An Introduction to Complex Analysis for Engineers

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Preface

These notes are intended to be of use to Third year Electrical and Electronic Engineers at the University of Western Australia coming to grips with Complex Function Theory.

There are many text books for just this purpose, and I have insufficient time to write a text book, so this is not a substitute for, say, Matthews and Howell's *Complex Analysis for Mathematics and Engineering*,[1], but perhaps a complement to it. At the same time, knowing how reluctant students are to use a textbook (except as a talisman to ward off evil) I have tried to make these notes sufficient, in that a student who reads them, understands them, and does the exercises in them, will be able to use the concepts and techniques in later years. It will also get the student comfortably through the examination. The shortness of the course, 20 lectures, for covering Complex Analysis, either presupposes genius (90% perspiration) on the part of the students or material skipped. These notes are intended to fill in some of the gaps that will inevitably occur in lectures. It is a source of some disappointment to me that I can cover so little of what is a beautiful subject, rich in applications and connections with other areas of mathematics. This is, then, a sort of sampler, and only touches the elements.

Styles of Mathematical presentation change over the years, and what was deemed acceptable rigour by Euler and Gauss fails to keep modern purists content. McLachlan, [2], clearly smarted under the criticisms of his presentation, and he goes to some trouble to explain in later editions that the book is intended for a different audience from the purists who damned him. My experience leads me to feel that the need for rigour has been developed to the point where the intuitive and geometric has been stunted. Both have a part in mathematics, which grows out of the conflict between them. But it seems to me more important to penetrate to the ideas in a sloppy, scruffy but serviceable way, than to reduce a subject to predicate calculus and omit the whole reason for studying it. There is no known means of persuading a hardheaded engineer that a subject merits his time and energy when it has been turned into an elaborate game. He, or increasingly she, wants to see two elements at an early stage: procedures for solving problems which make a difference and concepts which organise the procedures into something intelligible. Carried to excess this leads to avoidance of abstraction and consequent

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loss of power later; there is a good reason for the purist's desire for rigour. But it asks too much of a third year student to focus on the underlying logic and omit the geometry.

I have deliberately erred in the opposite direction. It is easy enough for the student with a taste for rigour to clarify the ideas by consulting other books, and to wind up as a logician if that is his choice. But it is hard to find in the literature any explicit commitment to getting the student to draw lots of pictures. It used to be taken for granted that a student would do that sort of thing, but now that the school syllabus has had Euclid expunged, the undergraduates cannot be expected to see drawing pictures or visualising surfaces as a natural prelude to calculation. There is a school of thought which considers geometric visualisation as immoral; and another which sanctions it only if done in private (and wash your hands before and afterwards). To my mind this imposes sterility, and constitutes an attempt by the bureaucrat to strangle the artist. [1] While I do not want to impose my informal images on anybody, if no mention is made of informal, intuitive ideas, many students never realise that there are any. All the good mathematicians I know have a rich supply of informal models which they use to think about mathematics, and it were as well to show students how this may be done. Since this seems to be the respect in which most of the text books are weakest, I have perhaps gone too far in the other direction, but then, I do not offer this as a text book. More of an antidote to some of the others.

I have talked to Electrical Engineers about Mathematics teaching, and they are strikingly consistent in what they want. Prior to talking to them, I feared that I'd find Engineers saying things like 'Don't bother with the ideas, forget about the pictures, just train them to do the sums'. There are, alas, Mathematicians who are convinced that this is how Engineers see the world, and I had supposed that there might be something in this belief. Silly me. In fact, it is simply quite wrong.

The Engineers I spoke to want Mathematicians to get across the abstract ideas in terms the students can grasp and use, so that the Engineers can subsequently rely on the student having those ideas as part of his or her

[1] The bureaucratic temper is attracted to mathematics while still at school, because it appears to be all about following rules, something the bureaucrat cherishes as the solution to the problems of life. Human beings on the other hand find this sufficiently repellant to be put off mathematics permanently, which is one of the ironies of education. My own attitude to the bureaucratic temper is rather that of Dave Allen's feelings about politicians. He has a soft spot for them. It's a bog in the West of Ireland.

thinking. Above all, they want the students to have clear pictures in their heads of what is happening in the mathematics. Since this is exactly what any competent Mathematician also wants his students to have, I haven't felt any need to change my usual style of presentation. This is informal and user-friendly as far as possible, with (because I am a Topologist by training and work with Engineers by choice) a strong geometric flavour.

I introduce Complex Numbers in a way which was new to me; I point out that a certain subspace of 2×2 matrices can be identifed with the plane $\mathbb{R}^2$, thus giving a simple rule for multiplying two points in $\mathbb{R}^2$: turn them into matrices, multiply the matrices, then turn the answer back into a point. I do it this way because (a) it demystifies the business of imaginary numbers, (b) it gives the Cauchy-Riemann conditions in a conceptually transparent manner, and (c) it emphasises that multiplication by a complex number is a similarity together with a rotation, a matter which is at the heart of much of the applicability of the complex number system. There are a few other advantages of this approach, as will be seen later on. After I had done it this way, Malcolm Hood pointed out to me that Copson, [3], had taken the same approach.[2]

Engineering students lead a fairly busy life in general, and the Sparkies have a particularly demanding load. They are also very practical, rightly so, and impatient of anything which they suspect is academic window-dressing. So far, I am with them all the way. They are, however, the main source of the belief among some mathematicians that peddling recipes is the only way to teach them. They do not feel comfortable with abstractions. Their goal tends to be examination passing. So there is some basic opposition between the students and me: I want them to be able to use the material in later years, they want to memorise the minimum required to pass the exam (and then forget it).

I exaggerate of course. For reasons owing to geography and history, this University is particularly fortunate in the quality of its students, and most of them respond well to the discovery that Mathematics makes sense. I hope that these notes will turn out to be enjoyable as well as useful, at least in retrospect.

But be warned:

[2] I am most grateful to Malcolm for running an editorial eye over these notes, but even more grateful for being a model of sanity and decency in a world that sometimes seems bereft of both.

‘ Well of course I didn’t do any at first ... then someone suggested I try just a little sum or two, and I thought “Why not? ... I can handle it”. Then one day someone said “Hey, man, that’s kidstuff - try some calculus” ... so I tried some differentials ... then I went on to integrals ... even the occasional volume of revolution ... but I can stop any time I want to ... I know I can. OK, so I do the odd bit of complex analysis, but only a few times ... that stuff can really screw your head up for days ... but I can handle it ... it’s OK really ... I can stop any time I want ...’ (tim@bierman.demon.co.uk (Tim Bierman))

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Chapter 1

Fundamentals

1.1 A Little History

If Complex Numbers had been invented thirty years ago instead of over three hundred, they wouldn't have been called 'Complex Numbers' at all. They'd have been called 'Planar Numbers', or 'Two-dimensional Numbers' or something similar, and there would have been none of this nonsense about 'imaginary' numbers. The square root of negative one is no more and no less imaginary than the square root of two. Or two itself, for that matter. All of them are just bits of language used for various purposes.

'Two' was invented for counting sheep. All the positive integers (whole numbers) were invented so we could count things, and that's all they were invented for. The negative integers were introduced so it would be easy to count money when you owed more than you had.

The rational numbers were invented for measuring lengths. Since we can transduce things like voltages and times to lengths, we can measure other things using the rational numbers, too.

The Real numbers were invented for wholly mathematical reasons: it was found that there were lengths such as the diagonal of the unit square which, in principle, couldn't be measured by the rational numbers. This is of not the slightest practical importance, because in real life you can measure only to some limited precision, but some people like their ideas to be clean and cool, so they went off and invented the real numbers, which included the

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rationals but also filled in the holes. So practical people just went on doing what they'd always done, but Pure Mathematicians felt better about them doing it. Daft, you might say, but let us be tolerant.

This has been put in the form of a story:

A (male) Mathematician and a (male) Engineer who knew each other, had both been invited to the same party. They were standing at one corner of the room and eyeing a particularly attractive girl in the opposite corner. 'Wow, she looks pretty good,' said the Engineer. 'I think I'll go over there and try my luck.'

'Impossible, and out of the question!' said the Mathematician, who was thinking much the same but wasn't as forthright.

'And why is it impossible?' asked the Engineer belligerently.

'Because,' said the Mathematician, thinking quickly, 'In order to get to her, you will first have to get halfway. And then you will have to get half of the rest of the distance, and then half of that. And so on; in short, you can never get there in a finite number of moves.'

The Engineer gave a cheerful grin.

'Maybe so,' he replied, 'But in a finite number of moves, I can get as close as I need to be for all practical purposes.'

And he made his moves.

***

The Complex Numbers were invented for purely mathematical reasons, just like the Reals, and were intended to make things neat and tidy in solving equations. They were regarded with deep suspicion by the more conservative folk for a century or so.

It turns out that they are very cool things to have for 'measuring' such things as periodic waveforms. Also, the functions which arise between them are very useful for talking about solutions of some Partial Differential Equations. So don't look down on Pure Mathematicians for wanting to have things clean and cool. It pays off in very unexpected ways. The Universe also seems to like things clean and cool. And most supersmart people, such as Gauss, like finding out about Electricity and Magnetism, working out how to handle

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calculations of orbits of asteroids *and* doing Pure Mathematics.

In these notes, I am going to rewrite history and give you the story about Complex Numbers and Functions as if they had been developed for the applications we now know they have. This will short-circuit some of the mystery, but will be regarded as shocking by the more conservative. The same sort of person who three hundred years ago wanted to ban them, is now trying to keep the confusion. It's a funny old world, and no mistake.

Your text books often have an introductory chapter explaining a bit of the historical development, and you should read this in order to be educated, but it isn't in the exam.

1.2 Why Bother With Complex Numbers and Functions?

In mastering the material in this book, you are going to have to do a lot of work. This will consist mainly of chewing a pencil or pen as you struggle to do some sums. Maths is like that. Hours of your life will pass doing this, when you could be watching the X-files or playing basketball, or whatever. There had better be some point to this, right?

There is, but it isn't altogether easy to tell you exactly what it is, because you can only really see the advantages in hindsight. You are probably quite glad now that you learnt to read when you were small, but it might have seemed a drag at the time. Trust me. It will all be worth it in the end.

If this doesn't altogether convince you, then talk to the Engineering Lecturers about what happens in their courses. Generally, the more modern and intricate the material, the more Mathematics it uses. Communication Engineering and Power Transmission both use Complex Functions; Filtering Theory in particular needs it. Control Theory uses the subject extensively. Whatever you think about Mathematicians, your lecturers in Engineering are practical people who wouldn't have you do this course if they thought they could use the time for teaching you more important things.

Another reason for doing it is that it is fun. You may find this hard to believe, but solving problems is like doing exercise. It keeps you fit and healthy and has its own satisfactions. I mean, on the face of it, someone who runs three

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kilometres every morning has to be potty: they could get there faster in a car, right? But some people do it and feel good about themselves because they've done it. Well, what works for your heart and lungs also applies to your brain. Exercising it will make you feel better. And Complex Analysis is one of the tougher and meatier bits of Mathematics. Tough minded people usually like it. But like physical exercise, it hurts the first time you do it, and to get the benefits you have to keep at it for a while.

I don't expect you to buy the last argument very easily. You're kept busy with the engineering courses which are much more obviously relevant, and I'm aware of the pressure you are under. Your main concern is making sure you pass the examination. So I am deliberately keeping the core material minimal.

I am going to start off by assuming that you have never seen any complex numbers in your life. In order to explain what they are I am going to do a bit of very easy linear algebra. The reasons for this will become clear fairly quickly.

1.3 What are Complex Numbers?

Complex numbers are points in the plane, together with a rule telling you how to multiply them. They are two-dimensional, whereas the Real numbers are one dimensional, they form a line. The fact that complex numbers form a plane is probably the most important thing to know about them.

Remember from first year that 2×2 matrices transform points in the plane. To be definite, take

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

for a point, or if you prefer *vector* in $\mathbb{R}^2$ and let

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

be a 2×2 matrix. Placing the matrix to the left of the vector:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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and doing matrix multiplication gives a new vector:

$$\begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$$

This is all old stuff which you ought to be good at by now[1].

Now I am going to look at a subset of the whole collection of 2×2 matrices: those of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for any real numbers a, b.

The following remarks should be carefully checked out:

- These matrices form a *linear subspace* of the four dimensional space of all 2×2 matrices. If you add two such matrices, the result still has the same form, the zero matrix is in the collection, and if you multiply any matrix by a real number, you get another matrix in the set.
- These matrices are also closed under multiplication: If you multiply any two such matrices, say

  $$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

  then the resulting matrix is still antisymmetric and has the top left entry equal to the bottom right entry, which puts it in our set.
- The identity matrix is in the set.
- Every such matrix has an inverse except when both a and b are zero, and the inverse is also in the set.
- The matrices in the set *commute* under multiplication. It doesn't matter which order you multiply them in.
- All the rotation matrices:

  $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  are in the set.

[1] If you are not very confident about this, (a) admit it to yourself and (b) dig out some old Linear Algebra books and practise a bit.

- The columns of any matrix in the set are *orthogonal*
- This subset of all 2×2 matrices is two dimensional.

Exercise 1.3.1 *Before going any further, go through every item on this list and check out that it is correct. This is important, because you are going to have to know every one of them, and verifying them is or ought to be easy.*

This particular collection of matrices IS the set of Complex Numbers. I define the complex numbers this way:

Definition 1.3.1 $\mathbb{C}$ *is the name of the two dimensional subspace of the four dimensional space of* 2×2 *matrices having entries of the form*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for any real numbers a, b. *Points of* $\mathbb{C}$ *are called, for historical reasons,* complex numbers.

There is nothing mysterious or mystical about them, they behave in a thoroughly straightforward manner, and all the properties of any other complex numbers you might have come across are all properties of my complex numbers, too.

You might be feeling slightly gobsmacked by this; where are all the imaginary numbers? Where is $\sqrt{-1}$? Have patience. We shall now gradually recover all the usual hocus-pocus.

First, the fact that the set of matrices is a two dimensional vector space means that we can treat it as if it were $\mathbb{R}^2$ for many purposes. To nail this idea down, define:

$$C : \mathbb{R}^2 \longrightarrow \mathbb{C}$$

by

$$\begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This sets up a one to one correspondence between the points of the plane and the matrices in $\mathbb{C}$. It is easy to check out:

Proposition 1.3.1 *C is a linear map*

It is clearly onto, one-one and an *isomorphism*. What this means is that there is no difference between the two objects as far as the linear space properties are concerned. Or to put it in an intuitive and dramatic manner: You can think of points in the plane $\begin{bmatrix} a \\ b \end{bmatrix}$ or you can think of matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and it makes no practical difference which you choose- at least as far as adding, subtracting or scaling them is concerned. To drive this point home, if you choose the vector representation for a couple of points, and I translate them into matrix notation, and if you add your vectors and I add my matrices, then your result translates to mine. Likewise if we take 3 times the first and add it to 34 times the second, it won't make a blind bit of difference if you do it with vectors or I do it with matrices, so long as we stick to the same translation rules. This is the force of the term *isomorphism*, which is derived from a Greek word meaning 'the same shape'. To say that two things are *isomorphic* is to say that they are basically the same, only the names have been changed. If you think of a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ as being a 'name' of a point in $\mathbb{R}^2$, and a two by two matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as being just a different name for the same point, you will have understood the very important idea of an isomorphism.

You might have an emotional attachment to one of these ways of representing points in $\mathbb{R}^2$, but that is your problem. It won't actually matter which you choose.

Of course, the matrix form uses up twice as much ink and space, so you'd be a bit weird to prefer the matrix form, but as far as the sums are concerned, it doesn't make any difference.

Except that you can multiply the matrices as well as add and subtract and scale them.

And what THIS means is that we have a way of multiplying points of $\mathbb{R}^2$.

Given the points $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ in $\mathbb{R}^2$, I decide that I prefer to think of them as matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$, then I multiply these together

to get (check this on a piece of paper)

$$\begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}$$

Now, if you have a preference for the more compressed form, you can't multiply your vectors, Or can you? Well, all you have to do is to translate your vectors into my matrices, multiply them and change them back to vectors. Alternatively, you can work out what the rules are once and store them in a safe place:

$$\begin{bmatrix} a \\ b \end{bmatrix} * \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$

Exercise 1.3.2 *Work through this carefully by translating the vectors into matrices then multiply the matrices, then translate back to vectors.*

Now there are lots of ways of multiplying points of $\mathbb{R}^2$, but this particular way is very cool and does some nice things. It isn't the most obvious way for multiplying points of the plane, but it is a zillion times as useful as the others. The rest of this book after this chapter will try to sell that idea.

First however, for those who are still worried sick that this seems to have nothing to do with $(a + ib)$, we need to invent a more compressed notation. I define:

Definition 1.3.2 *For all* $a, b \in \mathbb{R}, a + ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

So you now have *three* choices.

1. You can write $a+ib$ for a complex number; a is called the *real* part and b is called the *imaginary* part. This is just ancient history and faintly weird. I shall call this the *classical representation* of a complex number. The i is not a number, it is a sort of tag to keep the two components (a,b) separated.

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2. You can write $\begin{bmatrix} a \\ b \end{bmatrix}$ for a complex number. I shall call this the *point representation* of a complex number. It emphasises the fact that the complex numbers form a plane.

3. You can write
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
for the complex number. I shall call this the *matrix representation* for the complex number.

If we go the first route, then in order to get the right answer when we multiply

$$(a + ib) * (c + id) = ((ac - bd) + i(bc + ad))$$

(which has to be the right answer from doing the sum with matrices) we can sort of pretend that i is a number but that $i^2 = -1$. I suggest that you might feel better about this if you think of the matrix representation as the basic one, and the other two as shorthand versions of it designed to save ink and space.

Exercise 1.3.3 *Translate the complex numbers $(a+ib)$ and $(c+id)$ into matrix form, multiply them out and translate the answer back into the classical form.*

Now pretend that i is just an ordinary number with the property that $i^2 = -1$. Multiply out $(a + ib) * (c + id)$ as if everything is an ordinary real number, put $i^2 = -1$, and collect up the real and imaginary parts, now using the i as a tag. Verify that you get the same answer.

This certainly is one way to do things, and indeed it is traditional. But it requires the student to tell himself or herself that there is something deeply mysterious going on, and it is better not to ask too many questions. Actually, all that is going on is muddle and confusion, which is never a good idea unless you are a politician.

The only thing that can be said about these three notations is that they each have their own place in the scheme of things.

The first, $(a + ib)$, is useful when reading old fashioned books. It has the advantage of using least ink and taking up least space. Another advantage

is that it is easy to remember the rule for multiplying the points: you just carry on as if they were real numbers and remember that $i^2 = -1$. It has the disadvantage that it leaves you with a feeling that something inscrutable is going on, which is not the case.

The second is useful when looking at the geometry of complex numbers, something we shall do a lot. The way in which some of them are close to others, and how they move under transformations or maps, is best done by thinking of points in the plane.

The third is helpful when thinking about the multiplication aspects of complex numbers. Matrix multiplication is something you should be quite comfortable with.

Which is the *right* way to think of complex numbers? The answer is: **All of the above, simultaneously**. To focus on the geometry and ignore the algebra is a blunder, to focus on the algebra and forget the geometry is an even bigger blunder. To use a compact notation but to forget what it means is a sure way to disaster.

If you can be able to flip between all three ways of looking at the complex numbers and choose whichever is easiest and most helpful, then the subject is complicated but fairly easy. Try to find the one true way and cling to it and you will get marmelised. Which is most uncomfortable.

1.4 Some Soothing Exercises

You will probably be feeling a bit gobsmacked still. This is quite normal, and is cured by the following procedure: Do the next lot of exercises slowly and carefully. Afterwards, you will see that everything I have said so far is dead obvious and you will wonder why it took so long to say it. If, on the other hand you decide to skip them in the hope that light will dawn at a later stage, you risk getting more and more muddled about the subject. This would be a pity, because it is really rather neat.

There is a good chance you will try to convince yourself that it will be enough to put off doing these exercises until about a week before the exam. This will mean that you will not know what is going on for the rest of the course, but will spend the lectures copying down the notes with your brain out of

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gear. You won't enjoy this, you really won't.

So sober up, get yourself a pile of scrap paper and a pen, put a chair somewhere quiet and make sure the distractions are somewhere else. Some people are too dumb to see where their best interests lie, but you are smarter than that. Right?

Exercise 1.4.1 *Translate the complex numbers (1+i0), (0 +i1), (3-i2) into the other two forms. The first is often written* 1*, the second as i.*

Exercise 1.4.2 *Translate the complex numbers* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, *and* $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ *into the other two forms.*

Exercise 1.4.3 *Multiply the complex number*

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

by itself. Express in all three forms.

Exercise 1.4.4 *Multiply the complex numbers*

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Now do it for

$$\begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \begin{bmatrix} a \\ -b \end{bmatrix}$$

Translate this into the (a+ib) notation.

Exercise 1.4.5 *It is usual to define the* norm *of a point as its distance from the origin. The convention is to write*

$$\| \begin{bmatrix} a \\ b \end{bmatrix} \| = \sqrt{a^2 + b^2}$$

In the classical notation, we call it the modulus *and write*

$$|a + ib| = \sqrt{a^2 + b^2}$$

There is not the slightest reason to have two different names except that this is what we have always done.

Find a description of the complex numbers of modulus 1 in the point and matrix forms. Draw a picture in the first case.

Exercise 1.4.6 *You can also represent points in the plane by using polar coordinates. Work out the rules for multiplying* (r, θ) *by* (s, ϕ)*. This is a fourth representation, and in some ways the best. How many more, you may ask.*

Exercise 1.4.7 *Show that if you have two complex numbers of modulus 1, their product is of modulus 1. (Hint: This is very obvious in one representation and an amazing coincidence in another. Choose a representation for which it is obvious.)*

Exercise 1.4.8 *What can you say about the polar representation of a complex number of modulus 1?*

Exercise 1.4.9 *What can you say about the effect of multiplying by a complex number of modulus 1?*

Exercise 1.4.10 *Take a piece of graph paper, put axes in the centre and mark on some units along the axes so you go from about* $\begin{bmatrix} -5 \\ -5 \end{bmatrix}$ *in the bottom left corner to about* $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ *in the top right corner. We are going to see what happens to the complex plane when we multiply everything in it by a fixed complex number.*

I shall choose the complex number $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ *for reasons you will see later.*

Choose a point in the plane, $\begin{bmatrix} a \\ b \end{bmatrix}$ *(make the numbers easy) and mark it with a red blob. Now calculate* $(a + ib) * (1/\sqrt{2} + i/\sqrt{2})$ *and plot the result*

in green. Draw an arrow from the red point to the green one so you can see what goes where,

Now repeat for half a dozen points (a+ib). Can you explain what the map from $\mathbb{C}$ to $\mathbb{C}$ does?

Repeat using the complex number 2+0i (2 for short) as the multiplier.

Exercise 1.4.11 *By analogy with the real numbers, we can write the above map as*

$$w = (1/\sqrt{2} + i/\sqrt{2})z$$

which is similar to

$$y = (1/\sqrt{2})\ x$$

but is now a function from $\mathbb{C}$ to $\mathbb{C}$ instead of from $\mathbb{R}$ to $\mathbb{R}$.

Note that in functions from $\mathbb{R}$ to $\mathbb{R}$ we can draw the graph of the function and get a picture of it. For functions from $\mathbb{C}$ to $\mathbb{C}$ we cannot draw a graph! *We have to have other ways of visualising complex functions, which is where the subject gets interesting. Most of this course is about such functions.*

Work out what the simple (!) function $w = z^2$ does to a few points. This is about the simplest non-linear function you could have, and visualising what it does in the complex plane is very important. The fact that the real function $y = x^2$ has graph a parabola will turn out to be absolutely no help at all.

Sort this one out, and you will be in good shape for the more complicated cases to follow.

Warning: This will take you a while to finish. It's harder than it looks.

Exercise 1.4.12 *The rotation matrices*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

are the complex numbers of modulus one. If we think about the point representation of them, we get the points $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *or* $\cos\theta + i\sin\theta$ *in classical notation.*

The fact that such a matrix rotates the plane by the angle θ means that multiplying by a complex number of the form $\cos\theta + i\sin\theta$ just rotates the plane by an angle θ. This has a strong bearing on an earlier question.

If you multiply the complex number $\cos\theta + i\sin\theta$ by itself, you just get $\cos 2\theta + i\sin 2\theta$. Check this carefully.

What does this tell you about taking square roots of these complex numbers?

Exercise 1.4.13 *Write out the complex number $\sqrt{3}/2 + i$ in polar form, and check to see what happens when you multiply a few complex numbers by it. It will be easier if you put* everything *in polar form, and do the multiplications also in polars.*

Remember, I am giving you these different forms in order to make your life easier, not to complicate it. Get used to hopping between different representations and all will be well.

1.5 Some Classical Jargon

We write $1 + i0$ as 1, $a + i0$ as a, $0 + ib$ as ib. In particular, the origin $0 + i0$ is written 0.

You will often find 4 + 3i written when strictly speaking it should be 4 + i3. This is one of the differences that don't make a difference.

We use the following notation: $\Re(x + iy) = x$ which is read: 'The real part of the complex number x+iy is x.'

And

$\Im(x + iy) = y$ which is read : 'The imaginary part of the complex number x+iy is y.' The $\Im$ sign is a letter I in a font derived from German Blackletter. Some books use 'Re(x+iy)' in place of $\Re(x + iy)$ and 'Im(x+iy)' in place of $\Im(x + iy)$.

We also write

$\overline{x + iy} = x - iy$, and call $\bar{z}$ the *complex conjugate of* z for any complex number z.

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Notice that the complex conjugate of a complex number in matrix form is just the transpose of the matrix; reflect about the principal diagonal.

The following 'fact' will make some computations shorter:

$|z|^2 = z\bar{z}$

Verify it by writing out z as $x + iy$ and doing the multiplication.

Exercise 1.5.1 *Draw the triangle obtained by taking a line from the origin to the complex number x+iy, drawing a line from the origin along the X axis of length x, and a vertical line from (x,0) up to x+iy. Mark on this triangle the values* $|x + iy|$, $\Re(x + iy)$ *and* $\Im(x + iy)$.

Exercise 1.5.2 *Mark on the plane a point* $z = x + iy$. *Also mark on* $-z$ *and* $\bar{z}$.

Exercise 1.5.3 *Verify that* $\bar{\bar{z}} = z$ *for any* z.

The exercises will have shown you that it is easy to write out a complex number in Polar form. We can write

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

where $\theta = \arccos x = \arcsin y$, and $r = |z|$.

We write:

$\arg(z) = \theta$ in this case. There is the usual problem about adding multiples of 2π, we take the *principal value* of θ as you would expect. $\arg(0 + 0i)$ is not defined.

Exercise 1.5.4 *Calculate* $\arg(1 + i)$

I apologise for this jargon; it does help to make the calculations shorter after a bit of practice, and given that there have been four centuries of history to accumulate the stuff, it could be a lot worse.

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In general, I am more concerned with getting the ideas across than the jargon, which often obscures the ideas for beginners. Jargon is usually used to keep people from understanding what you are doing, which is childish, but the method only works on those who haven't seen it before. Once you figure out what it actually means, it is pretty simple stuff.

Exercise 1.5.5 *Show that*

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$$

Do it the long way by expanding z as $x + iy$ and the short way by cross multiplying. Is cross multiplying a respectable thing to do? Explain your position.

Note that $z\bar{z}$ is always real (the i component is zero). Use this for calculating

$$\frac{1}{4+3i} \text{ and } \frac{1}{5+12i}$$

Express your answers in the classical form $a+ib$.

Exercise 1.5.6 *Find $\frac{1}{z}$ when $z = r(\cos\theta + i\sin\theta)$ and express the answers in polar form.*

Exercise 1.5.7 *Find $\frac{1}{z}$ when*

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Express your answer in classical, point, polar and matrix forms.

Exercise 1.5.8 *Calculate*

$$\frac{2 - i3}{5 + i12}$$

Express your answer in classical, point, polar and matrix forms.

It should be clear from doing the exercises, that you can find a multiplicative inverse for any complex number except 0. Hence you can divide z by w for any complex numbers z and w except when $w = 0$.

This is most easily seen in the matrix form:

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Exercise 1.5.9 *Calculate the inverse matrix to*

$$z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and show it exists except when both a and b are zero

The classical jargon leads to some short and neat arguments which can all be worked out by longer calculations. Here is an example:

Proposition 1.5.1 (The Triangle Inequality) *For any two complex numbers z, w:*

$$|z + w| \leq |z| + |w|$$

Proof:

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\
&= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\
&= |z|^2 + 2\Re(z\bar{w}) + |w|^2 \\
&\leq |z|^2 + 2|\Re(z\bar{w})| + |w|^2 \\
&\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\
&\leq (|z| + |\bar{w}|)^2
\end{aligned}$$

Hence

$$|z + w| \leq |z| + |w|$$

since $|w| = |\bar{w}|$. □

Check through the argument carefully to justify each stage.

Exercise 1.5.10 *Prove that for any two complex numbers z, w, $|zw| = |z||w|$.*

1.6 The Geometry of Complex Numbers

The first thing to note is that as far as addition and scaling are concerned, we are in $\mathbb{R}^2$, so there is nothing new. You can easily draw the line segment

$$t(2 - i3) + (1 - t)(7 + i4), t \in [0, 1]$$

and if you do this in the point notation, you are just doing first year linear algebra again. I shall assume that you can do this and don't find it very exciting.

Life starts to get more interesting if we look at the geometry of multiplication. For this, the matrix form is going to make our life simpler.

First, note that any complex number can be put in the form $r(\cos\theta + i\sin\theta)$, which is a real number multiplying a complex number of modulus 1. This means that it is a multiple of some point lying on the unit circle, if we think in terms of points in the plane. If we take r positive, then this expression is unique up to multiples of 2π; if r is zero then it isn't. I shall NEVER take r negative in this course, and it is better to have nothing to do with those low-life who have been seen doing it after dark.

If we write this in matrix form, we get a much clearer picture of what is happening: the complex number comes out as the matrix:

$$r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

If you stop to think about what this matrix does, you can see that the r part merely stretches everything by a factor of r. If $r = 2$ then distances from the origin get doubled. Of course, if $0 < r < 1$ then the stretch is actually a compression, but I shall use the word 'stretch' in general.

The

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

part of the complex number merely rotates by an angle of θ in the positive (anti-clockwise) sense.

It follows that multiplying by a complex number is a mixture of a stretching by the *modulus* of the number, and a rotation by the *argument* of the number.

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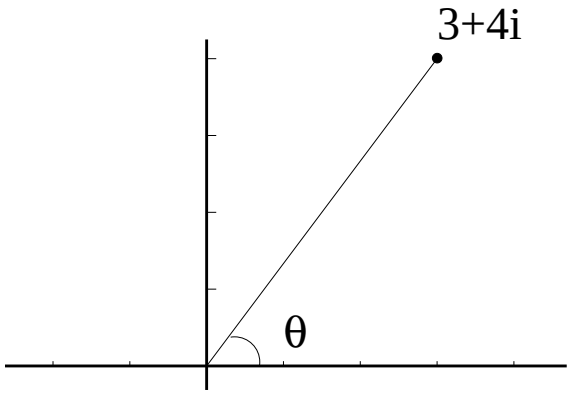

Figure 1.1: Extracting Roots

And this is all that happens, but it is enough to give us some quite pretty results, as you will see.

Example 1.6.1 *Find the fifth root of 3+i4*

Solution *The complex number can be drawn in the usual way as in figure 1.1, or written as the matrix*

$$5\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta = \arcsin 4/5$. *The simplest representation is probably in polars,* $(5, \arcsin 4/5)$, *or if you prefer*

$$5(\cos\theta + i\sin\theta)$$

A fifth root can be extracted by first taking the fifth root of 5. This takes care of the stretching. The rotation part or angular part is just one fifth of the angle. There are actually five distinct solutions:

$$5^{1/5}(\cos\phi + i\sin\phi)$$

for $\phi = \theta/5, (\theta+2\pi)/5, (\theta+4\pi)/5, (\theta+6\pi)/5, (\theta+8\pi)/5$, *and* $\theta = \arcsin 4/5 = \arccos 3/5$.

I have hopped into the polar and classical forms quite cheerfully. Practice does it.

