

CS 534: Computer Vision Camera Calibration

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Outlines

- Camera Calibration
- Linear Least-Squares and related stuff

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Camera Calibration, Why ?

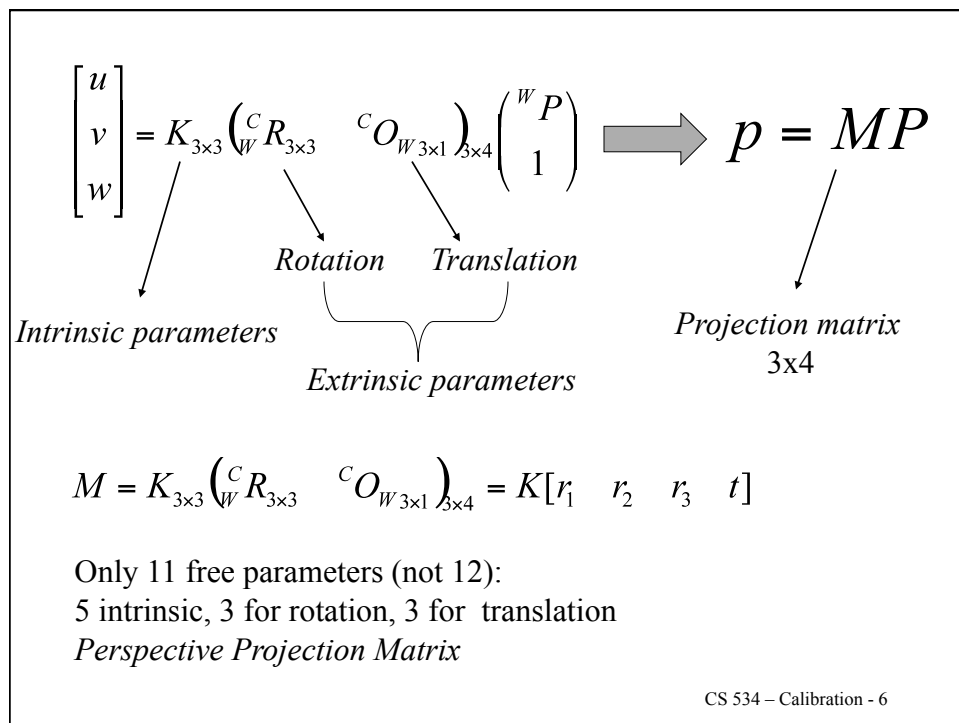
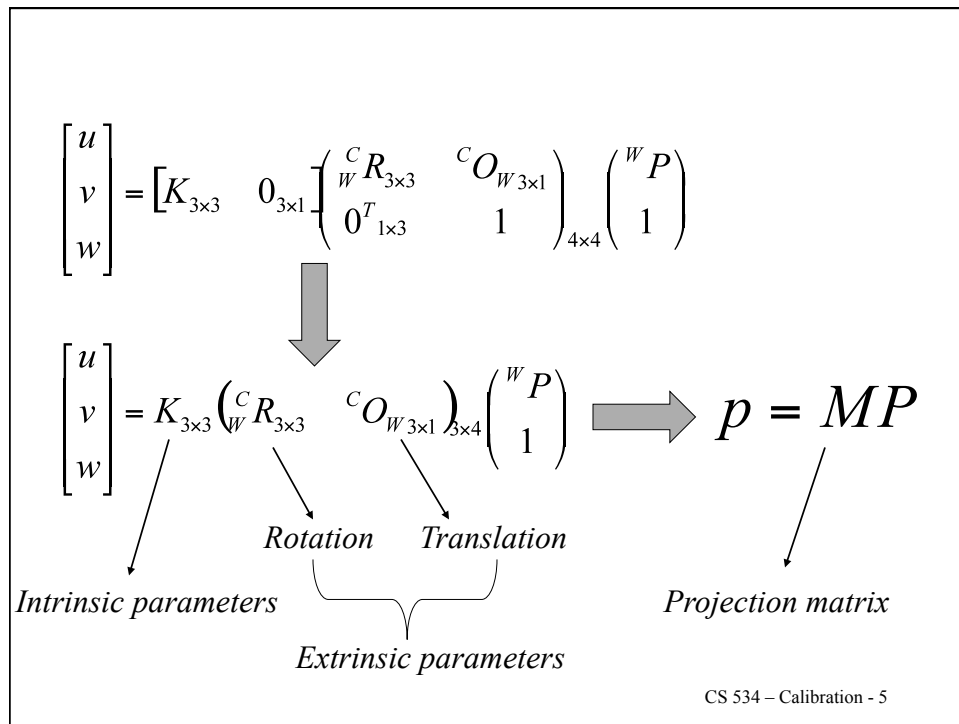
- Camera calibration is a necessary step in 3D computer vision.
- A calibrated camera can be used as a quantitative sensor
- It is essential in many applications to recover 3D quantitative measures about the observed scene from 2D images. Such as 3D Euclidean structure
- From a calibrated camera we can measure how far an object is from the camera, or the height of the object, etc. e.g., object avoidance in robot navigation.

