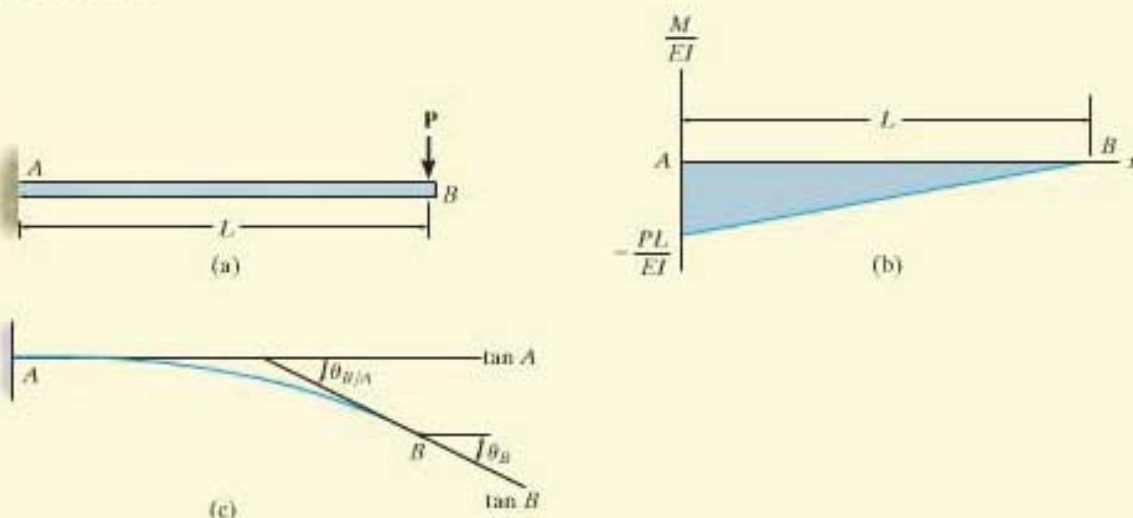


EXAMPLE 12.7

Determine the slope of the beam shown in Fig. 12-22*a* at point *B*.
 EI is constant.

**Fig. 12-22****SOLUTION**

M/EI Diagram. See Fig. 12-22*b*.

Elastic Curve. The force P causes the beam to deflect as shown in Fig. 12-22*c*. (The elastic curve is concave downward, since M/EI is negative.) The tangent at *B* is indicated since we are required to find θ_B . Also, the tangent at the support (*A*) is shown. This tangent has a *known* zero slope. By the construction, the angle between $\tan A$ and $\tan B$, that is, $\theta_{B/A}$, is equivalent to θ_B , or

$$\theta_B = \theta_{B/A}$$

Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between points *A* and *B*; that is,

$$\begin{aligned}\theta_B = \theta_{B/A} &= \frac{1}{2} \left(-\frac{PL}{EI} \right) L \\ &= -\frac{PL^2}{2EI} \quad \text{Ans.}\end{aligned}$$

The *negative sign* indicates that the angle measured from the tangent at *A* to the tangent at *B* is *clockwise*. This checks, since the beam slopes downward at *B*.

EXAMPLE 12.8

Determine the displacement of points B and C of the beam shown in Fig. 12-23a. EI is constant.

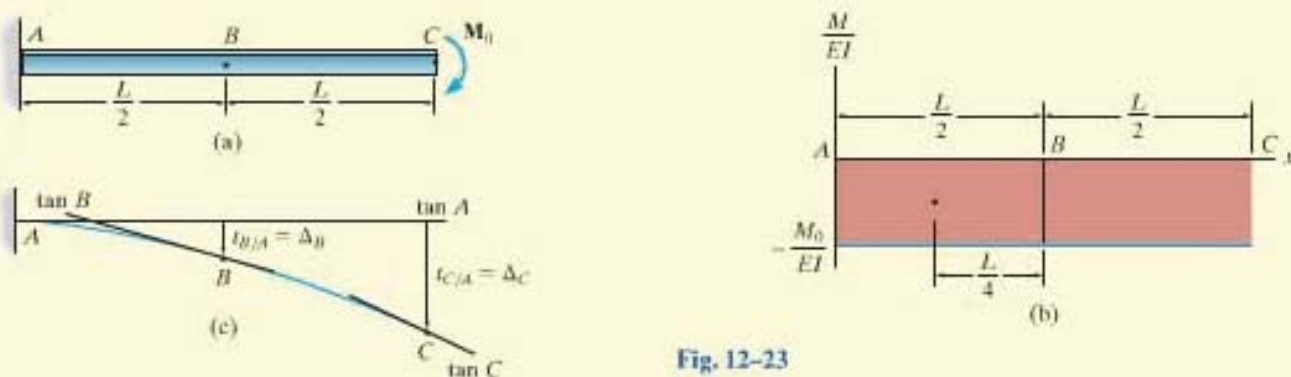


Fig. 12-23

SOLUTION

M/EI Diagram. See Fig. 12-23b.

Elastic Curve. The couple moment at C causes the beam to deflect as shown in Fig. 12-23c. The tangents at B and C are indicated since we are required to find Δ_B and Δ_C . Also, the tangent at the support (A) is shown since it is horizontal. The required displacements can now be related directly to the vertical distance between the tangents at B and A and C and A . Specifically,

$$\Delta_B = t_{B/A}$$

$$\Delta_C = t_{C/A}$$

Moment-Area Theorem. Applying Theorem 2, $t_{B/A}$ is equal to the moment of the shaded area under the M/EI diagram between A and B calculated about point B (the point on the elastic curve), since this is the point where the vertical distance is to be determined. Hence, from Fig. 12-23b,

$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right) \left[\left(-\frac{M_0}{EI}\right) \left(\frac{L}{2}\right) \right] = -\frac{M_0 L^2}{8EI} \quad \text{Ans.}$$

Likewise, for $t_{C/A}$ we must determine the moment of the area under the *entire* M/EI diagram from A to C about point C (the point on the elastic curve). We have

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right) \left[\left(-\frac{M_0}{EI}\right) (L) \right] = -\frac{M_0 L^2}{2EI} \quad \text{Ans.}$$

NOTE: Since both answers are *negative*, they indicate that points B and C lie *below* the tangent at A . This checks with Fig. 12-23c.

EXAMPLE 12.9

Determine the slope at point C of the shaft in Fig. 12-24a. EI is constant.

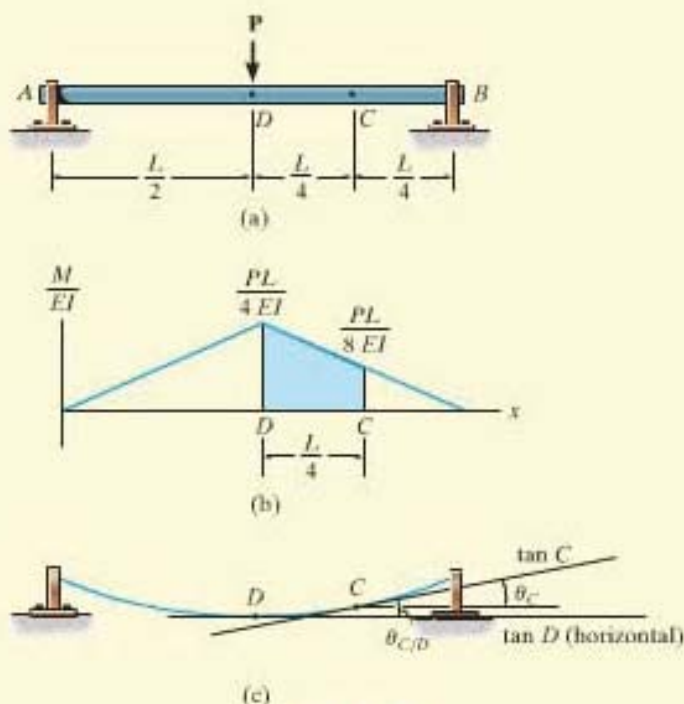


Fig. 12-24

SOLUTION

M/EI Diagram. See Fig. 12-24b.

Elastic Curve. Since the loading is applied symmetrically to the beam, the elastic curve is symmetric, and the tangent at D is horizontal, Fig. 12-24c. Also the tangent at C is drawn, since we must find the slope θ_C . By the construction, the angle $\theta_{C/D}$ between the tangents at D and C is equal to θ_C ; that is,

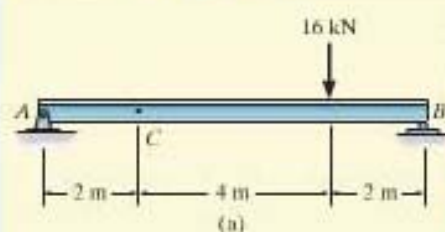
$$\theta_C = \theta_{C/D}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{C/D}$ is equal to the shaded area under the M/EI diagram between points D and C . We have

$$\theta_C = \theta_{C/D} = \left(\frac{PL}{8EI} \right) \left(\frac{L}{4} \right) + \frac{1}{2} \left(\frac{PL}{4EI} - \frac{PL}{8EI} \right) \left(\frac{L}{4} \right) = \frac{3PL^2}{64EI} \quad \text{Ans.}$$

What does the positive result indicate?

EXAMPLE 12.10



Determine the slope at point C for the steel beam in Fig. 12-25a. Take $E_{st} = 200 \text{ GPa}$, $I = 17(10^6) \text{ mm}^4$.

SOLUTION

M/EI Diagram. See Fig. 12-25b.

Elastic Curve. The elastic curve is shown in Fig. 12-25c. The tangent at C is shown since we are required to find θ_C . Tangents at the supports, A and B, are also constructed as shown. Angle $\theta_{C/A}$ is the angle between the tangents at A and C. The slope at A, θ_A , in Fig. 12-25c can be found using $|\theta_A| = |t_{B/A}|/L_{AB}$. This equation is valid since $t_{B/A}$ is actually very small, so that $t_{B/A}$ in meters can be approximated by the length of a circular arc defined by a radius of $L_{AB} = 8 \text{ m}$ and a sweep of θ_A in radians. (Recall that $s = \theta r$.) From the geometry of Fig. 12-25c, we have

$$|\theta_C| = |\theta_A| - |\theta_{C/A}| = \left| \frac{t_{B/A}}{8} \right| - |\theta_{C/A}| \quad (1)$$

Note that Example 12.9 could also be solved using this method.

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(2 \text{ m}) \left(\frac{8 \text{ kN} \cdot \text{m}}{EI} \right) = \frac{8 \text{ kN} \cdot \text{m}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B (the point on the elastic curve), since this is the point where the vertical distance is to be determined. We have

$$\begin{aligned} t_{B/A} &= \left(2 \text{ m} + \frac{1}{3}(6 \text{ m}) \right) \left[\frac{1}{2}(6 \text{ m}) \left(\frac{24 \text{ kN} \cdot \text{m}}{EI} \right) \right] \\ &\quad + \left(\frac{2}{3}(2 \text{ m}) \right) \left[\frac{1}{2}(2 \text{ m}) \left(\frac{24 \text{ kN} \cdot \text{m}}{EI} \right) \right] \\ &= \frac{320 \text{ kN} \cdot \text{m}^3}{EI} \end{aligned}$$

Substituting these results into Eq. 1, we get

$$\theta_C = \frac{320 \text{ kN} \cdot \text{m}^3}{(8 \text{ m})EI} - \frac{8 \text{ kN} \cdot \text{m}^2}{EI} = \frac{32 \text{ kN} \cdot \text{m}^2}{EI} \downarrow$$

We have calculated this result in units of kN and m, so converting EI into these units, we have

$$\theta_C = \frac{32 \text{ kN} \cdot \text{m}^2}{[200(10^6) \text{ kN/m}^2][17(10^{-6}) \text{ m}^4]} = 0.00941 \text{ rad} \downarrow \quad \text{Ans}$$

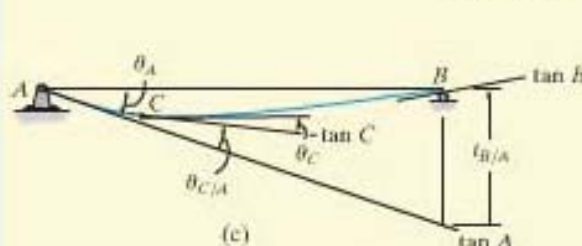
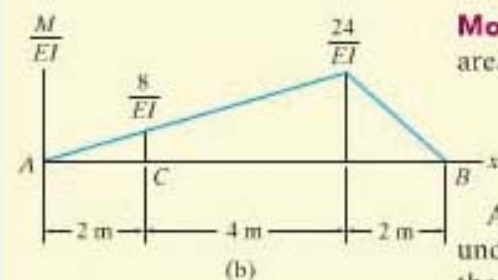


Fig. 12-25

EXAMPLE 12.11

Determine the displacement at C for the beam shown in Fig. 12-26a. EI is constant.

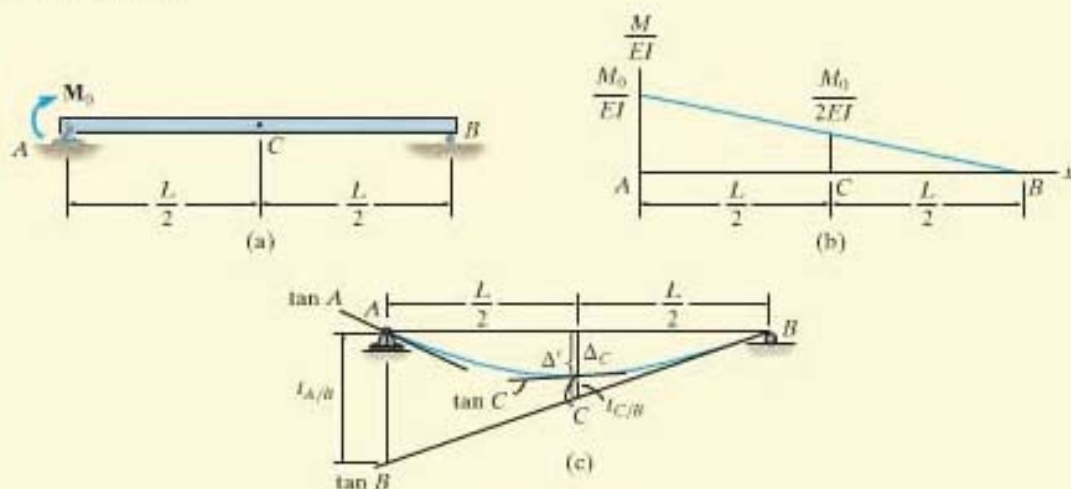


Fig. 12-26

SOLUTION

M/EI Diagram. See Fig. 12-26b.

Elastic Curve. The tangent at C is drawn on the elastic curve since we are required to find Δ_C , Fig. 12-26c. (Note that C is *not* the location of the maximum deflection of the beam, because the loading and hence the elastic curve are *not symmetric*.) Also indicated in Fig. 12-26c are the tangents at the supports A and B . It is seen that $\Delta_C = \Delta' - t_{C/B}$. If $t_{A/B}$ is determined, then Δ' can be found from proportional triangles, that is, $\Delta'/(L/2) = t_{A/B}/L$ or $\Delta' = t_{A/B}/2$. Hence,

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \quad (1)$$

Moment-Area Theorem. Applying Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$, we have

$$t_{A/B} = \left(\frac{1}{3}(L) \right) \left[\frac{1}{2}(L) \left(\frac{M_0}{EI} \right) \right] = \frac{M_0 L^2}{6EI}$$

$$t_{C/B} = \left(\frac{1}{3} \left(\frac{L}{2} \right) \right) \left[\frac{1}{2} \left(\frac{L}{2} \right) \left(\frac{M_0}{2EI} \right) \right] = \frac{M_0 L^2}{48EI}$$

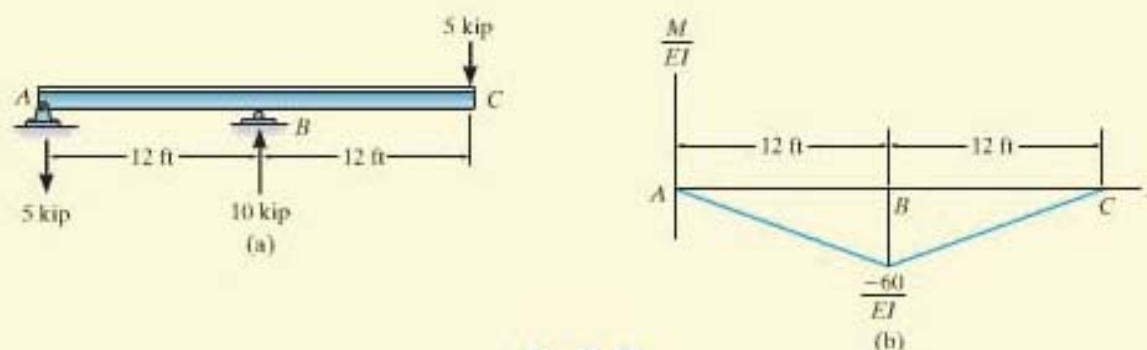
Substituting these results into Eq. 1 gives

$$\begin{aligned} \Delta_C &= \frac{1}{2} \left(\frac{M_0 L^2}{6EI} \right) - \left(\frac{M_0 L^2}{48EI} \right) \\ &= \frac{M_0 L^2}{16EI} \downarrow \end{aligned}$$

Ans.

EXAMPLE 12.12

Determine the displacement at point C for the steel overhanging beam shown in Fig. 12-27a. Take $E_{st} = 29(10^3)$ ksi, $I = 125$ in⁴.

**Fig. 12-27****SOLUTION**

M/EI Diagram. See Fig. 12-27b.

Elastic Curve. The loading causes the beam to deflect as shown in Fig. 12-27c. We are required to find Δ_C . By constructing tangents at C and at the supports A and B , it is seen that $\Delta_C = |t_{C/A}| - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles; that is, $\Delta'/24 = |t_{B/A}|/12$ or $\Delta' = 2|t_{B/A}|$. Hence

$$\Delta_C = |t_{C/A}| - 2|t_{B/A}| \quad (1)$$

Moment-Area Theorem. Applying Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$, we have

$$\begin{aligned}
 t_{C/A} &= (12 \text{ ft}) \left(\frac{1}{2} (24 \text{ ft}) \left(-\frac{60 \text{ kip} \cdot \text{ft}}{EI} \right) \right) \\
 &= \frac{8640 \text{ kip} \cdot \text{ft}^3}{EI} \\
 t_{B/A} &= \left(\frac{1}{3} (12 \text{ ft}) \right) \left[\frac{1}{2} (12 \text{ ft}) \left(-\frac{60 \text{ kip} \cdot \text{ft}}{EI} \right) \right] = -\frac{1440 \text{ kip} \cdot \text{ft}^3}{EI}
 \end{aligned}$$

Why are these terms negative? Substituting the results into Eq. 1 yields

$$\Delta_C = \frac{8640 \text{ kip} \cdot \text{ft}^3}{EI} - 2 \left(\frac{1440 \text{ kip} \cdot \text{ft}^3}{EI} \right) = \frac{5760 \text{ kip} \cdot \text{ft}^3}{EI} \downarrow$$

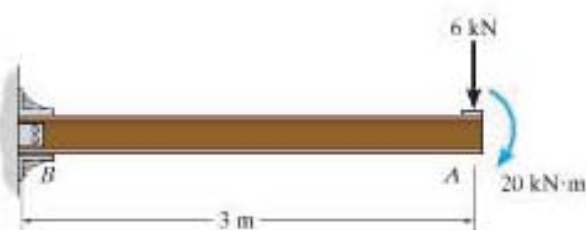
Realizing that the calculations were made in units of kip and ft, we have

$$\Delta_C = \frac{5760 \text{ kip} \cdot \text{ft}^3 (1728 \text{ in}^3/\text{ft}^3)}{[29(10^3) \text{ kip/in}^2](125 \text{ in}^4)} = 2.75 \text{ in.} \downarrow \quad \text{Ans}$$

FUNDAMENTAL PROBLEMS

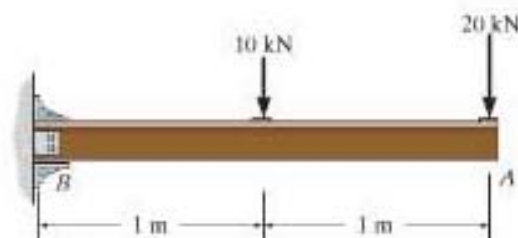
12

F12-7. Determine the slope and deflection of end *A* of the cantilevered beam. $E = 200 \text{ GPa}$ and $I = 65.0(10^{-6}) \text{ m}^4$.



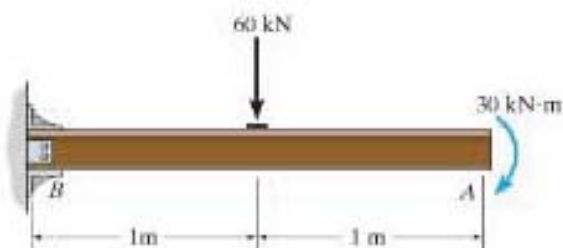
F12-7

F12-8. Determine the slope and deflection of end *A* of the cantilevered beam. $E = 200 \text{ GPa}$ and $I = 126(10^{-6}) \text{ m}^4$.



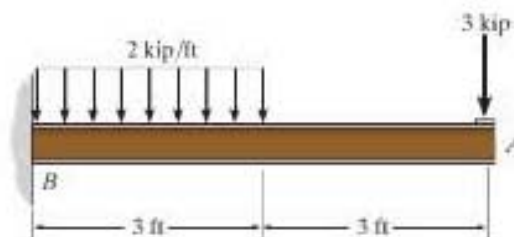
F12-8

F12-9. Determine the slope and deflection of end *A* of the cantilevered beam. $E = 200 \text{ GPa}$ and $I = 121(10^{-6}) \text{ m}^4$.



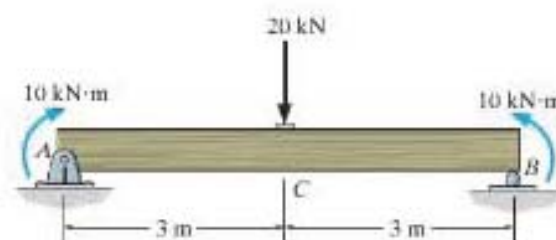
F12-9

F12-10. Determine the slope and deflection at *A* of the cantilevered beam. $E = 29(10^3) \text{ ksi}$, $I = 24.5 \text{ in}^4$.



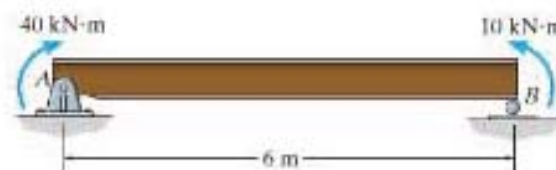
F12-10

F12-11. Determine the maximum deflection of the simply supported beam. $E = 200 \text{ GPa}$ and $I = 42.8(10^{-6}) \text{ m}^4$.



F12-11

F12-12. Determine the maximum deflection of the simply supported beam. $E = 200 \text{ GPa}$ and $I = 39.9(10^{-6}) \text{ m}^4$.

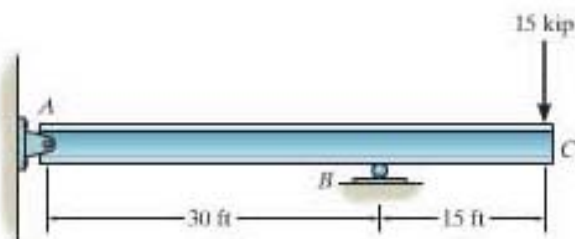


F12-12

12

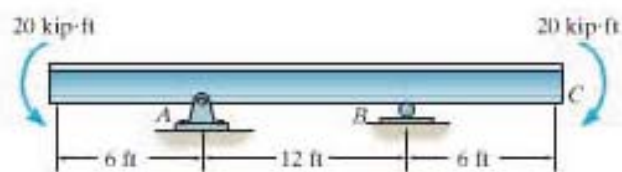
PROBLEMS

12-55. Determine the slope and deflection at C . EI is constant.



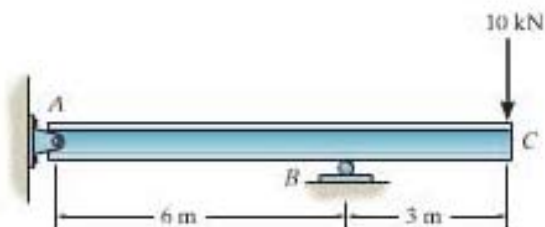
Prob. 12-55

12-58. Determine the slope at A and the maximum deflection. EI is constant.



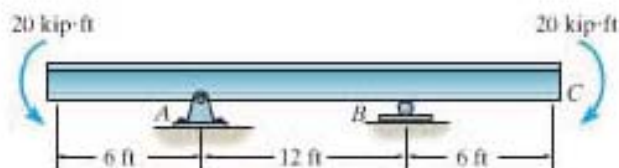
Prob. 12-58

***12-56.** Determine the slope and deflection at C . EI is constant.



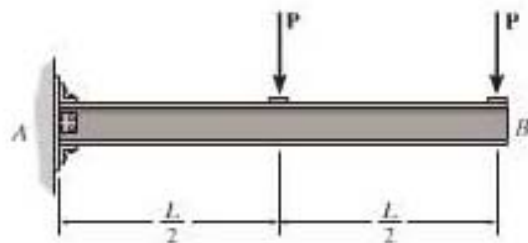
Prob. 12-56

12-59. Determine the slope and deflection at C . EI is constant.



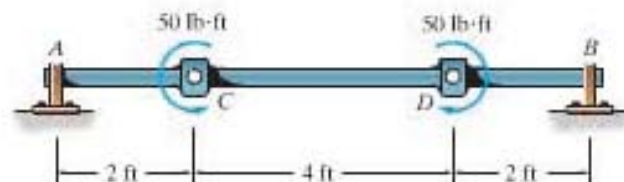
Prob. 12-59

***12-57.** Determine the deflection of end B of the cantilever beam. E is constant.



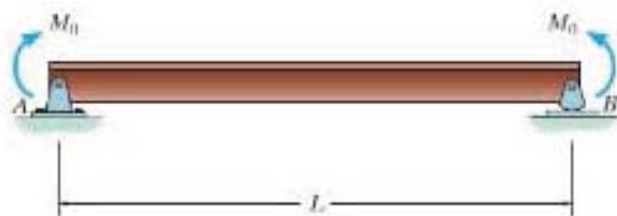
Prob. 12-57

***12-60.** If the bearings at A and B exert only vertical reactions on the shaft, determine the slope at A and the maximum deflection of the shaft. EI is constant.



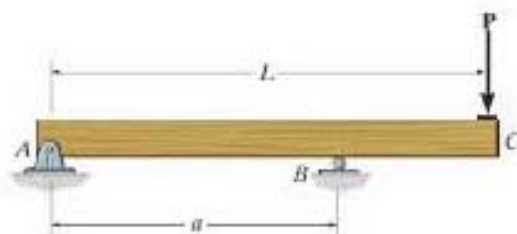
Prob. 12-60

- 12-61. Determine the maximum slope and the maximum deflection of the beam. EI is constant.



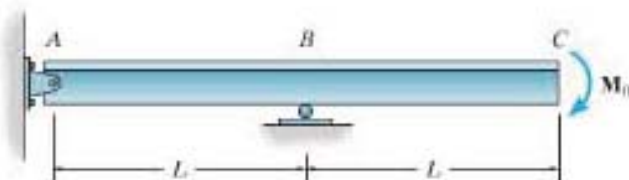
Prob. 12-61

- 12-65. Determine the position a of roller support B in terms of L so that the deflection at end C is the same as the maximum deflection of region AB of the overhang beam. EI is constant.



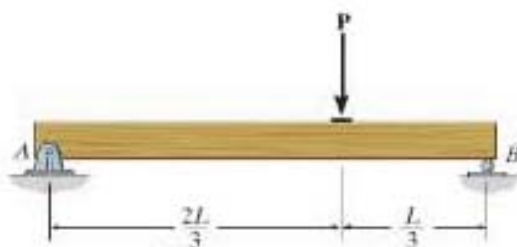
Prob. 12-65

- 12-62. Determine the deflection and slope at C . EI is constant.



Prob. 12-62

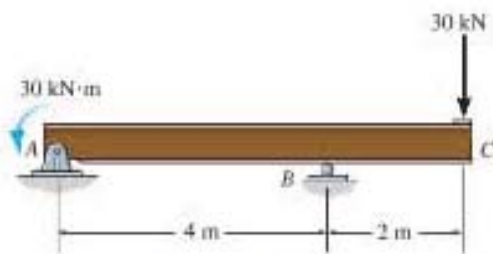
- 12-66. Determine the slope at A of the simply supported beam. EI is constant.



Prob. 12-66

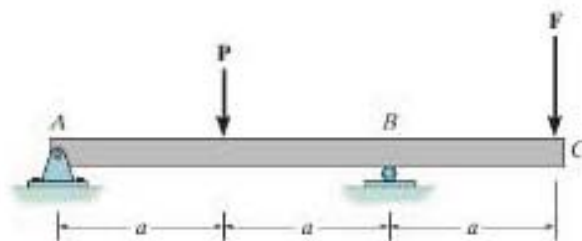
- 12-63. Determine the slope at A of the overhang beam. $E = 200 \text{ GPa}$ and $I = 45.5(10^6) \text{ mm}^4$.

- *12-64. Determine the deflection at C of the overhang beam. $E = 200 \text{ GPa}$ and $I = 45.5(10^6) \text{ mm}^4$.



Probs. 12-63/64

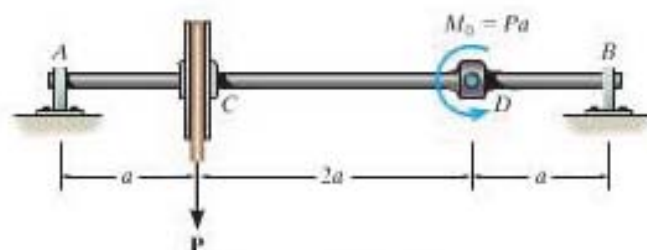
- 12-67. The beam is subjected to the load P as shown. Determine the magnitude of force F that must be applied at the end of the overhang C so that the deflection at C is zero. EI is constant.



Prob. 12-67

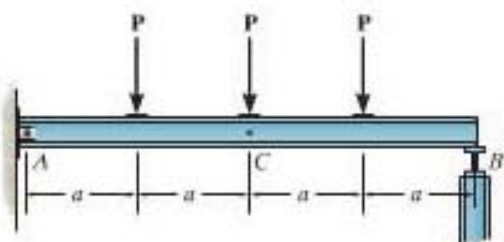
12

*12-68. If the bearings at A and B exert only vertical reactions on the shaft, determine the slope at A and the maximum deflection.



Prob. 12-68

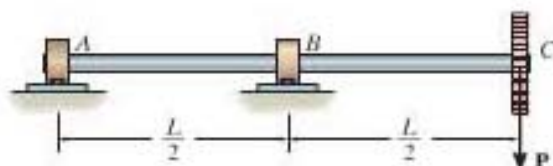
•12-69. The beam is subjected to the loading shown. Determine the slope at A and the displacement at C . Assume the support at A is a pin and B is a roller. EI is constant.



Prob. 12-69

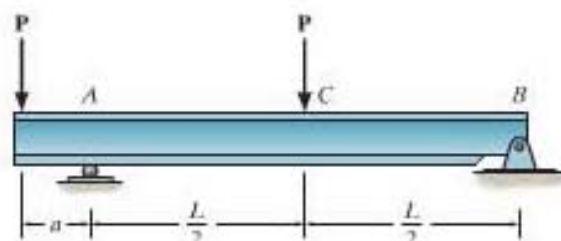
12-70. The shaft supports the gear at its end C . Determine the deflection at C and the slopes at the bearings A and B . EI is constant.

12-71. The shaft supports the gear at its end C . Determine its maximum deflection within region AB . EI is constant. The bearings exert only vertical reactions on the shaft.



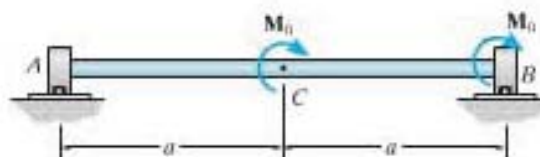
Probs. 12-70/71

*12-72. Determine the value of a so that the displacement at C is equal to zero. EI is constant.



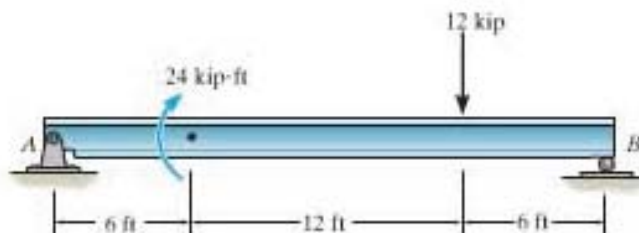
Prob. 12-72

•12-73. The shaft is subjected to the loading shown. If the bearings at A and B only exert vertical reactions on the shaft, determine the slope at A and the displacement at C . EI is constant.



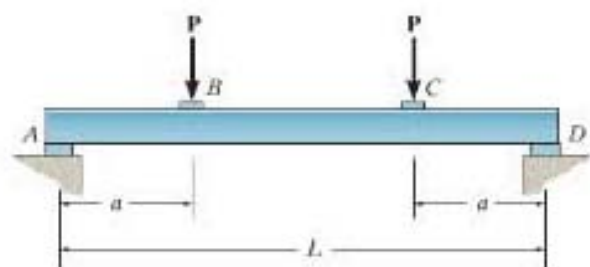
Prob. 12-73

12-74. Determine the slope at A and the maximum deflection in the beam. EI is constant.



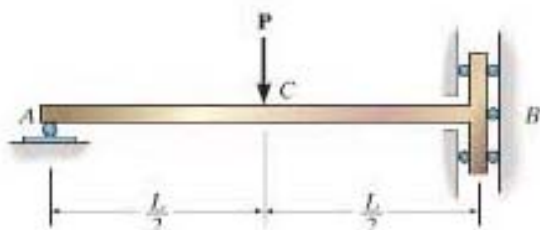
Prob. 12-74

12-75. The beam is made of a ceramic material. In order to obtain its modulus of elasticity, it is subjected to the elastic loading shown. If the moment of inertia is I and the beam has a measured maximum deflection Δ , determine E . The supports at A and D exert only vertical reactions on the beam.



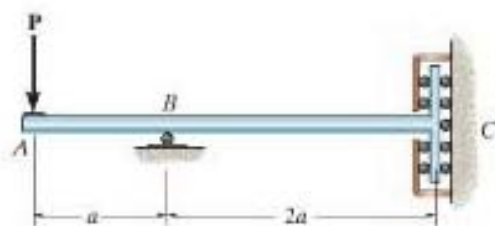
Prob. 12-75

***12-76.** The bar is supported by a roller constraint at B , which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at A and the deflection at C . EI is constant.



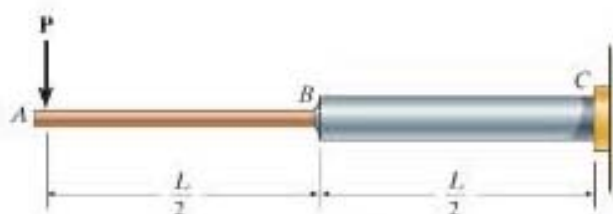
Prob. 12-76

***12-77.** The bar is supported by the roller constraint at C , which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope and displacement at A . EI is constant.



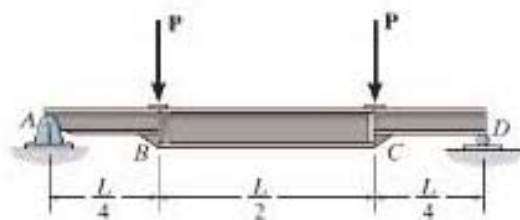
Prob. 12-77

12-78. The rod is constructed from two shafts for which the moment of inertia of AB is I and of BC is $2I$. Determine the maximum slope and deflection of the rod due to the loading. The modulus of elasticity is E .



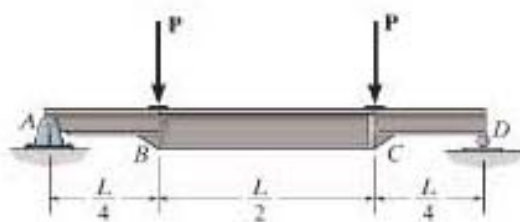
Prob. 12-78

12-79. Determine the slope at point D and the deflection at point C of the simply supported beam. The beam is made of material having a modulus of elasticity E . The moment of inertia of segments AB and CD of the beam is I , while the moment of inertia of segment BC of the beam is $2I$.



Prob. 12-79

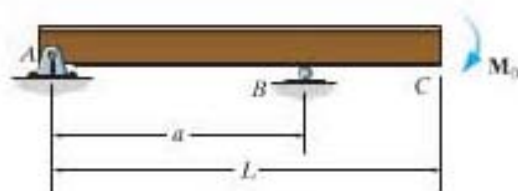
***12-80.** Determine the slope at point A and the maximum deflection of the simply supported beam. The beam is made of material having a modulus of elasticity E . The moment of inertia of segments AB and CD of the beam is I , while the moment of inertia of segment BC is $2I$.



Prob. 12-80

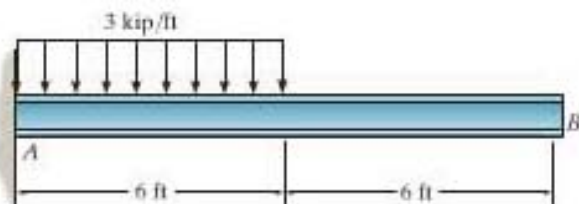
12

•12-81. Determine the position a of roller support B in terms of L so that deflection at end C is the same as the maximum deflection of region AB of the simply supported overhang beam. EI is constant.



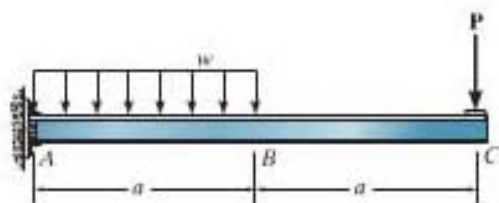
Prob. 12-81

12-82. The $W10 \times 15$ cantilevered beam is made of A-36 steel and is subjected to the loading shown. Determine the slope and displacement at its end B .



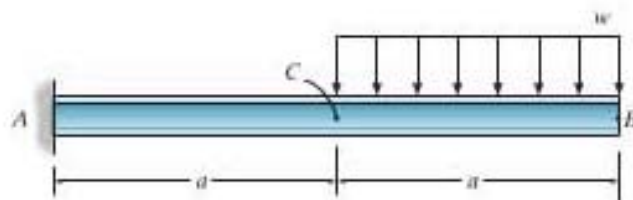
Prob. 12-82

12-83. The cantilevered beam is subjected to the loading shown. Determine the slope and displacement at C . Assume the support at A is fixed. EI is constant.



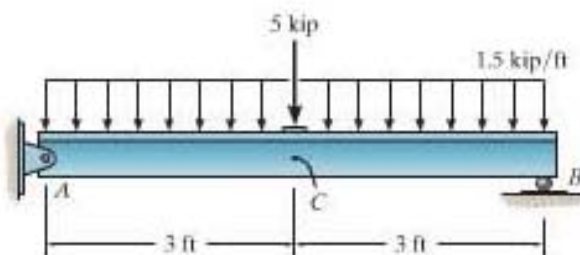
Prob. 12-83

*12-84. Determine the slope at C and deflection at B . EI is constant.



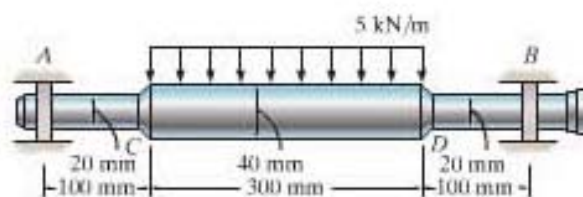
Prob. 12-84

•12-85. Determine the slope at B and the displacement at C . The member is an A-36 steel structural tee for which $I = 76.8 \text{ in}^4$.



Prob. 12-85

12-86. The A-36 steel shaft is used to support a rotor that exerts a uniform load of 5 kN/m within the region CD of the shaft. Determine the slope of the shaft at the bearings A and B . The bearings exert only vertical reactions on the shaft.



Prob. 12-86

12.5 Method of Superposition

12

The differential equation $EI d^4v/dx^4 = w(x)$ satisfies the two necessary requirements for applying the principle of superposition; i.e., the load $w(x)$ is linearly related to the deflection $v(x)$, and the load is assumed not to change significantly the original geometry of the beam or shaft. As a result, the deflections for a series of separate loadings acting on a beam may be superimposed. For example, if v_1 is the deflection for one load and v_2 is the deflection for another load, the total deflection for both loads acting together is the algebraic sum $v_1 + v_2$. Using tabulated results for various beam loadings, such as the ones listed in Appendix C, or those found in various engineering handbooks, it is therefore possible to find the slope and displacement at a point on a beam subjected to several different loadings by algebraically adding the effects of its various component parts.

The following examples illustrate how to use the method of superposition to solve deflection problems, where the deflection is caused not only by beam deformations, but also by rigid-body displacements, such as those that occur when the beam is supported by springs.



The resultant deflection at any point on this beam can be determined from the superposition of the deflections caused by each of the separate loadings acting on the beam.

EXAMPLE 12.13

Determine the displacement at point C and the slope at the support A of the beam shown in Fig. 12-28a. EI is constant.

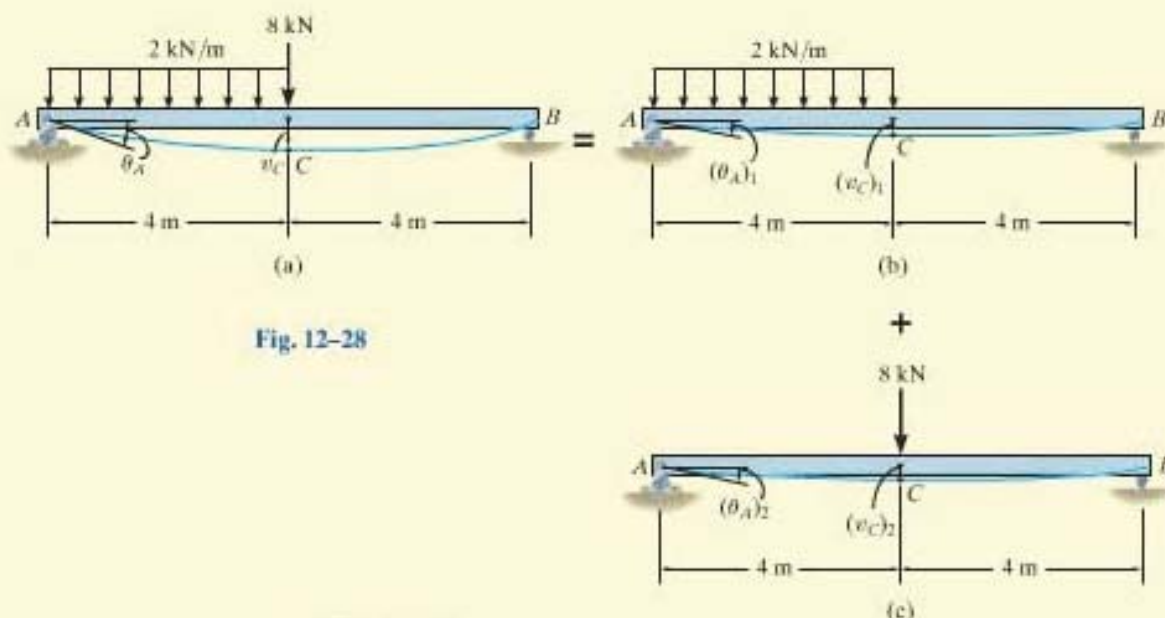


Fig. 12-28

SOLUTION

The loading can be separated into two component parts as shown in Figs. 12-28b and 12-28c. The displacement at C and slope at A are found using the table in Appendix C for each part.

For the distributed loading,

$$(\theta_A)_1 = \frac{3wL^3}{128EI} = \frac{3(2 \text{ kN/m})(8 \text{ m})^3}{128EI} = \frac{24 \text{ kN} \cdot \text{m}^2}{EI} \downarrow$$

$$(v_C)_1 = \frac{5wL^4}{768EI} = \frac{5(2 \text{ kN/m})(8 \text{ m})^4}{768EI} = \frac{53.33 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

For the 8-kN concentrated force,

$$(\theta_A)_2 = \frac{PL^2}{16EI} = \frac{8 \text{ kN}(8 \text{ m})^2}{16EI} = \frac{32 \text{ kN} \cdot \text{m}^2}{EI} \downarrow$$

$$(v_C)_2 = \frac{PL^3}{48EI} = \frac{8 \text{ kN}(8 \text{ m})^3}{48EI} = \frac{85.33 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

The displacement at C and the slope at A are the algebraic sums of these components. Hence,

$$(+\downarrow) \quad \theta_A = (\theta_A)_1 + (\theta_A)_2 = \frac{56 \text{ kN} \cdot \text{m}^2}{EI} \downarrow \quad \text{Ans.}$$

$$(+\downarrow) \quad v_C = (v_C)_1 + (v_C)_2 = \frac{139 \text{ kN} \cdot \text{m}^3}{EI} \downarrow \quad \text{Ans.}$$

EXAMPLE 12.14

Determine the displacement at the end C of the overhanging beam shown in Fig. 12-29a. EI is constant.

SOLUTION

Since the table in Appendix C *does not* include beams with overhangs, the beam will be separated into a simply supported and a cantilevered portion. First we will calculate the slope at B , as caused by the distributed load acting on the simply supported span, Fig. 12-29b.

$$(\theta_B)_1 = \frac{wL^3}{24EI} = \frac{5 \text{ kN/m}(4 \text{ m})^3}{24EI} = \frac{13.33 \text{ kN} \cdot \text{m}^2}{EI} \uparrow$$

Since this angle is *small*, $(\theta_B)_1 \approx \tan(\theta_B)_1$, and the vertical displacement at point C is

$$(v_C)_1 = (2 \text{ m}) \left(\frac{13.33 \text{ kN} \cdot \text{m}^2}{EI} \right) = \frac{26.67 \text{ kN} \cdot \text{m}^3}{EI} \uparrow$$

Next, the 10-kN load on the overhang causes a statically equivalent force of 10 kN and couple moment of 20 kN·m at the support B of the simply supported span, Fig. 12-29c. The 10-kN force does not cause a displacement or slope at B ; however, the 20-kN·m couple moment does cause a slope. The slope at B due to this moment is

$$(\theta_B)_2 = \frac{M_0 L}{3EI} = \frac{20 \text{ kN} \cdot \text{m}(4 \text{ m})}{3EI} = \frac{26.67 \text{ kN} \cdot \text{m}^2}{EI} \downarrow$$

so that the extended point C is displaced

$$(v_C)_2 = (2 \text{ m}) \left(\frac{26.7 \text{ kN} \cdot \text{m}^2}{EI} \right) = \frac{53.33 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

Finally, the cantilevered portion BC is displaced by the 10-kN force, Fig. 12-29d. We have

$$(v_C)_3 = \frac{PL^3}{3EI} = \frac{10 \text{ kN}(2 \text{ m})^3}{3EI} = \frac{26.67 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

Summing these results algebraically, we obtain the displacement of point C ,

$$(+\downarrow) \quad v_C = -\frac{26.7}{EI} + \frac{53.3}{EI} + \frac{26.7}{EI} = \frac{53.3 \text{ kN} \cdot \text{m}^3}{EI} \quad \text{Ans.}$$

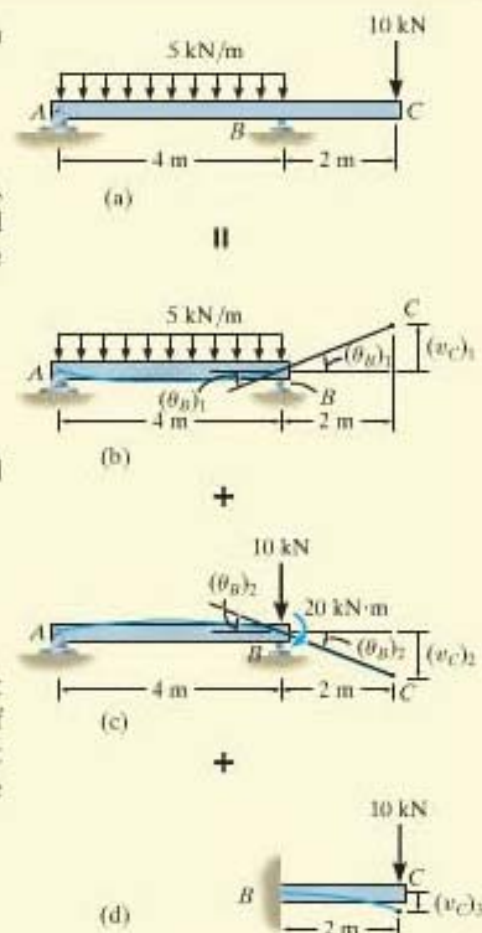


Fig. 12-29

EXAMPLE 12.15

Determine the displacement at the end C of the cantilever beam shown in Fig. 12–30. EI is constant.

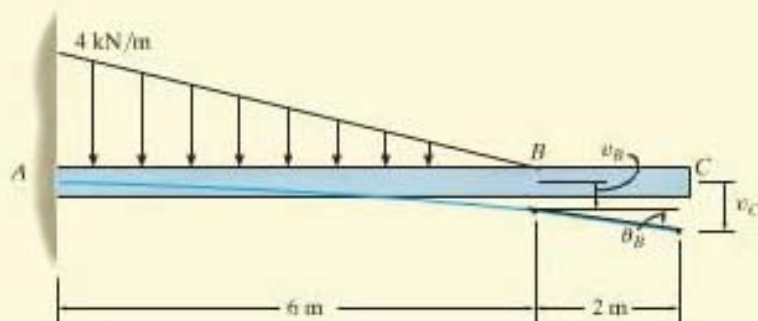


Fig. 12–30

SOLUTION

Using the table in Appendix C for the triangular loading, the slope and displacement at point B are

$$\theta_B = \frac{w_0 L^3}{24EI} = \frac{4 \text{ kN/m}(6 \text{ m})^3}{24EI} = \frac{36 \text{ kN} \cdot \text{m}^2}{EI}$$

$$v_B = \frac{w_0 L^4}{30EI} = \frac{4 \text{ kN/m}(6 \text{ m})^4}{30EI} = \frac{172.8 \text{ kN} \cdot \text{m}^3}{EI}$$

The unloaded region BC of the beam remains straight, as shown in Fig. 12–30. Since θ_B is small, the displacement at C becomes

$$\begin{aligned} (+\downarrow) \quad v_C &= v_B + \theta_B(L_{BC}) \\ &= \frac{172.8 \text{ kN} \cdot \text{m}^3}{EI} + \frac{36 \text{ kN} \cdot \text{m}^2}{EI}(2 \text{ m}) \\ &= \frac{244.8 \text{ kN} \cdot \text{m}^3}{EI} \downarrow \end{aligned}$$

Ans.

EXAMPLE 12.16

The steel bar shown in Fig. 12-31a is supported by two springs at its ends A and B . Each spring has a stiffness of $k = 15 \text{ kip/ft}$ and is originally unstretched. If the bar is loaded with a force of 3 kip at point C , determine the vertical displacement of the force. Neglect the weight of the bar and take $E_{st} = 29(10^3) \text{ ksi}$, $I = 12 \text{ in}^4$.

SOLUTION

The end reactions at A and B are calculated and shown in Fig. 12-31b. Each spring deflects by an amount

$$(v_A)_1 = \frac{2 \text{ kip}}{15 \text{ kip/ft}} = 0.1333 \text{ ft}$$

$$(v_B)_1 = \frac{1 \text{ kip}}{15 \text{ kip/ft}} = 0.0667 \text{ ft}$$

If the bar is considered to be *rigid*, these displacements cause it to move into the position shown in Fig. 12-31b. For this case, the vertical displacement at C is

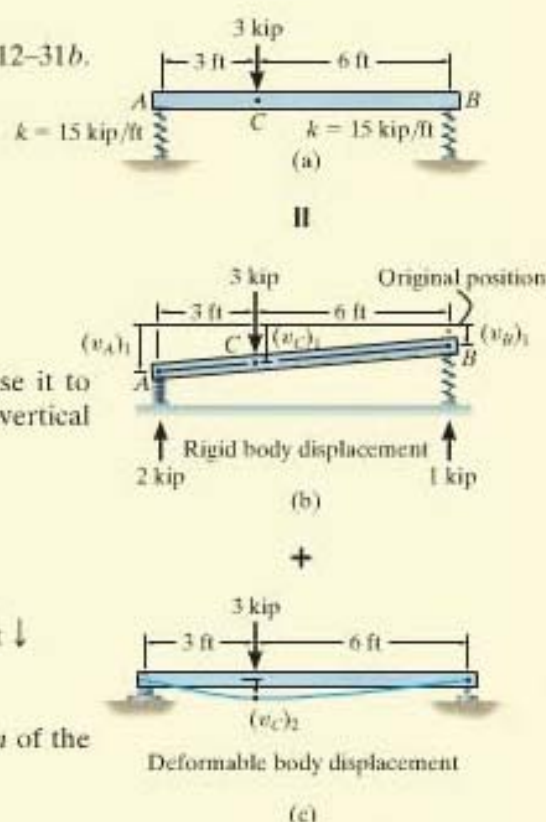
$$\begin{aligned} (v_C)_1 &= (v_B)_1 + \frac{6 \text{ ft}}{9 \text{ ft}}[(v_A)_1 - (v_B)_1] \\ &= 0.0667 \text{ ft} + \frac{2}{3}[0.1333 \text{ ft} - 0.0667 \text{ ft}] = 0.1111 \text{ ft} \downarrow \end{aligned}$$

We can find the displacement at C caused by the *deformation* of the bar, Fig. 12-31c, by using the table in Appendix C. We have

$$\begin{aligned} (v_C)_2 &= \frac{Pab}{6EIL}(L^2 - b^2 - a^2) \\ &= \frac{3 \text{ kip}(3 \text{ ft})(6 \text{ ft})[(9 \text{ ft})^2 - (6 \text{ ft})^2 - (3 \text{ ft})^2]}{6[29(10^3) \text{ kip/in}^2](144 \text{ in}^2/1 \text{ ft}^2)(12 \text{ in}^4)(1 \text{ ft}^4/20,736 \text{ in}^4)(9 \text{ ft})} \\ &= 0.0149 \text{ ft} \downarrow \end{aligned}$$

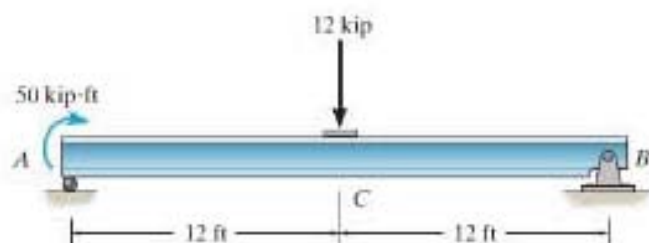
Adding the two displacement components, we get

$$(+\downarrow) v_C = 0.1111 \text{ ft} + 0.0149 \text{ ft} = 0.126 \text{ ft} = 1.51 \text{ in.} \downarrow$$

**Fig. 12-31***Ans.*

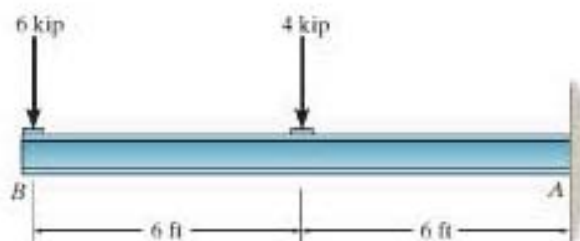
PROBLEMS

12-87. The $W12 \times 45$ simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C .



Prob. 12-87

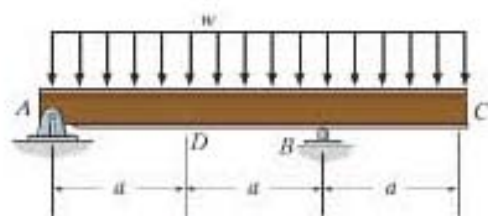
***12-88.** The $W10 \times 15$ cantilevered beam is made of A-36 steel and is subjected to the loading shown. Determine the displacement at B and the slope at A .



Prob. 12-88

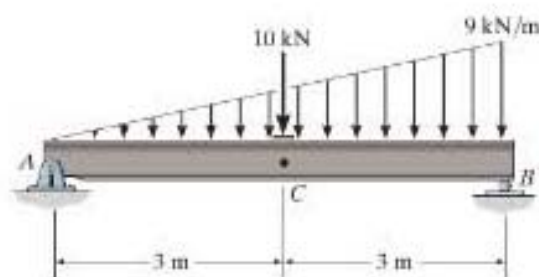
***12-89.** Determine the slope and deflection at end C of the overhang beam. EI is constant.

12-90. Determine the slope at A and the deflection at point D of the overhang beam. EI is constant.



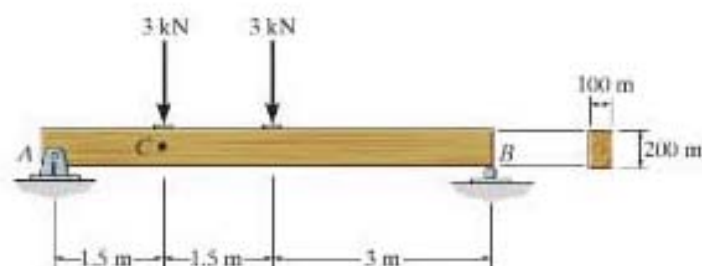
Probs. 12-89/90

12-91. Determine the slope at B and the deflection at point C of the simply supported beam. $E = 200$ GPa and $I = 45.5(10^6)$ mm⁴.



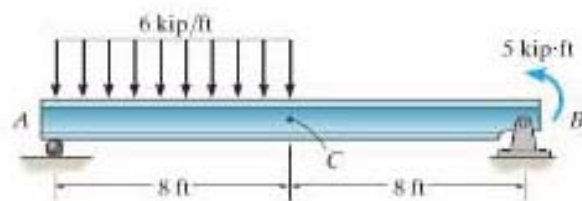
Prob. 12-91

***12-92.** Determine the slope at A and the deflection at point C of the simply supported beam. The modulus of elasticity of the wood is $E = 10$ GPa.



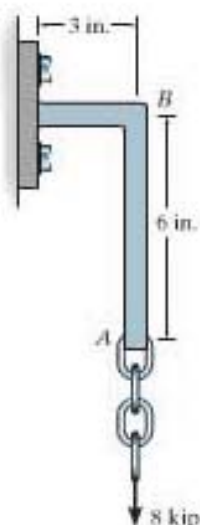
Prob. 12-92

***12-93.** The $W8 \times 24$ simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C .



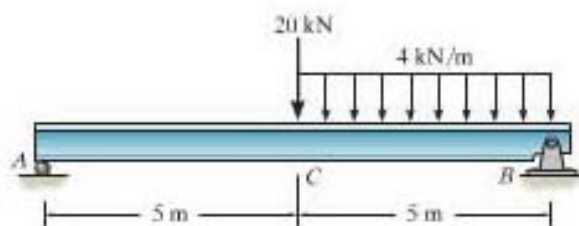
Prob. 12-93

12-94. Determine the vertical deflection and slope at the end A of the bracket. Assume that the bracket is fixed supported at its base, and neglect the axial deformation of segment AB . EI is constant.



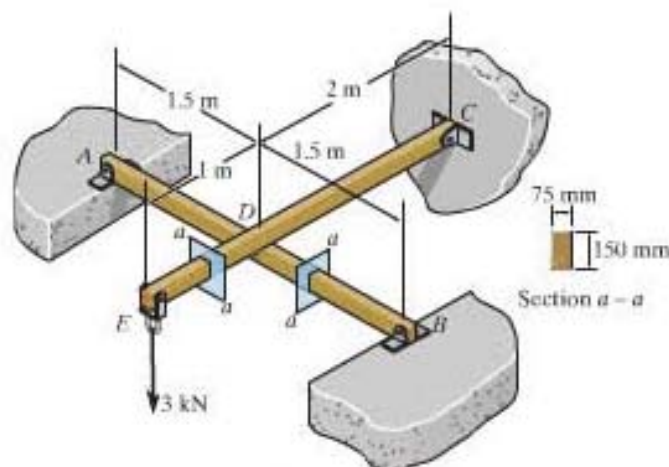
Prob. 12-94

12-95. The simply supported beam is made of A-36 steel and is subjected to the loading shown. Determine the deflection at its center C . $I = 0.1457(10^{-3}) \text{ m}^4$.



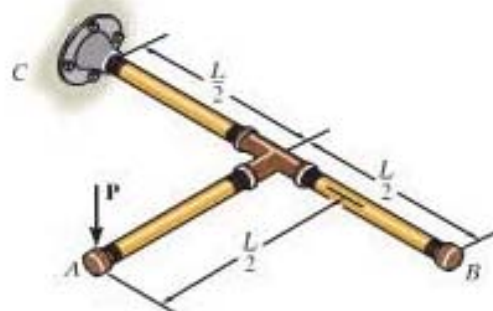
Prob. 12-95

***12-96.** Determine the deflection at end E of beam CDE . The beams are made of wood having a modulus of elasticity of $E = 10 \text{ GPa}$.



Prob. 12-96

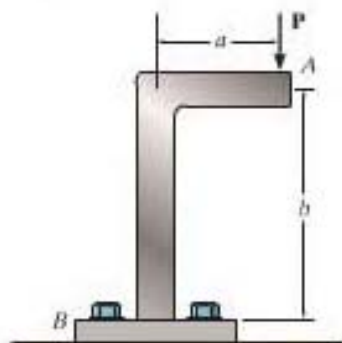
***12-97.** The pipe assembly consists of three equal-sized pipes with flexibility stiffness EI and torsional stiffness GJ . Determine the vertical deflection at point A .



Prob. 12-97

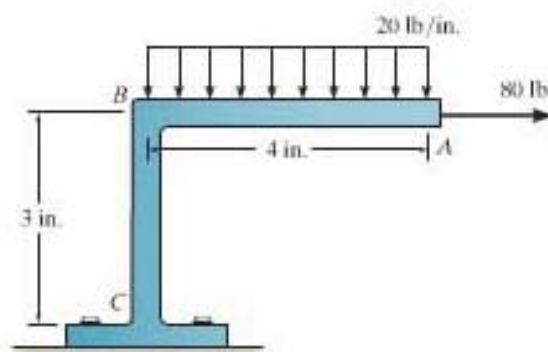
12

12-98. Determine the vertical deflection at the end A of the bracket. Assume that the bracket is fixed supported at its base B and neglect axial deflection. EI is constant.



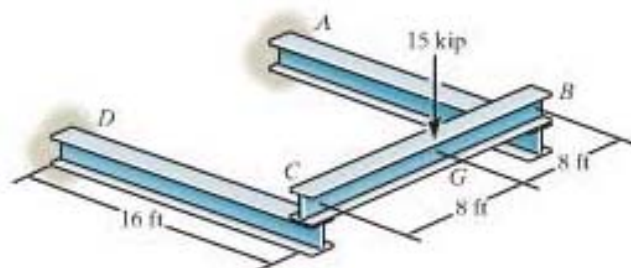
Prob. 12-98

12-99. Determine the vertical deflection and slope at the end A of the bracket. Assume that the bracket is fixed supported at its base, and neglect the axial deformation of segment AB . EI is constant.



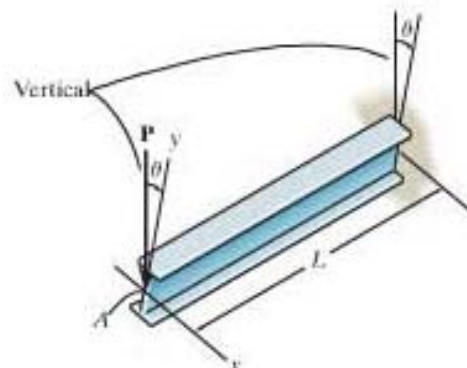
Prob. 12-99

***12-100.** The framework consists of two A-36 steel cantilevered beams CD and BA and a simply supported beam CB . If each beam is made of steel and has a moment of inertia about its principal axis of $I_x = 118 \text{ in}^4$, determine the deflection at the center G of beam CB .



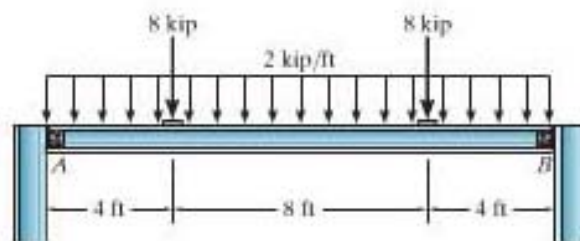
Prob. 12-100

***12-101.** The wide-flange beam acts as a cantilever. Due to an error it is installed at an angle θ with the vertical. Determine the ratio of its deflection in the x direction to its deflection in the y direction at A when a load P is applied at this point. The moments of inertia are I_x and I_y . For the solution, resolve P into components and use the method of superposition. *Note:* The result indicates that large lateral deflections (x direction) can occur in narrow beams, $I_y \ll I_x$, when they are improperly installed in this manner. To show this numerically, compute the deflections in the x and y directions for an A-36 steel $W10 \times 15$, with $P = 1.5 \text{ kip}$, $\theta = 10^\circ$, and $L = 12 \text{ ft}$.



Prob. 12-101

12-102. The simply supported beam carries a uniform load of 2 kip/ft . Code restrictions, due to a plaster ceiling, require the maximum deflection not to exceed $1/360$ of the span length. Select the lightest-weight A-36 steel wide-flange beam from Appendix B that will satisfy this requirement and safely support the load. The allowable bending stress is $\sigma_{\text{allow}} = 24 \text{ ksi}$ and the allowable shear stress is $\tau_{\text{allow}} = 14 \text{ ksi}$. Assume A is a pin and B a roller support.



Prob. 12-102

12.6 Statically Indeterminate Beams and Shafts

The analysis of statically indeterminate axially loaded bars and torsionally loaded shafts has been discussed in Secs. 4.4 and 5.5, respectively. In this section we will illustrate a general method for determining the reactions on statically indeterminate beams and shafts. Specifically, a member of any type is classified as *statically indeterminate* if the number of unknown reactions *exceeds* the available number of equilibrium equations.

The additional support reactions on the beam or shaft that are *not needed* to keep it in stable equilibrium are called *redundants*. The number of these redundants is referred to as the *degree of indeterminacy*. For example, consider the beam shown in Fig. 12-32*a*. If the free-body diagram is drawn, Fig. 12-32*b*, there will be four unknown support reactions, and since three equilibrium equations are available for solution, the beam is classified as being indeterminate to the first degree. Either A_y , B_y , or M_A can be classified as the redundant, for if any one of these reactions is removed, the beam remains stable and in equilibrium (A_x cannot be classified as the redundant, for if it were removed, $\Sigma F_x = 0$ would not be satisfied.) In a similar manner, the *continuous beam* in Fig. 12-33*a* is indeterminate to the second degree, since there are five unknown reactions and only three available equilibrium equations, Fig. 12-33*b*. Here the two redundant support reactions can be chosen among A_y , B_y , C_y , and D_y .



Fig. 12-32

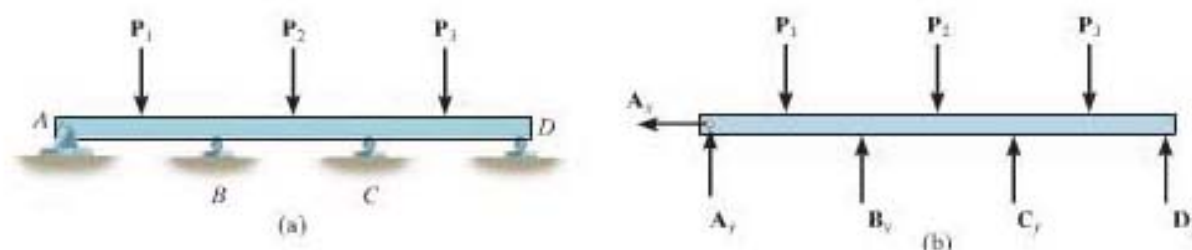


Fig. 12-33

To determine the reactions on a beam (or shaft) that is statically indeterminate, it is first necessary to specify the redundant reactions. We can determine these redundants from conditions of geometry known as *compatibility conditions*. Once determined, the redundants are then applied to the beam, and the remaining reactions are determined from the equations of equilibrium.

In the following sections we will illustrate this procedure for solution using the method of integration, Sec. 12.7; the moment-area method, Sec. 12.8; and the method of superposition, Sec. 12.9.

12.7 Statically Indeterminate Beams and Shafts—Method of Integration

The method of integration, discussed in Sec. 12.2, requires two integrations of the differential equation $d^2v/dx^2 = M/EI$ once the internal moment M in the beam is expressed as a function of position x . If the beam is statically indeterminate, however, M can also be expressed in terms of the *unknown* redundants. After integrating this equation twice, there will be two constants of integration along with the redundants to be determined. Although this is the case, these unknowns can always be found from the boundary and/or continuity conditions for the problem.

The following example problems illustrate specific applications of this method using the procedure for analysis outlined in Sec. 12.2.



An example of a statically indeterminate beam used to support a bridge deck.

EXAMPLE 12.17

The beam is subjected to the distributed loading shown in Fig. 12-34a. Determine the reaction at A. EI is constant.

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12-34a. Only one coordinate x is needed. For convenience we will take it directed to the right, since the internal moment is easy to formulate.

Moment Function. The beam is indeterminate to the first degree as indicated from the free-body diagram, Fig. 12-34b. We can express the internal moment M in terms of the redundant force at A using the segment shown in Fig. 12-34c. Here

$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

Slope and Elastic Curve. Applying Eq. 12-10, we have

$$EI \frac{d^2 v}{dx^2} = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI \frac{dv}{dx} = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EI v = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

The three unknowns A_y , C_1 , and C_2 are determined from the boundary conditions $x = 0$, $v = 0$; $x = L$, $dv/dx = 0$; and $x = L$, $v = 0$. Applying these conditions yields

$$x = 0, v = 0; \quad 0 = 0 - 0 + 0 + C_2$$

$$x = L, \frac{dv}{dx} = 0; \quad 0 = \frac{1}{2} A_y L^2 - \frac{1}{24} w_0 L^3 + C_1$$

$$x = L, v = 0; \quad 0 = \frac{1}{6} A_y L^3 - \frac{1}{120} w_0 L^4 + C_1 L + C_2$$

Solving,

$$A_y = \frac{1}{10} w_0 L$$

Ans.

$$C_1 = -\frac{1}{120} w_0 L^3 \quad C_2 = 0$$

NOTE: Using the result for A_y , the reactions at B can be determined from the equations of equilibrium, Fig. 12-34b. Show that $B_x = 0$, $B_y = 2w_0 L/5$, and $M_B = w_0 L^2/15$.

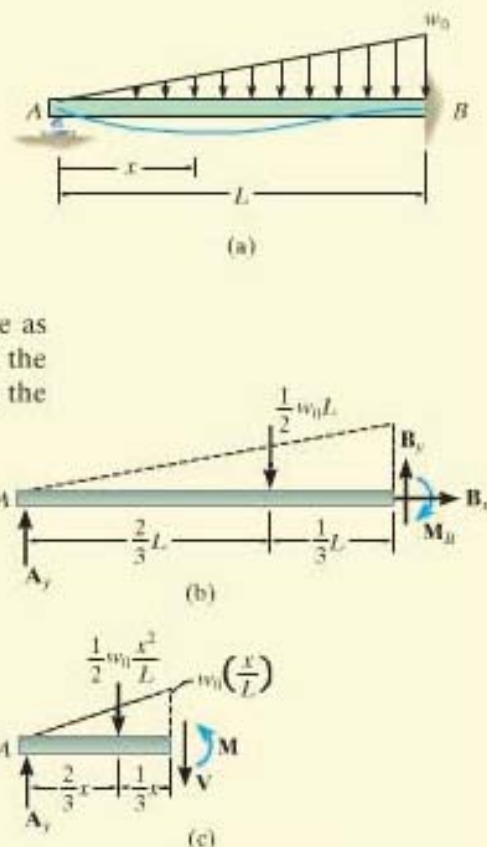


Fig. 12-34

EXAMPLE 12.18

The beam in Fig. 12-35a is fixed supported at both ends and is subjected to the uniform loading shown. Determine the reactions at the supports. Neglect the effect of axial load.

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12-35a. As in the previous problem, only one x coordinate is necessary for the solution since the loading is continuous across the span.

Moment Function. From the free-body diagram, Fig. 12-35b, the respective shear and moment reactions at A and B must be equal, since there is symmetry of both loading and geometry. Because of this, the equation of equilibrium, $\Sigma F_y = 0$, requires

$$V_A = V_B = \frac{wL}{2} \quad \text{Ans}$$

The beam is indeterminate to the first degree, where M' is redundant. Using the beam segment shown in Fig. 12-35c, the internal moment M can be expressed in terms of M' as follows:

$$M = \frac{wL}{2}x - \frac{w}{2}x^2 - M'$$

Slope and Elastic Curve. Applying Eq. 12-10, we have

$$EI \frac{d^2v}{dx^2} = \frac{wL}{2}x - \frac{w}{2}x^2 - M'$$

$$EI \frac{dv}{dx} = \frac{wL}{4}x^2 - \frac{w}{6}x^3 - M'x + C_1$$

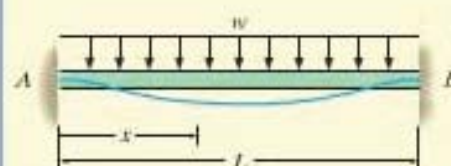
$$EIv = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{M'}{2}x^2 + C_1x + C_2$$

The three unknowns, M' , C_1 , and C_2 , can be determined from the three boundary conditions $v = 0$ at $x = 0$, which yields $C_2 = 0$; $dv/dx = 0$ at $x = 0$, which yields $C_1 = 0$; and $v = 0$ at $x = L$, which yields

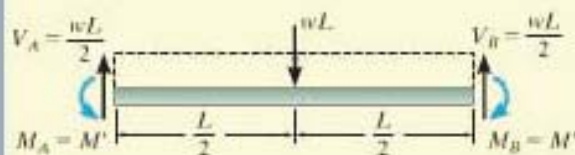
$$M' = \frac{wL^2}{12} \quad \text{Ans}$$

Using these results, notice that because of symmetry the remaining boundary condition $dv/dx = 0$ at $x = L$ is automatically satisfied.

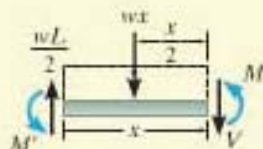
NOTE: It should be realized that this method of solution is generally suitable when only one x coordinate is needed to describe the elastic curve. If several x coordinates are needed, equations of continuity must be written, thus complicating the solution process.



(a)



(b)



(c)

Fig. 12-35

PROBLEMS

12

12-103. Determine the reactions at the supports A and B , then draw the moment diagram. EI is constant.



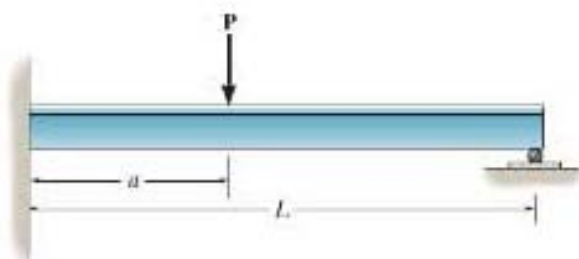
Prob. 12-103

12-106. Determine the reactions at the supports, then draw the shear and moment diagram. EI is constant.



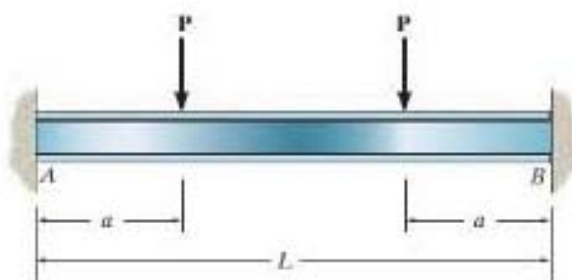
Prob. 12-106

***12-104.** Determine the value of a for which the maximum positive moment has the same magnitude as the maximum negative moment. EI is constant.



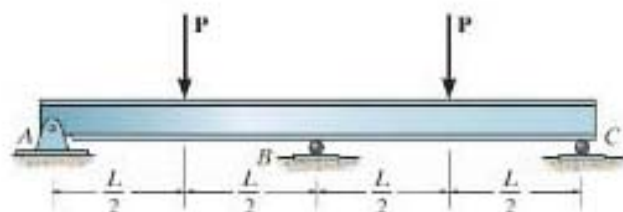
Prob. 12-104

12-107. Determine the moment reactions at the supports A and B . EI is constant.



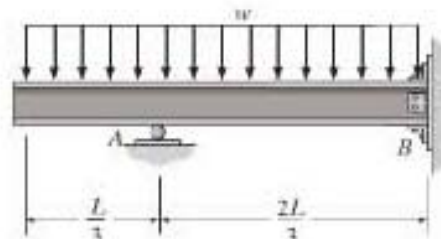
Prob. 12-107

•12-105. Determine the reactions at the supports A , B , and C ; then draw the shear and moment diagrams. EI is constant.



Prob. 12-105

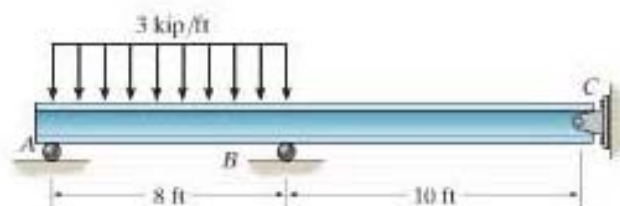
***12-108.** Determine the reactions at roller support A and fixed support B .



Prob. 12-108

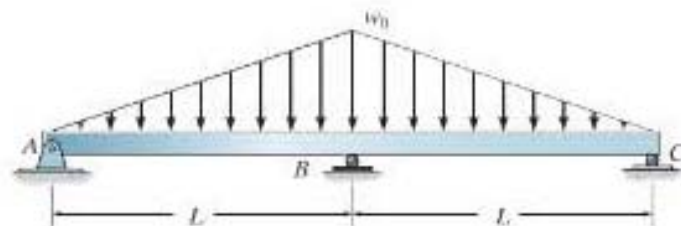
12

•12-109. Use discontinuity functions and determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



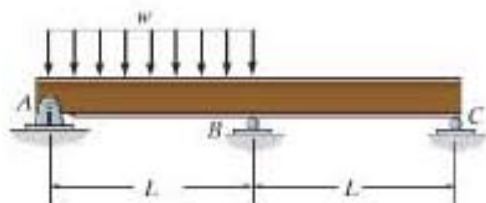
Prob. 12-109

12-110. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



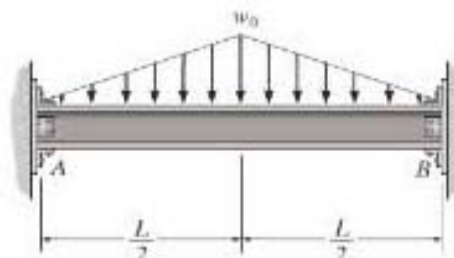
Prob. 12-110

12-111. Determine the reactions at pin support A and roller supports B and C. EI is constant.



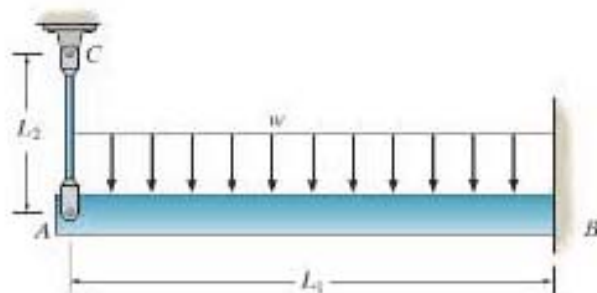
Prob. 12-111

*12-112. Determine the moment reactions at fixed supports A and B. EI is constant.



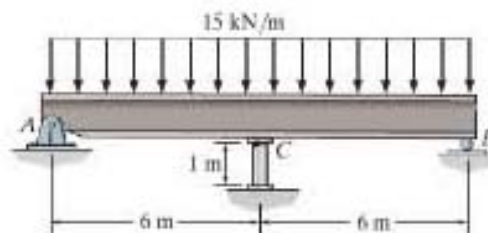
Prob. 12-112

•12-113. The beam has a constant $E_1 I_1$ and is supported by the fixed wall at B and the rod AC. If the rod has a cross-sectional area A_2 and the material has a modulus of elasticity E_2 , determine the force in the rod.



Prob. 12-113

12-114. The beam is supported by a pin at A, a roller at B, and a post having a diameter of 50 mm at C. Determine the support reactions at A, B, and C. The post and the beam are made of the same material having a modulus of elasticity $E = 200 \text{ GPa}$, and the beam has a constant moment of inertia $I = 255(10^6) \text{ mm}^4$.



Prob. 12-114

*12.8 Statically Indeterminate Beams and Shafts—Moment-Area Method

If the moment-area method is used to determine the unknown redundants of a statically indeterminate beam or shaft, then the M/EI diagram must be drawn such that the redundants are represented as unknowns on this diagram. Once the M/EI diagram is established, the two moment-area theorems can then be applied to obtain the proper relationships between the tangents on the elastic curve in order to meet the conditions of displacement and/or slope at the supports of the beam. In all cases the number of these compatibility conditions will be equivalent to the number of redundants, and so a solution for the redundants can be obtained.

Moment Diagrams Constructed by the Method of Superposition. Since application of the moment-area theorems requires calculation of both the area under the M/EI diagram and the centroidal location of this area, it is often convenient to use *separate* M/EI diagrams for *each* of the known loads and redundants rather than using the *resultant diagram* to calculate these geometric quantities. This is especially true if the resultant moment diagram has a complicated shape. The method for drawing the moment diagram in parts is based on the principle of superposition.

Most loadings on cantilevered *beams or shafts* will be a combination of the four loadings shown in Fig. 12-36. Construction of the associated moment diagrams, also shown in this figure, has been discussed in the examples of Chapter 6. Based on these results, we will now show how to use the method of superposition to represent the resultant moment diagram by a series of separate moment diagrams for the cantilevered beam shown in Fig. 12-37a. To do this, we will first replace the loads by a system of statically equivalent loads. For example, the three cantilevered beams shown in Fig. 12-37a are statically equivalent to the

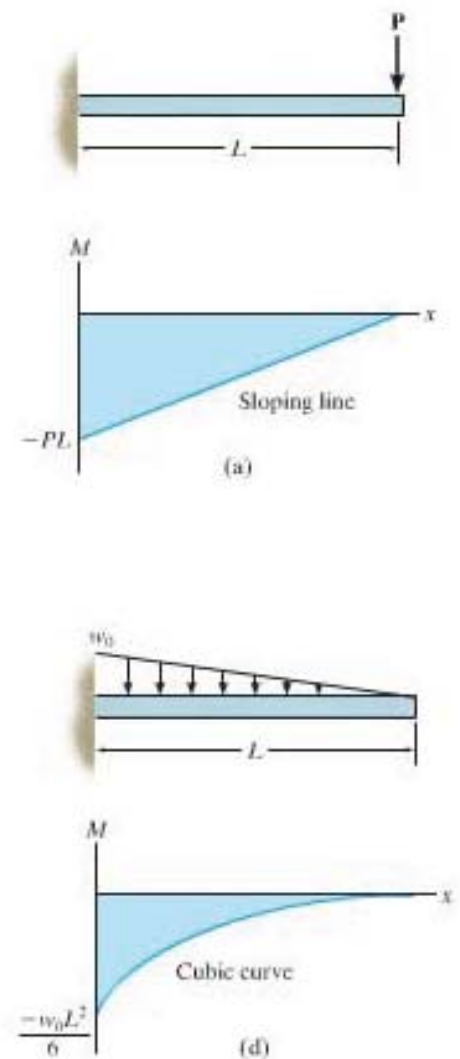


Fig. 12-36

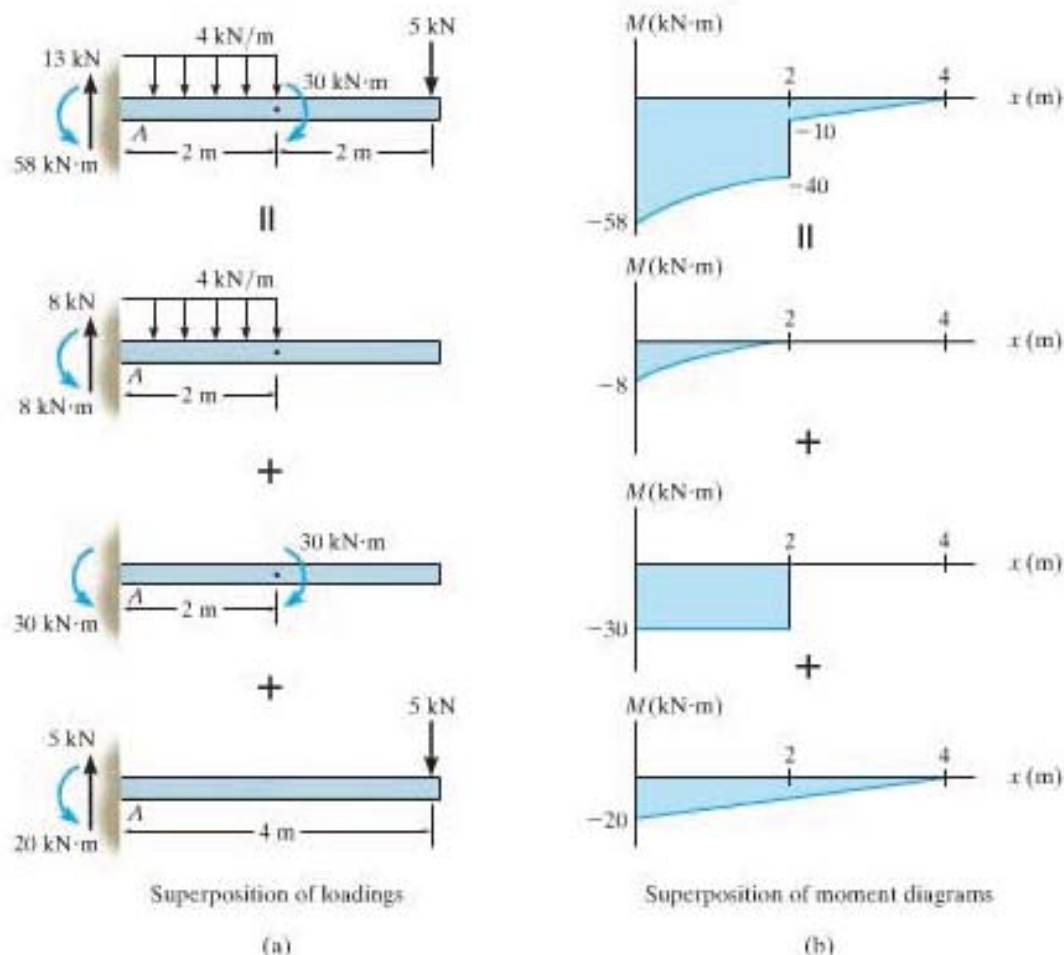


Fig. 12-37

resultant beam, since the load at each point on the resultant beam is equal to the superposition or addition of the loadings on the three separate beams. Thus, if the moment diagrams for each separate beam are drawn, Fig. 12-37b, the superposition of these diagrams will yield the moment diagram for the resultant beam, shown at the top. For example, from each of the separate moment diagrams, the moment at end A is $M_A = -8 \text{ kN} \cdot \text{m} - 30 \text{ kN} \cdot \text{m} - 20 \text{ kN} \cdot \text{m} = -58 \text{ kN} \cdot \text{m}$, as verified by the top moment diagram. This example demonstrates that it is sometimes easier to construct a series of separate statically equivalent moment diagrams for the beam, *rather* than constructing its more complicated resultant moment diagram. Obviously, the area and location of the centroid for each part are easier to establish than those of the centroid for the resultant diagram.

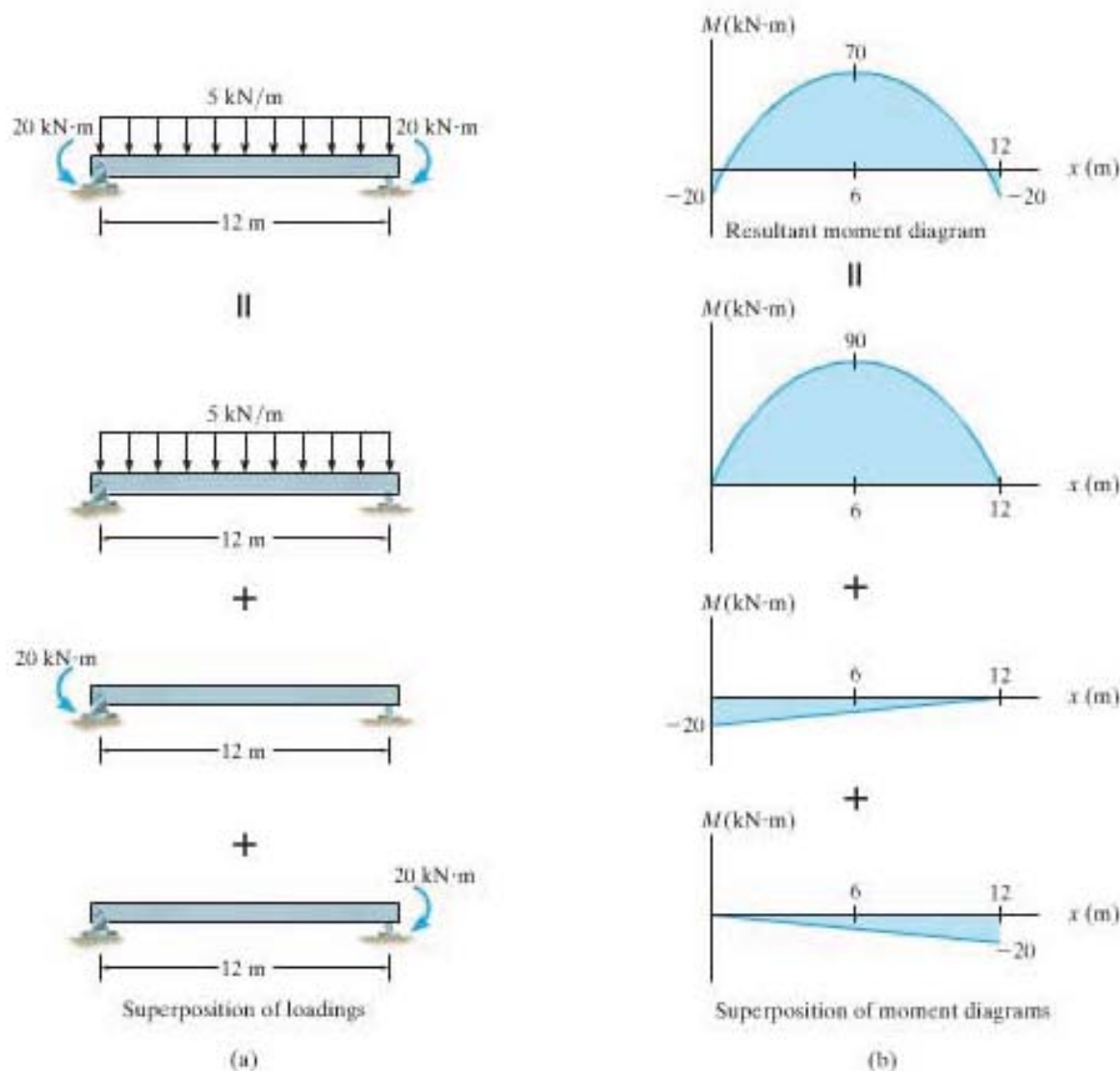


Fig. 12-38

In a similar manner, we can also represent the resultant moment diagram for a *simply supported beam* by using a superposition of moment diagrams for each loading acting on a series of simply supported beams. For example, the beam loading shown at the top of Fig. 12-38a is equivalent to the sum of the beam loadings shown below it. Consequently, the sum of the moment diagrams for each of these three loadings can be used rather than the resultant moment diagram shown at the top of Fig. 12-38b.

The examples that follow should also clarify some of these points and illustrate how to use the moment-area theorems to obtain the redundant reactions on statically indeterminate beams and shafts. The solutions follow the procedure for analysis outlined in Sec. 12.4.

EXAMPLE 12.19

The beam is subjected to the concentrated force shown in Fig. 12-39a. Determine the reactions at the supports. EI is constant.

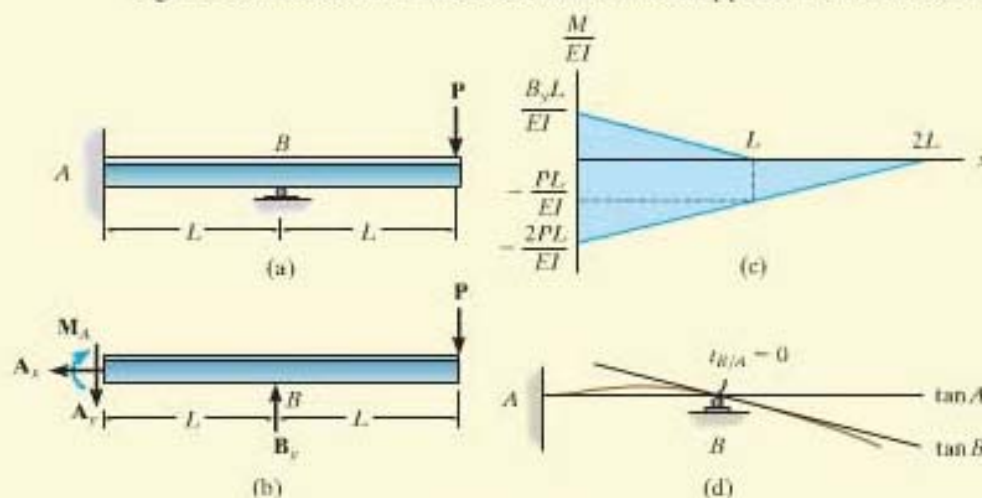


Fig. 12-39

SOLUTION

M/EI Diagram. The free-body diagram is shown in Fig. 12-39b. Using the method of superposition, the separate M/EI diagrams for the redundant reaction B_y and the load P are shown in Fig. 12-39c.

Elastic Curve. The elastic curve for the beam is shown in Fig. 12-39d. The tangents at the supports A and B have been constructed. Since $\Delta_B = 0$, then

$$t_{B/A} = 0$$

Moment-Area Theorem. Applying Theorem 2, we have

$$\begin{aligned} t_{B/A} &= \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{B_y L}{EI}\right)L\right] + \left(\frac{L}{2}\right)\left[\frac{-PL}{EI}(L)\right] \\ &\quad + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{-PL}{EI}\right)(L)\right] = 0 \\ B_y &= 2.5P \end{aligned}$$

Ans

Equations of Equilibrium. Using this result, the reactions at A on the free-body diagram, Fig. 12-39b, are

$$\rightarrow \Sigma F_x = 0; \quad A_x = 0 \quad \text{Ans}$$

$$\begin{aligned} + \uparrow \Sigma F_y = 0; \quad -A_y + 2.5P - P &= 0 \\ A_y &= 1.5P \end{aligned} \quad \text{Ans}$$

$$\begin{aligned} \curvearrowright \Sigma M_A = 0; \quad -M_A + 2.5P(L) - P(2L) &= 0 \\ M_A &= 0.5PL \end{aligned} \quad \text{Ans}$$

EXAMPLE 12.20

The beam is subjected to the couple moment at its end C as shown in Fig. 12-40a. Determine the reaction at B . EI is constant.

SOLUTION

M/EI Diagram. The free-body diagram is shown in Fig. 12-40b. By inspection, the beam is indeterminate to the first degree. In order to obtain a direct solution, we will choose B_y as the redundant. Using superposition, the M/EI diagrams for B_y and M_0 , each applied to a simply supported beam, are shown in Fig. 12-40c. (Note that for such a beam A_x , A_y , and C_y do not contribute to an M/EI diagram.)

Elastic Curve. The elastic curve for the beam is shown in Fig. 12-40d. The tangents at A , B , and C have been established. Since $\Delta_A = \Delta_B = \Delta_C = 0$, then the vertical distances shown must be proportional; i.e.,

$$t_{B/C} = \frac{1}{2} t_{A/C} \quad (1)$$

From Fig. 12-40c, we have

$$\begin{aligned} t_{B/C} &= \left(\frac{1}{3}L\right) \left[\frac{1}{2} \left(\frac{B_y L}{2EI} \right) (L) \right] + \left(\frac{2}{3}L\right) \left[\frac{1}{2} \left(\frac{-M_0}{2EI} \right) (L) \right] \\ &\quad + \left(\frac{L}{2}\right) \left[\left(\frac{-M_0}{2EI} \right) (L) \right] \\ t_{A/C} &= (L) \left[\frac{1}{2} \left(\frac{B_y L}{2EI} \right) (2L) \right] + \left(\frac{2}{3}(2L)\right) \left[\frac{1}{2} \left(\frac{-M_0}{EI} \right) (2L) \right] \end{aligned}$$

Substituting into Eq. 1 and simplifying yields

$$B_y = \frac{3M_0}{2L} \quad \text{Ans.}$$

Equations of Equilibrium. The reactions at A and C can now be determined from the equations of equilibrium, Fig. 12-40b. Show that $A_x = 0$, $C_y = 5M_0/4L$, and $A_y = M_0/4L$.

Note from Fig. 12-40e that this problem can also be worked in terms of the vertical distances,

$$t_{B/A} = \frac{1}{2} t_{C/A}$$

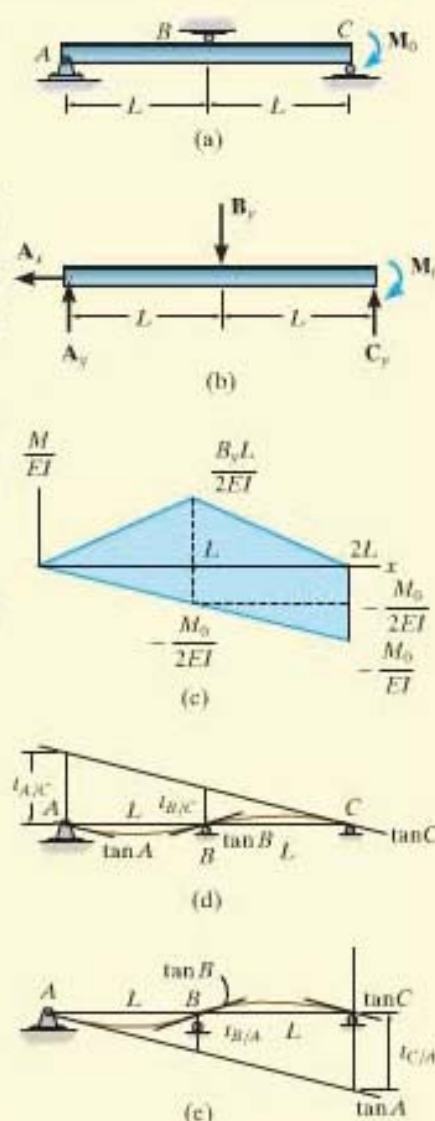
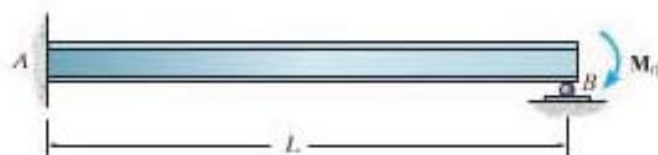


Fig. 12-40

PROBLEMS

12-115. Determine the moment reactions at the supports A and B , then draw the shear and moment diagrams. EI is constant.



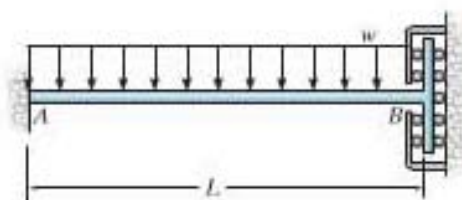
Prob. 12-115

12-118. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant.



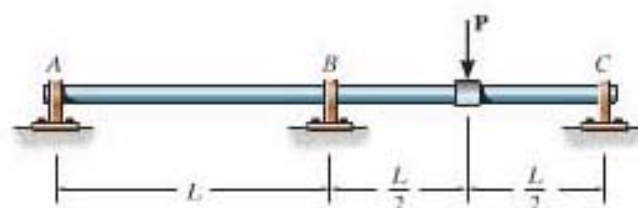
Prob. 12-118

***12-116.** The rod is fixed at A , and the connection at B consists of a roller constraint which allows vertical displacement but resists axial load and moment. Determine the moment reactions at these supports. EI is constant.



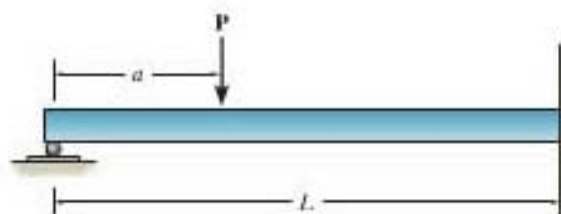
Prob. 12-116

12-119. Determine the reactions at the supports, then draw the shear and moment diagrams. EI is constant. Support B is a thrust bearing.



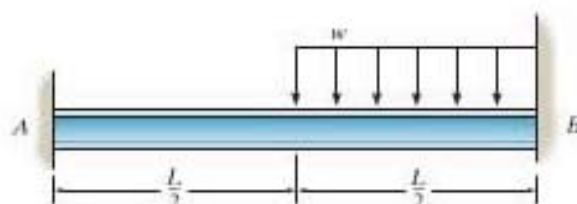
Prob. 12-119

***12-117.** Determine the value of a for which the maximum positive moment has the same magnitude as the maximum negative moment. EI is constant.



Prob. 12-117

***12-120.** Determine the moment reactions at the supports A and B . EI is constant.



Prob. 12-120

12.9 Statically Indeterminate Beams and Shafts—Method of Superposition

12

The method of superposition has been used previously to solve for the redundant loading on axially loaded bars and torsionally loaded shafts. In order to apply this method to the solution of statically indeterminate beams (or shafts), it is first necessary to identify the redundant support reactions as explained in Sec. 12.6. By removing them from the beam we obtain the so-called *primary beam*, which is statically determinate and stable, and is subjected *only* to the external load. If we add to this beam a succession of similarly supported beams, each loaded with a *separate* redundant, then by the principle of superposition, we obtain the actual loaded beam. Finally, in order to solve for the redundants, we must write the *conditions of compatibility* that exist at the supports where each of the redundants acts. Since the redundant forces are determined directly in this manner, this method of analysis is sometimes called the *force method*. Once the redundants are obtained, the other reactions on the beam can then be determined from the three equations of equilibrium.

To clarify these concepts, consider the beam shown in Fig. 12-41*a*. If we choose the reaction $\mathbf{B}_y$ at the roller as the redundant, then the primary beam is shown in Fig. 12-41*b*, and the beam with the redundant $\mathbf{B}_y$ acting on it is shown in Fig. 12-41*c*. The displacement at the roller is to be zero, and since the displacement of point B on the primary beam is v_B , and $\mathbf{B}_y$ causes point B to be displaced upward v'_B , we can write the compatibility equation at B as

$$(+\uparrow) \quad 0 = -v_B + v'_B$$

The displacements v_B and v'_B can be obtained using any one of the methods discussed in Secs. 12.2 through 12.5. Here we will obtain them directly from the table in Appendix C. We have

$$v_B = \frac{5PL^3}{48EI} \quad \text{and} \quad v'_B = \frac{B_y L^3}{3EI}$$

Substituting into the compatibility equation, we get

$$0 = -\frac{5PL^3}{48EI} + \frac{B_y L^3}{3EI}$$

$$B_y = \frac{5}{16}P$$

Now that $\mathbf{B}_y$ is known, the reactions at the wall are determined from the three equations of equilibrium applied to the free-body diagram of the beam, Fig. 12-41*d*. The results are

$$A_x = 0 \quad A_y = \frac{11}{16}P$$

$$M_A = \frac{3}{16}PL$$

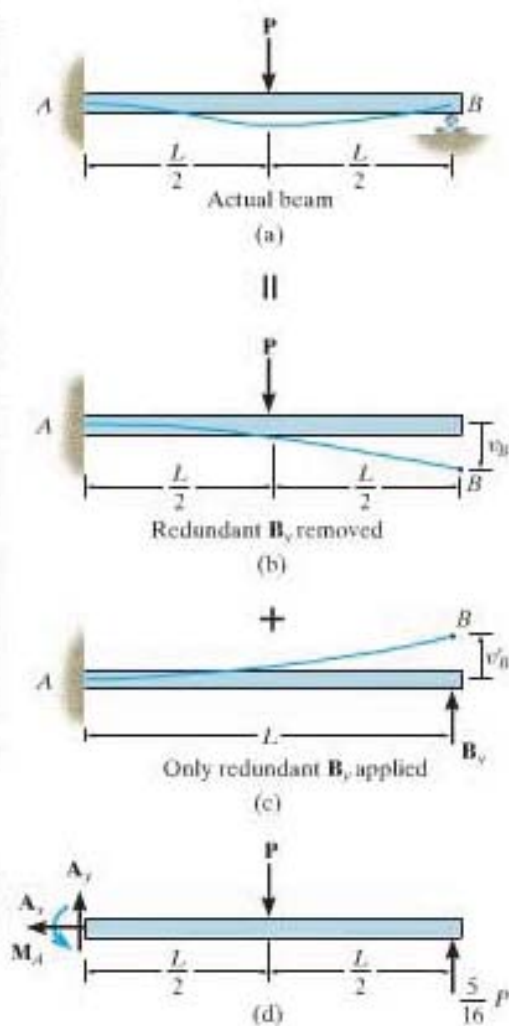


Fig. 12-41

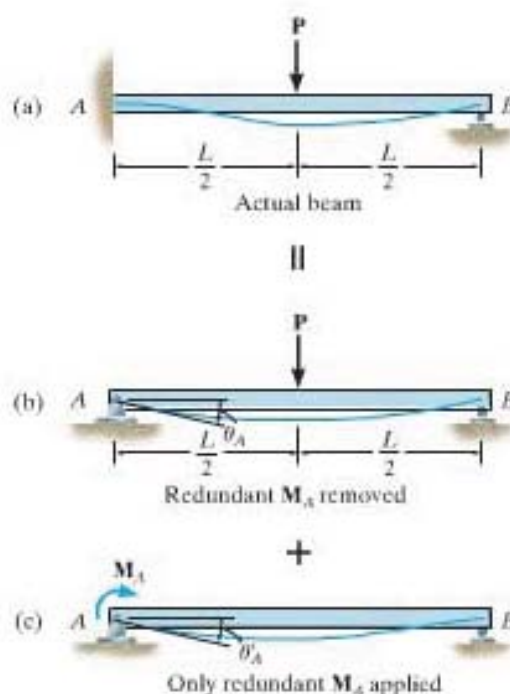


Fig. 12-42

As stated in Sec. 12.6, choice of the redundant is *arbitrary*, provided the primary beam remains stable. For example, the moment at A for the beam in Fig. 12-42a can also be chosen as the redundant. In this case the capacity of the beam to resist M_A is removed, and so the primary beam is then pin supported at A , Fig. 12-42b. To it we add the beam with the redundant at A acting on it, Fig. 12-42c. Referring to the slope at A caused by the load P as θ_A , and the slope at A caused by the redundant M_A as θ'_A , the compatibility equation for the slope at A requires

$$(\uparrow+) \quad 0 = \theta_A + \theta'_A$$

Again using the table in Appendix C, we have

$$\theta_A = \frac{PL^2}{16EI} \quad \text{and} \quad \theta'_A = \frac{M_AL}{3EI}$$

Thus,

$$0 = \frac{PL^2}{16EI} + \frac{M_AL}{3EI}$$

$$M_A = -\frac{3}{16}PL$$

This is the same result determined previously. Here the negative sign for M_A simply means that M_A acts in the opposite sense of direction of that shown in Fig. 12-42c.

