

Iterative image restoration algorithms

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Abstract. This tutorial paper discusses the use of successive-approximation-based iterative restoration algorithms for the removal of linear blurs and noise from images. Iterative algorithms are particularly attractive for this application because they allow for the incorporation of prior knowledge about the class of feasible solutions, because they can be used to remove nonstationary blurs, and because they are fairly robust with respect to errors in the approximation of the blurring operator. Regularization is introduced as a means for preventing the excessive noise magnification that is typically associated with ill-posed inverse problems such as the deblurring problem. Iterative algorithms with higher convergence rates and a multistep iterative algorithm are also discussed. A number of examples are presented.

Subject terms: image restoration; iterative algorithms; regularization; least-squares solution; projection onto convex sets.

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1. INTRODUCTION

Images are recorded to portray useful information about a phenomenon of interest. Unfortunately, a recorded image will

almost certainly be a degraded version of an original image or scene due to the imperfections of physical imaging systems and the particular physical limitations imposed in every application in which image data are recorded. The situation becomes more complicated due to random noise, which is inevitably mixed with the data and may originate from the image formation process, the transmission medium, the recording process, or any combination of these.

In many practical situations, the image degradation can be adequately modeled by a linear blur (motion, defocusing, atmospheric turbulence) and an additive white Gaussian noise process.¹ Then the degradation model is described by

$$\tilde{y} = y + n = Dx + n, \quad (1)$$

where the vectors $\tilde{y}$, y , x , and n represent, respectively, the lexicographically ordered noisy and blurred, blurred, and original images and the additive noise. The matrix D represents the linear spatially invariant or spatially varying distortion; it has as elements samples of the point spread function (PSF) of the imaging system. D can have a special structure dependent on the properties of the PSF. For the convolutional case, for example, D is a block Toeplitz matrix.

With this model, the purpose of digital image restoration is to operate on the degraded image $\tilde{y}$ to obtain an improved image that is as close to the original image x as possible, subject to a suitable optimality criterion. A number of techniques or filters or algorithms providing a solution to the image restoration problem have appeared in the literature.¹ These techniques can be grouped into² (a) direct, recursive, or iterative; (b) linear or nonlinear; (c) deterministic, hybrid, or stochastic; (d) spatial domain or frequency domain; and (e) adaptive or nonadaptive. Among the well-known image restoration filters are the Wiener and Kalman filters.^{1,3} The first filter represents a direct, linear, stochastic, spatial or frequency domain, linear, nonadaptive technique; the latter represents a recursive, linear, stochastic, spatial domain, nonadaptive technique.

In this tutorial paper we discuss the use of iterative techniques that are based on the method of successive approxima-

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tions (henceforth called iterative algorithms) for restoring noisy blurred images. We restrict our study to this class of iterative algorithms since they have been used more widely for signal restoration than other iterative algorithms.⁴ These techniques will be linear or nonlinear, hybrid, spatial or frequency domain, adaptive or nonadaptive. Iterative techniques exhibit certain advantages over the rest of the techniques and have become very popular in recent years. Among these advantages are the following^{2,4}: (1) there is no need to determine or implement the inverse of an operator; (2) knowledge about the solution can be incorporated into the restoration process; (3) the solution process can be monitored as it progresses; (4) constraints can be used to control the effects of noise.

This paper is arranged into several sections. Section 2 reviews the basic deterministic iterative restoration algorithms. The noise in the model of Eq. (1) is ignored in these algorithms. Their derivation, the conditions for convergence, and their rate of convergence are closely examined. The related method of projecting onto convex sets (POCS)⁵ is also discussed since it is used in Sec. 3.

In Sec. 3 we propose and analyze an iterative image restoration algorithm that is based on a technique for regularizing ill-posed problems. Knowledge about the noise as well as other properties of the solution are directly incorporated into the restoration process. The proposed algorithm has, on the one hand, the advantages of the iterative approaches mentioned above, and on the other hand it is computationally more advantageous than other iterative algorithms, such as the POCS. It is compared with related restoration algorithms, such as the constrained least squares (CLS) algorithm and the POCS. The set of feasible solutions to the restoration problem is described geometrically with the use of bounding ellipsoids. The form of the proposed algorithm is suitable for extending it to an adaptive iterative restoration algorithm, as described in Sec. 4. Properties of the human visual system are then incorporated into the restoration algorithm, resulting in visually better restored images. Experimental results obtained by using the algorithms presented in Secs. 3 and 4 for restoring simulated distorted images and photographically blurred images are presented in Sec. 5.

A drawback of iterative restoration algorithms is their linear rate of convergence. In extending the applicability of iterative algorithms, algorithms with higher rates of convergence are presented in Sec. 6. In Sec. 7 a multistep iterative algorithm derived from the single-step algorithm is discussed. The motivation for presenting such an algorithm is its suitability to VLSI implementation due to its characteristic of localized data transactions.

2. REVIEW OF DETERMINISTIC ITERATIVE RESTORATION ALGORITHMS

In this section we briefly review the derivation and convergence properties of some of the deterministic iterative signal restoration algorithms. These algorithms ignore the presence of noise in the degradation model of Eq. (1). A historical perspective is preserved in presenting these algorithms.

2.1. Basic iterative algorithm

2.1.1. Derivation

A class of these algorithms can be derived in a very straight-

forward way. Based on Eq. (1) with $\mathbf{n} = 0$, the following identity holds for all values of the gain parameter β :

$$\mathbf{x} = \mathbf{x} + \beta(\mathbf{y} - \mathbf{D}\mathbf{x}) . \quad (2)$$

Applying the method of successive approximations to this identity yields the iteration

$$\begin{aligned} \mathbf{x}_0 &= \beta\mathbf{y} , \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \beta(\mathbf{y} - \mathbf{D}\mathbf{x}_k) \\ &= \beta\mathbf{y} + (\mathbf{I} - \beta\mathbf{D})\mathbf{x}_k \\ &= \beta\mathbf{y} + \mathbf{G}_1\mathbf{x}_k = \mathbf{T}_1\mathbf{x}_k , \end{aligned} \quad (3)$$

where $\mathbf{I}$ is the identity operator. Perhaps the earliest reference to iteration (3) was by Van Cittert⁶ in the 1930s. In this case the distortion was spatially invariant and the gain β was equal to 1. Jansson et al.⁷ modified the Van Cittert algorithm by replacing β with a relaxation parameter that depends on the signal. Also, Kawata et al.^{8,9} used Eqs. (3) for image restoration with a fixed or a varying parameter β .

Clearly, for a spatially invariant distortion, iteration (3) takes the form

$$\begin{aligned} x_0(i,j) &= \beta y(i,j) , \\ x_{k+1}(i,j) &= x_k(i,j) + \beta(y(i,j) - d(i,j) ** x_k(i,j)) \\ &= \beta y(i,j) + (\delta(i,j) - \beta d(i,j)) ** x_k(i,j) \\ &= \beta y(i,j) + g_1(i,j) ** x_k(i,j) , \end{aligned} \quad (4)$$

where $\delta(i,j)$ is a two-dimensional impulse, $d(i,j)$ is the impulse response of the distorting system, and $**$ denotes 2-D discrete convolution.

2.1.2 Convergence

In dealing with iterative algorithms the convergence as well as the rate of convergence is very important. In this section we present some general convergence results to serve as reference material for the following sections. These results are presented for general operators, but equivalent representations in the Fourier domain are also shown.

The *contraction mapping theorem* usually serves as a basis for establishing convergence of iterative algorithms. According to this theorem, iteration (3) converges to a unique fixed point $\mathbf{x}^*$, that is, a point such that $\mathbf{T}_1\mathbf{x}^* = \mathbf{x}^*$, for any initial vector, if the operator or transformation $\mathbf{T}_1$ in Eqs. (3) is a contraction. This means that for any two vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ in the domain of $\mathbf{T}_1$ the following relation holds:

$$\|\mathbf{T}_1\mathbf{z}_1 - \mathbf{T}_1\mathbf{z}_2\| \leq \eta \|\mathbf{z}_1 - \mathbf{z}_2\| , \quad (5)$$

where η is strictly less than 1 and $\|\cdot\|$ denotes any norm. Since $\mathbf{T}_1$ is a linear operator, the sufficient convergence condition (5) results in

$$\|\mathbf{I} - \beta\mathbf{D}\| < 1 \text{ or } \|\mathbf{G}_1\| < 1 . \quad (6)$$

We emphasize that condition (6) is norm dependent; that is, a mapping may be contractive according to one norm but not

according to another. If the l_2 norm is used, then condition (6) is equivalent to the requirement that

$$\max_i |\sigma_i(G_1)| < 1, \quad (7)$$

where $|\sigma_i(G_1)|$ is the absolute value of the i th singular value of G_1 .¹⁰

The necessary and sufficient condition for iteration (3) to converge to a unique fixed point is that

$$\max_i |\lambda_i(G_1)| < 1 \text{ or } \max_i |1 - \beta \lambda_i(D)| < 1, \quad (8)$$

where $|\lambda_i(A)|$ represents the magnitude of the i th eigenvalue of the matrix A . Clearly, for a symmetric matrix D , conditions (7) and (8) are equivalent. Conditions (5) to (8) are used in defining the range of values of β for which convergence of iteration (3) is guaranteed.

