

## ABOUT THIS EDITION

Machinery performance depends on an engineer's understanding of measurement theory and technique. This edition presents the latest standards and industrial practices for the measurement of temperature, pressure, and flow rate. It shows how measurement techniques were developed and then applies them to many practical cases through examples. The book deals with all types of sensors and transducers, detailing construction, installation, supporting electronic circuitry, calibration, response time, and recovery time for each type of sensor.

The third edition incorporates important changes that have occurred during the past decade. It treats revisions in the International Practical Temperature Scale that have affected all thermometry. New thermocouple reference tables that correspond to standard equations issued by the National Bureau of Standards are given. Other new material includes: thermocouple analysis, temperature response, pressure tap errors, ASME discharge coefficients, and expansion factors. The chapter on measurement uncertainties and statistics has been completely revised in accord with the latest national standards because of the increasing importance of determining Best Values and Confidence Intervals.

Tables and graphs provide extensive supporting data. Engineers, physicists, technicians, and students will find this a thorough review of fundamentals and techniques for measuring temperature, pressure, and flow rate.

# **FUNDAMENTALS OF TEMPERATURE, PRESSURE, AND FLOW MEASUREMENTS**

## OTHER BOOKS BY THE AUTHOR

*Handbook of Generalized Gas Dynamics*

Plenum Press, 1966 (with W. G. Steltz)

*Handbook of Specific Losses in Flow Systems*

Plenum Press, 1966 (with N. A. Carlucci)

*Manual on the Use of Thermocouples in*

*Temperature Measurement*

STP470, STP470A & STP470B, American Society for  
Testing and Materials, 1970, 1974 & 1981 (Editor)

*Journey Away From God*

Fleming H. Revell Co., 1972

*Fundamentals of Pipe Flow*

Wiley-Interscience, 1980

*Fundamentals of Gas Dynamics*

John Wiley & Sons, 1983

# FUNDAMENTALS OF TEMPERATURE, PRESSURE, AND FLOW MEASUREMENTS

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THIRD EDITION

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**To My Wife  
Ruth**

*I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of Science, whatever the matter may be.*

*Sir William Thomson  
(Lord Kelvin)*

# Preface

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In this edition, the chapters on thermocouples, statistics, temperature response, pressure measurements in moving fluids, discharge coefficients, and flow installations have been especially strengthened. These changes and additions are in accord with the latest terminology used in the various standardizing groups of the United States and the international organizations.

In all chapters, I have tried to correct typographical errors and errors in fact, as well as to update the material wherever possible.

I would like to thank all the students, researchers, and teachers who have used this work, and who have been so generous in their remarks concerning this book. I would further like to acknowledge my debt to the members of the ASME Performance Test Codes and Fluid Meters Research Committees and of the ASTM Temperature Committees for their continued inspiration and for their supply of technical information, which I have used freely.

*Holly Hill, Pennsylvania  
March 1984*

ROBERT P. BENEDICT

# Preface to the Second Edition

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In preparing this edition, I have endeavored to include the major changes that have occurred in the areas of temperature, pressure, and flow measurements over the past decade. Specifically, the International Practical Temperature Scale has changed. This new scale (IPTS-68) is now the law of the land, and Chapter 4 has been completely rewritten to reflect this. Chapter 6, on Resistance Thermometry, also reflects changes in the IPTS. New Thermocouple Reference Tables were issued by the National Bureau of Standards in 1974. Accordingly, Chapter 7 has been revised to include these latest tables of temperature versus electromotive force for the thermocouple types most commonly used in industry. Also, along these same lines, the National Bureau of Standards has issued new methods for generating the new Reference Table values for computer applications. These power series relationships, giving emf as a function of temperature, are now included in Chapter 7. In Part II, much new material has been added in Chapter 17 to account for pressure tap errors, probe blockage effects, probe performance in pressure gradients, and so on. In Part III, on Flow Measurement, Chapters 21, 22, and 23 have been rewritten to include new material on ASME discharge coefficients, expansion factors, and the like. A new section, Chapter 10, has been added on the important subject of Statistics and Uncertainties as applied to engineering measurements. I have further attempted to correct any gross errors in the First Edition, and to provide a more complete and updated bibliography. Finally, many additional worked-out examples have been included in each chapter, and problems have been added at the end of each chapter to aid in the teaching of this subject material.

Once again, I wish to thank my wife for her continued patience and skill in typing this manuscript again. And finally, my best wishes go along to the students, teachers, and researchers who are the ultimate users of this work.

*Holly Hill, Pennsylvania  
August 1976*

ROBERT P. BENEDICT

# Preface to the First Edition

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Someone has said that science and the history of science are inseparable, and I believe it. How can one appreciate Michelson's and Morley's feat in determining the speed of light if the many attempts and frustrations of others to do the same are overlooked? How can one grasp the profundity of Max Planck's quantum theory if the paths of Maxwell, Kirchoff, and Wien are not followed first? So it is with thermal and flow measurements; unless we struggle through with Galileo, da Vinci, Newton, Bernoulli, Pitot, Joule, Kelvin, Weisbach, Herschel, and others in their efforts to sense temperature, pressure, and flow, we cannot appreciate our present-day array of fine instruments and techniques.

Behind every successful man, they say, there is an understanding woman. So behind every successful project, sale, or test involving machinery there is an engineer who understands measurements. Spend what you will on design and manufacture, in the end it is the field test that justifies your efforts. Were the heat rates met? Did the engine perform up to specs? Was the cooling arrangement adequate? The list could be long and varied.

This book is intended as a working reference for practicing engineers and associated workers involved in the measurement of temperature, pressure, and flow rate and as a text for engineering students. It should be especially useful to mechanical engineers in the fields of fluid mechanics, thermodynamics, and heat transfer. It provides a thorough review of techniques for measuring temperature, pressure, and flow rate and also includes practical data. Each topic is treated in depth, and applications are illustrated by many practical worked-out numerical examples. The book is self-contained in the sense that all necessary equations, corrections, schematic diagrams, curves, and tables of data are included.

The subject matter with which this book is concerned has been gathered through the years in the course of my work at the Westinghouse Steam and Aviation Gas Turbine Divisions. It has been strongly influenced by my participation in various ASME, ASTM, and ASA Committees on measurements. The material contained here is essentially that presented in a one-term course in

instrumentation which I have taught for a number of years at the Evening College of the Drexel Institute of Technology. There has always been much interest in this subject, as evidenced by the students' response to the material.

Parts of this work have appeared in a number of other publications. Three articles I have written in particular form the basis of this book. "Temperature and Its Measurement," July 1963; "Fluid Flow Measurement," January 1966; and "Pressure and Its Measurement," October 1967; all of which appeared in the Science and Engineering Series of *Electro-Technology*.

As we part company personally and get down to the business at hand, namely the study of thermal and flow instrumentation, I note with a twinge of conscience that the contributions of others have not been recognized adequately in this preface. I should like to acknowledge here that any merit or order that is found in this book must be credited largely to the work of others. Wherever possible references are made to the original workers at appropriate places in the text. In addition, I especially want to thank my wife for her patience and skill in typing the manuscript.

I hope that you, kind reader, will enjoy the perusal of this material as I have enjoyed gathering it.

Holly Hill, Pennsylvania  
October 1968

ROBERT P. BENEDICT

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## CHAPTER

# 10

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# Uncertainties and Statistics

*An intrinsic variability exists which prevents us from getting exactly the same observation or measurement on repeated trials.*

*J. Stuart Hunter (1968)*

## 10.1 GENERAL REMARKS

If there is one premise basic to instrumentation engineering, it is this: *no measurement is without error*. Hence neither the exact value of the quantity being measured nor the exact error associated with the measurement can be ascertained. In engineering, as in physics, the uncomfortable principle of indeterminacy exists. Yet as we have seen in our discussion of interpolation methods (Section 9.3), uncertainties can be useful and, like friction, are often a blessing in disguise. It is toward a methodical use of measurement uncertainties as a guide to approaching true values that this chapter is addressed.

The output in most experiments is a measurement. The reliability of the measurement depends not only on variations in controlled inputs, but also, in general, on variations in factors that are uncontrolled and perhaps unrecognized. Some of these factors that might unwittingly affect a measurement are the experimenter, the supporting equipment, and conditions of the environment. Thus in addition to errors caused by the device under test, and in addition to errors caused by variations in the quantity being measured, extraneous factors might introduce errors in the experiment that would cloud the results.

Errors caused by the experimenter and the supporting equipment can be *minimized* by securing independent readings by different observers and by the

use of different measuring equipment. Effects of those variables that are not part of the study can be further minimized by taking observations in a random order. This is called *randomization*.

The important task of measuring the remaining significant errors is approached by taking a number of independent observations of the output at fixed values of the controlled input. This is called *replication*. Replication is shown to be at the heart of statistics.