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Camera Calibration

- Find the intrinsic and extrinsic parameters of a camera
 - Extrinsic parameters: the camera's location and orientation in the world.
 - Intrinsic parameters: the relationships between pixel coordinates and camera coordinates.
- VERY large literature on the subject
- Work of Roger Tsai is influential
- Good calibration is important when we need to:
 - Reconstruct a world model.
 - Interact with the world: Robot, hand-eye coordination
- Basic idea:
 - Given a set of world points P_i and their image coordinates (u_p, v_p)
 - find the projection matrix M
 - and then find intrinsic and extrinsic parameters.

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Explicit Form of the Projection Matrix

$$\mathcal{M} = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}$$

Note: If $\mathcal{M} = (\mathcal{A} \quad \mathbf{b})$ then $|\mathbf{a}_3| = 1$.

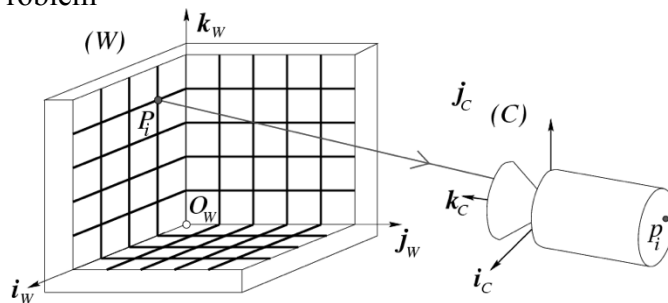
$$p = MP = \begin{pmatrix} m_1^T \\ m_2^T \\ m_3^T \end{pmatrix} P \quad \longrightarrow \quad \begin{aligned} u &= \frac{m_1^T \cdot P}{m_3^T \cdot P} \\ v &= \frac{m_2^T \cdot P}{m_3^T \cdot P} \end{aligned}$$

Replacing M by λM doesn't change u or v

M is only defined up to scale in this setting

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Calibration Problem



Given n points $P_1, \dots, P_n$ with *known* positions and their images $p_1, \dots, p_n$

Find $\mathbf{i}$ and $\mathbf{e}$ such that

$\mathbf{i}$ intrinsic parameters
 $\mathbf{e}$ extrinsic parameters

$$\begin{cases} u_i = \frac{m_1(\mathbf{i}, \mathbf{e}) \cdot P_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot P_i} \\ v_i = \frac{m_2(\mathbf{i}, \mathbf{e}) \cdot P_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot P_i} \end{cases} \quad \text{for } i = 1, \dots, n$$

$$\sum_{i=1}^n \left[\left(u_i - \frac{m_1(\mathbf{i}, \mathbf{e}) \cdot P_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot P_i} \right)^2 + \left(v_i - \frac{m_2(\mathbf{i}, \mathbf{e}) \cdot P_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot P_i} \right)^2 \right] \text{ is minimized}$$

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Calibration Techniques

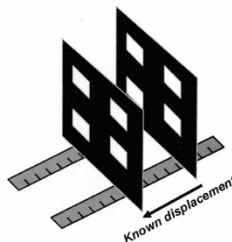
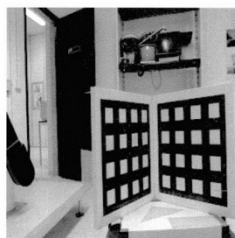
- Calibration using 3D calibration object
- Calibration using 2D planer pattern
- Calibration using 1D object (line-based calibration)
- Self Calibration: no calibration objects
- Vanishing points from for orthogonal direction
- Many other smart ideas

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Calibration Techniques

Calibration using 3D calibration object:

