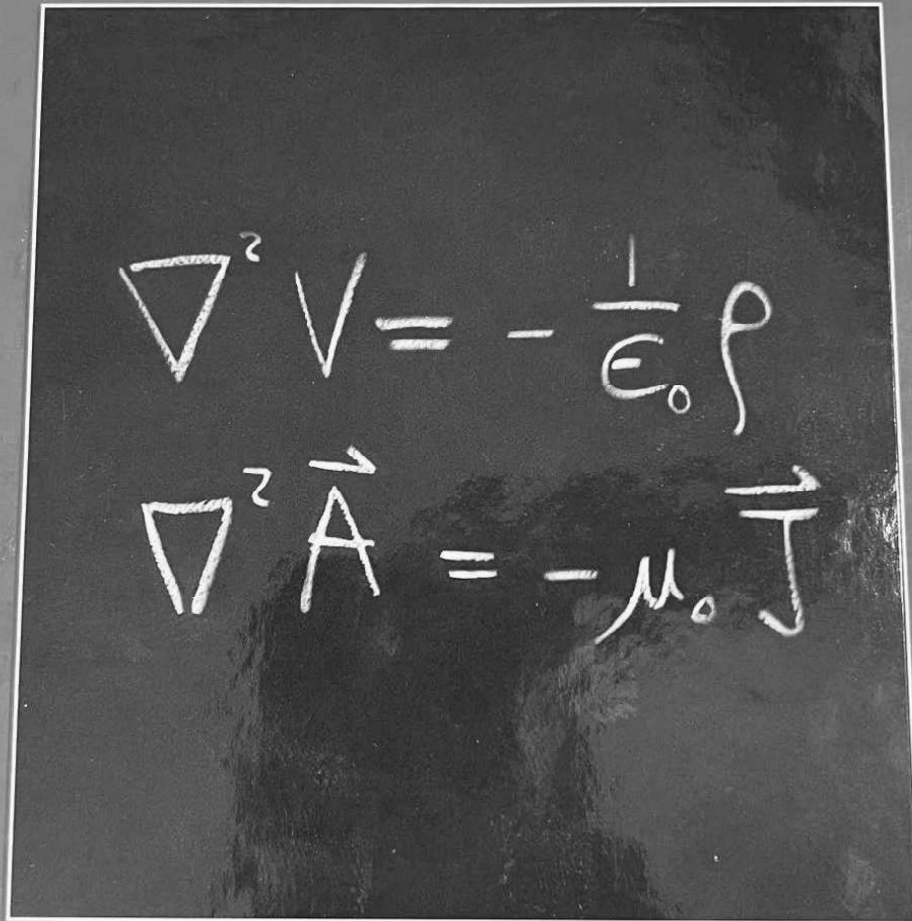


INTRODUCTION TO ELECTRODYNAMICS

Fourth Edition

A rectangular inset showing a chalkboard with two handwritten equations in white chalk. The top equation is $\nabla^2 V = -\frac{1}{\epsilon_0} \rho$ and the bottom equation is $\nabla^2 \vec{A} = -\mu_0 \vec{J}$.
$$\nabla^2 V = -\frac{1}{\epsilon_0} \rho$$
$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

DAVID J. GRIFFITHS

INTRODUCTION TO ELECTRODYNAMICS

Fourth Edition

David J. Griffiths
Reed College

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the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [Note: The dimensions of the loop cancel out; determine the actual numbers, in the units indicated.]

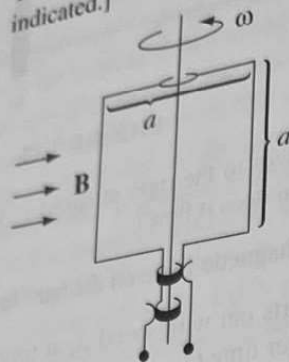


FIGURE 7.19

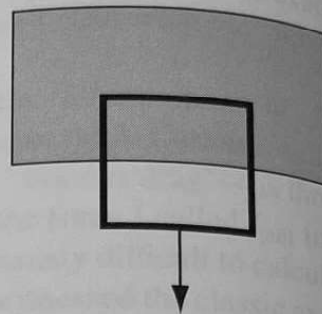


FIGURE 7.20

7.2 ■ ELECTROMAGNETIC INDUCTION

7.2.1 ■ Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.21a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.21b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.21c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

