrams for a beam based on the relations among distributed load, shear, and moment.

Support Reactions.

- Determine the support reactions and resolve the forces acting on the beam into components that are perpendicular and parallel to the beam's axis.

Shear Diagram.

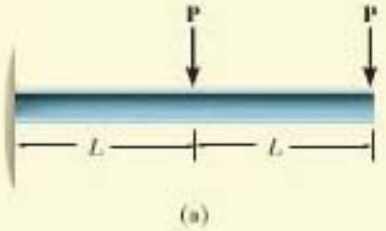
- Establish the V and x axes and plot the known values of the shear at the two *ends* of the beam.
- Notice how the values of the distributed load vary along the beam, and realize that each of these values indicates the way the shear diagram will slope ($dV/dx = w$). Here w is positive when it acts upward.
- If a numerical value of the shear is to be determined at a point, one can find this value either by using the method of sections and the equation of force equilibrium, or by using $\Delta V = \int w(x)\,dx$, which states that the *change in the shear* between any two points is equal to the *area under the load diagram* between the two points.

Moment Diagram.

- Establish the M and x axes and plot the known values of the moment at the *ends* of the beam.
- Notice how the values of the shear diagram vary along the beam, and realize that each of these values indicates the way the moment diagram will slope ($dM/dx = V$).
- At the point where the shear is zero, $dM/dx = 0$, and therefore this would be a point of maximum or minimum moment.
- If a numerical value of the moment is to be determined at the point, one can find this value either by using the method of sections and the equation of moment equilibrium, or by using $\Delta M = \int V(x)\,dx$, which states that the *change in moment* between any two points is equal to the *area under the shear diagram* between the two points.
- Since $w(x)$ must be *integrated* to obtain ΔV, and $V(x)$ is integrated to obtain $M(x)$, then if $w(x)$ is a curve of degree n, $V(x)$ will be a curve of degree $n + 1$ and $M(x)$ will be a curve of degree $n + 2$. For example, if $w(x)$ is uniform, $V(x)$ will be linear and $M(x)$ will be parabolic.

EXAMPLE 6.5

Draw the shear and moment diagrams for the beam shown in Fig. 6–11*a*.



(a)

SOLUTION

Support Reactions. The reaction at the fixed support is shown on the free-body diagram, Fig. 6–11*b*.

Shear Diagram. The shear at each end of the beam is plotted first, Fig. 6–11*c*. Since there is no distributed loading on the beam, the slope of the shear diagram is zero as indicated. Note how the force P at the center of the beam causes the shear diagram to jump downward an amount P, since this force acts downward.

Moment Diagram. The moments at the ends of the beam are plotted, Fig. 6–11*d*. Here the moment diagram consists of two sloping lines, one with a slope of $+2P$ and the other with a slope of $+P$.

The value of the moment in the center of the beam can be determined by the method of sections, or from the area under the shear diagram. If we choose the left half of the shear diagram,

$$M|_{x=L} = M|_{x=0} + \Delta M$$

$$M|_{x=L} = -3PL + (2P)(L) = -PL$$

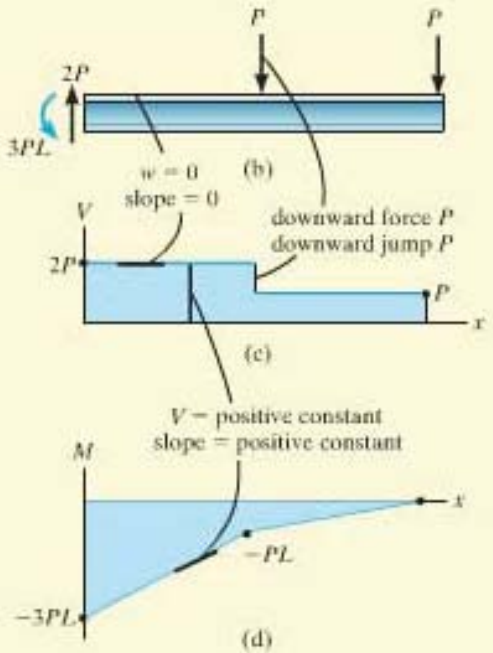
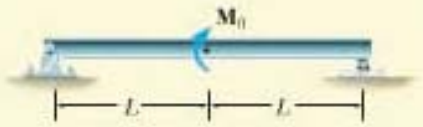


Fig. 6–11

EXAMPLE 6.6

Draw the shear and moment diagrams for the beam shown in Fig. 6–12*a*.



(a)

SOLUTION

Support Reactions. The reactions are shown on the free-body diagram in Fig. 6–12*b*.

Shear Diagram. The shear at each end is plotted first, Fig. 6–12*c*. Since there is no distributed load on the beam, the shear diagram has zero slope and is therefore a horizontal line.

Moment Diagram. The moment is zero at each end, Fig. 6–12*d*. The moment diagram has a constant negative slope of $-M_0/2L$ since this is the shear in the beam at each point. Note that the couple moment M_0 causes a jump in the moment diagram at the beam's center, but it does not affect the shear diagram at this point.

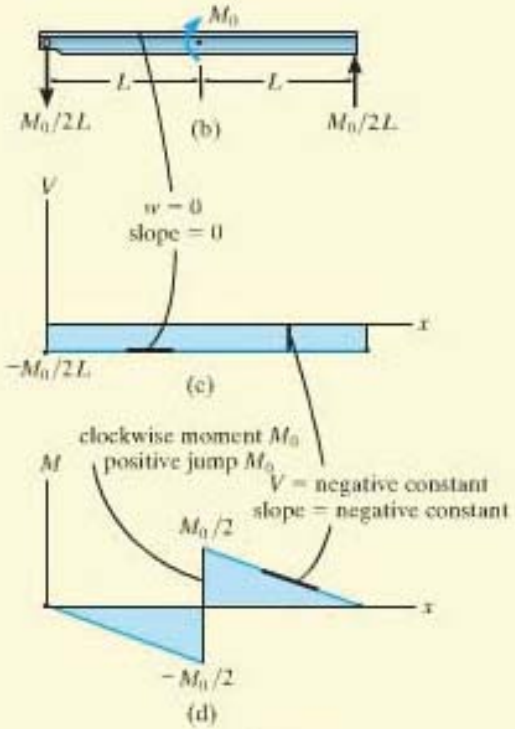


Fig. 6–12

EXAMPLE 6.7

Draw the shear and moment diagrams for each of the beams shown in Figs. 6–13*a* and 6–14*a*.

SOLUTION

Support Reactions. The reactions at the fixed support are shown on each free-body diagram, Figs. 6–13*b* and Fig. 6–14*b*.

Shear Diagram. The shear at each end point is plotted first, Figs. 6–13*c* and 6–14*c*. The distributed loading on each beam indicates the slope of the shear diagram and thus produces the shapes shown.

Moment Diagram. The moment at each end point is plotted first, Figs. 6–13*d* and 6–14*d*. Various values of the shear at each point on the beam indicate the slope of the moment diagram at the point. Notice how this variation produces the curves shown.

NOTE: Observe how the degree of the curves from w to V to M increases exponentially due to the integration of $dV = w\,dx$ and $dM = V dx$. For example, in Fig. 6–14, the linear distributed load produces a parabolic shear diagram and cubic moment diagram.

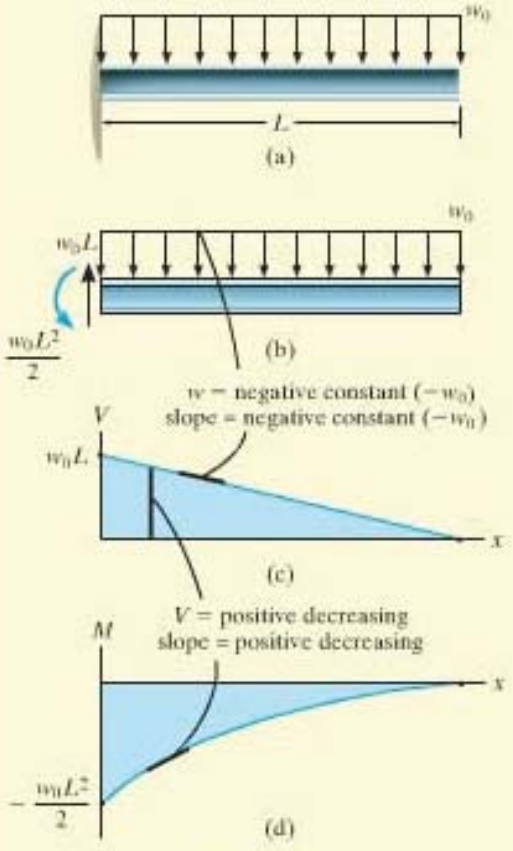


Fig. 6–13

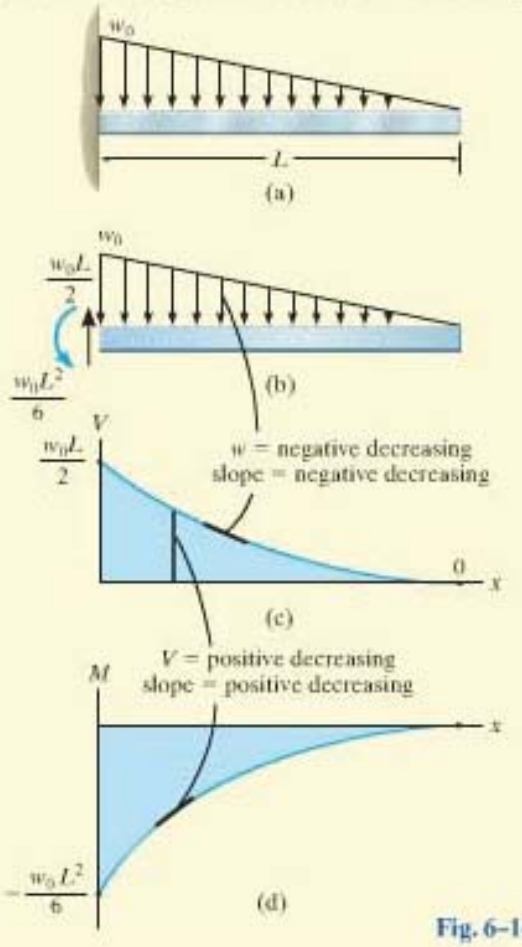
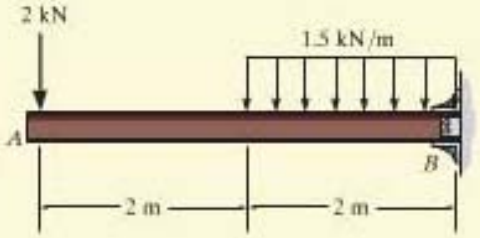


Fig. 6–14

EXAMPLE 6.8

Draw the shear and moment diagrams for the cantilever beam in Fig. 6–15*a*.



(a)

SOLUTION

Support Reactions. The support reactions at the fixed support B are shown in Fig. 6–15*b*.

Shear Diagram. The shear at end A is -2 kN. This value is plotted at $x = 0$, Fig. 6–15*c*. Notice how the shear diagram is constructed by following the slopes defined by the loading w. The shear at $x = 4$ m is -5 kN, the reaction on the beam. This value can be verified by finding the area under the distributed loading, Eq. 6–3.

$$V|_{x=4\,\mathrm{m}} = V|_{x=2\,\mathrm{m}} + \Delta V = -2\,\mathrm{kN} - (1.5\,\mathrm{kN/m})(2\,\mathrm{m}) = -5\,\mathrm{kN}$$

Moment Diagram. The moment of zero at $x = 0$ is plotted in Fig. 6–15*d*. Notice how the moment diagram is constructed based on knowing its slope, which is equal to the shear at each point. The change of moment from $x = 0$ to $x = 2$ m is determined from the area under the shear diagram. Hence, the moment at $x = 2$ m is

$$M|_{x=2\,\mathrm{m}} = M|_{x=0} + \Delta M = 0 + [-2\,\mathrm{kN}(2\,\mathrm{m})] = -4\,\mathrm{kN \cdot m}$$

This same value can be determined from the method of sections, Fig. 6–15*e*.

