

Super-Resolution of Images: Algorithms, Principles, Performance

B. R. Hunt

Department of Electrical and Computer Engineering, University of Arizona, Tucson, AZ 85721

ABSTRACT

A new facet of image restoration research has begun to emerge in recent years: super-resolution of images, which we define as the processing of an image so as to recover object information from beyond the spatial frequency bandwidth of the optical system that formed the image. Simple Fourier analysis would indicate that super-resolution is not possible. Therefore, it is important to reconcile this simplistic view with the existing algorithms that have been demonstrated to achieve super-resolution. In this article, we consider some of the algorithms that have demonstrated super-resolution and discuss the common principles that they share which makes it possible for them to recover some of the lost bandwidth of the object. We also consider the question of super-resolution performance, which is the measure of how much lost bandwidth can be recovered from a super-resolution algorithm, and how the performance is related to the algorithm principles that allow super-resolution to occur. We conclude with examples of super-resolution. © 1995 John Wiley & Sons, Inc.

I. INTRODUCTION

Image restoration was one of the earliest successful applications of image processing. Although it was never widely disclosed, the quality of the images transmitted back by the remote spacecraft of the U.S. National Aeronautics and Space Administration (NASA) in the 1960s was due to careful image restoration procedures. A great burst of activity in image restoration research followed, with a plateau in research occurring toward the end of the 1970s, as can be seen in the book by Andrews and Hunt [1]. Comparing the material in that book with more recent books (e.g., Pratt [2] or Gonzalez and Woods [3]) shows how little was added to the topic of image restoration in the era of the 1980s. Image restoration has become a topic of increased interest in the end of the 1980s and into our current era of the 1990s. Research papers in this area have begun to move into topics that were not treated extensively before. Some of this increased interest is due to new insights into the physical and mathematic foundations for image restoration. Another portion of increased interest is due to the widespread availability of high-performance computation at continually decreasing cost. Algorithmic solutions for image restoration that would not have been considered computationally feasible 20 years ago can be obtained in workstations or personal computers today.

One facet of image restoration to benefit from this new environment is the problem we shall refer to as *image super-*

resolution. At one time this topic was considered controversial, with a prominent book in the subject confidently stating that super-resolution was a "myth" [1].* We now know today that image super-resolution is not a myth, that published algorithms exist to achieve super-resolution (e.g., Sementilli et al. [4], Holmes [5], Gerchberg [6]), and analytical and practical evidence exists to predict the actual performance of super-resolution algorithms [4]. In this article, we introduce the basic motivations for super-resolution, discuss the basic reasons why super-resolution can be made to work, present super-resolution algorithms, and describe the expected performance of the same.

II. RESOLUTION LIMITATIONS

The motivation for super-resolution rises from one of the most basic properties of image formation in an optical system: diffraction. The phenomenon of diffraction limits the resolution that can be achieved by an optical system of finite aperture. The reasons for this require an extended discussion of the propagation of electromagnetic waves and will not be given in extended detail here. The result of the wave propagation analysis can be summarized in the following brief paragraphs.

Assume an optical system that is aberration free (e.g., see Goodman [7]), and for which image formation characteristics are governed primarily by simple propagation models of waves interacting with simple, geometrically ideal surfaces for lenses and mirrors. In such a case, it can be shown that the image formation process is a linear system, and that the two-dimensional Fourier transform description of the image formation process is:

$$G(u, v) = H(u, v)F(u, v), \quad (1)$$

where G is the (two-dimensional) Fourier transform of the *image*, F if the transform of the intensity distribution of the *object* that gave rise to the image, and H is the spatial frequency transfer function of the optical image formation system, and has a special name: the *optical transfer function* (OTF). Clearly, the OTF determines how spatial frequency information in the object is transferred into the image.

The OTF cannot take on arbitrary characteristics. Indeed, the characteristics of H are determined by the other properties of the optical system. The total amount of light energy

* The author of this article was a co-author of the book. This illustrates that if one is forced to eat his own words, it is better serving yourself than being force fed.

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which enters the optical system is limited by a physically real pupil (or limiting aperture) that exists somewhere in the optical system. Let this limiting pupil be described as an aperture function, $a(x, y)$. The remarkable fact about diffraction is that the OTF is controlled wholly by the aperture function in the aberration-free ("diffraction-limited") case. Specifically, the OTF $H(u, v)$ is calculated from $a(x, y)$ by:

$$H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x, y) a(x + u, y + v) dx dy, \quad (2)$$

i.e., the OTF is the autocorrelation of the aperture function. Equation (2) defines the OTF for incoherent image formation, with the OTF being equal to $a(x, y)$ in the coherent case. We obviously see an implied change of physical scale in Equation (2), as the x, y variables are in space coordinates and the u, v variables are in spatial frequency coordinates. The proper scaling can be derived from the fundamental parameters of the optical system [7].