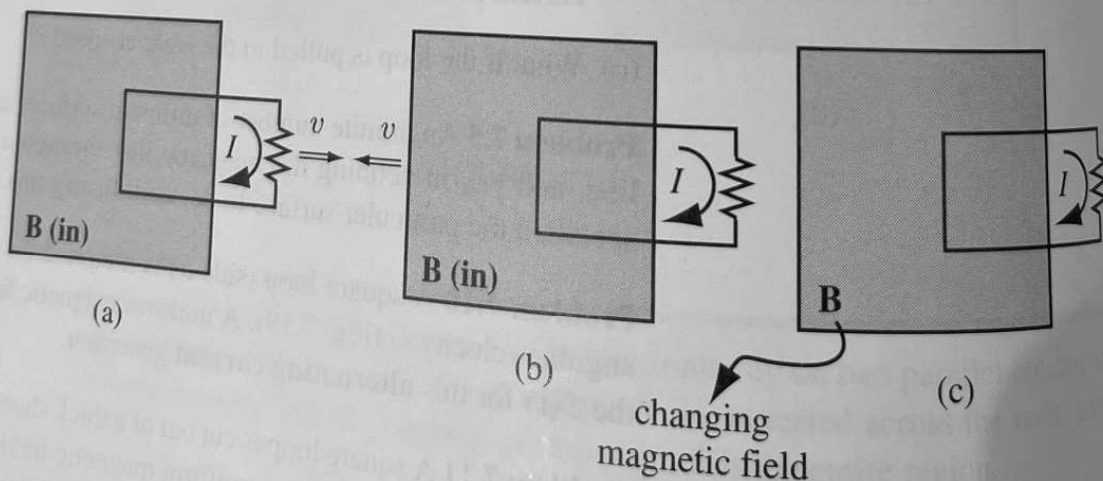


FIGURE 7.21

The first experiment, of course, is a straightforward case of motional emf; according to the flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

I don't think it will surprise you to learn that exactly the same emf arises in Experiment 2—all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity is a remarkable coincidence. For if the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces. In that case, what *is* responsible? What sort of field exerts a force on charges at rest? Well, *electric* fields do, of course, but in this case there doesn't seem to be any electric field in sight.

Faraday had an ingenious inspiration:

A changing magnetic field induces an electric field.

It is this induced⁸ electric field that accounts for the emf in Experiment 2.⁹ Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \quad (7.14)$$

then $\mathbf{E}$ is related to the change in $\mathbf{B}$ by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \quad (7.15)$$

This is **Faraday's law**, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (7.16)$$

⁸“Induce” is a subtle and slippery verb. It carries a faint odor of *causation* (“produce” would make this explicit) without quite committing itself. There is a sterile ongoing debate in the literature as to whether a changing magnetic field should be regarded as an independent “source” of electric fields (along with electric charge)—after all, the magnetic field *itself* is due to electric currents. It's like asking whether the postman is the “source” of my mail. Well, sure—he delivered it to my door. On the other hand, Grandma wrote the letter. Ultimately, ρ and $\mathbf{J}$ are the sources of *all* electromagnetic fields, and a changing magnetic field merely delivers electromagnetic news from currents elsewhere. But it is often convenient to think of a changing magnetic field “producing” an electric field, and it won't hurt you as long as you understand that this is the condensed version of a more complicated story. For a nice discussion, see S. E. Hill, *Phys. Teach.* **48**, 410 (2010).

⁹You might argue that the magnetic field in Experiment 2 is not really *changing*—just *moving*. What I mean is that if you sit at a *fixed location*, the field you experience changes as the magnet passes by.

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (or, in differential form, $\nabla \times \mathbf{E} = 0$) in the static case (constant $\mathbf{B}$) as, of course, it should.

In Experiment 3, the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (7.17)$$

will appear in the loop.

Many people call *this* "Faraday's law." Maybe I'm overly fastidious, but I find this confusing. There are really *two* totally different mechanisms underlying Eq. 7.17, and to identify them both as "Faraday's law" is a little like saying that because identical twins look alike we ought to call them by the same name. In Faraday's first experiment it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for Chapter 12. In the meantime, I shall reserve the term "Faraday's law" for electric fields induced by changing magnetic fields, and I do *not* regard Experiment 1 as an instance of Faraday's law.

Example 7.5. A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.22). Graph the emf induced in the ring, as a function of time.