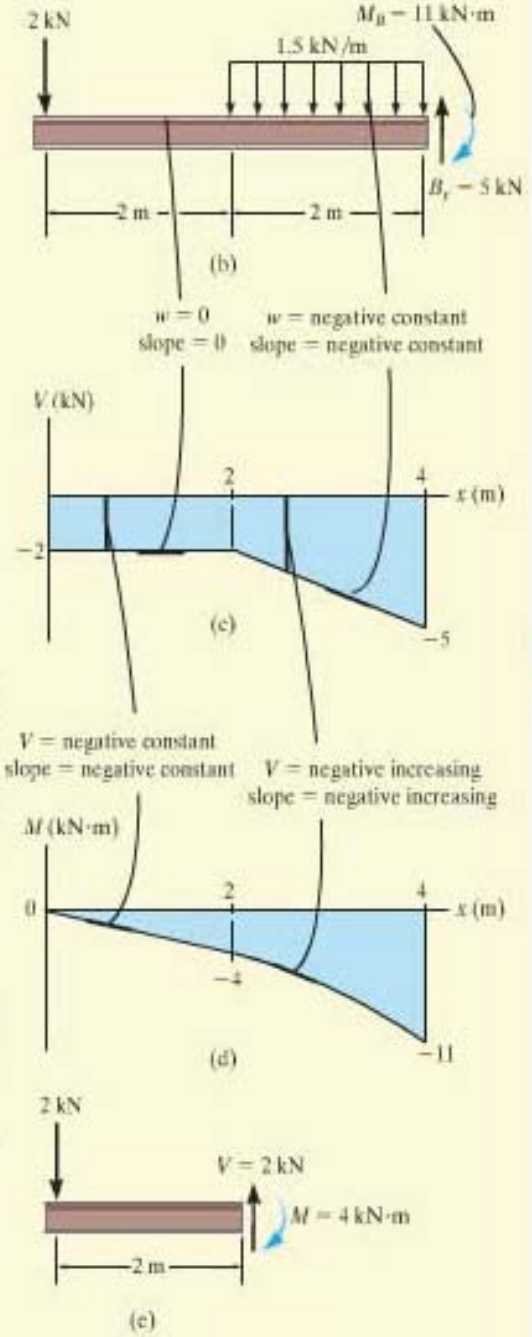
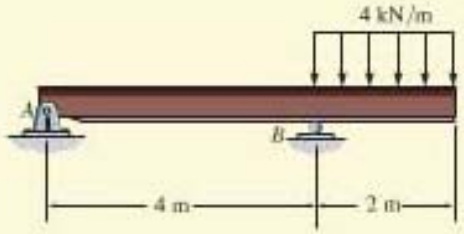


Fig. 6–15

EXAMPLE 6.9

Draw the shear and moment diagrams for the overhang beam in Fig. 6–16*a*.



(a)

SOLUTION

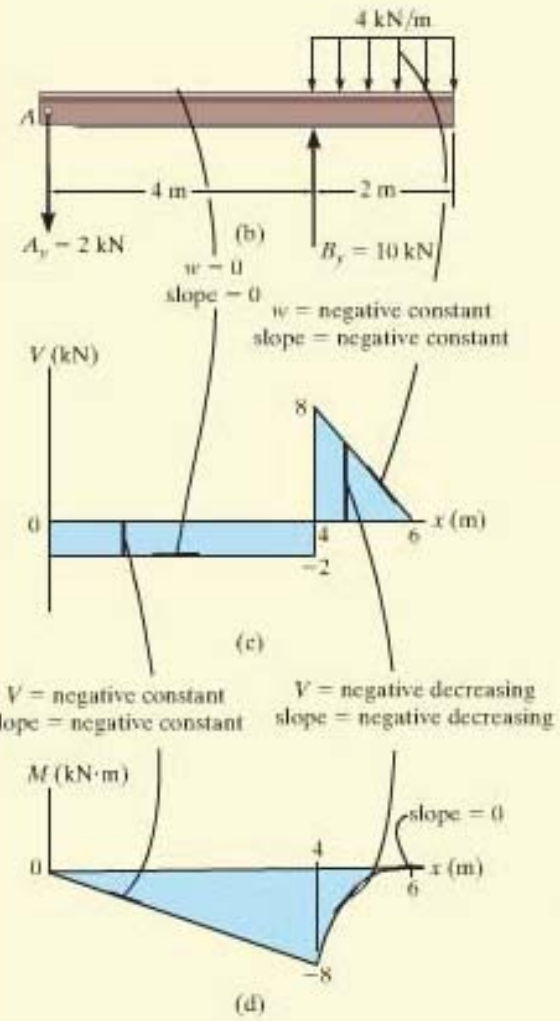
Support Reactions. The support reactions are shown in Fig. 6–16*b*.

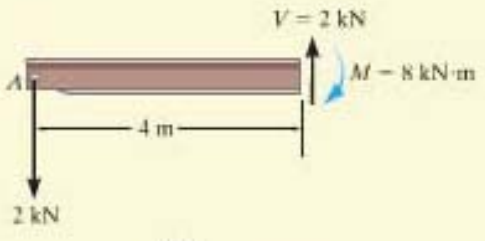
Shear Diagram. The shear of −2 kN at end A of the beam is plotted at $x = 0$, Fig. 6–16*c*. The slopes are determined from the loading and from this the shear diagram is constructed, as indicated in the figure. In particular, notice the positive jump of 10 kN at $x = 4$ m due to the force B_y, as indicated in the figure.

Moment Diagram. The moment of zero at $x = 0$ is plotted, Fig. 6–16*d*. Then following the behavior of the slope found from the shear diagram, the moment diagram is constructed. The moment at $x = 4$ m is found from the area under the shear diagram.

$$M|_{x=4\text{ m}} = M|_{x=0} + \Delta M = 0 + [-2\text{ kN}(4\text{ m})] = -8\text{ kN}\cdot\text{m}$$

We can also obtain this value by using the method of sections, as shown in Fig. 6–16*e*.



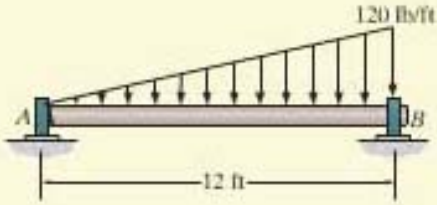


(e)

Fig. 6–16

EXAMPLE 6.10

The shaft in Fig. 6–17*a* is supported by a thrust bearing at A and a journal bearing at B. Draw the shear and moment diagrams.



(a)

SOLUTION

Support Reactions. The support reactions are shown in Fig. 6–17*b*.

Shear Diagram. As shown in Fig. 6–17*c*, the shear at $x = 0$ is $+240$. Following the slope defined by the loading, the shear diagram is constructed, where at B its value is -480 lb. Since the shear changes sign, the point where $V = 0$ must be located. To do this we will use the method of sections. The free-body diagram of the left segment of the shaft, sectioned at an arbitrary position x, is shown in Fig. 6–17*e*. Notice that the intensity of the distributed load at x is $w = 10x$, which has been found by proportional triangles, i.e., $120/12 = w/x$.

Thus, for $V = 0$,

$$+\uparrow \Sigma F_y = 0; \qquad 240 \text{ lb} - \tfrac{1}{2}(10x)x = 0$$

$$x = 6.93 \text{ ft}$$

Moment Diagram. The moment diagram starts at 0 since there is no moment at A; then it is constructed based on the slope as determined from the shear diagram. The maximum moment occurs at $x = 6.93$ ft, where the shear is equal to zero, since $dM/dx = V = 0$, Fig. 6–17*d*,

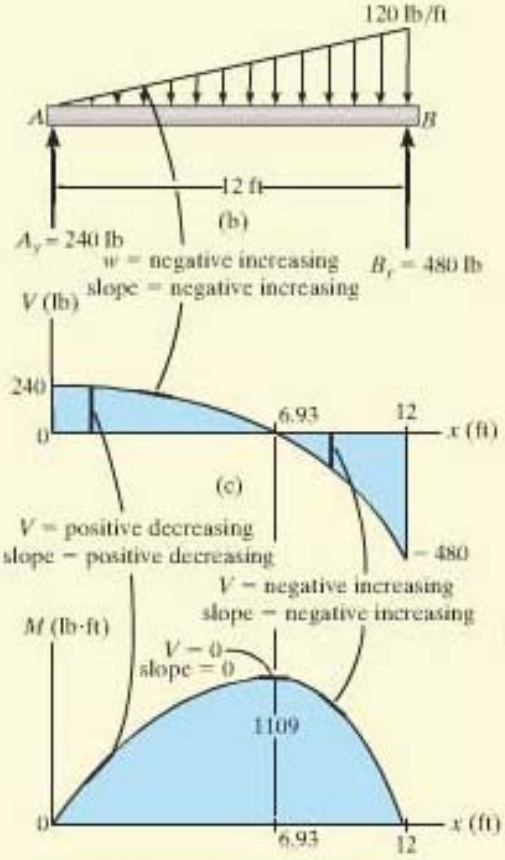
$$\circlearrowleft + \Sigma M = 0;$$

$$M_{\max} + \tfrac{1}{2}[(10)(6.93)]\,6.93\left(\tfrac{1}{3}(6.93)\right) - 240(6.93) = 0$$

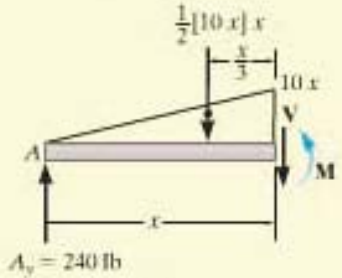
$$M_{\max} = 1109 \text{ lb} \cdot \text{ft}$$

Finally, notice how integration, first of the loading w which is linear, produces a shear diagram which is parabolic, and then a moment diagram which is cubic.

NOTE: Having studied these examples, test yourself by covering over the shear and moment diagrams in Examples 6–1 through 6–4 and see if you can construct them using the concepts discussed here.



(d)

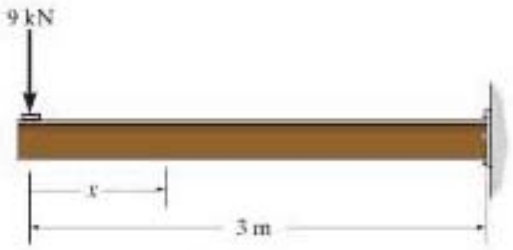


(e)

Fig. 6–17

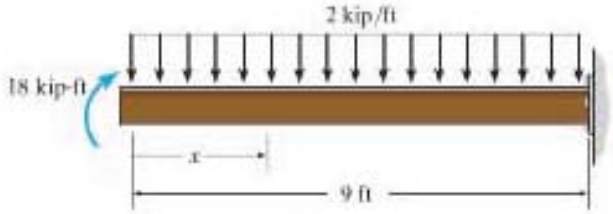
FUNDAMENTAL PROBLEMS

F6–1. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



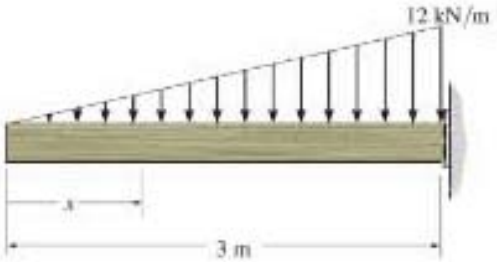
F6–1

F6–2. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



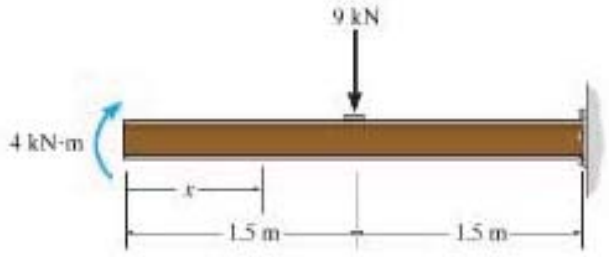
F6–2

F6–3. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



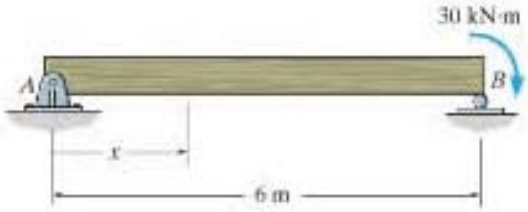
F6–3

F6–4. Express the shear and moment functions in terms of x, where $0 < x < 1.5$ m and 1.5 m $< x < 3$ m, and then draw the shear and moment diagrams for the cantilever beam.



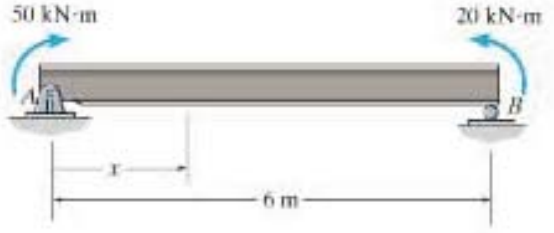
F6–4

F6–5. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.



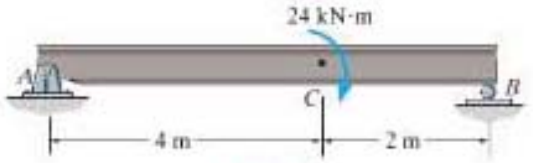
F6–5

F6–6. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.



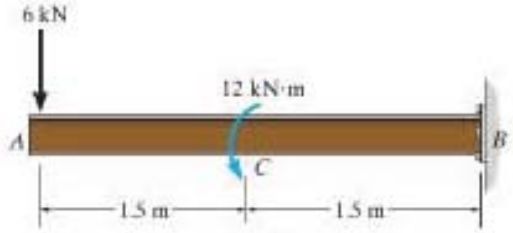
F6–6

F6–7. Draw the shear and moment diagrams for the simply supported beam.