Stating the above ideas in mathematical terms, each measurement  $X$  can be visualized as being accompanied by an error  $\delta$  such that the interval

$$X \pm \delta \quad (10.1)$$

will contain the true value  $\bar{X}'$  of the quantity being measured. The measurement error  $\delta$ , in turn, is usually expressed in terms of two components, a *random* error  $\epsilon$  and a *systematic* error  $\beta$ , such that

$$\delta = \beta + \epsilon \quad (10.2)$$

### 10.1.1 Random Errors

When repeated measurements are taken, random errors  $\epsilon$  will show up as scatter about the *average* of these measurements. The scatter is caused by characteristics of the measuring system and/or by changes in the quantity being measured. Random errors always will be observed as long as the readout equipment has adequate discrimination. The term *precision* is used to characterize random errors. Precision is quantified by the true standard deviation  $\sigma$  of the whole population of measurements or, more often, by its estimator  $S$ , the precision index of the data available. These statistical terms will be defined shortly by working equations. Now it is sufficient to understand that a large precision index means a lot of scatter in the data, and conversely, a small precision index means high precision.

### 10.1.2 Systematic Errors

Over and above the random errors involved in all measurements, there are also errors that are consistently either too high or too low with respect to the accepted true value. Such errors, which are termed fixed errors or systematic errors, are characterized by the term *bias*. Systematic error is quantified by the true bias  $\beta$  or, more often, by  $B$ , the estimate of the limit of the bias. When bias can be quantified, it is used as a correction factor to be applied to all measurements. A zero bias implies that there is no difference between the true value  $\bar{X}'$  and the true mean  $\mu$  of many observations. However, the zero-bias case is rare indeed; and experience indicates a strong tendency to underestimate systematic errors. All of these terms and symbols are shown geometrically in Figure 10.1.

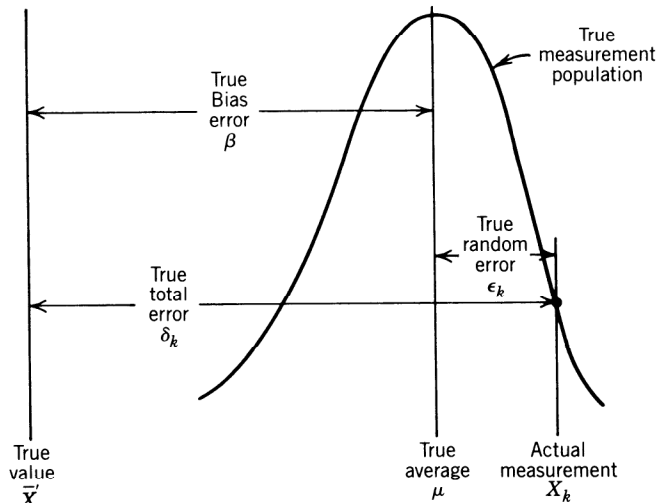


FIGURE 10.1 Geometric relations between various statistical terms.

Systematic errors can be minimized by various methods as, for example, by calibration (Figure 10.2). Calibrations are usually accomplished by comparing a test instrument to a standard instrument. Since such comparisons are not always direct or perfect, we may not succeed in totally determining the bias, that is, the bias may have a random component, but it is essentially fixed, and is never as random as precision errors.

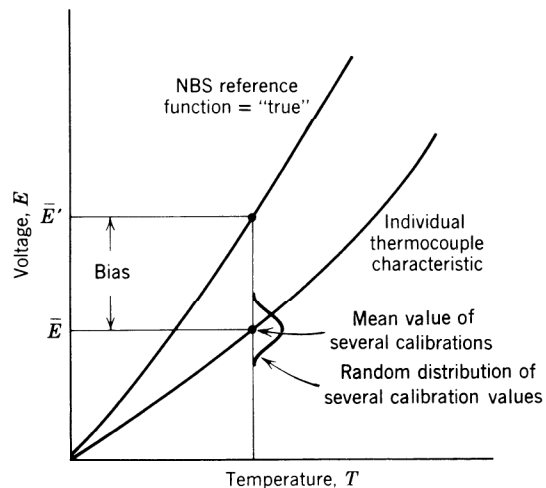


FIGURE 10.2 Systematic and random errors illustrated for the case of a thermocouple calibration.

## 10.2 STATISTICAL RELATIONS

There are cases in engineering practice, however, when we can presume that the bias *is* removed, that all errors *are* of the random type, and that hence the errors *can* be treated statistically [1], [2]. In this section we overlook for a time the fixed (bias) errors and consider only the random (precision) errors.

It is clear that, even in the absence of fixed errors, we are to be denied by the nature of things the ability to measure directly the true value of a variable. Thus it becomes our job to extract from the experimental data at least two vital bits of information. First we must form an estimate of the *best value* of the variable. This will be denoted by  $\bar{X}$ . Closely coupled with this requirement, we must give an estimate of the *intervals*, centered on  $\bar{X}$ , within which the true value is expected to lie. This will be denoted by the uncertainty margin that we tack on to  $\bar{X}$  [3].

### 10.2.1 Best Value at a Given Input

When an output  $X$  is measured many times at a given input, the mean value of  $X$  is simply

$$\bar{X} = \frac{1}{N} \sum_{k=1}^N X_k \quad (10.3)$$

where  $X_k$  is the value of the  $k$ th observation (called interchangeably the  $k$ th reading or measurement) and  $N$  is the number of observations in the sample. It is a mathematical fact that the arithmetic average defined by equation (10.3) is the best representation of the given set of  $X_k$ . Note that when the estimated best value of  $X$  is taken as  $\bar{X}$ , the sum of the squares of the deviations of the data from their estimate is a minimum. (This is essentially the least-squares principle.) However, whereas  $\bar{X}$  represents an unbiased estimate of the true arithmetic mean  $\mu$  of all possible values of  $X$ , there is no assurance that  $\bar{X}$  is the true value  $\bar{X}'$ , that is, in any actual measurement, the bias would have to be considered. (Thus good agreement, that is, high precision, in small sample replication does not imply that  $\bar{X}$  is close to  $\bar{X}'$ , that is, high accuracy.) Nevertheless, from any viewpoint the best estimate of the true value of the population mean  $\mu$  at a given input is the average  $\bar{X}$  of the available measurements (Figure 10.3).

### 10.2.2 Confidence Intervals

Having decided on the best available value of  $X$  (which is  $\bar{X}$ ), we inquire next as to its worth as an estimate of the true value of  $X$  (which is  $\mu$  for the case of zero bias). C. G. Darwin has noted in this regard [4]: “It seemed to me that there was a defect in the habit of thought of many in the engineering profession, and that some sort of campaign was needed to inculcate in people’s minds the idea

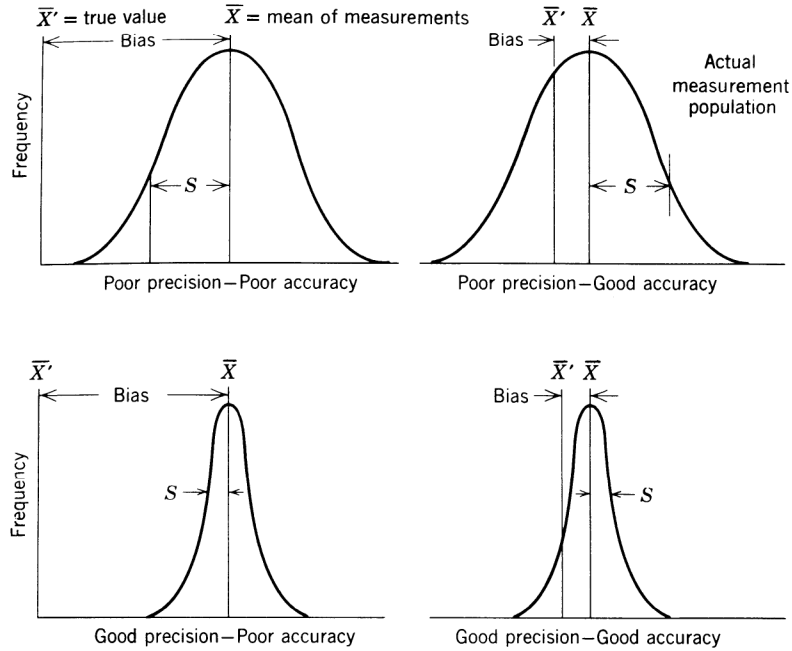


FIGURE 10.3 Distinction between precision and accuracy.

that every number has a fringe, that it is not to be regarded as exact but as so much plus or minus a bit, and that the size of this bit is one of its really important quantities.”

This plus or minus fringe that accompanies every measurement is called a confidence interval CI. Thus a confidence interval for the true value can be given by  $\bar{X} \pm \text{CI}(p)$ , where  $\bar{X}$  is the estimated best value of  $X$ , CI is the confidence interval, and  $(p)$  is the probability statement (and *not* a multiplier of CI).