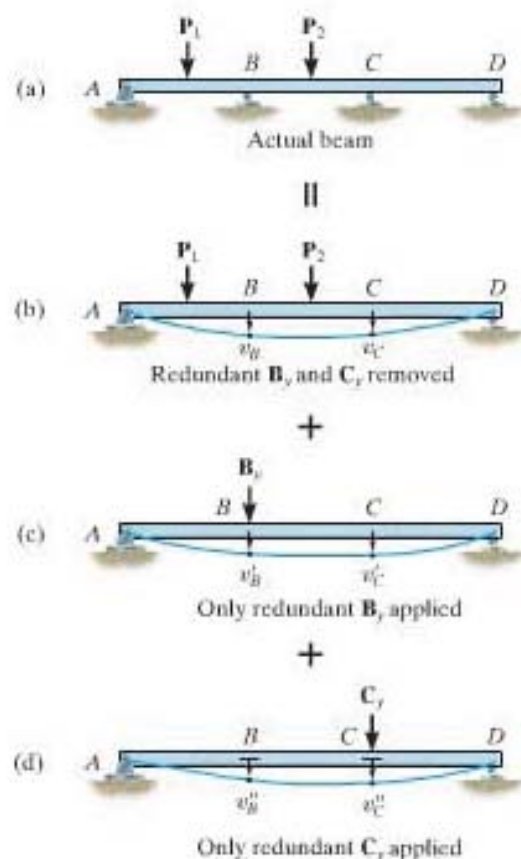


Fig. 12-43

Another example that illustrates this method is given in Fig. 12-43a. In this case the beam is indeterminate to the second degree and therefore *two* compatibility equations will be necessary for the solution. We will choose the forces at the roller supports B and C as redundants. The primary (statically determinate) beam deforms as shown in Fig. 12-43b when the redundants are removed. Each redundant force deforms this beam as shown in Figs. 12-43c and 12-43d, respectively. By superposition, the compatibility equations for the displacements at B and C are

$$\begin{aligned}
 (+\downarrow) \quad 0 &= v_B + v_B' + v_B'' \\
 (+\downarrow) \quad 0 &= v_C + v_C' + v_C''
 \end{aligned}
 \tag{12-20}$$

Here the displacement components v_B' and v_C' will be expressed in terms of the unknown B_y , and the components v_B'' and v_C'' will be expressed in terms of the unknown C_y . When these displacements have been determined and substituted into Eq. 12-20, these equations may then be solved simultaneously for the two unknowns B_y and C_y .

Procedure for Analysis

The following procedure provides a means for applying the method of superposition (or the force method) to determine the reactions on statically indeterminate beams or shafts.

Elastic Curve.

- Specify the unknown redundant forces or moments that must be removed from the beam in order to make it statically determinate and stable.
- Using the principle of superposition, draw the statically indeterminate beam and show it equal to a sequence of corresponding *statically determinate beams*.
- The first of these beams, the primary beam, supports the same external loads as the statically indeterminate beam, and each of the other beams “added” to the primary beam shows the beam loaded with a separate redundant force or moment.
- Sketch the deflection curve for each beam and indicate symbolically the displacement (slope) at the point of each redundant force (moment).

Compatibility Equations.

- Write a compatibility equation for the displacement (slope) at each point where there is a redundant force (moment).
- Determine all the displacements or slopes using an appropriate method as explained in Secs. 12.2 through 12.5.
- Substitute the results into the compatibility equations and solve for the unknown redundants.
- If a numerical value for a redundant is *positive*, it has the *same sense of direction* as originally assumed. Similarly, a *negative* numerical value indicates the redundant acts *opposite* to its assumed *sense of direction*.

Equilibrium Equations.

- Once the redundant forces and/or moments have been determined, the remaining unknown reactions can be found from the equations of equilibrium applied to the loadings shown on the beam’s free-body diagram.

The following examples illustrate application of this procedure. For brevity, all displacements and slopes have been found using the table in Appendix C.

EXAMPLE 12.21

Determine the reactions at the roller support B of the beam shown in Fig. 12-44a, then draw the shear and moment diagrams. EI is constant.

SOLUTION

Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The roller support at B will be chosen as the redundant so that B_y will be determined *directly*. Figures 12-44b and 12-44c show application of the principle of superposition. Here we have assumed that B_y acts upward on the beam.

Compatibility Equation. Taking positive displacement as downward, the compatibility equation at B is

$$(+\downarrow) \quad 0 = v_B - v'_B \quad (1)$$

These displacements can be obtained directly from the table in Appendix C.

$$\begin{aligned} v_B &= \frac{wL^4}{8EI} + \frac{5PL^3}{48EI} \\ &= \frac{2 \text{ kip/ft}(10 \text{ ft})^4}{8EI} + \frac{5(8 \text{ kip})(10 \text{ ft})^3}{48EI} = \frac{3333 \text{ kip} \cdot \text{ft}^3}{EI} \downarrow \\ v'_B &= \frac{PL^3}{3EI} = \frac{B_y(10 \text{ ft})^3}{3EI} = \frac{333.3 \text{ ft}^3 B_y}{EI} \uparrow \end{aligned}$$

Substituting into Eq. 1 and solving yields

$$\begin{aligned} 0 &= \frac{3333}{EI} - \frac{333.3 B_y}{EI} \\ B_y &= 10 \text{ kip} \end{aligned} \quad \text{Ans.}$$

Equilibrium Equations. Using this result and applying the three equations of equilibrium, we obtain the results shown on the beam's free-body diagram in Fig. 12-44d. The shear and moment diagrams are shown in Fig. 12-44e.

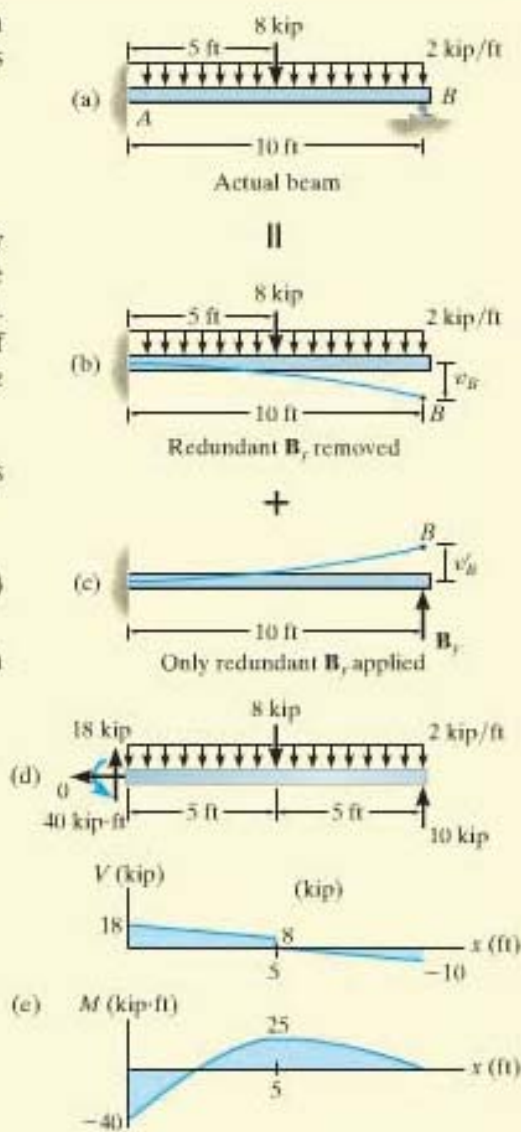


Fig. 12-44

EXAMPLE 12.22

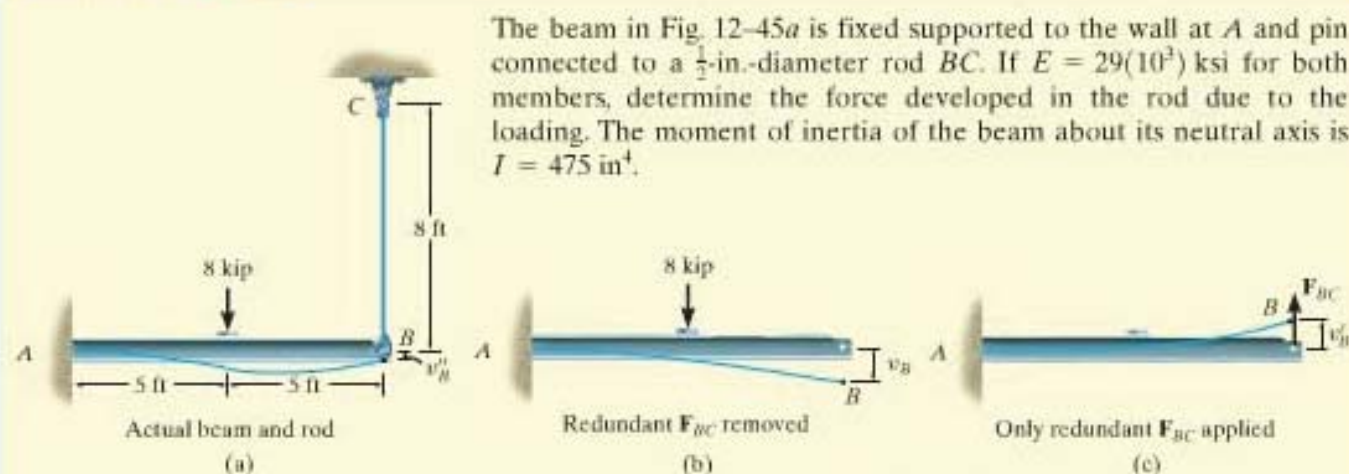


Fig. 12-45

SOLUTION I

Principle of Superposition. By inspection, this problem is indeterminate to the first degree. Here B will undergo an unknown displacement v_B , since the rod will stretch. The rod will be treated as the redundant and hence the force of the rod is removed from the beam at B , Fig. 12-45*b*, and then reapplied, Fig. 12-45*c*.

Compatibility Equation. At point B we require

$$(+\downarrow) \quad v_B'' = v_B - v_B' \quad (1)$$

The displacements v_B and v_B' are determined from the table in Appendix C. v_B'' is calculated from Eq. 4-2. Working in kilopounds and inches, we have

$$v_B'' = \frac{PL}{AE} = \frac{F_{BC}(8 \text{ ft})(12 \text{ in./ft})}{(\pi/4)(\frac{1}{2} \text{ in.})^2 [29(10^3) \text{ kip/in}^2]} = 0.01686 F_{BC} \downarrow$$

$$v_B = \frac{5PL^3}{48EI} = \frac{5(8 \text{ kip})(10 \text{ ft})^3 (12 \text{ in./ft})^3}{48[29(10^3) \text{ kip/in}^2](475 \text{ in}^4)} = 0.1045 \text{ in.} \downarrow$$

$$v_B' = \frac{PL^3}{3EI} = \frac{F_{BC}(10 \text{ ft})^3 (12 \text{ in./ft})^3}{3[29(10^3) \text{ kip/in}^2](475 \text{ in}^4)} = 0.04181 F_{BC} \uparrow$$

Thus, Eq. 1 becomes

$$(+\downarrow) \quad 0.01686 F_{BC} = 0.1045 - 0.04181 F_{BC}$$

$$F_{BC} = 1.78 \text{ kip}$$

Ans.

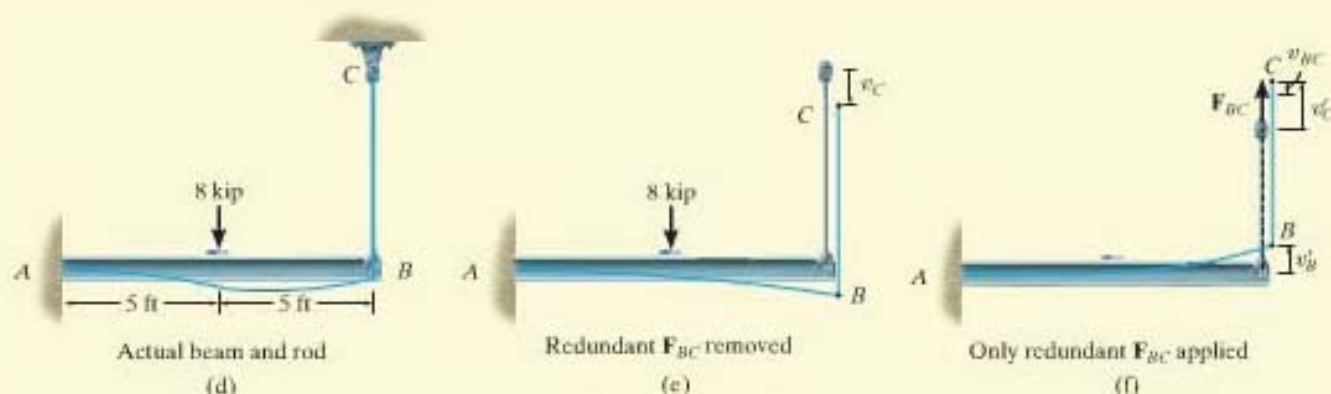


Fig. 12-45 (cont.)

SOLUTION II

Principle of Superposition. We can also solve this problem by removing the pin support at C and keeping the rod attached to the beam. In this case the 8-kip load will cause points B and C to be displaced downward the *same amount* v_C , Fig. 12-45e, since no force exists in rod BC . When the redundant force F_{BC} is applied at point C , it causes the end C of the rod to be displaced upward v'_C and the end B of the beam to be displaced upward v'_B , Fig. 12-45f. The difference in these two displacements, v_{BC} , represents the stretch of the rod due to F_{BC} , so that $v'_C = v_{BC} + v'_B$. Hence, from Figs. 12-45d, 12-45e, and 12-45f, the compatibility of displacement at point C is

$$(+\downarrow) \quad 0 = v_C - (v_{BC} + v'_B) \quad (2)$$

From Solution I, we have

$$v_C = v_B = 0.1045 \text{ in. } \downarrow$$

$$v_{BC} = v'_B = 0.01686 F_{BC} \uparrow$$

$$v'_B = 0.04181 F_{BC} \uparrow$$

Therefore, Eq. 2 becomes

$$(+\downarrow) \quad 0 = 0.1045 - (0.01686 F_{BC} + 0.04181 F_{BC})$$

$$F_{BC} = 1.78 \text{ kip}$$

Ans.

EXAMPLE 12.23

Determine the moment at B for the beam shown in Fig. 12-46a. EI is constant. Neglect the effects of axial load.

SOLUTION

Principle of Superposition. Since the axial load on the beam is neglected, there will be a vertical force and moment at A and B . Here there are only two available equations of equilibrium ($\Sigma M = 0$, $\Sigma F_y = 0$) and so the problem is indeterminate to the second degree. We will assume that $\mathbf{B}_y$ and $\mathbf{M}_B$ are redundant, so that by the principle of superposition, the beam is represented as a cantilever, loaded *separately* by the distributed load and reactions $\mathbf{B}_y$ and $\mathbf{M}_B$, Figs. 12-46b, 12-46c, and 12-46d.

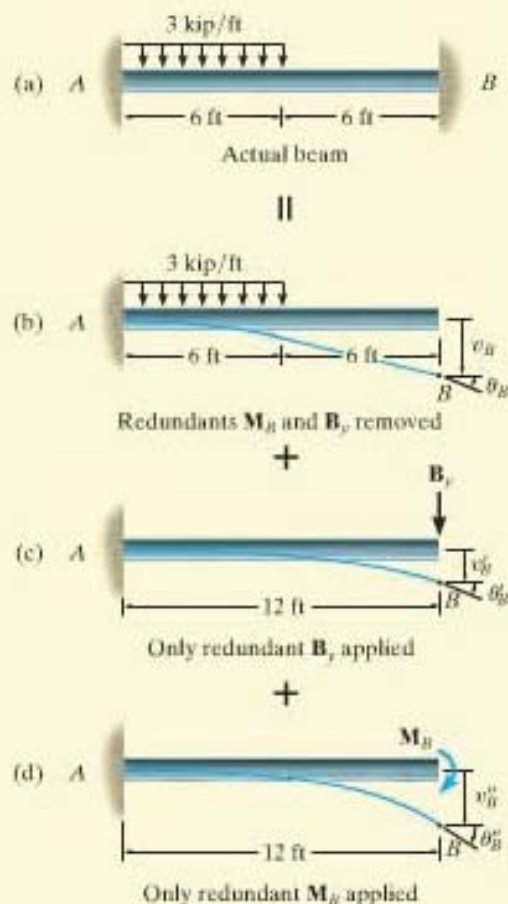


Fig. 12-46

Compatibility Equations. Referring to the displacement and slope at B , we require

$$(\uparrow+) \quad 0 = \theta_B + \theta'_B + \theta''_B \quad (1)$$

$$(+\downarrow) \quad 0 = v_B + v'_B + v''_B \quad (2)$$

Using the table in Appendix C to calculate the slopes and displacements, we have

$$\theta_B = \frac{wL^3}{48EI} = \frac{3 \text{ kip/ft} (12 \text{ ft})^3}{48EI} = \frac{108 \text{ kip} \cdot \text{ft}^2}{EI} \downarrow$$

$$v_B = \frac{7wL^4}{384EI} = \frac{7(3 \text{ kip/ft})(12 \text{ ft})^4}{384EI} = \frac{1134 \text{ kip} \cdot \text{ft}^3}{EI} \downarrow$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{B_y(12 \text{ ft})^2}{2EI} = \frac{72B_y}{EI} \downarrow$$

$$v'_B = \frac{PL^3}{3EI} = \frac{B_y(12 \text{ ft})^3}{3EI} = \frac{576B_y}{EI} \downarrow$$

$$\theta''_B = \frac{ML}{EI} = \frac{M_B(12 \text{ ft})}{EI} = \frac{12M_B}{EI} \downarrow$$

$$v''_B = \frac{ML^2}{2EI} = \frac{M_B(12 \text{ ft})^2}{2EI} = \frac{72M_B}{EI} \downarrow$$

Substituting these values into Eqs. 1 and 2 and canceling out the common factor EI , we get

$$(\uparrow+) \quad 0 = 108 + 72B_y + 12M_B$$

$$(+\downarrow) \quad 0 = 1134 + 576B_y + 72M_B$$

Solving these equations simultaneously gives

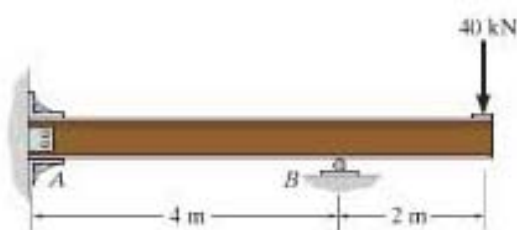
$$B_y = -3.375 \text{ kip}$$

$$M_B = 11.25 \text{ kip} \cdot \text{ft}$$

Ans.

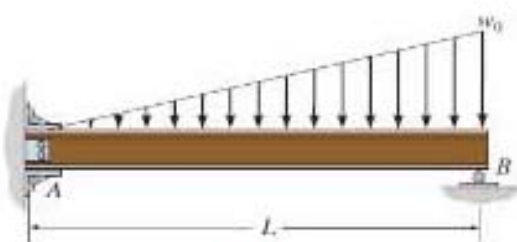
FUNDAMENTAL PROBLEMS

F12-13. Determine the reactions at the fixed support A and the roller B . EI is constant.



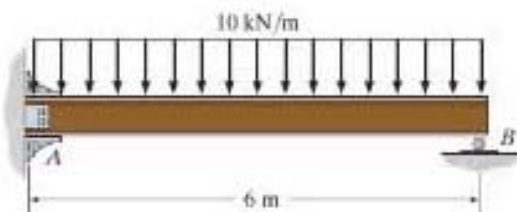
F12-13

F12-14. Determine the reactions at the fixed support A and the roller B . EI is constant.



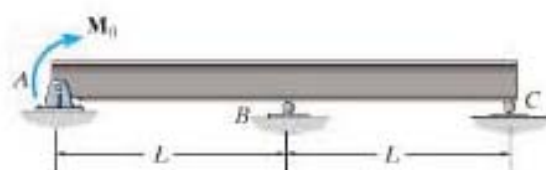
F12-14

F12-15. Determine the reactions at the fixed support A and the roller B . Support B settles 2 mm, $E = 200$ GPa, $I = 65.0(10^{-6})$ m⁴.



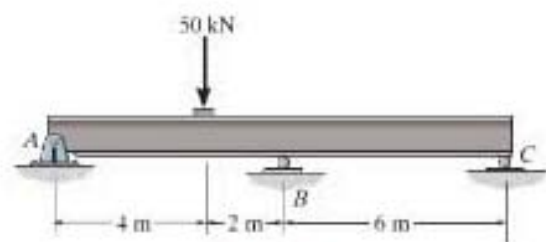
F12-15

F12-16. Determine the reaction at the roller B . EI is constant.



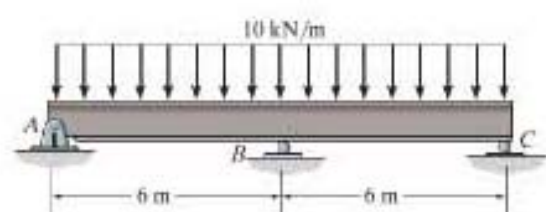
F12-16

F12-17. Determine the reaction at the roller B . EI is constant.



F12-17

F12-18. Determine the reaction at the roller support B if it settles 5 mm, $E = 200$ GPa and $I = 65.0(10^{-6})$ m⁴.

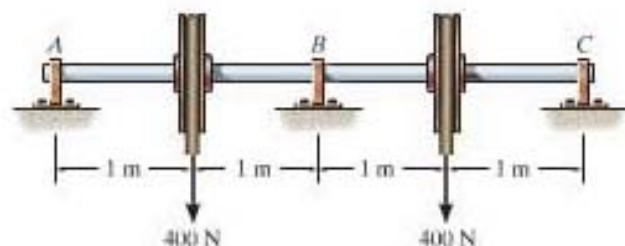


F12-18

PROBLEMS

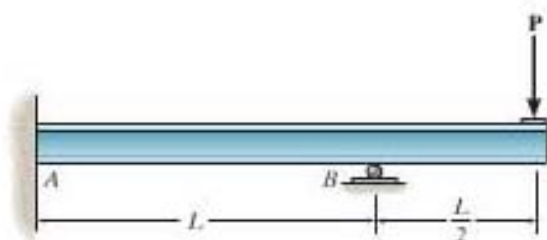
12

•12–121. Determine the reactions at the bearing supports A , B , and C of the shaft, then draw the shear and moment diagrams. EI is constant. Each bearing exerts only vertical reactions on the shaft.



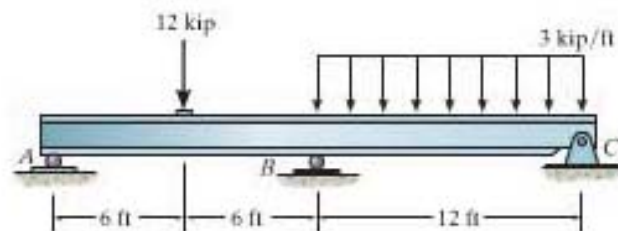
Prob. 12–121

12–122. Determine the reactions at the supports A and B . EI is constant.



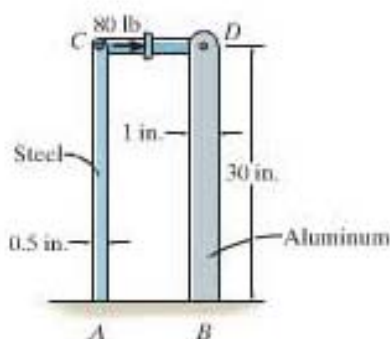
Prob. 12–122

12–123. Determine the reactions at the supports A , B , and C , then draw the shear and moment diagrams. EI is constant.



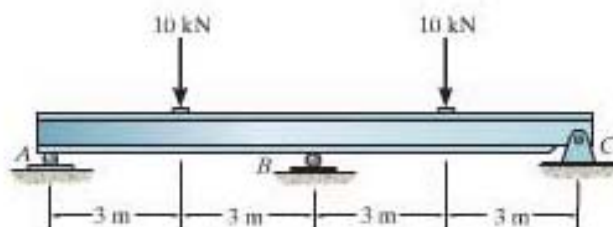
Prob. 12–123

•12–124. The assembly consists of a steel and an aluminum bar, each of which is 1 in. thick, fixed at its ends A and B , and pin connected to the rigid short link CD . If a horizontal force of 80 lb is applied to the link as shown, determine the moments created at A and B . $E_{st} = 29(10^3)$ ksi, $E_{al} = 10(10^3)$ ksi.



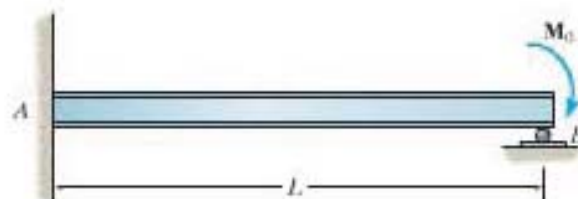
Prob. 12–124

•12–125. Determine the reactions at the supports A , B , and C , then draw the shear and moment diagrams. EI is constant.



Prob. 12–125

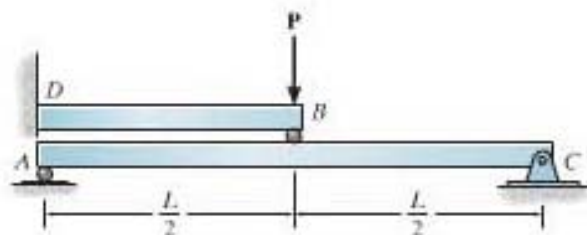
12–126. Determine the reactions at the supports A and B . EI is constant.



Prob. 12–126

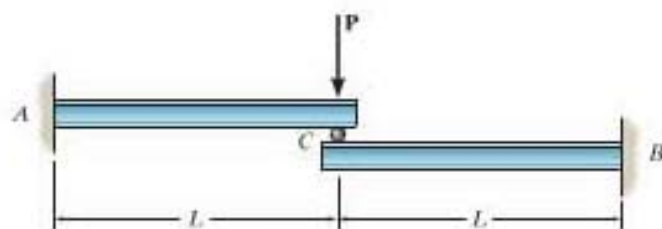
12

12-127. Determine the reactions at support C . EI is constant for both beams.



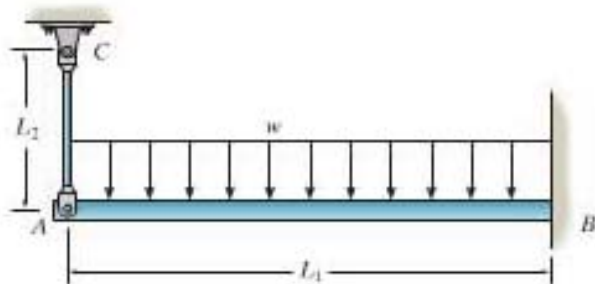
Prob. 12-127

***12-128.** The compound beam segments meet in the center using a smooth contact (roller). Determine the reactions at the fixed supports A and B when the load P is applied. EI is constant.



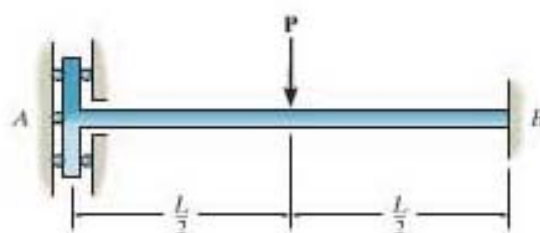
Prob. 12-128

***12-129.** The beam has a constant $E_1 I_1$ and is supported by the fixed wall at B and the rod AC . If the rod has a cross-sectional area A_2 and the material has a modulus of elasticity E_2 , determine the force in the rod.



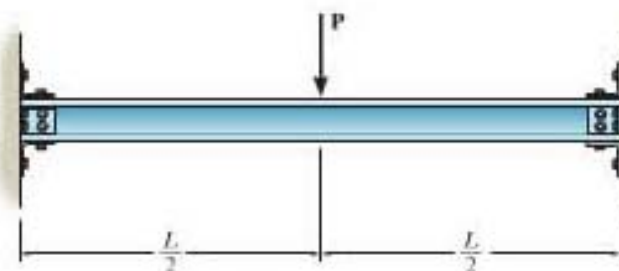
Prob. 12-129

12-130. Determine the reactions at A and B . Assume the support at A only exerts a moment on the beam. EI is constant.



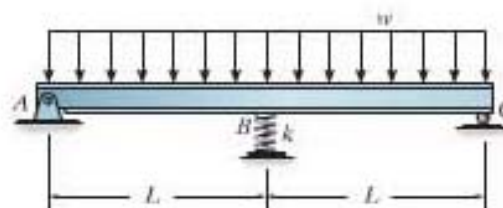
Prob. 12-130

12-131. The beam is supported by the bolted supports at its ends. When loaded these supports do not provide an actual fixed connection, but instead allow a slight rotation α before becoming fixed. Determine the moment at the connections and the maximum deflection of the beam.



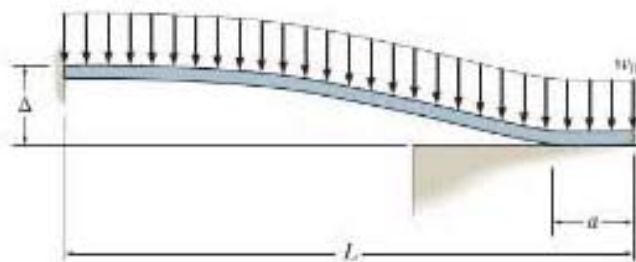
Prob. 12-131

***12-132.** The beam is supported by a pin at A , a spring having a stiffness k at B , and a roller at C . Determine the force the spring exerts on the beam. EI is constant.



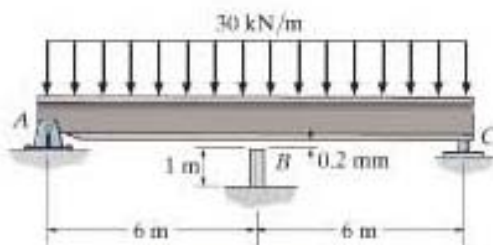
Prob. 12-132

•12–133. The beam is made from a soft linear elastic material having a constant EI . If it is originally a distance Δ from the surface of its end support, determine the distance a at which it rests on this support when it is subjected to the uniform load w_0 , which is great enough to cause this to happen.



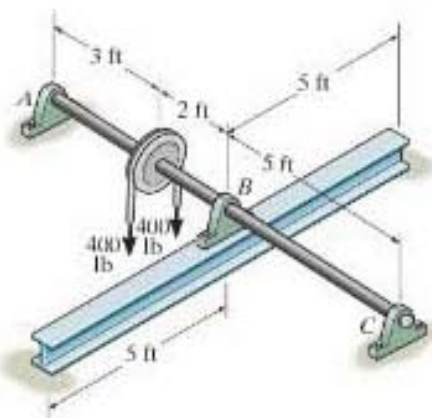
Prob. 12–133

12–134. Before the uniform distributed load is applied on the beam, there is a small gap of 0.2 mm between the beam and the post at B . Determine the support reactions at A , B , and C . The post at B has a diameter of 40 mm, and the moment of inertia of the beam is $I = 875(10^6) \text{ mm}^4$. The post and the beam are made of material having a modulus of elasticity of $E = 200 \text{ GPa}$.



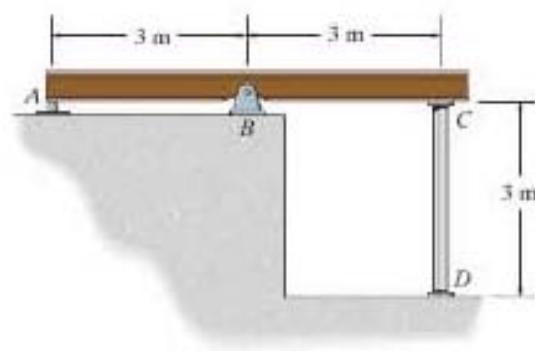
Prob. 12–134

12–135. The 1-in.-diameter A-36 steel shaft is supported by unyielding bearings at A and C . The bearing at B rests on a simply supported steel wide-flange beam having a moment of inertia of $I = 500 \text{ in}^4$. If the belt loads on the pulley are 400 lb each, determine the vertical reactions at A , B , and C .



Prob. 12–135

*12–136. If the temperature of the 75-mm-diameter post CD is increased by 60°C , determine the force developed in the post. The post and the beam are made of A-36 steel, and the moment of inertia of the beam is $I = 255(10^6) \text{ mm}^4$.

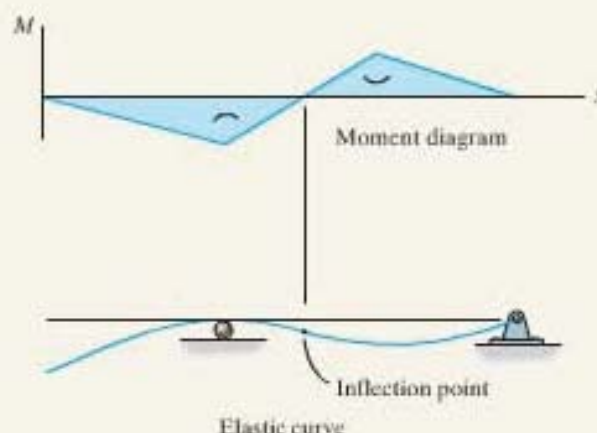


Prob. 12–136

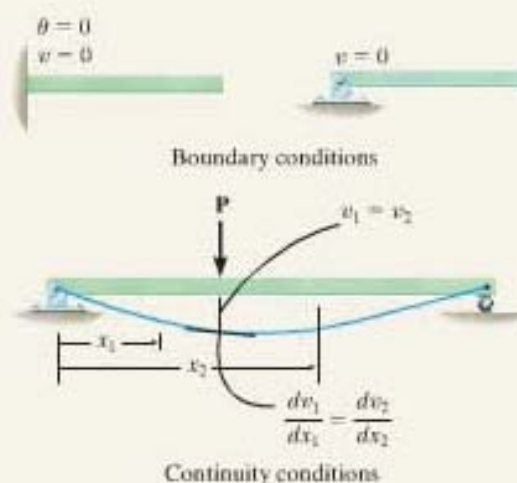
CHAPTER REVIEW

The elastic curve represents the centerline deflection of a beam or shaft. Its shape can be determined using the moment diagram. Positive moments cause the elastic curve to be concave upwards and negative moments cause it to be concave downwards. The radius of curvature at any point is determined from

$$\frac{1}{\rho} = \frac{M}{EI}$$

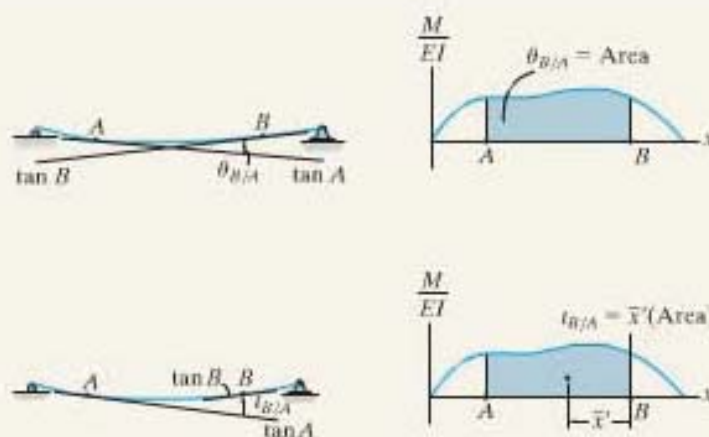


The equation of the elastic curve and its slope can be obtained by first finding the internal moment in the member as a function of x . If several loadings act on the member, then separate moment functions must be determined between each of the loadings. Integrating these functions once using $EI(d^2v/dx^2) = M(x)$ gives the equation for the slope of the elastic curve, and integrating again gives the equation for the deflection. The constants of integration are determined from the boundary conditions at the supports, or in cases where several moment functions are involved, continuity of slope and deflection at points where these functions join must be satisfied.



Discontinuity functions allow one to express the equation of the elastic curve as a continuous function, regardless of the number of loadings on the member. This method eliminates the need to use continuity conditions, since the two constants of integration can be determined solely from the two boundary conditions.

The moment-area method is a semi-graphical technique for finding the slope of tangents or the vertical distance between tangents at specific points on the elastic curve. It requires finding area segments under the M/EI diagram, or the moment of these segments about points on the elastic curve. The method works well for M/EI diagrams composed of simple shapes, such as those produced by concentrated forces and couple moments.

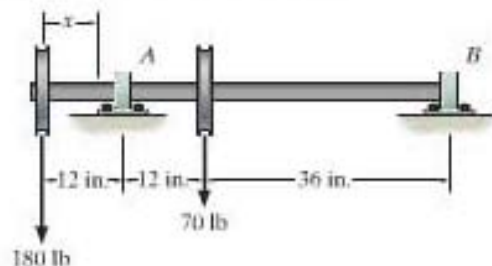


The deflection or slope at a point on a member subjected to combinations of loadings can be determined using the method of superposition. The table in Appendix C is available for this purpose.

Statically indeterminate beams and shafts have more unknown support reactions than available equations of equilibrium. To solve, one first identifies the redundant reactions. The method of integration or the moment-area theorems can then be used to solve for the unknown redundants. It is also possible to determine the redundants by using the method of superposition, where one considers the conditions of continuity at the redundant. Here the displacement due to the external loading is determined with the redundant removed, and again with the redundant applied and the external loading removed. The tables in Appendix C can be used to determine these necessary displacements.

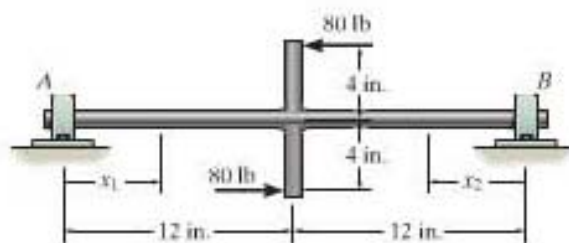
REVIEW PROBLEMS

•12-137. The shaft supports the two pulley loads shown. Using discontinuity functions, determine the equation of the elastic curve. The bearings at A and B exert only vertical reactions on the shaft. EI is constant.



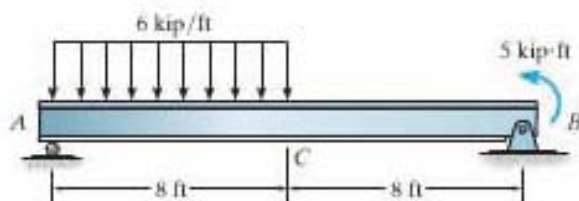
Prob. 12-137

12-138. The shaft is supported by a journal bearing at A , which exerts only vertical reactions on the shaft, and by a thrust bearing at B , which exerts both horizontal and vertical reactions on the shaft. Draw the bending-moment diagram for the shaft and then, from this diagram, sketch the deflection or elastic curve for the shaft's centerline. Determine the equations of the elastic curve using the coordinates x_1 and x_2 . EI is constant.



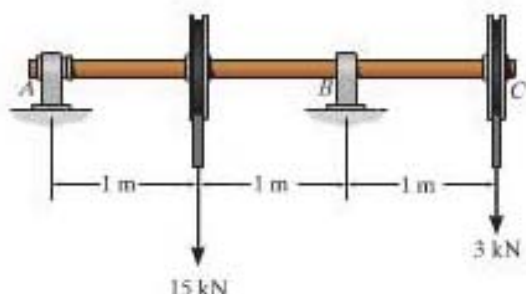
Prob. 12-138

12-139. The $W8 \times 24$ simply supported beam is subjected to the loading shown. Using the method of superposition, determine the deflection at its center C . The beam is made of A-36 steel.



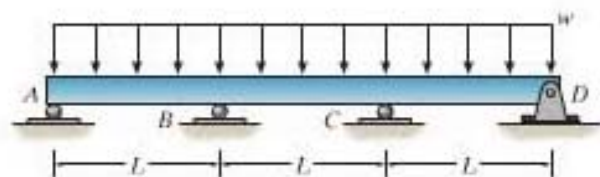
Prob. 12-139

***12-140.** Using the moment-area method, determine the slope and deflection at end C of the shaft. The 75-mm-diameter shaft is made of material having $E = 200$ GPa.



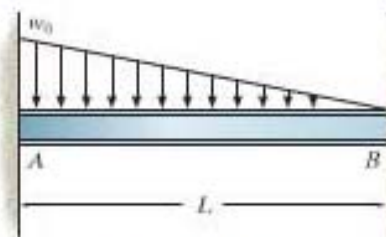
Prob. 12-140

•12-141. Determine the reactions at the supports. EI is constant. Use the method of superposition.



Prob. 12-141

12-142. Determine the moment reactions at the supports A and B . Use the method of integration. EI is constant.



Prob. 12-142



The columns used to support this water tank are braced at their mid-height in order to reduce their chance of buckling.

Buckling of Columns

CHAPTER OBJECTIVES

In this chapter, we will discuss the behavior of columns and indicate some of the methods used for their design. The chapter begins with a general discussion of buckling, followed by a determination of the axial load needed to buckle a so-called ideal column. Afterwards, a more realistic analysis is considered, which accounts for any bending of the column. Also, inelastic buckling of a column is presented as a special topic. At the end of the chapter we will discuss some of the methods used to design both concentrically and eccentrically loaded columns made of common engineering materials.

13.1 Critical Load

Whenever a member is designed, it is necessary that it satisfy specific strength, deflection, and stability requirements. In the preceding chapters we have discussed some of the methods used to determine a member's strength and deflection, while assuming that the member was always in stable equilibrium. Some members, however, may be subjected to compressive loadings, and if these members are long and slender the loading may be large enough to cause the member to deflect laterally or sidesway. To be specific, long slender members subjected to an axial compressive force are called **columns**, and the lateral deflection that occurs is called **buckling**. Quite often the buckling of a column can lead to a sudden and dramatic failure of a structure or mechanism, and as a result, special attention must be given to the design of columns so that they can safely support their intended loadings without buckling.

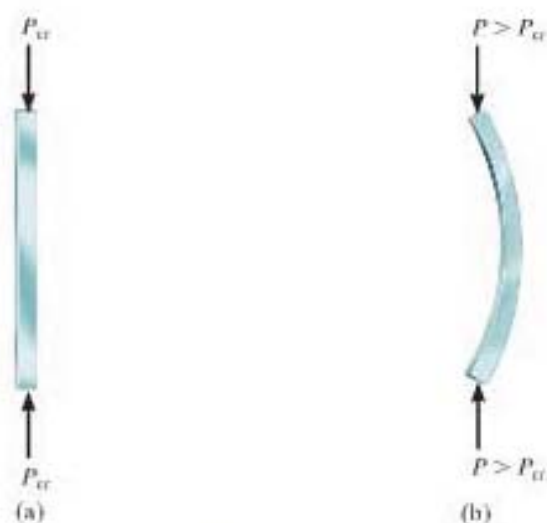


Fig. 13-1

The maximum axial load that a column can support when it is on the verge of buckling is called the **critical load**, P_{cr} , Fig. 13-1a. Any additional loading will cause the column to buckle and therefore deflect laterally as shown in Fig. 13-1b. In order to better understand the nature of this instability, consider a two-bar mechanism consisting of weightless bars that are rigid and pin connected as shown in Fig. 13-2a. When the bars are in the vertical position, the spring, having a stiffness k , is unstretched, and a *small* vertical force $\mathbf{P}$ is applied at the top of one of the bars. We can upset this equilibrium position by displacing the pin at A by a small amount Δ , Fig. 13-2b. As shown on the free-body diagram of the pin when the bars are displaced, Fig. 13-2c, the spring will produce a restoring force $F = k\Delta$, while the applied load $\mathbf{P}$ develops two horizontal components, $P_x = P \tan \theta$, which tend to push the pin (and the bars) further out of equilibrium. Since θ is small, $\Delta \approx \theta(L/2)$ and $\tan \theta \approx \theta$. Thus the *restoring* spring force becomes $F = k\theta L/2$, and the *disturbing* force is $2P_x = 2P\theta$.

If the restoring force is greater than the disturbing force, that is, $k\theta L/2 > 2P\theta$, then, noticing that θ cancels out, we can solve for P , which gives

$$P < \frac{kL}{4} \quad \text{stable equilibrium}$$

This is a condition for *stable equilibrium* since the force developed by the spring would be adequate to restore the bars back to their vertical position. However, if $kL\theta/2 < 2P\theta$, or

$$P > \frac{kL}{4} \quad \text{unstable equilibrium}$$

then the mechanism would be in *unstable equilibrium*. In other words, if this load $\mathbf{P}$ is applied, and a slight displacement occurs at A , the mechanism will tend to move out of equilibrium and not be restored to its original position.

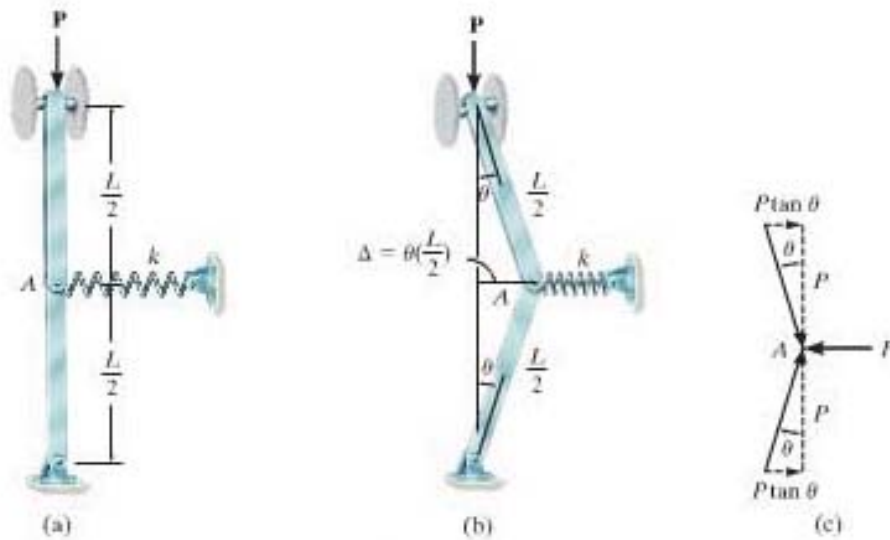


Fig. 13-2

The intermediate value of P , which requires $kL\theta/2 = 2P\theta$, is the *critical load*. Here

$$P_{cr} = \frac{kL}{4} \quad \text{neutral equilibrium}$$

This loading represents a case of the mechanism being in *neutral equilibrium*. Since P_{cr} is *independent* of the (small) displacement θ of the bars, any slight disturbance given to the mechanism will not cause it to move further out of equilibrium, nor will it be restored to its original position. Instead, the bars will *remain* in the deflected position.

These three different states of equilibrium are represented graphically in Fig. 13-3. The transition point where the load is equal to the critical value $P = P_{cr}$ is called the *bifurcation point*. At this point the mechanism will be in equilibrium for any *small* value of θ , measured either to the right or to the left of the vertical. Physically, P_{cr} represents the load for which the mechanism is on the verge of buckling. It is quite reasonable to determine this value by assuming *small displacements* as done here; however, it should be understood that P_{cr} may *not* be the largest value of P that the mechanism can support. Indeed, if a larger load is placed on the bars, then the mechanism may have to undergo a further deflection before the spring is compressed or elongated enough to hold the mechanism in equilibrium.

Like the two-bar mechanism just discussed, the critical buckling loads on columns supported in various ways can be obtained, and the method used to do this will be explained in the next section. Although in engineering design the critical load may be considered to be the largest load the column can support, realize that, like the two-bar mechanism in the deflected or buckled position, a column may actually support an

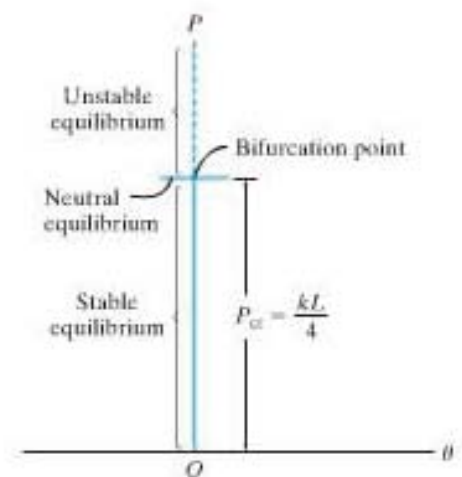


Fig. 13-3

even greater load than P_{cr} . Unfortunately, however, this loading may require the column to undergo a *large* deflection, which is generally not tolerated in engineering structures or machines. For example, it may take only a few newtons of force to buckle a meterstick, but the additional load it may support can be applied only after the stick undergoes a relatively large lateral deflection.

13.2 Ideal Column with Pin Supports



Some slender pin-connected members used in moving machinery, such as this short link, are subjected to compressive loads and thus act as columns.

In this section we will determine the critical buckling load for a column that is pin supported as shown in Fig. 13-4a. The column to be considered is an *ideal column*, meaning one that is perfectly straight before loading, is made of homogeneous material, and upon which the load is applied through the centroid of the cross section. It is further assumed that the material behaves in a linear-elastic manner and that the column buckles or bends in a single plane. In reality, the conditions of column straightness and load application are never accomplished; however, the analysis to be performed on an “ideal column” is similar to that used to analyze initially crooked columns or those having an eccentric load application. These more realistic cases will be discussed later in this chapter.

Since an ideal column is straight, theoretically the axial load P could be increased until failure occurs by either fracture or yielding of the material. However, when the critical load P_{cr} is reached, the column will be on the verge of becoming unstable, so that a small lateral force F , Fig. 13-4b, will cause the column to remain in the deflected position when F is removed, Fig. 13-4c. Any slight reduction in the axial load P from P_{cr} will allow the column to straighten out, and any slight increase in P , beyond P_{cr} , will cause further increases in lateral deflection.

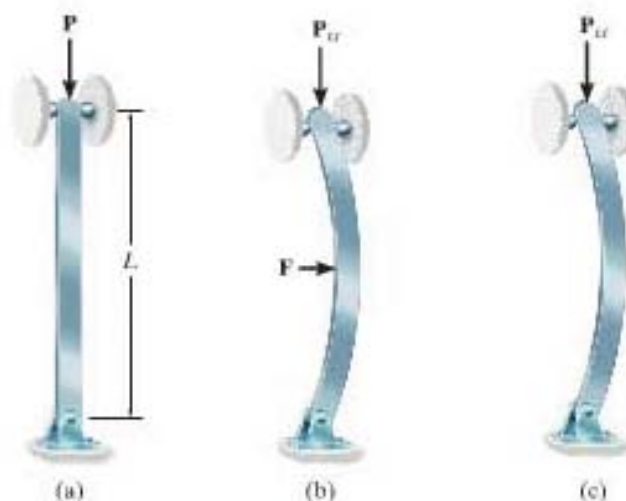


Fig. 13-4

Whether or not a column will remain stable or become unstable when subjected to an axial load will depend on its ability to restore itself, which is based on its resistance to bending. Hence, in order to determine the critical load and the buckled shape of the column, we will apply Eq. 12-10, which relates the internal moment in the column to its deflected shape, i.e.,

$$EI \frac{d^2 v}{dx^2} = M \quad (13-1)$$

Recall that this equation assumes that the slope of the elastic curve is small and that deflections occur only by bending. When the column is in its deflected position, Fig. 13-5a, the internal bending moment can be determined by using the method of sections. The free-body diagram of a segment in the deflected position is shown in Fig. 13-5b. Here both the deflection v and the internal moment M are shown in the *positive* direction according to the sign convention used to establish Eq. 13-1. Moment equilibrium requires $M = -Pv$. Thus Eq. 13-1 becomes

$$\begin{aligned} EI \frac{d^2 v}{dx^2} &= -Pv \\ \frac{d^2 v}{dx^2} + \left(\frac{P}{EI} \right) v &= 0 \end{aligned} \quad (13-2)$$

This is a homogeneous, second-order, linear differential equation with constant coefficients. It can be shown by using the methods of differential equations, or by direct substitution into Eq. 13-2, that the general solution is

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) \quad (13-3)$$

The two constants of integration are determined from the boundary conditions at the ends of the column. Since $v = 0$ at $x = 0$, then $C_2 = 0$. And since $v = 0$ at $x = L$, then

$$C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

This equation is satisfied if $C_1 = 0$; however, then $v = 0$, which is a *trivial solution* that requires the column to always remain straight, even though the load may cause the column to become unstable. The other possibility is for

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

which is satisfied if

$$\sqrt{\frac{P}{EI}}L = n\pi$$

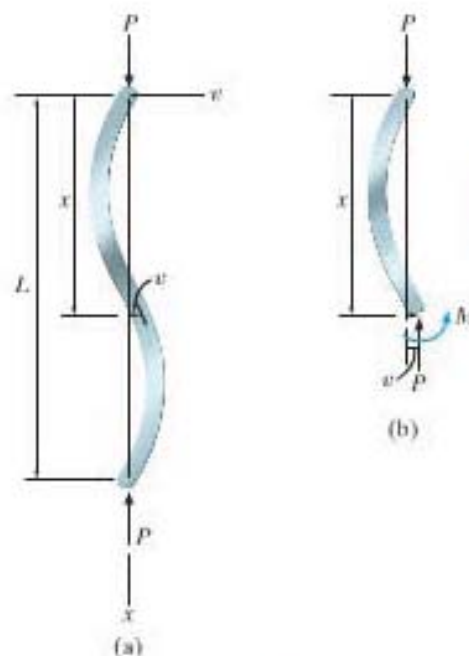


Fig. 13-5

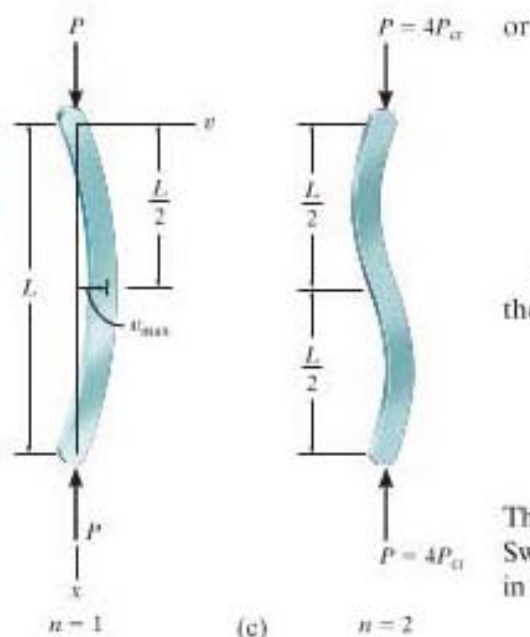


Fig. 13-5 (cont.)

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad n = 1, 2, 3, \dots \quad (13-4)$$

The *smallest* value of P is obtained when $n = 1$, so the *critical load* for the column is therefore*

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

This load is sometimes referred to as the *Euler load*, named after the Swiss mathematician Leonhard Euler, who originally solved this problem in 1757. The corresponding buckled shape is defined by the equation

$$v = C_1 \sin \frac{\pi x}{L}$$

Here the constant C_1 represents the maximum deflection, v_{max} , which occurs at the midpoint of the column, Fig. 13-5c. Specific values for C_1 cannot be obtained, since the exact deflected form for the column is unknown once it has buckled. It has been assumed, however, that this deflection is small.

Note that the critical load is independent of the strength of the material; rather it only depends on the column's dimensions (I and L) and the material's stiffness or modulus of elasticity E . For this reason, as far as elastic buckling is concerned, columns made, for example, of high-strength steel offer no advantage over those made of lower-strength steel, since the modulus of elasticity for both is approximately the same. Also note that the load-carrying capacity of a column will increase as the moment of inertia of the cross section increases. Thus, efficient columns are designed so that most of the column's cross-sectional area is located as far away as possible from the principal centroidal axes for the section. This is why hollow sections such as tubes are more economical than solid sections. Furthermore, wide-flange sections, and columns that are "built up" from channels, angles, plates, etc., are better than sections that are solid and rectangular.

* n represents the number of waves in the deflected shape of the column. For example, if $n = 2$, then *two* waves will appear, Fig. 13-5c. Here the critical load is $4 P_{cr}$, just prior to buckling, which practically speaking will not exist.

It is also important to realize that a column will buckle about the principal axis of the cross section having the **least moment of inertia** (the weakest axis). For example, a column having a rectangular cross section, like a meter stick, as shown in Fig. 13-6, will buckle about the a - a axis, not the b - b axis. As a result, engineers usually try to achieve a balance, keeping the moments of inertia the same in all directions. Geometrically, then, circular tubes would make excellent columns. Also, square tubes or those shapes having $I_x \approx I_y$ are often selected for columns.

Summarizing the above discussion, the buckling equation for a pin-supported long slender column can be rewritten, and the terms defined as follows:

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad (13-5)$$

where

P_{cr} = critical or maximum axial load on the column just before it begins to buckle. This load must *not* cause the stress in the column to exceed the proportional limit

E = modulus of elasticity for the material

I = *least* moment of inertia for the column's cross-sectional area

L = unsupported length of the column, whose ends are pinned

For purposes of design, the above equation can also be written in a more useful form by expressing $I = Ar^2$, where A is the cross-sectional area and r is the **radius of gyration** of the cross-sectional area. Thus,

$$P_{cr} = \frac{\pi^2 E(Ar^2)}{L^2}$$

$$\left(\frac{P}{A}\right)_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \quad (13-6)$$

Here

σ_{cr} = critical stress, which is an average normal stress in the column just before the column buckles. This stress is an *elastic stress* and therefore $\sigma_{cr} \leq \sigma_Y$

E = modulus of elasticity for the material

L = unsupported length of the column, whose ends are pinned

r = *smallest* radius of gyration of the column, determined from $r = \sqrt{I/A}$, where I is the *least* moment of inertia of the column's cross-sectional area A

The geometric ratio L/r in Eq. 13-6 is known as the **slenderness ratio**. It is a measure of the column's flexibility, and as will be discussed later, it serves to classify columns as long, intermediate, or short.



Fig. 13-6



Typical interior steel pipe columns used to support the roof of a single story building.

It is possible to graph Eq. 13-6 using axes that represent the critical stress versus the slenderness ratio. Examples of this graph for columns made of a typical structural steel and aluminum alloy are shown in Fig. 13-7. Note that the curves are hyperbolic and are valid only for critical stresses below the material's yield point (proportional limit), since the material must behave elastically. For the steel the yield stress is $(\sigma_Y)_{st} = 36 \text{ ksi}$ [$E_{st} = 29(10^3) \text{ ksi}$], and for the aluminum it is $(\sigma_Y)_{al} = 27 \text{ ksi}$ [$E_{al} = 10(10^3) \text{ ksi}$]. Substituting $\sigma_{cr} = \sigma_Y$ into Eq. 13-6, the *smallest* allowable slenderness ratios for the steel and aluminum columns are therefore $(L/r)_{st} = 89$ and $(L/r)_{al} = 60.5$, respectively. Thus, for a steel column, if $(L/r)_{st} \geq 89$, Euler's formula can be used to determine the critical load since the stress in the column remains elastic. On the other hand, if $(L/r)_{st} < 89$, the column's stress will exceed the yield point before buckling can occur, and therefore the Euler formula is not valid in this case.

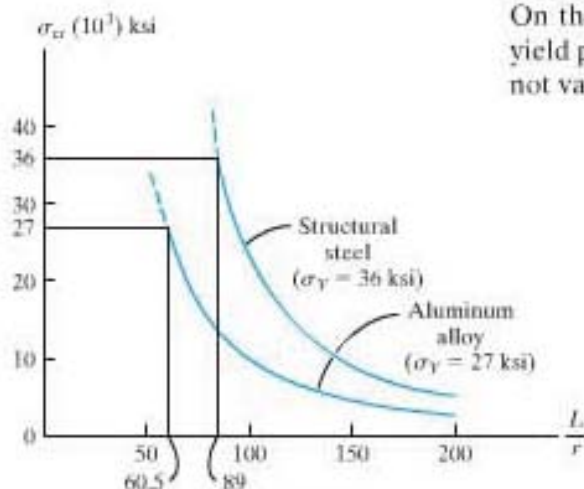


Fig. 13-7

Important Points

- *Columns* are long slender members that are subjected to axial compressive loads.
- The *critical load* is the maximum axial load that a column can support when it is on the verge of buckling. This loading represents a case of *neutral equilibrium*.
- An *ideal column* is initially perfectly straight, made of homogeneous material, and the load is applied through the centroid of the cross section.
- A pin-connected column will buckle about the principal axis of the cross section having the *least* moment of inertia.
- The *slenderness ratio* is L/r , where r is the smallest radius of gyration of the cross section. Buckling will occur about the axis where this ratio gives the greatest value.

EXAMPLE 13.1

The A-36 steel $W8 \times 31$ member shown in Fig. 13–8 is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.

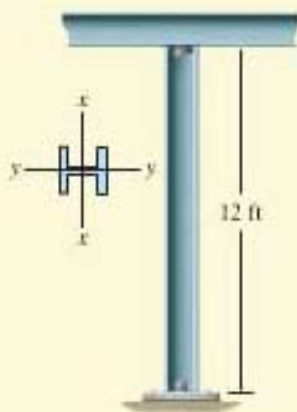


Fig. 13–8

SOLUTION

From the table in Appendix B, the column's cross-sectional area and moments of inertia are $A = 9.13 \text{ in}^2$, $I_x = 110 \text{ in}^4$, and $I_y = 37.1 \text{ in}^4$. By inspection, buckling will occur about the y - y axis. Why? Applying Eq. 13–5, we have

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 [29(10^3) \text{ kip/in}^2] (37.1 \text{ in}^4)}{[12 \text{ ft}(12 \text{ in./ft})]^2} = 512 \text{ kip}$$

When fully loaded, the average compressive stress in the column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{512 \text{ kip}}{9.13 \text{ in}^2} = 56.1 \text{ ksi}$$

Since this stress exceeds the yield stress (36 ksi), the load P is determined from simple compression:

$$36 \text{ ksi} = \frac{P}{9.13 \text{ in}^2}; \quad P = 329 \text{ kip} \quad \text{Ans.}$$

In actual practice, a factor of safety would be placed on this loading.

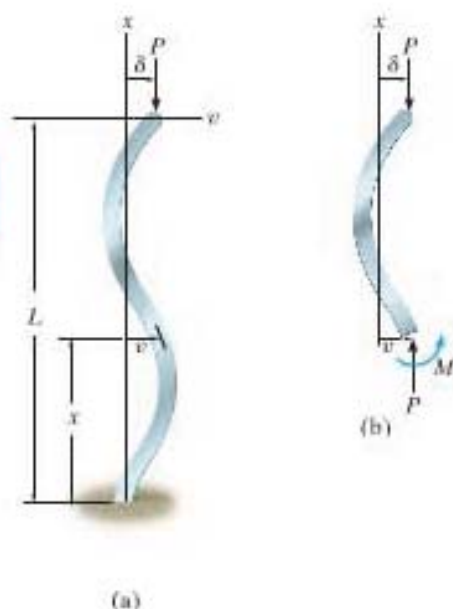


Fig. 13-9



The tubular columns used to support this water tank have been braced at three locations along their length to prevent them from buckling.

13.3 Columns Having Various Types of Supports

The Euler load was derived for a column that is pin connected or free to rotate at its ends. Oftentimes, however, columns may be supported in some other way. For example, consider the case of a column fixed at its base and free at the top, Fig. 13-9a. As the column buckles the load displaces δ and at x the displacement is v . From the free-body diagram in Fig. 13-9b, the internal moment at the arbitrary section is $M = P(\delta - v)$. Consequently, the differential equation for the deflection curve is

$$EI \frac{d^2 v}{dx^2} = P(\delta - v)$$

$$\frac{d^2 v}{dx^2} + \frac{P}{EI} v = \frac{P}{EI} \delta \quad (13-7)$$

Unlike Eq. 13-2, this equation is nonhomogeneous because of the nonzero term on the right side. The solution consists of both a complementary and a particular solution, namely,

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}} x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}} x\right) + \delta$$

The constants are determined from the boundary conditions. At $x = 0$, $v = 0$, so that $C_2 = -\delta$. Also,

$$\frac{dv}{dx} = C_1 \sqrt{\frac{P}{EI}} \cos\left(\sqrt{\frac{P}{EI}} x\right) - C_2 \sqrt{\frac{P}{EI}} \sin\left(\sqrt{\frac{P}{EI}} x\right)$$

At $x = 0$, $dv/dx = 0$, so that $C_1 = 0$. The deflection curve is therefore

$$v = \delta \left[1 - \cos\left(\sqrt{\frac{P}{EI}} x\right) \right] \quad (13-8)$$

Since the deflection at the top of the column is δ , that is, at $x = L$, $v = \delta$, we require

$$\delta \cos\left(\sqrt{\frac{P}{EI}} L\right) = 0$$

The trivial solution $\delta = 0$ indicates that no buckling occurs, regardless of the load P . Instead,

$$\cos\left(\sqrt{\frac{P}{EI}} L\right) = 0 \quad \text{or} \quad \sqrt{\frac{P}{EI}} L = \frac{n\pi}{2}, n = 1, 3, 5, \dots$$

The smallest critical load occurs when $n = 1$, so that

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \quad (13-9)$$

By comparison with Eq. 13-5, it is seen that a column fixed-supported at its base and free at its top will support only one-fourth the critical load that can be applied to a column pin-supported at both ends.

Other types of supported columns are analyzed in much the same way and will not be covered in detail here.* Instead, we will tabulate the results for the most common types of column support and show how to apply these results by writing Euler's formula in a general form.

Effective Length. As stated previously, the Euler formula, Eq. 13-5, was developed for the case of a column having ends that are pinned or free to rotate. In other words, L in the equation represents the unsupported distance between the points of zero moment. This formula can be used to determine the critical load on columns having other types of support provided " L " represents the distance between the zero-moment points. This distance is called the column's **effective length**, L_e . Obviously, for a pin-ended column $L_e = L$, Fig. 13-10a. For the fixed- and free-ended column, the deflection curve, Eq. 13-8, was found to be one-half that of a column that is pin connected and has a length of $2L$, Fig. 13-10b. Thus the effective length between the points of zero moment is $L_e = 2L$. Examples for two other columns with different end supports are also shown in Fig. 13-10. The column fixed at its ends, Fig. 13-10c, has inflection points or points of zero moment $L/4$ from each support. The effective length is therefore represented by the middle half of its length, that is, $L_e = 0.5L$. Lastly, the pin- and fixed-ended column, Fig. 13-10d, has an inflection point at approximately $0.7L$ from its pinned end, so that $L_e = 0.7L$.

Rather than specifying the column's effective length, many design codes provide column formulas that employ a dimensionless coefficient K called the **effective-length factor**. This factor is defined from

$$L_e = KL \quad (13-10)$$

Specific values of K are also given in Fig. 13-10. Based on this generality, we can therefore write Euler's formula as

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \quad (13-11)$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2} \quad (13-12)$$

Here (KL/r) is the column's **effective-slenderness ratio**. For example, if the column is fixed at its base and free at its end, we have $K = 2$, and therefore Eq. 13-11 gives the same result as Eq. 13-9.

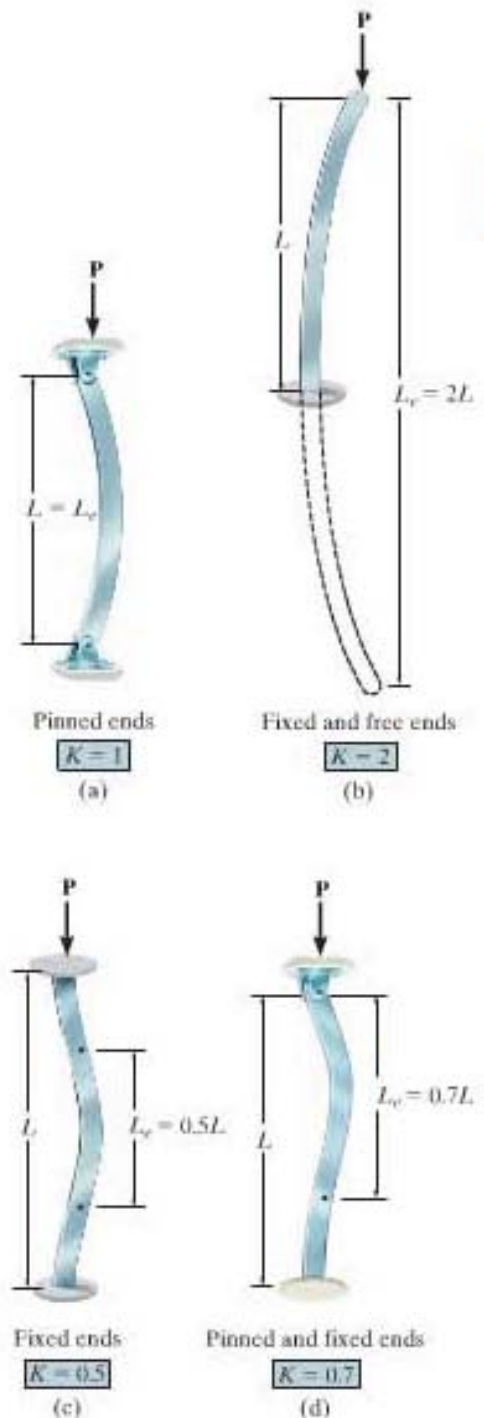


Fig. 13-10

*See Problems 13-43, 13-44, and 13-45.

EXAMPLE 13.2

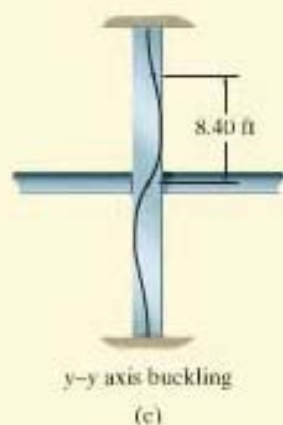
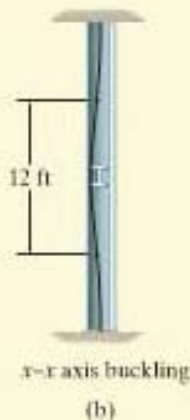
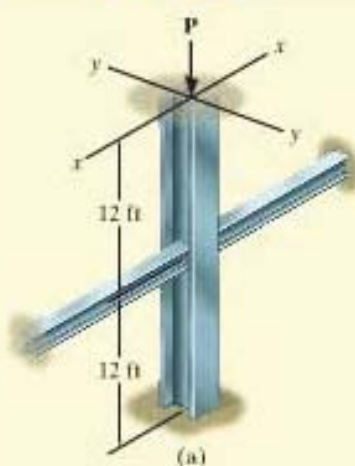


Fig. 13-11

A W6 $\times$ 15 steel column is 24 ft long and is fixed at its ends as shown in Fig. 13-11a. Its load-carrying capacity is increased by bracing it about the y - y (weak) axis using struts that are assumed to be pin connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 60$ ksi.

SOLUTION

The buckling behavior of the column will be *different* about the x - x and y - y axes due to the bracing. The buckled shape for each of these cases is shown in Figs. 13-11b and 13-11c. From Fig. 13-11b, the effective length for buckling about the x - x axis is $(KL)_x = 0.5(24 \text{ ft}) = 12 \text{ ft} = 144 \text{ in.}$, and from Fig. 13-11c, for buckling about the y - y axis, $(KL)_y = 0.7(24 \text{ ft}/2) = 8.40 \text{ ft} = 100.8 \text{ in.}$ The moments of inertia for a W6 $\times$ 15 are found from the table in Appendix B. We have $I_x = 29.1 \text{ in}^4$, $I_y = 9.32 \text{ in}^4$.

Applying Eq. 13-11,

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2 [29(10^3) \text{ ksi}] 29.1 \text{ in}^4}{(144 \text{ in.})^2} = 401.7 \text{ kip} \quad (1)$$

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2 [29(10^3) \text{ ksi}] 9.32 \text{ in}^4}{(100.8 \text{ in.})^2} = 262.5 \text{ kip} \quad (2)$$

By comparison, buckling will occur about the y - y axis.

The area of the cross section is 4.43 in^2 , so the average compressive stress in the column is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{262.5 \text{ kip}}{4.43 \text{ in}^2} = 59.3 \text{ ksi}$$

Since this stress is less than the yield stress, buckling will occur before the material yields. Thus,

$$P_{cr} = 263 \text{ kip} \quad \text{Ans.}$$

NOTE: From Eq. 13-12 it can be seen that buckling will always occur about the column axis having the *largest* slenderness ratio, since a large slenderness ratio will give a small critical stress. Thus, using the data for the radius of gyration from the table in Appendix B, we have

$$\left(\frac{KL}{r} \right)_x = \frac{144 \text{ in.}}{2.56 \text{ in.}} = 56.2$$

$$\left(\frac{KL}{r} \right)_y = \frac{100.8 \text{ in.}}{1.46 \text{ in.}} = 69.0$$

Hence, y - y axis buckling will occur, which is the same conclusion reached by comparing Eqs. 1 and 2.

EXAMPLE 13.3

The aluminum column is fixed at its bottom and is braced at its top by cables so as to prevent movement at the top along the x axis, Fig. 13-12a. If it is assumed to be fixed at its base, determine the largest allowable load P that can be applied. Use a factor of safety for buckling of F.S. = 3.0. Take $E_{al} = 70$ GPa, $\sigma_Y = 215$ MPa, $A = 7.5(10^{-3})$ m², $I_x = 61.3(10^{-6})$ m⁴, $I_y = 23.2(10^{-6})$ m⁴.

SOLUTION

Buckling about the x and y axes is shown in Fig. 13-12b and 13-12c, respectively. Using Fig. 13-10a, for x - x axis buckling, $K = 2$, so $(KL)_x = 2(5 \text{ m}) = 10 \text{ m}$. Also, for y - y axis buckling, $K = 0.7$, so $(KL)_y = 0.7(5 \text{ m}) = 3.5 \text{ m}$.

Applying Eq. 13-11, the critical loads for each case are

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2 [70(10^9) \text{ N/m}^2] (61.3(10^{-6}) \text{ m}^4)}{(10 \text{ m})^2}$$

$$= 424 \text{ kN}$$

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2 [70(10^9) \text{ N/m}^2] (23.2(10^{-6}) \text{ m}^4)}{(3.5 \text{ m})^2}$$

$$= 1.31 \text{ MN}$$

By comparison, as P is increased the column will buckle about the x - x axis. The allowable load is therefore

$$P_{\text{allow}} = \frac{P_{cr}}{\text{F.S.}} = \frac{424 \text{ kN}}{3.0} = 141 \text{ kN} \quad \text{Ans.}$$

Since

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{424 \text{ kN}}{7.5(10^{-3}) \text{ m}^2} = 56.5 \text{ MPa} < 215 \text{ MPa}$$

Euler's equation can be applied.

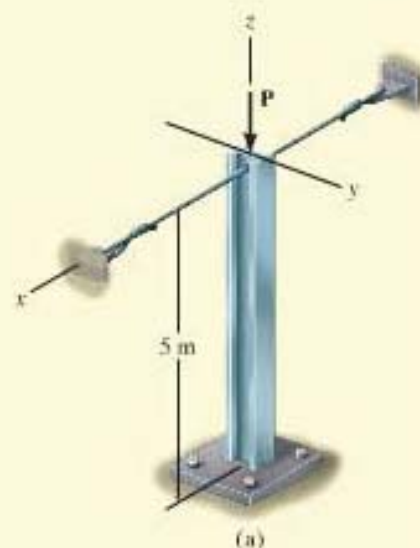
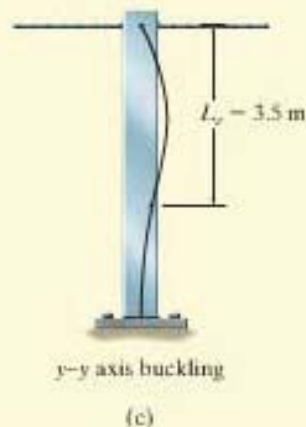
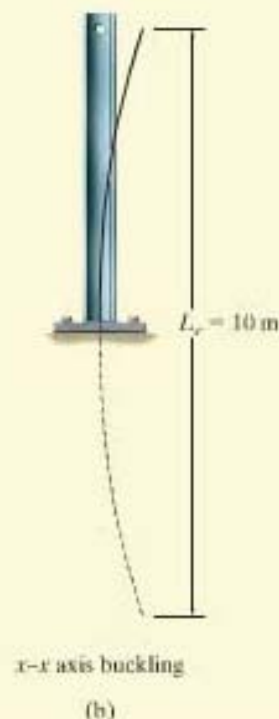


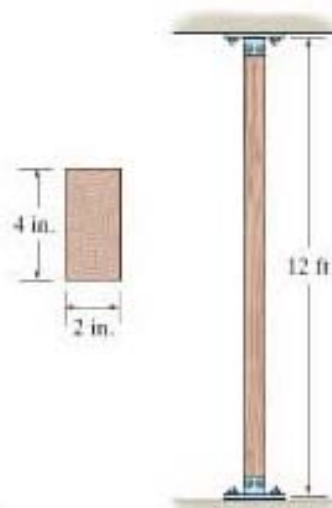
Fig. 13-12



FUNDAMENTAL PROBLEMS

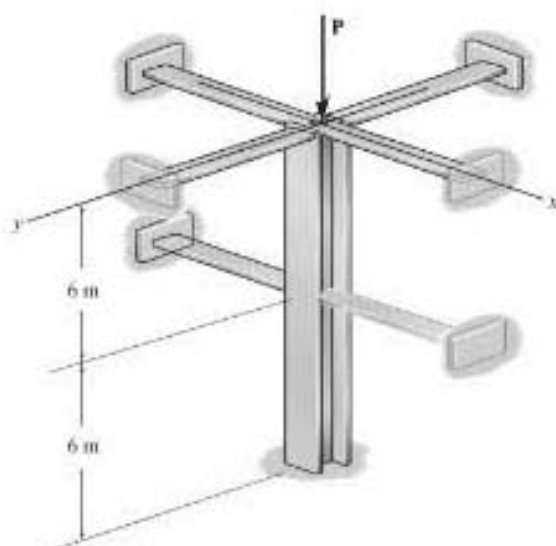
F13-1. A 50-in.-long rod is made from a 1-in.-diameter steel rod. Determine the critical buckling load if the ends are fixed supported. $E = 29(10^3)$ ksi, $\sigma_Y = 36$ ksi.

F13-2. A 12-ft wooden rectangular column has the dimensions shown. Determine the critical load if the ends are assumed to be pin-connected. $E = 1.6(10^3)$ ksi. Yielding does not occur.



F13-2

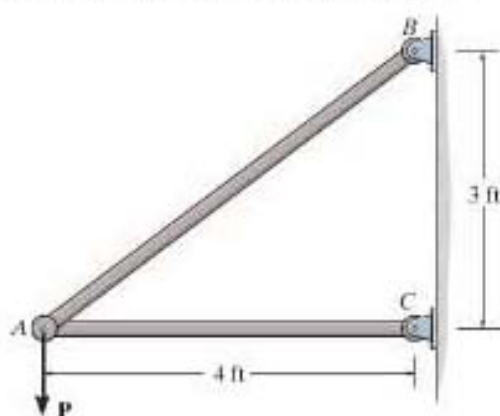
F13-3. The A-36 steel column can be considered pinned at its top and bottom and braced against its weak axis at the mid-height. Determine the maximum allowable force P that the column can support without buckling. Apply a F.S. = 2 against buckling. Take $A = 7.4(10^{-3})$ m², $I_x = 87.3(10^{-6})$ m⁴, and $I_y = 18.8(10^{-6})$ m⁴.



F13-3

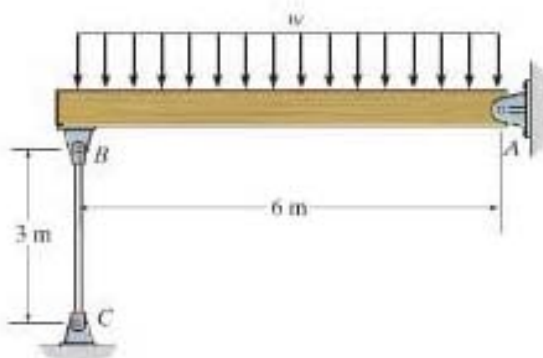
F13-4. A steel pipe is fixed supported at its ends. If it is 5 m long and has an outer diameter of 50 mm and a thickness of 10 mm, determine the maximum axial load P that it can carry without buckling. $E_s = 200$ GPa, $\sigma_Y = 250$ MPa.

F13-5. Determine the maximum force P that can be supported by the assembly without causing member AC to buckle. The member is made of A-36 steel and has a diameter of 2 in. Take F.S. = 2 against buckling.



F13-5

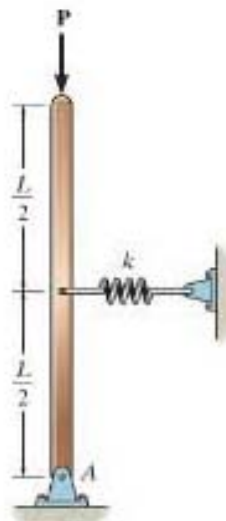
F13-6. The A-36 steel rod BC has a diameter of 50 mm and is used as a strut to support the beam. Determine the maximum intensity w of the uniform distributed load that can be applied to the beam without causing the strut to buckle. Take F.S. = 2 against buckling.



F13-6

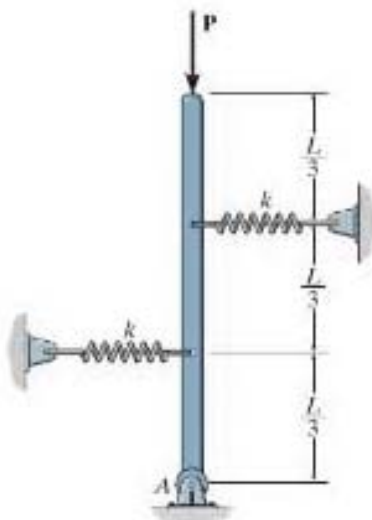
PROBLEMS

- 13-1. Determine the critical buckling load for the column. The material can be assumed rigid.



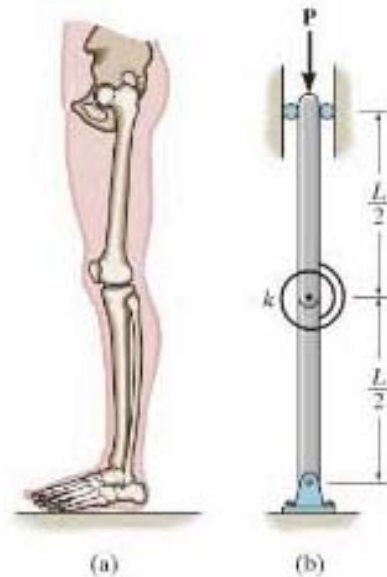
Prob. 13-1

- 13-2. Determine the critical load P_{cr} for the rigid bar and spring system. Each spring has a stiffness k .



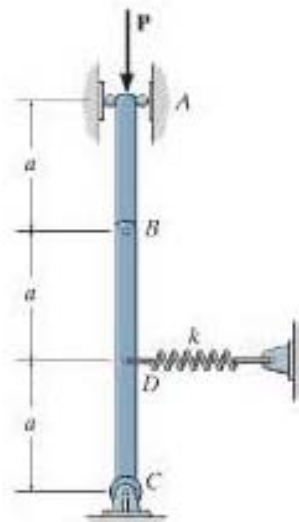
Prob. 13-2

- 13-3. The leg in (a) acts as a column and can be modeled (b) by the two pin-connected members that are attached to a torsional spring having a stiffness k (torque/rad). Determine the critical buckling load. Assume the bone material is rigid.



Prob. 13-3

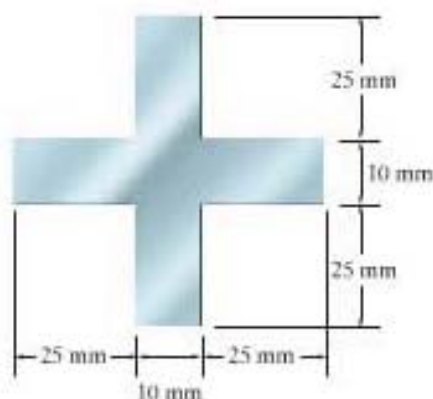
- *13-4. Rigid bars AB and BC are pin connected at B . If the spring at D has a stiffness k , determine the critical load P_{cr} for the system.



Prob. 13-4

•13-5. An A-36 steel column has a length of 4 m and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.

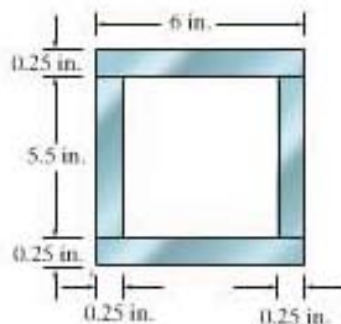
13-6. Solve Prob. 13-5 if the column is fixed at its bottom and pinned at its top.



Probs. 13-5/6

13-7. A column is made of A-36 steel, has a length of 20 ft, and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.

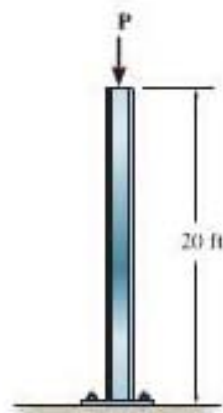
***13-8.** A column is made of 2014-T6 aluminum, has a length of 30 ft, and is fixed at its bottom and pinned at its top. If the cross-sectional area has the dimensions shown, determine the critical load.



Probs. 13-7/8

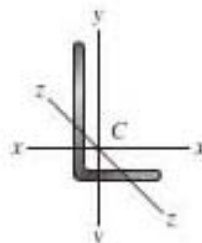
•13-9. The W14 $\times$ 38 column is made of A-36 steel and is fixed supported at its base. If it is subjected to an axial load of $P = 15$ kip, determine the factor of safety with respect to buckling.

13-10. The W14 $\times$ 38 column is made of A-36 steel. Determine the critical load if its bottom end is fixed supported and its top is free to move about the strong axis and is pinned about the weak axis.



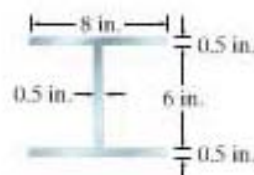
Probs. 13-9/10

13-11. The A-36 steel angle has a cross-sectional area of $A = 2.48$ in² and a radius of gyration about the x axis of $r_x = 1.26$ in. and about the y axis of $r_y = 0.879$ in. The smallest radius of gyration occurs about the z axis and is $r_z = 0.644$ in. If the angle is to be used as a pin-connected 10-ft-long column, determine the largest axial load that can be applied through its centroid C without causing it to buckle.



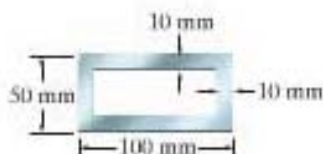
Prob. 13-11

***13-12.** An A-36 steel column has a length of 15 ft and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.



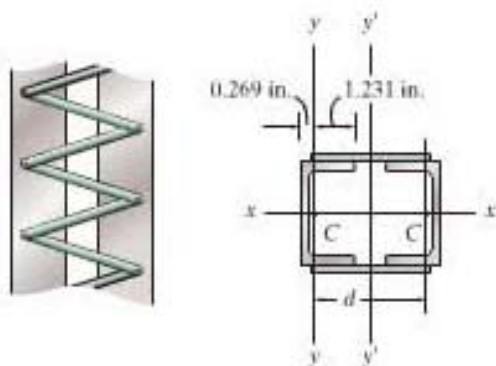
Prob. 13-12

- 13-13. An A-36 steel column has a length of 5 m and is fixed at both ends. If the cross-sectional area has the dimensions shown, determine the critical load.



Prob. 13-13

- 13-14. The two steel channels are to be laced together to form a 30-ft-long bridge column assumed to be pin connected at its ends. Each channel has a cross-sectional area of $A = 3.10 \text{ in}^2$ and moments of inertia $I_x = 55.4 \text{ in}^4$, $I_y = 0.382 \text{ in}^4$. The centroid C of its area is located in the figure. Determine the proper distance d between the centroids of the channels so that buckling occurs about the x - x and y - y' axes due to the same load. What is the value of this critical load? Neglect the effect of the lacing. $E_{st} = 29(10^3) \text{ ksi}$, $\sigma_Y = 50 \text{ ksi}$.



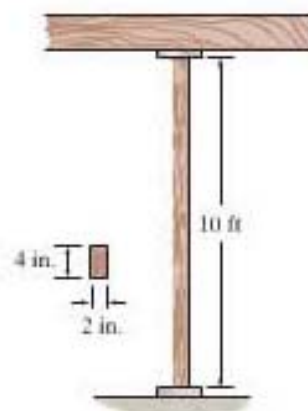
Prob. 13-14

- 13-15. An A-36-steel $W8 \times 24$ column is fixed at one end and free at its other end. If it is subjected to an axial load of 20 kip, determine the maximum allowable length of the column if F.S. = 2 against buckling is desired.

- *13-16. An A-36-steel $W8 \times 24$ column is fixed at one end and pinned at the other end. If it is subjected to an axial load of 60 kip, determine the maximum allowable length of the column if F.S. = 2 against buckling is desired.

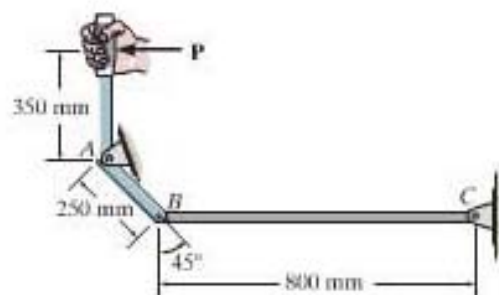
- 13-17. The 10-ft wooden rectangular column has the dimensions shown. Determine the critical load if the ends are assumed to be pin connected. $E_w = 1.6(10^3) \text{ ksi}$, $\sigma_Y = 5 \text{ ksi}$.

- 13-18. The 10-ft column has the dimensions shown. Determine the critical load if the bottom is fixed and the top is pinned. $E_w = 1.6(10^3) \text{ ksi}$, $\sigma_Y = 5 \text{ ksi}$.



Probs. 13-17/18

- 13-19. Determine the maximum force P that can be applied to the handle so that the A-36 steel control rod BC does not buckle. The rod has a diameter of 25 mm.

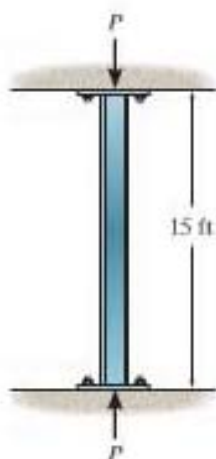


Prob. 13-19

***13-20.** The $W10 \times 45$ is made of A-36 steel and is used as a column that has a length of 15 ft. If its ends are assumed pin supported, and it is subjected to an axial load of 100 kip, determine the factor of safety with respect to buckling.

13

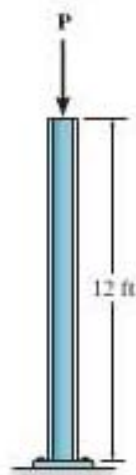
***13-21.** The $W10 \times 45$ is made of A-36 steel and is used as a column that has a length of 15 ft. If the ends of the column are fixed supported, can the column support the critical load without yielding?



Probs. 13-20/21

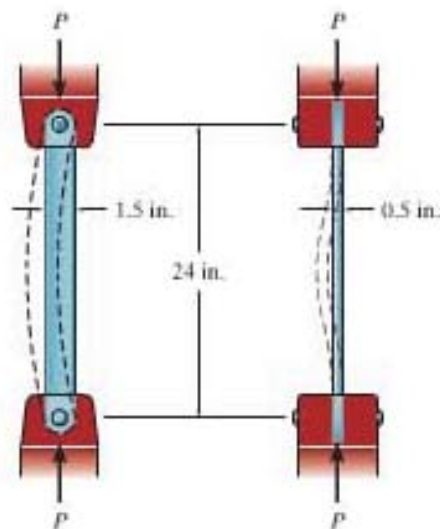
13-22. The $W12 \times 87$ structural A-36 steel column has a length of 12 ft. If its bottom end is fixed supported while its top is free, and it is subjected to an axial load of $P = 380$ kip, determine the factor of safety with respect to buckling.

13-23. The $W12 \times 87$ structural A-36 steel column has a length of 12 ft. If its bottom end is fixed supported while its top is free, determine the largest axial load it can support. Use a factor of safety with respect to buckling of 1.75.



Probs. 13-22/23

***13-24.** An L-2 tool steel link in a forging machine is pin connected to the forks at its ends as shown. Determine the maximum load P it can carry without buckling. Use a factor of safety with respect to buckling of $F.S. = 1.75$. Note from the figure on the left that the ends are pinned for buckling, whereas from the figure on the right the ends are fixed.



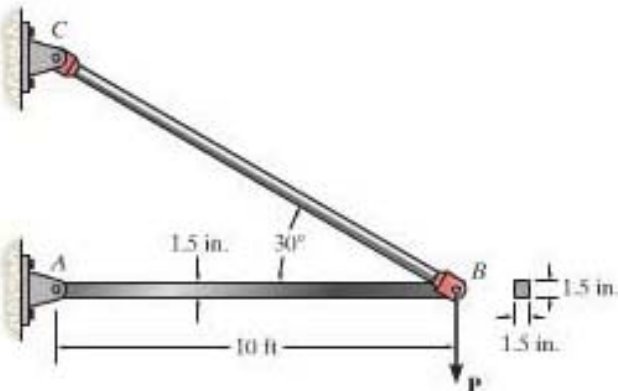
Prob. 13-24

***13-25.** The $W14 \times 30$ is used as a structural A-36 steel column that can be assumed pinned at both of its ends. Determine the largest axial force P that can be applied without causing it to buckle.



Prob. 13-25

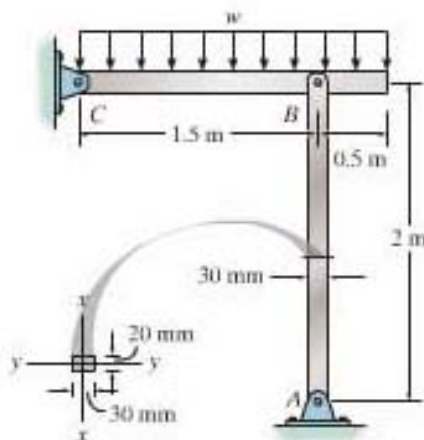
13-26. The A-36 steel bar AB has a square cross section. If it is pin connected at its ends, determine the maximum allowable load P that can be applied to the frame. Use a factor of safety with respect to buckling of 2.



Prob. 13-26

13-27. Determine the maximum allowable intensity w of the distributed load that can be applied to member BC without causing member AB to buckle. Assume that AB is made of steel and is pinned at its ends for x - x axis buckling and fixed at its ends for y - y axis buckling. Use a factor of safety with respect to buckling of 3. $E_{st} = 200$ GPa, $\sigma_Y = 360$ MPa.

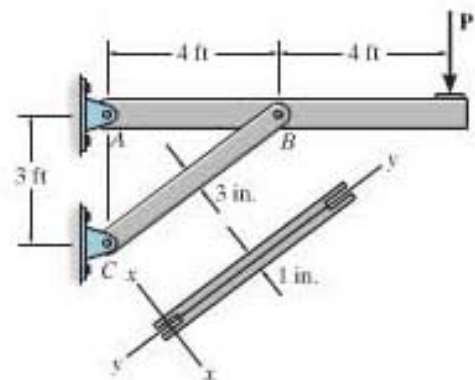
***13-28.** Determine if the frame can support a load of $w = 6$ kN/m if the factor of safety with respect to buckling of member AB is 3. Assume that AB is made of steel and is pinned at its ends for x - x axis buckling and fixed at its ends for y - y axis buckling. $E_{st} = 200$ GPa, $\sigma_Y = 360$ MPa.



Probs. 13-27/28

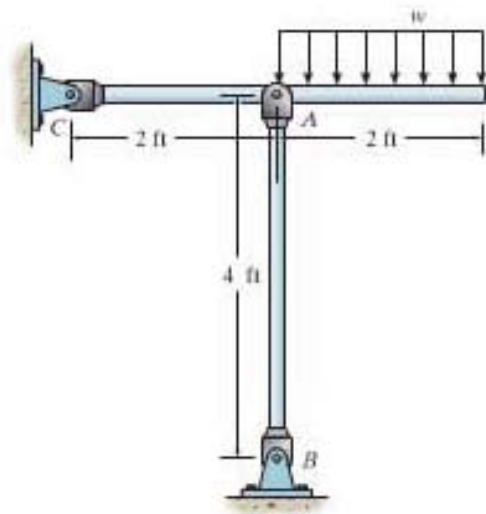
***13-29.** The beam supports the load of $P = 6$ kip. As a result, the A-36 steel member BC is subjected to a compressive load. Due to the forked ends on the member, consider the supports at B and C to act as pins for x - x axis buckling and as fixed supports for y - y axis buckling. Determine the factor of safety with respect to buckling about each of these axes.

13-30. Determine the greatest load P the frame will support without causing the A-36 steel member BC to buckle. Due to the forked ends on the member, consider the supports at B and C to act as pins for x - x axis buckling and as fixed supports for y - y axis buckling.



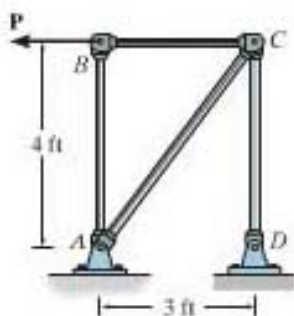
Probs. 13-29/30

13-31. Determine the maximum distributed load that can be applied to the bar so that the A-36 steel strut AB does not buckle. The strut has a diameter of 2 in. It is pin connected at its ends.



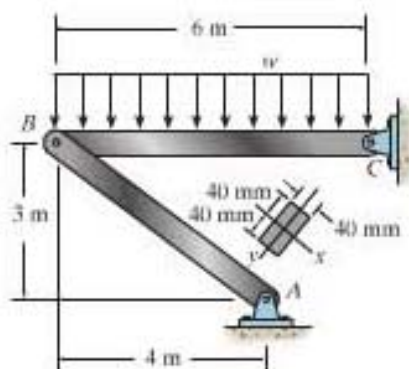
Prob. 13-31

***13-32.** The members of the truss are assumed to be pin connected. If member AC is an A-36 steel rod of 2 in. diameter, determine the maximum load P that can be supported by the truss without causing the member to buckle.



Prob. 13-32

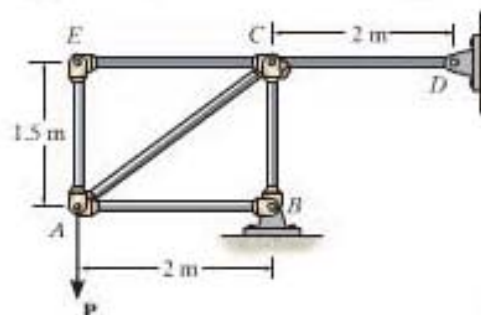
***13-33.** The steel bar AB of the frame is assumed to be pin connected at its ends for y - y axis buckling. If $w = 3 \text{ kN/m}$, determine the factor of safety with respect to buckling about the y - y axis due to the applied loading. $E_{st} = 200 \text{ GPa}$, $\sigma_Y = 360 \text{ MPa}$.



Prob. 13-33

13-34. The members of the truss are assumed to be pin connected. If member AB is an A-36 steel rod of 40 mm diameter, determine the maximum force P that can be supported by the truss without causing the member to buckle.

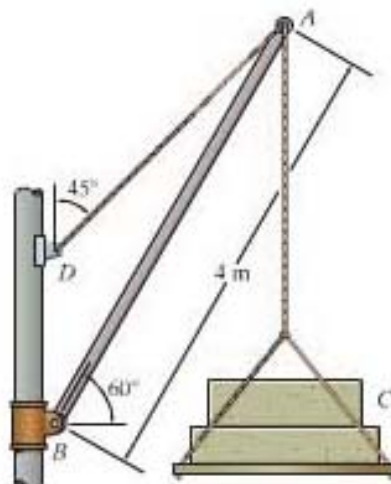
13-35. The members of the truss are assumed to be pin connected. If member CB is an A-36 steel rod of 40 mm diameter, determine the maximum load P that can be supported by the truss without causing the member to buckle.



Probs. 13-34/35

***13-36.** If load C has a mass of 500 kg, determine the required minimum diameter of the solid L2-steel rod AB to the nearest mm so that it will not buckle. Use F.S. = 2 against buckling.

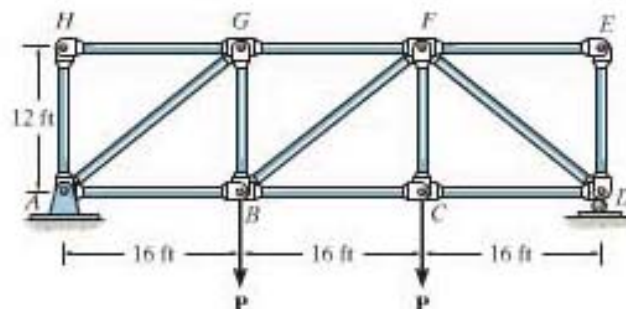
***13-37.** If the diameter of the solid L2-steel rod AB is 50 mm, determine the maximum mass C that the rod can support without buckling. Use F.S. = 2 against buckling.



Probs. 13-36/37

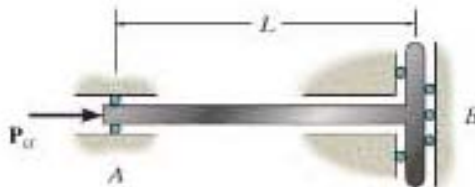
13-38. The members of the truss are assumed to be pin connected. If member GF is an A-36 steel rod having a diameter of 2 in., determine the greatest magnitude of load P that can be supported by the truss without causing this member to buckle.

13-39. The members of the truss are assumed to be pin connected. If member AG is an A-36 steel rod having a diameter of 2 in., determine the greatest magnitude of load P that can be supported by the truss without causing this member to buckle.



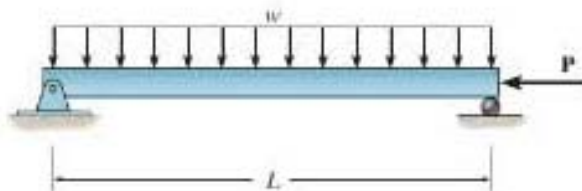
Probs. 13-38/39

***13-40.** The column is supported at B by a support that does not permit rotation but allows vertical deflection. Determine the critical load P_{cr} . EI is constant.



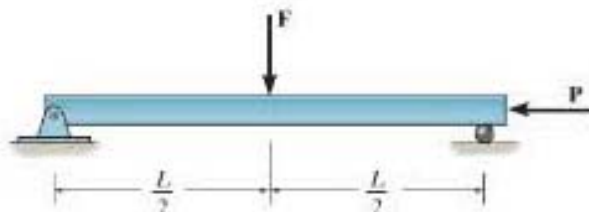
Prob. 13-40

•13-41. The ideal column has a weight w (force/length) and rests in the horizontal position when it is subjected to the axial load P . Determine the maximum moment in the column at midspan. EI is constant. *Hint:* Establish the differential equation for deflection, Eq. 13-1, with the origin at the midspan. The general solution is $v = C_1 \sin kx + C_2 \cos kx + (w/(2P))x^2 - (wL/(2P))x - (wEI/P^2)$ where $k^2 = P/EI$.



Prob. 13-41

13-42. The ideal column is subjected to the force F at its midpoint and the axial load P . Determine the maximum moment in the column at midspan. EI is constant. *Hint:* Establish the differential equation for deflection, Eq. 13-1. The general solution is $v = C_1 \sin kx + C_2 \cos kx - c^2 x/k^2$, where $c^2 = F/2EI$, $k^2 = P/EI$.



Prob. 13-42

13-43. The column with constant EI has the end constraints shown. Determine the critical load for the column.



Prob. 13-43

***13-44.** Consider an ideal column as in Fig. 13-10c, having both ends fixed. Show that the critical load on the column is given by $P_{cr} = 4\pi^2 EI/L^2$. *Hint:* Due to the vertical deflection of the top of the column, a constant moment M' will be developed at the supports. Show that $d^2v/dx^2 + (P/EI)v = M'/EI$. The solution is of the form $v = C_1 \sin(\sqrt{P/EI}x) + C_2 \cos(\sqrt{P/EI}x) + M'/P$.

•13-45. Consider an ideal column as in Fig. 13-10d, having one end fixed and the other pinned. Show that the critical load on the column is given by $P_{cr} = 20.19 EI/L^2$. *Hint:* Due to the vertical deflection at the top of the column, a constant moment M' will be developed at the fixed support and horizontal reactive forces R' will be developed at both supports. Show that $d^2v/dx^2 + (P/EI)v = (R'/EI)(L - x)$. The solution is of the form $v = C_1 \sin(\sqrt{P/EI}x) + C_2 \cos(\sqrt{P/EI}x) + (R'/P)(L - x)$. After application of the boundary conditions show that $\tan(\sqrt{P/EI}L) = \sqrt{P/EI}L$. Solve by trial and error for the smallest nonzero root.



The column supporting this crane is unusually long. It will be subjected not only to uniaxial load, but also a bending moment. To ensure it will not buckle, it should be braced at the roof as a pin connection.

*13.4 The Secant Formula

The Euler formula was derived assuming the load P is always applied through the centroid of the column's cross-sectional area and that the column is perfectly straight. This is actually quite unrealistic, since manufactured columns are never perfectly straight, nor is the application of the load known with great accuracy. In reality, then, columns never suddenly buckle; instead they begin to bend, although ever so slightly, immediately upon application of the load. As a result, the actual criterion for load application should be limited either to a specified deflection of the column or by not allowing the maximum stress in the column to exceed an allowable stress.

To study this effect, we will apply the load P to the column at a short *eccentric distance* e from its centroid, Fig. 13-13a. This loading on the column is statically equivalent to the axial load P and bending moment $M' = Pe$ shown in Fig. 13-13b. As shown, in both cases, the ends A and B are supported so that they are free to rotate (pin supported). As before, we will only consider small slopes and deflections and linear-elastic material behavior. Furthermore, the x - v plane is a plane of symmetry for the cross-sectional area.

From the free-body diagram of the arbitrary section, Fig. 13-13c, the internal moment in the column is

$$M = -P(e + v) \quad (13-13)$$

The differential equation for the deflection curve is therefore

$$EI \frac{d^2v}{dx^2} = -P(e + v)$$

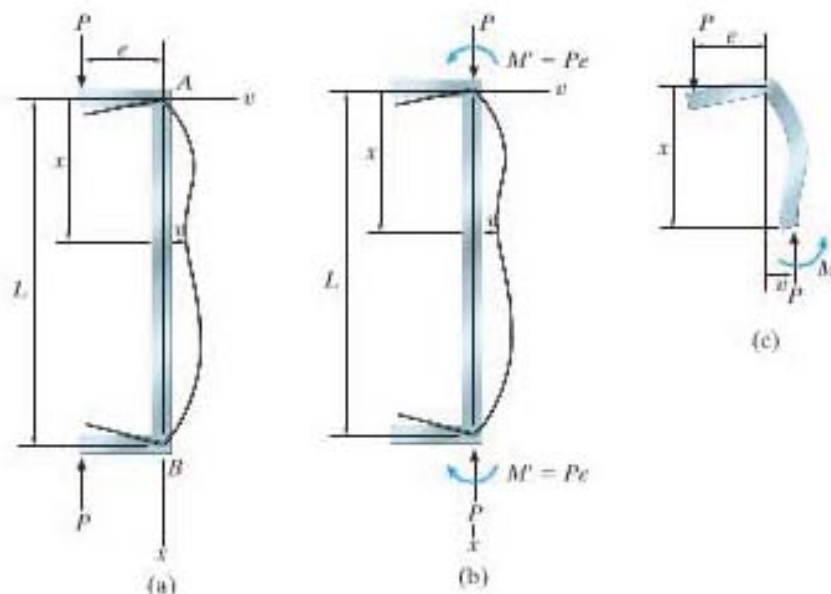


Fig. 13-13

or

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = -\frac{P}{EI}e$$

This equation is similar to Eq. 13-7 and has a general solution consisting of the complementary and particular solutions, namely,

$$v = C_1 \sin \sqrt{\frac{P}{EI}}x + C_2 \cos \sqrt{\frac{P}{EI}}x - e \quad (13-14)$$

To evaluate the constants we must apply the boundary conditions. At $x = 0$, $v = 0$, so $C_2 = e$. And at $x = L$, $v = 0$, which gives

$$C_1 = \frac{e[1 - \cos(\sqrt{P/EI}L)]}{\sin(\sqrt{P/EI}L)}$$

Since $1 - \cos(\sqrt{P/EI}L) = 2 \sin^2(\sqrt{P/EI}L/2)$ and $\sin(\sqrt{P/EI}L) = 2 \sin(\sqrt{P/EI}L/2) \cos(\sqrt{P/EI}L/2)$, we have

$$C_1 = e \tan\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right)$$

Hence, the deflection curve, Eq. 13-14, can be written as

$$v = e \left[\tan\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) \sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1 \right] \quad (13-15)$$

Maximum Deflection. Due to symmetry of loading, both the maximum deflection and maximum stress occur at the column's midpoint. Therefore, when $x = L/2$, $v = v_{\max}$, so

$$v_{\max} = e \left[\sec\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) - 1 \right] \quad (13-16)$$

Notice that if e approaches zero, then $v_{\max}$ approaches zero. However, if the terms in the brackets approach infinity as e approaches zero, then $v_{\max}$ will have a nonzero value. Mathematically, this would represent the behavior of an axially loaded column at failure when subjected to the critical load P_{cr} . Therefore, to find P_{cr} we require

$$\begin{aligned} \sec\left(\sqrt{\frac{P_{cr}}{EI}} \frac{L}{2}\right) &= \infty \\ \sqrt{\frac{P_{cr}}{EI}} \frac{L}{2} &= \frac{\pi}{2} \\ P_{cr} &= \frac{\pi^2 EI}{L^2} \end{aligned} \quad (13-17)$$

which is the same result found from the Euler formula, Eq. 13-5.