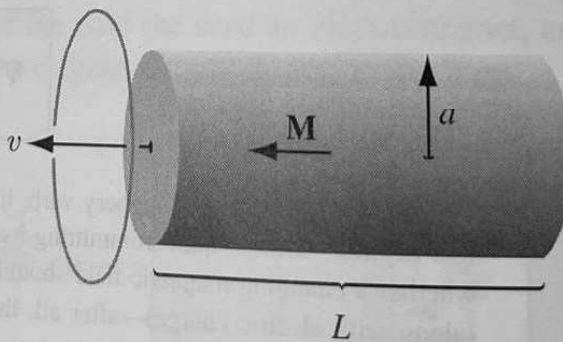


FIGURE 7.22

Solution

The magnetic field is the same as that of a long solenoid with surface current $\mathbf{K}_b = M \hat{\phi}$. So the field inside is $\mathbf{B} = \mu_0 \mathbf{M}$, except near the ends, where it starts to spread out. The flux through the ring is zero when the magnet is far away; it

builds up to a maximum of $\mu_0 M \pi a^2$ as the leading end passes through; and it drops back to zero as the trailing end emerges (Fig. 7.23a). The emf is (minus) the derivative of Φ with respect to time, so it consists of two spikes, as shown in Fig. 7.23b.

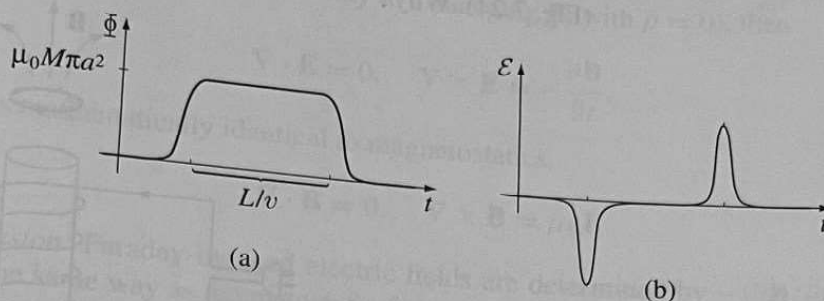


FIGURE 7.23

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in Ex. 7.5 we would like to know which *way* around the ring the induced current flows. In principle, the right-hand rule does the job (we called Φ positive to the left, in Fig. 7.22, so the positive direction for current in the ring is counterclockwise, as viewed from the left; since the first spike in Fig. 7.23b is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise). But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:¹⁰

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux *it* produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the *right*—it therefore flows *clockwise*.) Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops*, so the induced current flows *counterclockwise*, in an effort to restore it). Faraday induction is a kind of “inertial” phenomenon: A conducting loop “likes” to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (It doesn't *succeed* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

¹⁰Lenz's law applies to *motional* emfs, too, but for them it is usually easier to get the direction of the current from the Lorentz force law.

Example 7.6. The “jumping ring” demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air (Fig. 7.24). Why?

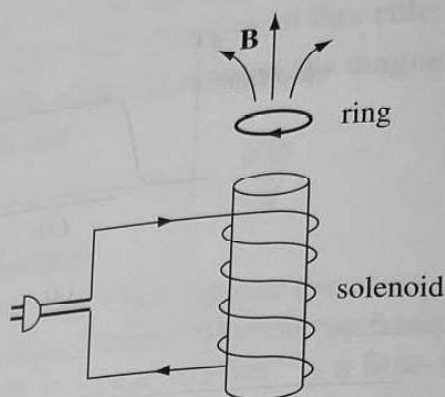


FIGURE 7.24

Solution

Before you turned on the current, the flux through the ring was zero. Afterward a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz's law, was in such a direction that its field tended to cancel this new flux. This means that the current in the loop is opposite to the current in the solenoid. And opposite currents repel, so the ring flies off.¹¹

Problem 7.12 A long solenoid, of radius a , is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t) \hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R , is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Problem 7.13 A square loop of wire, with sides of length a , lies in the first quadrant of the xy plane, with one corner at the origin. In this region, there is a nonuniform time-dependent magnetic field $\mathbf{B}(y, t) = ky^3 t^2 \hat{\mathbf{z}}$ (where k is a constant). Find the emf induced in the loop.