Exercise 1.6.1 *Draw the fifth roots on the figure (or a copy of it).*

Example 1.6.2 *Draw two straight lines at right angles to each other in the complex plane. Now choose a complex number, z, not equal to zero, and multiply every point on each line by z. I claim that the result has to be two straight lines, still cutting at right angles.*

Solution *The smart way is to point out that a scaling of the points along a straight line by a positive real number takes it to a straight line still, and rotating a straight line leaves it as a straight line. So the lines are still lines after the transformation. A rigid rotation won't change an angle, nor will a uniform scaling. So the claim has to be correct. In fact multiplication by a non-zero complex number, being just a uniform scaling and a rotation, must leave any angle between lines unchanged, not just right angles.*

The dumb way is to use algebra.

Let one line be the set of points

$$L = \{w \in \mathbb{C} : w = w_0 + tw_1, \exists t \in \mathbb{R}\}$$

for w_0 and w_1 some fixed complex numbers, and $t \in \mathbb{R}$. Then transforming this set by multiplying everything in it by z gives

$$zL = \{w \in \mathbb{C} : w = zw_0 + tzw_1, \exists t \in \mathbb{R}\}$$

which is still a straight line (through zw_0 in the direction of zw_1).

If the other line is

$$L' = \{w \in \mathbb{C} : w = w_0' + tw_1', \exists t \in \mathbb{R}\}$$

then the same applies to this line too.

If the lines L, L' are at right angles, then the directions w_1, w_1' are at right angles. If we take

$$w_1 = u + iv \quad \text{and} \quad w_1' = u' + iv'$$

then this means that we must have

$$uu' + vv' = 0$$

We need to show that zw_1 *and* zw_1' *are also at right angles. if* $z = x + iy$*, then we need to show*

$$uu' + vv' = 0 \;\Rightarrow (xu - yv)(xu' - yv') + (xv + yu)(xv' + yu') = 0$$

The right hand side simplifies to

$$(x^2 + y^2)(uu' + vv')$$

so it is true.

The above problem and the two solutions that go with it carry an important moral. It is this: If you can see what is going on, you can solve some problems instantly just by looking at them. And if you can't, then you just have to plug away doing algebra, with a serious risk of making a slip and wasting hours of your time as well as getting the wrong answer.

Seeing the patterns that make things happen the way they do is quite interesting, and it is boring to just plug away at algebra. So it is worth a bit of trouble trying to understand the stuff as opposed to just memorising rules for doing the sums.

If you can cheerfully hop to the matrix representation of complex numbers, some things are blindingly obvious that are completely obscure if you just learn the rules for multiplying complex numbers in the classical form. This is generally true in Mathematics, if you have several different ways of thinking about something, then you can often find one which makes your problems very easy. If you try to cling to the one true way, then you make a lot of work for yourself.

1.7 Conclusions

I have gone over the fundamentals of Complex Numbers from a somewhat different point of view from the usual one which can be found in many text

books. My reasons for this are starting to emerge already: the insight that you get into why things are the way they are will help solve some practical problems later.

There are lots of books on the subject which you might feel better about consulting, particularly if my breezy style of writing leaves you cold. The recommended text for the course is [1], and it contains everything I shall do, and in much the same order. It also contains more, and because you are doing this course to prepare you to handle other applications I am leaving to your lecturers in Engineering, it is worth buying for that reason alone. These notes are very specific to the course I am giving, and there's a lot of the subject that I shan't mention.

I found [4] a very intelligent book, indeed a very exciting book, but rather densely written. The authors, Carrier, Krook and Pearson, assume that you are extremely smart and willing to work very hard. This may not be an altogether plausible model of third year students. The book [3] by Copson is rather old fashioned but well organised. Jameson's book, [5], is short and more modern and is intended for those with more of a taste for rigour. Phillips, [6], gets through the material efficiently and fast, I liked Kodaira, [7], for its attention to the topological aspects of the subject, it does it more carefully than I do, but runs into the fundamental problems of rigour in the area: it is very, very difficult. McLachlan's book, [2], has lots of good applications and Esterman's [8] is a middle of the road sort of book which might suit some of you. It does the course, and it claims to be rigorous, using the rather debatable standards of the sixties. The book [9] by Jerrold Marsden is a bit more modern in approach, but not very different from the traditional. Finally, [10] by Ahlfors is a classic, with all that implies.

There are lots more in the library; find one that suits you.

The following is a proposition about Mathematics rather than in Mathematics:

Proposition 1.7.1 (Alder's Law about Learning Maths) *Confusion propagates. If you are confused to start with, things can only get worse.*

You will get more confused as things pile up on you. So it is necessary to get very clear about the basics.

The converse to Mike Alder's law about confusion is that if you sort out the

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basics, then you have a much easier life than if you don't.

So do the exercises, and suffer less.

Chapter 2

Functions from $\mathbb{C}$ to $\mathbb{C}$: Some Easy Examples

The complex numbers form what Mathematicians call (for no very good reason) a *field*, which is a collection of things you can add, subtract, multiply and (except in the case of 0) divide. There are some rules saying precisely what this means, for instance the associativity 'laws', but they are just the rules you already know for the real numbers. So every operation you can do on real numbers makes sense for complex numbers too.

After you learnt about the real numbers at school, you went on to discuss functions such as $y = mx + c$ and $y = x^2$. You may have started off by discussing functions as input-output machines, like slot machines that give you a bottle of coke in exchange for some coins, but you pretty quickly went on to discuss functions by looking at their graphs. This is the main way of thinking about functions, and for many people it is the only way they ever meet.

Which is a pity, because with complex functions it doesn't much help.

The graph of a function from $\mathbb{R}$ to $\mathbb{R}$ is a subset of $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$. The graph of a function from $\mathbb{C}$ to $\mathbb{C}$ will be a two-dimensional subset of $\mathbb{C} \times \mathbb{C}$ which is a surface sitting in four dimensions. Your chances with four dimensional spaces are not good. It is true that we can visualise the real part and imaginary part separately, because each of these is a function from $\mathbb{R}^2$ to $\mathbb{R}$ and has graph a surface. But this loses the relationship between the two components. So we need to go back to the input-output idea if we are to visualise complex

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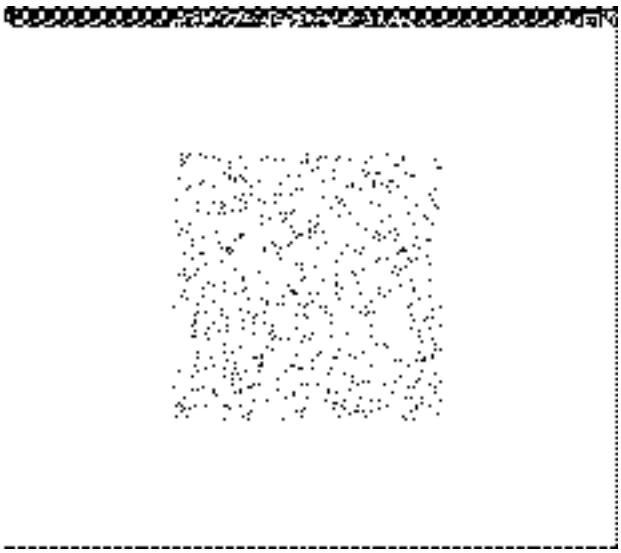

Figure 2.1: The random points in a square

functions.

2.1 A Linear Map

I have written a program which draws some random dots inside the square

$$\{x + iy \in \mathbb{C} : -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

which is shown in figure 2.1.

The second figure 2.2 shows what happens when each of the points is multiplied by the complex number $0.7 + i0.1$. The set is clearly stretched by a number less than 1 and rotated clockwise through a small angle.

This is about as close as we can get to visualising the map

$$w = (0.7 + i0.1)z$$

This is analogous to, say, $y = 0.7x$, which shrinks the line segment [-1,1] down to [-0.7,0.7] in a similar sort of way. We don't usually think of such a map as shrinking the real line, we usually think of a graph.

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Figure 2.2: After multiplication

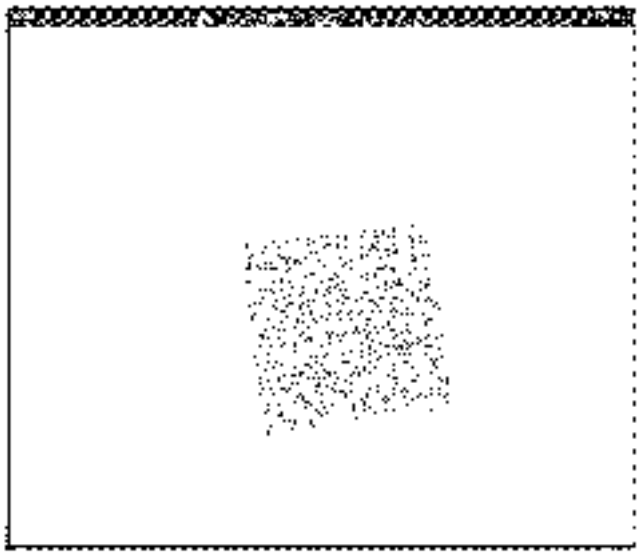

Figure 2.3: After multiplication and shifting

And this is about as simple a function as you could ask for.

For a slightly more complicated case, the next figure 2.3 shows the effect of

$$w = (0.7 + i0.1)z + (0.2 - i0.3)$$

which is rather predictable.

Functions of the form $f(z) = wz$ for some fixed w are the *linear* maps from $\mathbb{C}$ to $\mathbb{C}$. Functions of the form $f(z) = w_1 z + w_2$ for fixed w_1, w_2 are called *affine* maps. Old fashioned engineers still call the latter 'linear'; they shouldn't. The distinction is often important in engineering. The adding of some constant vector to every vector in the plane used to be called a *translation*. I prefer the term *shift*. So an affine map is just a linear map with a shift.

The terms 'function', 'transformation', 'map', 'mapping' all mean the same thing. I recommend *map*. It is shorter, and all important and much used terms should be short. I shall defer to tradition and call them complex functions much of the time. This is shorter than 'map from $\mathbb{C}$ to $\mathbb{C}$', which is necessary in general because you do need to tell people where you are coming from and where you are going to.

2.2 The function $w = z^2$

We can get some idea of what the function $w = z^2$ does by the same process. I have put rather more dots in the before picture, figure 2.4 and also made it smaller so you could see the 'after' picture at the same scale.

The picture in figure 2.5 shows what happens to the data points after we square them. Note the greater concentration in the centre.

Exercise 2.2.1 *Can you explain the greater concentration towards the origin?*

Exercise 2.2.2 *Can you work out where the sharp ends came from? Why are there only two pointy bits? Why are they along the Y-axis? How pointy are they? What is the angle between the opposite curves?*

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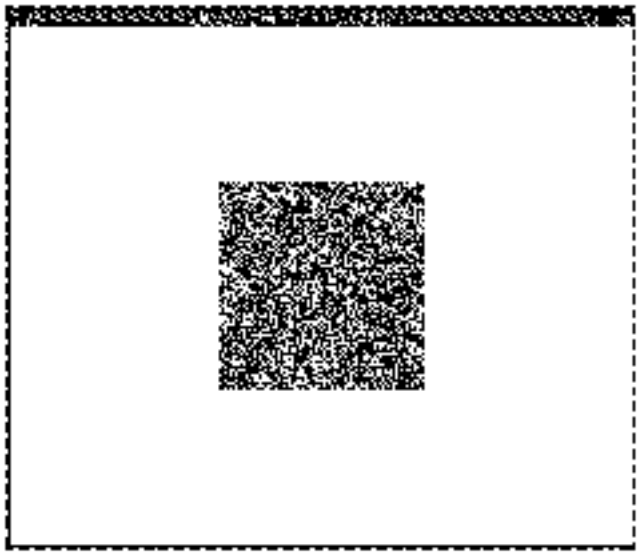

Figure 2.4: The square again

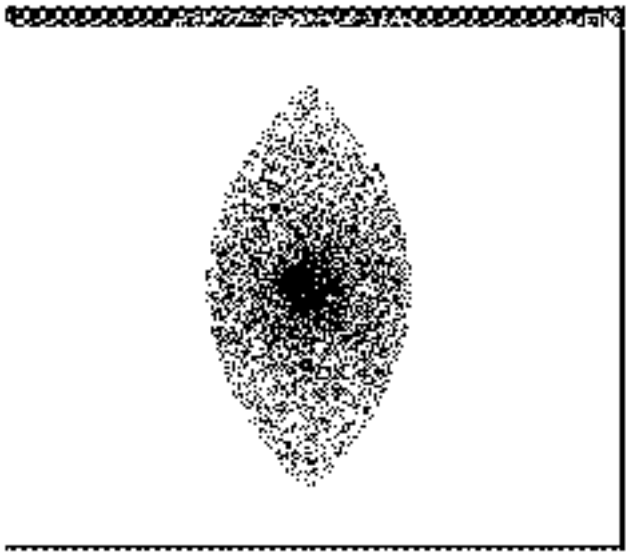

Figure 2.5: After Squaring the square

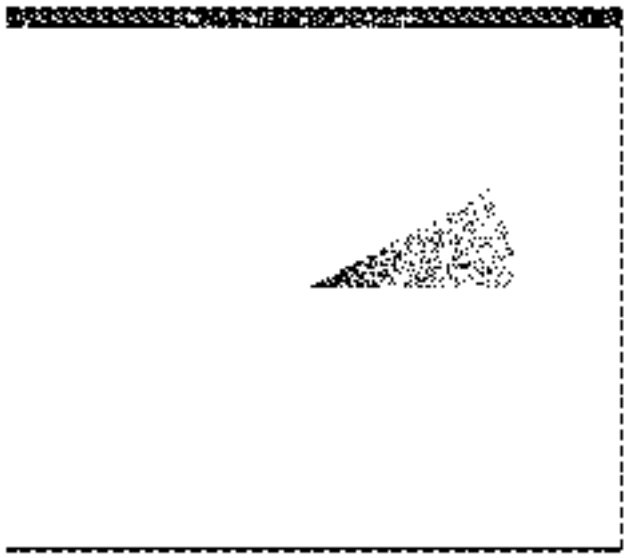

Figure 2.6: A sector of the unit disk

Exercise 2.2.3 *Try to get a clearer picture of what $w = z^2$ does by calculating some values. I suggest you look at the unit circle for a start, and see what happens there. Then check out to see how the radial distance from the origin (the modulus) of the points enters into the mapping.*

It is possible to give you some help with the last exercise: in figure 2.6 I have shown some points placed in a sector of the unit disk, and in figure 2.7 I have shown what happens when each point is squared. You should be able to calculate the squares for enough points on a calculator to see what is going on.

Your calculations can sometimes be much simplified by doing them in polars, and your points should be chosen judiciously rather than randomly.

As an alternative, those of you who can program a computer can do what I have done, and write a little program to do it for you. If you cannot program, you should learn how to do so, preferably in C or PASCAL. MATLAB can also do this sort of thing, I am told, but it seems to take longer to do easy things like this. An engineer who can't program is an anomaly. It isn't difficult, and it's a useful skill.

Exercise 2.2.4 *Can you see what would happen under the function $w = z^2$*

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Figure 2.7: After Squaring the Sector

if we took a sector of the disk in the second quadrant instead of the first?

Exercise 2.2.5 *Can you see what would happen to a sector in the first segment which had a radius from zero up to 2 instead of up to 1? If it only went up to 0.5?*

Example 2.2.1 *Can you see what happens to the X-axis under the same function? The Y-axis? A coordinate grid of horizontal and vertical lines?*

Solution

The program has been modified a bit to draw the grid points as shown in figure 2.8. (If you are viewing this on the screen, the picture may be grottied up a bit. It looks OK at high enough resolution). The squared grid points are shown in figure 2.9.

The rectangular grid gets transformed into a parabolic grid, and we can use this for specifying coordinates just as well as a rectangular one. There are some problems where this is a very smart move.

Note that the curves intersect at what looks suspiciously like a right angle. Is it?

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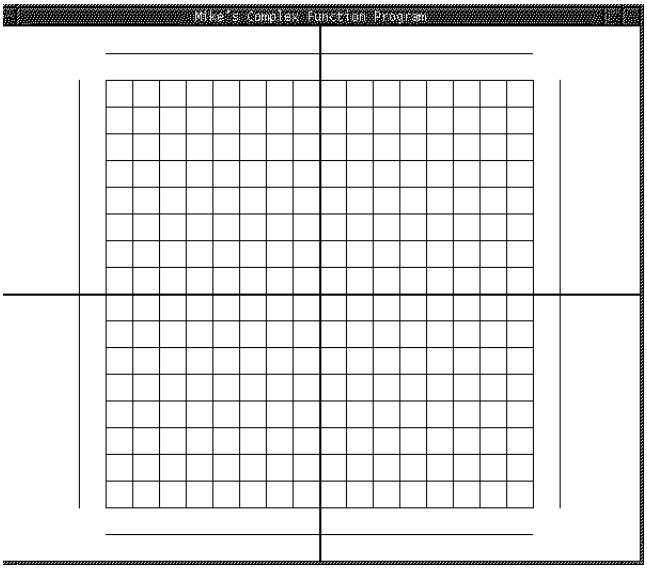

Figure 2.8: The usual Coordinate Grid

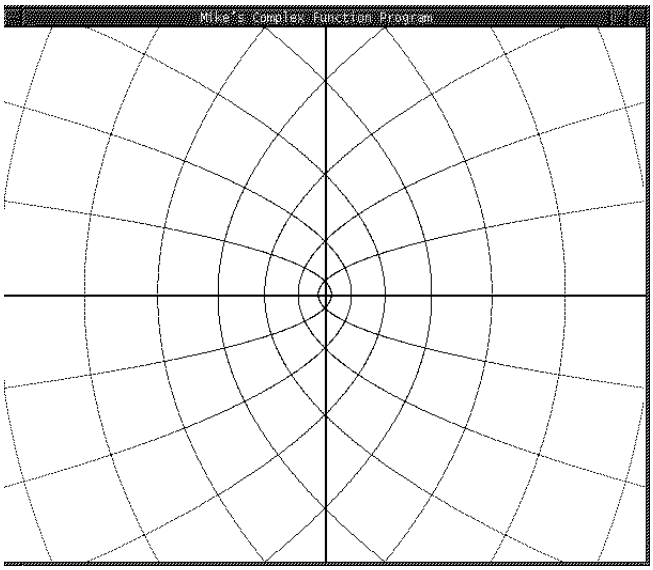

Figure 2.9: The result of squaring all the grid points: A NEW coordinate Grid

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Exercise 2.2.6 *Can you work out what would happen if we took instead the function $w = z^3$? For the case of a sector of the unit disk, or of a grid of points?*

It is rather important that you develop a feel for what simple functions do to the complex plane or bits of it. You are going to need as much expertise with Complex functions as you have with real functions, and so far we have only looked at a few of them.

In working out what they do, you have a choice: either learn to program so that you can do all the sums the easy way, or get very fast at doing them on a calculator, or use a lot of intelligence and thought in deciding how to choose some points that will tell you the most about the function. It is the last method which is best; you can fail to get much enlightenment from looking at a bunch of dots, but the process of figuring out what happens to lines and curves is very informative.

Example 2.2.2 *What is the image under the map $f(z) = z^2$ of the strip of width 1.0 and height 2.0 bounded by the X-axis and the Y-axis on two sides, and having the origin in the lower left corner and the point 1 + i 2 at the top right corner?*

Solution

Let's first draw a picture of the strip so we have a grasp of the before situation. I show this with dots in figure 2.10. I have changed the scale so that the answer will fit on the page.

Look at the bounding edges of our strip: there is a part of the X-axis between 0 and 1 for a start. Where does this go?

Well, the points are all of the form $x + i0$ for $0 \leq x \leq 1$. If you square a complex number with zero imaginary part, the result is real, and if you square a number between 0 and 1, it stays between 0 and 1, although it moves closer to 0. So this part of the edge stays in position, although it gets deformed towards the origin.

Now look at the vertical line which is on the Y-axis. These are the points:

$$\{iy : 0 \leq y \leq 2\}$$

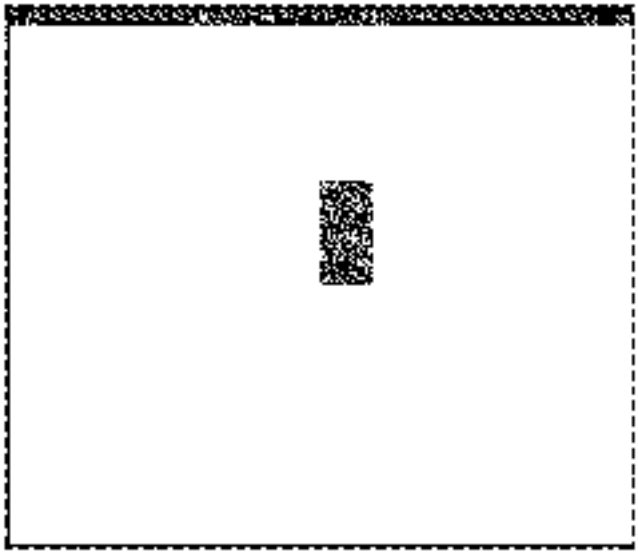

Figure 2.10: A vertical strip

If you square iy you get $-y^2$, and if $0 \leq y \leq 2$ you get the part of the X axis between -4 and 0. So the left and bottom edges of the strip have been straightened out to both lie along the X-axis.

We now look at the opposite edge, the points:

$$\{1 + iy : 0 \leq y \leq 2\}$$

We have

$$(1 + iy)(1 + iy) = (1 - y^2) + i(2y)$$

and if we write the result as $u + iv$ we get that $u = 1 - y^2$ and $v = 2y$. This is a parametric representation of a curve: eliminating $y = v/2$ we get

$$u = 1 - \frac{v^2}{4}$$

which is a parabola. Well, at last we get a parabola in there somewhere*!*

We only get the bit of it which has u lying between 1 and -3, with v lying between 0 and 4.

Draw the bits we have got so far!

Finally, what happens to the top edge of the strip? This is:

$$\{x + i2 : 0 \leq x \leq 1\}$$

which when squared gives

$$\{u + iv : u = x^2 - 4, v = 4x, 0 \leq x \leq 1\}$$

which is a part of the parabola

$$u = \frac{v^2}{16} - 4$$

with one end at $-4 + i0$ *and the other at* $-3 + i4$.

Check that it all joins up to give a region with three bounding curves, two of them parabolic and one linear.

Note how points get 'sucked in' towards the origin, and explain it to yourself.

The points inside the strip go inside the region, and everything inside the unit disk gets pulled in towards the origin, because the modulus of a square is smaller than the modulus of a point, when the latter is less than 1. Everything outside the unit disk gets shifted away from the origin for the same reason, and everything on the unit circle stays on it.

The output of the program is shown in figure 2.11 It should confirm your expectations based on a little thought.

Suppose I had asked what happens to the unit disk under the map $f(z) = z^2$? You should be able to see fairly quickly that it goes to the unit disk, but in a rather peculiar way: far from being the identity map, the perimeter is stretched out to twice its length and wrapped around the unit circle twice.

Some people find this hard to visualise, which gives them a lot of trouble; fortunately you are engineers and good at visualising things.

Looking just at the unit circle to see where that goes: imagine a loop made of chewing gum circling a can of beans.

If we take the loop, stretch it to twice its length and then put it back around the can, circling it twice, then we have performed the squaring map on it.

Figure 2.11: The Strip after Squaring

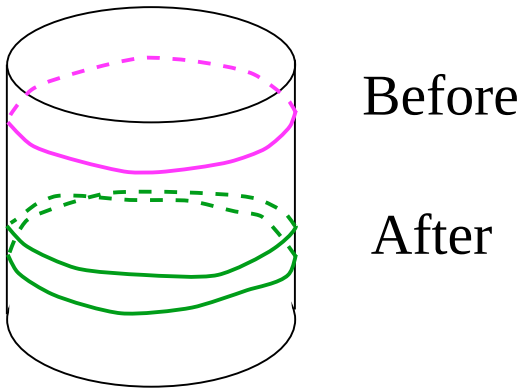

Figure 2.12: Squaring the Unit Circle

This is shown rather crudely in the 'after' part of figure 2.12. You have to imagine that we look at it from above to get the loop around the unit circle. Also, it should be smoother than my drawing. Don't shoot the artist, he's doing his best.

If you tried to 'do' the squaring function on a circular carpet representing the unit disk, you would have to first cut the carpet along the X-axis from the origin to $1+i0$. You need to take the top part of the cut, and push points close to the origin even closer. Then nail the top half of the cut section to the floor, and drag the rest of the carpet with you as you walk around the boundary. The carpet needs to be made of something stretchy, like chewing-gum[1]. When you have got back to your starting point, join up the tear you made and you have a double covering of every point under the carpet.

It is worth trying hard to visualise this, chewing-gum carpet and all.

Notice that there are two points which get sent to any point on the unit circle by the squaring map, which is simply an angle doubling. The same sort of thing is true for points inside and outside the disk: there are two points sent to $a + ib$ for any a, b. The only exception is 0, which has a unique square root, itself.

This is telling you that any non-zero complex number has two square roots. In particular, -1 has i and $-i$ as square roots. You should be able to visualise the squaring function taking a carpet made of chewing-gum and sending two points to every point.

This isn't exactly a formal proof of the claim that every non-zero complex number has precisely two distinct square roots; there is one, and it is long and subtle, because formalising our intuitions about carpets made of chewing-gum is quite tricky. This is done honestly in Topology courses. But the idea of the proof is as outlined.

I have tried to sketch the resulting surface just before it gets nailed down. It is impossible to draw it without it intersecting itself, which is an unfortunate property of $\mathbb{R}^3$ rather than any intrinsic feature of the surface itself. It is most easily thought of as follows; take two disks and glue them together at the centres. In figure 2.13, my disks have turned into cones touching at the vertices. Cut each disk from the centre to a single point on the perimeter in a straight line. This is the cut OP and OP' on the top disk, and the cut

[1] You need a quite horrid imagination to be good at maths.

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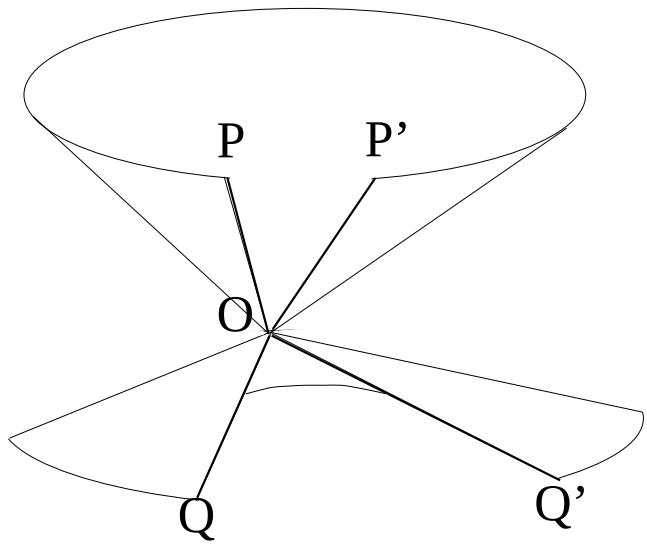

Figure 2.13: Squaring the Unit Disk

OQ, OQ' on the lower disk. Now join up the cuts, but instead of joining the bits on the same disks, join the opposite edges on opposite disks. So glue OP to OQ' and OP' to OQ. The fact that you cannot make it without it intersecting itself is because you are a poor, inadequate three dimensional being. If you were four dimensional, you could do it. See:

`http://maths.uwa.edu.au/~mike/PURE/`

and go to the fun pages. If you don't know what this means, you have never done any net surfing, and you need to.

This surface ought to extend to infinity radially; rather than being made from two disks, it should be made from two copies of the complex plane itself, with the gluings as described. It is known as a *Riemann Surface*.

2.3 The Square Root: $w = z^{\frac{1}{2}}$

The square root function, $f(z) = z^{\frac{1}{2}}$ is another function it pays to get a handle on. It is inverse to the square function, in the sense that if you square

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the square root of a a number you get the number back. This certainly works for the real numbers, although you may not *have* a square root if the number is negative. We have just convinced ourselves (by thinking about carpets) that every complex number except zero has precisely two square roots. So how do we get a well defined function from $\mathbb{C}$ to itself that takes a complex number to a square root?

In the case of the real numbers, we have that there are precisely two square roots, one positive and one negative, except when they coincide at zero. *The* square root is taken to be the positive one. The situation for the complex plane is not nearly so neat, and the reason is that as we go around the circle, looking for square roots, we go continuously from one solution to another.

Start off at $1 + i0$ and you will surely agree that the obvious value for its square root is itself. Proceed smoothly around the unit circle. To take a square root, simply halve the angle you have gone through.

By the time you get back, you have gone through 2π radians, and the preferred square root is now $-1 + i0$. So whereas the two solutions formed two branches in the case of the reals, and you could only get from one to the other by passing through zero, for $\mathbb{C}$ there are continuous paths from one solution to another which can go just about anywhere.

Remember that a function is an input-output machine, and if we input one value, we want a single value out. We *might* settle for a vector output in $\mathbb{C} \times \mathbb{C}$, but that doesn't work either, because the order won't stay fixed. We insist that a function should have a single unique output for every input, because all hell breaks loose if we try to have multiple outputs. Such things are studied by Mathematicians, who will do anything for a laugh, but it makes ideas such as continuity and differentiability horribly complicated. So the complications I have outlined to force the square root to be a proper function are designed to make your life simpler. In the real case, we can simply choose $\sqrt{x}$ and $-\sqrt{x}$ to be two neat functions that do what we want, at least when x is non-negative. In the complex plane, things are more complicated.

The solution proposed by Riemann was to say that the square root function should not be from $\mathbb{C}$ to $\mathbb{C}$, but should be defined on the Riemann surface illustrated in figure 2.13. This is cheating, but it cheats in a constructive and useful manner, so mathematicians don't complain that Riemann broke the rules and they won't play with him any more, they rather admire him

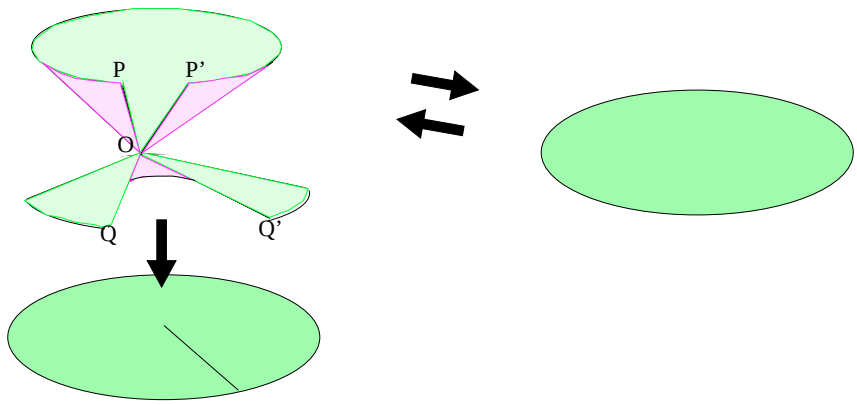

Figure 2.14: The Square function through the Riemann Surface

for pulling such a line[2].

If you build yourself a surface for the square function, then you project it down and squash the two sheets (cones in my picture) together to map it into $\mathbb{C}$, then you can see that there is a one-one, onto, continuous map from $\mathbb{C}$ to the surface, S, and then there is a projection of S on $\mathbb{C}$ which is two-one (except at the origin). So there *is* an inverse to the square function, but it goes from S to $\mathbb{C}$. This is Riemann's idea, and it is generally considered very cool by the smart money.

I have drawn the pictures again in figure 2.14; you can see the line in the lower left copy of $\mathbb{C}$ (or a bit of it) where I have glued OP to OQ' and OP' to OQ, and then both lines got glued together by the projection. The black arrow going down sends each copy of $\mathbb{C}$ to $\mathbb{C}$ by what amounts to the identity map. This is the projection map from S. The black arrow going from right to left and slightly uphill is the square function onto S. The top half of the complex plane is mapped by the square function to the top cone of S, and the bottom half of $\mathbb{C}$ is mapped to the lower cone.

[2]Well, the good mathematicians feel that way. They like *style*. Bad mathematicians don't like this sort of thing, but life is hard and unkind to bad mathematicians who spend a lot of the time feeling stupid and hating themselves for it. We should not add to their problems.

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The last black arrow going left to right is the square root function, and it is a perfectly respectable function now, precisely the inverse of the square function.

So when you write $f(z) = z^2$, you MUST be clear in your own mind whether you are talking about $f : \mathbb{C} \longrightarrow \mathbb{C}$ or $f : \mathbb{C} \longrightarrow S$. The second has an inverse square root function, and the former does not.