Of special interest is the case in which the matrix D is singular (D has at least one zero eigenvalue) since it represents a number of typical distortions of interest (for example, motion, defocusing, etc.). Then there is no value of β for which conditions (7) or (8) are satisfied. In this case the mapping G_1 (or equivalently T_1) is called a *nonexpansive mapping* [η in Eq. (5) is equal to 1]. Such a mapping may have any number of fixed points (zero to infinity). However, a very useful result is obtained if we further restrict the properties of D (this results in no loss of generality, as will become clear in the following sections). That is, if D is a symmetric, semipositive definite matrix (all of its eigenvalues are nonnegative), then according to Bialy's theorem,^{2,11} iteration (3) will converge to the minimum norm solution of Eq. (1), if this solution exists, plus the projection of x_0 onto the null space of D , denoted by $\mathcal{N}(D)$, for $0 < \beta < 2 \cdot \|D\|^{-1}$.

A necessary and sufficient convergence condition of iteration (4) also can be obtained by transforming it in the frequency domain. That is, it is easily found that²

$$X_k(u, v) = \beta Y(u, v) \sum_{i=0}^k G_1(u, v)^i, \quad (9)$$

where the upper case signals denote the 2-D discrete Fourier transform (DFT) of the corresponding lower case signal in Eqs. (4). Clearly, for the series in Eq. (9) to converge as $k \rightarrow \infty$, it is necessary and sufficient that

$$|G_1(u, v)| < 1 \text{ or } |1 - \beta D(u, v)| < 1 \quad (10)$$

for all discrete frequencies u, v . Condition (10) is equivalent to condition (8) since the eigenvalues of D in the convolution case (D is a block circulant matrix) are the DFT values of $d(i, j)$.^{2,12} In general, this property of block circulant matrices allows us to directly transform a matrix-vector equation to the discrete frequency domain. The limiting solution of iteration (4) is the well-known inverse filter, represented in the discrete frequency domain by Eq. (9) as $k \rightarrow \infty$.

2.2. Basic iterative algorithm with reblurring

2.2.1. Derivation

It is easily verified that for condition (10) to be satisfied, $D(u, v)$ should have a nonnegative real part.⁴ If this is not the case, then a simple approach in overcoming the convergence difficulty is to first filter (reblur) both the distorted image

$y(i, j)$ and the distortion impulse response $d(i, j)$ with an impulse response $d^*(-i, -j)$, where $*$ denotes complex conjugate,^{4,13} and then apply iteration (4). That is, the basic iterative algorithm (4) takes the form

$$\begin{aligned} x_0(i, j) &= \beta d^*(-i, -j) ** y(i, j), \\ x_{k+1}(i, j) &= x_k(i, j) + \beta d^*(-i, -j) ** (y(i, j) \\ &\quad - d(i, j) ** x_k(i, j)) \\ &= \beta d^*(-i, -j) ** y(i, j) + (\delta(i, j) \\ &\quad - \beta d^*(-i, -j) ** d(i, j)) ** x_k(i, j). \end{aligned} \quad (11)$$

Now the sufficient convergence condition, corresponding to condition (10), becomes

$$|1 - \beta |D(u, v)|^2| < 1, \quad (12)$$

and it can be always satisfied for

$$0 < \beta < 2 \cdot \max_{u, v} |D(u, v)|^{-2}. \quad (13)$$

As described above, iteration (11) was initially introduced to overcome convergence difficulties of the basic iterative restoration algorithm. However, the matrix-vector version of iteration (11) results from a least-squares approach to the solution of Eq. (1).^{2,14} More specifically, for the general case of a rectangular matrix D , in solving Eq. (1) for an x , the following functional is minimized:

$$\min_x M(x) = \min_x \|y - Dx\|^2. \quad (14)$$

A necessary condition for $M(x)$ to have a minimum is that its gradient with respect to x be equal to zero, which results in the normal equations

$$D^T D x = D^T y. \quad (15)$$

A solution to Eq. (15) can be successively approximated according to the iteration

$$\begin{aligned} x_0 &= \beta D^T y, \\ x_{k+1} &= x_k + \beta D^T (y - D x_k) \\ &= \beta D^T y + (1 - \beta D^T D) x_k \\ &= \beta D^T y + G_2 x_k = T_2 x_k. \end{aligned} \quad (16)$$

The procedure described outlines the method of *steepest descent* applied to the minimization of the functional $M(x)$ in Eq. (14). Clearly, a number of other optimization techniques can be used toward the minimization of $M(x)$, such as the conjugate gradient method^{15,16} and a projection method.¹⁷

2.2.2. Convergence and rate of convergence

The convergence analysis presented in the previous section holds for algorithms (11) and (16). The operators T_1 and G_1 should be replaced, respectively, by the operators T_2 and G_2 in expressions (5) to (10). The iterative algorithm with deblurring can be used now in removing distortions with a symmet-

ric impulse response but with a transfer function that takes positive and negative values (distortion due to motion, defocusing, etc). Such transfer functions $D(u, v)$ take the value of zero at certain discrete frequencies. However, for this case and the more general case of a singular $\mathbf{D}^T \mathbf{D}$, according to Bialy's theorem,² iteration (11) or (16) will converge to the minimum norm least squares solution of Eq. (1), denoted by $\mathbf{x}^+$, for $0 < \beta < 2 \cdot \|\mathbf{D}\|^2$, since $\mathbf{D}^T \mathbf{y}$ is in the range of $\mathbf{D}^T \mathbf{D}$, denoted by $\mathcal{R}(\mathbf{D}^T \mathbf{D})$.

The rate of convergence of iteration (16) is linear. If we denote by $\mathbf{D}^+$ the generalized inverse of $\mathbf{D}$, defined by $\mathbf{x}^+ = \mathbf{D}^+ \mathbf{y}$, then the rate of convergence of (16) is described by the relation^{18,19}

$$\frac{\|\mathbf{x}_k - \mathbf{x}^+\|}{\|\mathbf{x}^+\|} \leq c^{k+1}, \quad (17)$$

where

$$c = \max\{|1 - \beta\|\mathbf{D}\|^2|, |1 - \beta\|\mathbf{D}^+\|^{-2}|\}. \quad (18)$$

The expression for c in Eq. (18) also will be used in Sec. 6, where higher order iterative algorithms are presented.

2.3. Basic iterative algorithm with constraints

2.3.1. Derivation and convergence

Iterative signal restoration algorithms regained popularity in the 1970s due to the incorporation of prior knowledge about the solution into the restoration process.⁴ For example, we may know that $\mathbf{x}$ is a bandlimited signal or is space limited, or we may know on physical grounds that $\mathbf{x}$ can have only nonnegative values. A convenient way of expressing such prior knowledge is to define a constraint operator $\mathbf{C}_H$ such that

$$\mathbf{x} = \mathbf{C}_H \mathbf{x} \quad (19)$$

if and only if $\mathbf{x}$ satisfies the constraint. We call such a constraint a *hard constraint*.^{2,20} Using such a representation for a priori signal constraints, iteration (16), for example, can be written as⁴

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{D}^T \mathbf{y}, \\ \tilde{\mathbf{x}}_k &= \mathbf{C}_H \mathbf{x}_k, \\ \mathbf{x}_{k+1} &= \mathbf{T}_2 \tilde{\mathbf{x}}_k. \end{aligned} \quad (20)$$

The form of the constrained basic algorithm without reblurring is obtained by replacing $\mathbf{T}_2$ by $\mathbf{T}_1$ and $\mathbf{x}_0$ by $\beta \mathbf{y}$ in Eqs. (20). In general, $\mathbf{C}_H$ represents the concatenation of constraint operators, which are applied at each iteration prior to the application of the operator $\mathbf{T}_1$ or $\mathbf{T}_2$.

The recent popularity of constrained iterative restoration algorithms is also due to the fact that a number of other restoration problems, such as the bandlimited extrapolation problem⁴ and the reconstruction from phase or magnitude problem,^{4,21} can be solved by an iterative algorithm of the form of Eqs. (20) by appropriately describing the distortion and constraint operators. A review of the problems that can be solved by an algorithm of the form of Eqs. (20) is presented by Schafer et al.⁴

The contraction mapping theorem is again used as a basis

for establishing convergence of the constrained iterative algorithm. The resulting sufficient condition for convergence is that at least one of the operators $\mathbf{C}_H$ or $\mathbf{T}_2$ is contractive while the other is nonexpansive. If both $\mathbf{C}_H$ and $\mathbf{T}_2$ are nonexpansive but the conditions of Bialy's theorem are satisfied, then iteration (20) converges to a solution that is the minimum of $M(\mathbf{x})$ in Eq. (14), subject to the constraint.¹⁴ Usually, it may be harder to prove convergence and determine the convergence rate of the constrained iterative algorithm since some of the constraint operators (positivity constraint operator, for example) are nonlinear.

