

ADVANCES IN COMPUTER VISION AND IMAGE PROCESSING

A Research Annual

IMAGE RECONSTRUCTION FROM INCOMPLETE OBSERVATIONS

Editor: THOMAS S. HUANG
Coordinated Science Laboratory
University of Illinois

VOLUME 1 · 1984



JAI PRESS INC.

Greenwich, Connecticut

London, England

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36 Sherwood Place
Greenwich, Connecticut 06830*

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3 Henrietta Street
London WC2E 8LU
England*

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ISBN: 0-89232-280-2

Manufactured in the United States of America

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*Chapter 7*MULTIFRAME IMAGE RESTORATION
AND REGISTRATION*R. Y. Tsai and T. S. Huang*

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Chapter 7

MULTIFRAME IMAGE RESTORATION AND REGISTRATION

R. Y. Tsai and T. S. Huang

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Advances in Computer Vision and Image Processing, vol. 1, pages 317–339

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ISBN: 0-89232-280-2

ABSTRACT

A new method for multiframe image restoration and registration is presented. The observations are sequences of low-resolution, undersampled, discrete frames. The result is a restored high-resolution image. The restoration part is suitable for real-time implementation since the computation consists of only a few complex operations per pel of the resultant high resolution image. Tests on a class of waveforms with different bandwidths show that the performance is superior to that of the cubic spline technique in terms of the signal-to-noise ratio. The registration part exhibits two new features distinct from existing techniques: (1) The relative shifts of several frames can be estimated in one process. (2) The estimate is accurate even if the observation frames are severely undersampled, so long as the number of frames available is large enough.

1. INTRODUCTION

We are interested in restoring a high-resolution image from a sequence of low-resolution, undersampled, discrete frames of a moving object. The problem is motivated by the following: A Landsat satellite takes pictures over the same area on the ground every 18 days as it orbits around the earth. The pictures taken usually do not coincide exactly. The problem is how to obtain a high-resolution image using these low-resolution frames. The problem can be illustrated with Figure 1. First, the displacement of each frame relative to an arbitrarily chosen reference frame has to be measured. This is a registration problem. Next, samples on a set of grids with high sampling rate are to be estimated. This is a restoration problem. Although models have been formulated to allow other types of movement, for simplicity we shall assume the motion purely translational.

The restoration part, in the absence of degradation and noise, becomes an interpolation problem with recurrent, nonuniform sampling. The one-dimensional (1D) analog of this special case has been treated in References

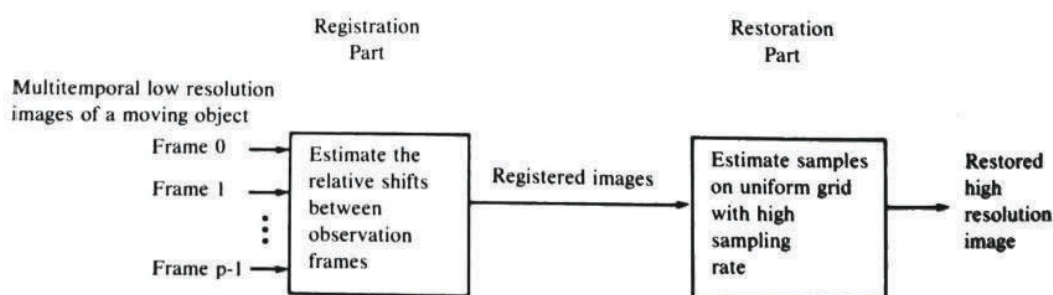


Figure 1. Flow diagram.

[4] and [6]. We shall present a new method that takes only a few multiplications per frequency sample for estimating the spectrum of the high-resolution image. Although forward and inverse discrete Fourier transforms have to be carried out, if we employ the Winograd fast Fourier transform (FFT) algorithm [5] the overall computational complexity is still within a few complex multiplications per pixel of the resultant high-resolution image.

The registration part exhibits two new features distinct from existing methods:

1. Conventional registration algorithms work with two frames at a time [1–3]. Our proposed method estimate the relative shifts of several frames in one process.
2. So long as the Nyquist rate of the original continuous waveform is less than p times half the sampling rate of the original discrete observations, where p is the number of frames available in the 1D case and the square root of that in the 2D case, the estimate is accurate even if the original observations are severely undersampled.

2. RESTORATION

2.1 Theory

For the 1D case, let the ideal image be $f(x)$ and, as we have p frames of discrete images available, let each frame be denoted by $\{f_{ki}\}_{i=0,\dots,N-1}$, where f_{ki} is the i th sample in the k th frame and N is the number of samples in each frame. Due to the translational motion assumption, and in the absence of degradation and noise, the k th frame are samples from $f(x)$ with uniform spacing T starting from $x = \delta_k$ such that $f_{ki} = f(iT + \delta_k)$, where $i = 0, 1, \dots, n - 1$ and $k = 0, \dots, p - 1$. Let $f_k(x)$ be such that $f_k(x) = f(x + \delta_k)$. Then the k th frame $\{f_{ki}\}$ can be regarded as being sampled from a continuous image $f_k(x)$ starting $x = 0$. From the shifting property of the continuous Fourier transform (CFT), we have

$$F_k(w) = e^{j\delta_k w} F(w), \quad (1)$$

where $F_k(w)$ and $F(w)$ are CFT of $f_k(x)$ and $f(x)$, respectively.

Let F_{kn} be the discrete Fourier transform (DFT) at discrete frequency n of the k th frame $\{f_{ki}\}$ such that

$$F_{kn} = \sum_{i=0}^{N-1} f_{ki} \exp\left(-j2\pi \frac{in}{N}\right), \quad n = 0, \dots, N - 1.$$

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$$F_{kn} = \sum_{i=0}^{N-1} f_{ki} \exp\left(-j2\pi \frac{in}{N}\right), \quad n = 0, \dots, N-1.$$

Then from the aliasing relationship between DFT and CFT, we have

$$F_{kn} = \frac{1}{T} \sum_{m=-\infty}^{\infty} F_k \left(\frac{n}{NT} + mw_s \right), \quad (2)$$

where w_s is the sampling frequency of each frame.

Let L be such that $F(w) = 0$ for $|w| \geq Lw_s$; then by combining (1) and (2), we arrive at the following matrix equation for each given discrete frequency n :

$$G_n = \phi_n F_n, \quad (3)$$

where G_n is a $p \times 1$ column vector with k th element F_{kn} ; F_n is a $2L \times 1$ column vector with i th element $F(n/NT + (i - L - 1)w_s)$; ϕ_n is a $p \times 2L$ rectangular matrix with (k, j) th element

$$T^{-1} \exp \left[j2\pi\delta_{k-1} \left(\frac{n}{NT} + (j - 1 - L)w_s \right) \right].$$

With δ_k either known a priori or estimated using some registration algorithm such as the one described in Section 3, below, Eq. (3) would be independent for different n and can be solved separately for the unknown column vector F_n , which contains the CFT of $f(x)$ at $2L$ equidistant frequency points with spacing w_s . When this is done for each n , we obtain the estimate of $F(w)$ at LN frequency points ranging from $-Lw_s$ to $Lw_s - 1/NT$ and with spacing $1/NT$. This can be used to estimate $f(x)$ from $x = 0$ to $(N - 1)T$, with spacing $T/2L$. The resolution is thus increased by a factor of $2L$.

Up to this point, a matrix equation (3) of size $p \times 2L$ has to be solved for every frequency pel, and without further simplification this would be too expensive. However, if we further observe the structure of ϕ_n in (3), we see that it can be decomposed as follows:

$$\phi_n = D_n H, \quad (4)$$

where D_n is a diagonal matrix with k th diagonal element

$$T^{-1} \exp \left\{ j2\pi\delta_{k-1} \left(\frac{n}{NT} - Lw_s \right) \right\}$$

and $[H]_{ij} = \exp\{j2\pi\delta_i(j - 1)w_s\}$.

Then from Eq. (3),

$$D_n^{-1} G_n = H F_n \quad (5)$$

where D_n^{-1} is a diagonal matrix with k th diagonal element

$$T^{-1} \exp \left\{ -j2\pi\delta_{k-1} \left(\frac{n}{NT} - Lw_s \right) \right\}.$$

Let $W_i = \exp\{j2\pi\delta_{i-1}w_s\}$; then H has the following form:

$$H = \begin{bmatrix} W_1^0 & W_1^1 & W_1^2 & \dots & W_1^{2L-1} \\ W_2^0 & W_2^1 & W_2^2 & \dots & W_2^{2L-1} \\ \vdots & \vdots & \vdots & & \vdots \\ W_p^0 & W_p^1 & W_p^2 & \dots & W_p^{2L-1} \end{bmatrix}.$$

Therefore, H is a rectangular Vandamonde matrix, and in the case when $p \geq 2L$, which is always desirable since the number of constraints would be less than the number of unknowns, rows of H are linearly independent if the shifts are such that $\delta_i \neq \delta_j + mT$, where $i \neq j$ and m is any integer. Thus Eq. (5) is always consistent. For the case when $p = 2L$, H becomes a square Vandamonde matrix and the (i, j) th element of H^{-1} is given by

$$[H^{-1}]_{ij} = \frac{(-1)^{p-i}}{\prod_{\substack{l=0 \\ l \neq j}}^{p-1} (W_j - W_l)} \sigma_{p-i, p-1}^j W_j, \quad (6)$$

where $\sigma_{i, p-1}^j$ is the sum of the product of i of the numbers $W_0, \dots, W_{j-1}, W_{j+1}, \dots, W_{p-1}$ without permutation or repetition.

