Michael Möser

# Engineering Acoustics

## An Introduction to Noise Control

Springer

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Mojawa v. Shokz, IPR2025-00843

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Translated by S. Zimmermann and R. Ellis

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Dedicated to my daughter Sarah

# Preface

This book presents the English translation of the author's German edition 'Technische Akustik' (the 8th edition published by Springer Verlag in 2009). In whatever language, 'Engineering Acoustics' sees itself as a teaching textbook that could serve as a tool for autodidactic studies and as a compendium of lectures and courses as well. Readers are addressed who already possess a certain training in physical and mathematical thinking and in expressing ideas and explanations using mathematical formulas. On the other hand no highly specified knowledge is vital: readers with no more than the usual skills – like taking derivatives and solving simple integrals – are assumed. The appendix gives a short introduction on the use of complex amplitudes in acoustics and the reasons for their use. It is in general one of the author's most important aims not only to describe *how* the topic and its description develops but also *why* a specific way is chosen. Often difficulties in understanding do not consist in comprehending the single steps but in the question why they are done in that - and in no other way.

Moreover the explanations do not restrict themselves to the mathematical formulas. No doubt that formulas give the most unambiguous description of matters, and they show problems and their solutions in quantity also, but more remains to be done. Only the illustrative explanation relying on the reader's imagination produces understanding and comprehension. Textbooks should make learning – often difficult enough – as easy as possible, and this certainly does not imply to reduce the level.

In many respects this book is obliged to Lothar Cremer. For example, parts of the author's own knowledge originate from Cremer's very first 'Vorlesungen über Technische Akustik'. Important discoveries of Cremer are included in this new edition and it's translations. Examples are, the optimum impedance for mufflers and the coincidence effect which leads to a satisfying explanation for sound transmission through walls; perhaps Cremer's most important discovery.

This book tries to present the foundations of that what nowadays seems necessary to make our environment quieter. All chapters between 'elastic iso-

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lation' – the 5th – and 'diffraction' – the 10th – directly or indirectly address the question, how to reduce the sound level in the most important environs of everyday life indoors and outdoors – in buildings and in the open air. This requires the understanding of some principal features first. To fully comprehend the physics of sound transmission through walls for example, implies the understanding of bending wave propagation on plates. Because of that reason chapters on 'the media' precede the chapters on the noise reduction methods. The (short) chapter on sound perception serves as an introduction. The last chapter deals with the most important receiving and source instruments: microphones and loudspeakers. Specific measurement procedures are already discussed in many other chapters. The chapter 'absorption' for example begins with a discussion of how to measure the absorption coefficient.

The translation of this book was done by Stefan Zimmermann and Rebecca Ellis. The cooperation with them was interesting, satisfying, and excellent. Many thanks to them for all their efforts and patience with me.

Berlin, February 2009 *Michael Möser*

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

# Contents


**1 Perception of sound** . . . . . 1
- 1.1 Octave and third-octave band filters . . . . . 7
- 1.2 Hearing levels . . . . . 9
- 1.3 A-Weighting . . . . . 10
- 1.4 Noise fluctuating over time . . . . . 12
- 1.5 Summary . . . . . 13
- 1.6 Further reading . . . . . 13
- 1.7 Practice exercises . . . . . 14

**2 Fundamentals of wave propagation** . . . . . 17
- 2.1 Thermodynamics of sound fields in gases . . . . . 18
- 2.2 One-dimensional sound fields . . . . . 24
  - 2.2.1 Basic equations . . . . . 24
  - 2.2.2 Progressive waves . . . . . 29
  - 2.2.3 Complex notation . . . . . 34
  - 2.2.4 Standing waves and resonance phenomena . . . . . 34
- 2.3 Three-dimensional sound fields . . . . . 38
- 2.4 Energy and power transport . . . . . 40
- 2.5 Intensity measurements . . . . . 44
  - 2.5.1 Time domain . . . . . 45
  - 2.5.2 Frequency domain . . . . . 45
  - 2.5.3 Measurement error and limitations . . . . . 49
  - 2.5.4 Standards . . . . . 53
- 2.6 Wave propagation in a moving medium . . . . . 53
- 2.7 Raised waves . . . . . 59
- 2.8 Summary . . . . . 62
- 2.9 Further reading . . . . . 63
- 2.10 Practice exercises . . . . . 63



Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

**3 Propagation and radiation of sound** . . . . . 67
3.1 Omnidirectional sound radiation of point sources . . . . . 67
3.2 Omnidirectional sound radiation of line sources . . . . . 68
3.3 Volume velocity sources . . . . . 70
3.4 Sound field of two sources . . . . . 72
3.5 Loudspeaker arrays . . . . . 85
3.5.1 One-dimensional piston . . . . . 87
3.5.2 Formation of main and side lobes . . . . . 91
3.5.3 Electronic beam steering . . . . . 96
3.5.4 Far-field conditions . . . . . 100
3.6 Sound radiation from plane surfaces . . . . . 103
3.6.1 Sound field on the axis of a circular piston . . . . . 106
3.7 Summary . . . . . 110
3.8 Further reading . . . . . 111
3.9 Practice exercises . . . . . 111

**4 Structure-borne sound** . . . . . 117
4.1 Introduction . . . . . 117
4.2 Bending waves in beams . . . . . 120
4.3 Propagation of bending waves . . . . . 124
4.4 Beam resonances . . . . . 126
4.4.1 Supported beams . . . . . 127
4.4.2 Bilaterally mounted beams . . . . . 129
4.4.3 Bilaterally suspended beams . . . . . 132
4.5 Bending waves in plates . . . . . 133
4.5.1 Plate vibrations . . . . . 136
4.6 Summary . . . . . 140
4.7 Further reading . . . . . 141
4.8 Practice exercises . . . . . 141

**5 Elastic isolation** . . . . . 143
5.1 Elastic bearings on rigid foundations . . . . . 145
5.2 Designing elastic bearings . . . . . 150
5.3 Influence of foundations with a compliance . . . . . 153
5.3.1 Foundation impedance . . . . . 153
5.3.2 The effect of foundation impedance . . . . . 154
5.4 Determining the transfer path . . . . . 161
5.5 Determining the loss factor . . . . . 162
5.6 Dynamic mass . . . . . 164
5.7 Conclusion . . . . . 166
5.8 Summary . . . . . 167
5.9 Further reading . . . . . 167
5.10 Practice exercises . . . . . 167

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

**6 Sound absorbers** . . . 171
6.1 Sound propagation in the impedance tube . . . 171
6.1.1 Tubes with rectangular cross sections . . . 176
6.1.2 Tubes with circular cross sections . . . 176
6.2 Measurements in the impedance tube . . . 178
6.2.1 Mini-max procedure . . . 180
6.3 Wall impedance . . . 183
6.4 Theory of locally reacting absorbers . . . 186
6.5 Specific absorbent structures . . . 191
6.5.1 The 'infinitely thick' porous sheet . . . 191
6.5.2 The porous sheet of finite thickness . . . 194
6.5.3 The porous curtain . . . 198
6.5.4 Resonance absorbers . . . 201
6.6 Oblique sound incidence . . . 208
6.7 Summary . . . 211
6.8 Further reading . . . 212
6.9 Practice exercises . . . 213

**7 Fundamentals of room acoustics** . . . 217
7.1 Diffuse sound field . . . 221
7.1.1 Reverberation . . . 224
7.1.2 Steady-state conditions . . . 226
7.1.3 Measurement of the absorption coefficient in the reverberation room . . . 231
7.2 Summary . . . 232
7.3 Further reading . . . 233
7.4 Practice exercises . . . 233

**8 Building acoustics** . . . 237
8.1 Measurement of airborne transmission loss . . . 239
8.2 Airborne transmission loss of single-leaf partitions . . . 241
8.3 Double-leaf partitions (flexible additional linings) . . . 252
8.4 Impact sound reduction . . . 259
8.4.1 Measuring impact sound levels . . . 259
8.4.2 Improvements . . . 260
8.5 Summary . . . 263
8.6 Further reading . . . 264
8.7 Practice exercises . . . 265

**9 Silencers** . . . 267
9.1 Changes in the cross-section of rigid ducts . . . 268
9.1.1 Abrupt change in cross-section . . . 268
9.1.2 Duct junctions . . . 270
9.1.3 Expansion chambers . . . 275
9.1.4 Chamber combinations . . . 279

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

9.2 Lined ducts . . . 284
9.2.1 Ducts with rigid walls . . . 286
9.2.2 Ducts with soft boundaries . . . 288
9.2.3 Silencers with arbitrary impedance boundaries . . . 290
9.3 Summary . . . 308
9.4 Further reading . . . 308
9.5 Practice exercises . . . 308

**10 Noise barriers** . . . 311
10.1 Diffraction by a rigid screen . . . 312
10.2 Approximation of insertion loss . . . 329
10.3 The importance of height in noise barriers . . . 332
10.4 Sound barriers . . . 333
10.5 Absorbent noise barriers . . . 335
10.6 Transmission through the barrier . . . 338
10.7 Conclusion . . . 339
10.8 Summary . . . 339
10.9 Further reading . . . 339
10.10 Practice exercises . . . 340
10.11 Appendix: MATLAB program for Fresnel integrals . . . 342

**11 Electro-acoustic converters for airborne sound** . . . 345
11.1 Condenser microphones . . . 347
11.2 Microphone directivity . . . 355
11.3 Electrodynamic microphones . . . 358
11.4 Electrodynamic loudspeakers . . . 362
11.5 Acoustic antennae . . . 365
11.5.1 Microphone arrays . . . 367
11.5.2 Two-dimensional sensor arrangements . . . 374
11.6 Summary . . . 380
11.7 Further reading . . . 380
11.8 Practice exercises . . . 381

**12 Fundamentals of Active Noise Control** . . . 383
12.1 The Influence of Replication Errors . . . 387
12.1.1 Perpendicularly Interfering Waves . . . 389
12.2 Reflection and Absorption . . . 390
12.3 Active Stabilization of Self-Induced Vibrations . . . 395
12.4 Summary . . . 403
12.5 Further Reading . . . 404
12.6 Practice Exercises . . . 404

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

**13 Aspects and Properties of Transmitters** . . . . . 407
13.1 Properties of Transmitters . . . . . 408
13.1.1 Linearity . . . . . 408
13.1.2 Time Invariance . . . . . 409
13.2 Description using the Impulse Response . . . . . 410
13.3 The Invariance Principle . . . . . 414
13.4 Fourier Decomposition . . . . . 415
13.4.1 Fourier Series . . . . . 416
13.4.2 Fourier Transform . . . . . 424
13.4.3 The Transmission Function and the Convolution Law . . 426
13.4.4 Symmetries . . . . . 428
13.4.5 Impulse Responses and Hilbert Transformation . . . . . 430
13.5 Fourier Acoustics: Wavelength Decomposition . . . . . 432
13.5.1 Radiation from Planes . . . . . 434
13.5.2 Emission of Bending Waves . . . . . 436
13.5.3 Acoustic Holography . . . . . 438
13.5.4 Three-dimensional Sound Fields . . . . . 439
13.6 Summary . . . . . 442
13.7 Further Reading . . . . . 443
13.8 Practice Exercises . . . . . 444

**A Level Arithmetics** . . . . . 449
A.1 Decadic Logarithm . . . . . 449
A.2 Level Inversion . . . . . 450
A.3 Level Summation . . . . . 451

**B Complex Pointers** . . . . . 453
B.1 Introduction to Complex Pointer Arithmetics . . . . . 453
B.2 Using Complex Pointers in Acoustics . . . . . 455

**C Solutions to the Practice Exercises** . . . . . 459
C.1 Practice Exercises Chapter 1 . . . . . 459
C.2 Practice Exercises Chapter 2 . . . . . 463
C.3 Practice Exercises Chapter 3 . . . . . 468
C.4 Practice Exercises Chapter 4 . . . . . 480
C.5 Practice Exercises Chapter 5 . . . . . 486
C.6 Practice Exercises Chapter 6 . . . . . 489
C.7 Practice Exercises Chapter 7 . . . . . 497
C.8 Practice Exercises Chapter 8 . . . . . 501
C.9 Practice Exercises Chapter 9 . . . . . 503
C.10 Practice Exercises Chapter 10 . . . . . 505
C.11 Practice Exercises Chapter 11 . . . . . 508
C.12 Practice Exercises Chapter 12 . . . . . 510
C.13 Practice Exercises Chapter 13 . . . . . 514
**Index** . . . . . 527

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

# 1

# Perception of sound

The perception of sound incidents requires the presence of some simple physical effects. A sound source oscillates and brings the surrounding air into motion. The compressability and mass of the air cause these oscillations to be transmitted to the listener's ear.

Small pressure fluctuations, referred to as sound pressure $p$, occur in air (or gas or fluid) and which are superimposed to the atmospheric pressure $p_0$. A spatially distributed sound field radiates from the source with different instantaneous sound pressures at each moment. The sound pressure is the most important quantity to describe sound fields and os always space- and time-dependent.

The observed sound incident at a point has two main distinguishing attributes: 'timbre' and 'loudness'. The physical quantity for loudness is sound pressure and the quantity for timbre is frequency $f$, measured in cycles per second, or Hertz (Hz). The frequency range of technical interest covers more than the range that is audible by the human ear, which is referred to as hearing level. The hearing range starts at about 16 Hz and ranges up to 16000 Hz (or 16 kHz). The infrasound, which is located below that frequency range, is less important for air-borne sound problems, but becomes relevant when dealing with vibrations of structures (e.g. in vibration control of machinery). Ultrasound begins above the audible frequency range. It is used in applications ranging from acoustic modelling techniques to medical diagnosis and non-destructive material testing.

The boundaries of the audible frequency range dealt with in this book cannot be defined precisely. The upper limit varies individually, depending on factors like age, and also in cases of extensive workplace noise exposure or the misuse of musical devices. The value of 16 kHz refers to a healthy, human being who is about 20 years old. With increasing age, the upper limit decreases by about 1 kHz per decade.

The lower limit is likewise not easy to define and corresponds to flickering. At very low frequencies a series of single sound incidents (e.g. a series of impulses) can be distinguished as well. If the frequency increases above the

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Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

flickering frequency of (about) 16 Hz, single incidents are no longer perceived individually, but seem to merge into a single noise. This transition can be found, for example, when it slowly starts to rain: the knocking of single rain drops at the windows can be heard until the noise at a certain density of rain merges into a continuous crackling. Note that the audible limit for the perception of flickering occurs at the same frequency at which a series of single images in a film start to appear as continuous motion.