Problem 7.14 As a lecture demonstration a short cylindrical bar magnet is dropped down a vertical aluminum pipe of slightly larger diameter, about 2 meters long. It takes several seconds to emerge at the bottom, whereas an otherwise identical piece of unmagnetized iron makes the trip in a fraction of a second. Explain why the magnet falls more slowly.¹²

¹¹For further discussion of the jumping ring (and the related “floating ring”), see C. S. Schneider and J. P. Ertel, *Am. J. Phys.* **66**, 686 (1998); P. J. H. Tjossem and E. C. Brost, *Am. J. Phys.* **79**, 353 (2011).
¹²For a discussion of this amazing demonstration see K. D. Hahn et al., *Am. J. Phys.* **66**, 1066 (1998) and G. Donoso, C. L. Ladera, and P. Martin, *Am. J. Phys.* **79**, 193 (2011).

7.2.2 ■ The Induced Electric Field

Faraday's law generalizes the electrostatic rule $\nabla \times \mathbf{E} = \mathbf{0}$ to the time-dependent régime. The *divergence* of $\mathbf{E}$ is still given by Gauss's law ($\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$). If $\mathbf{E}$ is a *pure* Faraday field (due exclusively to a changing $\mathbf{B}$, with $\rho = 0$), then

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This is mathematically identical to magnetostatics,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Conclusion: Faraday-induced electric fields are determined by $-(\partial \mathbf{B} / \partial t)$ in exactly the same way as magnetostatic fields are determined by $\mu_0 \mathbf{J}$. The analog to Biot-Savart is¹³ is

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial \mathbf{B} / \partial t) \times \hat{\mathbf{r}}}{r^2} d\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\mathbf{r}}}{r^2} d\tau, \quad (7.18)$$

and if symmetry permits, we can use all the tricks associated with Ampère's law in integral form ($\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$), only now it's *Faraday's law* in integral form:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \quad (7.19)$$

The rate of change of (magnetic) flux through the Amperian loop plays the role formerly assigned to $\mu_0 I_{\text{enc}}$.

Example 7.7. A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.25. If $\mathbf{B}$ is changing with time, what is the induced electric field?

Solution

$\mathbf{E}$ points in the circumferential direction, just like the *magnetic* field inside a long straight wire carrying a uniform *current* density. Draw an Amperian loop of radius s , and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt} (\pi s^2 B(t)) = -\pi s^2 \frac{dB}{dt}.$$

Therefore

$$\mathbf{E} = -\frac{s}{2} \frac{dB}{dt} \hat{\phi}.$$

If $\mathbf{B}$ is *increasing*, $\mathbf{E}$ runs *clockwise*, as viewed from above.

¹³Magnetostatics holds only for time-independent currents, but there is no such restriction on $\partial \mathbf{B} / \partial t$.

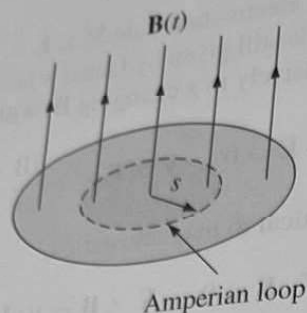


FIGURE 7.25

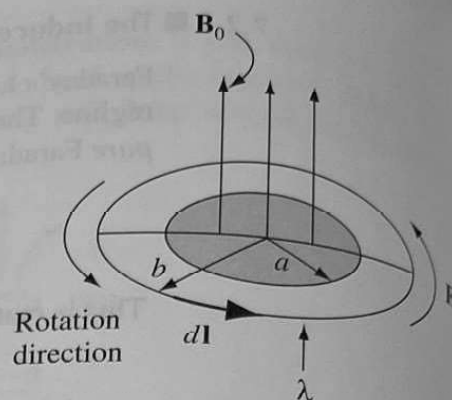


FIGURE 7.26

Example 7.8. A line charge λ is glued onto the rim of a wheel of radius b , which is then suspended horizontally, as shown in Fig. 7.26, so that it is free to rotate (the spokes are made of some nonconducting material—wood, maybe). In the central region, out to radius a , there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

Solution

The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that *its* field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Faraday's law, applied to the loop at radius b , says

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b} \frac{dB}{dt} \hat{\phi}.$$

The torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r} \times \mathbf{F})$, or $b\lambda E d\mathbf{l}$. The total torque on the wheel is therefore

$$N = b\lambda \left(-\frac{a^2}{2b} \frac{dB}{dt} \right) \oint d\mathbf{l} = -b\lambda \pi a^2 \frac{dB}{dt},$$

and the angular momentum imparted to the wheel is

$$\int N dt = -\lambda \pi a^2 b \int_{B_0}^0 dB = \lambda \pi a^2 b B_0.$$

It doesn't matter how quickly or slowly you turn off the field; the resulting angular velocity of the wheel is the same regardless. (If you find yourself wondering where the angular momentum *came* from, you're getting ahead of the story! Wait for the next chapter.)