To form these confidence intervals we need replicate data, and we note in this regard that these intervals will differ according to the size and number of sets available.

**Sets of Very Large  $N$ .** Many times in engineering, a tabulation of how often the various values of  $X$  occur in replication is well approximated by the Gauss–Laplace *normal distribution* relation [5]

$$f(X) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left[-\frac{1}{2} \left(\frac{X - \mu}{\sigma}\right)^2\right] \quad (10.4)$$

where the factor  $1/\sqrt{2\pi} \sigma$  has the normalizing effect of making the integral of  $f(X)$  over all values of  $X$  equal unity, and where  $\sigma$  represents the true standard deviation of  $X$ , which in turn is well approximated by

$$\sigma = \left[ \frac{1}{N} \sum_{k=1}^N (X_k - \mu)^2 \right]^{1/2} \quad (10.5)$$

The standard deviation of a normal distribution of  $X$  has the following characteristics:

1.  $\sigma$  measures the scatter of  $X$  at a given input, that is, it is a measure of the precision error  $\epsilon$ .
2.  $\sigma$  has the same units as  $X$ .
3.  $\sigma$  is the square root of the average of the sum of the squares of the deviations of all possible observations from the true arithmetic mean  $\mu$ .

For the normal distribution,  $\sigma$  has the direct interpretation that 68.26% of all possible values of  $X$  will fall within the  $\pm 1\sigma$  interval centered on  $\mu$ . For any engineering applications this is not good enough, and wider intervals must be employed to express greater confidence. For example, 95.46% of the data can be expected to fall within the  $\pm 2\sigma$  interval, and 99.73% within  $\pm 3\sigma$  (Figure 10.4).

We are assured that  $\bar{X}$  is a very good estimate of  $\mu$  by the large size of the sample. We may ask, however, how typical is a single observation of  $X$ . As we have just seen, one answer is

$$X \pm 3\sigma \text{ (at 99.73\%)} \quad (10.6)$$

Statement (10.6) indicates that the interval  $X \pm 3\sigma$  is expected to include  $\mu$  99.73% of the time. It further brings out the important point that, to be most meaningful, a measurement should be given in *three* parts [6], [7]. These are:

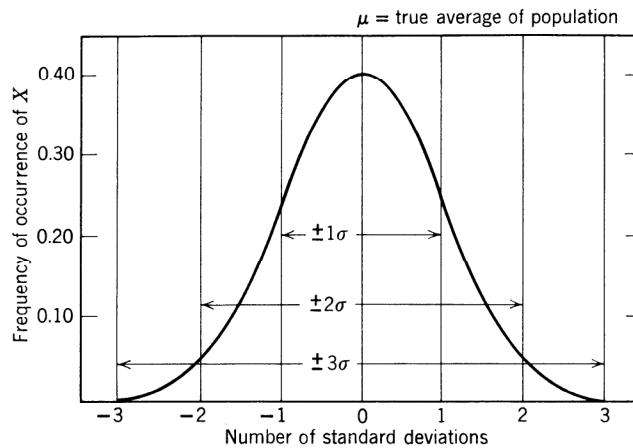


FIGURE 10.4 Normal distribution of  $X$ .

1. A magnitude (the indicated value of  $X$ ).
2. A confidence interval [which is your estimate of what the error might be;  $\pm 3\sigma$  in statement (10.6)].
3. A probability statement [an indication of your confidence that the true value will be within the confidence interval chosen; 99.73% in statement (10.6)].

**Single Set of Small  $N$ .** We now face the unpleasant fact that, in a practical experiment, we usually obtain only a relatively small sample from all possible values of  $X$ . This means that we cannot obtain the true arithmetic mean  $\mu$ , and hence we cannot form the true standard deviation  $\sigma$ . Instead of the inaccessible *deviations*  $(X_k - \mu)$  of equation (10.5), we can determine only the *residuals*  $(X_k - \bar{X})$ . We note in this regard that the sum of the squares of the residuals, being always a minimum according to the least-squares principle previously mentioned, is always less than the sum of the squares of the deviations. The precision index of the single sample is defined in terms of the residuals and is patterned after equation (10.5) as

$$S = \left[ \frac{1}{N-1} \sum_{k=1}^N (X_k - \bar{X})^2 \right]^{1/2} \quad (10.7)$$

where the factor  $(N-1)$  is used in place of the usual  $N$  in an attempt to compensate for the negative bias that results from using  $\bar{X}$  in place of  $\mu$  in forming the differences. However, a negative bias unfortunately still remains in the small sample estimate of the standard deviation, and  $S$ , the obtainable, does not equal  $\sigma$ , the desired.

*Student's  $t$  Distribution.* Recognizing this deficiency, a method was developed by the English chemist W. S. Gosset\* (writing in 1907 under the pseudonym "Student"), by which confidence intervals could be based on the precision index  $S$  of a single small sample. He introduced the "Student's  $t$  statistic whose values have been tabulated in terms of degrees of freedom  $\nu$  and the desired degree of confidence (quantified by the probability  $p$ ) (Table 10.1 and Figure 10.5).

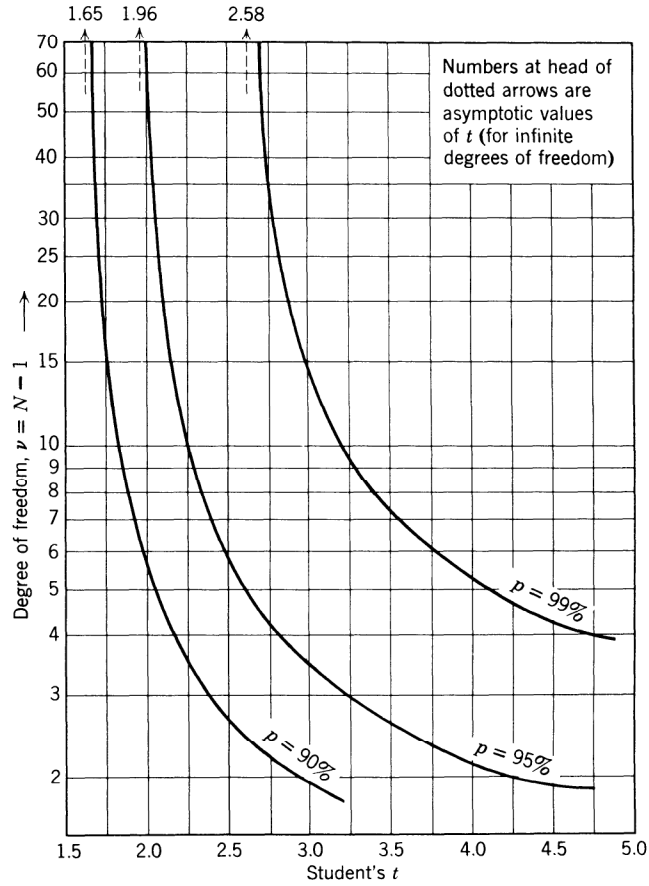
Careful perusal of these values will show that the  $t$  statistic inflates the confidence interval (i.e., the uncertainty margin) so as to reduce the effect of underestimating the standard deviation  $\sigma$  when a small sample is used to calculate  $S$  (Figure 10.6).

Degrees of freedom can be defined in general as the number of observations minus the number of constants calculated from the data. According to equation (10.3),  $\bar{X}$  has  $N$  degrees of freedom, whereas by equation (10.7),  $S$  has  $N-1$  degrees of freedom because one constant,  $\bar{X}$ , is used to calculate  $S$ .

\*William Sealy Gosset (1876–1937), a statistician for a famous Irish brewery, Guinness.

**TABLE 10.1 Student's  $t$  Statistic for a Normal Distribution**

| $\nu = N - 1$ | $t_{90\%}$ | $t_{95\%}$ | $t_{99\%}$ |
|---------------|------------|------------|------------|
| 1             | 6.314      | 12.706     | 63.657     |
| 2             | 2.920      | 4.303      | 9.925      |
| 3             | 2.353      | 3.182      | 5.841      |
| 4             | 2.132      | 2.776      | 4.604      |
| 5             | 2.015      | 2.571      | 4.032      |
| 6             | 1.943      | 2.447      | 3.707      |
| 7             | 1.895      | 2.365      | 3.499      |
| 8             | 1.860      | 2.306      | 3.355      |
| 9             | 1.833      | 2.262      | 3.250      |
| 10            | 1.812      | 2.228      | 3.169      |
| 11            | 1.796      | 2.201      | 3.106      |
| 12            | 1.782      | 2.179      | 3.055      |
| 13            | 1.771      | 2.160      | 3.012      |
| 14            | 1.761      | 2.145      | 2.977      |
| 15            | 1.753      | 2.131      | 2.947      |
| 16            | 1.746      | 2.120      | 2.921      |
| 17            | 1.740      | 2.110      | 2.898      |
| 18            | 1.734      | 2.101      | 2.878      |
| 19            | 1.729      | 2.093      | 2.861      |
| 20            | 1.725      | 2.086      | 2.845      |
| 21            | 1.721      | 2.080      | 2.831      |
| 22            | 1.717      | 2.074      | 2.819      |
| 23            | 1.714      | 2.069      | 2.807      |
| 24            | 1.711      | 2.064      | 2.797      |
| 25            | 1.708      | 2.060      | 2.787      |
| 26            | 1.706      | 2.056      | 2.779      |
| 27            | 1.703      | 2.052      | 2.771      |
| 28            | 1.701      | 2.048      | 2.763      |
| 29            | 1.699      | 2.045      | 2.756      |
| 30            | 1.697      | 2.042      | 2.750      |
| 40            | 1.684      | 2.021      | 2.704      |
| 60            | 1.671      | 2.000      | 2.660      |
| 120           | 1.658      | 1.980      | 2.617      |
| $\infty$      | 1.645      | 1.960      | 2.576      |

FIGURE 10.5 Student's  $t$  statistic.