2.3.2. Experiment 1

In this experiment we show a result obtained by using a deterministic iterative restoration algorithm. In all of the experimental results presented in this paper, the criterion $\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2 / \|\mathbf{x}_k\|^2 \leq 10^{-6}$ was used in terminating the iteration, the parameter β was equal to 1 for comparison purposes, and the space-limitation hard constraint to the 256×256 pixel image was used. Similarly, in all of the experiments the distortion is due to the translation of the object at constant velocity along a straight line during the exposure interval. The impulse response of the distorting system is equal to¹

$$d(n) = \begin{cases} \frac{1}{L}, & n = 0, 1, \dots, L-1, \\ 0, & \text{otherwise}, \end{cases} \quad (21)$$

where L is the extent of the motion. For such a $d(n)$, condition (8) is not satisfied; therefore iteration (4) cannot be used. Iteration (11) is used instead. The distorted image with $L = 16$ and the restored image are shown, respectively, in Figs. 1(a) and 1(b).

2.4. Method of POCS

The method of POCS describes an alternative approach in incorporating prior knowledge about the solution into the restoration process. It reappeared in the engineering literature in the early 1980s,^{5,22} and since then it has been successfully applied to the solution of different restoration problems (reconstruction from phase or magnitude, for example). According to the method of POCS, incorporating a priori knowledge into the solution can be interpreted as restricting the solution to be a member of a closed convex set that is defined as a set of vectors satisfying a particular property. If the constraint sets have a nonempty intersection, then a solution that belongs to the intersection set can be found by the method of POCS. Indeed, any solution in the intersection set is consistent with the a priori constraints and therefore is a feasible solution.

More specifically, let $Q_1, Q_2, \dots, Q_m$ be closed convex sets in a finite dimensional vector space, with $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$ their respective projectors. Then the iterative procedure

$$\mathbf{x}_{k+1} = \mathbf{P}_1 \mathbf{P}_2 \dots \mathbf{P}_m \mathbf{x}_k \quad (22)$$

converges to a vector that belongs in the intersection of the sets $Q_i, i = 1, 2, \dots, m$, for any starting vector $\mathbf{x}_0$. It is interesting to note that the resulting set intersection is also a closed convex set.

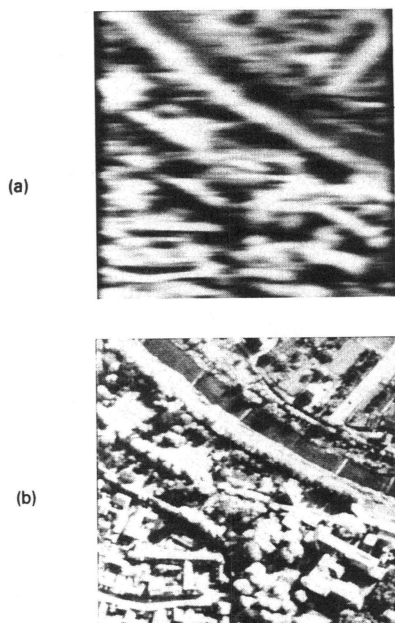


Fig. 1. (a) Motion blurred image with $L = 16$ pixels. (b) Image restored by the basic algorithm with reblurring.

Clearly, the application of a projection operator $\mathbf{P}$ and the constraint $\mathbf{C}_H$, discussed in the previous section, express the same idea. As a matter of fact, most of the $\mathbf{C}_H$ constraints are equivalent to a projection operator. Therefore, iteration (20) can also be written as

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{y}, \\ \mathbf{x}_{k+1} &= \mathbf{P} \mathbf{T}_2 \mathbf{x}_k. \end{aligned} \quad (23)$$

In establishing the convergence of iteration (22), in order for the mapping $\mathbf{P} \mathbf{T}_2$ to be a contraction, $\mathbf{T}_2$ needs to be a contraction, since $\mathbf{P}$ (or a concatenation of projection operators) is a nonexpansive mapping.⁵

3. REGULARIZED CONSTRAINED ITERATIVE RESTORATION ALGORITHM

As mentioned earlier, the noise in Eq. (1) was ignored in the derivation of the iterative restoration algorithms presented in Sec. 2. The noise has been incorporated into the iterative process in different ways. Schafer et al.⁴ and Richards et al.²³ have experimentally studied the effects of additive noise in the restored signal. Their approach was to preprocess the noisy data in order to suppress the broadband noise. With respect to removing the noise, the reblurring process or the solution of the normal equations has an effect similar to prefiltering with a noise rejecting filter. In Trussell²⁴ the iteration is terminated when the norm of the residual error $\|\tilde{\mathbf{y}} - \mathbf{D} \mathbf{x}_k\|$ is approximately equal to the norm of the noise. In Ref. 25 the knowledge about the noise is used in defining a convex set of signals for which the residual error is less than or equal to the norm of the noise. The restoration algorithm then consists of projecting onto this convex set.

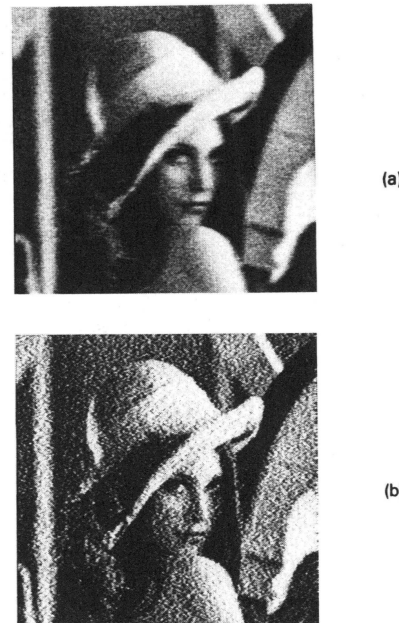


Fig. 2. (a) Noisy blurred image: $L = 9$, $\text{SNR} = 10$ dB. (b) Image restored by the basic algorithm with reblurring.

Clearly, if the noise is now included in the iterative algorithms [replace $\mathbf{y}$ by $\tilde{\mathbf{y}}$, according to Eq. (1)], then although the convergence properties of the algorithm will not be affected since they depend on the operator $\mathbf{D}$, the limiting solution will be different. If, for example, iteration (16) is used in restoring a noisy blurred image, then the algorithm converges to $\mathbf{x}^+ + \mathbf{D}^+ \mathbf{n}$. Depending on the signal-to-noise ratio (SNR), the term $\mathbf{D}^+ \mathbf{n}$, which represents the noise filtered by the inverse filter, may dominate the solution. As an example, consider the noisy blurred image in Fig. 2(a). The distortion is due to motion, with $L = 9$ and $\text{SNR} = 10$ dB. The image restored by iteration (11), after 50 iterations, is shown in Fig. 2(b). The noise is clearly amplified and dominates the solution. However, due to the finite number of iterations, the noise amplification was controlled. This ability to control the tradeoff between noise amplification and deblurring is one of the advantages of iterative restoration algorithms. A regularized algorithm that incorporates directly the knowledge about the noise into the restoration process is presented in the following section.

3.1. Ill-posed problems

If the image formation process is modeled in a continuous infinite dimensional space, $\mathbf{D}$ becomes an integral operator and Eq. (1) becomes a Fredholm integral equation of the first kind. Then the solution of Eq. (1) is almost always an ill-posed problem.^{2,22,26-30} This means that the unique least-squares solution of minimal norm of Eq. (1) does not depend continuously on the data, or that a bounded perturbation (noise) in the data results in an unbounded perturbation in the solution, or that the generalized inverse of $\mathbf{D}$ is unbounded.²⁷ The integral operator $\mathbf{D}$ has a countably infinite number of singu-

lar values that can be ordered with their limit approaching zero.²⁷ Since the finite dimensional discrete problem of image restoration results from the discretization of an ill-posed continuous problem, the matrix $\mathbf{D}$ has (in addition to possibly a number of zero singular values) a cluster of very small singular values. Clearly, the finer the discretization (the larger the size of $\mathbf{D}$), the closer the limit of the singular values is approximated. Therefore, although the finite dimensional inverse problem is well posed in the least-squares sense,²⁷ the ill-posedness of the continuous problem translates into an ill-conditioned matrix $\mathbf{D}$.

The problem of noise amplification now can be further explained by using the singular value decomposition (SVD) of the matrix $\mathbf{D}$. That is, the minimum norm least-squares solution of Eq. (1) can be written as¹⁰

$$\mathbf{x}^+ = \sum_{i=1}^r \frac{1}{\mu_i} (\mathbf{u}_i, \mathbf{y}) \mathbf{v}_i + \sum_{i=1}^r \frac{1}{\mu_i} (\mathbf{u}_i, \mathbf{n}) \mathbf{v}_i, \quad (24)$$

where $\mu_1, \dots, \mu_n$ are the singular values of $\mathbf{D}$, with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_r > \mu_{r+1} = \dots = \mu_N = 0$; r is the rank of $\mathbf{D}$; $\mathbf{u}_i$ and $\mathbf{v}_i$ are, respectively, the eigenvectors of $\mathbf{D}\mathbf{D}^T$ and $\mathbf{D}^T\mathbf{D}$; and $(\mathbf{z}, \mathbf{w})$ denotes the inner product of the vectors $\mathbf{z}$ and $\mathbf{w}$. Clearly, since $\mathbf{D}$ is an ill-conditioned matrix, some of its singular values will be very close to zero, so some of the weights μ_i^{-1} are very large numbers. If the i th inner product $(\mathbf{u}_i, \mathbf{n})$ is not zero (as is true when the noise is broadband), the noise [second term of Eq. (24)] is amplified.