2.3.1 Branch Cuts

Although the square function to the Riemann surface *followed by the projection to* $\mathbb{C}$ doesn't have a proper inverse, we can do the following: take half a plane in $\mathbb{C}$, map it to the Riemann surface, remove the boundary of the half plane, and project it down to $\mathbb{C}$. This has image a whole plane (the angle has been doubled), with a *cut* in it where the edge of the plane has been taken away. For example, if we take the region from 0 to π, but without the end angles 0 and π, the squaring map sends this to the whole complex plane with the positive X-axis removed. This map has an inverse, $(r, \theta) \rightsquigarrow (r^{1/2}, \frac{\theta}{2})$ which pulls it back to the half plane above the X-axis.

Another possibility is to take the half plane with positive real part, and square that. This gives us a branch cut along the negative real axis. We can then write

$$f_1(z) = f_1(r, \theta) = (r^{1/2}, \frac{\theta}{2})$$

for the inverse, which is called the *Principal Square Root.* It is called a *branch* of the square root function, thus confusing things in a way which is traditional. We say that this is defined for $-\pi < \theta < \pi$.

Suppose we take the half-plane with strictly *negative* real part: this also gets sent to the complex plane with the negative real axis removed. (We have to think of the angles, θ as being between $\pi/2$ and $3\pi/2$.) Now we get a square root of (r, θ) which is the negative of its value for the principal branch. I shall call this the *negative of the Principal branch.*

Exercise 2.3.1 *Draw the pictures of the before and after squaring, for the two branches just described. Confirm that* $(1, \pi/4)$ *is the unique square root of* $(1, \pi/2)$ *for the Principal branch, and that* $(1, 5\pi/4)$ *is the unique square root of* $(1, \pi/2)$ *for the negative of the Principal branch. Note that* $(1, \pi/4) = -(1, 5\pi/4)$.

Taking branches by choosing any half plane you want is possible, and for every such branch there is a branch cut, and a unique square root, For every such branch there is a negative branch obtained by squaring the opposite half plane, and having the same branch cut. This ensures that in one sense every complex number has two square roots, and yet forces us to restrict the domain to ensure that we only get them one at a time.

The point at the origin is called a *branch point*; I find the whole terminology of 'branches' unhelpful. It suggests rather that the Riemann surface comes in different lumps and you can go one way or the other, getting to different parts of the surface. For the Riemann surface associated with squaring and square rooting, it should be clear that there is no such thing. It certainly behaves in a rather odd way for those of us who are used to moving in three dimensions. It is rather like driving up one of those carp parks where you go upward in a spiral around some central column, only instead of going up to the top, if you go up twice you discover that, SPUNG! you are back where you started. Such behaviour in a car park would worry anyone except Dr. Who. The origin does have something special about it, but it is the only point that does.

The attempt to choose regions which are restricted in angular extent so that you can get a one-one map for the squaring function and so choose a particular square root is harmless, but it seems odd to call the resulting bits 'branches'. (Some books call them 'sheets', which is at least a bit closer to the picture of them in my mind.)

It is entirely up to you how you choose to do this cutting up of the space into bits. Of course, once you have taken a region, squared it, confirmed that the squaring map is one-one and taken your inverse, you still have to reckon with the fact that someone else could have taken a different region, squared that, and got the same set as you did. He would also have a square root, and it could be different from yours. If it was different, it would be the negative of yours.

Instead of different 'branches', you could think of there being two 'levels', corresponding to different levels of the car park, but it is completely up to you where you start a level, and you can go smoothly from one level to the next, and anyway levels 1 and 3 are the same.

This must be hard to get clear, because the explanations of it usually strike me as hopelessly muddled. I hope this one is better. The basic idea is fairly

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easy. Work through it carefully with a pencil and paper and draw lots of pictures.

2.3.2 Digression: Sliders

Things can and do get more complicated. Contemplate the following question:

Is $w = \sqrt{(z^2)}$ the same function as $w = z$?

The simplest answer is 'well it jolly well ought to be', but if you take $z = 1$ and square it and then take the square root, there is no particular reason to insist on taking the positive value. On the other hand, suppose we adopt the convention that we mean the positive square root for positive real numbers, in other words, on the positive reals, square root means what it used to mean. Are we forced to take the negative square root for negative numbers? No, we can take any one we please. But suppose I apply two rules:

1. For positive real values of z take the (positive) real root
2. If possible, make the function continuous

then there are no longer any choices at all. Because if we take a number such as $e^{i\theta}$ for some small positive θ, the square is $e^{i2\theta}$ and the only possibilities for the square root are $\pm e^{i\theta}$, which since r cannot be negative means $e^{i\theta}$ or $e^{i(\theta+\pi}$, and we will have to choose the former value to get continuity when $\theta = 0$. We can go around the unit circle and at each point we get a unique result: in particular $\sqrt{(-1)^2)} = -1$.

I could equally well have chosen the negative value everywhere, but with *both* the above conventions, I can say cheerfully that

$$\sqrt{z^2} = z$$

If I drop the continuity convention, then I can get a terrible mess, with signs selected any old way.

The argument for

$$(\sqrt{z})^2$$

is simpler. If you take z and look at its square roots, you are going up from $\mathbb{C}$ to the Riemann surface that is the double level spiral car-park space. you can go up to either level from any point (except the origin for which there is only one level). If I square the answer I will get back to my starting point, whatever it was. So

$$z = (\sqrt{z})^2$$

is unambiguously true, although it is expressing the identity function as a composite of a genuine function and a relation or 'multi-valued' function.

Now look at

$$w = \sqrt{z^2 + 1}$$

Again the square root will give an ambiguity, but I adopt the same two rules. So if $z = 1$, $w = \sqrt{2}$. At large enough values of z, we have that w is close to z. The same argument about going around a circle, this time a BIG circle, gives us a unique answer. $100i$ will have to go to about $100i$ and -100 will have to go to about -100.

It is by no means clear however that we can make the function continuous closer in to the origin. $f(0) = 1$ would seem to be forced if we approach zero from the right, but if we approach it from the left, we ought to get -1. So the two rules given above cannot both hold. Likewise, $\pm i$ both get sent to zero; If we have continuity far enough out, then we can send $10i$ to i times the positive value of $\sqrt{99}$. But what do we do for $0.5i$? Do we send it to $\sqrt{3/4}$ or $-\sqrt{3/4}$? Or do we just shrug our shoulders and say it is multivalued hereabouts?

If we just chop out the part of the imaginary axis between i and $-i$, we have a perfectly respectable map which is continuous, and sends $i(1+\delta)$ into $i\delta$ when $\delta > 0$, sends -1 to $-\sqrt{2}$, 1 to $+\sqrt{2}$. It has image the whole complex plane except for the part of the real axis between -1 and 1. Call this map f. You can visualise it quite clearly as pulling the real axis apart at the origin, with points close to zero on the right getting sent (almost) to 1 and points close to zero on the left getting sent (almost) to -1. The two points $\pm i$ get sucked in towards zero. Because of the slit in the plane, this is now a continuous map, although we haven't defined it on the points we threw out.

There is also a perfectly respectable map $-f$ which sends z to $-f(z)$. This has exactly the same domain and range space, $\mathbb{C}$ with a vertical slit in it, between $-i$ and i, and it has the same range space, $\mathbb{C}$ with a horizontal slit in it, between -1 and 1. It is just f followed by a rotation by 180°.

We now ask for a description of the Riemann surface for $\sqrt{z^2+1}$. You might think that asking about Riemann surfaces is an idle question prompted by nothing more than a desire to draw complicated surfaces, but it turns out to be important and very practical to try to construct these surfaces. The main reason is that we shall want to be able to integrate along curves in due course, and we don't want the curve torn apart.

The Riemann surface associated with the square and square root function was a surface which we pictured as sitting over the domain of the square root function, $\mathbb{C}$, and which projected down to it. Then we split the squaring map up so that it was made up of another map into the Riemann surface followed by the projection. Actually, the Riemann surface is just the graph of $w = z^2$, but instead of trying to picture it in four dimensions, we put it in three dimensions and tried not to think about the self-intersection this caused.

We could construct the above surface as follows: first think of the square root function. Take a sector of the plane, say the positive real axis and the angle between 0 and $\pi/2$. Now move it vertically up above the base plane. I choose one particular square root for the points in this sector, say I start with the ordinary real square root on the positive real line. This determines uniquely the value of the square root on the sector, since $w = z^2$ is one-one here, so the square root is just half the sector. I can do the same for another sector on which the square is still one-one, say the part where the imaginary component is positive. This will overlap the quarter plane I already have; I make sure that everything agrees with the values on the overlap. I keep going, but when I get back to the positive real axis, I discover that I have changed the value of the square root, so instead of joining the points, I lift the new edge up. I keep going around, and now I get different answers from before, but I can continue gluing bits together on the overlap. When I have gone around twice, I discover that the top edge now really ought to be joined up to the starting edge. So I do my Dr. Who act and identify the two edges.

The other way to look at it is to take two copies of the complex plane, and glue them together as in figure 2.14. We know there are two because of the square root, and we know that they are joined at the origin because there is only one square root of zero. We clearly pass from one plane to another at a branch cut, which can be anywhere, and then we go back again a full circuit later.

Now I shall do the same thing with $\sqrt{z^2+1}$. But before tackling this case,

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a short digression.

Example 2.3.1 *In the television series 'Sliders', the hero generated a disk shaped region which identified two different universes. Suppose there are two people intending to slide into a new universe and they see this disk opening into a tunnel in front of them. One of them walks around the back of the disk. If this one sees the other side of the disk and steps through it, and if the other person goes through the other side of the disk at the same time, is it true that they must come out in the same place? Do they bump into each other?*

Solution

It is probably easiest to think of this a dimension lower down. Take two sheets of paper, two universes. Draw a line segment on each. This is the 'door into Summer', the Stargate.

What we do is to identify the one edge of the line segment in one universe with the opposite edge in the other universe. To make this precise, take universe A to be the plane $(x, y, 1)$ for any pair of numbers x, y, and universe B to be the set of points $(x, y, 0)$. I shall make my 'gateway' the interval $(0, y, n)$ for $-1 \leq y \leq 1$, for both $n = 0, 1$.

Now I first cut out the interval of points in the 'stargate',

$$(0, y, n), \ -1 \leq y \leq 1; \quad n = 1, 2$$

I do this in both universes.

Now I pull the two edges of the slits apart a little bit. Then I put new boundaries on, one on each side. I have doubled up on points on the edge now, so there are two origins, a little way apart, in each universe. I call them 0 and $0'$ respectively, so I have duplicate points $(0, y, n)$ and $(0', y, n)$ for $-1 \leq y \leq 1; \quad n = 1, 2$. A crude sketch is shown in figure 2.15. Now I glue the left hand edge of the slit in one universe to the right hand edge of the slit in the other universe, and vice versa. So

$$(0, y, 0) = (0', y, 1) \ \ \& \ \ (0, y, 1) = (0', y, 0) \quad -1 \leq y \leq 1$$

This will make the path shown by the dotted line in figure 2.16 continuous.

I joined up the opposite two sides of the cut in each universe in the same way, but I don't have to. One thing I can do is to have another universe,

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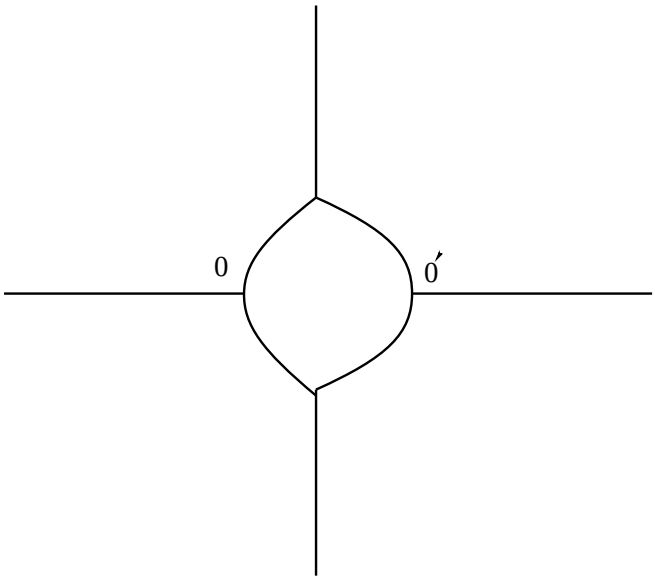

Figure 2.15: The construction of the Stargate

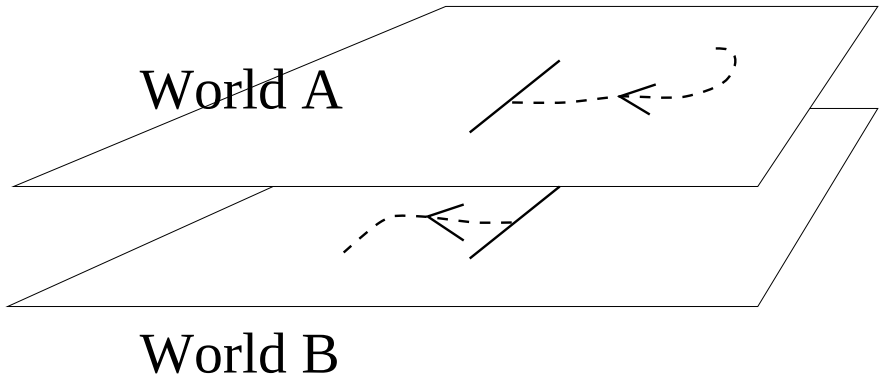

Figure 2.16: The Dotted line is a continuous path in the twin universes

and join to that *one. So if two of the slider gang go through the gate on opposite sides, they could emerge in the same universe on opposite sides of the connecting disk, or in two different universes. They won't bump into each other, they will be on opposite sides of the disk, but they may or may not be in the same universe.*

On the other hand, nipping back smartly where you have just come from, walking around the disk, and then going in the same side would get them together again.

On this model.

If someone ever does invent a gateway for travelling between universes, the mathematicians are ready for talking about it[3].

The reason for thinking about multidimensional car parks, sliders and bizarre topologies, is that it has everything to do with the Riemann surface for

$$w = f(z) = \sqrt{z^2 + 1}$$

We need to take both f and $-f$, and we have quite a large region in which we can have each branch of f single valued and 1-1, namely the whole plane with the slit from i to $-i$ removed. So we have two copies of $\mathbb{C}$ with slits in them.

We also have two similar looking copies of slitted $\mathbb{C}$s (but with horizontal slits) ready for the image of the new map.

We have to join the two copies of $\mathbb{C}$ across the slits. This is exactly what our picture of the two dimensional inter-universe sliders was doing.

In this case, we can label our two universes as f and $-f$. This is going to tell us what we are going to actually *do* with the linked pair of universes.

Points on the left hand side of the slit for universe f are defined to be close to the points on the right hand side of the slit for universe $-f$ and vice versa. So a path along the real axis from -1 towards 1 in the universe f slips smoothly into the universe $-f$ at the origin. You can retrace your path exactly. If you start off in Universe f at $+1$ going left, then you slide over into universe $-f$

[3]Actually they've been ready for well over a century. Riemann discussed this sort of thing in 1851. It took a while to get down to the level of popular television.

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at the origin. So in this case, it doesn't matter which way you go into the 'gate', you wind up in the same universe- there are only two. If you are a long way from the gate in either universe, you don't get to find out about the other universe at all. Continuous paths which don't go through the gate have to stay in the same universe.

Exercise 2.3.2 *Construct a complex function needing three 'universes' for the construction of its Riemann surface.*

To see that this *is* the Riemann surface, observe that if we travel in any path on the surface, the value of $\sqrt{z^2+1}$ varies continuously along the path.

Exercise 2.3.3 *Choose a path in the Riemann surface and confirm that the value of $\sqrt{z^2+1}$ varies continuously along the path. Do this for a few paths, some passing through the 'gate' described above.*

Exercise 2.3.4 *Describe the surface associated with the inverse function. Show that there is a one-one continuous map going in both directions between the two surfaces.*

It is worth pointing out that the Riemann surface can be constructed in several ways: there is nothing unique about the choice of branch cuts, for example. It is not so obvious that the Riemann surface is unique in the sense that there is always a way of deforming one into another. You don't have the background to go into this, so I shan't. But the text books often give the impression that branch cuts come automatically with the problem, whereas they are much less clear cut[4] than that.

It is clear that the z^2 cut along the positive real axis can be replaced by any ray from the origin. It might seem however that the slit between i and $-i$ is forced. This isn't so, but the proper investigation of these matters is quite difficult.

I have avoided defining Riemann surfaces, and simply considered them in rather special cases, because it needs some powerful ideas from Topology to do the job properly. This seems to be traditional in Complex Analysis, and it

[4] Aaaagh!

leaves students rather puzzled as to how to handle them in new cases. There isn't time in this course to do more than introduce them, but I hope you can see two things: first that quite simple real functions generalise to rather complicated complex functions, and second that the investigation of them is full of ideas that take you outside the universe you are used to. The fact that this is actually useful is one you will have to take for granted for a while.

2.4 Squares and Square roots: Summary

I have gone into the business of examining the square function and the square root function in agonising detail, because they illustrate many of the problems and opportunities of complex functions. They show that the little sweeties are (a) surprisingly complicated even when the real version of the function is boringly familiar, and (b) they are not so bad we can't make sense of them. Many hours of innocent fun can be had by exploring the behaviour of complex functions the real versions of which are simple and uninteresting. It is recommended that you play around with some yourself.

It makes sense to look at functions such as $f(z) = z^2$ because we have that $\mathbb{C}$ is a field, so we can do with $\mathbb{C}$ everything we could do with $\mathbb{R}$. So polynomials make sense. And so do infinite series, as we shall see later, so the trigonometric and exponential functions make sense, and just as we can ask for a square root of -1, so we can ask for a logarithm of it.

Exercise 2.4.1 *What would you expect to be the value of* $\ln(-1)$*?*

This is weird stuff by comparison with the innocent functions from $\mathbb{R}$ to $\mathbb{R}$, and it is a good idea to get the basics clear, which is the main reason for doing to death the square function. We can now move on to a few more easy functions to find out what they do. This should be approached in a spirit of fun and innocence. Who knows what bizarre things we shall find?

2.5 The function $f(z) = \frac{1}{z}$

The real function $f(x) = 1/x$ is a perfectly straightforward function which is defined everywhere except, of course, at $x = 0$. Since you can do in $\mathbb{C}$

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everything that you can do in $\mathbb{R}$, the function $f(z) = 1/z$ must also make sense except at $z = 0$.

We can say immediately that $f(z) = 1/z = \bar{z}/z\bar{z}$, so $1/z$ does two things: first it takes the conjugate of z, and then it scales by dividing by the square of the modulus. If this is 1, then the only effect is to reflect z in the X-axis. In order to make our lives easier, we decompose f into these two parts, the *inversion map* $inv(z) = z/z\bar{z}$ and the conjugation map $\bar{(z)} = \bar{z}$. and look at these separately.

To start to get a grip on the inv function, notice that in polar form, (r, θ) gets sent to $(1/r, \theta)$. A point on the unit circle will stay fixed, points on the axes stay on the axes. The origin gets sent off to infinity, points close to the origin get sent far away but preserve the angle. If we take the unit square in the plane, the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gets sent to $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

The top edge of the unit square, $y = 1, 0 \leq x \leq 1$, gets sent to a curve joining $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is left fixed by the map as it lies on the unit circle. The equation of the curve is given by

$$u = x/(x^2 + 1), v = 1/(x^2 + 1)$$

which can be written as

$$u = v\sqrt{1/v - 1}$$

The right edge behaves similarly.

The left edge is sent to the Y-axis for values greater than 1, the bottom edge to the X-axis with values from 1 to infinity.

The before and after pictures are figure 2.17 and figure 2.18 respectively.

Note that the point density is greatest closest to the origin. You should be able to see why this is so. (Hint: think of the derivative of $1/r$.)

If we take a disk, it can be discovered experimentally that the image is also a disk in most cases. Some before and after pictures are figure 2.19 and figure 2.20 respectively.

This is a harder one to calculate:

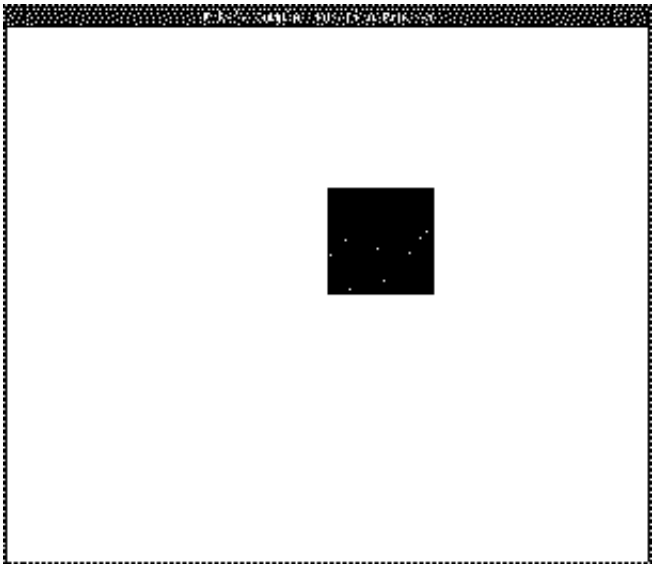

Figure 2.17: The Unit Square

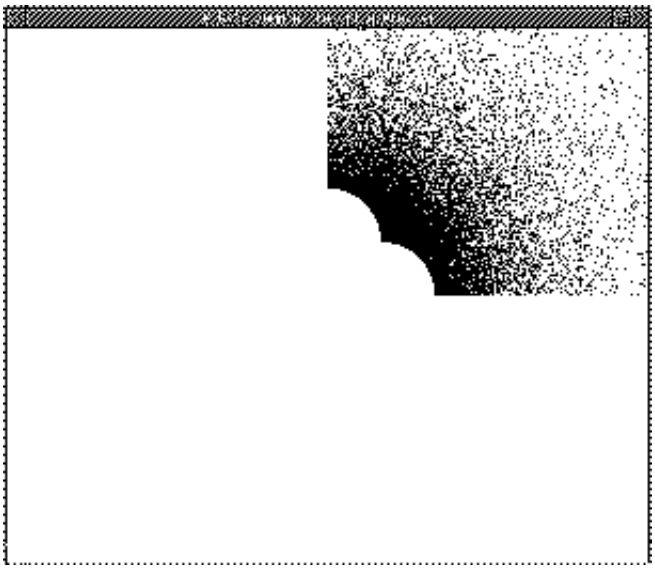

Figure 2.18: The Inversion of the unit square

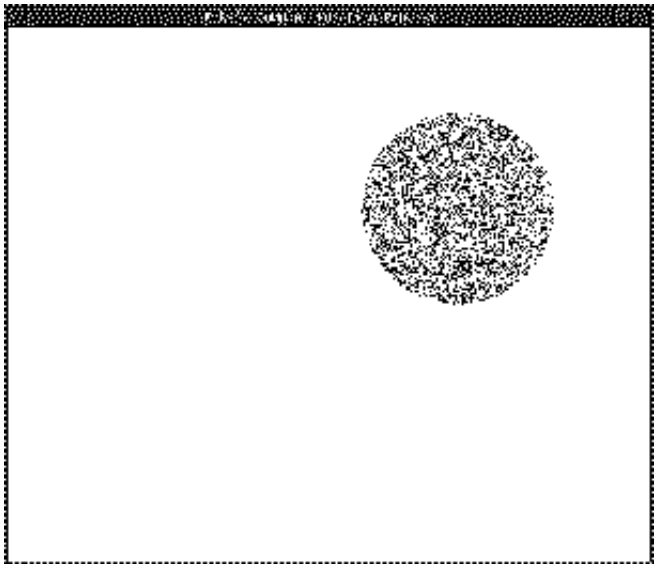

Figure 2.19: A disk

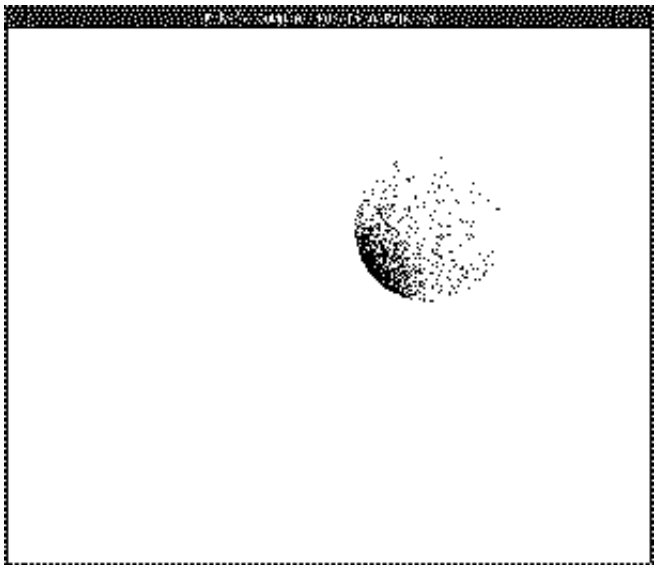

Figure 2.20: The Inversion of the disk

Exercise 2.5.1 *Can you find an expression for the inversion of the boundary of the disk?*

If you do some experimenting with a program that does inversions, you will discover that it looks very much as if the inversion of a circle is a circle except in the degenerate case where the circle passes through the origin. This is indeed the case.

In order to see this, write the circle with centre $\begin{bmatrix} a \\ b \end{bmatrix}$ and radius R in polar coordinates to get the equation

$$r^2 - 2ar\cos\theta - 2br\sin\theta = R^2 - a^2 - b^2$$

Now the angle is unchanged, so the inversion is the set of (s, θ) satisfying

$$1/s^2 - 2a\cos\theta/s - 2b\sin\theta/s = R^2 - a^2 - b^2$$

which after some rearrangement gives

$$s^2 - 2a\cos\theta/(a^2 + b^2 - R^2) - 2b\sin\theta/(a^2 + b^2 - R^2) = -1/((a^2 + b^2 - R^2)$$

This is a circle with centre at $\begin{bmatrix} a/(a^2 + b^2 - R^2) \\ b/(a^2 + b^2 - R^2) \end{bmatrix}$ and radius a rather horrible value which can be written down with some patience. If the original circle goes through the origin, the radius of the inverted circle is infinite, and its centre is also shifted off to infinity. This actually gives a straight line.

Exercise 2.5.2 *Verify the claim that the equation degenerates to a straight line when $R^2 = a^2 + b^2$.*

Sneaky Alternative Methods

There is a somewhat neater way of proving that inversions take circles to circles; it requires that we find a way of describing circles which is different from the usual one.

Suppose we write

$$\frac{|z - a|}{|z - b|} = r$$

for some positive real r, and complex a, b. If $r = 1$ this just gives the straight line bisecting the line segment from a to b. If $r \neq 1$, it gives a circle cutting

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the line segment between a and b and it is easy to write down its equation in more standard forms. Also, any circle can be written in this form.

Now putting $w = 1/z$ in this equation and doing a bit of messing around with algebra gives a new equation in the same form. Which is also a circle, or maybe a straight line.

Exercise 2.5.3 *Do the algebra to show that the representation is really that of a circle (or straight line if $r = 1$).*

Do the algebra to show that $w = 1/z$ in this equation gives a new circle (or possibly a straight line).

Exercise 2.5.4 *Show that the unit circle can be represented in the sneaky form with $r = 2$. Show that* any *circle can be written in this form with $r = 2$.*

Another way of representing any circle is in the form

$$A\bar{A}z\bar{z} + Bz + \bar{B}\bar{z} + C\bar{C} = 0$$

for complex numbers A, B, C.

If $A = 0$ this is a straight line, if $C = 0$ it passes through the origin.

For this form also, it is easy to confirm that inversion takes circles to circles, where a straight line is just a rather extremal case of a circle. Malcolm Hood told me this one.

These representations are sneaky and probably cheating, but it is telling you something important, namely, some representations for things will make some problems dead easy, and others make it horribly difficult. Thinking about this early on can save you a lot of grief.

Exercise 2.5.5 *Can you see why a parametric representation of the circle of the form $z = a + r\cos\theta + i(b + r\sin\theta)$ could be a serious blunder in trying to show that inversions take circles to circles?*

Remark:

The moral we draw from this little excursion is that being true and faithful to a human being is, possibly, a fine and splendid thing; being faithful and true to a principle or ideology might *be a fine and splendid thing, or it might be a sign of a sentimental nature gone wild. But being faithful and true to a brand of beans or a choice of representation of an object is to confuse the finger pointing at the moon with the moon itself, and a sure sign of total fatheadedness. The poor devil who believes deeply that the only true and proper way to represent a circle in the plane is by writing down*

$$(x-a)^2 + (y-b)^2 = r^2$$

is to be pitied as someone who has confused the language with the thing being talked about, and is fit only for politics. The more ways you have of talking and thinking about things, the easier it is to draw conclusions, and the harder it is to be led astray. It is also a lot more fun.

The converse is also true: the inversion of a straight line is a circle through the origin.

To see this, let $ax + by + c = 0$ be the equation of a straight line. Turn this into polars to get

$$ar\cos\theta + br\sin\theta + c = 0$$

Now put $r = 1/s$ to get the inversion:

$$(a/s)\cos\theta + (b/s)\sin\theta + c = 0$$

and rearrange to get

$$s^2 + (as/c)\cos\theta + (bs/c)\sin\theta = 0$$

which is a circle passing through the origin with centre at $\begin{bmatrix} -a/2c \\ -b/2c \end{bmatrix}$.

It is easy to see that the 'points at infinity' on each end of the line get sent to the origin.

This suggests that we could simplify the description by working not in the plane but in the space we would get by adjoining a 'point at infinity'.

We do this by putting a sphere of radius 1/2 sitting on the origin of $\mathbb{R}^3$, and identify the $z = 0$ plane with $\mathbb{C}$. Now to map from the sphere to the plane, take a line from the north pole of the sphere which is at the point $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

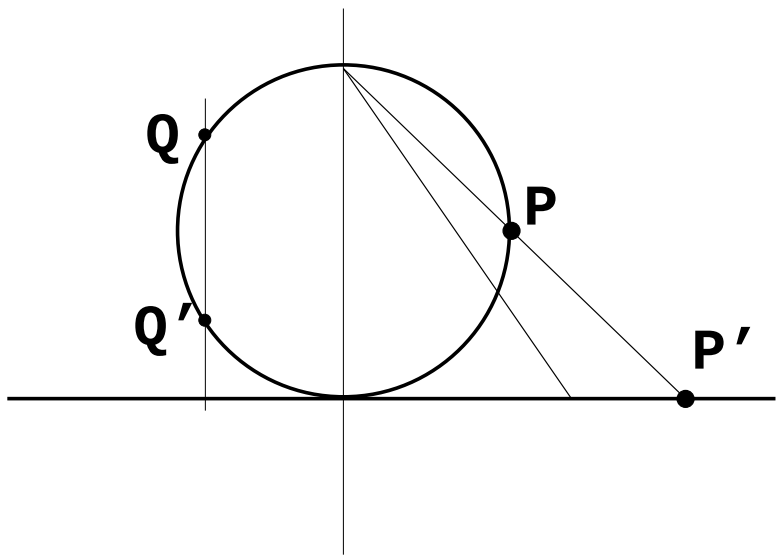

Figure 2.21: The Riemann Sphere

and draw it so it cuts the sphere in P and the plane at P'. Now this sets up a one-one correspondence between the points of the sphere other than the north pole and the points of the plane. The unit circle in the plane is sent to the equator of the sphere.

Now we put the 'point at infinity' of the plane in- at the north pole of the sphere.

An inversion of the plane now gives an inversion of the sphere, which sends the South pole (the origin) to the North pole: all we do is to project down so that the point Q goes directly to the point Q' vertically below it, and *vice-versa*. In other words, we reflect in the plane of the equator.

Exercise 2.5.6 *Verify that this rule ensures that a point in the plane is sent to its inversion when we go from the point up to the sphere, then reflect in the plane of the equator, then go back to the plane.*

Exercise 2.5.7 *Suppose we have a disk which contains the origin on its boundary. What would you expect the inversion of the disk to look like?*

Suppose we have a disk which contains the origin in its interior. What would you expect the inversion to look like?

Sketches of the general situation should take you only a few minutes to work out; it is probably easiest to visualise it on the Riemann Sphere.

Exercise 2.5.8 *What would you expect to get, qualitatively, if you invert a triangle shaped region of* $\mathbb{C}$*? Does it make a difference if the triangle contains the origin?*

Draw some pictures of some triangles and what you think their inversions would look like.

Note that if you do an inversion and then invert the result, you get back to where you started. In other words, the inversion is its own inverse map. Since the same is true of conjugation, the map $f(z) = 1/z$ also has this property.