It is interesting to note that in Eq. (5), H is independent of n . Thus its inverse need be computed only once and then applied to solve Eq. (5) for all n . That is,

$$F_n = H^{-1} D_n^{-1} G_n. \quad (7)$$

The number of operations for computing F_n using Eq. (7) is $P^2 + P$. Therefore, the computational complexity per frequency pel of the resultant high resolution image is $(P^2 + P)N/NP = P + 1$. If we employ the order n FFT algorithm (3), then the overall complexity would be $p + 3$ complex operations per pel. This is faster than the cubic spline technique and the techniques described in (1) and (2).

For the case where H is rectangular, a minimum norm least square solution of F_n can be obtained by computing the pseudoinverse of H only once and then multiplying upon the left-hand side of Eq. (5) for each n . The complexity per pel is $2L + 1$. The overall complexity per pel would be $2L + 3$.

For the 2D case, let the ideal image be $f(x, y)$ and the k th frame among the p frames available be denoted by $\{f_{ij}^k\}_{i=0, \dots, M-1, j=0, \dots, N-1}$, where f_{ij}^k is the (i, j) th sample in the k th frame and M, N are the number of samples of each frame in x and y directions, respectively. As in the 1D case, we have

$$f_{ij}^k = f(iT_1 + \delta_{k,1}, jT_2 + \delta_{k,2}),$$

where $\delta_{k,1}$ and $\delta_{k,2}$ are the x and y components of the relative shift of the k th frame with respect to the reference or the 0th frame and $\delta_{0,1} = \delta_{0,2} = 0$. T_1 and T_2 are the sampling intervals of each observation frame in x and y axes, respectively.

Let $f_k(x,y) = f(x + \delta_{k,1}, y + \delta_{k,2})$, and $F_k(u,v)$ and $F(u,v)$ be the 2D CFT of $f_k(x,y)$ and $f(x,y)$, respectively. Then, as in (1), we have

$$F_k(u,v) = e^{j2\pi(\delta_{k,1}u + \delta_{k,2}v)} F(u,v). \quad (8)$$

As in (2), the following equation is seen to hold:

$$F_{mn}^k = \frac{1}{T_1 T_2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F_k\left(\frac{m}{MT_1} + mw_{s1}, \frac{n}{NT_2} + nw_{s2}\right), \quad (9)$$

where F_{mn}^k is the 2D DFT of the k th frame at the discrete frequency pair (m,n) as follows:

$$F_{mn}^k = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f_{ij}^k \exp\left[-j2\pi\left(\frac{mi}{M} + \frac{nj}{N}\right)\right],$$

$$m = 0, \dots, M-1, \quad n = 0, \dots, N-1,$$

and $w_{si} = T_i^{-1}$, $i = 1, 2$.

Let L_i be such that $F(w_1, w_2) \rightarrow 0$ for $|w_i| \geq L_i w_{si}$, $i = 1, 2$, then by combining (8) and (9), we arrive at the following:

$$G_{mn} = \phi_{mn} F_{mn}^k, \quad (10)$$

where F_{mn} is a $4L_1 L_2 \times 1$ column vector with the i th element being

$$F\left(\frac{m}{MT_1} + rw_{s1}, \frac{n}{NT_2} + sw_{s2}\right),$$

$$\gamma = i \bmod(2L_1) - L_1, \quad s = \lfloor Li/2L_1 \rfloor - L_2,$$

and where G_{mn} is a $p \times 1$ column vector with the k th element being F_{mn}^k ; the (k,l) th element of ϕ_{mn} is given below:

$$[\phi_{mn}]_{kl} = \exp\left\{j2\pi\left[\delta_{k1}\left(\frac{m}{MT_1} - \frac{\gamma}{T_1}\right) + \delta_{k2}\left(\frac{n}{NT_2} - \frac{s}{T_2}\right)\right]\right\}. \quad (11)$$

As in (4), ϕ_{mn} can be factored as follows:

$$\phi_{mn} = D_{mn} H, \quad (12)$$

where D_{mn} is a diagonal matrix with the k th diagonal element being

$$(T_1 \cdot T_2)^{-1} \exp\left\{j2\pi\left[\delta_{k-1,1}\left(\frac{m}{MT_1} - L_1 w_{s1}\right) + \delta_{k-1,2}\left(\frac{n}{NT_2} - L_2 w_{s2}\right)\right]\right\}$$

and where H is a $p \times 4L_1L_2$ matrix with the (i, j) th element being $W_{il}^\gamma W_{l2}^S$; here $\gamma = j \bmod(2L_1)$, $S = \lfloor Lj/2L_1 \rfloor$, and W_{il} is defined below:

$$W_{il} = \exp\{j2\pi\delta_{i-1,l}w_{sl}\}, \quad l = 1, 2. \quad (13)$$

By plugging (12) into (11), we have

$$Q_{mn} = HF_{mn}, \quad (14)$$

where

$$Q_{mn} = D_{mn}^{-1}G_{mn}. \quad (15)$$

Just as in the 1D case, in Eq. (14) H is independent of m and n . Thus its inverse (or pseudoinverse if H is not square) need be computed only once and then applied to solve (14) for all m and n . That is,

$$F_{mn} = H^{-1}Q_{mn}. \quad (16)$$

With W_{il} given in (13), H has the following form:

$$H = [A^0B \quad A^1B \quad \cdots \quad A^{L_2-1}B] \quad (17)$$

where

$$A = \begin{bmatrix} W_{12} & & & \\ & W_{22} & & \\ & & \ddots & \\ & & & W_{p,2} \end{bmatrix} \quad (18)$$

and

$$B = \begin{bmatrix} W_{11}^0 & W_{11}^1 & \cdots & W_{11}^{2L_1-1} \\ W_{21}^0 & W_{21}^1 & \cdots & W_{21}^{2L_1-1} \\ \vdots & \vdots & & \vdots \\ W_{p,1}^0 & W_{p,1}^1 & \cdots & W_{p,1}^{2L_1-1} \end{bmatrix}. \quad (19)$$

It is clear that B is a Vandamonde matrix but H is not. Therefore unlike the 1D case, even if no frame coincides with any other, the matrix H in (17) could still be singular. However, this happens extremely rarely. For example, when there are four discrete frames available ($p = 4$) and H is square,

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_{11} & W_{12} & W_{11}W_{12} \\ 1 & W_{21} & W_{22} & W_{21}W_{22} \\ 1 & W_{31} & W_{32} & W_{31}W_{32} \end{bmatrix}. \quad (20)$$

It can be shown that H in (20) is singular if and only if the following holds:

$$\frac{(W_{21} - 1)(W_{11} - W_{31})}{(1 - W_{11})(W_{21} - W_{31})} = \frac{(W_{22} - 1)(W_{12} - W_{32})}{(1 - W_{12})(W_{22} - W_{32})}. \quad (21)$$

Obviously, when any of the two frames coincide, or when the horizontal shifts of two frames coincide while the vertical shifts of the other two frames are identical, (21) is satisfied. When it does happen that H is singular, as in the 1D case, a minimum norm least square estimate of F_{mn} can be obtained by computing the pseudoinverse of H and then multi-