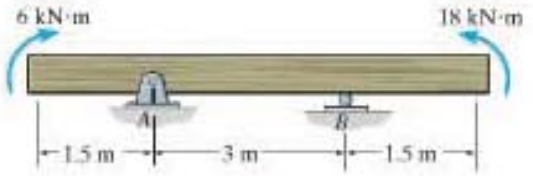
F6–7

F6–8. Draw the shear and moment diagrams for the cantilever beam.



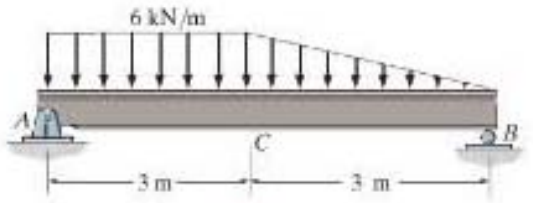
F6–8

F6–9. Draw the shear and moment diagrams for the double overhang beam.



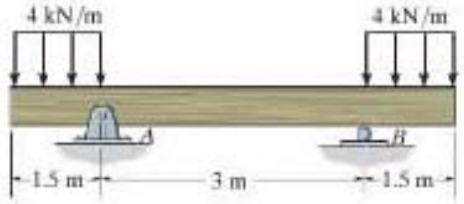
F6–9

F6–10. Draw the shear and moment diagrams for the simply supported beam.



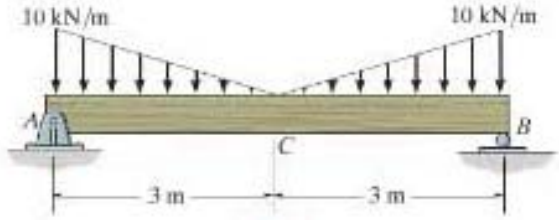
F6–10

F6–11. Draw the shear and moment diagrams for the double-overhang beam.



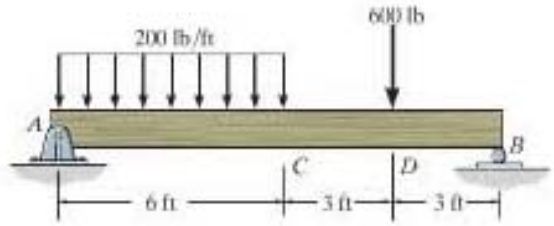
F6–11

F6–12. Draw the shear and moment diagrams for the simply supported beam.



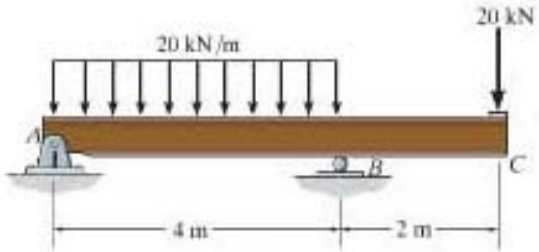
F6–12

F6–13. Draw the shear and moment diagrams for the simply supported beam.



F6–13

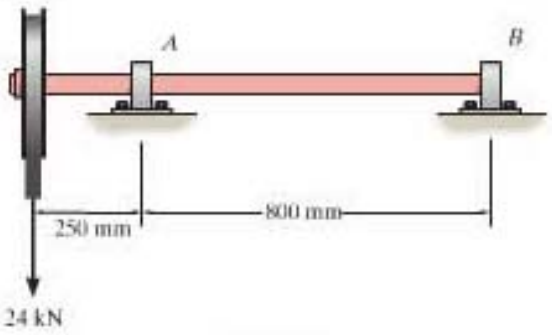
F6–14. Draw the shear and moment diagrams for the overhang beam.



F6–14

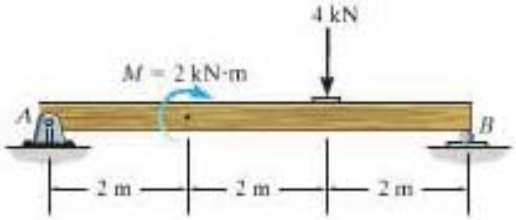
PROBLEMS

6–1. Draw the shear and moment diagrams for the shaft. The bearings at A and B exert only vertical reactions on the shaft.



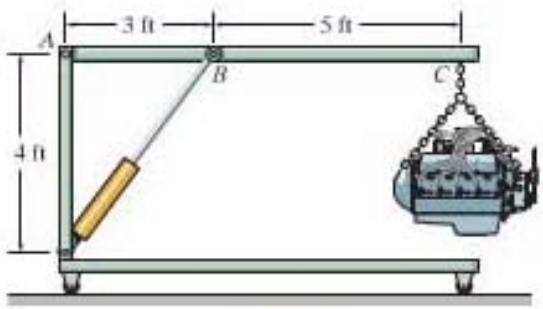
Prob. 6–1

6–2. Draw the shear and moment diagrams for the simply supported beam.



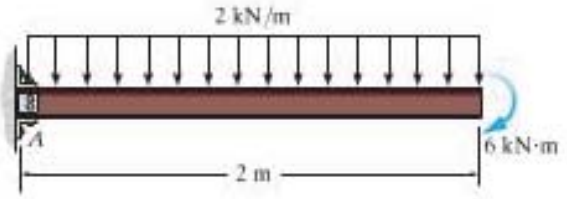
Prob. 6–2

6–3. The engine crane is used to support the engine, which has a weight of 1200 lb. Draw the shear and moment diagrams of the boom ABC when it is in the horizontal position shown.



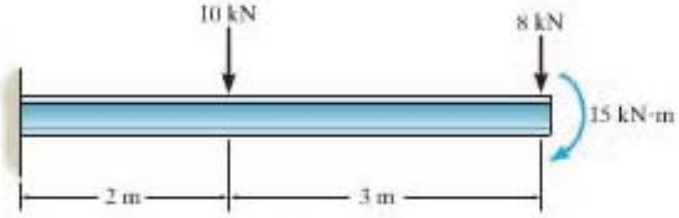
Prob. 6–3

***6–4.** Draw the shear and moment diagrams for the cantilever beam.



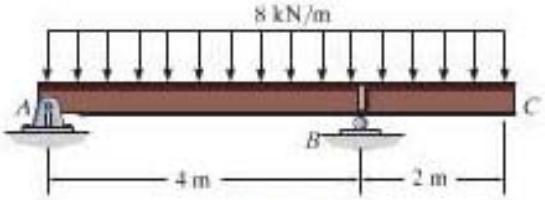
Prob. 6–4

6–5. Draw the shear and moment diagrams for the beam.



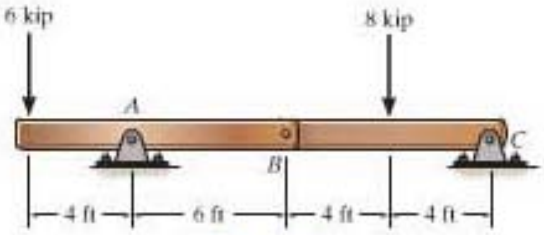
Prob. 6–5

6–6. Draw the shear and moment diagrams for the overhang beam.



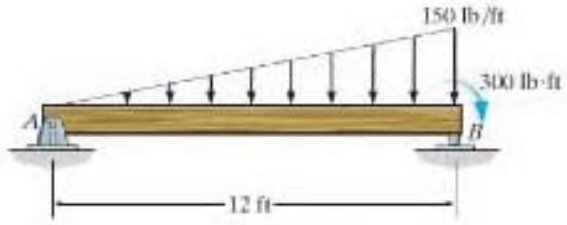
Prob. 6–6

6–7. Draw the shear and moment diagrams for the compound beam which is pin connected at B.



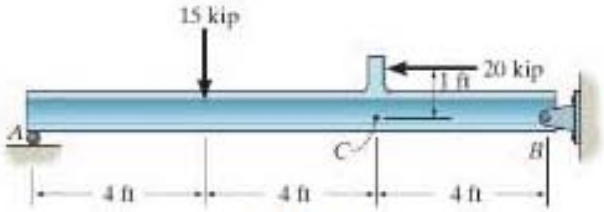
Prob. 6–7

***6–8.** Draw the shear and moment diagrams for the simply supported beam.



Prob. 6–8

6–9. Draw the shear and moment diagrams for the beam. *Hint:* The 20-kip load must be replaced by equivalent loadings at point C on the axis of the beam.



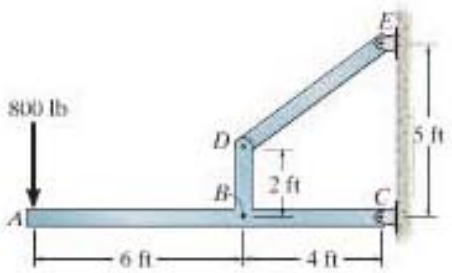
Prob. 6–9

6–10. Members ABC and BD of the counter chair are rigidly connected at B and the smooth collar at D is allowed to move freely along the vertical slot. Draw the shear and moment diagrams for member ABC.



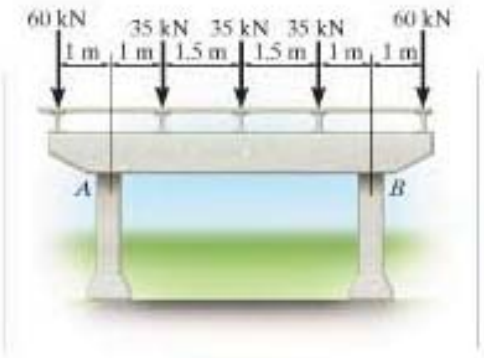
Prob. 6–10

6–11. The overhanging beam has been fabricated with a projected arm BD on it. Draw the shear and moment diagrams for the beam ABC if it supports a load of 800 lb. *Hint:* The loading in the supporting strut DE must be replaced by equivalent loads at point B on the axis of the beam.



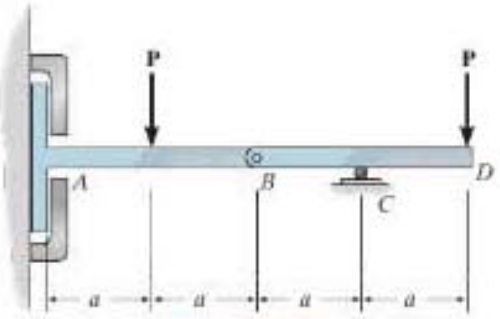
Prob. 6–11

***6–12.** A reinforced concrete pier is used to support the stringers for a bridge deck. Draw the shear and moment diagrams for the pier when it is subjected to the stringer loads shown. Assume the columns at A and B exert only vertical reactions on the pier.



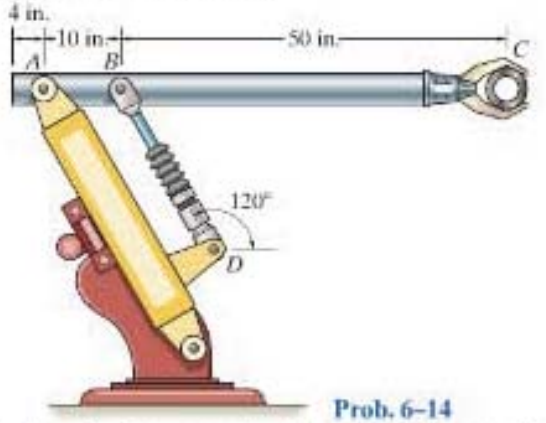
Prob. 6–12

6–13. Draw the shear and moment diagrams for the compound beam. It is supported by a smooth plate at A which slides within the groove and so it cannot support a vertical force, although it can support a moment and axial load.



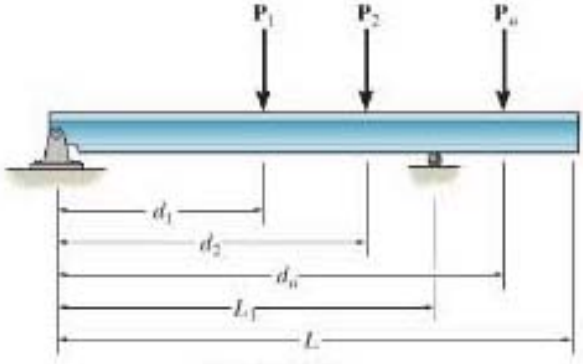
Prob. 6–13

6–14. The industrial robot is held in the stationary position shown. Draw the shear and moment diagrams of the arm ABC if it is pin connected at A and connected to a hydraulic cylinder (two-force member) BD. Assume the arm and grip have a uniform weight of 1.5 lb/in. and support the load of 40 lb at C.



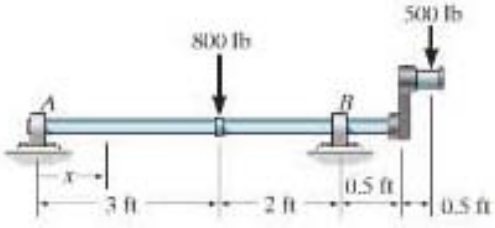
Prob. 6–14

6–15. Consider the general problem of the beam subjected to n concentrated loads. Write a computer program that can be used to determine the internal shear and moment at any specified location x along the beam, and plot the shear and moment diagrams for the beam. Show an application of the program using the values $P_1 = 500$ lb, $d_1 = 5$ ft, $P_2 = 800$ lb, $d_2 = 15$ ft, $L_1 = 10$ ft, $L = 15$ ft.