The term 'frequency' in acoustics is bound to pure tones, meaning a sinusoidal wave form in the time-domain. Such a mathematically well-defined incident can only rarely be observed in natural sound incidents. Even the sound of a musical instrument contains several colourations: the superposition of several harmonic (pure) tones produces the typical sound of the instrument (see Fig. 1.1 for examples). An arbitrary wave form can generally be

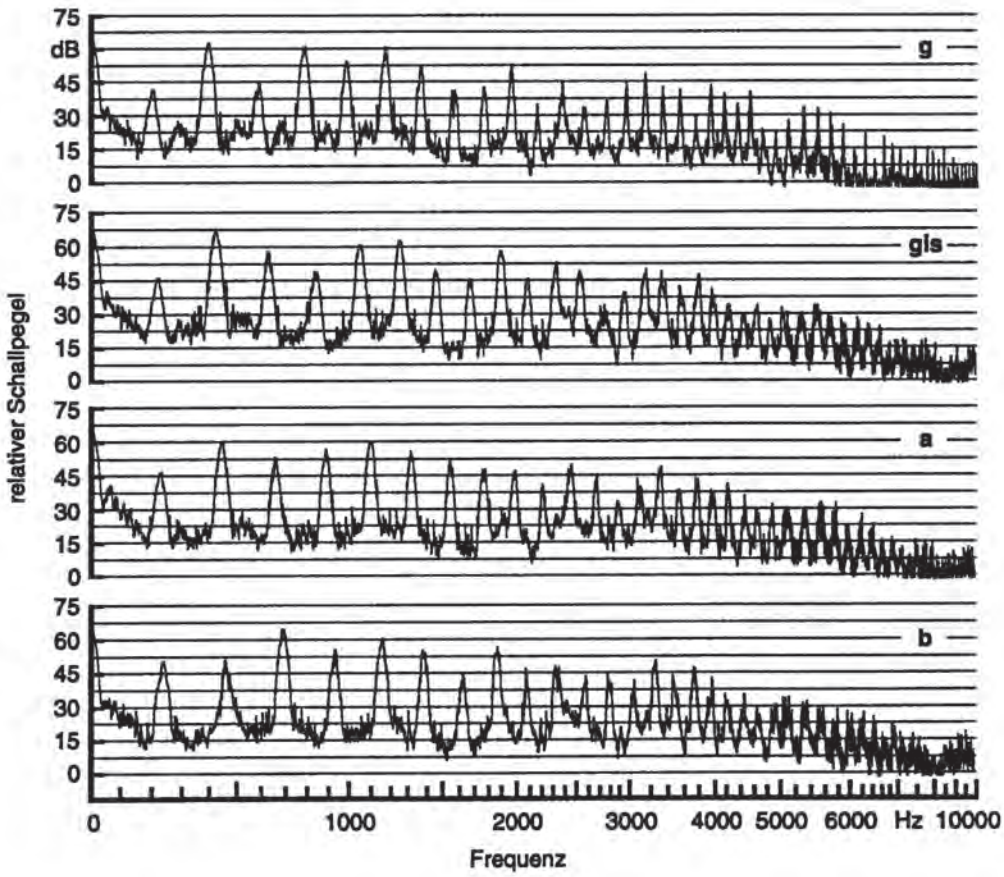


**Fig. 1.1.** Sound spectra of a violin played at different notes (from: Meyer, J.: "Akustik und musikalische Aufführungspraxis". Verlag Erwin Bochinsky, Frankfurt 1995). Relative sound pressure level versus frequency.

represented by its frequency components extracted through spectrum analy-

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

sis, similar to the analysis of light. Arbitrary signals can be represented by a sum of harmonics (with different amplitudes and frequencies). The association of decomposed time signals directly leads to the representation of the acoustic properties of transducers by their frequency response functions (as, for example, those of walls and ceilings in building acoustics, see Chap. 8). If, for instance, the frequency-dependent transmission loss of a wall is known, it is easy to imagine how it reacts to the transmission of certain sound incidents like, for example, speech. The transmission loss is nearly always bad at low frequencies and good at high frequencies: speech is therefore not only transmitted 'quieter' but also 'dull' through the wall. The more intuitive association that arbitrary signals can be represented by their harmonics will be sufficient throughout this book in the aforementioned hypothetical case. The expansion of a given signal into a series of harmonics, the so-called Fourier series and Fourier integrals, is based on a solid mathematical foundation of proof (see the last chapter of this book).

The subjective human impression of the sound pitch is perceived in such a way that a tonal difference of two pairs of tones is perceived equally if the ratio (and not the difference) of the two frequency pairs is equal. The tonal difference between the pair made of $f_{\mathrm{a1}}$ and $f_{\mathrm{a2}}$ and the pair made of $f_{\mathrm{b1}}$ and $f_{\mathrm{b2}}$ is perceived equally if the ratio

$$\frac{f_{\mathrm{a1}}}{f_{\mathrm{a2}}} = \frac{f_{\mathrm{b1}}}{f_{\mathrm{b2}}}$$

is valid. The transition from 100 Hz to 125 Hz and from 1000 Hz to 1250 Hz is, for example, perceived as an equal change in pitch. This law of 'relative tonal impression' is reflected in the subdivision of the scale into octaves (a doubling in frequency) and other intervals like second, third, fourth and fifth, etc. used for a long time in music. All of these stand for the ratio in frequency and not for the 'absolute increase in Hz'.

This law of 'tonal impression,' which more generally means that a stimulus $R$ has to be increased by a certain percentage to be perceived as an equal change in perception, is not restricted to the tonal impression of the human being. It is true for other human senses as well. In 1834, Weber conducted experiments using weights in 1834 and found that the difference between two masses laid on the hand of a test subject was only perceived equally, when a mass of 14 g was increased by 1 g and a mass of 28 g was increased by 2 g. This experiment and the aforementioned tonal perception leads to the assumption that the increment of a perception $\Delta E$ for these and other physical stimuli is proportional to the ratio of the absolute increase of the stimulus $\Delta R$ and the stimulus $R$

$$\Delta E = k\frac{\Delta R}{R}, \tag{1.1}$$

where $k$ is a proportionality constant. For the perception of pitch the stimulus $R = f$ represents the frequency, for the perception of weight $R = m$ represents the mass on the hand.

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This law of relative variation (1.1) is also true for the perception of loudness. If a test subject is repeatedly presented with sound incidents consisting of pairs of sound pressure incidents $p$ and $2p$ and $5p$ and $10p$ respectively, the perceived difference in loudness should be equal. The perception of both pitch and loudness should at least roughly follow the law of relative variation (1.1).

As mentioned earlier (1.1), there is a relativity law which governs variations in stimulus $\Delta R$ and in perception $\Delta E$. It is, of course, also interesting to examine the relation between $R$ and $E$. Given that it is, at best, problematic, if not presumably impossible, to quantify perceptions, the principal characteristics of the $E(R)$ function should be clarified. These 'perception characteristics' are easily constructed from the variation law, if two points of stimulus $R$ and perception $E$ are chosen as shown in Fig. 1.2. A threshold stimulus $R_0$ is defined, at which the perception starts: stimuli $R < R_0$ below the threshold are not perceivable. A minimal stimulus is needed to achieve perception at all. The second point is chosen arbitrarily to be twice the threshold $R = 2R_0$ and the (arbitrary) perception $E_0$ is assigned. The further characteristics re-

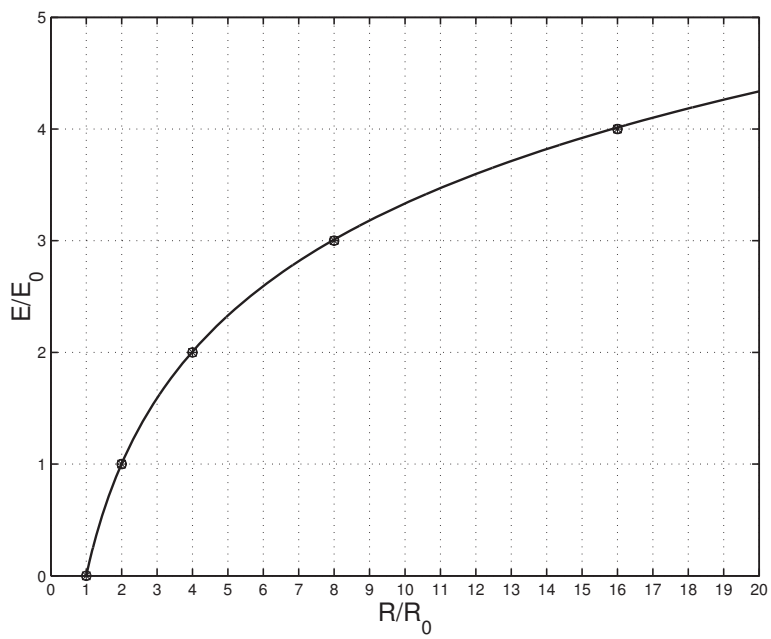


**Fig. 1.2.** Qualitative relation between stimulus $R$ and perception $E$

sult from examining the perceptions $2E_0$, $3E_0$, $4E_0$, etc. The perception $2E_0$ is assigned twice the stimulus of $E_0$, therefore related to $R = 4R_0$. Just as $E = 3E_0$ is related to the stimulus $R = 8R_0$ the perception $4E_0$ is related to $R = 16R_0$, etc. As can be seen from Fig. 1.2 the gradient of the curve $E = E(R)$ decreases with increasing stimulus $R$. The greater the perception, the greater the increase of the stimulus has to be to achieve another increment of perception (for example $E_0$).

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Mojawa v. Shokz, IPR2025-00843

The functional relation $E = E(R)$ can certainly be determined from the variation law (1.1) by moving towards infinitesimal small variations $\mathrm{d}E$ and $\mathrm{d}R$:

$$\mathrm{d}E = k\frac{\mathrm{d}R}{R} \ .$$

Integration yields

$$E = 2.3k \lg(R/R_0). \tag{1.2}$$

Bare in mind that the logarithms of different bases are proportional, e.g. $\ln x = 2.3 \lg x$. The perception of loudness is therefore proportional to the logarithm of the physical stimulus –in this case, sound pressure. This relation, validated at least roughly by numerous investigations, is also known as the Weber-Fechner-Law.

The sensual perception according to a logarithmic law (for the characteristics see Fig. 1.2 again) is a very sensible development of the 'human species'. Stimuli close to the threshold $R = R_0$ are emphasized and therefore 'well perceivable', whereas very large stimuli are highly attenuated in their perception; the logarithmic characteristics act as a sort of 'overload protection'. A wide range of physical values can thus be experienced (without pain) and several decades of physical orders of magnitude are covered. The history of the species shows that those perceptions necessary to survive in the given environment, which also cover a wide range of physical values, follow the Weber-Fechner-Law. This is not true for the comparatively smaller range of temperature perception. Variations of a tenth or a hundredth of a degree are by no means of interest to the individual. In contrast, the perception of light needs to cover several decades of order of magnitude. Surviving in the darkest night is as important as the ability to see in the sunlight of a very bright day. And the perception of weight covers a range starting from smallest masses of about 1 g up to loads of several 10000 g. The perception of loudness follows the logarithmic Weber-Fechner-Law, because the human ear is facing the problem of perceiving very quiet sounds, like the falling of the leaves in quiet surroundings, as well as very loud sounds, like the roaring sound of a waterfall in close vicinity. As a matter of fact, humans are able to perceive sound pressures in the range of $20 \cdot 10^{-6} N/m^2$ to approximately $200\,\mathrm{N/m^2}$, where the upper limit roughly depicts the pain threshold. About ten decades of loudness are covered, which represents an exceedingly large physical interval. To illustrate, this range in equivalent distances would cover an interval between 1 mm and 10 km. The amazing ear is able to perceive this range. Imagine the impossibility of an optical instrument (like a magnifying glass), to be able to operate in the millimeter range as well as in the kilometer range!

When technically quantifying sound pressure, it is more handy to use a logarithmic measure instead of the physical sound pressure itself to represent this wide range. The sound pressure level $L$ is internationally defined as

$$L = 20 \lg \left(\frac{p}{p_0}\right) = 10 \lg \left(\frac{p}{p_0}\right)^2 , \tag{1.3}$$

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Mojawa v. Shokz, IPR2025-00843

with $p_0 = 20\,10^{-6}\,\mathrm{N/m^2}$, as an expressive and easy to use measure. The reference value $p_0$ roughly corresponds to the hearing threshold (at a frequency of 1 kHz, because the hearing threshold is frequency-dependent, as will be shown in the next section), so that 0 dB denotes the 'just perceivable' or 'just not perceivable' sound event. If not otherwise stated, the sound pressure $p$ stands for the root mean square (rms-value) of the time domain signal. The specification in decibels (dB) is not related to a specific unit. It indicates the use of the logarithmic law. The factor 20 (or 10) in (1.3) is chosen in such a way that 1 dB corresponds to the difference threshold between two sound pressures: if two sound incidents differ by 1 dB they can just be perceived differently.

The physical sound pressure covering 7 decades is mapped to a 140 dB scale by assigning sound pressure levels, as can be seen in Table 1.1. Some examples for noise levels occurring in situations of every day life are also shown.

**Table 1.1.** Relationship between absolute sound pressure and sound pressure level

| Sound pressure $p$ ($\mathrm{N/m^2}$, rms) | Sound pressure level $L$ (dB) | Situation/description |
|---|---|---|
| $2\,10^{-5}$ | 0 | hearing threshold |
| $2\,10^{-4}$ | 20 | forest, slow winds |
| $2\,10^{-3}$ | 40 | library |
| $2\,10^{-2}$ | 60 | office |
| $2\,10^{-1}$ | 80 | busy street |
| $2\,10^{0}$ | 100 | pneumatic hammer, siren |
| $2\,10^{1}$ | 120 | jet plane during take-off |
| $2\,10^{2}$ | 140 | threshold of pain, hearing loss |

It should be noted that sound pressures related to the highest sound pressure levels are still remarkably smaller than the static atmospheric pressure of about $10^5\,\mathrm{N/m^2}$. The rms value of the sound pressure at 140 dB is only $200\,\mathrm{N/m^2}$ and therefore 1/500 of atmospheric pressure.