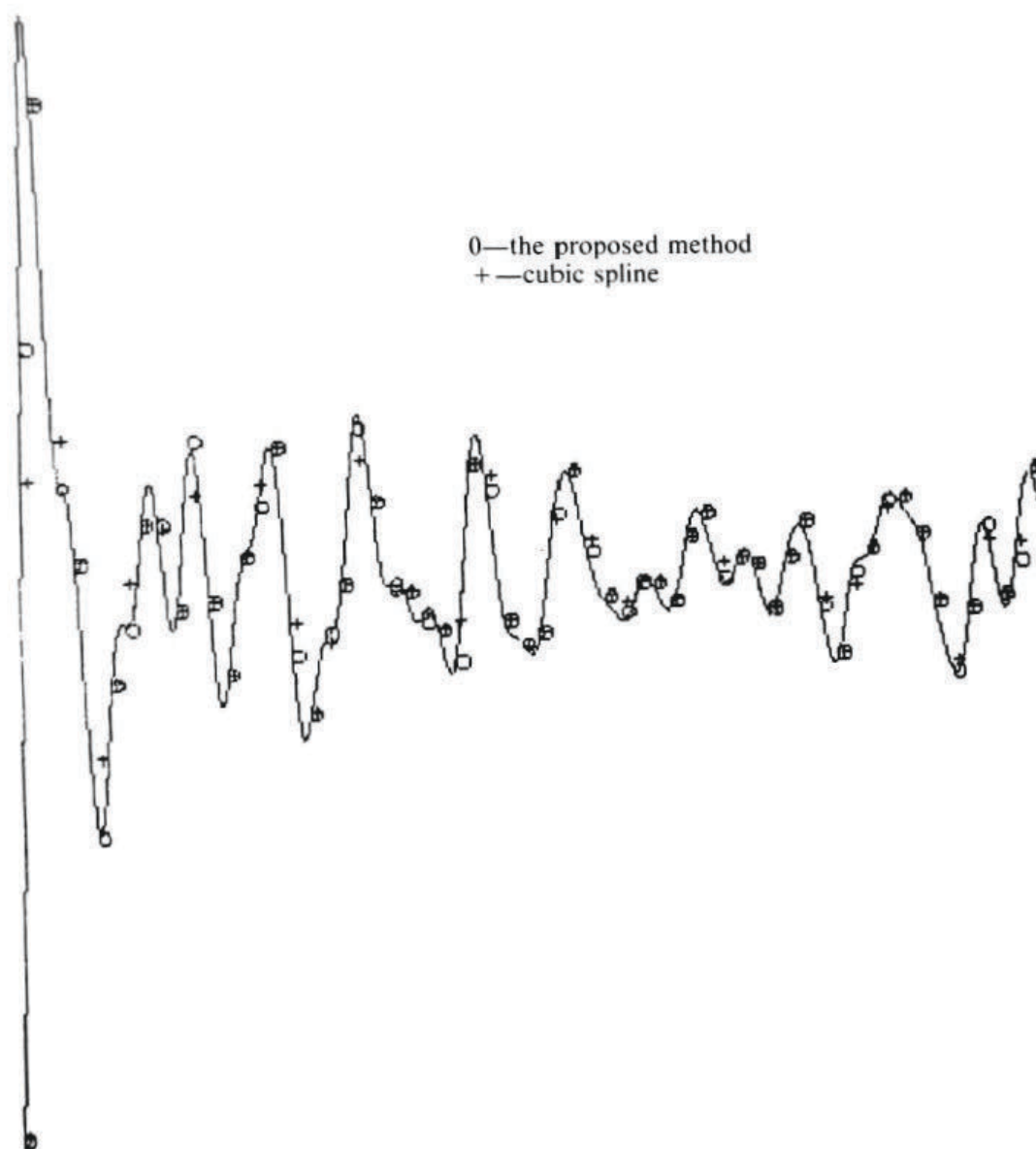


Figure 2. Estimated samples versus the ideal waveform.

plying upon the left-hand side of (14) for each m and n . The computational requirement per pel is the same as that in the 1D case.

2.2 Experimental Results

Test on the 1D case were made using a class of test waveforms with frequency spectra generated pseudorandomly and then scaled by a ramp with different slope so that waveforms with different bandwidths were available for testing purposes. The performance was compared with that of the cubic spline method, which can also be applied to this recurrent, nonuniform sampling case. The sampling frequency for the observation frame was 5 Hz, so that the test waveforms with Nyquist rate greater than 2.5 was undersampled. Two observation frames were used, and the sampling rate for the resultant high-resolution image was 10 Hz. Figure

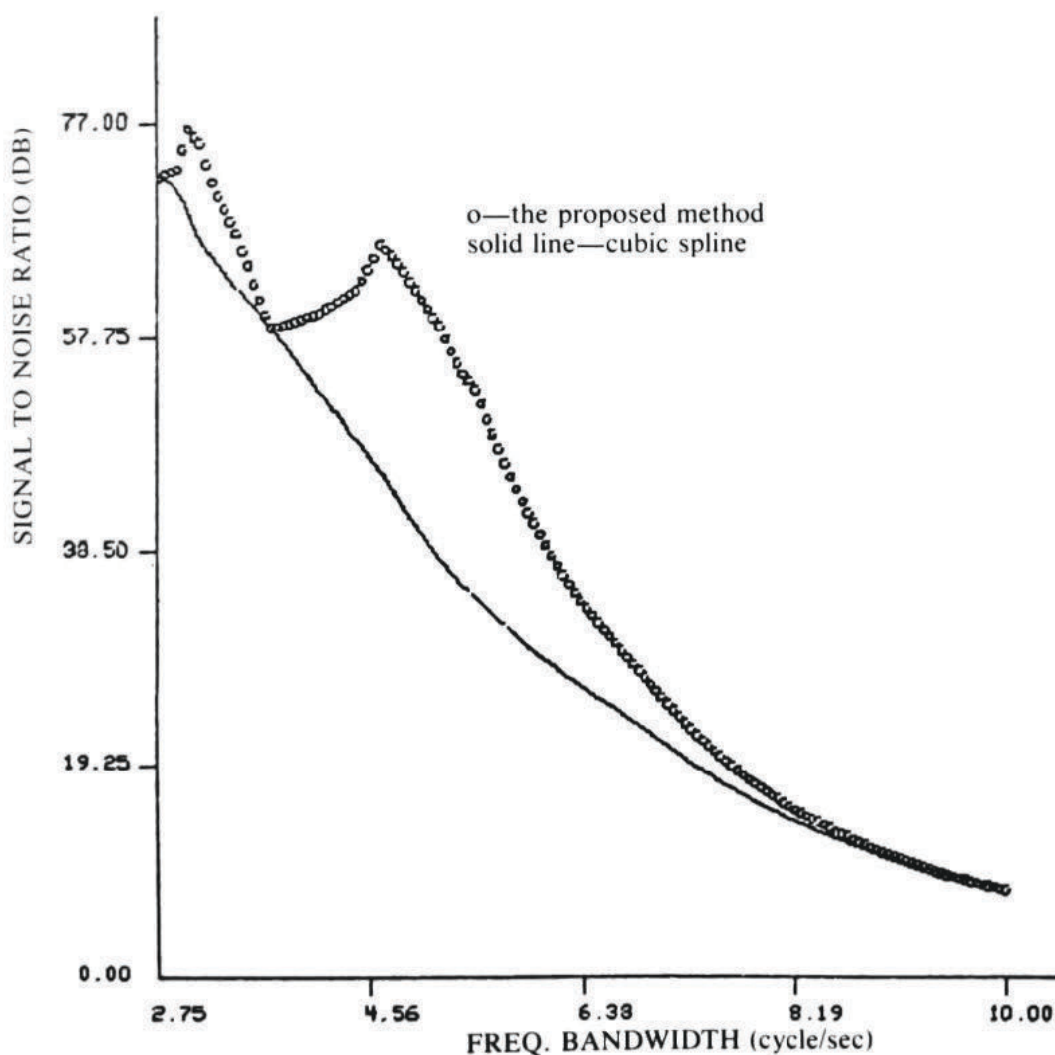


Figure 3. Comparison of SNR.

2 shows a particular example where the bandwidth of the test waveform is 4 Hz. The circles and the plus signs indicate the estimates using the proposed technique and the cubic spline technique, respectively. Figure 3 shows the signal-to-noise ratio (DB) versus the Nyquist rate of the test waveforms for both techniques. The proposed method is superior unless the Nyquist rate of the test waveform is less than one-quarter of the sampling rate for the resultant high-resolution image.

Tests on the 2D case were carried out on several images with numerous details. In these simulation tests, discrete images with high sampling rate were used as the background continuous images. Then several observation frames with a lower sampling rate were picked out from these background images, each one with the same sampling interval, say T , in both the horizontal direction and the vertical direction, but with different samples of shifts. Figure 4 shows four 64×64 low-resolution frames, each one sampled from a background 256×256 aerial image with sampling rate $(1/T, 1/T)$. The shifting parameters δ_{ij} are $\delta_{01} = \delta_{02} = 0$, $\delta_{11} = \delta_{22} = T/2$, $\delta_{12} = \delta_{31} = T/4$, $\delta_{21} = \delta_{32} = (3/4)T$, where δ_{k1} and δ_{k2} are the x and y components of the shift of the k th frame. The ideal image to be estimated is of size 128×128 , and is shown on the right-hand side of Figure 5. Using shifts estimated by the image registration technique to be described in Section 3, at 128×128 image is restored as shown on the left-hand side of Figure 5. The sampling frequency is $(2/T, 2/T)$. Note that this is



Figure 4. Four low-resolution input pictures.



Figure 5. Restored high-resolution image (left) versus the ideal image (right).

still less than two times the Nyquist rate of the original background continuous image.