- Calibration is performed by observing a calibration object whose geometry in 3D space is known with very good precision.
- Calibration object usually consists of two or three planes orthogonal to each other, e.g. calibration cube
- Calibration can also be done with a plane undergoing a precisely known translation (Tsai approach)
- (+) most accurate calibration, simple theory
- (-) more expensive, more elaborate setup

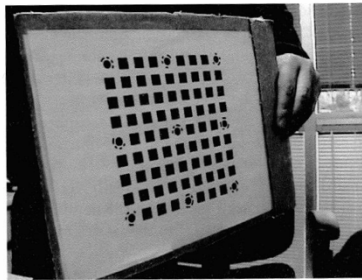


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Calibration Techniques

2D plane-based calibration

- Require observation of a planar pattern shown at few different orientations
- No need to know the plane motion
- Set up is easy, most popular approach.



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Calibration Techniques

1D line-based calibration:

- Relatively new technique.
- Calibration object is a set of collinear points, e.g., two points with known distance, three collinear points with known distances, four or more...
- Camera can be calibrated by observing a moving line around a fixed point, e.g. a string of balls hanging from the ceiling!
- Can be used to calibrate multiple cameras at once. Good for network of cameras.



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Calibration Techniques

Self-calibration:

- Techniques in this category do not use any calibration object
- Only image point correspondences are required
- Just move the camera in a static scene and obtain multiple images
- Correspondences between three images are sufficient to recover both the internal and external parameters

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Calibration Techniques

Which calibration technique to use ?

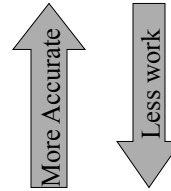
- Calibration with apparatus versus self-calibration:
 - Whenever possible, if you can pre-calibrate the camera, you should do it with a calibration object
 - Self-calibration cannot usually achieve the same accuracy as calibration with an object.
 - Sometimes pre-calibration is impossible (e.g., a scene reconstruction from an old movie), self-calibration is the only choice.

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Calibration Techniques

Which calibration technique to use ?

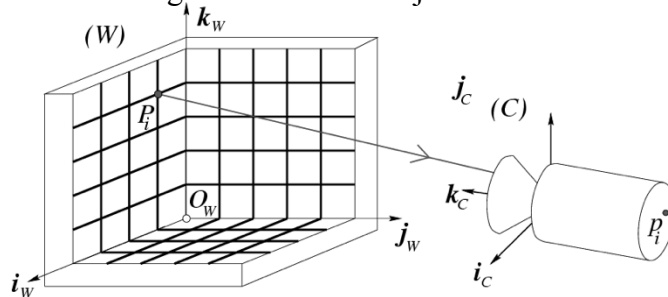
- using 3D calibration object
- using 2D planer pattern
- using 1D object (line-based calibration)
- Self calibration



- 2D planer pattern approaches seems to be a good compromise: good accuracy with simple setup
- 1D object is suitable for calibrating multiple cameras as once.

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Calibration Problem: using 3D calibration object

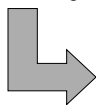


Given n points $P_1, \dots, P_n$ with *known* positions and their images $p_1, \dots, p_n$

Find $\mathbf{i}$ and $\mathbf{e}$ such that

$\mathbf{i}$ intrinsic parameters
 $\mathbf{e}$ extrinsic parameters

$$\begin{cases} u_i = \frac{m_1(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i} \\ v_i = \frac{m_2(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i} \end{cases} \quad \text{for } i = 1, \dots, n$$



$$\sum_{i=1}^n \left[\left(u_i - \frac{m_1(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i} \right)^2 + \left(v_i - \frac{m_2(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i}{m_3(\mathbf{i}, \mathbf{e}) \cdot \mathbf{P}_i} \right)^2 \right] \text{ is minimized}$$

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Camera Calibration and least-squares

- Camera Calibration can be posed as least-squares parameter estimation problem.
- Estimate the intrinsic and extrinsic parameters that minimize the mean-squared deviation between predicted and observed image features.
- Least-squares parameter estimation is a fundamental technique that is used extensively in computer vision.
- You can formulate many problems as error minimization between observed and predicted values.
 - Linear Least-Squares
 - Nonlinear Least-Squares
- Typically, use linear least-squares to obtain a solution and then use nonlinear least square to refine the solution and estimate more parameters (e.g. radial distortion)

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Linear Least-Squares

- Linear system of equations

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q &= b_2 \\
 \dots & \\
 a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pq}x_q &= b_p
 \end{aligned}
 \Leftrightarrow Ax = b$$

$$A_{p \times q} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}$$

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Linear Systems

$$\boxed{A} \quad \boxed{x} = \boxed{b}$$

Square system:

- unique solution
- Gaussian elimination

$$\begin{array}{|c|} \hline \\ \hline A \\ \hline \\ \hline \end{array} \quad \boxed{x} = \begin{array}{|c|} \hline \\ \hline b \\ \hline \\ \hline \end{array}$$

Rectangular system ??