The big advantage when using sound pressure levels is that they roughly represent a measure of the perceived loudness. However, think twice when calculating with sound pressure levels and be careful in your calculations. For instance: How high is the total sound pressure level of several single sources with known sound pressure levels? The derivation of the summation of sound pressure levels (where the levels are in fact *not* summed) gives an answer to the question for incoherent sources (and can be found more detailed in Appendix A)

$$L_{\mathrm{tot}} = 10\,\lg\left(\sum_{i=1}^{N} 10^{L_i/10}\right), \tag{1.4}$$

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where $N$ is the total number of incoherent sources with level $L_i$. Three vehicles, for example, with equal sound pressure levels produce a total sound pressure level

$$L_{\text{tot}} = 10\lg\left(3\,10^{L_i/10}\right) = 10\lg 10^{L_i/10} + 10\lg 3 = L_i + 4.8\,\text{dB}$$

which is 4.8 dB higher than the individual sound pressure level (and not three times higher than the individual sound pressure level).

## 1.1 Octave and third-octave band filters

In some cases a high spectral resolution is needed to decompose time domain signals. This may be the case when determining, for example, the possibly narrow-banded resonance peaks of a resonator, where one is interested in the actual bandwidth of the peak (see Chap. 5.5). Such a high spectral resolution can, for example, be achieved by the commonly used FFT-Analysis (FFT: Fast Fourier Transform). The FFT is not dealt with here, the interested reader can find more details for example in the work of Oppenheim and Schafer "Digital Signal Processing" (Prentice Hall, Englewood Cliffs New Jersey 1975).

In most cases, a high spectral resolution is neither desired nor necessary. If, for example, an estimate of the spectral composition of vehicle or railway noise is needed, it is wise to subdivide the frequency range into a small number of coarse intervals. Larger intervals do not express the finer details. They contain a higher random error rate and cannot be reproduced very accurately. Using broader frequency bands ensures a good reproducibility (provided that, for example, the traffic conditions do not change). Broadband signals are also often used for measurement purposes. This is the case in measurements of room acoustics and building acoustics, which use (mainly white) noise as excitation signal. Spectral details are not only of no interest, they furthermore would divert the attention from the validity of the results.

Measurements of the spectral components of time domain signals are realized using filters. These filters are electronic circuits which let a supplied voltage pass only in a certain frequency band. The filter is characterized by its bandwidth $\Delta f$, the lower and upper limiting frequency $f_l$ and $f_u$, respectively and the center frequency $f_c$ (Fig. 1.3). The bandwidth is determined by the difference of $f_u$ and $f_l$, $\Delta f = f_u - f_l$. Only filters with a constant relative bandwidth are used for acoustic purposes. The bandwidth is proportional to the center frequency of the filter. With increasing center frequency the bandwidth is also increasing. The most important representatives of filters with constant relative bandwidth are the octave and third-octave band filters. Their center frequency is determined by

$$f_c = \sqrt{f_l f_u}$$

The characteristic filter frequencies are known, if the ratio of the limiting frequencies $f_l$ and $f_u$ is given.

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Mojawa v. Shokz, IPR2025-00843

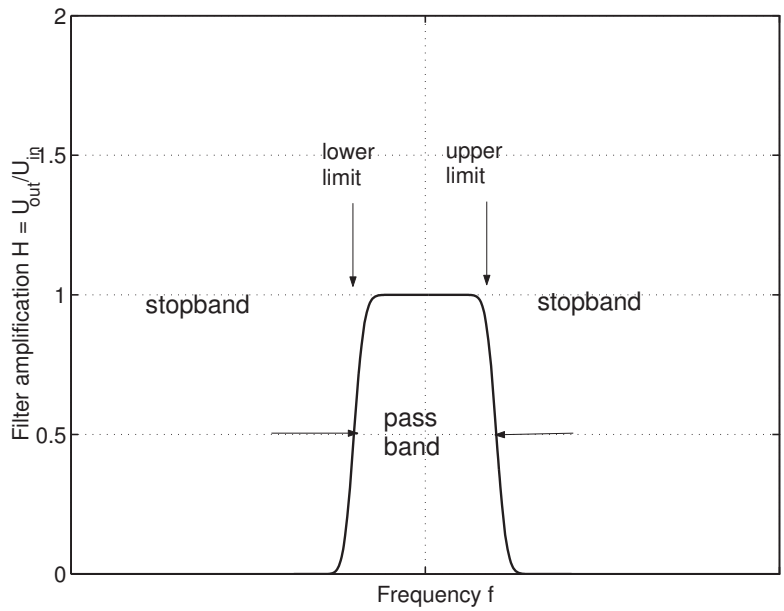


**Fig. 1.3.** Typical frequency response function of a filter (bandpass)

*Octave bandwidth*

$$f_u = 2f_l \ ,$$

which results in $f_c = \sqrt{2}f_l$ and $\Delta f = f_u - f_l = f_l = f_c/\sqrt{2}$.

*Third-octave bandwidth*

$$f_u = \sqrt[3]{2}f_l = 1.26f_l \ ,$$

which results in $f_c = \sqrt[6]{2}f_l = 1.12f_l$ and $\Delta f = 0.26f_l$.

The third-octave band filters are named that way, because three adjacent filters form an octave band filter ($\sqrt[3]{2}\sqrt[3]{2}\sqrt[3]{2} = 2$). The limiting frequencies are standardized in the international regulations EN 60651 and 60652.

When measuring sound levels one must state which filters were used during the measurement. The (coarser) octave band filters have a broader pass band than the (narrower) third-octave band filters which let contributions of a higher frequency range pass. Therefore octave band levels are always greater than third-octave band levels. The advantage of third-octave band measurements is the finer resolution (more data points in the same frequency range) of the spectrum.

By using the level summation (1.4), the octave band levels can be calculated using third-octave band measurements. In the same way, the levels of broader frequency bands may be calculated with the aid of level summation (1.4). The (non-weighted) linear level is often given. It contains all attributes of the frequency range between 16 Hz and 20000 Hz and can either be measured directly using an appropriate filter or determined by the level addition

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

of the third-octave or octave band levels in the frequency band (when converting from octave bands, $N = 11$ and the center frequencies of the filters are 16 Hz, 31.5 Hz, 63 Hz, 125 Hz, 250 Hz, 500 Hz, 1 kHz, 2 kHz, 4 kHz, 8 kHz and 16 kHz). The linear level is always higher than the individual levels, by which it is calculated.

## 1.2 Hearing levels

Results of acoustic measurements are also often specified using another single value called the 'A-weighted sound pressure level'. Some basic principles of the frequency dependence of the sensitivity of human hearing are now explained, as the measurement procedure for the A-weighted level is roughly based on this.

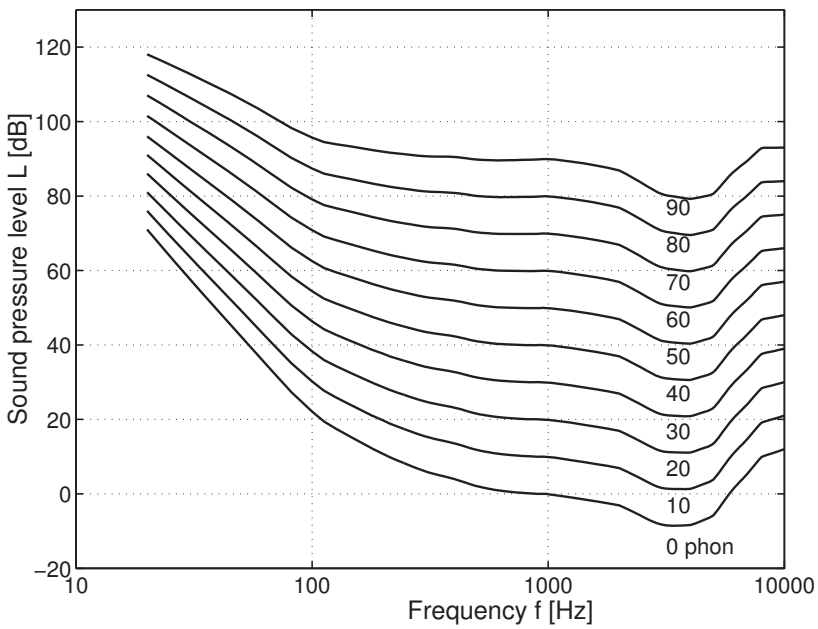


**Fig. 1.4.** Hearing levels

The sensitivity of the human ear is strongly dependent on the tonal pitch. The frequency dependence is depicted in Fig. 1.4. The figure is based on the findings from audiometric testing. The curves of perceived equal loudness (which have the unit 'phon') are drawn in a sound pressure level versus frequency plot. One can imagine the development of these curves as follows: a test subject compares a 1 kHz tone of a certain level to a second tone of another frequency and has to adjust the level of the second tone in such a way that it is perceived with equal loudness. The curve of one hearing level is obtained by varying the frequency of the second tone and is simply defined by the level of the 1 kHz tone. The array of curves obtained by varying the

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

level of the 1 kHz tone is called hearing levels. It reveals, for example, that a 50 Hz tone with an actual sound pressure level of 80 dB is perceived with the same loudness as a 1 kHz tone with 60 dB. The ear is more sensitive in the middle frequency range than at very high or very low frequencies.

## 1.3 A-Weighting

The relationship between the objective quantity sound pressure or sound pressure level, respectively, and the subjective quantity loudness is in fact quite complicated, as can be seen in the hearing levels shown in Fig. 1.4. The frequency dependence of the human ear's sensitivity, for example, is also level-dependent. The curves with a higher level are significantly flatter than the curves with smaller levels. The subjective perception 'loudness' is not only depending on frequency, but also on the bandwidth of the sound incident. The development of measurement equipment accounting for all properties of the human ear could only be realized with a very large effort.

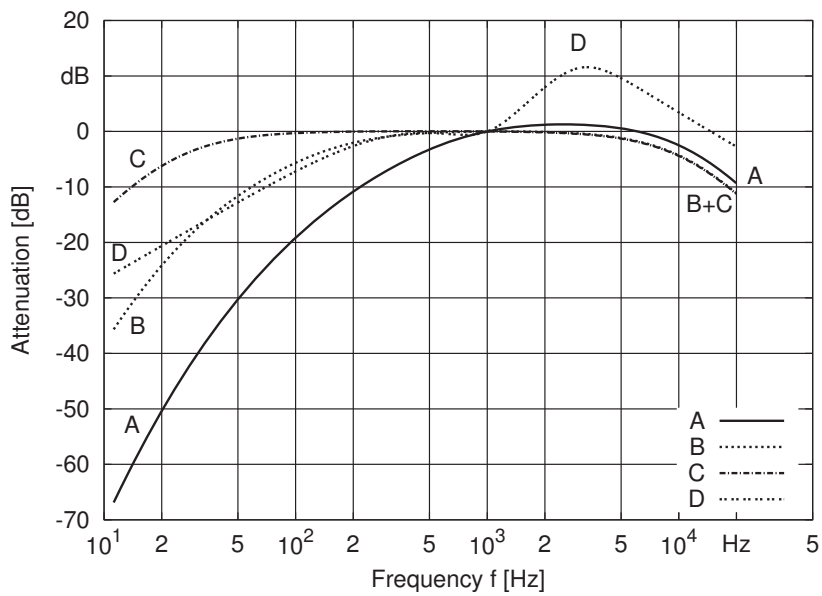


**Fig. 1.5.** Frequency response functions of A-, B-, C- and D-weighting filters

A frequency-weighted sound pressure level is used both nationally and internationally, which accounts for the basic aspects of the human ear's sensitivity and can be realized with reasonable effort. This so-called 'A-weighted sound pressure level' includes contributions of the whole audible frequency range. In practical applications the dB(A)-value is measured using the A-filter. The frequency response function of the A-filter is drawn in Fig. 1.5. The A-filter characteristics roughly represent the inverse of the hearing level

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

curve with 30 dB at 1 kHz. The lower frequencies and the very high frequencies are devaluated compared to the middle frequency range when determining the dB(A)-value. As a matter of fact, the A-weighted level can also be determined from measured third-octave band levels. The levels given in Fig. 1.5 are added to the third-octave band levels and the total sound pressure level, now A-weighted, is calculated according to the law of level summation (1.4). The A-weighting function is standardized in EN 60651.

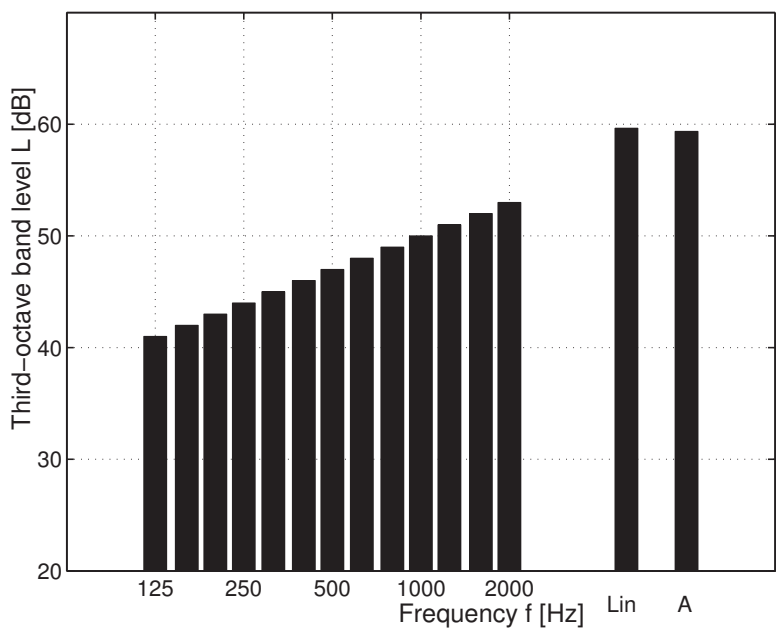


**Fig. 1.6.** Third-octave band, non-weighted and A-weighted levels of band-limited white noise

A practical example for the aforementioned level summation is given in Fig. 1.6 by means of a white noise signal. The third-octave band levels, the non-weighted (Lin) and the A-weighted (A) total sound pressure level are determined. The third-octave band levels increase by 1 dB for each band with increasing frequency. The linear (non-weighted) total sound pressure level is higher than each individual third-octave band level, the A-weighted level is only slightly smaller than the non-weighted level.

It should be noted that exceptions for certain noise problems (especially for vehicle and aircraft noise) exist, where other weighting functions (B, C and D) are used (see also Fig. 1.5). Regulations by law still commonly insist on the dB(A)-value.

Linearly determined single-number values, regardless of the filter used to produce them, are somewhat problematic, because considerable differences in individual perceptions do not become apparent. Fig. 1.4 clearly shows, for example, that 90 dB of level difference are needed at 1 kHz to increase the perception from 0 to 90 phon; at the lower frequency limit at 20 Hz only 50 dB

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

are needed. A simple frequency weighting is not enough to prevent certain possible inequities. On the other hand, simple and easy-to-use evaluation procedures are indispensable.

