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ADVANCED ENGINEERING MATHEMATICS

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CHAPTER

7 LINEAR ALGEBRA PART II: MATRICES AND DETERMINANTS

A matrix is a rectangular array of numbers. Such arrays occur in various branches of applied mathematics. In many cases they form the coefficients of linear transformations (examples in Sec. 7.1) or systems of linear equations (simultaneous linear equations) arising, for instance, from electrical networks, frameworks in mechanics, curve fitting in statistics, and transportation problems. Matrices are useful because they enable us to consider an array of many numbers as a single object, denote it by a single symbol, and perform calculations with these symbols in a very compact form. The "mathematical shorthand" thus obtained is very elegant and powerful and is suitable for various practical problems. It entered engineering mathematics over fifty years ago and is of increasing importance in various branches.

In this chapter we first introduce matrices and related concepts (Sec. 7.1), define the algebraic operations for matrices (Secs. 7.2–7.4), and consider systems of linear equations (solution by elimination in Sec. 7.5, existence in Sec. 7.7, solution by Cramer's rule in Sec. 7.11), determinants (Secs. 7.9 and 7.10), bilinear and other forms (Sec. 7.12), matrix eigenvalue problems (Secs. 7.13 and 7.14) and systems of differential equations (Sec. 7.15). Other important concepts in this chapter are the rank of a matrix (Secs. 7.6 and 7.11) and the inverse (Sec. 7.8).

Numerical methods are presented in Secs. 19.9–19.14 which are independent of the other sections in Chap. 19, so that they can be studied immediately after the corresponding material in the present chapter.

Prerequisite for this chapter: Chap. 6.
Sections which may be omitted in a shorter course: 7.12–7.15.
References: Appendix 1, Part C.
Answers to problems: Appendix 2.

7.1 Basic Concepts

Matrices are rectangular arrays of numbers enclosed in parentheses; for instance,

$$\begin{pmatrix} 2 & 0.4 & 8 \\ 5 & -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad (a_1 \quad a_2 \quad a_3).$$

GOGO EX2010, Page 3 of 6

The definition of a matrix will be given below, and rules of calculations with matrices will be defined and explained in the next sections.

Matrices occur in connection with linear transformations or systems of linear equations. For example, consider

$$\begin{aligned} 20x_1 - 3x_2 &= y_1 \\ 0.5x_1 + 7x_2 &= y_2 \end{aligned} \tag{1}$$

If y_1 and y_2 are given numbers, this is a system of two linear equations in two unknowns x_1, x_2. The coefficients are 20, -3, 0.5, 7 and, to recognize their position we may arrange them in the way they occur in the equations, that is, in the form of a matrix,

$$\begin{pmatrix} 20 & -3 \\ 0.5 & 7 \end{pmatrix}. \tag{2}$$

If x_1, x_2 and y_1, y_2 are variable quantities (for instance, coordinates corresponding to two different coordinate systems in the plane), then our pair of formulas (1) represents a linear transformation. This transformation is completely determined by its four coefficients, that is, by the matrix (2). We may denote the matrix by a single letter, for instance **A**, and write it in the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} = 20$, $a_{12} = -3$, $a_{21} = 0.5$, $a_{22} = 7$. The notation has the advantage that we can immediately see the position of a coefficient a_{jk} because the first subscript is the number of the row (horizontal line) and the second is the number of the column (vertical line) of a_{jk}. Let us now give the general definition of a matrix. (Rules of calculations with matrices will be defined and explained in the next sections.)

A rectangular array of (real or complex) numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a **matrix.** The numbers $a_{11}, \cdots, a_{mn}$ are called the **elements** of the matrix. The horizontal lines are called **rows** or *row vectors,* and the vertical lines are called **columns** or *column vectors* of the matrix. A matrix with m rows and n columns is called an $(m \times n)$ *matrix* (read "m by n matrix").

Matrices will be denoted by capital (upper case) bold-faced letters **A, B,** etc., or by (a_{jk}), (b_{ik}), etc., that is, by writing the general element of the matrix, enclosed in parentheses.

GOGO EX2010, Page 4 of 6

In the **double-subscript notation** *for the elements, the first subscript always denotes the row and the second subscript the column containing the given element.*

A matrix

$$(a_1 \quad a_2 \quad \cdots \quad a_n)$$

having only one row is called a **row matrix** or *row vector*. A matrix

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

having only one column is called a **column matrix** or *column vector*. Row and column matrices will be denoted by small (lower case) bold-faced letters. This agrees with our notation for vectors (Chap. 6), whose components may be written in the form of a row matrix or a column matrix.