If Eq. 13-16 is plotted as load P versus deflection $v_{\max}$ for various values of eccentricity e , the family of colored curves shown in Fig. 13-14 results. Here the critical load becomes an asymptote to the curves, and of

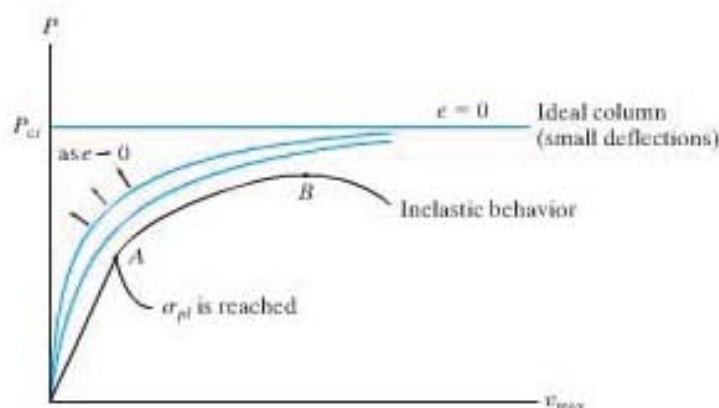


Fig. 13-14

course represents the unrealistic case of an ideal column ($e = 0$). As stated earlier, e is *never* zero due to imperfections in initial column straightness and load application; however, as $e \rightarrow 0$, the curves tend to approach the ideal case. Furthermore, these curves are appropriate only for *small deflections*, since the curvature was approximated by d^2v/dx^2 when Eq. 13-16 was developed. Had a more exact analysis been performed, all these curves would tend to turn upward, intersecting and then rising above the line $P = P_{cr}$. This, of course, indicates that a larger load P is needed to create larger column deflections. We have not considered this analysis here, however, since most often engineering design restricts the deflection of columns to small values.

It should also be noted that the colored curves in Fig. 13-14 apply only for linear-elastic material behavior. Such is the case if the column is long and slender. However, if a short or intermediate-length stocky column is considered, then the applied load, as it is increased, may eventually cause the material to yield, and the column will begin to behave in an *inelastic manner*. This occurs at point A for the black curve in Fig. 13-14. As the load is further increased, the curve never reaches the critical load, and instead the load reaches a maximum value at B . Afterwards, a sudden decrease in load-carrying capacity occurs as the column continues to yield and deflect by larger amounts.

Lastly, the colored curves in Fig. 13-14 also illustrate that a *nonlinear* relationship occurs between the load P and the deflection v . As a result, the principle of superposition *cannot be used* to determine the total deflection of a column caused by applying *successive loads* to the column. Instead, the loads must first be added, and then the corresponding deflection due to their resultant can be determined. Physically, the reason that successive loads and deflections cannot be superimposed is that the column's internal moment *depends on both* the load P and the deflection v , that is, $M = -P(e + v)$, Eq. 13-13.

The Secant Formula. The maximum stress in the column can be determined by realizing that it is caused by both the axial load and the moment, Fig. 13-15a. Maximum moment occurs at the column's midpoint, and using Eqs. 13-13 and 13-16, it has a magnitude of

$$M = |P(e + v_{\max})| \quad M = Pe \sec\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) \quad (13-18)$$

As shown in Fig. 13-15b, the maximum stress in the column is compressive, and it has a value of

$$\sigma_{\max} = \frac{P}{A} + \frac{Mc}{I}; \quad \sigma_{\max} = \frac{P}{A} + \frac{Pec}{I} \sec\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right)$$

Since the radius of gyration is defined as $r^2 = I/A$, the above equation can be written in a form called the *secant formula*:

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec\left(\frac{L}{2r} \sqrt{\frac{P}{EA}}\right) \right] \quad (13-19)$$

Here

$\sigma_{\max}$ = maximum *elastic stress* in the column, which occurs at the inner concave side at the column's midpoint. This stress is compressive

P = vertical load applied to the column. $P < P_{cr}$ unless $e = 0$; then $P = P_{cr}$ (Eq. 13-5)

e = eccentricity of the load P , measured from the centroidal axis of the column's cross-sectional area to the line of action of P

c = distance from the centroidal axis to the outer fiber of the column where the maximum compressive stress $\sigma_{\max}$ occurs

A = cross-sectional area of the column

L = unsupported length of the column *in the plane of bending*. For supports other than pins, the effective length $L_e = KL$ should be used. See Fig. 13-10

E = modulus of elasticity for the material

r = radius of gyration, $r = \sqrt{I/A}$, where I is calculated about the centroidal or bending axis

Like Eq. 13-16, Eq. 13-19 indicates that there is a nonlinear relationship between the load and the stress. Hence, the principle of superposition does not apply, and therefore the loads have to be added *before* the stress is determined. Furthermore, due to this nonlinear relationship, any factor of safety used for design purposes applies to the load and not to the stress.

For a given value of $\sigma_{\max}$, graphs of Eq. 13-19 can be plotted as the slenderness ratio KL/r versus the average stress P/A for various values of the *eccentricity ratio* ec/r^2 . A specific set of graphs for a structural-grade A-36 steel having a yield point of $\sigma_{\max} = \sigma_Y = 36$ ksi

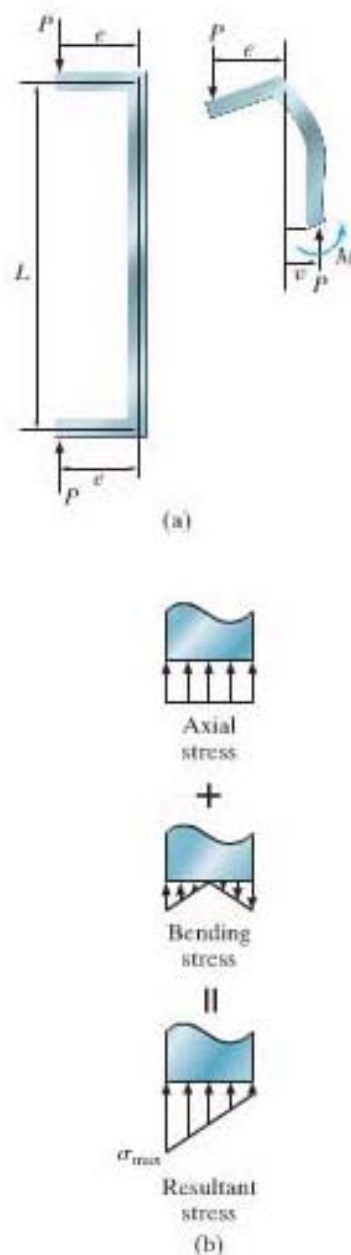


Fig. 13-15

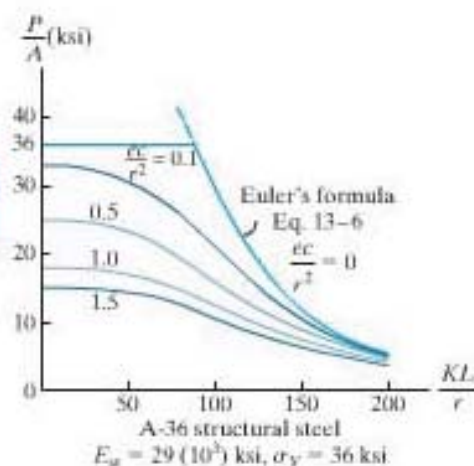


Fig. 13-16

and a modulus of elasticity of $E_{st} = 29(10^3)$ ksi is shown in Fig. 13-16. Note that when $e \rightarrow 0$, or when $ec/r^2 \rightarrow 0$, Eq. 13-19 gives $\sigma_{\max} = P/A$, where P is the critical load on the column, defined by Euler's formula. This results in Eq. 13-6, which has been plotted in Fig. 13-7 and repeated in Fig. 13-16. Since both Eqs. 13-6 and 13-19 are valid only for elastic loadings, the stresses shown in Fig. 13-16 cannot exceed $\sigma_Y = 36$ ksi, represented here by the horizontal line.

The curves in Fig. 13-16 indicate that differences in the eccentricity ratio have a marked effect on the load-carrying capacity of columns that have small slenderness ratios. However, columns that have large slenderness ratios tend to fail at or near the Euler critical load regardless of the eccentricity ratio. When using Eq. 13-19 for design purposes, it is therefore important to have a somewhat accurate value for the eccentricity ratio for shorter-length columns.

Design. Once the eccentricity ratio has been determined, the column data can be substituted into Eq. 13-19. If a value of $\sigma_{\max} = \sigma_Y$ is chosen, then the corresponding load P_Y can be determined from a trial-and-error procedure, since the equation is transcendental and cannot be solved explicitly for P_Y . As a design aid, computer software, or graphs such as those in Fig. 13-16, can also be used to determine P_Y directly.

Realize that P_Y is the load that will cause the column to develop a maximum compressive stress of σ_Y at its inner concave fibers. Due to the eccentric application of P_Y , this load will always be smaller than the critical load P_{cr} , which is determined from the Euler formula that assumes (unrealistically) that the column is axially loaded. Once P_Y is obtained, an appropriate factor of safety can then be applied in order to specify the column's safe load.

Important Points

- Due to imperfections in manufacturing or specific application of the load, a column will never suddenly buckle; instead, it begins to bend.
- The load applied to a column is related to its deflection in a nonlinear manner, and so the principle of superposition does not apply.
- As the slenderness ratio increases, eccentrically loaded columns tend to fail at or near the Euler buckling load.

EXAMPLE 13.4

The $W8 \times 40$ A-36 steel column shown in Fig. 13-17a is fixed at its base and braced at the top so that it is fixed from displacement, yet free to rotate about the y - y axis. Also, it can sway to the side in the y - z plane. Determine the maximum eccentric load the column can support before it either begins to buckle or the steel yields.

SOLUTION

From the support conditions it is seen that about the y - y axis the column behaves as if it were pinned at its top and fixed at the bottom and subjected to an axial load P , Fig. 13-17b. About the x - x axis the column is free at the top and fixed at the bottom, and it is subjected to both an axial load P and moment $M = P(9 \text{ in.})$, Fig. 13-17c.

y - y Axis Buckling. From Fig. 13-10d the effective length factor is $K_y = 0.7$, so $(KL)_y = 0.7(12) \text{ ft} = 8.40 \text{ ft} = 100.8 \text{ in.}$ Using the table in Appendix B to determine I_y for the $W8 \times 40$ section and applying Eq. 13-11, we have

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2 [29(10^3) \text{ ksi}] (49.1 \text{ in}^4)}{(100.8 \text{ in.})^2} = 1383 \text{ kip}$$

x - x Axis Yielding. From Fig. 13-10b, $K_x = 2$, so $(KL)_x = 2(12) \text{ ft} = 24 \text{ ft} = 288 \text{ in.}$ Again using the table in Appendix B to determine $A = 11.7 \text{ in}^2$, $c = 8.25 \text{ in.}/2 = 4.125 \text{ in.}$, and $r_x = 3.53 \text{ in.}$, and applying the secant formula, we have

$$\sigma_Y = \frac{P_x}{A} \left[1 + \frac{ec}{r_x^2} \sec \left(\frac{(KL)_x}{2r_x} \sqrt{\frac{P_x}{EA}} \right) \right]$$

Substituting the data and simplifying yields

$$421.2 = P_x [1 + 2.979 \sec(0.0700 \sqrt{P_x})]$$

Solving for P_x by trial and error, noting that the argument for the secant is in radians, we get

$$P_x = 88.4 \text{ kip} \quad \text{Ans.}$$

Since this value is less than $(P_{cr})_y = 1383 \text{ kip}$, failure will occur about the x - x axis.

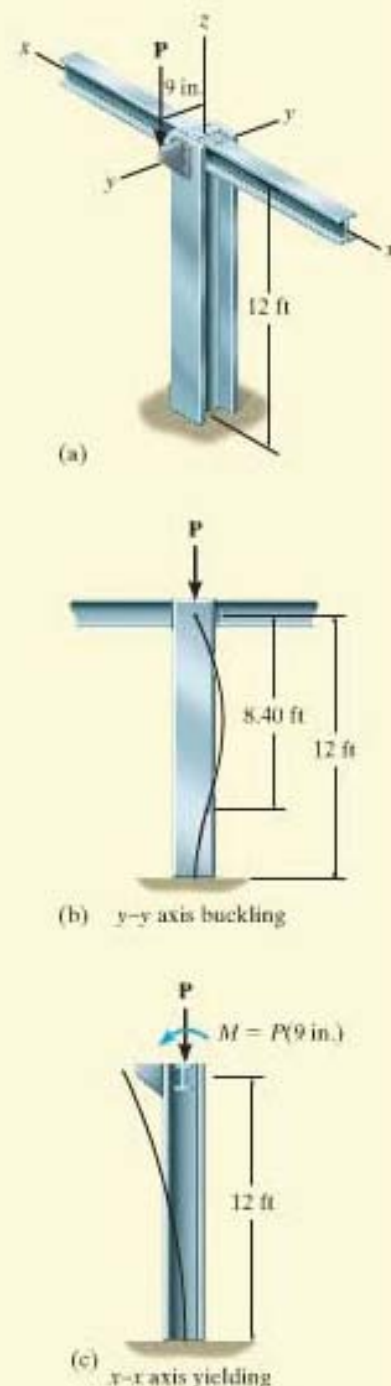


Fig. 13-17

*13.5 Inelastic Buckling



This crane boom failed by buckling caused by an overload. Note the region of localized collapse.

In engineering practice, columns are generally classified according to the type of stresses developed within the column at the time of failure. *Long slender columns* will become unstable when the compressive stress remains elastic. The failure that occurs is referred to as *elastic instability*. *Intermediate columns* fail due to *inelastic instability*, meaning that the compressive stress at failure is greater than the material's proportional limit. And *short columns*, sometimes called *posts*, do not become unstable; rather the material simply yields or fractures.

Application of the Euler equation requires that the stress in the column remain *below* the material's yield point (actually the proportional limit) when the column buckles, and so this equation applies only to long columns. In practice, however, most columns are selected to have intermediate lengths. The behavior of these columns can be studied by modifying the Euler equation so that it applies for inelastic buckling. To show how this can be done, consider the material to have a stress-strain diagram as shown in Fig. 13-18a. Here the proportional limit is σ_{pl} , and the modulus of elasticity, or slope of the line AB , is E .

If the column has a slenderness ratio that is *less* than $(KL/r)_{pl}$, then the critical stress in the column must be greater than σ_{pl} . For example, suppose a column has a slenderness ratio of $(KL/r)_1 < (KL/r)_{pl}$, with corresponding critical stress $\sigma_D > \sigma_{pl}$ needed to cause instability. When the column is *about to buckle*, the change in stress and strain that occurs in the column is within a *small range* $\Delta\sigma$ and $\Delta\epsilon$, so that the modulus of elasticity or stiffness for the material can be taken as the *tangent modulus* $E_t = \Delta\sigma/\Delta\epsilon$ defined as the slope of the σ - ϵ diagram at point D , Fig. 13-18a. In other words, at the time of failure, the column behaves as if it were made from a material that has a *lower stiffness* than when it behaves elastically, $E_t < E$.

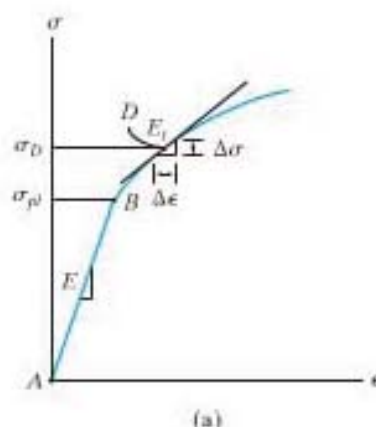


Fig. 13-18

In general, therefore, as the slenderness ratio (KL/r) decreases, the critical stress for a column continues to rise; and from the σ - ϵ diagram, the *tangent modulus* for the material decreases. Using this idea, we can modify Euler's equation to include these cases of inelastic buckling by substituting the material's tangent modulus E_t for E , so that

$$\sigma_{cr} = \frac{\pi^2 E_t}{(KL/r)^2} \quad (13-20)$$

This is the so-called *tangent modulus* or *Engesser equation*, proposed by F. Engesser in 1889. A plot of this equation for intermediate and short-length columns of a material defined by the σ - ϵ diagram in Fig. 13-18a is shown in Fig. 13-18b.

No actual column can be considered to be either perfectly straight or loaded along its centroidal axis, as assumed here, and therefore it is indeed very difficult to develop an expression that will provide a complete analysis of this phenomenon. As a result, other methods of describing the inelastic buckling of columns have been considered. One of these methods was developed by the aeronautical engineer F. R. Shanley and is called the *Shanley theory* of inelastic buckling. Although it provides a better description of the phenomenon than the tangent modulus theory, as explained here, experimental testing of a large number of columns, each of which approximates the ideal column, has shown that Eq. 13-20 is *reasonably accurate* in predicting the column's critical stress. Furthermore, the tangent modulus approach to modeling inelastic column behavior is relatively easy to apply.

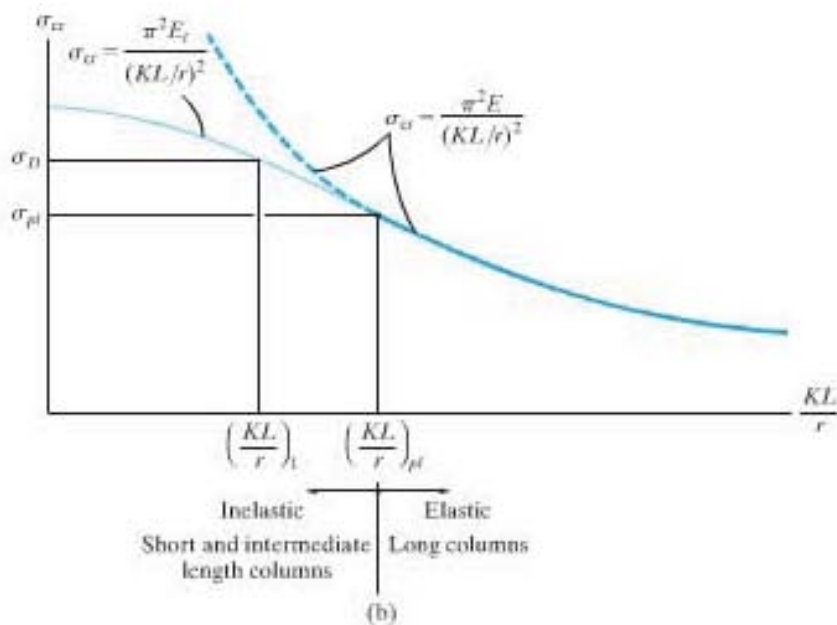


Fig. 13-18 (cont.)

EXAMPLE 13.5

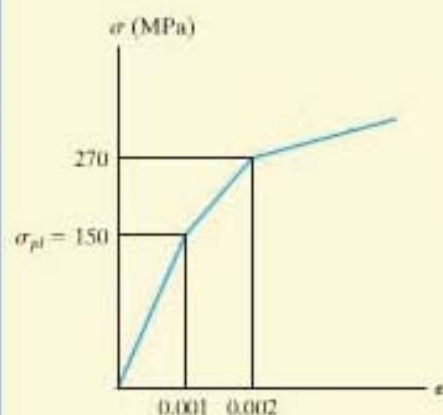


Fig. 13-19

A solid rod has a diameter of 30 mm and is 600 mm long. It is made of a material that can be modeled by the stress-strain diagram shown in Fig. 13-19. If it is used as a pin-supported column, determine the critical load.

SOLUTION

The radius of gyration is

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{(\pi/4)(15 \text{ mm})^4}{\pi(15 \text{ mm})^2}} = 7.5 \text{ mm}$$

and therefore the slenderness ratio is

$$\frac{KL}{r} = \frac{1(600 \text{ mm})}{7.5 \text{ mm}} = 80$$

Applying Eq. 13-20 we have,

$$\sigma_{cr} = \frac{\pi^2 E_t}{(KL/r)^2} = \frac{\pi^2 E_t}{(80)^2} = 1.542(10^{-3})E_t \quad (1)$$

First we will assume that the critical stress is elastic. From Fig. 13-19,

$$E = \frac{150 \text{ MPa}}{0.001} = 150 \text{ GPa}$$

Thus, Eq. 1 becomes

$$\sigma_{cr} = 1.542(10^{-3})[150(10^3)] \text{ MPa} = 231.3 \text{ MPa}$$

Since $\sigma_{cr} > \sigma_{y} = 150 \text{ MPa}$, inelastic buckling occurs.

From the second line segment of the σ - ϵ diagram, Fig. 13-19, we have

$$E_t = \frac{\Delta\sigma}{\Delta\epsilon} = \frac{270 \text{ MPa} - 150 \text{ MPa}}{0.002 - 0.001} = 120 \text{ GPa}$$

Applying Eq. 1 yields

$$\sigma_{cr} = 1.542(10^{-3})[120(10^3)] \text{ MPa} = 185.1 \text{ MPa}$$

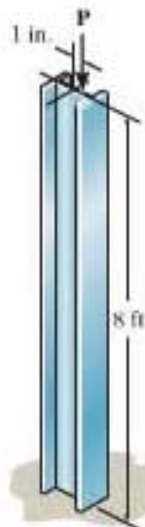
Since this value falls within the limits of 150 MPa and 270 MPa, it is indeed the critical stress.

The critical load on the rod is therefore

$$P_{cr} = \sigma_{cr} A = 185.1(10^6) \text{ Pa}[\pi(0.015 \text{ m})^2] = 131 \text{ kN} \quad \text{Ans}$$

PROBLEMS

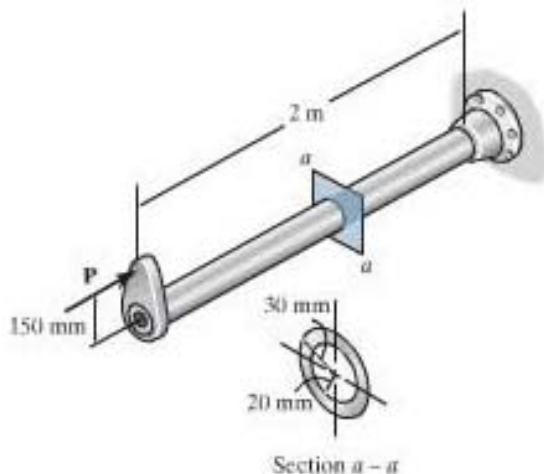
13-46. Determine the load P required to cause the A-36 steel $W8 \times 15$ column to fail either by buckling or by yielding. The column is fixed at its base and free at its top.



Prob. 13-46

13-47. The hollow red brass C83400 copper alloy shaft is fixed at one end but free at the other end. Determine the maximum eccentric force P the shaft can support without causing it to buckle or yield. Also, find the corresponding maximum deflection of the shaft.

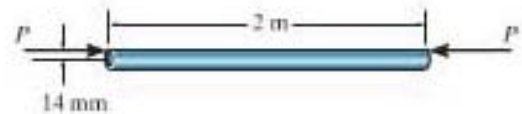
***13-48.** The hollow red brass C83400 copper alloy shaft is fixed at one end but free at the other end. If the eccentric force $P = 5$ kN is applied to the shaft as shown, determine the maximum normal stress and the maximum deflection.



Probs. 13-47/48

***13-49.** The tube is made of copper and has an outer diameter of 35 mm and a wall thickness of 7 mm. Using a factor of safety with respect to buckling and yielding of $F.S. = 2.5$, determine the allowable eccentric load P . The tube is pin supported at its ends. $E_{cu} = 120$ GPa, $\sigma_Y = 750$ MPa.

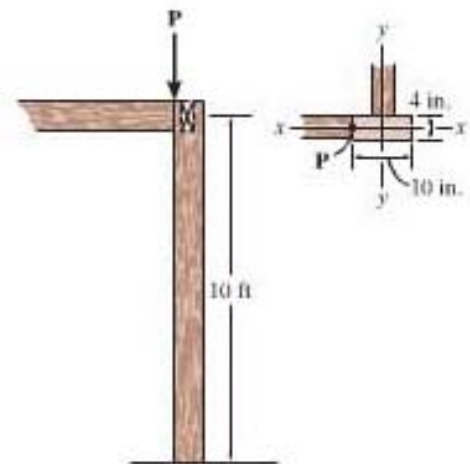
13-50. The tube is made of copper and has an outer diameter of 35 mm and a wall thickness of 7 mm. Using a factor of safety with respect to buckling and yielding of $F.S. = 2.5$, determine the allowable eccentric load P that it can support without failure. The tube is fixed supported at its ends. $E_{cu} = 120$ GPa, $\sigma_Y = 750$ MPa.



Probs. 13-49/50

13-51. The wood column is fixed at its base and can be assumed pin connected at its top. Determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.

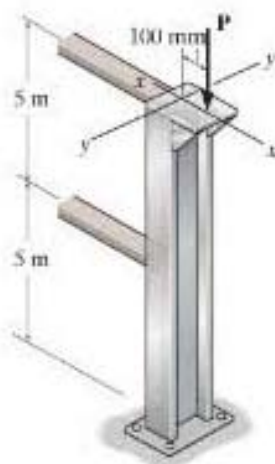
***13-52.** The wood column is fixed at its base and can be assumed fixed connected at its top. Determine the maximum eccentric load P that can be applied without causing the column to buckle or yield. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.



Probs. 13-51/52

13-53. The $W200 \times 22$ A-36-steel column is fixed at its base. Its top is constrained to rotate about the y - y axis and free to move along the y - y axis. Also, the column is braced along the x - x axis at its mid-height. Determine the allowable eccentric force P that can be applied without causing the column either to buckle or yield. Use F.S. = 2 against buckling and F.S. = 1.5 against yielding.

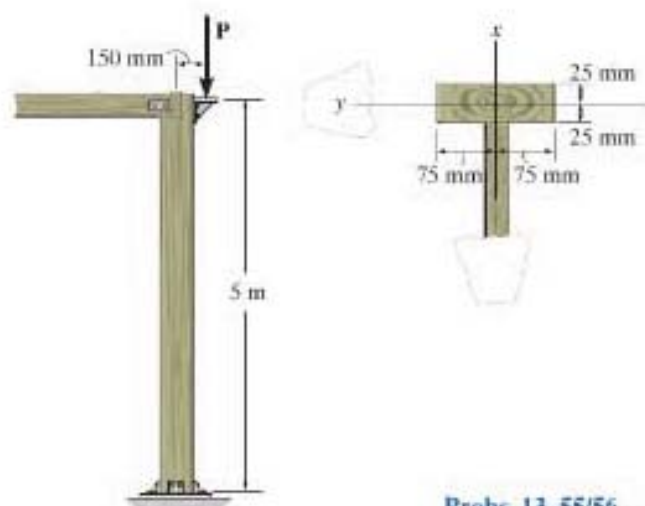
13-54. The $W200 \times 22$ A-36-steel column is fixed at its base. Its top is constrained to rotate about the y - y axis and free to move along the y - y axis. Also, the column is braced along the x - x axis at its mid-height. If $P = 25$ kN, determine the maximum normal stress developed in the column.



Probs. 13-53/54

13-55. The wood column is fixed at its base, and its top can be considered pinned. If the eccentric force $P = 10$ kN is applied to the column, investigate whether the column is adequate to support this loading without buckling or yielding. Take $E = 10$ GPa and $\sigma_Y = 15$ MPa.

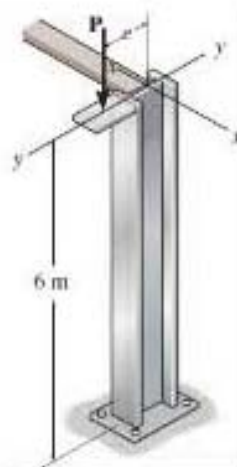
***13-56.** The wood column is fixed at its base, and its top can be considered pinned. Determine the maximum eccentric force P the column can support without causing it to either buckle or yield. Take $E = 10$ GPa and $\sigma_Y = 15$ MPa.



Probs. 13-55/56

***13-57.** The $W250 \times 28$ A-36-steel column is fixed at its base. Its top is constrained to rotate about the y - y axis and free to move along the y - y axis. If $e = 350$ mm, determine the allowable eccentric force P that can be applied without causing the column either to buckle or yield. Use F.S. = 2 against buckling and F.S. = 1.5 against yielding.

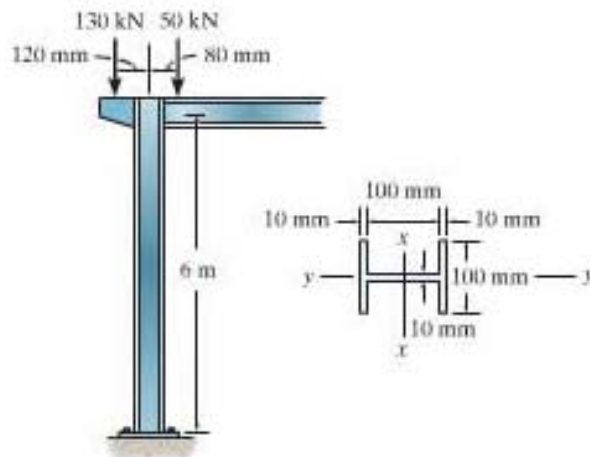
13-58. The $W250 \times 28$ A-36-steel column is fixed at its base. Its top is constrained to rotate about the y - y axis and free to move along the y - y axis. Determine the force P and its eccentricity e so that the column will yield and buckle simultaneously.



Probs. 13-57/58

13-59. The steel column supports the two eccentric loadings. If it is assumed to be pinned at its top, fixed at the bottom, and fully braced against buckling about the y - y axis, determine the maximum deflection of the column and the maximum stress in the column. $E_{st} = 200 \text{ GPa}$, $\sigma_Y = 360 \text{ MPa}$.

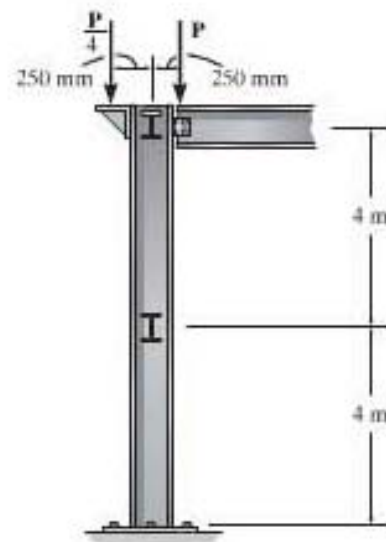
***13-60.** The steel column supports the two eccentric loadings. If it is assumed to be fixed at its top and bottom, and braced against buckling about the y - y axis, determine the maximum deflection of the column and the maximum stress in the column. $E_{st} = 200 \text{ GPa}$, $\sigma_Y = 360 \text{ MPa}$.



Probs. 13-59/60

13-61. The $W250 \times 45$ A-36-steel column is pinned at its top and fixed at its base. Also, the column is braced along its weak axis at mid-height. If $P = 250 \text{ kN}$, investigate whether the column is adequate to support this loading. Use $F.S. = 2$ against buckling and $F.S. = 1.5$ against yielding.

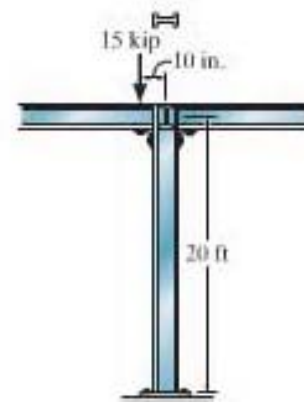
***13-62.** The $W250 \times 45$ A-36-steel column is pinned at its top and fixed at its base. Also, the column is braced along its weak axis at mid-height. Determine the allowable force P that the column can support without causing it either to buckle or yield. Use $F.S. = 2$ against buckling and $F.S. = 1.5$ against yielding.



Probs. 13-61/62

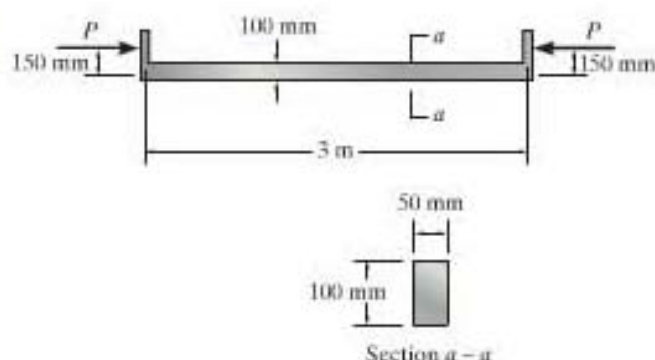
13-63. The $W14 \times 26$ structural A-36 steel member is used as a 20-ft-long column that is assumed to be fixed at its top and fixed at its bottom. If the 15-kip load is applied at an eccentric distance of 10 in., determine the maximum stress in the column.

***13-64.** The $W14 \times 26$ structural A-36 steel member is used as a column that is assumed to be fixed at its top and pinned at its bottom. If the 15-kip load is applied at an eccentric distance of 10 in., determine the maximum stress in the column.



Probs. 13-63/64

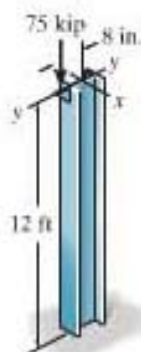
•13-65. Determine the maximum eccentric load P the 2014-T6-aluminum-alloy strut can support without causing it either to buckle or yield. The ends of the strut are pin-connected.



Prob. 13-65

13-66. The $W8 \times 48$ structural A-36 steel column is fixed at its bottom and free at its top. If it is subjected to the eccentric load of 75 kip, determine the factor of safety with respect to either the initiation of buckling or yielding.

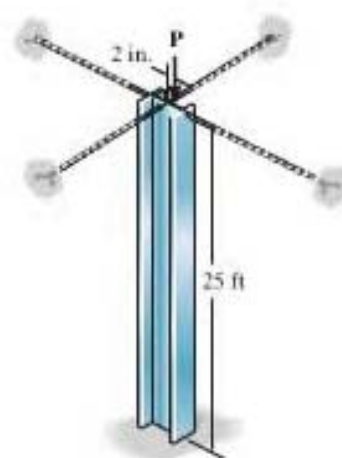
13-67. The $W8 \times 48$ structural A-36 steel column is fixed at its bottom and pinned at its top. If it is subjected to the eccentric load of 75 kip, determine if the column fails by yielding. The column is braced so that it does not buckle about the y - y axis.



Probs. 13-66/67

*13-68. Determine the load P required to cause the steel $W12 \times 50$ structural A-36 steel column to fail either by buckling or by yielding. The column is fixed at its bottom and the cables at its top act as a pin to hold it.

•13-69. Solve Prob. 13-68 if the column is an A-36 steel $W12 \times 16$ section.

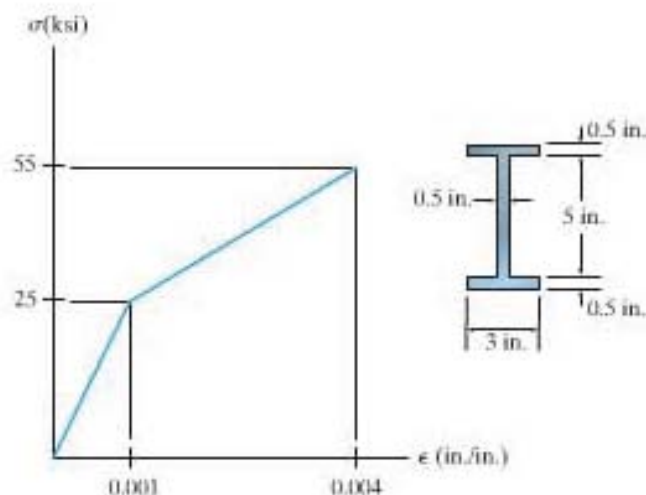


Probs. 13-68/69

13-70. A column of intermediate length buckles when the compressive stress is 40 ksi. If the slenderness ratio is 60, determine the tangent modulus.

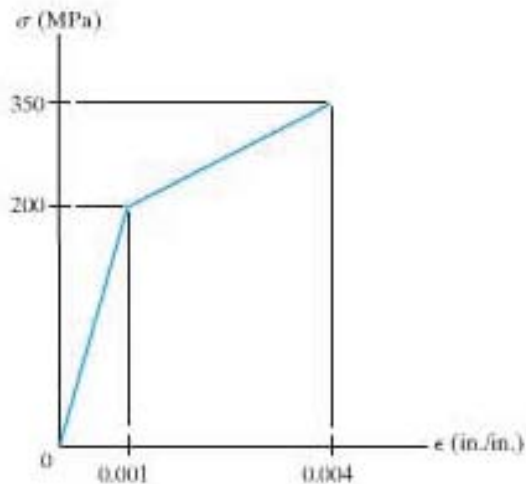
13-71. The 6-ft-long column has the cross section shown and is made of material which has a stress-strain diagram that can be approximated as shown. If the column is pinned at both ends, determine the critical load P_{cr} for the column.

*13-72. The 6-ft-long column has the cross section shown and is made of material which has a stress-strain diagram that can be approximated as shown. If the column is fixed at both ends, determine the critical load P_{cr} for the column.



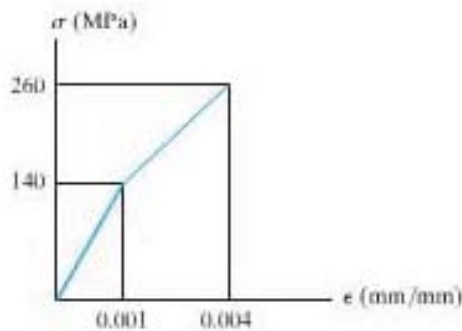
Probs. 13-71/72

- 13-73. The stress-strain diagram of the material of a column can be approximated as shown. Plot P/A vs. KL/r for the column.



Prob. 13-73

- 13-74. Construct the buckling curve, P/A versus L/r , for a column that has a bilinear stress-strain curve in compression as shown. The column is pinned at its ends.

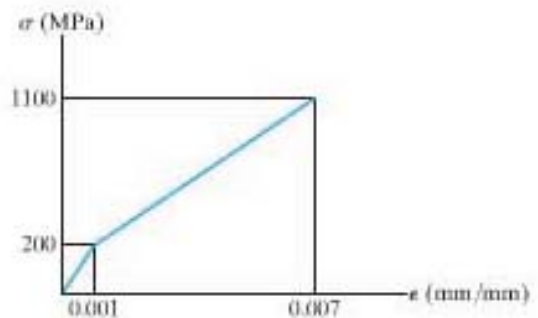


Prob. 13-74

- 13-75. The stress-strain diagram for a material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and a length of 1.5 m is made from this material, determine the critical load provided the ends are pinned. Assume that the load acts through the axis of the bar. Use Engesser's equation.

- 13-76. The stress-strain diagram for a material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and a length of 1.5 m is made from this material, determine the critical load provided the ends are fixed. Assume that the load acts through the axis of the bar. Use Engesser's equation.

- 13-77. The stress-strain diagram for a material can be approximated by the two line segments shown. If a bar having a diameter of 80 mm and length of 1.5 m is made from this material, determine the critical load provided one end is pinned and the other is fixed. Assume that the load acts through the axis of the bar. Use Engesser's equation.



Probs. 13-75/76/77

*13.6 Design of Columns for Concentric Loading



These long unbraced timber columns are used to support the roof of this building.

The theory presented thus far applies to columns that are perfectly straight, made of homogeneous material, and originally stress free. Practically speaking, though, as stated previously, columns are not perfectly straight, and most have residual stresses in them, primarily due to nonuniform cooling during manufacture. Also, the supports for columns are less than exact, and the points of application and directions of loads are not known with absolute certainty. In order to compensate for these effects, which actually vary from one column to the next, many design codes specify the use of column formulas that are empirical. By performing experimental tests on a large number of axially loaded columns, the results may be plotted and a design formula developed by curve-fitting the mean of the data.

An example of such tests for wide-flange steel columns is shown in Fig. 13–20. Notice the similarity between these results and those of the family of curves determined from the secant formula, Fig. 13–16. The reason for this similarity has to do with the influence of an “accidental” eccentricity ratio on the column’s strength. As stated in Sec. 13.4, this ratio has more of an effect on the strength of short and intermediate-length columns than on those that are long. Tests have indicated that ec/r^2 can range from 0.1 to 0.6 for most axially loaded columns.

In order to account for the behavior of different-length columns, design codes usually specify several formulas that will best fit the data within the short, intermediate, and long column range. Hence, each formula will apply only for a specific *range* of slenderness ratios, and so it is important that the engineer carefully observe the KL/r limits for which a particular formula is valid. Examples of design formulas for steel, aluminum, and wood columns that are currently in use will now be discussed. The purpose is to give some idea as to how columns are designed in practice. These formulas should not, however, be used for the design of actual columns, unless the code from which they are referenced is consulted.

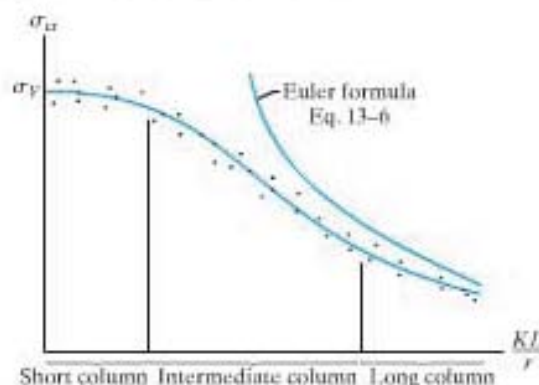


Fig. 13–20

Steel Columns. Columns made of structural steel can be designed on the basis of formulas proposed by the Structural Stability Research Council (SSRC). Factors of safety have been applied to these formulas and adopted as specifications for building construction by the American Institute of Steel Construction (AISC). Basically these specifications provide two formulas for column design, each of which gives the maximum allowable stress in the column for a specific range of slenderness ratios.*

For long columns the Euler formula is proposed, i.e., $\sigma_{\max} = \pi^2 E / (KL/r)^2$.

Application of this formula requires that a factor of safety F.S. = $\frac{23}{12} \approx 1.92$ be applied. Thus, for design,

$$\sigma_{\text{allow}} = \frac{12\pi^2 E}{23(KL/r)^2} \quad \left(\frac{KL}{r} \right)_c \leq \frac{KL}{r} \leq 200 \quad (13-21)$$

As stated, this equation is applicable for a slenderness ratio bounded by 200 and $(KL/r)_c$. A specific value of $(KL/r)_c$ is obtained by requiring the Euler formula to be used only for elastic material behavior. Through experiments it has been determined that compressive residual stresses can exist in rolled-formed steel sections that may be as much as one-half the yield stress. Consequently, if the stress in the Euler formula is greater than $\frac{1}{2}\sigma_Y$, the equation will not apply. Therefore the value of $(KL/r)_c$ is determined as follows:

$$\frac{1}{2}\sigma_Y = \frac{\pi^2 E}{(KL/r)_c^2} \quad \text{or} \quad \left(\frac{KL}{r} \right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} \quad (13-22)$$

Columns having slenderness ratios less than $(KL/r)_c$ are designed on the basis of an empirical formula that is parabolic and has the form

$$\sigma_{\max} = \left[1 - \frac{(KL/r)^2}{2(KL/r)_c^2} \right] \sigma_Y$$

Since there is more uncertainty in the use of this formula for longer columns, it is divided by a factor of safety defined as follows:

$$\text{F.S.} = \frac{5}{3} + \frac{3}{8} \frac{(KL/r)}{(KL/r)_c} - \frac{(KL/r)^3}{8(KL/r)_c^3}$$

Here it is seen that F.S. = $\frac{5}{3} \approx 1.67$ at $KL/r = 0$ and increases to F.S. = $\frac{23}{12} \approx 1.92$ at $(KL/r)_c$. Hence, for design purposes,

$$\sigma_{\text{allow}} = \frac{\left[1 - \frac{(KL/r)^2}{2(KL/r)_c^2} \right] \sigma_Y}{(5/3) + [(3/8)(KL/r)/(KL/r)_c] - [(KL/r)^3/8(KL/r)_c^3]} \quad (13-23)$$

Equations 13-21 and 13-23 are plotted in Fig. 13-21. When applying any of these equations, either FPS or SI units can be used for the calculations.

*The current AISC code enables engineers to use one of two methods for design, namely, Load and Resistance Factor Design, and Allowable Stress Design. The latter is explained here.

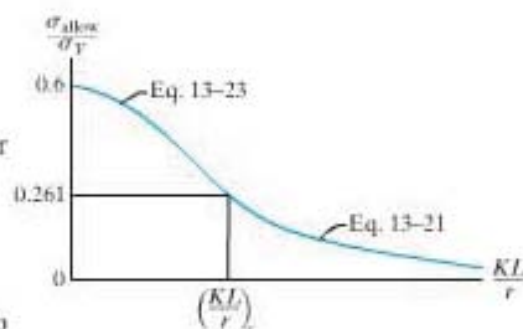


Fig. 13-21

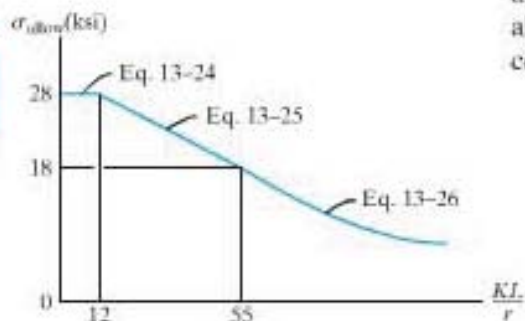


Fig. 13-22

Aluminum Columns. Column design for structural aluminum is specified by the Aluminum Association using three equations, each applicable for a specific range of slenderness ratios. Since several types of aluminum alloy exist, there is a unique set of formulas for each type. For a common alloy (2014-T6) used in building construction, the formulas are

$$\sigma_{\text{allow}} = 28 \text{ ksi} \quad 0 \leq \frac{KL}{r} \leq 12 \quad (13-24)$$

$$\sigma_{\text{allow}} = \left[30.7 - 0.23 \left(\frac{KL}{r} \right) \right] \text{ ksi} \quad 12 < \frac{KL}{r} < 55 \quad (13-25)$$

$$\sigma_{\text{allow}} = \frac{54,000 \text{ ksi}}{(KL/r)^2} \quad 55 \leq \frac{KL}{r} \quad (13-26)$$

These equations are plotted in Fig. 13-22. As shown, the first two represent straight lines and are used to model the effects of columns in the short and intermediate range. The third formula has the same form as the Euler formula and is used for long columns.

Timber Columns. Columns used in timber construction are designed on the basis of formulas published by the National Forest Products Association (NFPA) or the American Institute of Timber Construction (AITC). For example, the NFPA formulas for the allowable stress in short, intermediate, and long columns having a rectangular cross section of dimensions b and d , where d is the *smallest* dimension of the cross section, are

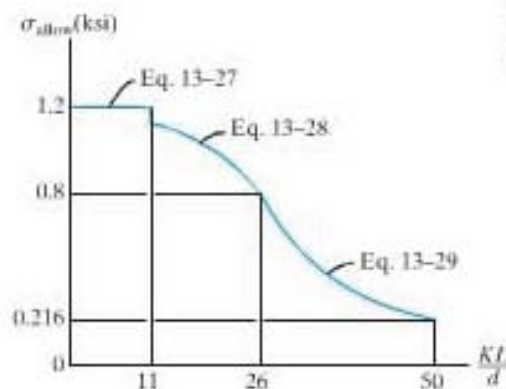


Fig. 13-23

$$\sigma_{\text{allow}} = 1.20 \text{ ksi} \quad 0 \leq \frac{KL}{d} \leq 11 \quad (13-27)$$

$$\sigma_{\text{allow}} = 1.20 \left[1 - \frac{1}{3} \left(\frac{KL/d}{26.0} \right)^2 \right] \text{ ksi} \quad 11 < \frac{KL}{d} \leq 26 \quad (13-28)$$

$$\sigma_{\text{allow}} = \frac{540 \text{ ksi}}{(KL/d)^2} \quad 26 < \frac{KL}{d} \leq 50 \quad (13-29)$$

Here wood has a modulus of elasticity of $E_w = 1.8(10^3) \text{ ksi}$ and an allowable compressive stress of 1.2 ksi parallel to the grain. In particular, Eq. 13-29 is simply Euler's equation having a factor of safety of 3. These three equations are plotted in Fig. 13-23.

Procedure for Analysis

Column Analysis.

- When using any formula to *analyze* a column, that is, to find its allowable load, it is first necessary to calculate the slenderness ratio in order to determine which column formula applies.
- Once the average allowable stress has been calculated, the allowable load on the column is determined from $P = \sigma_{\text{allow}} A$.

Column Design.

- If a formula is used to *design* a column, that is, to determine the column's cross-sectional area for a given loading and effective length, then a trial-and-check procedure generally must be followed when the column has a composite shape, such as a wide-flange section.
- One possible way to apply a trial-and-check procedure would be to *assume* the column's cross-sectional area, A' , and calculate the corresponding stress $\sigma' = P/A'$. Also, use an appropriate design formula to determine the allowable stress σ_{allow} . From this, calculate the *required* column area $A_{\text{req'd}} = P/\sigma_{\text{allow}}$.
- If $A' > A_{\text{req'd}}$, the design is safe. When making the comparison, it is practical to require A' to be close to but greater than $A_{\text{req'd}}$, usually within 2–3%. A redesign is necessary if $A' < A_{\text{req'd}}$.
- Whenever a trial-and-check procedure is repeated, the choice of an area is determined by the previously calculated required area. In engineering practice this method for design is usually shortened through the use of computer software or published tables and graphs.



These timber columns can be considered pinned at their bottom and fixed connected to the beams at their tops.

EXAMPLE 13.6

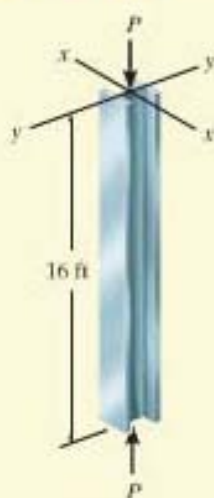


Fig. 13-24

An A-36 steel W10 $\times$ 100 member is used as a pin-supported column, Fig. 13-24. Using the AISC column design formulas, determine the largest load that it can safely support.

SOLUTION

The following data for a W10 $\times$ 100 is taken from the table in Appendix B.

$$A = 29.4 \text{ in}^2 \quad r_x = 4.60 \text{ in.} \quad r_y = 2.65 \text{ in.}$$

Since $K = 1$ for both x and y axis buckling, the slenderness ratio is largest if r_y is used. Thus,

$$\frac{KL}{r} = \frac{1(16 \text{ ft})(12 \text{ in./ft})}{2.65 \text{ in.}} = 72.45$$

From Eq. 13-22, we have

$$\begin{aligned} \left(\frac{KL}{r}\right)_c &= \sqrt{\frac{2\pi^2 E}{\sigma_y}} \\ &= \sqrt{\frac{2\pi^2 [29(10^3) \text{ ksi}]}{36 \text{ ksi}}} \\ &= 126.1 \end{aligned}$$

Here $0 < KL/r < (KL/r)_c$, so Eq. 13-23 applies.

$$\begin{aligned} \sigma_{\text{allow}} &= \frac{\left[1 - \frac{(KL/r)^2}{2(KL/r)_c^2}\right] \sigma_y}{(5/3) + [(3/8)(KL/r)/(KL/r)_c] - [(KL/r)^3/8(KL/r)_c^3]} \\ &= \frac{[1 - (72.45)^2/2(126.1)^2] 36 \text{ ksi}}{(5/3) + [(3/8)(72.45/126.1)] - [(72.45)^3/8(126.1)^3]} \\ &= 16.17 \text{ ksi} \end{aligned}$$

The allowable load P on the column is therefore

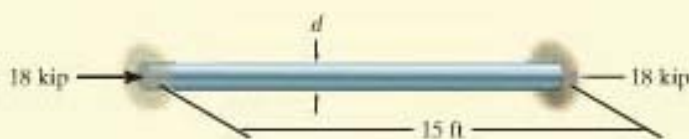
$$\sigma_{\text{allow}} = \frac{P}{A}, \quad 16.17 \text{ kip/in}^2 = \frac{P}{29.4 \text{ in}^2}$$

$$P = 476 \text{ kip}$$

Ans.

EXAMPLE 13.7

The steel rod in Fig. 13-25 is to be used to support an axial load of 18 kip. If $E_{st} = 29(10^3)$ ksi and $\sigma_Y = 50$ ksi, determine the smallest diameter of the rod as allowed by the AISC specification. The rod is fixed at both ends.

**Fig. 13-25****SOLUTION**

For a circular cross section the radius of gyration becomes

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{(1/4)\pi(d/2)^4}{(1/4)\pi d^2}} = \frac{d}{4}$$

Applying Eq. 13-22, we have

$$\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2 [29(10^3) \text{ ksi}]}{50 \text{ ksi}}} = 107.0$$

Since the rod's radius of gyration is unknown, KL/r is unknown, and therefore a choice must be made as to whether Eq. 13-21 or Eq. 13-23 applies. We will consider Eq. 13-21. For a fixed-end column $K = 0.5$, so

$$\begin{aligned} \sigma_{\text{allow}} &= \frac{12\pi^2 E}{23(KL/r)^2} \\ \frac{18 \text{ kip}}{(1/4)\pi d^2} &= \frac{12\pi^2 [29(10^3) \text{ kip/in}^2]}{23[0.5(15 \text{ ft})(12 \text{ in./ft})/(d/4)]^2} \\ \frac{22.92}{d^2} &= 1.152d^2 \\ d &= 2.11 \text{ in.} \end{aligned}$$

Use

$$d = 2.25 \text{ in.} = 2\frac{1}{4} \text{ in.} \quad \text{Ans.}$$

For this design, we must check the slenderness-ratio limits; i.e.,

$$\frac{KL}{r} = \frac{0.5(15 \text{ ft})(12 \text{ in./ft})}{(2.25 \text{ in.}/4)} = 160$$

Since $107.0 < 160 < 200$, use of Eq. 13-21 is appropriate.

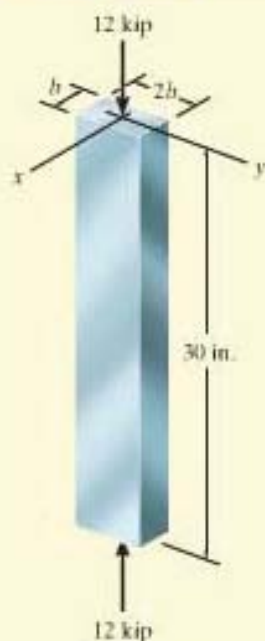
EXAMPLE 13.8

Fig. 13-26

A bar having a length of 30 in. is used to support an axial compressive load of 12 kip, Fig. 13-26. It is pin supported at its ends and made of a 2014-T6 aluminum alloy. Determine the dimensions of its cross-sectional area if its width is to be twice its thickness.

SOLUTION

Since $KL = 30$ in. is the same for both x and y axis buckling, the larger slenderness ratio is determined using the smaller radius of gyration, i.e., using $I_{\min} = I_y$:

$$\frac{KL}{r_y} = \frac{KL}{\sqrt{I_y/A}} = \frac{1(30)}{\sqrt{(1/12)2b(b^3)/[2b(b)]}} = \frac{103.9}{b} \quad (1)$$

Here we must apply Eq. 13-24, 13-25, or 13-26. Since we do not as yet know the slenderness ratio, we will begin by using Eq. 13-24.

$$\frac{P}{A} = 28 \text{ ksi}$$

$$\frac{12 \text{ kip}}{2b(b)} = 28 \text{ kip/in}^2$$

$$b = 0.463 \text{ in.}$$

Checking the slenderness ratio, we have

$$\frac{KL}{r} = \frac{103.9}{0.463} = 224.5 > 12$$

Try Eq. 13-26, which is valid for $KL/r \geq 55$,

$$\frac{P}{A} = \frac{54\,000 \text{ ksi}}{(KL/r)^2}$$

$$\frac{12}{2b(b)} = \frac{54\,000}{(103.9/b)^2}$$

$$b = 1.05 \text{ in.}$$

Ans.

From Eq. 1,

$$\frac{KL}{r} = \frac{103.9}{1.05} = 99.3 > 55 \quad \text{OK}$$

NOTE: It would be satisfactory to choose the cross section with dimensions 1 in. by 2 in.

EXAMPLE 13.9

A board having cross-sectional dimensions of 5.5 in. by 1.5 in. is used to support an axial load of 5 kip, Fig. 13-27. If the board is assumed to be pin supported at its top and bottom, determine its *greatest* allowable length L as specified by the NFPA.

**Fig. 13-27****SOLUTION**

By inspection, the board will buckle about the y axis. In the NFPA equations, $d = 1.5$ in. Assuming that Eq. 13-29 applies, we have

$$\begin{aligned}\frac{P}{A} &= \frac{540 \text{ ksi}}{(KL/d)^2} \\ \frac{5 \text{ kip}}{(5.5 \text{ in.})(1.5 \text{ in.})} &= \frac{540 \text{ ksi}}{(1 L/1.5 \text{ in.})^2} \\ L &= 44.8 \text{ in.} \quad \text{Ans.}\end{aligned}$$

Here

$$\frac{KL}{d} = \frac{1(44.8 \text{ in.})}{1.5 \text{ in.}} = 29.8$$

Since $26 < KL/d \leq 50$, the solution is valid.

PROBLEMS

13-78. Determine the largest length of a structural A-36 steel rod if it is fixed supported and subjected to an axial load of 100 kN. The rod has a diameter of 50 mm. Use the AISC equations.

13-79. Determine the largest length of a $W10 \times 45$ structural steel column if it is pin supported and subjected to an axial load of 290 kip. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 50$ ksi. Use the AISC equations.

***13-80.** Determine the largest length of a $W10 \times 12$ structural A-36 steel section if it is pin supported and is subjected to an axial load of 28 kip. Use the AISC equations.

•13-81. Using the AISC equations, select from Appendix B the lightest-weight structural A-36 steel column that is 14 ft long and supports an axial load of 40 kip. The ends are pinned. Take $\sigma_Y = 50$ ksi.

13-82. Using the AISC equations, select from Appendix B the lightest-weight structural A-36 steel column that is 12 ft long and supports an axial load of 40 kip. The ends are fixed. Take $\sigma_Y = 50$ ksi.

13-83. Using the AISC equations, select from Appendix B the lightest-weight structural A-36 steel column that is 24 ft long and supports an axial load of 100 kip. The ends are fixed.

***13-84.** Using the AISC equations, select from Appendix B the lightest-weight structural A-36 steel column that is 30 ft long and supports an axial load of 200 kip. The ends are fixed.

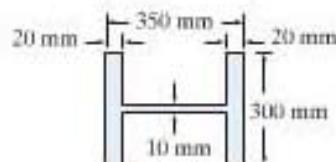
•13-85. A $W8 \times 24$ A-36-steel column of 30-ft length is pinned at both ends and braced against its weak axis at mid-height. Determine the allowable axial force P that can be safely supported by the column. Use the AISC column design formulas.

13-86. Check if a $W10 \times 39$ column can safely support an axial force of $P = 250$ kip. The column is 20 ft long and is pinned at both ends and braced against its weak axis at mid-height. It is made of steel having $E = 29(10^3)$ ksi and $\sigma_Y = 50$ ksi. Use the AISC column design formulas.

13-87. A 5-ft-long rod is used in a machine to transmit an axial compressive load of 3 kip. Determine its smallest diameter if it is pin connected at its ends and is made of a 2014-T6 aluminum alloy.

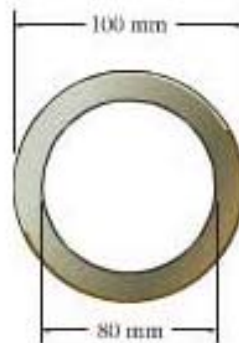
***13-88.** Check if a $W10 \times 45$ column can safely support an axial force of $P = 200$ kip. The column is 15 ft long and is pinned at both of its ends. It is made of steel having $E = 29(10^3)$ ksi and $\sigma_Y = 50$ ksi. Use the AISC column design formulas.

•13-89. Using the AISC equations, check if a column having the cross section shown can support an axial force of 1500 kN. The column has a length of 4 m, is made from A-36 steel, and its ends are pinned.



Prob. 13-89

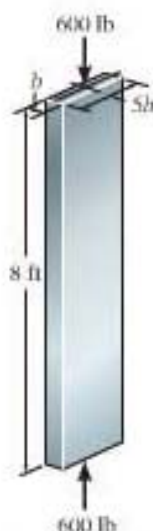
13-90. The A-36-steel tube is pinned at both ends. If it is subjected to an axial force of 150 kN, determine the maximum length that the tube can safely support using the AISC column design formulas.



Prob. 13-90

13-91. The bar is made of a 2014-T6 aluminum alloy. Determine its smallest thickness b if its width is $5b$. Assume that it is pin connected at its ends.

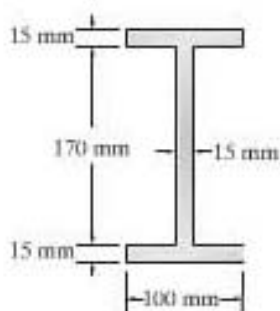
***13-92.** The bar is made of a 2014-T6 aluminum alloy. Determine its smallest thickness b if its width is $5b$. Assume that it is fixed connected at its ends.



Probs. 13-91/92

•13-93. The 2014-T6 aluminum column of 3-m length has the cross section shown. If the column is pinned at both ends and braced against the weak axis at its mid-height, determine the allowable axial force P that can be safely supported by the column.

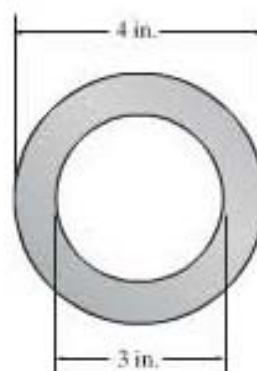
13-94. The 2014-T6 aluminum column has the cross section shown. If the column is pinned at both ends and subjected to an axial force $P = 100$ kN, determine the maximum length the column can have to safely support the loading.



Probs. 13-93/94

13-95. The 2014-T6 aluminum hollow section has the cross section shown. If the column is 10 ft long and is fixed at both ends, determine the allowable axial force P that can be safely supported by the column.

***13-96.** The 2014-T6 aluminum hollow section has the cross section shown. If the column is fixed at its base and pinned at its top, and is subjected to the axial force $P = 100$ kip, determine the maximum length of the column for it to safely support the load.

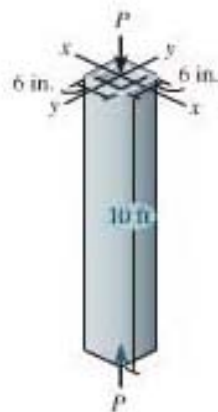


Probs. 13-95/96

•13-97. The tube is 0.25 in. thick, is made of a 2014-T6 aluminum alloy, and is fixed at its bottom and pinned at its top. Determine the largest axial load that it can support.

13-98. The tube is 0.25 in. thick, is made of a 2014-T6 aluminum alloy, and is fixed connected at its ends. Determine the largest axial load that it can support.

13-99. The tube is 0.25 in. thick, is made of 2014-T6 aluminum alloy and is pin connected at its ends. Determine the largest axial load it can support.



Probs. 13-97/98/99

***13-100.** A rectangular wooden column has the cross section shown. If the column is 6 ft long and subjected to an axial force of $P = 15$ kip, determine the required minimum dimension a of its cross-sectional area to the nearest $\frac{1}{16}$ in. so that the column can safely support the loading. The column is pinned at both ends.

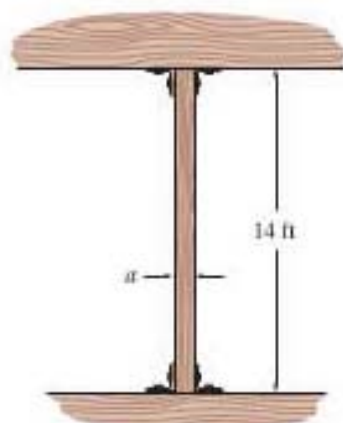
***13-101.** A rectangular wooden column has the cross section shown. If $a = 3$ in. and the column is 12 ft long, determine the allowable axial force P that can be safely supported by the column if it is pinned at its top and fixed at its base.

13-102. A rectangular wooden column has the cross section shown. If $a = 3$ in. and the column is subjected to an axial force of $P = 15$ kip, determine the maximum length the column can have to safely support the load. The column is pinned at its top and fixed at its base.



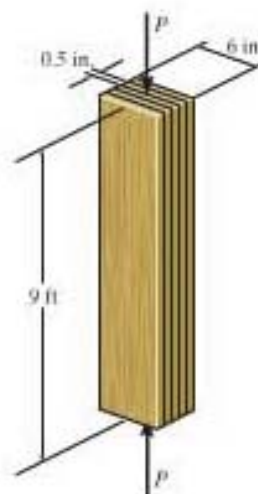
Probs. 13-100/101/102

13-103. The timber column has a square cross section and is assumed to be pin connected at its top and bottom. If it supports an axial load of 50 kip, determine its smallest side dimension a to the nearest $\frac{1}{2}$ in. Use the NFPA formulas.



Prob. 13-103

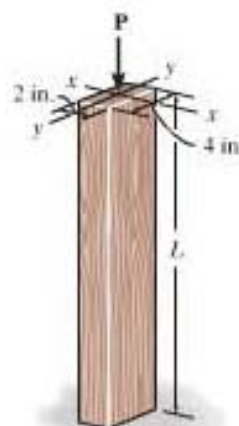
***13-104.** The wooden column shown is formed by gluing together the 6 in. $\times$ 0.5 in. boards. If the column is pinned at both ends and is subjected to an axial load $P = 20$ kip, determine the required number of boards needed to form the column in order to safely support the loading.



Prob. 13-104

***13-105.** The column is made of wood. It is fixed at its bottom and free at its top. Use the NFPA formulas to determine its greatest allowable length if it supports an axial load of $P = 2$ kip.

13-106. The column is made of wood. It is fixed at its bottom and free at its top. Use the NFPA formulas to determine the largest allowable axial load P that it can support if it has a length $L = 4$ ft.



Probs. 13-105/106

*13.7 Design of Columns for Eccentric Loading

Occasionally a column may be required to support a load acting either at its edge or on an angle bracket attached to its side, such as shown in Fig. 13-28a. The bending moment $M = Pe$, which is caused by the eccentric loading, must be accounted for when the column is designed. There are several acceptable ways in which this is done in engineering practice. We will discuss two of the most common methods.

Use of Available Column Formulas. The stress distribution acting over the cross-sectional area of the column shown in Fig. 13-28a is determined from a superposition of both the axial force P and the bending moment $M = Pe$. In particular, the maximum compressive stress is

$$\sigma_{\max} = \frac{P}{A} + \frac{Mc}{I} \quad (13-30)$$

A typical stress profile is shown in Fig. 13-28b. If we conservatively *assume* that the entire cross section is subjected to the uniform stress $\sigma_{\max}$ as determined from Eq. 13-30, then we can compare $\sigma_{\max}$ with σ_{allow} , which is determined using the formulas given in Sec. 13.6. Calculation of σ_{allow} is usually done using the *largest* slenderness ratio for the column, regardless of the axis about which the column experiences bending. This requirement is normally specified in design codes and will in most cases lead to a conservative design. If

$$\sigma_{\max} \leq \sigma_{\text{allow}}$$

then the column can carry the specified loading. If this inequality does not hold, then the column's area A must be increased, and a new $\sigma_{\max}$ and σ_{allow} must be calculated. This method of design is rather simple to apply and works well for columns that are short or of intermediate length.

Interaction Formula. When *designing* an eccentrically loaded column it is desirable to see how the bending and axial loads *interact*, so that a balance between these two effects can be achieved. To do this, we will consider the separate contributions made to the total column area by the axial force and moment. If the allowable stress for the axial load is $(\sigma_a)_{\text{allow}}$, then the required area for the column needed to support the load P is

$$A_a = \frac{P}{(\sigma_a)_{\text{allow}}}$$

Similarly, if the allowable bending stress is $(\sigma_b)_{\text{allow}}$, then since $I = Ar^2$, the required area of the column needed to support the eccentric moment is determined from the flexure formula, that is,

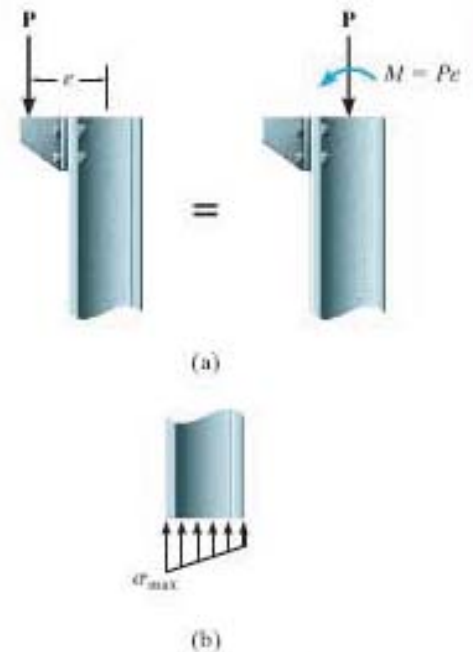


Fig. 13-28

$$A_b = \frac{Mc}{(\sigma_b)_{\text{allow}} r^2}$$

The total area A for the column needed to resist *both* the axial load and moment requires that

$$A_a + A_b = \frac{P}{(\sigma_a)_{\text{allow}}} + \frac{Mc}{(\sigma_b)_{\text{allow}} r^2} \leq A$$

or

$$\begin{aligned} \frac{P/A}{(\sigma_a)_{\text{allow}}} + \frac{Mc/Ar^2}{(\sigma_b)_{\text{allow}}} &\leq 1 \\ \frac{\sigma_a}{(\sigma_a)_{\text{allow}}} + \frac{\sigma_b}{(\sigma_b)_{\text{allow}}} &\leq 1 \end{aligned} \quad (13-31)$$

Here

σ_a = axial stress caused by the force P and determined from $\sigma_a = P/A$, where A is the cross-sectional area of the column

σ_b = bending stress caused by an eccentric load or applied moment M ; σ_b is found from $\sigma_b = Mc/I$, where I is the moment of inertia of the cross-sectional area calculated about the bending or centroidal axis

$(\sigma_a)_{\text{allow}}$ = allowable axial stress as defined by formulas given in Sec. 13.6 or by other design code specifications. For this purpose, always use the *largest* slenderness ratio for the column, regardless of the axis about which the column experiences bending

$(\sigma_b)_{\text{allow}}$ = allowable bending stress as defined by code specifications

Notice that, if the column is subjected only to an axial load, then the bending-stress ratio in Eq. 13-31 would be equal to zero and the design will be based only on the allowable axial stress. Likewise, when no axial load is present, the axial-stress ratio is zero and the stress requirement will be based on the allowable bending stress. Hence, each stress ratio indicates the contribution of axial load or bending moment. Since Eq. 13-31 shows how these loadings interact, this equation is sometimes referred to as the **interaction formula**. This design approach requires a trial-and-check procedure, where it is required that the designer *pick* an available column and then check to see if the inequality is satisfied. If it is not, a larger section is then picked and the process repeated. An economical choice is made when the left side is close to but less than 1.

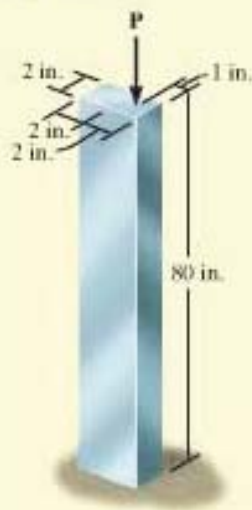
The interaction method is often specified in codes for the design of columns made of steel, aluminum, or timber. In particular, for allowable stress design, the American Institute of Steel Construction specifies the use of this equation only when the axial-stress ratio $\sigma_a/(\sigma_a)_{\text{allow}} \leq 0.15$. For other values of this ratio, a modified form of Eq. 13-31 is used.



Typical example of a column used to support an eccentric roof loading.

EXAMPLE 13.10

The column in Fig. 13-29 is made of aluminum alloy 2014-T6 and is used to support an eccentric load **P**. Determine the maximum magnitude of **P** that can be supported if the column is fixed at its base and free at its top. Use Eq. 13-30.

**Fig. 13-29****SOLUTION**

From Fig. 13-10*b*, $K = 2$. The largest slenderness ratio for the column is therefore

$$\frac{KL}{r} = \frac{2(80 \text{ in.})}{\sqrt{[(1/12)(4 \text{ in.})(2 \text{ in.})^3]/[(2 \text{ in.})(4 \text{ in.})]}} = 277.1$$

By inspection, Eq. 13-26 must be used ($277.1 > 55$). Thus,

$$\sigma_{\text{allow}} = \frac{54\,000 \text{ ksi}}{(KL/r)^2} = \frac{54\,000 \text{ ksi}}{(277.1)^2} = 0.7031 \text{ ksi}$$

The maximum compressive stress in the column is determined from the combination of axial load and bending. We have

$$\begin{aligned} \sigma_{\text{max}} &= \frac{P}{A} + \frac{(Pe)c}{I} \\ &= \frac{P}{2 \text{ in.}(4 \text{ in.})} + \frac{P(1 \text{ in.})(2 \text{ in.})}{(1/12)(2 \text{ in.})(4 \text{ in.})^3} \\ &= 0.3125P \end{aligned}$$

Assuming that this stress is *uniform* over the cross section, we require

$$\begin{aligned} \sigma_{\text{allow}} &= \sigma_{\text{max}}; & 0.7031 &= 0.3125P \\ & & P &= 2.25 \text{ kip} \end{aligned}$$

Ans.

EXAMPLE 13.11

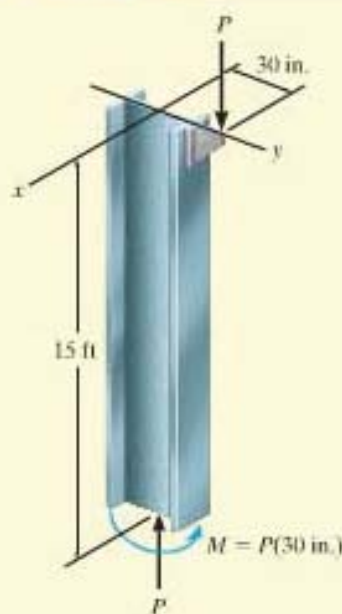


Fig. 13-30

The A-36 steel W6 $\times$ 20 column in Fig. 13-30 is pin connected at its ends and is subjected to the eccentric load P . Determine the maximum allowable value of P using the interaction method if the allowable bending stress is $(\sigma_b)_{\text{allow}} = 22$ ksi.

SOLUTION

Here $K = 1$. The necessary geometric properties for the W6 $\times$ 20 are taken from the table in Appendix B.

$$A = 5.87 \text{ in}^2 \quad I_x = 41.4 \text{ in}^4 \quad r_y = 1.50 \text{ in.} \quad d = 6.20 \text{ in.}$$

We will consider r_y because this will lead to the *largest* value of the slenderness ratio. Also, I_x is needed since bending occurs about the x axis ($c = 6.20 \text{ in.}/2 = 3.10 \text{ in.}$). To determine the allowable compressive stress, we have

$$\frac{KL}{r} = \frac{1[15 \text{ ft}(12 \text{ in./ft})]}{1.50 \text{ in.}} = 120$$

Since

$$\left(\frac{KL}{r}\right)_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2 [29(10^3) \text{ ksi}]}{36 \text{ ksi}}} = 126.1$$

then $KL/r < (KL/r)_c$ and so Eq. 13-23 must be used.

$$\begin{aligned} \sigma_{\text{allow}} &= \frac{[1 - (KL/r)^2/2(KL/r)_c^2]\sigma_Y}{(5/3) + [(3/8)(KL/r)/(KL/r)_c] - [(KL/r)^3/8(KL/r)_c^3]} \\ &= \frac{[1 - (120)^2/2(126.1)^2]36 \text{ ksi}}{(5/3) + [(3/8)(120)/(126.1)] - [(120)^3/8(126.1)^3]} \\ &= 10.28 \text{ ksi} \end{aligned}$$

Applying the interaction Eq. 13-31 yields

$$\begin{aligned} \frac{\sigma_a}{(\sigma_a)_{\text{allow}}} + \frac{\sigma_b}{(\sigma_b)_{\text{allow}}} &\leq 1 \\ \frac{P/5.87 \text{ in}^2}{10.28 \text{ ksi}} + \frac{P(30 \text{ in.})(3.10 \text{ in.})/(41.4 \text{ in}^4)}{22 \text{ ksi}} &= 1 \\ P &= 8.43 \text{ kip} \end{aligned}$$

Ans.

Checking the application of the interaction method for the steel section, we require

$$\frac{\sigma_a}{(\sigma_a)_{\text{allow}}} = \frac{8.43 \text{ kip}/(5.87 \text{ in.})}{10.28 \text{ kip/in}^2} = 0.140 < 0.15 \quad \text{OK}$$

EXAMPLE 13.12

The timber column in Fig. 13-31 is made from two boards nailed together so that the cross section has the dimensions shown. If the column is fixed at its base and free at its top, use Eq. 13-30 to determine the eccentric load **P** that can be supported.

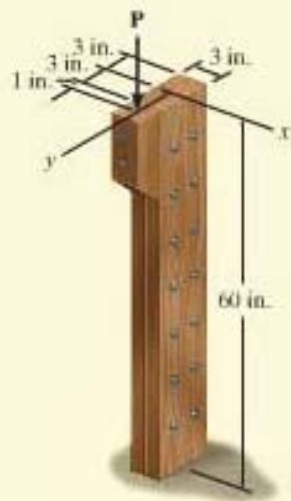


Fig. 13-31

SOLUTION

From Fig. 13-10*b*, $K = 2$. Here we must calculate KL/d to determine which equation from Eqs. 13-27 through 13-29 should be used. Since σ_{allow} is determined using the largest slenderness ratio, we choose $d = 3$ in. This is done to make this ratio as large as possible, and thereby yields the lowest possible allowable axial stress. We have

$$\frac{KL}{d} = \frac{2(60 \text{ in.})}{3 \text{ in.}} = 40$$

Since $26 < KL/d < 50$ the allowable axial stress is determined using Eq. 13-29. Thus,

$$\sigma_{\text{allow}} = \frac{540 \text{ ksi}}{(KL/d)^2} = \frac{540 \text{ ksi}}{(40)^2} = 0.3375 \text{ ksi}$$

Applying Eq. 13-30 with $\sigma_{\text{allow}} = \sigma_{\text{max}}$, we have

$$\begin{aligned} \sigma_{\text{allow}} &= \frac{P}{A} + \frac{Mc}{I} \\ 0.3375 \text{ ksi} &= \frac{P}{(3 \text{ in.})(6 \text{ in.})} + \frac{P(4 \text{ in.})(3 \text{ in.})}{(1/12)(3 \text{ in.})(6 \text{ in.})^3} \\ P &= 1.22 \text{ kip} \end{aligned}$$

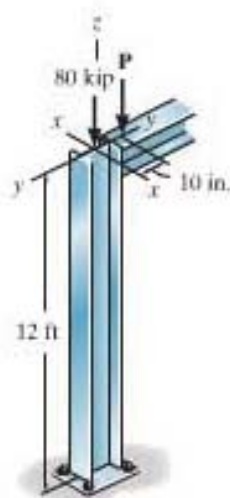
Ans

PROBLEMS

13

13-107. The $W14 \times 53$ structural A-36 steel column supports an axial load of 80 kip in addition to an eccentric load P . Determine the maximum allowable value of P based on the AISC equations of Sec. 13.6 and Eq. 13-30. Assume the column is fixed at its base, and at its top it is free to sway in the x - z plane while it is pinned in the y - z plane.

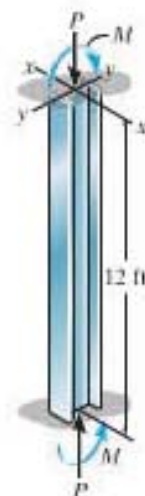
***13-108.** The $W12 \times 45$ structural A-36 steel column supports an axial load of 80 kip in addition to an eccentric load of $P = 60$ kip. Determine if the column fails based on the AISC equations of Sec. 13.6 and Eq. 13-30. Assume that the column is fixed at its base, and at its top it is free to sway in the x - z plane while it is pinned in the y - z plane.



Probs. 13-107/108

***13-109.** The $W14 \times 22$ structural A-36 steel column is fixed at its top and bottom. If a horizontal load (not shown) causes it to support end moments of $M = 10$ kip·ft, determine the maximum allowable axial force P that can be applied. Bending is about the x - x axis. Use the AISC equations of Sec. 13.6 and Eq. 13-30.

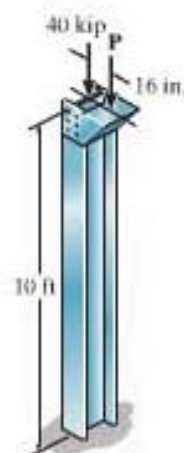
13-110. The $W14 \times 22$ column is fixed at its top and bottom. If a horizontal load (not shown) causes it to support end moments of $M = 15$ kip·ft, determine the maximum allowable axial force P that can be applied. Bending is about the x - x axis. Use the interaction formula with $(\sigma_b)_{allow} = 24$ ksi.



Probs. 13-109/110

13-111. The $W14 \times 43$ structural A-36 steel column is fixed at its bottom and free at its top. Determine the greatest eccentric load P that can be applied using Eq. 13-30 and the AISC equations of Sec. 13.6.

***13-112.** The $W10 \times 45$ structural A-36 steel column is fixed at its bottom and free at its top. If it is subjected to a load of $P = 2$ kip, determine if it is safe based on the AISC equations of Sec. 13.6 and Eq. 13-30.



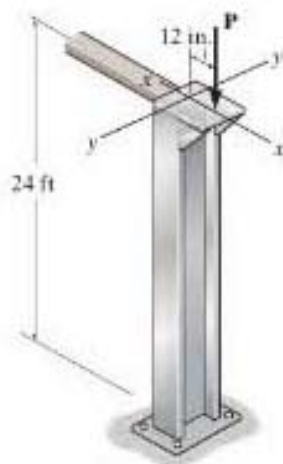
Probs. 13-111/112

•13-113. The A-36-steel W10 $\times$ 45 column is fixed at its base. Its top is constrained to move along the x - x axis but free to rotate about and move along the y - y axis. Determine the maximum eccentric force P that can be safely supported by the column using the allowable stress method.

13-114. The A-36-steel W10 $\times$ 45 column is fixed at its base. Its top is constrained to move along the x - x axis but free to rotate about and move along the y - y axis. Determine the maximum eccentric force P that can be safely supported by the column using an interaction formula. The allowable bending stress is $(\sigma_b)_{allow} = 15$ ksi.

13-115. The A-36-steel W12 $\times$ 50 column is fixed at its base. Its top is constrained to move along the x - x axis but free to rotate about and move along the y - y axis. If the eccentric force $P = 15$ kip is applied to the column, investigate if the column is adequate to support the loading. Use the allowable stress method.

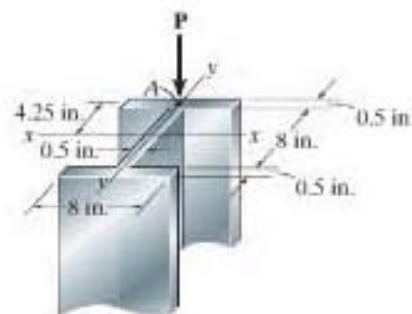
***13-116.** The A-36-steel W12 $\times$ 50 column is fixed at its base. Its top is constrained to move along the x - x axis but free to rotate about and move along the y - y axis. If the eccentric force $P = 15$ kip is applied to the column, investigate if the column is adequate to support the loading. Use the interaction formula. The allowable bending stress is $(\sigma_b)_{allow} = 15$ ksi.



Probs. 13-113/114/115/116

•13-117. A 16-ft-long column is made of aluminum alloy 2014-T6. If it is fixed at its top and bottom, and a compressive load P is applied at point A , determine the maximum allowable magnitude of P using the equations of Sec. 13.6 and Eq. 13-30.

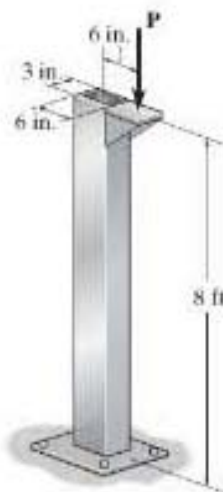
13-118. A 16-ft-long column is made of aluminum alloy 2014-T6. If it is fixed at its top and bottom, and a compressive load P is applied at point A , determine the maximum allowable magnitude of P using the equations of Sec. 13.6 and the interaction formula with $(\sigma_b)_{allow} = 20$ ksi.



Probs. 13-117/118

13-119. The 2014-T6 hollow column is fixed at its base and free at its top. Determine the maximum eccentric force P that can be safely supported by the column. Use the allowable stress method. The thickness of the wall for the section is $t = 0.5$ in.

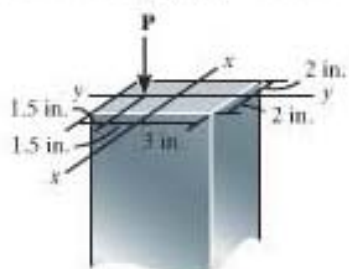
***13-120.** The 2014-T6 hollow column is fixed at its base and free at its top. Determine the maximum eccentric force P that can be safely supported by the column. Use the interaction formula. The allowable bending stress is $(\sigma_b)_{allow} = 30$ ksi. The thickness of the wall for the section is $t = 0.5$ in.



Probs. 13-119/120

•13-121. The 10-ft-long bar is made of aluminum alloy 2014-T6. If it is fixed at its bottom and pinned at the top, determine the maximum allowable eccentric load P that can be applied using the formulas in Sec. 13.6 and Eq. 13-30.

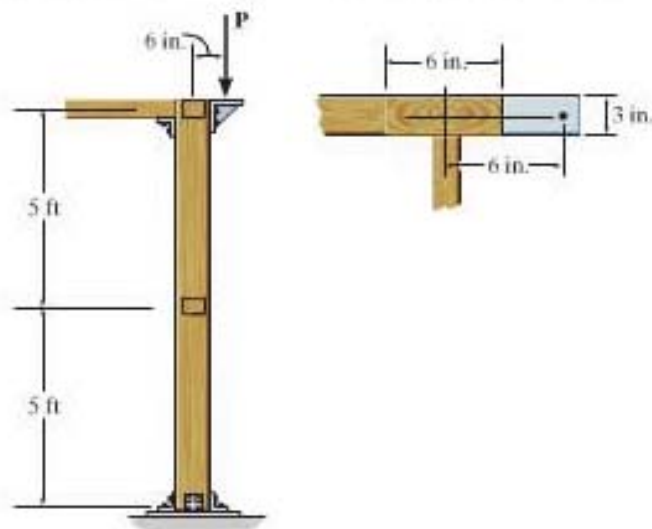
13-122. The 10-ft-long bar is made of aluminum alloy 2014-T6. If it is fixed at its bottom and pinned at the top, determine the maximum allowable eccentric load P that can be applied using the equations of Sec. 13.6 and the interaction formula with $(\sigma_b)_{allow} = 18 \text{ ksi}$.



Probs. 13-121/122

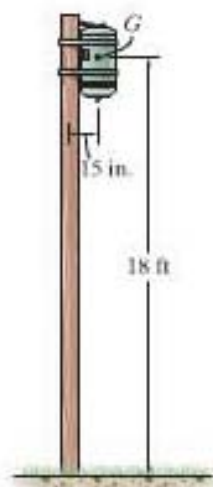
13-123. The rectangular wooden column can be considered fixed at its base and pinned at its top. Also, the column is braced at its mid-height against the weak axis. Determine the maximum eccentric force P that can be safely supported by the column using the allowable stress method.

*13-124. The rectangular wooden column can be considered fixed at its base and pinned at its top. Also, the column is braced at its mid-height against the weak axis. Determine the maximum eccentric force P that can be safely supported by the column using the interaction formula. The allowable bending stress is $(\sigma_b)_{allow} = 1.5 \text{ ksi}$.



Probs. 13-123/124

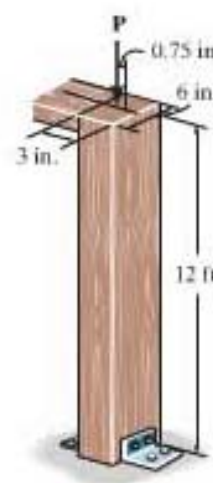
•13-125. The 10-in.-diameter utility pole supports the transformer that has a weight of 600 lb and center of gravity at G . If the pole is fixed to the ground and free at its top, determine if it is adequate according to the NFPA equations of Sec. 13.6 and Eq. 13-30.



Prob. 13-125

13-126. Using the NFPA equations of Sec. 13.6 and Eq. 13-30, determine the maximum allowable eccentric load P that can be applied to the wood column. Assume that the column is pinned at both its top and bottom.

13-127. Using the NFPA equations of Sec. 13.6 and Eq. 13-30, determine the maximum allowable eccentric load P that can be applied to the wood column. Assume that the column is pinned at the top and fixed at the bottom.



Probs. 13-126/127

CHAPTER REVIEW

Buckling is the sudden instability that occurs in columns or members that support an axial compressive load. The maximum axial load that a member can support just before buckling is called the critical load P_{cr} .

The critical load for an ideal column is determined from Euler's formula, where $K = 1$ for pin supports, $K = 0.5$ for fixed supports, $K = 0.7$ for a pin and a fixed support, and $K = 2$ for a fixed support and a free end.

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$



If the axial loading is applied eccentrically to the column, then the secant formula can be used to determine the maximum stress in the column.

$$\sigma_{max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right]$$

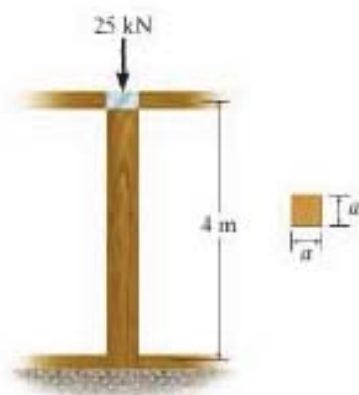
When the axial load causes yielding of the material, then the tangent modulus should be used with Euler's formula to determine the critical load for the column. This is referred to as Engesser's equation.

$$\sigma_{cr} = \frac{\pi^2 E_t}{(KL/r)^2}$$

Empirical formulas based on experimental data have been developed for use in the design of steel, aluminum, and timber columns.

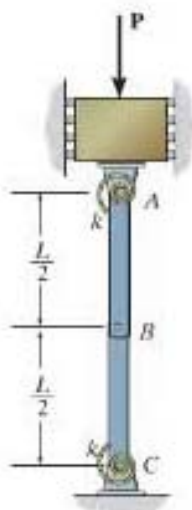
REVIEW PROBLEMS

***13-128.** The wood column is 4 m long and is required to support the axial load of 25 kN. If the cross section is square, determine the dimension a of each of its sides using a factor of safety against buckling of $F.S. = 2.5$. The column is assumed to be pinned at its top and bottom. Use the Euler equation. $E_w = 11 \text{ GPa}$, and $\sigma_Y = 10 \text{ MPa}$.



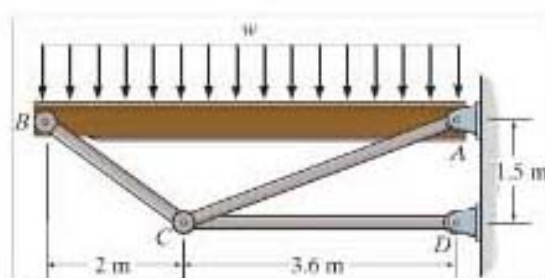
Prob. 13-128

•13-129. If the torsional springs attached to ends A and C of the rigid members AB and BC have a stiffness k , determine the critical load P_{cr} .



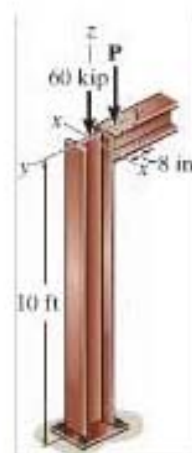
Prob. 13-129

13-130. Determine the maximum intensity w of the uniform distributed load that can be applied on the beam without causing the compressive members of the supporting truss to buckle. The members of the truss are made from A-36-steel rods having a 60-mm diameter. Use $F.S. = 2$ against buckling.



Prob. 13-130

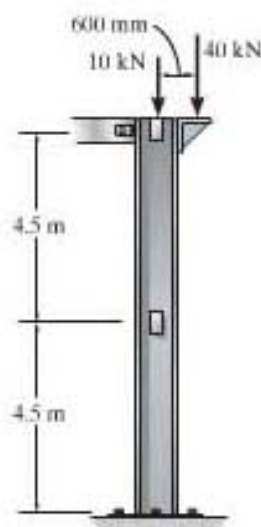
13-131. The $W10 \times 45$ steel column supports an axial load of 60 kip in addition to an eccentric load P . Determine the maximum allowable value of P based on the AISC equations of Sec. 13.6 and Eq. 13-30. Assume that in the x - z plane $K_x = 1.0$ and in the y - z plane $K_y = 2.0$. $E_{st} = 29(10^3) \text{ ksi}$, $\sigma_Y = 50 \text{ ksi}$.



Prob. 13-131

***13-132.** The A-36-steel column can be considered pinned at its top and fixed at its base. Also, it is braced at its mid-height along the weak axis. Investigate whether a W250 $\times$ 45 section can safely support the loading shown. Use the allowable stress method.

•13-133. The A-36-steel column can be considered pinned at its top and fixed at its base. Also, it is braced at its mid-height along the weak axis. Investigate whether a W250 $\times$ 45 section can safely support the loading shown. Use the interaction formula. The allowable bending stress is $(\sigma_b)_{allow} = 100$ MPa.



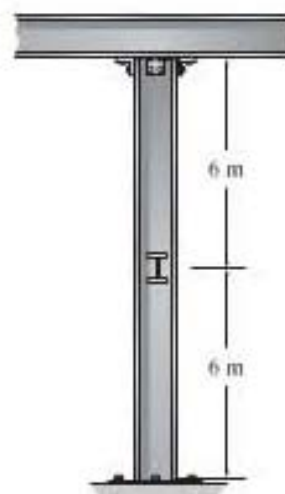
Probs. 13-132/133

13-134. The member has a symmetric cross section. If it is pin connected at its ends, determine the largest force it can support. It is made of 2014-T6 aluminum alloy.



Prob. 13-134

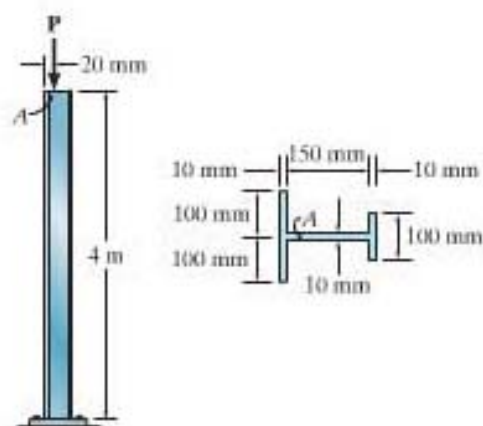
13-135. The W200 $\times$ 46 A-36-steel column can be considered pinned at its top and fixed at its base. Also, the column is braced at its mid-height against the weak axis. Determine the maximum axial load the column can support without causing it to buckle.



Prob. 13-135

***13-136.** The structural A-36 steel column has the cross section shown. If it is fixed at the bottom and free at the top, determine the maximum force P that can be applied at A without causing it to buckle or yield. Use a factor of safety of 3 with respect to buckling and yielding.

•13-137. The structural A-36 steel column has the cross section shown. If it is fixed at the bottom and free at the top, determine if the column will buckle or yield when the load $P = 10$ kN. Use a factor of safety of 3 with respect to buckling and yielding.



Probs. 13-136/137



As piles are driven in place, their ends are subjected to impact loading. The nature of impact and the energy derived from it must be understood in order to determine the stress developed within the pile.

Energy Methods

CHAPTER OBJECTIVES

In this chapter, we will show how to apply energy methods to solve problems involving deflection. The chapter begins with a discussion of work and strain energy, followed by a development of the principle of conservation of energy. Using this principle, the stress and deflection of a member are determined when the member is subjected to impact. The method of virtual work and Castigliano's theorem are then developed, and these methods are used to determine the displacement and slope at points on structural members and mechanical elements.

14.1 External Work and Strain Energy

The deflection of joints on a truss or points on a beam or shaft can be determined using energy methods. Before developing any of these methods, however, we will first define the work caused by an external force and couple moment and show how to express this work in terms of a body's strain energy. The formulations to be presented here and in the next section will provide the basis for applying the work and energy methods that follow throughout the chapter.

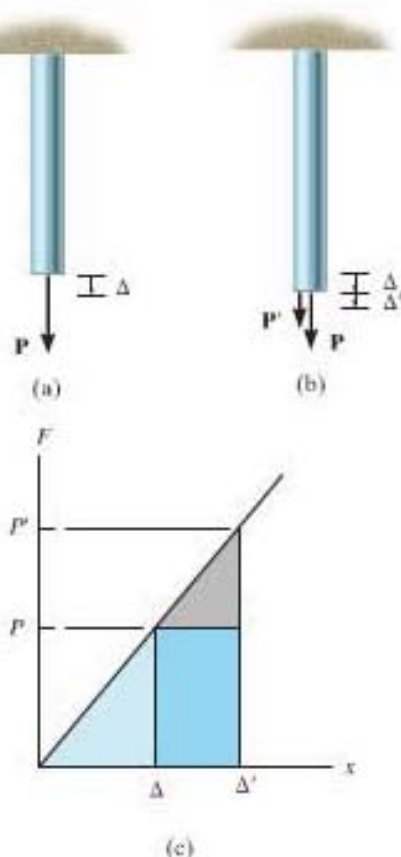


Fig. 14-1

Work of a Force. In mechanics, a force does *work* when it undergoes a displacement dx that is in the *same direction* as the force. The work done is a scalar, defined as $dU_e = F dx$. If the total displacement is Δ , the work becomes

$$U_e = \int_0^{\Delta} F dx \quad (14-1)$$

To show how to apply this equation, we will calculate the work done by an axial force applied to the end of the bar shown in Fig. 14-1a. As the magnitude of the force is *gradually* increased from zero to some limiting value $F = P$, the final displacement of the end of the bar becomes Δ . If the material behaves in a linear-elastic manner, then the force will be directly proportional to the displacement; that is, $F = (P/\Delta)x$. Substituting into Eq. 14-1 and integrating from 0 to Δ , we get

$$U_e = \frac{1}{2} P \Delta \quad (14-2)$$

Therefore, as the force is gradually applied to the bar, its magnitude builds from zero to some value P , and consequently, the work done is equal to the *average force magnitude*, $P/2$, times the total displacement Δ . We can represent this graphically as the light-blue shaded area of the triangle in Fig. 14-1c.

Suppose, however, that $\mathbf{P}$ is already applied to the bar and that *another force* $\mathbf{P}'$ is now applied, so that the end of the bar is displaced *further* by an amount Δ' , Fig. 14-1b. The work done by $\mathbf{P}'$ is equal to the gray shaded triangular area, but now the work done by $\mathbf{P}$ when the bar undergoes this further displacement is

$$U'_e = P \Delta' \quad (14-3)$$

Here the work represents the dark-blue shaded *rectangular area* in Fig. 14-1c. In this case $\mathbf{P}$ does not change its magnitude, since the bar's displacement Δ' is caused only by $\mathbf{P}'$. Therefore, work here is simply the force magnitude P times the displacement Δ' .

Work of a Couple Moment. A couple moment $\mathbf{M}$ does work when it undergoes an angular displacement $d\theta$ along its line of action. The work is defined as $dU_e = M d\theta$, Fig. 14-2. If the total angular displacement is θ rad, the work becomes

$$U_e = \int_0^\theta M d\theta \quad (14-4)$$

As in the case of force, if the couple moment is applied to a *body* having linear elastic material behavior, such that its magnitude is increased gradually from zero at $\theta = 0$ to M at θ , then the work is

$$U_e = \frac{1}{2} M \theta \quad (14-5)$$

However, if the couple moment is already applied to the body and other loadings further rotate the body by an amount θ' , then the work is

$$U'_e = M \theta'$$

Strain Energy. When loads are applied to a body, they will deform the material. Provided no energy is lost in the form of heat, the external work done by the loads will be converted into internal work called **strain energy**. This energy, which is *always positive*, is stored in the body and is caused by the action of either normal or shear stress.

Normal Stress. If the volume element shown in Fig. 14-3 is subjected to the normal stress σ_z , then the force created on the element's top and bottom faces is $dF_z = \sigma_z dA = \sigma_z dx dy$. If this force is applied gradually to the element, like the force $\mathbf{P}$ discussed previously, its magnitude is increased from zero to dF_z , while the element undergoes an elongation $d\Delta_z = \epsilon_z dz$. The work done by dF_z is therefore $dU_i = \frac{1}{2} dF_z d\Delta_z = \frac{1}{2} [\sigma_z dx dy] \epsilon_z dz$. Since the volume of the element is $dV = dx dy dz$, we have

$$dU_i = \frac{1}{2} \sigma_z \epsilon_z dV \quad (14-6)$$

Notice that dU_i is *always positive*, even if σ_z is compressive, since σ_z and ϵ_z will always be in the same direction.

In general then, if the body is subjected only to a uniaxial *normal stress* σ , the strain energy in the body is then

$$U_i = \int_V \frac{\sigma \epsilon}{2} dV \quad (14-7)$$

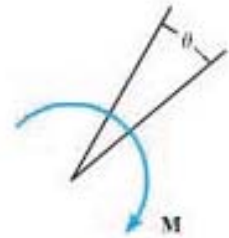


Fig. 14-2

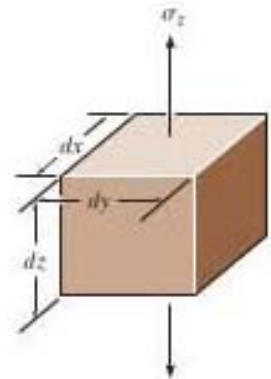


Fig. 14-3

Also, if the material behaves in a linear-elastic manner, then Hooke's law applies, and we can express the strain energy in terms of the normal stress as

$$U_i = \int_V \frac{\sigma^2}{2E} dV \quad (14-8)$$

14

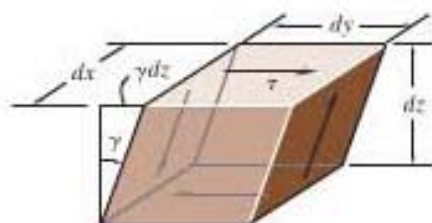


Fig. 14-4

Shear Stress. A strain-energy expression similar to that for normal stress can also be established for the material when it is subjected to shear stress. Consider the volume element shown in Fig. 14-4. Here the shear stress causes the element to deform such that only the shear force $dF = \tau(dx\,dy)$, acting on the top face of the element, is displaced $\gamma\,dz$ relative to the bottom face. The *vertical faces* only rotate, and therefore the shear forces on these faces do no work. Hence, the strain energy stored in the element is

$$dU_i = \frac{1}{2}[\tau(dx\,dy)]\gamma\,dz$$

or since $dV = dx\,dy\,dz$

$$dU_i = \frac{1}{2}\tau\gamma\,dV \quad (14-9)$$

The strain energy stored in the body is therefore

$$U_i = \int_V \frac{\tau\gamma}{2} dV \quad (14-10)$$

Like the case for normal strain energy, shear strain energy is always positive since τ and γ are always in the same direction. If the material is linear elastic, then, applying Hooke's law, $\gamma = \tau/G$, we can express the strain energy in terms of the shear stress as

$$U_i = \int_V \frac{\tau^2}{2G} dV \quad (14-11)$$

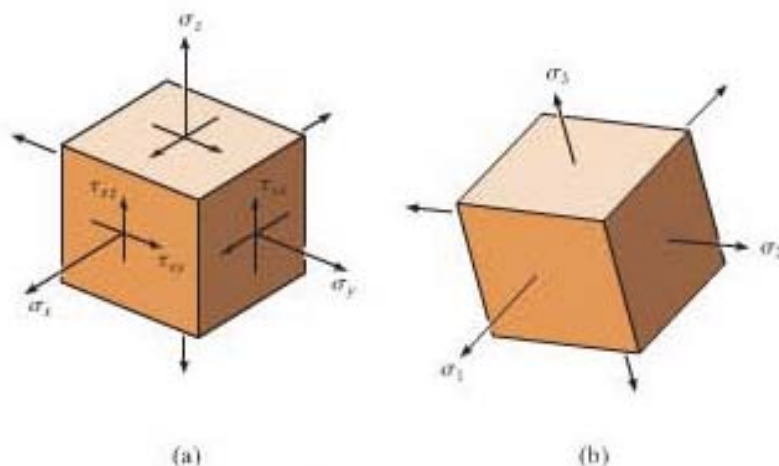


Fig. 14-5

In the next section, we will use Eqs. 14-8 and 14-11 to obtain formal expressions for the strain energy stored in members subjected to several types of loads. Once this is done we will then be able to develop the energy methods necessary to determine the displacement and slope at points on a body.

Multiaxial Stress. The previous development may be expanded to determine the strain energy in a body when it is subjected to a general state of stress, Fig. 14-5a. The strain energies associated with each of the normal and shear stress components can be obtained from Eqs. 14-6 and 14-9. Since energy is a scalar, the total strain energy in the body is therefore

$$U_i = \int_V \left[\frac{1}{2} \sigma_x \epsilon_x + \frac{1}{2} \sigma_y \epsilon_y + \frac{1}{2} \sigma_z \epsilon_z + \frac{1}{2} \tau_{xy} \gamma_{xy} + \frac{1}{2} \tau_{yz} \gamma_{yz} + \frac{1}{2} \tau_{xz} \gamma_{xz} \right] dV \quad (14-12)$$

The strains can be eliminated by using the generalized form of Hooke's law given by Eqs. 10-18 and 10-19. After substituting and combining terms, we have

$$U_i = \int_V \left[\frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \right] dV \quad (14-13)$$

If only the principal stresses σ_1 , σ_2 , σ_3 act on the element, Fig. 14-5b, this equation reduces to a simpler form, namely,

$$U_i = \int_V \left[\frac{1}{2E} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E} (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) \right] dV \quad (14-14)$$

This equation was used in Sec. 10.7 as a basis for developing the maximum-distortion-energy theory.