- under-constrained:
infinity of solutions
- over-constrained:
no solution

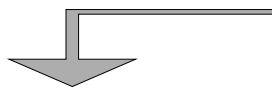
➔ Minimize $\|Ax - b\|^2$

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How do you solve overconstrained linear equations ??

- Over-constrained system, obtain a least-squared solution:

Solution that minimize the squared deviations between Ax and b :



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= b_2 \\ \dots & \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= b_p \end{aligned}$$

$$E = \sum_{i=1}^p (a_{i1}x_1 + \cdots + a_{iq}x_q - b_i)^2 = \|Ax - b\|^2 = (Ax - b)^T \cdot (Ax - b) = e^T \cdot e$$

$$\frac{\partial E}{\partial x_i} = 0 \quad i = 1, \dots, q$$

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How do you solve overconstrained linear equations ??

- Define $E = |\mathbf{e}|^2 = \mathbf{e} \cdot \mathbf{e}$ with

$$\mathbf{e} = A\mathbf{x} - \mathbf{b} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \mathbf{b}$$

$$= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots + x_n\mathbf{c}_n - \mathbf{b}$$

$\frac{\partial e}{\partial x_i} = \mathbf{c}_i$

- At a minimum,

$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \frac{\partial \mathbf{e}}{\partial x_i} \cdot \mathbf{e} + \mathbf{e} \cdot \frac{\partial \mathbf{e}}{\partial x_i} = 2 \frac{\partial \mathbf{e}}{\partial x_i} \cdot \mathbf{e} \\ &= 2 \frac{\partial}{\partial x_i} (x_1\mathbf{c}_1 + \dots + x_n\mathbf{c}_n - \mathbf{b}) \cdot \mathbf{e} = 2\mathbf{c}_i \cdot \mathbf{e} \\ &= 2\mathbf{c}_i^T (A\mathbf{x} - \mathbf{b}) = 0 \end{aligned}$$

- or

$$0 = \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_n^T \end{bmatrix} (A\mathbf{x} - \mathbf{b}) = A^T(A\mathbf{x} - \mathbf{b}) \Rightarrow A^T A\mathbf{x} = A^T \mathbf{b},$$

where $\mathbf{x} = A^\dagger \mathbf{b}$ and $A^\dagger = (A^T A)^{-1} A^T$ is the *pseudoinverse* of A !

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Singular Value Decomposition

- Chief tool for dealing with m by n systems and singular systems.
- SVD: If $\mathbf{A}$ is a real m by n matrix then there exist orthogonal matrices

$\mathbf{U}$ ($m \cdot m$) and $\mathbf{V}$ ($n \cdot n$) such that

$$\mathbf{U}^t \mathbf{A} \mathbf{V} = \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_p) \quad p = \min\{m, n\}$$

$$\mathbf{U}^t \mathbf{A} \mathbf{V} = \Sigma \quad \mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^t$$

- $\mathbf{U}$ and $\mathbf{V}$ are orthonormal matrices

$$\begin{array}{|c|} \hline \mathbf{A} \\ \hline m \times n \end{array} = \begin{array}{|c|} \hline \mathbf{U} \\ \hline m \times m \end{array} \begin{array}{|c|} \hline \Sigma \\ \hline m \times n \end{array} \begin{array}{|c|} \hline \mathbf{V}^t \\ \hline n \times n \end{array}$$

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Singular Value Decomposition