The answer to the question, how typical is a single observation of  $X$ , is, in terms of  $S$  and  $t$ ,

$$X \pm t_{\nu,p} S \quad (\text{to a given probability } p) \quad (10.8)$$

where  $\nu$  signifies the degrees of freedom in  $S$ . Statement (10.8) indicates that  $\mu$  will be included in the interval,  $X \pm tS$  to the probability  $p$ .

The counterpart of statement (10.8) was given in statement (10.6) in terms of  $\sigma$ . The interval of statement (10.6) can be generalized in terms of the normal deviate  $z$  as

$$X \pm z_p \sigma \quad (\text{to a given probability } p) \quad (10.9)$$

The plus and minus quantities in statements (10.8) and (10.9), that is,  $\pm tS$  and  $\pm z\sigma$ , should be recognized as the confidence intervals on the individual

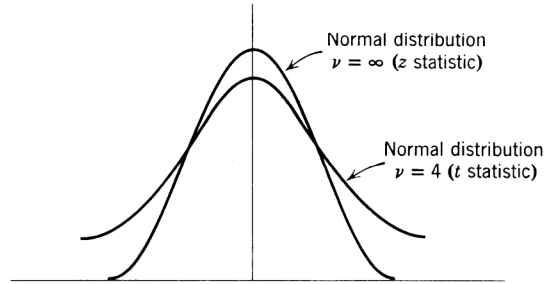


**FIGURE 10.6** Comparison of  $z$  and  $t$  statistics. There is a normal distribution for each degree of freedom  $\nu$ .

**TABLE 10.2** Values of the Normal Deviate

| Probability $p$ (%) | 80    | 90    | 95    | 98    | 99    | 99.8  |
|---------------------|-------|-------|-------|-------|-------|-------|
| Normal deviate $z$  | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 3.090 |

measurements of  $X$ . Appropriate values of  $z$  are given in Table 10.2. The  $z$  and  $t$  statistics are compared in Figure 10.6.

**Several Sets of Small  $N$ .** A useful measure of scatter in multiple-set experiments is called the *precision index of the mean*. This is patterned after equations (10.5) and (10.7) and is defined in general as

$$S_{\bar{X}} = \left[ \frac{\sum_{i=1}^M N_i (\bar{X}_i - \bar{\bar{X}})^2}{\sum_{i=1}^M N_i} \right]^{1/2} \quad (10.10)$$

where  $M$  is the number of sets involved, and of the  $i$ th set,  $\bar{X}_i$  is the mean and  $N_i$  is the number of observations. The factor  $\bar{\bar{X}}$  is the grand average of the  $M$  sets, defined in general as

$$\bar{\bar{X}} = \frac{\sum_{i=1}^M N_i \bar{X}_i}{\sum_{i=1}^M N_i} \quad (10.11)$$

$\bar{\bar{X}}$  is naturally to be used in all multiple-set experiments as the best estimate of the true value of the population mean at a given input.

When a common number of observations  $N$  applies to all  $M$  sets, equations (10.10) and (10.11) reduce to

$$S_{\bar{X},N} = \left[ \frac{1}{M} \sum_{i=1}^M (\bar{X}_i - \bar{\bar{X}})^2 \right]^{1/2} \quad (10.12)$$

and

$$\bar{\bar{X}}_N = \frac{1}{M} \sum_{i=1}^M \bar{X}_i = \frac{1}{MN} \sum_{k=1}^{MN} X_k \quad (10.13)$$

where the subscript  $N$  signifies a common set size.

It is an observable fact that the *means* of different sets of measurements from the same population are always much closer to each other than are the individual values of a single set. Equation (10.12) defines the precision index of a set of  $M$  values of  $\bar{X}$ , whereas equation (10.7) defines the precision index of a set of  $N$  values of  $X$ . Statistical theory gives us an important relation between these two statistics, namely,

$$S_{\bar{X}} = \frac{S_X}{\sqrt{N}} \quad (10.14)$$

where  $N$  is consistently the number of observations common to all  $M$  sets.

Equation (10.14) says in effect: The average value  $\bar{X}$  of a set has more precision than any of its parts  $X_k$  by the factor  $1/\sqrt{N}$ .

Patterned directly after equation (10.14) is the precision index of the grand average  $\bar{\bar{X}}$  of  $M$  sets of observations, which can be given as

$$S_{\bar{\bar{X}}} = \frac{S_{\bar{X}}}{\sqrt{M}} = \frac{S_X}{\sqrt{MN}} \quad (10.15)$$

Equation (10.15) presents one apparent problem: which  $S_X$  should be used in a multiple-set experiment? One answer is to choose any one  $S_{X,i}$  at random, but this leads to wide variation in  $S_{\bar{\bar{X}}}$ . A more satisfying answer, and the one recommended here, is to define a weighted average of the  $S_i$ , the weights being the appropriate degrees of freedom, that is, replace the  $S_X$  of equation (10.15) with  $\bar{\bar{S}}$  defined as

$$\bar{\bar{S}} = \left[ \frac{\sum_{i=1}^M \nu_i S_i^2}{\sum_{i=1}^M \nu_i} \right]^{1/2} = \left[ \frac{\nu_1 S_1^2 + \nu_2 S_2^2 + \cdots + \nu_M S_M^2}{\nu_1 + \nu_2 + \cdots + \nu_M} \right]^{1/2} \quad (10.16)$$

Since it is often common practice to keep all sets of the same size, that is, to keep  $N$  the same for all sets, and since this practice assures us that  $\bar{\bar{S}}$  is a minimum, equation (10.16) can be rewritten for the case of an  $N$  common to all  $M$  sets as

$$\bar{\bar{S}}_N = \left[ \frac{1}{M} \sum_{i=1}^M S_i^2 \right]^{1/2} \quad (10.17)$$

On the basis of the  $t$  statistic, best estimates of the interval which contains the true average  $\mu$  can be given in terms of the *mean* of a single set of measurements as

$$\bar{X} \pm \frac{t_{N-1,p} S_X}{\sqrt{N}} = \bar{X} \pm t_{N-1,p} S_{\bar{X}} \quad (10.18)$$

and in terms of the mean of  $M$  sets of measurements (each set containing the common number of measurements  $N$ ) as

$$\bar{\bar{X}} \pm \frac{t_{MN-M,p} \bar{S}}{\sqrt{MN}} \quad (10.19)$$

The subscripts of  $t$  indicate degrees of freedom  $\nu$  and probability  $p$ , and the  $\pm$  quantity indicates the confidence interval CI of the best value estimate. Note that degrees of freedom is now given by  $\nu = MN - M$  since not one  $\bar{X}$  but  $M$   $\bar{X}$  were obtained from the data.

Several examples are given here to illustrate the ideas embodied thus far in these statistical relations.

**Example 1.** For the observations  $X_i = 7, 8, 7, 6, 5, 6, 7, 8, 6, 9, 8$ , find the best estimate of the true mean  $\mu$ , the precision index of the observations, the precision index of the mean, and the 95% confidence interval statement for  $\bar{X}$  [8].

*Solution.* By equation (10.3),

$$\bar{X} = \frac{1}{11} \sum_{i=1}^{11} X_i = \frac{77}{11} = 7$$

which is the best estimate of  $\mu$ . By equation (10.7)

$$S_X = \left[ \frac{1}{N-1} \sum_{i=1}^{11} (X_i - 7)^2 \right]^{1/2} = \sqrt{\frac{14}{10}} = 1.18$$

which is the precision index of the observations. By equation (10.14),

$$S_{\bar{X}} = \frac{S_X}{\sqrt{N}} = \frac{1.18}{\sqrt{11}} = 0.357$$

which is the precision index of the mean.