14.2 Elastic Strain Energy for Various Types of Loading

Using the equations for elastic strain energy developed in the previous section, we will now formulate the strain energy stored in a member when it is subjected to an axial load, bending moment, transverse shear, and torsional moment. Examples will be given to show how to calculate the strain energy in members subjected to each of these loadings.

Axial Load. Consider a bar of variable yet slightly tapered cross section, Fig. 14-6. The *internal axial force* at a section located a distance x from one end is N . If the cross-sectional area at this section is A , then the normal stress on the section is $\sigma = N/A$. Applying Eq. 14-8, we have

$$U_i = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{N^2}{2EA^2} dV$$



Fig. 14-6

If we choose an element or differential slice having a volume $dV = A dx$, the general formula for the strain energy in the bar is therefore

$$U_i = \int_0^L \frac{N^2}{2AE} dx \quad (14-15)$$

For the more common case of a prismatic bar of constant cross-sectional area A , length L , and constant axial load N , Fig. 14-7, Eq. 14-15, when integrated, gives

$$U_i = \frac{N^2 L}{2AE} \quad (14-16)$$

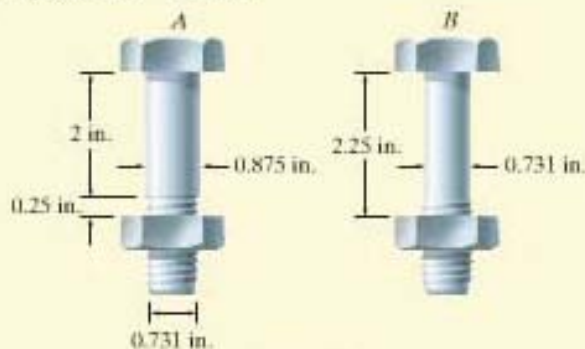


Fig. 14-7

Notice that the bar's elastic strain energy will *increase* if the length of the bar is increased, or if the modulus of elasticity or cross-sectional area is decreased. For example, an aluminum rod [$E_{al} = 10(10^3)$ ksi] will store approximately three times as much energy as a steel rod [$E_{st} = 29(10^3)$ ksi] having the same size and subjected to the same load. However, doubling the cross-sectional area of a rod will decrease its ability to store energy by one-half. The following example illustrates this point numerically.

EXAMPLE 14.1

One of the two high-strength steel bolts *A* and *B* shown in Fig. 14-8 is to be chosen to support a sudden tensile loading. For the choice it is necessary to determine the greatest amount of elastic strain energy that each bolt can absorb. Bolt *A* has a diameter of 0.875 in. for 2 in. of its length and a root (or smallest) diameter of 0.731 in. within the 0.25-in. threaded region. Bolt *B* has “upset” threads, such that the diameter throughout its 2.25-in. length can be taken as 0.731 in. In both cases, neglect the extra material that makes up the threads. Take $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 44$ ksi.

**Fig. 14-8****SOLUTION**

Bolt A. If the bolt is subjected to its maximum tension, the maximum stress of $\sigma_Y = 44$ ksi will occur within the 0.25-in. region. This tension force is

$$P_{\max} = \sigma_Y A = 44 \text{ ksi} \left[\pi \left(\frac{0.731 \text{ in.}}{2} \right)^2 \right] = 18.47 \text{ kip}$$

Applying Eq. 14-16 to each region of the bolt, we have

$$\begin{aligned} U_i &= \sum \frac{N^2 L}{2AE} \\ &= \frac{(18.47 \text{ kip})^2 (2 \text{ in.})}{2[\pi(0.875 \text{ in.}/2)^2][29(10^3) \text{ ksi}]} + \frac{(18.47 \text{ kip})^2 (0.25 \text{ in.})}{2[\pi(0.731 \text{ in.}/2)^2][29(10^3) \text{ ksi}]} \\ &= 0.0231 \text{ in.} \cdot \text{kip} \quad \text{Ans.} \end{aligned}$$

Bolt B. Here the bolt is assumed to have a uniform diameter of 0.731 in. throughout its 2.25-in. length. Also, from the calculation above, it can support a maximum tension force of $P_{\max} = 18.47$ kip. Thus,

$$U_i = \frac{N^2 L}{2AE} = \frac{(18.47 \text{ kip})^2 (2.25 \text{ in.})}{2[\pi(0.731 \text{ in.}/2)^2][29(10^3) \text{ ksi}]} = 0.0315 \text{ in.} \cdot \text{kip} \quad \text{Ans.}$$

NOTE: By comparison, bolt *B* can absorb 36% more elastic energy than bolt *A*, because it has a smaller cross section along its shank.

Bending Moment. Since a bending moment applied to a straight prismatic member develops *normal stress* in the member, we can use Eq. 14-8 to determine the strain energy stored in the member due to bending. For example, consider the axisymmetric beam shown in Fig. 14-9. Here the internal moment is M , and the normal stress acting on the arbitrary element a distance y from the neutral axis is $\sigma = My/I$. If the volume of the element is $dV = dA dx$, where dA is the area of its exposed face and dx is its length, the elastic strain energy in the beam is

$$U_i = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{1}{2E} \left(\frac{My}{I} \right)^2 dA dx$$

or

$$U_i = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

Realizing that the area integral represents the moment of inertia of the area about the neutral axis, the final result can be written as

$$U_i = \int_0^L \frac{M^2 dx}{2EI} \quad (14-17)$$

To evaluate the strain energy, therefore, we must first express the internal moment as a function of its position x along the beam, and then perform the integration over the beam's entire length.* The following examples illustrate this procedure.

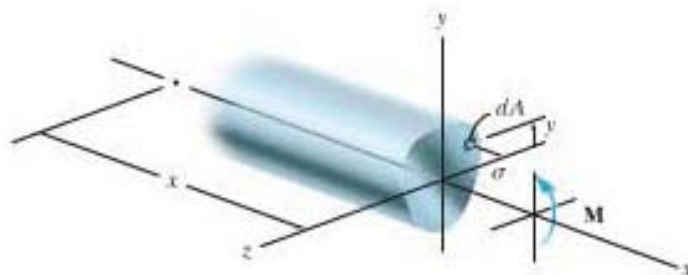


Fig. 14-9

*Recall that the flexure formula, as used here, can also be used with justifiable accuracy to determine the stress in slightly tapered beams. (See Sec. 6.4.) So in the general sense, I in Eq. 14-17 may also have to be expressed as a function of x .

EXAMPLE 14.2

Determine the elastic strain energy due to bending of the cantilevered beam in Fig. 14-10a. EI is constant.

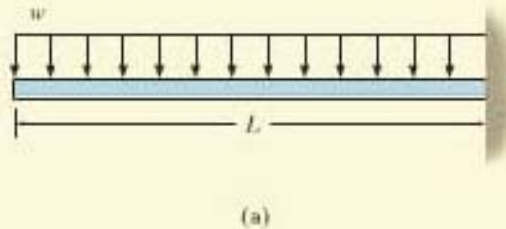
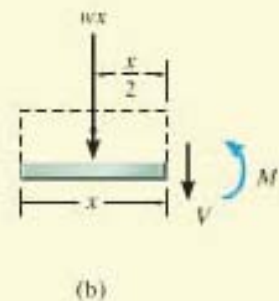


Fig. 14-10

SOLUTION

The internal moment in the beam is determined by establishing the x coordinate with origin at the left side. The left segment of the beam is shown in Fig. 14-10b. We have

$$\begin{aligned} \zeta + \Sigma M_{NA} = 0; \quad M + wx\left(\frac{x}{2}\right) &= 0 \\ M &= -w\left(\frac{x^2}{2}\right) \end{aligned}$$



Applying Eq. 14-17 yields

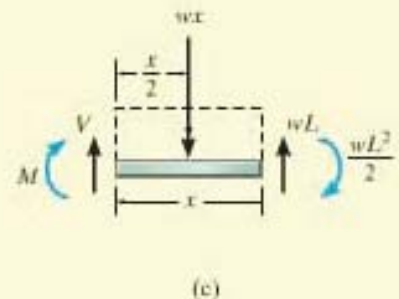
$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{[-w(x^2/2)]^2 dx}{2EI} = \frac{w^2}{8EI} \int_0^L x^4 dx$$

or

$$U_i = \frac{w^2 L^5}{40EI} \quad \text{Ans.}$$

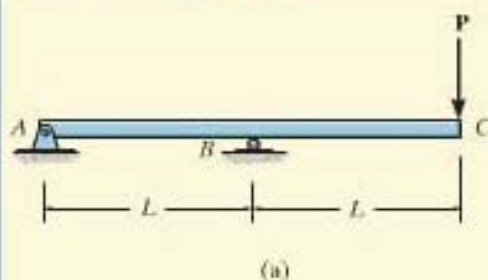
We can also obtain the strain energy using an x coordinate having its origin at the right side of the beam and extending positive to the left, Fig. 14-10c. In this case,

$$\begin{aligned} \zeta + \Sigma M_{NA} = 0; \quad -M - wx\left(\frac{x}{2}\right) + wL(x) - \frac{wL^2}{2} &= 0 \\ M &= -\frac{wL^2}{2} + wLx - w\left(\frac{x^2}{2}\right) \end{aligned}$$



Applying Eq. 14-17, we obtain the same result as before; however, more calculations are involved in this case.

EXAMPLE 14.3



Determine the bending strain energy in region AB of the beam shown in Fig. 14-11a. EI is constant.

SOLUTION

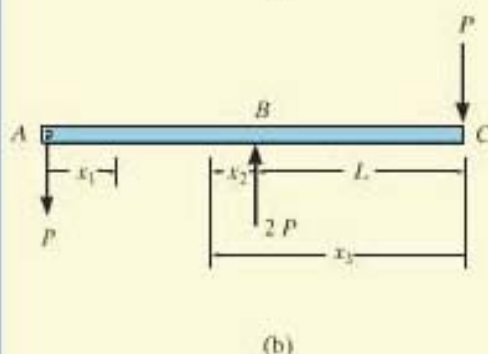
A free-body diagram of the beam is shown in Fig. 14-11b. To obtain the answer we can express the internal moment in terms of any one of the indicated three “ x ” coordinates and then apply Eq. 14-17. Each of these solutions will now be considered.

$0 \leq x_1 \leq L$. From the free-body diagram of the section in Fig. 14-11c, we have

$$\zeta + \sum M_{NA} = 0; \quad M_1 + Px_1 = 0$$

$$M_1 = -Px_1$$

$$U_i = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px_1)^2 dx_1}{2EI} = \frac{P^2 L^3}{6EI} \quad \text{Ans.}$$



$0 \leq x_2 \leq L$. Using the free-body diagram of the section in Fig. 14-11d gives

$$\zeta + \sum M_{NA} = 0; \quad -M_2 + 2P(x_2) - P(x_2 + L) = 0$$

$$M_2 = P(x_2 - L)$$

$$U_i = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{[P(x_2 - L)]^2 dx_2}{2EI} = \frac{P^2 L^3}{6EI} \quad \text{Ans.}$$

$L \leq x_3 \leq 2L$. From the free-body diagram in Fig. 14-11e, we have

$$\zeta + \sum M_{NA} = 0; \quad -M_3 + 2P(x_3 - L) - P(x_3) = 0$$

$$M_3 = P(x_3 - 2L)$$

$$U_i = \int \frac{M^2 dx}{2EI} = \int_L^{2L} \frac{[P(x_3 - 2L)]^2 dx_3}{2EI} = \frac{P^2 L^3}{6EI} \quad \text{Ans.}$$

NOTE: This and the previous example indicate that the strain energy for the beam can be found using *any* suitable x coordinate. It is only necessary to integrate over the range of the coordinate where the internal energy is to be determined. Here the choice of x_1 provides the simplest solution.

Fig. 14-11

Transverse Shear. The strain energy due to shear stress in a beam element can be determined by applying Eq. 14-11. Here we will consider the beam to be prismatic and to have an axis of symmetry about the y axis as shown in Fig. 14-12. If the internal shear at the section x is V , then the shear stress acting on the volume element of material, having an area dA and length dx , is $\tau = VQ/It$. Substituting into Eq. 14-11, the strain energy for shear becomes

$$U_i = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{1}{2G} \left(\frac{VQ}{It} \right)^2 dA dx$$

$$U_i = \int_0^L \frac{V^2}{2GI^2} \left(\int_A \frac{Q^2}{t^2} dA \right) dx$$

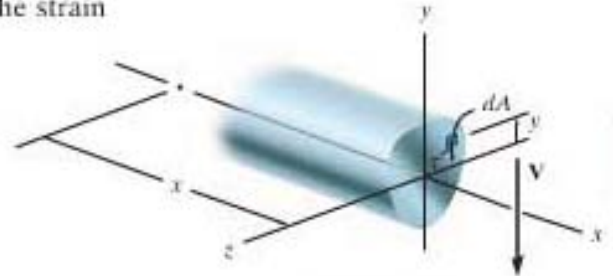


Fig. 14-12

The integral in parentheses can be simplified if we define the *form factor* for shear as

$$f_s = \frac{A}{I^2} \int_A \frac{Q^2}{t^2} dA \quad (14-18)$$

Substituting into the above equation, we get

$$U_i = \int_0^L \frac{f_s V^2 dx}{2GA} \quad (14-19)$$

The form factor defined by Eq. 14-18 is a dimensionless number that is unique for each specific cross-sectional area. For example, if the beam has a rectangular cross section of width b and height h , Fig. 14-13, then

$$t = b$$

$$dA = b dy$$

$$I = \frac{1}{12} b h^3$$

$$Q = \bar{y}' A' = \left(y + \frac{(h/2) - y}{2} \right) b \left(\frac{h}{2} - y \right) = \frac{b}{2} \left(\frac{h^2}{4} - y^2 \right)$$

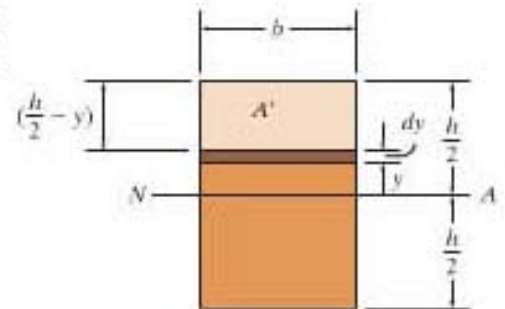


Fig. 14-13

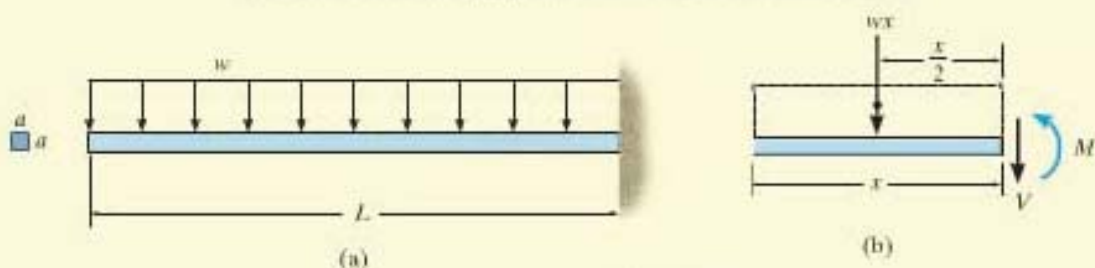
Substituting these terms into Eq. 14-18, we get

$$f_s = \frac{bh}{\left(\frac{1}{12} b h^3 \right)^2} \int_{-h/2}^{h/2} \frac{b^2}{4b^2} \left(\frac{h^2}{4} - y^2 \right)^2 b dy = \frac{6}{5} \quad (14-20)$$

The form factor for other sections can be determined in a similar manner. Once obtained, this factor is substituted into Eq. 14-19 and the strain energy for transverse shear can then be evaluated.

EXAMPLE 14.4

Determine the strain energy in the cantilevered beam due to shear if the beam has a square cross section and is subjected to a uniform distributed load w , Fig. 14-14a. EI and G are constant.

**Fig. 14-14****SOLUTION**

From the free-body diagram of an arbitrary section, Fig. 14-14b, we have

$$+\uparrow \Sigma F_y = 0; \quad -V - wx = 0$$

$$V = -wx$$

Since the cross section is square, the form factor $f_s = \frac{6}{5}$ (Eq. 14-20) and therefore Eq. 14-19 becomes

$$(U_i)_s = \int_0^L \frac{\frac{6}{5}(-wx)^2 dx}{2GA} = \frac{3w^2}{5GA} \int_0^L x^2 dx$$

or

$$(U_i)_s = \frac{w^2 L^3}{5GA} \quad \text{Ans.}$$

NOTE: Using the results of Example 14.2, with $A = a^2$, $I = \frac{1}{12}a^4$, the ratio of shear to bending strain energy is

$$\frac{(U_i)_s}{(U_i)_b} = \frac{w^2 L^3 / 5Ga^2}{w^2 L^5 / 40E(\frac{1}{12}a^4)} = \frac{2}{3} \left(\frac{a}{L} \right)^2 \frac{E}{G}$$

Since $G = E/2(1 + \nu)$ and $\nu \leq \frac{1}{2}$ (Sec. 10.6), then as an *upper* bound, $E = 3G$, so that

$$\frac{(U_i)_s}{(U_i)_b} = 2 \left(\frac{a}{L} \right)^2$$

It can be seen that this ratio will increase as L decreases. However, even for very short beams, where, say, $L = 5a$, the contribution due to shear strain energy is only 8% of the bending strain energy. For this reason, the shear strain energy stored in beams is usually neglected in engineering analysis.

Torsional Moment. To determine the internal strain energy in a circular shaft or tube due to an applied torsional moment, we must apply Eq. 14-11. Consider the slightly tapered shaft in Fig. 14-15. A section of the shaft taken a distance x from one end is subjected to an internal torque T . The shear stress distribution that causes this torque varies linearly from the center of the shaft. On the arbitrary element of area dA and length dx , the stress is $\tau = T\rho/J$. The strain energy stored in the shaft is thus

$$U_i = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{1}{2G} \left(\frac{T\rho}{J} \right)^2 dA dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

Since the area integral represents the polar moment of inertia J for the shaft at the section, the final result can be written as

$$U_i = \int_0^L \frac{T^2}{2GJ} dx \quad (14-21)$$

The most common case occurs when the shaft (or tube) has a constant cross-sectional area and the applied torque is constant, Fig. 14-16. Integration of Eq. 14-21 then gives

$$U_i = \frac{T^2 L}{2GJ} \quad (14-22)$$

From this equation we may conclude that, like an axially loaded member, the energy-absorbing capacity of a torsionally loaded shaft is *decreased* by increasing the diameter of the shaft, since this increases J .

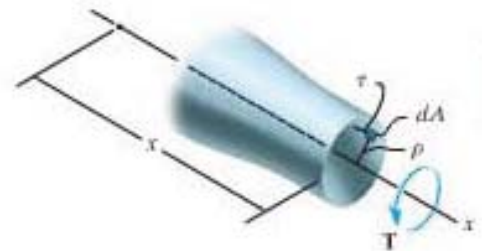


Fig. 14-15



Fig. 14-16

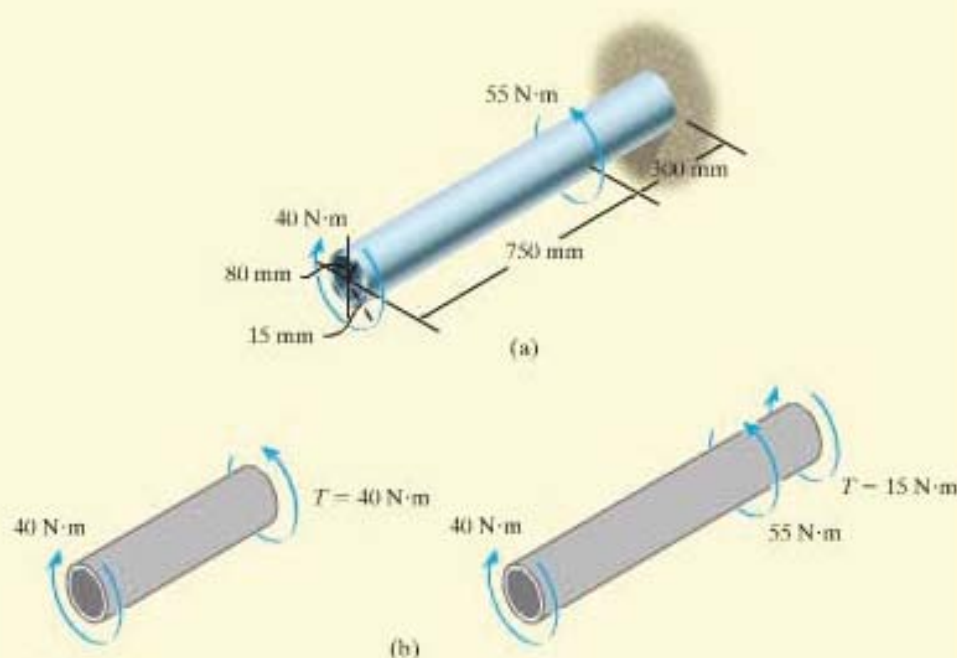
Important Points

- A *force* does work when it moves through a *displacement*. When a force is applied to a body and its magnitude is increased gradually from zero to F , the work is $U = (F/2)\Delta$, whereas if the force is constant when the displacement occurs then $U = F\Delta$.
- A *couple moment* does work when it displaces through a *rotation*.
- *Strain energy* is caused by the internal work of the normal and shear stresses. It is always a *positive* quantity.
- The strain energy can be related to the resultant internal loadings N , V , M , and T .
- As the beam becomes longer, the strain energy due to bending becomes much larger than the strain energy due to shear. For this reason, the *shear strain energy* in beams can generally be *neglected*.

The following example illustrates how to determine the strain energy in a circular shaft due to a torsional loading.

EXAMPLE 14.5

The tubular shaft in Fig. 14-17a is fixed at the wall and subjected to two torques as shown. Determine the strain energy stored in the shaft due to this loading. $G = 75 \text{ GPa}$.

**Fig. 14-17****SOLUTION**

Using the method of sections, the internal torque is first determined within the two regions of the shaft where it is constant, Fig. 14-17b. Although these torques ($40 \text{ N}\cdot\text{m}$ and $15 \text{ N}\cdot\text{m}$) are in opposite directions, this will be of no consequence in determining the strain energy, since the torque is squared in Eq. 14-22. In other words, the strain energy is always positive. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2} [(0.08 \text{ m})^4 - (0.065 \text{ m})^4] = 36.30(10^{-6}) \text{ m}^4$$

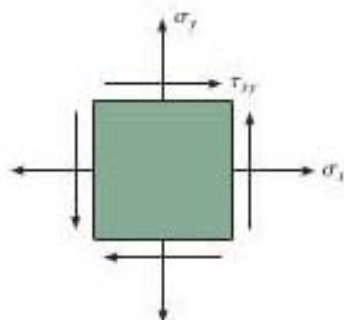
Applying Eq. 14-22, we have

$$\begin{aligned} U_i &= \sum \frac{T^2 L}{2GJ} \\ &= \frac{(40 \text{ N}\cdot\text{m})^2 (0.750 \text{ m})}{2[75(10^9) \text{ N/m}^2] 36.30(10^{-6}) \text{ m}^4} + \frac{(15 \text{ N}\cdot\text{m})^2 (0.300 \text{ m})}{2[75(10^9) \text{ N/m}^2] 36.30(10^{-6}) \text{ m}^4} \\ &= 233 \mu\text{J} \end{aligned}$$

Ans.

PROBLEMS

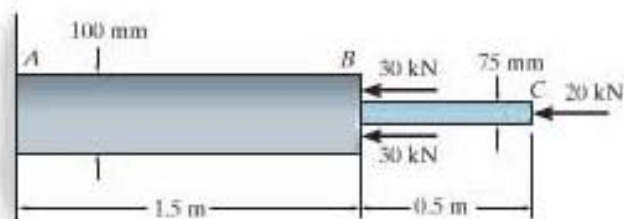
14-1. A material is subjected to a general state of plane stress. Express the strain energy density in terms of the elastic constants E , G , and ν and the stress components σ_x , σ_y , and τ_{xy} .



Prob. 14-1

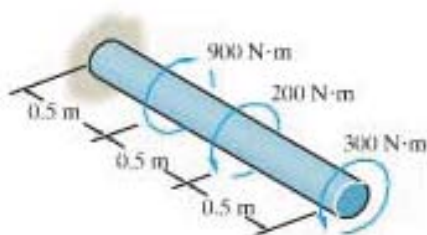
14-2. The strain-energy density must be the same whether the state of stress is represented by σ_x , σ_y , and τ_{xy} , or by the principal stresses σ_1 and σ_2 . This being the case, equate the strain-energy expressions for each of these two cases and show that $G = E/[2(1 + \nu)]$.

14-3. Determine the strain energy in the stepped rod assembly. Portion AB is steel and BC is brass. $E_{br} = 101$ GPa, $E_{st} = 200$ GPa, $(\sigma_Y)_{br} = 410$ MPa, $(\sigma_Y)_{st} = 250$ MPa.



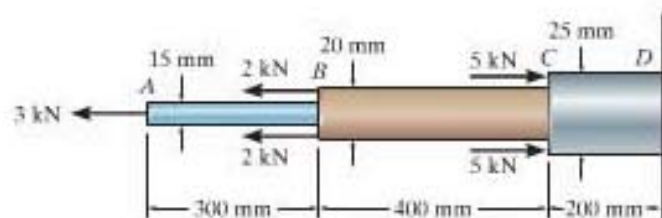
Prob. 14-3

***14-4.** Determine the torsional strain energy in the A-36 steel shaft. The shaft has a diameter of 40 mm.



Prob. 14-4

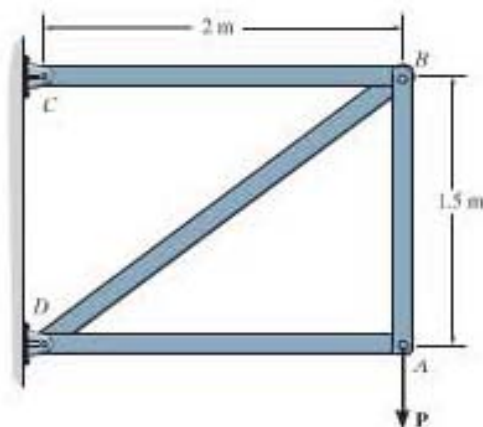
***14-5.** Determine the strain energy in the rod assembly. Portion AB is steel, BC is brass, and CD is aluminum. $E_{st} = 200$ GPa, $E_{br} = 101$ GPa, and $E_{al} = 73.1$ GPa.



Prob. 14-5

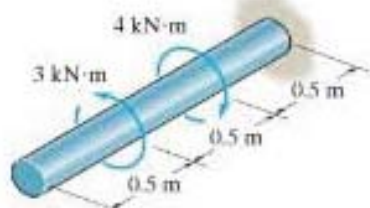
14-6. If $P = 60$ kN, determine the total strain energy stored in the truss. Each member has a cross-sectional area of $2.5(10^3)$ mm² and is made of A-36 steel.

14-7. Determine the maximum force P and the corresponding maximum total strain energy stored in the truss without causing any of the members to have permanent deformation. Each member has the cross-sectional area of $2.5(10^3)$ mm² and is made of A-36 steel.



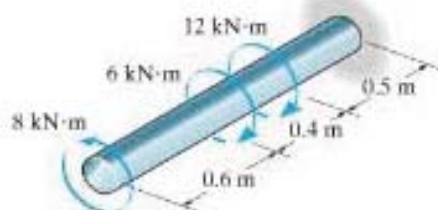
Probs. 14-6/7

***14-8.** Determine the torsional strain energy in the A-36 steel shaft. The shaft has a radius of 30 mm.



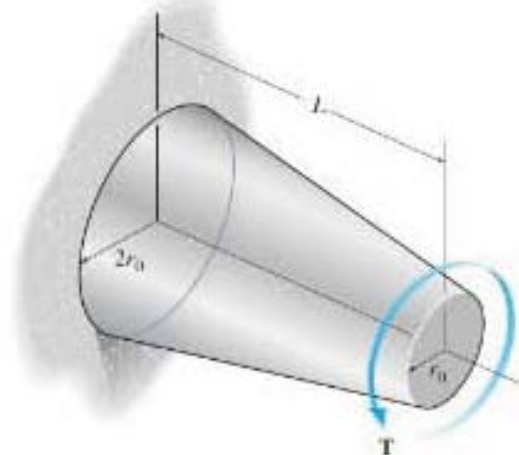
Prob. 14-8

***14-9.** Determine the torsional strain energy in the A-36 steel shaft. The shaft has a radius of 40 mm.



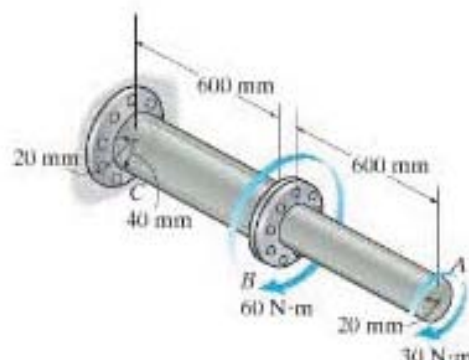
Prob. 14-9

14-10. Determine the torsional strain energy stored in the tapered rod when it is subjected to the torque T . The rod is made of material having a modulus of rigidity of G .



Prob. 14-10

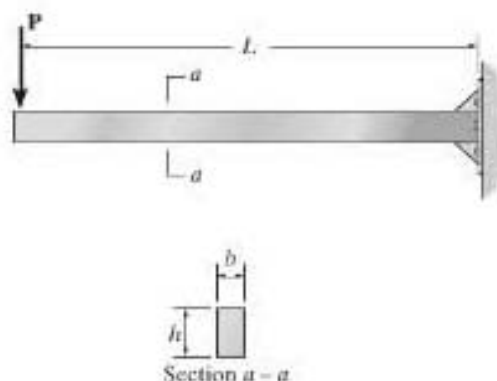
14-11. The shaft assembly is fixed at C . The hollow segment BC has an inner radius of 20 mm and outer radius of 40 mm, while the solid segment AB has a radius of 20 mm. Determine the torsional strain energy stored in the shaft. The shaft is made of 2014-T6 aluminum alloy. The coupling at B is rigid.



Prob. 14-11

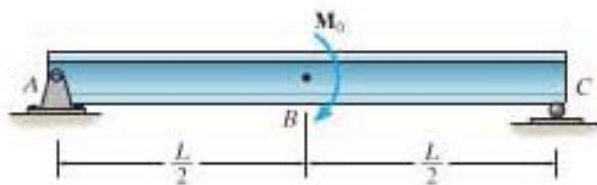
***14-12.** Consider the thin-walled tube of Fig. 5-28. Use the formula for shear stress, $\tau_{avg} = T/2tA_m$, Eq. 5-18, and the general equation of shear strain energy, Eq. 14-11, to show that the twist of the tube is given by Eq. 5-20, *Hint:* Equate the work done by the torque T to the strain energy in the tube, determined from integrating the strain energy for a differential element, Fig. 14-4, over the volume of material.

***14-13.** Determine the ratio of shearing strain energy to bending strain energy for the rectangular cantilever beam when it is subjected to the loading shown. The beam is made of material having a modulus of elasticity of E and Poisson's ratio of ν .



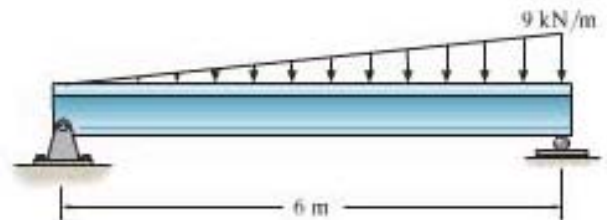
Prob. 14-13

14-14. Determine the bending strain energy in the beam due to the loading shown. EI is constant.



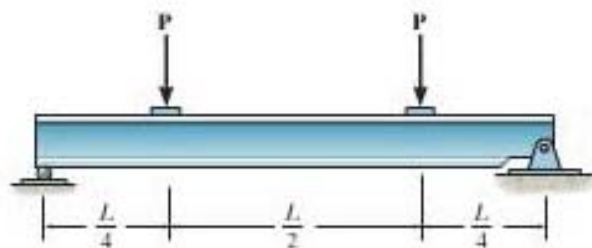
Prob. 14-14

14-17. Determine the bending strain energy in the A-36 steel beam. $I = 99.2 (10^6) \text{ mm}^4$.



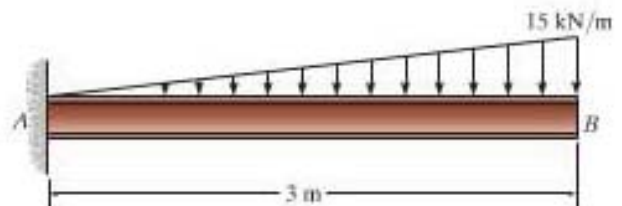
Prob. 14-17

14-15. Determine the bending strain energy in the beam. EI is constant.



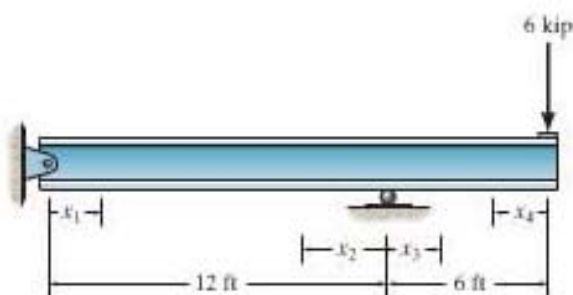
Prob. 14-15

14-18. Determine the bending strain energy in the A-36 steel beam due to the distributed load. $I = 122 (10^6) \text{ mm}^4$.



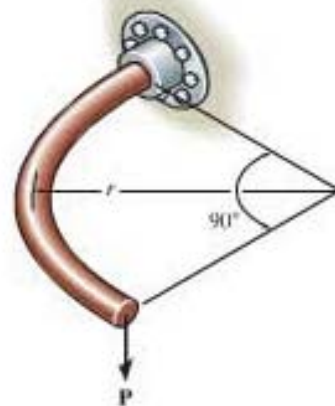
Prob. 14-18

***14-16.** Determine the bending strain energy in the A-36 structural steel W10 $\times$ 12 beam. Obtain the answer using the coordinates (a) x_1 and x_4 , and (b) x_2 and x_3 .



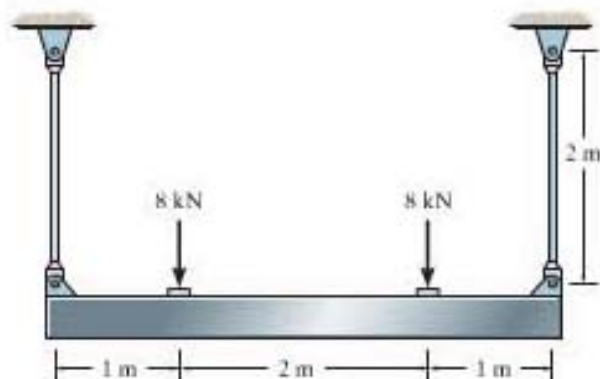
Prob. 14-16

14-19. Determine the strain energy in the horizontal curved bar due to torsion. There is a vertical force P acting at its end. JG is constant.



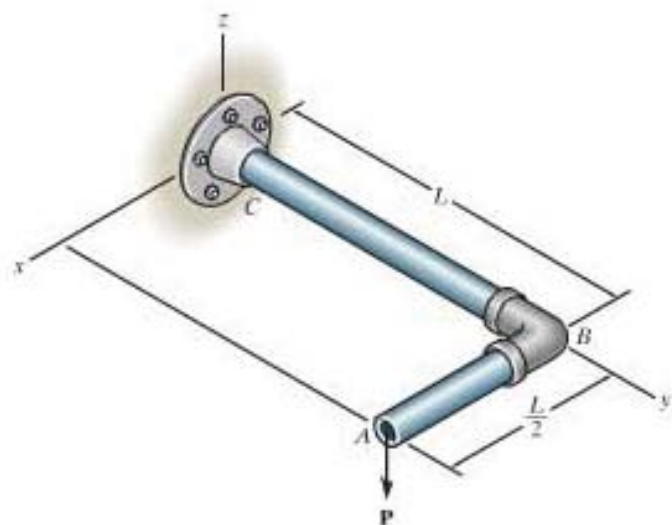
Prob. 14-19

***14-20.** Determine the bending strain energy in the beam and the axial strain energy in each of the two rods. The beam is made of 2014-T6 aluminum and has a square cross section 50 mm by 50 mm. The rods are made of A-36 steel and have a circular cross section with a 20-mm diameter.



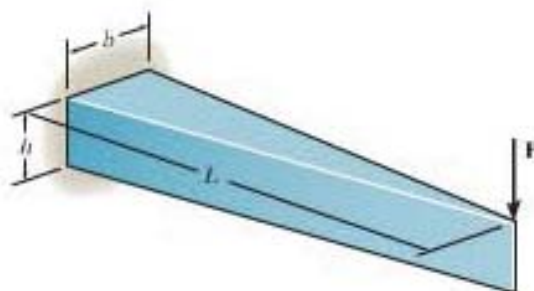
Prob. 14-20

***14-21.** The pipe lies in the horizontal plane. If it is subjected to a vertical force P at its end, determine the strain energy due to bending and torsion. Express the results in terms of the cross-sectional properties I and J , and the material properties E and G .



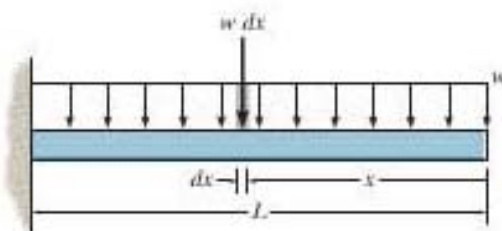
Prob. 14-21

14-22. The beam shown is tapered along its width. If a force P is applied to its end, determine the strain energy in the beam and compare this result with that of a beam that has a constant rectangular cross section of width b and height h .



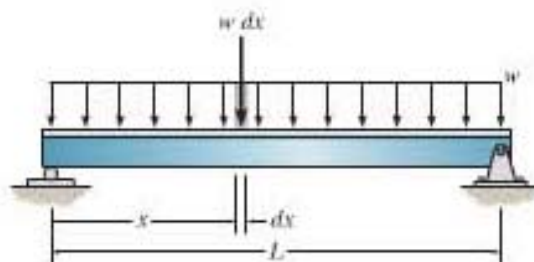
Prob. 14-22

14-23. Determine the bending strain energy in the cantilevered beam due to a uniform load w . Solve the problem two ways. (a) Apply Eq. 14-17. (b) The load $w dx$ acting on a segment dx of the beam is displaced a distance y , where $y = w(-x^4 + 4L^3x - 3L^4)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.



Prob. 14-23

***14-24.** Determine the bending strain energy in the simply supported beam due to a uniform load w . Solve the problem two ways. (a) Apply Eq. 14-17. (b) The load $w dx$ acting on the segment dx of the beam is displaced a distance y , where $y = w(-x^4 + 2Lx^3 - L^3x)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.



Prob. 14-24

14.3 Conservation of Energy

All energy methods used in mechanics are based on a balance of energy, often referred to as the conservation of energy. In this chapter, only mechanical energy will be considered in the energy balance; that is, the energy developed by heat, chemical reactions, and electromagnetic effects will be neglected. As a result, if a loading is applied *slowly* to a body, then physically the external loads tend to deform the body so that the loads do *external work* U_e as they are displaced. This external work on the body is transformed into *internal work* or strain energy U_i , which is stored in the body. Furthermore, when the loads are removed, the strain energy restores the body back to its original undeformed position, provided the material's elastic limit is not exceeded. The conservation of energy for the body can therefore be stated mathematically as

$$U_e = U_i \quad (14-23)$$

We will now show three examples of how this equation can be applied to determine the displacement of a point on a deformable member or structure. As the first example, consider the truss in Fig. 14-18 subjected to the load $\mathbf{P}$. Provided $\mathbf{P}$ is applied gradually, the external work done by $\mathbf{P}$ is determined from Eq. 14-2, that is, $U_e = \frac{1}{2}P\Delta$, where Δ is the vertical displacement of the truss at the joint where $\mathbf{P}$ is applied. Assuming that $\mathbf{P}$ develops an axial force $\mathbf{N}$ in a particular member, the strain energy stored in this member is determined from Eq. 14-16, that is, $U_i = N^2L/2AE$. Summing the strain energies for all the members of the truss, we can write Eq. 14-23 as

$$\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE} \quad (14-24)$$

Once the internal forces (N) in all the members of the truss are determined and the terms on the right calculated, it is then possible to determine the unknown displacement Δ .

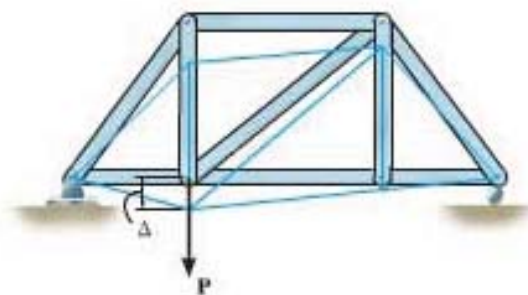


Fig. 14-18

As a second example, consider finding the vertical displacement Δ under the load $\mathbf{P}$ acting on the beam in Fig. 14-19. Again, the external work is $U_e = \frac{1}{2}P\Delta$. In this case the strain energy is the result of internal shear and moment loadings caused by $\mathbf{P}$. In particular, the contribution of strain energy due to shear is generally *neglected* in most beam deflection problems unless the beam is short and supports a very large load. (See Example 14.4.) Consequently, the beam's strain energy will be determined only by the internal bending moment M , and therefore, using Eq. 14-17, Eq. 14-23 can be written symbolically as

$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI} dx \quad (14-25)$$

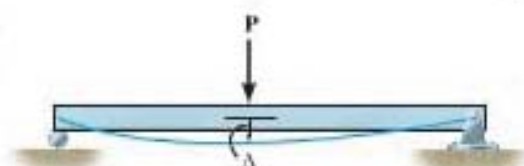


Fig. 14-19

Once M is expressed as a function of position x and the integral is evaluated, Δ can then be determined.

As a last example, we will consider a beam loaded by a couple moment $\mathbf{M}_0$ as shown in Fig. 14-20. This moment causes the rotational displacement θ at the point of application of the couple moment. Since the couple moment only does work when it *rotates*, using Eq. 14-5, the external work is $U_e = \frac{1}{2}M_0\theta$. Therefore Eq. 14-23 becomes

$$\frac{1}{2}M_0\theta = \int_0^L \frac{M^2}{2EI} dx \quad (14-26)$$

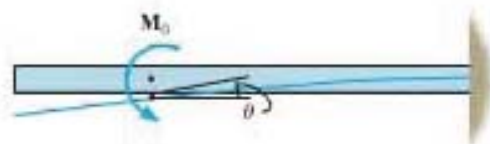


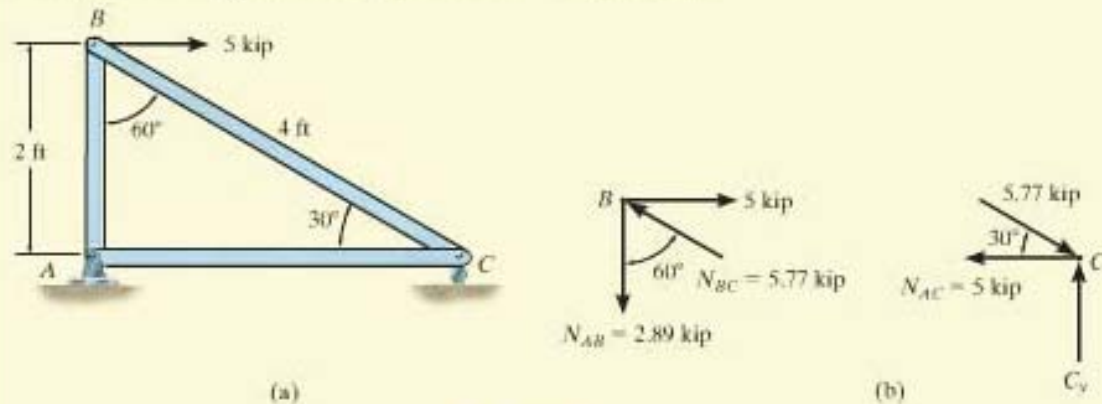
Fig. 14-20

Here the strain energy is the result of the internal bending moment M caused by application of the couple moment $\mathbf{M}_0$. Once M has been expressed as a function of x and the strain energy evaluated, then θ which measures the slope of the elastic curve can be determined.

In each of the above examples, it should be noted that application of Eq. 14-23 is *quite limited*, because only a *single* external force or couple moment must act on the member or structure. Also, the displacement can *only* be calculated at the point and in the direction of the external force or couple moment. If more than one external force or couple moment were applied, then the external work of each loading would involve its associated unknown displacement. As a result, *all* these unknown displacements could not be determined, since only the single Eq. 14-23 is available for the solution. Although application of the conservation of energy as described here has these restrictions, it does serve as an introduction to more general energy methods, which we will consider throughout the rest of this chapter.

EXAMPLE 14.6

The three-bar truss in Fig. 14-21a is subjected to a horizontal force of 5 kip. If the cross-sectional area of each member is 0.20 in^2 , determine the horizontal displacement at point B . $E = 29(10^3) \text{ ksi}$.

**Fig. 14-21****SOLUTION**

We can apply the conservation of energy to solve this problem because only a *single* external force acts on the truss and the required displacement happens to be in the *same direction* as the force. Furthermore, the reactive forces on the truss do no work since they are not displaced.

Using the method of joints, the force in each member is determined as shown on the free-body diagrams of the pins at B and C , Fig. 14-21b.

Applying Eq. 14-24, we have

$$\begin{aligned} \frac{1}{2} P \Delta &= \sum \frac{N^2 L}{2AE} \\ \frac{1}{2} (5 \text{ kip}) (\Delta_B)_h &= \frac{(2.89 \text{ kip})^2 (2 \text{ ft})}{2AE} + \frac{(-5.77 \text{ kip})^2 (4 \text{ ft})}{2AE} \\ &\quad + \frac{(5 \text{ kip})^2 (3.46 \text{ ft})}{2AE} \\ (\Delta_B)_h &= \frac{47.32 \text{ kip} \cdot \text{ft}}{AE} \end{aligned}$$

Notice that since N is squared, it does not matter if a particular member is in tension or compression. Substituting in the numerical data for A and E and solving, we get

$$\begin{aligned} (\Delta_B)_h &= \frac{47.32 \text{ kip} \cdot \text{ft} (12 \text{ in./ft})}{(0.2 \text{ in}^2) [29(10^3) \text{ kip/in}^2]} \\ &= 0.0979 \text{ in.} \rightarrow \end{aligned}$$

Ans.

EXAMPLE 14.7

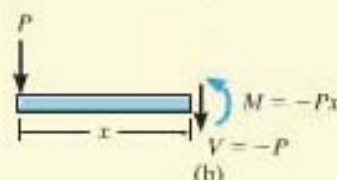
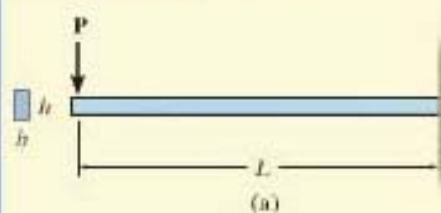


Fig. 14-22

The cantilevered beam in Fig. 14-22a has a rectangular cross section and is subjected to a load $\mathbf{P}$ at its end. Determine the displacement of the load. EI is constant.

SOLUTION

The internal shear and moment in the beam as a function of x are determined using the method of sections, Fig. 14-22b.

When applying Eq. 14-23 we will consider the strain energy due to both shear and bending. Using Eqs. 14-19 and 14-17, we have

$$\begin{aligned} \frac{1}{2} P \Delta &= \int_0^L \frac{f_s V^2 dx}{2GA} + \int_0^L \frac{M^2 dx}{2EI} \\ &= \int_0^L \frac{\left(\frac{6}{5}\right)(-P)^2 dx}{2GA} + \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{3P^2 L}{5GA} + \frac{P^2 L^3}{6EI} \quad (1) \end{aligned}$$

The first term on the right side of this equation represents the strain energy due to shear, while the second is the strain energy due to bending. As stated in Example 14.4, for most beams the shear strain energy is much smaller than the bending strain energy. To show when this is the case for the beam in Figure 14-22a, we require

$$\begin{aligned} \frac{3}{5} \frac{P^2 L}{GA} &\ll \frac{P^2 L^3}{6EI} \\ \frac{3}{5} \frac{P^2 L}{G(bh)} &\ll \frac{P^2 L^3}{6E\left[\frac{1}{12}(bh^3)\right]} \\ \frac{3}{5G} &\ll \frac{2L^2}{Eh^2} \end{aligned}$$

Since $E \approx 3G$ (see Example 14.4), then

$$0.9 \ll \left(\frac{L}{h}\right)^2$$

Hence if L is relatively long compared with h , the beam becomes slender and the shear strain energy can be neglected. In other words, the *shear strain energy* becomes important *only for short, deep beams*. For example, beams for which $L = 5h$ have approximately 28 times more bending strain energy than shear strain energy, so neglecting the shear strain energy represents an error of about 3.6%. With this in mind, Eq. 1 can be simplified to

$$\frac{1}{2} P \Delta = \frac{P^2 L^3}{6EI}$$

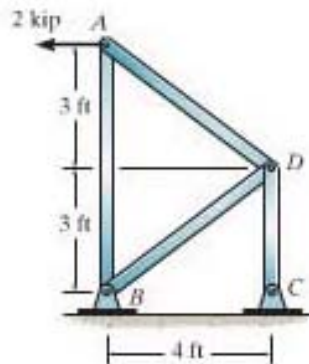
so that

$$\Delta = \frac{PL^3}{3EI}$$

Ans

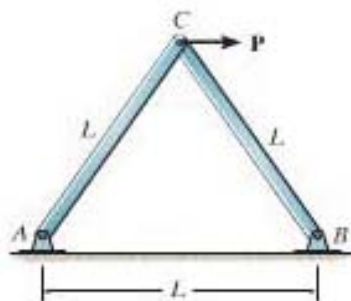
PROBLEMS

•14–25. Determine the horizontal displacement of joint A . Each bar is made of A-36 steel and has a cross-sectional area of 1.5 in^2 .



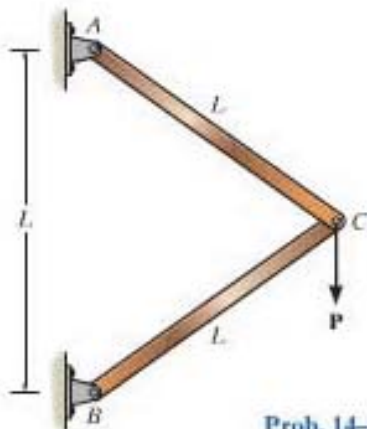
Prob. 14-25

14–26. Determine the horizontal displacement of joint C . AE is constant.



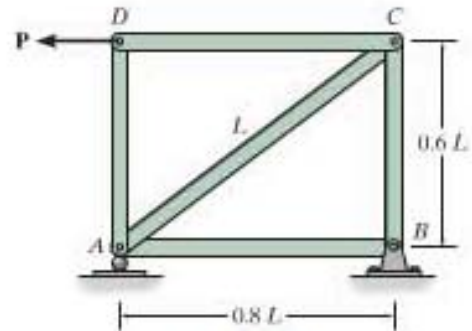
Prob. 14-26

14–27. Determine the vertical displacement of joint C . AE is constant.



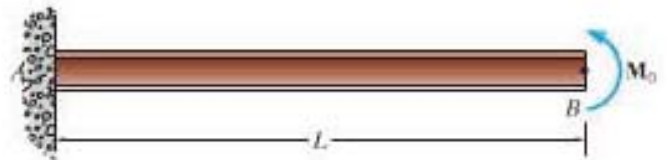
Prob. 14-27

*14–28. Determine the horizontal displacement of joint D . AE is constant.



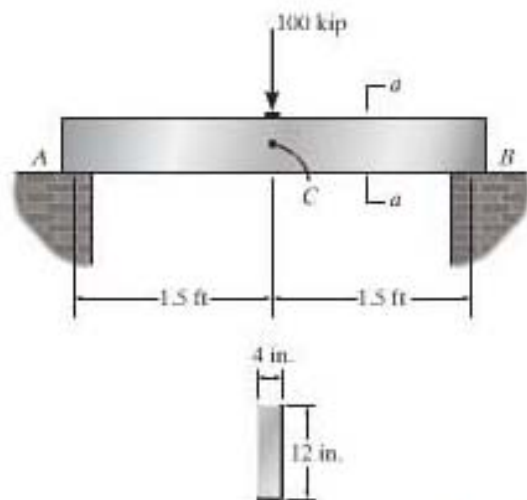
Prob. 14-28

•14–29. The cantilevered beam is subjected to a couple moment M_0 applied at its end. Determine the slope of the beam at B . EI is constant.



Prob. 14-29

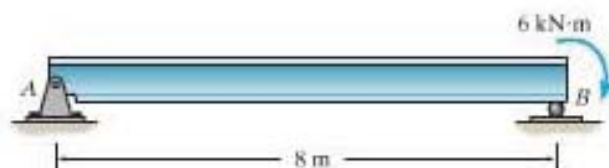
14–30. Determine the vertical displacement of point C of the simply supported 6061-T6 aluminum beam. Consider both shearing and bending strain energy.



Section a - a

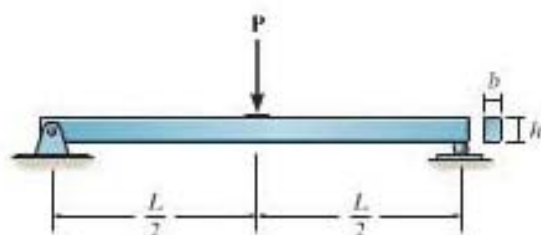
Prob. 14-30

14-31. Determine the slope at the end B of the A-36 steel beam. $I = 80(10^6) \text{ mm}^4$.



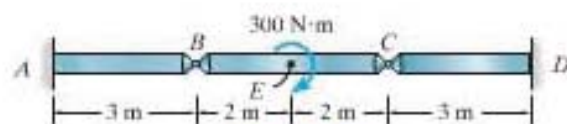
Prob. 14-31

***14-32.** Determine the deflection of the beam at its center caused by shear. The shear modulus is G .



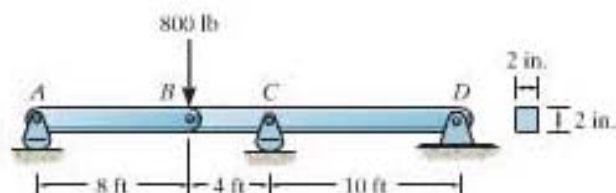
Prob. 14-32

•14-33. The A-36 steel bars are pin connected at B and C . If they each have a diameter of 30 mm, determine the slope at E .



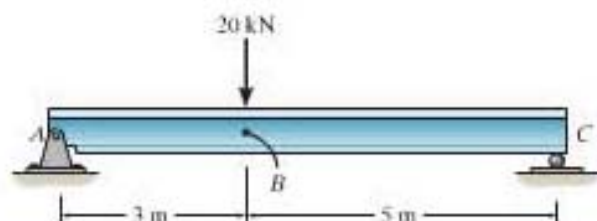
Prob. 14-33

14-34. The A-36 steel bars are pin connected at B . If each has a square cross section, determine the vertical displacement at B .



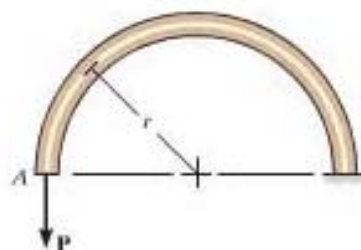
Prob. 14-34

14-35. Determine the displacement of point B on the A-36 steel beam. $I = 80(10^6) \text{ mm}^4$.



Prob. 14-35

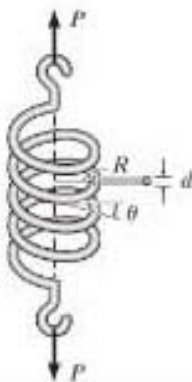
***14-36.** The rod has a circular cross section with a moment of inertia I . If a vertical force P is applied at A , determine the vertical displacement at this point. Only consider the strain energy due to bending. The modulus of elasticity is E .



Prob. 14-36

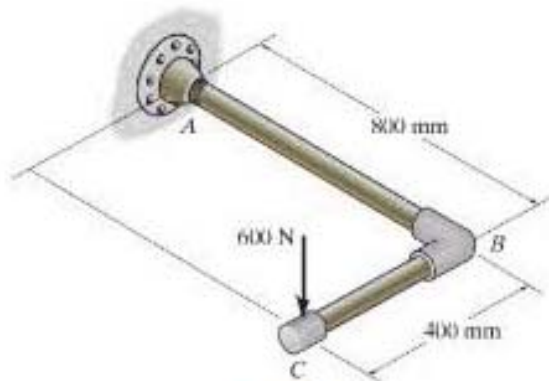
•14–37. The load $\mathbf{P}$ causes the open coils of the spring to make an angle θ with the horizontal when the spring is stretched. Show that for this position this causes a torque $T = PR \cos \theta$ and a bending moment $M = PR \sin \theta$ at the cross section. Use these results to determine the maximum normal stress in the material.

14–38. The coiled spring has n coils and is made from a material having a shear modulus G . Determine the stretch of the spring when it is subjected to the load $\mathbf{P}$. Assume that the coils are close to each other so that $\theta \approx 0^\circ$ and the deflection is caused entirely by the torsional stress in the coil.



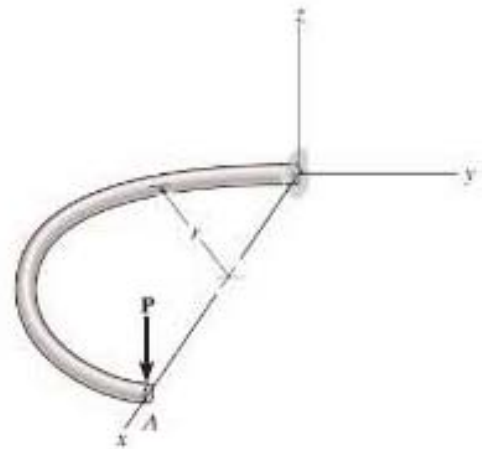
Probs. 14–37/38

14–39. The pipe assembly is fixed at A . Determine the vertical displacement of end C of the assembly. The pipe has an inner diameter of 40 mm and outer diameter of 60 mm and is made of A-36 steel. Neglect the shearing strain energy.



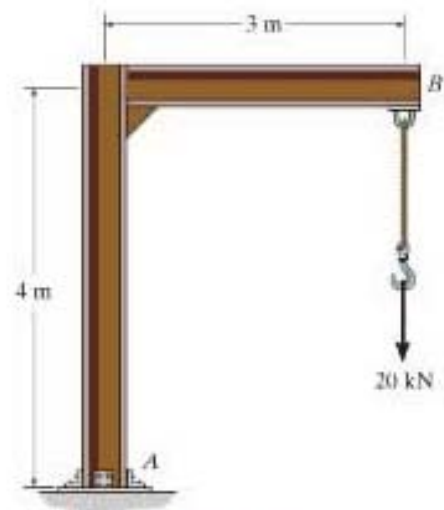
Prob. 14–39

*14–40. The rod has a circular cross section with a polar moment of inertia J and moment of inertia I . If a vertical force $\mathbf{P}$ is applied at A , determine the vertical displacement at this point. Consider the strain energy due to bending and torsion. The material constants are E and G .



Prob. 14–40

•14–41. Determine the vertical displacement of end B of the frame. Consider only bending strain energy. The frame is made using two A-36 steel W460 $\times$ 68 wide-flange sections.



Prob. 14–41

14.4 Impact Loading

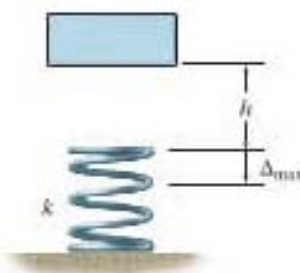


Fig. 14-23

Throughout this text we have considered all loadings to be applied to a body in a gradual manner, such that when they reach a maximum value the body remains static. Some loadings, however, are dynamic; that is, they vary with time. A typical example would be caused by the collision of objects. This is called an impact loading. Specifically, **impact** occurs when one object strikes another, such that large forces are developed between the objects during a very short period of time.

If we assume no energy is lost during impact, due to heat, sound or localized plastic deformations, then we can study the mechanics of impact using the conservation of energy. To show how this is done, we will first analyze the motion of a simple block-and-spring system as shown in Fig. 14-23. When the block is released from rest, it falls a distance h , striking the spring and compressing it a distance $\Delta_{\max}$ before momentarily coming to rest. If we neglect the mass of the spring and assume that the spring responds *elastically*, then the conservation of energy requires that the energy of the falling block be transformed into stored (strain) energy in the spring; or in other words, the work done by the block's weight, falling $h + \Delta_{\max}$, is equal to the work needed to displace the end of the spring by an amount $\Delta_{\max}$. Since the force in a spring is related to $\Delta_{\max}$ by the equation $F = k\Delta_{\max}$, where k is the spring stiffness, then applying the conservation of energy and Eq. 14-2, we have

$$\begin{aligned}
 U_e &= U_i \\
 W(h + \Delta_{\max}) &= \frac{1}{2}(k\Delta_{\max})\Delta_{\max} \\
 W(h + \Delta_{\max}) &= \frac{1}{2}k\Delta_{\max}^2 \quad (14-27) \\
 \Delta_{\max}^2 - \frac{2W}{k}\Delta_{\max} - 2\left(\frac{W}{k}\right)h &= 0
 \end{aligned}$$

This quadratic equation may be solved for $\Delta_{\max}$. The maximum root is

$$\Delta_{\max} = \frac{W}{k} + \sqrt{\left(\frac{W}{k}\right)^2 + 2\left(\frac{W}{k}\right)h}$$

If the weight W is supported statically by the spring, then the top displacement of the spring is $\Delta_{st} = W/k$. Using this simplification, the above equation becomes

$$\Delta_{\max} = \Delta_{st} + \sqrt{(\Delta_{st})^2 + 2\Delta_{st}h}$$

or

$$\Delta_{\max} = \Delta_{st} \left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)} \right] \quad (14-28)$$



This crash barrier is designed to absorb the impact energy of moving vehicles.

Once $\Delta_{\max}$ is calculated, the maximum force applied to the spring can be determined from

$$F_{\max} = k\Delta_{\max} \quad (14-29)$$

It should be realized, however, that this force and associated displacement occur only at an *instant*. Provided the block does not rebound off the spring, it will continue to vibrate until the motion dampens out and the block assumes the static position, Δ_{st} . Note also that if the block is held just above the spring, $h = 0$, and released, then, from Eq. 14-28, the maximum displacement of the block is

$$\Delta_{\max} = 2\Delta_{st}$$

In other words, when the block is released from the top of the spring (a dynamic load), the displacement is *twice* what it would be if it were set on the spring (a static load).

Using a similar analysis, it is also possible to determine the maximum displacement of the end of the spring if the block is sliding on a smooth horizontal surface with a known velocity v just before it collides with the spring, Fig. 14-24. Here the block's kinetic energy,* $\frac{1}{2}(W/g)v^2$, will be transformed into stored energy in the spring. Hence,

$$\begin{aligned} U_e &= U_i \\ \frac{1}{2}\left(\frac{W}{g}\right)v^2 &= \frac{1}{2}k\Delta_{\max}^2 \\ \Delta_{\max} &= \sqrt{\frac{Wv^2}{gk}} \end{aligned} \quad (14-30)$$

Since the static displacement at the top of the spring caused by the weight W resting on it is $\Delta_{st} = W/k$, then

$$\Delta_{\max} = \sqrt{\frac{\Delta_{st}v^2}{g}} \quad (14-31)$$

The results of this simplified analysis can be used to determine both the approximate deflection and the stress developed in a deformable member when it is subjected to impact. To do this we must make the necessary assumptions regarding the collision, so that the behavior of the colliding bodies is similar to the response of the block-and-spring models discussed above. Hence we will consider the moving body to be *rigid* like the block and the stationary body to be deformable like the spring. Also, it is assumed that the material behaves in a linear-elastic manner. When collision occurs, the bodies remain in contact until the elastic body reaches its maximum deformation, and during the motion the inertia or mass of the elastic body is neglected. Realize that each of these assumptions will lead to a *conservative* estimate of both the maximum stress and deflection of the elastic body. In other words, their values will be larger than those that actually occur.

*Recall from physics that kinetic energy is "energy of motion." For the translation of a body it is determined from $\frac{1}{2}mv^2$, where m is the body's mass, $m = W/g$.

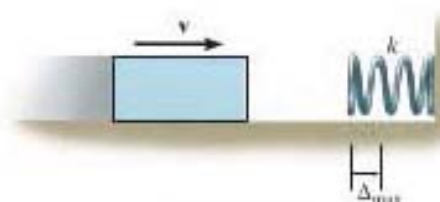


Fig. 14-24

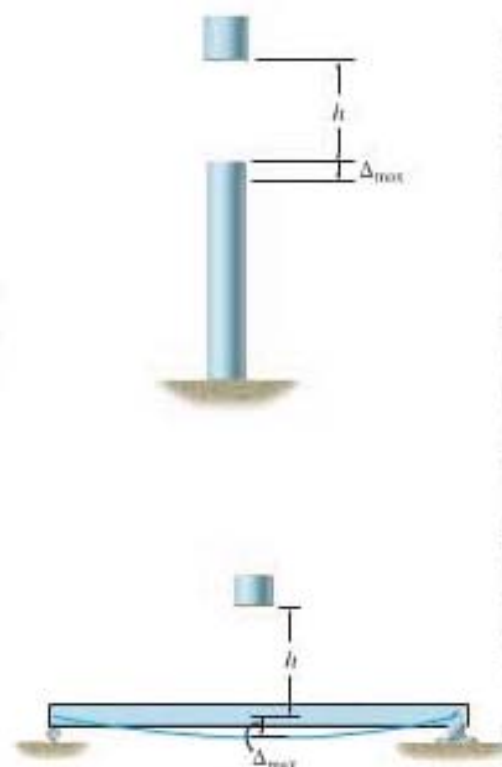


Fig. 14-25



The members of this crash guard must be designed to resist a prescribed impact loading in order to arrest the motion of a rail car.

A few examples of when this theory can be applied are shown in Fig. 14-25. Here a block of known weight is dropped onto a post and a beam, causing them to deform a maximum amount $\Delta_{\max}$. The energy of the falling block is transformed momentarily into axial strain energy in the post and bending strain energy in the beam.* In order to determine the deformation $\Delta_{\max}$, we could use the same approach as for the block-spring system, and that is to write the conservation-of-energy equation for the block and post or block and beam, and then solve for $\Delta_{\max}$. However, we can also solve these problems in a more direct manner by modeling the post and beam by an *equivalent spring*. For example, if a force $\mathbf{P}$ displaces the top of the post $\Delta = PL/AE$, then a spring having a stiffness $k = AE/L$ would be displaced the same amount by $\mathbf{P}$, that is, $\Delta = P/k$. In a similar manner, from Appendix C, a force $\mathbf{P}$ applied to the center of a simply supported beam displaces the center $\Delta = PL^3/48EI$, and therefore an equivalent spring would have a stiffness of $k = 48EI/L^3$. It is not necessary, however, to actually find the equivalent spring stiffness to apply Eq. 14-28 or 14-30. All that is needed to determine the dynamic displacement, $\Delta_{\max}$, is to calculate the *static displacement*, Δ_{st} , due to the weight $P_{st} = W$ of the block resting on the member.

Once $\Delta_{\max}$ is determined, the maximum dynamic force can then be calculated from $P_{\max} = k\Delta_{\max}$. If we consider $P_{\max}$ to be an *equivalent static load* then the maximum stress in the member can be determined using statics and the theory of mechanics of materials. Recall that this stress acts only for an *instant*. In reality, vibrational waves pass through the material, and the stress in the post or the beam, for example, does not remain constant.

The ratio of the equivalent static load $P_{\max}$ to the static load $P_{st} = W$ is called the *impact factor*, n . Since $P_{\max} = k\Delta_{\max}$ and $P_{st} = k\Delta_{st}$, then from Eq. 14-28, we can express it as

$$n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)} \quad (14-32)$$

This factor represents the magnification of a statically applied load so that it can be treated dynamically. Using Eq. 14-32, n can be calculated for any member that has a linear relationship between load and deflection. For a complicated system of connected members, however, impact factors are determined from experience and experimental testing. Once n is determined, the dynamic stress and deflection at the point of impact are easily found from the static stress σ_{st} and static deflection Δ_{st} caused by the load W , that is, $\sigma_{\max} = n\sigma_{st}$ and $\Delta_{\max} = n\Delta_{st}$.

*Strain energy due to shear is neglected for reasons discussed in Example 14.4.

Important Points

- *Impact* occurs when a large force is developed between two objects which strike one another during a short period of time.
- We can analyze the effects of impact by assuming the moving body is rigid, the material of the stationary body is linear elastic, no energy is lost during collision, the bodies remain in contact during collision, and the inertia of the elastic body is neglected.
- The dynamic load on a body can be determined by multiplying the static load by an *impact factor*.

EXAMPLE 14.8

The aluminum pipe shown in Fig. 14-26 is used to support a load of 150 kip. Determine the maximum displacement at the top of the pipe if the load is (a) applied gradually, and (b) applied by suddenly releasing it from the top of the pipe when $h = 0$. Take $E_{al} = 10(10^3)$ ksi and assume that the aluminum behaves elastically.

SOLUTION

Part (a). When the load is applied gradually, the work done by the weight is transformed into elastic strain energy in the pipe. Applying the conservation of energy, we have

$$\begin{aligned}
 U_e &= U_i \\
 \frac{1}{2}W\Delta_{st} &= \frac{W^2L}{2AE} \\
 \Delta_{st} &= \frac{WL}{AE} = \frac{150 \text{ kip}(12 \text{ in.})}{\pi[(3 \text{ in.})^2 - (2.5 \text{ in.})^2]10(10^3) \text{ kip/in}^2} \\
 &= 0.02083 \text{ in.} = 0.0208 \text{ in.} \quad \text{Ans}
 \end{aligned}$$

Part (b). Here Eq. 14-28 can be applied, with $h = 0$. Hence,

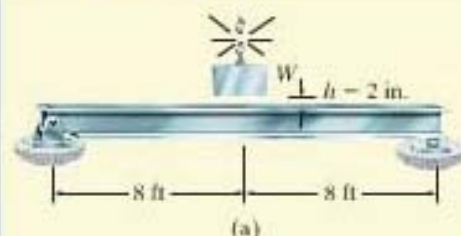
$$\begin{aligned}
 \Delta_{max} &= \Delta_{st} \left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)} \right] \\
 &= 2\Delta_{st} = 2(0.02083 \text{ in.}) \\
 &= 0.0417 \text{ in.} \quad \text{Ans}
 \end{aligned}$$

Hence, the displacement of the weight when applied dynamically is twice as great as when the load is applied statically. In other words, the impact factor is $n = 2$, Eq. 14-32.



Fig. 14-26

EXAMPLE 14.9



The A-36 steel beam shown in Fig. 14-27a is a W10 $\times$ 39. Determine the maximum bending stress in the beam and the beam's maximum deflection if the weight $W = 1.50$ kip is dropped from a height $h = 2$ in. onto the beam. $E_{st} = 29(10^3)$ ksi.

SOLUTION I

We will apply Eq. 14-28. First, however, we must calculate Δ_{st} . Using the table in Appendix C, and the data in Appendix B for the properties of a W10 $\times$ 39, we have

$$\Delta_{st} = \frac{WL^3}{48EI} = \frac{(1.50 \text{ kip})(16 \text{ ft})^3(12 \text{ in./ft})^3}{48[29(10^3) \text{ ksi}](209 \text{ in}^4)} = 0.03649 \text{ in.}$$

$$\begin{aligned} \Delta_{max} &= \Delta_{st} \left[1 + \sqrt{1 + 2 \left(\frac{h}{\Delta_{st}} \right)} \right] \\ &= 0.03649 \text{ in.} \left[1 + \sqrt{1 + 2 \left(\frac{2 \text{ in.}}{0.03649 \text{ in.}} \right)} \right] = 0.420 \text{ in.} \quad \text{Ans.} \end{aligned}$$

The equivalent static load that causes this displacement is therefore

$$P_{max} = \frac{48EI}{L^3} \Delta_{max} = \frac{48(29(10^3) \text{ ksi})(209 \text{ in}^4)}{(16 \text{ ft})^3 (12 \text{ in./ft})^3} (0.420 \text{ in.}) = 17.3 \text{ kip}$$

The internal moment caused by this load is maximum at the center of the beam, such that by the method of sections, Fig. 14-27b, $M_{max} = P_{max}L/4$. Applying the flexure formula to determine the bending stress, we have

$$\begin{aligned} \sigma_{max} &= \frac{M_{max}c}{I} = \frac{P_{max}Lc}{4I} = \frac{12E \Delta_{max}c}{L^2} \\ &= \frac{12[29(10^3) \text{ kip/in}^2](0.420 \text{ in.})(9.92 \text{ in./2})}{(16 \text{ ft})^2(12 \text{ in./ft})^2} = 19.7 \text{ ksi} \quad \text{Ans.} \end{aligned}$$

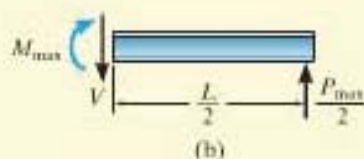


Fig. 14-27

SOLUTION II

It is also possible to obtain the dynamic or maximum deflection Δ_{max} from first principles. The external work of the falling weight W is $U_e = W(h + \Delta_{max})$. Since the beam deflects Δ_{max} , and $P_{max} = 48EI\Delta_{max}/L^3$, then

$$\begin{aligned} U_e &= U_i \\ W(h + \Delta_{max}) &= \frac{1}{2} \left(\frac{48EI\Delta_{max}}{L^3} \right) \Delta_{max} \\ (1.50 \text{ kip})(2 \text{ in.} + \Delta_{max}) &= \frac{1}{2} \left[\frac{48[29(10^3) \text{ kip/in}^2](209 \text{ in}^4)}{(16 \text{ ft})^3(12 \text{ in./ft})^3} \right] \Delta_{max}^2 \\ 20.55\Delta_{max}^2 - 1.50\Delta_{max} - 3.00 &= 0 \end{aligned}$$

Solving and choosing the positive root yields

$$\Delta_{max} = 0.420 \text{ in.} \quad \text{Ans.}$$

EXAMPLE 14.10

A railroad car that is assumed to be rigid and has a mass of 80 Mg is moving forward at a speed of $v = 0.2$ m/s when it strikes a steel 200-mm by 200-mm post at A , Fig. 14-28*a*. If the post is fixed to the ground at C , determine the maximum horizontal displacement of its top B due to the impact. Take $E_{st} = 200$ GPa.

SOLUTION

Here the kinetic energy of the railroad car is transformed into internal bending strain energy only for region AC of the post. (Region BA is not subjected to an internal loading.) Assuming that point A is displaced $(\Delta_A)_{max}$, then the force P_{max} that causes this displacement can be determined from the table in Appendix C. We have

$$P_{max} = \frac{3EI(\Delta_A)_{max}}{L_{AC}^3} \quad (1)$$

$$U_e = U_i; \quad \frac{1}{2}mv^2 = \frac{1}{2}P_{max}(\Delta_A)_{max}$$

$$\frac{1}{2}mv^2 = \frac{1}{2} \frac{3EI}{L_{AC}^3} (\Delta_A)_{max}^2; \quad (\Delta_A)_{max} = \sqrt{\frac{mv^2 L_{AC}^3}{3EI}}$$

Substituting in the numerical data yields

$$(\Delta_A)_{max} = \sqrt{\frac{80(10^3) \text{ kg}(0.2 \text{ m/s})^2(1.5 \text{ m})^3}{3[200(10^9) \text{ N/m}^2]\left[\frac{1}{12}(0.2 \text{ m})^4\right]}} = 0.01162 \text{ m} = 11.62 \text{ mm}$$

Using Eq. 1, the force P_{max} is therefore

$$P_{max} = \frac{3[200(10^9) \text{ N/m}^2]\left[\frac{1}{12}(0.2 \text{ m})^4\right](0.01162 \text{ m})}{(1.5 \text{ m})^3} = 275.4 \text{ kN}$$

With reference to Fig. 14-28*b*, segment AB of the post remains straight. To determine the maximum displacement at B , we must first determine the slope at A . Using the appropriate formula from the table in Appendix C to determine θ_A , we have

$$\theta_A = \frac{P_{max}L_{AC}^2}{2EI} = \frac{275.4(10^3) \text{ N}(1.5 \text{ m})^2}{2[200(10^9) \text{ N/m}^2]\left[\frac{1}{12}(0.2 \text{ m})^4\right]} = 0.01162 \text{ rad}$$

The maximum displacement at B is thus

$$\begin{aligned} (\Delta_B)_{max} &= (\Delta_A)_{max} + \theta_A L_{AB} \\ &= 11.62 \text{ mm} + (0.01162 \text{ rad}) 1(10^3) \text{ mm} = 23.2 \text{ mm} \quad \text{Ans.} \end{aligned}$$

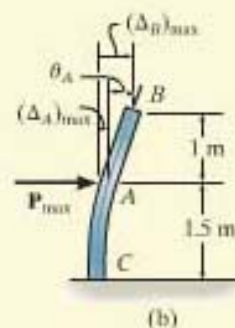
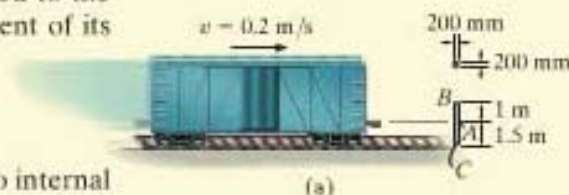


Fig. 14-28

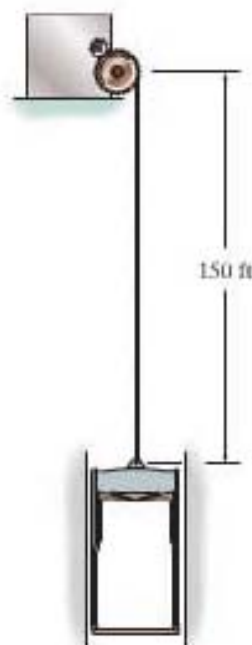
PROBLEMS

14-42. A bar is 4 m long and has a diameter of 30 mm. If it is to be used to absorb energy in tension from an impact loading, determine the total amount of elastic energy that it can absorb if (a) it is made of steel for which $E_{st} = 200$ GPa, $\sigma_Y = 800$ MPa, and (b) it is made from an aluminum alloy for which $E_{al} = 70$ GPa, $\sigma_Y = 405$ MPa.

14

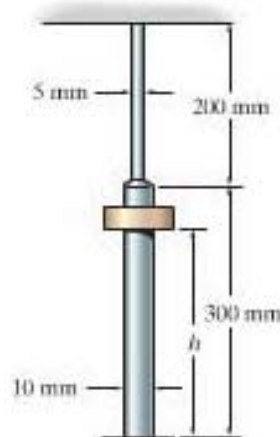
14-43. Determine the diameter of a red brass C83400 bar that is 8 ft long if it is to be used to absorb 800 ft·lb of energy in tension from an impact loading. No yielding occurs.

***14-44.** A steel cable having a diameter of 0.4 in. wraps over a drum and is used to lower an elevator having a weight of 800 lb. The elevator is 150 ft below the drum and is descending at the constant rate of 2 ft/s when the drum suddenly stops. Determine the maximum stress developed in the cable when this occurs. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 50$ ksi.



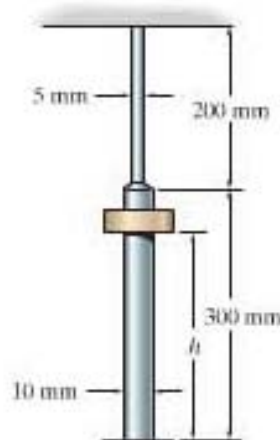
Prob. 14-44

•14-45. The composite aluminum bar is made from two segments having diameters of 5 mm and 10 mm. Determine the maximum axial stress developed in the bar if the 5-kg collar is dropped from a height of $h = 100$ mm. $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Prob. 14-45

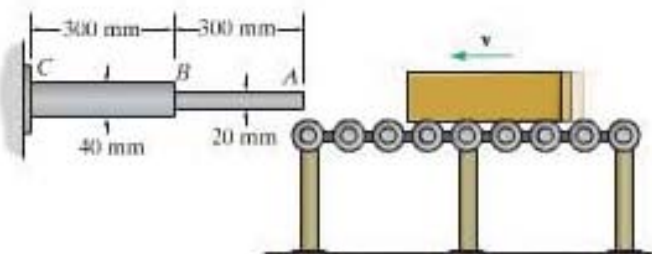
14-46. The composite aluminum bar is made from two segments having diameters of 5 mm and 10 mm. Determine the maximum height h from which the 5-kg collar should be dropped so that it produces a maximum axial stress in the bar of $\sigma_{max} = 300$ MPa. $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Prob. 14-46

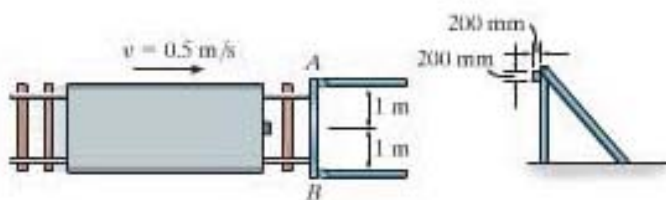
14-47. The 5-kg block is traveling with the speed of $v = 4$ m/s just before it strikes the 6061-T6 aluminum stepped cylinder. Determine the maximum normal stress developed in the cylinder.

***14-48.** Determine the maximum speed v of the 5-kg block without causing the 6061-T6 aluminum stepped cylinder to yield after it is struck by the block.



Probs. 14-47/48

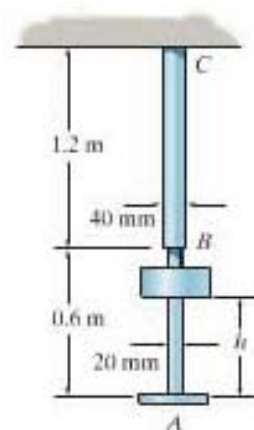
***14-49.** The steel beam AB acts to stop the oncoming railroad car, which has a mass of 10 Mg and is coasting towards it at $v = 0.5$ m/s. Determine the maximum stress developed in the beam if it is struck at its center by the car. The beam is simply supported and only horizontal forces occur at A and B . Assume that the railroad car and the supporting framework for the beam remains rigid. Also, compute the maximum deflection of the beam. $E_{st} = 200$ GPa, $\sigma_Y = 250$ MPa.



Prob. 14-49

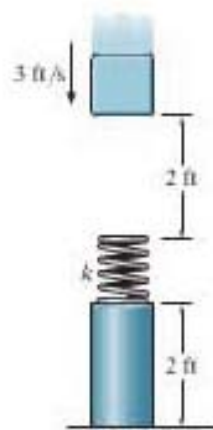
14-50. The aluminum bar assembly is made from two segments having diameters of 40 mm and 20 mm. Determine the maximum axial stress developed in the bar if the 10-kg collar is dropped from a height of $h = 150$ mm. Take $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.

14-51. The aluminum bar assembly is made from two segments having diameters of 40 mm and 20 mm. Determine the maximum height h from which the 60-kg collar can be dropped so that it will not cause the bar to yield. Take $E_{al} = 70$ GPa, $\sigma_Y = 410$ MPa.



Probs. 14-50/51

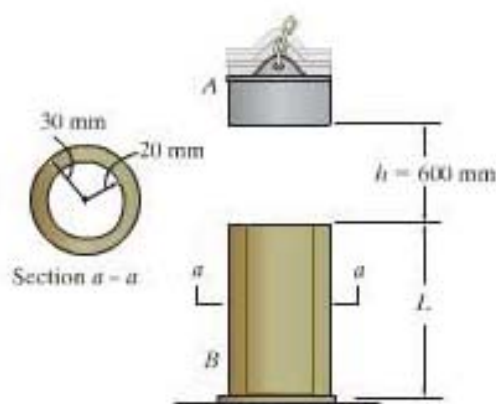
***14-52.** The 50-lb weight is falling at 3 ft/s at the instant it is 2 ft above the spring and post assembly. Determine the maximum stress in the post if the spring has a stiffness of $k = 200$ kip/in. The post has a diameter of 3 in. and a modulus of elasticity of $E = 6.80(10^3)$ ksi. Assume the material will not yield.



Prob. 14-52

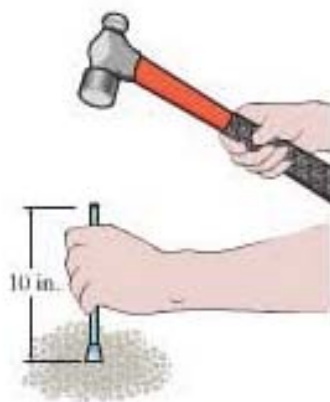
•14-53. The 50-kg block is dropped from $h = 600$ mm onto the bronze C86100 tube. Determine the minimum length L the tube can have without causing the tube to yield.

14-54. The 50-kg block is dropped from $h = 600$ mm onto the bronze C86100 tube. If $L = 900$ mm, determine the maximum normal stress developed in the tube.



Probs. 14-53/54

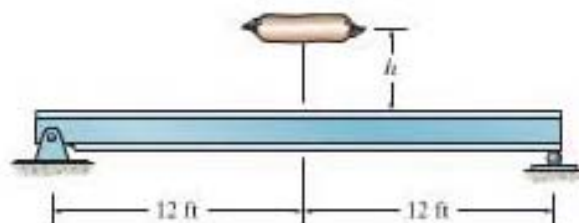
14-55. The steel chisel has a diameter of 0.5 in. and a length of 10 in. It is struck by a hammer that weighs 3 lb, and at the instant of impact it is moving at 12 ft/s. Determine the maximum compressive stress in the chisel, assuming that 80% of the impacting energy goes into the chisel. $E_{st} = 29(10^3)$ ksi, $\sigma_Y = 100$ ksi.



Prob. 14-55

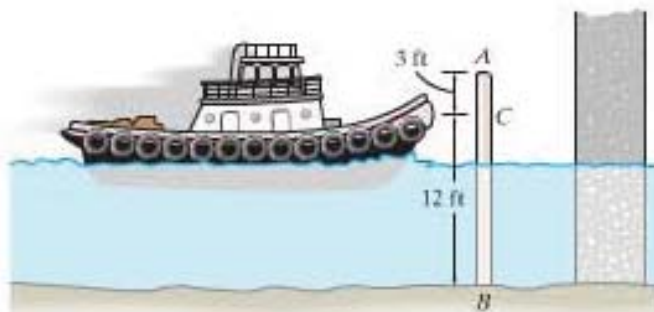
***14-56.** The sack of cement has a weight of 90 lb. If it is dropped from rest at a height of $h = 4$ ft onto the center of the W10 $\times$ 39 structural steel A-36 beam, determine the maximum bending stress developed in the beam due to the impact. Also, what is the impact factor?

•14-57. The sack of cement has a weight of 90 lb. Determine the maximum height h from which it can be dropped from rest onto the center of the W10 $\times$ 39 structural steel A-36 beam so that the maximum bending stress due to impact does not exceed 30 ksi.



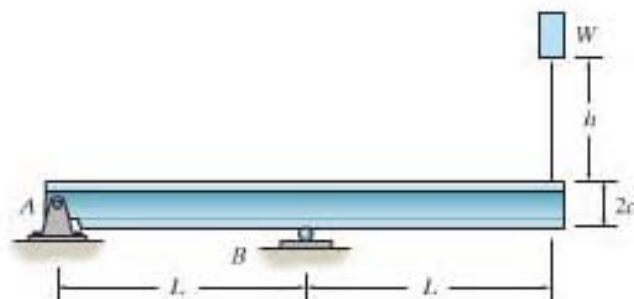
Probs. 14-56/57

14-58. The tugboat has a weight of 120 000 lb and is traveling forward at 2 ft/s when it strikes the 12-in.-diameter fender post AB used to protect a bridge pier. If the post is made from treated white spruce and is assumed fixed at the river bed, determine the maximum horizontal distance the top of the post will move due to the impact. Assume the tugboat is rigid and neglect the effect of the water.



Prob. 14-58

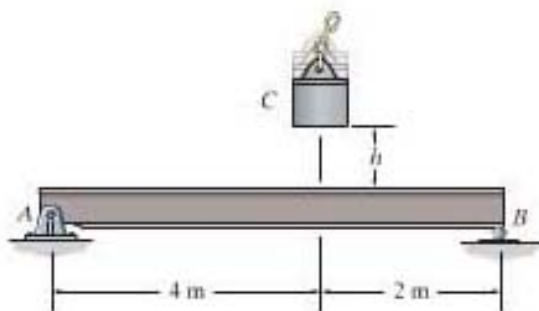
14-59. The wide-flange beam has a length of $2L$, a depth $2c$, and a constant EI . Determine the maximum height h at which a weight W can be dropped on its end without exceeding a maximum elastic stress $\sigma_{\max}$ in the beam.



Prob. 14-59

***14-60.** The 50-kg block C is dropped from $h = 1.5$ m onto the simply supported beam. If the beam is an A-36 steel W250 $\times$ 45 wide-flange section, determine the maximum bending stress developed in the beam.

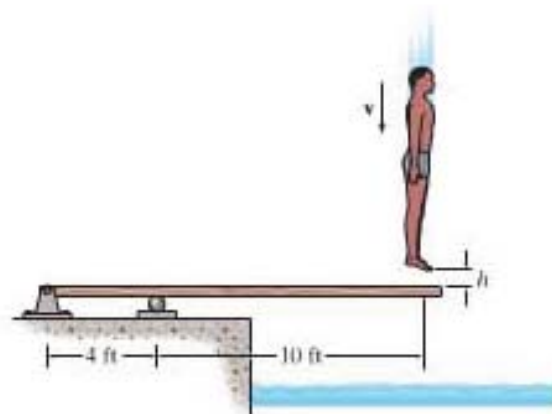
***14-61.** Determine the maximum height h from which the 50-kg block C can be dropped without causing yielding in the A-36 steel W310 $\times$ 39 wide flange section when the block strikes the beam.



Probs. 14-60/61

14-62. The diver weighs 150 lb and, while holding himself rigid, strikes the end of a wooden diving board ($h = 0$) with a downward velocity of 4 ft/s. Determine the maximum bending stress developed in the board. The board has a thickness of 1.5 in. and width of 1.5 ft. $E_w = 1.8(10^3)$ ksi, $\sigma_Y = 8$ ksi.

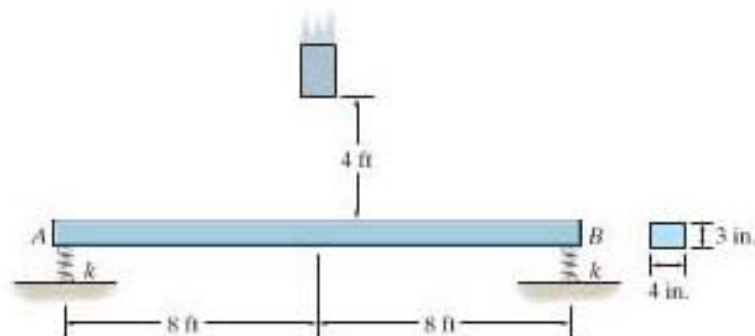
14-63. The diver weighs 150 lb and, while holding himself rigid, strikes the end of the wooden diving board. Determine the maximum height h from which he can jump onto the board so that the maximum bending stress in the wood does not exceed 6 ksi. The board has a thickness of 1.5 in. and width of 1.5 ft. $E_w = 1.8(10^3)$ ksi.



Probs. 14-62/63

***14-64.** The weight of 175 lb is dropped from a height of 4 ft from the top of the A-36 steel beam. Determine the maximum deflection and maximum stress in the beam if the supporting springs at A and B each have a stiffness of $k = 500$ lb/in. The beam is 3 in. thick and 4 in. wide.

***14-65.** The weight of 175 lb, is dropped from a height of 4 ft from the top of the A-36 steel beam. Determine the load factor n if the supporting springs at A and B each have a stiffness of $k = 300$ lb/in. The beam is 3 in. thick and 4 in. wide.

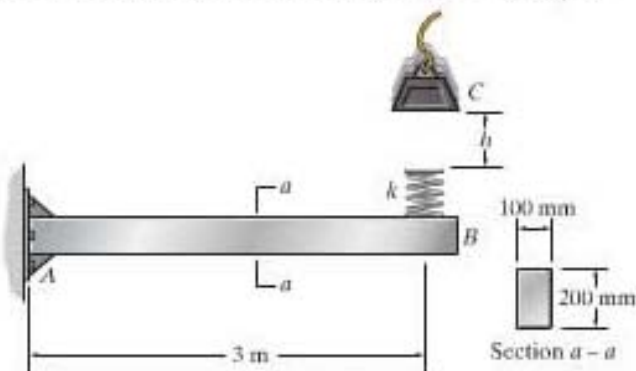


Probs. 14-64/65

14-66. Block C of mass 50 kg is dropped from height $h = 0.9$ m onto the spring of stiffness $k = 150$ kN/m mounted on the end B of the 6061-T6 aluminum cantilever beam. Determine the maximum bending stress developed in the beam.

14-67. Determine the maximum height h from which 200-kg block C can be dropped without causing the 6061-T6 aluminum cantilever beam to yield. The spring mounted on the end B of the beam has a stiffness of $k = 150$ kN/m.

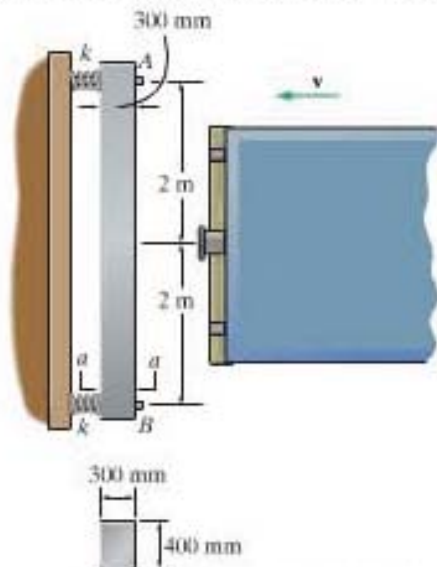
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Probs. 14-66/67

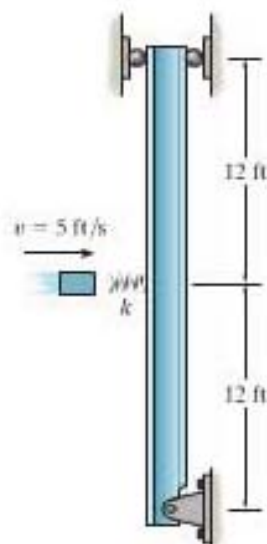
***14-68.** The 2014-T6 aluminum bar AB can slide freely along the guides mounted on the rigid crash barrier. If the railcar of mass 10 Mg is traveling with a speed of $v = 1.5$ m/s, determine the maximum bending stress developed in the bar. The springs at A and B have a stiffness of $k = 15$ MN/m.

***14-69.** The 2014-T6 aluminum bar AB can slide freely along the guides mounted on the rigid crash barrier. Determine the maximum speed v the 10-Mg railcar without causing the bar to yield when it is struck by the railcar. The springs at A and B have a stiffness of $k = 15$ MN/m.



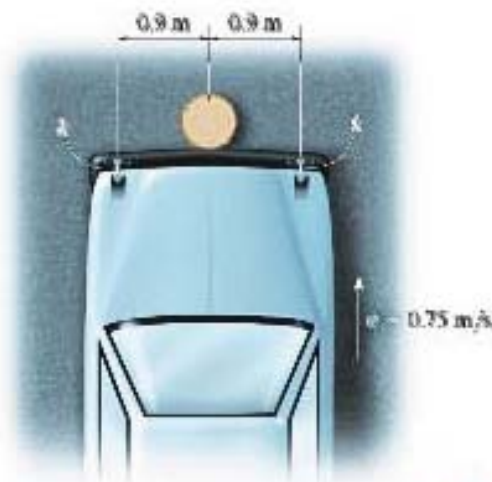
Probs. 14-68/69

14-70. The simply supported $W10 \times 15$ structural A-36 steel beam lies in the horizontal plane and acts as a shock absorber for the 500-lb block which is traveling toward it at 5 ft/s. Determine the maximum deflection of the beam and the maximum stress in the beam during the impact. The spring has a stiffness of $k = 1000$ lb/in.



Prob. 14-70

14-71. The car bumper is made of polycarbonate-polybutylene terephthalate. If $E = 2.0$ GPa, determine the maximum deflection and maximum stress in the bumper if it strikes the rigid post when the car is coasting at $v = 0.75$ m/s. The car has a mass of 1.80 Mg, and the bumper can be considered simply supported on two spring supports connected to the rigid frame of the car. For the bumper take $I = 300(10^6)$ mm⁴, $c = 75$ mm, $\sigma_Y = 30$ MPa and $k = 1.5$ MN/m.



Prob. 14-71

*14.5 Principle of Virtual Work

The principle of virtual work was developed by John Bernoulli in 1717, and like other energy methods of analysis, it is based on the conservation of energy. Although the principle of virtual work has many applications in mechanics, in this text we will use it to obtain the displacement and slope at a point on a deformable body.

To do this, we will consider the body to be of arbitrary shape as shown in Fig. 14-29b, and to be subjected to the “real loads” $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. It is assumed that these loads cause no movement of the supports; however, in general they can strain the material *beyond* the elastic limit. Suppose that it is necessary to determine the displacement Δ of point A on the body. Since there is no force acting at A , then Δ will *not* be included as an external “work term” in the equation when the conservation of energy principle is applied to the body. In order to get around this limitation, we will place an *imaginary* or “virtual” force $\mathbf{P}'$ on the body at point A , such that $\mathbf{P}'$ acts in the *same direction* as Δ . Furthermore, this load is applied to the body *before* the real loads are applied, Fig. 14-29a. For convenience, which will be made clear later, we will choose $\mathbf{P}'$ to have a “unit” magnitude; that is, $P' = 1$. It is to be emphasized that the term “*virtual*” is used to describe this load because it is *imaginary* and does not actually exist as part of the real loading.

This external virtual load, however, does create an internal virtual load $\mathbf{u}$ in a representative element or fiber of the body, as shown in Fig. 14-29a. As expected, P' and u can be related by the equations of equilibrium. Also, because of P' and u , the body and the element will each undergo a virtual (imaginary) displacement, although we will *not* be

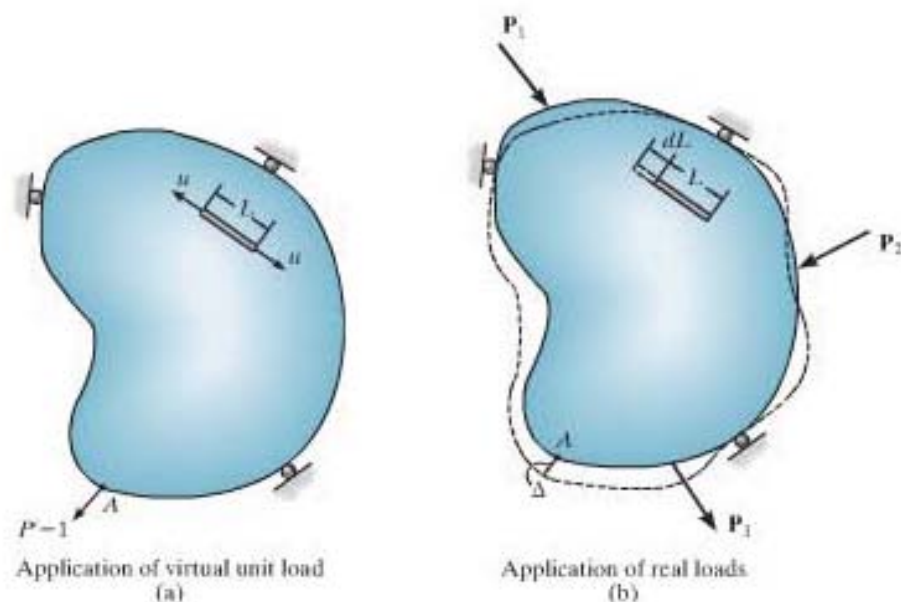


Fig. 14-29

concerned with their magnitudes. Once the virtual load is applied and *then* the body is subjected to the *real loads* $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, point A will be displaced a real amount Δ , which causes the element to be displaced dL , Fig. 14-29*b*. As a result, the external virtual force $\mathbf{P}'$ and internal virtual load $\mathbf{u}$ “ride along” or are displaced by Δ and dL , respectively. Consequently these loads perform *external virtual work* $1 \cdot \Delta$ on the body and *internal virtual work* $u \cdot dL$ on the element. Considering *only* the conservation of *virtual energy*, the external virtual work is then equal to the internal virtual work done on all the elements of the body. Therefore, we can write the virtual-work equation as

$$\begin{array}{c} \text{virtual loadings} \\ \overbrace{1 \cdot \Delta = \sum u \cdot dL} \\ \text{real displacements} \end{array} \quad (14-34)$$

Here

$P' = 1$ = external virtual unit load acting in the direction of Δ

u = internal virtual load acting on the element

Δ = external displacement caused by the real loads

dL = internal displacement of the element in the direction of $\mathbf{u}$, caused by the real loads

By choosing $P' = 1$, it can be seen that the solution for Δ follows directly, since $\Delta = \sum u dL$.

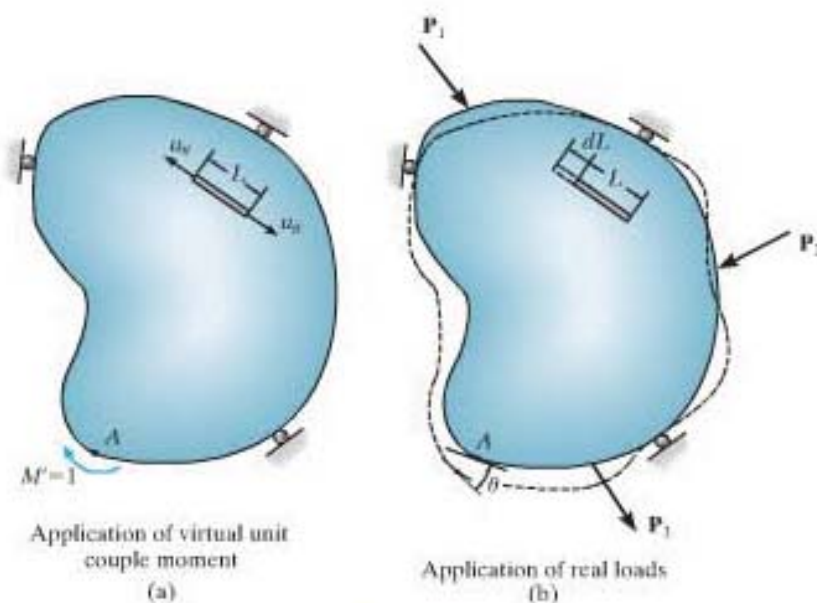


Fig. 14-30

In a similar manner, if the angular displacement or slope of the tangent at a point on the body is to be determined at A , Fig. 14-30*b*, then a virtual couple moment M' , having a "unit" magnitude, is applied at the point, Fig. 14-30*a*. As a result, this couple moment causes a virtual load u_θ in one of the elements of the body. Assuming the real loads P_1 , P_2 , P_3 deform the element an amount dL , the angular displacement θ can be found from the virtual-work equation

$$\begin{array}{c} \text{virtual loadings} \\ \left. 1 \cdot \theta = \sum u_\theta dL \right. \\ \text{real displacements} \end{array} \quad (14-35)$$

Here

$M' = 1$ = external virtual unit couple moment acting in the direction of θ

u_θ = internal virtual load acting on an element

θ = external angular displacement in radians caused by the real loads

dL = internal displacement of the element in the direction of u_θ , caused by the real loads

This method for applying the principle of virtual work is often referred to as the *method of virtual forces*, since a *virtual force* is applied, resulting in a determination of an external *real displacement*. The equation of virtual work in this case represents a statement of *compatibility requirements* for the body. Although it is not important here, realize that we can also apply the principle of virtual work as a *method of virtual displacements*. In this case, *virtual displacements* are imposed on the body when the body is subjected to *real loadings*. This method can be used to determine the external reactive force on the body or an unknown internal loading. When it is used in this manner, the equation of virtual work is a statement of the *equilibrium requirements* for the body.*

Internal Virtual Work. The terms on the right side of Eqs. 14-34 and 14-35 represent the internal virtual work developed in the body. The real internal displacements dL in these terms can be produced in several different ways. For example, these displacements may result from geometric fabrication errors, from a change in temperature, or more commonly from stress. In particular, no restriction has been placed on the magnitude of the external loading, so the stress may be large enough to cause yielding or even strain hardening of the material.

*See *Engineering Mechanics: Statics*, 12th edition, R.C. Hibbeler, Prentice Hall, Inc., 2009.

TABLE 14-1

Deformation caused by	Strain energy	Internal virtual work
Axial load N	$\int_0^L \frac{N^2}{2EA} dx$	$\int_0^L \frac{nN}{EA} dx$
Shear V	$\int_0^L \frac{f_s V^2}{2GA} dx$	$\int_0^L \frac{f_s vV}{GA} dx$
Bending moment M	$\int_0^L \frac{M^2}{2EI} dx$	$\int_0^L \frac{mM}{EI} dx$
Torsional moment T	$\int_0^L \frac{T^2}{2GJ} dx$	$\int_0^L \frac{tT}{GJ} dx$

If we assume that the material behavior is linear elastic and the stress does not exceed the proportional limit, we can then formulate the expressions for internal virtual work caused by stress using the equations of elastic strain energy developed in Sec. 14.2. They are listed in the center column of Table 14-1. Recall that each of these expressions assumes that the internal loading N , V , M , or T was applied gradually from zero to its full value. As a result, the work done by these resultants is shown in these expressions as *one-half* the product of the internal loading and its displacement. In the case of the virtual-force method, however, the “full” virtual internal loading is applied *before* the real loads cause displacements, and therefore the work of the virtual loading is simply the product of the virtual load and its real displacement. Referring to these internal virtual loadings (u) by the corresponding lowercase symbols n , v , m , and t , the virtual work due to axial load, shear, bending moment, and torsional moment is listed in the right-hand column of Table 14-1. Using these results, the virtual-work equation for a body subjected to a general loading can therefore be written as

$$1 \cdot \Delta = \int \frac{nN}{AE} dx + \int \frac{mM}{EI} dx + \int \frac{f_s vV}{GA} dx + \int \frac{tT}{GJ} dx \quad (14-36)$$

In the following sections we will apply the above equation to problems involving the displacement of joints on trusses, and points on beams and mechanical elements. We will also include a discussion of how to handle the effects of fabrication errors and differential temperature. For application it is important that a consistent set of units be used for all the terms. For example, if the real loads are expressed in kilonewtons and the body's dimensions are in meters, a 1-kN virtual force or 1-kN·m virtual couple should be applied to the body. By doing so a calculated displacement Δ will be in meters, and a calculated slope will be in radians.