- SVD: If A is a real m by n matrix then there exist orthogonal matrices U ($m \cdot m$) and V ($n \cdot n$) such that $U^t A V = \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_p)$ $p = \min\{m, n\}$

$$U^t A V = \Sigma \quad A = U \Sigma V^t$$
- **Singular values:** Non negative square roots of the eigenvalues of $A^t A$.
Denoted $\sigma_i, i=1, \dots, n$
- $A^t A$ is symmetric $\Rightarrow$ eigenvalues and singular values are real.
- Singular values arranged in decreasing order.
- Columns of V are the eigenvectors of $A^t A$

$$A^t A = (U \Sigma V^t)^t (U \Sigma V^t) = V \Sigma^t U^t U \Sigma V^t = V \Sigma^t \Sigma V^t = V \Sigma^2 V^{-1}$$

$$(A^t A) V = V \Sigma^2$$

$$(A^t A) v = v \lambda$$

$$\begin{array}{|c|} \hline A \\ \hline m \times n \end{array} = \begin{array}{|c|} \hline U \\ \hline m \times m \end{array} \begin{array}{|c|} \hline \Sigma \\ \hline m \times n \end{array} \begin{array}{|c|} \hline V^t \\ \hline n \times n \end{array}$$

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- For over-constrained system $Ax=b$:
- Use SVD to decompose $A=U\Sigma V^t$

$$Ax = b$$

$$U \Sigma V^T x = b$$

$$(V \Sigma^{-1} U^T) U \Sigma V^T x = (V \Sigma^{-1} U^T) b$$

$$x = V \Sigma^{-1} U^T b$$

$$\Sigma^{-1} = \text{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \dots, \frac{1}{\sigma_n})$$

For rank deficient matrices: we can obtain best rank-R approximation by setting the inverse of the smallest $n-R$ singular values to zeros

$$\Sigma^{-1} = \text{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \dots, \frac{1}{\sigma_R}, 0, \dots, 0)$$

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Homogeneous Linear Systems

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Square system:

- unique solution: 0
- unless $\text{Det}(A) \neq 0$

Rectangular system ??

$$\begin{bmatrix} A \\ 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

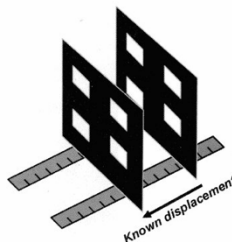
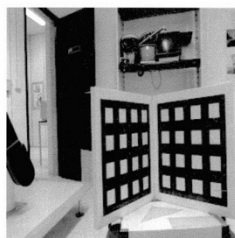
→ Minimize $\|Ax\|^2$
under the constraint $\|x\|^2 = 1$

x is the unit singular vector of A corresponding to the smallest singular value (the last column of V , where $A = U \Sigma V^T$ is the SVD of A)

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Calibration using 3D calibration object:

- Calibration is performed by observing a calibration object whose geometry in 3D space is known with very good precision.
- Calibration object usually consists of two or three planes orthogonal to each other, e.g. calibration cube
- Calibration can also be done with a plane undergoing a precisely known translation
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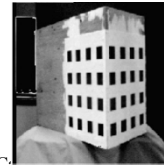
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Calibration using 3D object

- Calibration target: 2 planes at right angle with checkerboard patterns (Tsai grid)
- We know positions of pattern corners only with respect to a coordinate system of the target
- We position camera in front of target and find images of corners
- We obtain equations that describe point coordinates and contain intrinsic and extrinsic parameters of camera

Main Steps:

1. Detecting points of interest (e.g., corners of the checker pattern) in the 2D image and obtain their corresponding 3D measurement.
2. Find the best projection matrix M using linear least squares
3. Recover intrinsic and extrinsic parameters
4. Refine the parameters through nonlinear optimization



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Linear Camera Calibration

Given n points $P_1, \dots, P_n$ with *known* positions and their images $p_1, \dots, p_n$

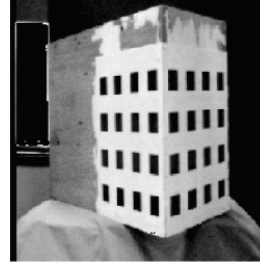
$$\Rightarrow \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{m}_1 \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \\ \frac{\mathbf{m}_2 \cdot \mathbf{P}_i}{\mathbf{m}_3 \cdot \mathbf{P}_i} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \mathbf{m}_1 - u_i \mathbf{m}_3 \\ \mathbf{m}_2 - v_i \mathbf{m}_3 \end{pmatrix} \mathbf{P}_i = 0$$