Note that it's the *electric* field that did the rotating. To convince you of this, I deliberately set things up so that the *magnetic* field is zero at the location of

the charge. The experimenter may tell you she never put in any electric field—all she did was switch off the magnetic field. But when she did that, an electric field automatically appeared, and it's this electric field that turned the wheel.

I must warn you, now, of a small fraud that tarnishes many applications of Faraday's law: Electromagnetic induction, of course, occurs only when the magnetic fields are *changing*, and yet we would like to use the apparatus of magnetostatics (Ampère's law, the Biot-Savart law, and the rest) to *calculate* those correct. But in practice the error is usually negligible, unless the field fluctuates extremely rapidly, or you are interested in points very far from the source. Even the case of a wire snipped by a pair of scissors (Prob. 7.18) is *static enough* for Ampère's law to apply. This régime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **quasistatic**. Generally speaking, it is only when we come to electromagnetic waves and radiation that we must worry seriously about the breakdown of magnetostatics itself.

Example 7.9. An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance s from the wire.¹⁴

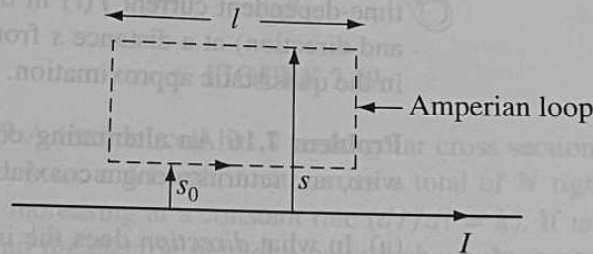


FIGURE 7.27

Solution

In the quasistatic approximation, the magnetic field is $(\mu_0 I / 2\pi s)$, and it circles around the wire. Like the $\mathbf{B}$ -field of a solenoid, $\mathbf{E}$ here runs parallel to the axis. For the rectangular “Amperian loop” in Fig. 7.27, Faraday's law gives:

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{l} &= E(s_0)l - E(s)l = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{a} \\ &= -\frac{\mu_0 l}{2\pi} \frac{dI}{dt} \int_{s_0}^s \frac{1}{s'} ds' = -\frac{\mu_0 l}{2\pi} \frac{dI}{dt} (\ln s - \ln s_0). \end{aligned}$$

¹⁴This example is artificial, and not just in the obvious sense of involving infinite wires, but in a more subtle respect. It assumes that the current is the same (at any given instant) all the way down the line. This is a safe assumption for the *short* wires in typical electric circuits, but not for *long* wires (**transmission lines**), unless you supply a distributed and synchronized driving mechanism. But never mind—the problem doesn't inquire how you would *produce* such a current; it only asks what *fields* would result if you *did*. Variations on this problem are discussed by M. A. Heald, *Am. J. Phys.* **54**, 1142 (1986).

Thus

$$\mathbf{E}(s) = \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln s + K \right] \hat{\mathbf{z}}, \quad (7.20)$$

where K is a constant (that is to say, it is independent of s —it might still be a function of t). The actual value of K depends on the whole history of the function $I(t)$ —we'll see some examples in Chapter 10.

Equation 7.20 has the peculiar implication that E blows up as s goes to infinity. That can't be true ... What's gone wrong? Answer: We have overstepped the limits of the quasistatic approximation. As we shall see in Chapter 9, electromagnetic "news" travels at the speed of light, and at large distances $\mathbf{B}$ depends not on the current *now*, but on the current *as it was* at some earlier time (indeed, a whole range of earlier times, since different points on the wire are different distances away). If τ is the time it takes I to change substantially, then the quasistatic approximation should hold only for

$$s \ll c\tau, \quad (7.21)$$

and hence Eq. 7.20 simply does not apply, at extremely large s .

Problem 7.15 A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

Problem 7.16 An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a .

- In what *direction* does the induced electric field point (radial, circumferential, or longitudinal)?
- Assuming that the field goes to zero as $s \rightarrow \infty$, find $\mathbf{E}(s, t)$.¹⁵

Problem 7.17 A long solenoid of radius a , carrying n turns per unit length, is looped by a wire with resistance R , as shown in Fig. 7.28.