By Table 10.1,

$$t_{\nu,p} = t_{10,0.95} = 2.228$$

which is the  $t$  statistic for the 95% confidence interval statement. Hence according to equation (10.18),  $\mu$  should be in the interval

$$\begin{aligned}\bar{X} \pm t_{n-1,p} S_{\bar{X}} &= 7 \pm 2.228(0.357) \\ &= 7 \pm 0.795, \quad 95\% \text{ of the time}\end{aligned}$$

In terms of  $\mu$ ,

$$6.2 \leq \mu \leq 7.8, \quad \text{with 95\% confidence}$$

**Example 2.** Express the best value and its 95% confidence interval for a single sample experiment of eight observations.

*Solution.* According to equation (10.18) and Table 10.1, at  $N = 8$ ,

$$\nu = 7 \text{ and } t_{7,0.95} = 2.365$$

Hence,

$$\bar{X} \pm \frac{2.365 S_X}{\sqrt{8}} \quad \text{at 95\%}$$

**Example 3.** If five sets of data of the type of Example 2 were taken, how much more confidence could be placed in the best value?

*Solution.* According to equation (10.19) and Table 10.1, at  $MN = 40$ ,

$$\nu = MN - M = 35 \text{ and } t_{\nu,p} = 2.031$$

Hence,

$$\bar{\bar{X}} \pm \frac{2.031 S_X}{\sqrt{40}} \quad \text{at 95\%}$$

[This formulation presumes that  $S_X$  remains constant. In reality it would vary as previously discussed, and one would use the weighted standard deviation  $\bar{\bar{S}}$  of equation (10.17).]

When the results of Examples 2 and 3 are ratioed, there results

$$\frac{2.365/\sqrt{8}}{2.031/\sqrt{40}} = \frac{0.836}{0.321} = 2.6$$

which indicates that the confidence interval could be tightened by a factor of about 3 for the multiple-set case.

**Example 4.** From three sets of five measurements each, the following table is derived:

| $M$ | $\bar{X}$ | $S$    |
|-----|-----------|--------|
| 1   | -0.2052   | 0.6719 |
| 2   | -0.2232   | 0.6395 |
| 3   | 0.1030    | 1.2168 |

Find the best estimate of  $\mu$  and its 95% confidence interval.

*Solution.* By equation (10.13),

$$\bar{\bar{X}} = \frac{1}{3}(-0.2052 - 0.2232 + 0.1030) = -0.1085$$

By equation (10.17),

$$\bar{\bar{S}} = [\frac{1}{3}(0.6719^2 + 0.6395^2 + 1.2168^2)]^{1/2} = 0.8834$$

By equation (10.19), for  $\nu = MN - M = 12$ ,

$$CI_{\bar{\bar{X}}} = \frac{2.179(0.8834)}{\sqrt{15}} = 0.4970$$

Hence  $\mu$  is within the interval  $-0.1085 \pm 0.4970$ , 95% of the time.

*Range.* In addition to the CI statements based on  $z\sigma$  [of equation (10.9)] and  $tS$  [of equation (10.8)], a third type of CI statement can be based on the range estimate of  $S$ . The range  $R$  is defined as the difference between the largest and smallest measurements in a set, and can be used to estimate the precision index  $S$ , that is,

$$S \approx \frac{\bar{R}}{(d_2^*)_N} \quad (10.20)$$

where  $\bar{R}$  is the average range of  $M$  sets, defined in turn by

$$\bar{R} = \frac{1}{M} \sum_{i=1}^M R_i \quad (10.21)$$

for  $N$  common to all  $M$  sets, and where  $d_2^*$  is tabulated as a function of  $N$  in Table 10.3.

There is a loss of degrees of freedom with this technique, and the estimate of  $S$  is less precise than those given above, but the range estimate of  $S$  as given by equation (10.20) is often convenient.

**TABLE 10.3** Values Associated with the Distribution of the Average Range

| Number<br>of<br>Sets $M$ | Number of Observations per Set $N$ |         |       |         |       |         |       |         |       |         |       |         |
|--------------------------|------------------------------------|---------|-------|---------|-------|---------|-------|---------|-------|---------|-------|---------|
|                          | 5                                  |         | 6     |         | 7     |         | 8     |         | 9     |         | 10    |         |
|                          | $\nu$                              | $d_2^*$ | $\nu$ | $d_2^*$ | $\nu$ | $d_2^*$ | $\nu$ | $d_2^*$ | $\nu$ | $d_2^*$ | $\nu$ | $d_2^*$ |
| 1                        | 3.8                                | 2.48    | 4.7   | 2.67    | 5.5   | 2.83    | 6.3   | 2.96    | 7.0   | 3.08    | 7.7   | 3.18    |
| 2                        | 7.5                                | 2.40    | 9.2   | 2.60    | 10.8  | 2.77    | 12.3  | 2.91    | 13.8  | 3.02    | 15.1  | 3.13    |
| 3                        | 11.1                               | 2.38    | 13.6  | 2.58    | 16.0  | 2.75    | 18.3  | 2.89    | 20.5  | 3.01    | 22.6  | 3.11    |
| 4                        | 14.7                               | 2.37    | 18.1  | 2.57    | 21.3  | 2.74    | 24.4  | 2.88    | 27.3  | 3.00    | 30.1  | 3.10    |
| 5                        | 18.4                               | 2.36    | 22.6  | 2.56    | 26.6  | 2.73    | 30.4  | 2.87    | 34.0  | 2.99    | 37.5  | 3.10    |
| 6                        | 22.0                               | 2.35    | 27.1  | 2.56    | 31.8  | 2.73    | 36.4  | 2.87    | 40.8  | 2.99    | 45.0  | 3.10    |
| 7                        | 25.6                               | 2.35    | 31.5  | 2.55    | 37.1  | 2.72    | 42.5  | 2.87    | 47.5  | 2.99    | 52.4  | 3.10    |
| 8                        | 29.3                               | 2.35    | 36.0  | 2.55    | 42.4  | 2.72    | 48.5  | 2.87    | 54.3  | 2.98    | 59.9  | 3.09    |
| 9                        | 32.9                               | 2.34    | 40.5  | 2.55    | 47.7  | 2.72    | 54.5  | 2.86    | 61.0  | 2.98    | 67.3  | 3.09    |
| 10                       | 36.5                               | 2.34    | 44.9  | 2.55    | 52.9  | 2.72    | 60.6  | 2.86    | 67.8  | 2.98    | 74.8  | 3.09    |
| $d_2$                    | 2.33                               |         | 2.53  |         | 2.70  |         | 2.85  |         | 2.97  |         | 3.08  |         |

Source: Adapted from Duncan [9].

Sometimes the range of a *single* set is used, via equation (10.20), to estimate  $S$  [10]–[12], and occasionally  $d_2$  (the infinity value of  $d_2^*$ ) is used in place of  $d_2^*$  in equation (10.20), where the degree of approximation can be determined via Table 10.3.

Avoiding the determination of  $S$  entirely, confidence interval statements for the case when the range is being used can be given in terms of a substitute  $t$  statistic (given in Table 10.4 and called the  $\tau$  statistic) as the counterparts of equations (10.18) and (10.19) as

$$\bar{X} \pm \tau_{N,p} R \quad (10.22)$$

for the *single* set, and

$$\bar{\bar{X}} \pm \frac{\tau_{N,p} \bar{R}}{\sqrt{M}} \quad (10.23)$$

in terms of the *mean* of  $M$  sets of measurements.

**Example 5.** Estimate the number of ranges by which  $\bar{X}$  can depart from  $\mu$  by reference to Table 10.4 and equation (10.22) for  $N = 2, 3$ , and 4.

**Solution.** For two observations in the sample,

$$\begin{aligned} \text{CI}_{\bar{X}} &= \tau_2 R = 6.353 \text{ ranges at } 95\% \\ &= 31.828 \text{ ranges at } 99\% \end{aligned}$$

**TABLE 10.4** Substitute  $t$  Statistic  
(the  $\tau$  Statistic)

| $N$ | $\tau_{90\%}$ | $\tau_{95\%}$ | $\tau_{99\%}$ |
|-----|---------------|---------------|---------------|
| 2   | 3.175         | 6.353         | 31.828        |
| 3   | 0.885         | 1.304         | 3.008         |
| 4   | 0.529         | 0.717         | 1.316         |
| 5   | 0.388         | 0.507         | 0.843         |
| 6   | 0.312         | 0.399         | 0.628         |
| 7   | 0.263         | 0.333         | 0.507         |
| 8   | 0.230         | 0.288         | 0.429         |
| 9   | 0.205         | 0.255         | 0.374         |
| 10  | 0.186         | 0.230         | 0.333         |
| 11  | 0.170         | 0.210         | 0.302         |
| 12  | 0.158         | 0.194         | 0.277         |
| 13  | 0.147         | 0.181         | 0.256         |
| 14  | 0.138         | 0.170         | 0.239         |
| 15  | 0.131         | 0.160         | 0.224         |
| 16  | 0.124         | 0.151         | 0.212         |
| 17  | 0.118         | 0.144         | 0.201         |
| 18  | 0.113         | 0.137         | 0.191         |
| 19  | 0.108         | 0.131         | 0.182         |
| 20  | 0.104         | 0.126         | 0.175         |

For three observations in the sample,

$$\begin{aligned} CI_{\bar{X}} &= 1.304 \text{ ranges at } 95\% \\ &= 3.008 \text{ ranges at } 99\% \end{aligned}$$

Here we note how dramatically the confidence interval tightens with one additional measurement. For four observations in the sample,

$$\begin{aligned} CI_{\bar{X}} &= \tau_4 R = 0.717 \text{ ranges at } 95\% \\ &= 1.316 \text{ ranges at } 99\% \end{aligned}$$

Thus we conclude that the use of three readings over two greatly improves our understanding of the dispersion of  $X$ , and the worth of additional measurements becomes primarily an economic question.