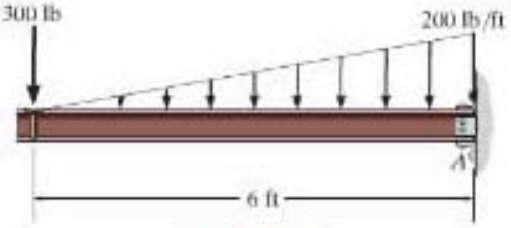
Prob. 6–15

***6–16.** Draw the shear and moment diagrams for the shaft and determine the shear and moment throughout the shaft as a function of x. The bearings at A and B exert only vertical reactions on the shaft.



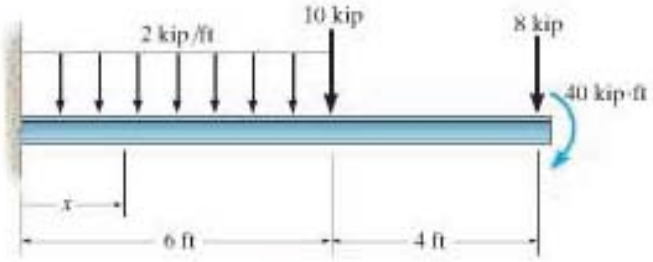
Prob. 6–16

•6–17. Draw the shear and moment diagrams for the cantilevered beam.



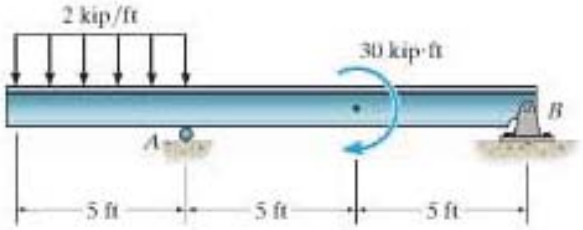
Prob. 6–17

6–18. Draw the shear and moment diagrams for the beam, and determine the shear and moment throughout the beam as functions of x.



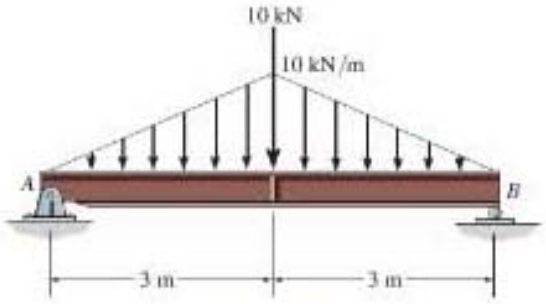
Prob. 6–18

6–19. Draw the shear and moment diagrams for the beam.



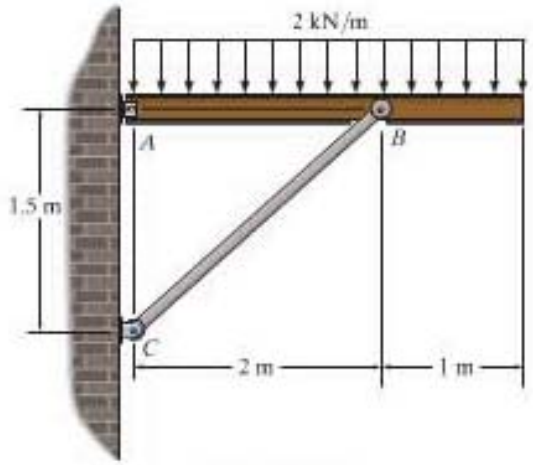
Prob. 6–19

***6–20.** Draw the shear and moment diagrams for the simply supported beam.



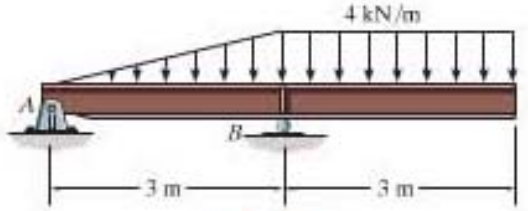
Prob. 6–20

•6–21. The beam is subjected to the uniform distributed load shown. Draw the shear and moment diagrams for the beam.



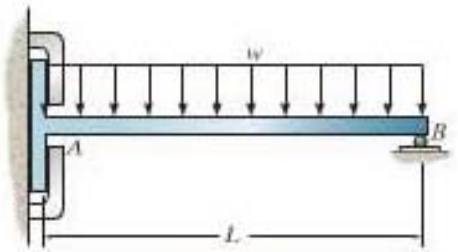
Prob. 6–21

6–22. Draw the shear and moment diagrams for the overhang beam.



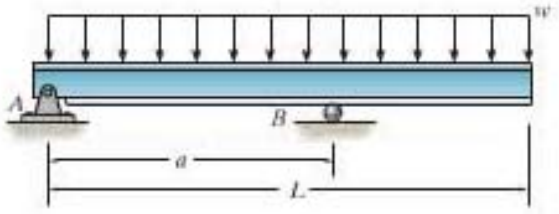
Prob. 6–22

6–23. Draw the shear and moment diagrams for the beam. It is supported by a smooth plate at A which slides within the groove and so it cannot support a vertical force, although it can support a moment and axial load.



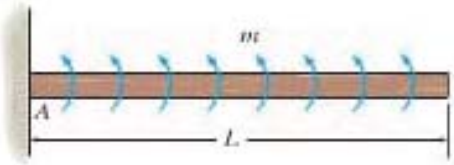
Prob. 6–23

***6–24.** Determine the placement distance a of the roller support so that the largest absolute value of the moment is a minimum. Draw the shear and moment diagrams for this condition.



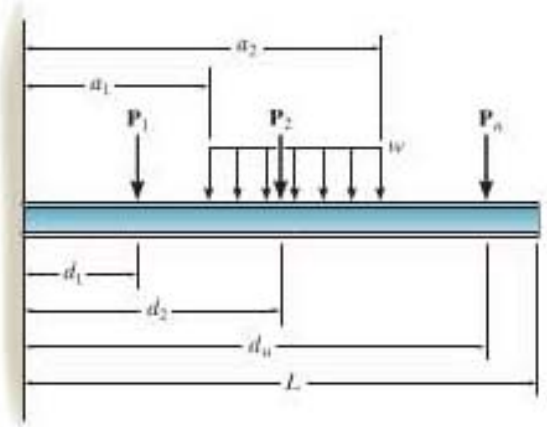
Prob. 6–24

6–25. The beam is subjected to the uniformly distributed moment m (moment/length). Draw the shear and moment diagrams for the beam.



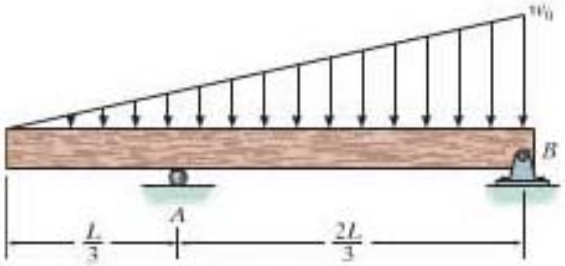
Prob. 6–25

6–26. Consider the general problem of a cantilevered beam subjected to n concentrated loads and a constant distributed loading w. Write a computer program that can be used to determine the internal shear and moment at any specified location x along the beam, and plot the shear and moment diagrams for the beam. Show an application of the program using the values $P_1 = 4$ kN, $d_1 = 2$ m, $w = 800$ N/m, $a_1 = 2$ m, $a_2 = 4$ m, $L = 4$ m.



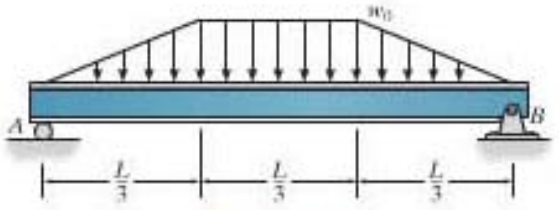
Prob. 6–26

6–27. Draw the shear and moment diagrams for the beam.



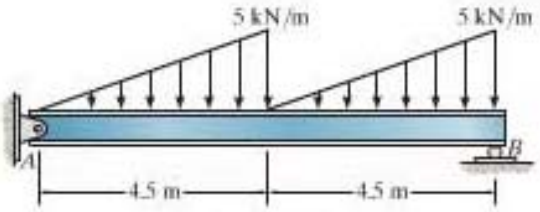
Prob. 6–27

***6–28.** Draw the shear and moment diagrams for the beam.



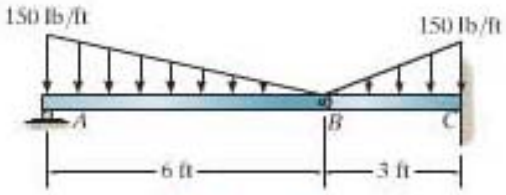
Prob. 6–28

•6–29. Draw the shear and moment diagrams for the beam.



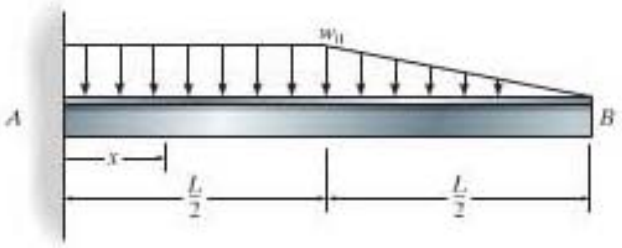
Prob. 6–29

6–30. Draw the shear and moment diagrams for the compound beam.



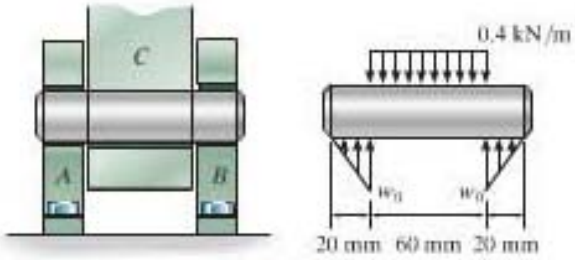
Prob. 6–30

6–31. Draw the shear and moment diagrams for the beam and determine the shear and moment in the beam as functions of x.



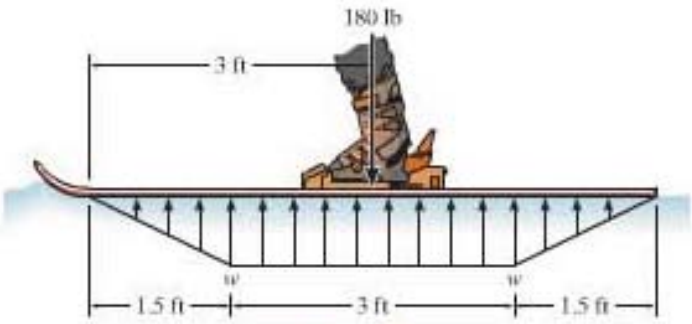
Prob. 6–31

***6–32.** The smooth pin is supported by two leaves A and B and subjected to a compressive load of 0.4 kN/m caused by bar C. Determine the intensity of the distributed load w_0 of the leaves on the pin and draw the shear and moment diagram for the pin.



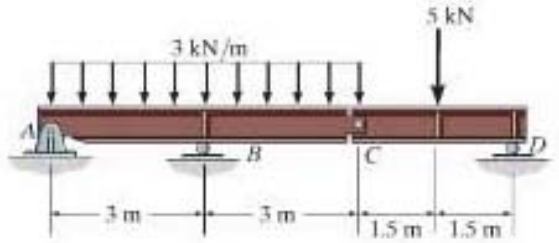
Prob. 6–32

•6–33. The ski supports the 180-lb weight of the man. If the snow loading on its bottom surface is trapezoidal as shown, determine the intensity w, and then draw the shear and moment diagrams for the ski.



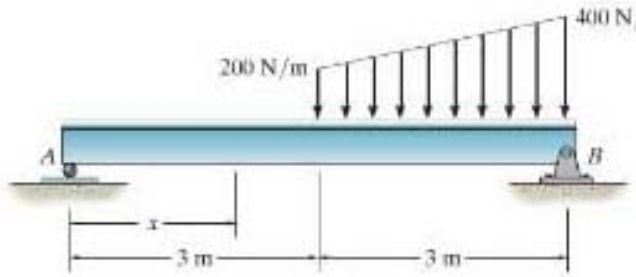
Prob. 6–33

6–34. Draw the shear and moment diagrams for the compound beam.



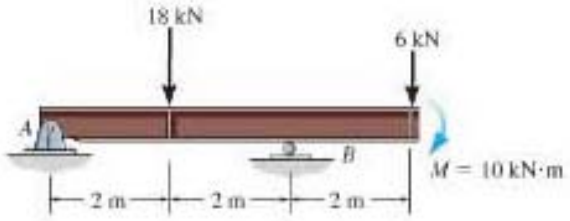
Prob. 6–34

6–35. Draw the shear and moment diagrams for the beam and determine the shear and moment as functions of x.