Exercise 2.5.9 *What happens if you invert a half-plane made by taking all the points on one side of a line through the origin? What if the half-plane is the set of points on one side of a line* not *through the origin?*

I haven't said anything much about the conjugation because it is really very trivial: just reflect everything in the X-axis.

2.6 The Möbius Transforms

The reciprocal transformation is a special case of a general class of complex functions called the *Fractional linear* or *Möbius transforms*. In the old days, they also were called *bilinear*, but this word now means something else and is no longer used by the even marginally fashionable.

The general form of the Möbius functions is:

$$w = f(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers. If $c = 0, d = 1$, we have the affine maps, and if $a = 0, b = 1, c = 1, d = 0$ we have the reciprocal map. It is tempting to represent each Möbius function by the corresponding matrix:

$$\frac{az+b}{cz+d} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which makes the identity matrix correspond nicely to the identity map $w = z$.

One reason it is tempting is that if we compose two Möbius functions we get another Möbius function and the matrix multiplication gives the corresponding coefficients. This is easily verified, and shows that providing $ad \neq bc$ the Möbius function

$$\frac{az+b}{cz+d}$$

has an inverse, and indeed it tells us what it is.

The sneaky argument for the inversion also goes through for Möbius functions, i.e. they take circles on the Riemann Sphere to other circles. It is clear that the Riemann Sphere is the natural place to discuss the Möbius functions since the point at ∞ is handled straightforwardly.

Exercise 2.6.1 *Verify that if $ad \neq bc$ the Möbius function can be defined for ∞ in a sensible manner. What if $ad = bc$?*

Exercise 2.6.2 *Confirm that any Möbius function takes circles to circles. What happens when $ad = bc$?*

A rather special case is when the image by a Möbius function of a circle is a straight line. It follows that the image of the interior of the disk bounded by the circle is a half-plane.

Example 2.6.1 *Find the image of the interior of the unit disk by the map*

$$w = f(z) = \frac{z-1}{z+1}$$

Solution

We see immediately that $z = -1$ goes to infinity, and so the bounding circle must be sent to a straight line, and the interior to a half-plane.

A quick check shows that the real axis stays real, and that $1 \rightsquigarrow 0$, $0 \rightsquigarrow -1, -0.5 \rightsquigarrow -3$, and the intersection of the real axis with the unit disk is sent to the negative real axis. It is easy to verify that $i \rightsquigarrow i, -i \rightsquigarrow -i$.

The inverse can be written down at sight (using the matrix representation!) and is

$$z = f^{-1}w = \frac{w+1}{-w+1}$$

which tells us that for $w = iv$ we have

$$z = \frac{1+iv}{1-iv} = \frac{(1-v^2)+2iv}{1+v^2}$$

which point lies on the unit circle. In other words, the inverse takes the imaginary axis to the unit circle, so the image by f of the unit circle is the imaginary axis. And since $0 \rightsquigarrow -1$ we conclude that the image by f of the interior of the unit disk has to be the half-plane having negative real part.

Any Möbius function has to be determined by its value at three points: it looks at first sight as though 4 will be required, but one could scale top and bottom by any complex number and still have the same function. This must be true, since if we have $z_1 \rightsquigarrow w_1, z_2 \rightsquigarrow w_2$, and $z_3 \rightsquigarrow w_3$ we have three linear equations in a, b, c, d and we can put $a = 1$ without loss of generality.

It follows that if you are given three points and their images you can determine the Möbius function which takes the three points where you now they need to go. There is a sneaky way of doing this which you will find in the books, but the method is not actually shorter than solving the linear equations in general, so I shall not burden your memory with it. It is possible, however, to use some intelligence in selecting the points:

Example 2.6.2 *Find a Möbius function which takes the interior of the unit disk to the half plane with positive imaginary part.*

Solution

We have to have the unit circle going to the real axis, so we might as well send 1 to 0. We can also send -1 to ∞. Finally, if we send 0 to i we have our three points.

The $-1 \rightsquigarrow \infty$ condition means that we have $cz + d = c(z+1)$ and $0 \rightsquigarrow i$ means we have $az + b = az + i$ while $1 \rightsquigarrow 0$ forces $a = -i$. So a suitable function is

$$f(z) = \frac{i(1-z)}{z+1}$$

Exercise 2.6.3 *Find a Möbius function which takes the interior of the disk*

$$|z - (1 + 2i)| < 3$$

to the half-plane with positive imaginary part.

Exercise 2.6.4 *Draw the images of the rays from the origin under the function*

$$w = \frac{z}{z-1} = 1 + \frac{1}{z-1}$$

Exercise 2.6.5 *Investigate the effect of composing some of the maps you have met so far. Show that a Möbius function can be written as a suitable composite of inversions and affine maps, and deduce directly that it has to take circles to circles.*

Exercise 2.6.6 *Calculate the image of the lines having imaginary part constant under the map*

$$f(z) = (z^2 - 1)^{1/2}$$

Exercise 2.6.7 *What is the Riemann surface for the above map?*

The Möbius functions are of some interest because they are closed under composition, and also for historical reasons. All books on Complex Analysis mention them. I should have been excommunicated if I had left them out, and I am already regarded as having heretical tendencies, so I have put them in. You are strongly encouraged to do the above exercises so that (a) you will be able to make an informed guess at some of the applications and (b) so that when you meet them in the examination you will approach them with confidence and a clear conscience.

2.7 The Exponential Function

The real exponential function is defined by

$$exp(x) = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$$

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or more formally as the infinite sum:

$$exp(x) = \sum_{n=0}^{\infty} x^n/n!$$

We write

$$exp(x) \text{ as } e^x$$

for reasons which will become apparent shortly.

2.7.1 Digression: Infinite Series

Expressing functions by infinite series is something you must get used to; the thing you need to realise is that almost *all* the functions that you use other than polynomials and ratios of polynomials are given by these infinite series (called *power series* in the above case, because they have different powers of x in them). When you calculate $\sin(12^o)$ or $\sqrt{768.3}$ or $\ln(35.4)$ by pressing the buttons on your calculator, it produces a number on the display. It gets it from taking the first k terms in a power series for the function. A better calculator will take more terms; you get the k you pay for.

It is very convenient to have a formula for sin, cos, e^x and all the other functions as an infinite series, because it is easy to add on some number of terms, and to stop when the increment is so small it doesn't make any difference to the answer. There is, however, a fundamental problem with this approach.

If you add the terms:

$$1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

or more formally if you calculate

$$\sum_{n=0}^{\infty} 1/2^n$$

you rapidly get something pretty close to 2. Ten terms gets you to within one tenth of a percent of the answer, which is, of course, 2.

Suppose you add up the first few terms of the series

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$$

or more explicitly you try to compute a finite number of terms of

$$\sum_{n=1}^{\infty} 1/n$$

You find that you seem to be getting to the answer rather slowly, but it is easy enough to put it on a computer and find a few thousand terms very quickly. If you do this, you discover that after about ten thousand terms, you are only getting increments in the fourth decimal place (of course!) and after a million terms, you get increments only in the sixth place. If your calculator is working to single precision and it does this sort of thing, it will conclude that the series sums to 16.695311. This is what I get on my computer if I sum ten million terms, which takes less than ten seconds. The question is: how far out is the result? Could it get up to 20 if we kept going on a higher precision machine?

The answer is that the result is about as far out after ten million terms as it is after two. The series actually diverges and goes off to infinity.

If you didn't know whether a series converged or diverged, it would be possible for you to calculate a number to six places of decimals in a few seconds, and to get a result which is absolutely and totally wrong, by assuming it converges because the increments have fallen below the precision of your machine. For this reason, it is of very considerable practical importance to be able to decide if a series converges. It is also useful to work out how fast it converges by getting a bound on the error as a function of the number of terms used in the sum.

If you have a power series expansion for a (real) function, then it will, of course have an x in it, and when you plug in a value for the x and add up the series, you get the value of $f(x)$. It may happen that the series converges nicely for some values of x and goes off its head for others. To see the kind of thing that could happen, take a look at the function

$$f(x) = \frac{1}{1+x}$$

You would have to be crazy to evaluate this by a power series, but there might be other functions which behave the way this one does, so bear with me.

It is not too hard to persuade yourself that the equation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

holds for at least some x. If you do the 'multiplication':

$$(1+x)(1-x+x^2-x^3+x^4-x^5+\cdots)$$

you get

$$(1-x+x^2-x^3+x^4-x^5+\cdots)+(x-x^2+x^3-x^4+\cdots)$$

and it certainly looks plausible that all terms cancel except for the initial 1. So cross multiplication seems to work. What more could you want?

If you put $x = 1/2$ you find, if you investigate the matter carefully, that the series does converge. If you put $x = -1$ the sum goes off to infinity, but it should anyway. If you put $x = 1$ you get

$$1-1+1-1+1-1+\cdots$$

which is supposed to sum to 1/2. There would seem to be some reasonable doubt about this.

Exercise 2.7.1 *Put $x = 2$. How do you feel about the resulting series converging to 1/3?*

I am, I hope, reminding you of first and second year material, and I hope even more that you have some recollections of how to test for convergence of infinite series. If not, look it up in a good book.

The best sort of result we can hope for is that a power series converges for every value of x, and that we can get a handle on estimating a bound for the error after n terms. This bound will usually depend on the x.

The situation for $\exp(x)$ is fairly good: the Taylor-MacLaurin theorem tells us that the error at the stage n is not bigger than the $(n+1)^{th}$ term, for negative x. This can, indeed, be made as small as desired by making n big enough. You can satisfy yourself by some heavy thought that the situation for positive x is also under control. So the formula

$$exp(x) = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$$

is one we can feel relatively secure about.

Exercise 2.7.2 *How many terms would you need to calculate exp(100) to four places of decimals? How about exp(-100)?*

2.7.2 Back to Real exp

The following exercise is important and will explain why we write $\exp(x)$ as e^x.

Exercise 2.7.3 *Write down the first four terms of* $\exp(x)$ *and the first four terms of* $\exp(y)$. *Multiply them together and collect up to show you have rather more than the first four terms of* $exp(x+y)$.

Produce an argument to convince a sceptical friend that you can say with confidence that

$$\forall x, y \in \mathbb{R} \exp(x) \exp(y) = \exp(x+y)$$

Another thing we can do is to show that if we differentiate the function $\exp(x)$, we recover the function. This assumes that if we have an infinite series and we differentiate it term by term, the resulting series will converge to the derivative of the function. You might like to brood on this to decide whether you think this is going to happen (a) always (b) sometimes (c) never. The answer cannot be (c) because the exponential function actually IS differentiable, and is indeed its own derivative, just as you would hope from differentiating the series termwise.

Exercise 2.7.4 *You recall, I hope, computing the Fourier series for a square wave. The series consists of differentiable functions, so you can differentiate the series expansion termwise. But you clearly can't differentiate the square wave at the discontinuities.*

What do you get if you differentiate the series termwise and take limits?

I said earlier that everything you could do for $\mathbb{R}$ you could also do for $\mathbb{C}$. You can certainly add and multiply complex numbers, and you can divide them except by zero. So the terms in the series

$$1 + z + z^2/2! + z^3/3! + z^4/4! + \cdots$$

are all respectable complex numbers. We can ask if the series converges to some complex number when we stick a particular value of z in.

We know that it works if the value of z is a real number. What if it isn't?

The answer is that all the arguments go through, and the series converges for every value of z. I am not going to give a formal proof of this as it is a fair amount of work and anyway, you are probably not much of a mind to do formal proofs. But it is instructive to work it out in a few cases. Let us therefore calculate $\exp(i)$.

We get the series:

$$exp(i) = 1 + i + (-1)/2! - i/3! + 1/4! - i/5! - 1/6! + \cdots$$

Separating the real and imaginary parts we get:

$$exp(i) = 1 - 1/2! + 1/4! - 1/6! + \cdots + i - i/3! + i/5! - i/7! + \cdots$$

or

$$exp(i) = (1 - 1/2! + 1/4! - 1/6! + \cdots) + i(1 - 1/3! + 1/5! - 1/7! + \cdots)$$

You may or may not recognise the separate series as representing terms you know. If you calculate the Taylor-MacLaurin expansions by

$$f(x) = f(0) + xf'(0)/1! + x^2 f''(0)/2! + x^3 f'''(0)/3! + \cdots$$

for the functions $\cos(x), \sin(x)$ you will immediately recognise

$$exp(i) = \cos 1 + i \sin 1$$

By putting ix in place of i you get:

$$exp(ix) = \cos(x) + i \sin(x)$$

This gives us, when $x = \pi$, Euler's Formula:

$$e^{i\pi} + 1 = 0$$

This links up the five most interesting numbers in Mathematics, $0, 1, e, i, \pi$, in the most remarkable formula there is. Since e seems to be all about what you get if you want a function f satisfying $f' = f$, and π is all about circles, it is decidedly mysterious.

Thinking about this gives you a creepy feeling up the back of the spine: it is as though you went exploring the Mandelbrot set and found a picture of an old bloke with a stern look and long white whiskers looking out at you. It might incline you to be better behaved henceforth. I have, therefore, some reservations about the next exercise:

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Exercise 2.7.5 (Don't do this if you watch the X-Files)

Euler's formula might either (a) be in no need of an explanation, just a proof, or (b) be explained by God having a silly sense of humour, just like most intelligent people or (c) have a more prosaic explanation.

The exponential function is a procedure for turning vector fields into flows; if you take the vector field which is given by

$$V\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

you call the matrix A and then the flow is given as

$$e^{tA}$$

This is basic to the theory of systems of ODE's. You can verify this particular case by exponentiating the matrix tA using the standard power series formula for the exponential of a real number x and replace x by tA. Since all you have to do with x is to multiply it by itself, divide by a non-zero real number, add a finite set of these things and take limits, and since all of these can be done with matrices, this all makes sense.

Draw a picture of the vector field and the resulting flow.

Identify the matrix as a complex number.

Deduce that $e^{it} = \cos t + i \sin t$ is little more than the observation that a tangent to a circle is always orthogonal to the radius, together with the observation that exponentiation is about solving ODE's by Euler's method taken to the limit.

If you watch the *X-Files* (The Truth is out there, the Lies are in the programme), you might prefer to have the mystery preserved. Actually, there is still heaps of mystery left, indeed it's the charm of Mathematics[5].

Since the argument that $\exp(x+y) = \exp(x)\exp(y)$ (the 'index law') goes through for the complex numbers just as it does for the reals, we can write

$$\exp(x+iy) = exp(x)(\cos(y) + i\sin(y))$$

[5]Engineers who want to preserve mystery are a bit of a worry.

And the index law justifies our writing

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

If we write our complex number out in Polar form, $z = (r, \theta)$ we have that $z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$

This is a quite common notation; it needs a bit of explanation and I have just given you one.

2.7.3 Back to Complex exp and Complex ln

We are now ready to look at the exponential map

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}$$

It is certainly much more complicated that the real exponential, and the extra complications will turn out to be very useful.

To see what it does, notice that exp takes the real axis to the positive real numbers. 0 goes to 1, and all the negative real numbers get squashed into the space between 0 and 1. It takes the imaginary axis and wraps it around the unit circle. The map e^{iy} is a periodic function: think of the imaginary axis as a long line made out of chewing gum, and note that the chewing gum line is picked up by exp and wrapped (without stretching) in the positive (anticlockwise) direction around the unit circle. The negative imaginary numbers are wrapped around in the opposite direction.

I have indicated the start of this process on the axes, as if we have almost got the exponential function but not quite, restricted to the axes. This is figure 2.22.

What happens to the rest of the plane? The image by exp of the axes will cover only the positive real axis and the unit circle, but the unit circle gets covered infinitely often. The number $1 + 0i$ also has an infinite number of points sent to it. Does anything at all get sent to the origin? To -2? These are all good questions to ask.

I start off to answer some of them in the following example of how to compute the effect of the complex exponential.

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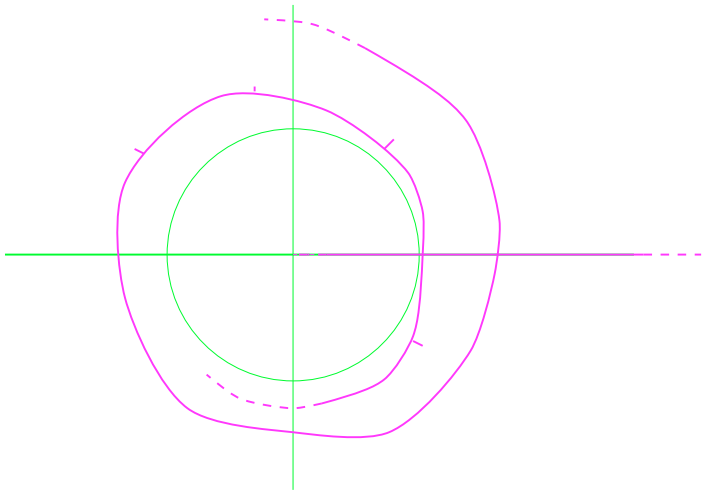

Figure 2.22: The start of the exponential function, restricted to the axes

Example 2.7.1 *What is the image by the exponential map of the unit square?*

Solution *The lower edge, the point* $x + 0i$ *for* $0 \leq x \leq 1$ *gets sent to the line segment* $e^x + 0i$ *since* $\cos 0 + i \sin 0 = 1$. *The top edge gets sent to* $e^x(\cos 1 + i \sin 1)$ *Since 1 is 1 radian, this goes to a line at about* 57^0 *and of radii from* 1 *to* e. *The left hand edge, the part of the imaginary axis between 0 and 1, goes to the corresponding arc of the unit circle, and the right hand edge of the unit square, the points* $1+iy$, *for* $0 \leq y \leq 1$ *goes to* $e(\cos y + i \sin y)$ *which is an arc of a circle of radius* e. *The result is shown in figure 2.23*

The figure following shows the results of applying the exponential map to the bigger square centred on the origin of side 2 units:

Exercise 2.7.6 *Mark in the image of the axes on figure 2.24*

The image by exp of the unit disk is shown in figure 2.25

I have marked on the X and Y axes to make it clearer where it is.

Exercise 2.7.7 *What is the inverse of* exp *of a point in the spotty region of figure 2.25 which is closest to the origin?*

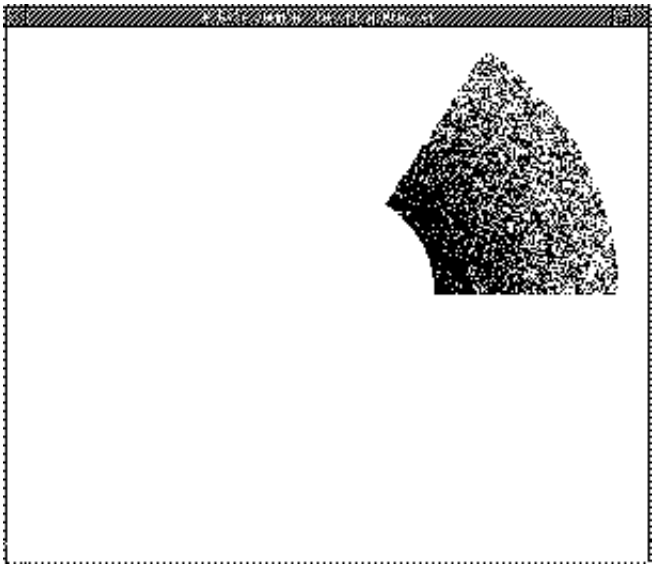

Figure 2.23: The image by the exponential function of the unit square

Figure 2.24: The image by the exponential function of the 4 times unit square

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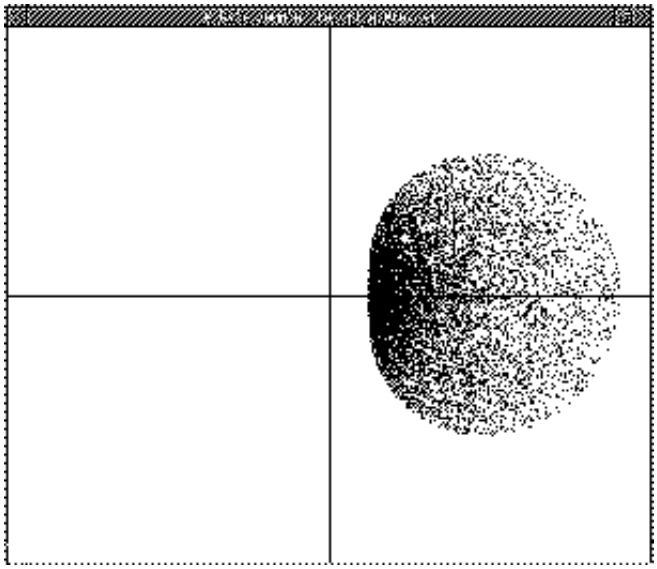

Figure 2.25: The image by the exponential function of the unit disk

Note that this is taking the complex logarithm of the point!

A little experimenting is called for, and is quite fun; strips which are vertical and big enough, get mapped onto disks about the origin with a hole at the centre. If the height of the strips is too small, they get mapped into sectors of disks with a hole at the centre. You can easily see that there is no way to actually get any point to cover the origin, although you can get as close as you like to it. If the strip is very high, you go around several times.

Suppose you wanted a logarithm for $-1+i$ which is sitting in the second quadrant. That is to say, you want something, anything which is mapped to it by exp. Then we have that

$$e^x(\cos y + i \sin y) = (-1+i)$$

It is easier to express (-1+i) in polars as $(\sqrt{2}, 3\pi/4)$. Then we have

$$e^x(\cos y + i \sin y) = \sqrt{2}(\cos(3\pi/4) + i \sin(3\pi/4))$$

which tells us that

$$x = \ln(\sqrt{2}), y = 3\pi/4$$

does the job.

So does

$$x = \ln(\sqrt{2}), y = (2n\pi + 3\pi/4)$$

for every integer n. There is a vertical line of points in $\mathbb{C}$, at x value $\ln(\sqrt{2})$ and y values separated by 2π which all get sent to the same point, $-1 + i$.

If we want the logarithm of -2, we write it as $2(\cos\pi + i\sin\pi)$ and see that $\ln 2 + i\pi$ does it nicely. So does $\ln 2 + i(2n+1)\pi$ for every integer n.

Exercise 2.7.8 *Check this claim by exponentiating* $\ln 2 + i(2n+1)\pi$

The conclusion that we come to is that every point in $\mathbb{C}$ except 0 has an infinite number of logarithms, so we have the same problem as for z^2, only much worse, if we insist on having a logarithm *function.* Our Riemann surface for the exponential and logarithm functions has not just two but an infinite number of leaves, joined together like an infinite ascending spiral staircase. The leaves all have their centres joined together: this one is a little difficult to draw. Think of a set of cones, one for each integer, all with a common vertex, nested inside each other, with cuts as in the diagram for z^2 giving a path from each cone to the one lower down- for ever.

Exercise 2.7.9 *Draw a bit of the Riemann surface for the exponential and logarithm functions. Show how you can make some branch cuts to get a piece of it which maps to* $\mathbb{C}$ *with the negative real axis removed. Show that there are infinitely many such pieces.*

It is common to define a *Principal Branch* of the logarithm, often called *Log*, by insisting that we restrict attention to answers which lie in the horizontal strip with $-\pi < y < \pi$. Alternatively, think of what exp does to such a strip.

The word 'branch' suggests to me either trees or banks, and neither seems to have much to do with a piece of the plane which is mapped to a piece of a thing like an infinite ascending spiral staircase, the central column of which is non-existent. It is, as explained earlier for the squaring function, rather old fashioned terminology. The exponential function onto the Riemann surface is a good test of your ability to visualise things. You know you are getting close when you feel dizzy just thinking about it.

The log function is a proper inverse to exp providing we regard exp as going from $\mathbb{C}$ to this Riemann surface. And if we don't, we get the usual mess, as seen in the case of the square function.

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Exercise 2.7.10 *Show that the function* $\mathrm{Log}(\frac{1+z}{1-z})$ *takes the interior of the unit disk to the horizontal strip* $-\pi/2 < y < \pi/2$.

2.8 Other powers

I can now define z^w for complex numbers z and w by

$$z^w = \exp(w \log(z))$$

which is 'multi-valued', i.e. not a function but an infinite family of them. Taking the Principal Branch makes this a function. This agrees with the ordinary definition when w is an integer.

Exercise 2.8.1 *Prove that last remark. Does it work for* w *any real number?*

Exercise 2.8.2 *Calculate* -1^i.

Since we can do in $\mathbb{C}$ *anything* we can do in $\mathbb{R}$ of an algebraic sort, we can find more exotic powers. The following exercise should be done in your head while walking to prove that you know your way around:

Exercise 2.8.3 *Calculate* i^i.

The next one can also be done internally if your concentration is in good nick:

Exercise 2.8.4 *Calculate* $(\frac{1-i}{\sqrt{2}})^{2i}$

This is good, clean fun. I have tried watching television and doing these sorts of calculations, and in my view the sums are more fun, although they may keep you awake at nights. You may be able to see why Gauss and Euler, two of the brightest men who ever lived, spent some time playing with the complex numbers a long, long time before they were really much use for anything. It's just nice to know that something like the square root of negative one raised to the power of itself is a perfectly respectable number. Actually a lot of perfectly respectable numbers. Find them all. One of them is a smidgin over 0.2.

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2.9 Trigonometric Functions

The argument through infinite series that showed that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and the argument that we can replace the usual series for the real exponential by simply putting in complex values, is asking to be carried the extra mile. Suppose we put a complex value in place of a real value for the functions sin and cos? Would we get respectable complex functions out? Yes indeed we do.

I define the complex trig functions as follows:

Definition 2.9.1

$$e^{iz} = \cos z + i\sin z$$

for any complex number z.

It follows immediately that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

and putting $z = x + iy$ and hence $iz = -y + ix$, $-iz = y - ix$ we get

$$\cos z = \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right)$$

or

$$\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Similarly we obtain:

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Example 2.9.1 *Solve:* $\sin z = i$.

Solution *We have* $\sin x\cosh y = 0$, *and since* $\cosh y \geq 1$ *it follows that* $x = n\pi$ *for some integer* n.

We also have $\cos x\sinh y = 1$, *hence* $\sinh y = \pm 1$ *follows.*

So $x = n\pi, y = \sinh^{-1}\pm 1$ *with the positive value when* n *is even and the negative when it is odd.* $x = 0, y = \ln(1 + \sqrt{2})$ *is a solution.*

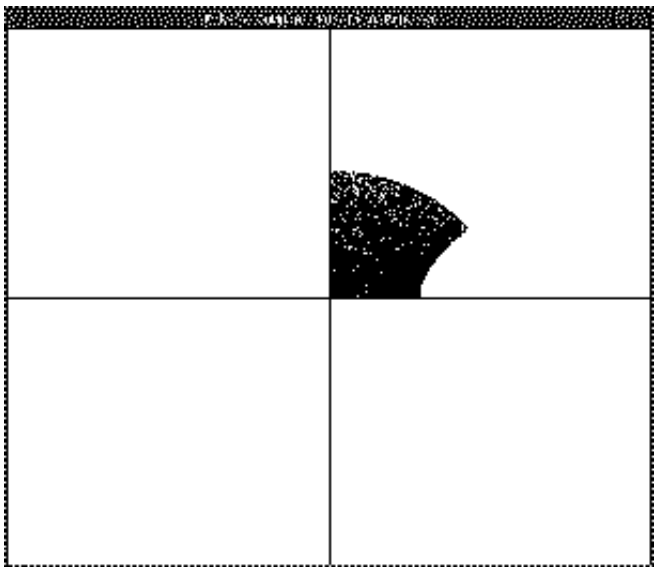

Figure 2.26: The image by the sine function of the unit square

Exercise 2.9.1 *Figure 2.26 shows the image of the unit square by the* sin *function. Show the top curved edge is a part of an ellipse, and the right curved edge is part of a hyperbola.*

It would appear that the edges of the image meet at right angles. Can you explain this?

Going back to the images we have for complex functions of squares and rectangles, you might notice that the images of square corners almost always come out as curves meting at right angles. There is one exception to this. Can you (a) give an explanation of the phenomenon and (b) account for the exception?

It follows from my definition that there are power series expansions of the usual sort for the trig functions $\sin z$ and $\cos z$. The tangent, secant, cotangent and cosecant functions are defined in the obvious ways. Inverse functions are defined in the obvious way also. The rest is algebra, but there's a lot of it.

Differentiating the trig functions proceeds from the definition:

$$\begin{aligned} e^{iz} &= \cos z + i \sin z \\ \Rightarrow \quad ie^{iz} &= \cos' z + i \sin' z \end{aligned}$$

$$= -\sin z + i\cos z$$

where the second line is obtained by differentiating the top line, and the last line is obtained by multiplying the top line by i. This tells us that the derivative of cos is $-\sin$ and the derivative of sin is cos, as in the real case. The definitions also imply that $\cos z$ is just the usual function when z is real, and likewise for sin.

The inverse trig functions can be obtained from the definitions:

Example 2.9.2 *If $w = \arccos z$, obtain an expression for w in terms of the functions defined earlier.*

Solution

We have

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

or

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

Solving the quadratic (over $\mathbb{C}$!)

$$e^{iw} = \frac{2z + \sqrt{4z^2 - 4}}{2} = z + \sqrt{z^2 - 1}$$

Hence

$$w = -i\ \log(z + \sqrt{z^2 - 1})$$

We have all the problems of multiple values in both the square root and the log functions.

Exercise 2.9.2 *Find* $\arcsin 3$.

It is worth exploring the derivatives of these functions, if only so as to be able to do some nasty integrals later by knowing they have easy antiderivatives[6].

[6]This sort of thing used to be a cottage industry in the seventeenth and eighteenth centuries: mathematicians would issue public challenges to solve horrible integration problems which they made up by doing a lot of differentiations. This is cheating, something Mathematicians are good at.

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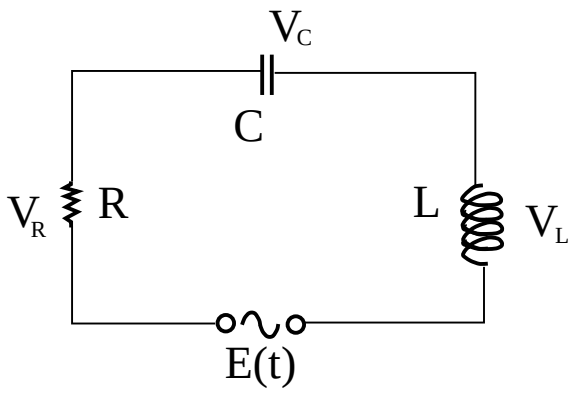

Figure 2.27: A simple LCR circuit

Exercise 2.9.3 *Compute the derivatives of as many of the trig functions and their inverses as you can.*

There is a standard application of the use of complex functions to LCR circuits which it would be a pity to pass up:

Example 2.9.3 (LCR circuits)

The figure shows a series LCR circuit with applied EMF $E(t)$; the voltage drop across each component is shown by V_R, V_C, V_L respectively. We have

$$E(t) = V_R + V_C + V_L \tag{2.1}$$

at every time t.

It is well known that the current I in a resistance satisfies Ohms Law, so we have immediately

$$V_R = IR \tag{2.2}$$

and since what goes in must come out, the current I through each component is the same.

The current and voltage drop across an inductance or choke is given by

$$V_L = L\dot{I} \tag{2.3}$$

since the impedance is due to the self induced magnetic field which by Faraday's Laws is proportional to the rate of change of current.

Finally, the voltage drop across a capacitor or condenser is proportional to the charge on the plates, so we have

$$V_C = \frac{1}{C}\int_0^t I(\tau)d\tau \tag{2.4}$$

If we have a periodic driving EMF as would arise naturally from any generator, we can write

$$I(t) = I_0 \cos(\omega t) \tag{2.5}$$

where ω is the frequency.

I now assume that the current is the real part of a complex current I^*, which will make keeping track of things simpler.

Then

$$I^*(t) = I_0 e^{i(\omega t)} \tag{2.6}$$

and similarly for complex voltages:

$$V_R^* = I^* R; \quad V_L^* = i\omega L I^*; \quad V_C^* = \frac{1}{i\omega C} I^*$$

Adding up the voltages of equation 2.1 we get:

$$E^* = \left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] I^*$$

and the quantity

$$R + i\left(\omega L - \frac{1}{\omega C}\right)$$

is called the complex impedance *usually denoted by Z.*

Then Ohm's Law holds for complex voltages and currents.

This notation may seem puzzling; it is little more than a notation, but it allows us to carry through phase information (since the phase of the voltage is changed by inductances or capacitances) which is of very considerable practical significance in Power distribution, for example. But I shall leave this to your Engineering lecturers to develop.