3. REGISTRATION

3.1 Theory

The registration problem is to estimate the relative shifts between frames. Suppose the four circles in Figure 6 indicate the first frame and the triangles the second frame, while the original continuous image contains so much detail that the Nyquist rate is as big as the sampling rate of the set of grid in Figure 6. In this case, any existing registration technique would fail to estimate the relative shift between these two frames simply because the amount of information available is not enough. Suppose, however, that there are more frames available, for example, 20 or 30 frames, so that the amount of information is enough. But since existing registration techniques can only work with two frames at a time, they cannot take advantage of the fact that the combined sampling grid is proper. (By proper we mean the sampling rate is up to the Nyquist rate.) We propose a new registration technique that can estimate the relative

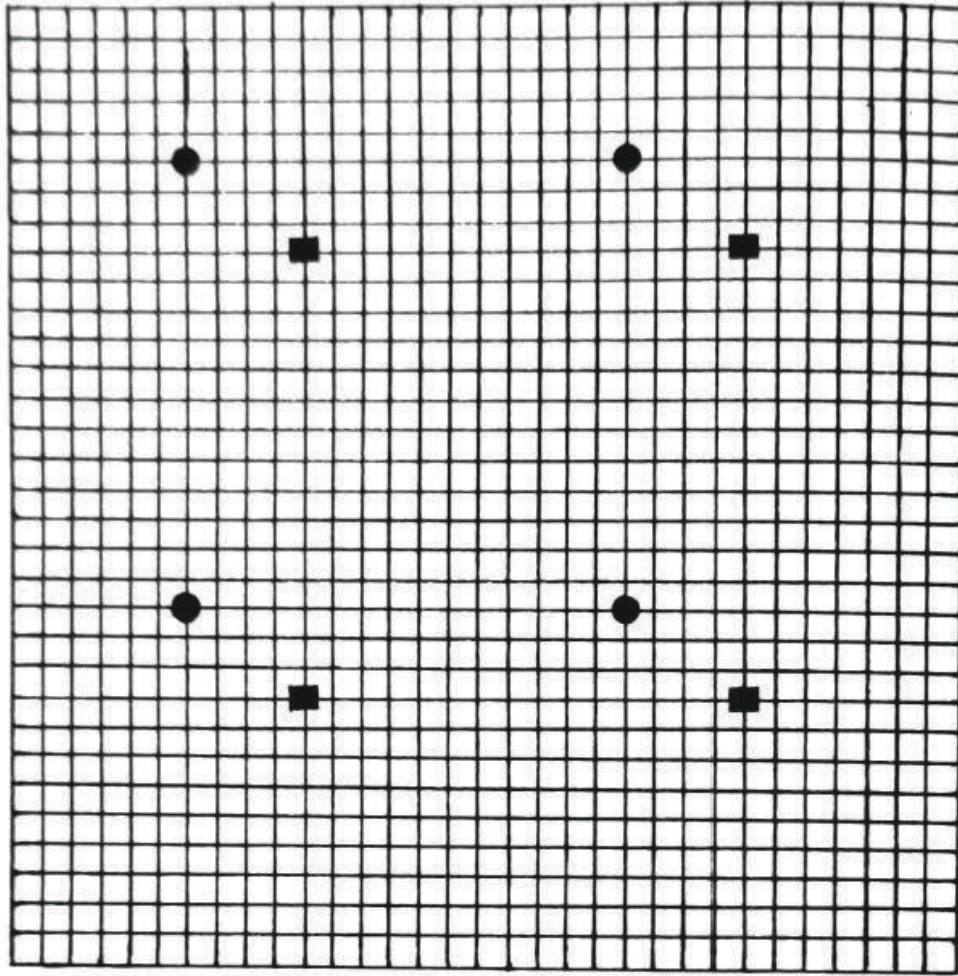


Figure 6. Two severely undersampled frames versus the proper sampling grid.

shifts between all of these frames in one process. Therefore, there are two novel features in the proposed registration technique:

1. More than two frames can be registered in one process.
2. Even if the individual frames are severely undersampled, as long as the number of frames is large enough so that the combined sampling grid is proper, the relative shifts can be estimated accurately.

Let ϕ be the total energy of the estimated $F(w)$ [using Eq. (7)] in the frequency range outside the bandwidth of the ideal $F(w)$. Then ϕ will be a function of the shifting parameter δ_i , since different δ_i can yield different estimates of $F(w)$ when we use Eq. (7). The proposed registration algorithm is to find the δ_i that minimizes ϕ . For example, suppose the ideal $F(w)$ is given in Figure 7. By using Eq. (7), with the exact shifting pa-

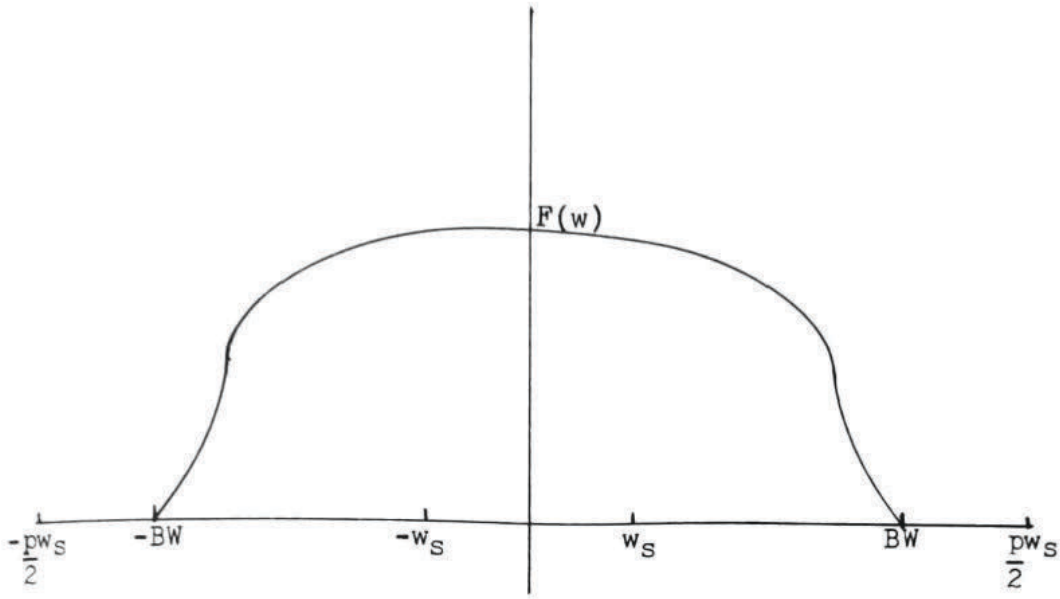


Figure 7. Ideal Fourier transform.

rameters, $F(w)$ can be estimated exactly up to $w = (P/2)w_s$. If, however, the shifting parameters are incorrect, then the estimated $F(w)$ using Eq. (7) may be given in Figure 8. The energy of the estimated $F(w)$ in the frequency range outside $w = BW$ is no longer zero. This is based upon the following theorem:

THEOREM. For the 1D case, suppose each of the p observation frames $\{f_{ki}\}_{i=0, \dots, N-1} k = 0, \dots, p-1$ are sampled with fixed sampling interval

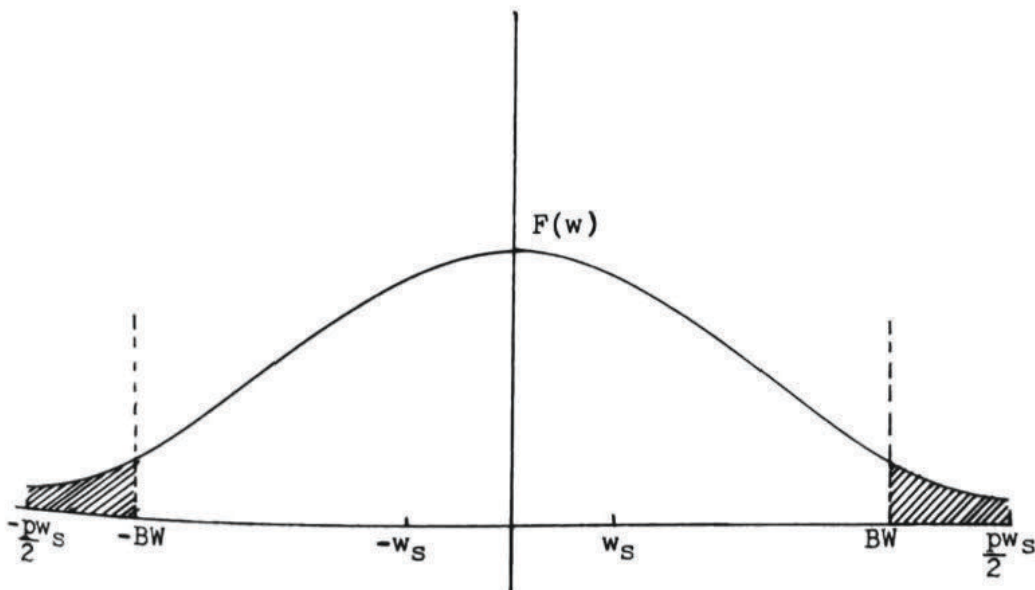


Figure 8. Estimated Fourier transform using incorrect shifts.

T but are infinite in extent, that is, N approaches infinity. Let the bandwidth of the ideal signal $f(x)$ be BW . If $BW < P/2T$ and Eq. (7) in the restoration part is used to estimate the Fourier transform $F(w)$ of $f(x)$, but with one of the shifting parameters, δ_k , incorrect, and assuming the discrepancy is small relative to the ideal value (this is sufficient for achieving subpixel accuracy), then the estimated $F(w)$ for $w > BW$ will be non-zero (that is, the bandwidth will be increased) provided that the DFT of the k th frame at the frequency different from w by an integral multiple of w_s does not vanish.