**Example 6.** For two sets of five measurements each, the following table is derived.

| $M$ | $R$   | $\bar{X}$ |
|-----|-------|-----------|
| 1   | 1.516 | 0.4506    |
| 2   | 0.815 | 0.0478    |

Find the best estimate of  $\mu$  and its confidence interval.

*Solution.* By equation (10.13),

$$\bar{\bar{X}} = \frac{1}{2}(0.4506 + 0.0478) = 0.2492$$

By equation (10.21),

$$\bar{R} = \frac{1}{2}(1.516 + 0.815) = 1.1655$$

By equation (10.23),

$$CI_{\bar{X}} = \frac{\tau_N \bar{R}}{\sqrt{M}} = \frac{0.507(1.1655)}{\sqrt{2}} = 0.4178$$

Hence  $\mu$  is within the interval  $0.2492 \pm 0.4178$  95% of the time.

**Samples of Small  $N$  with  $\sigma$  Known.** Often  $\sigma$  can be considered known in the sense of being established by experience. In such favorable situations, even a single small sample can yield reliable estimates of  $\mu$ . For example, in terms of  $\sigma$ , the counterparts of equations (10.18), (10.19) and of equations (10.22), (10.23) are

$$\bar{X} \pm \frac{z\sigma}{\sqrt{N}} \quad (10.24)$$

for the *single* set and

$$\bar{\bar{X}} \pm \frac{z\sigma}{\sqrt{MN}} \quad (10.25)$$

for *multiple* sets.

All of the confidence intervals developed thus far are summarized in Table 10.5.

**Example 7.** A certain temperature measurement yields an average value of  $150.75^\circ\text{F}$  based on four readings. By experience, it is felt that no temperature measurement is off more than  $\pm 0.5^\circ\text{F}$ , with 95% assurance. What confidence interval statement can be made concerning the true temperature?

**TABLE 10.5 Summary of Various Confidence Interval Statements**

| Available<br>Statistic | Multiplying<br>Statistic | Confidence Interval             |                                         |
|------------------------|--------------------------|---------------------------------|-----------------------------------------|
|                        |                          | On Single-Set<br>Mean $\bar{X}$ | On Multiple-Set<br>Mean $\bar{\bar{X}}$ |
| $\sigma$               | $z$                      | $z_p \frac{\sigma}{\sqrt{N}}$   | $z_p \frac{\sigma}{\sqrt{MN}}$          |
| $S$                    | $t$                      | $t_{N-1,p} \frac{S}{\sqrt{N}}$  | $t_{MN-M,p} \frac{S}{\sqrt{MN}}$        |
| $R$                    | $\tau$                   | $\tau_{N,p} R$                  | $\tau_{N,p} \frac{\bar{R}}{\sqrt{M}}$   |

*Solution.* Based on a *single* measurement, according to equation (10.9) and Table 10.2,  $\bar{T}'$  should be in the interval

$$T \pm 1.96\sigma \quad 95\% \text{ of the time}$$

Based on the *mean* of four measurements, according to equation (10.24),  $\bar{\bar{T}}'$  should be in the interval

$$150.75 \pm \frac{1.96\sigma}{\sqrt{4}} \quad 95\% \text{ of the time}$$

or, with  $1.96\sigma = 0.5^\circ\text{F}$ , we can write the interval as

$$150.75 \pm \frac{0.5}{2}$$

Thus  $\bar{\bar{T}}'$  can be given as

$$150.5^\circ\text{F} \leq \bar{\bar{T}}' \leq 151.0^\circ\text{F} \quad (95\%)$$

In words, the most believable value of  $T$  is  $150.75^\circ\text{F}$ . Furthermore, 95% of the time the true temperature is believed to lie between  $150.5$  and  $151.0^\circ\text{F}$ .

### 10.3 UNCERTAINTY OF A SINGLE PARAMETER

In Section 10.1 both systematic and random errors are considered. In Section 10.2 we dwell at length on the confidence interval method for expressing random errors statistically. However, since random errors are derived by statistical methods, while systematic errors must be estimated by nonstatistical methods,

it follows that these two types of errors should not be joined together lightly. Indeed, the best procedure to follow in describing the uncertainty in a measured parameter is to quote the random and systematic errors separately, and let it go at that.

However, there are often times in engineering when a *single number* is required to describe the uncertainty of a set of measurements. That is, the random and systematic errors accompanying repeated measurements of a given parameter (like temperature or pressure) are often combined to yield a single number for the uncertainty. The number formed by combining the bias error  $B$  and the precision error  $t_{95}S_{\bar{X}}$  is called the uncertainty  $U$ .

Since there can be no rigorous basis in statistics for the required relation defining uncertainty, its formulation must remain arbitrary. So it is not surprising that two definitions for uncertainty are in common usage [13].

The most conservative model for uncertainty is

$$U_{\text{ADD}} = B + t_{95}S_{\bar{X}} \quad (\text{at } 99\%) \quad (10.26)$$

where the subscript ADD signifies that the bias and precision errors are simply added, and the 99% figure in parentheses indicates the percent coverage of the true value expected of this model.

A more realistic estimate for uncertainty, presuming that there will be some beneficial canceling of the errors or, to say it another way, assuming that all errors will not be in the same direction, is based on the familiar root-sum-square model\* as

$$U_{\text{RSS}} = [B^2 + (t_{95}S_{\bar{X}})^2]^{1/2} \quad (\text{at } 95\%) \quad (10.27)$$

where the subscript RSS signifies that the bias and precision errors are combined by taking the square root of the sum of the squares of these errors, and the 95% figure in parentheses indicates the percent coverage expected by this model.

Either model indicates the expected error limit of a measured parameter for a given coverage, that is, the uncertainty  $U$  is our best estimate of the total error  $\delta$  of equation (10.2). The coverage indicates the expected probability that the interval defined by the best value  $\bar{X}$  of the parameter plus or minus the uncertainty,  $\pm U$ , will include the true value  $\bar{X}'$ . The uncertainty interval is shown in Figure 10.7.

In engineering work, the RSS model of equation (10.27) is used most often, and is the approach recommended here for combining systematic and random errors. Sometimes the  $U_{\text{ADD}}$  model is called the *maximum* uncertainty, and sometimes the  $U_{\text{RSS}}$  model is called the *probable* uncertainty.

\*Hayward [14, p. 19] has indicated: "The real justification for adding uncertainty components in quadrature (i.e., by RSS) is that it seems to work. Experience has shown that the arithmetic addition of components often leads to a large overestimate of total uncertainty."

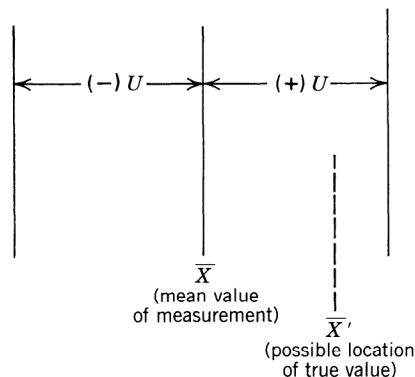


FIGURE 10.7 Uncertainty interval coverage.

**Example 8.** A given parameter  $P$  is measured with an estimated bias  $B$  of 2 units and a calculated precision index  $S$  of 1 unit when the number of observations  $N$  is 10. Give the uncertainty interval statements based on 99% and 95% coverage.

**Solution.** By equation (10.26), at  $t_{95\%,9} = 2.262$ , we have  $U_{\text{ADD}} = 2 + 2.262 \times 1 = 4.262$  at 99%, and  $\bar{P} \pm 4.262$  represents the interval believed to contain the true value of the parameter with 99% coverage.