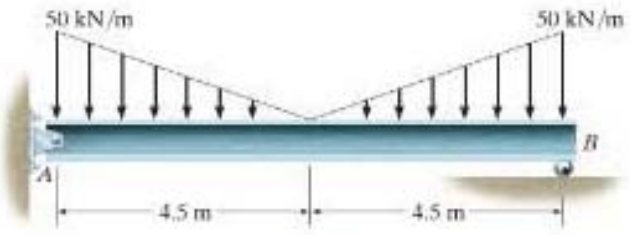
Prob. 6–35

***6–36.** Draw the shear and moment diagrams for the overhang beam.



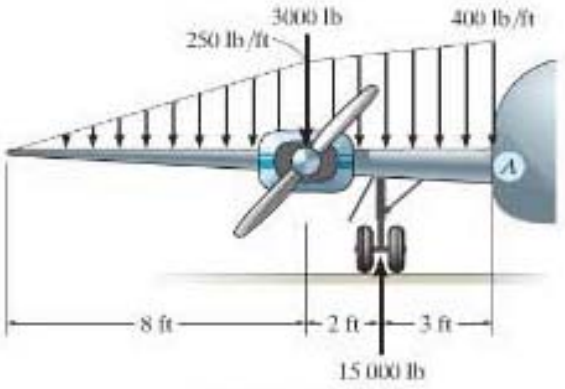
Prob. 6–36

6–37. Draw the shear and moment diagrams for the beam.



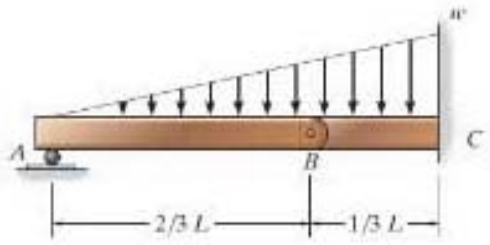
Prob. 6–37

6–38. The dead-weight loading along the centerline of the airplane wing is shown. If the wing is fixed to the fuselage at A, determine the reactions at A, and then draw the shear and moment diagram for the wing.



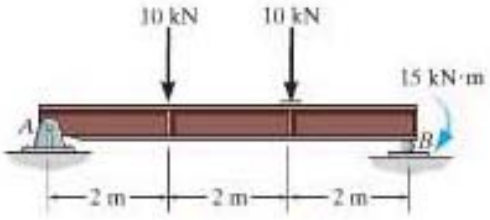
Prob. 6–38

6–39. The compound beam consists of two segments that are pinned together at B. Draw the shear and moment diagrams if it supports the distributed loading shown.



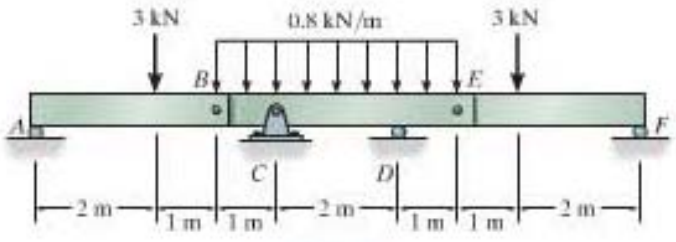
Prob. 6–39

***6–40.** Draw the shear and moment diagrams for the simply supported beam.



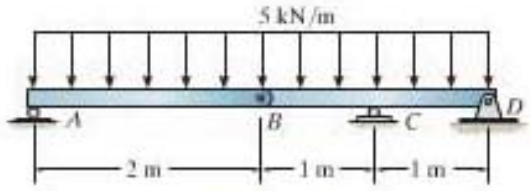
Prob. 6–40

6–41. Draw the shear and moment diagrams for the compound beam. The three segments are connected by pins at B and E.



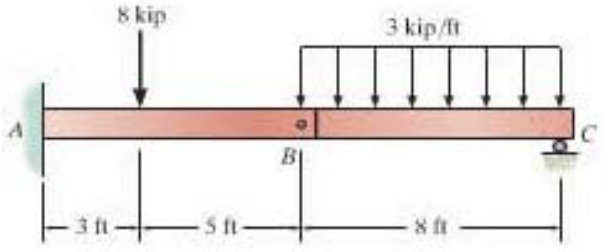
Prob. 6–41

6–42. Draw the shear and moment diagrams for the compound beam.



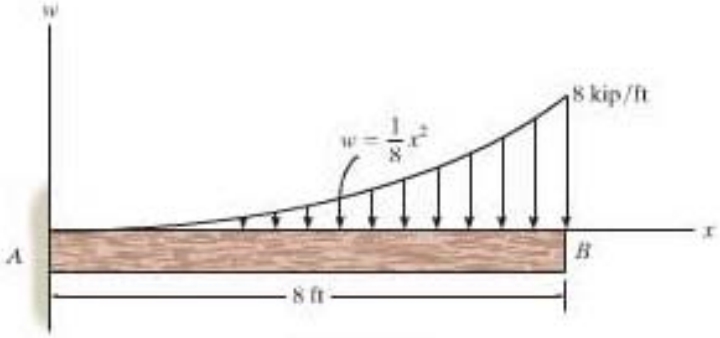
Prob. 6–42

6–43. Draw the shear and moment diagrams for the beam. The two segments are joined together at B.



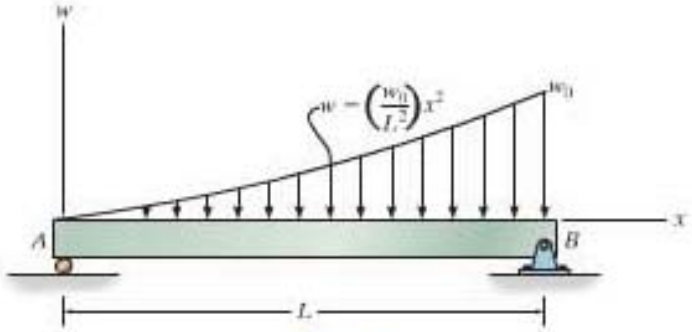
Prob. 6–43

***6–44.** Draw the shear and moment diagrams for the beam.



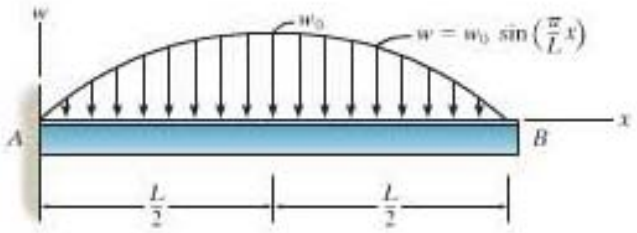
Prob. 6–44

•6–45. Draw the shear and moment diagrams for the beam.



Prob. 6–45

6–46. Draw the shear and moment diagrams for the beam.



Prob. 6–46

6.3 Bending Deformation of a Straight Member

In this section, we will discuss the deformations that occur when a straight prismatic beam, made of homogeneous material, is subjected to bending. The discussion will be limited to beams having a cross-sectional area that is symmetrical with respect to an axis, and the bending moment is applied about an axis perpendicular to this axis of symmetry as shown in Fig. 6–18. The behavior of members that have unsymmetrical cross sections, or are made from several different materials, is based on similar observations and will be discussed separately in later sections of this chapter.

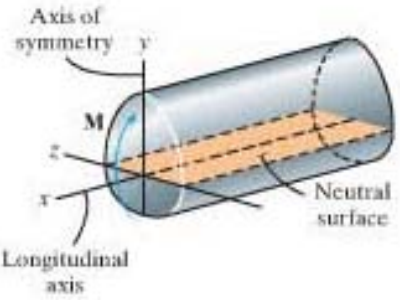


Fig. 6–18

By using a highly deformable material such as rubber, we can illustrate what happens when a straight prismatic member is subjected to a bending moment. Consider, for example, the undeformed bar in Fig. 6–19*a*, which has a square cross section and is marked with longitudinal and transverse grid lines. When a bending moment is applied, it tends to distort these lines into the pattern shown in Fig. 6–19*b*. Notice that the longitudinal lines become *curved* and the vertical transverse lines *remain straight* and yet undergo a *rotation*.

The bending moment causes the material within the *bottom* portion of the bar to *stretch* and the material within the *top* portion to *compress*. Consequently, between these two regions there must be a surface, called the *neutral surface*, in which longitudinal fibers of the material will not undergo a change in length, Fig. 6–18.

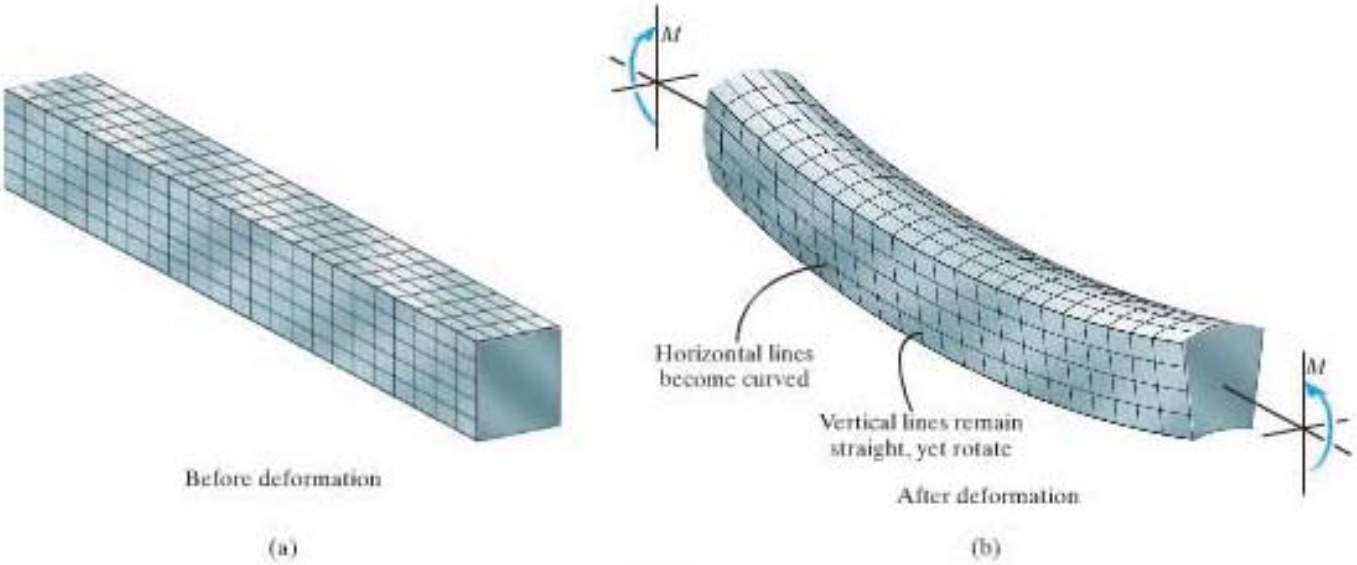
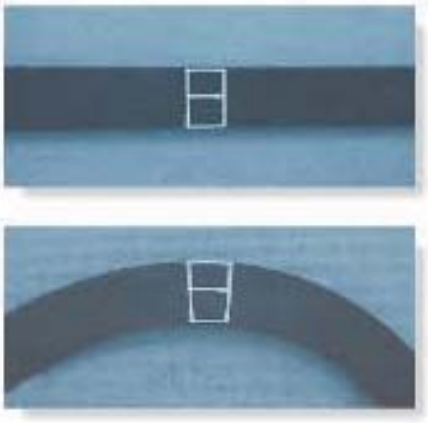


Fig. 6–19



Note the distortion of the lines due to bending of this rubber bar. The top line stretches, the bottom line compresses, and the center line remains the same length. Furthermore the vertical lines rotate and yet remain straight.

From these observations we will make the following three assumptions regarding the way the stress deforms the material. First, the *longitudinal axis* x, which lies within the neutral surface, Fig. 6–20*a*, does *not* experience any *change in length*. Rather the moment will tend to deform the beam so that this line *becomes a curve* that lies in the x–y plane of symmetry, Fig. 6–20*b*. Second, all ***cross sections*** of the beam ***remain plane*** and perpendicular to the longitudinal axis during the deformation. And third, *any deformation* of the *cross section* within its own plane, as noticed in Fig. 6–19*b*, will be *neglected*. In particular, the z axis, lying in the plane of the cross section and about which the cross section rotates, is called the *neutral axis*, Fig. 6–20*b*.