Since you ought to be getting the idea by now as to what to look for, I shall finish the chapter in a spirit of optimism, believing that you have sorted out at least a few functions from $\mathbb{C}$ to $\mathbb{C}$ and that you have some ideas of how to go about investigating others if they are sprung on you in an examination. I leave you to think about some possibilities by working out which real functions have not yet been extended to complex functions. There is a lot of room for some experimenting here to investigate the behaviour of lots of functions I haven't mentioned as well as lots that I have. Life being short, I have to leave it to you to do some investigation. You will find it more fun than most of what's on television.

In the next chapter we continue to work out parallels between $\mathbb{R}$ and $\mathbb{C}$ and the functions between them, but we take a big jump in generality. We ask what it would mean to differentiate a complex function.

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Chapter 3

C - Differentiable Functions

3.1 Two sorts of Differentiability

Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a function, taking $x + iy$ to $u + iv$. We know that if it is differentiable regarded as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, then the derivative is a matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

If you learnt nothing else from second year Mathematics, you may still be able to hold your head up high if you grasped the idea that the above matrix is the two dimensional version of the slope of the tangent line in dimension one. It gives the linear part (corresponding to the slope) of the affine map which best approximates f at each point.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function, then df/dx at any value of t is some real number, m. Well, what we really mean is that the map $y = mx + f(t) - mt$ is the affine map which is the best approximation to f at t. It has slope m, and the constants have been fixed up to ensure that it passes through the point $(t, f(t))$.

This is the old diagram from school-days, figure 3.1.

In a precisely parallel way, the matrix of partial derivatives gives the linear part of the best affine approximation to the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. But at

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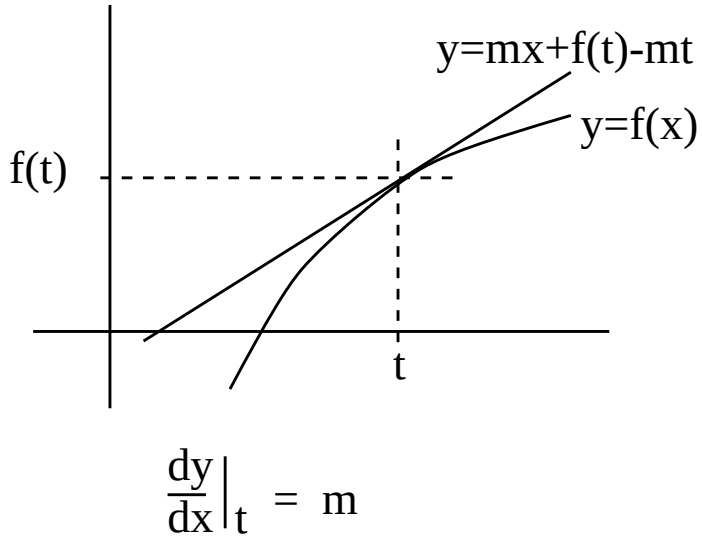

Figure 3.1: The Best Affine Approximation to a (real) differentiable function

any point $x + iy$, if f is differentiable in the *complex* sense, this must be just a linear complex map, i.e. it multiplies by some complex number. So the matrix must be in our set of complex numbers. In other words, for every value of x, it looks like

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for some real numbers a,b, which change with x.

This forces us to have the famous *Cauchy Riemann* equations:

$$\partial u/\partial x = \partial v/\partial y \quad \text{and} \quad \partial u/\partial y = -\partial v/\partial x$$

It is important to understand what they are saying; there are plenty of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which are *real* differentiable and will have the matrix of partial derivatives not satisfying the CR conditions. But these will not correspond to being a linear approximation in the sense of complex numbers. There is no *complex* derivative in this case. For the complex derivative to exist in strict analogy with the real case, the matrix must be antisymmetric and have the top left and bottom right values equal. This is a very considerable restriction, and means that many real differentiable functions will fail to be complex differentiable.

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Exercise 3.1.1 *Let $^-$ denote the conjugation map which takes z to $\bar{z}$. This is a very differentiable map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Write down its derivative matrix.*

Is conjugation complex differentiable anywhere?

On the other hand, the definition of the derivative for a real function such as $f(x) = x^2$ in the real case was

$$\frac{dy}{dx}|_t = \lim_{\Delta \to 0} \frac{f(t+\Delta) - f(t)}{\Delta}$$

We know that at $t = 1$ and $f(x) = x^2$ we have

$$\frac{dy}{dx}|_1 = \lim_{\Delta \to 0} \frac{(1+\Delta)^2 - 1^2}{\Delta}$$

and of course

$$\begin{aligned} & \lim_{\Delta \to 0} \frac{(1+\Delta)^2 - 1^2}{\Delta} \\ = & \lim_{\Delta \to 0} \frac{2\Delta + \Delta^2}{\Delta} \\ = & \lim_{\Delta \to 0} 2 + \Delta \\ = & 2 \end{aligned}$$

Now all this makes sense in the complex numbers. So if we want the derivative of $f(z) = z^2$ at $1 + i$, we have

$$\begin{aligned} f'(1+i) &= \lim_{\Delta \to 0} \frac{(1+i+\Delta)^2 - (1+i)^2}{\Delta} \\ &= \lim_{\Delta \to 0} \frac{(1+i)^2 + \Delta^2 + 2\Delta(1+i) - (1+i)^2}{\Delta} \\ &= \lim_{\Delta \to 0} 2(1+i) + \Delta \\ &= 2(1+i) \end{aligned}$$

Here, Δ is some complex number, but this has no effect on the argument. By going through the above reasoning with z in place of $1 + i$, you can see that the derivative of $f(z) = z^2$ is $2z$, regardless of whether z is real or complex.

If we write the function $f(z) = z^2$ as

$$x + iy \rightsquigarrow u + iv = x^2 - y^2 + i(2xy)$$

we see that $\partial u/\partial x = \partial v/\partial y = 2x$ and $\partial u/\partial y = -\partial v/\partial x$, so the CR equations are satisfied. And the derivative is

$$\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

as a matrix, and hence $2x + i2y$ as a complex number. So everything fits together neatly.

Moreover, the same argument holds for all polynomial functions. The arguments to show the rules for the derivative of sums, differences, products and quotients all still work. You can either go back, dig in your memories and check, or take my word for it if you are the naturally credulous sort that school-teachers and con-men approve so heartily.

It might be worth pointing out that the reason Mathematicians like abstraction, and talk of doing vector spaces over arbitrary fields for instance, is that they are lazy. If you do it once and find out exactly what properties your arguments depend upon, you won't have to go over it all again a little later when you come to a new case. I have just done exactly that bit of unnecessary repetition with my investigation of the derivative of z^2, but had you been prepared to buy the abstraction, we could have worked over arbitrary fields in first year, and you would have *known* exactly what properties were needed to get these results. The belief that Mathematicians (particularly Pure Mathematicians) are impractical dreamers is held only by those too dumb to grasp the practicality of not wasting your time repeating the same idea in new words[1].

Virtually everything that works for $\mathbb{R}$ also works for $\mathbb{C}$ then. This includes such tricks as *L'Hopital's rule* for finding limits:

Example 3.1.1 *Find*

$$\lim_{z \to i} \frac{z^4 - 1}{z - i}$$

[1] It is quite common for stupid people to claim that they have oodles of 'common sense' or 'practicality'. My father assured me that I was much less practical and sensible than he was when he found he couldn't do my Maths homework. I believed him until one day in my teens I found he had fixed a blown fuse by replacing it with a six inch nail. I concluded that if this was common sense, I'd rather have the uncommon sort.

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Solution

If $z = i$ we get the indeterminate form 0/0 so we take the derivative of both numerator and denominator to get

$$\lim_{z \to i} \frac{4z^3}{1} = 4i^3 = -4i$$

which we can confirm by putting $z^4 - 1 = (z - i)(z + i)(z^2 - 1)$.

The Cauchy Riemann equations are necessary for a function to be complex differentiable, but they are not sufficient. As with the case of $\mathbb{R}$ differentiable maps, we need the partial derivatives to be continuous, and for complex differentiability they must also be continuous *and* satisfy the CR conditions.

Example 3.1.2 *Is $f(z) = |z|^2$ differentiable anywhere?*

Solution

The $\mathbb{R}$-derivative is the matrix:

$$\begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

This cannot satisfy the CR conditions except at the origin. So f is not differentiable except possibly at the origin. If it were differentiable at the origin it would have to be with derivative the zero matrix. Taking

$$\lim_{\Delta \to 0} \frac{f(\Delta) - f(0)}{\Delta}$$

we get

$$\begin{aligned} & \lim_{x+iy \to 0} \frac{x^2 + y^2}{x + iy} \\ = & \lim_{x+iy \to 0} x - iy \\ = & \ 0 \end{aligned}$$

Since if $x + iy$ is getting closer to zero, so is its conjugate. Hence f has a derivative, zero, at the origin but nowhere else.

The function $f(z) = |z|^2$ is of course a very nice real valued function, which is to say it has zero imaginary part regarded as a complex function. And as

a complex function, it fails to be differentiable except at a single point. As a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, it has $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$, both of which are as differentiable as you can get. This should persuade you that complex differentiability is something altogether more than real differentiability.

What does it mean to have an expression like

$$\lim_{\Delta \to w} f(\Delta) = z$$

over the complex numbers? That is, are there any new problems associated with Δ, z and w being points in the plane? The only issue is that of the direction in which we approach the critical point w. In one dimension, we have the same issue: the limit from the left and the limit from the right can be different, in which case we say that the limit does not exist. Similarly, if the limit as $\Delta \to w$ depends on which way we choose to home in on w, we say that there is no limit. In particular problems, coming in to zero down the Y-axis can give a different answer from coming in along the X-axis, or along the line $y = x$. There are some very bizarre functions, few of which arise in real life, but you need to know that the functions you are familiar with are not the only ones there are. You have led sheltered lives.

In the case where the CR equations for some function $f : \mathbb{C} \longrightarrow \mathbb{C}$ are satisfied, and the partial derivatives not only exist but are continuous, we have that the complex derivative of f exists and is given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

in classical form.

There is a polar form of the CR equations. It is fairly easy to work it out, I give it as a pair of exercises:

Exercise 3.1.2 *By writing*

$$\partial u/\partial r = (\partial u/\partial x)\ (\partial x/\partial r) + (\partial u/\partial y)\ (\partial y/\partial r)$$

And similarly for $\partial u/\partial\theta$, $\partial v/\partial r$ and $\partial v/\partial\theta$, Show the CR equations require:

$$\partial v/\partial\theta = r\ \partial u/\partial r,\ \partial u/\partial\theta = -r\ \partial v/\partial r$$

Exercise 3.1.3 *Verify that $\partial\theta/\partial x = \sin\theta/r$; derive the corresponding expression for $\partial\theta/\partial y$ and deduce that*

$$\partial u/\partial x + i\ \partial v/\partial x = (\cos\theta - i\sin\theta)(\partial u/\partial r + i\ \partial v/\partial r)$$

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which is the partial derivative in polars.

Exercise 3.1.4 *Find the other form of the derivative in polars involving θ instead of r in the partial derivatives.*

Exercise 3.1.5 *We can argue that the formulae:*

$$\partial v/\partial\theta = r\ \partial u/\partial r,\ \partial u/\partial\theta = -r\ \partial v/\partial r$$

are 'obvious' by writing $\partial x \sim \partial r$ and $\partial y \sim r\ \partial\theta$ on the basis that r, θ are just rotated versions of any coordinate frame locally, and regarding ∂v and ∂u as infinitesimals obtained by taking infinitesimal independent increments ∂r and $r\partial\theta$. Perhaps for this reason it is common to write the polar form as:

$$\frac{1}{r}\frac{\partial v}{\partial\theta} = \frac{\partial u}{\partial r}, \quad \frac{1}{r}\frac{\partial u}{\partial\theta} = -\frac{\partial v}{\partial r}$$

This is the sort of reasoning that Euler or Gauss would have thought useful and gives some Pure Mathematicians the screaming ab-dabs. It can be regarded as a convenient heuristic for remembering the polar form, or it can be regarded as showing that infinitesimals ought to have a place in Mathematics because they work. Although, to be fair to Pure Mathematicians, second rate, sloppy thinking with infinitesimals can lead to total garbage. For example, if you had tried to put $\partial x \sim r\ \partial\theta$ and $\partial y \sim \partial r$ you would have got the wrong answer. Can you see why this is not *a good idea?*

It is possible, as we have seen, to have a function which is complex differentiable at only one point, This is rather a bizarre case. Functions like $f(z) = z^2$ are differentiable everywhere. If a function f is differentiable at every point in an open ball centred on some point z_0, then it is a particularly well behaved function at that point:

Definition 3.1.1 *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is (complex) differentiable at every point in a ball centred on z_0, we say that f is* analytic *or* holomorphic *at z_0.*

Definition 3.1.2 *A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be* entire *if it is analytic at every point of $\mathbb{C}$.*

Definition 3.1.3 *A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to have a* singularity *at z_1 if it is not analytic at this point. This includes the case when it is not defined there.*

Definition 3.1.4 *A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be* meromorphic *if it is analytic on its domain and this domain is $\mathbb{C}$ except for a discrete set of singular points.*

There is a somewhat tighter definition of *meromorphic* given in many texts, which I shall come to later.

I hate to load you down with jargon, but this is long standing terminology, and you need to know it so that you don't panic when it is sprung on you in later years. Very often the singularities of a complex function tell you an awful lot about it, and they come up in Engineering and Physics repeatedly.

There is another definition of the term 'analytic' which makes sense for real valued functions, and is concerned with them agreeing with their Taylor expansions at every point. The two definitions are in fact very closely related, but this is a little too advanced for me to get into here. I mention it in case you have come across the other definition and are confused. The term 'complex analytic' is sometimes used for the form I have given. Some authors insist on using 'holomorphic' until they have shown that holomorphic functions are in fact analytic in the sense of agreeing with their Taylor expansion (a Theorem of some importance). Then the theorem states that holomorphic complex functions are analytic.

The following results are mostly obvious or easy to prove and are exact analogues of the real case:

Proposition 3.1.1 *If f and g are functions analytic on a domain E, (i.e. analytic at every point of E) then*

1. *$f+g$ is analytic on E*
2. *f-g is analytic on E*
3. *wf is analytic on E for any complex or real number w*
4. *fg is analytic on E*

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5. *f/g is analytic on E except at the zeros of g*

□

Proposition 3.1.2 *If $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ are analytic functions, then the composite function $f \circ g : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic.* □

If f or g have point singularities but are otherwise analytic, then the composite is analytic except at the obvious singularities. These results will be used extensively, and because analytic functions have some remarkable properties they need to be absorbed.

3.2 Harmonic Functions

The fact that a function f from $\mathbb{R}^2$ to $\mathbb{R}^2$ is *complex* differentiable puts some very strong conditions on it. These conditions turn out to have connections with Laplace's equation which must be the most important Partial Differential Equation (PDE) there is.

Recall the various PDE's you came across last year, in particular the diffusion or heat equation and the wave equation. In steady state cases you had functions satisfying Laplace's Equation arising in many cases. For those in doubt, go to

`http://maths.uwa.edu.au/~mike/m252alder.html`

for some notes on second year calculus and PDE material. You should download the vector calculus notes which have a part on Stoke's Theorem, and a smaller part on PDEs at the end. I haven't the time to explain PDEs to you again, so you should read this stuff if you are confused and muddled about PDEs.

I remind you that a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is said to be *harmonic* or to *satisfy Laplace's Equation*, if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Thus $x^2 - y^2$ is harmonic, while $x^2 + y^2$ is not. Notice that harmonic functions on $\mathbb{R}^2$ have graphs which have opposite curvature in orthogonal directions. So hyperboloids are in there with a chance of being harmonic, while paraboloids can't ever be. Harmonic functions have the remarkable property that if you draw a circle around a point and find the average value of the function around the circle, it is always equal to the value of the function at the centre of the circle. This holds true for every circle which has f defined everywhere inside it and on its boundary. This gives a neat way of solving Laplace's equation for f on some region of the plane when we are given the values of f on the boundary (a Dirichlet Problem for the region). All we do is to fix f on the boundary, give it random values on the interior, and then go through a cycle of replacing the value at points inside the region with an average of the values of neighbouring points on some finite grid. This only gives an approximation, but that is all you ever get anyway.

One of the ways of trying to understand functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ is to think of them as a pair of functions from $\mathbb{R}^2$ to $\mathbb{R}$, the first giving the function $u(x, y)$ and the second $v(x, y)$. This means that we can draw the graphs of each function. While not entirely useless, this is not always illuminating. It does have its merits however, when considering harmonic functions.

The reason is simple: if $\partial u/\partial x = \partial v/\partial y$ then $\partial^2 u/\partial x^2 = \partial^2 v/\partial x \partial y$, which is equal to $\partial^2 v/\partial y \partial x$ providing the mixed partial derivatives are equal. This will be the case if f is analytic.

And if $\partial u/\partial y = -\partial v/\partial x$ then $\partial^2 u/\partial y^2 = -\partial^2 v/\partial y \partial x$.

Hence provided f is analytic we have:

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$$

which is to say, u is harmonic.

It is trivial to check that v is harmonic by the same argument applied to v.

Not only are both functions harmonic, they are said to be *conjugate* harmonic functions because they are related by the CR equations. For conjugate harmonic functions a number of special properties hold: for example, their product, and the difference of their squares, are also harmonic.

The argument is rather neat: If f is analytic, so is f^2. This follows because the product of analytic functions is analytic. If $f = u + iv$ then $f^2 = (u^2 - v^2) + i(2uv)$. Hence $2uv$ is harmonic, and so is any multiple of it for

obvious reasons. Similarly $u^2 - v^2$ is harmonic, and so is its negative. They are, of course, conjugate.

It is not true generally that the product of harmonic functions is harmonic.

Exercise 3.2.1 *Find two harmonic functions the product of which is not harmonic.*

Quite a lot of investigation has gone on into working out which functions are harmonic and which aren't. The reason for this is that if you are looking for a solution to Laplace's Equation, then it helps if you don't have to look too far, and if you have a 'dictionary' of them, you can save yourself some time. The fact that they come streaming out of complex analytic functions makes compiling such a dictionary easy.

Given a harmonic function, we can easily construct a conjugate harmonic function to get back to a complex analytic function. Up to an additive constant, the conjugate is unique. An example will make the procedure clear.

Example 3.2.1 *It is easy to verify that*

$$u(x, y) = \cosh(x)\sin(y)$$

is harmonic.

Differentiating with respect to x,

$$\partial u/\partial x = \sinh(x)\sin(y) = \partial v/\partial y$$

giving, by integration,

$$v = -\sinh(x)\cos(y) + \phi(x)$$

Repeating this but differentiating with respect to y this time, we get:

$$v = -\sinh(x)\cos(y) + \psi(y)$$

From which we deduce that

$$v = -\sinh(x)\cos(y) + C$$

is a conjugate (for any real number C), and $u+iv$ is easily seen to be analytic.

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Exercise 3.2.2 *If $f : \mathbb{C} \longrightarrow \mathbb{R}$ is harmonic and $g : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic, show that $f \circ g$ is harmonic. We say that* analytic maps preserve solutions to Laplace's Equation, *or* Laplace's Equation is invariant under analytic transforms.

3.2.1 Applications

Let us think about fluid flow. (The fluid might be the 'flux' of an electric field, so don't imagine this has nothing to do with your field of study!)

We write a vector field in the plane as

$$V \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ w \end{bmatrix}$$

where u and w are the components of some vector attached to $\begin{bmatrix} x \\ y \end{bmatrix}$.

Now if the fluid is irrotational, the 'curl' of $u\ dx + w\ dy$ is zero:

$$(\partial w/\partial x - \partial u/\partial y) \ = 0$$

that is:

$$\partial w/\partial x = \partial u/\partial y \tag{3.1}$$

This tells us that there is a potential function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $u = \partial\phi/\partial x$ and $w = \partial\phi/\partial y$

If there are no sources or sinks, then we also have that the divergence is zero:

$$\partial u/\partial x + \partial w/\partial y = 0$$

or

$$\partial u/\partial x = -\partial w/\partial y \tag{3.2}$$

If you have trouble with this, take it out of two dimensions into three by going to $\mathbb{R}^3$ where this makes more sense and assuming the dz component of the vector field is zero.

Now equations 3.1 and 3.2 look rather like the CR conditions, but a sign has gone wrong. This explains why I used w. I fix things up by saying that V is

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the wrong function to be concerned with, I really need $\bar{V}$, the conjugate. I can then let

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

be defined as

$$f(x + iy) = u + iv$$

where $v = -w$. Now we get

$$\partial v/\partial x = -\partial u/\partial y \tag{3.3}$$

and

$$\partial u/\partial x = \partial v/\partial y \tag{3.4}$$

This tells us that if V is an irrotational vector field with no sources and sinks, then $f = \bar{V}$ is a differentiable complex function, and indeed an analytic complex function if V is differentiable and the partial derivatives are continuous.

This in turn tells us that the components of f are harmonic.

The function ϕ is the real part of an *antiderivative* of f There is an imaginary part as well, ψ for later reference.

Example 3.2.2 *Suppose $V(x + iy)$ is the vector field $2x - i2y$. Find the potential function.*

Solution

$\bar{V} = f(x + iy) = 2(x + iy)$, *i.e.*

$$f(z) = 2z$$

This is well known to be the derivative of

$$F(z) = z^2$$

This has real part $x^2 - y^2$ (and imaginary part $2xy$). So

$$\phi(x + iy) = x^2 - y^2$$

is the required potential function.

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Of course, we could have got the same answer by standard methods, but this is rather neat.

We shall discover later that the curves $\phi(x + iy) = C$, the *equipotentials* decompose the plane into a family of curves for various values of C, which are orthogonal to the curves $\psi(x + iy) = D$ for various values of D. This means that we can look upon the latter curves as the *streamlines* of the flow. It should be obvious to you for physical reasons that the flow should always be orthogonal to the curves of constant potential. If it isn't obvious, *ask*.

In other words, the solutions to the vector field regarded as a system of ODEs can be obtained directly from integrating a complex function. Thinking about this leads to the conclusion that this is not too surprising, but again, it is rather neat.

3.3 Conformal Maps

There was an exercise in chapter two which invited you to notice that if you took any of the functions you had been working with at the time, all of which were analytic almost everywhere, then the image by such a function of a rectangle gave something which had corners. Moreover, although the edges of the rectangle were sent to curves, the curves intersected at right angles. The only exception was the case when $f(z) = z^2$ and the corner was at the origin.

The question was asked, why is this happening and why is there an exception in the one case?

If you are really smart you will have seen the answer: if you take a corner where the edges are lines intersecting at right angles, then if the map f is analytic at the corner, it may be approximated by its derivative there. And this means that in a sufficiently small neighbourhood, the map is approximable as an affine map, multiplication by a complex number together with a shift. And multiplication by a complex number is just a rotation and a similarity. None of these will stop a right angle being a right angle. The only exception is when the derivative is zero, when all bets are off.

It is clear that not just right angles are preserved by analytic functions; any angle is preserved. This is rather a striking restriction, forced by the

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properties of complex numbers and derivatives.

This property of a complex function is called *isogonality*[2] or *conformality*, with the latter sometimes being restricted to the case where the sense of the angle is preserved. For our purposes, the term conformal means that angles are preserved everywhere, which is guaranteed if the map is analytic and has derivative non-zero everywhere.

Exercise 3.3.1 *For which complex numbers w is multiplication by w going to preserve the sense of two intersecting lines?*

Exercise 3.3.2 *Give an example of a conformal map in this sense which is not analytic.*

There are a lot of applications of Complex Function Theory which depend on this property; I do not, alas, have time to do more than warn you of what your lecturers in Engineering may exploit at some later time.

It is very commonly desired to transform some one shape in the plane into some other shape, by a conformal map. Some very remarkable such transforms are known; see [11] for a dictionary of very unlikely looking conformal maps. See [9] for the Schwartz-Christoffel transformations, which take the half plane to any polygon, and are conformal on the interior.

It is a remarkable fact that

Theorem 3.1 (The Riemann Mapping Theorem)

If U is some connected and simply connected region of the complex plane (i.e. it is in one piece and has no holes in it), and if it is open *(i.e. every point in the U has a disk centred on it also contained in U) then providing U is not the whole plane, there is a 1-1 conformal mapping of U onto the interior of the unit disk.* □

3

[2]From the greek *isos* meaning equal and *agon* an angle, as in *pentagon* and *polygon*.

[3]Malcolm suggested that I point out that the selection of the interior of the unit disk is for ease of stating the theorem. It works for a much larger range of regions; it is particularly useful on occasion to take a half plane as the 'universal' region onto which all manner of unlikely regions can be taken by conformal maps.

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It follows that for any two open regions of $\mathbb{C}$ which are connected and simply connected, there is an invertible conformal map which takes one to the other.

This may seem somewhat unlikely, but it has been proved. See [10] for details.

Chapter 4

Integration

4.1 Discussion

Since we have discussed differentiating complex functions, it is now natural to turn to the problem of integrating them.

Brooding on what it might mean to integrate a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ we might conclude that there are two factors which need to be considered.

The first is that integration ought to still be a one-sided inverse to differentiation; differentiating an indefinite integral of a complex function should yield the function back again. The second is that integration ought still to be something to do with adding up numbers associated with little boxes and taking limits as the boxes get smaller.

We have just been discussing writing out a vector field as the conjugate of a complex function, so there is a good prospect that we can integrate complex functions over curves, by thinking of them as vector fields. In second year you managed to make sense of integrating vector fields over curves and surfaces, and should now feel cheerful about doing this in the plane. So your experience of integration already extends to two and three dimensions, and you recall, I hope, the planar form of the Fundamental Theorem of Calculus known as Green's Theorem. If you don't, look it up in your notes, you're going to need it.

On the other hand, we could just take the real and imaginary parts separately,

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and integrate each of these in the usual way as a function of two variables. This would give us some sort of complex number associated with a function and a region in $\mathbb{C}$. If we were to try to 'integrate' the function $2z$ in this way, to get an indefinite integral, we would get $x^2y + iy^2x$, which is not complex differentiable except at the origin. If the FTC is to hold, differentiating an indefinite integral ought to get us back to the thing integrated, and here it does no such thing. So we conclude that this is not a particularly useful way to define a complex integral.

Now the derivative of a complex function is a complex function, so the integral of a complex function should also be a complex function. So integrating functions from $\mathbb{C}$ to $\mathbb{C}$ to get other functions from $\mathbb{C}$ to $\mathbb{C}$ must be more like integrating functions from $\mathbb{R}$ to $\mathbb{R}$ than integrating or vector fields. This leads to the issue: what do we integrate over? If we integrate over regions in $\mathbb{C}$, then any version of the Fundamental Theorem of Calculus has to be some variant of Green's Theorem, and must be concerned with relating the integral over the region of one function with the integral over the boundary of another. So we seem to need to integrate complex functions over curves if we need to integrate them over regions. And we know how to integrate along curves, because a complex function $f(z)$ is a vector field in an obvious way.

Another argument for thinking that curves are the things to integrate complex functions over is that if we have an expression like

$$\int f(z)dz$$

then the dz ought surely to be $dx + i\ dy$ and this is an infinitesimal complex number, representable perhaps as a very, very small arrow. And not as a very, very small square.

Intuitive arguments of this sort can merely be suggestive, since they are derived from our experience on a different world, the world of real functions. There is a school of thought which would ban such arguments on the grounds that they can lead us astray, but it is more useful to go *somewhere* on the strength of a risky analogy than to go nowhere because it is safer. Anyway, it isn't.

We therefore investigate to see if integrating a complex function along a curve is generally a reasonable thing to do.

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4.2 The Complex Integral

Given $f(z) = 2z$ let us try to integrate it along the straight line path from 0 to $1 + i$.

As for second year integration along curves, I shall parametrise the curve: $x = t, y = t, t \in [0, 1]$ takes us uniformly from 0 to $1+i$. I put $dz = dx + i\ dy$; then $dx = dt = dy$ so we have

$$\int_0^1 2(t + it)(1 + i)\ dt$$

which is

$$(1 + i)^2 \int_0^1 2tdt = (1 + i)^2 = 2i$$

Note that we could have got the same answer by writing

$$\int_0^{1+i} 2z\ dz = [z^2]_0^{1+i} = (1 + i)^2 = 2i$$

This is using the fact that we know that $2z$ has an antiderivative, and we put our faith in the Fundamental Theorem of Calculus. It seems to work in this case.

More generally, suppose I gave you a curve in the complex plane, by giving you a function $c : [0, 1] \longrightarrow \mathbb{C}$, and a complex function f. It makes sense to do the usual business of confusing functions with values and write

$$c(t) = x(t) + i\ y(t)$$

I shall assume that both x and y are differentiable functions of t.

I can reasonably argue that now I have

$$dx = \dot{x}\ dt; \quad i\ dy = i\ \dot{y}\ dt$$

and I can define

$$\int_c f = \int_{t=0}^{t=1} f(x(t) + i\ y(t))(\dot{x} + i\ \dot{y})\ dt$$

Example 4.2.1 *Integrate the function $f(z) = 2x+2iy$ around the unit circle, starting and finishing at 1. Compare with the value of integrating around the same size circle shifted to have centre $a + ib$. What happens if the circle is made bigger or smaller?*

Solution

Put $z = e^{it}$ to get the unit circle; $dz = ie^{it}\ dt$, so the integral is

$$\int_0^{2\pi} 2e^{it}ie^{it}\ dt$$

$$= 2i\int_0^{2\pi} e^{2it}dt = 0$$

If the circle is of radius r and centre $a + ib$ we have $z = re^{it} + a + ib$, and $dz = ire^{it}\ dt$ so we obtain

$$\int_0^{2\pi} 2(re^{it} + a + ib)ire^{it}\ dt$$

$$= 2ri\int_0^{2\pi} e^{2it}dt \quad + \quad r(a + ib)\int_0^{2\pi} e^{it}dt = 0 + 0 = 0$$

Exercise 4.2.1 *I can integrate along curves which are straight lines and indeed are along or parallel to the axes. If I integrate along the X-axis, with the obvious parametrisation $x(t) = t$, I can calculate things such as*

$$\int_0^{\pi/2} e^{it}\ dt$$

Do it two ways: first by finding an antiderivative to f and directly.

Explain carefully why you would expect these to agree.

You will recall something (I hope) of the integration of vector fields over curves from second year. You may remember that the value of the integral of a vector field along a curve depends only on the set of points on the curve, and not on the parametrisation of the curve. This is physically obvious:

The idea of integrating a vector field along a curve is that of driving along a track and measuring the extent to which the gravity (Vector Field) helps you

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when you are going down hill and costs you when you are going up hill. You compute the projection of the force on the direction in which you are going and multiply the value of the force by the distance you go in a very short (infinitesimal) time. Now travelling at different speeds will make a difference to the infinitesimal distances, but they must all add up to the total distance along the track. And the value of the assistance given by the force field doesn't depend on the time.

The above argument is heuristic and would put some Pure Mathematicians in a cold sweat until they noticed that a proof of the theorem can be made which follows this heuristic argument quite closely.

So if we take

$$\int_C P\ dx + Q\ dy$$

and parametrise C by $c : [0,1] \longrightarrow \mathbb{R}^2$ with c given by $t \rightsquigarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ we have that the integral becomes

$$\int_0^1 P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt}\ \ dt$$

And the answer will not be changed by altering the parametrisation, which was only introduced to save us the hassle of chopping the curve up into little bits and calculating the projection of the force on each little bit, and then adding them all up; and then doing it again (and again!) for smaller, littler little bits and taking the limit.

Now the integration of a complex function $u + iv$ is in many ways similar to this.

In integrating a complex valued function along a curve, we have

$$\begin{aligned} & \int_c (u + i\ v)(dx + i\ dy) \\ = & \int_c (u\ dx - v\ dy) + i(v\ dx + u\ dy) \\ = & \int_0^1 [u(x(t) + iy(t))\ \dot{x} - v(x(t) + iy(t))\ \dot{y}]\ dt \\ + & \ i \int_0^1 [v(x(t) + iy(t))\ \dot{x} + u(x(t) + iy(t))\ \dot{y}]\ dt \end{aligned}$$

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So the real part is the integral of the vector field $\begin{bmatrix} u \\ -v \end{bmatrix}$ over the curve, and the imaginary part is the integral of the vector field $\begin{bmatrix} v \\ u \end{bmatrix}$ over the curve.

Now since both of these are going to be independent of the parametrisation of the curve for the same reasons as usual, it follows immediately that the path integral in $\mathbb{C}$ is independent of the parametrisation.