Proof: Let n and i be such that $n/NT + (i - L - 1)w_s > BW$, and let the i th element of the column vectors F_n and G_n in (7) be denoted by F_n^i and G_n^i , respectively. Then $F_n^i = F(n/NT + (i - L - 1)w_s) = 0$. From (7) and (6), we have

$$\begin{aligned} F_n^i &= \sum_{j=1}^p [H^{-1}]_{ij} [D_n^{-1}]_{jj} G_n^j \\ &= \sum_{j=1}^p \frac{(-1)^{p-i}}{\prod_{\substack{l=0 \\ l \neq j}}^{p-1} (W_j - W_l)} \sigma_{p-i, p-1} W_j T^{-1} \\ &\quad \times \exp\left\{-j2\pi\delta_{j-1} \left(\frac{n}{NT} - Lw_s\right)\right\} F_{jn}. \end{aligned} \quad (22)$$

Let $\hat{F}_n^i$ be the estimated F_n^i using (22) but with δ_k replaced by $\delta_k + \Delta$. It is to be shown that for nonzero Δ small relative to δ_k , $\hat{F}_n^i$ will be nonzero provided that $F_{kn} \neq 0$, where F_{kn} is the DFT of the k th frame at frequency n/NT . It is easy to see that with δ_k replaced by $\delta_k + \Delta$, only the k th term in the summation in (22) is changed. Thus

$$\begin{aligned} \hat{F}_n^i - F_n^i &= \hat{F}_n^i \\ &= \left[\frac{(-1)^{p-i}}{\prod_{\substack{l=0 \\ l \neq k+1}}^{p-1} (W'_{k+1} - W_l)} \sigma_{p-i, p-1}^{k'} W'_{k+1} T^{-1} \right. \\ &\quad \times \exp\left\{-j2\pi(\delta_k + \Delta) \left(\frac{n}{NT} - Lw_s\right)\right\} \\ &\quad - \frac{(-1)^{p-i}}{\prod_{\substack{l=0 \\ l \neq k+1}}^{p-1} (W_{k+1} - W_l)} \sigma_{p-i, p-1}^k W_{k+1} T^{-1} \\ &\quad \times \exp\left\{-j2\pi\delta_k \left(\frac{n}{NT} - Lw_s\right)\right\} \Big] F_{kn}, \end{aligned} \quad (23)$$

where $W'_{k+1} = \exp\{j2\pi(\delta_k + \Delta)w_s\}$. If $F_{kn} \neq 0$, then it is clear that $\hat{F}_n^i = 0$ if and only if the following holds:

$$\exp\left\{-j2\pi\Delta\left[\frac{n}{NT} - (L+1)w_s\right]\right\} = \frac{\prod_{\substack{l=0 \\ l \neq k}}^{p-1} (e^{j2\pi(\delta_k + \Delta)w_s} - W_l)}{\prod_{\substack{l=0 \\ l \neq k}}^{p-1} (e^{j2\pi\delta_k w_s} - W_l)}. \quad (24)$$

Since $\Delta \ll \delta_k$, Δ on the right-hand side of (24) can be dropped while Δ on the left cannot. Therefore, $\hat{F}_n^k = 0$ if and only if

$$\exp\left\{-j2\pi\Delta\left[\frac{n}{NT} - (L+1)w_s\right]\right\} = 1 \quad (25)$$

Since $w_s = 1/T$, (25) is equivalent to the following:

$$2\pi\frac{\Delta}{T}\left(\frac{n}{N} - L - 1\right) = 2\pi m,$$

where m is arbitrary integer. But $n/N - L - 1 < 0$ and $\Delta \ll T$. Thus $m = 0$ and $\Delta = 0$. Therefore, it is proved that $\hat{F}_n^k = 0$ if and only if $\Delta = 0$. $\square$

Although the above theorem is exact for small Δ , Eq. (24) holds for general Δ . There may be some large Δ that can satisfy (24). But unless the original image is trivial and contains extremely rare symmetry, this can never happen. Besides, local convergence is sufficient to guarantee subpixel accuracy. Also, in our foregoing theorem, only one shifting parameter is assumed to be incorrect, and the frame numbers for these frames are $k_1, k_2, \dots, k_m$, respectively. If more than one of the δ_i values are incorrect, then the estimated $F(w)$ may vanish for a particular $w > BW$, but cannot vanish for all $w > BW$. This is justified below:

Suppose the shifting parameters for m frames are incorrect. Then it is obvious from the proof of the theorem that

$$\hat{F}_n^i - F_n^i = \sum_{\gamma=1}^m \alpha_{n,i}^{k_\gamma} F_{k_\gamma n}, \quad (26)$$

where $\alpha_{n,i}^{k_\gamma}$ denotes the expression inside the outer bracket in (23), which is the change of $\hat{F}_n^i$ divided by $F_{k_\gamma n}$ due to the change of the shifting parameters for the k_γ th frame from δ_{k_γ} to $\delta_{k_\gamma} + \Delta_{k_\gamma}$.

From the theorem, $\alpha_{n,i}^{k_\gamma} = 0$ iff $\Delta_{k_\gamma} = 0$. However it may be possible that (26) is zero even if $\Delta_{k_\gamma} \neq 0$. But this cannot happen simultaneously for all n and i such that $n/NT + (i - L + 1)w_s > BW$ since, were this be true, we would have m unknowns $\Delta_{k_1}, \Delta_{k_2}, \dots, \Delta_{k_\gamma}$ satisfying a set

of simultaneous equations with the number of equations several orders of magnitude greater than m . Therefore, we can obtain an optimum estimate of the shifting parameters by minimizing ϕ using any iterative non-linear minimization algorithm.

3.2 Experimental Results

Tests of this algorithm on a class of 1D waveforms used in the restoration part were extremely encouraging. Figure 9 is a plot of ϕ versus δ_1 for the case when the Nyquist rate of the original waveform is 3.75, which is one-half greater than half the sampling rate of the original observation frames, i.e., it is undersampled. Two frames were used, and the ideal relative shift is 0.15. The minimum of ϕ occurs at $\delta_1 = 0.149950$. The two peaks in Figure 9 occur because for these cases $\delta_1 = \delta_0 + kT$, where k is any integer and H in Eq. (3) becomes singular. Two other local minima

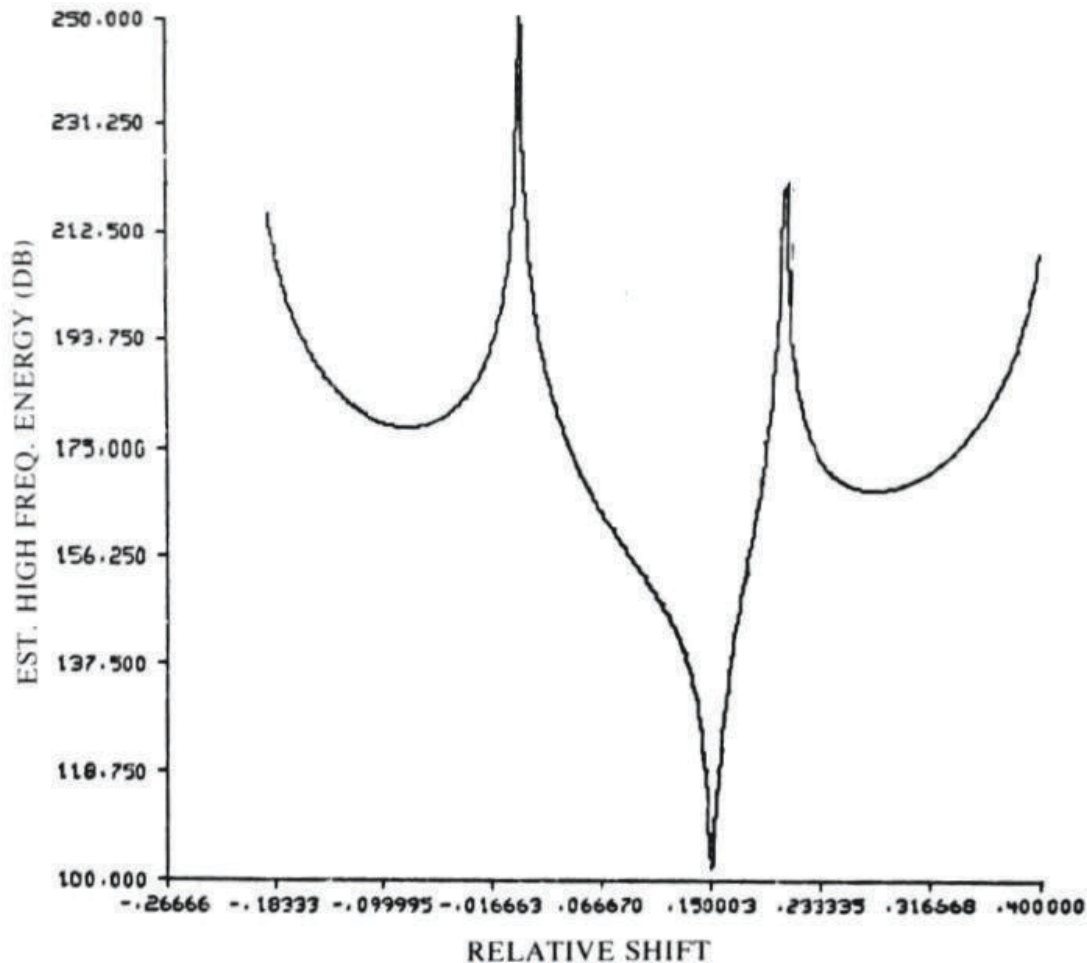


Figure 9. Funtion ϕ versus the relative shift with the condition in theorem satisfied.