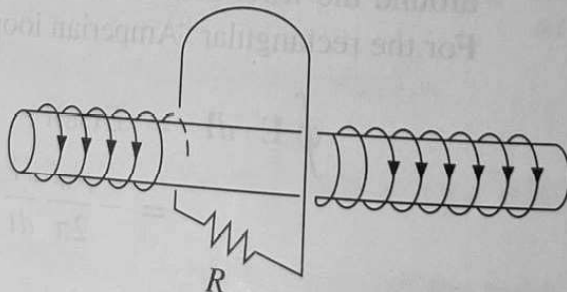


FIGURE 7.28

¹⁵This is not at all the way electric fields *actually* behave in coaxial cables, for reasons suggested in the previous footnote. See Sect. 9.5.3, or J. G. Chervenik, *Am. J. Phys.*, **54**, 946 (1986), for a more realistic treatment.

- (a) If the current in the solenoid is increasing at a constant rate ($dI/dt = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor?
- (b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop (toward the left, to a place far from the loop), what total charge passes through the resistor?

Problem 7.18 A square loop, side a , resistance R , lies a distance s from an infinite straight wire that carries current I (Fig. 7.29). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t) = \begin{cases} (1 - \alpha t)I, & \text{for } 0 \leq t \leq 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

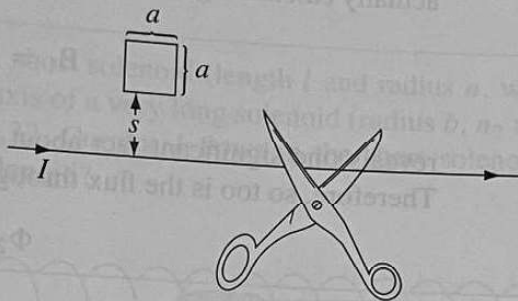


FIGURE 7.29

Problem 7.19 A toroidal coil has a rectangular cross section, with inner radius a , outer radius $a + w$, and height h . It carries a total of N tightly wound turns, and the current is increasing at a constant rate ($dI/dt = k$). If w and h are both much less than a , find the electric field at a point z above the center of the toroid. [Hint: Exploit the analogy between Faraday fields and magnetostatic fields, and refer to Ex. 5.6.]

Problem 7.20 Where is $\partial \mathbf{B} / \partial t$ nonzero, in Figure 7.21(b)? Exploit the analogy between Faraday's law and Ampère's law to sketch (qualitatively) the electric field.

Problem 7.21 Imagine a uniform magnetic field, pointing in the z direction and filling all space ($\mathbf{B} = B_0 \hat{\mathbf{z}}$). A positive charge is at rest, at the origin. Now somebody turns off the magnetic field, thereby inducing an electric field. In what direction does the charge move?¹⁶

7.2.3 ■ Inductance

Suppose you have two loops of wire, at rest (Fig. 7.30). If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass

¹⁶This paradox was suggested by Tom Colbert. Refer to Problem 2.55.

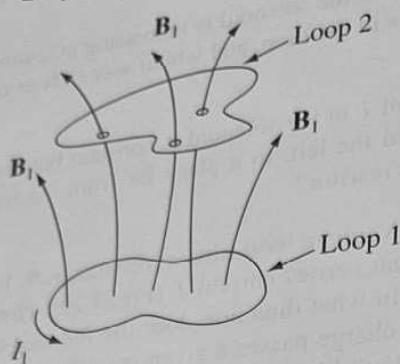


FIGURE 7.30

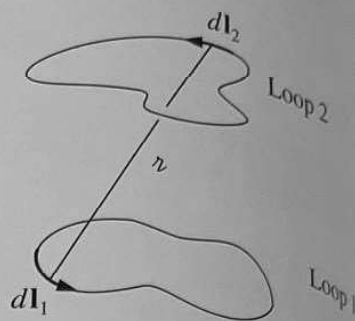


FIGURE 7.31

through loop 2; let Φ_2 be the flux of $\mathbf{B}_1$ through 2. You might have a tough time actually calculating $\mathbf{B}_1$, but a glance at the Biot-Savart law,

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}}{r^2},$$

reveals one significant fact about this field: *It is proportional to the current I_1 .* Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

Thus

$$\Phi_2 = M_{21} I_1,$$

(7.22)

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a cute formula for the mutual inductance, which you can derive by expressing the flux in terms of the vector potential, and invoking Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

Now, according to Eq. 5.66,

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{r},$$

and hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{r} \right) \cdot d\mathbf{l}_2.$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}.$$

(7.23)

This is the **Neumann formula**; it involves a double line integral—one integration around loop 1, the other around loop 2 (Fig. 7.31). It's not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.
2. The integral in Eq. 7.23 is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12}. \quad (7.24)$$

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2.* We may as well drop the subscripts and call them both M .