*14.6 Method of Virtual Forces Applied to Trusses

In this section, we will apply the method of virtual forces to determine the displacement of a truss joint. To illustrate the principles, the vertical displacement of joint A of the truss shown in Fig. 14-31*b* will be determined. To do this, we must place a virtual unit force at this joint, Fig. 14-31*a*, so that when the real loads $\mathbf{P}_1$ and $\mathbf{P}_2$ are applied to the truss, they cause the external virtual work $1 \cdot \Delta$. The internal virtual work in each member is $n\Delta L$. Since each member has a constant cross-sectional area A , and n and N are constant throughout the member's length, then from Table 14-1, the internal virtual work for each member is

$$\int_0^L \frac{nN}{AE} dx = \frac{nNL}{AE}$$

Therefore, the virtual-work equation for the entire truss is

$$1 \cdot \Delta = \sum \frac{nNL}{AE} \quad (14-37)$$

Here

1 = external virtual unit load acting on the truss joint in the direction of Δ

Δ = joint displacement caused by the real loads on the truss

n = internal virtual force in a truss member caused by the external virtual unit load

N = internal force in a truss member caused by the real loads

L = length of a member

A = cross-sectional area of a member

E = modulus of elasticity of a member

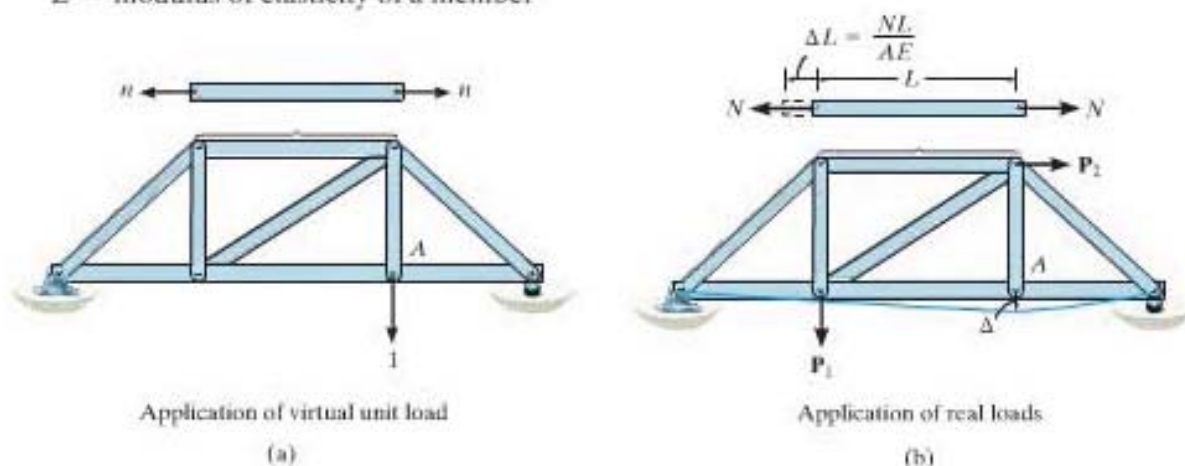


Fig. 14-31

Temperature Change. Truss members can change their length due to a change in temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in temperature, the change in length of a member is $\Delta L = \alpha \Delta T L$ (Eq. 4-4). Hence, we can determine the displacement of a selected truss joint due to this temperature change from Eq. 14-34, written as

$$1 \cdot \Delta = \sum n \alpha \Delta T L \quad (14-38)$$

Here

- 1 = external virtual unit load acting on the truss joint in the direction of Δ
- Δ = joint displacement caused by the temperature change
- n = internal virtual force in a truss member caused by the external virtual unit load
- α = coefficient of thermal expansion of material
- ΔT = change in temperature of member
- L = length of member

Fabrication Errors. Occasionally errors in fabricating the lengths of the members of a truss may occur. If this happens, the displacement Δ in a particular direction of a truss joint from its expected position can be determined from direct application of Eq. 14-34 written as

$$1 \cdot \Delta = \sum n \Delta L \quad (14-39)$$

Here

- 1 = external virtual unit load acting on the truss joint in the direction of Δ
- Δ = joint displacement caused by the fabrication errors
- n = internal virtual force in a truss member caused by the external virtual unit load
- ΔL = difference in length of the member from its intended length caused by a fabrication error

A combination of the right-hand sides of Eqs. 14-37 through 14-39 will be necessary if external loads act on the truss and some of the members undergo a temperature change or have been fabricated with the wrong dimensions.

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint on a truss using the method of virtual forces.

Virtual Forces n .

- Place the virtual unit load on the truss at the joint where the displacement is to be determined. The load should be directed along the line of action of the displacement.
- With the unit load so placed and all the real loads *removed* from the truss, calculate the internal n force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

Real Forces N .

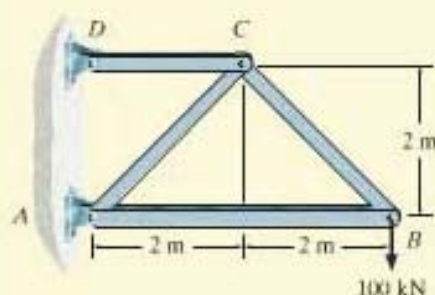
- Determine the N forces in each member. These forces are caused only by the real loads acting on the truss. Again, assume that tensile forces are positive and compressive forces are negative.

Virtual-Work Equation.

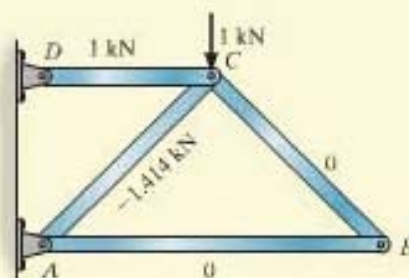
- Apply the equation of virtual work to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding n and N forces when substituting these terms into the equation.
- If the resultant sum $\sum nNL/AE$ is positive, the displacement Δ is in the same direction as the virtual unit load. If a negative value results, Δ is opposite to the virtual unit load.
- When applying $1 \cdot \Delta = \sum n\alpha \Delta TL$, an *increase* in temperature, ΔT , will be *positive*; whereas a *decrease* in temperature will be *negative*.
- For $1 \cdot \Delta = \sum n \Delta L$, when a fabrication error causes an increase in the length of a member, ΔL is *positive*, whereas a *decrease* in length is *negative*.
- When applying this method, attention should be paid to the units of each numerical quantity. Notice, however, that the virtual unit load can be assigned any arbitrary unit: pounds, kips, newtons, etc., since the n forces will have these *same* units, and as a result, the units for both the virtual unit load and the n forces will cancel from both sides of the equation.

EXAMPLE 14.11

Determine the vertical displacement of joint C of the steel truss shown in Fig. 14-32a. The cross-sectional area of each member is $A = 400 \text{ mm}^2$ and $E_{st} = 200 \text{ GPa}$.

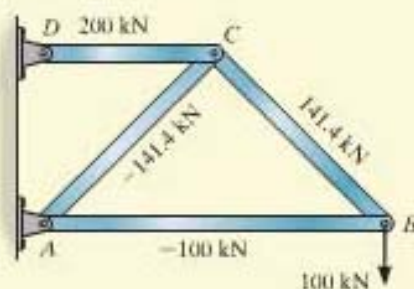


(a)



Virtual forces

(b)



Real forces

(c)

Fig. 14-32**SOLUTION**

Virtual Forces n . Since the vertical displacement at joint C is to be determined, *only* a vertical 1-kN virtual load is placed at joint C ; and the force in each member is calculated using the method of joints. The results of this analysis are shown in Fig. 14-32b. Using our sign convention, positive numbers indicate tensile forces and negative numbers indicate compressive forces.

Real Forces N . The applied load of 100 kN causes forces in the members that are also calculated using the method of joints. The results of this analysis are shown in Fig. 14-32c.

Virtual-Work Equation. Arranging the data in tabular form, we have

Member	n	N	L	nNL
AB	0	-100	4	0
BC	0	141.4	2.828	0
AC	-1.414	-141.4	2.828	565.7
CD	1	200	2	400
				$\Sigma 965.7 \text{ kN}^2 \cdot \text{m}$

Thus,

$$1 \text{ kN} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{AE}$$

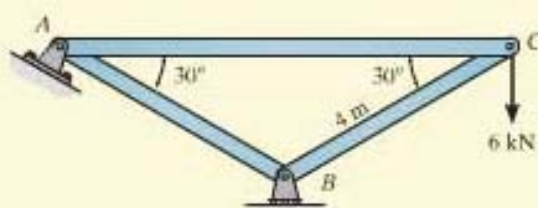
Substituting the numerical values for A and E , we have

$$1 \text{ kN} \cdot \Delta_{C_v} = \frac{965.7 \text{ kN}^2 \cdot \text{m}}{[400(10^{-6}) \text{ m}^2] 200(10^6) \text{ kN/m}^2}$$

$$\Delta_{C_v} = 0.01207 \text{ m} = 12.1 \text{ mm} \quad \text{Ans.}$$

EXAMPLE 14.12

Determine the horizontal displacement of the roller at B of the truss shown in Fig. 14-33a. Due to radiant heating, member AB is subjected to an *increase* in temperature of $\Delta T = +60^\circ\text{C}$, and this member has been fabricated 3 mm too short. The members are made of steel, for which $\alpha_{st} = 12(10^{-6})/^\circ\text{C}$ and $E_{st} = 200 \text{ GPa}$. The cross-sectional area of each member is 250 mm^2 .



(a)

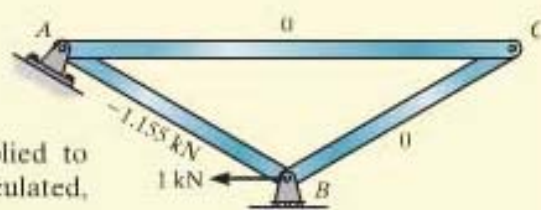
SOLUTION

Virtual Forces n . A horizontal 1-kN virtual load is applied to the truss at joint B , and the forces in each member are calculated, Fig. 14-33b.

Real Forces N . Since the n forces in members AC and BC are zero, the N forces in these members do *not* have to be determined. Why? For completeness, though, the entire “real” force analysis is shown in Fig. 14-33c.

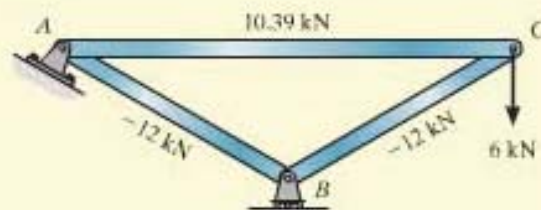
Virtual-Work Equation. The loads, temperature, and the fabrication error all affect the displacement of point B ; therefore, Eqs. 14-37, 14-38, and 14-39 must be combined, which gives

$$\begin{aligned}
 1 \text{ kN} \cdot \Delta_{B_h} &= \sum \frac{nNL}{AE} + \sum n\alpha \Delta T L + \sum n\Delta L \\
 &= 0 + 0 + \frac{(-1.155 \text{ kN})(-12 \text{ kN})(4 \text{ m})}{[250(10^{-6}) \text{ m}^2][200(10^6) \text{ kN/m}^2]} \\
 &\quad + 0 + 0 + (-1.155 \text{ kN})[12(10^{-6})/^\circ\text{C}](60^\circ\text{C})(4 \text{ m}) \\
 &\quad + (-1.155 \text{ kN})(-0.003 \text{ m}) \\
 \Delta_{B_h} &= 0.00125 \text{ m} \\
 &= 1.25 \text{ mm} \leftarrow
 \end{aligned}$$



Virtual forces

(b)



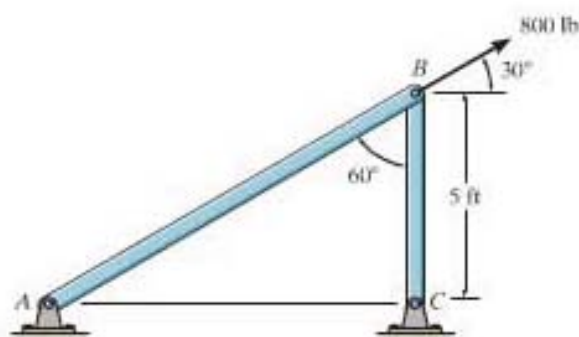
Real forces

(c)

Fig. 14-33*Ans.*

PROBLEMS

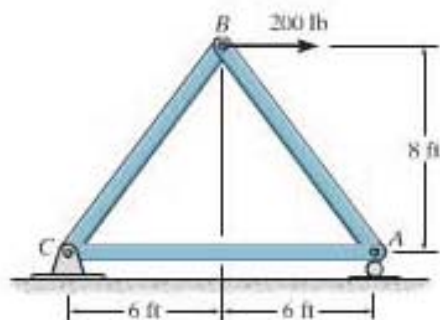
***14-72.** Determine the horizontal displacement of joint B on the two-member frame. Each A-36 steel member has a cross-sectional area of 2 in^2 .



Prob. 14-72

***14-73.** Determine the horizontal displacement of point B . Each A-36 steel member has a cross-sectional area of 2 in^2 .

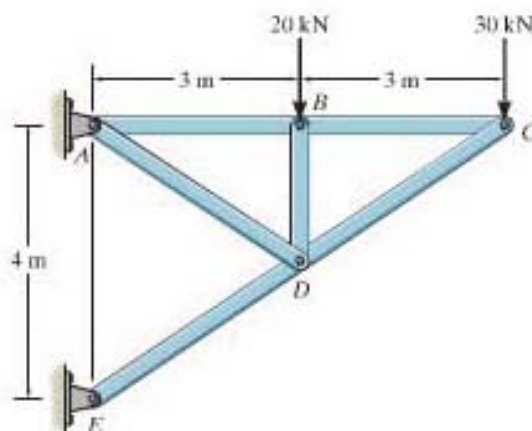
14-74. Determine the vertical displacement of point B . Each A-36 steel member has a cross-sectional area of 2 in^2 .



Probs. 14-73/74

14-75. Determine the vertical displacement of joint C on the truss. Each A-36 steel member has a cross-sectional area of $A = 300 \text{ mm}^2$.

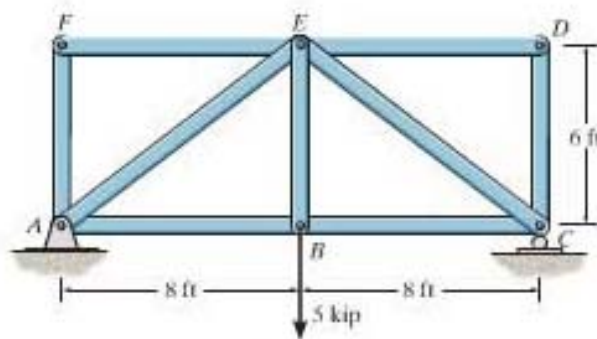
***14-76.** Determine the vertical displacement of joint D on the truss. Each A-36 steel member has a cross-sectional area of $A = 300 \text{ mm}^2$.



Probs. 14-75/76

***14-77.** Determine the vertical displacement of point B . Each A-36 steel member has a cross-sectional area of 4.5 in^2 .

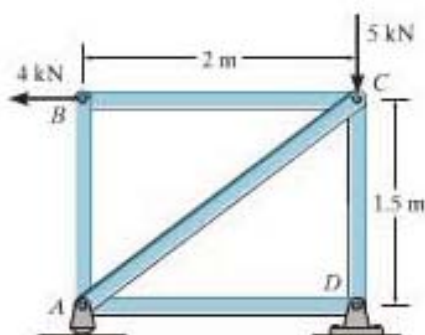
14-78. Determine the vertical displacement of point E . Each A-36 steel member has a cross-sectional area of 4.5 in^2 .



Probs. 14-77/78

14-79. Determine the horizontal displacement of joint B of the truss. Each A-36 steel member has a cross-sectional area of 400 mm^2 .

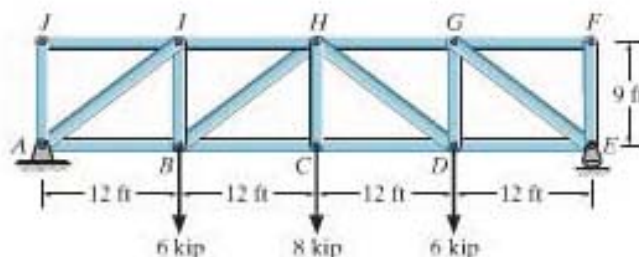
***14-80.** Determine the vertical displacement of joint C of the truss. Each A-36 steel member has a cross-sectional area of 400 mm^2 .



Probs. 14-79/80

14-83. Determine the vertical displacement of joint C . Each A-36 steel member has a cross-sectional area of 4.5 in^2 .

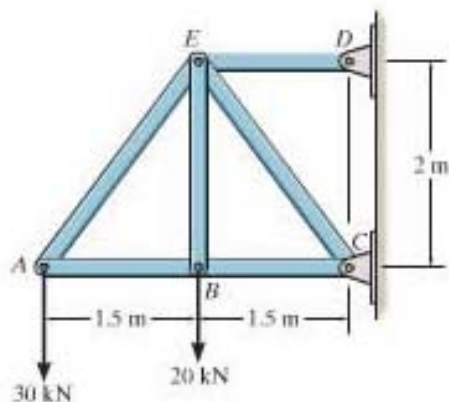
***14-84.** Determine the vertical displacement of joint H . Each A-36 steel member has a cross-sectional area of 4.5 in^2 .



Probs. 14-83/84

***14-81.** Determine the vertical displacement of point A . Each A-36 steel member has a cross-sectional area of 400 mm^2 .

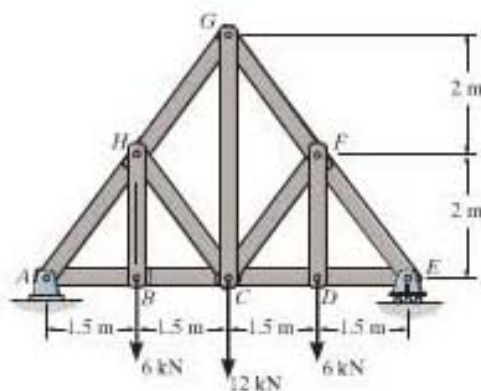
14-82. Determine the vertical displacement of point B . Each A-36 steel member has a cross-sectional area of 400 mm^2 .



Probs. 14-81/82

***14-85.** Determine the vertical displacement of joint C . The truss is made from A-36 steel bars having a cross-sectional area of 150 mm^2 .

14-86. Determine the vertical displacement of joint G . The truss is made from A-36 steel bars having a cross-sectional area of 150 mm^2 .



Probs. 14-85/86

*14.7 Method of Virtual Forces Applied to Beams

In this section we will apply the method of virtual forces to determine the displacement and slope at a point on a beam. To illustrate the principles, the vertical displacement Δ of point A on the beam shown in Fig. 14-34*b* will be determined. To do this we must place a vertical unit load at this point, Fig. 14-34*a*, so that when the “real” distributed load w is applied to the beam it will cause the internal virtual work $1 \cdot \Delta$. Because the load causes both a shear V and moment M within the beam, we must actually consider the internal virtual work due to both of these loadings. In Example 14.7, however, it was shown that beam deflections due to shear are negligible compared with those caused by bending, particularly if the beam is long and slender. Since this type of beam is most often used in practice, we will only consider the virtual strain energy due to bending, Table 14-1. Hence, the real load causes the element dx to deform so its sides rotate by an angle $d\theta = (M/EI)dx$, which causes internal virtual work $m d\theta$. Applying Eq. 14-34, the virtual-work equation for the entire beam, we have

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx \quad (14-40)$$

Here

1 = external virtual unit load acting on the beam in the direction of Δ

Δ = displacement caused by the real loads acting on the beam

m = internal virtual moment in the beam, expressed as a function of x and caused by the external virtual unit load

M = internal moment in the beam, expressed as a function of x and caused by the real loads

E = modulus of elasticity of the material

I = moment of inertia of the cross-sectional area about the neutral axis

In a similar manner, if the slope θ of the tangent at a point on the beam's elastic curve is to be determined, a virtual unit couple moment must be applied at the point, and the corresponding internal virtual moment m_θ has to be determined. If we apply Eq. 14-35 for this case and neglect the effect of shear deformations, we have

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx \quad (14-41)$$

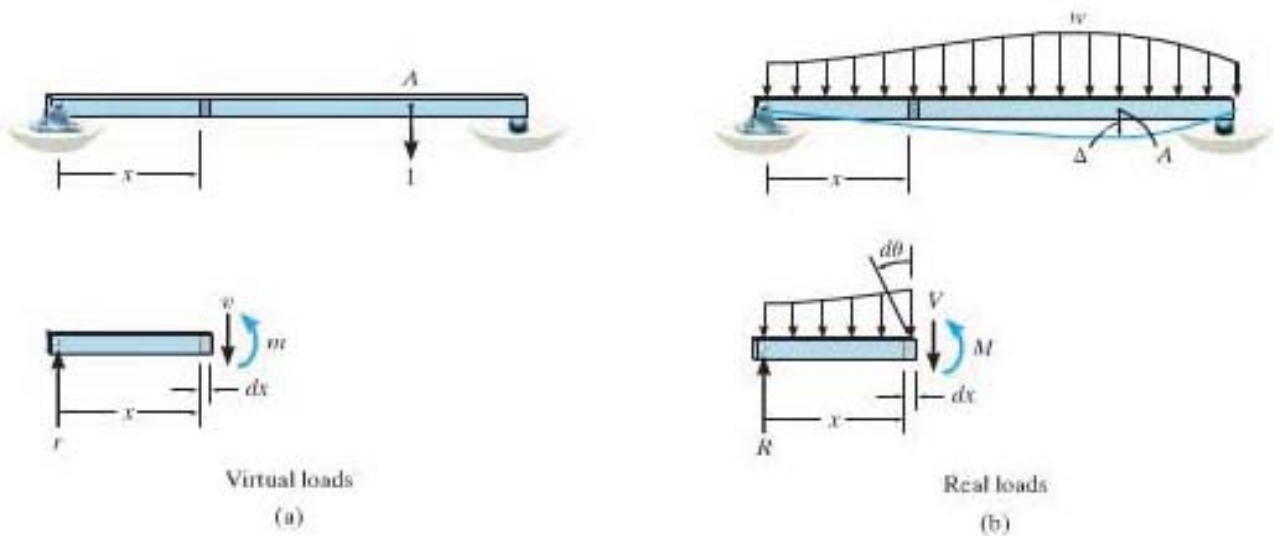


Fig. 14-34

When applying these equations, keep in mind that the integrals on the right side represent the amount of virtual bending strain energy that is stored in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration *cannot* be performed across the beam's entire length. Instead, separate x coordinates must be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each x have the same origin; however, the x selected for determining the real moment M in a particular region must be the *same* x as that selected for determining the virtual moment m or m_0 within the same region. For example, consider the beam shown in Fig. 14-35. In order to determine the displacement at D , we can use x_1 to determine the strain energy in region AB , x_2 for region BC , x_3 for region DE , and x_4 for region DC . In any case, each x coordinate should be selected so that both M and m (or m_0) can easily be formulated.

Unlike beams, as discussed here, some members may also be subjected to significant virtual strain energy caused by axial load, shear, and torsional moment. When this is the case, we must include in the above equations the energy terms for these loadings as formulated in Eq. 14-36,



Fig. 14-35

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the method of virtual forces.

Virtual Moments m or m_θ .

- Place a *virtual unit load* on the beam at the point and directed along the line of action of the desired displacement.
- If the slope is to be determined, place a *virtual unit couple moment* at the point.
- Establish appropriate x coordinates that are valid within regions of the beam where there is no discontinuity of both real and virtual load.
- With the virtual load in place, and all the real loads *removed* from the beam, calculate the internal moment m or m_θ as a function of each x coordinate.
- Assume that m or m_θ acts in the positive direction according to the established beam sign convention for positive moment, Fig. 6-3.

Real Moments.

- Using the *same* x coordinates as those established for m or m_θ , determine the internal moments M caused by the real loads.
- Since positive m or m_θ was assumed to act in the conventional “positive direction,” it is important that positive M acts in this *same direction*. This is necessary since positive or negative internal virtual work depends on the directional sense of *both* the virtual load, defined by $\pm m$ or $\pm m_\theta$, and displacement, caused by $\pm M$.

Virtual-Work Equation.

- Apply the equation of virtual work to determine the desired displacement Δ or slope θ . It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam is positive, Δ or θ is in the same direction as the virtual unit load or virtual unit couple moment. If a negative value results, Δ or θ is opposite to the virtual unit load or couple moment.

EXAMPLE 14.13

Determine the displacement of point B on the beam shown in Fig. 14-36a. EI is constant.

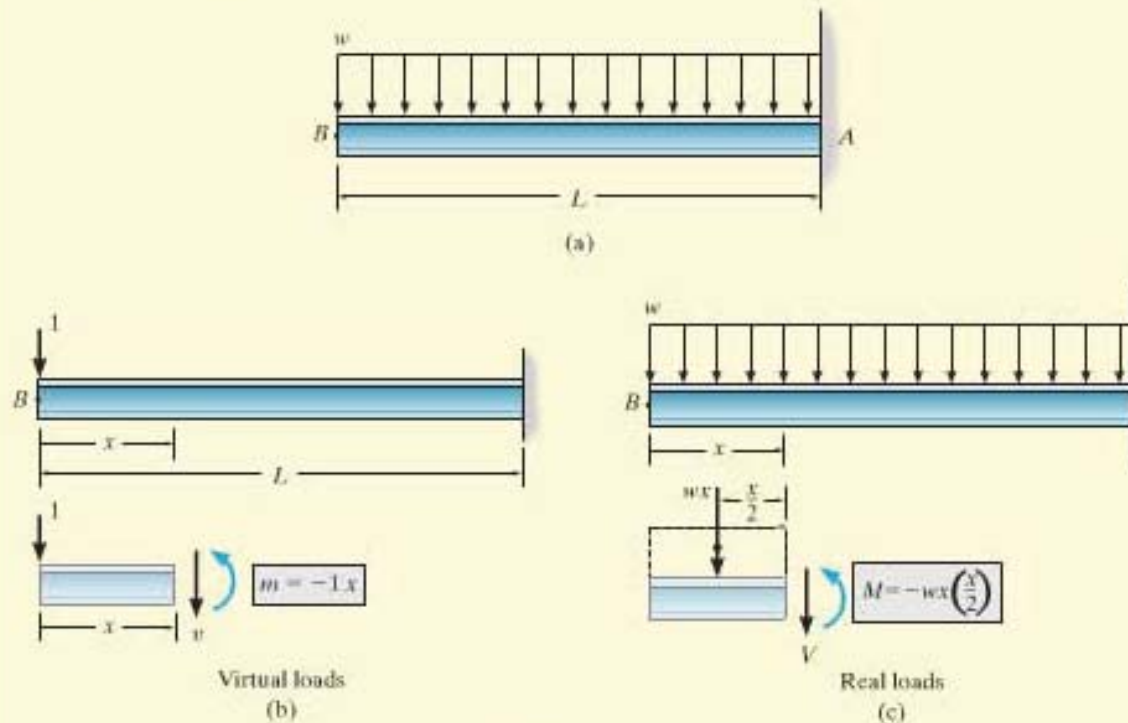


Fig. 14-36

SOLUTION

Virtual Moment m . The vertical displacement of point B is obtained by placing a virtual unit load at B , Fig. 14-36b. By inspection, there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a *single* x coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at B , so that the reactions at A do not have to be determined in order to find the internal moments m and M . Using the method of sections, the internal moment m is shown in Fig. 14-36b.

Real Moment M . Using the *same* x coordinate, the internal moment M is shown in Fig. 14-36c.

Virtual-Work Equation. The vertical displacement at B is thus

$$1 \cdot \Delta_B = \int_0^L \frac{mM}{EI} dx = \int_0^L \frac{(-1x)(-wx^2/2)}{EI} dx$$

$$\Delta_B = \frac{wL^4}{8EI} \quad \text{Ans.}$$

EXAMPLE 14.14

Determine the slope at point B of the beam shown in Fig. 14-37a. EI is constant.

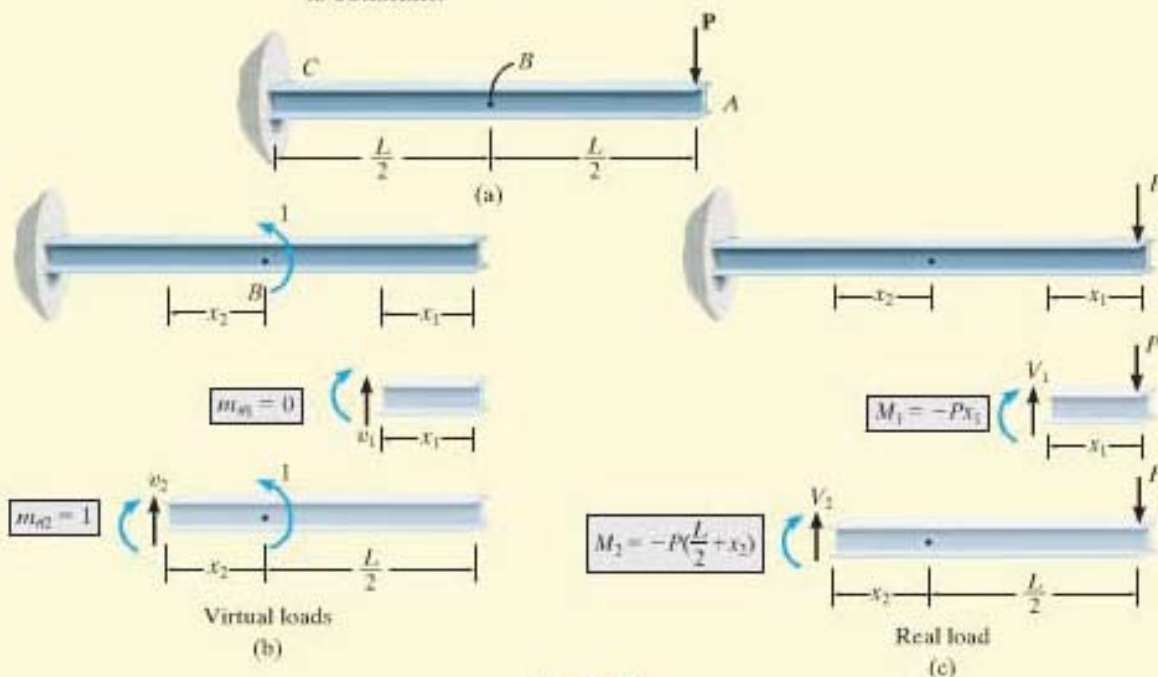


Fig. 14-37

SOLUTION

Virtual Moments m_{θ} . The slope at B is determined by placing a virtual unit couple moment at B , Fig. 14-37b. Two x coordinates must be selected in order to determine the total virtual strain energy in the beam. Coordinate x_1 accounts for the strain energy within segment AB , and coordinate x_2 accounts for the strain energy in segment BC . Using the method of sections the internal moments m_{θ} within each of these segments are shown in Fig. 14-37b.

Real Moments M . Using the *same coordinates* x_1 and x_2 (Why?), the internal moments M are shown in Fig. 14-37c.

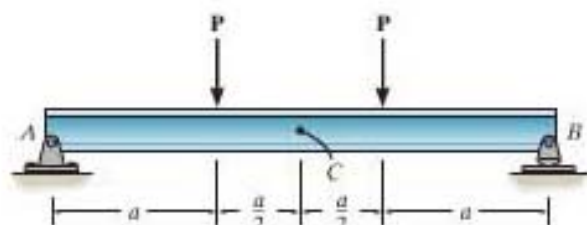
Virtual-Work Equation. The slope at B is thus

$$\begin{aligned}
 1 \cdot \theta_B &= \int \frac{m_{\theta} M}{EI} dx \\
 &= \int_0^{L/2} \frac{0(-Px_1)}{EI} dx_1 + \int_0^{L/2} \frac{1\{-P[(L/2) + x_2]\}}{EI} dx_2 \\
 \theta_B &= -\frac{3PL^2}{8EI} \quad \text{Ans.}
 \end{aligned}$$

The *negative sign* indicates that θ_B is *opposite* to the direction of the virtual couple moment shown in Fig. 14-37b.

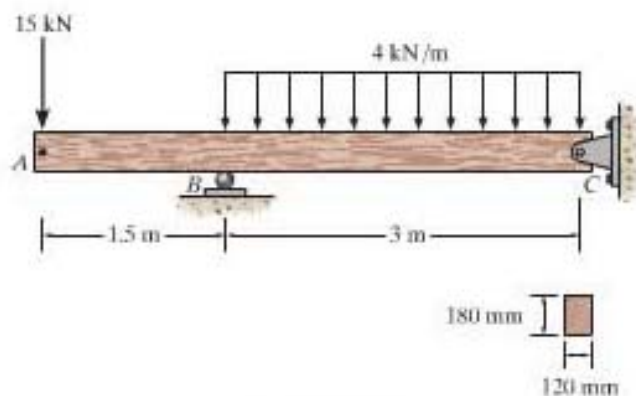
PROBLEMS

14-87. Determine the displacement at point C . EI is constant.



Prob. 14-87

***14-88.** The beam is made of southern pine for which $E_p = 13 \text{ GPa}$. Determine the displacement at A .

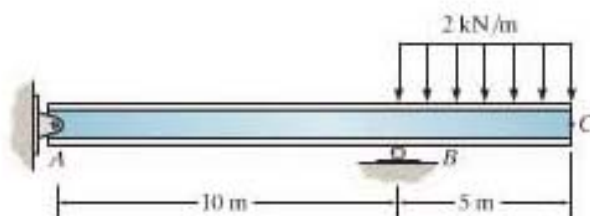


Prob. 14-88

***14-89.** Determine the displacement at C of the A-36 steel beam. $I = 70(10^6) \text{ mm}^4$.

14-90. Determine the slope at A of the A-36 steel beam. $I = 70(10^6) \text{ mm}^4$.

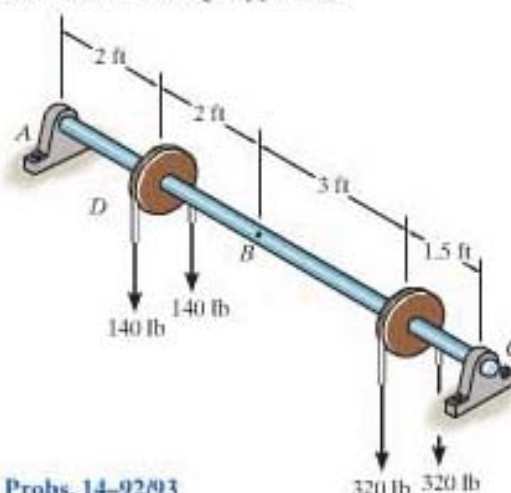
14-91. Determine the slope at B of the A-36 steel beam. $I = 70(10^6) \text{ mm}^4$.



Probs. 14-89/90/91

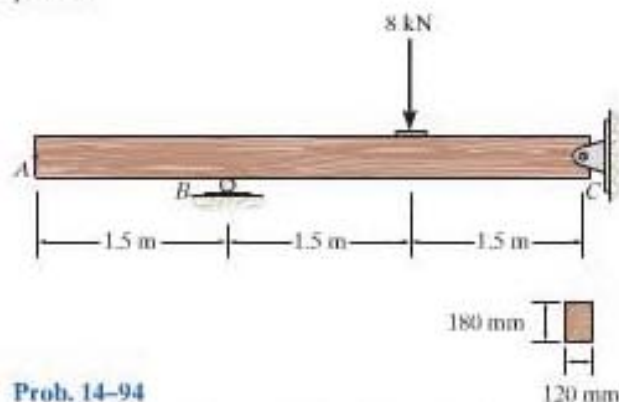
***14-92.** Determine the displacement at B of the 1.5-in.-diameter A-36 steel shaft.

***14-93.** Determine the slope of the 1.5-in.-diameter A-36 steel shaft at the bearing support A .



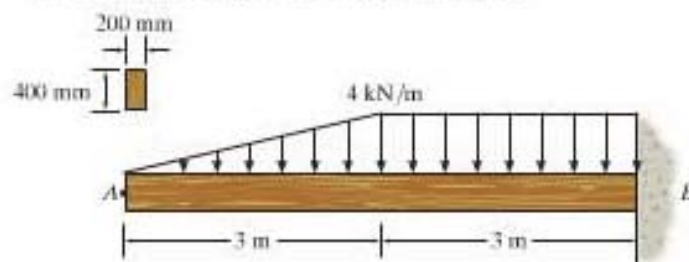
Probs. 14-92/93

14-94. The beam is made of Douglas fir. Determine the slope at C .



Prob. 14-94

14-95. The beam is made of oak, for which $E_o = 11 \text{ GPa}$. Determine the slope and displacement at A .

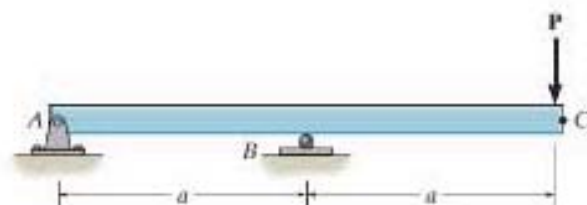


Prob. 14-95

*14-96. Determine the displacement at point C . EI is constant.

*14-97. Determine the slope at point C . EI is constant.

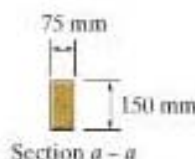
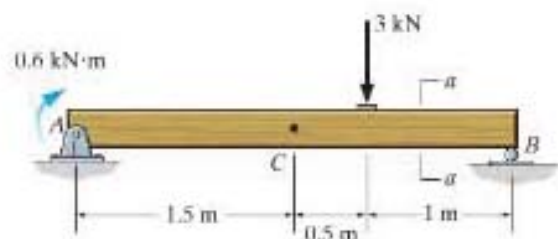
14-98. Determine the slope at point A . EI is constant.



Probs. 14-96/97/98

14-99. Determine the slope at point A of the simply supported Douglas fir beam.

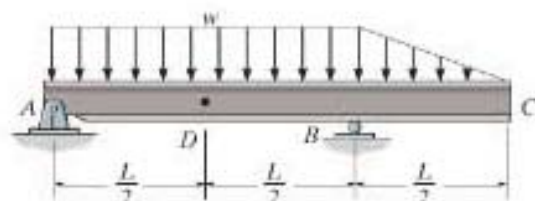
*14-100. Determine the displacement at C of the simply supported Douglas fir beam.



Probs. 14-99/100

*14-101. Determine the slope of end C of the overhang beam. EI is constant.

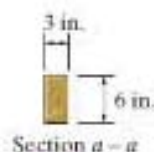
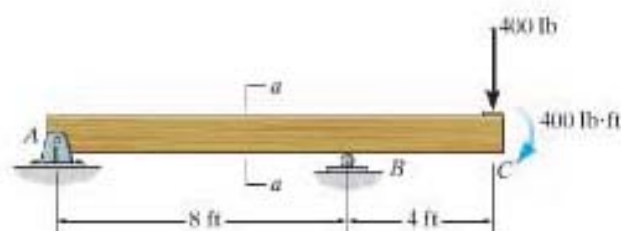
14-102. Determine the displacement of point D of the overhang beam. EI is constant.



Probs. 14-101/102

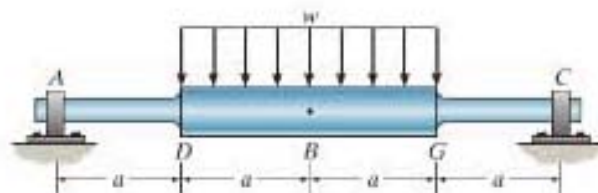
14-103. Determine the displacement of end C of the overhang Douglas fir beam.

*14-104. Determine the slope at A of the overhang white spruce beam.



Probs. 14-103/104

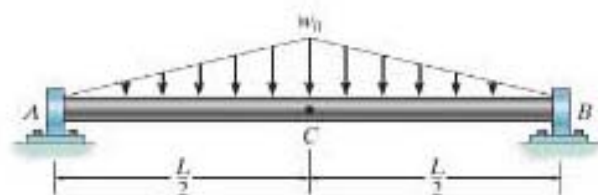
*14-105. Determine the displacement at point B . The moment of inertia of the center portion DG of the shaft is $2I$, whereas the end segments AD and GC have a moment of inertia I . The modulus of elasticity for the material is E .



Prob. 14-105

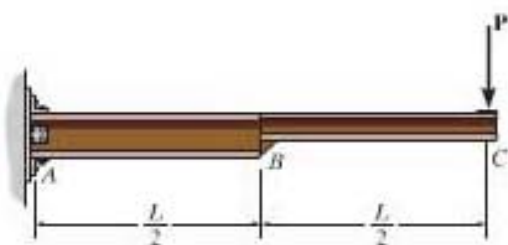
14-106. Determine the displacement of the shaft at C . EI is constant.

14-107. Determine the slope of the shaft at the bearing support A . EI is constant.



Probs. 14-106/107

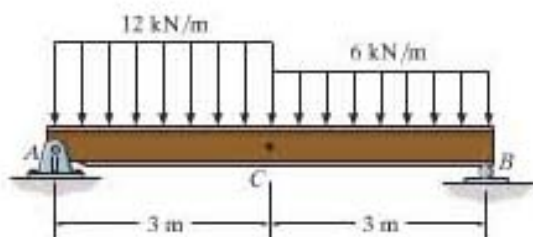
***14-108.** Determine the slope and displacement of end C of the cantilevered beam. The beam is made of a material having a modulus of elasticity of E . The moments of inertia for segments AB and BC of the beam are $2I$ and I , respectively.



Prob. 14-108

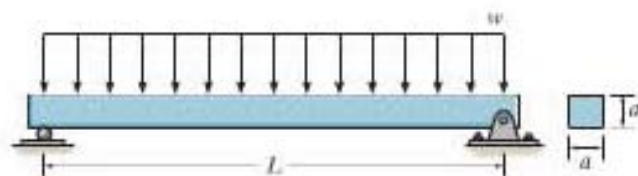
•14-109. Determine the slope at A of the A-36 steel W200 $\times$ 46 simply supported beam.

14-110. Determine the displacement at point C of the A-36 steel W200 $\times$ 46 simply supported beam.



Probs. 14-109/110

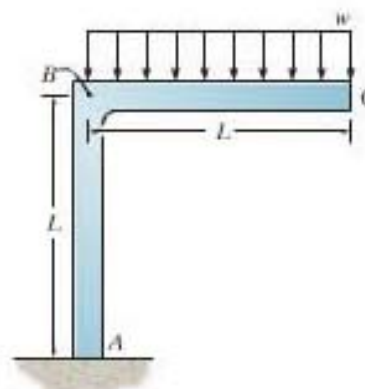
14-111. The simply supported beam having a square cross section is subjected to a uniform load w . Determine the maximum deflection of the beam caused only by bending, and caused by bending and shear. Take $E = 3G$.



Prob. 14-111

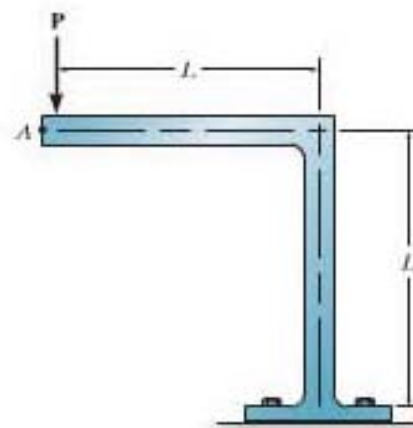
***14-112.** The frame is made from two segments, each of length L and flexural stiffness EI . If it is subjected to the uniform distributed load determine the vertical displacement of point C . Consider only the effect of bending.

•14-113. The frame is made from two segments, each of length L and flexural stiffness EI . If it is subjected to the uniform distributed load, determine the horizontal displacement of point B . Consider only the effect of bending.



Probs. 14-112/113

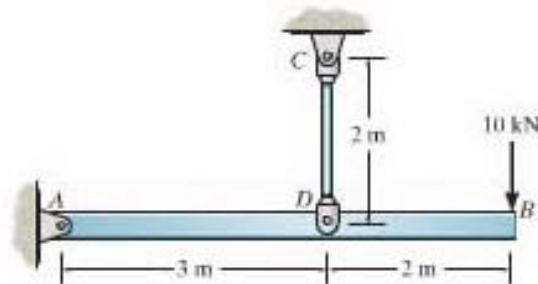
14-114. Determine the vertical displacement of point A on the angle bracket due to the concentrated force P . The bracket is fixed connected to its support. EI is constant. Consider only the effect of bending.



Prob. 14-114

14-115. Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of A-36 steel, determine the vertical displacement of point B due to the loading of 10 kN.

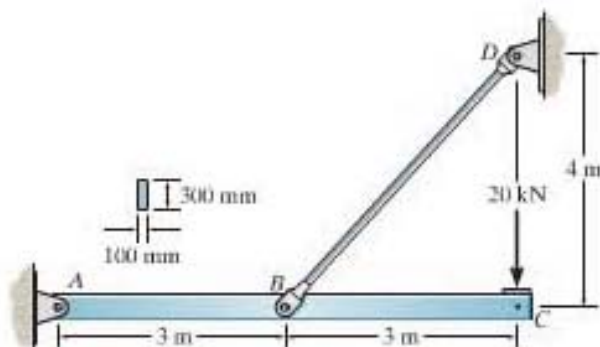
***14-116.** Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of A-36 steel, determine the slope at A due to the loading of 10 kN.



Probs. 14-115/116

14-117. Bar ABC has a rectangular cross section of 300 mm by 100 mm. Attached rod DB has a diameter of 20 mm. If both members are made of A-36 steel, determine the vertical displacement of point C due to the loading. Consider only the effect of bending in ABC and axial force in DB .

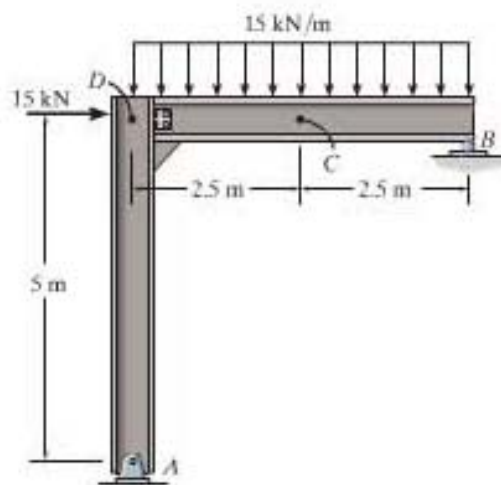
14-118. Bar ABC has a rectangular cross section of 300 mm by 100 mm. Attached rod DB has a diameter of 20 mm. If both members are made of A-36 steel, determine the slope at A due to the loading. Consider only the effect of bending in ABC and axial force in DB .



Probs. 14-117/118

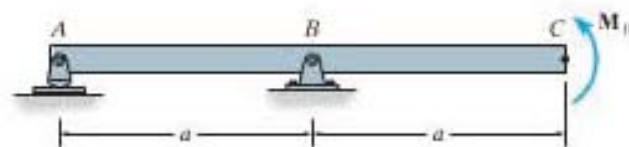
14-119. Determine the vertical displacement of point C . The frame is made using A-36 steel W250 $\times$ 45 members. Consider only the effects of bending.

***14-120.** Determine the horizontal displacement of end B . The frame is made using A-36 steel W250 $\times$ 45 members. Consider only the effects of bending.



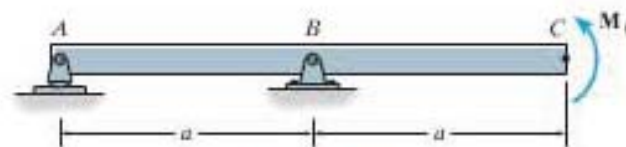
Probs. 14-119/120

***14-121.** Determine the displacement at point C . EI is constant.



Prob. 14-121

14-122. Determine the slope at B . EI is constant.



Prob. 14-122

*14.8 Castigliano's Theorem

In 1879, Alberto Castigliano, an Italian railroad engineer, published a book in which he outlined a method for determining the displacement and slope at a point in a body. This method, which is referred to as Castigliano's second theorem, applies only to bodies that have constant temperature and material with linear-elastic behavior. If the displacement at a point is to be determined, the theorem states that the displacement is equal to the first partial derivative of the strain energy in the body with respect to a force acting at the point and in the direction of displacement. In a similar manner, the slope of the tangent at a point in a body is equal to the first partial derivative of the strain energy in the body with respect to a couple moment acting at the point and in the direction of the slope angle.

To derive Castigliano's second theorem, consider a body of any arbitrary shape, which is subjected to a series of n forces $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$, Fig. 14-38. According to the conservation of energy, the external work done by these forces is equal to the internal strain energy stored in the body. However, the external work is a function of the external loads, $U_e = \sum \int P dx$, Eq. 14-1, so the internal work is also a function of the external loads. Thus,

$$U_i = U_e = f(P_1, P_2, \dots, P_n) \quad (14-42)$$

Now, if any one of the external forces, say P_j , is increased by a differential amount dP_j , the internal work will also be increased, such that the strain energy becomes

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_j} dP_j \quad (14-43)$$

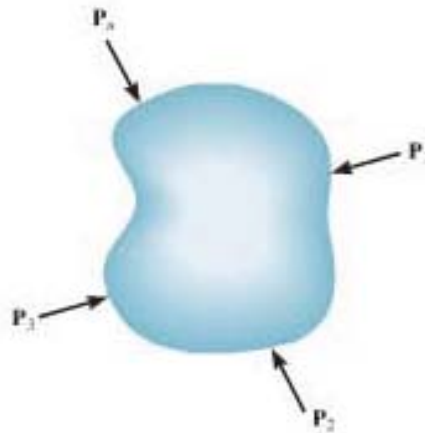


Fig. 14-38

This value, however, will not depend on the sequence in which the n forces are applied to the body. For example, we could apply $d\mathbf{P}_j$ to the body *first*, then apply the loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$. In this case, $d\mathbf{P}_j$ would cause the body to displace a differential amount $d\Delta_j$ in the direction of $d\mathbf{P}_j$. By Eq. 14-2 ($U_e = \frac{1}{2}P_j\Delta_j$), the increment of strain energy would be $\frac{1}{2}dP_j d\Delta_j$. This quantity, however, is a second-order differential and may be neglected. Further application of the loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ causes $d\mathbf{P}_j$ to move through the displacement Δ_j so that now the strain energy becomes

$$U_i + dU_i = U_i + dP_j\Delta_j \quad (14-44)$$

Here, as above, U_i is the internal strain energy in the body, caused by the loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$, and $dP_j\Delta_j$ is the *additional* strain energy caused by $d\mathbf{P}_j$.

In summary, Eq. 14-43 represents the strain energy in the body determined by first applying the loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$, then $d\mathbf{P}_j$; Eq. 14-44 represents the strain energy determined by first applying $d\mathbf{P}_j$ and then the loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$. Since these two equations must be equal, we require

$$\Delta_j = \frac{\partial U_i}{\partial P_j} \quad (14-45)$$

which proves the theorem; i.e., the displacement Δ_j in the direction of $\mathbf{P}_j$ is equal to the first partial derivative of the strain energy with respect to $\mathbf{P}_j$.

Castigliano's second theorem, Eq. 14-45, is a statement regarding the body's *compatibility requirements*, since it is a condition related to displacement. Also, the above derivation requires that *only conservative forces* be considered for the analysis. These forces can be applied in any order, and furthermore, they do work that is independent of the path and therefore create no energy loss. As long as the material has linear-elastic behavior, the applied forces will be conservative and the theorem is valid. Castigliano's first theorem is similar to his second theorem; however, it relates the load P_j to the partial derivative of the strain energy with respect to the corresponding displacement, that is, $P_j = \partial U_i / \partial \Delta_j$. The proof is similar to that given above. This theorem is another way of expressing the *equilibrium requirements* for the body; however, it has limited application and therefore it will not be discussed here.

*14.9 Castigliano's Theorem Applied to Trusses

Since a truss member is only subjected to an axial load, the strain energy for the member is given by Eq. 14-16, $U_i = N^2 L / 2AE$. Substituting this equation into Eq. 14-45 and omitting the subscript i , we have

$$\Delta = \frac{\partial}{\partial P} \sum \frac{N^2 L}{2AE}$$

It is generally easier to perform the differentiation prior to summation. Also, L , A , and E are constant for a given member, and therefore we can write

$$\Delta = \sum N \left(\frac{\partial N}{\partial P} \right) \frac{L}{AE} \quad (14-46)$$

Here

Δ = displacement of the truss joint

P = an external force of *variable magnitude* applied to the truss joint in the direction of Δ

N = internal axial force in a member caused by *both* force P and the actual loads on the truss

L = length of a member

A = cross-sectional area of a member

E = modulus of elasticity of the material

By comparison, Eq. 14-46 is similar to that used for the method of virtual forces, Eq. 14-37 ($1 \cdot \Delta = \sum nNL/AE$), except that n is replaced by $\partial N / \partial P$. These terms, n and $\partial N / \partial P$, are the *same*, since they represent the change of the member's axial force with respect to the load P or, in other words, the axial force per unit load.

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint on a truss using Castigliano's second theorem.

External Force P .

- Place a force P on the truss at the joint where the displacement is to be determined. This force is assumed to have a *variable magnitude* and should be directed along the line of action of the displacement.

Internal Forces N .

- Determine the force N in each member in terms of both the actual (numerical) loads and the (variable) force P . Assume that tensile forces are positive and compressive forces are negative.
- Find the respective partial derivative $\partial N / \partial P$ for each member.
- After N and $\partial N / \partial P$ have been determined, assign P its numerical value if it has actually replaced a real force on the truss. Otherwise, set P equal to zero.

Castigliano's Second Theorem.

- Apply Castigliano's second theorem to determine the desired displacement Δ . It is important to retain the algebraic signs for corresponding values of N and $\partial N / \partial P$ when substituting these terms into the equation.
- If the resultant sum $\sum N(\partial N / \partial P)L/AE$ is positive, Δ is in the same direction as P . If a negative value results, Δ is opposite to P .

EXAMPLE 14.15

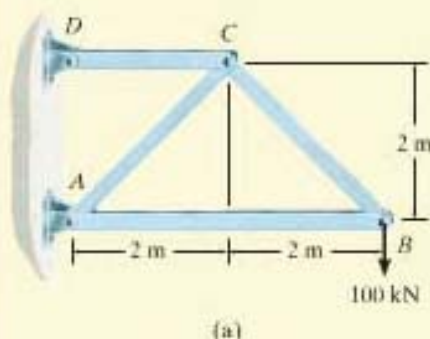


Fig. 14-39

Determine the vertical displacement of joint C of the steel truss shown in Fig. 14-39a. The cross-sectional area of each member is $A = 400 \text{ mm}^2$, and $E_{st} = 200 \text{ GPa}$.

SOLUTION

External Force P . A vertical force P is applied to the truss at joint C , since this is where the vertical displacement is to be determined, Fig. 14-39b.

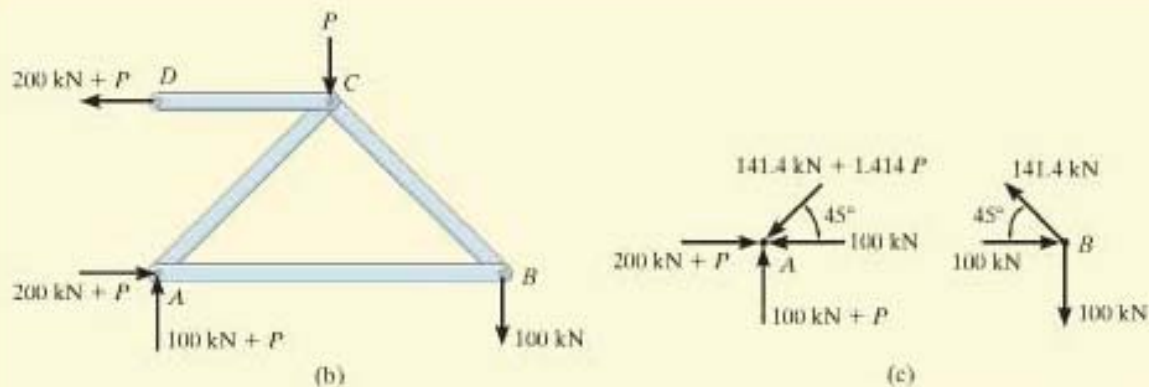


Fig. 14-39 (cont.)

Internal Forces N . The reactions at the truss supports A and D are calculated and the results are shown in Fig. 14-39*b*. Using the method of joints, the N forces in each member are determined, Fig. 14-39*c*.^{*} For convenience, these results along with their partial derivatives $\partial N/\partial P$ are listed in tabular form. Note that since P does not actually exist as a real load on the truss, we require $P = 0$.

Member	N	$\frac{\partial N}{\partial P}$	$N(P = 0)$	L	$N\left(\frac{\partial N}{\partial P}\right)L$
AB	-100	0	-100	4	0
BC	141.4	0	141.4	2.828	0
AC	-(141.4 + 1.414 <i>P</i>)	-1.414	-141.4	2.828	565.7
CD	200 + <i>P</i>	1	200	2	400
					$\Sigma 965.7 \text{ kN} \cdot \text{m}$

Castigliano's Second Theorem. Applying Eq. 14-46, we have

$$\Delta_{C_v} = \Sigma N \left(\frac{\partial N}{\partial P} \right) \frac{L}{AE} = \frac{965.7 \text{ kN} \cdot \text{m}}{AE}$$

Substituting the numerical values for A and E , we get

$$\begin{aligned} \Delta_{C_v} &= \frac{965.7 \text{ kN} \cdot \text{m}}{[400(10^{-6}) \text{ m}^2] 200(10^6) \text{ kN/m}^2} \\ &= 0.01207 \text{ m} = 12.1 \text{ mm} \end{aligned} \quad \text{Ans.}$$

This solution should be compared with that of Example 14.11, using the virtual-work method.

^{*}It may be more convenient to analyze the truss with just the 100-kN load on it, then analyze the truss with the P load on it. The results can then be summed algebraically to give the N forces.

PROBLEMS

14-123. Solve Prob. 14-72 using Castigliano's theorem.

*14-124. Solve Prob. 14-73 using Castigliano's theorem.

•14-125. Solve Prob. 14-75 using Castigliano's theorem.

14-126. Solve Prob. 14-76 using Castigliano's theorem.

14-127. Solve Prob. 14-77 using Castigliano's theorem.

*14-128. Solve Prob. 14-78 using Castigliano's theorem.

•14-129. Solve Prob. 14-79 using Castigliano's theorem.

14-130. Solve Prob. 14-80 using Castigliano's theorem.

14-131. Solve Prob. 14-81 using Castigliano's theorem.

*14-132. Solve Prob. 14-82 using Castigliano's theorem.

•14-133. Solve Prob. 14-83 using Castigliano's theorem.

14-134. Solve Prob. 14-84 using Castigliano's theorem.

*14.10 Castigliano's Theorem Applied to Beams

The internal strain energy for a beam is caused by both bending and shear. However, as pointed out in Example 14.7, if the beam is long and slender, the strain energy due to shear can be neglected compared with that of bending. Assuming this to be the case, the internal strain energy for a beam is given by $U_i = \int M^2 dx / 2EI$, Eq. 14-17. Omitting the subscript i , Castigliano's second theorem, $\Delta_i = \partial U_i / \partial P_i$, becomes

$$\Delta = \frac{\partial}{\partial P} \int_0^L \frac{M^2 dx}{2EI}$$

Rather than squaring the expression for internal moment, integrating, and then taking the partial derivative, it is generally easier to differentiate prior to integration. Provided E and I are constant, we have

$$\Delta = \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI} \quad (14-47)$$

Here

Δ = displacement of the point caused by the real loads acting on the beam

P = an external force of *variable magnitude* applied to the beam at the point and in the direction of Δ

M = internal moment in the beam, expressed as a function of x and caused by both the force P and the actual loads on the beam

E = modulus of elasticity of the material

I = moment of inertia of cross-sectional area about the neutral axis

If the slope of the tangent θ at a point on the elastic curve is to be determined, the partial derivative of the internal moment M with respect to an *external couple moment* M' acting at the point must be found. For this case,

$$\theta = \int_0^L M \left(\frac{\partial M}{\partial M'} \right) \frac{dx}{EI} \quad (14-48)$$

The above equations are similar to those used for the method of virtual forces, Eqs. 14-40 and 14-41, except m and m_θ replace $\partial M / \partial P$ and $\partial M / \partial M'$, respectively.

In addition, if axial load, shear, and torsion cause significant strain energy within the member, then the effects of all these loadings should be included when applying Castigliano's theorem. To do this we must use the strain-energy functions developed in Sec. 14.2, along with their associated partial derivatives. The result is

$$\Delta = \Sigma N \left(\frac{\partial N}{\partial P} \right) \frac{L}{AE} + \int_0^L f_s V \left(\frac{\partial V}{\partial P} \right) \frac{dx}{GA} + \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI} + \int_0^L T \left(\frac{\partial T}{\partial P} \right) \frac{dx}{GJ} \quad (14-49)$$

The method of applying this general formulation is similar to that used to apply Eqs. 14-47 and 14-48.

Procedure for Analysis

The following procedure provides a method that may be used to apply Castigliano's second theorem.

External Force P or Couple Moment M' .

- Place a force P on the beam at the point and directed along the line of action of the desired displacement.
- If the slope of the tangent is to be determined at the point, place a couple moment M' at the point.
- Assume that both P and M' have a variable magnitude.

Internal Moments M .

- Establish appropriate x coordinates that are valid within regions of the beam where there is no discontinuity of force, distributed load, or couple moment.
- Determine the internal moments M as a function of x , the actual (numerical) loads, and P or M' , and then find the partial derivatives $\partial M/\partial P$ or $\partial M/\partial M'$ for each coordinate x .
- After M and $\partial M/\partial P$ or $\partial M/\partial M'$ have been determined, assign P or M' its numerical value if it has actually replaced a real force or couple moment. Otherwise, set P or M' equal to zero.

Castigliano's Second Theorem.

- Apply Eq. 14-47 or 14-48 to determine the desired displacement Δ or θ . It is important to retain the algebraic signs for corresponding values of M and $\partial M/\partial P$ or $\partial M/\partial M'$.
- If the resultant sum of all the definite integrals is positive, Δ or θ is in the same direction as P or M' . If a negative value results, Δ or θ is opposite to P or M' .

EXAMPLE 14.16

Determine the displacement of point B on the beam shown in Fig. 14-40a. EI is constant.

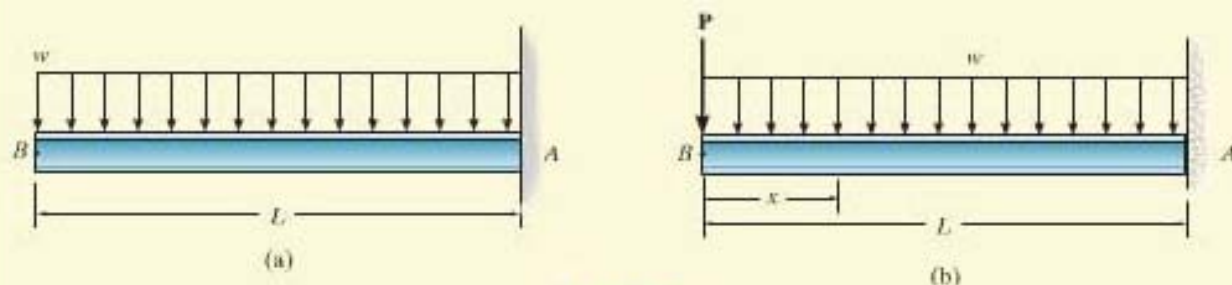


Fig. 14-40

SOLUTION

External Force P . A vertical force P is placed on the beam at B as shown in Fig. 14-40b.

Internal Moments M . A single x coordinate is needed for the solution, since there are no discontinuities of loading between A and B . Using the method of sections, Fig. 14-40c, the internal moment and its partial derivative are determined as follows:

$$\downarrow + \sum M_{NA} = 0; \quad M + wx\left(\frac{x}{2}\right) + P(x) = 0$$

$$M = -\frac{wx^2}{2} - Px$$

$$\frac{\partial M}{\partial P} = -x$$

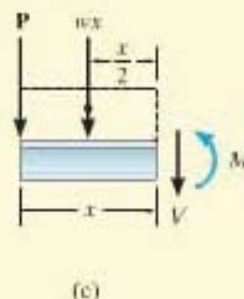
Setting $P = 0$ gives

$$M = -\frac{wx^2}{2} \quad \text{and} \quad \frac{\partial M}{\partial P} = -x$$

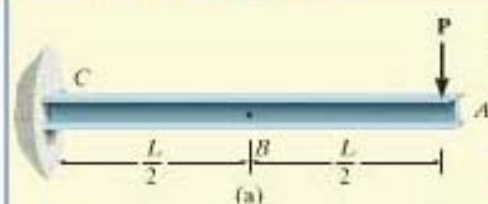
Castigliano's Second Theorem. Applying Eq. 14-47, we have

$$\begin{aligned} \Delta_B &= \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI} = \int_0^L \frac{(-wx^2/2)(-x) dx}{EI} \\ &= \frac{wL^4}{8EI} \end{aligned} \quad \text{Ans.}$$

The similarity between this solution and that of the virtual-work method, Example 14.13, should be noted.



EXAMPLE 14.17

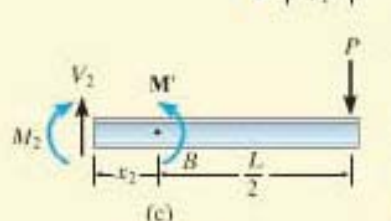
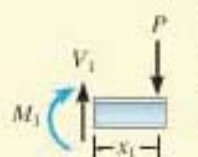
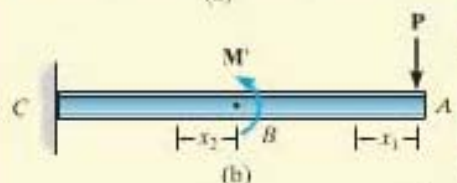


Determine the slope at point B of the beam shown in Fig. 14-41a. EI is constant.

SOLUTION

External Couple Moment M' . Since the slope at point B is to be determined, an external couple moment M' is placed on the beam at this point, Fig. 14-41b.

Internal Moments M . Two coordinates, x_1 and x_2 , must be used to completely describe the internal moments within the beam since there is a discontinuity, M' , at B . As shown in Fig. 14-41b, x_1 ranges from A to B and x_2 ranges from B to C . Using the method of sections, Fig. 14-41c, the internal moments and the partial derivatives for x_1 and x_2 are determined as follows:



$$\downarrow + \Sigma M_{NA} = 0; \quad M_1 = -Px_1, \quad \frac{\partial M_1}{\partial M'} = 0$$

$$\downarrow + \Sigma M_{NA} = 0; \quad M_2 = M' - P\left(\frac{L}{2} + x_2\right), \quad \frac{\partial M_2}{\partial M'} = 1$$

Castigliano's Second Theorem. Setting $M' = 0$ and applying Eq. 14-48, we have

$$\theta_B = \int_0^L M \left(\frac{\partial M}{\partial M'} \right) \frac{dx}{EI} = \int_0^{L/2} \frac{(-Px_1)(0) dx_1}{EI} + \int_0^{L/2} \frac{-P[(L/2) + x_2](1) dx_2}{EI} = -\frac{3PL^2}{8EI} \quad \text{Ans.}$$

Note the similarity between this solution and that of Example 14.14.

PROBLEMS

14-135. Solve Prob. 14-87 using Castigliano's theorem.

*14-136. Solve Prob. 14-88 using Castigliano's theorem.

*14-137. Solve Prob. 14-90 using Castigliano's theorem.

14-138. Solve Prob. 14-92 using Castigliano's theorem.

14-139. Solve Prob. 14-93 using Castigliano's theorem.

*14-140. Solve Prob. 14-96 using Castigliano's theorem.

14-141. Solve Prob. 14-97 using Castigliano's theorem.

14-142. Solve Prob. 14-98 using Castigliano's theorem.

14-143. Solve Prob. 14-112 using Castigliano's theorem.

*14-144. Solve Prob. 14-114 using Castigliano's theorem.

*14-145. Solve Prob. 14-121 using Castigliano's theorem.

CHAPTER REVIEW

When a force (couple moment) acts on a deformable body it will do external work when it displaces (rotates). The internal stresses produced in the body also undergo displacement, thereby creating elastic strain energy that is stored in the material. The conservation of energy states that the external work done by the loading is equal to the internal elastic strain energy produced by the stresses in the body.

$$U_e = U_i$$

The conservation of energy can be used to solve problems involving elastic impact, which assumes the moving body is rigid and all the strain energy is stored in the stationary body. This leads to use of an impact factor n , which is a ratio of the dynamic load to the static load. It is used to determine the maximum stress and displacement of the body at the point of impact.

$$n = 1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{st}}\right)}$$

$$\sigma_{\max} = n\sigma_{st}$$

$$\Delta_{\max} = n\Delta_{st}$$

The principle of virtual work can be used to determine the displacement of a joint on a truss or the slope and the displacement of points on a beam. It requires placing an external virtual unit force (virtual unit couple moment) at the point where the displacement (rotation) is to be determined. The external virtual work that is produced by the external loading is then equated to the internal virtual strain energy in the structure.

$$1 \cdot \Delta = \sum \frac{nNL}{AE}$$

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

Castigliano's second theorem can also be used to determine the displacement of a joint on a truss or the slope and the displacement at a point on a beam. Here a variable force P (couple moment M') is placed at the point where the displacement (slope) is to be determined. The internal loading is then determined as a function of P (M') and its partial derivative with respect to P (M') is determined. Castigliano's second theorem is then applied to obtain the desired displacement (rotation).

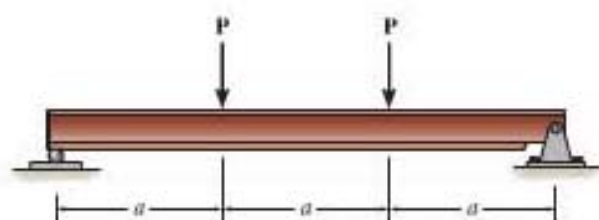
$$\Delta = \sum N \left(\frac{\partial N}{\partial P} \right) \frac{L}{AE}$$

$$\Delta = \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI}$$

$$\theta = \int_0^L M \left(\frac{\partial M}{\partial M'} \right) \frac{dx}{EI}$$

REVIEW PROBLEMS

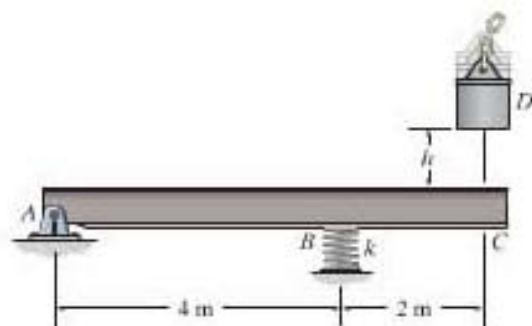
14-146. Determine the bending strain energy in the beam due to the loading shown. EI is constant.



Prob. 14-146

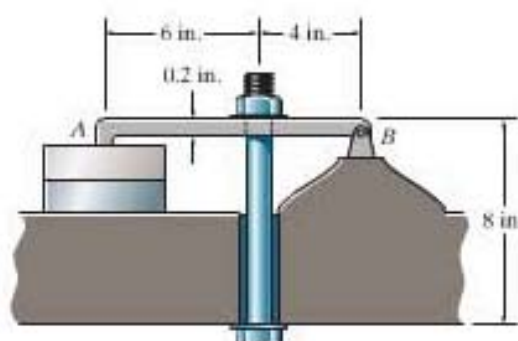
14-147. The 200-kg block D is dropped from a height $h = 1$ m onto end C of the A-36 steel $W200 \times 36$ overhang beam. If the spring at B has a stiffness $k = 200$ kN/m, determine the maximum bending stress developed in the beam.

***14-148.** Determine the maximum height h from which the 200-kg block D can be dropped without causing the A-36 steel $W200 \times 36$ overhang beam to yield. The spring at B has a stiffness $k = 200$ kN/m.



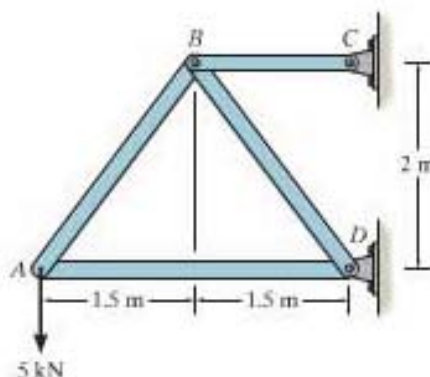
Probs. 14-147/148

•14-149. The L2 steel bolt has a diameter of 0.25 in., and the link AB has a rectangular cross section that is 0.5 in. wide by 0.2 in. thick. Determine the strain energy in the link AB due to bending, and in the bolt due to axial force. The bolt is tightened so that it has a tension of 350 lb. Neglect the hole in the link.



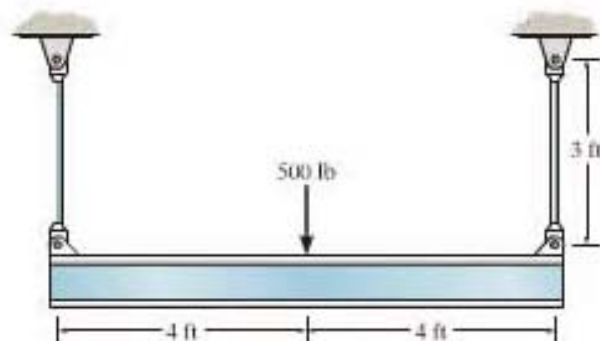
Prob. 14-149

14-150. Determine the vertical displacement of joint A . Each bar is made of A-36 steel and has a cross-sectional area of 600 mm². Use the conservation of energy.



Prob. 14-150

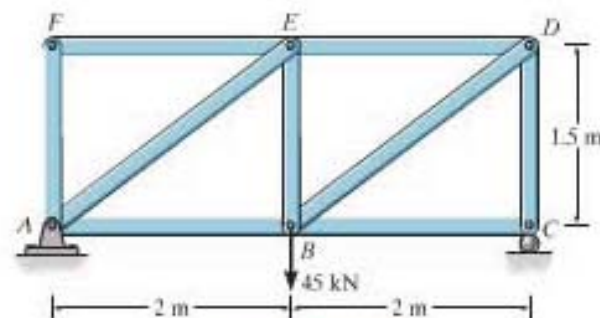
14-151. Determine the total strain energy in the A-36 steel assembly. Consider the axial strain energy in the two 0.5-in.-diameter rods and the bending strain energy in the beam for which $I = 43.4 \text{ in}^4$.



Prob. 14-151

***14-152.** Determine the vertical displacement of joint E . For each member $A = 400 \text{ mm}^2$, $E = 200 \text{ GPa}$. Use the method of virtual work.

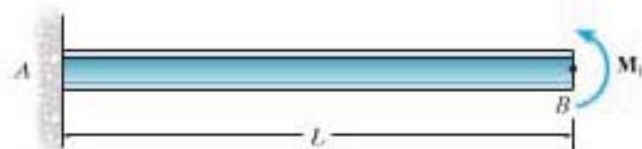
•14-153. Solve Prob. 14-152 using Castigliano's theorem.



Probs. 14-152/153

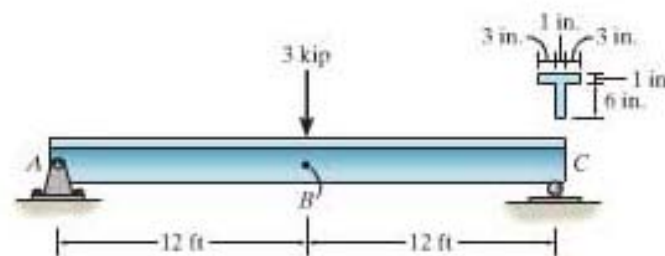
14-154. The cantilevered beam is subjected to a couple moment M_0 applied at its end. Determine the slope of the beam at B . EI is constant. Use the method of virtual work.

14-155. Solve Prob. 14-154 using Castigliano's theorem.



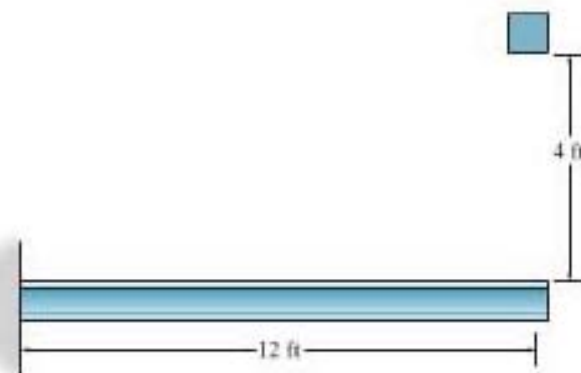
Probs. 14-154/155

***14-156.** Determine the displacement of point B on the aluminum beam. $E_{al} = 10.6(10^3) \text{ ksi}$. Use the conservation of energy.



Prob. 14-156

14-157. A 20-lb weight is dropped from a height of 4 ft onto the end of a cantilevered A-36 steel beam. If the beam is a $W12 \times 50$, determine the maximum stress developed in the beam.



Prob. 14-157

Geometric Properties of an Area

A.1 Centroid of an Area

The *centroid* of an area refers to the point that defines the geometric center for the area. If the area has an arbitrary shape, as shown in Fig. A-1a, the x and y coordinates defining the location of the centroid C are determined using the formulas

$$\bar{x} = \frac{\int_A x \, dA}{\int_A dA} \quad \bar{y} = \frac{\int_A y \, dA}{\int_A dA} \quad (\text{A-1})$$

The numerators in these equations are formulations of the “moment” of the area element dA about the y and the x axis, respectively, Fig. A-1b; the denominators represent the total area A of the shape.

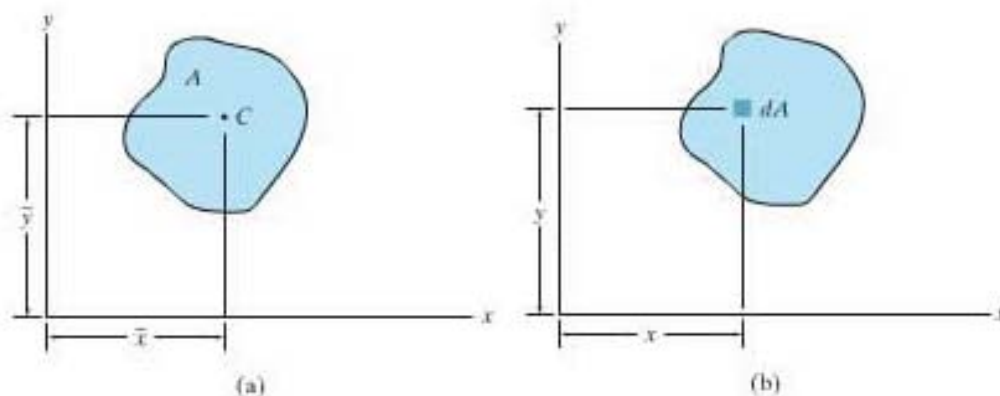


Fig. A-1

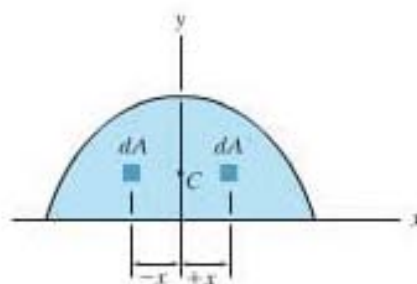


Fig. A-2

The location of the centroid for some areas may be partially or completely specified by using symmetry conditions. In cases where the area has an axis of symmetry, the centroid for the area will lie along this axis. For example, the centroid C for the area shown in Fig. A-2 must lie along the y axis, since for every elemental area dA a distance $+x$ to the right of the y axis, there is an identical element a distance $-x$ to the left. The total moment for all the elements about the axis of symmetry will therefore cancel; that is, $\int x dA = 0$ (Eq. A-1), so that $\bar{x} = 0$. In cases where a shape has two axes of symmetry, it follows that the centroid lies at the intersection of these axes, Fig. A-3. Based on the principle of symmetry, or using Eq. A-1, the locations of the centroid for common area shapes are listed on the inside front cover.

Composite Areas. Often an area can be sectioned or divided into several parts having simpler shapes. Provided the area and location of the centroid of each of these “composite shapes” are known, one can eliminate the need for integration to determine the centroid for the entire area. In this case, equations analogous to Eq. A-1 must be used, except that finite summation signs replace the integrals; i.e.,

$$\bar{x} = \frac{\sum \bar{x} A}{\sum A} \quad \bar{y} = \frac{\sum \bar{y} A}{\sum A} \quad (\text{A-2})$$

Here $\bar{x}$ and $\bar{y}$ represent the *algebraic distances* or x, y coordinates for the centroid of each composite part, and $\sum A$ represents the sum of the areas of the composite parts or simply the *total area*. In particular, if a hole, or a geometric region having no material, is located within a composite part, the hole is considered as an additional composite part having a *negative area*. Also, as discussed above, if the total area is symmetrical about an axis, the centroid of the area lies on the axis.

The following example illustrates application of Eq. A-2.

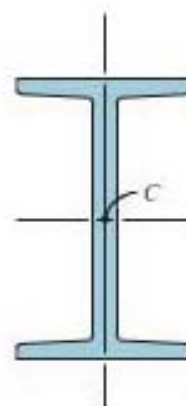
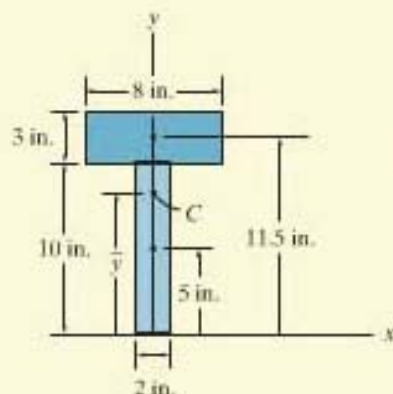


Fig. A-3

EXAMPLE A.1

Locate the centroid C of the cross-sectional area for the T-beam shown in Fig. A-4a.



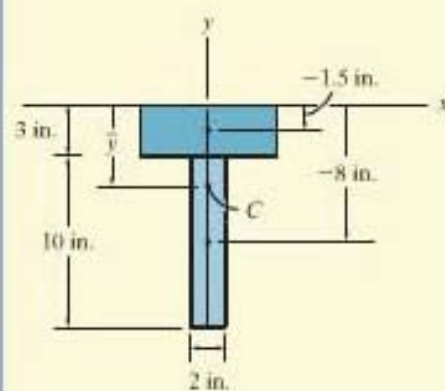
(a)

SOLUTION I

The y axis is placed along the axis of symmetry so that $\bar{x} = 0$, Fig. A-4a. To obtain $\bar{y}$ we will establish the x axis (reference axis) through the base of the area. The area is segmented into two rectangles as shown, and the centroidal location $\bar{y}$ for each is established. Applying Eq. A-2, we have

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A} = \frac{[5 \text{ in.}](10 \text{ in.})(2 \text{ in.}) + [11.5 \text{ in.}](3 \text{ in.})(8 \text{ in.})}{(10 \text{ in.})(2 \text{ in.}) + (3 \text{ in.})(8 \text{ in.})}$$

$$= 8.55 \text{ in.} \quad \text{Ans.}$$



(b)

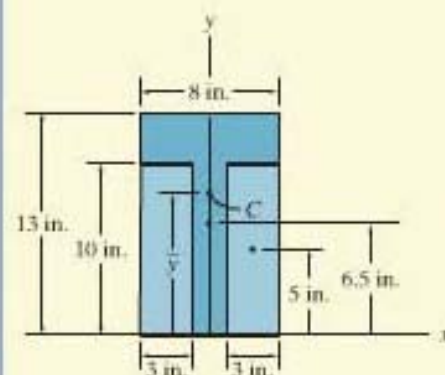
SOLUTION II

Using the same two segments, the x axis can be located at the top of the area, Fig. A-4b. Here

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A} = \frac{[-1.5 \text{ in.}](3 \text{ in.})(8 \text{ in.}) + [-8 \text{ in.}](10 \text{ in.})(2 \text{ in.})}{(3 \text{ in.})(8 \text{ in.}) + (10 \text{ in.})(2 \text{ in.})}$$

$$= -4.45 \text{ in.} \quad \text{Ans.}$$

The negative sign indicates that C is located *below* the x axis, which is to be expected. Also note that from the two answers $8.55 \text{ in.} + 4.45 \text{ in.} = 13.0 \text{ in.}$, which is the depth of the beam.



(c)

SOLUTION III

It is also possible to consider the cross-sectional area to be one large rectangle *less* two small rectangles shown shaded in Fig. A-4c. Here we have

$$\bar{y} = \frac{\sum \bar{y}A}{\sum A} = \frac{[6.5 \text{ in.}](13 \text{ in.})(8 \text{ in.}) - 2[5 \text{ in.}](10 \text{ in.})(3 \text{ in.})}{(13 \text{ in.})(8 \text{ in.}) - 2(10 \text{ in.})(3 \text{ in.})}$$

$$= 8.55 \text{ in.} \quad \text{Ans.}$$

Fig. A-4

A.2 Moment of Inertia for an Area

The moment of inertia of an area often appears in formulas used in mechanics of materials. It is a geometric property that is calculated about an axis, and for the x and y axes shown in Fig. A-5, it is defined as

$$\begin{aligned} I_x &= \int_A y^2 dA \\ I_y &= \int_A x^2 dA \end{aligned} \quad (\text{A-3})$$

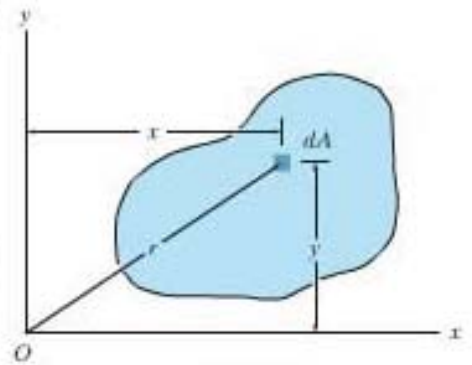


Fig. A-5

These integrals have no physical meaning, but they are so named because they are similar to the formulation of the moment of inertia of a mass, which is a dynamical property of matter.

We can also calculate the moment of inertia of an area about the pole O or z axis, Fig. A-5. This is referred to as the *polar moment of inertia*,

$$J_O = \int_A r^2 dA = I_x + I_y \quad (\text{A-4})$$

Here r is the perpendicular distance from the pole (z axis) to the element dA . The relationship between J_O and I_x , I_y is possible since $r^2 = x^2 + y^2$, Fig. A-5.

From the above formulations it is seen that I_x , I_y , and J_O will *always* be *positive*, since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4 , mm^4 , or ft^4 , in^4 .

Using the above equations, the moments of inertia for some common area shapes have been calculated about their *centroidal axes* and are listed on the inside front cover.

Parallel-Axis Theorem for an Area. If the moment of inertia for an area is known about a centroidal axis, we can determine the moment of inertia of the area about a corresponding parallel axis using the *parallel-axis theorem*. To derive this theorem, consider finding the moment of inertia of the shaded area shown in Fig. A-6 about the x axis. In this case, a differential element dA is located at the arbitrary distance y' from the centroidal x' axis, whereas the *fixed distance* between the

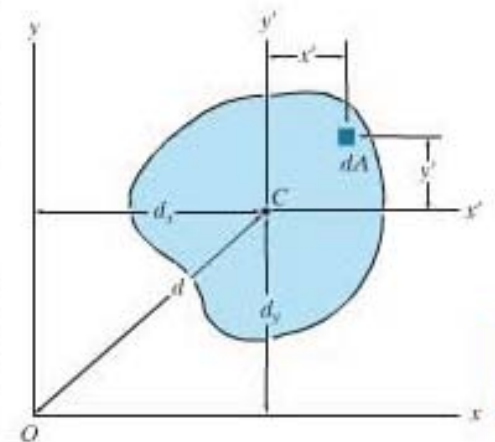


Fig. A-6

parallel x and x' axes is d_y . Since the moment of inertia of dA about the x axis is $dI_x = (y' + d_y)^2 dA$, then for the entire area,

$$I_x = \int_A (y' + d_y)^2 dA = \int_A y'^2 dA + 2d_y \int_A y' dA + d_y^2 \int_A dA$$

The first term on the right represents the moment of inertia of the area about the x' axis, $\bar{I}_{x'}$. The second term is zero since the x' axis passes through the area's centroid C , that is, $\int y' dA = \bar{y}' A = 0$ since $\bar{y}' = 0$. The final result is therefore

$$I_x = \bar{I}_{x'} + Ad_y^2 \quad (\text{A-5})$$

A similar expression can be written for I_y , that is,

$$I_y = \bar{I}_{y'} + Ad_x^2 \quad (\text{A-6})$$

And finally, for the polar moment of inertia about an axis perpendicular to the x - y plane and passing through the pole O (z axis), Fig. A-6, we have

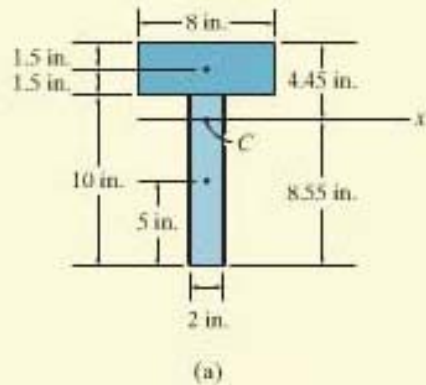
$$J_O = \bar{J}_C + Ad^2 \quad (\text{A-7})$$

The form of each of the above equations states that *the moment of inertia of an area about an axis is equal to the area's moment of inertia about a parallel axis passing through the "centroid" plus the product of the area and the square of the perpendicular distance between the axes.*

Composite Areas. Many cross-sectional areas consist of a series of connected simpler shapes, such as rectangles, triangles, and semicircles. In order to properly determine the moment of inertia of such an area about a specified axis, it is first necessary to divide the area into its composite parts and indicate the perpendicular distance from the axis to the parallel centroidal axis for each part. Using the table on the inside front cover of the book, the moment of inertia of each part is determined about the centroidal axis. If this axis does not coincide with the specified axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the specified axis. The moment of inertia of the entire area about this axis is then determined by summing the results of its composite parts. In particular, if a composite part has a "hole," the moment of inertia for the composite is found by "subtracting" the moment of inertia for the hole from the moment of inertia of the entire area including the hole.

EXAMPLE A.2

Determine the moment of inertia of the cross-sectional area of the T-beam shown in Fig. A-7a about the centroidal x' axis.

**Fig. A-7****SOLUTION I**

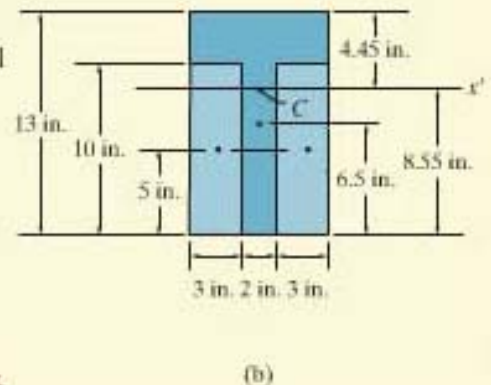
The area is segmented into two rectangles as shown in Fig. A-7a, and the distance from the x' axis and each centroidal axis is determined. Using the table on the inside front cover, the moment of inertia of a rectangle about its centroidal axis is $I = \frac{1}{12}bh^3$. Applying the parallel-axis theorem, Eq. A-5, to each rectangle and adding the results, we have

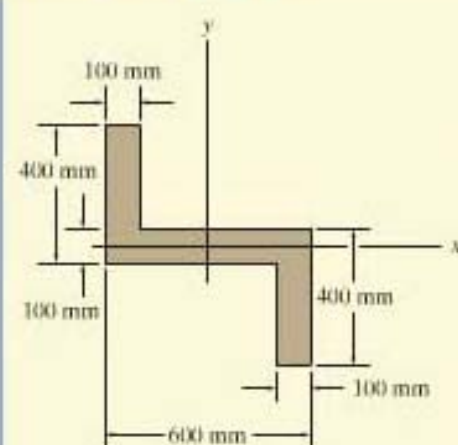
$$\begin{aligned}
 I &= \Sigma(\bar{I}_{x'} + Ad_y^2) \\
 &= \left[\frac{1}{12}(2 \text{ in.})(10 \text{ in.})^3 + (2 \text{ in.})(10 \text{ in.})(8.55 \text{ in.} - 5 \text{ in.})^2 \right] \\
 &\quad + \left[\frac{1}{12}(8 \text{ in.})(3 \text{ in.})^3 + (8 \text{ in.})(3 \text{ in.})(4.45 \text{ in.} - 1.5 \text{ in.})^2 \right] \\
 I &= 646 \text{ in}^4 \qquad \text{Ans}
 \end{aligned}$$

SOLUTION II

The area can be considered as one large rectangle less two small rectangles, shown shaded in Fig. A-7b. We have

$$\begin{aligned}
 I &= \Sigma(\bar{I}_{x'} + Ad_y^2) \\
 &= \left[\frac{1}{12}(8 \text{ in.})(13 \text{ in.})^3 + (8 \text{ in.})(13 \text{ in.})(8.55 \text{ in.} - 6.5 \text{ in.})^2 \right] \\
 &\quad - 2 \left[\frac{1}{12}(3 \text{ in.})(10 \text{ in.})^3 + (3 \text{ in.})(10 \text{ in.})(8.55 \text{ in.} - 5 \text{ in.})^2 \right] \\
 I &= 646 \text{ in}^4 \qquad \text{Ans}
 \end{aligned}$$



EXAMPLE A.3

(a)

Determine the moments of inertia of the beam's cross-sectional area shown in Fig. A-8a about the x and y centroidal axes.

SOLUTION

The cross section can be considered as three composite rectangular areas A , B , and D shown in Fig. A-8b. For the calculation, the centroid of each of these rectangles is located in the figure. From the table on the inside front cover, the moment of inertia of a rectangle about its centroidal axis is $I = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles A and D , the calculations are as follows:

Rectangle A:

$$I_x = \bar{I}_x + Ad_y^2 = \frac{1}{12}(100 \text{ mm})(300 \text{ mm})^3 + (100 \text{ mm})(300 \text{ mm})(200 \text{ mm})^2$$

$$= 1.425(10^9) \text{ mm}^4$$

$$I_y = \bar{I}_y + Ad_x^2 = \frac{1}{12}(300 \text{ mm})(100 \text{ mm})^3 + (100 \text{ mm})(300 \text{ mm})(250 \text{ mm})^2$$

$$= 1.90(10^9) \text{ mm}^4$$

Rectangle B:

$$I_x = \frac{1}{12}(600 \text{ mm})(100 \text{ mm})^3 = 0.05(10^9) \text{ mm}^4$$

$$I_y = \frac{1}{12}(100 \text{ mm})(600 \text{ mm})^3 = 1.80(10^9) \text{ mm}^4$$

Rectangle D:

$$I_x = \bar{I}_x + Ad_y^2 = \frac{1}{12}(100 \text{ mm})(300 \text{ mm})^3 + (100 \text{ mm})(300 \text{ mm})(200 \text{ mm})^2$$

$$= 1.425(10^9) \text{ mm}^4$$

$$I_y = \bar{I}_y + Ad_x^2 = \frac{1}{12}(300 \text{ mm})(100 \text{ mm})^3 + (100 \text{ mm})(300 \text{ mm})(250 \text{ mm})^2$$

$$= 1.90(10^9) \text{ mm}^4$$

The moments of inertia for the entire cross section are thus

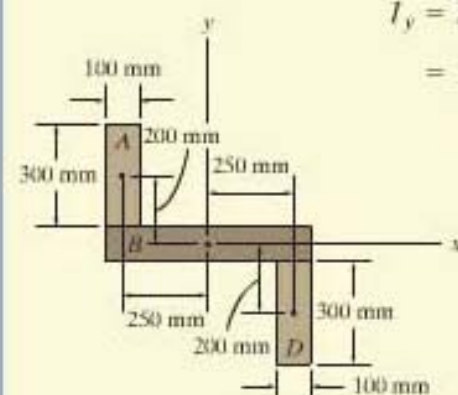
$$I_x = 1.425(10^9) + 0.05(10^9) + 1.425(10^9)$$

$$= 2.90(10^9) \text{ mm}^4$$

Ans.

$$I_y = 1.90(10^9) + 1.80(10^9) + 1.90(10^9)$$

$$= 5.60(10^9) \text{ mm}^4$$

Ans.

(b)

Fig. A-8

A.3 Product of Inertia for an Area

In general, the moment of inertia for an area is different for every axis about which it is computed. In some applications of mechanical or structural design it is necessary to know the orientation of those axes that give, respectively, the maximum and minimum moments of inertia for the area. The method for determining this is discussed in Sec. A.4. To use this method, however, one must first determine the product of inertia for the area as well as its moments of inertia for given x, y axes.

The **product of inertia** for the area A shown in Fig. A-9 is defined as

$$I_{xy} = \int_A xy \, dA \quad (\text{A-8})$$

Like the moment of inertia, the product of inertia has units of length raised to the fourth power, e.g., m^4 , mm^4 or ft^4 , in^4 . However, since x or y may be a negative quantity, while dA is always positive, the product of inertia may be positive, negative, or zero, depending on the location and orientation of the coordinate axes. For example, the product of inertia I_{xy} for an area will be **zero** if either the x or y axis is an axis of *symmetry* for the area. To show this, consider the shaded area in Fig. A-10, where for every element dA located at point (x, y) there is a corresponding element dA located at $(x, -y)$. Since the products of inertia for these elements are, respectively, $xy \, dA$ and $-xy \, dA$, their algebraic sum or the integration of all the elements of area chosen in this way will cancel each other. Consequently, the product of inertia for the total area becomes zero.

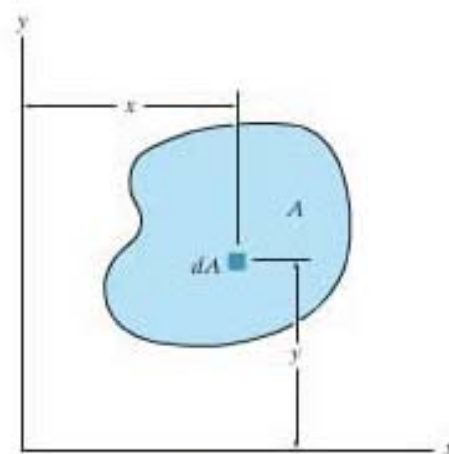


Fig. A-9

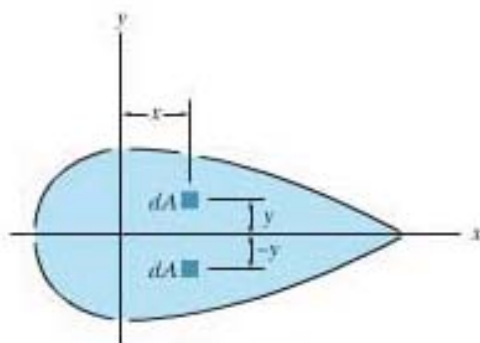


Fig. A-10

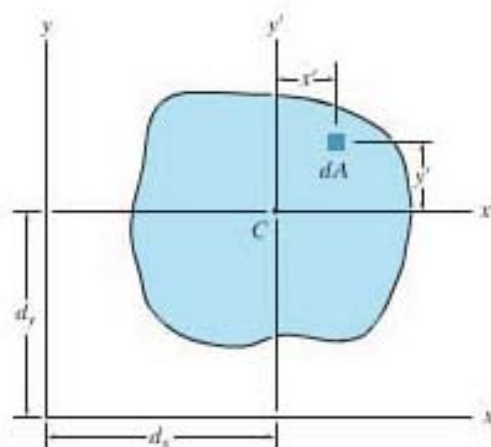


Fig. A-11

Parallel-Axis Theorem. Consider the shaded area shown in Fig. A-11, where x' and y' represent a set of centroidal axes, and x and y represent a corresponding set of parallel axes. Since the product of inertia of dA with respect to the x and y axes is $dI_{xy} = (x' + d_x)(y' + d_y) dA$, then for the entire area,

$$\begin{aligned} I_{xy} &= \int_A (x' + d_x)(y' + d_y) dA \\ &= \int_A x' y' dA + d_x \int_A y' dA + d_y \int_A x' dA + d_x d_y \int_A dA \end{aligned}$$

The first term on the right represents the product of inertia of the area with respect to the centroidal axis, $\bar{I}_{x'y'}$. The second and third terms are zero since the moments of the area are taken about the centroidal axis. Realizing that the fourth integral represents the total area A , we therefore have

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y \quad (\text{A-9})$$

The similarity between this equation and the parallel-axis theorem for moments of inertia should be noted. In particular, it is important that here the *algebraic signs* for d_x and d_y be maintained when applying Eq. A-9.

EXAMPLE A.4

Determine the product of inertia of the beam's cross-sectional area, shown in Fig. A-12a, about the x and y centroidal axes.

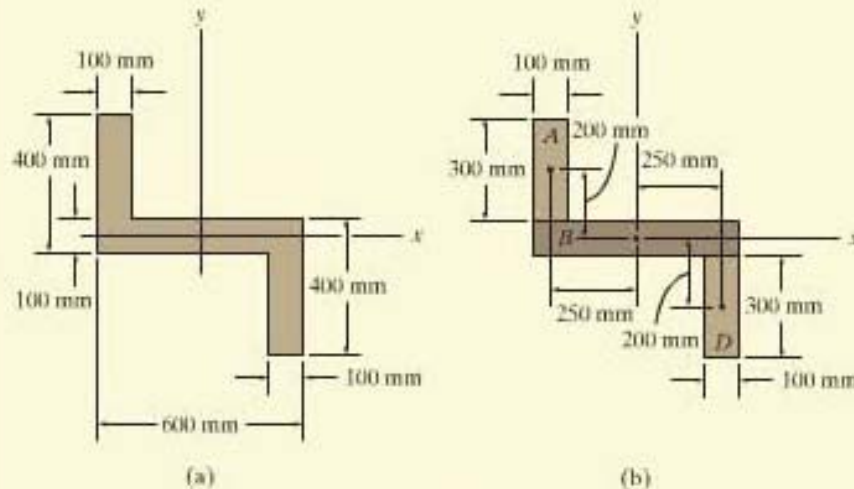


Fig. A-12

SOLUTION

As in Example A.3, the cross section can be considered as three composite rectangular areas A , B , and D , Fig. A-12b. The coordinates for the centroid of each of these rectangles are shown in the figure. Due to symmetry, the product of inertia of *each rectangle* is *zero* about a set of x' , y' axes that pass through the rectangle's centroid. Hence, application of the parallel-axis theorem to each of the rectangles yields

Rectangle A:

$$\begin{aligned} I_{xy} &= \bar{I}_{x'y'} + Ad_xd_y \\ &= 0 + (300 \text{ mm})(100 \text{ mm})(-250 \text{ mm})(200 \text{ mm}) \\ &= -1.50(10^9) \text{ mm}^4 \end{aligned}$$

Rectangle B:

$$\begin{aligned} I_{xy} &= \bar{I}_{x'y'} + Ad_xd_y \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Rectangle D:

$$\begin{aligned} I_{xy} &= \bar{I}_{x'y'} + Ad_xd_y \\ &= 0 + (300 \text{ mm})(100 \text{ mm})(250 \text{ mm})(-200 \text{ mm}) \\ &= -1.50(10^9) \text{ mm}^4 \end{aligned}$$

The product of inertia for the entire cross section is thus

$$\begin{aligned} I_{xy} &= [-1.50(10^9) \text{ mm}^4] + 0 + [-1.50(10^9) \text{ mm}^4] \\ &= -3.00(10^9) \text{ mm}^4 \end{aligned}$$

Ans

A.4 Moments of Inertia for an Area about Inclined Axes

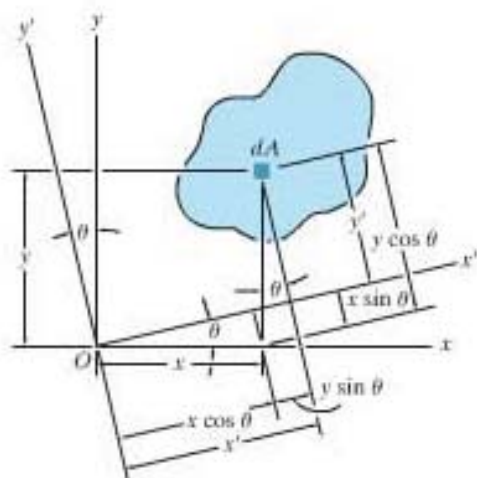


Fig. A-13

In mechanical or structural design, it is sometimes necessary to calculate the moments and product of inertia $I_{x'}$, $I_{y'}$ and $I_{x'y'}$ for an area with respect to a set of inclined x' and y' axes when the values for θ , I_x , I_y , and I_{xy} are known. As shown in Fig. A-13, the coordinates to the area element dA from each of the two coordinate systems are related by the transformation equations

$$x' = x \cos \theta + y \sin \theta$$

$$y' = y \cos \theta - x \sin \theta$$

Using these equations, the moments and product of inertia of dA about the x' and y' axes become

$$dI_{x'} = y'^2 dA = (y \cos \theta - x \sin \theta)^2 dA$$

$$dI_{y'} = x'^2 dA = (x \cos \theta + y \sin \theta)^2 dA$$

$$dI_{x'y'} = x'y' dA = (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dA$$

Expanding each expression and integrating, realizing that $I_x = \int y^2 dA$, $I_y = \int x^2 dA$, and $I_{xy} = \int xy dA$, we obtain

$$I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - 2I_{xy} \sin \theta \cos \theta$$

$$I_{y'} = I_x \sin^2 \theta + I_y \cos^2 \theta + 2I_{xy} \sin \theta \cos \theta$$

$$I_{x'y'} = I_x \sin \theta \cos \theta - I_y \sin \theta \cos \theta + I_{xy}(\cos^2 \theta - \sin^2 \theta)$$

These equations may be simplified by using the trigonometric identities $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, in which case

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

(A-10)

Principal Moments of Inertia. Note that $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ depend on the angle of inclination, θ , of the x' , y' axes. We will now determine the orientation of these axes about which the moments of inertia for the area, $I_{x'}$ and $I_{y'}$, are maximum and minimum. This particular set of axes is called the **principal axes** of inertia for the area, and the corresponding moments of inertia with respect to these axes are called the **principal moments of inertia**. In general, there is a set of principal axes for every chosen origin O ; however, in mechanics of materials the area's centroid is the most important location for O .

The angle $\theta = \theta_p$, which defines the orientation of the principal axes for the area, can be found by differentiating the first of Eq. A-10 with respect to θ and setting the result equal to zero. Thus,

$$\frac{dI_{x'}}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right) \sin 2\theta - 2I_{xy} \cos 2\theta = 0$$

Therefore, at $\theta = \theta_p$,

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} \quad (\text{A-11})$$

This equation has two roots, θ_{p_1} and θ_{p_2} , which are 90° apart and so specify the inclination of each principal axis.

The sine and cosine of $2\theta_{p_1}$ and $2\theta_{p_2}$ can be obtained from the triangles shown in Fig. A-14, which are based on Eq. A-11. If these trigonometric relations are substituted into the first or second of Eq. A-10 and simplified, the result is

$$I_{\max} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad (\text{A-12})$$

Depending on the sign chosen, this result gives the maximum or minimum moment of inertia for the area. Furthermore, if the above trigonometric relations for θ_{p_1} and θ_{p_2} are substituted into the third of Eq. A-10, it will be seen that $I_{x'y'} = 0$; that is, the *product of inertia with respect to the principal axes is zero*. Since it was indicated in Sec. A.3 that the product of inertia is zero with respect to any symmetrical axis, it therefore follows that *any symmetrical axis and the one perpendicular to it represent principal axes of inertia for the area*. Also, notice that the equations derived in this section are similar to those for stress and strain transformation developed in Chapters 9 and 10, respectively.

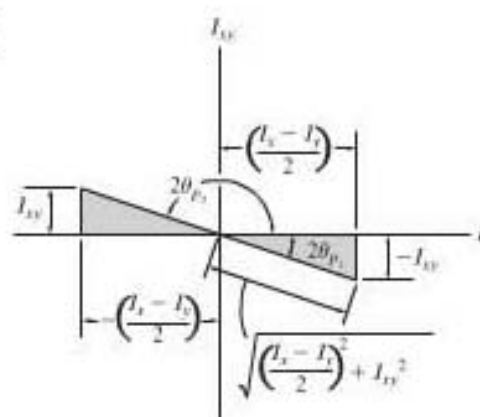


Fig. A-14

EXAMPLE A.5

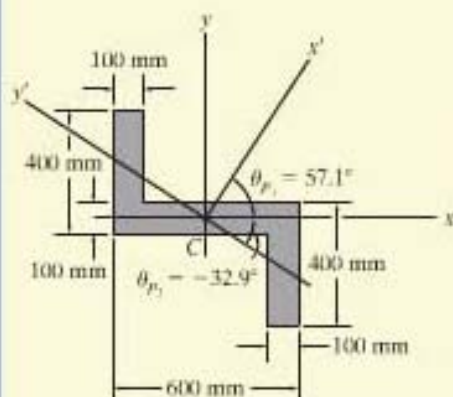


Fig. A-15

Determine the principal moments of inertia for the beam's cross-sectional area shown in Fig. A-15 with respect to an axis passing through the centroid C .