In order to show how this distortion will strain the material, we will isolate a small segment of the beam located a distance x along the beam's length and having an undeformed thickness Δx, Fig. 6–20*a*. This element, taken from the beam, is shown in profile view in the undeformed and

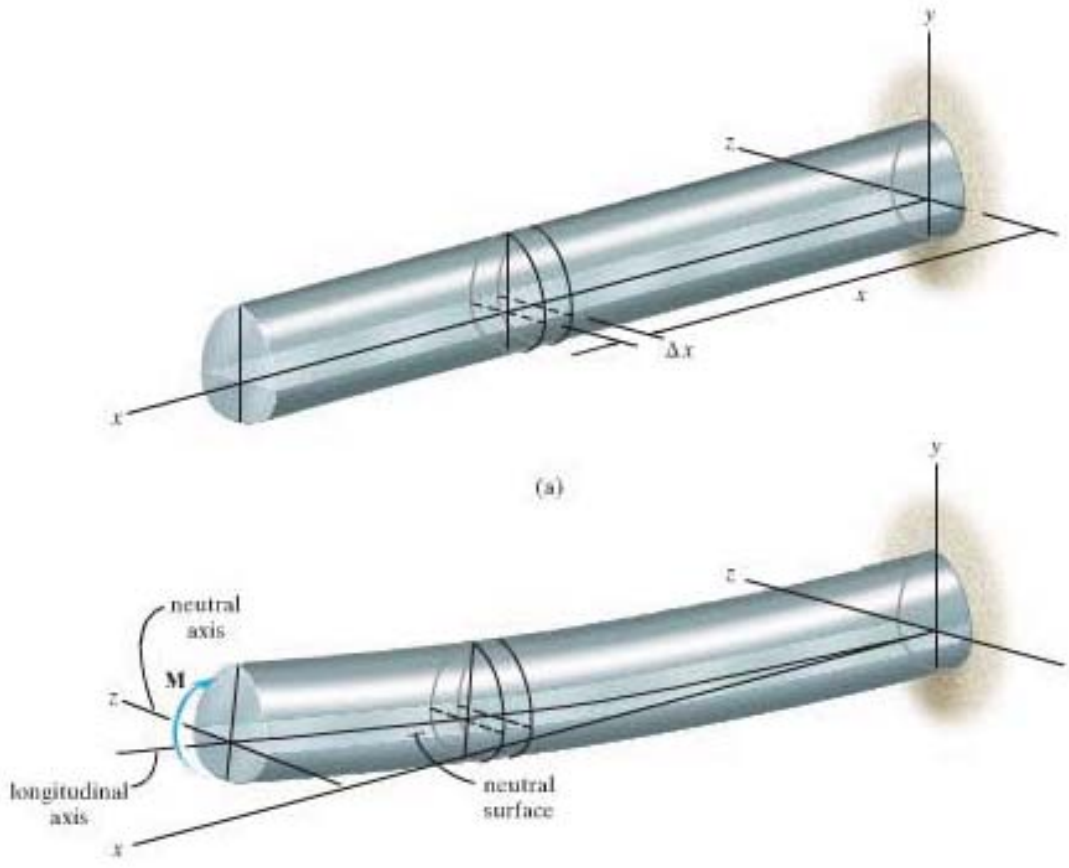


Fig. 6–20

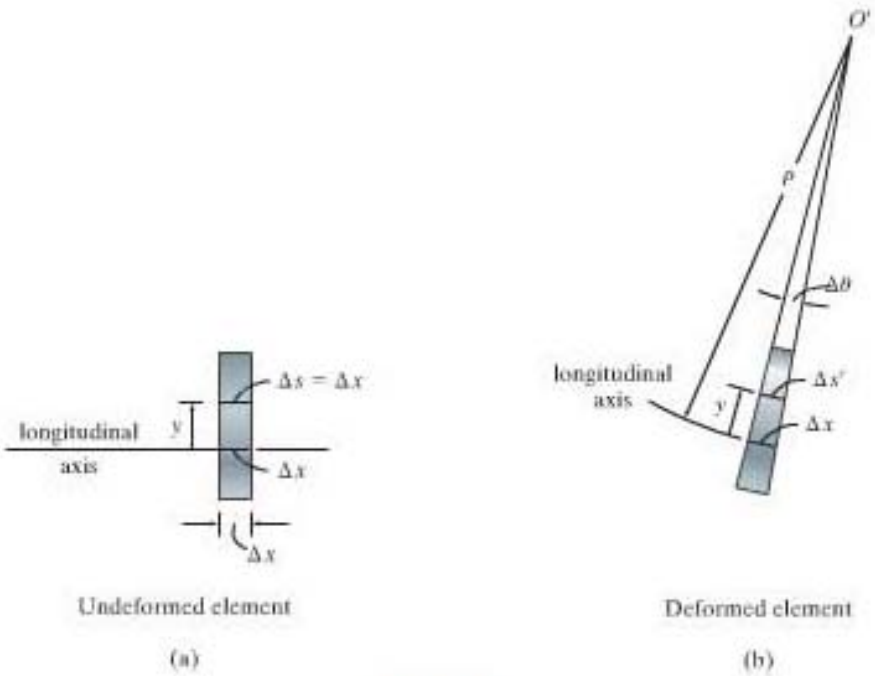


Fig. 6–21

deformed positions in Fig. 6–21. Notice that any line segment Δx, located on the neutral surface, does not change its length, whereas any line segment Δs, located at the arbitrary distance y above the neutral surface, will contract and become $\Delta s'$ after deformation. By definition, the normal strain along Δs is determined from Eq. 2–2, namely,

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

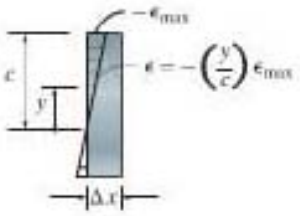
We will now represent this strain in terms of the location y of the segment and the radius of curvature ρ of the longitudinal axis of the element. Before deformation, $\Delta s = \Delta x$, Fig. 6–21*a*. After deformation Δx has a radius of curvature ρ, with center of curvature at point O', Fig. 6–21*b*. Since $\Delta\theta$ defines the angle between the sides of the element, $\Delta x = \Delta s = \rho\Delta\theta$. In the same manner, the deformed length of Δs becomes $\Delta s' = (\rho - y)\Delta\theta$. Substituting into the above equation, we get

$$\epsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

or

$$\epsilon = -\frac{y}{\rho} \tag{6–7}$$

This important result indicates that the longitudinal normal strain of any element within the beam depends on its location y on the cross



Normal strain distribution

Fig. 6–22

section and the radius of curvature of the beam's longitudinal axis at the point. In other words, for any specific cross section, the ***longitudinal normal strain will vary linearly*** with y from the neutral axis. A contraction $(-\epsilon)$ will occur in fibers located above the neutral axis $(+y)$, whereas elongation $(+\epsilon)$ will occur in fibers located below the axis $(-y)$. This variation in strain over the cross section is shown in Fig. 6–22. Here the maximum strain occurs at the outermost fiber, located a distance of $y = c$ from the neutral axis. Using Eq. 6–7, since $\epsilon_{max} = c/\rho$, then by division,

$$\frac{\epsilon}{\epsilon_{max}} = -\left(\frac{y/\rho}{c/\rho}\right)$$

So that

$$\epsilon = -\left(\frac{y}{c}\right)\epsilon_{max} \tag{6–8}$$

This normal strain depends only on the assumptions made with regards to the *deformation*. When a moment is applied to the beam, therefore, it will only cause a *normal stress* in the longitudinal or x direction. All the other components of normal and shear stress will be zero. It is this uniaxial state of stress that causes the material to have the longitudinal normal strain component ϵ_x, defined by Eq. 6–8. Furthermore, by Poisson's ratio, there must *also* be associated strain components $\epsilon_y = -\nu\epsilon_x$ and $\epsilon_z = -\nu\epsilon_x$, which deform the plane of the cross-sectional area, although here we have neglected these deformations. Such deformations will, however, cause the *cross-sectional dimensions* to become smaller below the neutral axis and larger above the neutral axis. For example, if the beam has a square cross section, it will actually deform as shown in Fig. 6–23.

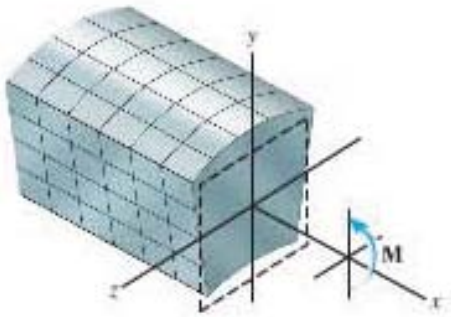


Fig. 6–23

6.4 The Flexure Formula

In this section, we will develop an equation that relates the stress distribution in a beam to the internal resultant bending moment acting on the beam's cross section. To do this we will assume that the material behaves in a linear-elastic manner and therefore a ***linear variation of normal strain***, Fig. 6–24*a*, must then be the result of a ***linear variation in normal stress***, Fig. 6–24*b*. Hence, like the normal strain variation, σ will vary from zero at the member's neutral axis to a maximum value, σ_{max}, a distance c farthest from the neutral axis. Because of the proportionality of triangles, Fig. 6–23*b*, or by using Hooke's law, $\sigma = E\epsilon$, and Eq. 6–8, we can write

$$\sigma = -\left(\frac{y}{c}\right)\sigma_{max} \tag{6–9}$$

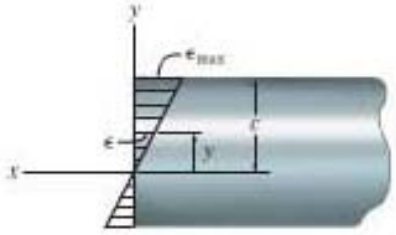
This equation describes the stress distribution over the cross-sectional area. The sign convention established here is significant. For positive **M**, which acts in the $+z$ direction, positive values of y give negative values for σ, that is, a compressive stress since it acts in the negative x direction. Similarly, negative y values will give positive or tensile values for σ. If a volume element of material is selected at a specific point on the cross section, only these tensile or compressive normal stresses will act on it. For example, the element located at $+y$ is shown in Fig. 6–24*c*.

We can locate the position of the neutral axis on the cross section by satisfying the condition that the *resultant force* produced by the stress distribution over the cross-sectional area must be equal to *zero*. Noting that the force $dF = \sigma\, dA$ acts on the arbitrary element dA in Fig. 6–24*c*, we require

$$F_R = \Sigma F_x; \qquad 0 = \int_A dF = \int_A \sigma\, dA$$

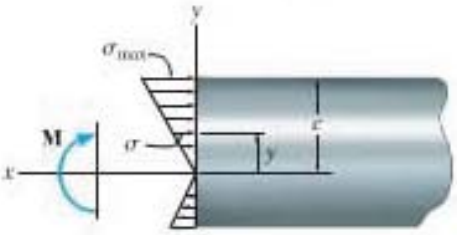
$$= \int_A -\left(\frac{y}{c}\right)\sigma_{max}\, dA$$

$$= \frac{-\sigma_{max}}{c}\int_A y\, dA$$



Normal strain variation (profile view)

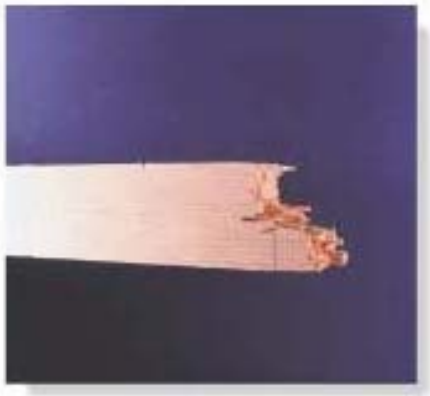
(a)



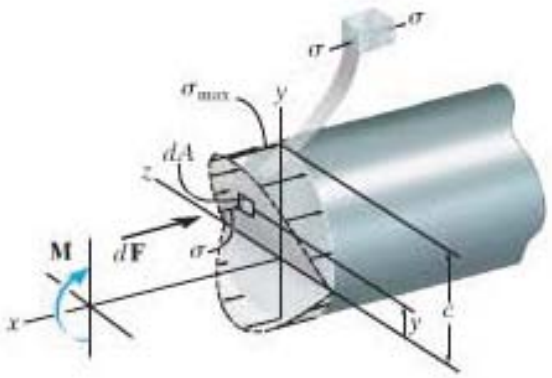
Bending stress variation (profile view)

(b)

Fig. 6–24



This wood specimen failed in bending due to its fibers being crushed at its top and torn apart at its bottom.



Bending stress variation

(c)

Fig. 6–24 (cont.)

Since σ_{max}/c is not equal to zero, then

$$\int_A y\,dA = 0 \tag{6–10}$$

In other words, the first moment of the member's cross-sectional area about the neutral axis must be zero. This condition can only be satisfied if the *neutral axis* is also the horizontal *centroidal axis* for the cross section.* Consequently, once the centroid for the member's cross-sectional area is determined, the location of the neutral axis is known

We can determine the stress in the beam from the requirement that the resultant internal moment M must be equal to the moment produced by the stress distribution about the neutral axis. The moment of $d\mathbf{F}$ in Fig. 6–24*c* about the neutral axis is $dM = y\,dF$. Since $dF = \sigma\,dA$, using Eq. 6–9, we have for the entire cross section,

$$(M_R)_z = \Sigma M_z; \quad M = \int_A y\,dF = \int_A y(\sigma\,dA) = \int_A y\left(\frac{y}{c}\sigma_{max}\right)dA$$

or

$$M = \frac{\sigma_{max}}{c}\int_A y^2\,dA \tag{6–11}$$

*Recall that the location $\bar{y}$ for the centroid of the cross-sectional area is defined from the equation $\bar{y} = \int y\,dA/\int dA$. If $\int y\,dA = 0$, then $\bar{y} = 0$, and so the centroid lies on the reference (neutral) axis. See Appendix A.