## 1.4 Noise fluctuating over time

It is easy to determine the noise level of constant, steady noise, such as from an engine with constant rpm, a vacuum cleaner, etc. Due to their uniform formations, such noise can be sufficiently described by the A-level (or third-octave level, if so desired.)

How, on the other hand, can one measure intermittent signals, such as speech, music or traffic noise? Of course, one can use the level-over-time notation, but this description falls short, because a notation of various noise events along a time continuum makes an otherwise simple quantitative comparison of a variety of noise scenarios, such as traffic on different highways, quite difficult. In order to obtain simple comparative values, one must take the mean value over a realistic average time period.

The most conventional and simplest method is the so-called 'energy-equivalent continuous sound level' $L_{eq}$. It reflects the sound pressure square over a long mean time

$$L_{eq} = 10\lg\left(\frac{1}{T}\int_0^T \frac{p_{eff}^2(t)}{p_0^2}\right)dt = 10\lg\left(\frac{1}{T}\int_0^T 10^{L(t)/10}\right)dt \tag{1.5}$$

($p_0 = 20\ 10^{-6}\,N/m^2$). Hereby $p_{eff}(t)$ indicates the time domain function of the rms-value and $L(t) = 10lg(p_{eff}(t)/p_0)^2$, the level gradient over time. The square of a time-dependent signal function is also referred to as 'signal energy', the energy-equivalent continuous sound level denotes the average signal energy; this explains the somewhat verbose terminology. For sound pressure signals obtained using an A-filter, third-octave filters, or the like, we use an A-weighted energy-equivalent level.

Depending on the need or application, any amount of integration time $T$ can be used, ranging from a few seconds to several hours. There are comprehensive bodies of legislation which outline norms set by a maximum level $L_{eq}$, which is permissible over a certain time reference period ranging up to several hours. For instance, the reference time period 'night' is normally between the hours of 10 p.m. and 6 a.m., an eight-hour time period. Normally, for measuring purposes, a much smaller mean time frame is used in order to reduce the effect of background noise. The level $L_{eq}$ is then reconstructed based on the number of total noise events and applied over a longer period of time. For example, let $L_{eq}$ be an energy-equivalent sound level to be verified for a city railway next to a street. The mean time period for this measurement would then be approximately how long it takes for the train

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

to pass by the street. That is, we are looking for the energy-equivalent continuous sound level for an average period that corresponds to the time it takes for the train to pass by the street, such as of 30 seconds $L_{eq}(30s)$. Suppose the train passes by every 5 minutes without a break. In this case, we can easily calculate the long-term $L_{eq}$ (measured over several hours, as may correspond to the reference periods 'day' or 'night', for instance) by $L_{eq}(long) = L_{eq}(30s) - 10lg(5min/30s) = L_{eq}(30s) - 10\,dB$.

Applying mean values is often the most sensible and essential method for determining or verifying maximum permissible noise levels. However, mean values, per definition, omit single events over a time-based continuum and can thus blur the distinctions between possibly very different situations. A light rail train passing by once every hour could well emit similar sound levels $L_{eq}$ as characterized by the permanent noise level of a busy street over a very long reference period, for example. The long-term effects of both sources combined may, in fact, result in one of the sources being completely obliterated from the $L_{eq}$ altogether (refer to Exercise 5).

The energy-equivalent continuous sound level serves as the simplest way to characterize sound levels fluctuating over time. Cumulative frequency levels can be determined by the peak time-related characteristic measurements, a method which is instrumental in the statistical analysis of sound levels.

## 1.5 Summary

Sound perception is governed by relativity. Changes are perceived to be the same when the stimulus increases by a certain percentage. This has led to the conclusion of the Weber-Fechner law, according to which perception is proportional to the logarithm of the stimulus. The physical sound pressures are therefore expressed through their logarithmic counterparts using sound levels of a pseudo-unit, the decibel ($dB$). The entire span of sound pressure relevant for human hearing, encompassing about 7 powers of ten, is reflected in a clearly defined scale from about $0\,dB$ (hearing threshold) up to approximately $140\,dB$ (threshold of pain). A-weighting is scaled to the human ear in order to roughly capture the frequency response function of hearing. A-weighted sound levels are expressed in the pseudo-unit $dB(A)$.

Sounds at intermittent time intervals are quantified using mean time values. One such quantification method approximates 'energy-equivalent permanent sound levels.'

## 1.6 Further reading

Stanley A. Gelfand's "Hearing – an Introduction to Psychological and Physiological Acoustics" (Marcel Dekker, New York 1998) is a physiologically oriented work and contains a detailed description of the anatomy of the human ear and the conduction of stimuli.

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

## 1.7 Practice exercises

### Problem 1

An A-weighted sound level of $50\,dB(A)$ originating from a neighboring factory was registered at an emission control center. A pump is planned for installation $50\,m$ away from the emission control center. How high can the A-weighted decibel level, resulting from the pump alone, be registered at the emission control center so that the overall sound level does not exceed $55\,dB(A)$?

### Problem 2

A noise contains only the frequency components listed in the table below:

| $f/Hz$ | $L_{Thirdoctave}/dB$ | $\Delta_i/dB$ |
|---|---|---|
| 400 | 78 | -4,8 |
| 500 | 76 | -3,2 |
| 630 | 74 | -1,9 |
| 800 | 75 | -0,8 |
| 1000 | 74 | 0 |
| 1250 | 73 | 0,6 |

Calculate

- both non-weighted octave levels
- the non-weighted overall decibel level and
- the A-weighted overall decibel level.

The corresponding A-weights are given in the last column of the table.

### Problem 3

A noise consisting of what is known as white noise is defined by a $1\,dB$-increase from third-octave to third-octave (see Figure 1.6). How much does the octave level increase from octave to octave? How much higher is the overall decibel level in proportion to the smallest third-octave when $N$ third-octaves are contained in the noise? State the numerical value for $N = 10$.

### Problem 4

A noise consisting of what is known as pink noise is defined by equal decibel levels for all the third-octaves it contains. How much does the octave level increase from octave to octave and what are their values? How much higher is the overall decibel level when there are $N$ thirds contained in the noise? State the numerical value for $N = 10$.

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

**Problem 5**

The energy-equivalent permanent sound level is registered at $55\,dB(A)$ at an emission control center near a street during the reference time period 'day,' lasting 16 hours. A new high-speed train track is scheduled to be constructed near the sight. The 2-minute measurement of a sound level $L_{eq}$ of a passing train is $75\,dB(A)$. The train passes by every 2 hours.

How high is the energy-equivalent permanent sound level measured over a longer period of time (in this case, reference time interval 'day')

- a) of the train alone and
- b) of both sound sources combined?

**Problem 6**

A city train travels every 5 minutes from 6 a.m. to 10 p.m. At night, between 10 p.m. and 2 a.m., it travels every 20 minutes, with a break from 2-6 a.m. in between. A single train passes within 30 seconds and for this time duration, the sound level registers at $L_{eq}(30s) = 78\,dB(A)$. How high is the energy-equivalent permanent sound level for the reference time intervals 'day' and 'night'?

**Problem 7**

The sound pressure level $L$ of a particular event, such as the emission of a city train as in the previous example, can under certain circumstances only be measured against a given background noise, such as traffic. Assume that the background noise differs from the sound event to be measured at a decibel level $\Delta L$. How high is the actual combined noise level? State the general equation for measuring errors and the numerical value for $\Delta L = 6\,dB$, $\Delta L = 10\,dB$ and $\Delta L = 20\,dB$.

**Problem 8**

As in Problem 7, sound emission is to be measured in the presence of a noise disturbance. How far away does the the noise source have to be in order to obtain a measuring error of $0.1\,dB$?

**Problem 9**

Sometimes filters of a relatively constant bandwidth are employed to take finer measurements in sixth-octaves increments as opposed to the customary third-octave increments. State the equations for

- the consecutive center frequencies
- the bandwidth and
- the limiting frequencies.

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Mojawa v. Shokz, IPR2025-00843

### Problem 10

In a calculated measurement where an octave and all of its constituent thirds are given, it appears that one of the thirds may have been a measurement error. How can the result be checked against the other three values that are assumed to be correct?

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

# 4

# Structure-borne sound

## 4.1 Introduction

Vibrations and waves in solid structures, like, for example, vibrations in plates and beams, walls, ships and buildings, etc., are summarized by the term 'structure-borne sound'. Structure-borne sound has major importance with respect to solving noise control problems: the air-borne sound into (or from) the aforementioned solid structures is caused by motion in the structure's surface. In many cases it is the structure-borne sound that is responsible for the resulting sound in air (or sound in liquids). Even the transmission through walls, ceilings and windows, etc. is essentially a structural problem.

A vital and fundamental difference exists between sound waves in air and sound waves in structures. A gas (or a liquid) reacts with a change in pressure if its volume is changed. A mere change of the geometrical shape of the gas mass has no influence on the pressure at all (apart from losses due to friction). The boundaries between elements of volume in a gas therefore only transmit forces normal to their surface.

As illustrated by means of a simple example, a bendable thin beam (such as a ruler),- solid structures not only try to resist a compression of the volume they fill, but also a deformation of their shape. The boundaries in solid structures therefore transmit tangential forces, or shear tension, as well. By means of the example, the bendable beam, the existence of forces normal to its axes can easily be observed: these shear forces keep the beam in its bendable shape. It would otherwise be impossible for it to stay in this shape.

Instead of only the normal component of the tension, which appears in gases, three components of forces have to be taken into account at the boundaries when dealing with elements of volume in solid structures (Fig. 4.1). Just as one uses the force exerted onto the surface (pressure) for the purposes of definition in the case of airborne sound, one uses tension to formulate the the force laws under the circumstance of structure-borne sound. The tension is equal to the ratio of force to surface area. Furthermore, a distinction has to be

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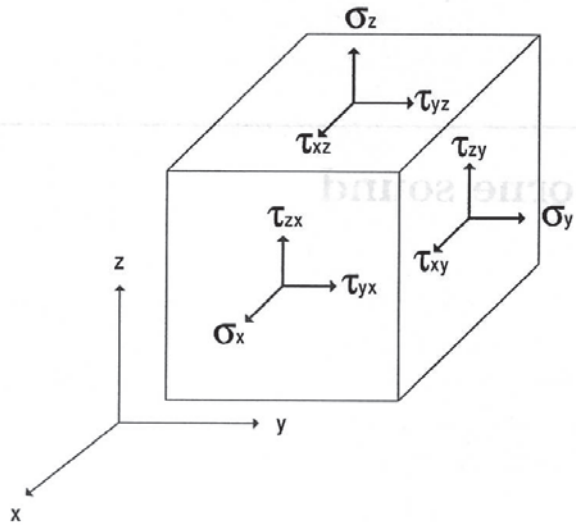


**Fig. 4.1.** Normal tension and shear tension on the element of volume in a solid structure

made between normal tension (normal to the imaginary boundary) and shear tension (tangential to the boundary).

All external tension components result in an elastic deformation of the structure which reacts with a resilient oscillation to its equilibrium position when the external tension is removed. The observed oscillation can be explained by a continuous conversion of potential energy which is stored in a change of shape and volume, into kinetic energy of the involved masses and vice versa. This 'reciprocal transformation' of the stored energy is not only taking place continually with time, but is also spatially distributed in such a way that vibrations occur as waveforms. In beams, for instance, the three-axial state of tension leads to different sorts of waves for each direction of motion. In beams

- bending waves, where the displacement is normal to the beam axis and therefore also normal to the propagation direction of the wave (Fig. 4.2a),
- the likewise transverse torsional waves, produced by torsion of the beam cross-section and
- the longitudinal waves caused by strain, where the displacement mainly appears along the beam axis (Fig. 4.2b)

occur.

The circumstances become even more complicated by the fact that bending waves with displacement can occur in both of the directions pointing normal to the beam axis. Only in the case of circular cross-sections (or quadratic cross-sections) do both types of bending waves show the same behavior. As can easily be observed in a beam with a flat, prolated cross-section (a ruler), the bending stiffness generally depends on the direction of the tension.

In addition, there are many more waveforms in beams and plates if the finite cross-sectional dimensions are taken into account. In the same manner

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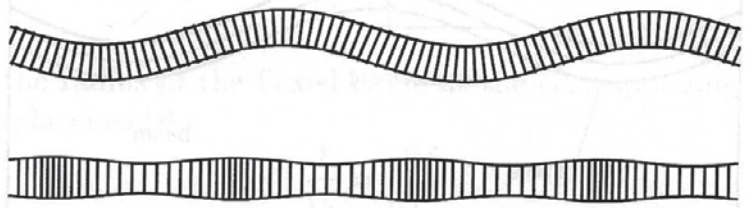
Mojawa v. Shokz, IPR2025-00843

as in air-borne sound, cross-sectional distributions of the sound (so-called modes) develop, each having an individual wave form (an example is given in Fig. 4.3). The waves mentioned above are only a special case of modes. It is quite obvious that the full breadth of structure-borne waves which can occur in beams and plates cannot be dealt with here. The interested reader should especially refer to the book "Structure Borne Sound" by Lothar Cremer and Manfred Heckl (translated by B.A.T. Petersson, Springer, Berlin and New York 2004) for a detailed description of the wave forms. Here, it is sufficient to limit the explanation to the bare necessities.



(a)

(b)

**Fig. 4.2.** (a) Bending waves and (b) longitudinal waves in beams

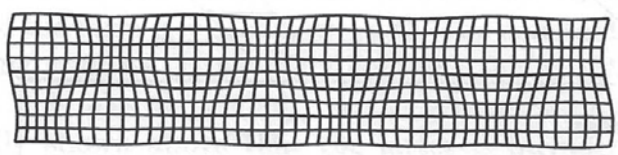


**Fig. 4.3.** Higher order wave (mode) in a thick plate

In engineering acoustics, the propagation of bending waves is mainly of interest because this wave form – also occurring in laterally extended plates – is the most essential one for a simple reason: the plate displacements are normal to the plate (or beam) surface and will preferably result in a radiation of air-borne sound rather than longitudinal strain or torsional waves with their motion mainly tangential to the surface. In addition to that, a second reason for the dominating role of bending waves can be deduced by intuition. Bending provides a much smaller resistance to the external exciting force (in most practically relevant cases) as does the reaction ot strain, for example. It can therefore be assumed that bending waves are more easily excitable and are thus the dominating type of vibration.