3.2. Two regularization methods

A regularization method replaces an ill-posed problem by a well-posed problem whose solution is an acceptable approximation to the solution of the given ill-posed problem.^{31,32} A class of regularization methods associates both the family of admissible solutions and the observation noise with random processes.³³ Another class of regularization methods regards the solution as a deterministic quantity. Two methods from the second class that have been used extensively for the regularization of various ill-posed problems are presented next.

3.2.1. Tikhonov's method or CLS method

According to Hunt¹² and Tikhonov and Arsenin,³² the problem of solving Eq. (1) is replaced by the minimization problem

$$\begin{aligned} &\text{minimize } \|\mathbf{C}\mathbf{x}\| \\ &\text{subject to } \|\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}}\| = \epsilon, \end{aligned} \quad (25)$$

where ϵ is an estimate on the data accuracy (noise norm) and $\mathbf{C}$ is a linear operator, whose properties are analyzed in a later section. The similar minimization problem

$$\begin{aligned} &\text{minimize } \|\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}}\| \\ &\text{subject to } \|\mathbf{C}\mathbf{x}\| = E \end{aligned} \quad (26)$$

can be used in solving Eq. (1) if the constant E is known.

In solving the minimization problems (25) and (26), the standard approach is the use of the Lagrangian method. This results in solving

$$(\mathbf{D}^T\mathbf{D} + \lambda\mathbf{C}^T\mathbf{C})\mathbf{x} = \mathbf{D}^T\tilde{\mathbf{y}} \quad (27)$$

for $\mathbf{x}$. The Lagrange multiplier λ needs to be chosen so that the constraints in Eqs. (25) and (26) are satisfied.

3.2.2. Miller's method

According to Miller,³¹ the problem of solving Eq. (1) is replaced by the problem of searching for vectors $\mathbf{x}$ that satisfy both the constraints

$$\|\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}}\| \leq \epsilon, \quad (28)$$

$$\|\mathbf{C}\mathbf{x}\| \leq E, \quad (29)$$

where ϵ is an estimate of the data accuracy (noise norm) and E is a prescribed constant.

We follow two different directions in finding a solution to the problem described by Miller's regularization method [conditions (28) and (29)]. This results in different restoration algorithms. The solution resulting from these algorithms as well as the CLS algorithm are compared in Sec. 3.5.

3.2.2.1. POCS approach

Observe that conditions (28) and (29) each represent a closed convex set, more specifically, an N -dimensional ellipsoid, where N is the dimensionality of the vector $\tilde{\mathbf{y}}$. More specifically, conditions (28) and (29) represent, respectively, the sets Q_1 and Q_2 , defined by¹⁴

$$(\mathbf{x} - \mathbf{x}^+)^T \frac{\mathbf{D}^T\mathbf{D}}{\epsilon^2} (\mathbf{x} - \mathbf{x}^+) \leq 1, \quad (30)$$

$$\mathbf{x}^T \frac{\mathbf{C}^T\mathbf{C}}{E^2} \mathbf{x} \leq 1, \quad (31)$$

where $\mathbf{x}^+ = \mathbf{D}^+\tilde{\mathbf{y}}$.

Iteration (22) can now be applied in finding a solution to the restoration problem regularized according to Miller's approach. The projections $\mathbf{P}_1\mathbf{x}$ and $\mathbf{P}_2\mathbf{x}$ are defined by¹⁴

$$\mathbf{P}_1\mathbf{x} = \mathbf{x} + \lambda_1(\mathbf{I} + \lambda_1\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T(\tilde{\mathbf{y}} - \mathbf{D}\mathbf{x}), \quad (32)$$

$$\mathbf{P}_2\mathbf{x} = [\mathbf{I} - \lambda_2(\mathbf{I} + \lambda_2\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T\mathbf{C}]\mathbf{x}, \quad (33)$$

where λ_1 and λ_2 need to be chosen so that conditions (30) and (31) are satisfied, respectively. Clearly, a number of other projection operators can be used in Eq. (22) that force the signal to exhibit certain known a priori properties expressed by convex sets.

3.2.2.2. Functional minimization approach

A second approach in finding a solution satisfying conditions (28) and (29), due to Miller,³¹ is the following: The two constraints (28) and (29) are combined quadratically into the constraint

$$M(\alpha, \mathbf{x}) = \|\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}}\|^2 + \alpha\|\mathbf{C}\mathbf{x}\|^2 \leq 2\epsilon^2, \quad (34)$$

where α (the regularization parameter) is set equal to $(\epsilon/E)^2$. If there is an $\mathbf{x}$ satisfying (28) and (29), then it will also satisfy (34). Conversely, if a solution $\mathbf{x}$ satisfies (34), then (28) and (29) will be also satisfied except for a factor of at most $\sqrt{2}$ (Ref. 31, lemma 3), which is insignificant for practical purposes.

Among the vectors satisfying (34) are the vectors $\mathbf{x}_M$ that minimize the functional $M(\alpha, \mathbf{x})$. If there exists at least one solution that satisfies (34), then the vector $\mathbf{x}_M$ will satisfy (34) as well (Ref. 31, lemma 4). Therefore, Eq. (34) forms an a posteriori test for the correctness of the bounds ϵ and E . If the vector $\mathbf{x}_M$ satisfies the more strict constraint $M(\alpha, \mathbf{x}) \leq \epsilon^2$, then $\mathbf{x}_M$ satisfies (28) and (29).

The necessary condition for $M(\alpha, \mathbf{x})$ to have a minimum is that its gradient with respect to $\mathbf{x}$ be equal to zero, which results in the equation

$$(\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}) \mathbf{x} = \mathbf{D}^T \tilde{\mathbf{y}}. \quad (35)$$

The sufficient condition for $M(\alpha, \mathbf{x})$ to have a minimum is that the Hessian of $M(\alpha, \mathbf{x})$, which is equal to $\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}$, be positive definite.

Equation (35) also represents the CLS filter of Eq. (27), where α is a Lagrange multiplier. One of the advantages of Miller's method over the CLS method is that there is no need for the iterative determination of the Lagrange multiplier. This is due to the assumption of additional knowledge, namely, of the ratio of the bounds ϵ and E . An additional advantage of the functional minimization approach discussed in this section over the POCS approach is that only the ratio (ϵ/E) is required and not the individual bounds ϵ and E .

3.3. Formulation of the algorithm

In solving Eq. (35), any matrix inversion technique or a technique for solving for $\mathbf{x}$ directly can be used.¹⁰ If the matrices $\mathbf{D}$ and $\mathbf{C}$ are circulant, then the solution can be obtained in the discrete frequency domain by using the discrete Fourier transform (DFT). However, the dimensions of the matrices and vectors involved are usually large when dealing with images. Furthermore, with direct methods it is usually impossible to incorporate knowledge about the solution into the algorithm. Therefore, an iterative approach to solving Eq. (35) is more attractive. The successive approximation of the solution according to the constrained basic iterative algorithm of Sec. 2.3 results in³⁴

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{D}^T \tilde{\mathbf{y}}, \\ \mathbf{x}_{k+1} &= (\mathbf{I} - \beta \alpha \mathbf{C}^T \mathbf{C}) \tilde{\mathbf{x}}_k + \beta \mathbf{D}^T (\tilde{\mathbf{y}} - \mathbf{D} \tilde{\mathbf{x}}_k) \\ &= \mathbf{C}_S \tilde{\mathbf{x}}_k + \beta \mathbf{D}^T (\tilde{\mathbf{y}} - \mathbf{D} \tilde{\mathbf{x}}_k), \\ \tilde{\mathbf{x}}_k &= \mathbf{P} \mathbf{x}_k, \end{aligned} \quad (36)$$

where $\mathbf{C}_S = \mathbf{I} - \beta \alpha \mathbf{C}^T \mathbf{C}$. Iteration (36) is henceforth referred to as the *regularized iteration*. The constraint $\mathbf{C}_S$ is called a *soft* or *statistical* constraint since it depends on the regularization parameter α , which in turn depends on the amount of noise on the data.^{2,20} $\mathbf{C}_S$ is not, in general, a projection operator. For noise-free data, $\mathbf{C}_S$ is disabled and iteration (36) reduces to the basic constrained algorithm with reblurring, discussed in Sec. 2.3. Iteration (36) is the generalization of the iteration discussed by Schafer et al.⁴ since the information about the noise is incorporated directly into the iteration by the operator $\mathbf{C}_S$. The conditions for convergence of iteration (36) were discussed in Sec. 2.