SOLUTION

The moments and product of inertia of the cross section with respect to the x , y axes have been determined in Examples A.3 and A.4. The results are

$$I_x = 2.90(10^9) \text{ mm}^4 \quad I_y = 5.60(10^9) \text{ mm}^4 \quad I_{xy} = -3.00(10^9) \text{ mm}^4$$

Using Eq. A-11, the angles of inclination of the principal axes x' and y' are

$$\begin{aligned} \tan 2\theta_p &= \frac{-I_{xy}}{(I_x - I_y)/2} = \frac{3.00(10^9)}{[2.90(10^9) - 5.60(10^9)]/2} = -2.22 \\ 2\theta_{p1} &= 114.2^\circ \quad \text{and} \quad 2\theta_{p2} = -65.8^\circ \end{aligned}$$

Thus, as shown in Fig. A-15,

$$\theta_{p1} = 57.1^\circ \quad \text{and} \quad \theta_{p2} = -32.9^\circ$$

The principal moments of inertia with respect to the x' and y' axes are determined by using Eq. A-12.

$$\begin{aligned} I_{x'_{\max}} &= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \\ &= \frac{2.90(10^9) + 5.60(10^9)}{2} \pm \sqrt{\left[\frac{2.90(10^9) - 5.60(10^9)}{2}\right]^2 + [-3.00(10^9)]^2} \\ &= 4.25(10^9) \pm 3.29(10^9) \end{aligned}$$

or

$$I_{\max} = 7.54(10^9) \text{ mm}^4 \quad I_{\min} = 0.960(10^9) \text{ mm}^4 \quad \text{Ans.}$$

Specifically, the maximum moment of inertia, $I_{\max} = 7.54(10^9) \text{ mm}^4$, occurs with respect to the x' axis (major axis), since *by inspection* most of the cross-sectional area is farthest away from this axis. To show this, substitute the data with $\theta = 57.1^\circ$ into the first of Eq. A-10.

A.5 Mohr's Circle for Moments of Inertia

Equations A-10 through A-12 have a semi-graphical solution that is convenient to use and generally easy to remember. Squaring the first and third of Eq. A-10 and adding, it is found that

$$\left(I_{x'} - \frac{I_x + I_y}{2}\right)^2 + I_{x'y'}^2 = \left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2 \quad (\text{A-13})$$

In any given problem, $I_{x'}$ and $I_{x'y'}$ are *variables*, and I_x , I_y , and I_{xy} are *known constants*. Thus, the above equation may be written in compact form as

$$(I_{x'} - a)^2 + I_{x'y'}^2 = R^2$$

When this equation is plotted, the resulting graph represents a *circle* of radius

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

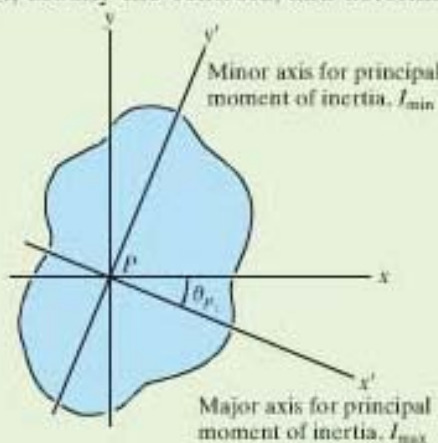
and having its center located at point $(a, 0)$, where $a = (I_x + I_y)/2$. The circle so constructed is called *Mohr's circle*. Its application is similar to that used for stress and strain transformation developed in Chapters 9 and 10, respectively.

Procedure for Analysis

The main purpose for using Mohr's circle here is to have a convenient means of transforming I_x , I_y , and I_{xy} into the principal moments of inertia for the area. The following procedure provides a method for doing this.

Calculate I_x , I_y , I_{xy}

Establish the x , y axes for the area, with the origin located at the point P of interest, usually the centroid, and determine I_x , I_y , and I_{xy} , Fig. A-16a.

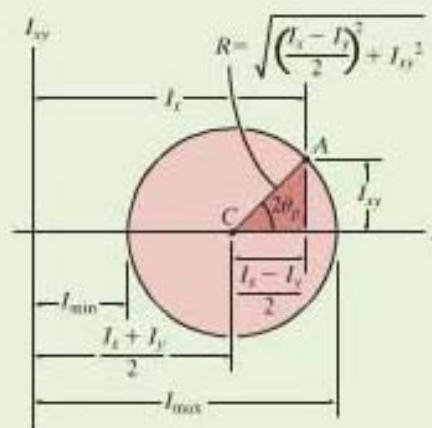


(a) Fig. A-16

Procedure for Analysis (continued)

Construct the Circle.

Establish a rectangular coordinate system such that the horizontal axis represents the moment of inertia I , and the vertical axis represents the product of inertia I_{xy} , Fig. A-16b. Determine the center of the circle, C , which is located a distance $(I_x + I_y)/2$ from the origin, and plot the "reference point" A having coordinates (I_x, I_{xy}) . By definition, I_x is always positive, whereas I_{xy} can either be positive or negative. Connect the reference point A with the center of the circle, and determine the distance CA by trigonometry. This distance represents the radius of the circle, Fig. A-16b. Finally, draw the circle.



(b)

Fig. A-16 (cont.)

Principal Moments of Inertia.

The points where the circle intersects the I axis give the values of the principal moments of inertia I_{min} and I_{max} . Here the *product of inertia will be zero at these points*, Fig. A-16b.

To find the orientation of the major principal axis, determine by trigonometry the angle $2\theta_p$, *measured from the radius CA to the positive I axis*, Fig. A-16b. This angle represents twice the angle from the x axis to the axis of maximum moment of inertia I_{max} , Fig. A-16a. Both the angle on the circle, $2\theta_p$, and the angle on the area, θ_p , *must be measured in the same sense*, as shown in Fig. A-16. The minor axis is for minimum moment of inertia I_{min} , which is always perpendicular to the major axis defining I_{max} .

EXAMPLE A.6

Use Mohr's circle to determine the principal moments of inertia for the beam's cross-sectional area, shown in Fig. A-17a, with respect to principal axes passing through the centroid C .

SOLUTION

Compute I_x , I_y , I_{xy} . The moments of inertia and the product of inertia have been determined in Examples A.3 and A.4 with respect to the x , y axes shown in Fig. A-17a. The results are $I_x = 2.90(10^9) \text{ mm}^4$, $I_y = 5.60(10^9) \text{ mm}^4$, and $I_{xy} = -3.00(10^9) \text{ mm}^4$.

Construct the Circle. The I and I_{xy} axes are shown in Fig. A-17b. The center of the circle, C , lies at a distance $(I_x + I_y)/2 = (2.90 + 5.60)/2 = 4.25$ from the origin. When the reference point $A(2.90, -3.00)$ is connected to point C , the radius CA is determined from the shaded triangle CBA using the Pythagorean theorem:

$$CA = \sqrt{(1.35)^2 + (-3.00)^2} = 3.29$$

The circle is constructed in Fig. A-17c.

Principal Moments of Inertia. The circle intersects the I axis at points $(7.54, 0)$ and $(0.960, 0)$. Hence

$$I_{\max} = 7.54(10^9) \text{ mm}^4 \quad \text{Ans.}$$

$$I_{\min} = 0.960(10^9) \text{ mm}^4 \quad \text{Ans.}$$

As shown in Fig. A-17c, the angle $2\theta_{p_1}$ is determined from the circle by measuring *counterclockwise* from CA to the *positive* I axis. Hence,

$$2\theta_{p_1} = 180^\circ - \tan^{-1}\left(\frac{|BA|}{|BC|}\right) = 180^\circ - \tan^{-1}\left(\frac{3.00}{1.35}\right) = 114.2^\circ$$

$$\theta_{p_1} = 57.1^\circ$$

The major principal axis (for $I_{\max} = 7.54(10^9) \text{ mm}^4$) is therefore oriented at an angle $\theta_{p_1} = 57.1^\circ$, measured *counterclockwise*, from the *positive* x axis. The minor axis is perpendicular to this axis. The results are shown in Fig. A-17a.

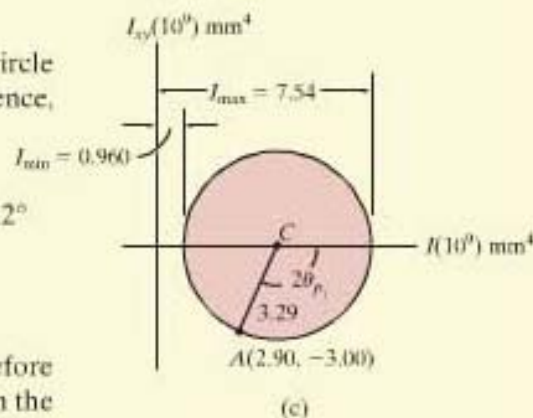
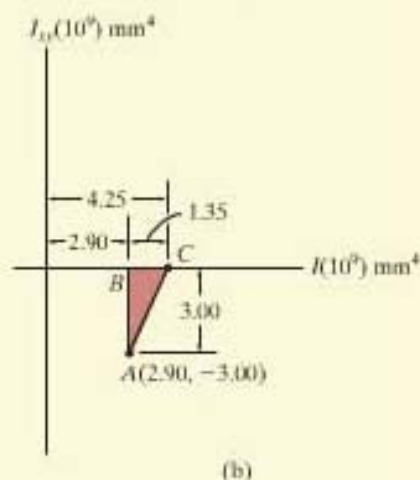
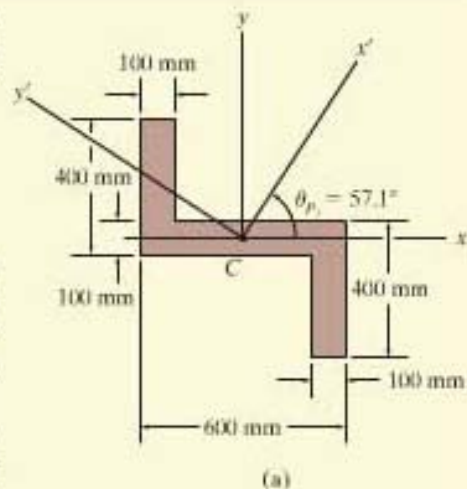


Fig. A-17

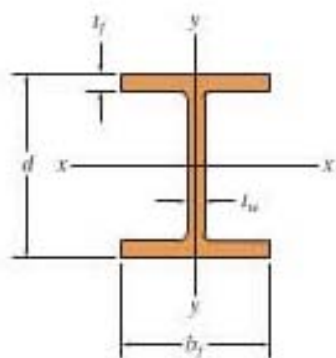
APPENDIX

B

Geometric Properties
of Structural Shapes

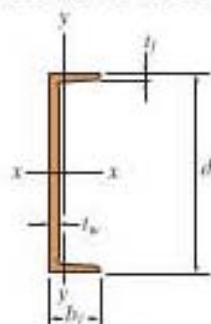
Wide-Flange Sections or W Shapes FPS Units

Designation	Area A	Depth d	Web thickness t_w	Flange		x - x axis			y - y axis		
				width	thickness	I	S	r	I	S	r
				b_f	t_f						
in. $\times$ lb/ft	in ²	in.	in.	in.	in.	in ⁴	in ³	in.	in ⁴	in ³	in.
W24 $\times$ 104	30.6	24.06	0.500	12.750	0.750	3100	258	10.1	259	40.7	2.91
W24 $\times$ 94	27.7	24.31	0.515	9.065	0.875	2700	222	9.87	109	24.0	1.98
W24 $\times$ 84	24.7	24.10	0.470	9.020	0.770	2370	196	9.79	94.4	20.9	1.95
W24 $\times$ 76	22.4	23.92	0.440	8.990	0.680	2100	176	9.69	82.5	18.4	1.92
W24 $\times$ 68	20.1	23.73	0.415	8.965	0.585	1830	154	9.55	70.4	15.7	1.87
W24 $\times$ 62	18.2	23.74	0.430	7.040	0.590	1550	131	9.23	34.5	9.80	1.38
W24 $\times$ 55	16.2	23.57	0.395	7.005	0.505	1350	114	9.11	29.1	8.30	1.34
W18 $\times$ 65	19.1	18.35	0.450	7.590	0.750	1070	117	7.49	54.8	14.4	1.69
W18 $\times$ 60	17.6	18.24	0.415	7.555	0.695	984	108	7.47	50.1	13.3	1.69
W18 $\times$ 55	16.2	18.11	0.390	7.530	0.630	890	98.3	7.41	44.9	11.9	1.67
W18 $\times$ 50	14.7	17.99	0.355	7.495	0.570	800	88.9	7.38	40.1	10.7	1.65
W18 $\times$ 46	13.5	18.06	0.360	6.060	0.605	712	78.8	7.25	22.5	7.43	1.29
W18 $\times$ 40	11.8	17.90	0.315	6.015	0.525	612	68.4	7.21	19.1	6.35	1.27
W18 $\times$ 35	10.3	17.70	0.300	6.000	0.425	510	57.6	7.04	15.3	5.12	1.22
W16 $\times$ 57	16.8	16.43	0.430	7.120	0.715	758	92.2	6.72	43.1	12.1	1.60
W16 $\times$ 50	14.7	16.26	0.380	7.070	0.630	659	81.0	6.68	37.2	10.5	1.59
W16 $\times$ 45	13.3	16.13	0.345	7.035	0.565	586	72.7	6.65	32.8	9.34	1.57
W16 $\times$ 36	10.6	15.86	0.295	6.985	0.430	448	56.5	6.51	24.5	7.00	1.52
W16 $\times$ 31	9.12	15.88	0.275	5.525	0.440	375	47.2	6.41	12.4	4.49	1.17
W16 $\times$ 26	7.68	15.69	0.250	5.500	0.345	301	38.4	6.26	9.59	3.49	1.12
W14 $\times$ 53	15.6	13.92	0.370	8.060	0.660	541	77.8	5.89	57.7	14.3	1.92
W14 $\times$ 43	12.6	13.66	0.305	7.995	0.530	428	62.7	5.82	45.2	11.3	1.89
W14 $\times$ 38	11.2	14.10	0.310	6.770	0.515	385	54.6	5.87	26.7	7.88	1.55
W14 $\times$ 34	10.0	13.98	0.285	6.745	0.455	340	48.6	5.83	23.3	6.91	1.53
W14 $\times$ 30	8.85	13.84	0.270	6.730	0.385	291	42.0	5.73	19.6	5.82	1.49
W14 $\times$ 26	7.69	13.91	0.255	5.025	0.420	245	35.3	5.65	8.91	3.54	1.08
W14 $\times$ 22	6.49	13.74	0.230	5.000	0.335	199	29.0	5.54	7.00	2.80	1.04



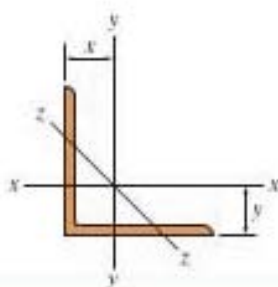
Wide-Flange Sections or W Shapes FPS Units

Designation	Area A	Depth d	Web thickness t_w	Flange		x-x axis			y-y axis		
				width b_f	thickness t_f	I	S	r	I	S	r
in. × lb/ft	in ²	in.	in.	in.	in.	in ⁴	in ³	in.	in ⁴	in ³	in.
W12 × 87	25.6	12.53	0.515	12.125	0.810	740	118	5.38	241	39.7	3.07
W12 × 50	14.7	12.19	0.370	8.080	0.640	394	64.7	5.18	56.3	13.9	1.96
W12 × 45	13.2	12.06	0.335	8.045	0.575	350	58.1	5.15	50.0	12.4	1.94
W12 × 26	7.65	12.22	0.230	6.490	0.380	204	33.4	5.17	17.3	5.34	1.51
W12 × 22	6.48	12.31	0.260	4.030	0.425	156	25.4	4.91	4.66	2.31	0.847
W12 × 16	4.71	11.99	0.220	3.990	0.265	103	17.1	4.67	2.82	1.41	0.773
W12 × 14	4.16	11.91	0.200	3.970	0.225	88.6	14.9	4.62	2.36	1.19	0.753
W10 × 100	29.4	11.10	0.680	10.340	1.120	623	112	4.60	207	40.0	2.65
W10 × 54	15.8	10.09	0.370	10.030	0.615	303	60.0	4.37	103	20.6	2.56
W10 × 45	13.3	10.10	0.350	8.020	0.620	248	49.1	4.32	53.4	13.3	2.01
W10 × 39	11.5	9.92	0.315	7.985	0.530	209	42.1	4.27	45.0	11.3	1.98
W10 × 30	8.84	10.47	0.300	5.810	0.510	170	32.4	4.38	16.7	5.75	1.37
W10 × 19	5.62	10.24	0.250	4.020	0.395	96.3	18.8	4.14	4.29	2.14	0.874
W10 × 15	4.41	9.99	0.230	4.000	0.270	68.9	13.8	3.95	2.89	1.45	0.810
W10 × 12	3.54	9.87	0.190	3.960	0.210	53.8	10.9	3.90	2.18	1.10	0.785
W8 × 67	19.7	9.00	0.570	8.280	0.935	272	60.4	3.72	88.6	21.4	2.12
W8 × 58	17.1	8.75	0.510	8.220	0.810	228	52.0	3.65	75.1	18.3	2.10
W8 × 48	14.1	8.50	0.400	8.110	0.685	184	43.3	3.61	60.9	15.0	2.08
W8 × 40	11.7	8.25	0.360	8.070	0.560	146	35.5	3.53	49.1	12.2	2.04
W8 × 31	9.13	8.00	0.285	7.995	0.435	110	27.5	3.47	37.1	9.27	2.02
W8 × 24	7.08	7.93	0.245	6.495	0.400	82.8	20.9	3.42	18.3	5.63	1.61
W8 × 15	4.44	8.11	0.245	4.015	0.315	48.0	11.8	3.29	3.41	1.70	0.876
W6 × 25	7.34	6.38	0.320	6.080	0.455	53.4	16.7	2.70	17.1	5.61	1.52
W6 × 20	5.87	6.20	0.260	6.020	0.365	41.4	13.4	2.66	13.3	4.41	1.50
W6 × 16	4.74	6.28	0.260	4.030	0.405	32.1	10.2	2.60	4.43	2.20	0.966
W6 × 15	4.43	5.99	0.230	5.990	0.260	29.1	9.72	2.56	9.32	3.11	1.46
W6 × 12	3.55	6.03	0.230	4.000	0.280	22.1	7.31	2.49	2.99	1.50	0.918
W6 × 9	2.68	5.90	0.170	3.940	0.215	16.4	5.56	2.47	2.19	1.11	0.905



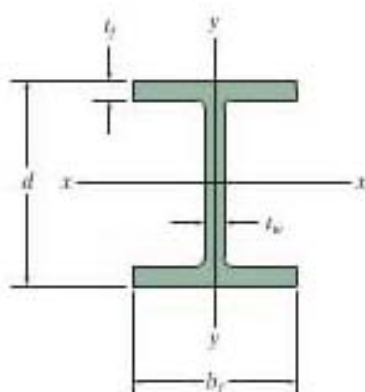
American Standard Channels or C Shapes FPS Units

Designation	Area A	Depth d	Web thickness t _w		Flange				x-x axis			y-y axis		
					width b _f		thickness t _f		I	S	r	I	S	r
in. × lb/ft	in ²	in.	in.		in.		in.		in ⁴	in ³	in.	in ⁴	in ³	in.
C15 × 50	14.7	15.00	0.716	11/16	3.716	3 3/4	0.650	5/8	404	53.8	5.24	11.0	3.78	0.86
C15 × 40	11.8	15.00	0.520	1/2	3.520	3 1/2	0.650	5/8	349	46.5	5.44	9.23	3.37	0.886
C15 × 33.9	9.96	15.00	0.400	3/8	3.400	3 3/8	0.650	5/8	315	42.0	5.62	8.13	3.11	0.904
C12 × 30	8.82	12.00	0.510	1/2	3.170	3 1/8	0.501	1/2	162	27.0	4.29	5.14	2.06	0.763
C12 × 25	7.35	12.00	0.387	3/8	3.047	3	0.501	1/2	144	24.1	4.43	4.47	1.88	0.780
C12 × 20.7	6.09	12.00	0.282	5/16	2.942	3	0.501	1/2	129	21.5	4.61	3.88	1.73	0.799
C10 × 30	8.82	10.00	0.673	11/16	3.033	3	0.436	7/16	103	20.7	3.42	3.94	1.65	0.669
C10 × 25	7.35	10.00	0.526	1/2	2.886	2 7/8	0.436	7/16	91.2	18.2	3.52	3.36	1.48	0.676
C10 × 20	5.88	10.00	0.379	3/8	2.739	2 3/4	0.436	7/16	78.9	15.8	3.66	2.81	1.32	0.692
C10 × 15.3	4.49	10.00	0.240	1/4	2.600	2 5/8	0.436	7/16	67.4	13.5	3.87	2.28	1.16	0.713
C9 × 20	5.88	9.00	0.448	7/16	2.648	2 5/8	0.413	7/16	60.9	13.5	3.22	2.42	1.17	0.642
C9 × 15	4.41	9.00	0.285	5/16	2.485	2 1/2	0.413	7/16	51.0	11.3	3.40	1.93	1.01	0.66
C9 × 13.4	3.94	9.00	0.233	1/4	2.433	2 3/8	0.413	7/16	47.9	10.6	3.48	1.76	0.962	0.669
C8 × 18.75	5.51	8.00	0.487	1/2	2.527	2 1/2	0.390	3/8	44.0	11.0	2.82	1.98	1.01	0.599
C8 × 13.75	4.04	8.00	0.303	5/16	2.343	2 3/8	0.390	3/8	36.1	9.03	2.99	1.53	0.854	0.615
C8 × 11.5	3.38	8.00	0.220	1/4	2.260	2 1/4	0.390	3/8	32.6	8.14	3.11	1.32	0.781	0.625
C7 × 14.75	4.33	7.00	0.419	7/16	2.299	2 1/4	0.366	3/8	27.2	7.78	2.51	1.38	0.779	0.564
C7 × 12.25	3.60	7.00	0.314	5/16	2.194	2 1/4	0.366	3/8	24.2	6.93	2.60	1.17	0.703	0.571
C7 × 9.8	2.87	7.00	0.210	3/16	2.090	2 1/8	0.366	3/8	21.3	6.08	2.72	0.968	0.625	0.581
C6 × 13	3.83	6.00	0.437	7/16	2.157	2 1/8	0.343	5/16	17.4	5.80	2.13	1.05	0.642	0.525
C6 × 10.5	3.09	6.00	0.314	5/16	2.034	2	0.343	5/16	15.2	5.06	2.22	0.866	0.564	0.529
C6 × 8.2	2.40	6.00	0.200	3/16	1.920	1 7/8	0.343	5/16	13.1	4.38	2.34	0.693	0.492	0.537
C5 × 9	2.64	5.00	0.325	5/16	1.885	1 7/8	0.320	5/16	8.90	3.56	1.83	0.632	0.450	0.489
C5 × 6.7	1.97	5.00	0.190	3/16	1.750	1 3/4	0.320	5/16	7.49	3.00	1.95	0.479	0.378	0.493
C4 × 7.25	2.13	4.00	0.321	5/16	1.721	1 3/4	0.296	5/16	4.59	2.29	1.47	0.433	0.343	0.450
C4 × 5.4	1.59	4.00	0.184	3/16	1.584	1 3/8	0.296	5/16	3.85	1.93	1.56	0.319	0.283	0.449
C3 × 6	1.76	3.00	0.356	3/8	1.596	1 5/8	0.273	1/4	2.07	1.38	1.08	0.305	0.268	0.416
C3 × 5	1.47	3.00	0.258	1/4	1.498	1 1/2	0.273	1/4	1.85	1.24	1.12	0.247	0.233	0.410
C3 × 4.1	1.21	3.00	0.170	3/16	1.410	1 3/4	0.273	1/4	1.66	1.10	1.17	0.197	0.202	0.404



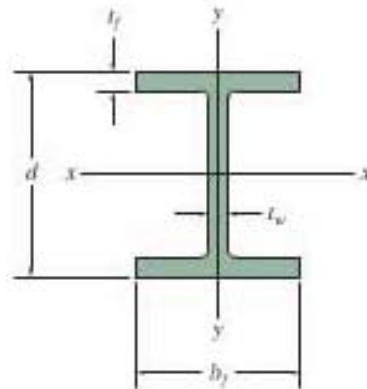
Angles Having Equal Legs FPS Units

Size and thickness	Weight per foot	Area A	x-x axis				y-y axis				z-z axis
			I	S	r	y	I	S	r	x	r
in.	lb	in ²	in ⁴	in ³	in.	in.	in ⁴	in ³	in.	in.	in.
L8 × 8 × 1	51.0	15.0	89.0	15.8	2.44	2.37	89.0	15.8	2.44	2.37	1.56
L8 × 8 × 3/4	38.9	11.4	69.7	12.2	2.47	2.28	69.7	12.2	2.47	2.28	1.58
L8 × 8 × 1/2	26.4	7.75	48.6	8.36	2.50	2.19	48.6	8.36	2.50	2.19	1.59
L6 × 6 × 1	37.4	11.0	35.5	8.57	1.80	1.86	35.5	8.57	1.80	1.86	1.17
L6 × 6 × 3/4	28.7	8.44	28.2	6.66	1.83	1.78	28.2	6.66	1.83	1.78	1.17
L6 × 6 × 1/2	19.6	5.75	19.9	4.61	1.86	1.68	19.9	4.61	1.86	1.68	1.18
L6 × 6 × 3/8	14.9	4.36	15.4	3.53	1.88	1.64	15.4	3.53	1.88	1.64	1.19
L5 × 5 × 3/4	23.6	6.94	15.7	4.53	1.51	1.52	15.7	4.53	1.51	1.52	0.975
L5 × 5 × 1/2	16.2	4.75	11.3	3.16	1.54	1.43	11.3	3.16	1.54	1.43	0.983
L5 × 5 × 3/8	12.3	3.61	8.74	2.42	1.56	1.39	8.74	2.42	1.56	1.39	0.990
L4 × 4 × 3/4	18.5	5.44	7.67	2.81	1.19	1.27	7.67	2.81	1.19	1.27	0.778
L4 × 4 × 1/2	12.8	3.75	5.56	1.97	1.22	1.18	5.56	1.97	1.22	1.18	0.782
L4 × 4 × 3/8	9.8	2.86	4.36	1.52	1.23	1.14	4.36	1.52	1.23	1.14	0.788
L4 × 4 × 1/4	6.6	1.94	3.04	1.05	1.25	1.09	3.04	1.05	1.25	1.09	0.795
L3 1/2 × 3 1/2 × 1/2	11.1	3.25	3.64	1.49	1.06	1.06	3.64	1.49	1.06	1.06	0.683
L3 1/2 × 3 1/2 × 3/4	8.5	2.48	2.87	1.15	1.07	1.01	2.87	1.15	1.07	1.01	0.687
L3 1/2 × 3 1/2 × 1/4	5.8	1.69	2.01	0.794	1.09	0.968	2.01	0.794	1.09	0.968	0.694
L3 × 3 × 1/2	9.4	2.75	2.22	1.07	0.898	0.932	2.22	1.07	0.898	0.932	0.584
L3 × 3 × 3/8	7.2	2.11	1.76	0.833	0.913	0.888	1.76	0.833	0.913	0.888	0.587
L3 × 3 × 1/4	4.9	1.44	1.24	0.577	0.930	0.842	1.24	0.577	0.930	0.842	0.592
L2 1/2 × 2 1/2 × 1/2	7.7	2.25	1.23	0.724	0.739	0.806	1.23	0.724	0.739	0.806	0.487
L2 1/2 × 2 1/2 × 3/8	5.9	1.73	0.984	0.566	0.753	0.762	0.984	0.566	0.753	0.762	0.487
L2 1/2 × 2 1/2 × 1/4	4.1	1.19	0.703	0.394	0.769	0.717	0.703	0.394	0.769	0.717	0.491
L2 × 2 × 3/8	4.7	1.36	0.479	0.351	0.594	0.636	0.479	0.351	0.594	0.636	0.389
L2 × 2 × 1/4	3.19	0.938	0.348	0.247	0.609	0.592	0.348	0.247	0.609	0.592	0.391
L2 × 2 × 1/8	1.65	0.484	0.190	0.131	0.626	0.546	0.190	0.131	0.626	0.546	0.398



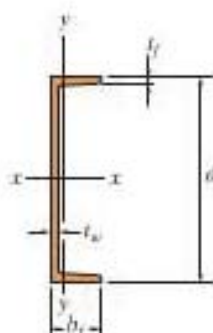
Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange		x - x axis			y - y axis		
				width b_f	thickness t_f	I	S	r	I	S	r
mm $\times$ kg/m	mm ²	mm	mm	mm	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm
W610 $\times$ 155	19 800	611	12.70	324.0	19.0	1 290	4 220	255	108	667	73.9
W610 $\times$ 140	17 900	617	13.10	230.0	22.2	1 120	3 630	250	45.1	392	50.2
W610 $\times$ 125	15 900	612	11.90	229.0	19.6	985	3 220	249	39.3	343	49.7
W610 $\times$ 113	14 400	608	11.20	228.0	17.3	875	2 880	247	34.3	301	48.8
W610 $\times$ 101	12 900	603	10.50	228.0	14.9	764	2 530	243	29.5	259	47.8
W610 $\times$ 92	11 800	603	10.90	179.0	15.0	646	2 140	234	14.4	161	34.9
W610 $\times$ 82	10 500	599	10.00	178.0	12.8	560	1 870	231	12.1	136	33.9
W460 $\times$ 97	12 300	466	11.40	193.0	19.0	445	1 910	190	22.8	236	43.1
W460 $\times$ 89	11 400	463	10.50	192.0	17.7	410	1 770	190	20.9	218	42.8
W460 $\times$ 82	10 400	460	9.91	191.0	16.0	370	1 610	189	18.6	195	42.3
W460 $\times$ 74	9 460	457	9.02	190.0	14.5	333	1 460	188	16.6	175	41.9
W460 $\times$ 68	8 730	459	9.14	154.0	15.4	297	1 290	184	9.41	122	32.8
W460 $\times$ 60	7 590	455	8.00	153.0	13.3	255	1 120	183	7.96	104	32.4
W460 $\times$ 52	6 640	450	7.62	152.0	10.8	212	942	179	6.34	83.4	30.9
W410 $\times$ 85	10 800	417	10.90	181.0	18.2	315	1 510	171	18.0	199	40.8
W410 $\times$ 74	9 510	413	9.65	180.0	16.0	275	1 330	170	15.6	173	40.5
W410 $\times$ 67	8 560	410	8.76	179.0	14.4	245	1 200	169	13.8	154	40.2
W410 $\times$ 53	6 820	403	7.49	177.0	10.9	186	923	165	10.1	114	38.5
W410 $\times$ 46	5 890	403	6.99	140.0	11.2	156	774	163	5.14	73.4	29.5
W410 $\times$ 39	4 960	399	6.35	140.0	8.8	126	632	159	4.02	57.4	28.5
W360 $\times$ 79	10 100	354	9.40	205.0	16.8	227	1 280	150	24.2	236	48.9
W360 $\times$ 64	8 150	347	7.75	203.0	13.5	179	1 030	148	18.8	185	48.0
W360 $\times$ 57	7 200	358	7.87	172.0	13.1	160	894	149	11.1	129	39.3
W360 $\times$ 51	6 450	355	7.24	171.0	11.6	141	794	148	9.68	113	38.7
W360 $\times$ 45	5 710	352	6.86	171.0	9.8	121	688	146	8.16	95.4	37.8
W360 $\times$ 39	4 960	353	6.48	128.0	10.7	102	578	143	3.75	58.6	27.5
W360 $\times$ 33	4 190	349	5.84	127.0	8.5	82.9	475	141	2.91	45.8	26.4



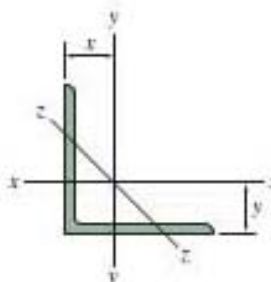
Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t _w	Flange		x-x axis			y-y axis		
				width b _f	thickness t _f	I	S	r	I	S	r
mm × kg/m	mm ²	mm	mm	mm	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm
W310 × 129	16 500	318	13.10	308.0	20.6	308	1940	137	100	649	77.8
W310 × 74	9 480	310	9.40	205.0	16.3	165	1060	132	23.4	228	49.7
W310 × 67	8 530	306	8.51	204.0	14.6	145	948	130	20.7	203	49.3
W310 × 39	4 930	310	5.84	165.0	9.7	84.8	547	131	7.23	87.6	38.3
W310 × 33	4 180	313	6.60	102.0	10.8	65.0	415	125	1.92	37.6	21.4
W310 × 24	3 040	305	5.59	101.0	6.7	42.8	281	119	1.16	23.0	19.5
W310 × 21	2 680	303	5.08	101.0	5.7	37.0	244	117	0.986	19.5	19.2
W250 × 149	19 000	282	17.30	263.0	28.4	259	1840	117	86.2	656	67.4
W250 × 80	10 200	256	9.40	255.0	15.6	126	984	111	43.1	338	65.0
W250 × 67	8 560	257	8.89	204.0	15.7	104	809	110	22.2	218	50.9
W250 × 58	7 400	252	8.00	203.0	13.5	87.3	693	109	18.8	185	50.4
W250 × 45	5 700	266	7.62	148.0	13.0	71.1	535	112	7.03	95	35.1
W250 × 28	3 620	260	6.35	102.0	10.0	39.9	307	105	1.78	34.9	22.2
W250 × 22	2 850	254	5.84	102.0	6.9	28.8	227	101	1.22	23.9	20.7
W250 × 18	2 280	251	4.83	101.0	5.3	22.5	179	99.3	0.919	18.2	20.1
W200 × 100	12 700	229	14.50	210.0	23.7	113	987	94.3	36.6	349	53.7
W200 × 86	11 000	222	13.00	209.0	20.6	94.7	853	92.8	31.4	300	53.4
W200 × 71	9 100	216	10.20	206.0	17.4	76.6	709	91.7	25.4	247	52.8
W200 × 59	7 580	210	9.14	205.0	14.2	61.2	583	89.9	20.4	199	51.9
W200 × 46	5 890	203	7.24	203.0	11.0	45.5	448	87.9	15.3	151	51.0
W200 × 36	4 570	201	6.22	165.0	10.2	34.4	342	86.8	7.64	92.6	40.9
W200 × 22	2 860	206	6.22	102.0	8.0	20.0	194	83.6	1.42	27.8	22.3
W150 × 37	4 730	162	8.13	154.0	11.6	22.2	274	68.5	7.07	91.8	38.7
W150 × 30	3 790	157	6.60	153.0	9.3	17.1	218	67.2	5.54	72.4	38.2
W150 × 22	2 860	152	5.84	152.0	6.6	12.1	159	65.0	3.87	50.9	36.8
W150 × 24	3 060	160	6.60	102.0	10.3	13.4	168	66.2	1.83	35.9	24.5
W150 × 18	2 290	153	5.84	102.0	7.1	9.19	120	63.3	1.26	24.7	23.5
W150 × 14	1 730	150	4.32	100.0	5.5	6.84	91.2	62.9	0.912	18.2	23.0



American Standard Channels or C Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange		$x-x$ axis			$y-y$ axis		
				width	thickness	I	S	r	I	S	r
				b_f	t_f						
mm $\times$ kg/m	mm ²	mm	mm	mm	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm	10 ⁶ mm ⁴	10 ³ mm ³	mm
C380 $\times$ 74	9 480	381.0	18.20	94.4	16.50	168	882	133	4.58	61.8	22.0
C380 $\times$ 60	7 610	381.0	13.20	89.4	16.50	145	761	138	3.84	55.1	22.5
C380 $\times$ 50	6 430	381.0	10.20	86.4	16.50	131	688	143	3.38	50.9	22.9
C310 $\times$ 45	5 690	305.0	13.00	80.5	12.70	67.4	442	109	2.14	33.8	19.4
C310 $\times$ 37	4 740	305.0	9.83	77.4	12.70	59.9	393	112	1.86	30.9	19.8
C310 $\times$ 31	3 930	305.0	7.16	74.7	12.70	53.7	352	117	1.61	28.3	20.2
C250 $\times$ 45	5 690	254.0	17.10	77.0	11.10	42.9	338	86.8	1.61	27.1	17.0
C250 $\times$ 37	4 740	254.0	13.40	73.3	11.10	38.0	299	89.5	1.40	24.3	17.2
C250 $\times$ 30	3 790	254.0	9.63	69.6	11.10	32.8	258	93.0	1.17	21.6	17.6
C250 $\times$ 23	2 900	254.0	6.10	66.0	11.10	28.1	221	98.4	0.949	19.0	18.1
C230 $\times$ 30	3 790	229.0	11.40	67.3	10.50	25.3	221	81.7	1.01	19.2	16.3
C230 $\times$ 22	2 850	229.0	7.24	63.1	10.50	21.2	185	86.2	0.803	16.7	16.8
C230 $\times$ 20	2 540	229.0	5.92	61.8	10.50	19.9	174	88.5	0.733	15.8	17.0
C200 $\times$ 28	3 550	203.0	12.40	64.2	9.90	18.3	180	71.8	0.824	16.5	15.2
C200 $\times$ 20	2 610	203.0	7.70	59.5	9.90	15.0	148	75.8	0.637	14.0	15.6
C200 $\times$ 17	2 180	203.0	5.59	57.4	9.90	13.6	134	79.0	0.549	12.8	15.9
C180 $\times$ 22	2 790	178.0	10.60	58.4	9.30	11.3	127	63.6	0.574	12.8	14.3
C180 $\times$ 18	2 320	178.0	7.98	55.7	9.30	10.1	113	66.0	0.487	11.5	14.5
C180 $\times$ 15	1 850	178.0	5.33	53.1	9.30	8.87	99.7	69.2	0.403	10.2	14.8
C150 $\times$ 19	2 470	152.0	11.10	54.8	8.70	7.24	95.3	54.1	0.437	10.5	13.3
C150 $\times$ 16	1 990	152.0	7.98	51.7	8.70	6.33	83.3	56.4	0.360	9.22	13.5
C150 $\times$ 12	1 550	152.0	5.08	48.8	8.70	5.45	71.7	59.3	0.288	8.04	13.6
C130 $\times$ 13	1 700	127.0	8.25	47.9	8.10	3.70	58.3	46.7	0.263	7.35	12.4
C130 $\times$ 10	1 270	127.0	4.83	44.5	8.10	3.12	49.1	49.6	0.199	6.18	12.5
C100 $\times$ 11	1 370	102.0	8.15	43.7	7.50	1.91	37.5	37.3	0.180	5.62	11.5
C100 $\times$ 8	1 030	102.0	4.67	40.2	7.50	1.60	31.4	39.4	0.133	4.65	11.4
C75 $\times$ 9	1 140	76.2	9.04	40.5	6.90	0.862	22.6	27.5	0.127	4.39	10.6
C75 $\times$ 7	948	76.2	6.55	38.0	6.90	0.770	20.2	28.5	0.103	3.83	10.4
C75 $\times$ 6	781	76.2	4.32	35.8	6.90	0.691	18.1	29.8	0.082	3.32	10.2

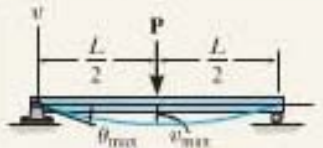
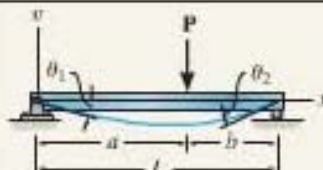
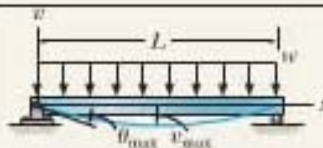
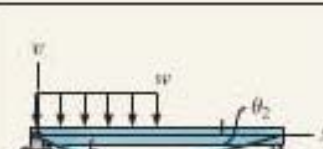
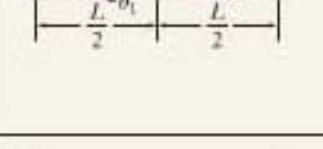


Angles Having Equal Legs SI Units

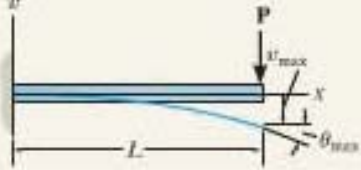
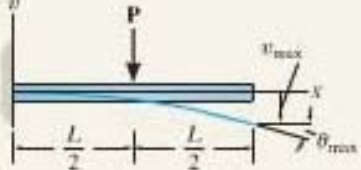
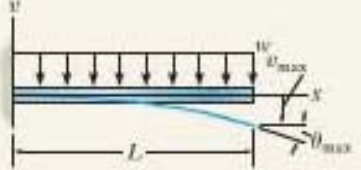
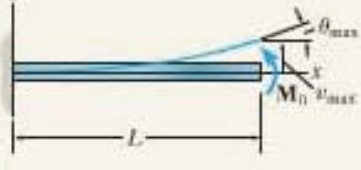
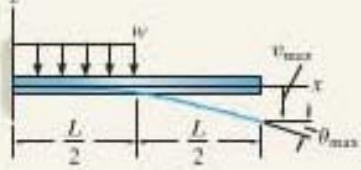
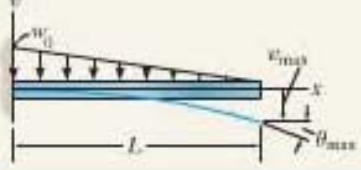
Size and thickness	Mass per Meter	Area	x-x axis				y-y axis				z-z axis	
			<i>l</i>	<i>S</i>	<i>r</i>	<i>y</i>	<i>l</i>	<i>S</i>	<i>r</i>	<i>x</i>	<i>r</i>	<i>r</i>
mm	kg	mm ²	10 ⁶ mm ⁴	10 ⁶ mm ³	mm	mm	10 ⁶ mm ⁴	10 ⁶ mm ³	mm	mm	mm	mm
L203 × 203 × 25.4	75.9	9 680	36.9	258	61.7	60.1	36.9	258	61.7	60.1	39.6	39.6
L203 × 203 × 19.0	57.9	7 380	28.9	199	62.6	57.8	28.9	199	62.6	57.8	40.1	40.1
L203 × 203 × 12.7	39.3	5 000	20.2	137	63.6	55.5	20.2	137	63.6	55.5	40.4	40.4
L152 × 152 × 25.4	55.7	7 100	14.6	139	45.3	47.2	14.6	139	45.3	47.2	29.7	29.7
L152 × 152 × 19.0	42.7	5 440	11.6	108	46.2	45.0	11.6	108	46.2	45.0	29.7	29.7
L152 × 152 × 12.7	29.2	3 710	8.22	75.1	47.1	42.7	8.22	75.1	47.1	42.7	30.0	30.0
L152 × 152 × 9.5	22.2	2 810	6.35	57.4	47.5	41.5	6.35	57.4	47.5	41.5	30.2	30.2
L127 × 127 × 19.0	35.1	4 480	6.54	73.9	38.2	38.7	6.54	73.9	38.2	38.7	24.8	24.8
L127 × 127 × 12.7	24.1	3 060	4.68	51.7	39.1	36.4	4.68	51.7	39.1	36.4	25.0	25.0
L127 × 127 × 9.5	18.3	2 330	3.64	39.7	39.5	35.3	3.64	39.7	39.5	35.3	25.1	25.1
L102 × 102 × 19.0	27.5	3 510	3.23	46.4	30.3	32.4	3.23	46.4	30.3	32.4	19.8	19.8
L102 × 102 × 12.7	19.0	2 420	2.34	32.6	31.1	30.2	2.34	32.6	31.1	30.2	19.9	19.9
L102 × 102 × 9.5	14.6	1 840	1.84	25.3	31.6	29.0	1.84	25.3	31.6	29.0	20.0	20.0
L102 × 102 × 6.4	9.8	1 250	1.28	17.3	32.0	27.9	1.28	17.3	32.0	27.9	20.2	20.2
L89 × 89 × 12.7	16.5	2 100	1.52	24.5	26.9	26.9	1.52	24.5	26.9	26.9	17.3	17.3
L89 × 89 × 9.5	12.6	1 600	1.20	19.0	27.4	25.8	1.20	19.0	27.4	25.8	17.4	17.4
L89 × 89 × 6.4	8.6	1 090	0.840	13.0	27.8	24.6	0.840	13.0	27.8	24.6	17.6	17.6
L76 × 76 × 12.7	14.0	1 770	0.915	17.5	22.7	23.6	0.915	17.5	22.7	23.6	14.8	14.8
L76 × 76 × 9.5	10.7	1 360	0.726	13.6	23.1	22.5	0.726	13.6	23.1	22.5	14.9	14.9
L76 × 76 × 6.4	7.3	927	0.514	9.39	23.5	21.3	0.514	9.39	23.5	21.3	15.0	15.0
L64 × 64 × 12.7	11.5	1 450	0.524	12.1	19.0	20.6	0.524	12.1	19.0	20.6	12.4	12.4
L64 × 64 × 9.5	8.8	1 120	0.420	9.46	19.4	19.5	0.420	9.46	19.4	19.5	12.4	12.4
L64 × 64 × 6.4	6.1	766	0.300	6.59	19.8	18.2	0.300	6.59	19.8	18.2	12.5	12.5
L51 × 51 × 9.5	7.0	877	0.202	5.82	15.2	16.2	0.202	5.82	15.2	16.2	9.88	9.88
L51 × 51 × 6.4	4.7	605	0.146	4.09	15.6	15.1	0.146	4.09	15.6	15.1	9.93	9.93
L51 × 51 × 3.2	2.5	312	0.080	2.16	16.0	13.9	0.080	2.16	16.0	13.9	10.1	10.1

Slopes and Deflections of Beams

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{\max} = \frac{-PL^2}{16EI}$	$v_{\max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \leq x \leq L/2$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v \Big _{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \leq x \leq a$
	$\theta_1 = \frac{-M_0L}{6EI}$ $\theta_2 = \frac{M_0L}{3EI}$	$v_{\max} = \frac{-M_0L^2}{\sqrt{243}EI}$ at $x = 0.5774L$	$v = \frac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{\max} = \frac{-wL^3}{24EI}$	$v_{\max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v \Big _{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{\max} = -0.006563 \frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \leq x \leq L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \leq x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{\max} = -0.00652 \frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{\max} = \frac{-PL^2}{2EI}$	$v_{\max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{\max} = \frac{-PL^2}{8EI}$	$v_{\max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{6EI}(\frac{1}{2}L - x) \quad 0 \leq x \leq L/2$ $v = \frac{-PL^2}{24EI}(3x - \frac{1}{2}L) \quad L/2 \leq x \leq L$
	$\theta_{\max} = \frac{-wL^3}{6EI}$	$v_{\max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{\max} = \frac{M_0L}{EI}$	$v_{\max} = \frac{M_0L^2}{2EI}$	$v = \frac{M_0x^2}{2EI}$
	$\theta_{\max} = \frac{-wL^3}{48EI}$	$v_{\max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 2Lx + \frac{3}{2}L^2) \quad 0 \leq x \leq L/2$ $v = \frac{-wL^3}{192EI}(4x - L/2) \quad L/2 \leq x \leq L$
	$\theta_{\max} = \frac{-w_0L^3}{24EI}$	$v_{\max} = \frac{-w_0L^4}{30EI}$	$v = \frac{-w_0x^2}{120EI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

Fundamental Problems Partial Solutions and Answers

Chapter 1

F1-1	Entire beam: $\zeta + \Sigma M_B = 0;$	$60 - 10(2) - A_y(2) = 0$	$A_y = 20 \text{ kN}$	
	Left segment: $\rightarrow \Sigma F_x = 0;$	$N_C = 0$		<i>Ans.</i>
	$+\uparrow \Sigma F_y = 0;$	$20 - V_C = 0$	$V_C = 20 \text{ kN}$	<i>Ans.</i>
	$\zeta + \Sigma M_C = 0;$	$M_C + 60 - 20(1) = 0$	$M_C = -40 \text{ kN} \cdot \text{m}$	<i>Ans.</i>
F1-2	Entire beam: $\downarrow + \Sigma M_A = 0;$	$B_y(3) - 100(1.5)(0.75) - 200(1.5)(2.25) = 0$ $B_y = 262.5 \text{ N}$		
	Right segment: $\rightarrow \Sigma F_x = 0;$	$N_C = 0$		<i>Ans.</i>
	$+\uparrow \Sigma F_y = 0;$	$V_C + 262.5 - 200(1.5) = 0$	$V_C = 37.5 \text{ N}$	<i>Ans.</i>
	$\zeta + \Sigma M_C = 0;$	$262.5(1.5) - 200(1.5)(0.75) - M_C = 0$ $M_C = 169 \text{ N} \cdot \text{m}$		<i>Ans.</i>
F1-3	Entire beam: $\rightarrow \Sigma F_x = 0;$	$B_x = 0$		
	$\downarrow + \Sigma M_A = 0;$	$20(2)(1) - B_y(4) = 0$	$B_y = 10 \text{ kN}$	
	Right segment: $\rightarrow \Sigma F_x = 0;$	$N_C = 0$		<i>Ans.</i>
	$+\uparrow \Sigma F_y = 0;$	$V_C - 10 = 0$	$V_C = 10 \text{ kN}$	<i>Ans.</i>
	$\zeta + \Sigma M_C = 0;$	$-M_C - 10(2) = 0$	$M_C = -20 \text{ kN} \cdot \text{m}$	<i>Ans.</i>
F1-4	Entire beam: $\zeta + \Sigma M_B = 0;$	$\frac{1}{2}(10)(3)(2) + 10(3)(4.5) - A_y(6) = 0$	$A_y = 27.5 \text{ kN}$	
	Left segment: $\rightarrow \Sigma F_x = 0;$	$N_C = 0$		<i>Ans.</i>
	$+\uparrow \Sigma F_y = 0;$	$27.5 - 10(3) - V_C = 0$	$V_C = -2.5 \text{ kN}$	<i>Ans.</i>
	$\zeta + \Sigma M_C = 0;$	$M_C + 10(3)(1.5) - 27.5(3) = 0$	$M_C = 37.5 \text{ kN} \cdot \text{m}$	<i>Ans.</i>
F1-5	Entire beam: $\rightarrow \Sigma F_x = 0;$	$A_x = 0$		
	$\zeta + \Sigma M_B = 0;$	$300(6)(3) - \frac{1}{2}(300)(3)(1) - A_y(6) = 0$	$A_y = 825 \text{ lb}$	
	Left segment: $\rightarrow \Sigma F_x = 0;$	$N_C = 0$		<i>Ans.</i>
	$+\uparrow \Sigma F_y = 0;$	$825 - 300(3) - V_C = 0$	$V_C = -75 \text{ lb}$	<i>Ans.</i>
	$\zeta + \Sigma M_C = 0;$	$M_C + 300(3)(1.5) - 825(3) = 0$	$M_C = 1125 \text{ lb} \cdot \text{ft}$	<i>Ans.</i>

F1-6 Entire beam:

$$\downarrow + \Sigma M_A = 0; \quad F_{BD} \left(\frac{3}{5} \right) (4) - 5(6)(3) = 0 \quad F_{BD} = 37.5 \text{ kN}$$

$$\rightarrow \Sigma F_x = 0; \quad 37.5 \left(\frac{4}{5} \right) - A_x = 0 \quad A_x = 30 \text{ kN}$$

$$+\uparrow \Sigma F_y = 0; \quad A_y + 37.5 \left(\frac{3}{5} \right) - 5(6) = 0 \quad A_y = 7.5 \text{ kN}$$

Left segment:

$$\rightarrow \Sigma F_x = 0; \quad N_C - 30 = 0 \quad N_C = 30 \text{ kN} \quad \text{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \quad 7.5 - 5(2) - V_C = 0 \quad V_C = -2.5 \text{ kN} \quad \text{Ans.}$$

$$\downarrow + \Sigma M_C = 0; \quad M_C + 5(2)(1) - 7.5(2) = 0 \quad M_C = 5 \text{ kN} \cdot \text{m} \quad \text{Ans.}$$

F1-7 Beam:

$$\Sigma M_A = 0; T_{CD} = 2w$$

$$\Sigma F_y = 0; T_{AB} = w$$

Rod AB:

$$\sigma = \frac{P}{A}; 300(10^3) = \frac{w}{10^3}$$

$$w = 3 \text{ N/m}$$

Rod CD:

$$\sigma = \frac{P}{A}; 300(10^3) = \frac{2w}{15}$$

$$w = 2.25 \text{ N/m}$$

F1-8 $A = \pi(0.1^2 - 0.08^2) = 3.6(10^{-3})\pi \text{ m}^2$

$$\sigma_{\text{avg}} = \frac{P}{A} = \frac{300(10^3)}{3.6(10^{-3})\pi} = 26.5 \text{ MPa}$$

F1-9 $A = 3[4(1)] = 12 \text{ in}^2$

$$\sigma_{\text{avg}} = \frac{P}{A} = \frac{15}{12} = 1.25 \text{ ksi}$$

F1-10 Consider the cross section to be a rectangle and two triangles.

$$\bar{y} = \frac{\Sigma \bar{y} A}{\Sigma A} = \frac{0.15[(0.3)(0.12)] + (0.1)\left[\frac{1}{2}(0.16)(0.3)\right]}{0.3(0.12) + \frac{1}{2}(0.16)(0.3)}$$

$$= 0.13 \text{ m} = 130 \text{ mm} \quad \text{Ans.}$$

$$\sigma_{\text{avg}} = \frac{P}{A} = \frac{600(10^3)}{0.06} = 10 \text{ MPa} \quad \text{Ans.}$$

F1-11

$$A_A = A_C = \frac{\pi}{4}(0.5^2) = 0.0625\pi \text{ in}^2, A_B = \frac{\pi}{4}(1^2) = 0.25\pi \text{ in}^2$$

$$\sigma_A = \frac{N_A}{A_A} = \frac{3}{0.0625\pi} = 15.3 \text{ ksi (T)} \quad \text{Ans.}$$

$$\sigma_B = \frac{N_B}{A_B} = \frac{-6}{0.25\pi} = -7.64 \text{ ksi} = 7.64 \text{ ksi (C)} \quad \text{Ans.}$$

$$\sigma_C = \frac{N_C}{A_C} = \frac{2}{0.0625\pi} = 10.2 \text{ ksi (T)} \quad \text{Ans.}$$

F1-12

$$F_{AD} = 50(9.81) \text{ N} = 490.5 \text{ N}$$

$$\text{Ans.} \quad +\uparrow \Sigma F_y = 0; \quad F_{AC} \left(\frac{3}{5} \right) - 490.5 = 0 \quad F_{AC} = 817.5 \text{ N}$$

$$\text{Ans.} \quad \rightarrow \Sigma F_x = 0; \quad 817.5 \left(\frac{4}{5} \right) - F_{AB} = 0 \quad F_{AB} = 654 \text{ N}$$

$$A_{AB} = \frac{\pi}{4}(0.008^2) = 16(10^{-6})\pi \text{ m}^2$$

$$\text{Ans.} \quad (\sigma_{AB})_{\text{avg}} = \frac{F_{AB}}{A_{AB}} = \frac{654}{16(10^{-6})\pi} = 13.0 \text{ MPa} \quad \text{Ans.}$$

F1-13 Ring C:

$$+\uparrow \Sigma F_y = 0; \quad 2F \cos 60^\circ - 200(9.81) = 0 \quad F = 1962 \text{ N}$$

$$(\sigma_{\text{allow}})_{\text{avg}} = \frac{F}{A}; \quad 150(10^6) = \frac{1962}{\frac{\pi}{4}d^2}$$

$$d = 0.00408 \text{ m} = 4.08 \text{ mm}$$

$$\text{Use } d = 5 \text{ mm} \quad \text{Ans.}$$

F1-14 Entire frame:

$$\begin{aligned}\Sigma F_y &= 0; A_y = 600 \text{ lb} \\ \Sigma M_B &= 0; A_x = 800 \text{ lb} \\ F_A &= \sqrt{(600)^2 + (800)^2} = 1000 \text{ lb} \\ (\tau_A)_{\text{avg}} &= \frac{F_A/2}{A} = \frac{1000/2}{\frac{\pi}{4}(0.25)^2} = 10.2 \text{ ksi}\end{aligned}$$

F1-15 Double shear:

$$\begin{aligned}\Sigma F_y &= 0; 4V - 10 = 0 \quad V = 2.5 \text{ kip} \\ A &= \frac{\pi \left(\frac{3}{4}\right)^2}{4} = 0.140625\pi \text{ in}^2 \\ \tau_{\text{avg}} &= \frac{V}{A} = \frac{2.5}{0.140625\pi} = 5.66 \text{ ksi}\end{aligned}$$

F1-16 Single shear:

$$\begin{aligned}\Sigma F_x &= 0; P - 3V = 0 \quad V = \frac{P}{3} \\ A &= \frac{\pi}{4}(0.004^2) = 4(10^{-6})\pi \text{ m}^2 \\ (\tau_{\text{avg}})_{\text{allow}} &= \frac{V}{A}; \quad 60(10^6) = \frac{\frac{P}{3}}{4(10^{-6})\pi}\end{aligned}$$

$$P = 2.262(10^3) \text{ N} = 2.26 \text{ kN} \quad \text{Ans.}$$

F1-17 $\rightarrow \Sigma F_x = 0; V - P \cos 60^\circ = 0 \quad V = 0.5P$

$$\begin{aligned}A &= \left(\frac{0.05}{\sin 60^\circ}\right)(0.025) = 1.4434(10^{-3}) \text{ m}^2 \\ (\tau_{\text{avg}})_{\text{allow}} &= \frac{V}{A}; \quad 600(10^3) = \frac{0.5P}{1.4434(10^{-3})}\end{aligned}$$

$$P = 1.732(10^3) \text{ N} = 1.73 \text{ kN} \quad \text{Ans.}$$

F1-18 The resultant force on the pin is

$$F = \sqrt{30^2 + 40^2} = 50 \text{ kN.}$$

Here we have double shear:

$$\begin{aligned}V &= \frac{F}{2} = \frac{50}{2} = 25 \text{ kN} \\ A &= \frac{\pi}{4}(0.03^2) = 0.225(10^{-3})\pi \text{ m}^2\end{aligned}$$

$$\tau_{\text{avg}} = \frac{V}{A} = \frac{25(10^3)}{0.225(10^{-3})\pi} = 35.4 \text{ MPa} \quad \text{Ans.}$$

F1-19 $\rightarrow \Sigma F_x = 0; 30 - N = 0 \quad N = 30 \text{ kN}$

$$\begin{aligned}\sigma_{\text{allow}} &= \frac{\sigma_Y}{\text{F.S.}} = \frac{250}{1.5} = 166.67 \text{ MPa} \\ \sigma_{\text{allow}} &= \frac{N}{A}; \quad 166.67(10^6) = \frac{30(10^3)}{\frac{\pi}{4}d^2}\end{aligned}$$

$$d = 15.14 \text{ mm}$$

Use $d = 16 \text{ mm}$

Ans.

F1-20

$$\begin{aligned}\rightarrow \Sigma F_x &= 0; N_{AB} - 30 = 0 \quad N_{AB} = 30 \text{ kip} \\ \rightarrow \Sigma F_x &= 0; N_{BC} - 15 - 15 - 30 = 0 \quad N_{BC} = 60 \text{ kip} \\ \sigma_{\text{allow}} &= \frac{\sigma_Y}{\text{F.S.}} = \frac{50}{1.5} = 33.33 \text{ ksi}\end{aligned}$$

Segment AB:

$$\begin{aligned}\sigma_{\text{allow}} &= \frac{N_{AB}}{A_{AB}}; \quad 33.33 = \frac{30}{h_1(0.5)} \\ h_1 &= 1.8 \text{ in.}\end{aligned}$$

Segment BC:

$$\begin{aligned}\sigma_{\text{allow}} &= \frac{N_{BC}}{A_{BC}}; \quad 33.33 = \frac{60}{h_2(0.5)} \\ h_2 &= 3.6 \text{ in.}\end{aligned}$$

$$\text{Use } h_1 = 1\frac{7}{8} \text{ in. and } h_2 = 3\frac{5}{8} \text{ in.}$$

Ans.

F1-21 $N = P$

$$\sigma_{\text{allow}} = \frac{\sigma_Y}{\text{F.S.}} = \frac{250}{2} = 125 \text{ MPa}$$

$$A_r = \frac{\pi}{4}(0.04^2) = 1.2566(10^{-3}) \text{ m}^2$$

$$A_{\text{rod}} = 2(0.06 - 0.03)(0.05) = 3(10^{-3}) \text{ m}^2$$

The rod will fail first.

$$\sigma_{\text{allow}} = \frac{N}{A_r}; \quad 125(10^6) = \frac{P}{1.2566(10^{-3})}$$

$$P = 157.08(10^3) \text{ N} = 157 \text{ kN}$$

Ans.

F1-22 $\rightarrow \Sigma F_x = 0; 80 - 2V = 0 \quad V = 40 \text{ kN}$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{100}{2.5} = 40 \text{ MPa}$$

$$\tau_{\text{allow}} = \frac{V}{A}; \quad 40(10^6) = \frac{40(10^3)}{\frac{\pi}{4}d^2}$$

$$d = 0.03568 \text{ m} = 35.68 \text{ mm}$$

Use $d = 36 \text{ mm}$

Ans.

F1-23 $V = P$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{120}{2.5} = 48 \text{ MPa}$$

Area of shear plane for bolt head and plate:

$$A_b = \pi dt = \pi(0.04)(0.075) = 0.003\pi \text{ m}^2$$

$$A_p = \pi dt = \pi(0.08)(0.03) = 0.0024\pi \text{ m}^2$$

Since the area of shear plane for the plate is smaller,

$$\tau_{\text{allow}} = \frac{V}{A_p}; \quad 48(10^6) = \frac{P}{0.0024\pi}$$

$$P = 361.91(10^3) \text{ N} = 362 \text{ kN} \quad \text{Ans.}$$

F1-24

$$\downarrow + \Sigma M_B = 0; \quad \frac{1}{2}(300)(9)(6) - 6V(9) = 0 \quad V = 150 \text{ lb}$$

$$\tau_{\text{allow}} = \frac{\tau_{\text{fail}}}{\text{F.S.}} = \frac{16}{2} = 8 \text{ ksi}$$

$$\tau_{\text{allow}} = \frac{V}{A}; \quad 8(10^3) = \frac{150}{\frac{\pi}{4}d^2}$$

$$d = 0.1545 \text{ in.}$$

$$\text{Use } d = \frac{3}{16} \text{ in.}$$

Chapter 2

$$\text{F2-1} \quad \frac{\delta_C}{600} = \frac{0.2}{400}; \quad \delta_C = 0.3 \text{ mm}$$

$$\epsilon_{CD} = \frac{\delta_C}{L_{CD}} = \frac{0.3}{300} = 0.001 \text{ mm/mm} \quad \text{Ans.}$$

F2-2

$$\theta = \left(\frac{0.02^\circ}{180^\circ} \right) \pi \text{ rad} = 0.3491(10^{-3}) \text{ rad}$$

$$\delta_B = \theta L_{AB} = 0.3491(10^{-3})(600) = 0.2094 \text{ mm}$$

$$\delta_C = \theta L_{AC} = 0.3491(10^{-3})(1200) = 0.4189 \text{ mm}$$

$$\epsilon_{BD} = \frac{\delta_B}{L_{BD}} = \frac{0.2094}{400} = 0.524(10^{-3}) \text{ mm/mm} \quad \text{Ans.}$$

$$\epsilon_{CE} = \frac{\delta_C}{L_{CE}} = \frac{0.4189}{600} = 0.698(10^{-3}) \text{ mm/mm} \quad \text{Ans.}$$

F2-3

$$\alpha = \frac{2}{400} = 0.005 \text{ rad} \quad \beta = \frac{4}{300} = 0.01333 \text{ rad}$$

$$(\gamma_A)_{xy} = \frac{\pi}{2} - \theta$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha + \beta \right)$$

$$= \alpha - \beta$$

$$= 0.005 - 0.01333$$

$$= -0.00833 \text{ rad} \quad \text{Ans.}$$

F2-4

$$L_{BC} = \sqrt{300^2 + 400^2} = 500 \text{ mm}$$

$$L_{B'C'} = \sqrt{(300 - 3)^2 + (400 + 5)^2} = 502.2290 \text{ mm}$$

$$\alpha = \frac{3}{405} = 0.007407 \text{ rad}$$

$$(\epsilon_{BC})_{\text{avg}} = \frac{L_{B'C'} - L_{BC}}{L_{BC}} = \frac{502.2290 - 500}{500}$$

$$= 0.00446 \text{ mm/mm} \quad \text{Ans.}$$

$$(\gamma_A)_{xy} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} + \alpha \right) = -\alpha = -0.00741 \text{ rad} \quad \text{Ans.}$$

F2-5

$$L_{AC} = \sqrt{L_{CD}^2 + L_{AD}^2} = \sqrt{300^2 + 300^2} = 424.2641 \text{ mm}$$

$$L_{A'C'} = \sqrt{L_{C'D'}^2 + L_{A'D'}^2} = \sqrt{306^2 + 296^2} = 425.7370 \text{ mm}$$

Ans.

$$\frac{\theta}{2} = \tan^{-1} \left(\frac{L_{C'D'}}{L_{A'D'}} \right); \quad \theta = 2 \tan^{-1} \left(\frac{306}{296} \right) = 1.6040 \text{ rad}$$

$$(\epsilon_{AC})_{\text{avg}} = \frac{L_{A'C'} - L_{AC}}{L_{AC}} = \frac{425.7370 - 424.2641}{424.2641}$$

$$= 0.00347 \text{ mm/mm} \quad \text{Ans.}$$

$$(\gamma_E)_{xy} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - 1.6040 = -0.0332 \text{ rad} \quad \text{Ans.}$$

Chapter 3

F3-1 Material has uniform properties throughout. *Ans.*

F3-2 Proportional limit is A. *Ans.*
Ultimate stress is D. *Ans.*

F3-3 The initial slope of the $\sigma - \epsilon$ diagram. *Ans.*

F3-4 True. *Ans.*

F3-5 False. Use the *original* cross-sectional area and length. *Ans.*

F3-6 False. It will normally decrease. *Ans.*

$$\text{F3-7} \quad \epsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\delta = \epsilon L = \frac{PL}{AE} = \frac{100(10^3)(0.100)}{\frac{\pi}{4}(0.015)^2 200(10^9)}$$

$$= 0.283 \text{ mm} \quad \text{Ans.}$$

$$\text{F3-8} \quad \epsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\delta = \epsilon L = \frac{PL}{AE}$$

$$0.003 = \frac{(10\,000)(8)}{12E}$$

$$E = 2.22(10^6) \text{ psi} \quad \text{Ans.}$$

$$\begin{aligned}\text{F3-9} \quad \epsilon &= \frac{\sigma}{E} = \frac{P}{AE} \\ \delta &= \epsilon L = \frac{PL}{AE} = \frac{6(10^3)4}{\frac{\pi}{4}(0.01)^2 100(10^9)} \\ &= 3.06 \text{ mm}\end{aligned}$$

Ans.

$$\text{F3-10} \quad \sigma = \frac{P}{A} = \frac{100(10^3)}{\frac{\pi}{4}(0.02)^2} = 318.31 \text{ MPa}$$

Since $\sigma < \sigma_Y = 450 \text{ MPa}$, Hooke's Law is applicable.

$$\begin{aligned}E &= \frac{\sigma_Y}{\epsilon_Y} = \frac{450(10^6)}{0.00225} = 200 \text{ GPa} \\ \epsilon &= \frac{\sigma}{E} = \frac{318.31(10^6)}{200(10^9)} = 0.001592 \text{ mm/mm} \\ \delta &= \epsilon L = 0.001592(50) = 0.0796 \text{ mm}\end{aligned}$$

Ans.

$$\text{F3-11} \quad \sigma = \frac{P}{A} = \frac{150(10^3)}{\frac{\pi}{4}(0.02)^2} = 477.46 \text{ MPa}$$

Since $\sigma > \sigma_Y = 450 \text{ MPa}$, Hooke's Law is not applicable. From the geometry of the stress-strain diagram,

$$\begin{aligned}\frac{\epsilon - 0.00225}{0.03 - 0.00225} &= \frac{477.46 - 450}{500 - 450} \\ \epsilon &= 0.017493\end{aligned}$$

When the load is removed, the strain recovers along a line parallel to the original elastic line.

$$\text{Here } E = \frac{\sigma_Y}{\epsilon_Y} = \frac{450(10^6)}{0.00225} = 200 \text{ GPa.}$$

The elastic recovery is

$$\begin{aligned}\epsilon_r &= \frac{\sigma}{E} = \frac{477.46(10^6)}{200(10^9)} = 0.002387 \text{ mm/mm} \\ \epsilon_p &= \epsilon - \epsilon_r = 0.017493 - 0.002387 \\ &= 0.01511 \text{ mm/mm} \\ \delta_p &= \epsilon_p L = 0.01511(50) = 0.755 \text{ mm}\end{aligned}$$

Ans.

$$\begin{aligned}\text{F3-12} \quad \epsilon_{BC} &= \frac{\delta_{BC}}{L_{BC}} = \frac{0.2}{300} = 0.6667(10^{-3}) \text{ mm/mm} \\ \sigma_{BC} &= E\epsilon_{BC} = 200(10^9)(0.6667(10^{-3})) \\ &= 133.33 \text{ MPa}\end{aligned}$$

Since $\sigma_{BC} < \sigma_Y = 250 \text{ MPa}$, Hooke's Law is valid.

$$\begin{aligned}\sigma_{BC} &= \frac{F_{BC}}{A_{BC}}, \quad 133.33(10^6) = \frac{F_{BC}}{\frac{\pi}{4}(0.003)^2} \\ F_{BC} &= 942.48 \text{ N} \\ (+\Sigma M_A = 0); \quad 942.48(0.4) - P(0.6) &= 0 \\ P &= 628.31 \text{ N} = 628 \text{ N}\end{aligned}$$

Ans.

$$\text{F3-13} \quad \sigma = \frac{P}{A} = \frac{10(10^3)}{\frac{\pi}{4}(0.015)^2} = 56.59 \text{ MPa}$$

$$\begin{aligned}\epsilon_{\text{long}} &= \frac{\sigma}{E} = \frac{56.59(10^6)}{70(10^9)} = 0.808(10^{-3}) \\ \epsilon_{\text{lat}} &= -\nu\epsilon_{\text{long}} = -0.35(0.808(10^{-3})) \\ &= -0.283(10^{-3})\end{aligned}$$

$$\delta d = (-0.283(10^{-3}))(15 \text{ mm}) = -4.24(10^{-3}) \text{ mm}$$

Ans.

$$\text{F3-14} \quad \sigma = \frac{P}{A} = \frac{50(10^3)}{\frac{\pi}{4}(0.02)^2} = 159.15 \text{ MPa}$$

$$\epsilon_a = \frac{\delta}{L} = \frac{1.40}{600} = 0.002333 \text{ mm/mm}$$

$$E = \frac{\sigma}{\epsilon_a} = \frac{159.15(10^6)}{0.002333} = 68.2 \text{ GPa}$$

Ans.

$$\epsilon_c = \frac{d' - d}{d} = \frac{19.9837 - 20}{20} = -0.815(10^{-3}) \text{ mm/mm}$$

$$\nu = -\frac{\epsilon_c}{\epsilon_a} = -\frac{-0.815(10^{-3})}{0.002333} = 0.3493 = 0.349$$

$$G = \frac{E}{2(1 + \nu)} = \frac{68.21}{2(1 + 0.3493)} = 25.3 \text{ GPa}$$

Ans.

$$\text{F3-15} \quad \alpha = \frac{0.5}{150} = 0.003333 \text{ rad}$$

$$\begin{aligned}\gamma &= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right) \\ &= \alpha = 0.003333 \text{ rad}\end{aligned}$$

$$\tau = G\gamma = [26(10^9)](0.003333) = 86.67 \text{ MPa}$$

$$\begin{aligned}\tau &= \frac{V}{A}, \quad 86.67(10^6) = \frac{P}{0.15(0.02)} \\ P &= 260 \text{ kN}\end{aligned}$$

Ans.

$$\text{F3-16} \quad \alpha = \frac{3}{150} = 0.02 \text{ rad}$$

$$\gamma = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right) = \alpha = 0.02 \text{ rad}$$

When P is removed, the shear strain recovers along a line parallel to the original elastic line.

$$\gamma_r = \gamma_Y = 0.005 \text{ rad}$$

$$\gamma_p = \gamma - \gamma_r = 0.02 - 0.005 = 0.015 \text{ rad}$$

Ans.

Chapter 4

$$\text{F4-1} \quad A = \frac{\pi}{4}(0.02)^2 = 0.1(10^{-3})\pi \text{ m}^2$$

$$\begin{aligned}\delta_C &= \frac{1}{AE}[40(10^3)(400) + [-60(10^3)(600)]] \\ &= \frac{-20(10^6) \text{ N} \cdot \text{mm}}{AE} \\ &= -0.318 \text{ mm}\end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F4-2} \quad A_{AB} &= A_{CD} = \frac{\pi}{4}(0.02^2) = 0.1(10^{-3})\pi \text{ m}^2 \\
 A_{BC} &= \frac{\pi}{4}(0.04^2 - 0.03^2) = 0.175(10^{-3})\pi \text{ m}^2 \\
 \delta_{D/A} &= \frac{[-10(10^3)](400)}{[0.1(10^{-3})\pi][68.9(10^9)]} \\
 &\quad + \frac{[10(10^3)](400)}{[0.175(10^{-3})\pi][68.9(10^9)]} \\
 &\quad + \frac{[-20(10^3)](400)}{[0.1(10^{-3})\pi][68.9(10^9)]} \\
 &= -0.449 \text{ mm}
 \end{aligned}$$

$$\begin{aligned}
 \text{F4-3} \quad A &= \frac{\pi}{4}(0.03^2) = 0.225(10^{-3})\pi \text{ m}^2 \\
 \delta_C &= \frac{1}{0.225(10^{-3})\pi[200(10^9)]} \left\{ \left[-90(10^3) \right. \right. \\
 &\quad \left. \left. - 2\left(\frac{4}{5}\right)30(10^3) \right](0.4) + [-90(10^3)](0.6) \right\} \\
 &= -0.772(10^{-3}) \text{ m} = -0.772 \text{ mm}
 \end{aligned}$$

$$\begin{aligned}
 \text{F4-4} \quad \delta_{A/B} &= \frac{PL}{AE} = \frac{[60(10^3)](0.8)}{[0.1(10^{-3})\pi][200(10^9)]} \\
 &= 0.7639(10^{-3}) \text{ m} \downarrow \\
 \delta_B &= \frac{F_{sp}}{k} = \frac{60(10^3)}{50(10^6)} = 1.2(10^{-3}) \text{ m} \downarrow \\
 + \downarrow \quad \delta_A &= \delta_B + \delta_{A/B} \\
 \delta_A &= 1.2(10^{-3}) + 0.7639(10^{-3}) \\
 &= 1.9639(10^{-3}) \text{ m} = 1.96 \text{ mm} \downarrow
 \end{aligned}$$

$$\begin{aligned}
 \text{F4-5} \quad A &= \frac{\pi}{4}(0.02^2) = 0.1(10^{-3})\pi \text{ m}^2 \\
 \text{Internal load } P(x) &= 30(10^3)x \\
 \delta_A &= \int \frac{P(x)dx}{AE} \\
 &= \frac{1}{[0.1(10^{-3})\pi][73.1(10^9)]} \int_0^{0.9 \text{ m}} 30(10^3)x \, dx \\
 &= 0.529(10^{-3}) \text{ m} = 0.529 \text{ mm}
 \end{aligned}$$

$$\begin{aligned}
 \text{F4-6} \quad \text{Distributed load } P(x) &= \frac{45(10^3)}{0.9}x = 50(10^3)x \text{ N/m} \\
 \text{Internal load } P(x) &= \frac{1}{2}(50(10^3))x(x) = 25(10^3)x^2 \\
 \delta_A &= \int_0^L \frac{P(x)dx}{AE} \\
 &= \frac{1}{[0.1(10^{-3})\pi][73.1(10^9)]} \int_0^{0.9 \text{ m}} [25(10^3)x^2]dx \\
 &= 0.265 \text{ mm}
 \end{aligned}$$

Chapter 5

$$\begin{aligned}
 \text{F5-1} \quad J &= \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4 \\
 \tau_A = \tau_{\max} &= \frac{T_C}{J} = \frac{5(10^3)(0.04)}{1.28(10^{-6})\pi} = 49.7 \text{ MPa} \\
 \tau_B &= \frac{T_{PB}}{J} = \frac{5(10^3)(0.03)}{1.28(10^{-6})\pi} = 37.3 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-2} \quad J &= \frac{\pi}{2}(0.06^4 - 0.04^4) = 5.2(10^{-6})\pi \text{ m}^4 \\
 \tau_B = \tau_{\max} &= \frac{T_C}{J} = \frac{10(10^3)(0.06)}{5.2(10^{-6})\pi} = 36.7 \text{ MPa} \\
 \tau_A &= \frac{T_{PA}}{J} = \frac{10(10^3)(0.04)}{5.2(10^{-6})\pi} = 24.5 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-3} \quad J_{AB} &= \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4 \\
 J_{BC} &= \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4 \\
 (\tau_{AB})_{\max} &= \frac{T_{AB}c_{AB}}{J_{AB}} = \frac{[2(10^3)](0.04)}{0.875(10^{-6})\pi} = 29.1 \text{ MPa} \\
 (\tau_{BC})_{\max} &= \frac{T_{BC}c_{BC}}{J_{BC}} = \frac{[6(10^3)](0.04)}{1.28(10^{-6})\pi} \\
 &= 59.7 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-4} \quad T_{AB} &= 0, T_{BC} = 600 \text{ N}\cdot\text{m}, T_{CD} = 0 \\
 J &= \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4 \\
 \tau_{\max} &= \frac{T_C}{J} = \frac{600(0.02)}{80(10^{-9})\pi} = 47.7 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-5} \quad J_{BC} &= \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4 \\
 (\tau_{BC})_{\max} &= \frac{T_{BC}c_{BC}}{J_{BC}} = \frac{2100(0.04)}{0.875(10^{-6})\pi} \\
 &= 30.6 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-6} \quad t &= 5(10^3) \text{ N}\cdot\text{m/m} \\
 \text{Internal torque is } T &= 5(10^3)(0.8) = 4000 \text{ N}\cdot\text{m} \\
 J &= \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4 \\
 \tau_A &= \frac{T_Ac}{J} = \frac{4000(0.04)}{1.28(10^{-6})\pi} = 39.8 \text{ MPa}
 \end{aligned}$$

$$\begin{aligned}
 \text{F5-7} \quad J &= \frac{\pi}{2}(0.03^4) = 0.405(10^{-6})\pi \text{ m}^4 \\
 \phi_{A/C} &= \frac{1}{[0.405(10^{-6})\pi][75(10^9)]} \{ [-2(10^3)](0.6) \\
 &\quad + 1(10^3)(0.4) \} \\
 &= -0.00838 \text{ rad} = -0.480^\circ
 \end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F5-8} \quad J &= \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4 \\
 \phi_{B/A} &= \frac{600(0.45)}{[80(10^{-9})\pi][75(10^9)]} \\
 &= 0.01432 \text{ rad} = 0.821^\circ
 \end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F5-9} \quad J &= \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4 \\
 \phi_{A/B} &= \frac{T_{AB} L_{AB}}{JG} = \frac{3(10^3)(0.9)}{[0.875(10^{-6})\pi][26(10^9)]} \\
 &= 0.03778 \text{ rad} \\
 \phi_B &= \frac{T_B}{k} = \frac{3(10^3)}{90(10^3)} = 0.03333 \text{ rad} \\
 \phi_A &= \phi_B + \phi_{A/B} \\
 &= 0.03333 + 0.03778 \\
 &= 0.07111 \text{ rad} = 4.07^\circ
 \end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F5-10} \quad J &= \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4 \\
 \phi_{B/A} &= \frac{0.2}{[80(10^{-9})\pi][75(10^9)]} [600 + (-300) \\
 &\quad + 200 + 500] \\
 &= 0.01061 \text{ rad} = 0.608^\circ
 \end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F5-11} \quad J &= \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4 \\
 t &= 5(10^3) \text{ N}\cdot\text{m}/\text{m} \\
 \text{Internal torque is } 5(10^3)x \text{ N}\cdot\text{m} \\
 \phi_{A/B} &= \int_0^L \frac{T(x)dx}{JG} \\
 &= \frac{1}{[1.28(10^{-6})\pi][75(10^9)]} \int_0^{0.8 \text{ m}} 5(10^3)x dx \\
 &= 0.00531 \text{ rad} = 0.304^\circ
 \end{aligned}$$

Ans.

$$\begin{aligned}
 \text{F5-12} \quad J &= \frac{\pi}{2}(0.04^4) = 1.28(10^{-6})\pi \text{ m}^4 \\
 \text{Distributed torque is } t &= \frac{15(10^3)}{0.6}(x) \\
 &= 25(10^3)x \text{ N}\cdot\text{m}/\text{m}
 \end{aligned}$$

$$\begin{aligned}
 \text{Internal torque is } T(x) &= \frac{1}{2}(25x)(x) \\
 &= 12.5(10^3)x^2 \text{ N}\cdot\text{m}
 \end{aligned}$$

$$\begin{aligned}
 \phi_{A/C} &= \int_0^L \frac{T(x)dx}{JG} + \frac{T_{BC} L_{BC}}{JG} \\
 &= \frac{1}{[1.28(10^{-6})\pi][75(10^9)]} \left[\int_0^{0.6 \text{ m}} 12.5(10^3)x^2 dx + 4500(0.4) \right] \\
 &= 0.008952 \text{ rad} = 0.513^\circ
 \end{aligned}$$

Ans.

Chapter 6

$$\begin{aligned}
 \text{F6-1} \quad + \uparrow \Sigma F_y &= 0; \quad -V - 9 = 0 \quad V = -9 \text{ kN} \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M + 9x = 0 \quad M = [-9x] \text{ kN}\cdot\text{m} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6-2} \quad + \uparrow \Sigma F_y &= 0; \quad -V - 2x = 0 \quad V = [-2x] \text{ kip} \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M + 2x\left(\frac{x}{2}\right) - 18 = 0 \\
 M &= [18 - x^2] \text{ kip}\cdot\text{ft} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6-3} \quad + \uparrow \Sigma F_y &= 0; \quad -V - \frac{1}{2}(4x)(x) = 0 \\
 V &= [-2x^2] \text{ kN} \quad \text{Ans.} \\
 \Sigma M_O &= 0; \quad M + \left[\frac{1}{2}(4x)(x) \right] \left(\frac{x}{3} \right) = 0 \\
 M &= \left\{ -\frac{2}{3}x^3 \right\} \text{ kN}\cdot\text{m} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6-4} \quad 0 \leq x < 1.5 \text{ m} \\
 + \uparrow \Sigma F_y &= 0; \quad V = 0 \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M - 4 = 0 \quad M = 4 \text{ kN}\cdot\text{m} \quad \text{Ans.} \\
 1.5 \text{ m} < x \leq 3 \text{ m} \\
 + \uparrow \Sigma F_y &= 0; \quad -V - 9 = 0 \quad V = -9 \text{ kN} \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M + 9(x - 1.5) - 4 = 0 \\
 M &= [17.5 - 9x] \text{ kN}\cdot\text{m} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6-5} \quad \downarrow + \Sigma M_B &= 0; \quad A_y(6) - 30 = 0 \quad A_y = 5 \text{ kN} \\
 + \uparrow \Sigma F_y &= 0; \quad -V - 5 = 0 \quad V = -5 \text{ kN} \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M + 5x = 0 \quad M = [-5x] \text{ kN}\cdot\text{m} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F6-6} \quad \downarrow + \Sigma M_B &= 0; \quad A_y(6) + 20 - 50 = 0 \\
 A_y &= 5 \text{ kN} \\
 + \uparrow \Sigma F_y &= 0; \quad -V - 5 = 0 \quad V = -5 \text{ kN} \quad \text{Ans.} \\
 \downarrow + \Sigma M_O &= 0; \quad M + 5x - 50 = 0 \\
 M &= [50 - 5x] \text{ kN}\cdot\text{m} \quad \text{Ans.}
 \end{aligned}$$

F6-7 Shear diagram. $V = -4$, $x = 0$. Zero slope to $x = 6$.
 Moment diagram. $M = 0$, $x = 0$. Constant negative slope to $M = -16$, $x = 4^-$. $M = 8$, $x = 4^+$. Constant negative slope to $M = 0$, $x = 6$.

F6-8 Shear diagram. $V = -6$, $x = 0$. Zero slope to $x = 3$.
 Moment diagram. $M = 0$, $x = 0$. Constant negative slope to $M = -9$, $x = 1.5^-$. $M = -21$, $x = 1.5^+$. Constant negative slope to $M = -30$, $x = 3$.

F6-9 Shear diagram. $V = 0$, $x = 0$. Zero slope to $x = 1.5^-$. $V = 4$, $x = 1.5^+$. Zero slope to $x = 4.5^-$. $V = 0$, $x = 4.5^+$. Zero slope to $V = 0$, $x = 6$.
 Moment diagram. $M = 6$, $x = 0$. Zero slope to $x = 1.5$. $M = 6$, $x = 1.5$. Constant positive slope to $x = 4.5$. $M = 18$, $x = 4.5$. Zero slope to $M = 18$, $x = 6$.

F6-10 Shear diagram. $V = 16.5$, $x = 0$. Constant negative slope to $x = 3$. $V = 0$, $x = 2.75$. $V = -1.5$, $x = 3$. Negative decreasing slope, $V = -10.5$, $x = 6$.
 Moment diagram. $M = 0$, $x = 0$. Positive decreasing slope. $M = 22.7$, $x = 2.75$. Negative decreasing slope to $M = 0$, $x = 6$.

F6-11 Shear diagram. $V = 0$, $x = 0$. Negative constant slope, $V = -6$, $x = 1.5^-$. $V = 0$, $x = 1.5^+$. Zero slope to $x = 4.5$. $V = 0$, $x = 4.5^-$. $V = 6$, $x = 4.5^+$. Constant negative slope, $V = 0$, $x = 6$.
 Moment diagram. $M = 0$, $x = 0$. Negative increasing slope, $M = -4.5$, $x = 1.5$. Zero slope, $M = -4.5$, $x = 4.5$. Positive decreasing slope, $M = 0$, $x = 6$.

F6-12 Shear diagram. $V = 15$, $x = 0$. Negative decreasing slope to zero slope at $x = 3$. $V = 0$, $x = 0$. Negative increasing slope to $V = -15$, $x = 6$.
 Moment diagram. $M = 0$, $x = 0$. Positive decreasing slope to zero slope at $x = 3$. $M = 15$, $x = 3$. Negative increasing slope. $M = 0$, $x = 6$.

F6-13 Shear diagram. $V = 1050$, $x = 0$. Constant negative slope, $V = 0$, $x = 5.25$. $V = -150$, $x = 6$. Zero slope, $V = -150$, $x = 9^-$. $V = -750$, $x = 9^+$. Zero slope, $V = -750$, $x = 12$.
 Moment diagram. $M = 0$, $x = 0$. Positive decreasing slope to $x = 5.25$. $M = 2756$, $x = 5.25$. Negative increasing slope to $M = 2700$, $x = 6$.

Constant negative slope, $M = 2250$, $x = 9$.
 Constant negative slope, $M = 0$, $x = 12$.

F6-14 Shear diagram. $V = 30$, $x = 0$. Constant negative slope, $V = 0$, $x = 1.5$. $V = -50$, $x = 4^-$. $V = 20$, $x = 4^+$. Zero slope, $V = 20$, $x = 6$.

Moment diagram. $M = 0$, $x = 0$. Positive decreasing slope, to zero slope at $x = 1.5$. $M = 22.5$, $x = 1.5$. Negative increasing slope, $M = -40$, $x = 4$. Positive constant slope, $M = 0$, $x = 6$.

$$\begin{aligned} \text{F6-15} \quad I &= 2 \left[\frac{1}{12} (0.02)(0.2^3) \right] + \frac{1}{12} (0.26)(0.02^3) \\ &= 26.84(10^{-6}) \text{ m}^4 \\ \sigma_{\max} &= \frac{Mc}{I} = \frac{20(10^3)(0.1)}{26.84(10^{-6})} = 74.5 \text{ MPa} \quad \text{Ans} \end{aligned}$$

$$\begin{aligned} \text{F6-16} \quad \bar{y} &= \frac{0.3}{3} = 0.1 \text{ m} \\ I &= \frac{1}{36} (0.3)(0.3^3) = 0.225(10^{-3}) \text{ m}^4 \\ (\sigma_{\max})_c &= \frac{Mc}{I} = \frac{50(10^3)(0.3 - 0.1)}{0.225(10^{-3})} \\ &= 44.4 \text{ MPa (C)} \quad \text{Ans} \\ (\sigma_{\max})_t &= \frac{My}{I} = \frac{50(10^3)(0.1)}{0.225(10^{-3})} = 22.2 \text{ MPa (T)} \quad \text{Ans} \end{aligned}$$

$$\begin{aligned} \text{F6-17} \quad I &= \frac{1}{12} (0.2)(0.3^3) - \frac{1}{12} (0.18)(0.26^3) \\ &= 0.18636(10^{-3}) \text{ m}^4 \\ \sigma_{\max} &= \frac{Mc}{I} = \frac{50(10^3)(0.15)}{0.18636(10^{-3})} = 40.2 \text{ MPa} \quad \text{Ans} \end{aligned}$$

$$\begin{aligned} \text{F6-18} \quad I &= 2 \left[\frac{1}{12} (0.03)(0.4^3) \right] + 2 \left[\frac{1}{12} (0.14)(0.03^3) + 0.14(0.03)(0.15^2) \right] \\ &= 0.50963(10^{-3}) \text{ m}^4 \\ \sigma_{\max} &= \frac{Mc}{I} = \frac{10(10^3)(0.2)}{0.50963(10^{-3})} = 3.92 \text{ MPa} \quad \text{Ans} \end{aligned}$$

$$\begin{aligned} \text{F6-19} \quad I &= \frac{1}{12} (0.05)(0.4)^3 + 2 \left[\frac{1}{12} (0.025)(0.3)^3 \right] \\ &= 0.37917(10^{-3}) \text{ m}^4 \\ \sigma_A &= \frac{My_A}{I} = \frac{5(10^3)(-0.15)}{0.37917(10^{-3})} = 1.98 \text{ MPa (T)} \quad \text{Ans} \end{aligned}$$

$$\mathbf{F6-20} \quad M_y = 50\left(\frac{4}{5}\right) = 40 \text{ kN} \cdot \text{m}$$

$$M_z = 50\left(\frac{3}{5}\right) = 30 \text{ kN} \cdot \text{m}$$

$$I_y = \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3}) \text{ m}^4$$

$$I_z = \frac{1}{12}(0.2)(0.3^3) = 0.45(10^{-3}) \text{ m}^4$$

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_A = -\frac{[30(10^3)](-0.15)}{0.45(10^{-3})} + \frac{[40(10^3)](0.1)}{0.2(10^{-3})}$$

$$= 30 \text{ MPa (T)}$$

Ans.

$$\sigma_B = -\frac{[30(10^3)](0.15)}{0.45(10^{-3})} + \frac{[40(10^3)](0.1)}{0.2(10^{-3})}$$

$$= 10 \text{ MPa (T)}$$

Ans.

$$\tan \alpha = \frac{I_z}{I_y} \tan \theta$$

$$\tan \alpha = \left[\frac{0.45(10^{-3})}{0.2(10^{-3})} \right] \left(\frac{4}{3} \right)$$

$$\alpha = 71.6^\circ$$

Ans.

F6-21 Maximum stress occurs at D or A .

$$(\sigma_{\max})_D = \frac{(50 \cos 30^\circ)(12)(3)}{\frac{1}{12}(4)(6)^3} + \frac{(50 \sin 30^\circ)(12)(2)}{\frac{1}{12}(6)(4)^3}$$

$$= 40.4 \text{ psi}$$

Ans.

Chapter 7

$$\mathbf{F7-1} \quad I = 2\left[\frac{1}{12}(0.02)(0.2^3) \right] + \frac{1}{12}(0.26)(0.02^3)$$

$$= 26.84(10^{-6}) \text{ m}^4$$

$$Q_A = 0.055(0.09)(0.02) = 99(10^{-6}) \text{ m}^3$$

$$\tau_A = \frac{VQ_A}{It} = \frac{100(10^3)[99(10^{-6})]}{[26.84(10^{-6})](0.02)}$$

$$= 18.4 \text{ MPa}$$

Ans.

$$\mathbf{F7-2} \quad I = \frac{1}{12}(0.1)(0.3^3) + \frac{1}{12}(0.2)(0.1^3) = 0.24167(10^{-3}) \text{ m}^4$$

$$Q_A = y'_1 A'_1 + y'_2 A'_2$$

$$= \left[\frac{1}{2}(0.05) \right] (0.05)(0.3) + 0.1(0.1)(0.1)$$

$$= 1.375(10^{-3}) \text{ m}^3$$

$$Q_B = y'_3 A'_3 = 0.1(0.1)(0.1) = 1(10^{-3}) \text{ m}^3$$

$$\tau_A = \frac{VQ}{It} = \frac{600(10^3)[1.375(10^{-3})]}{[0.24167(10^{-3})](0.3)} = 11.4 \text{ MPa Ans.}$$

$$\tau_B = \frac{VQ}{It} = \frac{600(10^3)[1(10^{-3})]}{[0.24167(10^{-3})](0.1)} = 24.8 \text{ MPa Ans.}$$

$$\mathbf{F7-3} \quad V_{\max} = 4.5 \text{ kip}$$

$$I = \frac{1}{12}(3)(6^3) = 54 \text{ in}^4$$

$$Q_{\max} = y' A' = 1.5(3)(3) = 13.5 \text{ in}^3$$

$$(\tau_{\max})_{\text{abs}} = \frac{V_{\max} Q_{\max}}{It} = \frac{4.5(10^3)(13.5)}{54(3)} = 375 \text{ psi}$$

Ans.

$$\mathbf{F7-4} \quad I = 2\left[\frac{1}{12}(0.03)(0.4^3) \right] + 2\left[\frac{1}{12}(0.14)(0.03)^3 \right]$$

$$+ 0.14(0.03)(0.15^2) = 0.50963(10^{-3}) \text{ m}^4$$

$$Q_{\max} = 2y'_1 A'_1 + y'_2 A'_2 = 2(0.1)(0.2)(0.03)$$

$$+ (0.15)(0.14)(0.03) = 1.83(10^{-3}) \text{ m}^3$$

$$\tau_{\max} = \frac{VQ_{\max}}{It} = \frac{20(10^3)[1.83(10^{-3})]}{0.50963(10^{-3})[2(0.3)]} = 1.20 \text{ MPa}$$

Ans.

$$\mathbf{F7-5} \quad I = \frac{1}{12}(0.05)(0.4^3) + 2\left[\frac{1}{12}(0.025)(0.3)^3 \right]$$

$$= 0.37917(10^{-3}) \text{ m}^4$$

$$Q_{\max} = 2y'_1 A'_1 + y'_2 A'_2 = 2(0.075)(0.025)(0.15)$$

$$+ (0.1)(0.05)(0.2) = 1.5625(10^{-3}) \text{ m}^3$$

$$\tau_{\max} = \frac{VQ_{\max}}{It} = \frac{20(10^3)[1.5625(10^{-3})]}{[0.37917(10^{-3})][2(0.025)]}$$

$$= 1.65 \text{ MPa}$$

Ans.

$$\mathbf{F7-6} \quad I = \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3}) \text{ m}^4$$

$$Q = y' A' = 0.05(0.1)(0.3) = 1.5(10^{-3}) \text{ m}^3$$

$$q_{\text{allow}} = 2\left(\frac{F}{s} \right) = \frac{2[15(10^3)]}{s} = \frac{30(10^3)}{s}$$

$$q_{\text{allow}} = \frac{VQ}{It} = \frac{30(10^3)}{s} = \frac{50(10^3)[1.5(10^{-3})]}{0.2(10^{-3})}$$

$$s = 0.08 \text{ m} = 80 \text{ mm}$$

Ans.

$$\begin{aligned}
 \text{F7-7} \quad I &= \frac{1}{12}(0.3)(0.2^3) = 0.2(10^{-3}) \text{ m}^4 \\
 Q &= y'A' = 0.05(0.1)(0.3) = 1.5(10^{-3}) \text{ m}^3 \\
 q_{\text{allow}} &= 2\left(\frac{F}{s}\right) = \frac{2[15(10^3)]}{0.1} = 300(10^3) \text{ N/m} \\
 q_{\text{allow}} &= \frac{VQ}{I}, \quad 300(10^3) = \frac{V[1.5(10^{-3})]}{0.2(10^{-3})} \\
 V &= 40(10^3) \text{ N} = 40 \text{ kN} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F7-8} \quad I &= \frac{1}{12}(0.2)(0.34^3) - \frac{1}{12}(0.19)(0.28^3) \\
 &= 0.3075(10^{-3}) \text{ m}^4 \\
 Q &= y'A' = 0.16(0.02)(0.2) = 0.64(10^{-3}) \text{ m}^3 \\
 q_{\text{allow}} &= 2\left(\frac{F}{s}\right) = \frac{2[30(10^3)]}{s} = \frac{60(10^3)}{s} \\
 q_{\text{allow}} &= \frac{VQ}{I}, \quad \frac{60(10^3)}{s} = \frac{300(10^3)[0.64(10^{-3})]}{0.3075(10^{-3})} \\
 s &= 0.09609 \text{ m} = 96.1 \text{ mm} \\
 \text{Use } s &= 96 \text{ mm} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F7-9} \quad I &= 2\left[\frac{1}{12}(0.025)(0.3^3)\right] + 2\left[\frac{1}{12}(0.05)(0.2^3) + 0.05(0.2)(0.15^2)\right] \\
 &= 0.62917(10^{-3}) \text{ m}^4 \\
 Q &= y'A' = 0.15(0.2)(0.05) = 1.5(10^{-3}) \text{ m}^3 \\
 q_{\text{allow}} &= 2\left(\frac{F}{s}\right) = \frac{2[8(10^3)]}{s} = \frac{16(10^3)}{s} \\
 q_{\text{allow}} &= \frac{VQ}{I}, \quad \frac{16(10^3)}{s} = \frac{20(10^3)[1.5(10^{-3})]}{0.62917(10^{-3})} \\
 s &= 0.3356 \text{ m} = 335.56 \text{ mm} \\
 \text{Use } s &= 335 \text{ mm} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F7-10} \quad I &= \frac{1}{12}(1)(6^3) + 4\left[\frac{1}{12}(0.5)(4^3) + 0.5(4)(3^2)\right] \\
 &= 100.67 \text{ in}^4 \\
 Q &= y'A' = 3(4)(0.5) = 6 \text{ in}^3 \\
 q_{\text{allow}} &= \frac{F}{s} = \frac{6}{s} \\
 q_{\text{allow}} &= \frac{VQ}{I}, \quad \frac{6}{s} = \frac{15(6)}{100.67} \\
 s &= 6.711 \text{ in.} \\
 \text{Use } s &= 6\frac{5}{8} \text{ in.} \quad \text{Ans.}
 \end{aligned}$$

Chapter 8

$$\begin{aligned}
 \text{F8-1} \quad + \uparrow \Sigma F_z &= (F_R)_z; -500 - 300 = P \\
 P &= -800 \text{ kN} \\
 \Sigma M_x &= 0; \quad 300(0.05) - 500(0.1) = M_x \\
 M_x &= -35 \text{ kN} \cdot \text{m} \\
 \Sigma M_y &= 0; \quad 300(0.1) - 500(0.1) = M_y \\
 M_y &= -20 \text{ kN} \cdot \text{m} \\
 A &= 0.3(0.3) = 0.09 \text{ m}^2 \\
 I_x &= I_y = \frac{1}{12}(0.3)(0.3^3) = 0.675(10^{-3}) \text{ m}^4 \\
 \sigma_A &= \frac{-800(10^3)}{0.09} + \frac{[20(10^3)](0.15)}{0.675(10^{-3})} + \frac{[35(10^3)](0.15)}{0.675(10^{-3})} \\
 &= 3.3333 \text{ MPa} = 3.33 \text{ MPa (T)} \quad \text{Ans.} \\
 \sigma_B &= \frac{-800(10^3)}{0.09} + \frac{[20(10^3)](0.15)}{0.675(10^{-3})} - \frac{[35(10^3)](0.15)}{0.675(10^{-3})} \\
 &= -12.22 \text{ MPa} = 12.2 \text{ MPa (C)} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F8-2} \quad + \uparrow \Sigma F_y &= 0; \quad V - 400 = 0 \quad V = 400 \text{ kN} \\
 \downarrow + \Sigma M_A &= 0; \quad -M - 400(0.5) = 0 \quad M = -200 \text{ kN} \cdot \text{m} \\
 I &= \frac{1}{12}(0.1)(0.3^3) = 0.225(10^{-3}) \text{ m}^4 \\
 Q_A &= y'A' = 0.1(0.1)(0.1) = 1(10^{-3}) \text{ m}^3 \\
 \sigma_A &= \frac{My}{I} = \frac{[200(10^3)](-0.05)}{0.225(10^{-3})} \\
 &= -44.44 \text{ MPa} = 44.4 \text{ MPa (C)} \quad \text{Ans.} \\
 \tau_A &= \frac{VQ}{It} = \frac{400(10^3)[1(10^{-3})]}{0.225(10^{-3})(0.1)} = 17.8 \text{ MPa} \quad \text{Ans.}
 \end{aligned}$$

$$\begin{aligned}
 \text{F8-3} \quad \text{Left reaction is } 20 \text{ kN.} \\
 \text{Left segment:} \\
 + \uparrow \Sigma F_y &= 0; \quad 20 - V = 0 \quad V = 20 \text{ kN} \\
 \downarrow + \Sigma M_x &= 0; \quad M - 20(0.5) = 0 \quad M = 10 \text{ kN} \cdot \text{m} \\
 I &= \frac{1}{12}(0.1)(0.2^3) - \frac{1}{12}(0.09)(0.18^3) \\
 &= 22.9267(10^{-6}) \text{ m}^4 \\
 Q_A &= y_1'A_1' + y_2'A_2' = 0.07(0.04)(0.01) \\
 &\quad + 0.095(0.1)(0.01) = 0.123(10^{-3}) \text{ m}^3 \\
 \sigma_A &= \frac{My_A}{I} = \frac{[10(10^3)](0.05)}{22.9267(10^{-6})} \\
 &= -21.81 \text{ MPa} = 21.8 \text{ MPa (C)} \quad \text{Ans.} \\
 \tau_A &= \frac{VQ_A}{It} = \frac{20(10^3)[0.123(10^{-3})]}{[22.9267(10^{-6})](0.01)} \\
 &= 10.7 \text{ MPa} \quad \text{Ans.}
 \end{aligned}$$

F8-4 At the section through centroidal axis:

$$N = P$$

$$V = 0$$

$$M = (2 + 1)P = 3P$$

$$\sigma = \frac{P}{A} + \frac{Mc}{I}$$

$$30 = \frac{P}{2(0.5)} + \frac{(3P)(1)}{\frac{1}{12}(0.5)(2)^3}$$

$$P = 3 \text{ kip}$$

Ans.

F8-5 At section through B:

$$N = 500 \text{ lb}, V = 400 \text{ lb}$$

$$M = 400(10) = 4000 \text{ lb} \cdot \text{in.}$$

Axial load:

$$\sigma_x = \frac{P}{A} = \frac{500}{4(3)} = 41.667 \text{ psi (T)}$$

Shear load:

$$\tau_{xy} = \frac{VQ}{It} = \frac{400[(1.5)(3)(1)]}{\frac{1}{12}(3)(4)^3(3)} = 37.5 \text{ psi}$$

Bending moment:

$$\sigma_x = \frac{My}{I} = \frac{4000(1)}{\frac{1}{12}(3)(4)^3} = 250 \text{ psi (C)}$$

Thus

$$\sigma_x = 41.667 - 250 = 208 \text{ psi (C)}$$

Ans.

$$\sigma_y = 0$$

Ans.

$$\tau_{xy} = 37.5 \text{ psi}$$

Ans.

F8-6 Top segment:

$$\Sigma F_y = 0; \quad V_y + 1000 = 0 \quad V_y = -1000 \text{ N}$$

$$\Sigma F_x = 0; \quad V_x - 1500 = 0 \quad V_x = 1500 \text{ N}$$

$$\Sigma M_z = 0; \quad T_z - 1500(0.4) = 0 \quad T_z = 600 \text{ N} \cdot \text{m}$$

$$\Sigma M_y = 0; \quad M_y - 1500(0.2) = 0 \quad M_y = 300 \text{ N} \cdot \text{m}$$

$$\Sigma M_x = 0; \quad M_x - 1000(0.2) = 0 \quad M_x = 200 \text{ N} \cdot \text{m}$$

$$I_y = I_x = \frac{\pi}{4}(0.02^4) = 40(10^{-9})\pi \text{ m}^4$$

$$J = \frac{\pi}{2}(0.02^4) = 80(10^{-9})\pi \text{ m}^4$$

$$(Q_y)_A = \frac{4(0.02)}{3\pi} \left[\frac{\pi}{2}(0.02^2) \right] = 5.3333(10^{-6}) \text{ m}^3$$

$$\sigma_A = \frac{M_x y}{I_x} + \frac{M_y z}{I_y} = \frac{200(0)}{40(10^{-9})\pi} + \frac{300(0.02)}{40(10^{-9})\pi}$$

$$= 47.7 \text{ MPa (T)}$$

Ans.

$$[(\tau_{xy})_T]_A = \frac{T_z c}{J} = \frac{600(0.02)}{80(10^{-9})\pi} = 47.746 \text{ MPa}$$

$$[(\tau_{xy})_V]_A = \frac{V_y(Q_y)_A}{I_x t} = \frac{1000[5.3333(10^{-6})]}{40(10^{-9})\pi(0.04)} \\ = 1.061 \text{ MPa}$$

Combining these two shear stress components,

$$(\tau_{xy})_A = 47.746 + 1.061 = 48.8 \text{ MPa} \quad \text{Ans.}$$

F8-7 Right Segment:

$$\Sigma F_z = 0; \quad V_z - 6 = 0 \quad V_z = 6 \text{ kN}$$

$$\Sigma M_y = 0; \quad T_y - 6(0.3) = 0 \quad T_y = 1.8 \text{ kN} \cdot \text{m}$$

$$\Sigma M_x = 0; \quad M_x - 6(0.3) = 0 \quad M_x = 1.8 \text{ kN} \cdot \text{m}$$

$$I_x = \frac{\pi}{4}(0.05^4 - 0.04^4) = 0.9225(10^{-6})\pi \text{ m}^4$$

$$J = \frac{\pi}{2}(0.05^4 - 0.04^4) = 1.845(10^{-6})\pi \text{ m}^4$$

$$(Q_z)_A = y_2^2 A_2^* - y_1^2 A_1^* \\ = \frac{4(0.05)}{3\pi} \left[\frac{\pi}{2}(0.05^2) \right] - \frac{4(0.04)}{3\pi} \left[\frac{\pi}{2}(0.04^2) \right] \\ = 40.6667(10^{-6}) \text{ m}^3$$

$$\sigma_A = \frac{M_x z}{I_x} = \frac{1.8(10^3)}{0.9225(10^{-6})\pi} = 0 \quad \text{Ans.}$$

$$[(\tau_{xy})_T]_A = \frac{T_y c}{J} = \frac{[1.8(10^3)](0.05)}{1.845(10^{-6})\pi} = 15.53 \text{ MPa}$$

$$[(\tau_{xy})_V]_A = \frac{V_z(Q_z)_A}{I_x t} = \frac{6(10^3)[40.6667(10^{-6})]}{[0.9225(10^{-6})\pi](0.02)} \\ = 4.210 \text{ MPa}$$

Combining these two shear stress components,

$$(\tau_{xy})_A = 15.53 - 4.210 = 11.3 \text{ MPa} \quad \text{Ans.}$$

F8-8 Left Segment:

$$\Sigma F_z = 0; \quad V_z - 900 - 300 = 0 \quad V_z = 1200 \text{ N}$$

$$\Sigma M_y = 0; \quad T_y + 300(0.1) - 900(0.1) = 0 \quad T_y = 60 \text{ N} \cdot \text{m}$$

$$\Sigma M_x = 0; \quad M_x + (900 + 300)0.3 = 0 \quad M_x = -360 \text{ N} \cdot \text{m}$$

$$I_x = \frac{\pi}{4}(0.025^4 - 0.02^4) = 57.65625(10^{-9})\pi \text{ m}^4$$

$$J = \frac{\pi}{2}(0.025^4 - 0.02^4) = 0.1153125(10^{-6})\pi \text{ m}^4$$

$$(Q_y)_A = 0$$

$$\sigma_A = \frac{M_x y}{I_x} = \frac{(360)(0.025)}{57.65625(10^{-9})\pi} = 49.7 \text{ MPa} \quad \text{Ans.}$$

$$[(\tau_{xy})_T]_A = \frac{T_y \rho_A}{J} = \frac{60(0.025)}{0.1153125(10^{-6})\pi} = 4.14 \text{ MPa} \quad \text{Ans.}$$

$$[(\tau_{xy})_V]_A = \frac{V_z(Q_z)_A}{I_x t} = 0 \quad \text{Ans.}$$

Chapter 9

F9-1 $\theta = 120^\circ$ $\sigma_x = 500 \text{ kPa}$ $\sigma_y = 0$ $\tau_{xy} = 0$

Apply Eqs. 9-1, 9-2.

$$\sigma_{x'} = 125 \text{ kPa}$$

$$\tau_{x'y'} = 217 \text{ kPa}$$

Ans.