$$\begin{array}{ll} m_{11}X_i + m_{12}Y_i + m_{13}Z_i + m_{14} & -m_{31}u_iX_i - m_{32}u_iY_i - m_{33}u_iZ_i - m_{34}u_i = 0 \\ m_{21}X_i + m_{22}Y_i + m_{23}Z_i + m_{24} & -m_{31}v_iX_i - m_{32}v_iY_i - m_{33}v_iZ_i - m_{34}v_i = 0 \end{array}$$

$$\Rightarrow \mathcal{P}\mathbf{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{P}_1^T & \mathbf{0}^T & -u_1\mathbf{P}_1^T \\ \mathbf{0}^T & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ \dots & \dots & \dots \\ \mathbf{P}_n^T & \mathbf{0}^T & -u_n\mathbf{P}_n^T \\ \mathbf{0}^T & \mathbf{P}_n^T & -v_n\mathbf{P}_n^T \end{pmatrix}_{2n \times 12} \text{ and } \mathbf{m} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{pmatrix}_{12 \times 1}$$

Observations:

- This is a linear problem with respect to M but nonlinear with respect to the parameters (intrinsic and extrinsic)
- Each 3D-2D point match provides 2 equations
- We need at least 6 points in general configuration (12 equations for 11 parameters)



$$\mathcal{P}\mathbf{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{P}_1^T & \mathbf{0}^T & -u_1\mathbf{P}_1^T \\ \mathbf{0}^T & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ \dots & \dots & \dots \\ \mathbf{P}_n^T & \mathbf{0}^T & -u_n\mathbf{P}_n^T \\ \mathbf{0}^T & \mathbf{P}_n^T & -v_n\mathbf{P}_n^T \end{pmatrix}_{2n \times 12} \text{ and } \mathbf{m} \stackrel{\text{def}}{=} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}_{12 \times 1}$$

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$$\mathcal{P}\mathbf{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{P}_1^T & \mathbf{0}^T & -u_1\mathbf{P}_1^T \\ \mathbf{0}^T & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ \dots & \dots & \dots \\ \mathbf{P}_n^T & \mathbf{0}^T & -u_n\mathbf{P}_n^T \\ \mathbf{0}^T & \mathbf{P}_n^T & -v_n\mathbf{P}_n^T \end{pmatrix}_{2n \times 12} \text{ and } \mathbf{m} \stackrel{\text{def}}{=} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}_{12 \times 1}$$

Problem: $\min_m \|\mathcal{P}\mathbf{m}\|^2 = 0$ Subject to: $\|\mathbf{m}\| = 1$

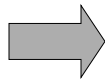
- When $n \geq 6 \Rightarrow$ homogenous linear least-squares can be used to compute unit vector $\mathbf{m} \Rightarrow M$
- *Solution:* The eigenvector of $\mathcal{P}^T\mathcal{P}$ corresponding to the smallest eigen value
- Degenerate case: All points lie on the same plane.
Points should not be on the same plane
- When possible, have at least 5 times as many equations as unknowns (28 points)

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Once M is known, you still got to recover the intrinsic and extrinsic parameters !!!

This is a decomposition problem, not an estimation problem.

$$\boxed{\rho} \mathcal{M} = \begin{pmatrix} \alpha \mathbf{r}_1^T - \alpha \cot \theta \mathbf{r}_2^T + u_0 \mathbf{r}_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} \mathbf{r}_2^T + v_0 \mathbf{r}_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}$$



- Intrinsic parameters
- Extrinsic parameters

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- Once the projection matrix M is known, we can uniquely recover the intrinsic and extrinsic parameters of the camera

$$M = K[R \quad t]$$

$$B = KR$$

$$b = Kt$$

$$K = \begin{bmatrix} \alpha & \gamma & u_o \\ 0 & \beta & v_o \\ 0 & 0 & 1 \end{bmatrix}$$

$$BB^T = KK^T = \begin{bmatrix} k_u & k_c & u_o \\ k_c & k_v & v_o \\ u_o & v_o & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + \gamma^2 + u_o^2 & u_o v_o + \gamma \beta & u_o \\ u_o v_o + \gamma \beta & \beta^2 + v_o^2 & v_o \\ u_o & v_o & 1 \end{bmatrix}$$

$$\beta = \sqrt{k_v - v_o^2},$$

$$\gamma = \frac{k_c - u_o v_o}{\beta}$$

$$\alpha = \sqrt{k_u - u_o^2 - \gamma^2}$$

$$R = K^{-1}B,$$

$$t = K^{-1}b$$

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Another approach to decomposition