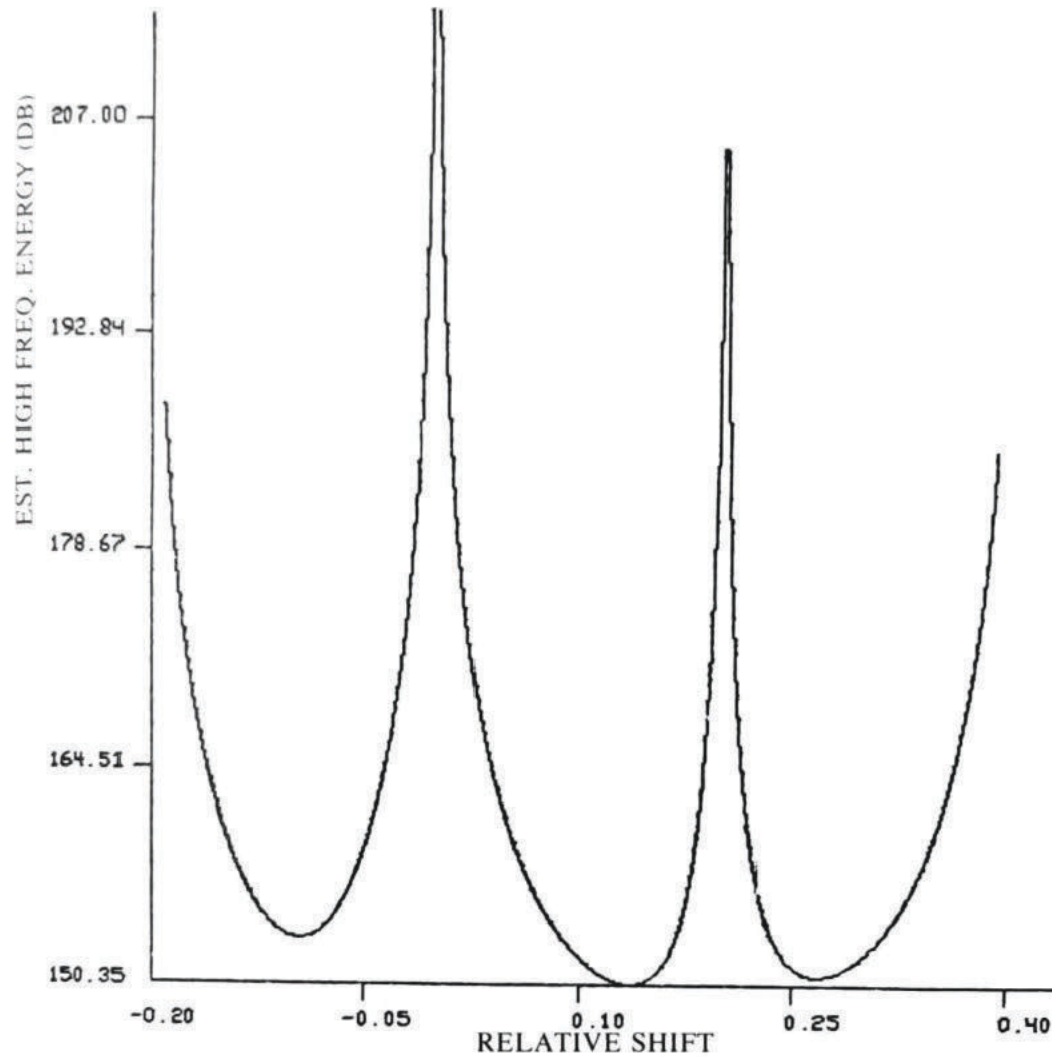


Figure 10. Function ϕ versus the relative shift with the condition in theorem not satisfied.

occur at $\delta_1 < 0$ and $\delta_1 > T$. This phenomenon can also be explained by observing Eq. (3). Therefore, it is suggested that a rough global search be carried out first, followed by searching for local minima using any nonlinear minimization scheme. If the number of frames available is not big enough such that the condition $BW < p/2T$ is not met, ϕ is not defined. We can still estimate δ_i using the same algorithm but changing the definition of ϕ to be the total energy of the estimated $F(w)$ in the high-frequency range. In this case, our theorem cannot be applied exactly, but intuitively it might work. A particular example is depicted in Figure 10. It is clear that the minimum is not as sharp as that in Figure 9 and the relative error is about 15%.

Tests for the case of four frames were equally satisfactory. Figure 11 shows the estimated ϕ versus the shifting parameters for a particular

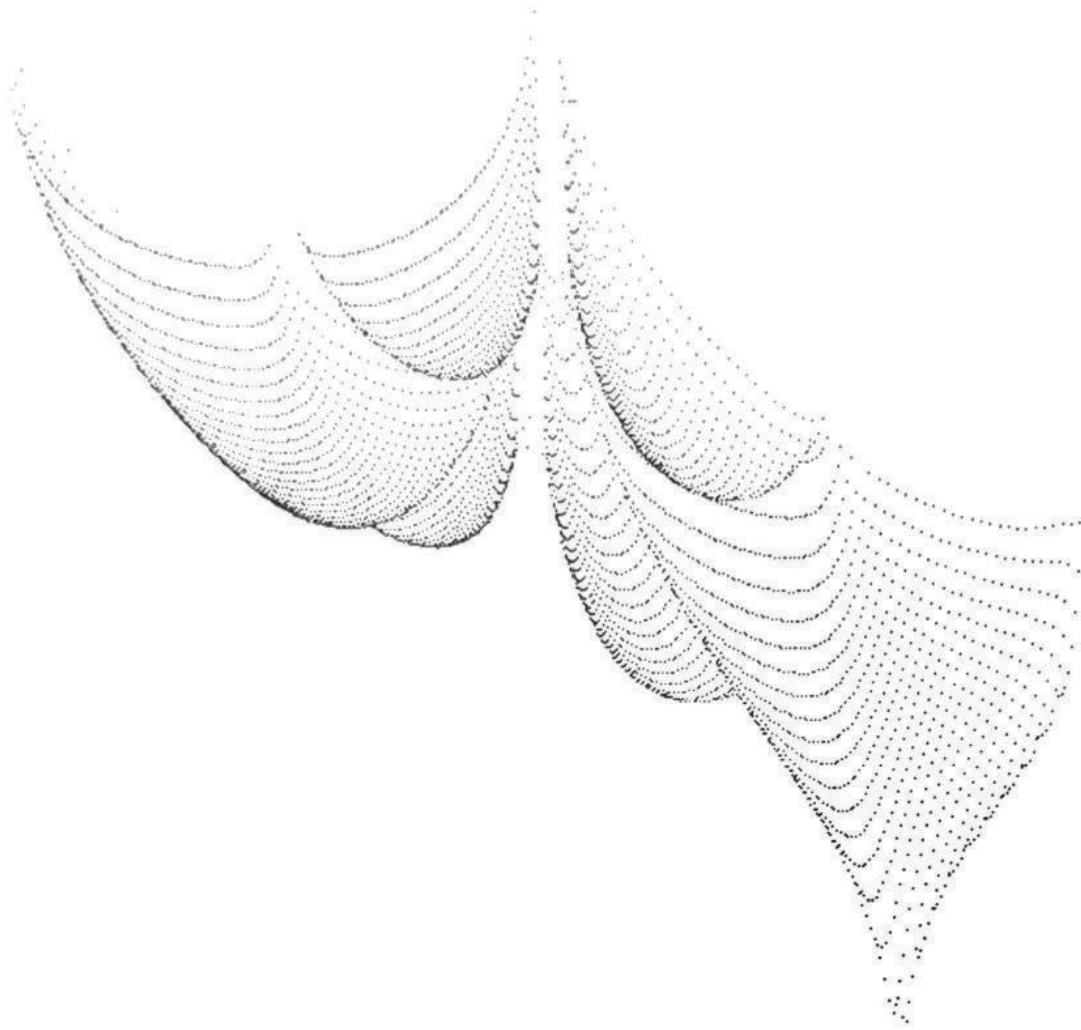


Figure 11. Function ϕ versus δ_1 and δ_3 with δ_0 and δ_2 left fixed for the case of four frames ($p = 4$).

example where the sampling frequency of each frame 2.5 Hz and BW of $F(w)$ is 3.75 Hz. For the purpose of displaying the result, δ_2 is fixed at 0.2, while δ_1 and δ_3 are unknowns. In this case ϕ is a 2D function with δ_1 and δ_3 as variables. As expected, the minimum occurs almost exactly at the ideal value. The relative error is about 10^{-4} . Figure 12 is the same as Figure 11 except that ϕ is viewed from a different angle.