The most important fact about Equation (2) is the implied spatial-frequency cutoff. The aperture function is a region that transmits propagating electromagnetic waves. Within the aperture, transmission is perfect and $a(x, y) = 1$. Outside the aperture the transmission $a(x, y) = 0$, and no wave can propagate. The autocorrelation of Equation (2) guarantees that there will be a region of support for the OTF H [i.e., a region in which $H(u, v) > 0$] and that $H(u, v) = 0$ for any u, v not in the region of support. Thus, the OTF goes to zero outside of a boundary that is defined from the autocorrelation of the aperture function, and no spatial frequency information from the object can be passed into the image by the OTF for spatial frequencies lying outside the region of support. This is why diffraction limits the resolution of images. The actual limit to the maximum spatial frequency that can be passed into the image is given by the expression:

$$f_c = \frac{\lambda \ell}{D}, \quad (3)$$

where λ is the wavelength of the light, ℓ is the focal length of the optics, D is the diameter of a circular limiting aperture, and f_c is the spatial frequency cutoff, i.e., the boundary of the region of support. This definition of the spatial frequency cutoff can also be used to scale the variables u, v in Equation (2) [7].

From this description of the effects of diffraction, we can precisely define the meaning and objectives of super-resolution. As seen in Equation (1), the OTF will eliminate all object spatial frequencies outside the region of support of the OTF. The objective of super-resolution is to recover spatial frequency information of the object that lies beyond the spatial frequency cutoff imposed by the OTF. Because the cutoff defines the resolution of the system, a processing operation that achieves this result is referred to as super-resolution.

An initial inspection of Equation (1) may lead to the conclusion that super-resolution is not possible. A simple rearrangement of Equation (1) gives:

$$F(u, v) = \frac{G(u, v)}{H(u, v)}. \quad (4)$$

This defines the object spatial frequency only for those values

of $H(u, v) \neq 0$. Clearly, outside the region of support of the OTF the object spatial frequencies are ambiguous. That is, because the OTF $H(u, v) = 0$ outside the region of support, any finite value for $F(u, v)$ is possible as being consistent with Equations (1) and (4). In other words, inspection of the basic imaging equation, given the property $H(u, v) = 0$ outside the region of support, implies that information about the object is ambiguous for all spatial frequencies of the object outside the region of support of the OTF. It is this interpretation of the relation between image and object in a diffraction limited system that leads to the assertion that super-resolution is not possible—that it is a "myth."

Fortunately, there is more to the story than the simple description visible in Equations (1) and (4). There is more information available to recover $F(u, v)$ from the equations and that additional information is the heart of what makes super-resolution possible.

III. THE BASIS FOR SUPER RESOLUTION

To understand why super-resolution can be achieved, we return to Equation (1) and ask: Have we used all available information in moving to the simple solution implied by Equation (4)? That the answer to this question is no can be seen from the analytic continuation theorem, and important result in the Theory of Complex Variables. If a complex function, $s(u)$, is analytic, then it can be expanded in a power series everywhere in the complex plane. In particular, the knowledge of $s(u)$ in one finite region of the complex plane makes it possible to determine $s(u)$ everywhere else in the complex plane (e.g., Kreyszig [8]). Thus, knowing that an object possesses an analytic Fourier transform points to the prospect of achieving super-resolution, because the values of $F(u, v)$ that lie in the region of support can be computed by Equation (4), with a power series complex variable expansion used to construct values of F outside the region of support. This was first noted by Harris [9], and a particularly simple and compact discussion is contained in Goodman's book [7, p. 134–136]. The requirement of analyticity for the Fourier transform of the object is easily satisfied by an object which is compact in the object space domain, i.e., the object is confined to within an interval of finite size.

Analytic continuation for super-resolution demonstrates the key feature of the question at the beginning of the previous paragraph: using all available information about the object. If we assume the object is analytic, we see a possibility for recovering information from outside the diffraction limit. Ignoring this additional information leads only to the solution of Equation (4), with all the ambiguities for the object that lie outside the region of support of the OTF. We hasten to add that analytic continuation does not yield a practical way to achieve super-resolution; it is easy to determine that the solution for the undetermined coefficients of the series expansion can be very difficult to achieve, particularly in the presence of noise [7]. We use this concept, however, to demonstrate two critical principles that form the basis for super-resolution:

- The spatial frequencies that are captured by image formation below the diffraction limit contain some of the information necessary to reconstruct spatial frequencies above the diffraction limit;

- Using additional information about the object (e.g., compact, hence possessing an analytic Fourier transform) provides a means to use the information below the diffraction limit to reconstruct information above that limit.