Ans.

F9-2 $\theta = -45^\circ$ $\sigma_x = 0$ $\sigma_y = -400 \text{ kPa}$

$$\tau_{xy} = -300 \text{ kPa}$$

Apply Eqs. 9-1, 9-3, 9-2.

$$\sigma_{x'} = 100 \text{ kPa}$$

$$\sigma_{y'} = -500 \text{ kPa}$$

$$\tau_{x'y'} = 200 \text{ kPa}$$

Ans.

Ans.

Ans.

F9-3 $\sigma_x = 80 \text{ MPa}$ $\sigma_y = 0$ $\tau_{xy} = 30 \text{ MPa}$

Apply Eqs. 9-5, 9-4.

$$\sigma_1 = 90 \text{ MPa} \quad \sigma_2 = -10 \text{ MPa}$$

$$\theta_p = 18.43^\circ \text{ and } 108.43^\circ$$

From Eq. 9-1,

$$\begin{aligned} \sigma_{x'} &= \frac{80 + 0}{2} + \frac{80 - 0}{2} \cos 2(18.43^\circ) \\ &\quad + 30 \sin 2(18.43^\circ) \\ &= 90 \text{ MPa} = \sigma_1 \end{aligned}$$

Thus,

$$(\theta_p)_1 = 18.4^\circ \text{ and } (\theta_p)_2 = 108^\circ$$

Ans.

F9-4 $\sigma_x = 100 \text{ kPa}$ $\sigma_y = 700 \text{ kPa}$

$$\tau_{xy} = -400 \text{ kPa}$$

Apply Eqs. 9-7, 9-8.

$$\tau_{\max}^{\text{in-plane}} = 500 \text{ kPa}$$

$$\sigma_{\text{avg}} = 400 \text{ kPa}$$

Ans.

Ans.

F9-5 At the cross section through B:

$$N = 4 \text{ kN} \quad V = 2 \text{ kN}$$

$$M = 2(2) = 4 \text{ kN} \cdot \text{m}$$

$$\begin{aligned} \sigma_B &= \frac{P}{A} + \frac{Mc}{I} = \frac{4(10^3)}{0.03(0.06)} + \frac{4(10^3)(0.03)}{\frac{1}{12}(0.03)(0.06)^3} \\ &= 224 \text{ MPa (T)} \end{aligned}$$

Note $\tau_B = 0$ since $Q = 0$.

Thus

$$\sigma_1 = 224 \text{ MPa}$$

$$\sigma_2 = 0$$

Ans.

F9-6 $A_y = B_y = 12 \text{ kN}$

Segment AC:

$$V_C = 0 \quad M_C = 24 \text{ kN} \cdot \text{m}$$

$$\tau_C = 0 \text{ (since } V_C = 0)$$

$$\sigma_C = 0 \text{ (since } C \text{ is on neutral axis)}$$

$$\sigma_1 = \sigma_2 = 0$$

Ans.

F9-7 $\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{500 + 0}{2} = 250 \text{ kPa}$

The coordinates of the center C of the circle and the reference point A are

$$A(500, 0) \quad C(250, 0)$$

$$R = CA = 500 - 250 = 250 \text{ kPa}$$

$\theta = 120^\circ$ (counterclockwise). Rotate the radial line CA counterclockwise $2\theta = 240^\circ$ to the coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$.

$$\alpha = 240^\circ - 180^\circ = 60^\circ$$

$$\sigma_{x'} = 250 - 250 \cos 60^\circ = 125 \text{ kPa}$$

Ans.

$$\tau_{x'y'} = 250 \sin 60^\circ = 217 \text{ kPa}$$

Ans.

F9-8 $\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{80 + 0}{2} = 40 \text{ kPa}$

The coordinates of the center C of the circle and the reference point A are

$$A(80, 30) \quad C(40, 0)$$

$$R = CA = \sqrt{(80 - 40)^2 + 30^2} = 50 \text{ MPa}$$

$$\sigma_1 = 40 + 50 = 90 \text{ MPa}$$

Ans.

$$\sigma_2 = 40 - 50 = -10 \text{ MPa}$$

Ans.

$$\tan 2(\theta_p)_1 = \frac{30}{80 - 40} = 0.75$$

$$(\theta_p)_1 = 18.4^\circ \text{ (counterclockwise)}$$

Ans.

F9-9 $J = \frac{\pi}{2}(0.04^4 - 0.03^4) = 0.875(10^{-6})\pi \text{ m}^4$

$$\tau = \frac{Tc}{J} = \frac{4(10^3)(0.04)}{0.875(10^{-6})\pi} = 58.21 \text{ MPa}$$

$$\sigma_x = \sigma_y = 0 \text{ and } \tau_{xy} = -58.21 \text{ MPa}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = 0$$

The coordinates of the reference point A and the center C of the circle are

$$A(0, -58.21) \quad C(0, 0)$$

$$R = CA = 58.21 \text{ MPa}$$

$$\sigma_1 = 0 + 58.21 = 58.2 \text{ MPa}$$

Ans.

$$\sigma_2 = 0 - 58.21 = -58.2 \text{ MPa}$$

Ans.

F9-10

$$+\uparrow \Sigma F_y = 0;$$

$$V - 30 = 0$$

$$V = 30 \text{ kN}$$

$$+\circlearrowleft \Sigma M_O = 0;$$

$$-M - 30(0.3) = 0$$

$$M = -9 \text{ kN} \cdot \text{m}$$

Ans.

$$I = \frac{1}{12}(0.05)(0.15^3) = 14.0625(10^{-6}) \text{ m}^4$$

$$Q_A = y'A' = 0.05(0.05)(0.05) = 0.125(10^{-3}) \text{ m}^3$$

$$\sigma_A = -\frac{My_A}{I} = \frac{[-9(10^3)](0.025)}{14.0625(10^{-6})} = 16 \text{ MPa (T)}$$

$$\tau_A = \frac{VQ_A}{It} = \frac{30(10^3)[0.125(10^{-3})]}{14.0625(10^{-6})(0.05)} = 5.333 \text{ MPa}$$

$$\sigma_x = 16 \text{ MPa}, \sigma_y = 0, \text{ and } \tau_{xy} = -5.333 \text{ MPa}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{16 + 0}{2} = 8 \text{ MPa}$$

The coordinates of the reference point A and the center C of the circle are

$$A(16, -5.333) \quad C(8, 0)$$

$$R = CA = \sqrt{(16-8)^2 + (-5.333)^2} = 9.615 \text{ MPa}$$

$$\sigma_1 = 8 + 9.615 = 17.6 \text{ MPa}$$

$$\sigma_2 = 8 - 9.615 = -1.61 \text{ MPa}$$

Ans.

Ans.

F9-11

$$\downarrow + \Sigma M_B = 0; \quad 60(1) - A_y(1.5) = 0 \quad A_y = 40 \text{ kN}$$

$$+\uparrow \Sigma F_y = 0; \quad 40 - V = 0 \quad V = 40 \text{ kN}$$

$$\downarrow + \Sigma M_O = 0; \quad M - 40(0.5) = 0 \quad M = 20 \text{ kN} \cdot \text{m}$$

$$I = \frac{1}{12}(0.1)(0.2^3) - \frac{1}{12}(0.09)(0.18^3) = 22.9267(10^{-6}) \text{ m}^4$$

$$Q_A = y'A' = 0.095(0.01)(0.1) = 95(10^{-6}) \text{ m}^3$$

$$\sigma_A = -\frac{My_A}{I} = \frac{[20(10^3)](0.09)}{22.9267(10^{-6})} = -78.51 \text{ MPa}$$

$$= 78.51 \text{ MPa (C)}$$

$$\tau_A = \frac{VQ_A}{It} = \frac{40(10^3)[95(10^{-6})]}{[22.9267(10^{-6})](0.01)} = 16.57 \text{ MPa}$$

$$\sigma_x = -78.51 \text{ MPa}, \sigma_y = 0, \text{ and } \tau_{xy} = -16.57 \text{ MPa}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-78.51 + 0}{2} = -39.26 \text{ MPa}$$

The coordinates of the reference point A and the center C of the circle are

$$A(-78.51, -16.57) \quad C(-39.26, 0)$$

$$R = CA = \sqrt{[-78.51 - (-39.26)]^2 + (-16.57)^2} = 42.61 \text{ MPa}$$

$$\tau_{\text{max in-plane}} = |R| = 42.6 \text{ MPa}$$

Ans.

Chapter 11

F11-1

$$V_{\text{max}} = 12 \text{ kN} \quad M_{\text{max}} = 18 \text{ kN} \cdot \text{m}$$

$$\sigma_{\text{allow}} = \frac{M_{\text{max}} c}{I}; \quad 10(10^6) = \frac{18(10^3)(a)}{\frac{2}{3}a^4}$$

$$a = 0.1392 \text{ m} = 139.2 \text{ mm}$$

Ans.

Use $a = 140 \text{ mm}$

$$I = \frac{2}{3}(0.14^4) = 0.2561(10^{-3}) \text{ m}^4$$

$$Q_{\text{max}} = \frac{0.14}{2}(0.14)(0.14) = 1.372(10^{-3}) \text{ m}^3$$

$$\tau_{\text{max}} = \frac{V_{\text{max}} Q_{\text{max}}}{It} = \frac{12(10^3)[1.372(10^{-3})]}{[0.2561(10^{-3})](0.14)}$$

$$= 0.459 \text{ MPa} < \tau_{\text{allow}} = 1 \text{ MPa (OK)}$$

F11-2

$$V_{\text{max}} = 3 \text{ kip} \quad M_{\text{max}} = 12 \text{ kip} \cdot \text{ft}$$

$$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi d^4}{64}$$

$$\sigma_{\text{allow}} = \frac{M_{\text{max}} c}{I}; \quad 20 = \frac{12(12)\left(\frac{d}{2}\right)}{\frac{\pi d^4}{64}}$$

$$d = 4.19 \text{ in.}$$

Ans.

$$\text{Use } d = 4\frac{1}{4} \text{ in.}$$

$$I = \frac{\pi}{64}(4.25^4) = 16.015 \text{ in}^4$$

$$Q_{\text{max}} = \frac{4(4.25/2)}{3\pi}\left[\frac{1}{2}\left(\frac{\pi}{4}\right)(4.25^2)\right] = 6.397 \text{ in}^3$$

$$\tau_{\text{max}} = \frac{V_{\text{max}} Q_{\text{max}}}{It} = \frac{3(6.397)}{16.015(4.25)}$$

$$= 0.282 \text{ ksi} < \tau_{\text{allow}} = 10 \text{ ksi (OK)}$$

F11-3

$$V_{\text{max}} = 10 \text{ kN} \quad M_{\text{max}} = 5 \text{ kN} \cdot \text{m}$$

$$I = \frac{1}{12}(a)(2a)^3 = \frac{2}{3}a^4$$

$$\sigma_{\text{allow}} = \frac{M_{\text{max}} c}{I}; \quad 12(10^6) = \frac{5(10^3)(a)}{\frac{2}{3}a^4}$$

$$a = 0.0855 \text{ m} = 85.5 \text{ mm}$$

Ans.

Use $a = 86 \text{ mm}$

$$I = \frac{2}{3}(0.086^4) = 36.4672(10^{-6}) \text{ m}^4$$

$$Q_{\max} = \frac{0.086}{2}(0.086)(0.086) \\ = 0.318028(10^{-3}) \text{ m}^3$$

$$\tau_{\max} = \frac{V_{\max} Q_{\max}}{It} = \frac{10(10^3)[0.318028(10^{-3})]}{[36.4672(10^{-6})](0.086)} \\ = 1.01 \text{ MPa} < \tau_{\text{allow}} = 1.5 \text{ MPa (OK)}$$

F11-4

$$V_{\max} = 4.5 \text{ kip} \quad M_{\max} = 6.75 \text{ kip} \cdot \text{ft}$$

$$I = \frac{1}{12}(4)(h^3) = \frac{h^3}{3}$$

$$\sigma_{\text{allow}} = \frac{M_{\max} c}{I}; \quad 2 = \frac{6.75(12)\left(\frac{h}{2}\right)}{\frac{h^3}{3}} \\ h = 7.794 \text{ in.}$$

$$Q_{\max} = y'A' = \frac{h}{4}\left(\frac{h}{2}\right)(4) = \frac{h^2}{2} \\ \tau_{\max} = \frac{V_{\max} Q_{\max}}{It}; \quad 0.2 = \frac{4.5\left(\frac{h^2}{2}\right)}{\frac{h^3}{3}(4)} \\ h = 8.4375 \text{ in. (controls)}$$

$$\text{Use } h = 8\frac{1}{2} \text{ in.}$$

F11-5

$$V_{\max} = 25 \text{ kN} \quad M_{\max} = 20 \text{ kN} \cdot \text{m}$$

$$I = \frac{1}{12}(b)(3b)^3 = 2.25b^4$$

$$\sigma_{\text{allow}} = \frac{M_{\max} c}{I}; \quad 12(10^6) = \frac{20(10^3)(1.5b)}{2.25b^4} \\ b = 0.1036 \text{ m} = 103.6 \text{ mm}$$

$$\text{Use } b = 104 \text{ mm}$$

$$I = 2.25(0.104^4) = 0.2632(10^{-3}) \text{ m}^4$$

$$Q_{\max} = 0.75(0.104)[1.5(0.104)(0.104)] = 1.2655(10^{-3}) \text{ m}^3$$

$$\tau_{\max} = \frac{V_{\max} Q_{\max}}{It} = \frac{25(10^3)[1.2655(10^{-3})]}{[0.2632(10^{-3})](0.104)} \\ = 1.156 \text{ MPa} < \tau_{\text{allow}} = 1.5 \text{ MPa (OK)}$$

F11-6

$$V_{\max} = 150 \text{ kN} \quad M_{\max} = 150 \text{ kN} \cdot \text{m}$$

$$S_{\text{req'd}} = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{150(10^3)}{150(10^6)} = 0.001 \text{ m}^3 = 1000(10^3) \text{ mm}^3$$

Select W410 $\times$ 67 [$S_x = 1200(10^3) \text{ mm}^3$, $d = 410 \text{ mm}$, and $t_w = 8.76 \text{ mm}$].

Ans.

$$\tau_{\max} = \frac{V}{t_w d} = \frac{150(10^3)}{0.00876(0.41)} \\ = 41.76 \text{ MPa} < \tau_{\text{allow}} = 75 \text{ MPa (OK)}$$

Chapter 12**F12-1**

$$\downarrow + \Sigma M_O = 0; \quad M(x) = 30 \text{ kN} \cdot \text{m}$$

$$EI \frac{d^2 v}{dx^2} = 30$$

$$EI \frac{dv}{dx} = 30x + C_1$$

$$EI v = 15x^2 + C_1 x + C_2$$

$$\text{At } x = 3 \text{ m, } \frac{dv}{dx} = 0,$$

$$C_1 = -90 \text{ kN} \cdot \text{m}^2$$

$$\text{At } x = 3 \text{ m, } v = 0,$$

$$C_2 = 135 \text{ kN} \cdot \text{m}^3$$

$$\frac{dv}{dx} = \frac{1}{EI}(30x - 90)$$

$$v = \frac{1}{EI}(15x^2 - 90x + 135)$$

$$\text{For end A, } x = 0$$

$$\theta_A = \left. \frac{dv}{dx} \right|_{x=0} = -\frac{90(10^3)}{200(10^9)[65.0(10^{-6})]} = -0.00692 \text{ rad}$$

Ans.

$$v_A = v|_{x=0} = \frac{135(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.01038 \text{ m} = 10.4 \text{ mm}$$

Ans.

Ans.

F12-2

$$\downarrow + \Sigma M_O = 0; \quad M(x) = (-10x - 10) \text{ kN} \cdot \text{m}$$

$$EI \frac{d^2 v}{dx^2} = -10x - 10$$

$$EI \frac{dv}{dx} = -5x^2 - 10x + C_1$$

$$EI v = -\frac{5}{3}x^3 - 5x^2 + C_1 x + C_2$$

$$\text{At } x = 3 \text{ m, } \frac{dv}{dx} = 0,$$

$$EI(0) = -5(3^2) - 10(3) + C_1 \quad C_1 = 75 \text{ kN} \cdot \text{m}^2$$

$$\text{At } x = 3 \text{ m, } v = 0,$$

$$EI(0) = -\frac{5}{3}(3^3) - 5(3^2) + 75(3) + C_2 \quad C_2 = -135 \text{ kN} \cdot \text{m}^3$$

$$\frac{dv}{dx} = \frac{1}{EI}(-5x^2 - 10x + 75)$$

$$v = \frac{1}{EI}\left(-\frac{5}{3}x^3 - 5x^2 + 75x - 135\right)$$

For end A, $x = 0$

$$\theta_A = \left.\frac{dv}{dx}\right|_{x=0} = \frac{1}{EI}[-5(0) - 10(0) + 75]$$

$$= \frac{75(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.00577 \text{ rad}$$

Ans.

$$v_A = v|_{x=0} = \frac{1}{EI}\left[-\frac{5}{3}(0^3) - 5(0^2) + 75(0) - 135\right]$$

$$= -\frac{135(10^3)}{200(10^9)[65.0(10^{-6})]} = -0.01038 \text{ m} = -10.4 \text{ mm}$$

Ans.

F12-3

$$\zeta + \Sigma M_O = 0; \quad M(x) = \left(-\frac{3}{2}x^2 - 10x\right) \text{ kN} \cdot \text{m}$$

$$EI \frac{d^2x}{dx^2} = -\frac{3}{2}x^2 - 10x$$

$$EI \frac{dv}{dx} = -\frac{1}{2}x^3 - 5x^2 + C_1$$

$$\text{At } x = 3 \text{ m, } \frac{dv}{dx} = 0.$$

$$EI(0) = -\frac{1}{2}(3^3) - 5(3^2) + C_1 \quad C_1 = 58.5 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}\left(-\frac{1}{2}x^3 - 5x^2 + 58.5\right)$$

For end A, $x = 0$

$$\theta_A = \left.\frac{dv}{dx}\right|_{x=0} = \frac{58.5(10^3)}{200(10^9)[65.0(10^{-6})]} = 0.0045 \text{ rad}$$

Ans.

F12-4

$$A_y = 600 \text{ lb}$$

$$\zeta + \Sigma M_O = 0; \quad M(x) = (600x - 50x^2) \text{ lb} \cdot \text{ft}$$

$$EI \frac{d^2x}{dx^2} = 600x - 50x^2$$

$$EI \frac{dv}{dx} = 300x^2 - 16.667x^3 + C_1$$

$$EIv = 100x^3 - 4.1667x^4 + C_1x + C_2$$

$$\text{At } x = 0, v = 0.$$

$$EI(0) = 100(0^3) - 4.1667(0^4) + C_1(0) + C_2 \quad C_2 = 0$$

$$\text{At } x = 12 \text{ ft, } v = 0.$$

$$EI(0) = 100(12^3) - 4.1667(12^4) + C_1(12)$$

$$C_1 = -7200 \text{ lb} \cdot \text{ft}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}(300x^2 - 16.667x^3 - 7200)$$

$$v = \frac{1}{EI}(100x^3 - 4.1667x^4 - 7200x)$$

$$v_{\max} \text{ occurs where } \frac{dv}{dx} = 0.$$

$$300x^2 - 16.667x^3 - 7200 = 0$$

$$x = 6 \text{ ft}$$

Ans.

$$v = \frac{1}{EI}[100(6^3) - 4.1667(6^4) - 7200(6)]$$

$$= \frac{-27\,000(12 \text{ in./ft})^3}{1.5(10^6)\left[\frac{1}{12}(3)(6^3)\right]}$$

$$= -0.576 \text{ in.}$$

Ans.

F12-5

$$\zeta + \Sigma M_O = 0; \quad M(x) = (40 - 5x) \text{ kN} \cdot \text{m}$$

$$EI \frac{d^2x}{dx^2} = 40 - 5x$$

$$EI \frac{dv}{dx} = 40x - 2.5x^2 + C_1$$

$$EIv = 20x^2 - 0.8333x^3 + C_1x + C_2$$

$$\text{At } x = 0, v = 0.$$

$$EI(0) = 20(0^2) - 0.8333(0^3) + C_1(0) + C_2 \quad C_2 = 0$$

$$\text{At } x = 6 \text{ m, } v = 0.$$

$$EI(0) = 20(6^2) - 0.8333(6^3) + C_1(6) + 0$$

$$C_1 = -90 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}(40x - 2.5x^2 - 90)$$

$$v = \frac{1}{EI}(20x^2 - 0.8333x^3 - 90x)$$

$$v_{\max} \text{ occurs where } \frac{dv}{dx} = 0.$$

$$40x - 2.5x^2 - 90 = 0$$

$$x = 2.7085 \text{ m}$$

$$v = \frac{1}{EI}[20(2.7085^2) - 0.8333(2.7085^3) - 90(2.7085)]$$

$$= -\frac{113.60(10^3)}{200(10^9)[39.9(10^{-6})]} = -0.01424 \text{ m} = -14.2 \text{ mm}$$

Ans.

F12-6

$$\zeta + \Sigma M_O = 0; \quad M(x) = (10x + 10) \text{ kN} \cdot \text{m}$$

$$EI \frac{d^2x}{dx^2} = 10x + 10$$

$$EI \frac{dv}{dx} = 5x^2 + 10x + C_1$$

$$\text{Due to symmetry, } \frac{dv}{dx} = 0 \text{ at } x = 3 \text{ m.}$$

$$EI(0) = 5(3^2) + 10(3) + C_1 \quad C_1 = -75 \text{ kN} \cdot \text{m}^2$$

$$\frac{dv}{dx} = \frac{1}{EI}[5x^2 + 10x - 75]$$

$$\text{At } x = 0,$$

$$\frac{dv}{dx} = \frac{-75(10^3)}{200(10^9)(39.9(10^{-6}))} = -9.40(10^{-3}) \text{ rad}$$

Ans.

F12-7

Since B is a fixed support, $\theta_B = 0$.

$$\theta_A = |\theta_{A/B}| = \frac{1}{2} \left(\frac{38}{EI} + \frac{20}{EI} \right) (3) = \frac{87 \text{ kN} \cdot \text{m}^2}{EI}$$

$$= \frac{87(10^3)}{200(10^9)(65(10^{-6}))} = 0.00669 \text{ rad}$$

Ans.

$$\Delta_A = |t_{A/B}| = (1.5) \left[\frac{20}{EI} (3) \right] + 2 \left[\frac{1}{2} \left(\frac{18}{EI} \right) (3) \right]$$

$$= \frac{144(10^3)}{200(10^9)(65(10^{-6}))} = 0.01108 \text{ m} = 11.1 \text{ mm}$$

Ans.

F12-8

Since B is a fixed support, $\theta_B = 0$.

$$\theta_A = |\theta_{A/B}| = \frac{1}{2} \left(\frac{50}{EI} + \frac{20}{EI} \right) (1) + \frac{1}{2} \left(\frac{20}{EI} \right) (1) = \frac{45 \text{ kN} \cdot \text{m}^2}{EI}$$

$$= \frac{45(10^3)}{200(10^9)(126(10^{-6}))} = 0.00179 \text{ rad}$$

Ans.

$$\Delta_A = |t_{A/B}| =$$

$$(1.6667) \left[\frac{1}{2} \left(\frac{30}{EI} \right) (1) \right] + 1.5 \left[\frac{20}{EI} (1) \right] + 0.6667 \left[\frac{1}{2} \left(\frac{20}{EI} \right) (1) \right]$$

$$= \frac{61.667 \text{ kN} \cdot \text{m}^3}{EI} = \frac{61.667(10^3)}{200(10^9)(126(10^{-6}))}$$

$$= 0.002447 \text{ m} = 2.48 \text{ mm}$$

Ans.

F12-9

Since B is a fixed support, $\theta_B = 0$.

$$\theta_A = |\theta_{A/B}| = \frac{1}{2} \left[\frac{60}{EI} (1) \right] + \frac{30}{EI} (2) = \frac{90 \text{ kN} \cdot \text{m}^2}{EI}$$

$$= \frac{90(10^3)}{200(10^9)(121(10^{-6}))} = 0.00372 \text{ rad}$$

Ans.

$$\Delta_A = |t_{A/B}| = 1.6667 \left[\frac{1}{2} \left(\frac{60}{EI} \right) (1) \right] + (1) \left[\frac{30}{EI} (2) \right]$$

$$= \frac{110 \text{ kN} \cdot \text{m}^3}{EI}$$

$$= \frac{110(10^3)}{200(10^9)(121(10^{-6}))} = 0.004545 \text{ m} = 4.55 \text{ mm}$$

Ans.

F12-10

Since B is a fixed support, $\theta_B = 0$.

$$\theta_A = |\theta_{A/B}| = \frac{1}{2} \left(\frac{18}{EI} \right) (6) + \frac{1}{3} \left(\frac{9}{EI} \right) (3) = \frac{63 \text{ kip} \cdot \text{ft}^2}{EI}$$

$$= \frac{63(12^2)}{29(10^3)(245)} = 0.00128 \text{ rad}$$

Ans.

$$\Delta_A = |t_{A/B}| = 4 \left[\frac{1}{2} \left(\frac{18}{EI} \right) (6) \right] + (3 + 2.25) \left[\frac{1}{3} \left(\frac{9}{EI} \right) (3) \right]$$

$$= \frac{263.25 \text{ kip} \cdot \text{ft}^3}{EI} = \frac{263.25(12^3)}{29(10^3)(245)} = 0.0640 \text{ in.}$$

Ans.

F12-11

Due to symmetry, the slope at the midspan of the beam (point C) is zero, i.e., $\theta_C = 0$.

$$\Delta_{\max} = \Delta_C = |t_{A/C}| = (2) \left[\frac{1}{2} \left(\frac{30}{EI} \right) (3) \right] + 1.5 \left[\frac{10}{EI} (3) \right]$$

$$= \frac{135 \text{ kN} \cdot \text{m}^3}{EI}$$

$$= \frac{135(10^3)}{200(10^9)(42.8(10^{-6}))} = 0.0158 \text{ m} = 15.8 \text{ mm} \downarrow$$

Ans.

F12-12

$$t_{A/B} = 2 \left[\frac{1}{2} \left(\frac{30}{EI} \right) (6) \right] + 3 \left[\frac{10}{EI} (6) \right] = \frac{360}{EI}$$

$$\theta_B = \frac{|t_{A/B}|}{L} = \frac{360}{6} = \frac{60}{EI}$$

The maximum deflection occurs at point C where the slope of the elastic curve is zero.

$$\theta_B = \theta_{B/C}$$

$$\frac{60}{EI} = \left(\frac{10}{EI} \right) x + \frac{1}{2} \left(\frac{5x}{EI} \right) x$$

$$2.5x^2 + 10x - 60 = 0$$

$$x = 3.2915 \text{ m}$$

$$\Delta_{\max} = |t_{B/C}| =$$

$$\frac{2}{3} (3.2915) \left\{ \frac{1}{2} \left[\frac{5(3.2915)}{EI} \right] (3.2915) \right\} + \frac{1}{2} (3.2915) \left[\frac{10}{EI} (3.2915) \right]$$

$$= \frac{113.60 \text{ kN} \cdot \text{m}^3}{EI}$$

$$= \frac{113.60(10^3)}{200(10^9)(39.9(10^{-6}))} = 0.01424 \text{ m} = 14.2 \text{ mm} \downarrow$$

Ans.

F12-13

$$(v_B)_1 = \frac{Px^2}{6EI}(3L - x) = \frac{40(4^2)}{6EI}[3(6) - 4] = \frac{1493.33}{EI} \downarrow$$

$$(v_B)_2 = \frac{PL^3}{3EI} - \frac{B_y(4^3)}{3EI} = \frac{21.33B_y}{EI} \uparrow$$

$$(+ \uparrow) v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{1493.33}{EI} + \frac{21.33B_y}{EI}$$

$$B_y = 70 \text{ kN}$$

$$\rightarrow \Sigma F_x = 0; \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0; \quad 70 - 40 - A_y = 0 \quad A_y = 30 \text{ kN}$$

$$\downarrow + \Sigma M_A = 0; \quad 70(4) - 40(6) - M_A = 0$$

$$M_A = 40 \text{ kN} \cdot \text{m}$$

Ans.

Ans.

Ans.

Ans.

F12-14

To use the deflection tables, consider loading as a superposition of uniform distributed load minus a triangular load.

$$(v_B)_1 = \frac{w_0 L^4}{8EI} \downarrow \quad (v_B)_2 = \frac{w_0 L^4}{30EI} \uparrow \quad (v_B)_3 = \frac{B_y L^3}{3EI} \uparrow$$

$$(+ \uparrow) \quad v_B = 0 = (v_B)_1 + (v_B)_2 + (v_B)_3$$

$$0 = -\frac{w_0 L^4}{8EI} + \frac{w_0 L^4}{30EI} + \frac{B_y L^3}{3EI}$$

$$B_y = \frac{11w_0 L}{40}$$

$$\rightarrow \Sigma F_x = 0; \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0; \quad A_y + \frac{11w_0 L}{40} - \frac{1}{2}w_0 L = 0$$

$$A_y = \frac{9w_0 L}{40}$$

$$\downarrow + \Sigma M_A = 0; \quad M_A + \frac{11w_0 L}{40}(L) - \frac{1}{2}w_0 L\left(\frac{2}{3}L\right) = 0$$

$$M_A = \frac{7w_0 L^2}{120}$$

Ans.

Ans.

Ans.

Ans.

F12-15

$$(v_B)_1 = \frac{wL^4}{8EI} = \frac{[10(10^3)](6^4)}{8[200(10^9)][65.0(10^{-6})]} = 0.12461 \text{ m} \downarrow$$

$$(v_B)_2 = \frac{B_y L^3}{3EI} = \frac{B_y(6^3)}{3[200(10^9)][65.0(10^{-6})]} = 5.5385(10^{-6})B_y \uparrow$$

$$(+ \downarrow) \quad v_B = (v_B)_1 + (v_B)_2$$

$$0.002 = 0.12461 - 5.5385(10^{-6})B_y$$

$$B_y = 22.14(10^3) \text{ N} = 22.1 \text{ kN}$$

$$\rightarrow \Sigma F_x = 0; \quad A_x = 0$$

Ans.

Ans.

$$+\uparrow \Sigma F_y = 0; \quad A_y + 22.14 - 10(6) = 0 \quad A_y = 37.9 \text{ kN}$$

Ans.

$$\downarrow + \Sigma M_A = 0; \quad M_A + 22.14(6) - 10(6)(3) = 0$$

$$M_A = 47.2 \text{ kN} \cdot \text{m}$$

Ans.

F12-16

$$(v_B)_1 = \frac{M_O L}{6EI(2L)}[(2L)^2 - L^2] = \frac{M_O L^2}{4EI} \downarrow$$

$$(v_B)_2 = \frac{B_y(2L)^3}{48EI} = \frac{B_y L^3}{6EI} \uparrow$$

$$(+ \uparrow) \quad v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{M_O L^2}{4EI} + \frac{B_y L^3}{6EI}$$

$$B_y = \frac{3M_O}{2L}$$

Ans.

F12-17

$$(v_B)_1 = \frac{Phx}{6EIL}(L^2 - b^2 - x^2) = \frac{50(4)(6)}{6EI(12)}(12^2 - 4^2 - 6^2)$$

$$= \frac{1533.3 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

$$(v_B)_2 = \frac{B_y L^3}{48EI} = \frac{B_y(12^3)}{48EI} = \frac{36B_y}{EI} \uparrow$$

$$(+ \uparrow) \quad v_B = 0 = (v_B)_1 + (v_B)_2$$

$$0 = -\frac{1533.3 \text{ kN} \cdot \text{m}^3}{EI} + \frac{36B_y}{EI}$$

$$B_y = 42.6 \text{ kN}$$

Ans.

F12-18

$$(v_B)_1 = \frac{5wL^4}{384EI} = \frac{5[10(10^3)](12^4)}{384[200(10^9)][65.0(10^{-6})]} = 0.20769 \downarrow$$

$$(v_B)_2 = \frac{B_y L^3}{48EI} = \frac{B_y(12^3)}{48[200(10^9)][65.0(10^{-6})]} = 2.7692(10^{-6})B_y \uparrow$$

$$(+ \uparrow) \quad v_B = (v_B)_1 + (v_B)_2$$

$$0.005 = 0.20769 - 2.7692(10^{-6})B_y$$

$$B_y = 73.19(10^3) \text{ N} = 73.2 \text{ kN}$$

Ans.

Chapter 13

F13-1

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(29(10^3))(\frac{\pi}{4}(0.5)^4)}{[0.5(50)]^2} = 22.5 \text{ kip}$$

Ans.

$$\sigma = \frac{P}{A} = \frac{22.5}{\pi(0.5)^2} = 28.6 \text{ ksi} < \sigma_Y \quad \text{OK}$$

F13-2

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 (1.6)(10^3) \left[\frac{1}{12} (4)(2)^3 \right]}{[1(12)(12)]^2} = 2.03 \text{ kip}$$

F13-3

For buckling about the x axis, $K_x = 1$ and $L_x = 12 \text{ m}$.

$$P_{cr} = \frac{\pi^2 EI_x}{(K_x L_x)^2} = \frac{\pi^2 [200(10^9)] [87.3(10^{-6})]}{[1(12)]^2} = 1.197(10^6) \text{ N}$$

For buckling about the y axis, $L_y = 6 \text{ m}$ and $K_y = 1$

$$P_{cr} = \frac{\pi^2 EI_y}{(K_y L_y)^2} = \frac{\pi^2 [200(10^9)] [18.8(10^{-6})]}{[1(6)]^2} = 1.031(10^6) \text{ N (controls)}$$

$$P_{allow} = \frac{P_{cr}}{\text{F.S.}} = \frac{1.031(10^6)}{2} = 515 \text{ kN}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1.031(10^6)}{7.4(10^{-3})} = 139.30 \text{ MPa} < \sigma_Y = 250 \text{ MPa} \quad (\text{OK})$$

F13-4

$$A = \pi((0.025)^2 - (0.015)^2) = 1.257(10^{-3}) \text{ m}^2$$

$$I = \frac{1}{4} \pi((0.025)^4 - (0.015)^4) = 267.04(10^{-9}) \text{ m}^4$$

$$P = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 (200(10^9)) (267.04(10^{-9}))}{[0.5(5)]^2} = 84.3 \text{ kN} \quad \text{Ans.}$$

$$\sigma = \frac{P}{A} = \frac{84.3(10^3)}{1.257(10^{-3})} = 67.1 \text{ MPa} < 250 \text{ MPa} \quad (\text{OK})$$

F13-5

$$+\uparrow \Sigma F_y = 0; \quad F_{AB} \left(\frac{3}{5} \right) - P = 0 \quad F_{AB} = 1.6667P \text{ (T)}$$

$$\rightarrow \Sigma F_x = 0; \quad 1.6667P \left(\frac{4}{5} \right) - F_{AC} = 0$$

$$F_{AC} = 1.3333P \text{ (C)}$$

$$A = \frac{\pi}{4} (2^2) = \pi \text{ in}^2 \quad I = \frac{\pi}{4} (1^4) = \frac{\pi}{4} \text{ in}^4$$

$$P_{cr} = F_{AC} \text{ (F.S.)} = 1.3333P(2) = 2.6667P$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 [29(10^3)] \left[\frac{\pi}{4} \right]}{[1(4)(12)]^2}$$

$$P = 36.59 \text{ kip} = 36.6 \text{ kip} \quad \text{Ans.}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{2.6667(36.59)}{\pi} = 31.06 \text{ ksi} < \sigma_Y = 36 \text{ ksi} \quad (\text{OK})$$

F13-6

$$+\circlearrowleft \Sigma M_A = 0; \quad w(6)(3) - F_{BC}(6) = 0 \quad F_{BC} = 3w$$

$$A = \frac{\pi}{4} (0.05^2) = 0.625(10^{-3}) \pi \text{ m}^2 \quad I = \frac{\pi}{4} (0.025^4) = 97.65625(10^{-9}) \pi \text{ m}^4$$

$$P_{cr} = F_{BC} \text{ (F.S.)} = 3w(2) = 6w$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2 [200(10^9)] [97.65625(10^{-9}) \pi]}{[1(3)]^2}$$

$$w = 11.215(10^3) \text{ N/m} = 11.2 \text{ kN/m} \quad \text{Ans.}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{6[11.215(10^3)]}{0.625(10^{-3}) \pi} = 34.27 \text{ MPa} < \sigma_Y = 250 \text{ MPa} \quad (\text{OK})$$

Answers to Selected Problems

Chapter 1

- 1-1. (a) $F_A = 13.8 \text{ kip}$, (b) $F_A = 34.9 \text{ kN}$
- 1-2. $T_C = 250 \text{ N} \cdot \text{m}$, $T_D = 0$
- 1-3. $T_B = 150 \text{ lb} \cdot \text{ft}$, $T_C = 500 \text{ lb} \cdot \text{ft}$
- 1-5. $9.00(4) - A_y(12) = 0$, $A_y = 3.00 \text{ kip}$,
 $B_y = 6.00 \text{ kip}$, $N_D = 0$, $V_D = 0.750 \text{ kip}$,
 $M_D = 13.5 \text{ kip} \cdot \text{ft}$, $N_E = 0$, $V_E = -9.00 \text{ kip}$,
 $M_E = -24.0 \text{ kip} \cdot \text{ft}$
- 1-6. $N_C = -30.0 \text{ kN}$, $V_C = -8.00 \text{ kN}$,
 $M_C = 6.00 \text{ kN} \cdot \text{m}$
- 1-7. $P = 0.533 \text{ kN}$, $N_C = -2.00 \text{ kN}$,
 $V_C = -0.533 \text{ kN}$, $M_C = 0.400 \text{ kN} \cdot \text{m}$
- 1-9. $B_y = 3.00 \text{ kN}$, $N_D = 0$, $V_D = -1.875 \text{ kN}$,
 $M_D = 3.94 \text{ kN} \cdot \text{m}$
- 1-10. $N_A = 0$, $V_A = 450 \text{ lb}$, $M_A = -1.125 \text{ kip} \cdot \text{ft}$,
 $N_B = 0$, $V_B = 850 \text{ lb}$, $M_B = -6.325 \text{ kip} \cdot \text{ft}$,
 $V_C = 0$, $N_C = -1.20 \text{ kip}$, $M_C = -8.125 \text{ kip} \cdot \text{ft}$
- 1-11. $V_A = 77.3 \text{ lb}$, $N_A = 20.7 \text{ lb}$, $M_A = 14.5 \text{ lb} \cdot \text{in}$.
- 1-13. $N_B = -0.4 \text{ kip}$, $V_B = 0.960 \text{ kip}$,
 $-M_B - 0.16(2) - 0.8(4.25) + 0.4(1.5) = 0$,
 $M_B = -3.12 \text{ kip} \cdot \text{ft}$
- 1-14. $N_C = -0.4 \text{ kip}$, $V_C = 1.08 \text{ kip}$,
 $M_C = -6.18 \text{ kip} \cdot \text{ft}$, $N_D = 0$, $V_D = 1.45 \text{ kip}$,
 $M_D = -15.7 \text{ kip} \cdot \text{ft}$
- 1-15. $V_C = 60 \text{ N}$, $N_C = 0$, $M_C = 0.9 \text{ N} \cdot \text{m}$
- 1-17. $N_B = 5.303 \text{ kN}$, $N_{a-a} = -3.75 \text{ kN}$,
 $V_{a-a} = 1.25 \text{ kN}$, $M_{a-a} = 3.75 \text{ kN} \cdot \text{m}$,
 $N_{b-b} = -1.77 \text{ kN}$, $V_{b-b} = 3.54 \text{ kN}$,
 $M_{b-b} = 3.75 \text{ kN} \cdot \text{m}$
- 1-18. $N_C = -80 \text{ lb}$, $V_C = 0$, $M_C = -480 \text{ lb} \cdot \text{in}$.
- 1-19. $N_C = 0$, $V_C = 4.50 \text{ kip}$, $M_C = 31.5 \text{ kip} \cdot \text{ft}$
- 1-21. $N_{a-a} = 779 \text{ N}$, $V_{a-a} = 450 \text{ N}$,
 $900(0.2) - M_{a-a} = 0$, $M_{a-a} = 180 \text{ N} \cdot \text{m}$
- 1-22. $N_G = 9.81 \text{ kN}$, $V_G = 0$, $M_G = 0$
- 1-23. $N_H = -2.71 \text{ kN}$, $V_H = -20.6 \text{ kN}$,
 $M_H = -4.12 \text{ kN} \cdot \text{m}$
- 1-25. $(V_B)_x = 105 \text{ lb}$, $(V_B)_y = 0$, $(N_B)_z = 0$,
 $(M_B)_x = 0$, $(M_B)_y = 105(7.5) = 0$,
 $(M_B)_z = 788 \text{ lb} \cdot \text{ft}$, $(T_B)_z = 105(0.5) = 0$,
 $(T_B)_z = 52.5 \text{ lb} \cdot \text{ft}$
- 1-26. $(V_C)_x = -250 \text{ N}$, $(N_C)_y = 0$,
 $(V_C)_z = -240 \text{ N}$,
 $(M_C)_x = -108 \text{ N} \cdot \text{m}$,
 $(T_C)_y = 0$, $(M_C)_z = -138 \text{ N} \cdot \text{m}$
- 1-27. $(N_B)_x = 0$, $(V_B)_y = 0$, $(V_B)_z = 70.6 \text{ N}$,
 $(T_B)_x = 9.42 \text{ N} \cdot \text{m}$, $(M_B)_y = 6.23 \text{ N} \cdot \text{m}$,
 $(M_B)_z = 0$
- 1-29. $P \cos \theta - N_A = 0$, $N_A = P \cos \theta$,
 $V_A - P \sin \theta = 0$, $V_A = P \sin \theta$,
 $M_A - P[r(1 - \cos \theta)] = 0$,
 $M_A = Pr(1 - \cos \theta)$
- 1-31. $\sigma = 1.82 \text{ MPa}$
- 1-33. $V = P \cos \theta$, $N = P \sin \theta$,
 $\sigma = \frac{P}{A} \sin^2 \theta$, $\tau_{\text{avg}} = \frac{P}{2A} \sin 2\theta$
- 1-34. $\sigma_D = 13.3 \text{ MPa (C)}$, $\sigma_E = 70.7 \text{ MPa (T)}$
- 1-35. Joint A: $\sigma_{AB} = 10.7 \text{ ksi (T)}$,
 $\sigma_{AE} = 8.53 \text{ ksi (C)}$,
 Joint E: $\sigma_{ED} = 8.53 \text{ ksi (C)}$,
 $\sigma_{EB} = 4.80 \text{ ksi (T)}$,
 Joint B: $\sigma_{BC} = 23.5 \text{ ksi (T)}$,
 $\sigma_{BD} = 18.7 \text{ ksi (C)}$
- 1-37. $dF = 7.5(10^6) x^{1/2} dx$,
 $P = 40 \text{ MN}$, $d = 2.40 \text{ m}$
- 1-38. $\sigma = 66.7 \text{ psi}$, $\tau = 115 \text{ psi}$
- 1-39. $\sigma_{\text{avg}} = 5 \text{ MPa}$
- 1-41. $D_y = 650 \text{ lb}$, $E_x = 500 \text{ lb}$, $E_y = 350 \text{ lb}$,
 $C_y = 150 \text{ lb}$, $B_x = 150 \text{ lb}$,
 $F_B = F_C = 594.24 \text{ lb}$,
 $(\tau_B)_{\text{avg}} = 12.1 \text{ ksi}$
- 1-42. $(\tau_D)_{\text{avg}} = 6.62 \text{ ksi}$, $(\tau_E)_{\text{avg}} = 6.22 \text{ ksi}$
- 1-43. $(\tau_D)_{\text{avg}} = 13.2 \text{ ksi}$, $(\tau_E)_{\text{avg}} = 12.4 \text{ ksi}$
- 1-45. Joint B: $\sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{625}{1.5} = 417 \text{ psi (C)}$,
 $\sigma_{BC} = 469 \text{ psi (T)}$,
 Joint A: $\sigma_{AC} = 833 \text{ psi (T)}$
- 1-46. $\sigma = 339 \text{ MPa}$
- 1-47. $\tau_A = 138 \text{ MPa}$

- 1-49. $A_x = 9.5263P$, $A_y = 5.5P$, $F_A = 11P$,
 $P = 3.70 \text{ kN}$
- 1-50. $(\sigma_{x-x})_{avg} = 66.7 \text{ kPa}$, $(\tau_{x-y})_{avg} = 115 \text{ kPa}$
- 1-51. $P = 4 \text{ kip}$, $(\tau_{x-y})_{avg} = 250 \text{ psi}$
- 1-53. $V_h = P/4$, $V_p = P/4$, $P = 9.05 \text{ kN}$
- 1-54. $(\sigma_{avg})_s = 56.6 \text{ MPa}$, $(\sigma_{avg})_h = 31.8 \text{ MPa}$
- 1-55. $(\sigma_{avg})_{AB} = 118 \text{ MPa}$, $(\sigma_{avg})_{BC} = 58.8 \text{ MPa}$
- 1-57. $V = 12.19 \text{ kip}$, $N = 15.603 \text{ kip}$,
 Inclined plane: $\sigma' = 62.6 \text{ ksi}$, $\tau'_{avg} = 48.9 \text{ ksi}$,
 Cross section: $\sigma = 101 \text{ ksi}$, $\tau_{avg} = 0$
- 1-58. $P = 68.3 \text{ kN}$
- 1-59. $\sigma = 3.125 \text{ ksi}$, $\tau_{avg} = 1.80 \text{ ksi}$
- 1-61. $A_x = 1.732P$, $A_y = P$, $F_A = 2P$,
 $P = 15.3 \text{ kN}$
- 1-62. $(\tau_A)_{avg} = 3.71 \text{ ksi}$
- 1-63. $(\tau_B)_{avg} = 1.59 \text{ ksi}$
- 1-65. $V = 636.40 \text{ N}$, $\tau_{avg} = 509 \text{ kPa}$
- 1-66. $P = 62.5 \text{ kN}$
- 1-67. $\sigma = \frac{w_0}{2aA}(2a^2 - x^2)$
- 1-69. $A = 7.069 \text{ in}^2$, $N = 720 \text{ lb}$, $\sigma = 102 \text{ psi}$
- 1-70. $r = r_1 e^{\left(\frac{\rho g r_1^2}{2P}\right)z}$
- 1-71. $\sigma_{x-x} = 1.39 \text{ ksi}$, $\tau_{h-h} = 600 \text{ psi}$
- 1-73. $h = 2.74 \text{ in.}$, $h = 2\frac{3}{4} \text{ in.}$
- 1-74. $d = 5.71 \text{ mm}$
- 1-75. $d = 13.5 \text{ mm}$
- 1-77. Shear limitation, $t = 167 \text{ mm}$,
 Tension limitation, $b = 33.3 \text{ mm}$
- 1-78. $a = 6\frac{1}{2} \text{ in.}$
- 1-79. $d_A = 27.6 \text{ mm}$
- 1-81. $V = 1.696 \text{ kip}$, $P = 3.39 \text{ kip}$, $N = 2.827 \text{ kip}$,
 $P = 3.26 \text{ kip}$
- 1-82. $d_{BD} = 7.00 \text{ mm}$, $d_{AB} = 6.50 \text{ mm}$,
 $d_{BC} = 6.00 \text{ mm}$
- 1-83. $P = 4.43 \text{ kN}$
- 1-85. $T = 1178.10 \text{ lb}$, $F_{AB} = 1442.9 \text{ lb}$, $W = 431 \text{ lb}$
- 1-86. $d = 1\frac{1}{16} \text{ in.}$
- 1-87. $P = 90 \text{ kN}$, $A = 6.19(10^{-3}) \text{ m}^2$,
 $P_{max} = 155 \text{ kN}$
- 1-89. $21.0(10^3) = \frac{5(10^3)}{\frac{\pi}{4}d^2}$, $d = \frac{5}{8} \text{ in.}$,
 $5(10^3) = \frac{5(10^3)}{\pi(1)(h)}$, $h = \frac{3}{8} \text{ in.}$
- 1-90. $d_B = 7.08 \text{ mm}$, $d_C = 6.29 \text{ mm}$
- 1-91. $(F.S.)_B = 2.24$, $(F.S.)_C = 2.13$
- 1-93. $\sigma_{rod} = 13.26 \text{ ksi}$, $(F.S.)_{rod} = 2.71$,
 $\tau_{pin} = 11.79 \text{ ksi}$, $(F.S.)_{pin} = 1.53$
- 1-94. $w = 0.530 \text{ kip/ft}$
- 1-95. $a_A = 130 \text{ mm}$, $a_B = 300 \text{ mm}$
- 1-97. $F_{CD} = 6.70 \text{ kN}$, $F_{AB} = 8.30 \text{ kN}$,
 $d_{AB} = 6.02 \text{ mm}$, $d_{CD} = 5.41 \text{ mm}$
- 1-98. $h = 1.74 \text{ in.}$
- 1-99. $P = 55.0 \text{ kN}$
- 1-101. $N_{x-x} = 259.81 \text{ kN}$, $V_{x-x} = 150 \text{ kN}$,
 $(\sigma_{x-x})_{avg} = 7.16 \text{ MPa}$, $(\tau_{x-y})_{avg} = 4.13 \text{ MPa}$
- 1-102. $\sigma_s = 208 \text{ MPa}$, $(\tau_{avg})_A = 4.72 \text{ MPa}$,
 $(\tau_{avg})_B = 45.5 \text{ MPa}$
- 1-103. $t = \frac{1}{4} \text{ in.}$, $d_A = 1\frac{1}{8} \text{ in.}$, $d_B = \frac{13}{16} \text{ in.}$
- 1-105. $F = 3678.75 \text{ N}$, $\tau_{avg} = 61.3 \text{ MPa}$
- 1-106. $\sigma_{x-x} = 200 \text{ kPa}$, $\tau_{x-y} = 115 \text{ kPa}$
- 1-107. $\sigma_{40} = 3.98 \text{ MPa}$, $\sigma_{30} = 7.07 \text{ MPa}$,
 $\tau_{avg} = 5.09 \text{ MPa}$

Chapter 2

- 2-1. $\epsilon = 0.167 \text{ in./in.}$
- 2-2. $\epsilon = 0.0472 \text{ in./in.}$
- 2-3. $\epsilon_{CE} = 0.00250 \text{ mm/mm}$, $\epsilon_{BD} = 0.00107 \text{ mm/mm}$
- 2-5. $\delta_B = 4 \text{ mm}$, $(\epsilon_{avg})_{BD} = 0.00267 \text{ mm/mm}$,
 $(\epsilon_{avg})_{CE} = 0.005 \text{ mm/mm}$
- 2-6. $\gamma = 0.197 \text{ rad}$
- 2-7. $\epsilon_{avg} = 0.0689 \text{ in./in.}$
- 2-9. $AB = 500 \text{ mm}$, $AB' = 501.75 \text{ mm}$,
 $\Delta_D = 4.38 \text{ mm}$
- 2-10. $(\gamma_{xy})_A = 0.206 \text{ rad}$, $(\gamma_{xy})_B = -0.206 \text{ rad}$
- 2-11. $(\epsilon_{avg})_{AB} = -0.0889 \text{ mm/mm}$,
 $(\epsilon_{avg})_{BD} = -0.1875 \text{ mm/mm}$
- 2-13. $AD' = 400.01125 \text{ mm}$, $AB' = 300.00667$,
 $D'B' = 496.6014 \text{ mm}$, $DB = 500 \text{ mm}$,
 $\epsilon_{DB} = -0.00680 \text{ mm/mm}$,
 $\epsilon_{AD} = 0.0281(10^{-3}) \text{ mm/mm}$
- 2-14. $x = -0.192 \text{ in.}$, $y = -0.218 \text{ in.}$

- 2-15. $\epsilon_{AB} = 0.152 \text{ in./in.}$, $\epsilon_{AC} = 0.0274 \text{ in./in.}$
 2-17. $L_{DB} = L\sqrt{1 + (2\epsilon_{AB}\cos^2\theta + 2\epsilon_{CB}\sin^2\theta)}$
 $\epsilon_{DB} = \epsilon_{AB}\cos^2\theta + \epsilon_{CB}\sin^2\theta$
 2-18. $(\gamma_B)_{xy} = 11.6(10^{-3}) \text{ rad}$
 $(\gamma_A)_{xy} = -11.6(10^{-3}) \text{ rad}$
 2-19. $(\gamma_C)_{xy} = -11.6(10^{-3}) \text{ rad}$
 $(\gamma_D)_{xy} = 11.6(10^{-3}) \text{ rad}$
 2-21. $L_{BD} = 0.6155 \text{ m}$, $\epsilon_{avg} = 0.0258 \text{ mm/mm}$
 2-22. $(\gamma_A)_{xy} = 5.24(10^{-3}) \text{ rad}$
 2-23. $\epsilon_{AC} = 16.7(10^{-3}) \text{ mm/mm}$,
 $\epsilon_{BD} = 11.3(10^{-3}) \text{ mm/mm}$
 2-25. $A'B' = 5.08416 \text{ m}$, $AB = 5.00 \text{ m}$,
 $\epsilon_{AB} = 16.8(10^{-3}) \text{ mm/mm}$
 2-26. $(\epsilon_{avg})_{AC} = 0.0112 \text{ mm/mm}$,
 $(\epsilon_{avg})_{CD} = 0.125 \text{ mm/mm}$,
 $(\gamma_{xy})_F = 0.245 \text{ rad}$,
 $(\epsilon_{avg})_{BE} = 0.0635 \text{ mm/mm}$
 2-27. $(\epsilon_{avg})_{AD} = 0.132 \text{ mm/mm}$,
 $(\epsilon_{avg})_{CF} = -0.0687 \text{ mm/mm}$
 2-29. $\epsilon = 0.05 \cos\theta$, $\Delta L = \int_0^{90^\circ} (0.05 \cos\theta)(2 d\theta)$,
 $\Delta L = 0.100 \text{ ft}$
 2-30. $\Delta L = 0.16 \text{ ft}$
 2-31. $\epsilon_{avg} = 0.479 \text{ ft/ft}$
 2-33. $\epsilon_{AB} = \left[1 + \frac{2(v_B \sin\theta - u_A \cos\theta)}{L} \right]^{\frac{1}{2}} - 1$,
 $\epsilon_{AB} = \frac{v_B \sin\theta}{L} - \frac{u_A \cos\theta}{L}$

Chapter 3

- 3-1. $E_{approx} = \frac{1.31 - 0}{0.0004 - 0} = 3.275(10^5) \text{ ksi}$
 3-2. $E = 55.3(10^3) \text{ ksi}$, $u_x = 9.96 \frac{\text{in} \cdot \text{lb}}{\text{in}^3}$
 3-3. $(u_x)_{approx} = 85.0 \frac{\text{in} \cdot \text{lb}}{\text{in}^3}$
 3-5. $(u_x)_{approx} = 117 \text{ MJ/m}^3$
 3-6. $E = 8.83(10^3) \text{ ksi}$
 3-7. $A = 0.209 \text{ in}^2$, $P = 1.62 \text{ kip}$
 3-9. $\sigma = 1.50 \text{ ksi}$, $\epsilon = 0.035 \text{ in./in.}$, $\delta = 0.228 \text{ in.}$
 3-10. $E = 30.0(10^3) \text{ ksi}$, $P_Y = 11.8 \text{ kip}$, $P_{ult} = 19.6 \text{ kip}$
 3-11. Elastic Recovery = 0.003 in./in. ,
 $\Delta L = 0.094 \text{ in.}$

- 3-13. $\sigma = 11.43 \text{ ksi}$, $\epsilon = 0.000400 \text{ in./in.}$,
 $E = 28.6(10^3) \text{ ksi}$
 3-14. $\delta_{BD} = 0.0632 \text{ in.}$
 3-15. $P = 570 \text{ lb}$
 3-17. $\sigma_{pl} = 44 \text{ ksi}$, $\sigma_Y = 60 \text{ ksi}$, $E = 11.0(10^3) \text{ ksi}$
 3-18. $(U)_r = 88 \frac{\text{in} \cdot \text{lb}}{\text{in}^3}$, $[(U)_r]_{approx} = 6.50(10^3) \frac{\text{in} \cdot \text{lb}}{\text{in}^3}$
 3-19. $\sigma = 2.22 \text{ MPa}$
 3-21. $\sigma_{AB} = 31.83 \text{ MPa}$, $\epsilon_{AB} = 0.009885 \text{ mm/mm}$,
 $\sigma_{CD} = 7.958 \text{ MPa}$, $\epsilon_{CD} = 0.002471 \text{ mm/mm}$,
 $\alpha = 0.708^\circ$
 3-22. $P = 11.3 \text{ kN}$
 3-23. From the stress-strain diagram, the copolymer will satisfy both stress and strain requirements.
 3-25. $\sigma = 1.697 \text{ MPa}$,
 $\delta = 0.126 \text{ mm}$, $\Delta d = -0.00377 \text{ mm}$
 3-26. (a) $\delta = -0.577(10^{-3}) \text{ in.}$
 (b) $d' = 0.5000673 \text{ in.}$
 3-27. $\nu = 0.300$
 3-29. $\epsilon_{long} = -0.0002667$, $\epsilon_{lat} = 0.0000880$, $\nu = 0.330$,
 $h' = 2.000176 \text{ in.}$
 3-30. $\epsilon_y = -0.0150 \text{ in./in.}$, $\epsilon_x = 0.00540 \text{ in./in.}$,
 $\gamma_{xy} = -0.00524 \text{ rad}$
 3-31. $P = 53.0 \text{ kip}$, $E = 28.6(10^3) \text{ ksi}$
 3-33. $\tau_{avg} = 4166.67 \text{ Pa}$, $\gamma = 0.02083 \text{ rad}$,
 $\delta = 0.833 \text{ mm}$
 3-34. $\delta = \frac{Pa}{2bhG}$
 3-35. $G_{al} = 4.31(10^3) \text{ ksi}$
 3-37. $E = 5.5 \text{ psi}$, $u_t = 19.25 \text{ psi}$, $u_r = 11 \text{ psi}$
 3-38. $\delta = -0.0173 \text{ mm}$, $d' = 20.0016 \text{ mm}$
 3-39. $x = 1.53 \text{ m}$, $d'_A = 30.008 \text{ mm}$
 3-41. $\tau = 148.89 \text{ kPa}$, $G = 1.481 \text{ MPa}$, $\delta_B = 3.02 \text{ mm}$
 3-42. $\epsilon_{DE} = 0.00116 \text{ in./in.}$, $W = 112 \text{ lb}$,
 $\epsilon_{BC} = 0.00193 \text{ in./in.}$
 3-43. $\epsilon_b = 0.00227 \text{ mm/mm}$,
 $\epsilon_r = 0.000884 \text{ mm/mm}$

Chapter 4

- 4-1. $\delta_A = \frac{-5.00(10^3)(8)}{\frac{\pi}{4}(0.4^2 - 0.3^2)200(10^9)} = -3.64(10^{-3}) \text{ mm}$
 4-2. $\delta_{AD} = 0.766(10^{-3}) \text{ in.}$
 4-3. $\delta_D = 0.850 \text{ mm}$
 4-5. $\delta_A = 6.14 \text{ mm}$

- 4-6. $\delta_A = 0.0128$ in.
 4-7. $\delta_P = 0.0350$ in. $\downarrow$
 4-9. $\delta_C = 0.0055172$ in., $\delta_A = 0.0110344$ in.,
 $\delta_{F/E} = 0.0020690$ in., $\delta'_E = 0.0036782$ in.,
 $\delta_F = 0.0113$ in.
 4-10. $\theta = 0.00878^\circ$
 4-11. $\delta_j = 0.0260$ in.
 4-13. $\delta = \frac{1}{AE} \int_0^L (\gamma Ax + P) dx = \frac{\gamma L^2}{2E} + \frac{PL}{AE}$
 4-14. $\delta_{A/B} = -0.864$ mm
 4-15. $\delta_{A/B} = -1.03$ mm
 4-17. $\delta = -0.4310(10^{-3})P$, $P = 46.4$ kip
 4-18. $\delta_F = 0.0230$ in. $\downarrow$
 4-19. $\theta = 0.439(10^{-3})$ rad
 4-21. $\delta_D = 0.1374$ mm, $\delta_{A/B} = 0.3958$ mm,
 $\delta_C = 0.5332$ mm, $\delta_{tot} = 33.9$ mm
 4-22. $W = 9.69$ kN
 4-25. $\delta = 0.360$ mm
 4-26. $\delta = \frac{\gamma L^2}{6E}$
 4-27. $\delta = -\frac{P}{2a\pi r_0^2 E} (1 - e^{-2aL})$
 4-29. $A = \pi r^2 = \pi(r_0 \cos \theta)^2 = \pi r_0^2 \cos^2 \theta$,
 $y = r_0 \sin \theta$; $dy = r_0 \cos \theta d\theta$,
 When $y = \frac{r_0}{4}$; $\theta = 14.48^\circ$,
 $\delta = \frac{0.511P}{\pi r_0 E}$
 4-30. $p_0 = 250$ kN/m, $\delta = 2.93$ mm
 4-31. $\sigma_{st} = 3.14$ ksi, $\sigma_{con} = 0.455$ ksi
 4-33. $P_{st} = 57.47$ kN, $P_{con} = 22.53$ kN,
 $\sigma_{st} = 48.8$ MPa, $\sigma_{con} = 5.85$ MPa
 4-34. $\sigma_{br} = 0.341$ ksi, $\sigma_{st} = 0.654$ ksi
 4-35. $d = 2.39$ in.
 4-37. $F_D = 107.89$ kN, $\delta_{A/B} = 0.335$ mm
 4-38. $\sigma_{st} = 1.66$ ksi, $\sigma_{con} = 0.240$ ksi,
 $\delta = 0.0055$ in.
 4-39. $A_{st} = 18.2$ in², $\delta = 0.00545$ in.
 4-41. $P_{con} = 36.552 P_{st}$,
 $\sigma_{con} = 8.42$ MPa, $\sigma_{st} = 67.3$ MPa
 4-42. $d = 24.6$ mm
 4-43. $\sigma_{AB} = 26.5$ MPa, $\sigma_{EF} = 33.8$ MPa
 4-45. $F_b = 10.17(10^3)$ N, $F_t = 29.83(10^3)$ N,
 $\sigma_b = 32.4$ MPa, $\sigma_t = 34.5$ MPa
 4-46. $F_D = 20.4$ kN, $F_A = 180$ kN
 4-47. $T_{AB} = 361$ lb, $T_{A'B'} = 289$ lb
 4-49. $y = 3 - 0.025x$, $F_A = 4.09$ kip, $F_B = 2.91$ kip
 4-50. $x = 28.9$ in., $P = 60.4$ kip
 4-51. $T_{CD} = 27.2$ kip, $T_{CD} = 9.06$ kip
 4-53. $F_{st} = 1.822$ kip, $F_{al} = 3.644$ kip,
 $\sigma_{rod} = 9.28$ ksi, $\sigma_{al} = 1.16$ ksi
 4-54. $\theta = 698^\circ$
 4-55. $\sigma_{BE} = 96.3$ MPa, $\sigma_{AD} = 79.6$ MPa,
 $\sigma_{CF} = 113$ MPa
 4-57. $F_{CD} = 614.73$ lb, $F_{BC} = 454.69$ lb, $\theta = 0.0633^\circ$
 4-58. $F_A = 5.79$ kN, $F_B = 9.64$ kN, $F_C = 11.6$ kN
 4-59. $\theta = 1.14(10^{-3})^\circ$
 4-61. Assume failure of AB and EF: $F = 42\,300$ N,
 Assume failure of CD: $F_{CD} = 81\,000$ N,
 $w = 45.9$ kN/m
 4-62. $\sigma_D = 13.4$ MPa, $\sigma_{BC} = 9.55$ MPa
 4-63. $\theta = 0.00365^\circ$
 4-65. $0.02 = \delta_j + \delta_h$, $P = 1.16$ kN
 4-66. $a = 0.120$ mm
 4-67. $\sigma_{AB} = \frac{7P}{12A}$, $\sigma_{CD} = \frac{P}{3A}$, $\sigma_{EF} = \frac{P}{12A}$
 4-69. $0 = \Delta_T - \delta$, $F = 4.20$ kN
 4-70. $F = 0.509$ kip
 4-71. $F = 116$ kip
 4-73. $0 = \delta_T - \delta_F$, $F = 19.14A$, $\sigma = 19.1$ ksi
 4-74. $F = 7.60$ kip
 4-75. $\delta = 0.348$ in., $F = 19.5$ kip
 4-77. $0 = \Delta_T - \delta_F$, $F = \frac{\alpha AE}{2}(T_B - T_A)$
 4-78. $F_B = 183$ kN, $F_A = 383$ kN
 4-79. $P = 188$ kN
 4-81. $620.136 = 75F_{AB} + 48F_{AD}$,
 $F_{AD} = 6.54$ kip, $F_{AC} = F_{AB} = 4.09$ kip
 4-82. $\delta_A = 0.0407$ in. $\uparrow$
 4-83. $F_{AC} = 10.0$ lb, $F_{AD} = 136$ lb
 4-85. $F = 107\,442.47$ N, $T = 172^\circ$ C
 4-86. $\sigma_s = 40.1$ MPa, $\sigma_h = 29.5$ MPa
 4-87. $\sigma_{max} = 190$ MPa
 4-89. $K = 2.45$, $w = 2.49$ in.
 4-90. $P = 77.1$ kN, $\delta = 0.429$ mm
 4-91. $P = 1.21$ kip
 4-93. Maximum normal stress at fillet: $K = 1.4$,
 Maximum normal stress at the hole: $K = 2.65$,
 $\sigma_{max} = 88.3$ MPa
 4-94. $P = 15$ kip, $K = 1.60$

- 4-95. $P = 16.8 \text{ kip}$, $K = 1.29$
 4-97. $P_{al} = 156.91 \text{ kip}$, $P_{al} = 143.09 \text{ kip}$,
 $\sigma_{al} = 36.0 \text{ ksi}$, $\sigma_{al} = 19.8 \text{ ksi}$
 4-98. $\delta_C = 0.432 \text{ in.}$
 4-99. $w = 21.9 \text{ kN/m}$, $\delta_G = 4.24 \text{ mm}$
 4-101. $F_{CD} = 1800 \text{ N}$, $F_{AB} = 3600 \text{ N}$,
 $F_{AB} = 3.14 \text{ kN}$, $F_{CD} = 2.72 \text{ kN}$,
 $\delta_{CD} = 0.324 \text{ mm}$, $\delta_{AB} = 0.649 \text{ mm}$
 4-102. (a) $P = 2.62 \text{ kN}$, (b) $P = 3.14 \text{ kN}$
 4-103. $P = \sigma_Y A(2 \cos \theta + 1)$, $\delta_A = \frac{\sigma_Y L}{E \cos \theta}$
 4-105. $\sigma_{CF} = 250 \text{ MPa (T)}$, $F_{CF} = 122\,718.46 \text{ N}$,
 $F_{BE} = 91\,844.61 \text{ N}$, $F_{AD} = 15\,436.93 \text{ N}$,
 $(\sigma_{CF})_t = 17.7 \text{ MPa (C)}$,
 $(\sigma_{BE})_t = 53.2 \text{ MPa (T)}$,
 $(\sigma_{AD})_t = 35.5 \text{ MPa (C)}$
 4-106. $\sigma_A = 53.33 \text{ ksi}$, $\delta = 8.69 \text{ in.}$
 4-107. $w = 10.9 \text{ kip/ft}$
 4-109. $(F_{al})_y = 56.55 \text{ kN}$, $F_{al} = 146.9 \text{ kN}$,
 $d_B = 17.8 \text{ mm}$
 4-110. (a) $\delta_D = 0.375 \text{ in.}$
 (b) $\delta_D = 6.40 \text{ in.}$
 4-111. $P = 126 \text{ kip}$, $\Delta\delta = 0.00720 \text{ in.} \leftarrow$
 4-113. $\delta = \frac{1}{A^2 c^2} \int_0^L (\gamma A x)^2 dx$, $\delta = \frac{\gamma^3 L^3}{3c^2}$
 4-114. $F_B = 2.13 \text{ kip}$, $F_A = 2.14 \text{ kip}$
 4-115. $P = 4.85 \text{ kip}$
 4-117. $P = 46.4 \text{ kip}$, $\sigma_{AB} = 145 \text{ ksi}$
 4-118. $P = 56.5 \text{ kN}$, $\delta_{B/A} = 0.0918 \text{ mm}$
 4-119. $\theta = \frac{3E_2 L(T_2 - T_1)(\alpha_2 - \alpha_1)}{d(5E_2 + E_1)}$

Chapter 5

- 5-1. (a) $T = 7.95 \text{ kip} \cdot \text{in.}$, (b) $T' = 6.38 \text{ kip} \cdot \text{in.}$,
 $\tau_{p=0.5 \text{ in.}} = \frac{6.381(0.5)}{\frac{\pi}{2}(0.75^4 - 0.5^4)} = 8.00 \text{ ksi}$
 5-2. (a) $r' = 0.841r$, (b) $r' = 0.841r$
 5-3. $\tau_B = 6.04 \text{ MPa}$, $\tau_A = 6.04 \text{ MPa}$
 5-5. $\tau_{\max} = 26.7 \text{ MPa}$
 5-6. $(\tau_{BC})_{\max} = 5.07 \text{ ksi}$, $(\tau_{DE})_{\max} = 3.62 \text{ ksi}$
 5-7. $(\tau_{EF})_{\max} = 0$, $(\tau_{CD})_{\max} = 2.17 \text{ ksi}$
 5-9. $J = 2.545(10^{-6}) \text{ m}^4$,
 $\tau_{\max} = 11.9 \text{ MPa}$
 5-10. $n = \frac{2r^3}{Rd^2}$
 5-11. $\tau_{AB} = 7.82 \text{ ksi}$, $\tau_{BC} = 2.36 \text{ ksi}$
 5-13. $F = 600 \text{ N}$, $T_A = 30.0 \text{ N} \cdot \text{m}$,
 $(\tau_{EA})_{\max} = 5.66 \text{ MPa}$, $(\tau_{CD})_{\max} = 8.91 \text{ MPa}$
 5-14. $(\tau_{\max})_{\text{abs}} = 10.2 \text{ MPa}$
 5-15. $d = 33 \text{ mm}$
 5-17. $T_A = 960 \text{ lb} \cdot \text{in.}$, $J = 0.03125\pi \text{ in.}^4$,
 $\tau_{\max} = 4.89 \text{ ksi}$
 5-18. $\tau_{\max} = 7.33 \text{ ksi}$
 5-19. $(\tau_{\max})_{AB} = 41.4 \text{ MPa}$, $(\tau_{\max})_{BC} = 82.8 \text{ MPa}$
 5-21. $T_{AB} = (2000x - 1200) \text{ N} \cdot \text{m}$,
 $d = 0.9 \text{ m}$, $\tau_{\min} = 0$,
 $d = 0$, $\tau_{\max} = 42.4 \text{ MPa}$
 5-22. $d = 57 \text{ mm}$
 5-25. $T_{\max} = 260.42 \text{ lb} \cdot \text{ft}$, $\tau_{\text{abs}} = 3.59 \text{ ksi}$
 5-26. $\tau_{\max} = \frac{T}{2\pi r_i^2 h}$
 5-27. $(\tau_{AB})_{\max} = 23.9 \text{ MPa}$, $(\tau_{BC})_{\max} = 15.9 \text{ MPa}$
 5-29. $T_A + \frac{1}{2}J_A L - T_B = 0$, $T_B = \frac{2T_A + J_A L}{2}$,
 $\tau_{\max} = \frac{(2T_A + J_A L)r_o}{\pi(r_o^4 - r_i^4)}$
 5-30. $c = (2.98x) \text{ mm}$
 5-31. $(\tau_{AB})_{\max} = 1.04 \text{ MPa}$, $(\tau_{BC})_{\max} = 3.11 \text{ MPa}$
 5-33. $P = 1100 \text{ ft} \cdot \text{lb/s}$,
 $T = 280.11 \text{ lb} \cdot \text{in.}$,
 $\tau_{\max} = 1.43 \text{ ksi}$
 5-34. $d = \frac{7}{8} \text{ in.}$
 5-35. $\omega = 21.7 \text{ rad/s}$
 5-37. $P = 990\,000 \text{ ft} \cdot \text{lb/s}$,
 $T = 6302.54 \text{ lb} \cdot \text{ft}$,
 $\tau_{\max} = 6.02 \text{ ksi}$
 5-38. $d_A = 12.4 \text{ mm}$, $d_B = 16.8 \text{ mm}$
 5-39. $(\tau_{\max})_{CF} = 12.5 \text{ MPa}$,
 $(\tau_{\max})_{BC} = 7.26 \text{ MPa}$
 5-41. $T = 625 \text{ N} \cdot \text{m}$,
 $t = 2.5 \text{ mm}$
 5-42. $\omega = 17.7 \text{ rad/s}$
 5-43. $d = 2\frac{1}{2} \text{ in.}$
 5-45. $T = 525.21 \text{ lb} \cdot \text{ft}$,
 $r_i = 1.1460 \text{ in.}$,
 $t = 0.104 \text{ in.}$

- 5-46. $d = \frac{7}{8}$ in.
- 5-47. $\tau_{\max} = 44.3$ MPa, $\phi = 11.9^\circ$
- 5-49. $T_{AB} = -85$ N·m,
 $T_{BC} = -85$ N·m,
 $\phi_{A/D} = 0.879^\circ$
- 5-50. $\tau_{\max} = 2.83$ ksi,
 $\phi = 4.43^\circ$
- 5-51. $d = 2.75$ in.
- 5-53. $T_{BC} = -80$ N·m,
 $T_{CD} = -60$ N·m,
 $T_{DA} = -90$ N·m,
 $\phi_B = |5.74^\circ|$
- 5-54. $\phi_D = 1.01^\circ$
- 5-55. $\phi_C = 0.227^\circ$
- 5-57. $T_M = 175.07$ lb·ft, $T_C = 109.42$ lb·ft,
 $T_D = 65.65$ lb·ft, $d = 1\frac{1}{4}$ in.
- 5-58. $\tau_{\max}^{\text{abs}} = 3.17$ ksi,
 $\phi_{C/D} = 0.0661^\circ$
- 5-59. $\phi_{B/D} = 1.15^\circ$
- 5-61. $\phi_E = 0.01778$ rad,
 $\phi_F = 0.02667$ rad,
 $\phi_B = 1.53^\circ$
- 5-62. $\phi_A = 1.78^\circ$
- 5-63. $(\tau_{BC})_{\max} = 10.2$ ksi,
 $(\tau_{BA})_{\max} = 1.86$ ksi,
 $\phi_C = 2.66^\circ$
- 5-65. $\phi_B = 0.001852$ rad,
 $\phi_{C/B} = -0.0001119$ rad,
 $\phi_C = 0.113^\circ$
- 5-66. $\phi_A = 2.66^\circ$,
 $\phi_C = 2.30^\circ$
- 5-67. $\phi_A = \phi_B + \phi_{A/B}$,
 $\phi_C = \phi_B + \phi_{C/B}$,
 $T_1 = 2.19$ kN·m,
 $T_2 = 3.28$ kN·m
- 5-69. $\phi_B = 0.01194$ rad,
 $\phi_C = 0.008952$ rad,
 $\phi_E = 1.20^\circ$
- 5-70. $\phi_D = 1.42^\circ$
- 5-73. $J(x) = \frac{\pi r^4}{2L^4}(L+x)^4$,
 $\phi = \frac{7TL}{12\pi r^4 G}$
- 5-74. $t_o = \frac{4pd}{L}$,
 $\phi = \frac{4PLd}{3\pi r^4 G}$
- 5-75. $\phi = \frac{2L(I_0L + 3T_A)}{3\pi(r_o^4 - r_i^4)G}$
- 5-77. $T_A = 200$ N·m,
 $T_B = 100$ N·m,
 $(\tau_{AC})_{\max} = 8.15$ MPa,
 $(\tau_{CB})_{\max} = 4.07$ MPa
- 5-78. $\tau_{AC} = 9.77$ MPa
- 5-79. $\tau_{AC} = 29.3$ ksi
- 5-81. $T_A = 1.498$ kN·m,
 $T_B = 0.502$ kN·m,
 $\tau_{CD} = 24.9$ MPa
- 5-82. $\phi_C = 0.116^\circ$,
 $(\tau_w)_{\max} = 395$ psi,
 $(\gamma_{st})_{\max} = 34.3(10^{-6})$ rad,
 $(\tau_{br})_{\max} = 96.1$ psi,
 $(\gamma_{rd})_{\max} = 17.2(10^{-6})$ rad
- 5-83. $(\tau_{BC})_{\max} = 1.47$ ksi,
 $(\tau_{BD})_{\max} = 1.96$ ksi,
 $\phi = 0.338^\circ$
- 5-85. $T_R = (300x - 2.5x^2)$ lb·in., $\phi_B = 1.75^\circ$,
 $\tau_{\max}^{\text{abs}} = 13.3$ ksi
- 5-86. $T_B = 222$ N·m,
 $T_A = 55.6$ N·m
- 5-87. $\phi_E = 1.66^\circ$
- 5-89. $F = 4.412$ kip,
 $T_E = 4.412$ kip·ft,
 $T_A = 12.79$ kip·ft,
 $\phi_B = 0.955^\circ$
- 5-90. $(\tau_{BD})_{\max} = 4.35$ ksi,
 $(\tau_{AC})_{\max} = 2.17$ ksi
- 5-91. $\tau_{\max}^{\text{abs}} = 5.50$ ksi
- 5-93. $J(x) = \frac{\pi c^4}{2L^4}(L+x)^4$,
 $T_B = \frac{37}{189}T$,
 $T_A = \frac{152}{189}T$

$$5-94. \quad T_B = \frac{7t_0 L}{12},$$

$$T_A = \frac{3t_0 L}{4}$$

$$5-95. \quad (\tau_c)_{\max} = 525 \text{ psi},$$

$$(\tau_r)_{\max} = 713 \text{ psi},$$

$$\phi_c = 0.0582^\circ,$$

$$\phi_r = 0.0657^\circ$$

$$5-97. \quad (\tau_{\max})_c = \frac{16T}{\pi d^3},$$

$$(\tau_{\max})_c = \frac{16T}{\pi k^2 d^3},$$

$$\text{Factor of increase in shear stress} = \frac{1}{k^2}$$

$$5-98. \quad (\tau_{BC})_{\max} = 0.955 \text{ MPa},$$

$$(\tau_{AC})_{\max} = 1.59 \text{ MPa},$$

$$\phi_{B/A} = 0.207^\circ$$

$$5-99. \quad (\tau_{BC})_{\max} = 0.955 \text{ MPa},$$

$$(\tau_{AC})_{\max} = 1.59 \text{ MPa},$$

$$\phi_{B/C} = [0.0643^\circ]$$

$$5-101. \quad \text{For segment } AB, T = 3180.86 \text{ N} \cdot \text{m},$$

$$\text{For segment } BC, T = 11\,366.94 \text{ N} \cdot \text{m},$$

$$T = 2.80 \text{ kN} \cdot \text{m}$$

$$5-102. \quad T_B = 32 \text{ lb} \cdot \text{ft},$$

$$T_A = 48 \text{ lb} \cdot \text{ft},$$

$$\phi_C = 0.0925^\circ$$

$$5-103. \quad (\tau_{\max})_A = 308 \text{ MPa}$$

$$5-105. \quad T = 1663.2 \text{ lb} \cdot \text{in.}, F = 104 \text{ lb}$$

$$5-106. \quad \tau_{\max} = 2.31 \text{ ksi},$$

$$\delta_F = 0.0303 \text{ in.}$$

$$5-107. \quad t = 0.104 \text{ in.}$$

$$5-109. \quad A_m = 1.4498 \text{ in}^2,$$

$$A_m' = 2.4002 \text{ in}^2,$$

$$\text{Factor} = 1.66$$

$$5-110. \quad \text{The factor of increase} = 2.85$$

$$5-111. \quad q_{\alpha} = \frac{\pi}{4} q_{\alpha'}$$

$$5-113. \quad A_m = 7959.50 \text{ in}^2,$$

$$t = \frac{5}{16} \text{ in.},$$

$$\phi = 0.0536^\circ$$

$$5-114. \quad \phi = 5134 \text{ kip} \cdot \text{ft}$$

$$5-115. \quad (\tau_{avg})_A = 15.6 \text{ MPa},$$

$$(\tau_{avg})_B = 10.4 \text{ MPa}$$

$$5-117. \quad A_m = 1.8927 \text{ m}^2,$$

$$T = 4.73 \text{ MN} \cdot \text{m},$$

$$\phi = 0.428^\circ/\text{m}$$

$$5-118. \quad \tau_{avg} = 119 \text{ MPa},$$

$$\phi = 0.407^\circ/\text{m}$$

$$5-119. \quad (\tau_{avg})_A = (\tau_{avg})_B = 357 \text{ kPa}$$

$$5-121. \quad K = 1.28,$$

$$r = 7.98 \text{ mm},$$

No, it is not possible.

$$5-122. \quad P = 101 \text{ kW}$$

$$5-123. \quad (\tau_{\max})_f = 50.6 \text{ MPa}$$

$$5-125. \quad K = 1.40, r = 0.075 \text{ in.}$$

$$5-127. \quad T = 2.71 \text{ kip} \cdot \text{ft},$$

$$T_P = 2.79 \text{ kip} \cdot \text{ft}$$

$$5-129. \quad T = 20.8 \text{ kN} \cdot \text{m},$$

$$\phi = 34.4^\circ,$$

$$G = 40 \text{ GPa},$$

$$\phi' = 0.3875 \text{ rad},$$

$$\phi_r = 12.2^\circ$$

$$5-130. \quad T = 14.4 \text{ kip} \cdot \text{ft}$$

$$5-131. \quad T_Y = 12.6 \text{ kN} \cdot \text{m},$$

$$T_P = 16.8 \text{ kN} \cdot \text{m}$$

$$5-133. \quad T = 110 \text{ lb} \cdot \text{ft}$$

$$5-134. \quad T_P = \frac{2}{3} \pi \tau_Y (c_o^3 - c_i^3),$$

$$\phi = \frac{\tau_Y L}{c_i G},$$

$$\gamma_{\max} = \frac{c_o \tau_Y}{c_i G}$$

$$5-135. \quad T_C = 9.3 \text{ kN} \cdot \text{m}, T_A = 5.70 \text{ kN} \cdot \text{m}$$

$$5-137. \quad \rho_Y = 0.00625 \text{ m},$$

$$\tau_1 = 8(10^9) \rho,$$

$$\tau_2 = 4(10^9) \rho + 25(10^6),$$

$$T = 3.27 \text{ kN} \cdot \text{m},$$

$$\phi = 34.4^\circ$$

$$5-138. \quad T = 39.2 \text{ kip} \cdot \text{ft}, \phi_P = 0.413^\circ$$

$$5-139. \quad T = 41.2 \text{ kip} \cdot \text{ft}, \text{ at } 3 \text{ in.}, \tau = 2.44 \text{ ksi},$$

$$\text{at } 1.5 \text{ in.}, \tau = -3.78 \text{ ksi}$$

$$5-141. \quad \text{Elastic, } T_t = 9256.95 \text{ N} \cdot \text{m}, T_c = 5743.05 \text{ N} \cdot \text{m},$$

$$T_t > (T_Y)_t, \text{ Plastic, } T_t = 7.39 \text{ kN} \cdot \text{m},$$

$$T_c = 7.61 \text{ kN} \cdot \text{m}$$

$$5-142. \quad \tau_{\max} = \frac{19T}{12\pi r^3}$$

$$5-145. \quad r_o = 0.0625 \text{ m}, r_i = 0.0575 \text{ m},$$

$$\text{Eq. 5-7: } \tau_{P=0.06 \text{ m}} = 88.27 \text{ MPa},$$

$$\text{Eq. 5-18: } \tau_{avg} = 88.42 \text{ MPa},$$

$$\text{Eq. 5-15: } \phi = 4.495^\circ,$$

$$\text{Eq. 5-20: } \phi = 4.503^\circ$$

- 5-146. $T = 331 \text{ N} \cdot \text{m}$
 5-147. $t = 8 \text{ mm}$
 5-149. $T = 71.5 \text{ N} \cdot \text{m}$, $\tau_{\max} = 23.3 \text{ MPa}$
 5-150. $\tau_{\max} = 82.0 \text{ MPa}$
 5-151. $F = 26.2 \text{ N}$, $\phi = 1.86^\circ$

Chapter 6

- 6-1. $x = 0.25^-$, $V = -24$, $M = -6$
 6-2. $x = 2^-$, $V = 1$, $M = 2$, $x = 4^-$, $V = 1$, $M = 6$
 6-3. $x = 3^-$, $V = -2000$, $M = -6000$
 6-5. $x = 2^+$, $V = 8$, $M = -39$
 6-6. $x = 1.5$, $V = 0$, $M = 9$, $x = 4^-$,
 $V = -20$, $M = -16$
 6-7. $x = 4^-$, $V = -6$, $M = -24$
 6-9. $x = 4^+$, $V = -3.33$, $M = 46.7$
 6-10. $x = 1.5^-$, $V = 150$, $M = 225$
 6-11. $x = 6^-$, $V = -800$, $M = -4800$
 6-13. $x = 3a^-$, $V = -P$, $M = -Pa$
 6-14. $x = 14^+$, $V = 115$, $M = -3875$
 6-17. $x = 6$, $V = -900$, $M = -3000$,
 $V = [-300 - 16.67x^2] \text{ lb}$,
 $M = [-300x - 5.556x^3] \text{ lb} \cdot \text{ft}$
 6-18. $V = [30.0 - 2x] \text{ kip}$,
 $M = [-x^2 + 30.0x - 216] \text{ kip} \cdot \text{ft}$,
 $V = 8.00 \text{ kip}$,
 $M = [8.00x - 120] \text{ kip} \cdot \text{ft}$
 6-19. $x = 5^-$, $V = -10$, $M = -25$
 6-21. $x = 0.75$, $V = 0$, $M = 0.5625$,
 $F_{BC} = 7.5 \text{ kN}$,
 $A_y = 1.5 \text{ kN}$
 6-22. $x = 3^-$, $V = -10$, $M = -18$
 6-23. $x = L$, $V = -wL$, $M = 0$
 6-25. $x = L$, $V = 0$, $M = 0$
 6-27. $x = (L/3)^-$, $V = -w_0L/18$,
 $M = -0.00617w_0L^2$
 6-29. $x = 4.11$, $V = 0$, $M = 25.7$,
 At $x = 4.108 \text{ m}$, $M = 25.67 \text{ kN} \cdot \text{m}$,
 At $x = 4.5 \text{ m}$, $M = 25.31 \text{ kN} \cdot \text{m}$
 6-30. $x = 2.54$, $V = 0$, $M = 346$
 6-31. $V = \frac{w_0}{4}(3L - 4x)$,
 $M = \frac{w_0}{24}(-12x^2 + 18Lx - 7L^2)$,
 $V = \frac{w_0}{L}(L - x)^2$, $M = -\frac{w_0}{3L}(L - x)^3$
 6-33. $w = 40.0 \text{ lb/ft}$,
 $V = 30.0 \text{ lb}$,
 $M = 15.0 \text{ lb} \cdot \text{ft}$
 6-34. $x = 3^-$, $V = -11.5$, $M = -21$
 6-35. $V = 200 \text{ N}$, $M = (200x) \text{ N} \cdot \text{m}$,
 $V = \left\{ -\frac{100}{3}x^2 + 500 \right\} \text{ N}$,
 $M = \left\{ -\frac{100}{9}x^3 + 500x - 600 \right\} \text{ N} \cdot \text{m}$
 6-37. $x = 4.5$, $V = 0$, $M = 169$
 6-38. $A_y = 9.375 \text{ kip}$, $A_x = 0$
 6-39. $x = L$, $V = -\frac{23}{54}wL$, $M = -\frac{5}{54}wL^2$
 6-41. $x = 4^-$, $V = -2.8$, $M = -2.4$
 6-42. $x = 1$, $V = 0$, $M = 2.50$
 6-43. $x = 14$, $V = 0$, $M = 24$
 6-45. $x = 0.630L$, $V = 0$, $M = 0.0394w_0L^2$,
 $M = \frac{w_0Lx}{12} - \frac{w_0x^4}{12L^2}$
 6-46. $x = 0$, $V = 2w_0L/\pi$, $M = -w_0L^2/\pi$
 6-47. $\sigma_{\max} = 120 \text{ MPa}$,
 $\sigma_{\max} = 90 \text{ MPa}$
 6-49. $\bar{y} = 3.40 \text{ in.}$,
 $I_{NA} = 91.73 \text{ in}^4$,
 $(\sigma_t)_{\max} = 3.72 \text{ ksi}$, $(\sigma_c)_{\max} = 1.78 \text{ ksi}$
 6-50. $\sigma_A = 6.81 \text{ MPa}$,
 $\sigma_B = 1.01 \text{ MPa}$,
 $\sigma_C = 4.14 \text{ MPa}$
 6-51. $M = 771 \text{ N} \cdot \text{m}$
 6-53. $I = 91.14583(10^{-6}) \text{ m}^4$,
 $M = 36.5 \text{ kN} \cdot \text{m}$, $\sigma_{\max} = 40.0 \text{ MPa}$
 6-54. $\sigma_{\max} = 2.06 \text{ MPa}$
 6-55. $F = 4.56 \text{ kN}$
 6-57. $I = 17.8133(10^{-6}) \text{ m}^4$,
 $\sigma_{\max} = 49.4 \text{ MPa}$
 6-58. $(\sigma_{\max})_T = 23.8 \text{ ksi (T)}$,
 $(\sigma_{\max})_C = 20.0 \text{ ksi (C)}$
 6-59. $M = 101 \text{ kip} \cdot \text{ft}$
 6-61. $\bar{y} = 9.3043 \text{ in.}$, $I = 1093.07 \text{ in}^4$,
 $\sigma_A = 2.0544 \text{ ksi}$, $\sigma_D = 0.2978 \text{ ksi}$,
 $(F_R)_C = 11.8 \text{ kip}$
 6-62. $\sigma_A = 6.21 \text{ MPa (C)}$,
 $\sigma_B = 5.17 \text{ MPa (T)}$
 6-63. $a = 1.68r$

- 6-65. $I = 1863 \text{ in}^4$, $\sigma_{\max} = 193 \text{ psi}$, $\sigma = 155 \text{ psi}$
- 6-66. $F_R = 3.13 \text{ kip}$
- 6-67. $\sigma_{\max} = 158 \text{ MPa}$
- 6-69. $I_o = 0.21645(10^{-3}) \text{ m}^4$,
 $I_b = 0.36135(10^{-3}) \text{ m}^4$,
 $\sigma_{\max} = 74.7 \text{ MPa}$
- 6-70. $\sigma_{\max} = 22.1 \text{ ksi}$
- 6-71. $\sigma_{\max} = 12.2 \text{ ksi}$
- 6-73. $I = 152.344 \text{ in}^4$, $\sigma_{\max} = 10.0 \text{ ksi}$
- 6-74. $\sigma_{\max} = 21.1 \text{ ksi}$
- 6-75. $\sigma_{\text{allow}} = 52.8 \text{ MPa}$
- 6-77. $I = 152.344 \text{ in}^4$, $w = 1.65 \text{ kip/ft}$
- 6-78. $\sigma_{\max} = 66.8 \text{ ksi}$
- 6-79. $\sigma_{\max} = 15.6 \text{ ksi}$
- 6-81. $w = 3.75 \text{ kip/ft}$, $\sigma_{\max} = 1.25 \text{ ksi}$
- 6-82. $t = 5\frac{1}{2} \text{ in.}$
- 6-83. $\sigma_{\max} = 129 \text{ MPa}$
- 6-85. $b = 53.1 \text{ mm}$
- 6-86. $\sigma_{\max} = 19.1 \text{ ksi}$
- 6-87. $d = 2 \text{ in.}$
- 6-89. $M_{\max} = 7.50 \text{ kN} \cdot \text{m}$, $a = 66.9 \text{ mm}$
- 6-90. $\sigma_{\max} = \frac{23w_0 I_x^2}{36bh^2}$
- 6-91. $\sigma_{\max} = 119 \text{ MPa}$
- 6-93. $\bar{y} = 0.012848 \text{ m}$, $I = 0.79925(10^{-6}) \text{ m}^4$,
 $\epsilon_{\max} = 0.711(10^{-3}) \text{ mm/mm}$
- 6-94. $d = 116 \text{ mm}$
- 6-95. $d = 199 \text{ mm}$
- 6-97. $M_{\max} = 2P$, $P = 119 \text{ lb}$
- 6-98. $\sigma_{\max} = 7.59 \text{ ksi}$
- 6-99. $\sigma_{\max} = 5.60 \text{ ksi}$
- 6-101. $I = 204.84375 \text{ in}^4$, $\sigma_{\max} = 19.8 \text{ ksi}$
- 6-102. $\sigma_B = 13.3 \text{ ksi}$, $\sigma_A = 11.8 \text{ ksi}$
- 6-103. $\omega = 11.25 \text{ kN/m}$
- 6-105. $I = \frac{2}{3}b^4$, $b = 7\frac{1}{2} \text{ in.}$
- 6-106. $\sigma_{\max} = 147 \text{ psi}$
- 6-107. $c = \frac{h\sqrt{E_c}}{\sqrt{E_f} + \sqrt{E_c}}$,
 $(\sigma_{\max})_f = \frac{3M}{bh^2} \left(\frac{\sqrt{E_f} + \sqrt{E_c}}{\sqrt{E_f}} \right)$
- 6-109. $M_y = -14.14 \text{ kip} \cdot \text{ft}$, $M_z = -14.14 \text{ kip} \cdot \text{ft}$,
 $I_y = 736 \text{ in}^4$, $I_z = 1584 \text{ in}^4$,
 $\sigma_{\max} = 2.01 \text{ ksi (T)}$, $\sigma_{\max} = 2.01 \text{ ksi (C)}$,
 $\alpha = 65.1^\circ$
- 6-110. $M = 119 \text{ kip} \cdot \text{ft}$
- 6-111. $\bar{y} = 57.4 \text{ mm}$, $\sigma_A = 1.30 \text{ MPa (C)}$,
 $\sigma_B = 0.587 \text{ MPa (T)}$,
 $\alpha = -3.74^\circ$
- 6-114. $\sigma_A = 8.95 \text{ ksi}$
- 6-115. $\sigma_B = 7.81 \text{ ksi}$
- 6-117. $M_z = -1039.23 \text{ N} \cdot \text{m}$, $M_y = -600.0 \text{ N} \cdot \text{m}$,
 $I_z = 28.44583(10^{-6}) \text{ m}^4$,
 $I_y = 13.34583(10^{-6}) \text{ m}^4$, $\sigma_A = 7.60 \text{ MPa (T)}$
- 6-118. $\sigma_B = 131 \text{ MPa (C)}$, $\alpha = -66.5^\circ$
- 6-119. $M = 1186 \text{ kN} \cdot \text{m}$
- 6-121. $M_{\max} = 427.2 \text{ N} \cdot \text{m}$, $\sigma_{\max} = 161 \text{ MPa}$
- 6-122. $\sigma_A = 293 \text{ kPa (C)}$
- 6-123. $\sigma_A = 293 \text{ kPa (C)}$
- 6-125. $z'_A = 1.155 \text{ in.}$, $y'_A = -2.828 \text{ in.}$,
 $\sigma_A = 21.0 \text{ ksi (C)}$
- 6-126. $\sigma_A = 21.0 \text{ ksi}$
- 6-127. $h = 41.3 \text{ mm}$,
 $M = 6.60 \text{ kN} \cdot \text{m}$
- 6-129. $M_{\max} = 25.3125 \text{ kip} \cdot \text{ft}$, $\bar{y} = 2.3030 \text{ in.}$,
 $I = 30.8991 \text{ in}^4$, $(\sigma_{\max})_{\text{at}} = 22.6 \text{ ksi}$,
 $(\sigma_{\max})_{\text{at}} = 13.3 \text{ ksi}$
- 6-130. $w = 0.875 \text{ kip/ft}$
- 6-131. $(\sigma_{\max})_{\text{at}} = 8.51 \text{ ksi}$, $(\sigma_{\max})_{\text{at}} = 0.558 \text{ ksi}$
- 6-133. $\bar{y} = 2.5247 \text{ in.}$, $I_{NA} = 85.4170 \text{ in}^4$,
 $M = 16.4 \text{ kip} \cdot \text{ft}$
- 6-134. $(\sigma_{\text{at}})_{\max} = 20.1 \text{ MPa}$
- 6-135. $(\sigma_{\text{at}})_{\max} = 1.40 \text{ ksi}$, $(\sigma_{\text{at}})_{\max} = 77.0 \text{ psi}$
- 6-137. $\bar{y} = 0.1882 \text{ m}$, $I = 18.08(10^{-6}) \text{ m}^4$,
 $(\sigma_{\max})_{\text{at}} = 154 \text{ MPa}$, $(\sigma_{\max})_{\text{at}} = 171 \text{ MPa}$
- 6-138. $d = 531 \text{ mm}$, $M = 98.6 \text{ kN} \cdot \text{m}$
- 6-139. $(\sigma_{\max})_{\text{psc}} = 1.53 \text{ ksi}$
- 6-141. $M_{\max} = 40 \text{ kip} \cdot \text{ft}$, $A_{\text{at}} = 2.3562 \text{ in}^2$,
 $h' = 5.517 \text{ in.}$, $I = 1358.78 \text{ in}^4$,
 $(\sigma_{\text{at}})_{\max} = 1.95 \text{ ksi}$, $(\sigma_{\text{at}})_{\max} = 18.3 \text{ ksi}$

- 6-142. $M = 97.5 \text{ kip} \cdot \text{ft}$
- 6-145. $A = 0.0028125\pi \text{ m}^2$, $\int_A \frac{dA}{r} = 0.053049301 \text{ m}$,
 $M = 14.0 \text{ kN} \cdot \text{m}$
- 6-146. $P = 55.2 \text{ kN}$
- 6-147. $(\sigma_{\max})_t = 4.51 \text{ MPa}$,
 $(\sigma_{\max})_c = -5.44 \text{ MPa}$
- 6-149. $\Sigma A = 0.00425 \text{ m}^2$, $\bar{r} = 0.5150 \text{ m}$,
 $\Sigma \int_A \frac{dA}{r} = 8.348614(10^{-3}) \text{ m}$,
 $\sigma_c = 2.66 \text{ MPa (T)}$
- 6-150. $(\sigma_{\max})_t = 204 \text{ psi (T)}$, $(\sigma_{\max})_c = 120 \text{ psi (C)}$
- 6-151. $\sigma_A = 10.6 \text{ ksi (T)}$, $\sigma_B = 12.7 \text{ ksi (C)}$
- 6-153. $\bar{r} = 1.235 \text{ m}$, $\Sigma \int_A \frac{dA}{r} = 6.479051(10^{-3}) \text{ m}$,
 $A = 0.008 \text{ m}^2$, $\sigma_B = 26.2 \text{ MPa (C)}$
- 6-154. $\sigma_t = 2.01 \text{ MPa (T)}$
- 6-155. $P = 3.09 \text{ N}$
- 6-157. $K = 2.60$, $M = 15.0 \text{ kip} \cdot \text{ft}$
- 6-158. $\sigma_{\max} = 12.0 \text{ ksi}$
- 6-159. $r = 5.00 \text{ mm}$
- 6-161. $K = 1.92$, $P = 122 \text{ lb}$
- 6-162. $\sigma_{\max} = 29.5 \text{ ksi}$
- 6-163. $L = 950 \text{ mm}$
- 6-165. $I_z = 82.78333(10^{-6}) \text{ m}^4$, $M_p = 211.25 \text{ kN} \cdot \text{m}$,
 $\sigma_{\text{top}} = \sigma_{\text{bottom}} = 43.5 \text{ MPa}$
- 6-166. $k = \frac{3h}{2} \left[\frac{4bt(h-t) + t(h-2t)^2}{bh^3 - (b-t)(h-2t)^3} \right]$
- 6-167. $k = 1.71$
- 6-169. $M_p = 289062.5 \text{ N} \cdot \text{m}$, $I = 91.14583(10^{-6}) \text{ m}^4$,
 $\sigma'_{\text{top}} = \sigma'_{\text{bottom}} = 67.1 \text{ MPa}$
- 6-170. $k = 1.17$
- 6-171. $k = 1.70$
- 6-173. $I_x = 26.8(10^{-6}) \text{ m}^4$, $M_p = 0.00042\sigma_Y$,
 $M_Y = 0.000268\sigma_Y$, $k = 1.57$
- 6-174. $\sigma_Y = \sigma_B = 142 \text{ MPa}$
- 6-175. $k = 1.71$
- 6-177. $M_Y = 87.83\sigma_Y$, $M_p = 121.33\sigma_Y$, $k = 1.38$
- 6-178. $k = \frac{16r_o(r_o^3 - r_i^3)}{3\pi(r_o^4 - r_i^4)}$
- 6-179. $k = 2$
- 6-181. $d = \frac{2}{3}h$, $M = \frac{11ah^2}{54}\sigma_Y$
- 6-182. (a) $w_0 = 18.0 \text{ kip/ft}$, (b) $w_0 = 22.8 \text{ kip/ft}$
- 6-183. (a) $P = 37.3 \text{ kip}$, (b) $P = 45.5 \text{ kip}$
- 6-185. $\sigma = 50\sigma d - 3500(10^6)d = 0$, $M = 94.7 \text{ N} \cdot \text{m}$
- 6-186. (a) $M = 35.0 \text{ kip} \cdot \text{ft}$, (b) $M = 59.8 \text{ kip} \cdot \text{ft}$
- 6-187. $M = 251 \text{ N} \cdot \text{m}$
- 6-189. $\sigma = 82 \text{ ksi}$, $M = 73.5 \text{ kip} \cdot \text{ft}$
- 6-190. $F_R = 5.88 \text{ kN}$
- 6-191. $(\sigma_{\max})_t = 3.43 \text{ MPa (T)}$,
 $(\sigma_{\max})_c = 1.62 \text{ MPa (C)}$
- 6-193. $n = 18.182$, $I = 0.130578(10^{-3}) \text{ m}^4$,
 $M = 14.9 \text{ kN} \cdot \text{m}$
- 6-194. $M = 26.4 \text{ kN} \cdot \text{m}$
- 6-195. (a) $\sigma_{\max} = 0.410 \text{ MPa}$, (b) $\sigma_{\max} = 0.410 \text{ MPa}$
- 6-197. $\int_A \frac{dA}{r} = 0.012908358 \text{ m}$, $A = 6.25(10^{-3}) \text{ m}^2$,
 $\sigma_A = 225 \text{ kPa (C)}$, $\sigma_B = 265 \text{ kPa (T)}$
- 6-198. $V = 20 - 2x$, $M = -x^2 + 20x - 166$
- 6-199. $x = 0.6$, $V = -233$, $M = -50$
- 6-201. $\sigma = \frac{6M}{a^3}(\cos \theta + \sin \theta)$, $\frac{d\sigma}{d\theta} = 0$,
 $\theta = 45^\circ$, $\alpha = 45^\circ$