In defining α we propose the use of $E = \|\mathbf{C}\|_2 \cdot \|\mathbf{x}\|_2$ since $\|\mathbf{C}\mathbf{x}\|_2 \leq \|\mathbf{C}\|_2 \cdot \|\mathbf{x}\|_2$. Then

$$\begin{aligned} \alpha &= \left(\frac{\epsilon}{E} \right)^2 = \left(\frac{\|\mathbf{n}\|_2}{\|\mathbf{C}\|_2 \cdot \|\mathbf{x}\|_2} \right)^2 = \|\mathbf{C}\|_2^{-2} \cdot \frac{\sigma_n^2}{\sigma_x^2 + \bar{\mathbf{x}}^2} \\ &= \|\mathbf{C}\|_2^{-2} \cdot \frac{1}{\text{SNR} + (\bar{\mathbf{x}}^2 / \sigma_n^2)}, \end{aligned} \quad (37)$$

where σ_x^2 and σ_n^2 are, respectively, the variances of the signal and the zero mean noise and $\bar{\mathbf{x}}$ is the mean of $\mathbf{x}$. The constraint $\mathbf{C}$ is usually normalized so that $\|\mathbf{C}\|_2 = 1$. Then according to Eq. (37), α is inversely proportional to the SNR, a quantity that is known or that can be computed from the data.¹

3.4. Properties and choice of the constraint operator $\mathbf{C}$

The matrix $\mathbf{C}$ in condition (29) should be chosen to describe some known properties of the original signal while rendering the resulting system of linear equations (35) better conditioned than the original system [Eq. (1)] or the system of normal equations [Eq. (35) with $\alpha = 0$]. The following definition of the P-condition number is used:

$$\mathbf{P}(\mathbf{D}) = \|\mathbf{D}\|_2 \cdot \|\mathbf{D}^+\|_2, \quad (38)$$

where $\|\cdot\|_2$ denotes the ℓ_2 norm. Then, $\mathbf{C}$ should be chosen in such a way that the condition numbers satisfy

$$\mathbf{P}(\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}) < \mathbf{P}(\mathbf{D}), \quad (39)$$

or in the worse case

$$\mathbf{P}(\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}) < \mathbf{P}(\mathbf{D}^T \mathbf{D}), \quad (40)$$

since $\mathbf{P}(\mathbf{D}^T \mathbf{D}) > \mathbf{P}(\mathbf{D})$ due to $\mathbf{P}(\mathbf{D}) > 1$. Conceptually, what inequalities (39) and (40) tell us is that it would be desirable for the constraint matrix $\mathbf{C}$ to leave the large singular values of $\mathbf{D}$ unchanged while it moves the small singular values of $\mathbf{D}$ away from zero without introducing new small singular values. According to this, $\mathbf{C}$ should be a singular matrix so that the large singular values of $\mathbf{D}$ are not altered. Therefore, if $\mathbf{C}$ is chosen as discussed above, r' , the dimensionality of the range of $\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}$, will be at least as large as r , the dimensionality of the range of $\mathbf{D}^T \mathbf{D}$.

Conditions (39) and (40) are easily verified if we assume that the matrices $\mathbf{D}^T \mathbf{D}$ and $\mathbf{C}^T \mathbf{C}$ commute, which means that the two matrices have the same complete set of eigenvectors.¹⁰ Then

$$\mathbf{P}(\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{C}) = \frac{\max_i (\mu_i^2 + \alpha \sigma_i^2)}{\min_i (\mu_i^2 + \alpha \sigma_i^2)}. \quad (41)$$

In this case, the minimum norm least-squares solution of Eq. (35), $\mathbf{x}_M$, is equal to²

$$\mathbf{x}_M = \sum_{i=1}^{r'} \frac{\mu_i}{\mu_i^2 + \alpha \sigma_i^2} (\mathbf{u}_i, \mathbf{y}) \mathbf{v}_i + \sum_{i=1}^{r'} \frac{\mu_i}{\mu_i^2 + \alpha \sigma_i^2} (\mathbf{u}_i, \mathbf{n}) \mathbf{v}_i. \quad (42)$$

Equation (42) should be compared with Eq. (24) in describing the effect of the regularization.

The assumption that the matrices $\mathbf{D}^T \mathbf{D}$ and $\mathbf{C}^T \mathbf{C}$ commute may be somewhat restrictive, although it holds for a large class of problems of interest, namely, when $\mathbf{D}$ and $\mathbf{C}$ represent linear space-invariant systems. In this case, the matrices $\mathbf{D}^T \mathbf{D}$

and $C^T C$ can be represented by block circulant matrices and Eq. (35) takes the form^{12,35}

$$(|D(u, v)|^2 + \alpha |C(u, v)|^2) X(u, v) = D^*(u, v) \tilde{Y}(u, v), \quad (43)$$

where the notation introduced in Eq. (9) is used. From Eq. (43) the desirable properties of the constraint become clear. That is, since μ_i is a decreasing sequence with i , σ_i should be an increasing sequence with i . In other words, if $D(u, v)$ is the frequency response of a low-pass filter (a very common type of distortion), then $C(u, v)$ must be the frequency response of a high-pass filter. Conceptually, the constraint C is chosen so that the energy of the restored signal at high frequencies, mainly due to the amplification of the broadband noise, is suppressed. In general, if $D(u, v)$ is a bandpass filter, $C(u, v)$ must be a bandstop filter, with their stop- and passbands interchanged.

In general, since $C^T C$ is chosen by the designer, he/she can require that it has the same eigenvectors with $D^T D$. However, if $D^T D$ does not have a special structure, such as the block Toeplitz structure, the determination of the eigenvalues and eigenvectors of a large matrix may be computationally difficult. Therefore, if the above analysis is difficult or if it is chosen to avoid it, a smoothness requirement on the solution can be imposed by requiring that C be a high-pass filter. In agreement with this, C was chosen to be a p th order differential operator (high-pass filter) when the CLS method was implemented.^{12,28-30,32}

A heuristic approach is taken next in proposing a constraint C_s instead of C . According to the preceding discussion, C should represent a high-pass filter due to the low-pass nature of most distortions or due to a smoothness requirement on the solution. Therefore, $C_s = I - \beta \alpha C^T C$ should represent a low-pass filter. To be successful in suppressing the noise without severely distorting the original signal, information about the smoothness of the original signal is necessary. This information can be incorporated into the algorithm by the signal and noise covariance matrices. Based on this argument, the use of a Wiener noise smoothing filter is proposed as the constraint C_s . The determination of α is part of this choice for C_s since α depends on the SNR according to Eq. (37). Although this choice for C_s assumes a stochastic formulation of the restoration problem, we are not using complicated spectrum estimation techniques in defining the Wiener filter, as is explained in Sec. 5.

3.5. Analysis of the algorithm and comparison with other restoration approaches

In this section the solution obtained by the proposed regularized iterative algorithm is geometrically characterized. The algorithm is also compared with the restoration methods of CLS and POCS. Figure 3 depicts the two ellipsoids Q_1 and Q_2 , represented by conditions (30) and (31), respectively, in a two-dimensional space. Their respective centers are $\mathbf{x}^+ = D^+ \tilde{\mathbf{y}}$ and $\mathbf{x} = 0$, corresponding to $\alpha = 0$ and $\alpha = \infty$. Their intersection, denoted by Q_0 , represents the set of feasible solutions to the restoration problem. Q_0 is not necessarily an ellipsoid; therefore, it is difficult to characterize. Instead, an ellipsoid bounding Q_0 can be used. Such a bounding ellipsoid Q_b is defined by³⁶

$$Q_b = \{\mathbf{x}: (\mathbf{x} - \mathbf{c}_b)^T \Gamma_b^{-1} (\mathbf{x} - \mathbf{c}_b) \leq 1\}. \quad (44)$$

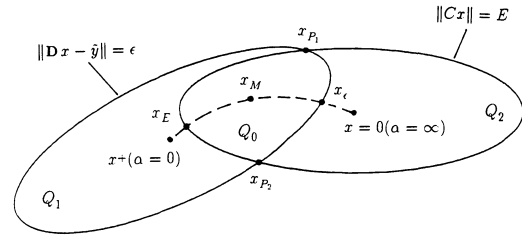


Fig. 3. Representation of the intersection of the two ellipsoids Q_1 and Q_2 .

For the ellipsoids Q_1 and Q_2 , it is easily found that¹⁴

$$\mathbf{c}_b = \left[D^T D + \frac{\gamma_2}{\gamma_1} \left(\frac{\epsilon}{E} \right)^2 C^T C \right]^{-1} D^T \tilde{\mathbf{y}}, \quad (45)$$

$$\Gamma_b^{-1} =$$

$$\left[\frac{\epsilon^2}{\gamma_1} - \|\tilde{\mathbf{y}}\|^2 + (\tilde{\mathbf{y}}, D\mathbf{c}_b) \right]^{-1} \left[D^T D + \frac{\gamma_2}{\gamma_1} \left(\frac{\epsilon}{E} \right)^2 C^T C \right], \quad (46)$$

where $\gamma_1 + \gamma_2 \leq 1$ and $(\mathbf{z}, \mathbf{w})$ denotes the inner product of the two vectors $\mathbf{z}$ and $\mathbf{w}$. For $\gamma_1 = \gamma_2 = 1/2$, the center $\mathbf{c}_b$ of Eq. (45) is the solution of Eq. (35); that is, $\mathbf{x}_M = \mathbf{c}_b$. It is easily verified that in this case the bounding ellipsoid of Eq. (44) and the ellipsoid represented by condition (34) coincide. That is, the Miller solution is the center of the ellipsoid represented by condition (34) that bounds the intersection Q_0 . The solutions $\mathbf{x}_E$, $\mathbf{x}_{E2}$, $\mathbf{x}_{P1}$ and $\mathbf{x}_{P2}$ are also shown in Fig. 3. The vectors $\mathbf{x}_E$ and $\mathbf{x}_{E2}$ represent CLS solutions. In the first case only ϵ is known, while in the second case only E is known. Any vector belonging on the boundary of Q_0 can be obtained as a fixed point of the POCS restoration provided that $\mathbf{x}_0 \notin Q_0$, where $\mathbf{x}_0$ is the initial estimate of $\mathbf{x}$. The vectors $\mathbf{x}_{P1}$ and $\mathbf{x}_{P2}$ are two of these fixed points. If $\mathbf{x}_0 \in Q_0$, then $\mathbf{x}_0$ is the solution. By varying α continuously from zero to infinity, the vectors on the dashed curve in Fig. 3 can be obtained as the minimizers of $M(\alpha, \mathbf{x})$ in Eq. (34). Therefore, depending on the particular restoration method used, any vector belonging in the set of feasible solutions Q_0 can be obtained.