Example 7.10. A short solenoid (length l and radius a , with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b , n_2 turns per unit length) as shown in Fig. 7.32. Current I flows in the short solenoid. What is the flux through the long solenoid?

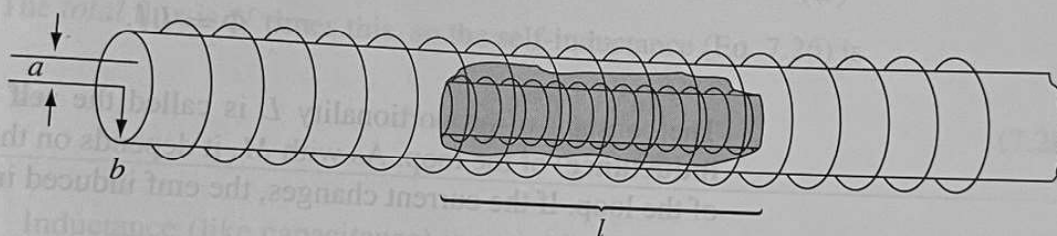


FIGURE 7.32

Solution

Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different flux through each turn of the outer solenoid. It would be a *miserable* task to compute the total flux this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current I through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

(Eq. 5.59), so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2.$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I.$$

This is also the flux a current I in the *short* solenoid would put through the *long* one, which is what we set out to find. Incidentally, the mutual inductance, in this case, is

$$M = \mu_0 \pi a^2 n_1 n_2 l.$$

Suppose, now, that you vary the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}. \quad (7.25)$$

(In quoting Eq. 7.22—which was based on the Biot-Savart law—I am tacitly assuming that the currents change slowly enough for the system to be considered quasistatic.) What a remarkable thing: Every time you change the current in loop 1, an induced current flows in loop 2—even though there are no wires connecting them!

Come to think of it, a changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop *itself* (Fig 7.33). Once again, the field (and therefore also the flux) is proportional to the current:

$$\Phi = LI. \quad (7.26)$$

The constant of proportionality L is called the **self inductance** (or simply the **inductance**) of the loop. As with M , it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in the loop is

$$\mathcal{E} = -L \frac{dI}{dt}. \quad (7.27)$$

Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

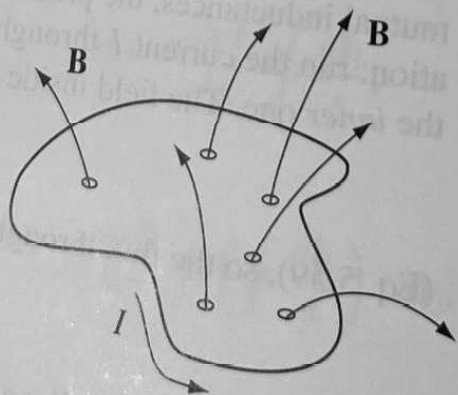


FIGURE 7.33

Example 7.11. Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a , outer radius b , height h), that carries a total of N turns.

Solution

The magnetic field inside the toroid is (Eq. 5.60)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

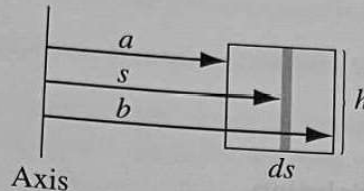


FIGURE 7.34

The flux through a single turn (Fig. 7.34) is

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s} ds = \frac{\mu_0 N I h}{2\pi} \ln \left(\frac{b}{a} \right).$$

The *total* flux is N times this, so the self-inductance (Eq. 7.26) is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \left(\frac{b}{a} \right). \quad (7.28)$$

Inductance (like capacitance) is an intrinsically *positive* quantity. Lenz's law, which is enforced by the minus sign in Eq. 7.27, dictates that the emf is in such a direction as to *oppose* any *change in current*. For this reason, it is called a **back emf**. Whenever you try to alter the current in a wire, you must fight against this back emf. Inductance plays somewhat the same role in electric circuits that *mass* plays in mechanical systems: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

Example 7.12. Suppose a current I is flowing around a loop, when someone suddenly cuts the wire. The current drops “instantaneously” to zero. This generates a whopping back emf, for although I may be small, dI/dt is enormous. (That's why you sometimes draw a spark when you unplug an iron or toaster—electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.)