The specifics of existing super-resolution algorithms can be related to these two principles, as we shall see below in reviewing several algorithms.

IV. EXISTING SUPER-RESOLUTION ALGORITHMS

One of the earliest algorithms explicitly identified for super-resolution is the Gerchberg algorithm [6]. It is a simple algorithm, exemplified by the block diagram in Figure 1. It is assumed that the object is known to be confined to a certain interval (hence, satisfying the analyticity requirement discussed earlier), and that other spatial domain constraints on the object are also known, e.g., the object is positive. The constraints that exist on the object are imposed on the image in the space domain. The modified image is then Fourier-transformed. Any existing constraints on the Fourier domain are then imposed on the Fourier data. Typically, the only constraint that can be imposed is the knowledge of the Fourier transform below the diffraction limit. The imposition of space domain constraints modifies the transform below the diffraction limit while simultaneously extending the spatial frequencies above the diffraction limit. Thus, the modified frequencies below the diffraction limit are replaced by the original Fourier transform values in unmodified form, i.e., the values of $G(u, v)/H(u, v)$ from Equation (4) below the diffraction limit. The modified Fourier transform is inverse-transformed to the space domain where a new cycle begins. Although the procedure seems ad hoc, it can be justified, and shown to be convergent (see Rushforth [10] or Jones [11].)

A modification to the Gerchberg algorithm, which computes the spatial frequency components above the diffraction limit directly, was recently demonstrated by Walsh and

Delaney [12], and is also related to usage of singular value decomposition and the pseudoinverse of the OTF.

An entirely different super-resolution algorithm was discovered in three independent ways by Richardson [13], Lucy [14], and Shepp and Vardi [15]. Richardson and Lucy justified the algorithm from a simple Bayes analysis, whereas Shepp and Vardi used a more elaborate maximum likelihood model for Poisson statistics in the emission of positrons in medical imaging for positron emission tomography. The basic computation of the algorithm is given by the equation:

$$f^{(n+1)}(x, y) = f^n(x, y) \left[\left(\frac{g(x, y)}{h(x, y) * f^n(x, y)} \right) \odot h(x, y) \right]. \quad (5)$$

This is an iterative computation with the estimate of the object at iteration $n + 1$ being computed from the estimate at iteration n . The image data is $g(x, y)$. The point-spread function is $h(x, y)$, and is the inverse Fourier transform of the OTF, $H(u, v)$. The symbol $*$ denotes convolution, and the symbol $\odot$ is for correlation. Although the precise understanding of how this algorithm achieves super-resolution is not as simple as the Gerchberg algorithm, it is easy to show the following:

- Extension of the object beyond the diffraction limit occurs because of the nonlinearity in the product of f^n with the other quantities in square brackets.
- Because h is always positive [being the inverse transform of a positive definite quantity, i.e., the definition of the OTF in Equation (2)], and g is always positive, using an initial estimate for f that is positive leads to each iteration being positive. Thus, the solution is constrained to be positive, which eliminates all ambiguous solutions for the object with negative values. Examples of the super-resolution capabilities of this algorithm can be seen in the paper by Holmes [5], which displays simulations of recovery of object information from beyond the diffraction limit.

A super-resolution algorithm which is the logical extension of Richardson–Lucy is the Poisson MAP algorithm of Hunt and Sementilli [18]. This algorithm assumes Poisson statistics for both image and object and constructs the Bayes maximum a-posteriori estimate in the form of the iteration:

$$f^{n+1}(x, y) = f^n(x, y) \times \exp \left[\left(\frac{g(x, y)}{h(x, y) * f^n(x, y)} - 1 \right) \odot h(x, y) \right], \quad (6)$$

where the meaning of the symbols is the same as previously used for Equation (5). The mechanisms for super-resolution are similar to Equation (5):

- Positivity is assured by the exponential; this is also an extra source of nonlinearity that extends information below the diffraction limit above the diffraction limit;
- A similar bandwidth extension process occurs for the product with f^n .

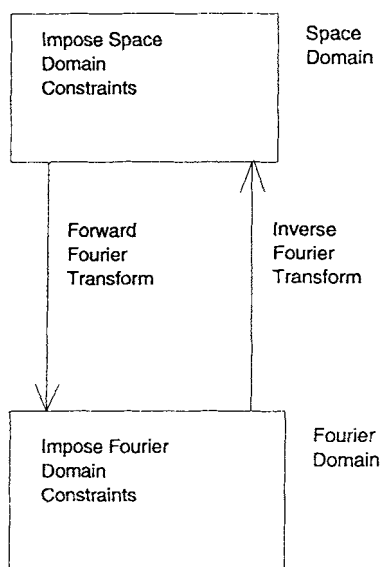


Figure 1. Block diagram of the Gerchberg's Algorithm.