The obvious question then is which of the feasible solutions is optimum. This is an unanswered question at this point from a theoretical point of view. Clearly, the smaller the size of Q_0 , the closer the different solutions are to each other. On the other hand, the noisier the data, the larger the intersection Q_0 since Q_1 becomes larger, even though Q_2 may remain unchanged. The true solution is on the boundary of Q_1 . Therefore, $\mathbf{x}_E$, $\mathbf{x}_{P1}$, and $\mathbf{x}_{P2}$ have a better chance of being optimum solutions. However, since the location of the true solution is not known and depends on $\tilde{\mathbf{y}}$, solutions inside Q_0 may be preferable to the solutions on the common boundary of Q_1 and Q_0 .

A more detailed geometric analysis of the convex sets describing the regularized solution is presented in Ref. 14. For example, a measure of the size of Q_0 by considering the bounding ellipsoid Q_b is obtained, questions regarding the nonintersection of Q_1 and Q_2 are discussed, and the characterization of the regularized solution when more than two convex sets are involved is presented.

4. TWO ADAPTIVE REGULARIZED IMAGE RESTORATION ALGORITHMS

Although the most tractable criterion—mean squared error—has formed the basis for most of the published work in image restoration, this criterion does not agree with the way the human visual system functions. The resulting restoration filter is low-pass and gives rise to unacceptable blurring of lines and edges in the image. When the human observer is the receiver of the restored image, the properties of the visual system should be incorporated into the restoration algorithm to obtain visually optimal results. The response of the visual system can be incorporated into the soft constraint introduced in the previous section. The properties of the visual system are represented by noise masking and visibility functions, which are briefly discussed next.

4.1. Properties of the human visual system

Psychophysical experiments confirm that noise in flat regions of the image will give rise to spurious features or textures to the observer and that at sharp transitions in image intensity, the contrast sensitivity of the human visual system decreases with the sharpness of the transition and increases approximately exponentially as a function of spatial distance from the transition.^{37,38} This masking effect in the visual system results in lower noise visibility in the vicinity of edges.

Based on this information, Anderson and Netravali³⁷ first defined the noise masking function $M(i, j)$ at coordinate (i, j) as a measure of spatial detail. Then they performed subjective tests and obtained the visibility function $f(i, j)$ at coordinate (i, j) , which expresses the relationship between the visibility of noise and the masking function.

The use of the local variance as a measure of the spatial detail was proposed in Refs. 39 and 40. For an image $x(i, j)$, the local mean $m_x(i, j)$ and local variance $\alpha_x^2(i, j)$ at coordinate (i, j) are defined as

$$m_x(i, j) = \frac{1}{(2P+1)(2Q+1)} \sum_{m=i-P}^{i+P} \sum_{n=j-Q}^{j+Q} x(m, n), \quad (47)$$

$$M(i, j) = \alpha_x^2(i, j) = \frac{1}{(2P+1)(2Q+1)} \times \sum_{m=i-P}^{i+P} \sum_{n=j-Q}^{j+Q} [x(m, n) - m_x(i, j)]^2, \quad (48)$$

where $(2P+1)(2Q+1)$ is the extent of the analysis window, which is symmetric about the point (i, j) . The local variance, which has been used for contrast stretching and noise filtering in a different context,^{38,41} is a good choice for a generalized masking function since it does not distinguish among the vertical, horizontal, or any orientation slopes or between positive and negative slopes.

Following Ref. 37, the visibility function is now defined as

$$f(i, j) = \frac{1}{\theta M(i, j) + 1}, \quad (49)$$

where θ is a tuning parameter that must be adjusted experimentally for each class of images. The visibility function is normalized and takes values between zero and one. It is clear from Eq. (49) that for the areas with high spatial activity [large

value of $M(i, j)$], the visibility function goes to zero (noise is not visible), while for flat areas [$M(i, j)$ small], the visibility function goes to one (noise is visible). The information provided by the visibility function has been also used by Anderson and Netravali³⁷ and Knutsson et al.⁴² in designing noise smoothing filters and by Rajala and deFigueiredo⁴³ in designing restoration filters.

4.2. Algorithm I

An adaptive iterative regularized constrained image restoration algorithm is derived in this section.^{2,39} According to this approach, each pixel is assigned to one of L classes R_i , $i = 1, \dots, L$, based on its value of the visibility function defined by Eq. (49). That is, the (m, n) th pixel is assigned to class R_i if $b_{i-1} < f(m, n) \leq b_i$, where the constants b_i , $i = 0, \dots, L$, need to be specified. Then for the class R_i the following problem is solved: search for vectors $\mathbf{x}$ that satisfy the constraints

$$\|\mathbf{I}_i(\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}})\| \leq \epsilon_i = f_i \epsilon \left(\frac{N_i}{N^2} \right), \quad (50)$$

$$\|\mathbf{I}_i \mathbf{C} \mathbf{x}\| \leq E \left(\frac{N_i}{N^2} \right), \quad (51)$$

where N^2 is the total number of pixels in the image; N_i is the number of pixels in the class R_i ; $\mathbf{I}_i$ is an $N^2 \times N^2$ diagonal matrix with $(N^2 - N_i)$ elements equal to 0 and N_i elements equal to 1 at the corresponding locations of the pixels that belong in R_i ; and f_i is the value of the visibility function associated with the class R_i . The functional minimization approach presented in Sec. 3.2.2.2 is followed again here in finding an $\mathbf{x}$ that satisfies conditions (50) and (51). That is, the functional

$$M(\alpha_i, \mathbf{x}) = \|\mathbf{I}_i(\mathbf{D}\mathbf{x} - \tilde{\mathbf{y}})\|^2 + \alpha_i \|\mathbf{I}_i \mathbf{C} \mathbf{x}\|^2 \quad (52)$$

is minimized, where $\alpha_i = f_i^2 (\epsilon/E)^2 = f_i^2 \alpha$. The necessary condition for a minimum results in

$$(\mathbf{D}^T \mathbf{I}_i^T \mathbf{I}_i \mathbf{D} + \alpha_i \mathbf{C}^T \mathbf{I}_i^T \mathbf{I}_i \mathbf{C}) \mathbf{x} = \mathbf{D}^T \mathbf{I}_i^T \mathbf{I}_i \tilde{\mathbf{y}}. \quad (53)$$

There are infinitely many $\mathbf{x}$'s that satisfy (53) since the entries of $\mathbf{x}$ that do not belong to R_i (except from a number of entries that constitute the boundaries of R_i , whose thickness is determined by the extent of the distortion) can be equal to an arbitrary number. However, such an $\mathbf{x}$ has to satisfy L different equations of the form of (53) since $i = 1, \dots, L$. Therefore, by combining the L different equations we get (notice that $\mathbf{I}_i^T \mathbf{I}_i = \mathbf{I}_i$)

$$\begin{aligned} \mathbf{D}^T (\mathbf{I}_1 + \dots + \mathbf{I}_L) \mathbf{D} \mathbf{x} + \alpha \mathbf{C}^T (f_1^2 \mathbf{I}_1 + \dots + f_L^2 \mathbf{I}_L) \mathbf{C} \mathbf{x} \\ = \mathbf{D}^T (\mathbf{I}_1 + \dots + \mathbf{I}_L) \tilde{\mathbf{y}}. \end{aligned} \quad (54)$$

We define the visibility matrix $\mathbf{F}$ as

$$\mathbf{F}^T \mathbf{F} = f_1^2 \mathbf{I}_1 + \dots + f_L^2 \mathbf{I}_L, \quad (55)$$

and Eq. (54) results in

$$(\mathbf{D}^T \mathbf{D} + \alpha \mathbf{C}^T \mathbf{F}^T \mathbf{F} \mathbf{C}) \mathbf{x} = \mathbf{D}^T \tilde{\mathbf{y}} \quad (56)$$

since $\mathbf{I} = \mathbf{I}_1 + \dots + \mathbf{I}_L$. Clearly, the number of classes L can become as large as the dimension of the vector $\mathbf{x}$. The basic iterative algorithms of Secs. 2.1 and 2.3 applied to Eq. (56) result in the iteration^{2,39}

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{D}^T \tilde{\mathbf{y}}, \\ \mathbf{x}_{k+1} &= (\mathbf{I} - \beta \alpha \mathbf{C}^T \mathbf{F}^T \mathbf{F} \mathbf{C}) \tilde{\mathbf{x}}_k + \beta \mathbf{D}^T (\tilde{\mathbf{y}} - \mathbf{D} \tilde{\mathbf{x}}_k) \\ &= \mathbf{C}_s \tilde{\mathbf{x}}_k + \beta \mathbf{D}^T (\tilde{\mathbf{y}} - \mathbf{D} \tilde{\mathbf{x}}_k), \\ \tilde{\mathbf{x}}_k &= \mathbf{P} \mathbf{x}_k. \end{aligned} \quad (57)$$

Clearly, the computation of the visibility matrix $\mathbf{F}$ should be based on the restored image. One could obtain a restored image first with the use of any restoration algorithm, compute $\mathbf{F}$, and then run the adaptive algorithm. A less computationally expensive alternative is to obtain an $\mathbf{F}$ at each iteration based on the available form of the restored image. Then $\mathbf{F}$ should be replaced by $\mathbf{F}_k$ and $\mathbf{C}_s$ by $\mathbf{C}_{s,k}$ in iteration (57). Equation (56) has also been derived in Refs. 44 and 45 with the use of weighted norms. Another weight matrix besides $\mathbf{F}$ is used in their formulation, as is done in Ref. 46.