You may also remember from second year that there are 'nice' vector fields which are derived from a potential field and have the much stronger property that the integral along any curve of the field gives a result which depends only on the end points of the curve, and is hence zero for closed curves. And there are 'nasty' vector fields where this ain't so. If you write down a vector field 'at random', then it is 'nasty', for any sensible definition of 'at random'. It is cheering therefore to be able to tell you that the vector fields in the plane arising from analytic functions are all nice. This is the *Cauchy-Goursat Theorem*:

Theorem 4.1 (Cauchy-Goursat)

If we integrate a function f which is analytic in a domain $E \subseteq \mathbb{C}$ around a piecewise smooth simple closed curve contained in E, the result is zero.

Idea of Proof: If f is analytic, then it satisfies the CR equations. Write $f(x+iy) = u + iv$.

We want

$$\int_C [u\ dx - v\ dy] + i\,[v\ dx + u\ dy] \tag{4.1}$$

Now Let D be the region having the simple closed curve as its boundary: $\partial D = C$. From Green's Theorem we have:

$$\int_{\partial D} F = \int_D dF$$

and if $F = u\ dx - v\ dy$, which is the real part of the complex integral 4.1, we have

$$\int_C [u\ dx - v\ dy] = \int_D -\partial v/\partial x - \partial u/\partial y = 0$$

by the CR equations.

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Similarly, the imaginary part is also zero. □

It follows immediately that if $p : [0,1] \longrightarrow \mathbb{C}$ is a piecewise smooth path from 0 to w in $\mathbb{C}$, and if f is a complex function which is analytic on a ball big enough to contain 0 and w, $\int_0^1 f(p(t))(\dot{x}+i\dot{y})\,dt$ gives a result which depends on w but not p. This is obvious, because if we could find a path with the same end points but a different value for the integral, we could go out along one path, back along the other, and have a non-zero outcome, contradicting the last theorem.

We use this to define an indefinite integral:

Theorem 4.2 (Antiderivatives)

For any f which is analytic on a domain E, define

$$F : \mathbb{C} \longrightarrow \mathbb{C}, \text{ by } F(w) = \int_0^1 f(c(t))(\dot{x} + i\dot{y})\,dt$$

where c is any smooth path which has $c(0) = 0$ and $c(1) = w$.

Then F is analytic and $F' = f$

Proof

The proof is usually a fiddly argument from first principles. Since you will have done similar things for the existence of the potential function for conservative fields, and this is pretty much the same idea, I shall skip it. □

Corollary 4.2.1 *If f is analytic and $c : [0,1] \longrightarrow \mathbb{C}$ is any smooth path in C, then if F is the antiderivative provided by the above theorem,*

$$\int_c f(z)dz = F(z(0)) - F(z(1))$$

Proof:

This follows immediately from the construction of F. □

It should be apparent that there is no need to start my construction of F from the origin; anywhere else would do. The two antiderivatives would differ by a (complex) constant.

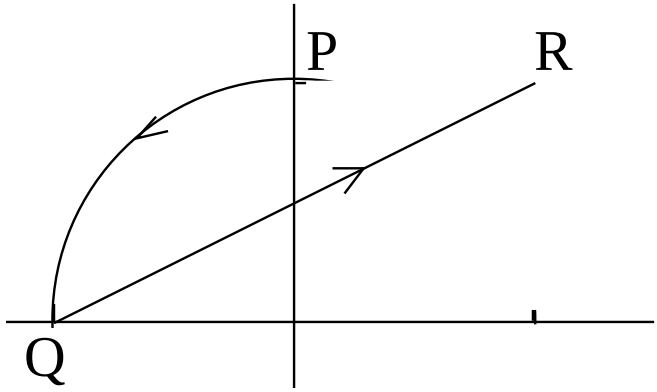

Figure 4.1: A path from i to $1+i$

This has given us a fairly satisfactory idea of what is involved in doing integration for analytic complex functions. The key result is that integrating an analytic function around a simple closed loop C gives zero. This has implications for evaluating integrals around nasty curves:

Example 4.2.2 *Evaluate the contour integral*

$$\int_C \frac{1+2z^2}{z}$$

where C is the curve starting at $P = i$ and going to $Q = -1$ along the unit circular arc centred at the origin in the anticlockwise direction, followed by a straight line from Q to R at $1+i$.

Solution

The diagram figure 4.1 shows the curve we have to integrate over. If we were to join the endpoints by going to i from $1+i$, the resulting closed curve would not contain a singularity of the functions, which is analytic (being a ratio of analytic functions), and the integral around the curve would therefore be zero.

The integral therefore does not depend on the path, and the straight line path

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from i to $1+i$ given by $c : [0,1] \longrightarrow \mathbb{C}$, $t \rightsquigarrow t+i$ gives the same answer as the integral over the much more complicated curve asked for.

In fact we can rewrite the integral as

$$\int_C \frac{dz}{z} + \int_C 2z\ dz \tag{4.2}$$

and since the path does not do a circuit of a singularity, this is

$$\begin{aligned} & [\mathrm{Log} z]_0^1 + [z^2]_0^1 \\ = \ & \mathrm{Log}(1+i) - \mathrm{Log}(i) + (1+i)^2 - i^2 \\ = \ & \mathrm{Log}(1-i) + 1 + 2i \\ = \ & \log(\sqrt{2}) - i\pi/4 + 1 + 2i \\ = \ & 1 + \log(\sqrt{2}) + i(2 - \pi/4) \end{aligned}$$

□

Exercise 4.2.2 *Rework the last solution by substituting $z = t + i$ in equation 4.2 and integrating along the path to confirm that we agree on the answer.*

Exercise 4.2.3 *Suppose instead of going by the anticlockwise route along the unit circle, the curve went the clockwise route and hence circumnavigated the origin. How would you evaluate, quickly, the new path integral?*

Things can be very different when f stops being analytic, for example when it has a singularity in the region enclosed by C. For a start, there is no guarantee that integrating around such a loop will give zero, and it often does not. For seconds, there is no guarantee that such a function will have an antiderivative.

This is the start of some rather curious phenomena which will be investigated in a separate section.

4.3 Contour Integration

For some reason known only to historians, the term *contour* is used in Complex Analysis to denote a curve, usually a simple closed curve, almost always

a piecewise differentiable curve, in the plane. The term 'simple' means that it does not cross itself, and we can always integrate over pieces that are smooth, and add up the results (since integration is just adding up anyway!). So we can include polygons as among the family of curves we can integrate over. And integrating around such curves is called *contour integration*. If you had to guess what it meant, you might come up with a lot of possibilities before you hit on the actual meaning according to complex function theorists. More bloody jargon, in short. Still, I suppose it is useful for frightening Law students and other low forms of life who have never performed even the simplest contour integrals. So that you won't be mistaken for such low life, we shall now perform one. Watch closely.

Example 4.3.1 (Contour Integral)

Integrate $1/z$ *around the unit circle, starting and finishing at 1.*

Solution

The fact that the function $1/z$ *is not even defined at* 0 *and hence cannot be differentiable there means that we cannot cheerfully claim that the answer is zero, anyway, it isn't. First we do it the clunky way:*

Put $x + iy(t) = \cos t + i \sin t$ *as a parametrisation of the unit circle, with* $t \in [0, 2\pi]$.

$dx + i\ dy = -\sin t + i\ \cos t$, *and* $1/z = \bar{z}/z\bar{z}$ *and on the unit circle* $z\bar{z} = 1$. *This gives:*

$$\begin{aligned} & \int_{S^1} 1/z\ dz \\ = & \int_0^{2\pi} (\cos t - i \sin t)(-\sin t + i\ \cos t)\ dt \\ = & \int_0^{2\pi} i(\sin^2 t + \cos^2 t)\ dt \\ = & 2\pi i \end{aligned}$$

Next we do it more neatly: $z = e^{it}$ *parametrises the circle.* $dz = ie^{it}\ dt$ *follows. So*

$$\int_{S^1} 1/z\ dz$$

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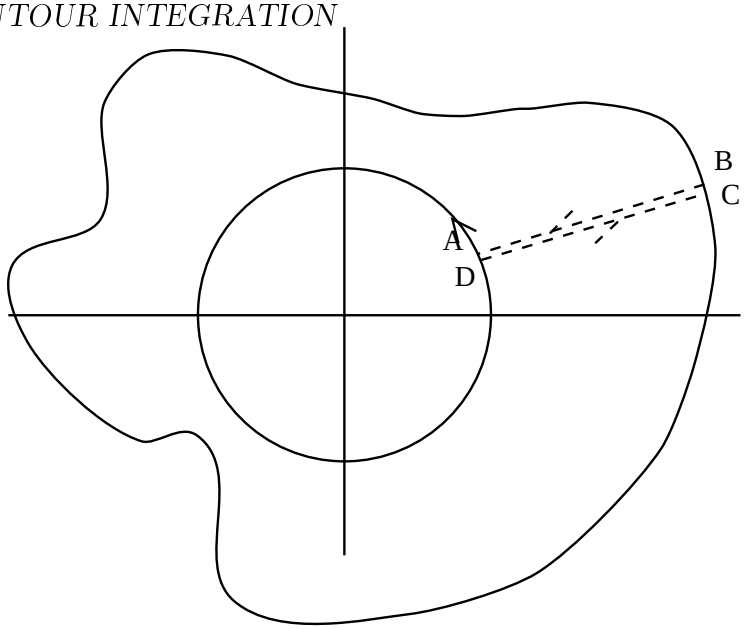

Figure 4.2: Any loop enclosing the (single) singularity has the same integral

$$
\begin{aligned}
&= \int_0^{2\pi} \frac{1}{e^{it}} i e^{it}\ dt \\
&= i \int_0^{2\pi}\ dt \\
&= 2\pi i
\end{aligned}
$$

□

This result measures some property of the singularity.

To see this, note that if I had gone around the origin in a different loop but in the same direction, once, I should have got exactly the same answer.

Exercise 4.3.1 *Just to confirm this, go around a square with vertices at* $\pm 1 \pm i$.

And to see *this*, look at figure 4.2 which shows another loop going once around the origin.

If we went around the circle, from A to D, but then went along the line DC, then around the outer loop clockwise, then in to the circle by BA, we should

get a value of $\int_C 1/z\ dz = 0$, since the function $1/z$ is analytic on the curve and its interior (the region between the circle and the outer curve).

But the line CD and the line AB will cancel out if the two line segments coincide, since they are traversed in opposite directions. Hence the integral over the inner circle and the outer loop (traversed clockwise) sum to zero. So the integral over the circle and over the outer loop traversed in the same direction must be equal.

Thus I have shown:

Proposition 4.3.1 *Let f be an analytic function with a singularity at a point. Then the integral of f around any loop making one circuit of the singularity is the same as the integral of f around any other loop making a single circuit of the singularity in the same direction.*

Definition 4.3.1 *We say that $1/z$ has a* pole *at the origin. More generally, f has a pole at w if*

$$\lim_{z \to w} |f(z)| = \infty$$

So $1/(z-1)(z-i)$ has a pole at $z = 1$ and another at $z = i$. $1/z^2$ also has a pole at 0.

Exercise 4.3.2 *Find the integral for a loop around the singularity at 0 of $1/z^2$.*

The above exercises should leave you prepared for the following:

Proposition 4.3.2 *The function $1/(z - z_0)$ has a pole at z_0 and the integral of any single loop around z_0 traversed anticlockwise is $2\pi i$.*

For any integer $n \neq 1$, and any simple closed loop c around z_0 traversed anticlockwise around z_0,

$$\int_c 1/(z - z_0)^n\ dz = 0$$

Proof:

Let c be the loop $z_0 + e^{it}$, so $dz = ie^{it}$. Then

$$\begin{aligned}\int_c 1/(z-z_0)^n \, dz &= i\int_0^{2\pi} \frac{e^{it}}{e^{int}} dt \\ &= 2\pi i \text{ if } n = 1 \\ &= 0 \text{ if } n \neq 1\end{aligned}$$

□

This allows us to use partial fractions to work out the integrals for loops around a range of functions with singularities enclosed by the loops.

Example 4.3.2 *Calculate the integral of $\frac{z}{z^2-1}$ around the circle centred on the origin of radius 2, in the anticlockwise direction.*

Solution

$$\frac{z}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$$

So

$$\begin{aligned}\int_C \frac{z}{z^2-1} \, dz &= \frac{1}{2}\left(\int_C \frac{1}{z-1} + \frac{1}{z+1} \, dz\right) \\ &= \frac{1}{2}\int_C \frac{1}{z-1} \, dz + \frac{1}{2}\int_C \frac{1}{z-1} \, dz \\ &= \frac{1}{2}2\pi i + \frac{1}{2}2\pi i \\ &= 2\pi i\end{aligned}$$

Note that the loop contains both the singularities.

Exercise 4.3.3 *I claim that the integral of an analytic function around a loop containing k singularities is the same as the sum of the integrals of loops around each one separately.*

Produce an argument to show that my claim is correct, or produce a counter-example to show I am blathering.

It is important to realise that all this works for functions which are analytic except at a set of discrete singularities. It fails miserably when the function is not analytic:

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Example 4.3.3 *Integrate the function $\bar{z}$ anticlockwise around the unit circle and also around the square with vertices at $\pm 1 \pm i$, in the same sense.*

Solution *The circle first: $z = \cos t + i \sin t \Rightarrow dz = i(\cos t + i \sin t)$ so:*

$$\int_0^{2\pi} (\cos t - i \sin t) i (\cos t + i \sin t)\ dt = 2\pi i$$

The right hand edge of the square: $z = 1 + ti \Rightarrow dz = i\ dt$:

$$\int_{-1}^{1} (1 - ti) i\ dt = 2i$$

The opposite edge: $z = -1 - ti \Rightarrow dz = -i\ dt$ so:

$$\int_{-1}^{1} (-1 + ti) - i\ dt = 2i$$

The bottom edge: $z = t - i \Rightarrow dz = dt$ so:

$$\int_{-1}^{1} (t + i)\ dt = 2i$$

And the top edge: $z = -t + i \Rightarrow dz = -dt$ so:

$$\int_{-1}^{1} -(-t - i) dt = 2i$$

So the result for the square is $8i$ and for the circle $2\pi i$ □

It is immediate that the contour integral of a path in one direction is always the negative of the integral in the opposite direction, and that this works for functions which are not analytic as well as for those which are, since the independence of parametrisation holds for all integrable functions, analytic or not.

Exercise 4.3.4 *Prove that reversing the direction of travel reverses the sign of the answer for any integrable function.*

It is also obvious that the integral along two paths which follow is the sum of the integrals around each path separately, something we used in the last example. This follows from the definition of the path integral- we are adding up lots of little bits anyway.

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4.4 Some Inequalities

It is important to be able to obtain rough estimates of path integrals, so as to be able to decide whether you have got a reasonable sort of answer or have made a blunder somewhere. For this reason, the following inequalities are useful:

Proposition 4.4.1 *If $c : [0,1] \to \mathbb{C}$ is a smooth path in $\mathbb{C}$*

$$\left| \int_0^1 c(t)dt \right| \leq \int_0^1 |c(t)|dt \tag{4.3}$$

Proof:

If $\int_0^1 c(t)dt = Re^{i\theta}$, the left hand side of 4.3 is just R.

We have that

$$R = \int_0^1 e^{-i\theta} c(t)\ dt$$

and since the left hand side is real we have also:

$$R = \int_0^1 \Re[e^{-i\theta} c(t)]\ dt$$

But

$$\int_0^1 \Re[e^{-i\theta} c(t)]dt \leq \int_0^1 |e^{-i\theta} c(t)|dt$$

since for all t, and any function g, $\Re(g(t)) \leq |g(t)|$.

Then since $|zw| = |z||w|$ and $|e^{-i\theta}| = 1$ we have

$$R = \left| \int_0^1 c(t)dt \right| \leq \int_0^1 |c(t)|dt$$

□

It is not necessary for the path c to be smooth, but it needs to be continuous. Note that we are integrating the constant function 1 over the path.

We can strengthen this as follows:

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Proposition 4.4.2 *Let c be a smooth path in $\mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ a continuous function. Let L be the length of the path and M be the maximum value of $|f|$ on c. Then*

$$\left| \int_c f(z)dz \right| \leq ML$$

Proof:

$$\left| \int_c f(z)dz \right| = \left| \int_0^1 f(z)\dot{z}\ dt \right|$$

By the preceding result we have:

$$\left| \int_0^1 f(z)\dot{z}\ dt \right| \leq \int_0^1 |f(z)\dot{z}\ dt| = \int_0^1 |f(z)||\dot{z}|\ dt$$

And

$$\int_0^1 |f(z)||\dot{z}|\ dt \leq M \int_0^1 |\dot{z}|\ dt = ML$$

□

This is a rather coarse inequality, and we can get better estimates by partitioning c and looking for better bounds on the parts.

Example 4.4.1 *Estimate the modulus of the integral of $\bar{z}$ from $1-i$ to $1+i$*

We have that the length is 2 and the maximum value of $|\bar{z}|$ along the path is $\sqrt{2}$ at the end points. So

$$\left| \int_c \bar{z}dz \right| \leq 2\sqrt{2}$$

From an earlier example we know that the actual value is 2. □

4.5 Some Solid and Useful Theorems

Theorem 4.3 (The Cauchy Integral Formula)

If f is analytic in a region $E \subseteq \mathbb{C}$, and if C is any closed simple curve in E, then for any $w \in E$,

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz$$

Proof:

We certainly have that

$$\int_C \frac{f(w)}{z-w}dz = f(w)\int_C \frac{1}{z-w}dz = f(w)\ 2\pi i$$

Since the integral $\int_C \frac{f(w)}{z-w}dz$ *will remain constant no matter how small the loop is, the limit as* C *shrinks to zero of the integral exists and is* $f(w)2\pi i$*. But this is also the limit of*

$$\int_C \frac{f(z)}{z-w}dz$$

which is also independent of the loop size.

Hence

$$\int_C \frac{f(z)}{z-w}dz = \int_C \frac{f(w)}{z-w}dz = f(w)\int_C \frac{1}{z-w}dz = f(w)\ 2\pi i$$

and the result is proved. □

Theorem 4.4 (The Cauchy Integral Formula for Derivatives)

If f *is analytic in a region* $E \subseteq \mathbb{C}$*, and if* C *is a simple closed curve in* E *then for any* z_0 *enclosed by* C*, the* n^{th} *derivative of* f *exists and is given by:*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Proof:

We have for the original Cauchy formula:

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}dz$$

for any $w \in \mathbb{C}$*.*

Parametrising the loop C *by* $z(t)$*, we can write this as*

$$f(w) = \frac{1}{2\pi i}\int_0^1 \frac{f(z(t))\dot{z}(t)}{z-w}\ dt$$

We now treat the integral as a function of w and t and use Leibnitz Rule which says we can differentiate through an integral sign to get

$$f'(w) = \frac{1}{2\pi i}\int_0^1 \frac{\partial}{\partial w}\left(\frac{f(z(t)\dot{z}(t)}{z-w}\right)\, dt$$

This gives immediately:

$$f'(w) = \frac{1}{2\pi i}\int_0^1 \frac{f(z(t))\dot{z}(t)}{(z-w)^2}\, dt$$

We simply carry on doing this to get the required result. □

An important corollary is:

Corollary 4.4.1 *If f is analytic in a region E, then it has derivatives of all orders in E and every derivative is also analytic in E.* □

This makes it clear that complex analytic functions are very special and quite different from continuously differentiable real functions. If you can differentiate a complex function everywhere in a region, you can differentiate the derivative in the region, and so on indefinitely.

A second corollary follows also:

Corollary 4.4.2 *If $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is harmonic, then it has partial derivatives of all orders, and all are harmonic functions.* □

There is a converse to the Cauchy-Goursat theorem:

Theorem 4.5 (Morera's Theorem)

If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous and satisfies the condition that for every closed loop c

$$\int_c f(z)\, dz = 0$$

then f is analytic.

Proof:

We can construct an antiderivative of f, F say, by the usual process of integrating f from the origin (or some other convenient location) to the point w to define $F(w)$. Then F has derivative f, which is by hypothesis continuous, so F is analytic. Hence it has derivatives of all orders, each of which is also analytic; f is the first of them. □

We can also show the mean value theorem that says that for any circle centred on a point w in the domain of an analytic function f, the mean value of all the values of f on the circle is the value at the centre:

Theorem 4.6 (Gauss' Mean Value Theorem)

If f is analytic and w is any point, then for the circle $w + Re^{i\theta}$ we have

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w + Re^{i\theta})d\theta$$

Proof:

By Cauchy's integral formula we have

$$f(w) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - w}dz$$

where c can be taken to be $w + Re^i\theta$ for $\theta \in [0, 2\pi]$. Substituting for z and dz we get

$$f(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(w + Re^{i\theta})iRe^{i\theta}}{Re^{i\theta}}d\theta$$

and some cancelling gives the result. □

It is important to see that the integral $\int_0^{2\pi} f(w + Re^{i\theta})d\theta$ is NOT a path integral. If it were, it would be zero. We are not multiplying by a dz, which being an infinitesimal complex number has a direction associated with it, but by a $d\theta$ which is a 'real infinitesimal'.

You may have been told that infinitesimals are wicked. This is obsolete. Modern mathematicians just take them to be elements of a thing called the 'tangent bundle' and treat them pretty much the same way the great classical

mathematicians did. Since I cannot explain the rationale properly in less than a lecture course on manifolds, I shall rely on your vague intuitions.

The result of Gauss leads to another important property of analytic functions:

Theorem 4.7 (The Maximum Modulus Principle)

If f is analytic and non-constant in a connected region E, then $|f(z)|$ attains its maximum on the boundary of E.

Proof: Suppose that $|f(z)|$ has a maximum at an interior point w. Then we could find a circle $C = w + Re^{i\theta}$ centred on w such that

(0) The disk with boundary C is in E,

(1) for every $z \in C$, $|f(z)| \leq |f(w)|$, and

(2) $|f(w)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(w + Re^{i\theta})d\theta\right|$

but we have

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(w + Re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(w + Re^{i\theta})|d\theta$$

But by (1) we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(w + Re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(w)|d\theta = |f(w)|$$

The two inequalities must mean that

$$\int_0^{2\pi} |f(w)| - |f(w) + Re^{i\theta})|d\theta = 0$$

which can only happen if

$$|f(w)| = |f(w) + Re^{i\theta})|$$

for every point on the circle. But this must hold for every circle centred on w of smaller radius than R, so $|f(z)|$ must be constant in a disk shaped neighbourhood of w.

Now we cover E with disks. each disk contained in E, with a disk centred at every point of E. Since $|f(z)|$ is constant in the first disk we can take any

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disk C' of radius R' intersecting the first disk, and observe that there is a point w' inside both disks and if we go through items (0), (1) and (2) above replacing C by C', R by R' and w by w' everything still holds. From which we conclude that $|f(z)|$ must also be constant (with the same value) on the second disk.

This can be extended for all disks, and so $|f(z)|$ is constant on E. This contradicts the hypothesis. So $|f(z)|$ cannot have an interior point of E as its maximum. □

Example 4.5.1 *Find the maximum value of $|z^2 + 3z - 1|$ on the unit disk $|z| \leq 1$.*

Solution *By the Maximum Modulus Principle, the value must be a maximum on the boundary, $|z| = 1$. We can therefore put $z = \cos\theta + i\sin\theta$, and try to maximise*

$$(\cos 2\theta + 3\cos\theta - 1)^2 + (\sin 2\theta + 3\sin\theta)^2$$

since the maximum of a positive function occurs at the same place as the maximum of its square. This simplifies by elementary trigonometry to

$$11 - 2\cos 2\theta$$

which has a maximum at $\theta = \pm\pi/2$. So $z = \pm i$ is the location of the maximum which has value $\sqrt{13}$. This may be confirmed by plugging $z = \pm i$ into the original function and computing the modulus.

Theorem 4.8 (Cauchy's Inequalities)

For f analytic in a region containing the disk D of radius R centered on w, and $|f(z)| \leq B$ for all $z \in D$ being a bound on $|f(z)|$ on D, then the nth derivative of f, f^n has modulus bound:

$$|f^n(w)| \leq \frac{n!B}{R^n}$$

for all positive integers n.

Proof:

We have

$$f^n(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}}dz$$

Hence

$$|f^n(w)| = \left| \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(w + Re^{i\theta}(iRe^{i\theta}}{R^{n+1}e^{i(n+1)\theta}} \, d\theta \right|$$

and

$$|f^n(w)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(w + Re^{i\theta}| \, d\theta$$

and since

$$\int_0^{2\pi} |f(w + Re^{i\theta}| \, d\theta \leq 2\pi B$$

the result follows. □

The extension of the Maximum Modulus Principle to the whole of $\mathbb{C}$ is obvious; if f is entire (analytic on all of $\mathbb{C}$), then $|f(z)|$ cannot have a maximum at all, except in the rather uninteresting case where it is constant. Of course, it might, in principle, be the case that although not achieving any maximum, it 'saturates, that is, it gets closer and closer to some least upper bound. This doesn't happen either:

Theorem 4.9 (Liouville's Theorem)

If f is an entire function which has $|f(z)|$ bounded, then f is constant.

Proof: Take a circle of radius R around any point $w \in \mathbb{C}$.

By Cauchy's inequality for the first derivative we have

$$|f'(w)| \leq \frac{B}{R}$$

where B is the bound for $|f(z)|$ on all of $\mathbb{C}$. Since this holds for all circles of radius R, we see that

$$f'(w) = 0$$

This has to hold for all $w \in \mathbb{C}$. So f must be constant.

□

It is clear that these results for complex functions have implications for the real and complex parts which are harmonic, and since any harmonic function can be extended to a complex function by computing the conjugate harmonic

function, we can deduce corresponding results for harmonic functions. For example, we can deduce that the mean of the values on a circle is the value of the function at the centre, and that the only bounded harmonic functions defined on $\mathbb{R}^2$ are constant. When trying to solve Laplace's equation, every little helps.

Exercise 4.5.1 *Show that if u is a harmonic function of two variables, it has the property that the mean value of u on a circle centred at w is $u(w)$.*

Exercise 4.5.2 *Show that if u is a harmonic function of two variables and E is a region in $\mathbb{R}^2$, then the maximum value of $|u(x,y)|$ is attained on the boundary of E.*

(It helps give some insight into the theorems for complex functions to see what they say about the harmonic functions which are their components: this makes particular sense with constraints on the modulus.)

Finally, the Fundamental Theorem of Algebra is going back to the roots of Complex Analysis. It says that every polynomial of degree n has n roots, generally complex, although some may be the same. So we count multiplicities. Another way of putting this is that we can factorise any polynomial of degree n into n linear factors $(z-r_1)(z-r_2)\cdots(z-r_n)$, where the roots r_j are generally complex. Now this is pretty much what the Complex Numbers were invented for, in particular so that we could always factorise quadratics.

But there is more to the theorem than saying that if we take a real polynomial, i.e. one with real coefficients, then we can factorise it into complex roots. What if we allow ourselves complex coefficients? Well, it still works. We can factorise *all* polynomials over $\mathbb{C}$ into linear factors

This is the Fundamental Theorem of Algebra (FTA):

Theorem 4.10 (Fundamental Theorem of Algebra)

A complex polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $n \geq 1$ can be factorised, uniquely up to order of terms as

$$a_n(z-r_1)(z-r_2)\cdots(z-r_n)$$

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Proof:

We show first that P has at least one zero, that is there exists $w \in \mathbb{C}$ such that $P(w) = 0$.

If not then $\frac{1}{P(z)}$ is an entire function.

Now it is easy to see that

$$lim_{|z|\to\infty}|\frac{1}{P(z)}| = 0$$

since the $a_n z^n$ term of P dominates in $\mathbb{C}$ for the same reason that it does in $\mathbb{R}$. So we can find a disk of radius R centred on the origin, such that

$$|z| > R \Rightarrow |\frac{1}{P(z)}| \leq 1$$

Now on the disk, $|\frac{1}{P(z)}|$ is a continuous function and the disk is compact so there is some bound B which is attained by $|\frac{1}{P(z)}|$ on the disk. Actually on its boundary.

It follows that $|\frac{1}{P(z)}|$ is bounded by the larger of B and 1 everywhere on $\mathbb{C}$. Hence, by Liouville's Theorem, $\frac{1}{P(z)}$ is constant, which is clearly not the case.

So $P(z)$ has at least one zero, r_1. Which means that $(z - r_1)$ is a factor of $P(z)$, for we could certainly divide $(z - r_1)$ into $P(z)$ by the usual rules, and there could not be a non-zero remainder.

So $P(z) = (z - r_1)P_{n-1}(z)$ for a new polynomial of degree $n-1$. This reduces the degree of the polynomial by one. But the same argument as above applies here also. So we can keep reducing the degree of the polynomial until it is one, when the result is obvious.

□

It is also true that if the coefficients of P are all real, then the roots must come in conjugate pairs or be real. This certainly holds for P quadratic, for if

$$z^2 + az + b = (z - r_1)(z - r_2)$$

we have immediately that

$$r_1 + r_2 = a, \quad r_1 r_2 = b$$

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and the first equation tells us that the imaginary parts of r_1, r_2 must have equal and opposite values, and the second implies that the real parts have to be the same.

It is easy to verify that if we multiply a quadratic with real coefficients by a linear term $z - r$ then we can get a cubic with real coefficients only if r is real.

Exercise 4.5.3 *Complete the argument to show that if a polynomial in z has real coefficients, the roots must be real or come in conjugate pairs.*

You are getting, in rather a lump, the results of about a century of exploration of Complex Functions by some of the brightest guys in Europe. The impact of it all, can be more than a bit mind numbing. Indeed if you don't feel smashed by the weight of it all you have probably missed out on the meaning. This is very dense, solid stuff that needs a lot of thinking about to really absorb. You are being told a lot of properties of these very special functions.

Exercise 4.5.4 *Can you think of a well-known class of real functions which have the property that they satisfy Liouville's Theorem?*

You may be left wondering how they discovered all these results. Well, this was before television, and mucking about with complex functions is rather fun if you happen to be brilliant. There was certainly a lot to be found out. And of all the ways of passing an idle hour known to man, just mucking about with complex functions to see what happens has turned out to be one of the most productive.

A very practical problem for people wanting to survive the exam is: how do you get to know and feel comfortable with all these theorems?

The answer is, (1) you use them for solving problems, and (2) you work through the proofs to see what the ideas are. Much of it is quite intuitive; for instance the proof of the Cauchy Integral Formula depends strongly on integrals around loops not changing as they shrink closer to a point inside the loops. This in turn means that the functions have to be analytic except at the point we are shrinking towards. This tells you what the assumptions in the theorem are, which stops you doing something daft with the result. Settling down somewhere quiet with a pen and lots of blank paper and making up

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your own problems, or working through a text book and doing the problems there, is the best and surest way of feeling good about Mathematics. You discover the reasons why Cauchy and Euler and Gauss did the original work: there is a sense of triumph in getting something as fundamental as this sorted out. It isn't easy, but when was anything worthwhile ever easy?

Our present culture is very different from the one which produced the great results of Mathematics and Science. It has taught you to regard anyone who enjoys this sort of activity as qualifying for the title of King Nerd. A cynic might say that the ideals of our culture are designed to reassure thickos, who believe deeply that being really good at throwing balls in buckets makes you a hero, while playing around with ideas makes you a nerd. This is because there are a lot of thickos who are incapable of seeing the point of playing with ideas, and you don't want a bunch of thickos going around feeling insecure and inferior. Better by far if they focus their minds on watching other people throw balls in buckets.

For the non-thickos:

Exercise 4.5.5 *Show that if f is a non-constant complex function and $|f(z)| > 1$ for all $z \in E$, and f is analytic in E, some region in $\mathbb{C}$, then $|f(z)|$ has its* **minimum** *value on the boundary of E.*

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Chapter 5

Taylor and Laurent Series

5.1 Fundamentals

In chapter 2, section 2.7.1, I mentioned briefly the importance of infinite series, particularly power series, in estimating values of functions. What it comes down to is that we can easily add, subtract, multiply and except in the case of zero, divide real numbers, and this is essentially all we can do with them. The same applies to complex numbers. The only operation that makes sense otherwise is taking limits, and again this makes sense for complex numbers also. It follows that if we want to calculate $\sin(2)$ or some other function value, it *must* be possible to compute the answer, to increasing accuracy, in terms of some finite number of repeated additions, multiplications, subtractions and divisions, or there isn't any meaning to the expression. We can accept that we may never get an exact answer in a finite number of operations, but if we can't get an estimate *and know the size of the uncertainty* with a finite number of standard operations, and if we cannot guarantee that we can reduce the uncertainty below any amount that is desired by doing more simple arithmetic, then $\sin(2)$ simply doesn't have any meaning. The same holds for all the other functions. Even the humble square root of 2 exists only because we have a means of computing it to any desired precision. And the only way of doing this must involve only additions, subtractions, multiplications and divisions, because this is all there is. Your calculator or computer must therefore use some form of truncated infinite series in order to compute $\sqrt{2}$ or $\arctan 1/4$ or whatever. A more expensive calculator may use more terms, or it may use a smarter series which converges faster, or it

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may do some preprocessing using properties of the function, such as reducing trig functions by taking a remainder after subtracting off multiples of 2π to evaluate $\sin(100)$. But it must come down to infinite series except for the cases where it can be calculated exactly in a finite number of operations.