The discussion of bending waves starts with the simplest case of beams. The obtained results will be transferred to plates which are more relevant in practice.

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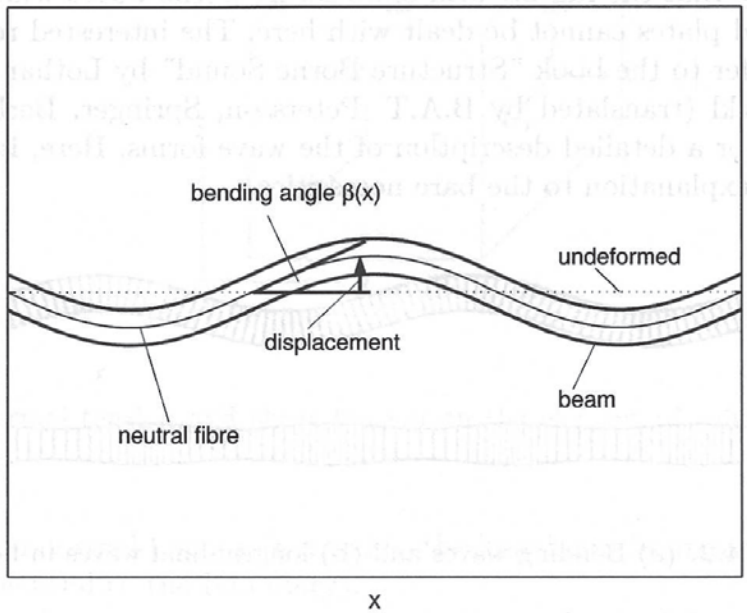
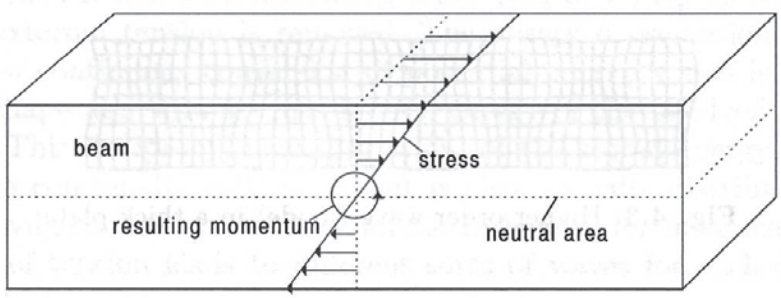
## 4.2 Bending waves in beams




(a)




(b)

**Fig. 4.4.** Elastic bending of a beam. (**a**) Displacement and bending angle. (**b**) tension and resulting momentum

Static bending of beams and plates has understandably been a considerably important topic in the statics of structures and has been elucidated in past studies. One can therefore refer to the results of static bending science and readily use it here. The kinetic quantities are described by the beam displacement $\xi(x)$ and the bending angle $\beta(x)$ (Fig. 4.4a). Only small bending angles are of interest here, to which

$$\frac{\partial \xi}{\partial x} = \tan\beta \approx \beta \tag{4.1}$$

applies. Apart from these, the normal tension and the shear tension in the beam cross-section are also important. If it is assumed, as is conventional in

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Mojawa v. Shokz, IPR2025-00843

static bending science, that the shear tension increases linearly from a (shear tension free) neutral beam fibre (Fig. 4.4a and 4.4b). The shear tension $\varepsilon_x$ can be summarized in a resulting moment $M$, which acts on the neutral fiber

$$M = \int_S \varepsilon_x y \mathrm{d}S \, .$$

The stronger the beam is flexed by bending the larger this bending moment becomes. A reasonable assumption is

$$M \sim \frac{1}{r_\mathrm{k}} \, ,$$

where $r_\mathrm{k}$ is the radius of the flexed circle at the corresponding beam point. For small displacements

$$\frac{1}{r_\mathrm{k}} = \frac{\partial^2 \xi}{\partial x^2}$$

applies, as is generally known. Using the proportional constant $B$, we obtain

$$M = -B \frac{\partial^2 \xi}{\partial x^2} \tag{4.2}$$

where the sign of $M$ is specified in such a way that $M$ and $\xi$ have equal signs for waves of the form $\xi = \xi_0 cos(kx - \omega t)$. The constant $B$ is accordingly called the bending stiffness. It not only contains the specific stiffness of the material $E$, but it is also dependent on the geometry of the cross-section. It was graphically shown above that the latter is included in the bending stiffness. Bending science shows that (Fig. 4.5a)

$$B = E \int_S y^2 \mathrm{d}y \mathrm{d}z = EI \tag{4.3}$$

is given. The specific stiffness $E$ is called elastic modulus, also known as Young's modulus. It may be determined from the stiffness $s$ of a material block which can be regarded as a spring, having the cross-sectional area $S$ and the thickness $h$ (Fig. 4.5b). If the block is loaded with the force $F$, then it is compressed by $\Delta x$ according to Hook's law:

$$F = s\Delta x = \frac{ES}{h} \Delta x \tag{4.4}$$

The elastic modulus $E = sh/S$ therefore may be determined from the stiffness of a probe and its dimensions.

The quantity $I$, describing the geometry of the cross-section, is called the second moment of area. In this book, only rectangular cross-sections are of interest. Using the beam thickness $h$ (pointing towards the direction of the exciting force) and the width $b$, $I$ is obtained for the

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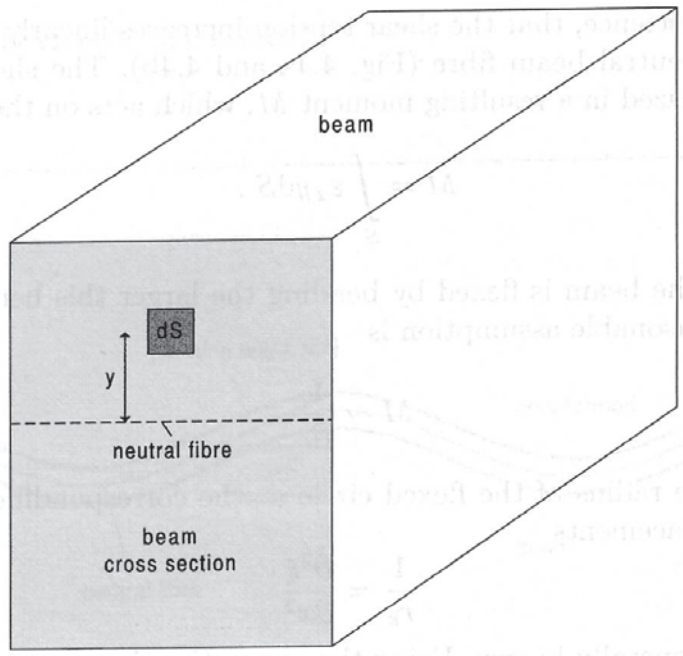
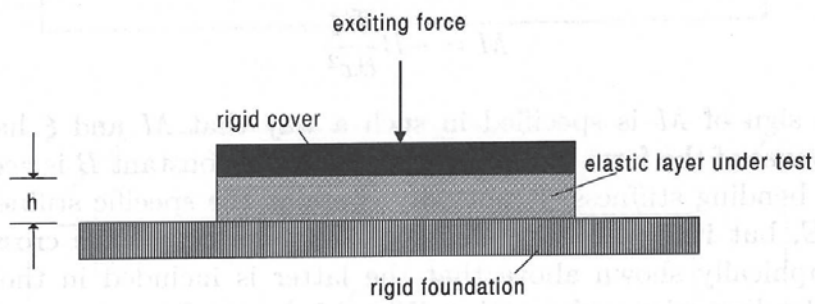
Mojawa v. Shokz, IPR2025-00843




(a)




(b)

**Fig. 4.5.** (a) Definition of the second moment of area (b) Measurement setup for determination of the elastic modulus $E$

Rectangular cross-section

$$I = \frac{bh^3}{12} \, .$$

A second example, using the radius $a$, is mentioned for the
Circular cross-section

$$I = \frac{\pi}{4} a^4 \, .$$

Other values for the second moment of area can be found, for example in Dubbel's book "Taschenbuch für den Maschinenbau" (Springer, Berlin 2001), which is also a very useful book for acousticians.

Similar to summarizing the normal tension in a bending moment, a shear force can be assigned to the shear tension $\tau$, pointing in the direction of the displacement:

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

$$F = \int_S \tau_{xy} \mathrm{d}y\mathrm{d}z \,. \tag{4.5}$$

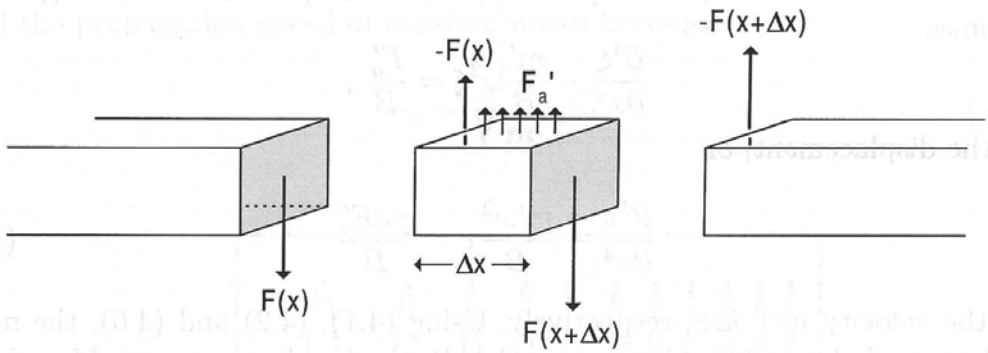



**Fig. 4.6.** Free body diagram of a beam element

As static bending science shows, the shear force, acting tangentially on the cross-section and pointing downwards, results in

$$F = -\frac{\partial M}{\partial x} \tag{4.6}$$

when using the bending moment. Due to 'actio=reactio', the transversal force pair $F(x + \Delta x)$ and $-F(x)$ now acts on the free body of the beam element (Fig. 4.6) with the length $\Delta x$. If an external force $F_a = F_a' \Delta x$ is taken into account, pointing in the direction of the displacement (e.g. the beam is excited by a hammer-blow) and uniformly distributed along the length $\Delta x$, Newton's law requires that

$$\Delta x \varrho S \ddot{\xi} = F(x) - F(x + \Delta x) + \Delta x F_a' \,.$$

Or, after proceeding to the limiting case $\Delta x \to 0$,

$$m' \ddot{\xi} + \frac{\partial F}{\partial x} = F_a' \,, \tag{4.7}$$

where $m'$ is the beam mass and $F_a'$ is the external force per corresponding unit length. (4.6) and (4.2) specify

$$\frac{\partial F}{\partial x} = -\frac{\partial^2 M}{\partial x^2} = B \frac{\partial^4 \xi}{\partial x^4} \,,$$

and therefore the equation

$$m' \ddot{\xi} + B \frac{\partial^4 \xi}{\partial x^4} = F_a'$$

is finally obtained from (4.7) for bending waves. It is similar only to the wave equation in gases inasmuch as the second time derivative appears. The

principal differences between bending waves and air waves become apparent, when pure tones

$$\xi(x,t) = \mathrm{Re}\left\{\xi(x)\mathrm{e}^{j\omega t}\right\}$$

are considered. For complex amplitudes, the wave equation for bending waves becomes

$$\frac{\partial^4 \xi}{\partial x^4} - \frac{m'}{B}\omega^2 \xi = \frac{F_a'}{B}\,,$$

for the displacement, or

$$\frac{\partial^4 v}{\partial x^4} - \frac{m'\omega^2}{B} v = \frac{j\omega F_a'}{B} \tag{4.8}$$

for the velocity $v = j\omega\xi$, respectively. Using (4.1), (4.2) and (4.6), the more seldom needed angular velocity $w = \mathrm{d}\beta/\mathrm{d}t$, the bending moment $M$ and the shear force $F$ are given by

$$w = \frac{\partial v}{\partial x}\,, \tag{4.9}$$

$$M = -\frac{B}{j\omega}\frac{\partial^2 v}{\partial x^2} \quad \text{and} \tag{4.10}$$

$$F = \frac{B}{j\omega}\frac{\partial^3 v}{\partial x^3}\,. \tag{4.11}$$

Angular velocity, moment and shear force can be calculated from the velocity accordingly.

## 4.3 Propagation of bending waves

The main characteristics of bending waves can be explained using the basic ansatz for waves

$$v = v_0 \mathrm{e}^{-jk_\mathrm{B}x}$$

in the homogeneous differential equation

$$\frac{\partial^4 v}{\partial x^4} - \frac{m'}{B}\omega^2 v = 0\,, \tag{4.12}$$

which is valid outside of local external forces. Inserting the ansatz into (4.12) yields the bending wavenumber

$$k_\mathrm{B}^4 = \frac{m'}{B}\omega^2\,. \tag{4.13}$$

Using

$$k_\mathrm{B} = \frac{2\pi}{\lambda_\mathrm{B}} = \frac{\omega}{c_\mathrm{B}}$$

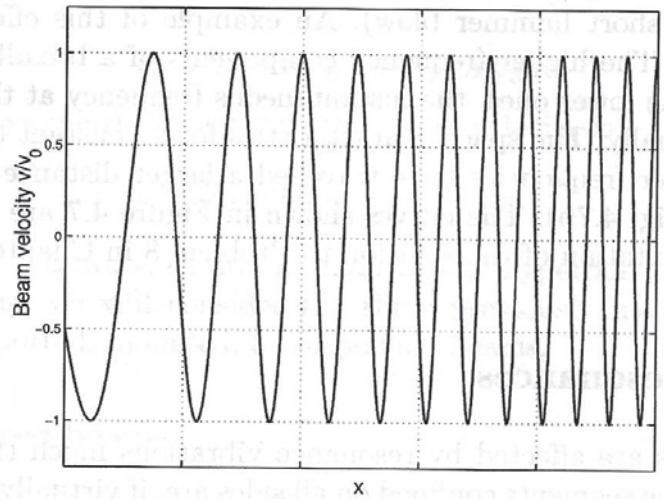
Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

the bending wavelength becomes

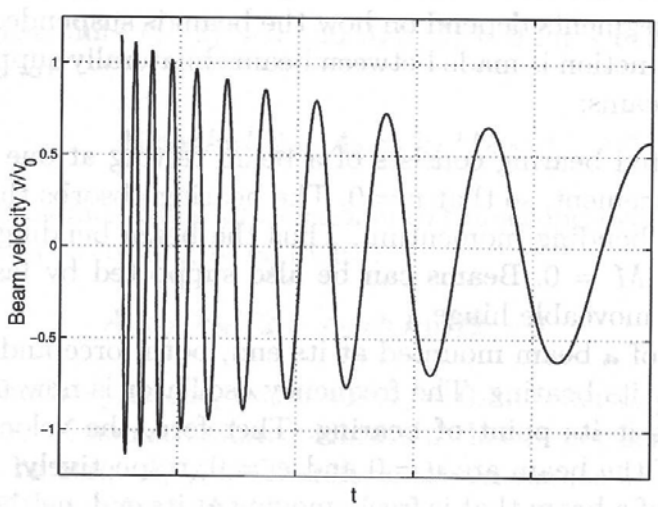
$$\lambda_B = 2\pi \sqrt[4]{\frac{B}{m'}} \frac{1}{\sqrt{\omega}} \tag{4.14}$$

and the propagation speed of bending waves becomes

$$c_B = \sqrt[4]{\frac{B}{m'}} \sqrt{\omega} \,. \tag{4.15}$$




(a)




(b)

**Fig. 4.7.** Impulse response of the beam velocity. (**a**) Space characteristics for constant time. (**b**) Time characteristics for constant position

Thus, a bending wavelength $\lambda_B$ is obtained which only decreases with the square-root of increasing frequency and has a frequency-dependent propagation speed. These facts illustrate the fundamental difference between air waves

and bending waves. If, instead of pure tones, we consider time dependencies consisting of several spectral components, the frequency-dependent propagation speed causes the spectral components to 'run away from each other' the larger the distance is that they traverse and the more their frequencies diverge. This also means that the spectral composition at two different positions on the beam is different and it follows that, at two different positions, totally different time characteristics of the beam velocity are found. The time signal is distorted along the propagation direction of the bending wave.