By equation (10.27), again at  $t_{95\%,9} = 2.262$ , we have

$$U_{\text{RSS}} = [2^2 + (2.262 \times 1)^2]^{1/2} = 3.019 \quad (\text{at } 95\%)$$

and  $\bar{P} \pm 3.019$  represents the uncertainty interval believed to contain the true value of the parameter with 95% coverage.

When the number of measurements of a given parameter is extremely small, or when no statistical information is available (as when an instrument is used in an uncalibrated state), one must estimate the uncertainty  $U$  in place of the calculated uncertainty of equations (10.26) and (10.27). In such cases the uncertainty represents the experimenter's best estimate of the maximum error to be reasonably associated with the parameter. Thus one could say, for example, that based on experience, the uncertainty of a temperature measurement is  $\pm 1^\circ\text{F}$  (95% of the time) without reference to any particular set of measurements, without computing  $S$ , and without application of the  $t$  statistic. Or one could say that all flow measurements made with uncalibrated nozzles can be counted on to  $\pm 1.5\%$  (95% of the time), and use this as the uncertainty.

Although it is true that the uncertainty thus conceived includes both systematic and random errors, the idea of separating these errors and dealing with them separately is too arbitrary to be practical, and we recommend considering such uncertainties as describing systematic errors (i.e., biases) alone.

Uncertainties can be given on an absolute or on a relative basis. An *absolute* uncertainty is one that is expressed in the same units as the mean value of the

parameter (such as  $150 \pm 0.5^\circ\text{F}$ ). A *relative* uncertainty is one that is formed by dividing the absolute uncertainty by the mean value of the parameter (such as  $150^\circ\text{F} \pm 0.5/150 = 150^\circ\text{F} \pm 1/3\%$ ).

#### 10.4 PROPAGATION OF MEASUREMENT ERRORS INTO A RESULT

Often a result  $r$  is derived by combining a number of independent parameters  $P_i$  according to some functional relationship:

$$r = f(\bar{P}_1, \bar{P}_2, \dots, \bar{P}_J) \quad (10.28)$$

where

$$\bar{P}_i = \frac{1}{N} \sum_{k=1}^N P_{ik} \quad (10.29)$$

and the subscript  $J$  indicates the number of parameters involved. In such cases an important question must be considered, namely, how are the measurement errors of independent parameters propagated into the result? To ask this in another way, what are the precision and bias errors of a derived result?

The uncertainty interval of interest is now

$$r \pm U_r \quad (10.30)$$

where, patterned after equations (10.26) and (10.27), the uncertainty of the result is defined as either

$$U_{r,\text{ADD}} = B_r + t_{95}S_r \quad (\text{at } 99\%) \quad (10.31)$$

or

$$U_{r,\text{RSS}} = [B_r^2 + (t_{95}S_r)^2]^{1/2} \quad (\text{at } 95\%) \quad (10.32)$$

A recognized concept of statistics provides the answer to one of our questions. The precision index of the result is given by

$$S_r = \left[ \left( \frac{\partial r}{\partial P_1} S_{\bar{P}_1} \right)^2 + \left( \frac{\partial r}{\partial P_2} S_{\bar{P}_2} \right)^2 + \dots + \left( \frac{\partial r}{\partial P_J} S_{\bar{P}_J} \right)^2 \right]^{1/2} \quad (10.33)$$

on an absolute basis, and by

$$\frac{S_r}{r} = \left[ \sum_{i=1}^J \left( \frac{\partial r/r}{\partial P_i/\bar{P}_i} \frac{S_{\bar{P}_i}}{\bar{P}_i} \right)^2 \right]^{1/2} \quad (10.34)$$

on a relative basis. Note once again the use of the root-sum-square principle.

The so-called sensitivity factors of equations (10.33) and (10.34), namely, the absolute

$$\theta_i = \frac{\partial r}{\partial P_i} \quad (10.35)$$

and the relative

$$\theta'_i = \frac{\partial r/r}{\partial P_i/\bar{P}_i} \quad (10.36)$$

must be evaluated (analytically or numerically) and used as multipliers of the precision indices  $S_i$  of each of the parameters.

When all parameters have sample sizes greater than 30, the number of degrees of freedom of  $S_r$  is  $\approx 2$ , as seen by reference to Table 10.1. For all other cases the number of degrees of freedom of  $S_r$  is determined by the Welch–Satterthwaite equation [15], [16]:

$$\nu_r = \frac{S_r^4}{\sum_{i=1}^J \frac{(\theta_i S_{\bar{P}_i})^4}{\nu_{\bar{P}_i}}} = \frac{(S_r/r)^4}{\sum_{i=1}^J \frac{(\theta'_i S_{\bar{P}_i}/\bar{P}_i)^4}{\nu_{\bar{P}_i}}} \quad (10.37)$$

where  $\nu_{\bar{P}} = N - 1$ .

If equation (10.37) results in a noninteger number, it should be rounded downward to the next integer.

By analogy with equations (10.33) and (10.34), but with little basis in statistics, the bias of a result [17] is given by

$$B_r = \left[ \sum_{i=1}^J (\theta_i B_{\bar{P}_i})^2 \right]^{1/2} \quad (10.38)$$

on an absolute basis, and by

$$\frac{B_r}{r} = \left[ \sum_{i=1}^J \left( \theta'_i \frac{B_{\bar{P}_i}}{\bar{P}_i} \right)^2 \right]^{1/2} \quad (10.39)$$

on a relative basis, where  $B_{\bar{P}_i}$  are the biases of each parameter.

It is important to note that the bias errors and the precision errors of each of the independent parameters are kept *separate* until the final determination of the uncertainty of the result.

All of the concepts of this section are illustrated by Examples 9–11. These same concepts are succinctly summarized in Reference [18].

**Example 9.** A certain result  $r$  is related to its variables  $P_i$  by the equation

$$r = \frac{P_1 P_2^2}{P_3^{1/2}} \quad (10.40)$$

It follows from equations (10.35) and (10.36) that

$$\theta_1 = \frac{\partial r}{\partial P_1} = \frac{P_2^2}{P_3^{1/2}}, \quad \theta_2 = \frac{\partial r}{\partial P_2} = \frac{2P_1 P_2}{P_3^{1/2}}, \quad \theta_3 = \frac{\partial r}{\partial P_3} = \frac{P_1 P_2^2}{2P_3^{3/2}}$$

and

$$\theta'_1 = \frac{\partial r/r}{\partial P_1/P_1} = 1, \quad \theta'_2 = 2 \quad \theta'_3 = \frac{1}{2}$$

(Note, on a relative basis, that the sensitivity factors are simply the exponents of the parameters.)

The mean values, the sensitivity factors, and the biases (based on the experimenter's judgment) for each of the parameters are tabulated below.

| Mean Value<br>of Parameter $\bar{P}$ ,<br>(with units) | Relative<br>Sensitivity $\theta'$ ,<br>(pure number) | Absolute<br>Sensitivity $\theta$ ,<br>(with units) | Relative<br>Bias $B/P$ ,<br>(percent) | Absolute<br>Bias $B$ ,<br>(with units) |
|--------------------------------------------------------|------------------------------------------------------|----------------------------------------------------|---------------------------------------|----------------------------------------|
| $\bar{P}_1 = 1$                                        | 1                                                    | 5                                                  | 0.5                                   | 0.005                                  |
| $\bar{P}_2 = 5$                                        | 2                                                    | 2                                                  | 1                                     | 0.05                                   |
| $\bar{P}_3 = 25$                                       | 0.5                                                  | 0.1                                                | 2                                     | 0.5                                    |
| Given                                                  | Derived                                              | Derived                                            | Given                                 | Derived                                |

It follows from equation (10.40) that  $r = 1 \times 5^2/25^{1/2} = 5$ . Determine the bias of the result in absolute and relative terms.

**Solution.** By equation (10.38),

$$\begin{aligned} B_r &= [(\theta_1 \bar{B}_1)^2 + (\theta_2 \bar{B}_2)^2 + (\theta_3 \bar{B}_3)^2]^{1/2} \\ &= [(5 \times 0.005)^2 + (2 \times 0.05)^2 + (0.1 \times 0.5)^2]^{1/2} \\ &= 0.11456 \quad (\text{with units}) \end{aligned}$$

By equation (10.39),

$$\frac{B_r}{r} = \left[ \left( \theta'_1 \frac{\bar{B}_1}{P_1} \right)^2 + \left( \theta'_2 \frac{\bar{B}_2}{P_2} \right)^2 + \left( \theta'_3 \frac{\bar{B}_3}{P_3} \right)^2 \right]^{1/2}$$

$$\begin{aligned}
&= [(1 \times 0.5)^2 + (2 \times 1)^2 + (0.5 \times 2)^2]^{1/2} \\
&= 2.2913\%
\end{aligned}$$

As a check on consistency,  $B_r$  can be determined from  $B_r/r$  as

$$B_r = \frac{B_r}{r} \times r = 0.022913 \times 5 = 0.11456$$

Note that, in general, it is easier to work on a relative basis, since the sensitivities can often be determined by inspection.