The integral represents the *moment of inertia* of the cross-sectional area about the neutral axis. We will symbolize its value as I. Hence, Eq. 6–11 can be solved for σ_{max} and written as

$$\sigma_{max} = \frac{Mc}{I} \tag{6–12}$$

Here

σ_{max} = the maximum normal stress in the member, which occurs at a point on the cross-sectional area *farthest away* from the neutral axis

M = the resultant internal moment, determined from the method of sections and the equations of equilibrium, and calculated about the neutral axis of the cross section

c = the perpendicular distance from the neutral axis to a point farthest away from the neutral axis. This is where σ_{max} acts

I = the moment of inertia of the cross-sectional area about the neutral axis

Since $\sigma_{max}/c = -\sigma/y$, Eq. 6–9, the normal stress at the intermediate distance y can be determined from an equation similar to Eq. 6–12. We have

$$\sigma = -\frac{My}{I} \tag{6–13}$$

Note that the negative sign is necessary since it agrees with the established x, y, z axes. By the right-hand rule, M is positive along the $+z$ axis, y is positive upward, and σ therefore must be negative (compressive) since it acts in the negative x direction, Fig. 6–24*c*.

Either of the above two equations is often referred to as the ***flexure formula***. It is used to determine the normal stress in a straight member, having a cross section that is symmetrical with respect to an axis, and the moment is applied perpendicular to this axis. Although we have assumed that the member is prismatic, we can in most cases of engineering design also use the flexure formula to determine the normal stress in members that have a *slight taper*. For example, using a mathematical analysis based on the theory of elasticity, a member having a rectangular cross section and a length that is tapered 15° will have an actual maximum normal stress that is about 5.4% *less* than that calculated using the flexure formula.

Important Points

- The cross section of a straight beam *remains plane* when the beam deforms due to bending. This causes tensile stress on one portion of the cross section and compressive stress on the other portion. In between these portions, there exists the *neutral axis* which is subjected to *zero stress*.
- Due to the deformation, the *longitudinal strain* varies *linearly* from zero at the neutral axis to a maximum at the outer fibers of the beam. Provided the material is homogeneous and linear elastic, then the *stress* also varies in a *linear* fashion over the cross section.
- The neutral axis passes through the *centroid* of the cross-sectional area. This result is based on the fact that the resultant normal force acting on the cross section must be zero.
- The flexure formula is based on the requirement that the resultant internal moment on the cross section is equal to the moment produced by the normal stress distribution about the neutral axis.

Procedure for Analysis

In order to apply the flexure formula, the following procedure is suggested.

Internal Moment.

- Section the member at the point where the bending or normal stress is to be determined, and obtain the internal moment M at the section. The centroidal or neutral axis for the cross section must be known, since M *must* be calculated about this axis.
- If the absolute maximum bending stress is to be determined, then draw the moment diagram in order to determine the maximum moment in the member.

Section Property.

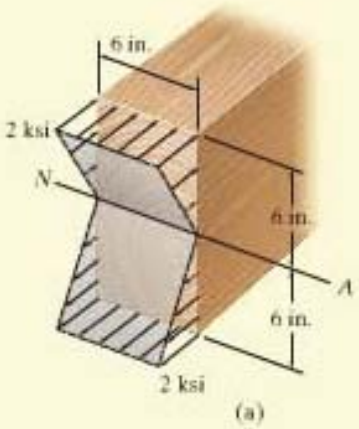
- Determine the moment of inertia of the cross-sectional area about the neutral axis. Methods used for its calculation are discussed in Appendix A, and a table listing values of I for several common shapes is given on the inside front cover.

Normal Stress.

- Specify the distance y, measured perpendicular to the neutral axis to the point where the normal stress is to be determined. Then apply the equation $\sigma = -My/I$, or if the maximum bending stress is to be calculated, use $\sigma_{max} = Mc/I$. When substituting the data, make sure the units are consistent.
- The stress acts in a direction such that the force it creates at the point contributes a moment about the neutral axis that is in the same direction as the internal moment $\mathbf{M}$, Fig. 6–24*c*. In this manner the stress distribution acting over the entire cross section can be sketched, or a volume element of the material can be isolated and used to graphically represent the normal stress acting at the point.

EXAMPLE 6.11

A beam has a rectangular cross section and is subjected to the stress distribution shown in Fig. 6–25*a*. Determine the internal moment **M** at the section caused by the stress distribution (a) using the flexure formula, (b) by finding the resultant of the stress distribution using basic principles.



SOLUTION

Part (a). The flexure formula is $\sigma_{max} = Mc/I$. From Fig. 6–25*a*, $c = 6$ in. and $\sigma_{max} = 2$ ksi. The neutral axis is defined as line NA, because the stress is zero along this line. Since the cross section has a rectangular shape, the moment of inertia for the area about NA is determined from the formula for a rectangle given on the inside front cover; i.e.,

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(6\text{ in.})(12\text{ in.})^3 = 864\text{ in}^4$$

Therefore,

$$\sigma_{max} = \frac{Mc}{I}; \qquad 2\text{ kip/in}^2 = \frac{M(6\text{ in.})}{864\text{ in}^4}$$

$$M = 288\text{ kip}\cdot\text{in.} = 24\text{ kip}\cdot\text{ft} \qquad \textit{Ans.}$$

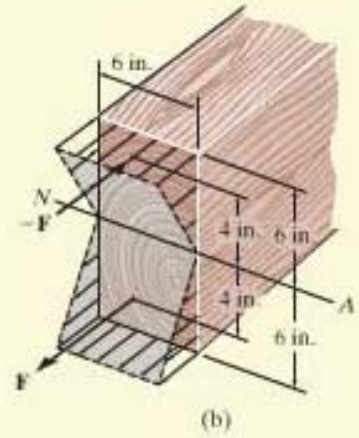


Fig. 6–25

Part (b). The resultant force for each of the two *triangular* stress distributions in Fig. 6–25*b* is graphically equivalent to the *volume* contained within each stress distribution. Thus, each volume is

$$F = \frac{1}{2}(6\text{ in.})(2\text{ kip/in}^2)(6\text{ in.}) = 36\text{ kip}$$

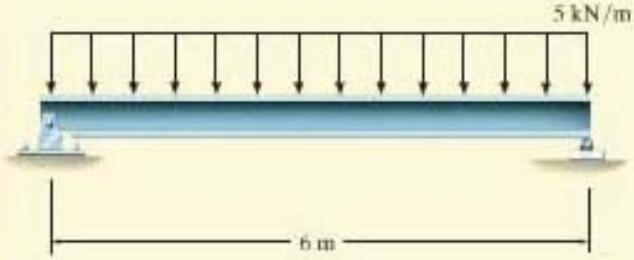
These forces, which form a couple, act in the same direction as the stresses within each distribution, Fig. 6–25*b*. Furthermore, they act through the *centroid* of each volume, i.e., $\frac{2}{3}(6\text{ in.}) = 4$ in. from the neutral axis of the beam. Hence the distance between them is 8 in. as shown. The moment of the couple is therefore

$$M = 36\text{ kip}\,(8\text{ in.}) = 288\text{ kip}\cdot\text{in.} = 24\text{ kip}\cdot\text{ft} \qquad \textit{Ans.}$$

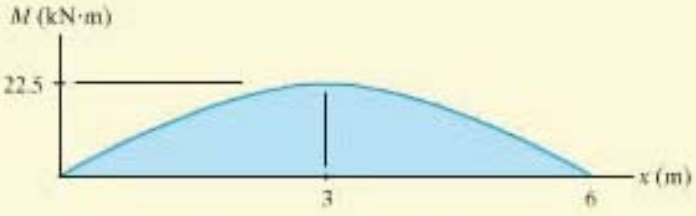
NOTE: This result can also be obtained by choosing a horizontal strip of area $dA = (6\text{ in.})\,dy$ and using integration by applying Eq. 6–11.

EXAMPLE 6.12

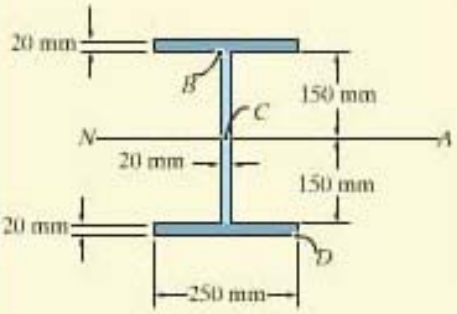
The simply supported beam in Fig. 6–26*a* has the cross-sectional area shown in Fig. 6–26*b*. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.



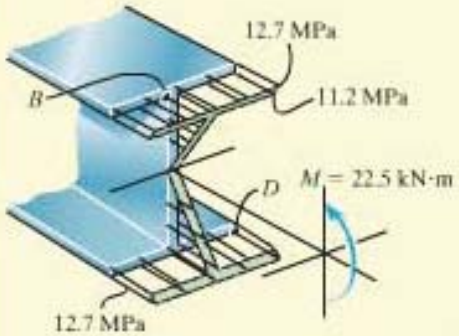
(a)



(c)



(b)



(d)

Fig. 6–26

SOLUTION

Maximum Internal Moment. The maximum internal moment in the beam, $M = 22.5\ \text{kN} \cdot \text{m}$, occurs at the center.

Section Property. By reasons of symmetry, the neutral axis passes through the centroid C at the midheight of the beam, Fig. 6–26*b*. The area is subdivided into the three parts shown, and the moment of inertia of each part is calculated about the neutral axis using the parallel-axis theorem. (See Eq. A–5 of Appendix A.) Choosing to work in meters, we have

$$I = \Sigma(\bar{I} + Ad^2)$$

$$= 2\left[\frac{1}{12}(0.25\ \text{m})(0.020\ \text{m})^3 + (0.25\ \text{m})(0.020\ \text{m})(0.160\ \text{m})^2\right] + \left[\frac{1}{12}(0.020\ \text{m})(0.300\ \text{m})^3\right]$$

$$= 301.3(10^{-6})\ \text{m}^4$$

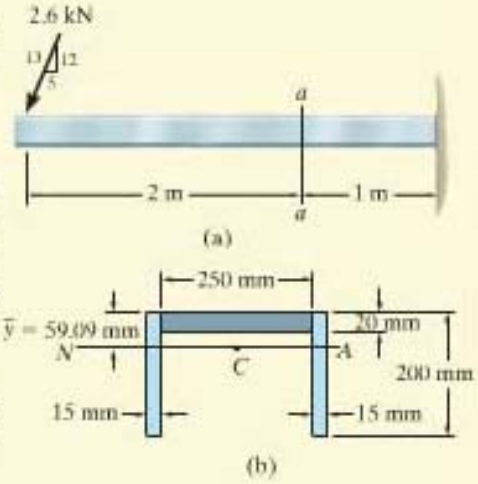
$$\sigma_{\max} = \frac{Mc}{I}; \qquad \sigma_{\max} = \frac{22.5(10^3)\ \text{N} \cdot \text{m}(0.170\ \text{m})}{301.3(10^{-6})\ \text{m}^4} = 12.7\ \text{MPa} \quad \textit{Ans.}$$

A three-dimensional view of the stress distribution is shown in Fig. 6–26*d*. Notice how the stress at points B and D on the cross section develops a force that contributes a moment about the neutral axis that has the same direction as $\mathbf{M}$. Specifically, at point B, $y_B = 150\ \text{mm}$, and so

$$\sigma_B = -\frac{My_B}{I}; \qquad \sigma_B = -\frac{22.5(10^3)\ \text{N} \cdot \text{m}(0.150\ \text{m})}{301.3(10^{-6})\ \text{m}^4} = -11.2\ \text{MPa}$$

EXAMPLE 6.13

The beam shown in Fig. 6–27*a* has a cross-sectional area in the shape of a channel, Fig. 6–27*b*. Determine the maximum bending stress that occurs in the beam at section *a–a*.



SOLUTION

Internal Moment. Here the beam's support reactions do not have to be determined. Instead, by the method of sections, the segment to the left of section *a–a* can be used, Fig. 6–27*c*. In particular, note that the resultant internal axial force **N** passes through the centroid of the cross section. Also, realize that *the resultant internal moment must be calculated about the beam's neutral axis* at section *a–a*.

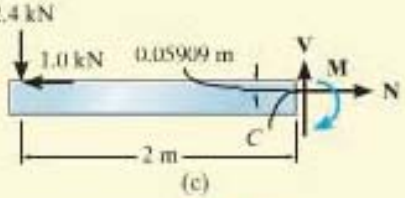
To find the location of the neutral axis, the cross-sectional area is subdivided into three composite parts as shown in Fig. 6–27*b*. Using Eq. A–2 of Appendix A, we have

$$\bar{y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{2[0.100\text{ m}](0.200\text{ m})(0.015\text{ m}) + [0.010\text{ m}](0.02\text{ m})(0.250\text{ m})}{2(0.200\text{ m})(0.015\text{ m}) + 0.020\text{ m}(0.250\text{ m})}$$

$$= 0.05909\text{ m} = 59.09\text{ mm}$$

This dimension is shown in Fig. 6–27*c*.