This effect is called dispersion. It can be recognized, for example, in the 'impulse response' (i.e. the beam velocity, if the beam at $x = 0$ is excited at $t = 0$ by a short hammer blow). An example of this effect is shown in Fig. 4.7 above. The higher frequency components of a broadband excitation arrive before the lower ones, the instantaneous frequency at the arrival point decreases gradually. The space characteristics for a constant time behave accordingly: higher frequencies have travelled a larger distance than the lower components (Fig. 4.7b). The curves shown in Figure 4.7 are depicted in the graphic representation of the solution to Problem 8 in Chapter 13.

## 4.4 Beam resonances

Beam segments are affected by resonance vibrations much the same way as finite gas volume segments confined on all sides are, if virtually no wave energy is lost at the terminations. Resonance frequencies and types of vibrations that occur in beam segments depend on how the beam is suspended or mounted at the ends. A distinction is made between beams bilaterally supported, mounted or suspended beams:

- The supported bearing consists of a beam resting at one point. This impedes displacement, so that $v = 0$. The bearing absorbs the bending force, but not the bending momentum. Thus the beam bending momentum diminishes to $M = 0$. Beams can be also supported by fastening them to any type of moveable hinge.
- In the case of a beam mounted at its end, both force and momentum are absorbed by its bearing. The frequency oscillator is now not able to move or rotate at it its point of bearing. Therefore, the velocity and angular velocities of the beam are $v = 0$ and $w = 0$, respectively.
- In the case of a beam that is freely moving at its end, neither its directional nor its rotational motion is hindered. That is, neither force nor momentum is absorbed. For this reason, $M = 0$ and $F = 0$ are true for suspended beams.

A mathematical consideration of beam resonances generally necessitates an ansatz for beam velocity consisting of four linear-independent solutions of the bending wave function (4.8). Because – as mentioned before – it is necessary to find the non-stimulated vibrations, we will consider the homogenous bending

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

wave equation ($F_a' = 0$ in eq.(4.8)). The purpose of the ansatz is to satisfy the constraints at the ends of the beam, resulting, of course, in a system of four equations. Symmetrical constructions simplify the analysis a bit. Such constructions consist of beam segments characterized by the same constraints at both ends. Here, one can make a separate distinction between point- and axis-symmetric wave forms. As shown in the following, this allows us to only use two equations to describe this situation. However, both have to be solved twice. The beam ends exist in this scenario at the points $x = -l/2$ and $x = +l/2$.

For axis-symmetric wave forms, the sound velocity is

$$v_g = A_1 \cos k_B x + A_2 \, ch \, k_B x \tag{4.16}$$

and for point-symmetric wave forms the sound velocity is

$$v_u = A_1 \sin k_B x + A_2 \, sh \, k_B x \, . \tag{4.17}$$

$A_1$ and $A_2$ are constants, $ch$ and $sh$ indicate the hyperbolic cosine and sine. In the following, we will consider the three previously mentioned cases of bilaterally supported, mounted, or suspended beams.

### 4.4.1 Supported beams

#### Axis-symmetric wave forms of bilaterally supported beams

The bearing constraints $v(l/2) = 0$ and $M(l/2) = 0$ for a beam supported at the end $x = l/2$ are

$$A_1 \cos k_B l/2 + A_2 \, ch \, k_B l/2 = 0 \tag{4.18}$$

and, because the momentum is proportional to the second-order spatial derivative of the velocity according to eq.(4.10):

$$A_1 \cos k_B l/2 - A_2 \, ch \, k_B l/2 = 0 \, . \tag{4.19}$$

By subtracting both equations, we obtain $A_2 = 0$. In order to provide the pre-existing condition of non-stimulated vibration that is not zero, the constant $A_1$ can also not be zero, $A_1 \neq 0$. Therefore, the resonance condition exists in $\cos k_B l/2 = 0$. This results in $k_B l/2 = \pi/2 + n\pi$ or

$$k_B l = (2n+1)\pi \tag{4.20}$$

($n = 0, 1, 2, ...$). For the resonance frequencies we use $k_B = \sqrt[4]{m'\omega^2/B}$ to obtain

$$f = (2n+1)^2 \frac{\pi}{2l^2} \sqrt{\frac{B}{m'}} \, . \tag{4.21}$$

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

The shape of the wave form function in the resonance is referred to as a mode. For the modes, we use the ansatz (4.16) where $k_B l/2 = \pi/2 + n\pi$ and $A_2 = 0$ to obtain

$$v_{g,mod} = \cos\left((2n+1)\pi\frac{x}{l}\right). \tag{4.22}$$

Wave modes are scalable. For this reason, we can arbitrarily set $A_1 = 1$.

### Point-symmetric wave forms of bilaterally supported beams

The parameters $v(l/2) = 0$ and $M(l/2) = 0$ for the beam supported at the end $x = l/2$ result in

$$A_1 \sin k_B l/2 + A_2\, sh\, k_B l/2 = 0 \tag{4.23}$$

and – since, as in the above, the momentum is proportional to the second-order spatial derivative of the velocity according to eq.(4.9):

$$A_1 \sin k_B l/2 - A_2\, sh\, k_B l/2 = 0\,. \tag{4.24}$$

Again, we obtain $A_2 = 0$ by subtracting both equations. In order to define the pre-existing condition of a non-stimulated vibration that is not zero, the constant $A_1$ must also have a value other then zero, $A_1 \neq 0$. Therefore, the resonance constraint exists in $\sin k_B l/2 = 0$. This results in $k_B l/2 = n\pi$ or

$$k_B l = 2n\pi \tag{4.25}$$

$(n = 1, 2, ...)$. This leads to the definition of the resonance frequencies in

$$f = (2n)^2 \frac{\pi}{2l^2}\sqrt{\frac{B}{m'}}\,. \tag{4.26}$$

According to the ansatz (4.17) where $k_B l/2 = n\pi$ and $A_2 = 0$, the modes exist in

$$v_{u,mod} = \sin\left(2n\pi\frac{x}{l}\right). \tag{4.27}$$

Here, too, we can scale the modes to $A_1 = 1$ for the sake of simplicity.

### Summary of bilaterally supported beams

The $k_B l$ pertaining to the resonances are alternately even- and odd-numbered multiples of $\pi$ according to equations (4.20) and (4.25). In sum:

$$k_B l = m\pi \tag{4.28}$$

Due to $k_B = \sqrt[4]{m'\omega^2/B}$, the resonance frequencies altogether result in

$$f = m^2 \frac{\pi}{2l^2}\sqrt{\frac{B}{m'}} \tag{4.29}$$

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

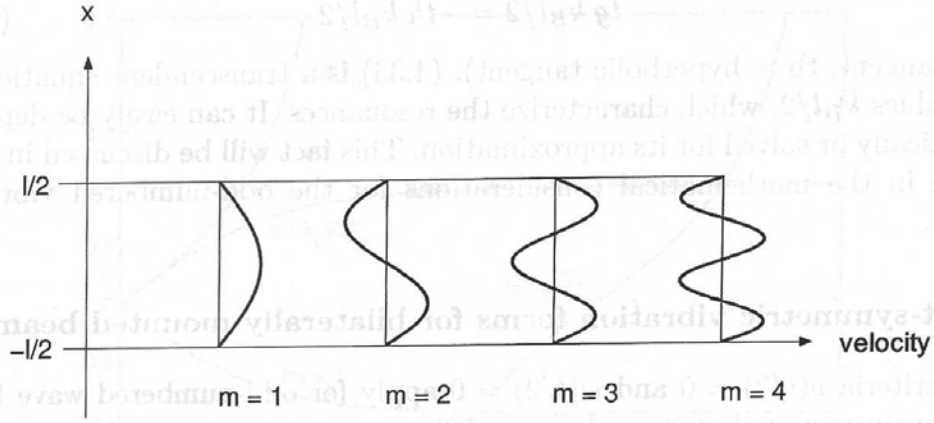



**Fig. 4.8.** Vibration modes of bilaterally supported beams

where $m = 1, 2, 3, ...$, and whereby odd-numbered $m$ indicate axis-symmetric modes with forms as given in eq.(4.22) and even-numbered $m$ indicate point-symmetric modes with forms given in eq.(4.27). As seen above, the distance between two resonances increases quadratically. The number of resonance incidences in a given finite frequency band decrease as the center frequency of the band increases. Unlike one-dimensional gaseous wave guides, the frequency density decreases for bilaterally supported beams. In addition, the bending resonance frequencies are inversely proportional to the square of the length of the wave guide $l$. This is also different from sound travelling through air, where the resonance frequencies are inversely proportional to the length itself.

Figure 4.8 provides a visual overview of the first four vibration modes.

### 4.4.2 Bilaterally mounted beams

#### Axis-symmetric wave Forms for bilaterally mounted beams

The criteria $v(l/2) = 0$ and $w(l/2) = 0$ for a beam mounted at its end $x = l/2$ result in

$$A_1 \cos k_B l/2 + A_2 \, ch \, k_B l/2 = 0 \tag{4.30}$$

and (because the angular velocity is proportional to the spatial derivative of the velocity according to eq.(4.9)):

$$A_1 \sin k_B l/2 - A_2 \, sh \, k_B l/2 = 0 \, . \tag{4.31}$$

Non-zero solutions $A_1$ and $A_2$ only exist when the determinant of the homogenous system of both equations vanishes. Only then do non-stimulated vibrations – resonances – occur. For this reason, the equation which defines the resonance frequencies is

$$\cos k_B l/2 \, sh \, k_B l/2 + \sin k_B l/2 \, ch \, k_B l/2 = 0 \, , \tag{4.32}$$

or in

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

$$tg\, k_B l/2 = -th\, k_B l/2 \tag{4.33}$$

(tg=tangent, th = hyperbolic tangent). (4.33) is a transcendent equation for the values $k_B l/2$, which characterize the resonances. It can easily be depicted graphically or solved for its approximation. This fact will be discussed in more detail in the mathematical considerations for the odd-numbered vibration forms.

### Point-symmetric vibration forms for bilaterally mounted beams

The criteria $v(l/2) = 0$ and $w(l/2) = 0$ apply for odd-numbered wave forms in a beam mounted at its end at $x = l/2$

$$A_1 \sin k_B l/2 + A_2\, sh\, k_B l/2 = 0 \tag{4.34}$$

and (because the angular velocity is proportional to the spatial derivative of the velocity according to eq.(4.9)):

$$A_1 \cos k_B l/2 + A_2\, ch\, k_B l/2 = 0\,. \tag{4.35}$$

Non-zero solutions $A_1$ and $A_2$ are again only possible if the determinant of the homogenous system of both equations vanishes. Only then do non-stimulated vibrations – resonances – occur. For this reason, the equation which defines the resonance frequencies is

$$\sin k_B l/2\, ch\, k_B l/2 - \cos k_B l/2\, sh\, k_B l/2 = 0\,, \tag{4.36}$$

or

$$tg\, k_B l/2 = th\, k_B l/2\,. \tag{4.37}$$

### Summary of bilaterally mounted beams

When applying equations (4.33) and (4.37), the resonances can be derived from the point where the tangent function intersects with the positive and the negative hyperbolic tangents, alternately (refer to Figure 4.9). At the first point of intersection which has the smallest argument $k_B l/2$, the tangent hyperbole already is close to the value 1. We can use the resonance criterium

$$tg\, k_B l/2 = \pm 1 \tag{4.38}$$

to achieve quite a close approximation. The equation results in

$$k_B l/2 = 3\pi/4;\ 5\pi/4;\ 7\pi/4;\ \ldots$$

(as can be seen in Figure 4.9) or

$$k_B l = (2m+1)\pi/2 \tag{4.39}$$

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Mojawa v. Shokz, IPR2025-00843

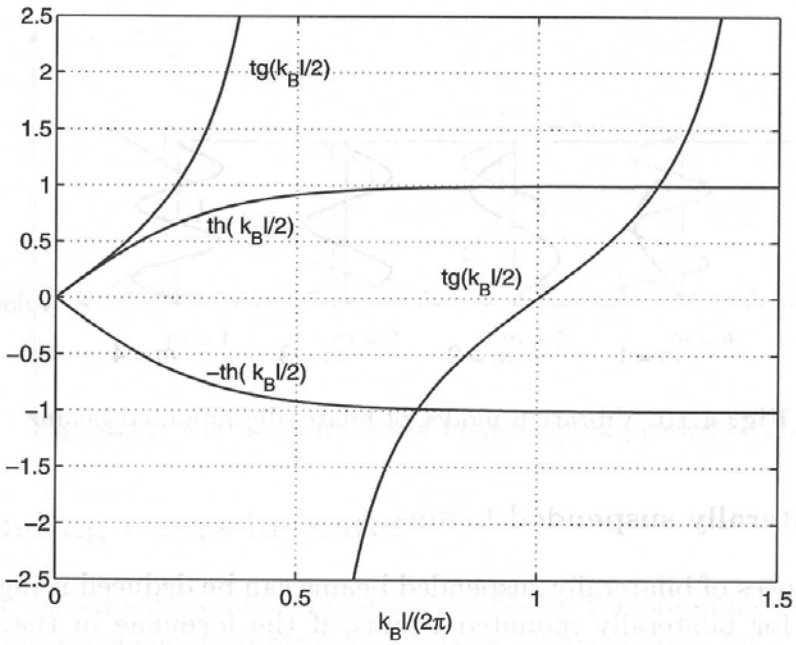


**Fig. 4.9.** Graphic solution of the eigenvalue equations (4.33) and (4.37)

$(m = 1, 2, ...)$. For the resonance frequencies, it follows that

$$f = (m + \frac{1}{2})^2 \frac{\pi}{2l^2} \sqrt{\frac{B}{m'}} \,. \tag{4.40}$$

For all practical considerations, the application of the above approximation is adequate. An even more precise assessment of the first point of intersection between the tangent and hyperbolic tangent for $m = 1$ produces the more accurate value $k_B l = 1.506\pi$ instead of $k_B l = 1.5\pi$. The margin of error for the lowest resonance frequency is therefore less than 1 percent.