It follows that series expansions for functions is absolutely fundamental, and that the question of when they converge is also crucial. A calculator that tried to compute something by using the series

$$1 + 1/x + 1/2x^2 + 1/3x^3 + \cdots$$

would run into trouble at $x = 1$, but it would produce an answer- one which is meaningless. Somebody has to design the calculator and that someone has to know when garbage is going in if garbage is not to come out.

The idea of an infinite series representations of a function then is simply that of always being able to add on an extra little bit which will make the result closer to the 'true' answer, and knowing something about the precision we have attained at each step. And that is *all* infinite series are about.

This comes out in the jargon as:

$$f(z) = \sum_1^\infty a_k z^k$$

or something similar, where we have a way of calculating the a_k. And what this means is that if

$$S_n(z) = \sum_1^n a_k z^k$$

is the sum of the first n terms, the sequence $S_n(z)$ has a limit for every z.

And what *this* means is that there is for each z some complex number w such that if you stipulate a precision ε, a small positive real number, then there is some N, a critical number of steps, such that after that many steps, the partial sum S_n for $n > N$ is always within the desired accuracy of the answer w. In algebra:

$$n > N \Rightarrow |S_n - w| < \varepsilon$$

Putting this together, we say that

$$f(z) = \sum_1^\infty a_k z^k \quad \Leftrightarrow \quad \forall \varepsilon \in \mathbb{R}^+,\ \exists N \in \mathbb{N} : n > N \Rightarrow |f(z) - \sum_1^n a_k z^k| < \varepsilon$$

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This blows the mind at first sight, but it only says compactly what the preceding half page said. Read it as: '$f(z)$ is expressed as the infinite sum $\sum_1^\infty a_k z^k$ means that for any accuracy ε desired in the answer, we can always find some number of terms N, such that if we calculate the sum to at least N terms, we are guaranteed to be within ε of the answer.'

Note that this makes as much sense for z complex as it does for z real.

What is essential is that you read such expressions for meaning and don't simply switch off your brain and goggle at it. It shouldn't be necessary to say this, it should have come with every small bit of Mathematics that you ever did, but cruel experience has taught me that too many people stop thinking about meaning and start trying to memorise lines of symbols instead. I have been to too many Engineering Honours seminars to have any faith in students having grasped the fundamentals, and without the fundamentals it turns into ritualistic nonsense rather fast.

From the above definition, it should be very clear that if I give you a new function of a complex variable, I must either tell you how to calculate those a_ks, or equivalently I must tell you how to calculate it in terms of other functions you already know, where you have been given the corresponding a_ks.

When you first met the cos and sin functions, they were probably defined in terms of the x and y coordinates of a point on the unit circle. If they weren't, they should have been. This is all very well, but you ought to have asked how to calculate them. You cannot expect your hand calculator to work out $\cos(2)$ by drawing bloody big circles. At some later stage, you met the Taylor-MacLaurin series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and this should have cheered you up somewhat. This is something your calculator *can* do. The first question that you should be all agog to find out the answer to is, how did we get the series and is it actually right? And the second question any reasonably suspicious engineer should ask is, does it always converge? And the third question is, given that it converges and to the right answer, how many steps does it take to get a reasonably accurate answer? How many steps do we need to get within 10^{-4} of the true result, for example? This last is a severely practical matter: a computer can do some millions of floating point operations in a second, and TF1 can do about 10^{12} flops. But the definition of convergence only says that an N has to exist

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for any ε, it doesn't say that it has to be some piddling little number like 10^{100} or less. There must be a function such that when ε is 10, N is $10^{10^{10}}$. This means that we would never know the value of $f(1)$ to within an order of magnitude before the stars turn into black cinders. One would like to do a little better than that for $\sin(1)$.

There are satisfactory answers to these questions for the function $\sin(x)$. It is worth understanding how it was done for $\sin(x)$ so you can do the same thing for other functions, in particular for $\sin(z)$ when z is complex. I have already assured you that there *is* a power series for $\sin(z)$, and you may have learnt it. But knowing how to get it is rather more useful.

5.2 Taylor Series

Definition 5.2.1 *If f is an infinitely differentiable function at some point z_0, then the* Taylor Series *for f about z_0 is*

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f^2(z_0)}{2!}(z - z_0)^2 + \frac{f^3(z_0)}{3!}(z - z_0)^3 + \cdots$$

or in more condensed notation:

$$\sum_0^\infty \frac{f^k(z_0)}{k!}(z - z_0)^k$$

where $f^0 = f$ and $0! = 1! = 1$.

Note what I didn't say: I didn't claim that the series was equal to $f(z)$. In general it isn't. For example, the function might be zero on $[-1, 1]$ and do anything you liked outside that interval. Then if we took $z_0 = 0$ we would get zero for every coefficient in the series, which would not tell us anything about $f(2)$. Why should it? Things are even worse than this however:

Exercise 5.2.1 *The real function f is defined as follows: for $x \leq 0$, $f(x) = 0$. For all other x, $f(x) = e^{101}e^{-\frac{1}{x^2}}$.*

Verify that f is infinitely differentiable everywhere. Verify that all derivatives are zero at the origin. Deduce that the series about 0, evaluated at 1 is zero and the value of $f(1)$, e^{100}, differs from it by rather a lot.

Draw the graph.

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I also didn't claim that the Taylor series for f converges. We have from contemplating the above exercise the depressing conclusion that even when it converges, it doesn't necessarily converge to anything of the slightest interest.

Exercise 5.2.2 *There is a perfectly respectable function*

$$e^{e^x}$$

Compute its Taylor series about the origin. Likewise, investigate the Taylor Series for

$$e^{e^{e^x}}$$

Does it converge?

The question of whether Taylor Series have to converge could keep you busy for quite a while, but I shall pass over this issue rather quickly. The situation for analytic functions of a complex variable is so cheering by comparison that it needs to be stated quickly as an antidote to the depression brought on by thinking about the real case.

Theorem 5.1 (Taylor's Theorem)

If f is a function of a complex variable which is analytic in a disk of radius R centred on w, then the Taylor series for f about w converges to f:

$$\begin{aligned} f(z) &= f(w) + f'(w)(z-w) + \frac{f^2(w)}{2!}(z-w)^2 + \frac{f^3(w)}{3!}(z-w)^3 + \cdots \\ &= \sum_0^\infty \frac{f^k(w)}{k!}(z-w)^k \end{aligned}$$

provided $|z-w| < R$.

No Proof □

It is usual to tell you that the convergence of the series is *uniform* on subdisks of the given disk, which means that the N you find for some accuracy ε depends only on ε and not on z. Unfortunately, this merely means that on each subdisk there is for every ε a 'worst case z' and we can pick the N for that case and it will work for all. Of course, the worst case may be

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terrible, and the case we actually care about have much smaller N, so this is of limited practical value sometimes.

Although Taylor's Theorem brings us a ray of cheer, note that it gives us no practical information about how fast the series converges, although this may be available in particular cases. I shall skip telling you how to find this out; it is treated in almost all books on Complex Function Theory.

It is worth pointing out that the power series expansion of a function is unique:

Theorem 5.2 (Uniqueness of Power Series)

$$\sum_0^\infty a_k z^k = \sum_0^\infty b_k z^k \Rightarrow \forall k, a_k = b_k$$

No Proof. □

This doesn't mean that the Taylor series for a function about different points can't look different.

We know that $\sin z$ has derivative $\cos z$ which in turn has derivative $-\sin z$. We also know the values of $\sin 0(0)$ and $\cos 0(1)$. This is enough:

$$\sin(z) = \sin(0) + \sin'(0)z + \frac{\sin^2(0)}{2!}z^2 + \cdots$$

So

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

as advertised.

The Taylor Series about 0 is often (but not always) a good choice, and is called the *MacLaurin Series*.

It is normally the case that if a Power series converges in a disk but not at some point on the boundary, it will diverge for every point outside the disk. This doesn't mean that there isn't a perfectly good power series about some other point.

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Example 5.2.1 *Take*

$$f(z) = \frac{1}{1-z}$$

Then it is easy to see that

$$f^k(z) = \frac{k!}{(1-z)^{k+1}}$$

and hence that $f^k(0) = k!$. *It follows that the Taylor series is*

$$f(z) = 1 + z + z^2 + z^3 + \cdots = \sum_0^\infty z^k$$

Now it is clear that this converges in a disk centred at 0 of radius 1, and rather obvious that it doesn't converge at $z = 1$. *At* $z = 2i$ *we get*

$$1 + 2i - 4 - 8i + 16 + \cdots$$

which also diverges. If however we expand about i *and evaluate at* $2i$ *we get*

$$\frac{1}{1-z} = \frac{1}{1-i} + i\frac{1}{(1-i)^2} + i^2\frac{1}{(1-i)^3} + \cdots$$

which is

$$\frac{e^{i\pi/4}}{\sqrt{2}} + i\frac{e^{i2\pi/4}}{(\sqrt{2})^2} + i^2\frac{e^{i3\pi/4}}{(\sqrt{2})^3} + \cdots$$

Now if we look at the modulus of each term we get:

$$\frac{1}{\sqrt{2}} + \frac{1}{(\sqrt{2})^2} + \frac{1}{(\sqrt{2})^3} + \cdots$$

which is a geometric series with ratio less than 1 and hence converges.

If you were to 'straighten out' the series of complex numbers being added up so that they all lay along the positive reals, then we would have a convergent series. Now if you rotate them back into position a term at a time, you would have to still have the series converge in the plane. (This is an intuitive argument to show that absolute convergence implies convergence for series. It is not hard to make it rigorous.) So the series converges.

Exercise 5.2.3 *What is the radius of convergence (i.e. the radius of the largest disk such that the series converges in the interior of the disk) for the function* $\sin z$ *expanded about the point* i*? Draw a picture and take a flying guess first, then prove your guess is correct.*

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It is true that, on any common domain of convergence, power series can be added, subtracted, multiplied and divided. The last operation may introduce poles at the zeros of the divisor, just as for polynomials. All the others result in new (convergent) power series. In fact thinking about power series as 'infinitely long polynomials where the higher terms matter less and less' is not a bad start. It clearly goes a bit wrong with division however.

5.3 Laurent Series

You may have noticed a certain interest in functions which are reciprocals of polynomials. The reason of course is that they are easy to compute, just as polynomials are. It is worth looking at functions which are ratios of polynomials also, and indeed functions which are ratios of other functions we already know. We shall come back to this later, but for the moment consider a function such as

$$\frac{e^z}{z^2}$$

This is a ratio of analytic functions and is hence analytic except at the zeros of the denominator. There are two roots, both the same, so we have a singularity at $z = 0$. We can divide out the power series to get

$$\frac{e^z}{z^2} \sim \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots$$

I haven't wanted to say these are equal: this would beg the question. But the question more or less raises itself, are these equal? Or to put it another way, does the expression on the right converge if $|z| > 0$, and if so, does it converge to e^z/z^2?

The answer is 'yes' to both parts.

More generally, suppose we have a function which is analytic in the annulus $r < |z - c| < R$, for some point c, the centre of the annulus. Then it will in general have an expansion in terms of integral powers, some or all of which may be negative. This is called a *Laurent Series* for the function.

More formally:

Definition 5.3.1 (Laurent Series)

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For any w, the integer power series

$$\sum_{-\infty}^{\infty} a_k (z-c)^k, \quad k \in \mathbb{Z}$$

is defined to be

$$\sum_{0}^{\infty} a_k (z-c)^k + \sum_{1}^{\infty} a_{-k} (\frac{1}{z-c})^k$$

when both of these series converge.

Theorem 5.3 (Laurent's Theorem)

If f is analytic on the annulus $r < |z-c| < R$, for some point c, then f is equal to the Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-c)^k$$

where the coefficients a_k can be computed from:

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-c)^{1+k}} \, dz$$

if k is positive or zero, and

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-c)^{1-k}} \, dz$$

when k is negative. C is any simple closed loop around the centre c which is contained in the annulus and goes in the positive sense.

No Proof: See Mathews and Howell or any standard text. □

Exercise 5.3.1 *Try showing a similar result for the Taylor series for an analytic function; i.e. try to get an expression for the Taylor series coefficients in terms of path integrals.*

It is the case that Laurent series about any point, like Power series, are unique when they converge.

The following result is extremely useful:

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Theorem 5.4 (Differentiability of Laurent Series)

The Laurent series for a function analytic in an annulus if differentiated termwise gives the derivative of the function.

No Proof: □

Since the case where all the negative coefficients are zero reduces to the case of the Taylor series, this is also true for Taylor Series. It is not generally true that if a function is given by a sequence of approximating functions, the derivative is given by the sequence of derivatives. After all,

$$\frac{1}{n}\sin nx$$

gets closer and closer to the zero function as n increases. But the derivatives $\cos nx$ certainly do not get closer to anything.

This tells us yet again that the analytic functions are very special and that they behave in particularly pleasant ways, all things considered.

5.4 Some Sums

The series

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots = \sum_0^\infty (-1)^n z^n$$

converges if $|z| < 1$. This is 'Well Known fact' number 137 or thereabouts.

We can get a Laurent Series for this as follows: find a series for $1/(1+1/w)$ by the usual trick of doing (very) long division to get

$$\frac{1}{1+\frac{1}{w}} = w - w^2 + w^3 - w^4 + \cdots = \sum_1^\infty (-1)^{n+1} w^n$$

and then put $z = 1/w$ to get:

$$\frac{1}{1+z} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots = \sum_1^\infty (-1)^{n+1} \frac{1}{z^n}$$

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This converges for $|z| > 1$. It clearly goes bung when $z = -1$, and equally clearly is a geometric series with ratio less than 1 provided $|z| > 1$.

There are therefore two different Laurent series for $\frac{1}{1+z}$, one inside the unit disk, one outside. One is actually a Taylor Series, which is just a special case.

Suppose we have a function like

$$\frac{1}{1-z} + \frac{1}{z-2i} = \frac{1-2i}{z^2-(1+2i)z+2i}$$

This has singularities at $z = 1$ and $z - 2i$, where the modulus of the function goes through the roof.

The function can be expanded about the origin to get:

$$1 + z + z^2 + z^3 + \cdots + \frac{i}{2} - \frac{z}{4} + \frac{z^2}{8} + \cdots$$

which converges inside the unit disk.

In the annulus given by $1 < |z| < 2$ it can be written as

$$-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots + \frac{i}{2} - \frac{z}{4} + \frac{z^2}{8} + \cdots$$

And in the annulus $|z| > 2$ it can be written:

$$-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots + \frac{1}{z} + \frac{2i}{z^2} - \frac{4}{z^3} - \frac{8i}{z^4} + \frac{16}{z^5} + \cdots$$

Exercise 5.4.1 *Confirm the above or find my error.*

Substitutions for terms are valid providing care is taken about the radius of convergence of both series.

Laurent series expansions about the origin have been produced by some simple division. The uniqueness of the Laurent expansions tells us that these have to be the right answers. Next we consider some simple tricks for getting expansions about other points:

Example 5.4.1 *Find a Laurent expansion of* $\frac{1}{1+z}$ *about* i

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We write

$$\frac{1}{1+z} = \frac{1}{(1+i)+(z-i)} = (\frac{1}{1+i})(\frac{1}{1+\frac{z-i}{1+i}})$$

Then we have the Taylor expansion

$$\frac{1}{1+\frac{z-i}{1+i}} = 1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \frac{(z-i)^3}{(1+i)^3} + \cdots$$

valid for $|\frac{z-i}{1+i}| < 1$ *i.e. for* $|z-i| < \sqrt{2}$. *We also have*

$$\frac{1}{1+\frac{z-i}{1+i}} = \frac{1+i}{z-i} - \frac{(1+i)^2}{(z-i)^2} + \cdots$$

valid when $|z-i| > \sqrt{2}$.

Example 5.4.2 *Find a Laurent expansion for*

$$\frac{(1-z)^3}{z-2}$$

about 1.

$$\frac{(1-z)^3}{z-2} = \frac{(z-1)^3}{2-z} = \frac{(z-1)^3}{1-(z-1)}$$

Putting $w = z-1$ *we get*

$$w^3\frac{1}{1-w} = w^3(-\frac{1}{w} - \frac{1}{w^2} - \frac{1}{w^3} - \cdots)$$

To give the final result

$$-((z-1)^2 + (z-1) + 1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots)$$

which is valid for $|z-1| > 1$.

A small amount of ingenuity may be required to beat the expressions into the correct shape; practice does it.

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Exercise 5.4.2 *Find the Laurent expansion for*

$$\frac{1-z}{z-3}$$

about 1, valid for $|z-1| > 2$.

Exercise 5.4.3 *Make up a problem of this type and solve it.*

Exercise 5.4.4 *Go through the exercises on page 230, 232 of Mathews and Howell.*

5.5 Poles and Zeros

We have been looking at functions which are analytic at all points except some set of 'bad' or *singular* points. In the case where every point in some neighbourhood of a singular point is analytic, we say that we have an *isolated singularity.* Almost all examples have such singularities *poles* of the function, places where the modulus goes through the roof no matter how high the roof is: put more formally, points w such that

$$\lim_{z \to w} |f(z)| = \infty$$

We can distinguish different types of singularity: there are those that look like $1/z$, those that look like $1/z^2$, those that look like Log (at the origin) and there are those that are just not defined at some point w but could have been if we wanted to. The last are called *removable singularities* because we can remove them. For example, if I give you the real function $f(x) = x^2/x$, you might in a careless mood just cancel the x and assume that it is the same as the function $f(x) = x$. This is so easily done, you can do it by accident, but strictly speaking, you have a new function. It so happens that it agrees with the old function everywhere except at zero, where the original function is not defined. Moreover, the new function is differentiable everywhere, while the old function has a singularity at the origin. But not the sort of singularity which should worry a reasonable man, the gap can be plugged in only one way that will make the resulting function smooth and indeed infinitely differentiable. And if I'd made the x a z and said it was

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a complex function, exactly the same applies. Of course, it may take a bit more effort to see that a singularity of a function is removable. But if there is a new function which is analytic at w and which agrees with the old function in a neighbourhood of w, the w is a removable singularity.

To be more formal in our definitions, we can say that if w is an isolated singularity of a function f, then f has a Laurent expansion about w, and the cases are as follows:

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-w)^k$$

1. If $a_k = 0$ for all negative k, then f has a *removable singularity.*
2. If $a_k = 0$ for all negative k less than negative n, and $a_{-n} \neq 0$ then we say that w is a pole of order n. Thus $1/z$ has a pole of order 1. (Its Laurent expansion has every other coefficient zero!)
3. If there are infinitely many negative k non-zero, then we say that w is a pole of infinite order. The singularity is said to be *essential.*

Exercise 5.5.1 *Give examples of all types of poles.*

Exercise 5.5.2 *Verify that*

$$\frac{\sin z}{z}$$

has a removable singularity at 0*, and remove it (i.e. define a value*[1] *for the function at* 0*).*

We can do the same kind of thing with zeros as we have done with poles. If w is a point such that $f(z)$ is analytic at w then if $f(w) = 0$ we say that w is a *zero* of f. It is an isolated zero if there is a neighbourhood of w such that $f(z)$ is non zero throughout the neighbourhood except at w. An isolated zero of f, w is said to be of order n if the Taylor series for f centred on w

$$f(z) = \sum_{0}^{\infty} a_k (z-w)^k$$

has $a_k = 0$ for every k less than n, and $a_n \neq 0$.

[1] Never forget that cancelling $\frac{\sin \cancel{z}}{\cancel{z}}$ is a sin.

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Example 5.5.1 *The function $z^2 \cos z$ has a zero of order 2 at the origin.*

We have the following easy theorem:

Theorem 5.5 *If f is analytic in a neighbourhood of w and has a zero of order n at w, then there is a function g which is analytic in the neighbourhood of w, is non-zero at w and has*

$$f(z) = (z - w)^n g(z)$$

Proof:

Write out the Taylor expansion about w for f and divide by $(z-w)^n$ to get a Laurent expansion for some function g. This will in fact be a Taylor series, the constant term of which is non-zero, and it must converge everywhere the original Taylor series converged. □

Exercise 5.5.3 *Show the converse: if f can be expressed as*

$$f(z) = (z - w)^n g(z)$$

for some analytic function g which is non-zero at w, then f has a zero at w of order n.

Corollary 5.5.1 *If f and g are analytic and have zeros at w of n and m, then the product function has a zero at w of order $n + m$.* □

Very similar to the above theorem is the corresponding result for poles. I leave it as an exercise:

Exercise 5.5.4 *Prove that if a function f has an isolated pole of order n at w then there is a neighbourhood W of w and a function g analytic on W and having $g(w) \neq 0$ such that*

$$f(z) = \frac{g(z)}{(z - w)^n}$$

I defined a *meromorphic function* earlier as one that had isolated singularities. I really ought to have said isolated *poles*, and moreover, isolated poles of finite order.

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Exercise 5.5.5 *What is the difference?*

Because the poles and zeros of a meromorphic function tell us a lot about the function, it is important to be able to say something about them. (The correspondence pages of one of the major Electrical Engineering Journals used to be called 'Poles and Zeros'.) In Control Theory for example, knowing the locations of poles and zeros is critical in coming to conclusions about the stability of the system.

Example 5.5.2 *Locate the poles and zeros of the function*

$$\frac{\tan z}{z^2}$$

Solution *The poles will be the zeros of* $\cos z$ *together with the origin; writing*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

we get the ratio of power series:

$$\begin{aligned} & \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)} \\ = & \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{z - \frac{z^3}{2!} + \frac{z^5}{4!} - \cdots} \end{aligned}$$

Doing the (very) long division:

$$\begin{array}{lll} z - \dfrac{z^3}{2!} + \dfrac{z^5}{4!} - \cdots & \overline{)1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots(} & \dfrac{1}{z} \\ & 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots & \\ & \overline{)0 + \frac{z^2}{3} - \frac{4z^4}{5!} + \cdots(} & \dfrac{z}{3} \\ & \frac{z^2}{3} - \frac{z^4}{3!} + \cdots & \end{array}$$

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If you can't do long division, check the result by cross multiplication and take it on faith, or find out how to do long division.

This gives the first few terms of the Laurent Series:

$$\frac{1}{z} + \frac{z}{3} + \frac{16z^3}{5!} + \cdots$$

which tells us that there is a pole of order 1 at the origin, which should not come as a surprise to the even moderately alive.

Away from the origin, we have zeros at the locations of the zeros of $\sin z$*, namely at* $n\pi$ *where* n *is an integer, and poles at the zeros of* $\cos z$*, i.e. at* $n\pi/2$ *for* n *an integer. Each of these poles and zeros will have order one. This follows by observing that if you differentiate* sin *you get* cos *and when* $sinz = 0, \cos z = \pm 1$ *and* vice versa.

You have probably already realised:

Theorem 5.6 *If* f *is analytic and has an isolated zero of order* n *at* w*, then* $1/f$ *is meromorphic in a neighbourhood of* w *and has a pole of order* n *at* w*.* □

Exercise 5.5.6 *Work out the possible poles and zeros and orders thereof, for the ratio of two meromorphic functions with known poles and zeros and orders thereof.*

Exercise 5.5.7 (Riemann's Singularity Theorem)

f *is a function known to be analytic in a punctured disk* D *centred on* w *that has* w *removed, with a singularity at* w*, and* $|f|$ *is bounded on the punctured disk. Show that the singularity is removable.*

[Hint: investigate $g(z) = (z - w)^2 f(z)$ *]*

In conclusion, the trick of writing out Laurent Series for functions is a smart way of learning a lot about their local behaviour, and there are scads of results you can do which we don't have time in this course to look at. Which is a little sad, and I hope the other material in your engineering courses is as interesting to explore as this stuff is.

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Chapter 6

Residues

Definition 6.0.1 *Given a Laurent Series for the function f about w,*

$$f(z) = \sum_{-\infty}^{\infty} a_k (z-w)^k$$

the value a_{-1} is called the residue *of f at w.*

We write Res[f,w] for the value a_{-1}.

Example 5.5.2 makes it easy to see that $\text{Res}[\tan z/z^2, 0] = 1$.

There are some nifty tricks for calculating residues. Why we want to calculate them will appear later, take it that there are good reasons.

For example:

Example 6.0.3 *Calculate [Res f,0] for*

$$f(z) = \frac{2}{z^3 - (i+1)z^2 + iz}$$

Solution:

We can clearly factorise the denominator easily into

$$z(z-1)(z-i)$$

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so the coefficient of $1/z$ *is going to come from*

$$\frac{2}{(z-1)(z-i)}$$

which when $z = 0$ *is just*

$$\frac{2}{i} = -2i$$

Exercise 6.0.8 *Do the last example the long way around and confirm that you get the same answer.*

And now one reason why we would like to be able to compute residues:

Theorem 6.1 *If* f *has a singularity at* w *and is otherwise analytic in a neighbourhood of* w*, and if* c *is a simple closed loop going around* w *once in an anticlockwise sense, then*

$$\oint_c f(z)dz = 2\pi i \ \mathrm{Res}[f, w]$$

Proof:

This comes immediately from the definition of the Laurent series. □

And the obvious extension for multiple singularities:

Theorem 6.2 (Cauchy's Residue Theorem)

If c *is a simple closed curve in* $\mathbb{C}$ *and the function* f *is meromorphic on the region enclosed by* c *and* c *itself, with singularities at* $w_1, w_2, \cdots w_n$ *in the region enclosed by* c*, then*

$$\oint_c f(z)dz = 2\pi i \sum_1^n \mathrm{Res}[f, w_k]$$

where the integration is taken in the positive sense.

Proof:

The usual argument which replaces the given curve by circuits around each singularity will do the job. □

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There is a clever way to compute residues for poles of order greater than one:

Theorem 6.3 *If f has a pole of order k at w,*

$$\mathrm{Res}[f, w] = \frac{1}{(k-1)!} \lim_{z \to w} [(z-w)^k f(z)]^{[k-1]}$$

where the exponent $[q]$ refers to the q-fold derivative.

Proof:

We have

$$f(z) = \frac{a_{-k}}{(z-w)^k} + \frac{a_{-k+1}}{(z-w)^{k-1}} + \cdots + \frac{a_{-1}}{(z-w)} + a_0 + a_1(x-w) + a_2(x-w)^2 + \cdots$$

since f has a pole of order k. Then the function $g(z) = (z-w)^k f(z)$ is analytic at w and has derivatives of all orders, and they go:

$$\begin{aligned}
g(z) &= (z-w)^k f(z) \\
&= a_{-k} + a_{-k+1}(z-w) + a_{-k+2}(z-w)^2 + \cdots + a_{-1}(z-w)^{k-1} + a_0(z-w)^k + \cdots \\
g'(z) &= a_{-k+1} + 2a_{-k+2}(z-w) + \cdots + (k-1)a_{-1}(z-w)^{k-2} + \cdots \\
&\vdots \\
g^{[k-1]}(z) &= (k-1)!\, a_{-1} + k!\, a_1(z-w) + \cdots
\end{aligned}$$

And as $z \to w$, the higher terms all vanish to give the result □

Example 6.0.4 *Calculate*

$$\oint_c \frac{dz}{z^4 + (1-i)z^3 - iz^2}$$

Where c is the circle centred on the origin of radius 2.

Solution *The long way around is fairly long. So we use residues and the last theorem.*

First we rewrite the function f:

$$f(z) = \frac{1}{(z^2)(z+1)(z-i)}$$

and observe that it has poles at zero, -1 and i. All are within the circle c. The (z+1) pole has coefficient

$$\frac{1}{(z^2)(z-i)}$$

which at $z=-1$ is $-1/(1+i)=(i-1)/2$.

The $z-i$ pole has coefficient

$$\frac{1}{(z^2)(z+1)}$$

which at $z=i$ is $-1/(1+i)=(i-1)/2$.

And finally we compute the residue at 0 which is

$$\lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{(z+1)(z-i)}\right)$$

$$=\lim_{z\to 0}-[\frac{2z+(1-i)}{((z+1)(z-i))^2}]=1-i$$

The integral is therefore

$$2\pi i[(1-i)+(i-1)/2+(i-1)/2)]=0$$

Exercise 6.0.9 *Do it the long way around to convince yourself that I haven't blundered.*

The quick way saves some messing around with partial fractions. In fact we can use the above results to calculate the partial fractions; it is a neat trick which you will find in Mathews and Howell, pp 249-250. If you ever need to compute a lot of partial fractions, look it up[1].

[1]Some people love knowing little smart tricks like this. It used to be thought the best thing about Mathematics: you can use sneaky little tricks for impressing the peasantry. Some schoolteachers use them to impress teenagers. This tells you a lot about such folk.

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6.1 Trigonometric Integrals

This is something more than just a trick, because it gives a practical method for solving some very nasty definite integrals of trigonometric functions. What we do is to transform them into path integrals and use residues to evaluate them. An example will make this clear:

Example 6.1.1 *Evaluate*

$$\int_0^{2\pi} \frac{d\theta}{3\cos\theta + 5}$$

We are going to transform into an integral around the unit circle S^1. In this case we have

$$z = e^{i\theta}, dz = iz\ d\theta, 1/z = e^{-i\theta}$$

From which we deduce that

$$\cos\theta = \frac{z + 1/z}{2}$$

Substituting in the given integral we get

$$-i \oint_{S^1} \frac{dz}{z[\frac{3}{2}(z + 1/z) + 5]}$$

This can be rewritten

$$\begin{aligned} & -i \oint_{S^1} \frac{dz}{\frac{3}{2}z^2 + \frac{3}{2} + 5z} \\ = & -2i \oint_{S^1} \frac{dz}{(3z+1)(z+3)} \\ = & -\frac{2i}{3} \oint_{S^1} \frac{dz}{(z+1/3)(z+3)} \end{aligned}$$

The pole at $z = -3$ is outside the unit circle so we evaluate the residue at $-1/3$ and we know the coefficient there is $1/(z+3) = 3/8$. The integral is therefore

$$-2\pi i \frac{2i}{3}\ (3/8) = \pi/2$$

The substitution for $\sin\theta = \frac{1}{2i}(z - 1/z)$ is obvious.

It is clear that we can reduce a trigonometric integral from 0 to 2π to a rational function in a great many cases, and thus use the residue theory to get a result. This is (a) cute and (b) useful.

6.2 Infinite Integrals of rational functions

There is a very nice application of the above ideas to integrating real functions from $-\infty$ to ∞. These are some of the so called 'improper' integrals, so called because respectable integrals have real numbers at the limits of the integrals, and functions which are bounded on those bounded intervals.

In first year, you did the Riemann Integral, and it was all about chopping the domain interval up into little bits and taking limits of sums of heights of functions over the little bits. I remind you that we define, for any real number b,

$$\int_b^\infty f(x)\ dx = \lim_{y\to\infty} \int_b^y f(x)\ dx$$

when the limit exists, and

$$\int_{-\infty}^a f(x)\ dx = \lim_{y\to-\infty} \int_y^a f(x)\ dx$$

for any real number a. Then the doubly infinite integral exists if, for some a,

$$\lim_{y\to\infty} \int_a^y f(x)\ dx$$

and

$$\lim_{y\to-\infty} \int_y^a f(x)\ dx$$

both exist, in which case

$$\int_{-\infty}^\infty f(x)\ dx = \lim_{y\to\infty} \int_a^y f(x)\ dx + \lim_{y\to-\infty} \int_y^a f(x)\ dx$$

This is the Riemann Integral for the case when the domain is unbounded. There are plenty of cases where it doesn't exist. For example,

$$\int_{-\infty}^\infty x\ dx$$

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clearly does not exist.

On the other hand, there is a case for saying that for the function $f(x) = x$, the area above the X-axis on the positive side always cancels out the area below it on the negative side, so we *ought* to have

$$\int_{-\infty}^{\infty} x \, dx = 0$$

There are two approaches to this sort of problem; one is the rather repressive one favoured by most schoolteachers and all bureaucrats, which is to tell you what the rules are and to insist that you follow them. The other is favoured by engineers, mathematicians and all those with a bit of go in them, and it is to make up a new kind of integral which behaves the way your intuitions think is reasonable[2].