4.3. Algorithm II

A heuristic approach based on the "signal equivalent approach"⁴⁷ for removing the additive noise for a blur-free image is followed here in deriving an adaptive restoration algorithm. According to this approach, the additive noise is decomposed into a visible and a nonvisible part with the use of the visibility function. Then the signal to be recovered is the sum of the undistorted signal and the nonvisible noise part. The resulting constraint $\mathbf{C}_{s,k}$ in Eq. (57) is⁴⁰

$$\mathbf{C}_{s,k}(u, v, i, j) = f_k(i, j) \cdot W(u, v) + [1 - f_k(i, j)], \quad (58)$$

where $\mathbf{C}_{s,k}(u, v, i, j)$ denotes the DFT of the soft constraint at the spatial location (i, j) , and $W(u, v)$ is the frequency response of the stationary Wiener filter proposed to be used as a soft constraint in Sec. 3.4. Clearly, the constraint is disabled at the edges while in the flat regions it becomes the stationary Wiener noise smoothing filter. Other approaches in adapting $\mathbf{C}_{s,k}$ to the local spatial activity with the use of the local variance or the visibility function are discussed in Ref. 2.

5. EXPERIMENTAL RESULTS

In this section we present experimental results obtained with the use of the nonadaptive and adaptive regularized iterative algorithms. Although the criterion we used for terminating the iteration was given in experiment 1, the iteration could also be terminated as soon as a vector in the intersection of the sets Q_1 and Q_2 is found, provided that these sets are appropriately chosen. Alternatively, an interactive approach can be followed, according to which the iteration is terminated when the visually best solution is obtained. We consider the possibility of obtaining different solutions, depending on the convergence criterion used, an advantage of iterative restoration techniques.

When a Wiener noise smoothing filter was used as a soft constraint, a commonly used bistationary, separable covariance image function was assumed.¹ Then, since it is assumed that the SNR is available (or directly estimated from the data),



(a)



(b)



(c)



(d)

Fig. 4. (a) Noisy motion blurred image; $L = 9$, $\text{SNR} = 20$ dB. (b)–(d) Restorations of Fig. 4(a) by the regularized algorithm: (b) $c_s(0) = 0.9$ and $c_s(-1) = c_s(1) = 0.05$; $\Delta_{\text{SNR}} = 7.42$ dB. (c) $\mathbf{C}$ is a 2-D Laplacian filter; $\Delta_{\text{SNR}} = 7.96$ dB. (d) $\mathbf{C}$ was defined by the relation $\mathbf{C}(u, v) = \mathbf{D}(u, v) - 1$; $\Delta_{\text{SNR}} = 6.67$ dB.

the impulse response of the Wiener filter was computed and normalized in advance for different SNRs and different values for the vertical and horizontal correlation coefficients. Depending on the value of the SNR, a small number of different filters was tried. One of the restored images was



Fig. 5. Image of Fig. 2(a) restored by the regularized iterative algorithm; impulse response of the soft constraint: $c_s(0) = 0.7$ and $c_s(-1) = c_s(1) = 0.15$; $\Delta_{\text{SNR}} = 5.6$ dB.

selected on the basis of its visual quality. The parameter θ in computing the visibility function in Eq. (49) and the constants P and Q in computing the local variance in Eq. (48) were adjusted experimentally.

When the original undistorted image is available in a simulation experiment, the performance of the restoration iteration can be evaluated by measuring the improvement in SNR, Δ_{SNR} , after k iterations. It is defined by

$$\Delta_{\text{SNR}} = 10 \log_{10} \frac{\|\tilde{y} - x\|_2^2}{\|x_k - x\|_2^2}. \quad (59)$$

Distortion due to motion was simulated according to Eq. (21) with $L = 9$. Noise was added, resulting in SNRs of 10, 20, and 30 dB. The blurred images with SNR = 10 dB and 20 dB are shown in Figs. 2(a) and 4(a), respectively.

Restorations by the regularized algorithm of Eq. (36) are considered first. The restored image of Fig. 2(a) is shown in Fig. 5 with the truncated and normalized impulse response of the 1-D Wiener filter equal to $c_s(0) = 0.7$ and $c_s(-1) = c_s(1) = 0.15$. The SNR improvement is 5.6 dB. The noise amplification is controlled, as can be seen by comparing Figs. 2(b) and 5. Three restorations of Fig. 4(a) are shown in Figs. 4(b) through 4(d). Figure 4(b) was obtained with $c_s(0) = 0.9$ and $c_s(-1) = c_s(1) = 0.05$. The SNR improvement is 7.42 dB. The simplicity of the 1-D Wiener filter makes it very attractive from a computational point of view. A 2-D Laplace filter¹ was used next as a constraint C , in obtaining Fig. 4(c), with $\Delta_{\text{SNR}} = 7.96$ dB. Finally, a restoration example is shown when the constraint C is chosen according to the analysis of Sec. 3.4. The simple relation $C(u, v) = D(u, v) - 1$ was used in defining C . A restoration of Fig. 4(a) is shown in Fig. 4(d), with $\Delta_{\text{SNR}} = 6.67$ dB.

Results obtained by the adaptive iterative algorithms are shown next. Restorations of Figs. 2(a) and 4(a) obtained by the adaptive algorithm with the soft constraint adjusted according to Eq. (58) are shown, respectively, in Fig. 6 ($\Delta_{\text{SNR}} = 6.8$ dB) and Fig. 7(a) ($\Delta_{\text{SNR}} = 7.9$ dB). A 2-D Wiener filter with 7×7 support was used in obtaining Fig. 7(a). Finally, a restoration of Fig. 4(a) obtained by the adaptive algorithm (57) with C a Laplacian is shown in Fig. 7(b) ($\Delta_{\text{SNR}} = 8.29$ dB). Table I lists the improvements for the various SNRs, the values of α and θ used, and the number of iterations run. The parameters P and Q were equal to 1 in all of the experiments. Table I also lists the values of $P(D^T D)$ and



Fig. 6. Image of Fig. 2(a) restored by the adaptive regularized algorithm of Eq. (58); $\Delta_{\text{SNR}} = 6.8$ dB.



(a)



(b)

Fig. 7. (a) Image of Fig. 4(a) restored by the adaptive regularized algorithm of Eq. (58) and a 2-D Wiener filter; $\Delta_{\text{SNR}} = 7.9$ dB. (b) Image of Fig. 4(a) restored by the adaptive regularized algorithm of Eqs. (57); $\Delta_{\text{SNR}} = 8.29$ dB.

$P(D^T D + \alpha C^T C)$, demonstrating that condition (40) is satisfied.

As a general comment, the size of the intersection is changed by varying α . As α increases, the size of the intersection decreases and the restored image becomes smoother (closer to the CLS solution). On the other hand, as α decreases, the size of the intersection increases and the restored image becomes noisier and closer to the minimum norm least-squares solution x^+ . For low SNRs the α 's chosen according to Eq. (37) result in very noisy solutions (large intersection), and a larger α may need to be chosen.

As a final example we show the restoration of a real photographically blurred image. The motion blurred image, Fig. 8(a), was obtained through the courtesy of Kodak Research Laboratories. The blur is one-dimensional across each image

TABLE I. Improvement for various SNRs and constraints by using the nonadaptive and adaptive regularized iterative restoration algorithms.

SNR(dB)	Nonadaptive					adaptive		
	$C_S = \text{Wiener}$	C = Laplacian						
	$\Delta_{SNR}(k)$	$\Delta_{SNR}(k)$	$P(D^T D)$	$P(D^T D + \alpha C^T C)$	$M(\alpha, \mathbf{x}_M)$	$\Delta_{SNR}(k)$	α	θ
10	5.60(19)	6.13(42)	.223x10 ⁶	.300x10 ³	.116x10 ⁸	5.84(43)	2.0	0.001
20	7.42(23)	7.96(36)	.223x10 ³	.996x10 ³	.111x10 ⁷	8.29(36)	0.5	0.001
30	9.61(43)	9.62(44)	.223x10 ⁶	.849x10 ⁵	.817x10 ⁵	9.67(43)	0.0017	0.1

line; therefore, there are undistorted and distorted image lines only. The length of the blur L in Eq. (21) was estimated to be equal to nine pixels using a cepstrum-like technique,¹ while the SNR was estimated to be equal to 20 dB. A restoration by iteration (36) is shown in Fig. 8(b).