The axis-symmetric modal forms can be obtained by solving the eq.(4.31) for $A_2$ and subsequently applying eq.(4.16) to the ansatz:

$$v_{g,mod} = \cos k_B x + \frac{\sin k_B l/2}{sh\, k_B l/2}\, ch\, k_B x \,. \tag{4.41}$$

Since these modes consist of axis-symmetric vibrations in resonance, we proceed by inserting the values $k_B l/2 = 3\pi/4; 7\pi/4; 11\pi/4....$ Again, it has been scaled to $A_1 = 1$.

The odd-numbered modal forms can be obtained by solving eq.(4.35) for $A_2$ and subsequently plugging eq. (4.17) into the ansatz:

$$v_{u,mod} = \sin k_B x - \frac{\cos k_B l/2}{ch\, k_B l/2}\, sh\, k_B x \,. \tag{4.42}$$

Since these modes are point-symmetric vibrations in resonance, we use the values $k_B l/2 = 5\pi/4; 9\pi/4; 13\pi/4....$ Here too, it was scaled to $A_1 = 1$.

The first four modal forms are depicted in Figure 4.10.

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Mojawa v. Shokz, IPR2025-00843

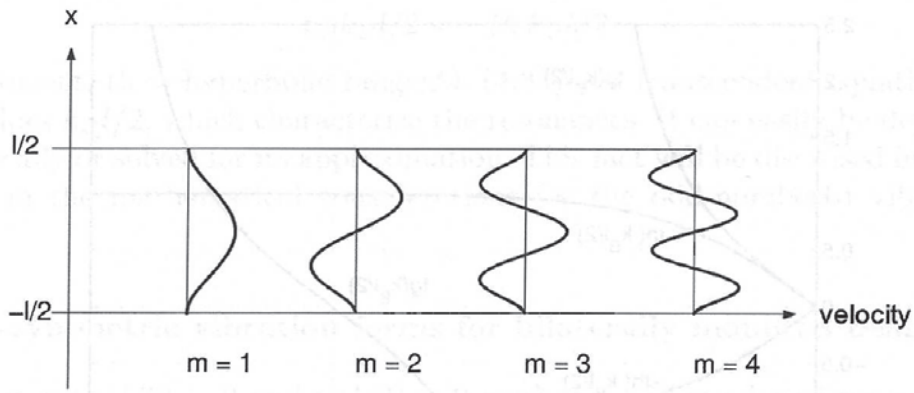


**Fig. 4.10.** Vibration modes of bilaterally mounted beams

### 4.4.3 Bilaterally suspended beams

The resonances of bilaterally suspended beams can be deduced using the same criteria as for bilaterally mounted beams, if the formulae in the equations (4.16) and (4.17) are applied for the bending momentum rather than the vibrational speed, that is, if (4.16) and (4.17) are replaced by

$$M_g = A_1 \cos k_B x + A_2 \, ch \, k_B x \tag{4.43}$$

and, for the point-symmetric forms of the beam velocity,

$$M_u = A_1 \sin k_B x + A_2 \, sh \, k_B x. \tag{4.44}$$

The constraint $F(l/2) = 0$ is synonymous with the first spatial derivative being eliminated at the onset of the momentum. Along with the parameter $M(l/2) = 0$, this results in the equations (4.30) and (4.31) for the axis-symmetric case, and (4.34) and (4.35) for the point-symmetric case. For this reason, the resonance frequencies of bilaterally suspended beams are equal to those of bilaterally mounted beams. This also means that (4.39) (and of course (4.40)) apply to the resonance frequencies for bilaterally suspended beams.

The vibration forms (as represented in the vibrational speed) are, of course, not identical to those for mounted beams. The suspended vibration modes are derived from the two-fold integration of equation (4.41) and result in

$$v_{g,mod} = \cos k_B x - \frac{\sin k_B l/2}{sh \, k_B l/2} \, ch \, k_B x, \tag{4.45}$$

(where $k_B l/2 = 3\pi/4; 7\pi/4; 11\pi/4...$) and by integrating equation (4.42) twice,

$$v_{u,mod} = \sin k_B x + \frac{\cos k_B l/2}{ch \, k_B l/2} \, sh \, k_B x \tag{4.46}$$

(where $k_B l/2 = 5\pi/4; 9\pi/4; 13\pi/4...$). Figure 4.11 shows the first four modes here as well.

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Mojawa v. Shokz, IPR2025-00843

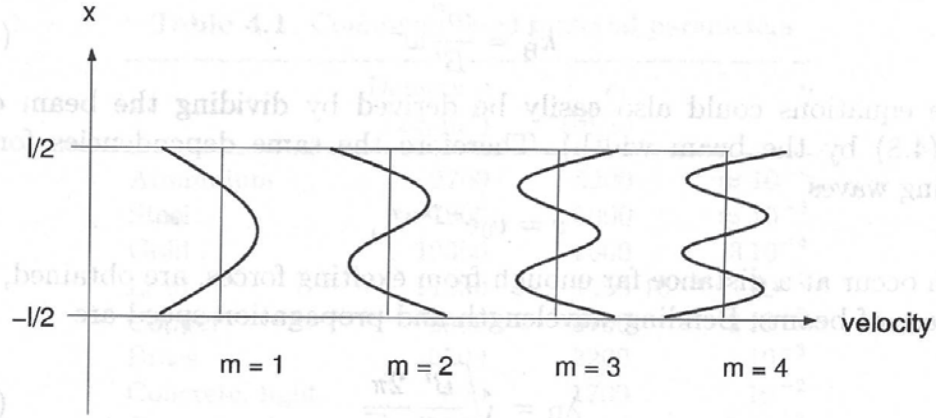


**Fig. 4.11.** Vibration modes of bilaterally suspended beams

## 4.5 Bending waves in plates

As can be shown with similar investigations as for the one-dimensional beam, the bending wave equation for homogeneous plates is given by

$$\frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} - \frac{m''}{B'}\omega^2 v = \frac{j\omega p}{B'} \tag{4.47}$$

where $p$ denotes an external force of area (e.g. a pressure) acting on the plate. This time $m''$ represents the mass per unit area

$$m'' = \varrho h \tag{4.48}$$

($\varrho$ is the plate density and $h$ the thickness) which is obtained using the bending stiffness of the beam (for a rectangular cross-section) per unit length in the same way as the bending stiffness of the plate

$$B' = \frac{E}{1-\mu^2}\frac{h^3}{12}\,. \tag{4.49}$$

The denominator containing Poisson's ratio $\mu$ accounts for the fact that plate elements of volume are somewhat stiffer than beam elements. Whereas the material of the beam can elude laterally to a small extent (normal to the beam and the direction of displacement), this possibility is not given in the plate. Depending on the material, it is approximately $\mu \approx 0.3$, so that $\mu^2 \ll 1$ is negligible in so far that material parameters can be specified accurately.

In the case of one-dimensional wave propagation of the plate motion, excited, for instance, by a line force or an impinging sound wave, the bending wave equation becomes

$$\frac{1}{k_B^4}\frac{d^4 v}{dx^4} - v = \frac{jp}{m''\omega}\,, \tag{4.50}$$

with

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

$$k_{\mathrm{B}}^4 = \frac{m''}{B'}\omega^2 \tag{4.51}$$

(these equations could also easily be derived by dividing the beam equation (4.8) by the beam width). Therefore the same dependencies for free bending waves

$$v = v_0 \mathrm{e}^{-jk_{\mathrm{B}}x} ,$$

which occur at a distance far enough from exciting forces, are obtained, as in the case of beams. Bending wavelength and propagation speed are

$$\lambda_{\mathrm{B}} = \sqrt[4]{\frac{B'}{m''}}\frac{2\pi}{\sqrt{\omega}} \tag{4.52}$$

$$c_{\mathrm{B}} = \sqrt[4]{\frac{B'}{m''}}\sqrt{\omega} . \tag{4.53}$$

The propagation speed of bending waves in plates is also frequency-dependent and the transmission of the displacement is dispersive. The wavelength is likewise inversely proportional to the square-root of the frequency. As a matter of principle, a radial symmetric sound field produced by a point force has the same wave form as plane waves in plates.

Due to energy balance, the amplitude has to be inversely proportional to the square-root of the distance $r$ from the source. For not-too-short distances, it is

$$v = \frac{A}{\sqrt{r}}\mathrm{e}^{-jk_{\mathrm{B}}r} .$$

In the course of the discussion of bending waves in beams, the basic physical principles were of interest, whereas this time the focus shall be the relationships and the order of magnitudes which can be found in practice. It should first be stated that the use of the bending stiffness is not very handy in practical calculations. It is thus replaced by the ratio $m''/B'$, which consists of more useful material parameters, while at the same time $\mu^2 \ll 1$ is neglected:

$$\frac{m''}{B'} = \frac{\varrho h 12}{E h^3} = \frac{12}{c_{\mathrm{L}}^2 h^2} \tag{4.54}$$

where

$$c_{\mathrm{L}} = \sqrt{\frac{E}{\varrho}}$$

is the longitudinal wave propagation speed in beams, consisting of the same material, as can be shown. $c_{\mathrm{L}}$, thickness $h$ and mass of area $m''$ are most commonly used for the acoustic description of plates. The influence of the material (expressed as $c_{\mathrm{L}}$), thickness $h$ and frequency $f$ cannot be overlooked directly in (4.52) and (4.53), because the parameters $B'$ and $m''$ not only depend on the material but also on the thickness. The dependencies can clearly be shown by inserting (4.54) into (4.52) and (4.53). We thus obtain the handy equations

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

**Table 4.1.** Commonly used material parameters

| | Density $\varrho$ (kg/m$^3$) | $c_L$ (m/s) | $\eta$ |
|---|---|---|---|
| Aluminium | 2700 | 5200 | $\approx 10^{-4}$ |
| Steel | 7800 | 5000 | $\approx 10^{-4}$ |
| Gold | 19300 | 2000 | $3\,10^{-4}$ |
| Lead | 11300 | 1250 | $10^{-3} \ldots 10^{-1}$ |
| Copper | 8900 | 3700 | $2\,10^{-3}$ |
| Brass | 8500 | 3200 | $10^{-3}$ |
| Concrete, light | 600 | 1700 | $10^{-2}$ |
| Concrete, dense | 2300 | 3400 | $5\,10^{-3}$ |
| Bricks (+mortar) | 2000 | 2500–3000 | $10^{-2}$ |
| Plywood | 600 | 3000 | $10^{-2}$ |
| Oak | 700–1000 | 1500–3500 | $10^{-2}$ |
| Spruce | 400–700 | 1200–2500 | $10^{-2}$ |
| Plaster | 1200 | 2400 | $8\,10^{-3}$ |
| Beaverboard | 600–700 | 2700 | $10^{-2}$ |
| Perspex | 1150 | 2200 | $3\,10^{-2}$ |
| Sand, light | 1500 | 100–200 | $10^{-1}$ |
| Sand, dense | 1700 | 200–500 | $10^{-2}$ |
| Glass | 2500 | 4900 | $2\,10^{-3}$ |

$$\lambda_B \approx 1.35\sqrt{\frac{hc_L}{f}}$$

and for the propagation speed

$$c_B \approx 1.35\sqrt{hc_L f}\,.$$

to describe bending wavelength, respectively.

The parameters for the materials most commonly used in practice are given in Table 4.1. The loss factors correspond to pure internal damping, thus radiation losses and the damping at the connections of some compounds (e.g. bolt connections) have to be added when appropriate.

The longitudinal propagation speeds of different materials do not differ very much, as can be seen in Table 4.1: the range of 2000 m/s to 5000 m/s is roughly covered, whereas the range of thicknesses used in acoustics is a lot larger. Metal sheets of 0.5 mm thickness in vehicles are of the same interest as concrete walls of 0.5 m thickness, which represents a considerable ratio of 1:1000. The range of wavelengths is accordingly large. For 1000 Hz, for instance,

$$\lambda_B(0.5\,\text{mm metal sheet}) = 7\,\text{cm}$$
$$\lambda_B(25\,\text{cm light concrete}) = 90\,\text{cm}$$
$$\text{for comparison:}\quad \lambda_0(\text{air}) = 34\,\text{cm}$$

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

is obtained. As it becomes obvious by the examples, shorter wavelengths than in air occur in thin plates and longer wavelengths in thick plates.

The differences between 'long-wave' and 'short-wave' components for the transmission loss are of major importance, as it will be explained in Chap. 8 of this book. Generally, the property 'shorter wavelength than air' (or 'longer wavelength than air') can be assigned to a frequency interval. The two wavelengths $\lambda_0$ (for air) and $\lambda_B$ (for the bending waves)

$$\lambda_0 = c/f \quad \text{and} \quad \lambda_B \approx 1.35\sqrt{\frac{hc_L}{f}}$$

are equal at a typical 'critical' frequency $f_{cr}$. Taking the square of the wavelengths and setting both equations to be equal shows that

$$f_{cr} = \frac{c^2}{1.82hc_L}$$

is obtained for the critical frequency $f_{cr}$. This is also called 'limiting frequency' or 'coincidence frequency'. For frequencies

- $f < f_{cr}$ below the critical frequency, the bending waves are shorter than the air waves, for frequencies
- $f > f_{cr}$ above the critical frequency, the bending waves are longer than the air waves.