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = K_{3 \times 3} \begin{pmatrix} {}^C R_{3 \times 3} & {}^C O_{3 \times 1} \end{pmatrix}_{3 \times 4} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}$$

The translation vector can be written as:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = K_{3 \times 3} \begin{pmatrix} {}^C R_{3 \times 3} & -{}^C R_{3 \times 3} {}^W O_{3 \times 1} \end{pmatrix}_{3 \times 4} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = K_{3 \times 3} {}^C R_{3 \times 3} \begin{pmatrix} I_{3 \times 3} & -{}^W O_{3 \times 1} \end{pmatrix}_{3 \times 4} \begin{pmatrix} {}^W P \\ 1 \end{pmatrix}$$

$${}^B P = {}^B R {}^A P + {}^B O_A$$

$${}^W O_W = {}^W R {}^C O_W + {}^W O_C$$

$$0 = {}^W R {}^C O_W + {}^W O_C$$

$${}^W R {}^C O_W = -{}^W O_C$$

$${}^C O_W = -{}^C R {}^W O_C$$

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Getting Camera Translation

$$M_{3 \times 4} = K_{3 \times 3} {}^C R_{3 \times 3} \begin{pmatrix} I_{3 \times 3} & -{}^W O_{3 \times 1} \end{pmatrix}_{3 \times 4}$$

Upper triangle matrix Orthogonal matrix Translation Vector:
Where in the world is the camera center

Note that:

$$\begin{pmatrix} 1 & 0 & 0 & -X_c \\ 0 & 1 & 0 & -Y_c \\ 0 & 0 & 1 & -Z_c \end{pmatrix} \begin{pmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{pmatrix} = 0$$

If we solve the system $M \cdot t = 0$ we can get the translation vector! (t is a null vector for M)

Find null vector t of M using SVD

- t is the unit singular vector of M corresponding to the smallest singular value (the last column of V , where $M = U \Sigma V^T$ is the SVD of M)

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Getting Camera Orientation

$$M_{3 \times 4} = K_{3 \times 3}^c R_{3 \times 3} (I_{3 \times 3} - {}^w O_{C3 \times 1} {}^c O_{1 \times 3})$$

$K_{3 \times 3}^c$: Upper triangle matrix
 $R_{3 \times 3}$: Orthogonal matrix
 $- {}^w O_{C3 \times 1} {}^c O_{1 \times 3}$: Translation Vector: Where in the world is the camera center

- Left 3x3 submatrix M' of M is of form $M' = K R$
- K is an upper triangular matrix
- R is an orthogonal matrix
- Any non-singular square matrix M' can be decomposed into the product of an upper-triangular matrix K and an orthogonal matrix R using the RQ factorization
- Similar to QR factorization but order of 2 matrices is reversed $A = Q.R$

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Rotation matrices

- 3-D rotation is result of three consecutive rotations around the coordinate axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
 - the angles of the rotations are the parameters of the rotation

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \quad R_2(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad R_3(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_1 R_2 R_3 = \begin{bmatrix} \cos \beta \cos \gamma & -\cos \beta \sin \gamma & \sin \beta \\ \sin \alpha \sin \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & -\sin \alpha \cos \beta \\ -\cos \alpha \sin \beta \cos \gamma - \sin \alpha \sin \gamma & \cos \alpha \sin \beta \sin \gamma + \sin \alpha \cos \gamma & \cos \alpha \cos \beta \end{bmatrix}$$

- The order in which we perform the multiplications matters!
 - Six ways to represent a rotation matrix

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Rotation matrices

$$R = R_1 R_2 R_3 = \begin{bmatrix} \cos \beta \cos \gamma & -\cos \beta \sin \gamma & \sin \beta \\ \sin \alpha \sin \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & -\sin \alpha \cos \beta \\ -\cos \alpha \sin \beta \cos \gamma - \sin \alpha \sin \gamma & \cos \alpha \sin \beta \sin \gamma + \sin \alpha \cos \gamma & \cos \alpha \cos \beta \end{bmatrix}$$