**Example 10.** Using the relationship (10.40) between result and parameters, and the values tabulated in Example 9, estimate the precision error of the result in absolute and relative terms if the following information applies:

| Parameter | Number of<br>Measurements Taken $N$ ,<br>(pure number) | Relative<br>Precision Index $\overline{S}/P$ ,<br>(percent) | Absolute<br>Precision Index $\overline{S}$ ,<br>(with units) |
|-----------|--------------------------------------------------------|-------------------------------------------------------------|--------------------------------------------------------------|
| 1         | 20                                                     | 0.5                                                         | 0.005                                                        |
| 2         | 10                                                     | 0.2                                                         | 0.010                                                        |
| 3         | 5                                                      | 0.1                                                         | 0.025                                                        |
|           | Given                                                  | Given                                                       | Derived                                                      |

*Solution.* By equation (10.33),

$$\begin{aligned}
S_r &= [(\theta_1 \overline{S}_1)^2 + (\theta_2 \overline{S}_2)^2 + (\theta_3 \overline{S}_3)^2]^{1/2} \\
&= [(5 \times 0.005)^2 + (2 \times 0.01)^2 + (0.1 \times 0.025)^2]^{1/2} \\
&= 0.03211 \quad (\text{with units})
\end{aligned}$$

By equation (10.34),

$$\begin{aligned}
\frac{S_r}{r} &= \left[ \left( \theta'_1 \frac{\overline{S}_1}{P_1} \right)^2 + \left( \theta'_2 \frac{\overline{S}_2}{P_2} \right)^2 + \left( \theta'_3 \frac{\overline{S}_3}{P_3} \right)^2 \right]^{1/2} \\
&= [(1 \times 0.5)^2 + (2 \times 0.2)^2 + (0.5 \times 0.1)^2]^{1/2} \\
&= 0.64226\%
\end{aligned}$$

As a check on consistency,  $S_r$  can be determined from  $S_r/r$  as

$$S_r = \frac{S_r}{r} \times r = 0.0064226 \times 5 = 0.03211$$

To get the precision error of the result (i.e.,  $t_{95}S_r$ ), it is necessary to determine first the number of degrees of freedom of  $S_r$  via equation (10.37):

$$\begin{aligned}
 \nu_r &= \frac{S_r^4}{\frac{(\theta_1 \bar{S}_1)^4}{N_1 - 1} + \frac{(\theta_2 \bar{S}_2)^4}{N_2 - 1} + \frac{(\theta_3 \bar{S}_3)^4}{N_3 - 1}} = \frac{(S_r/r)^4}{\frac{(\theta'_1 \bar{S}_1/P_1)^4}{N_1 - 1} + \frac{(\theta'_2 \bar{S}_2/P_2)^4}{N_2 - 1} + \frac{(\theta'_3 \bar{S}_3/P_3)^4}{N_3 - 1}} \\
 &= \frac{0.03211^4}{\frac{(5 \times 0.005)^4}{19} + \frac{(2 \times 0.01)^4}{9} + \frac{(0.1 \times 0.025)^4}{4}} \\
 &= \frac{0.0064226^4}{\frac{(1 \times 0.005)^4}{19} + \frac{(2 \times 0.002)^4}{9} + \frac{(0.5 \times 0.001)^4}{4}} \\
 &\simeq 27.7
 \end{aligned}$$

and rounded downwards,  $\nu_r = 27$ .

By Table 10.1, we have for the result  $t_{95,27} = 2.052$ , and the precision error PE of the result on an absolute basis is

$$\text{PE}_r = t_{95,27}S_r = 2.052 \times 0.03211 = 0.06589 \quad (\text{with units})$$

On a relative basis,

$$\frac{\text{PE}_r}{r} = t_{95,27} \frac{S_r}{r} = 2.052 \times 0.64226 = 1.3179\%$$

**Example 11.** If the instrumentation used in determining the measurements of the three parameters in equation (10.40) was such that both the systematic and the random errors of Examples 9 and 10 applied, what would be the maximum and probable values of the uncertainty of the result on an absolute and relative basis?

*Solution.* On an absolute basis, by equation (10.31),

$$\begin{aligned}
 U_{r,\text{ADD}} &= U_{r,\text{max}} = B_r + t_{95}S_r \quad (\text{at } 99\%) \\
 U_{r,\text{max}} &= 0.11456 + 0.06589 = 0.18045 \quad (\text{with units})
 \end{aligned}$$

By equation (10.32),

$$\begin{aligned}
 U_{r,\text{RSS}} &= U_{r,\text{probable}} = [B_r^2 + (t_{95}S_r)^2]^{1/2} \quad (\text{at } 95\%) \\
 U_{r,\text{probable}} &= [(0.11456)^2 + (0.06589)^2]^{1/2} = 0.13216 \quad (\text{with units})
 \end{aligned}$$

On a relative basis;

$$\frac{U_{r,\max}}{r} = \frac{B_r}{r} + \frac{t_{95}S_r}{r} = 2.2913 + 1.3179 = 3.61\%$$

$$\frac{U_{r,\text{probable}}}{r} = \left[ \left( \frac{B_r}{r} \right)^2 + \left( \frac{t_{95}S_r}{r} \right)^2 \right]^{1/2} = 2.64\%$$

As a check on consistency,

$$U_{r,\max} = \frac{U_{r,\max}}{r} \times r = 0.0361 \times 5 = 0.1805$$

and

$$U_{r,\text{probable}} = \frac{U_{r,\text{probable}}}{r} \times r = 0.0264 \times 5 = 0.132$$

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## NOMENCLATURE

### Roman

|                 |                                    |
|-----------------|------------------------------------|
| $B$             | Bias limit                         |
| CI              | Confidence interval                |
| $d_2, d_2^*$    | Statistics used with range         |
| $M$             | Number of sets of observations     |
| $N$             | Number of observations in set      |
| $p$             | Probability                        |
| $P$             | Value of parameter                 |
| $\bar{P}$       | Mean value of parameter            |
| $r$             | Result                             |
| $R$             | Range of observations              |
| $\bar{R}$       | Mean value of range                |
| $S$             | Precision index of measurement     |
| $\bar{S}$       | Precision index of mean            |
| $\bar{\bar{S}}$ | Pooled precision index             |
| $t$             | Student's $t$ statistic            |
| $T$             | Temperature                        |
| $U$             | Uncertainty                        |
| $X$             | Value of measurement               |
| $\bar{X}$       | Mean of single set of measurements |
| $\bar{\bar{X}}$ | Mean of $M$ sets of measurements   |
| $\bar{X}'$      | True value of quantity             |
| $z$             | Normal deviate                     |

### Greek

|            |                   |
|------------|-------------------|
| $\beta$    | True bias         |
| $\delta$   | True error        |
| $\epsilon$ | True random error |

- $\theta$  Absolute sensitivity  
 $\theta'$  Relative sensitivity  
 $\mu$  True mean value of population  
 $\nu$  Degrees of freedom  
 $\sigma$  True standard deviation of population  
 $\tau$  Substitute  $t$  statistic

*Subscripts*

- $i$  General index  
 $k$  General measurement  
 $J$  Number of parameters  
 $X$  Individual statistic  
 $\bar{X}$  Mean statistic

## PROBLEMS

- 10.1** A thermocouple in a fixed-point bath yields the following replicate data in terms of deviates  $X_i$  from the standard reference table:

| $N$   | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $X_i$ | 0.061 | 0.061 | 0.061 | 0.062 | 0.062 | 0.050 | 0.059 | 0.056 | 0.058 |

Find the mean value, the range, the precision index, and the 95% confidence interval by both range and precision index estimates for the *individual* readings.

- 10.2** If the thermocouple of Problem 10.1 was a type J with a slope of 0.03085 mV/°F, express the confidence intervals of Problem 10.1 in terms of temperature.
- 10.3** Given three sets of ten readings each, find the 95% confidence interval of the mean: *a.* If  $\sigma = 0.5$ . *b.* If  $\bar{S} = 0.5$ . *c.* If  $\bar{R} = 0.5$ .
- 10.4** The mass flow rate is given in terms of  $Ap/\sqrt{T}$ . If  $A = 10$ ,  $p = 20$ , and  $T = 484$  in compatible units, and the biases are, respectively; 0.1%, 0.2%, and 1%, find the absolute and relative biases in  $\dot{m}$ .
- 10.5** If conditions of Problem 10.4 apply, and in addition  $S/A = 0.1\%$  at  $N_A = 5$ ,  $S/p = 0.2\%$  at  $N_p = 10$ , and  $S/T = 1\%$  at  $N_T = 20$ , find the absolute and relative precision indexes in  $\dot{m}$ .
- 10.6** For the conditions of Problems 10.4 and 10.5, find the probable uncertainty in  $\dot{m}$ .