Some examples of the critical frequency are given in the following table:

| Plate consisting of | $f_{cr}$(Hz) |
|---|---|
| 0.5 mm metal sheet | 25000 |
| 4 mm glass | 3000 |
| 5 cm plaster | 530 |
| 25 cm heavy concrete | 75 |

Here, it is clear that thick and massive walls and ceilings have a limiting frequency at the lower edge of the frequency range of interest, whereas the critical frequencies of windows and metal sheets are located at the upper edge.

Thin walls built, for example, into flats or offices, have a limiting frequency right in the middle of the frequency range of interest, thus they can suffer from poor transmission loss.

### 4.5.1 Plate vibrations

As is the case with beams, the vibrations and modal forms of plates are also dependent on the material their bearings consist of. With the exception of the scenario described below, where a plate is supported on all sides by its bearing, handling problems with plate resonance is by no way an easy task.

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

Usually, finding workable solutions entails complex and lengthy quantitative considerations, which have filled pages in lengthy treatises especially devoted to discussing this problem. Readers interested in more in-depth information on this topic are encouraged to take a look at Leissa, A.W.: 'Vibration of Plates', Office of Technology Utilization, National Aeronautics and Space Administration, Washington 1969 and Blevins, R. D.: 'Formulas for natural frequency and mode shape', Van Nostrand Reinhold, New York 1979. The latter work also contains sections discussing the vibration of beams, in particular, combined with various factors affecting the left and right ends.

For plates supported on all four edges $x = 0$, $y = 0$, $x = l_x$ and $y = l_y$, modal forms exist in

$$v = \sin(n_x \pi x / l_x) \sin(n_y \pi y / l_y) \tag{4.55}$$

($n_x = 1, 2, 3, ...$ and $n_y = 1, 2, 3, ...$). The plate dimensions are defined by $l_x$ and $l_y$, the plate surface is therefore $S = l_x l_y$. The plate modes (1,1), (1,2) and (2,2) are shown in Figures 4.12, 4.13 and 4.14.

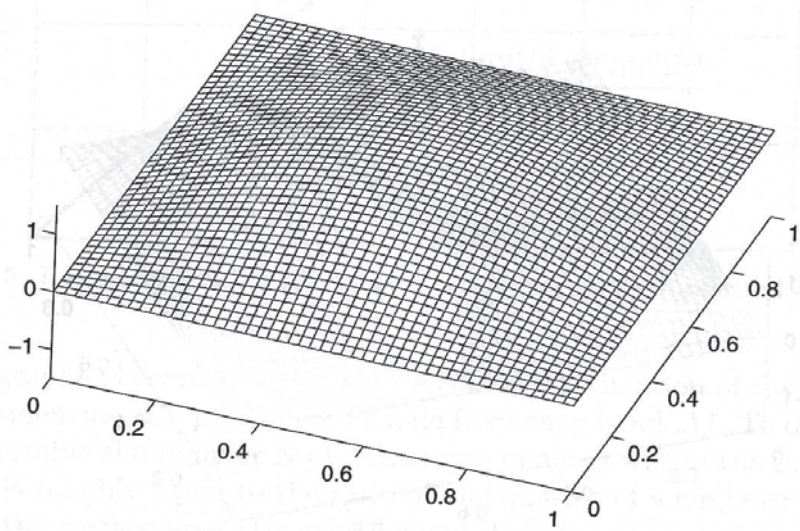


**Fig. 4.12.** Vibration mode $n_x = 1$ and $n_y = 1$

The resonance frequencies are obtained by plugging the modal forms into the bending wave equation (homogenous with $p = 0$) (4.47):

$$k_B^4 = \frac{m''}{B'}\omega^2 = (\frac{n_x \pi}{l_x})^4 + 2(\frac{n_x \pi}{l_x})^2(\frac{n_y \pi}{l_y})^2 + (\frac{n_y \pi}{l_y})^4 = [(\frac{n_x \pi}{l_x})^2 + (\frac{n_y \pi}{l_y})^2]^2 , \tag{4.56}$$

therefore

$$f = \frac{\pi}{2}[(\frac{n_x}{l_x})^2 + (\frac{n_y}{l_y})^2]\sqrt{\frac{B'}{m''}} . \tag{4.57}$$

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

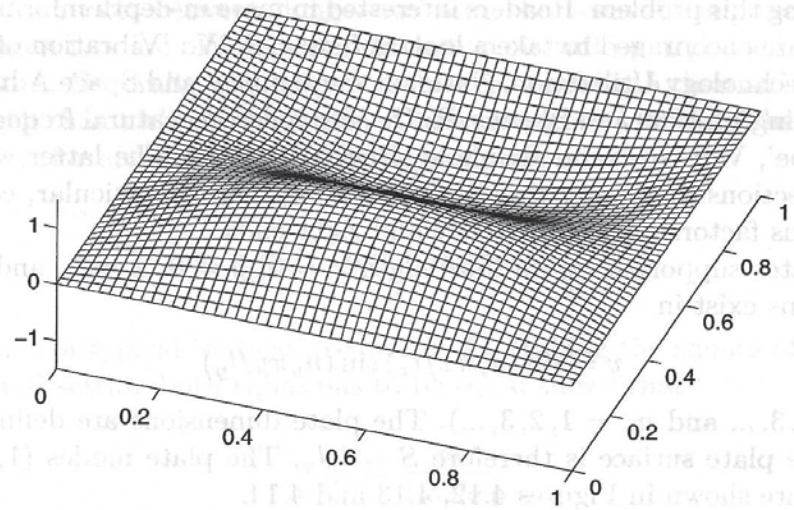


**Fig. 4.13.** Vibration mode $n_x = 1$ and $n_y = 2$

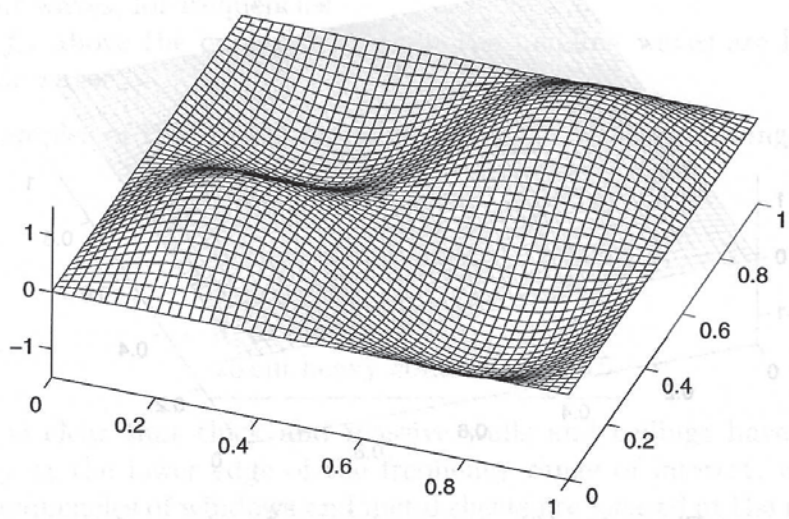


**Fig. 4.14.** Vibration mode $n_x = 2$ and $n_y = 2$

The resonance frequencies can be easily represented in graph form by taking the root of the last equation, thereby obtaining a grid point for each frequency root. These grid points exist in a constant eigenfrequency grid with grid edge lengths (Figure 4.15)

$$\sqrt{\frac{\pi}{2}}\sqrt[4]{\frac{B'}{m''}}\frac{1}{l_x}$$

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

and

$$\sqrt{\frac{\pi}{2}}\sqrt[4]{\frac{B'}{m''}}\frac{1}{l_y}\,.$$

Every grid node pertains to the root of each resonance frequency.

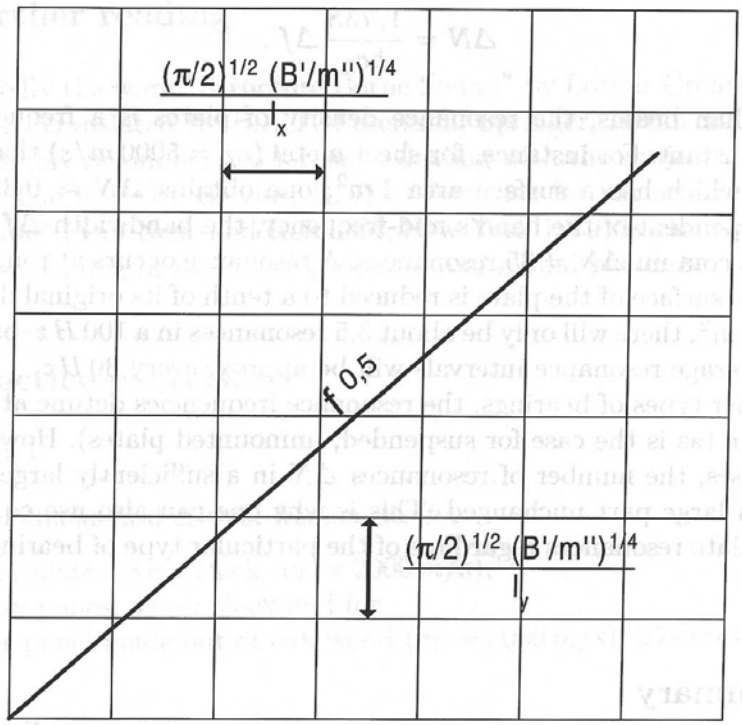


**Fig. 4.15.** Resonance grid of bending vibrations on plates supported at all edges

This graphical representation allows a simple estimation of the total resonance frequencies $\Delta N$ present in a single frequency band $\Delta f$. To do this, one must determine the number N of existing resonances within the interval 0 to $f$, which is roughly equal to the ratio of one quarter of a circle with a radius of $\sqrt{f}$ to the surface area of a grid element, that is

$$N = \frac{\frac{\pi}{4}f}{\frac{\pi}{2}\sqrt{\frac{B'}{m''}}\frac{1}{l_x l_y}} = \frac{1}{2}fS\sqrt{\frac{m''}{B'}}\,. \tag{4.58}$$

The differential quotient $\Delta N/\Delta f$ can be roughly substituted by

$$\frac{\Delta N}{\Delta f} \approx \frac{dN}{df} = \frac{1}{2}S\sqrt{\frac{m''}{B'}}\,,$$

thereby resulting in

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

$$\Delta N = \frac{1}{2} S \sqrt{\frac{m''}{B'}} \Delta f \,. \tag{4.59}$$

Furthermore, for practical calculations, it makes more sense to express the relationship of $m''$ and $B'$ in terms of the longitudinal wave speed $c_L$ and the thickness of the plate $h$ according to eq.(4.54). This subsequently yields

$$\Delta N = \frac{1,73S}{hc_L} \Delta f \,. \tag{4.60}$$

Different than beams, the resonance density of plates is a frequency-independent constant. For instance, for sheet metal ($c_L = 5000\,m/s$) that is $1\,mm$ thick and which has a surface area $1\,m^2$, one obtains $\Delta N \approx 0.35\,\Delta f/Hz$. Thus, independent of the band's mid-frequency, the bandwidth $\Delta f = 100Hz$ will always contain $\Delta N = 35$ resonances. A resonance occurs at roughly every $3\,Hz$. If the surface of the plate is reduced to a tenth of its original dimensions at $S = 0,1\,m^2$, there will only be about 3.5 resonances in a $100\,Hz$–bandwidth, and the average resonance intervals will be approx. every $30\,Hz$.

For other types of bearings, the resonance frequencies detune at the edges of the plate (as is the case for suspended, unmounted plates). However, even in such cases, the number of resonances $\Delta N$ in a sufficiently large band $\Delta f$ remains in large part unchanged. This is why one can also use eq. (4.60) to estimate plate resonances regardless of the particular type of bearing arrangement.

## 4.6 Summary

The most essential structure-borne sound waves constitute bending waves occurring on plate and beam structures. As opposed to wave propagation in gases and liquids, flexural wavelengths depend on the frequency, growing proportionally to the square root of $f$. For mechanical sound signals consisting of multiple frequencies, each spectral component is transported at different speeds, leading to a dispersion effect in the signal shape. Due to this dispersion, bending wavelengths are inversely proportional to the square root of the frequency. As a result, bending wavelengths $\lambda_B$ below a certain critical frequency $f_{cr}$ are smaller than sound wavelengths in air $\lambda$. Conversely, $\lambda_B > \lambda$ is true for all $f$ above $f_{cr}$. This fact plays an important role in both bending wave phenomena as well as sound insulation in plate-like structures such as walls, ceilings, windows, etc. (see Chapters 3 and 8). The threshold frequency is inversely proportional to the thickness of the building material $f_{cr} \sim 1/h$. The critical frequency is high for thin building materials and low for thick materials.

Resonance occurs in beam and plate structures of finite lengths, given that only a small amount of energy is lost beyond the material's boundaries. Resonance frequencies depend on the material's bearings. In beams, the distance

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843

between resonance frequencies increases with increasing frequency, that is, the density of resonances decreases. In plate structures, the resonance density is a constant, independent of frequency. The vibration patterns of the resonances are referred to as modes.

## 4.7 Further reading

Undoubtedly the work "Structure Borne Sound" by Lothar Cremer and Manfred Heckl (translated by B.A.T. Petersson, Springer, Berlin and New York 2004) is highly recommended for those dealing with the subject mentioned in the title. The book by Blevins, R. D.: "Formulas for natural frequency and mode shape" (Van Nostrand Reinhold, New York 1979) is also recommended for further reading in resonance processes and modal patterns.

## 4.8 Practice exercises

### Problem 1

Find the coincidental critical frequencies for

- Plaster plates, $8\,cm$ thick ($c_L = 2000\,m/s$),
- window panes, $4\,mm$ thick and for
- a door panel made out of oak wood ($c_L = 3000\,m/s$), $25\,mm$ thick.

### Problem 2

Find the first 5 resonance frequencies of aluminum beams with a thickness of $5\,mm$, and $50\,cm$ and one $100\,cm$ long. Consider two cases: bilaterally supported and bilaterally mounted.

### Problem 3

Find the four lowest resonance frequencies

- of a window pane $4\,mm$ thick, measuring $50\,cm$ by $100\,cm$,
- a $10\,cm$–thick plaster wall ($c_L = 2000\,m/s$), measuring $3\,m$ by $3\,m$ and
- a $2\,mm$–enforced steel plate, measuring $20\,cm$ by $25\,cm$.

### Problem 4

Find the resonance frequencies and the modal patterns of the mounted beam to the left and the supported beam to the right.

Shokz, Ex. 2019
Mojawa v. Shokz, IPR2025-00843