Nothing so dramatic occurs when you plug *in* a toaster or iron. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and

continuous buildup. Suppose, for instance, that a battery (which supplies a constant emf $\mathcal{E}_0$) is connected to a circuit of resistance R and inductance L (Fig. 7.35). What current flows?

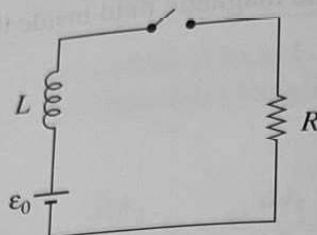


FIGURE 7.35

Solution

The total emf in this circuit is $\mathcal{E}_0$ from the battery plus $-L(dI/dt)$ from the inductance. Ohm's law, then, says¹⁷

$$\mathcal{E}_0 - L \frac{dI}{dt} = IR.$$

This is a first-order differential equation for I as a function of time. The general solution, as you can show for yourself, is

$$I(t) = \frac{\mathcal{E}_0}{R} + ke^{-(R/L)t},$$

where k is a constant to be determined by the initial conditions. In particular, if you close the switch at time $t = 0$, so $I(0) = 0$, then $k = -\mathcal{E}_0/R$, and

$$I(t) = \frac{\mathcal{E}_0}{R} [1 - e^{-(R/L)t}]. \quad (7.29)$$

This function is plotted in Fig. 7.36. Had there been no inductance in the circuit, the current would have jumped immediately to $\mathcal{E}_0/R$. In practice, *every* circuit has *some* self-inductance, and the current approaches $\mathcal{E}_0/R$ asymptotically. The quantity $\tau \equiv L/R$ is the **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly two-thirds) of its final value.

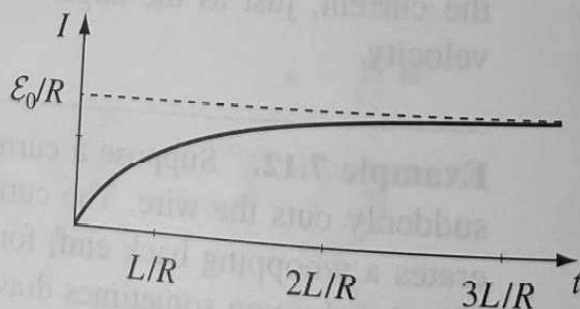


FIGURE 7.36

¹⁷Notice that $-L(dI/dt)$ goes on the *left* side of the equation—it is part of the emf that establishes the voltage across the resistor.

Problem 7.22 A small loop of wire (radius a) is held a distance z above the center of a large loop (radius b), as shown in Fig. 7.37. The planes of the two loops are parallel, and perpendicular to the common axis.

- Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)
- Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)
- Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

Problem 7.23 A square loop of wire, of side a , lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

Problem 7.24 Find the self-inductance per unit length of a long solenoid, of radius R , carrying n turns per unit length.

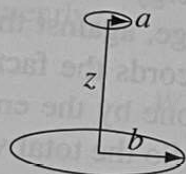


FIGURE 7.37

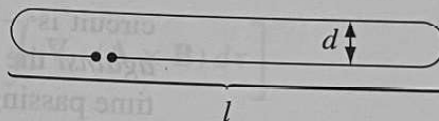


FIGURE 7.38

Problem 7.25 Try to compute the self-inductance of the “hairpin” loop shown in Fig. 7.38. (Neglect the contribution from the ends; most of the flux comes from the long straight section.) You’ll run into a snag that is characteristic of many self-inductance calculations. To get a definite answer, assume the wire has a tiny radius ϵ , and ignore any flux through the wire itself.

Problem 7.26 An alternating current $I(t) = I_0 \cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.

- In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_R(t)$, in the resistor.
- Calculate the back emf in the coil, due to the current $I_R(t)$. What is the ratio of the amplitudes of this back emf and the “direct” emf in (a)?

Problem 7.27 A capacitor C is charged up to a voltage V and connected to an inductor L , as shown schematically in Fig. 7.39. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L ?