6. CLASS OF HIGHER ORDER ITERATIVE ALGORITHMS

One of the drawbacks of the iterative algorithms presented in the previous sections is their linear rate of convergence. Therefore, if the available processing time is of central importance, algorithms with higher convergence rates or efficient implementations of the linear algorithms are of interest.

A number of iterative algorithms with various rates of convergence have appeared in the literature in recent years.⁴⁸⁻⁵⁰ In Refs. 18 and 19 a unified approach is presented in obtaining a class of iterative algorithms with different rates of convergence, based on a representation of the generalized inverse of a matrix. That is, the algorithm

$$x_0 = \beta D^T y,$$

$$D_0 = \beta D^T D,$$

$$\Phi_k = \sum_{i=0}^{p-1} (I - D_k)^i, \quad (60)$$

$$D_{k+1} = \Phi_k D_k,$$

$$x_{k+1} = \Phi_k x_k,$$

converges to the minimum norm least-squares solution of Eq. (1), with $n = 0$. Algorithm (60) is not restricted to the noiseless case. It can be applied in solving Eq. (35), in which case D is substituted by $(D^T D + \alpha C^T C)$. If iteration (60) is thought of as corresponding to the basic iteration with reblurring [Eq. (16)], then an iteration similar to (60) that corresponds to the basic iteration without reblurring [Eq. (3)] has also been derived.^{18, 19, 50}

Algorithm (60) exhibits a p th order of convergence. That is, the following relation holds¹⁹:

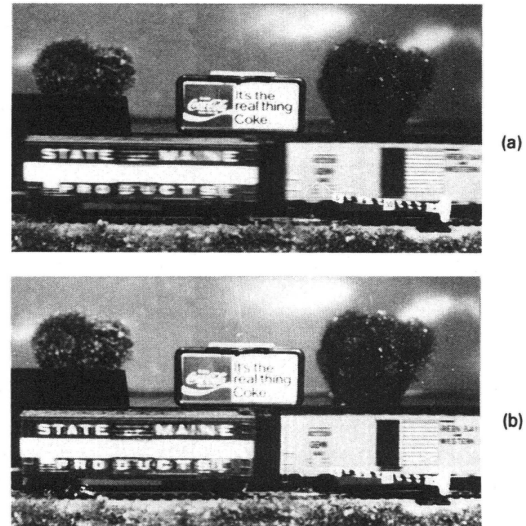


Fig. 8. (a) Noisy blurred image: estimated $L = 9$ and SNR = 20 dB. (b) Image restored by the regularized algorithm.

$$\frac{\|x_k - x^+\|}{\|x^+\|} < c^{p^k}, \quad (61)$$

where the convergence factor c is described by Eq. (18). Figure 9 shows curves, based on a particular experiment, of the normalized residual error for the linear algorithm and the higher order algorithms for different values of p . The number of iterations required by each algorithm to reach the same solution point are easily compared.

It is observed that the matrix sequences $\{\Phi_k\}$ or D_k can be computed in advance or *off-line*. When the matrix D is block circulant, substantial computational savings result by using iteration (60) over the linear algorithms. Questions dealing with the best order p of algorithm (60) that should be used in a given application, as well as comparisons of the tradeoff between speed of computation and computational load, are addressed in Refs. 18, 19, and 50. One drawback of the higher order algorithms is that the constraints are not very useful

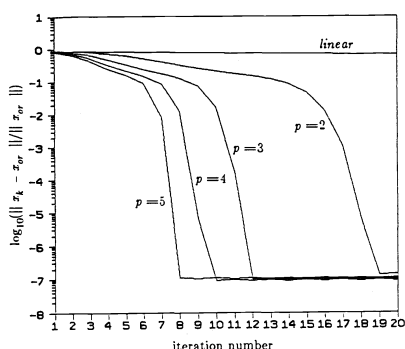


Fig. 9. Normalized residual error versus number of iterations for the linear and the higher order algorithms.

since their application may lead to erroneous results. Combined adaptive or nonadaptive linear and higher order algorithms have been proposed in overcoming this difficulty.¹⁹

7. MULTISTEP ITERATIVE IMAGE RESTORATION ALGORITHM

In the iterative algorithms presented in the previous sections the evaluation of the restored image at the k th iteration step depends only on the result of the previous iteration. Therefore, these algorithms are called single-step iterative algorithms. If $\mathbf{x}_k$ depends on the result of more than one previous iteration, then we are referring to a multistep iterative algorithm. The principal motivation in studying such an algorithm is that it exhibits desirable characteristics for VLSI implementation.⁵¹ Such characteristics are the localized data transactions and the fact that it requires considerably less time to implement each step of the multistep iteration than the single-step iteration. Let us rewrite iteration (36) as

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{D}^T \tilde{\mathbf{y}}, \\ \mathbf{x}_k &= (\mathbf{I} - \beta \mathbf{D}^T \mathbf{D} - \alpha \beta \mathbf{C}^T \mathbf{C}) \mathbf{x}_{k-1} + \beta \mathbf{D}^T \tilde{\mathbf{y}} \\ &= \mathbf{W} \mathbf{x}_{k-1} + \mathbf{b}, \end{aligned} \quad (62)$$

where $\mathbf{W} = \mathbf{I} - \beta \mathbf{D}^T \mathbf{D} - \alpha \beta \mathbf{C}^T \mathbf{C}$ and $\mathbf{b} = \beta \mathbf{D}^T \tilde{\mathbf{y}}$. A number of multistep iterations can be derived from the single-step iteration (62). We assume without loss of generality that the impulse responses of the distortion and the constraint filters denoted respectively by $d(i,j)$ and $c(i,j)$ and forming the matrices $\mathbf{D}$ and $\mathbf{C}$ have support $(2L+1) \times (2P+1)$ pixels, where $L \geq P$. Then the impulse response of the composite filter $w(i,j)$, $i = -2L, \dots, 2L$, $j = -2P, \dots, 2P$ in Eqs. (62) is equal to

$$w(i,j) = \delta(i,j) - \beta d(-i,-j) ** d(i,j) - \beta \alpha c(-i,-j) ** c(i,j). \quad (63)$$

The following additive decomposition of $w(i,j)$ has been proposed⁵¹:

$$w(i,j) = w_1(i,j) + w_2(i,j) + \dots + w_{2L}(i,j), \quad (64)$$

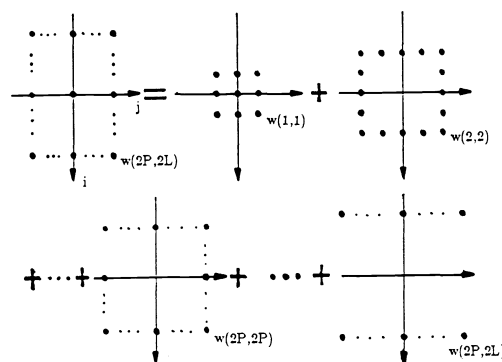


Fig. 10. Decomposition of the impulse response of the restoration filter.

where the functions $w_\ell(i,j)$, $\ell = 1, \dots, 2L$, are depicted in Fig. 10. Then iteration (62) takes the form

$$\begin{aligned} \mathbf{x}_0 &= \beta \mathbf{D}^T \tilde{\mathbf{y}}, \\ \mathbf{x}_k &= \mathbf{W}_1 \mathbf{x}_{k-1} + \mathbf{W}_2 \mathbf{x}_{k-2} + \dots + \mathbf{W}_{2P} \mathbf{x}_{k-2P} + \dots \\ &\quad + \mathbf{W}_{2L} \mathbf{x}_{k-2L} + \mathbf{b}. \end{aligned} \quad (65)$$

The convergence and the rate of convergence of iteration (65) are studied in Ref. 51. In general, the convergence of the single-step iteration does not guarantee the convergence of the multistep algorithm.⁵²

8. SUMMARY

This tutorial paper has discussed many recent developments in the field of iterative image restoration. Although it has concentrated on the removal of linear, spatially invariant distortions (deconvolution), the iterative algorithms presented can be used for the removal of spatially varying as well as nonlinear distortions.^{4,53} This paper discussed the relationship between the basic iterative algorithm and various other methods and its extensions to adaptive and nonadaptive regularized iterative algorithms and iterative algorithms with higher convergence rates. Furthermore, a multistep iteration was presented that is suitable for VLSI implementation.

The example of Fig. 8 brings up an important point: to make image restoration applicable to practical situations, the unknown blurs have to be estimated from the noisy blurred images themselves. Furthermore, these blurs, even if they are locally spatially invariant [as is the case in Fig. 8(a)], are usually spatially varying if the whole image is considered. Therefore, a shift has occurred from the pure image restoration problem toward the combined blur identification and restoration problem.

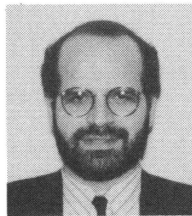
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