The condition for the validity of our theorem is that the bandwidth $BW < p/2T$. For the case mentioned above, $p/2T = 5$. Therefore, the performance of the algorithm should deteriorate if $BW > 5$. Experimental results agree with this fact, as can be seen in Figure 13, which is a plot of the normalized average registration error (%) versus the frequency bandwidth. The normalized average registration error (NARE) is defined

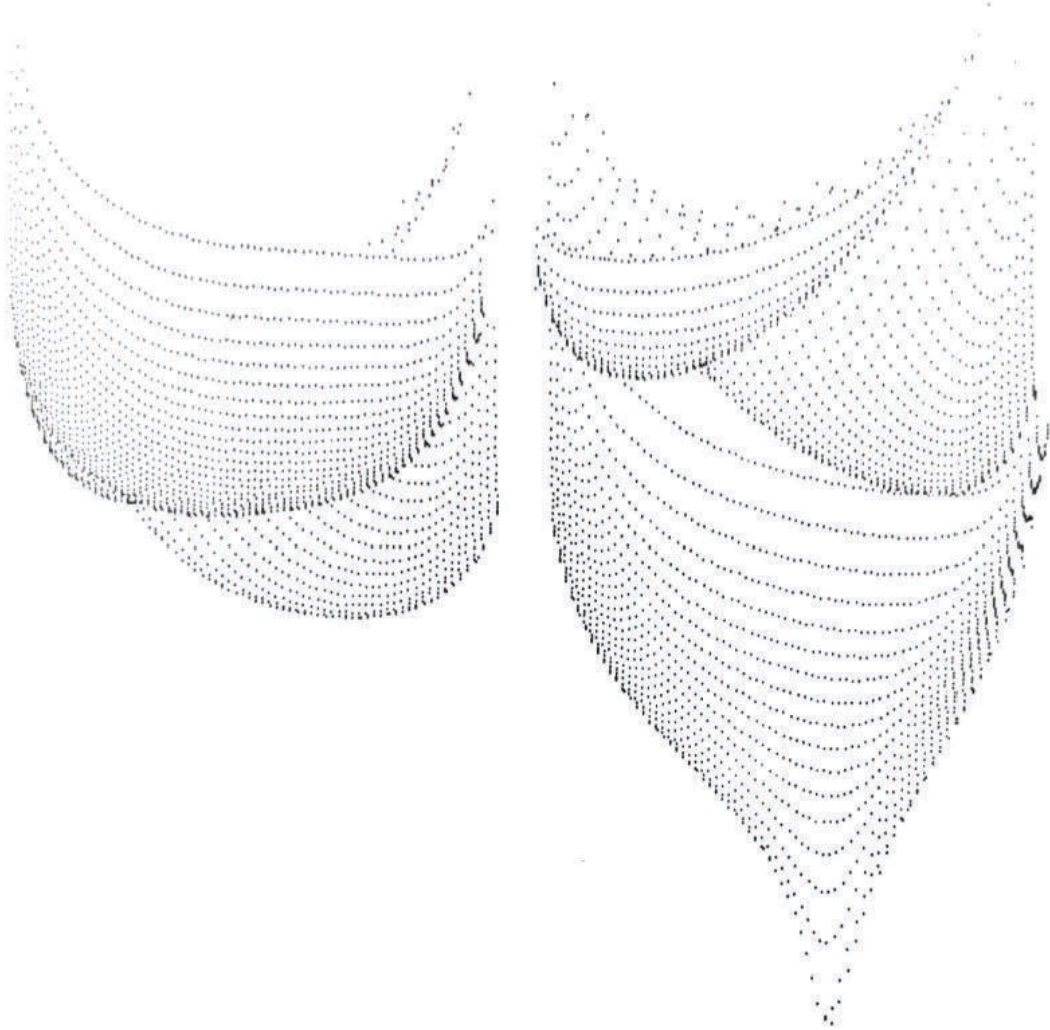


Figure 12. Same as Figure 11 but with different viewing angle.

as follows:

$$\text{NARE} = \left[\frac{1}{P-1} \sum_{i=1}^{P-1} \frac{(\delta_i - \hat{\delta}_i)^2}{T^2} \right]^{1/2}.$$

This fact implies that even if each individual frame may be severely undersampled, so long as the bandwidth of the ideal signal is less than p times half the sampling frequency of the observation frame, the algorithm will still give an accurate estimate of the relative shifts. This is a great advantage over any of the conventional technique, which works well only for images not severely undersampled.

Tests on the 2D case were carried out on several images. The experimental results show that if the condition $\text{BW} < P/2T$ is roughly satisfied in both directions then NARE is less than 0.8%, but if $\text{BW} > p/2T$ then

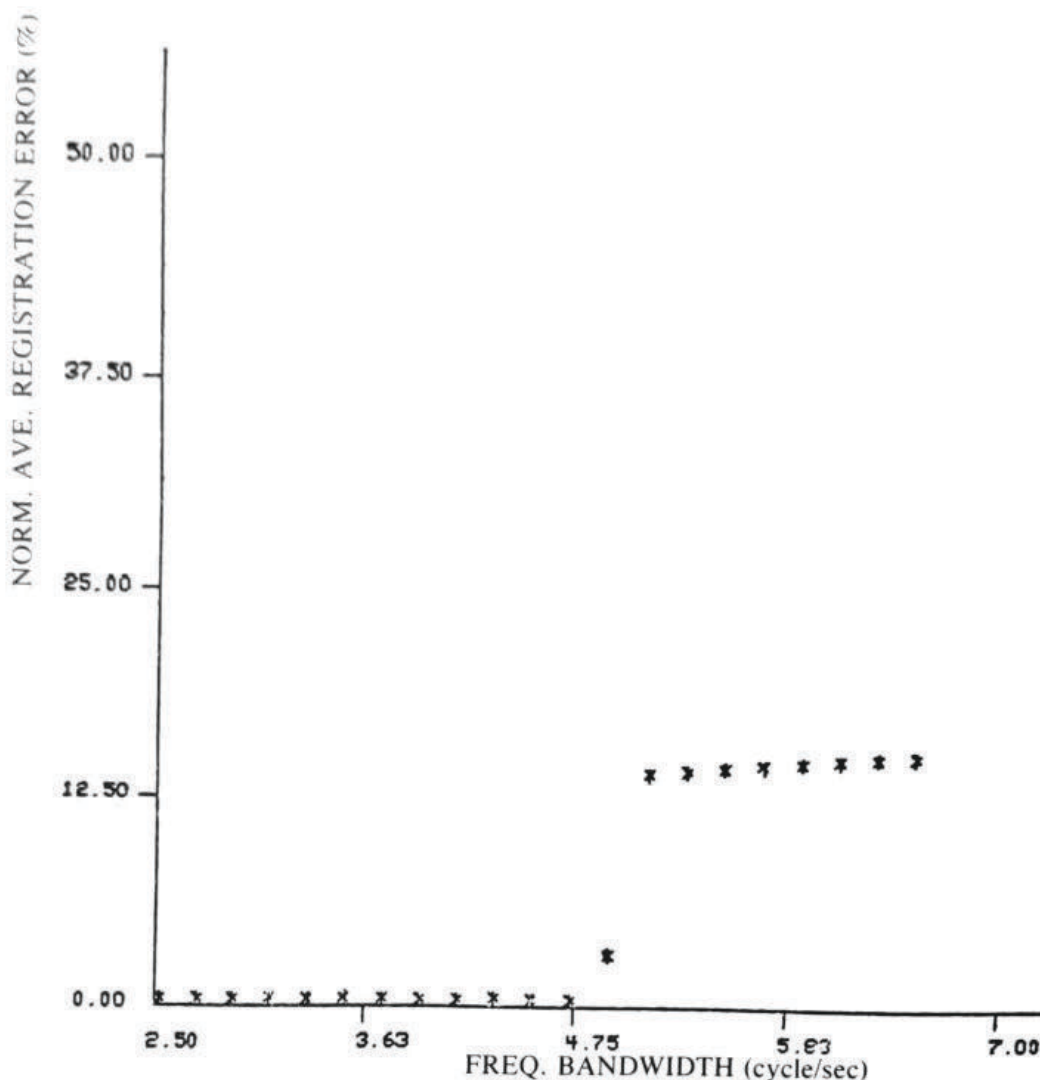


Figure 13. NARE versus BW.

NARE is around 7–12%. For example, for the aerial image described in the restoration section, if the observation frame is 64×64 then the condition $BW < P/2T$ will not be met. In this case, NARE is about 7.2%. If the sampling interval is reduced to half, that is, the observation frame is 128×128 instead of 64×64 , then the condition $BW < P/2T$ is approximately met. NARE now becomes 0.76%. For the former case, the log magnitude of the estimated Fourier transform using the correct shifting parameters is given on the left-hand side of Figure 14. The low-frequency parts are at the four corners, and the origin of the frequency domain is at the upper left corner. The right-hand side of Figure 14 shows the result obtained using incorrect shifts. As expected, the estimated high-frequency energy using the incorrect shifts is much higher than that using the correct shifts. The left-hand picture in Figure 15 shows the restored image using