The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, the matrix is called a **real matrix.**

A matrix having the same number of rows and columns is called a **square matrix,** and the number of rows is called its **order.** A square matrix of order n is also called an $(n \times n)$ *matrix* (read "n by n matrix"). The diagonal containing the elements $a_{11}, a_{22}, \cdots, a_{nn}$ of such a matrix is called the **principal diagonal.** Square matrices are of particular importance.

A definition of **equality of matrices** is suggested by that for vectors in Chap. 6 (cf. Sec. 6.2, at the end). Two matrices $\mathbf{A} = (a_{jk})$ and $\mathbf{B} = (b_{jk})$ are *equal* if and only if $\mathbf{A}$ and $\mathbf{B}$ have the same number of rows and the same number of columns and corresponding elements are equal, that is,

$$a_{jk} = b_{jk} \qquad \text{for all occurring } j \text{ and } k.$$

Then we write

$$\mathbf{A} = \mathbf{B}.$$

Any matrix obtained by omitting some rows and columns from a given $(m \times n)$ matrix $\mathbf{A}$ is called a **submatrix** of $\mathbf{A}$.

As a matter of convention and convenience the notion "submatrix" includes $\mathbf{A}$ itself (as the matrix obtained from $\mathbf{A}$ by omitting no rows or columns).

Example 1. Submatrices of a matrix

The matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

contains three (2×2) submatrices, namely,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}, \quad \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

GOGO EX2010, Page 5 of 6

two (1×3) submatrices (the two row vectors), three (2×1) submatrices (the column vectors), six (1×2) submatrices, namely,

$$(a_{11} \quad a_{12}), \quad (a_{11} \quad a_{13}), \quad (a_{12} \quad a_{13}),$$

$$(a_{21} \quad a_{22}), \quad (a_{21} \quad a_{23}), \quad (a_{22} \quad a_{23}),$$

and six (1×1) submatrices, (a_{11}), (a_{12}), $\cdots$, (a_{23}).

7.2 Addition of Matrices, Multiplication by Numbers

Since we want to do calculations with matrices, we shall now introduce algebraic operations for matrices. We shall define two such operations, which are called *addition of matrices* and *multiplication of matrices by numbers*. These definitions are motivated and suggested by various applications (to be considered later in this chapter) as well as by the corresponding definitions for vectors in Sec. 6.3, which they are supposed to include as special cases.

Addition of matrices is defined only for matrices having the same number of rows and the same number of columns, and is then defined as follows. The **sum** of two $(m \times n)$ matrices $\mathbf{A} = (a_{jk})$ and $\mathbf{B} = (b_{jk})$ is the $(m \times n)$ matrix $\mathbf{C} = (c_{jk})$ with elements

$$\textbf{(1)} \qquad c_{jk} = a_{jk} + b_{jk} \qquad \begin{matrix} j = 1, \cdots, m \\ k = 1, \cdots, n \end{matrix}$$

and is written

$$\mathbf{C} = \mathbf{A} + \mathbf{B}.$$

We see that $\mathbf{A} + \mathbf{B}$ is obtained by adding corresponding elements of $\mathbf{A}$ and $\mathbf{B}$.

Example 1. Addition of matrices
If

$$\mathbf{A} = \begin{pmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix}, \quad \text{then} \quad \mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{pmatrix}. \quad \blacksquare$$

We define the $(m \times n)$ **zero matrix** $\mathbf{0}$ to be the $(m \times n)$ matrix with all elements zeros. Then we see from the definition that matrix addition enjoys properties which are quite similar to those of the addition of real numbers,

$$\textbf{(2)} \qquad \begin{aligned} &\text{(a)} && \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\ &\text{(b)} && (\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W}) \qquad \text{(written } \mathbf{U} + \mathbf{V} + \mathbf{W}\text{)} \\ &\text{(c)} && \mathbf{A} + \mathbf{0} = \mathbf{A} \\ &\text{(d)} && \mathbf{A} + (-\mathbf{A}) = \mathbf{0} \end{aligned}$$

GOGO EX2010, Page 6 of 6