We therefore define a new improper integral for the case where the function is bounded and the domain is the whole real line:

$$C \int_{-\infty}^{\infty} f(x) \, dx = \lim_{y \to \infty} \int_{-y}^{y} f(x) \, dx$$

This means that $C \int_{\mathbb{R}} x = 0$, although $\int_{\mathbb{R}} x$ does not exist.

The C stands for Cauchy, but I shan't call this the Cauchy Integral because that term could cause confusion. It is often called the *Cauchy Principal Value*, but this leads one to think it is something possessed by an integral which does not exist.

Note that if the integral does exist, then so does the $C \int$ and they have the same value. The converse is obviously false.

In what follows, I shall just use the integral sign, without sticking a C in front of it, to denote this Cauchy Principal Value. We are using the new, symmetrised integral instead of the Riemann integral: they are the same when the Riemann integral is defined, but the new symmetrised integral exists for functions where there is no Riemann Integral[3].

[2]This may turn out to be impossible, in which case your intuitions need a bit of straightening out. You should not assume that absolutely *anything* goes; only the things that make sense work.

[3]This sort of thing happens a good deal more than you might have been led to believe

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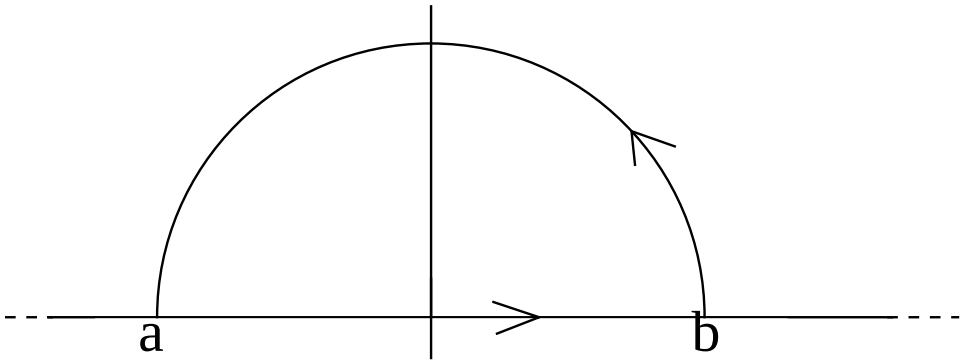

Figure 6.1: The integral around the outer semicircle tends to zero if f dies away fast enough

The idea of the application of Complex Analysis to evaluation of such integrals is indicated by the diagram figure 6.1.

The idea is that the integral along the line segment from a to b plus the integral around the semicircle is a loop integral which can be evaluated by using residues. But as a gets more negative and b more positive, the line integral gets closer to the integral

$$\int_{-\infty}^{\infty} f(x)\, dx$$

and the arc gets further and further away from the origin. Now if $f(z) \to 0$ fast enough to overcome the arc length getting longer, the integral around the arc tends to zero. So for some functions at least, we can integrate f over the real line by making f the real part of a complex function and counting residues in the top half of the plane.

I hope you will agree that this is a very cool idea and deserves to work.

Example 6.2.1 *Evaluate the area under the 'poor man's gaussian':*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

in first year. There are, still to come, *Riemann-Stieltjes* integrals and *Lebesgue* integrals, both of which also extend Riemann integrals in useful ways. But not in this course.

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Solution 1

The old fashioned way is to substitute $x = \tan\theta$ *when we get the indefinite integral* $\arctan\theta$*, and evaluating from* $x = 0$ *to* $x = \infty$ *is to go from* $\theta = 0$ *to* $\theta = \pi/2$*. So the answer is just* π*.*

Solution 2

We argue that we the result will be the same as

$$\oint_c \frac{dz}{1+z^2}$$

where c is the infinite semi-circle in the positive plane, because the path length of the semi-circle will go up linearly with the radius of the semi-circle, but the value of the integral will go down as the square at each point, so the limit of the integral around the semi-circle will be zero, and the whole contribution must come from the part along the real axis.

Now there is a pole at $\pm i$*, and the pole at* $-i$ *is outside the region. So we factorise*

$$\frac{1}{1+z^2} = (\frac{1}{z+i})\ (\frac{1}{z-i})$$

whereupon the Laurent expansion about $z = i$ *has coefficient*

$$\frac{1}{2i}$$

and the integral is

$$2\pi i\frac{1}{2i} = \pi$$

This gives us the right answer, increasing confidence in the reasoning.

It is about the same amount of work whichever way you do this particular case, but the general situation is that you probably won't know what substitution to make. The contour integral approach means you don't have to know.

Example 6.2.2 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-4x+5)^2}$$

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Solution

First we recognise that this is

$$\oint_H \frac{dz}{(z^2 - 4z + 5)^2}$$

where H is a semicircle in the upper half plane big enough to contain all poles of the function with positive imaginary part.

We factorise

$$\frac{1}{(z^2 - 4z + 5)^2} = \left(\frac{1}{(z - (2 + i))^2}\right) \left(\frac{1}{(z - (2 - i))^2}\right)$$

and note that there is one pole at $2 + i$ of order 2 in the upper half plane.

We evaluate the residue by taking

$$\lim_{z \to 2+i} \frac{d}{dz} \left(\frac{1}{(z - (2 - i))^2}\right) = \frac{-2}{((2 + i) - (2 - i))^3} = -i/4$$

Then it follows that the integral is $2\pi i$ times this, i.e. $\pi/2$.

Exercise 6.2.1 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^3}$$

Exercise 6.2.2 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x\ dx}{(x^2 + 1)^2}$$

Verify your answer by drawing the graph of the function.

Exercise 6.2.3 *Make up a few integrals of this type and solve them. If this is beyond you, try the problems in Mathews and Howell, p 260.*

The ideas should be now sufficiently clear to allow you to see the nature of the arguments required to prove:

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Theorem 6.4 *If*

$$f(x) = \frac{P(x)}{Q(x)}$$

for real non-zero polynomials P, Q and if the degree of Q is at least two more than the degree of P, then

$$\int_{-\infty}^{\infty} f(z)\ dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}[f(z), w_j]$$

where there are k poles $w_1, \cdots, w_k$ of $f(z)$ in the top half plane of $\mathbb{C}$. □

6.3 Trigonometric and Polynomial functions

We can also deal with the case of some mixtures of trigonometric and polynomials in improper integrals. Expressions such as

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\ \cos ax\ dx,\ \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\ \sin ax\ dx$$

can be integrated.

Since $\cos x = \Re(e^{ix}),\ \sin x = \Im(e^{ix})$, we have the result:

Theorem 6.5 *When P and Q are real polynomials with degree of Q at least two greater than the degree of P, the integral of the function*

$$f(x) = \frac{P(x)}{Q(x)}\ \cos(ax)$$

for $a > 0$ is given by extending f to the complex plane by putting

$$f(z) = \frac{P(z)\ e^{iaz}}{Q(z)}$$

whereupon

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\ \cos(ax)\ dx = -2\pi \sum_{j=1}^{k} \Im\left(\mathrm{Res}[f(z), w_j]\right)$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(ax)\, dx = 2\pi \sum_{j=1}^{k} \Re\left(\mathrm{Res}[f(z), w_j]\right)$$

where $w_1, \cdots, w_k$ are the poles in the top half of the complex plane. □

Example 6.3.1 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$$

Solution

We have the solution is

$$-2\pi\, \Im\left(\mathrm{Res}\left[\frac{e^{iz}}{(z-i)(z+i)}, i\right]\right)$$

The residue is the coefficient of $1/(z-i)$ which is, at $z = i$,

$$\frac{e^{i(i)}}{2i} = -i\, \frac{e^{-1}}{2}$$

So the imaginary part is $-e^{-1}/2$ and multiplying by -2π gives the final value

$$\frac{\pi}{e}$$

A sketch of the graph of this function shows that the result is reasonable.

Exercise 6.3.1 *Sketch the graph and estimate the above integral.*

We can see immediately from a sketch of the graph that the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\, dx = 0$$

From the antisymmetry of the function. This also comes out of the above example immediately.

The restriction that the degree of P has to be at least two less than the degree of Q looks sensible for the case of polynomials, but for the case of mixtures

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with trigonometric functions, we can do better: the integral $\int_1^a 1/x$, will grow without bound as $a \to \infty$, but the integral of $(\sin x)/x$ will not, because the positive and negative bits will partly cancel. A careful argument shows that we can in practice get away with the degree of P being only at least *one* less than the degree of Q. The problems that are associated with this are that we can run into trouble with the limits. It is essentially the same problem as that we experience when summing an infinite series with alternating terms: grouping the terms differently can give you different results. We can therefore get away with a difference of one in the degrees of the polynomials provided we change the definition of the improper integral to impose some symmetry in the way we take limits, in other words we use the Cauchy version of the integral, or the *Cauchy Principal Value.*

6.4 Poles on the Real Axis

A problem which can easily arise is when there is a pole actually on the x-axis. We have a different sort of improper integral in this case, and if $f(x)$ goes off to infinity at b in the interval $[a, c]$, we say that the integral $\int_a^c$ is defined provided that

$$\lim_{y \to b^-} \int_a^y$$

exists,

$$\lim_{y \to b^+} \int_y^c$$

exists, and the improper integral over $[a, c]$ is defined to be

$$\lim_{y \to b^-} \int_a^y + \lim_{y \to b^+} \int_y^c$$

In the same way as we symmetrised the definition of the other improper integrals, we can take a Cauchy Principal Value for these also, and we have that for a pole at $b \in [a, c]$. the Cauchy Principal Value of

$$\int_a^c f(x)\ dx$$

exists and is

$$\lim_{\Delta \to 0^+} \left(\int_a^{b-\Delta} f(x)\ dx + \int_{b+\Delta}^c f(x)\ dx \right)$$

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providing the limit exists. Again, the integral may not actually exist, but still have a Cauchy Principal Value. One could wish for more carefully thought out terminology. If it does exist, then the Cauchy Principal Value is the value of the integral. This is rather a drag to keep writing out, so I shall just go on writing an ordinary integral sign. So mentally, you should adapt the definition of the integral from the Riemann integral to the symmetrised integral and all will be well.

If the (symmetrised) integral exists, then it can be evaluated as for the case where the poles are off the axis, except in one respect: we count the residue from a pole on the axis as only 'half a residue'. We have the more general case:

Theorem 6.6 *If*

$$f(x) = \frac{P(x)}{Q(x)}$$

for real non-zero polynomials P, Q and if the degree of Q is at least two more than the degree of P, and if u_1, u_2, u_ℓ are isolated zeros of order one of P, then

$$\int_{-\infty}^{\infty} f(z)\ dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}[f(z), w_j] + \pi i \sum_{j=1}^{\ell} \mathrm{Res}[f(z), u_j]$$

where there are k poles $w_1, \cdots, w_k$ of $f(z)$ in the top half plane of $\mathbb{C}$, and ℓ poles $u_1, \cdots, u_\ell$ of $f(z)$ on the real axis. □

Similarly for the trigonometric functions:

Theorem 6.7 *When P and Q are real polynomials with degree of Q at least* one *greater than the degree of P, and when Q has ℓ isolated zeros of order one, the integral of the function*

$$f(x) = \frac{P(x)}{Q(x)}\ \cos(ax)$$

for $a > 0$ is given by extending f to the complex plane by putting

$$f(z) = \frac{P(z)e^{iaz}}{Q(z)}$$

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whereupon

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx = -2\pi \sum_{j=1}^{k} \Im\left(\mathrm{Res}[f(z), w_j]\right) - \pi \sum_{j=1}^{\ell} \Im\left(\mathrm{Res}[f(z), u_j]\right)$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(ax)\, dx = 2\pi \sum_{j=1}^{k} \Re\left(\mathrm{Res}[f(z), w_j]\right) + \pi \sum_{j=1}^{\ell} \Re\left(\mathrm{Res}[f(z), u_j]\right)$$

where $w_1, \cdots, w_k$ are the poles in the top half of the complex plane, and $u_1, \cdots, u_\ell$ are the poles on the real axis. □

Example 6.4.1 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

Solution

The above theorem tells us that the integral is

$$\pi \Re\left(\mathrm{Res}[\frac{e^{iz}}{z}, 0]\right)$$

Now at $z = 0$ the residue is just the coefficient $e^{i0} = 1$ so the result is π.

Example 6.4.2 *The same calculation shows that*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0$$

It is easy to see that this is true for the Cauchy symmetrised integral, but not for the unreconstructed Riemann integral, which does not exist.

Exercise 6.4.1 *Sketch the graph of*

$$\frac{\cos x}{x}$$

and verify that the integral

$$\int_{0}^{\infty} \frac{\cos x}{x}\, dx$$

does not exist.

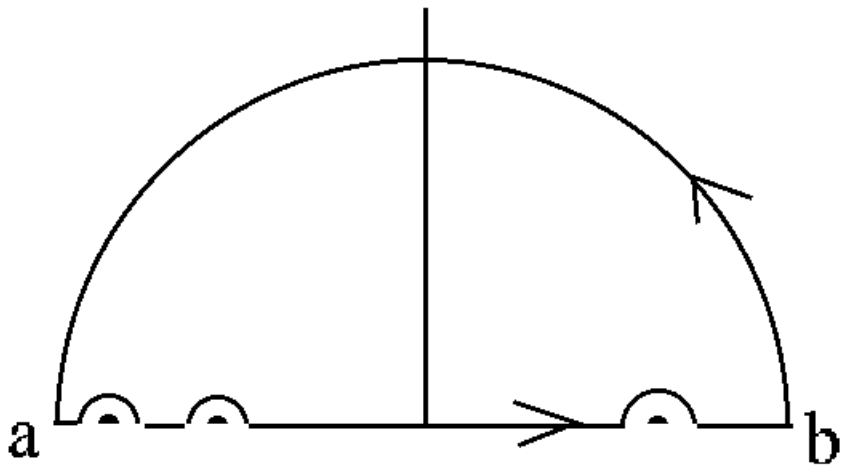

Figure 6.2: The bites provide half the residues

The last two theorems depend for the proof on taking little bites out of the path around the poles in the top half plane in the neighbourhood of each of the singularities on the real axis. The diagram of figure 6.2 gives the game away. Those of you with the persistence should try to prove the results. For those without, there are proofs in all the standard texts.

We have to take the limits as the radii of the bites get smaller and the big semi-circle gets bigger. The reason for the half of the $2\pi i$ Res[f,w] should be obvious.

6.5 More Complicated Functions

We can do something for functions which contain square roots and the like. It should, after all, be possible to use the same ideas for:

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$$

The problem we face immediately is that the square root function is well defined on the associated Riemann surface, and we have to worry about the problem of the so called 'multi-valued functions'.

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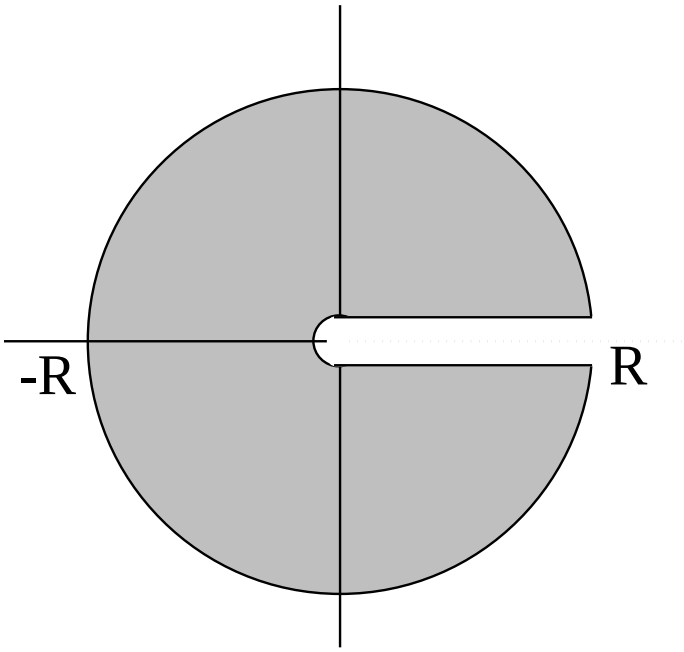

Figure 6.3: A contour for z^q, $0 < q < 1$.

We may do this in a variety of ways, but it is convenient to look at the function z^q for $0 < q < 1$, and to observe that if we define this for $r > 0$ and for $0 < \theta < 2\pi$, then

$$z^q = e^{q \log(re^{i\theta})} = e^{q \log r} e^{qi\theta} = r^q \ (\cos q\theta + i \sin q\theta)$$

It is clear that this is analytic and one-one on the region

$$r > 0, \quad 0 < \theta < 2\pi$$

We are, in effect, introducing a branch cut along the positive real axis. Put $q = 1/2$ for the branch of the square root function given in the first paragraph of section 2.3.1. The square root will pull the plane with the positive real axis removed back to the top half plane, just as the square took the top half plane and wrapped it around to the plane with the positive real axis removed.

Suppose we take a contour which avoids the branch cut as shown in figure 6.3 but which encloses all the poles in the positive half plane of some rational function (ratio of polynomials)

$$\frac{P(z)}{Q(z)}$$

I shall divide up the total contour c into four parts:

1. OC, the outer (almost) circle, going from $Re^{i\Delta}$ for Δ some small positive real number, to $Re^{i(2\pi-\Delta)}$.
2. IC, the inner semi-circle which has some small radius, r, and which is centred on the origin.
3. T, the top line segment which is just above the positive real axis and
4. B, the line segment just below the real axis.

Now we consider the function

$$f(z) = z^q \frac{P(z)}{Q(z)}$$

It is clear that if there are k poles $w_1, \cdots, w_k$ in the entire plane, none of which are on the positive real axis, then for some value of R and r,

$$\oint_c f(z)\ dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}[f, w_j]$$

We can approximate the left hand side by

$$\oint_{RS^1} f(z)\ dz + \int_0^R x^q \frac{P(x)}{Q(x)}\ dx - \int_0^R x^q\ e^{qi2\pi} \frac{P(x)}{Q(x)}\ dx + \int_{IC} f(z)\ dz$$

Now as the semi-circle SC gets smaller, the radius goes down, and the value of z^q also goes down; provided that $\frac{P(z)}{Q(z)}$ does not have a pole of order greater than one, i.e. provided $Q(z)$ has no zero of order greater than one at the origin, then the reducing size of the circle ensures that the last term goes to zero.

Similarly, if the degree of Q is at least two more than the degree of P, the integral over OC will also go to zero.

This gives us the result:

$$\int_0^\infty x^q \frac{P(x)}{Q(x)} = \frac{2\pi i}{1 - e^{qi2\pi}} \sum_{j=1}^{k} \mathrm{Res}[f, w_j]$$

This is the idea of the proof of:

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Theorem 6.8 *For any polynomials $P(x)$ and $Q(x)$ with the degree of Q at least two more than the degree of P, and for any real $q : 0 < q < 1$, then provided Q has a zero of at most one at the origin and no zeros on the positive reals, and if the zeros of Q, i.e. the poles of P/Q are $w_1, \cdots, w_k$, we have that*

$$\int_0^\infty x^q \frac{P(x)}{Q(x)} = \frac{1}{1 - e^{qi2\pi}} \sum_{j=1}^{k} \mathrm{Res}[f, w_j]$$

□

It is easy enough to use this result:

Example 6.5.1 *Evaluate*

$$\int_0^\infty \frac{\sqrt{x}}{1 + x^2}\, dx$$

Solution

There are two poles of the complex function

$$\frac{\sqrt{z}}{1 + z^2}$$

one at i and one at $-i$. The residues are, respectively

$$\frac{\sqrt{i}}{2i}, \quad \frac{\sqrt{-i}}{-2i}$$

which sum to

$$\frac{1}{2\sqrt{2}} [(1 - i) - (-1 - i)] = \frac{-i}{\sqrt{2}}$$

We take care to choose the square roots for the given branch cut. I have chosen to take the square roots in the top half of the plane in both cases.

We multiply by $2\pi i$ and divide by $1 - e^{i\pi} = 2$ to get

$$\frac{\pi}{\sqrt{2}}$$

Exercise 6.5.1 *Solve the above problem by putting $x = y^2$ and using the earlier method. Confirm that you get the same answer.*

The same ideas can be used to handle other integrals. Some of the regions to be integrated over and the functions used are far from obvious, and a great deal of ingenuity and experience is generally required to tackle new cases.

I am reluctant to show you some of the special tricks which work (and which you wouldn't have thought of in a million years [4]) because your only possible response is to ask if you should learn it for an exam. And knowing special tricks isn't much use in general. Nor do you have the time to spend on acquiring the general expertise. So if you should ever be told that some particularly foul integral can be evaluated by contour integration, you can demand to be told the function and the contour, and you can check it for yourself, but it is unlikely that you will hit upon some of the known special results in the time you have available. So I shall stop here, but warn you that there are many developments which I am leaving out.

6.6 The Argument Principle; Rouché's Theorem

The following results are of some importance in several areas including control theory.

Recall that f is *meromorphic* if it has only isolated poles of finite order and is otherwise analytic. It follows from classical real analysis that in any bounded region of $\mathbb{C}$, a meromorphic function has only a finite number of poles. The idea is that any infinite set of poles inside a bounded region would have to have a limit point which would also have to be a pole, and it wouldn't be isolated.

If f is not the zero function, then f can have only isolated zeros, since we can take a Taylor expansion about any zero, w, and get an expression $(z-w)^n T$, where n is the order of the zero and T is a power series with non-zero leading term and hence f is non zero in some punctured disk centred on w. Then standard compactness arguments give the required result. It follows that except for the case where f is the zero function, $1/f$ is also meromorphic when f is.

It is convenient to think of a pole of order k as k poles of order 1 on top of

[4] And neither would I

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each other. Thus we can regard the order of a pole at w as the multiplicity of a single pole. Similarly with zeros.

Suppose c is a simple closed curve which does not intersect any poles or zeros of a meromorphic function f. Then we have the following:

Theorem 6.9 (The Argument Principle)

$$\frac{1}{2\pi i}\oint_c \frac{f'(z)}{f(z)}\,dz = Z - P$$

where Z is the number of zeros of f inside c, P is the number of poles inside c, both counted with their multiplicity, and c has the positive orientation.

Proof:

We can write

$$f(z) = \frac{(z-a_1)(z-a_2)\cdots(z-a_Z)}{(z-b_1)(z-b_2)\cdots(z-b_P)}g(z)$$

for some analytic function g having no zeros in or on c.

Taking logarithms and differentiating, or differentiating and rearranging if you have the patience, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1}+\frac{1}{z-a_2}+\cdots+\frac{1}{z-a_Z}-\frac{1}{z-b_1}-\frac{1}{z-b_2}-\cdots-\frac{1}{z-b_Z}+\frac{g'(z)}{g(z)}$$

Since g and g' are analytic and g has no zeros on or inside c, the last term is analytic, and the result follows from the residue theorem. □

It may not be entirely obvious why this is called 'the argument principle', or in older texts 'The Principle of the Argument'. The reason is that we can write

$$\frac{1}{2\pi i}\oint_c \frac{f'(z)}{f(z)}\,dz$$

as

$$\frac{1}{2\pi i}\left[\mathrm{Log}(f(z))\right]_c$$

where the evaluation means that we put in the same point z twice, for some starting point on c, and the same finishing point, say 0 and 2π. Of course

we don't get zero, because Log is multi-valued, which really means that we are at a different place on the Riemann surface.

Expanding the expression for Log$f(z)$ we get

$$\frac{1}{2\pi i}\left[\log|f(z)| + i\arg(f(z))\right]_c$$

and the $\log|f(z)|$ part *does* return to its original value as we complete the loop c. But the argument part does not. So we get

$$Z - P = \frac{1}{2\pi}\left[arg(f(z))\right]_c$$

meaning that the difference between Z and P is the number of times $f(z)$ winds around the origin while θ goes from 0 to 2π and z winds around c once.

We note that if f is analytic, then $P = 0$. So for analytic f and a simple closed loop c not intersecting a zero of f we have

$$\frac{1}{2\pi i}\oint_c \frac{f'(z)}{f(z)}\,dz = Z$$

Example 6.6.1 *If $f(z) = z$ and c is the unit circle, then we wind around the origin, which is a zero of order 1 once. There are no poles because f is analytic (!), and the winding number is 1, which is right. If we had $f(z) = z^2$, with the same c, then the image of c winds around the origin twice, which gives us a count of 2. Geometrically, what you are doing is standing at the origin, and looking along the line towards $f(z)$, as z traces around c. When z returns (for the first time) to its starting point, you must be looking in the same direction. So you count the number of whole turns you have made. We get a count of two for z^2, and this is right is because we have a zero of order 2 at the origin now. So if you think of the original path as doing a single wind around a region, and f as wrapping the region around itself, you just count the 'winding number' to get a count of the number of zeros (multiplied by the order).*

If you had $z - 1$ for $f(z)$ then this has a zero at 1. If we take $c(z) = 1 + e^{i\theta}$ we go around the zero once with c, and $f(c)$ goes around the origin once. If we took c to be $1/2\ e^{i\theta}$ for the same f, we don't contain any zeros of f in c and the winding number around the origin is also zero- we don't go around

the origin at all, we never get closer than $1/2$ *to it. So we turn a little way and then unturn.*

If we have two zeros, each of of order one, inside c*, then* f *sends both of them to the origin. As* θ *goes from* 0 *to* 2π *and* z *goes around* c*, the angle or argument of* $f(z)$ *seen from the origin goes around twice. You have, after all, something like* $(z-a)(z-b)$*, and this looks like* z^2 *with wobbly bits. The wobbly bits don't change the angle you turn through, precisely because* c *contains both the zeros.*

The following exercise will convince you that this is sensible faster than any amount of brooding or persuasion (or even logic):

Exercise 6.6.1 *Take* $f(z) = z(z-1/2)$ *and* c *the unit circle. Draw* $f(c)$*, and confirm that it circumnavigates the origin twice.*

Now take $f(z) = z(z-1.5)$ *and the same* c*. Draw* $f(c)$*, and confirm that it circumnavigates the origin only once.*

The pictures obtained from the above exercise will make the next theorem easy to grasp:

Theorem 6.10 (Rouché's Theorem)

If f *and* g *are two functions analytic inside and on a closed simple loop* c *on which* f *has no zeros, and if* $|g(z)| < |f(z)|$ *on* c*, then* f *and* $f+g$ *have the same number of zeros inside* c*.*

Idea of Proof:

Suppose c *traces out a simple closed curve enclosing some region of* $\mathbb{C}$*, and that* f *has* n *zeros inside* c*, and* g *has, we suppose,* m *zeros in the same region enclosed by* c*. Then*

$$F(z) = 1 + \frac{g(z)}{f(z)}$$

has m *zeros and* n *poles inside* c*. It is also analytic on* c *itself. We therefore have that* $n-m$ *is the number of times* F *winds around the the origin.*

But if $|g(z)| < |f(z)|$, it follows that $|1 - F(z)| < 1$, in other words, F maps c to a curve which doesn't actually wind around the origin at all. It winds around 1, and doesn't get too far from it. So $n - m = 0$ and the result holds. See figure 6.4 for the picture. □

Making the above result rigorous is a little tricky; but actually defining carefully what we mean by the region enclosed by a simple loop is also tricky. Even proving that a simple closed curve divides $\mathbb{C}$ into two regions, both of them connected, one inside and the other outside, is also tricky. It takes quite a chunk of a topology course to do it justice, and the difference between a topologist and a classical mathematician is that the topologist can actually make his or her intuitions precise if pushed, and the classical mathematician has to fall back on 'you know what I mean' at some point. I am sticking to a classical way of doing things here, indeed I am being even sloppier than many of the classical mathematicians felt was safe. This is because, having a background in topology, I have the cool, calm confidence of a Christian with four aces, as Mark Twain once put it. Engineers vary a lot in the degree of faith they can generate for intuitive arguments such as this. Many mathematicians cannot stomach them at all because they cannot see how to make them rigorous. If you are an algebraist and have no geometric intuitions, you are likely to find the above argument heretical and a case for burning Mike Alder at the stake. Most statisticians seem to feel the same way. You can take a vote among the Sparkies, but whichever way it comes out, catching me will be difficult: don't think it hasn't been tried.

And now for some applications of the result:

Example 6.6.2 *Find the number of zeros of $z^7 - 5z^3 + 2z - 1$ inside the unit circle*

Solution

Put $g(z) = z^7 + 2z - 1$ and $f(z) = -5z^3$. Now $|g(z)|$ on the unit circle is not greater than

$$|z^7| + 2|z| + |1| = 4$$

and $|f(z)| = 5$. So the number of zeros of $(f + g)(z) = z^7 - 5z^3 + 2z - 1$ is the same as the number of zeros of $f(z) = -5z^3$. Counting multiplicities, this is three, which is the answer.

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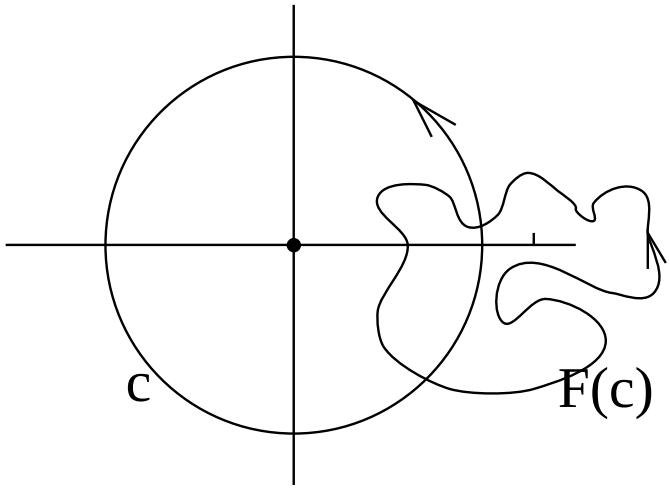

Figure 6.4: A picture to show the winding number of F is zero.

Example 6.6.3 *Find the number of poles of*

$$\frac{z^3 + z^2}{z^7 - 5z^3 + 2z - 1}$$

inside the unit circle.

Solution

Since the numerator is analytic, this is just the same as the number of zeros of $z^7 - 5z^3 + 2z - 1$ *which is three.*

Exercise 6.6.2 *Make up some more like this. Find someone who hasn't done Rouché's Theorem and ask them to find the answer. Gloat a bit afterwards.*

The methods used here are topological, depending on things which can be deformed, such as the closed loops, providing they do not pass over a pole or zero of the function. This makes them hard to prove properly but very powerful to use. More modern mathematics has pushed this a very long way, and it has astonishing applications in some unexpected areas. My favourite one is to do with finding lines in images: there is a standard method called the Hough Transform, which parametrises the space of lines in the plane by

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giving m and c for each line, to give a two dimensional space of lines. The Hough Transform takes a point (a, b) of the image, uses it to get a relation between m and c, $b = -am + c$ and draws the resulting curve, actually a straight line, in the line $(m - c)$ space. It then repeats for every other point of the image; where the curves all intersect you have the m, c belonging to the line.

The m, c parametrisation fails to find vertical lines, and there are two ways to go: one is to do it twice, once with m, c and once with $1/m, -c/m$, as when $x = my + c$. This takes twice as long to compute, and it is a rather slow algorithm anyway.

The other way to go is to find a better parametrisation: You can write $x \cos\theta + y \sin\theta = r$ for an alternative pair of numbers (r, θ) to specify the line. This handles the vertical lines perfectly well, but runs into trouble when the line passes through the origin. So engineers and computer scientists tried for other parametrisations of the space of lines which would get them off the hook.

Well, there isn't one, and this is a matter of the topology of the space of lines, and is not an easy argument for an engineer (and an impossible one for a computer scientist). Complex function theory was one of the lines of thought which developed into topology around the turn of the century.

6.7 Concluding Remarks

I have only scratched the surface of this immensely important area of mathematics, as an inspection of the books mentioned below will indicate. What can be done with PDEs, including the amazing Joukowsky Transform, is another story and one I cannot say anything about.

Many people have spent their entire lives inside Complex Function Theory and felt it worthwhile. I suppose some people have spent their entire lives hitting balls with sticks and thought *that* worthwhile. The difference is that the first activity tells people how to design aeroplanes and control systems and build filtering circuits and a million other things, while hitting balls with sticks doesn't seem to generate much except sweat and an appetite. And, I suppose, a large income for people who organise the television rights.

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It's a funny old world, and no mistake.

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