Applying the moment equation of equilibrium about the neutral axis, we have



$$\zeta + \Sigma M_{NA} = 0; \quad 2.4\text{ kN}(2\text{ m}) + 1.0\text{ kN}(0.05909\text{ m}) - M = 0$$

$$M = 4.859\text{ kN}\cdot\text{m}$$

Fig. 6–27

Section Property. The moment of inertia about the neutral axis is determined using the parallel-axis theorem applied to each of the three composite parts of the cross-sectional area. Working in meters, we have

$$I = \left[\frac{1}{12}(0.250\text{ m})(0.020\text{ m})^3 + (0.250\text{ m})(0.020\text{ m})(0.05909\text{ m} - 0.010\text{ m})^2\right]$$

$$+ 2\left[\frac{1}{12}(0.015\text{ m})(0.200\text{ m})^3 + (0.015\text{ m})(0.200\text{ m})(0.100\text{ m} - 0.05909\text{ m})^2\right]$$

$$= 42.26(10^{-6})\text{ m}^4$$

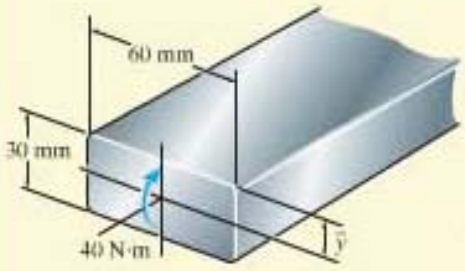
Maximum Bending Stress. The maximum bending stress occurs at points farthest away from the neutral axis. This is at the bottom of the beam, $c = 0.200\text{ m} - 0.05909\text{ m} = 0.1409\text{ m}$. Thus,

$$\sigma_{max} = \frac{Mc}{I} = \frac{4.859(10^3)\text{ N}\cdot\text{m}(0.1409\text{ m})}{42.26(10^{-6})\text{ m}^4} = 16.2\text{ MPa} \quad \textit{Ans.}$$

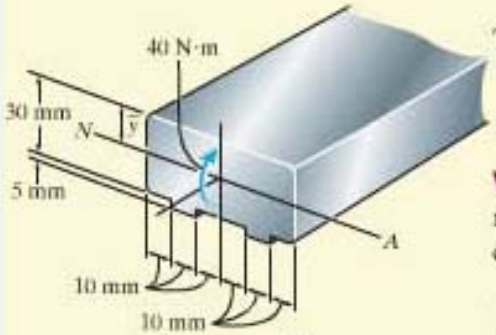
Show that at the top of the beam the bending stress is $\sigma' = 6.79$ MPa.

NOTE: The normal force of $N = 1$ kN and shear force $V = 2.4$ kN will also contribute additional stress on the cross section. The superposition of all these effects will be discussed in Chapter 8.

EXAMPLE 6.14



(a)



(b)

Fig. 6–28

The member having a rectangular cross section, Fig. 6–28*a*, is designed to resist a moment of 40 N · m. In order to increase its strength and rigidity, it is proposed that two small ribs be added at its bottom, Fig. 6–28*b*. Determine the maximum normal stress in the member for both cases.

SOLUTION

Without Ribs. Clearly the neutral axis is at the center of the cross section, Fig. 6–28*a*, so $\bar{y} = c = 15\text{ mm} = 0.015\text{ m}$. Thus,

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.060\text{ m})(0.030\text{ m})^3 = 0.135(10^{-6})\text{ m}^4$$

Therefore the maximum normal stress is

$$\sigma_{max} = \frac{Mc}{I} = \frac{(40\text{ N}\cdot\text{m})(0.015\text{ m})}{0.135(10^{-6})\text{ m}^4} = 4.44\text{ MPa} \qquad \textit{Ans.}$$

With Ribs. From Fig. 6–28*b*, segmenting the area into the large main rectangle and the bottom two rectangles (ribs), the location $\bar{y}$ of the centroid and the neutral axis is determined as follows:

$$\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A}$$

$$= \frac{[0.015\text{ m}](0.030\text{ m})(0.060\text{ m}) + 2[0.0325\text{ m}](0.005\text{ m})(0.010\text{ m})}{(0.03\text{ m})(0.060\text{ m}) + 2(0.005\text{ m})(0.010\text{ m})}$$

$$= 0.01592\text{ m}$$

This value does not represent c. Instead

$$c = 0.035\text{ m} - 0.01592\text{ m} = 0.01908\text{ m}$$

Using the parallel-axis theorem, the moment of inertia about the neutral axis is

$$I = \left[\frac{1}{12}(0.060\text{ m})(0.030\text{ m})^3 + (0.060\text{ m})(0.030\text{ m})(0.01592\text{ m} - 0.015\text{ m})^2\right]$$
$$+ 2\left[\frac{1}{12}(0.010\text{ m})(0.005\text{ m})^3 + (0.010\text{ m})(0.005\text{ m})(0.0325\text{ m} - 0.01592\text{ m})^2\right]$$
$$= 0.1642(10^{-6})\text{ m}^4$$

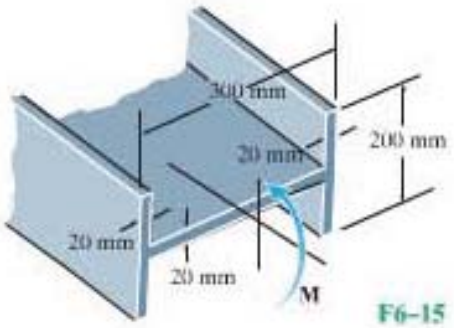
Therefore, the maximum normal stress is

$$\sigma_{max} = \frac{Mc}{I} = \frac{40\text{ N}\cdot\text{m}(0.01908\text{ m})}{0.1642(10^{-6})\text{ m}^4} = 4.65\text{ MPa} \qquad \textit{Ans.}$$

NOTE: This surprising result indicates that the addition of the ribs to the cross section will *increase* the normal stress rather than decrease it, and for this reason they should be omitted.

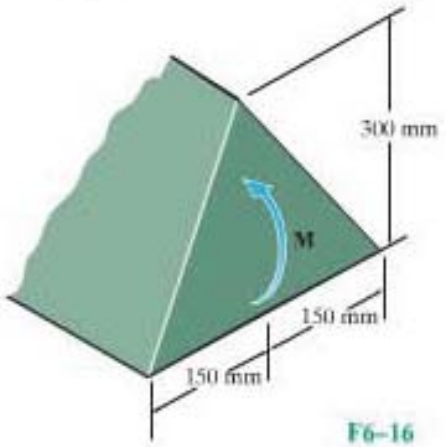
FUNDAMENTAL PROBLEMS

F6–15. If the beam is subjected to a bending moment of $M = 20\ \text{kN} \cdot \text{m}$, determine the maximum bending stress in the beam.



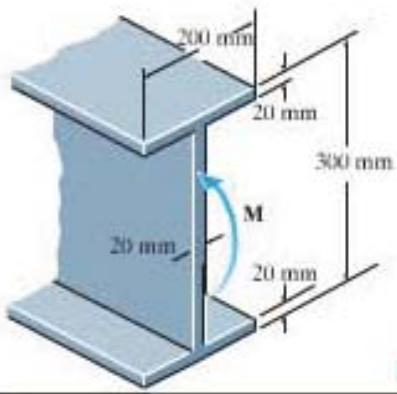
F6–15

F6–16. If the beam is subjected to a bending moment of $M = 50\ \text{kN} \cdot \text{m}$, sketch the bending stress distribution over the beam's cross section.



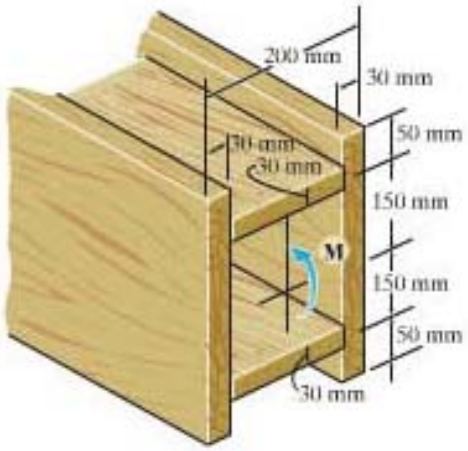
F6–16

F6–17. If the beam is subjected to a bending moment of $M = 50\ \text{kN} \cdot \text{m}$, determine the maximum bending stress in the beam.



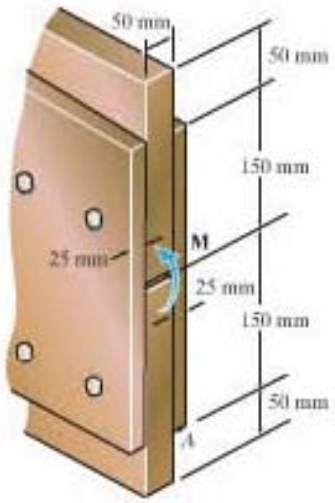
F6–17

F6–18. If the beam is subjected to a bending moment of $M = 10\ \text{kN} \cdot \text{m}$, determine the maximum bending stress in the beam.



F6–18

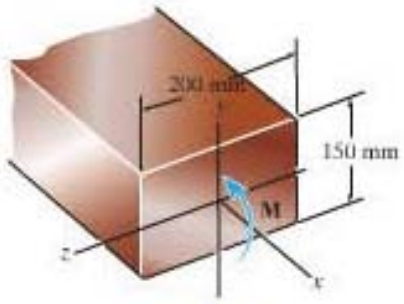
F6–19. If the beam is subjected to a bending moment of $M = 5\ \text{kN} \cdot \text{m}$, determine the bending stress developed at point A.



F6–19

PROBLEMS

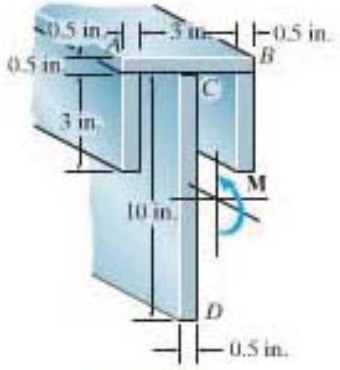
6–47. A member having the dimensions shown is used to resist an internal bending moment of $M = 90\,\text{kN}\cdot\text{m}$. Determine the maximum stress in the member if the moment is applied (a) about the z axis (as shown) (b) about the y axis. Sketch the stress distribution for each case.



Prob. 6–47

***6–48.** Determine the moment M that will produce a maximum stress of 10 ksi on the cross section.

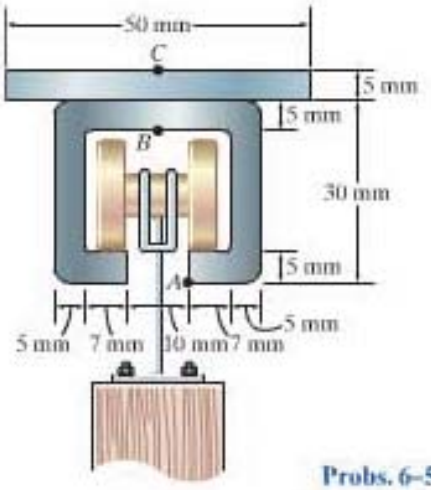
•6–49. Determine the maximum tensile and compressive bending stress in the beam if it is subjected to a moment of $M = 4\,\text{kip}\cdot\text{ft}$.



Probs. 6–48/49

6–50. The channel strut is used as a guide rail for a trolley. If the maximum moment in the strut is $M = 30\,\text{N}\cdot\text{m}$, determine the bending stress at points A, B, and C.

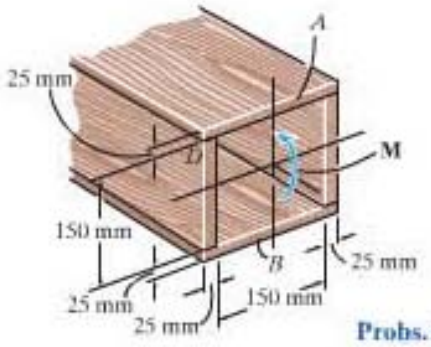
6–51. The channel strut is used as a guide rail for a trolley. If the allowable bending stress for the material is $\sigma_{allow} = 175\,\text{MPa}$, determine the maximum bending moment the strut will resist.



Probs. 6–50/51

***6–52.** The beam is subjected to a moment **M**. Determine the percentage of this moment that is resisted by the stresses acting on both the top and bottom boards, A and B, of the beam.

•6–53. Determine the moment **M** that should be applied to the beam in order to create a compressive stress at point D of $\sigma_D = 30\,\text{MPa}$. Also sketch the stress distribution acting over the cross section and compute the maximum stress developed in the beam.



Probs. 6–52/53