Chapter 7

- 7-1. $I = 0.2501(10^{-3}) \text{ m}^4$, $Q_A = 0.64(10^{-3}) \text{ m}^3$,
 $\tau_A = 2.56 \text{ MPa}$
- 7-2. $\tau_{\max} = 3.46 \text{ MPa}$
- 7-3. $V_w = 19.0 \text{ kN}$
- 7-5. $\bar{y} = 3.30 \text{ in.}$, $I_{NA} = 390.60 \text{ in}^4$,
 $Q = 65.34 - 6y^2$, $V_f = 3.82 \text{ kip}$
- 7-6. $\tau_A = 1.99 \text{ MPa}$, $\tau_B = 1.65 \text{ MPa}$
- 7-7. $\tau_{\max} = 4.62 \text{ MPa}$
- 7-9. $\bar{y} = 1.1667 \text{ in.}$, $I = 6.75 \text{ in}^4$, $V = 32.1 \text{ kip}$
- 7-10. $\tau_{\max} = 4.48 \text{ ksi}$
- 7-11. $V = 100 \text{ kN}$
- 7-13. $\bar{y} = 0.080196 \text{ m}$, $I = 4.8646(10^{-6}) \text{ m}^4$,
 $\tau_{\max} = 4.22 \text{ MPa}$
- 7-14. $V = 190 \text{ kN}$
- 7-15. The factor = $\frac{4}{3}$
- 7-17. $I = 0.175275(10^{-3}) \text{ m}^4$,
 $Q_{\max} = 1.09125(10^{-3}) \text{ m}^3$,
 $\tau_{\max} = 37.4 \text{ MPa}$
- 7-18. $V = 723 \text{ kN}$
- 7-19. $(\tau_f)_{\max} = 9.24 \text{ MPa}$,
 $(\tau_w)_{\max} = 37.4 \text{ MPa}$

$$7-21. I = 4\pi \text{ in}^4, Q = \frac{2}{3}(4 - y^2)^{3/2}, \tau_A = 2.39 \text{ ksi}$$

$$7-22. \tau_B = 4.41 \text{ MPa}$$

$$7-23. \tau_{\max} = 4.85 \text{ MPa}$$

$$7-25. V_C = -13.75 \text{ kN}, I = 27.0(10^{-6}) \text{ m}^4, \\ Q_{\max} = 0.216(10^{-3}) \text{ m}^3, \tau_{\max} = 3.67 \text{ MPa}$$

$$7-26. \tau_{\max} = 280 \text{ psi}$$

$$7-27. \tau_C = 1.43 \text{ ksi}, \tau_D = 1.17 \text{ ksi}$$

$$7-33. I = 32.0 \text{ in}^4, Q = 12.0 \text{ in}^3, F = 675 \text{ lb}$$

$$7-34. V = 1.35 \text{ kip}$$

$$7-35. V = 1.80 \text{ kip}, s = \frac{1}{8} \text{ in.}$$

$$7-37. I_{NA} = 93.25 \text{ in}^4, Q = 10.125 \text{ in}^3, V = 34.5 \text{ kip}$$

$$7-38. \tau_n = 35.2 \text{ MPa}$$

$$7-39. F = 12.5 \text{ kN}$$

$$7-41. I_{NA} = 2902 \text{ in}^4, Q = 168 \text{ in}^3, \\ Q_{\max} = 208.5 \text{ in}^3, \\ P = 6.91 \text{ kip}$$

$$7-42. V = 8.82 \text{ kip}, s = \frac{1}{8} \text{ in.}$$

$$7-43. (\tau_{\text{rail}})_{\text{avg}} = 119 \text{ MPa}$$

$$7-45. I_{NA} = 72.0(10^{-6}) \text{ m}^4, Q = 0.450(10^{-3}) \text{ m}^3, \\ P = 6.60 \text{ kN}$$

$$7-47. s = 8.66 \text{ in.}, s' = 1.21 \text{ in.}$$

$$7-50. q_A = 228 \text{ kN/m}, q_B = 462 \text{ kN/m}$$

$$7-51. q_C = 0, q_D = 601 \text{ kN/m}$$

$$7-53. I_{NA} = 125.17(10^{-6}) \text{ m}^4, Q_C = 0.5375(10^{-3}) \text{ m}^3, \\ q_C = 38.6 \text{ kN/m}$$

$$7-54. q_A = 1.39 \text{ kN/m}, q_B = 1.25 \text{ kN/m}$$

$$7-55. q_{\max} = 1.63 \text{ kN/m}$$

$$7-57. \bar{y} = 3.70946 \text{ in.}, I = 145.98 \text{ in}^4, \\ q_{\max} = 414 \text{ lb/in.}$$

$$7-58. q_A = 215 \text{ kN/m}$$

$$7-59. q_{\max} = 232 \text{ kN/m}$$

$$7-61. \bar{y} = 2.8362 \text{ in.}, I = 92.569 \text{ in}^4, \\ q_A = 196 \text{ lb/in.}, q_B = 452 \text{ lb/in.}, \\ q_{\max} = 641 \text{ lb/in.}$$

$$7-62. \tau = \frac{V}{\pi R^2 t} \sqrt{R^2 - y^2}$$

$$7-63. e = \frac{3(b_1^2 - b_2^2)}{h + 6(b_1 + b_2)}$$

$$7-65. I = \frac{10}{3} a^3 t, Q_1 = \frac{t}{2} y^2, \\ Q_2 = \frac{at}{2} (a + 2x), e = \frac{7}{10} a$$

$$7-66. e = \frac{2\sqrt{3}}{3} a$$

$$7-67. e = 0$$

$$7-69. Pe = F(h) + 2V(b),$$

$$I = \frac{t}{12} (2h^3 + 6bh^2 - (h - 2h_1)^3),$$

$$e = \frac{b(6h_1h^2 + 3h^2b - 8h_1^3)}{2h^3 + 6bh^2 - (h - 2h_1)^3}$$

$$7-70. e = \frac{4r(\sin \alpha - \alpha \cos \alpha)}{2\alpha - \sin 2\alpha}$$

$$7-71. V_{AB} = 9.96 \text{ kip}$$

$$7-73. \bar{y} = 0.08798 \text{ m.}$$

$$I_{NA} = 86.93913(10^{-6}) \text{ m}^4, \\ Q_A = 0, Q_C = 0.16424(10^{-3}) \text{ m}^3, \\ q_A = 0, q_B = 1.21 \text{ kN/m}, \\ q_C = 3.78 \text{ kN/m}$$

$$7-74. V = 4.10 \text{ kip}$$

$$7-75. V = 749 \text{ lb}$$

Chapter 8

$$8-1. t = 18.8 \text{ mm}$$

$$8-2. r_o = 75.5 \text{ in.}$$

$$8-3. \text{Case (a): } \sigma_1 = 1.04 \text{ ksi}, \sigma_2 = 0$$

$$\text{Case (b): } \sigma_1 = 1.04 \text{ ksi}, \sigma_2 = 520 \text{ psi}$$

$$8-5. \sigma = 133 \text{ MPa}, P_b = 35.56(10^3) \pi \text{ N}, \\ \sigma_b = 228 \text{ MPa}$$

$$8-6. t = 26.7 \text{ mm}, n = 820 \text{ bolts}$$

$$8-7. \text{(a) } \sigma_1 = 127 \text{ MPa}, \text{(b) } \sigma'_1 = 79.1 \text{ MPa}, \\ \text{(c) } (\tau_{\text{avg}})_t = 322 \text{ MPa}$$

$$8-9. t_c = 40 \text{ mm}, t_s = 20 \text{ mm}, \\ (P_b)_{\text{allow}} = 122.72(10^3) \text{ N}, n_s = 308 \text{ bolts}$$

$$8-10. s = 33.3 \text{ in.}$$

$$8-11. \sigma_b = 432 \text{ psi}, \sigma_h = 8.80 \text{ ksi}$$

$$8-13. \delta_F - \delta_T = 0, \sigma_c = 2.69 \text{ ksi}$$

$$8-14. \delta_r = \frac{pr_i^2}{E(r_o - r_i)}$$

$$8-15. p = \frac{E(r_2 - r_3)}{\frac{r_2^2}{r_2 - r_1} + \frac{r_3^2}{r_4 - r_3}}$$

$$8-17. \sigma_{fl} = \frac{pr}{t + t'} + \frac{T}{wt'},$$

$$\sigma_w = \frac{pr}{t + t'} - \frac{T}{wt}$$

$$8-18. d = 66.7 \text{ mm}$$

$$8-19. \sigma_L = 66.7 \text{ MPa (C)}, \sigma_R = 33.3 \text{ MPa (T)}$$

- 8-21. $\sigma_A = \frac{P}{A} + \frac{Mc}{I} = 123 \text{ MPa}$, $\sigma_B = 62.5 \text{ MPa}$
- 8-22. $\sigma_{\max} = 1.07 \text{ MPa}$
- 8-23. $\sigma_{\max} = 1.07 \text{ MPa}$
- 8-25. $N = 606.218 \text{ lb}$, $V = 350 \text{ lb}$,
 $M = 175 \text{ lb} \cdot \text{in.}$,
 $\sigma_B = 5.35 \text{ ksi}$, $\tau_B = 0$
- 8-26. $w = 79.7 \text{ mm}$
- 8-27. $P = 109 \text{ kN}$
- 8-29. $A = 0.375 \text{ in}^2$, $Q_A = 0.0234375 \text{ in}^3$,
 $I = 0.0078125 \text{ in}^4$,
 $\sigma_A = 533 \text{ psi (T)}$, $\sigma_B = 1067 \text{ psi (C)}$,
 $\tau_A = 600 \text{ psi}$, $\tau_B = 0$
- 8-30. $\sigma_A = 504 \text{ kPa (C)}$, $\tau_A = 14.9 \text{ kPa}$
- 8-31. $d = 66.7 \text{ mm}$
- 8-33. $I = 1.0667(10^{-3}) \text{ in}^4$, $Q_B = 0$,
 $Q_C = 4(10^{-3}) \text{ in}^3$, $\sigma_B = 5.56 \text{ ksi (T)}$,
 $\tau_B = 0$, $\sigma_C = 62.5 \text{ psi (C)}$, $\tau_C = 162 \text{ psi}$
- 8-34. $\sigma_D = 0$, $\tau_D = 667 \text{ psi}$, $\sigma_E = 23.3 \text{ ksi (T)}$, $\tau_E = 0$
- 8-35. $\sigma_A = -9.41 \text{ ksi}$, $\tau_A = 0$, $\sigma_B = 2.69 \text{ ksi}$,
 $\tau_B = 0.869 \text{ ksi}$
- 8-37. $A = 25\pi(10^{-6}) \text{ m}^2$, $I_z = 0.15625\pi(10^{-9}) \text{ m}^4$,
 $J = 0.3125\pi(10^{-9}) \text{ m}^4$, $\sigma_B = 1.53 \text{ MPa (C)}$,
 $\tau_B = 100 \text{ MPa}$
- 8-38. $T = 2.16 \text{ kip}$
- 8-39. $T = 2.16 \text{ kip}$
- 8-41. $A = 9.00(10^{-3}) \text{ m}^2$, $I = 82.8(10^{-6}) \text{ m}^4$,
 $Q_B = 0$, $\sigma_B = 0.522 \text{ MPa (C)}$, $\tau_B = 0$
- 8-42. $\sigma = 17.9 \text{ MPa (C)}$, $\tau = 1.06 \text{ MPa}$
- 8-43. $\sigma = 23.9 \text{ MPa (C)}$, $\tau = 0.796 \text{ MPa}$
- 8-45. $A = 18.0 \text{ in}^2$, $I_y = 13.5 \text{ in}^4$, $I_z = 54.0 \text{ in}^4$,
 $\sigma_A = 1.00 \text{ ksi (C)}$, $\sigma_B = 3.00 \text{ ksi (C)}$
- 8-46. $\sigma_{\max} = \frac{1.33P}{a^2} \text{ (C)}$, $\sigma_{\min} = \frac{P}{3a^2} \text{ (T)}$
- 8-47. $\sigma_{\max} = \frac{0.368P}{r^2} \text{ (C)}$, $\sigma_{\min} = \frac{0.0796P}{r^2} \text{ (T)}$
- 8-49. $R = 0.080889 \text{ m}$, $N = -24.525 \text{ N}$,
 $M = 14.2463 \text{ N} \cdot \text{m}$,
 $\sigma_A = 89.1 \text{ MPa (C)}$, $\sigma_B = 79.3 \text{ kPa (T)}$
- 8-50. $(\sigma_t)_{\max} = 8.37 \text{ ksi}$, $(\sigma_c)_{\max} = -6.95 \text{ ksi}$
- 8-51. $6e_y + 18e_z < 5a$
- 8-53. $A = 13.5 \text{ m}^2$, $I_x = 22.78125 \text{ m}^4$,
 $I_y = 10.125 \text{ m}^4$, $y = 0.75 - 1.5x$
- 8-54. $\sigma_A = 9.88 \text{ kPa (T)}$, $\sigma_B = 49.4 \text{ kPa (C)}$,
 $\sigma_C = 128 \text{ kPa (C)}$, $\sigma_D = 69.1 \text{ kPa (C)}$
- 8-55. $\sigma_A = 11.9 \text{ MPa (T)}$, $\tau_A = -0.318 \text{ MPa}$
- 8-57. $N_y = 800 \text{ lb}$, $V_z = -600 \text{ lb}$,
 $V_x = -500 \text{ lb}$, $T_y = -7200 \text{ lb} \cdot \text{in.}$,
 $M_x = 4800 \text{ lb} \cdot \text{in.}$, $\sigma = 17.6 \text{ ksi (T)}$,
 $\tau = 4.84 \text{ ksi}$
- 8-58. $\sigma = 5.86 \text{ ksi (C)}$, $\tau = 4.80 \text{ ksi}$
- 8-59. $\sigma_{\max} = 71.0 \text{ MPa (C)}$
- 8-61. $A = 0.7854 \text{ in}^2$, $J = 0.098175 \text{ in}^4$,
 $I = 0.049087 \text{ in}^4$, $(Q_A)_x = 0$,
 $(Q_A)_z = 0.08333 \text{ in}^3$, $\sigma_A = 16.2 \text{ ksi (T)}$,
 $\tau_A = -2.84 \text{ ksi}$
- 8-62. $\sigma_B = 7.80 \text{ ksi (T)}$, $\tau_B = 3.40 \text{ ksi}$
- 8-63. $\sigma_C = 15.6 \text{ ksi (T)}$, $\sigma_D = 124 \text{ ksi (T)}$,
 $\tau_D = 62.4 \text{ ksi}$, $\tau_C = -52.4 \text{ ksi}$
- 8-65. $T = -519.62 \text{ lb} \cdot \text{in.}$, $M_y = 250 \text{ lb} \cdot \text{in.}$,
 $M_z = -433.01 \text{ lb} \cdot \text{in.}$, $I_y = I_z = 0.53689 \text{ in}^4$,
 $J = 1.07379 \text{ in}^4$, $(Q_z)_A = 0$,
 $(Q_z)_B = 0.38542 \text{ in}^3$,
 $\sigma_A = 605 \text{ psi (T)}$, $\tau_A = 327 \text{ psi}$
- 8-66. $\sigma_B = 466 \text{ psi (C)}$, $\tau_B = 422 \text{ psi}$
- 8-67. $\sigma = -\frac{2P}{bh^3}(h^2 + 18e_y y)$, $-\frac{h}{6} \leq e_y \leq \frac{h}{12}$
- 8-69. $I = 0.1256637(10^{-6}) \text{ m}^4$,
 $A = 1.256637(10^{-3}) \text{ m}^2$, $Q_B = 0$,
 $\sigma_B = -21.7 \text{ MPa}$, $\tau_B = 0$
- 8-70. $\tau_A = 0$, $\sigma_A = 30.2 \text{ ksi (C)}$
- 8-71. $\sigma_B = 0$, $\tau_B = 0.377 \text{ ksi}$
- 8-73. $R = 1.74103 \text{ in.}$, $e = 0.0089746 \text{ in.}$,
 $I = 0.9765625(10^{-3})\pi \text{ in}^4$, $Q_B = 0.0104167 \text{ in}^3$,
 $\sigma = 1.62 \text{ psi (T)}$, $\tau = 384 \text{ psi}$
- 8-74. $\sigma_A = -21.3 \text{ psi}$, $\sigma_B = -12.2 \text{ psi}$
- 8-75. $\sigma_E = 802 \text{ kPa}$, $\tau_E = 69.8 \text{ kPa}$
- 8-77. $\int_A \frac{dA}{r} = 0.035774 \text{ in.}$, $A = 0.049087 \text{ in}^2$,
 $(\sigma_t)_{\max} = 49.0 \text{ ksi (T)}$, $(\sigma_c)_{\max} = 40.8 \text{ ksi (C)}$
- 8-78. $(\sigma_t)_{\max} = 28.8 \text{ ksi (T)}$, $(\sigma_c)_{\max} = 24.0 \text{ ksi (C)}$
- 8-79. $\sigma_{\max} = 236 \text{ psi (C)}$
- 8-81. $p = 12(10^6) \text{ MPa}$, $F = 30(10^3)\pi$,
 $P = 94.2 \text{ kN}$
- 8-82. $\sigma_{\max} = 44.0 \text{ ksi (T)}$
- 8-83. $\sigma_1 = 7.07 \text{ MPa}$, $\sigma_2 = 0$
- 8-85. $p = 3.60 \text{ MPa}$, $F_b = 6.3617(10^6) \text{ N}$,
 $n = 113 \text{ bolts}$
- 8-86. $\sigma_1 = 50.0 \text{ MPa}$, $\sigma_2 = 25.0 \text{ MPa}$, $F_b = 133 \text{ kN}$

Chapter 9

- 9-2. $\sigma_x = -3.48$ ksi, $\tau_{xy} = 4.63$ ksi
- 9-3. $\sigma_x = -678$ psi, $\tau_{xy} = 41.5$ psi
- 9-5. $\sigma_x = -650$ psi, $\sigma_y = 400$ psi, $\tau_{xy} = 0$, $\theta = 30^\circ$,
 $\sigma_{x'} = -388$ psi, $\tau_{x'y'} = 455$ psi
- 9-6. $\sigma_x = 49.7$ MPa, $\tau_{xy} = -34.8$ MPa
- 9-7. $\sigma_x = 49.7$ MPa, $\tau_{xy} = -34.8$ MPa
- 9-9. $\theta = +135^\circ$, $\sigma_x = 80$ MPa,
 $\sigma_y = 0$, $\tau_{xy} = 45$ MPa,
 $\sigma_{x'} = -5$ MPa, $\tau_{x'y'} = 40$ MPa
- 9-10. $\sigma_x = -2.71$ ksi, $\tau_{xy} = 4.17$ ksi
- 9-11. $\sigma_x = -2.71$ ksi, $\tau_{xy} = 4.17$ ksi
- 9-13. $\theta = -60^\circ$, $\sigma_x = 200$ psi,
 $\sigma_y = -350$ psi, $\tau_{xy} = 75$ psi,
 $\sigma_{x'} = -277$ psi, $\sigma_{y'} = 127$ psi, $\tau_{x'y'} = 201$ psi
- 9-14. (a) $\sigma_1 = 4.21$ ksi, $\sigma_2 = -34.2$ ksi,
 $\theta_{p_1} = 19.3^\circ$, $\theta_{p_2} = -70.7^\circ$
 (b) $\tau_{\max} = 19.2$ ksi, $\sigma_{\text{avg}} = -15$ ksi,
 $\theta_s = -25.7^\circ$, 64.3°
- 9-15. $\sigma_1 = -19.0$ MPa, $\sigma_2 = -121$ MPa,
 $(\theta_p)_1 = 39.3^\circ$, $(\theta_p)_2 = -50.7^\circ$,
 $\tau_{\max} = 51.0$ MPa, $\theta_s = -5.65^\circ$, 84.3°
- 9-17. $\sigma_x = 125$ MPa, $\sigma_y = -75$ MPa,
 $\tau_{xy} = -50$ MPa,
 $\sigma_1 = 137$ MPa, $\sigma_2 = -86.8$ MPa,
 $(\theta_p)_1 = -13.3^\circ$, $(\theta_p)_2 = 76.7^\circ$,
 $\tau_{\max} = 112$ MPa, $\sigma_{\text{avg}} = 25$ MPa
- 9-18. $\sigma_x = -193$ MPa, $\sigma_y = -357$ MPa,
 $\tau_{xy} = 102$ MPa
- 9-19. $\sigma_1 = 224$ MPa, $\sigma_2 = -64.2$ MPa,
 $(\theta_p)_1 = -61.8^\circ$, $(\theta_p)_2 = 28.2^\circ$,
 $\tau_{\max} = 144$ MPa, $\theta_s = -16.8^\circ$, 73.2° ,
 $\sigma_{\text{avg}} = 80$ MPa
- 9-21. $\sigma_x = 51.962$ ksi, $\tau_{xy} = 30$ ksi,
 $\sigma_1 = 80.1$ ksi, $\sigma_2 = 19.9$ ksi
- 9-22. $\sigma_1 = 4.93$ MPa, $\sigma_2 = -111$ MPa,
 $(\theta_p)_1 = 78.1^\circ$, $(\theta_p)_2 = -11.9^\circ$
- 9-23. $I = 0.45(10^{-3}) \text{ m}^4$, $Q_A = 1.6875(10^{-3}) \text{ m}^3$,
 $\sigma_x = 0.507$ MPa, $\tau_{xy} = 0.958$ MPa
- 9-25. $N = 400$ N, $M = 100$ N·m,
 $A = 0.1(10^{-3})\pi \text{ m}^2$, $I = 2.5(10^{-9})\pi \text{ m}^4$,
 $\sigma_1 = 0$, $\sigma_2 = -126$ MPa,
 $\tau_{\max} = 63.0$ MPa, $\sigma_{\text{avg}} = -63.0$ MPa
- 9-26. $\sigma_1 = 29.8$ ksi, $\sigma_2 = 0$,
 $\tau_{\max} = 14.9$ ksi, $\theta_s = -45^\circ$, 45° ,
 $\sigma_{\text{avg}} = 14.9$ ksi
- 9-27. $\sigma_1 = 0$, $\sigma_2 = -22.9$ ksi,
 $\tau_{\max} = 11.5$ ksi, $\theta_s = 45^\circ$, 135° ,
 $\sigma_{\text{avg}} = -11.5$ ksi
- 9-29. $V = 70.5$ kN, $M = 39.15$ kN·m,
 $I = 49.175(10^{-6}) \text{ m}^4$, $Q_A = 0.255(10^{-3}) \text{ m}^3$,
 $\sigma_1 = 64.9$ MPa, $\sigma_2 = -5.15$ MPa,
 $(\theta_p)_1 = 15.7^\circ$, $(\theta_p)_2 = -74.3^\circ$
- 9-30. Point A: $\sigma_1 = 1.50$ ksi, $\sigma_2 = -0.0235$ ksi,
 Point B: $\sigma_1 = 0.0723$ ksi, $\sigma_2 = -0.683$ ksi
- 9-31. $\sigma_1 = 6.38$ MPa, $\sigma_2 = -0.360$ MPa,
 $(\theta_p)_1 = 13.4^\circ$, $(\theta_p)_2 = 103^\circ$
- 9-33. $I = 0.3125(10^{-6}) \text{ m}^4$, $Q_A = 0$,
 $Q_B = 9.375(10^{-6}) \text{ m}^3$,
 $\sigma_1 = 0$, $\sigma_2 = -192$ MPa,
 $\sigma_1 = 24.0$, $\sigma_2 = -24.0$ MPa,
 $\theta_{p_1} = -15.0^\circ$, $\theta_{p_2} = 45.0^\circ$
- 9-34. $\sigma_1 = 0$, $\sigma_2 = -1.34$ ksi,
 $\tau_{\max} = 668$ psi, $\theta_s = 45^\circ$, -45°
- 9-35. $\tau_{\max} = 5$ kPa, $\sigma_{\text{avg}} = 0$
- 9-37. $A = \frac{\pi}{4}d^2$, $I = \frac{\pi}{64}d^4$, $Q_A = 0$,
 $\sigma_1 = \frac{4}{\pi d^2} \left(\frac{2PL}{d} - F \right)$, $\sigma_2 = 0$,
 $\tau_{\max} = \frac{2}{\pi d^2} \left(\frac{2PL}{d} - F \right)$
- 9-38. $\tau_{x'y'} = -47.5$ kPa
- 9-39. $\sigma_y = 82.3$ kPa
- 9-41. $\bar{y} = 0.0991$ m, $I = 7.4862(10^{-6}) \text{ m}^4$,
 $A = 3.75(10^{-3}) \text{ m}^2$,
 $\sigma_1 = 21.2$ MPa, $\sigma_2 = -0.380$ MPa, $\theta_p = -7.63^\circ$
- 9-42. $\sigma_1 = 2.97$ ksi, $\sigma_2 = -4.12$ ksi,
 $\tau_{\max} = 3.55$ ksi
- 9-43. $\sigma_1 = 1.27$ MPa, $\sigma_2 = -62.4$ MPa,
 $(\theta_p)_1 = 81.9^\circ$, $(\theta_p)_2 = -8.11^\circ$

- 9-45. $V = 2 \text{ kip}$, $M = 13 \text{ kip} \cdot \text{ft}$,
 $I = 86.6667 \text{ in}^4$, $Q_A = 0$, $\tau_{\max}^{\text{in-plane}} = 2.70 \text{ ksi}$,
 $\sigma_{\text{avg}} = -2.70 \text{ ksi}$, $\theta_s = 45^\circ$, -45°
- 9-46. $\sigma_1 = 219 \text{ psi}$, $\sigma_2 = -219 \text{ psi}$,
 $(\theta_p)_1 = 45^\circ$, $(\theta_p)_2 = -45^\circ$
- 9-47. $\sigma_1 = 5.50 \text{ MPa}$, $\sigma_2 = -0.611 \text{ MPa}$
- 9-49. $I_z = 0.350(10^{-3}) \text{ m}^4$, $I_y = 68.75(10^{-6}) \text{ m}^4$,
 $(Q_A)_y = 0$, $\sigma_1 = 0$, $\sigma_2 = -77.4 \text{ MPa}$,
 $\tau_{\max}^{\text{in-plane}} = 38.7 \text{ MPa}$
- 9-50. $\sigma_1 = 0$, $\sigma_2 = -20 \text{ kPa}$, $\tau_{\max}^{\text{in-plane}} = 10 \text{ kPa}$
- 9-51. $\sigma_{x'} = -388 \text{ psi}$, $\tau_{x'y'} = 455 \text{ psi}$
- 9-53. $R = 19.21 \text{ ksi}$,
 $\sigma_1 = 4.21 \text{ ksi}$, $\sigma_2 = -34.2 \text{ ksi}$, $\theta_{p2} = 19.3^\circ$,
 $\tau_{\max}^{\text{in-plane}} = 19.2 \text{ ksi}$, $\sigma_{\text{avg}} = -15 \text{ ksi}$, $\theta_{s2} = 64.3^\circ$
- 9-54. $\sigma_1 = 53.0 \text{ MPa}$, $\sigma_2 = -68.0 \text{ MPa}$,
 $\theta_{p1} = 14.9^\circ$ counterclockwise,
 $\tau_{\max}^{\text{in-plane}} = 60.5 \text{ MPa}$, $\sigma_{\text{avg}} = -7.50 \text{ MPa}$,
 $\theta_{s1} = 30.1^\circ$ clockwise
- 9-55. $\sigma_{x'} = -19.9 \text{ ksi}$, $\tau_{x'y'} = 7.70 \text{ ksi}$,
 $\sigma_{y'} = 9.89 \text{ ksi}$
- 9-58. $\sigma_{x'} = -421 \text{ MPa}$, $\tau_{x'y'} = -354 \text{ MPa}$,
 $\sigma_{y'} = 421 \text{ MPa}$
- 9-59. $\sigma_{x'} = 4.99 \text{ ksi}$, $\tau_{x'y'} = -1.46 \text{ ksi}$,
 $\sigma_{y'} = -3.99 \text{ ksi}$
- 9-61. $\sigma_{\text{avg}} = -155 \text{ MPa}$, $R = 569.23 \text{ MPa}$,
 $\sigma_{x'} = -299 \text{ MPa}$, $\tau_{x'y'} = 551 \text{ MPa}$,
 $\sigma_{y'} = -11.1 \text{ MPa}$
- 9-62. $\sigma_{x'} = 0.250 \text{ ksi}$, $\tau_{x'y'} = 3.03 \text{ ksi}$, $\sigma_{y'} = -3.25 \text{ ksi}$
- 9-63. $\sigma_{\text{avg}} = 7.50 \text{ ksi}$
 (a) $\sigma_1 = 16.5 \text{ ksi}$, $\sigma_2 = -1.51 \text{ ksi}$,
 $\theta_{p1} = 16.8^\circ$ (Clockwise)
 (b) $\tau_{\max}^{\text{in-plane}} = -9.01 \text{ ksi}$,
 $\theta_s = 28.2^\circ$ (Counterclockwise)
- 9-65. $R = 192.094$, $\sigma_1 = 342 \text{ psi}$,
 $\sigma_2 = -42.1 \text{ psi}$, $\theta_p = 19.3^\circ$,
 $\sigma_{\text{avg}} = 150 \text{ psi}$, $\tau_{\max}^{\text{in-plane}} = 192 \text{ psi}$,
 $\theta_s = -25.7^\circ$
- 9-66. (a) $\sigma_1 = 88.1 \text{ MPa}$,
 $\sigma_2 = -13.1 \text{ MPa}$
 (b) $\tau_{\max}^{\text{in-plane}} = 50.6 \text{ MPa}$,
 $\sigma_{\text{avg}} = 37.5 \text{ MPa}$,
 $\theta_s = 4.27^\circ$
- 9-67. (a) $\sigma_1 = 646 \text{ MPa}$,
 $\sigma_2 = -496 \text{ MPa}$,
 $\theta_{p1} = 30.6^\circ$ (Counterclockwise)
 (b) $\tau_{\max}^{\text{in-plane}} = 571 \text{ MPa}$,
 $\theta_s = 14.4^\circ$ (Clockwise)
- 9-69. $\sigma_{x'} = 11.0 \text{ kPa}$,
 $\tau_{x'y'} = -22.6 \text{ kPa}$
- 9-70. $\sigma_{x'} = -12.5 \text{ kPa}$,
 $\tau_{x'y'} = 21.7 \text{ kPa}$
- 9-71. $\sigma_1 = 68.6 \text{ psi}$, $\sigma_2 = -206 \text{ psi}$
- 9-73. $A = 18.0 \text{ in}^2$, $I = 54.0 \text{ in}^4$, $Q_A = 10.125 \text{ in}^3$,
 $\sigma_1 = 1.50 \text{ ksi}$, $\sigma_2 = -0.0235 \text{ ksi}$
- 9-74. $\sigma_1 = 0.0723 \text{ ksi}$, $\sigma_2 = -0.683 \text{ ksi}$
- 9-75. $\sigma_1 = 4.38 \text{ ksi}$, $\sigma_2 = -1.20 \text{ ksi}$,
 $\tau_{\max}^{\text{in-plane}} = 2.79 \text{ ksi}$
- 9-77. $\sigma_1 = \sigma_2 = 4.80 \text{ ksi}$
- 9-78. $\sigma_{x'} = 500 \text{ MPa}$, $\tau_{x'y'} = 167 \text{ MPa}$
- 9-79. $\tau = 0.2222 \text{ MPa}$, $\sigma_{\text{avg}} = 0.5556 \text{ MPa}$,
 $R = 0.5984 \text{ MPa}$,
 $\sigma_{x'} = 470 \text{ kPa}$,
 $\tau_{x'y'} = 592 \text{ kPa}$
- 9-81. $N = 900 \text{ N}$, $V = 900 \text{ N}$, $M = 675 \text{ N} \cdot \text{m}$,
 $A = 1.4(10^{-3}) \text{ m}^2$, $I = 1.7367(10^{-6}) \text{ m}^4$,
 $\sigma_1 = 9.18 \text{ MPa}$, $\sigma_2 = -0.104 \text{ MPa}$,
 $(\theta_p)_1 = 6.08^\circ$ (Counterclockwise)
- 9-82. $\sigma_1 = 32.5 \text{ MPa}$, $\sigma_2 = -0.118 \text{ MPa}$,
 $(\theta_p)_1 = 3.44^\circ$ (Counterclockwise)
- 9-83. $\sigma_1 = 7.52 \text{ MPa}$, $\sigma_2 = 0$,
 $\tau_{\max}^{\text{in-plane}} = 3.76 \text{ MPa}$,
 $\theta_s = 45^\circ$ (Counterclockwise)
- 9-85. $\sigma_{\min} = -300 \text{ psi}$, $\sigma_{\text{int}} = 0$, $\sigma_{\max} = 400 \text{ psi}$
- 9-86. $\sigma_1 = 0$, $\sigma_2 = 137 \text{ MPa}$,
 $\sigma_3 = -46.8 \text{ MPa}$,
 $\tau_{\text{abs}}^{\max} = 91.8 \text{ MPa}$
- 9-87. $\sigma_{\max} = 158 \text{ psi}$, $\sigma_{\min} = -8.22 \text{ psi}$,
 $\sigma_{\text{int}} = 0 \text{ psi}$, $\tau_{\text{abs}}^{\max} = 83.2 \text{ psi}$

9-89. $\sigma_1 = 222 \text{ MPa}$, $\sigma_2 = 0 \text{ MPa}$,

$\sigma_3 = -102 \text{ MPa}$,

$\tau_{\text{abs max}} = 162 \text{ MPa}$

9-90. $\sigma_1 = 6.73 \text{ ksi}$, $\sigma_2 = 0$, $\sigma_3 = -4.23 \text{ ksi}$,

$\tau_{\text{abs max}} = 5.48 \text{ ksi}$

9-93. $\sigma_{\text{max}} = 100 \text{ MPa}$, $\sigma_{\text{int}} = 50 \text{ MPa}$, $\sigma_{\text{min}} = 0$,

$\tau_{\text{abs max}} = 50 \text{ MPa}$

9-94. $\sigma_{\text{max}} = 582 \text{ psi}$, $\sigma_{\text{int}} = 0$, $\sigma_{\text{min}} = -926 \text{ psi}$,

$\tau_{\text{abs max}} = 755 \text{ psi}$

9-95. $\sigma_{\text{max}} = 10.9 \text{ ksi}$, $\sigma_{\text{int}} = \sigma_{\text{min}} = 0$,

$\tau_{\text{abs max}} = 5.46 \text{ ksi}$

9-97. $P = 0.900(10^6) \text{ N} \cdot \text{m/s}$, $T_0 = 60.0(10^3) \text{ N} \cdot \text{m}$,

$A = 0.015625\pi \text{ m}^2$, $J = 0.3835(10^{-3}) \text{ m}^4$,

$\tau_{\text{max in-plane}} = 23.2 \text{ MPa}$

9-98. $\sigma_1 = 119 \text{ psi}$, $\sigma_2 = -119 \text{ psi}$

9-99. $\sigma_1 = 329 \text{ psi}$, $\sigma_2 = -72.1 \text{ psi}$

9-101. $\sigma_1 = \frac{2}{\pi d^2} \left(-F + \sqrt{F^2 + \frac{64T_0^2}{d^2}} \right)$,

$\sigma_2 = \frac{2}{\pi d^2} \left(F + \sqrt{F^2 + \frac{64T_0^2}{d^2}} \right)$,

$\tau_{\text{max in-plane}} = \frac{2}{\pi d^2} \sqrt{F^2 + \frac{64T_0^2}{d^2}}$

9-102. $\sigma_{x'} = -63.3 \text{ MPa}$, $\tau_{x'y'} = 35.7 \text{ MPa}$

9-103. $\sigma_1 = 3.29 \text{ MPa}$,

$\sigma_2 = -4.30 \text{ MPa}$,

$(\theta_p)_2 = 41.1^\circ$ (Clockwise)

9-105. $Q_C = 31.25(10^{-6}) \text{ m}^3$, $I = 2.0833(10^{-6}) \text{ m}^4$,

$\tau = 26.4 \text{ kPa}$, $\sigma_1 = 26.4 \text{ kPa}$, $\sigma_2 = -26.4 \text{ kPa}$,

$\theta_{p1} = -45^\circ$; $\theta_{p2} = 45^\circ$

9-106. $\sigma_{x'} = -22.9 \text{ kPa}$, $\tau_{x'y'} = -13.2 \text{ kPa}$

10-6. $\epsilon_{x'} = 145(10^{-6})$, $\gamma_{x'y'} = 583(10^{-6})$,

$\epsilon_{y'} = 155(10^{-6})$

10-7. $\epsilon_{x'} = 215(10^{-6})$, $\gamma_{x'y'} = -248(10^{-6})$,

$\epsilon_{y'} = 185(10^{-6})$

10-9. $\epsilon_1 = 188(10^{-6})$, $\epsilon_2 = -128(10^{-6})$,

$(\theta_p)_1 = -9.22^\circ$, $(\theta_p)_2 = 80.8^\circ$,

$\gamma_{\text{max in-plane}} = 316(10^{-6})$,

$\theta_s = 35.8^\circ$, -54.2° , $\epsilon_{\text{avg}} = 30(10^{-6})$

10-10. $\epsilon_{x'} = 103(10^{-6})$, $\epsilon_{y'} = 46.7(10^{-6})$,

$\gamma_{x'y'} = 718(10^{-6})$

10-11. $\epsilon_1 = -79.3(10^{-6})$, $\epsilon_2 = -221(10^{-6})$,

$(\theta_p)_1 = 22.5^\circ$, $(\theta_p)_2 = -67.5^\circ$,

$\gamma_{\text{max in-plane}} = 141(10^{-6})$, $\theta_s = -22.5^\circ$, 67.5° ,

$\epsilon_{\text{avg}} = -150(10^{-6})$

10-13. $\epsilon_1 = 17.7(10^{-6})$, $\epsilon_2 = -318(10^{-6})$,

$(\theta_p)_1 = 76.7^\circ$, $(\theta_p)_2 = -13.3^\circ$,

$\gamma_{\text{max in-plane}} = 335(10^{-6})$, $\theta_s = 31.7^\circ$, 122° ,

$\epsilon_{\text{avg}} = -150(10^{-6})$

10-14. (a) $\epsilon_1 = 368(10^{-6})$, $\epsilon_2 = 182(10^{-6})$,

$\theta_{p1} = -52.8^\circ$, $\theta_{p2} = 37.2^\circ$

(b) $\gamma_{\text{max in-plane}} = 187(10^{-6})$,

$\theta_s = -7.76^\circ$, 82.2° , $\epsilon_{\text{avg}} = 275(10^{-6})$

10-17. $R = 167.71(10^{-6})$, $\epsilon_1 = 138(10^{-6})$,

$\epsilon_2 = -198(10^{-6})$, $\theta_p = 13.3^\circ$

10-18. $(\gamma_{xy})_{\text{max in-plane}} = 335(10^{-6})$, $\epsilon_{\text{avg}} = -30(10^{-6})$,

$\theta_s = -31.7^\circ$

10-19. $\epsilon_{x'} = -309(10^{-6})$, $\epsilon_{y'} = -541(10^{-6})$,

$\gamma_{x'y'} = -423(10^{-6})$

10-21. (a) $R = 93.408$, $\epsilon_1 = 368(10^{-6})$,

$\epsilon_2 = 182(10^{-6})$, $\theta_{p1} = -52.8^\circ$, $\theta_{p2} = 37.2^\circ$

(b) $\gamma_{\text{max in-plane}} = 187(10^{-6})$,

$\theta_s = -7.76^\circ$, 82.2° , $\epsilon_{\text{avg}} = 275(10^{-6})$

10-22. (a) $\epsilon_1 = 773(10^{-6})$, $\epsilon_2 = 76.8(10^{-6})$

(b) $\gamma_{\text{max in-plane}} = 696(10^{-6})$

(c) $\gamma_{\text{abs max}} = 773(10^{-6})$

Chapter 10

10-2. $\epsilon_{x'} = 248(10^{-6})$, $\gamma_{x'y'} = -233(10^{-6})$,

$\epsilon_{y'} = -348(10^{-6})$

10-3. $P = 111 \text{ hp}$

10-5. (a) $\epsilon_1 = 441(10^{-6})$, $\epsilon_2 = -641(10^{-6})$,

$\theta_{p1} = -24.8^\circ$, $\theta_{p2} = 65.2^\circ$

(b) $\gamma_{\text{max in-plane}} = 1.08(10^{-3})$,

$\epsilon_{\text{avg}} = -100(10^{-6})$

10-23. (a) $\epsilon_1 = 192(10^{-6})$, $\epsilon_2 = -152(10^{-6})$

(b, c) $\gamma_{\text{abs}} = \gamma_{\text{max in-plane}} = 344(10^{-6})$

10-25. $\epsilon_1 = 308(10^{-6})$, $\epsilon_2 = -108(10^{-6})$,

$(\theta_p)_2 = 8.05^\circ$ (Clockwise), $R = 208.17(10^{-6})$,

$\gamma_{\text{max in-plane}} = 416(10^{-6})$, $\epsilon_{\text{avg}} = 100(10^{-6})$,

$\theta_s = 36.9^\circ$ (Counterclockwise)

10-26. $\epsilon_1 = 451(10^{-6})$, $\epsilon_2 = -451(10^{-6})$,

$(\theta_p)_1 = 28.2^\circ$ (Counterclockwise),

$\gamma_{\text{max in-plane}} = 902(10^{-6})$, $\epsilon_{\text{avg}} = 0$,

$\theta_s = 16.8^\circ$ (Clockwise)

10-27. $\epsilon_1 = 339(10^{-6})$, $\epsilon_2 = -489(10^{-6})$,

$(\theta_p)_1 = 32.5^\circ$ (Counterclockwise),

$\gamma_{\text{max in-plane}} = 828(10^{-6})$,

$\epsilon_{\text{avg}} = -75(10^{-6})$,

$\theta_s = 12.5^\circ$ (Clockwise)

10-33. $\sigma_3 = 0$, $\sigma_1 = 10.2$ ksi, $\sigma_2 = 7.38$ ksi

10-34. $\gamma_{\text{max in-plane}} = 41.1(10^{-6})$

10-35. $\epsilon_{\text{max}} = 30.5(10^{-6})$, $\epsilon_{\text{int}} = \epsilon_{\text{min}} = -10.7(10^{-6})$

10-37. (a) $K_f = 3.33$ ksi,

(b) $K_g = 5.13(10^3)$ ksi

10-38. $\epsilon_{\text{max}} = 546(10^{-6})$, $\epsilon_{\text{int}} = 364(10^{-6})$,

$\epsilon_{\text{min}} = -910(10^{-6})$

10-39. $\rho = 3.43$ MPa,

$\tau_{\text{max in-plane}} = 0$,

$\tau_{\text{abs max}} = 85.7$ MPa

10-41. $\sigma_z = \frac{12 My}{bh^3}$, $\epsilon_y = \frac{12 \nu My}{Eb h^3}$,

$\Delta L_{AB} = \frac{3 \nu M}{2 E b h}$,

$\Delta L_{CD} = \frac{6 \nu M}{E h^2}$

10-42. $\epsilon_x = 2.35(10^{-3})$, $\epsilon_y = -0.972(10^{-3})$,

$\epsilon_z = -2.44(10^{-3})$

10-43. $P = 390$ lb

10-45. $t_h = 0.206$ in.

10-46. $\sigma_1 = 8.37$ ksi, $\sigma_2 = 6.26$ ksi

10-47. $\epsilon_1 = 833(10^{-6})$, $\epsilon_2 = 168(10^{-6})$,

$\epsilon_3 = -763(10^{-6})$

10-49. $\epsilon_x = 0.3125(10^{-3})$, and $\epsilon_y = 0.25(10^{-3})$, $\sigma_z = 0$,

$\sigma_x = 15.5$ ksi (C), $\sigma_y = 16.8$ ksi (C)

10-50. $w_y = -184$ kN/m, $w_x = 723$ kN/m

10-51. $E = 67.7$ GPa, $\nu = 0.335$,

$G = 25.0$ GPa

10-53. $0 = \sigma_x - 0.35\sigma_y - 0.35\sigma_z + 26.2$,

$0 = \sigma_y - 0.35\sigma_x - 0.35\sigma_z + 26.2$,

$0 = \sigma_z - 0.35\sigma_x - 0.35\sigma_y + 6.20$,

$\sigma_x = \sigma_y = -70.0$ ksi, $\sigma_z = -55.2$ ksi

10-54. $\epsilon_x = \epsilon_y = 0$, $\epsilon_z = 5.44(10^{-3})$

10-57. $E_{\text{eff}} = \frac{E}{1 - \nu^2}$

10-58. $k = 1.35$

10-59. $\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_z^2$

10-61. $\omega = 80 \pi$ rad/s, $T = \frac{3300}{x}$ lb·in,

$d = 0.833$ in.

10-62. $d = 0.794$ in.

10-63. $d = 1.88$ in.

10-65. $\sigma = \frac{450}{a^3}$, $a = 1.78$ in.

10-66. $T_e = \sqrt{\frac{4}{3}M^2 + T^2}$

10-67. $M_e = \sqrt{M^2 + \frac{3}{4}T^2}$

10-69. $\sigma_1 = 50 + 197.23 = 247$ MPa,

$\sigma_2 = 50 - 197.23 = -147$ MPa,

Yes,

10-70. $M_e = \sqrt{M^2 + T^2}$

10-71. No,

10-73. $\sigma = -9.549$ ksi, $\tau = 30.56$ ksi,

Yes,

10-74. No,

10-75. F.S. = 1.43

10-77. $\sigma_1 = 7.314$ ksi, $\sigma_2 = -15.314$ ksi,

F.S. = 1.59

10-78. F.S. = 1.80

10-79. $\sigma_2 = 38.9$ ksi

10-81. $\sigma_2 = \frac{\sigma_1}{2}$, $\sigma_x = 121$ ksi

- 10-82. $\sigma_Y = 94.3 \text{ ksi}$
 10-83. $\sigma_Y = 91.0 \text{ ksi}$
 10-85. $\sigma_1 = 8.8489 \text{ ksi}$, $\sigma_2 = -10.8489 \text{ ksi}$,
 $\sigma_Y = 19.7 \text{ ksi}$
 10-86. (a) $p = \frac{1}{r}\sigma_Y$
 (b) $p = \frac{2t}{\sqrt{3}r}\sigma_Y$
 10-89. $(\tau_{\max})_0 = 8626.28T$, $(\tau_{\max})_s = 9947.18T$,
 $\sigma_1 = 9947.18T$, $\sigma_2 = -9947.18T$,
 $T = 838 \text{ kN} \cdot \text{m}$
 10-90. $T = 9.67 \text{ kN} \cdot \text{m}$
 10-91. $d = 1.50 \text{ in.}$
 10-93. $\sigma_{\text{allow}} = 166.67(10^6) \text{ Pa}$,
 (a) $t = 22.5 \text{ mm}$
 (b) $t = 19.5 \text{ mm}$
 10-94. $\epsilon = \frac{pr}{2Et}(1 - \nu)$
 10-95. (a) $\epsilon_1 = 482(10^{-6})$, $\epsilon_2 = 168(10^{-6})$
 (b) $\gamma_{\text{max in-plane}} = 313(10^{-6})$
 (c) $\gamma_{\text{abs max}} = 482(10^{-6})$
 10-97. $\sigma_1 = 350.42 \text{ MPa}$, $\sigma_2 = -65.42 \text{ MPa}$, No.
 10-98. $\epsilon_{\text{avg}} = 83.3(10^{-6})$,
 (a) $\epsilon_1 = 880(10^{-6})$, $\epsilon_2 = -713(10^{-6})$,
 $\theta_p = 54.8^\circ$ (Clockwise)
 (b) $\gamma_{\text{max in-plane}} = -1593(10^{-6})$,
 $\theta_s = 9.78^\circ$ (Clockwise)
 10-99. $T = 736 \text{ N} \cdot \text{m}$
 10-101. $\epsilon_1 = 996(10^{-6})$, $\epsilon_2 = 374(10^{-6})$,
 $(\theta_p)_1 = 15.8^\circ$ (Clockwise),
 $\gamma_{\text{max in-plane}} = -622(10^{-6})$, $\epsilon_{\text{avg}} = 685(10^{-6})$,
 $\theta_s = 29.2^\circ$ (Counterclockwise)
 10-102. $\epsilon_{x'} = 480(10^{-6})$, $\gamma_{x'y'} = 23.2(10^{-6})$,
 $\epsilon_{y'} = 120(10^{-6})$
 10-103. $\epsilon_{\text{avg}} = 300(10^{-6})$, $\epsilon_1 = 480(10^{-6})$,
 $\epsilon_2 = 120(10^{-6})$, $(\theta_p)_1 = 28.2^\circ$ (Clockwise),
 $\gamma_{\text{max in-plane}} = -361(10^{-6})$,
 $\theta_s = 16.8^\circ$ (Counterclockwise)

Chapter 11

- 11-1. $I_x = 0.16276b^4$, $Q_{\max} = 0.1953125b^3$,
 $b = 211 \text{ mm}$, $h = 264 \text{ mm}$
 11-2. Use W12 $\times$ 22
 11-3. Use $b = 4.25 \text{ in.}$
 11-5. $S_{\text{req'd}} = 15.0 \text{ in}^3$,
 Use W12 $\times$ 16
 11-6. Member AB: Use W10 $\times$ 12,
 Member BC: Use W6 $\times$ 9
 11-7. $w = 6.12 \text{ kN/m}$
 11-9. $S_{\text{req'd}} = 29.45 \text{ in}^3$,
 $\tau_{\max} = 2.67 \text{ ksi}$,
 Use W12 $\times$ 26
 11-10. Use W14 $\times$ 43
 11-11. $P = 2.49 \text{ kN}$
 11-13. $S_{\text{req'd}} = 32.73 \text{ in}^3$,
 Use W12 $\times$ 26
 11-14. Use W14 $\times$ 30
 11-15. $b = 15.5 \text{ in.}$
 11-17. $S = 65.23 \text{ in}^3$, $\sigma_{\max} = 26.5 \text{ ksi}$. No, the beam
 fails due to bending stress criteria.
 11-18. $d = 11.4 \text{ mm}$
 11-19. $d_i = 13.0 \text{ mm}$
 11-21. $\sigma_{\max} = 17.46 \text{ ksi}$, Yes.
 11-22. Use $h = 9\frac{1}{8} \text{ in.}$
 11-23. $w = 3.02 \text{ kN/m}$,
 $s_{\text{ends}} = 16.7 \text{ mm}$,
 $s_{\text{mid}} = 50.2 \text{ mm}$
 11-25. $h = 0.643 \text{ in.}$,
 Yes, the joist will safely support the load.
 11-26. Use W14 $\times$ 22
 11-27. $w = 10.8 \text{ kN/m}$
 11-29. $P = 83.33h^2$, $h = 7.20 \text{ in.}$,
 $P = 4.32 \text{ kip}$
 11-30. Use $s = 3\frac{3}{4} \text{ in.}$, $s' = 5\frac{3}{4} \text{ in.}$,
 $s'' = 11\frac{1}{2} \text{ in.}$. Yes, it can support the load.
 11-31. $\sigma_{\max} = \frac{3PL}{8bh_0^2}$
 11-33. $S = \frac{b_0 r^2}{3L}x$, $\sigma = \frac{3PL}{2b_0 d^2}$. The bending stress
 is constant throughout the span.

$$11-34. \quad h = \frac{h_0}{L^{3/2}}(3L^2x - 4x^3)^{1/2}$$

$$11-35. \quad \sigma_{\max} = \frac{0.155w_0L^2}{bh_0^2}$$

$$11-37. \quad \sigma_{\max} = \frac{Px}{b_0d^2/6}, \quad d = d_0\sqrt{\frac{x}{L}}$$

$$11-38. \quad b = \frac{b_0}{L^2}x^2$$

$$11-39. \quad \text{Use } d = 20 \text{ mm}$$

$$11-41. \quad T = 100 \text{ N} \cdot \text{m}, \quad c = 0.01421,$$

$$\text{Use } d = 29 \text{ mm}$$

$$11-42. \quad T = 100 \text{ N} \cdot \text{m}, \quad \text{Use } d = 33 \text{ mm}$$

$$11-43. \quad \text{Use } d = 1\frac{1}{4} \text{ in.}$$

$$11-45. \quad M = 496.1 \text{ N} \cdot \text{m}, \quad c = 0.0176 \text{ m}, \quad \text{Use } d = 36 \text{ mm}$$

$$11-46. \quad d = 34.3 \text{ mm}$$

$$11-47. \quad \text{Use } d = 21 \text{ mm}$$

$$11-49. \quad M = 1274.75 \text{ N} \cdot \text{m}, \quad \text{Use } d = 44 \text{ mm}$$

$$11-50. \quad \text{Use } d = 41 \text{ mm}$$

$$11-51. \quad \text{Use W10} \times 12$$

$$11-53. \quad S = \frac{bh_0^2}{6L^2}(2x + L)^2, \quad \sigma_{\max} = \frac{wL^2}{4bh_0^2}$$

$$11-54. \quad \text{Use } d = 21 \text{ mm}$$

$$11-55. \quad \text{Use } d = 19 \text{ mm}$$

Chapter 12

$$12-1. \quad \sigma_{\max} = \frac{c}{\rho}E = 100 \text{ MPa}$$

$$12-2. \quad \sigma = 582 \text{ MPa}$$

$$12-3. \quad W = 113 \text{ lb}$$

$$12-5. \quad M(x_1) = -\frac{P}{2}x_1, \quad M(x_2) = -Px_2,$$

$$v_1 = \frac{P}{12EI}(-x_1^3 + L^2x_1),$$

$$v_2 = \frac{P}{24EI}(-4x_2^3 + 7L^2x_2 - 3L^3)$$

$$12-6. \quad v_1 = \frac{Px_1}{12EI}(-x_1^3 + L^2),$$

$$v_2 = \frac{P}{12EI}(2x_2^3 - 9Lx_2^2 + 10L^2x_2 - 3L^3),$$

$$v_{\max} = \frac{PL^3}{8EI}$$

$$12-7. \quad v_{\max} = \frac{P}{3EI_{AB}}\left\{\left(1 - \frac{I_{AB}}{I_{AC}}\right)L^3 - L^3\right\}$$

$$12-9. \quad M_1 = \frac{Pb}{L}x_1, \quad M_2 = Pa\left(1 - \frac{x_2}{L}\right),$$

$$v_1 = \frac{Pb}{6EIL}(x_1^3 - (L^2 - b^2)x_1),$$

$$v_2 = \frac{Pa}{6EIL}(3x_2^3L - x_2^3 - (2L^2 + a^2)x_2 + a^2L)$$

$$12-10. \quad \theta_{\max} = \frac{M_0L}{3EI},$$

$$v_{\max} = \frac{\sqrt{3}M_0L^2}{27EI}$$

$$12-11. \quad v_1 = \frac{P}{9EI}(x_1^3 - 5a^2x_1),$$

$$v_2 = \frac{P}{18EI}(-x_2^3 + 9ax_2^2 - 19a^2x_2 + 3a^3),$$

$$v_{\max} = \frac{0.484Pa^3}{EI} \downarrow$$

$$12-13. \quad \theta_A = -\frac{3}{8}\frac{PL^2}{EI}, \quad v_C = -\frac{PL^3}{6EI}$$

$$12-14. \quad v_{\max} = \frac{3PL^3}{256EI} \downarrow$$

$$12-15. \quad \theta_A = \frac{Pab}{2EI}, \quad v_1 = \frac{P}{6EI}[-x_1^3 + 3a(a+b)x_1 - a^2(2a+3b)],$$

$$v_2 = \frac{Pax_2}{2EI}(-x_2 + b), \quad v_C = \frac{Pab^2}{8EI}$$

$$12-17. \quad M(x_1) = \frac{M_0}{L}x_1, \quad M(x_2) = M_0,$$

$$\theta_A = \frac{M_0L}{6EI}, \quad \nabla, \quad v_1 = \frac{M_0}{6EIL}(x_1^3 - L^2x_1),$$

$$v_2 = \frac{M_0}{24EI}(12x_2^2 - 20Lx_2 + 7L^2),$$

$$v_C = \frac{7M_0L^2}{24EI} \uparrow$$

$$12-18. \quad \theta_A = -\frac{M_0L}{6EI}, \quad v = \frac{M_0}{6EIL}(3Lx^2 - 2x^3 - L^2x),$$

$$v_{\max} = \frac{0.0160M_0L^2}{EI} \downarrow, \quad v_{\max} = \frac{0.0160M_0L^2}{EI} \uparrow$$

$$12-19. \quad \theta_B = \frac{M_0L}{6EI}, \quad \nabla, \quad v_{x=L/2} = 0$$

$$12-21. \quad M(x_1) = -\frac{wL}{8}x_1, \quad M(x_2) = -\frac{w}{2}x_2^2,$$

$$v_C = \frac{11wL^4}{384EI} \downarrow$$

$$12-22. \quad \theta_{\max} = 0.00466 \text{ rad}, \quad v_{\max} = 0.369 \text{ in.} \downarrow$$

$$12-23. \theta_{\max} = \frac{w_0 L^3}{45EI}$$

$$12-25. M(x) = \left(36x - \frac{1}{3}x^3\right) \text{ kN} \cdot \text{m},$$

$$\theta_A = \frac{540 \text{ kN} \cdot \text{m}^2}{EI} \curvearrowright,$$

$$v = \frac{1}{EI} \left(6x^3 - \frac{1}{60}x^5 - 540x\right) \text{ kN} \cdot \text{m}^3,$$

$$v_{\max} = \frac{2074 \text{ kN} \cdot \text{m}^3}{EI} \downarrow$$

$$12-26. \theta_B = -\frac{wa^3}{6EI},$$

$$v_1 = \frac{w}{24EI} (-x_1^4 + 4ax_1^3 - 6a^2x_1^2),$$

$$v_2 = \frac{wa^3}{24EI} (-4x_2 + a),$$

$$v_B = \frac{wa^3}{24EI} (-4L + a)$$

$$12-27. \theta_A = 0.0611 \text{ rad}, v_A = -3.52 \text{ in.}$$

$$12-29. I(x) = \frac{bh^3}{12L^3} x^3, \theta_A = \frac{2\gamma L^3}{h^2 E},$$

$$v_A|_{x=0} = \frac{\gamma L^4}{h^2 E}$$

$$12-30. I = \left(\frac{2Lc}{L}\right)x,$$

$$\theta_A = \frac{\gamma L^3}{3r^2 E},$$

$$v_A|_{x=0} = \frac{\gamma L^4}{6r^2 E}$$

$$12-31. v_{\max} = \frac{PL^3}{2EI_0}$$

$$12-33. I = \left(\frac{2Ic}{L}\right)x, v_C = -\frac{PL^3}{32EI_C}$$

$$12-34. \sigma_{\max} = \frac{3PL}{2nbt^2}$$

$$12-35. v_{\max} = -3.64 \text{ mm}$$

$$12-37. M = 180x - 150(x - 0.25) - 60(x - 0.5) \\ - 150(x - 0.75), v_C = v_E = -0.501 \text{ mm}, \\ v_D = -0.698 \text{ mm}, v_F = -0.501 \text{ mm}$$

$$12-38. v = \frac{1}{EI} [-1.67x^3 - 6.67(x - 20)^3 + \\ 18.3(x - 40)^3 + 4000x] \text{ lb} \cdot \text{in}^3$$

$$12-39. v = \frac{1}{EI} [4.1667x^3 - 5(x - 2)^3 \\ - 2.5(x - 4)^3 - 93.333x], \\ v_{\max} = 13.3 \text{ mm} \downarrow$$

$$12-41. M = M_0 \left\langle x - \frac{L}{3} \right\rangle^0 - M_0 \left\langle x - \frac{2}{3}L \right\rangle, \\ v = \frac{M_0}{6EI} \left[3 \left\langle x - \frac{L}{3} \right\rangle^2 \right. \\ \left. - 3 \left\langle x - \frac{2}{3}L \right\rangle^2 - Lx \right], \\ v_{\max} = \frac{5M_0 L^2}{72EI} \downarrow$$

$$12-42. \theta_A = \frac{PL^2}{9EI}, \\ v = \frac{P}{18EI} \left[3x^3 - 3 \left\langle x - \frac{L}{3} \right\rangle^3 \right. \\ \left. - 3 \left\langle x - \frac{2}{3}L \right\rangle^3 - 2L^2x \right], \\ v_{\max} = \frac{23PL^3}{648EI} \downarrow$$

$$12-43. v = \frac{1}{EI} \left[6.25x^3 - 33.75x^2 - \frac{1}{6}x^5 \right. \\ \left. + \frac{1}{6}(x - 1.5)^5 + \frac{5}{4}(x - 1.5)^4 \right], \\ v_{\max} = 12.9 \text{ mm} \downarrow$$

$$12-45. M = 24.6x - 1.5x^2 + 1.5(x - 4)^2 - 50(x - 7), \\ \theta_A = -\frac{279 \text{ kN} \cdot \text{m}^2}{EI}, \\ v|_{x=7\text{m}} = -\frac{835 \text{ kN} \cdot \text{m}^3}{EI}$$

$$12-46. v = \frac{1}{EI} \left[3.75x^3 - \frac{10}{3}(x - 1.5)^3 \right. \\ \left. - 0.625(x - 3)^4 + \frac{1}{24}(x - 3)^5 - 77.625x \right], \\ v_{\max} = 11.0 \text{ mm} \downarrow$$

$$12-47. \theta_B = -0.705^\circ, v_B = -51.7 \text{ mm}$$

$$12-49. M = 2400x - 600(x - 9) \\ - 250x^2 + 250(x - 6)^2, \\ v = \frac{1}{EI} \left[400x^3 - 100(x - 9)^3 \right. \\ \left. - \frac{125}{6}x^4 + \frac{125}{6}(x - 6)^4 - 23625x \right], \\ v_{\max} = 1.76 \text{ in} \downarrow$$

$$12-50. \frac{dv}{dx} = \frac{1}{EI} \{0.100x^2 - 0.333x^3 + 0.333(x-5)^3 + 8.90(x-5)^2 + 9.58\} \text{ kN} \cdot \text{m}^2,$$

$$v = \frac{1}{EI} \{0.0333x^3 - 0.0833x^4 + 0.833(x-5)^4 + 2.97(x-5)^3 + 9.58x\} \text{ kN} \cdot \text{m}^3$$

$$12-51. v = \frac{1}{EI} \{-0.25x^4 + 0.208(x-1.5)^3 + 0.25(x-1.5)^4 + 4.625(x-4.5)^3 + 25.1x - 36.4\} \text{ kN} \cdot \text{m}^3$$

$$12-53. \theta_A = \frac{302 \text{ kip} \cdot \text{ft}^2}{EI},$$

$$v_C = \frac{3110 \text{ kip} \cdot \text{ft}^3}{EI}$$

$$12-54. v = \frac{1}{EI} \{-0.00556x^3 + 12.9(x-9)^3 + 0.00556(x-9)^5 - 256x + 2637\} \text{ kip} \cdot \text{ft}^3$$

$$12-55. \theta_C = \frac{3937.5}{EI}, \Delta_C = \frac{50\,625}{EI}$$

$$12-57. \theta_B = |\theta_{B/A}| = \frac{5PL^2}{8EI},$$

$$\Delta_B = |t_{B/A}| = \frac{7PL^3}{16EI} \downarrow,$$

$$12-58. \theta_A = \frac{120 \text{ kip} \cdot \text{ft}^2}{EI} \angle,$$

$$v_{\max} = v_C = \frac{1080 \text{ kip} \cdot \text{ft}}{EI} \downarrow$$

$$12-59. \theta_C = \frac{240 \text{ kip} \cdot \text{ft}^2}{EI} \nabla,$$

$$v_C = \frac{1080 \text{ kip} \cdot \text{ft}^3}{EI} \downarrow$$

$$12-61. \theta_{C/A} = \frac{M_0 L}{2EI}, \theta_{\max} = \theta_A = \frac{M_0 L}{2EI},$$

$$\Delta_{\max} = |t_{B/C}| = \frac{M_0 L^2}{8EI}$$

$$12-62. \Delta_C = \frac{5M_0 L^2}{6EI}, \theta_C = \frac{4M_0 L}{3EI}$$

$$12-63. \theta_A = 0.00879 \text{ rad}$$

$$12-65. \Delta_C = |t_{C/A}| = \frac{|t_{B/A}|}{a} L, \theta_A = \frac{|t_{B/A}|}{a},$$

$$x = \frac{\sqrt{3}}{3} a, a = 0.858L$$

$$12-66. \theta_A = \frac{4PL^2}{81EI}$$

$$12-67. F = \frac{P}{4}$$

$$12-69. \theta_A = \theta_{A/C} = \frac{5Pa^2}{2EI}, \Delta_C = t_{A/C} = \frac{19Pa^3}{6EI} \downarrow$$

$$12-70. \Delta_C = \frac{PL^3}{12EI}, \theta_A = \frac{PL^2}{24EI}, \theta_B = \frac{PL^2}{12EI}$$

$$12-71. \Delta_{\max} = \frac{0.00802PL^3}{EI}$$

$$12-73. t_{B/A} = -\frac{5M_0 a^2}{6EI}, t_{C/A} = -\frac{M_0 a^2}{6EI},$$

$$\theta_A = \frac{|t_{B/A}|}{L} = \frac{5M_0 a}{12EI},$$

$$\Delta_C = \left| \frac{1}{2} t_{B/A} \right| - |t_{C/A}| = \frac{M_0 a^2}{4EI} \uparrow$$

$$12-74. v_{\max} = \frac{3048 \text{ kip} \cdot \text{ft}^3}{EI} \downarrow$$

$$12-75. E = \frac{Pa}{24\Delta I} (3L^2 - 4a^2)$$

$$12-77. \theta_{A/C} = -\frac{5Pa^2}{2EI}, t_{B/C} = -\frac{2Pa^3}{EI},$$

$$t_{A/C} = -\frac{13Pa^3}{3EI}, \theta_A = |\theta_{A/C}| = \frac{5Pa^2}{2EI},$$

$$\Delta_A = |t_{A/C}| - |t_{B/C}| = \frac{7Pa^3}{3EI} \downarrow$$

$$12-78. \theta_{\max} = \frac{5PL^2}{16EI}, \Delta_{\max} = \frac{3PL^3}{16EI}$$

$$12-79. \theta_D = \frac{PL^2}{16EI}, \Delta_C = \frac{5PL^3}{384EI}$$

$$12-81. \Delta_C = |t_{C/A}| = \frac{|t_{B/A}|}{a} L, \theta_A = \frac{|t_{B/A}|}{a} = \frac{M_0 a}{6EI},$$

$$x = \frac{\sqrt{3}}{3} a, \Delta_D = |t_{A/D}| = \frac{\sqrt{3}M_0 a^2}{27EI},$$

$$a = 0.865L$$

$$12-82. \theta_B = 0.00778 \text{ rad } \swarrow, v_B = 0.981 \text{ in. } \downarrow$$

$$12-83. \theta_C = \frac{a^2}{6EI}(12P + wa),$$

$$\Delta_C = \frac{a^3}{24EI}(64P + 7wa) \downarrow$$

$$12-85. \theta_B = |\theta_{B/C}| = 0.00160 \text{ rad,}$$

$$\Delta_C = |\Delta_{A/C}| = 0.0371 \text{ in. } \downarrow$$

$$12-86. \theta_A = 0.175^\circ$$

$$12-87. \Delta_C = 0.895 \text{ in. } \downarrow$$

$$12-89. (\theta_C)_1 = (\theta_B)_1 = \frac{wa^3}{3EI}, (\Delta_C)_1 = \frac{wa^4}{3EI} \uparrow,$$

$$(\theta_C)_2 = \frac{wa^3}{6EI}, (\Delta_C)_2 = \frac{wa^4}{8EI} \downarrow,$$

$$(\Delta_C)_3 = (\theta_B)_3(a) = \frac{wa^4}{3EI} \downarrow,$$

$$\theta_C = \frac{wa^3}{6EI}, \Delta_C = \frac{wa^4}{8EI} \downarrow$$

$$12-90. \theta_A = \frac{wa^3}{6EI}, \Delta_B = \frac{wa^4}{12EI} \downarrow$$

$$12-91. \theta_B = 0.00722 \text{ rad, } \Delta_C = 13.3 \text{ mm } \downarrow$$

$$12-93. (\Delta_C)_1 = \frac{2560}{EI} \downarrow,$$

$$\Delta_2(x) = \frac{Mx}{6LEI}(L^2 - x^2),$$

$$(\Delta_C)_2 = \frac{80}{EI} \downarrow, \Delta_C = 1.90 \text{ in.}$$

$$12-94. \Delta_A = \frac{72}{EI}, \theta_A = \frac{36}{EI}$$

$$12-95. \Delta_C = 23.2 \text{ m } \downarrow$$

$$12-97. \Delta_B = \frac{PL^3}{24EI}, (\Delta_A)_1 = \frac{PL^3}{24EI}, \theta = \frac{PL^2}{4JG},$$

$$(\Delta_A)_2 = \frac{PL^3}{8JG}, \Delta_A = PL^3 \left(\frac{1}{12EI} + \frac{1}{8JG} \right)$$

$$12-98. \Delta_A = \frac{Pa^3(3b + a)}{3EI}$$

$$12-99. \theta_A = \frac{1053 \text{ lb} \cdot \text{in}^2}{EI},$$

$$(\Delta_A)_B = \frac{4000 \text{ lb} \cdot \text{in}^3}{EI} \downarrow$$

$$12-101. y_{\max} = \frac{P \cos \theta L^3}{3EI_x}; x_{\max} = \frac{P \sin \theta L^3}{3EI_y},$$

$$\frac{x_{\max}}{y_{\max}} = \frac{I_x}{I_y} \tan \theta, y_{\max} = 0.736 \text{ in.,}$$

$$x_{\max} = 3.09 \text{ in.}$$

$$12-102. \text{Use W14} \times 34$$

$$12-103. A_y = \frac{3M_0}{2L}, B_y = \frac{3M_0}{2L}, M_B = \frac{M_0}{2}$$

$$12-105. M(x_1) = C_y x_1, M(x_2) = C_y x_2 - P x_2 + \frac{PL}{2},$$

$$C_y = \frac{5}{16}P, B_y = \frac{11}{8}P, A_y = \frac{5}{16}P$$

$$12-106. M_A = \frac{PL}{2}, A_y = \frac{3P}{2}, B_y = \frac{5P}{2}$$

$$12-107. M_A = \frac{Pa}{L}(L - a), M_B = \frac{Pa}{L}(L - a)$$

$$12-109. M(x) = C_y x + B_y(x - 10) - \frac{3}{2}(x - 10)^2,$$

$$B_y = 14.4 \text{ kip, } C_y = 1.07 \text{ kip } \downarrow, A_y = 10.7 \text{ kip}$$

$$12-110. A_x = 0, C_y = \frac{w_0 L}{10}, B_y = \frac{4w_0 L}{5}, A_y = \frac{w_0 L}{10}$$

$$12-111. A_x = 0, A_y = \frac{7wL}{16}, C_y = \frac{wL}{16}, B_y = \frac{5wL}{8}$$

$$12-113. M_a = \frac{wL_1^2}{2} - T_{AC}L_1,$$

$$T_{AC} = \frac{3A_2 E_2 w L_1^4}{8(A_2 E_2 L_1^3 + 3E_1 I_1 L_2)}$$

$$12-114. A_x = 0, F_C = 112 \text{ kN,}$$

$$A_y = 34.0 \text{ kN, } B_y = 34.0 \text{ kN}$$

$$12-115. A_x = 0, B_y = \frac{3M_0}{2L}, A_y = \frac{3M_0}{2L}, M_A = \frac{M_0}{2}$$

$$12-117. (t_{A/B})_1 = \frac{-P(L - a)^2(2L + a)}{6EI},$$

$$(t_{A/B})_2 = \frac{A_y L^3}{3EI}, A_y = \frac{P(L - a)^2(2L + a)}{2L^3},$$

$$a = 0.414L$$

$$12-118. A_y = \frac{3M_0}{2L}, C_y = \frac{3M_0}{2L}, B_y = \frac{3M_0}{L}, C_x = 0$$

$$12-119. B_x = 0, B_y = \frac{11P}{16}, C_y = \frac{13P}{32}, A_y = \frac{3P}{32}$$

$$12-121. v_B' = \frac{366.67 \text{ N} \cdot \text{m}^3}{EI} \downarrow,$$

$$v_B'' = \frac{1.3333 B_y \text{ m}^3}{EI} \uparrow,$$

$$B_y = 550 \text{ N, } A_y = 125 \text{ N, } C_y = 125 \text{ N}$$

$$12-122. B_y = \frac{7P}{4}, A_y = \frac{3P}{4}, M_A = \frac{PL}{4}$$

$$12-123. C_x = 0, B_y = 30.75 \text{ kip,}$$

$$A_y = 2.625 \text{ kip, } C_y = 14.625 \text{ kip}$$

$$12-125. C_x = 0, (v_p)_1 = (v_p)_2 = \frac{247.5 \text{ kN} \cdot \text{m}^3}{EI} \downarrow,$$

$$(v_B)_y = \frac{36B_y}{EI} \uparrow, B_y = 13.75 \text{ kN},$$

$$A_y = C_y = 3.125 \text{ kN}$$

$$12-126. A_y = 0, B_y = \frac{3M_0}{2L}, A_y = \frac{3M_0}{2L}, M_A = \frac{M_0}{2}$$

$$12-127. C_x = 0, C_y = \frac{P}{3}$$

$$12-129. (\Delta_A)' = \frac{wL_1^4}{8E_1I_1}, \Delta_A = \frac{T_{AC}L_2}{A_2E_2}, \delta_A = \frac{T_{AC}L_1^3}{3E_1I_1},$$

$$T_{AC} = \frac{3wA_2E_2L_1^4}{8[3E_1I_1L_2 + A_2E_2L_1]}$$

$$12-130. M_A = \frac{PL}{8}, M_B = \frac{3PL}{8}, B_x = 0, B_y = P$$

$$12-131. M = \left(\frac{PL}{8} - \frac{2EI}{L} \alpha \right), \Delta_{\max} = \frac{PL^3}{192EI} + \frac{\alpha L}{4}$$

$$12-133. \Delta = \frac{w_0(L-a)^4}{8EI} - \frac{R(L-a)^3}{3EI},$$

$$R = \left(\frac{8\Delta EI w_0^3}{9} \right)^{\frac{1}{4}}, a = L - \left(\frac{72\Delta EI}{w_0} \right)^{\frac{1}{4}}$$

$$12-134. A_y = 0, F_B = 220 \text{ kN},$$

$$A_y = C_y = 70.1 \text{ kN}$$

$$12-135. B_y = 634 \text{ lb}, A_y = 243 \text{ lb}, C_y = 76.8 \text{ lb}$$

$$12-137. M = -180x + 277.5(x-12) - 70(x-24),$$

$$v = \frac{1}{EI} [-30x^3 + 46.25(x-12)^3 - 11.7(x-24)^3 + 38700x - 412560]$$

$$12-138. v_1 = \frac{1}{EI} (4.44x_1^3 - 640x_1) \text{ lb} \cdot \text{in}^3,$$

$$v_2 = \frac{1}{EI} (-4.44x_2^3 + 640x_2) \text{ lb} \cdot \text{in}^3$$

$$12-139. \Delta_C = 1.90 \text{ in.} \downarrow$$

$$12-141. \Delta_B = \Delta_C = \frac{11wL^4}{12EI}, \Delta_{BB} = \Delta_{CC} = \frac{4B_yL^3}{9EI},$$

$$\Delta_{BC} = \Delta_{CB} = \frac{7B_yL^3}{18EI}, B_y = C_y = \frac{11wL}{10},$$

$$A_y = \frac{2wL}{5}, D_y = \frac{2wL}{5}, D_x = 0$$

$$12-142. M_B = \frac{w_0L^2}{30}, M_A = \frac{w_0L^2}{20}$$

$$12-143. v_A = \frac{w_0L^4}{E(h_0)^3} \downarrow$$

$$12-145. \Delta_D = \frac{6400 \text{ lb} \cdot \text{ft}^3}{EI} \downarrow, (\Delta_C)_1 = \frac{3200 \text{ lb} \cdot \text{ft}^3}{EI} \downarrow,$$

$$(\Delta_C)_2 = \frac{27000 \text{ lb} \cdot \text{ft}^3}{EI} \downarrow, \Delta_C = 0.644 \text{ in.} \downarrow$$

$$12-146. M_{\max} = \frac{\pi^2 b t \gamma \omega^2 r^3}{108g}$$

Chapter 13

$$13-1. F = 2P\theta, F_s = \frac{kL\theta}{2}, P_{cr} = \frac{kL}{4}$$

$$13-2. P_{cr} = \frac{5}{9} kL$$

$$13-3. P_{cr} = \frac{4k}{L}$$

$$13-5. A = 1.10(10^{-3}) \text{ m}^2,$$

$$I_x = I_y = 0.184167(10^{-6}) \text{ m}^4, P_{cr} = 22.7 \text{ kN}$$

$$13-6. P_{cr} = 46.4 \text{ kN}$$

$$13-7. P_{cr} = 158 \text{ kip}$$

$$13-9. P_{cr} = 33.17 \text{ kip, F.S.} = 2.21$$

$$13-10. P_{cr} = 271 \text{ kip}$$

$$13-11. P_{cr} = 20.4 \text{ kip}$$

$$13-13. I = 0.86167(10^{-6}) \text{ m}^4, P_{cr} = 272 \text{ kN}$$

$$13-14. d = 8.43 \text{ in.}, P_{cr} = 245 \text{ kip}$$

$$13-15. L = 15.1 \text{ ft}$$

$$13-17. A = 8.00 \text{ in}^2, I_y = 10.667 \text{ in}^4,$$

$$I_y = 2.6667 \text{ in}^4, P_{cr} = 2.92 \text{ kip}$$

$$13-18. P_{cr} = 5.97 \text{ kip}$$

$$13-19. P = 29.9 \text{ kN}$$

$$13-21. P_{cr} = 1886.92 \text{ kip, No.}$$

$$13-22. \text{F.S.} = 2.19$$

$$13-23. P = 475 \text{ kip}$$

$$13-25. A = 8.85 \text{ in}^2, I_y = 19.6 \text{ in}^4, P = 62.3 \text{ kip}$$

$$13-26. P = 2.42 \text{ kip}$$

$$13-27. w = 5.55 \text{ kN/m}$$

$$13-29. F_{BC} = 20 \text{ kip}, P_{cr} = 178.9 \text{ kip},$$

$$x-x \text{ axis: F.S.} = 8.94, y-y \text{ axis: F.S.} = 3.98$$

$$13-30. P = 23.9 \text{ kip}$$

$$13-31. w = 32.5 \text{ kip/ft}$$

$$13-33. F_{AB} = 15 \text{ kN}, A = 3.2(10^{-3}) \text{ m}^2,$$

$$I_y = 0.4267(10^{-6}) \text{ m}^4, P_{cr} = 33.69 \text{ kN},$$

$$\text{F.S.} = 2.25$$

13-34. $P = 46.5 \text{ kN}$

13-35. $P = 110 \text{ kN}$

13-37. $F_{AB} = 26.8014m$, $A = 0.625(10^{-3})\pi \text{ m}^2$,
 $I = 97.65625(10^{-9})\pi \text{ m}^4$, $m = 7.06 \text{ kg}$

13-38. $P = 4.57 \text{ kip}$

13-39. $P = 2.34 \text{ kip}$

13-41. $M(x) = \frac{w}{2}(x^2 - Lx) - Pv$,
$$v_{\max} = \frac{wEI}{P^2} \left[\sec\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) - \frac{PL^2}{8EI} - 1 \right]$$
$$M_{\max} = -\frac{wEI}{P} \left[\sec\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) - 1 \right]$$

13-42. $v_{\max} = \frac{F}{2P} \left[\sqrt{\frac{EI}{P}} \tan\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right) - \frac{L}{2} \right]$,
$$M_{\max} = -\frac{F}{2} \sqrt{\frac{EI}{P}} \tan\left(\sqrt{\frac{P}{EI}} \frac{L}{2}\right)$$

13-43. $P_{cr} = \frac{\pi^2 EI}{4L^2}$

13-45. $M(x) = R'(L - x) - Pv$

13-46. $P = 26.5 \text{ kip}$

13-47. $P = 5.87 \text{ kN}$, $v_{\max} = 42.1 \text{ mm}$

13-49. $A = 0.61575(10^{-3}) \text{ m}^2$,
 $I = 64.1152(10^{-9}) \text{ m}^4$, $P_{\max} = P_{cr} = 18.98 \text{ kN}$,
 $P = 6.75 \text{ kN}$

13-50. $P = 20.1 \text{ kN}$

13-51. $P = 73.5 \text{ kip}$

13-53. $P_{cr} = 98.70 \text{ kN}$, $P_{\text{allow}} = 49.35 \text{ kN}$,
 $P_{\text{allow}} = 26.3 \text{ kN}$

13-54. $\sigma_{\max} = 130 \text{ MPa}$

13-55. The column is adequate.

13-57. Strong axis yielding controls.
 $P_{\text{allow}} = 89.0 \text{ kN}$

13-58. $P_{cr} = 199 \text{ kN}$, $e = 175 \text{ mm}$

13-59. $\sigma_{\max} = 199 \text{ MPa}$, $v_{\max} = 24.3 \text{ mm}$

13-61. The column is adequate.

13-62. $e = 0.15 \text{ m}$, $P_{\text{allow}} = 346.92 \text{ kN}$,
 $P_{\text{allow}} = 268 \text{ kN}$

13-63. $\sigma_{\max} = 6.22 \text{ ksi}$

13-65. $(KL)_x = (KL)_y = 3 \text{ m}$, $P_{cr} = 83.5 \text{ kN}$

13-66. F.S. = 1.56

13-67. The column does not fail by yielding.

13-69. $(KL)_y = (KL)_x = 210 \text{ in.}$, $P = 18.3 \text{ kip}$

13-70. $E_c = 14.6 (10^3) \text{ ksi}$

13-71. $P_{cr} = 110 \text{ kip}$

13-73. $\frac{KL}{r} < 49.7$; $\frac{P}{A} = \frac{0.4935(10^6)}{\left(\frac{KL}{r}\right)^2} \text{ MPa}$,

$$49.7 < \frac{KL}{r} < 99.3$$
; $\frac{P}{A} = 200 \text{ MPa}$,

$$\frac{KL}{r} > 99.3$$
; $\frac{P}{A} = \frac{1.974(10^6)}{\left(\frac{KL}{r}\right)^2} \text{ MPa}$

13-75. $P_{cr} = 1323 \text{ kN}$

13-77. $E_1 = 200 \text{ GPa}$, $E_2 = 150 \text{ GPa}$,
 $r = 0.02 \text{ m}$, $P_{cr} = 2700 \text{ kN}$

13-78. $L = 3.56 \text{ m}$

13-79. $L = 10.9 \text{ ft}$

13-81. $\sigma_{\text{allow}} = 11.28 \text{ ksi}$, Use W6 $\times$ 15

13-82. Use W6 $\times$ 9

13-83. Use W8 $\times$ 24

13-85. $\left(\frac{KL}{r}\right)_x = 105.26$, $\left(\frac{KL}{r}\right)_y = 111.80$,
 $P_{\text{allow}} = 80.9 \text{ kip}$

13-86. Yes.

13-87. $d = 1.42 \text{ in.}$

13-89. $A = 0.0151 \text{ m}^2$, $I_y = 90.025833(10^{-6}) \text{ m}^4$.
The column is adequate.

13-90. $L = 4.46 \text{ m}$

13-91. $b = 0.704 \text{ in.}$

13-93. $A = 5.55(10^{-3}) \text{ m}^2$, $I_x = 31.86625(10^{-6}) \text{ m}^4$,
 $I_y = 2.5478(10^{-6}) \text{ m}^4$, $P_{\text{allow}} = 422 \text{ kN}$

13-94. $L = 3.08 \text{ m}$

13-95. $P_{\text{allow}} = 108 \text{ kip}$

13-97. $A = 5.57 \text{ in}^2$, $I = 31.7448 \text{ in}^4$, $P_{\text{allow}} = 129 \text{ kip}$

13-98. $P_{\text{allow}} = 143 \text{ kip}$

13-99. $P_{\text{allow}} = 109 \text{ kip}$

13-101. $\sigma_{\text{allow}} = 0.4783 \text{ ksi}$, $P_{\text{allow}} = 8.61 \text{ kip}$

13-102. $L = 8.89 \text{ ft}$

13-103. Use $a = 7\frac{1}{2} \text{ in.}$

13-105. $\frac{KL}{d} = 1.00L$, $L = 3.87 \text{ ft}$

13-106. $P_{\text{allow}} = 1.875 \text{ kip}$

13-107. $P = 7.83 \text{ kip}$

$$13-109. \frac{KL}{r_y} = 69.231, (\sigma_a)_{allow} = 16.510 \text{ ksi}, \\ P = 80.3 \text{ kip}$$

$$13-110. P = 79.4 \text{ kip}$$

$$13-111. P = 4.07 \text{ kip}$$

$$13-113. \left(\frac{KL}{r} \right)_x = 133.33 \text{ (controls)}, \\ \left(\frac{KL}{r} \right)_y = 100.30, P = 8.60 \text{ kip}$$

$$13-114. P = 14.6 \text{ kip}$$

$$13-115. \text{The column is not adequate.}$$

$$13-117. A = 12 \text{ in}^2, I_x = 166 \text{ in}^4, I_y = 42.75 \text{ in}^4, \\ P = 95.7 \text{ kip}$$

$$13-118. P = 98.6 \text{ kip}$$

$$13-119. P = 2.79 \text{ kip}$$

$$13-121. A = 24.0 \text{ in}^2, I_x = 72.0 \text{ in}^4, I_y = 32.0 \text{ in}^4, \\ P = 98.0 \text{ kip}$$

$$13-122. P = 132 \text{ kip}$$

$$13-123. P = 2.48 \text{ kip}$$

$$13-125. \frac{KL}{d} = 43.2 \text{ in.}, \text{Yes.}$$

$$13-126. P = 1.69 \text{ kip}$$

$$13-127. P = 3.44 \text{ kip}$$

$$13-129. B_x = 0, B_y = \frac{2M}{L \sin \theta}, M = \frac{PL}{2} \sin \theta, \\ M = \frac{PL}{2} \theta, P_{cr} = \frac{2k}{L}$$

$$13-130. w = 4.63 \text{ kN/m}$$

$$13-131. P = 25.0 \text{ kip}$$

$$13-133. \left(\frac{KL}{r} \right)_x = 56.25, \left(\frac{KL}{r} \right)_y = 128.21 \text{ (controls)}, \\ \text{Yes.}$$

$$13-134. P_{allow} = 57.6 \text{ kip}$$

$$13-135. P_{cr} = 839 \text{ kN}$$

$$13-137. \bar{x} = 0.06722 \text{ m}, I_y = 20.615278(10^{-6}) \text{ m}^4, \\ I_z = 7.5125(10^{-4}) \text{ m}^4, \text{No.}$$

Chapter 14

$$14-1. \frac{U_i}{V} = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G}$$

$$14-3. (U_i)_a = 3.28 \text{ J}$$

$$14-5. N_{AB} = 3 \text{ kN}, N_{BC} = 7 \text{ kN}, N_{CD} = -3 \text{ kN}, \\ U_i = 0.372 \text{ J}$$

$$14-6. (U_i)_a = 43.2 \text{ J}$$

$$14-7. P = 375 \text{ kN}, (U_i)_a = 1.69 \text{ kJ}$$

$$14-9. T = 8.0 \text{ kN} \cdot \text{m}, T = 2.0 \text{ kN} \cdot \text{m}, \\ T = -10.0 \text{ kN} \cdot \text{m}, U_i = 149 \text{ J}$$

$$14-10. (U_i)_i = \frac{7T^2L}{24\pi r_0^4 G}$$

$$14-11. (U_i)_i = 0.0637 \text{ J}$$

$$14-13. V = -P, M = -Px, \frac{(U_i)_y}{(U_i)_h} = \frac{3(1+\nu)}{5} \left(\frac{h}{L} \right)^2$$

$$14-14. U_i = \frac{M_0^2 L}{24EI}$$

$$14-15. (U_i)_h = \frac{P^2 L^3}{48EI}$$

$$14-17. M(x) = \left(9x - \frac{1}{4}x^3 \right) \text{ kN} \cdot \text{m}, (U_i)_h = 33.6 \text{ J}$$

$$14-18. (U_i)_h = 29.3 \text{ J}$$

$$14-19. U_i = \frac{P^2 r^3}{JG} \left(\frac{3\pi}{8} - 1 \right)$$

$$14-21. T_y = \frac{PL}{2}, M_x = Py, U_i = P^2 L^3 \left[\frac{3}{16EI} + \frac{1}{8JG} \right]$$

$$14-22. U_i = \frac{3P^2 L^3}{bh^3 E}, 1.5 \text{ times as great as for a uniform cross section}$$

$$14-23. (a) U_i = \frac{w^2 L^5}{40EI}, (b) U_i = \frac{w^2 L^5}{40EI}$$

$$14-25. F_{AD} = 2.50 \text{ kip (T)}, F_{AB} = 1.50 \text{ kip (C)}, \\ F_{DB} = 2.50 \text{ kip (C)}, F_{DC} = 3.00 \text{ kip (T)}, \\ (\Delta_A)_h = 0.0142 \text{ in.}$$

$$14-26. (\Delta_C)_h = \frac{2PL}{AE}$$

$$14-27. \Delta_C = \frac{2PL}{AE}$$

$$14-29. U_e = \frac{1}{2} (M_0 \theta_B), \theta_B = \frac{M_0 L}{EI}$$

$$14-30. U_i = 0.726 \text{ in} \cdot \text{kip}, \Delta_C = 0.0145 \text{ in.}$$

$$14-31. \theta_B = 0.00100 \text{ rad}$$

$$14-33. U_i = \frac{65625}{EI}, U_e = 150 \theta_E, \\ \theta_E = 3.15^\circ$$

$$14-34. \Delta_B = 2.67 \text{ in.}$$

$$14-35. \Delta_B = 11.7 \text{ mm}$$

$$14-37. T = PR \cos \theta, M = PR \sin \theta,$$

$$\sigma_{\max} = \frac{16PR}{\pi d^3} [\sin \theta + 1]$$

$$14-38. \Delta = \frac{64nPR^3}{d^4G}$$

$$14-39. \Delta_C = 2.13 \text{ mm}$$

$$14-41. M_1 = -20(10^3) \text{ N}\cdot\text{m}, M_2 = 60(10^3) \text{ N}\cdot\text{m},$$

$$\Delta_B = 15.2 \text{ mm}$$

$$14-42. (a) U_i = 4.52 \text{ kJ}, (b) U_i = 3.31 \text{ kJ}$$

$$14-43. d = 5.35 \text{ in.}$$

$$14-45. \Delta_{st} = 9.8139(10^{-6}) \text{ m}, \sigma_{\max} = 359 \text{ MPa}$$

$$14-46. h = 69.6 \text{ mm}$$

$$14-47. \sigma_{\max} = 216 \text{ MPa}$$

$$14-49. \Delta_{st} = 0.613125(10^{-3}) \text{ m}, k = 160(10^6) \text{ N/m},$$

$$\sigma_{\max} = 237 \text{ MPa}, \Delta_{\max} = 3.95 \text{ mm}$$

$$14-50. \sigma_{\max} = 85.7 \text{ MPa}$$

$$14-51. h = 0.571 \text{ m}$$

$$14-53. \sigma_{st} = 0.3123 \text{ MPa}, I_x = 325 \text{ mm}^4$$

$$14-54. \sigma_{\max} = 207 \text{ MPa}$$

$$14-55. \sigma_{\max} = 43.6 \text{ ksi}$$

$$14-57. \sigma_{st} = 153.78 \text{ psi}, n = 195.08, h = 11.6 \text{ ft}$$

$$14-58. (\Delta_A)_{\max} = 15.4 \text{ in.}$$

$$14-59. h = \frac{\sigma_{\max} L^2}{3Ec} \left[\frac{\sigma_{\max} I}{WLc} - 2 \right]$$

$$14-61. n = 209.13, h = 2.23 \text{ m}$$

$$14-62. \sigma_{\max} = 5.88 \text{ ksi}$$

$$14-63. h = 3.73 \text{ in.}$$

$$14-65. k_{\text{beam}} = 1.7700 \text{ kip/in.}, n = 16.7$$

$$14-66. \sigma_{\max} = 47.8 \text{ MPa}$$

$$14-67. h = 6.57 \text{ m}$$

$$14-69. k_b = 49.3425(10^6) \text{ N/m}, \Delta_b = 0.050342 \text{ m},$$

$$v = 5.75 \text{ m/s}$$

$$14-70. \Delta_{\text{beam}} = 0.481 \text{ in.}, \sigma_{\max} = 10.1 \text{ ksi}$$

$$14-71. \Delta_{\max} = 23.3 \text{ mm}, \sigma_{\max} = 4.89 \text{ MPa}$$

$$14-73. 1 \text{ lb} \cdot (\Delta_B)_B = \frac{40\,533.33 \text{ lb}^2 \cdot \text{in.}}{AE},$$

$$(\Delta_B)_B = 0.699(10^{-3}) \text{ in.} \rightarrow$$

$$14-74. (\Delta_B)_C = 0.0931(10^{-3}) \text{ in.} \downarrow$$

$$14-75. \Delta_{C'} = 20.4 \text{ mm}$$

$$14-77. 1 \text{ kip} \cdot (\Delta_B)_B = \frac{1620 \text{ kip}^2 \cdot \text{in.}}{AE},$$

$$(\Delta_B)_B = 0.0124 \text{ in.} \downarrow$$

$$14-78. (\Delta_E)_E = 0.00966 \text{ in.} \downarrow$$

$$14-79. \Delta_{B_1} = 0.367 \text{ mm}$$

$$14-81. 1 \text{ N} \cdot (\Delta_A)_A = \frac{498.125(10^3) \text{ N}^2 \cdot \text{m}}{AE},$$

$$(\Delta_A)_A = 6.23 \text{ mm} \downarrow$$

$$14-82. (\Delta_B)_B = 3.79 \text{ mm} \downarrow$$

$$14-83. \Delta_{C'} = 0.163 \text{ in.}$$

$$14-85. 1 \text{ N} \cdot (\Delta_C)_C = \frac{174.28125(10^3)}{0.15(10^{-3})[200(10^9)]},$$

$$(\Delta_C)_C = 5.81 \text{ mm} \downarrow$$

$$14-86. (\Delta_G)_G = 3.41 \text{ mm} \downarrow$$

$$14-87. \Delta_C = \frac{23Pa^3}{24EI}$$

$$14-89. \Delta_C = \frac{572.92 \text{ kN} \cdot \text{m}^3}{EI},$$

$$M(x_1) = 2.50x_1, M(x_2) = x_2^2,$$

$$m(x_1) = 0.50x_1, m(x_2) = x_2,$$

$$\Delta_{C'} = 40.9 \text{ mm} \downarrow$$

$$14-90. \theta_A = 0.00298 \text{ rad}$$

$$14-91. \theta_B = 0.00595 \text{ rad}$$

$$14-93. M = 327.06x_1, M = 654.12 + 47.06x_2,$$

$$m_\theta = 1 - 0.1176x_1, m_\theta = 0.1176x_2,$$

$$\theta_A = 2.73^\circ$$

$$14-94. \theta_{C'} = 5.89(10^{-3}) \text{ rad}$$

$$14-95. \Delta_A = 27.4 \text{ mm} \downarrow, \theta_A = 5.75(10^{-3}) \text{ rad}$$

$$14-97. M_1 = Px, M_2 = Px_2,$$

$$m_\theta = \frac{x_1}{a}, m_\theta = 1,$$

$$\theta_{C'} = \frac{5Pa^2}{6EI}$$

$$14-98. \theta_A = \frac{Pa^2}{6EI}$$

$$14-99. \theta_A = 0.00700 \text{ rad}$$

$$14-101. M(x_1) = \frac{w}{24} (11Lx_1 - 12x_1^2), M(x_2) = \frac{w}{3L} x_2^3,$$

$$m_\theta(x_1) = -\frac{x_1}{L}, m_\theta(x_2) = 1, \theta_{C'} = \frac{13wL^3}{576EI}$$

$$14-102. \Delta_D = \frac{wL^4}{96EI} \downarrow$$

$$14-103. \Delta_{C'} = 0.557 \text{ in.} \downarrow$$

$$14-105. M(x_2) = wa(a + x_2) - \frac{w}{2}x_2^2, M(x_1) = wax_1,$$

$$m(x_2) = \frac{1}{2}(x_2 + a), m(x_1) = \frac{x_1}{2}, \Delta_B = \frac{65wa^4}{48EI} \downarrow$$

$$14-106. \Delta_C = \frac{w_0L^4}{120EI}$$

$$14-107. \theta_A = \frac{5w_0L^3}{192EI}$$

$$14-109. M(x_1) = (31.5x_1 - 6x_1^2) \text{ kN} \cdot \text{m},$$

$$M(x_2) = (22.5x_2 - 3x_2^2) \text{ kN} \cdot \text{m},$$

$$m(x_1) = (1 - 0.1667x_1) \text{ kN} \cdot \text{m},$$

$$m(x_2) = (0.1667x_2) \text{ kN} \cdot \text{m},$$

$$\theta_A = 0.00927 \text{ rad}$$

$$14-110. \Delta_C = 16.7 \text{ mm} \downarrow$$

14-111. Bending and shear:

$$\Delta = \left(\frac{w}{G}\right)\left(\frac{L}{a}\right)^2 \left[\left(\frac{5}{96}\right)\left(\frac{L}{a}\right)^2 + \frac{3}{20} \right],$$

$$\text{Bending only: } \Delta = \frac{5w}{96G}\left(\frac{L}{a}\right)^4$$

$$14-113. M(x_1) = \frac{w}{2}x_1^2, M(x_2) = \frac{wL^2}{2},$$

$$m(x_1) = 0, m(x_2) = 1.0L - 1.0x_2,$$

$$(\Delta_B)_s = \frac{wL^4}{4EI} \rightarrow$$

$$14-114. \Delta_A = \frac{4PL^3}{3EI}$$

$$14-115. \Delta_B = 43.5 \text{ mm} \downarrow$$

$$14-117. \Delta_C = 17.9 \text{ mm} \downarrow$$

$$14-118. \theta_A = 0.991(10^{-3}) \text{ rad}$$

$$14-119. (\Delta_C)_s = 16.8 \text{ mm} \downarrow$$

$$14-121. M = \frac{M_0}{a}, M = M_0,$$

$$m = 1x, m = 1x, \Delta_C = \frac{5M_0a^2}{6EI}$$

$$14-122. \theta_B = \frac{M_0a}{3EI}$$

$$14-123. \Delta_{B_s} = 0.00191 \text{ in.}$$

$$14-125. \Delta_C = \frac{1225.26(10^3)}{300(10^{-6})(200)(10^9)} = 20.4 \text{ mm}$$

$$14-126. \Delta_{D_s} = 4.88 \text{ mm}$$

$$14-127. (\Delta_B)_s = 0.0124 \text{ in.} \downarrow$$

$$14-129. \Delta_{B_s} = \frac{29.375(10^5)}{400(10^{-6})(200)(10^9)} = 0.367 \text{ mm}$$

$$14-130. \Delta_{C_s} = 0.0375 \text{ mm}$$

$$14-131. (\Delta_A)_s = 6.23 \text{ mm} \downarrow$$

$$14-133. \Delta_{C_s} = \frac{21\,232}{4.5(29)(10^3)} = 0.163 \text{ in.}$$

$$14-134. \Delta_{H_s} = 0.156 \text{ in.}$$

$$14-135. \Delta_C = \frac{23Pa^3}{24EI}$$

$$14-137. \theta_A = \frac{41.667(10^3)}{200(10^9)[70(10^{-6})]} = 0.00298 \text{ rad}$$

$$14-138. \Delta_B = 1.54 \text{ in.}$$

$$14-139. \theta_A = 2.73^\circ$$

$$14-141. \theta_C = \frac{5Pa^2}{6EI}$$

$$14-142. \theta_A = \frac{Pa^2}{6EI}$$

$$14-143. (\Delta_C)_s = \frac{5wL^4}{8EI} \downarrow$$

$$14-145. M = \frac{M_0}{a}x, M = M_0, M = Px, M = Px,$$

$$\Delta_C = \frac{5M_0a^2}{6EI}$$

$$14-146. U_i = \frac{5P^2a^3}{6EI}$$

$$14-147. \sigma_{\max} = 116 \text{ MPa}$$

14-149. Bending strain energy:

$$(U_b)_i = \frac{1.176(10^6)}{29(10^6)(\frac{1}{12})(0.5)(0.2^3)} = 10.1 \text{ ft} \cdot \text{lb.}$$

Axial force strain energy:

$$(U_a)_i = \frac{(350)^2(8)}{2(29)(10^6)(\frac{\pi}{4})(0.25^2)} = 0.344 \text{ in} \cdot \text{lb}$$

$$14-150. \Delta_A = 0.536 \text{ mm}$$

$$14-151. (U_i)_i = 2.23 \text{ in} \cdot \text{lb}$$

$$14-153. \Delta_{E_s} = \frac{236.25(10^3)}{400(10^{-6})(200)(10^9)} = 2.95 \text{ mm}$$

$$14-154. \theta_B = \frac{M_0L}{EI}$$

$$14-155. \theta_B = \frac{M_0L}{EI}$$

$$14-157. \sigma_{\max} = 10.5 \text{ ksi}$$

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Average Mechanical Properties of Typical Engineering Materials^a
(U.S. Customary Units)

Materials	Specific Weight γ (lb/in ³)	Modulus of Elasticity E (10 ³) ksi	Modulus of Rigidity G (10 ³) ksi	Yield Strength (ksi)		Ultimate Strength (ksi)		% Elongation in 2 in. specimen	Poisson's Ratio ν	Coef. of Therm. Expansion α (10 ⁻⁶)/°F
Metallic										
Aluminum ☐ 2014-T6	0.101	10.6	3.9	60	60	68	68	10	0.35	12.8
Wrought Alloys ☐ 6061-T6	0.098	10.0	3.7	37	37	42	42	12	0.35	13.1
Cast Iron ☐ Gray ASTM 20	0.260	10.0	3.9	—	—	26	47	0.6	0.28	6.70
Alloys ☐ Malleable ASTM A-197	0.263	25.0	9.8	—	—	40	83	5	0.28	6.60
Copper ☐ Red Brass C83400	0.316	14.6	5.4	11.4	11.4	35	35	35	0.35	9.80
Alloys ☐ Bronze C86100	0.319	15.0	5.6	50	50	95	95	20	0.34	9.60
Magnesium Alloy ☐ Am 1004-T61	0.086	6.48	2.5	22	22	40	40	1	0.30	14.3
Steel ☐ Structural A36	0.284	29.0	11.0	36	36	58	58	30	0.32	6.60
Alloys ☐ Stainless 304	0.284	28.0	11.0	30	30	75	75	40	0.27	9.60
☐ Tool L2	0.295	29.0	11.0	102	102	116	116	22	0.32	6.50
Titanium Alloy ☐ Ti-6Al-4V	0.160	17.4	6.4	134	134	145	145	16	0.36	5.30
Nonmetallic										
Concrete ☐ Low Strength	0.086	3.20	—	—	—	—	—	—	0.15	6.0
☐ High Strength	0.086	4.20	—	—	—	—	—	—	0.15	6.0
Plastic ☐ Kevlar 49	0.0524	19.0	—	—	—	104	70	2.8	0.34	—
Reinforced ☐ 30% Glass	0.0524	10.5	—	—	—	13	19	—	0.34	—
Wood										
Select Structural ☐ Douglas Fir	0.017	1.90	—	—	—	0.30 ^d	3.78 ^d	—	0.29 ^e	—
Grade ☐ White Spruce	0.130	1.40	—	—	—	0.30 ^d	5.18 ^d	—	0.31 ^e	—

^a Specific values may vary for a particular material due to alloy or mineral composition, mechanical working of the specimen, or heat treatment. For a more exact value reference books for the material should be consulted.

^b The yield and ultimate strengths for ductile materials can be assumed equal for both tension and compression.

^c Measured perpendicular to the grain.

^d Measured parallel to the grain.

^e Deformation measured perpendicular to the grain when the load is applied along the grain.

Average Mechanical Properties of Typical Engineering Materials^a
(SI Units)

Materials	Density ρ (N/g/m ³)	Modulus of Elasticity E (GPa)	Modulus of Rigidity G (GPa)	Yield Strength (MPa) σ_Y			Ultimate Strength (MPa) σ_u			% Elongation in 50 mm specimen	Poisson's Ratio ν	Coef. of Therm. Expansion α (10 ⁻⁶)/°C
Metallic												
Aluminum ☐ 2014-T6	2.79	73.1	27	414	414	172	469	469	290	10	0.35	23
Wrought Alloys ☐ 6061-T6	2.71	68.9	26	255	255	131	290	290	186	12	0.35	24
Cast Iron ☐ Gray ASTM 20	7.19	67.0	27	—	—	—	179	669	—	0.6	0.28	12
Alloys ☐ Malleable ASTM A-197	7.28	172	68	—	—	—	276	572	—	5	0.28	12
Copper ☐ Red Brass C83400	8.74	101	37	70.0	70.0	—	241	241	—	35	0.35	18
Alloys ☐ Bronze C96100	8.83	103	38	345	345	—	655	655	—	20	0.34	17
Magnesium Alloy ☐ [Am 1004-T61]	1.83	44.7	18	152	152	—	276	276	152	1	0.30	26
Steel ☐ Structural A36	7.85	200	75	250	250	—	400	400	—	30	0.32	12
Alloys ☐ Stainless 304	7.86	193	75	207	207	—	517	517	—	40	0.27	17
☐ Tool L2	8.16	200	75	703	703	—	800	800	—	22	0.32	12
Titanium Alloy ☐ [Ti-6Al-4V]	4.43	120	44	924	924	—	1,000	1,000	—	16	0.36	9.4
Nonmetallic												
Concrete ☐ Low Strength	2.38	22.1	—	—	—	12	—	—	—	—	0.15	11
☐ High Strength	2.38	29.0	—	—	—	38	—	—	—	—	0.15	11
Plastic ☐ Nylon 49	1.45	131	—	—	—	—	717	483	203	2.8	0.34	—
Reinforced ☐ 50% Glass	1.49	72.4	—	—	—	—	90	131	—	—	0.34	—
Wood ☐ Douglas Fir	0.47	13.1	—	—	—	—	2.1 ^b	26 ^d	6.2 ^d	—	0.29 ^c	—
Select Structural ☐ White Spruce	3.60	9.65	—	—	—	—	2.8 ^c	36 ^d	6.7 ^d	—	0.31 ^c	—
Grade												

^a Specific values may vary for a particular material due to alloy or mineral composition, mechanical working of the specimen, or heat treatment. For a more exact value reference books for the material should be consulted.

^b The yield and ultimate strengths for ductile materials can be assumed equal for both tension and compression.

^c Measured perpendicular to the grain.

^d Measured parallel to the grain.

^e Deformation measured perpendicular to the grain when the load is applied along the grain.

Fundamental Equations of Mechanics of Materials

Axial Load

Normal Stress

$$\sigma = \frac{P}{A}$$

Displacement

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E}$$

$$\delta = \sum \frac{PL}{AE}$$

$$\delta_T = \alpha \Delta T L$$

Torsion

Shear stress in circular shaft

$$\tau = \frac{Tr}{J}$$

where

$$J = \frac{\pi}{2} c^4 \text{ solid cross section}$$

$$J = \frac{\pi}{2} (c_o^4 - c_i^4) \text{ tubular cross section}$$

Power

$$P = T\omega = 2\pi fT$$

Angle of twist

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G}$$

$$\phi = \sum \frac{TL}{JG}$$

Average shear stress in a thin-walled tube

$$\tau_{avg} = \frac{T}{2lA_m}$$

Shear Flow

$$q = \tau_{avg}l = \frac{T}{2A_m}$$

Bending

Normal stress

$$\sigma = \frac{My}{I}$$

Unsymmetric bending

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}, \quad \tan \alpha = \frac{I_z}{I_y} \tan \theta$$

Shear

Average direct shear stress

$$\tau_{avg} = \frac{V}{A}$$

Transverse shear stress

$$\tau = \frac{VQ}{It}$$

Shear flow

$$q = \tau t = \frac{VQ}{I}$$

Stress in Thin-Walled Pressure Vessel

Cylinder

$$\sigma_1 = \frac{pr}{t} \quad \sigma_2 = \frac{pr}{2t}$$

Sphere

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Stress Transformation Equations

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Principal Stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum in-plane shear stress

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Absolute maximum shear stress

$$\tau_{abs,max} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

$$\sigma_{avg} = \frac{\sigma_{max} + \sigma_{min}}{2}$$

Geometric Properties of Area Elements

Material Property Relations

Poisson's ratio

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}$$

Generalized Hooke's Law

$$\epsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\epsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy}, \quad \gamma_{yz} = \frac{1}{G} \tau_{yz}, \quad \gamma_{zx} = \frac{1}{G} \tau_{zx}$$

where

$$G = \frac{E}{2(1 + \nu)}$$

Relations Between w , V , M

$$\frac{dV}{dx} = -w(x), \quad \frac{dM}{dx} = V$$

Elastic Curve

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$EI \frac{d^4 v}{dx^4} = -w(x)$$

$$EI \frac{d^3 v}{dx^3} = V(x)$$

$$EI \frac{d^2 v}{dx^2} = M(x)$$

Buckling

Critical axial load

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

Critical stress

$$\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}, \quad r = \sqrt{I/A}$$

Secant formula

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right]$$

Energy Methods

Conservation of energy

$$U_e = U_i$$

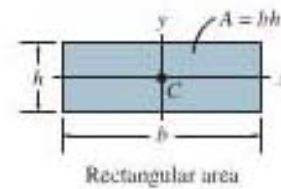
Strain energy

$$U_i = \frac{N^2 L}{2AE} \quad \text{constant axial load}$$

$$U_i = \int_0^L \frac{M^2 dx}{EI} \quad \text{bending moment}$$

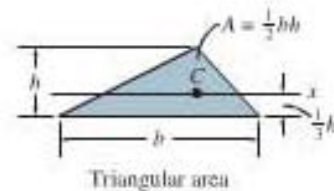
$$U_i = \int_0^L \frac{f_s V^2 dx}{2GA} \quad \text{transverse shear}$$

$$U_i = \int_0^L \frac{T^2 dx}{2GJ} \quad \text{torsional moment}$$

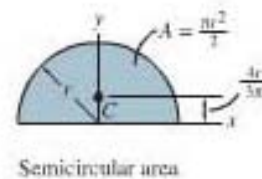
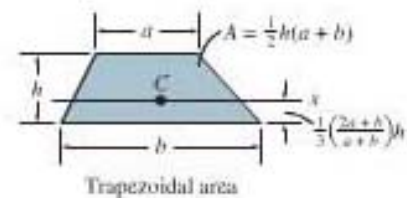


$$I_x = \frac{1}{12} bh^3$$

$$I_y = \frac{1}{12} hb^3$$

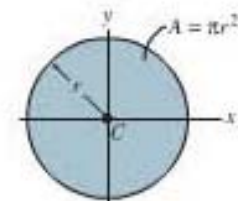


$$I_x = \frac{1}{36} bh^3$$



$$I_x = \frac{1}{8} \pi r^4$$

$$I_y = \frac{1}{8} \pi r^4$$



$$I_x = \frac{1}{4} \pi r^4$$

$$I_y = \frac{1}{4} \pi r^4$$