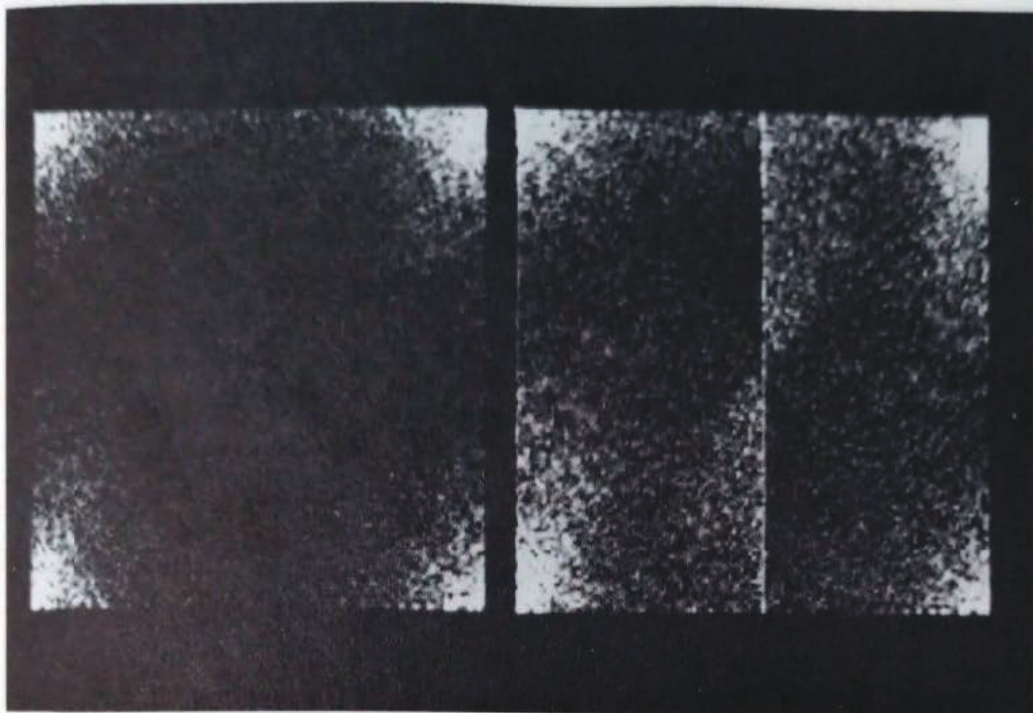


Figure 14. Estimated Fourier transform with correct shifts (left) and incorrect shifts (right).

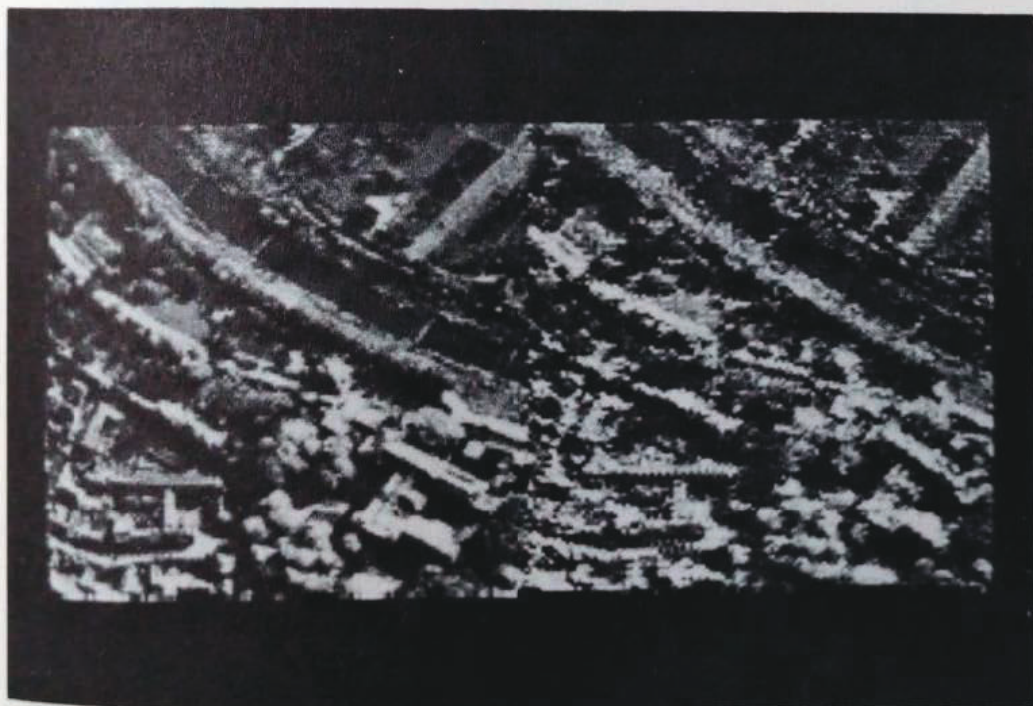


Figure 15. Restored image with correct shifts (left) and incorrect shifts (right).

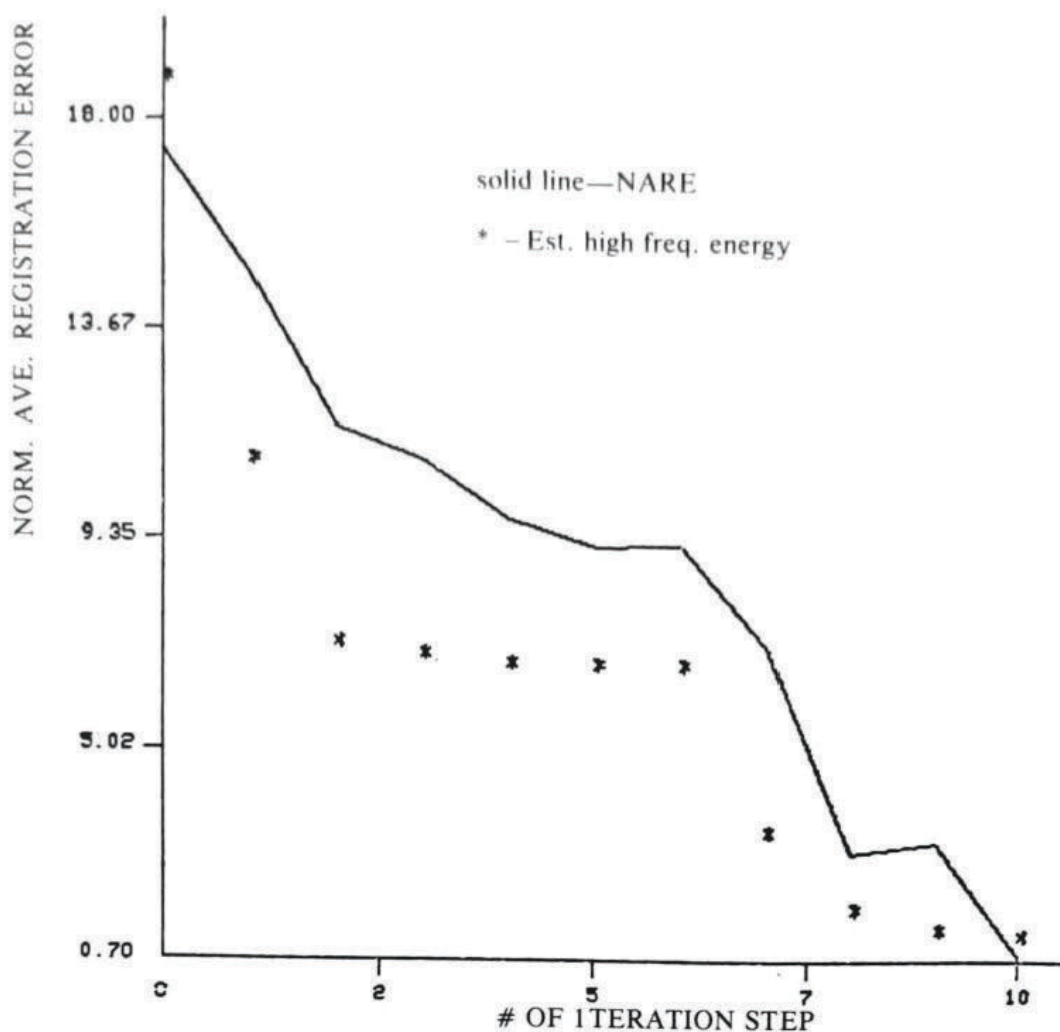


Figure 16. NARE versus ITERATION STEP.

the correct shifts, while the right-hand picture shows that using the incorrect shifts.

Like the 1D case, there are several occasions that local minima may occur: (1) $\hat{\delta}_{ij} \approx \delta_{ij} + kT$, where δ_{ij} is the ideal shift and k is any integer; (2) interchange of δ_{i1} and δ_{i2} ; (3) interchange of $(\delta_{i1}, \delta_{i2})$ and $(\delta_{j1}, \delta_{j2})$. A rough global search was first performed. Then a quasi-Newton minimization scheme was used to search for the minimum of ϕ . Since for each evaluation of ϕ only a narrow high-frequency band of $F(w)$ need be computed, the computation is not very expensive. Figure 16 is a plot of NARE (%) and ϕ (DB) versus the iteration step of the quasi-Newton method.

4. CONCLUSION

Both theoretical and experimental results of a new method for multiframe image restoration and registration are described. The restoration part is

suitable for real-time implementation since the computation consists of only a few complex operations per pel of the resultant high-resolution image. Tests on a class of one-dimensional waveforms with different bandwidths show that the performance is superior to that of the cubic spline technique in terms of the signal-to-noise ratio. Results on the two-dimensional case are also satisfactory. The registration part has two new features: (1) The relative shifts of several frames can be estimated in one process. (2) The estimate is accurate even if the observation frames are severely undersampled, as long as the number of frames available is big enough. Tests for both 1D and 2D cases were successful.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant NSF MCS82-06926 and in part by INRS-Telecommunications, University of Quebec, Montreal, Canada.

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