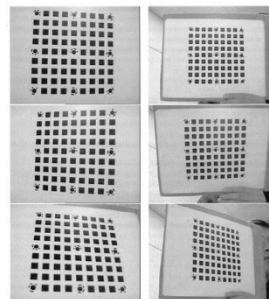
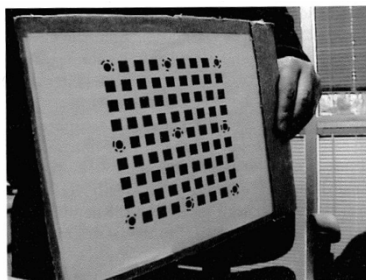
- The order in which we perform the multiplications matters!
 - Six ways to represent a rotation matrix
- It is easy to recover the rotation angles (α, β, γ) from R
 - $R_{1,3}$ gives β
 - Then γ can be recovered from $R_{1,1}$, and α from $R_{2,3}$
- R is orthornormal $RR^T = R^T R = I$
- So, even though there are 9 entries in R , there are still only the 3 parameters
- This makes estimating R difficult in practice.

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Camera Calibration with a 2D planer object

2D plane-based calibration

- Require observation of a planar pattern shown at few different orientations
- No need to know the plane motion
- Set up is easy, just print the calibration pattern an stick it to surface.
- most popular approach.



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Camera Calibration with a 2D planer object

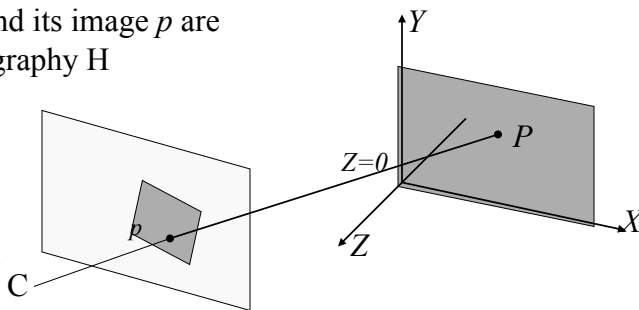
- Assume model plane is on $Z=0$ of the world coordinate system

$$s \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \begin{bmatrix} r_1 & r_2 & r_3 \\ t \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = K \begin{bmatrix} r_1 & r_2 & t \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \Rightarrow sp = H_{3 \times 3} P$$

A model point P and its image p are related by a homography H

$$H = K \begin{bmatrix} r_1 & r_2 & t \end{bmatrix}$$

H is a 3×3 matrix defined up to a scale factor (only 8 degrees of freedom)



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- Given an image of the model plane, a homography can be estimated, Let's denote by $H = [h_1 \ h_2 \ h_3]$

$$[h_1 \ h_2 \ h_3] = \lambda K \begin{bmatrix} r_1 & r_2 & t \end{bmatrix}$$

$$K^{-1} [h_1 \ h_2 \ h_3] = \lambda \begin{bmatrix} r_1 & r_2 & t \end{bmatrix}$$

$$K^{-1} h_1 = \lambda r_1$$

$$K^{-1} h_2 = \lambda r_2$$

Since r_1 and r_2 are orthonormal, we can obtain the following two important constraints relating K and H

$$h_1^T K^{-T} K^{-1} h_2 = 0$$

$$h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2$$

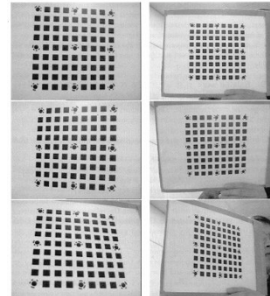
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- Given a set of images of the same plane we can establish a set of homographies H_i , one homography from each image.
- Given the constraints between H and K , we can solve for K

$$h_1^T K^{-T} K^{-1} h_2 = 0$$

$$h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2$$

- Note $K^{-T} K^{-1}$ is a 3x3 symmetric matrix; we have six unknowns.
- Each homography gives two linear equations in 6 unknowns
- We need at least 3 images of the plane to estimate the parameters in K
- Once K is known, we can estimate the extrinsic parameters:



$$r_1 = \lambda K^{-1} h_1, \quad r_2 = \lambda K^{-1} h_2, \quad r_3 = r_1 \times r_2, \quad t = \lambda K^{-1} h_3$$

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Sources

- Forsyth and Ponce, “Computer Vision a Modern approach” 3.1,3.2
- Medioni and Kang “Emerging Topics in Computer Vision” Prentice Hall 2004- Chapter 2 by Zhengyou Zhang.
- Fougheras, “Three-dimensional Computer Vision” MIT press.
- Slides by J. Ponce UIUC

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