Examples of recovering information from beyond the diffraction limit can be seen in the paper by Sementilli et al. [4].

V. THE IMPORTANCE OF OBJECT SPACE INTERPOLATION

There is a practical detail of super-resolution that is often overlooked in the theoretical analyses of such algorithms. We refer to the necessity of maintaining a sufficiently small sample spacing in the object estimate created by a super-resolution algorithm. To understand the significance of this issue, consider the nature of Nyquist sampling for image data. The Nyquist theorem states that a band-limited signal must be sampled at more than twice the highest frequency component for the original signal to be recovered from the samples. Optical images are truly band-limited, as a consequence of the phenomenon of diffraction. From Equation (3) we see that we can establish the Nyquist sampling rate from the cutoff spatial frequency as:

$$\Delta x \leq \frac{1}{2f_c} = \frac{D}{2\lambda\ell}, \quad (7)$$

where Δx is the distance between adjacent samples. If we choose

$$\Delta x = \frac{D}{2\lambda\ell}, \quad (8)$$

then Nyquist or critical sampling is present.

Nyquist sampling is the preferred sampling for any discrete data set. Undersampling, i.e., the case of choosing:

$$\Delta x > \frac{D}{2\lambda\ell} \quad (9)$$

is to be avoided because of the errors induced into the sampled data by aliasing, which is the result of replications of the original signal spectrum (the replications caused by the sampling process itself) adding up on each other (see, for example, the book by Oppenheim and Schaffer for detailed discussion [17]). On the other hand, if we choose a sample spacing which is less than Nyquist, i.e.,

$$\Delta x < \frac{D}{2\lambda\ell}, \quad (10)$$

then we obtain data free from any errors and can reconstruct the original signal. However, this case, which is called oversampling, collects more samples than necessary to represent the original signal without error. These additional samples have a cost, and the extra cost to store or transmit these samples is usually avoided by choosing to sample at the Nyquist spacing instead of oversampling. For example, if we sample at one-half the Nyquist spacing in both x and y coordinates of an image, we face storage and transmission of four times as many samples as the Nyquist case.

For the efficiency of data costs, virtually any image data set will be sampled at the Nyquist rate/Nyquist spacing. We are aware of no widely used image sensor that oversamples the image data in the focal plane. Many sensors, indeed, are undersampled (such as the Wide-Field Planetary Camera of the Hubble Space Telescope).

Assume that image data that are Nyquist sampled are to be used in a super-resolution algorithm. What is the result of successfully recovering object information that lies beyond the

diffraction limit? Because this new spatial frequency information lies beyond the sampling frequency of the original image, it cannot be represented without error in the sampling scheme of the original image data. Indeed, it will be aliased down below the sampling frequency of the original data and create errors, thereby limiting the accuracy of attempts to reconstruct the original object information that lies below the diffraction limit. A bit of reflection makes clear that the object must be reconstructed on a pixel sampling grid that is more finely spaced than the sample spacing of the original image data. Another way of stating this is that the reconstructed object must interpolate the image data. For example, if we use Equation (8) to define the original sample spacing, and if we choose to double the spatial frequency bandwidth by a suitable super-resolution process, then we must reconstruct the object on a grid of sample spacing:

$$\frac{\Delta x}{2} = \frac{D}{4\lambda\ell}, \quad (11)$$

That is, the pixels of the object estimate are one half the size, in each dimension, of the pixels in the original image data. This is a consequence of making the choice to double the spatial frequency bandwidth:

$$2f_c = \frac{2\lambda\ell}{D}, \quad (12)$$

which relates directly to a Nyquist sample spacing of Equation (11) for twice the bandwidth.

The significance of proper object interpolation is not noted in the literature of super-resolution. This is probably the result of most algorithmic tests being conducted by filtering full bandwidth signals to a lower bandwidth by simulation of a diffraction-limited OTF. The result is a signal with much additional "room" in the frequency domain for bandwidth reconstruction beyond the diffraction limit to occur. See, for example, the examples in the article by Holmes [5]. However, when working with real image data that have been Nyquist sampled, it is necessary to construct a strategy for object interpolation that is inherently carried out in the fundamental processes of super-resolution. One way to do this is to use two different models for the image and object, with the sample spacing for the object model being less than the sample spacing for the image model. This is the approach used in the Poisson MAP algorithm to result in an algorithm that operates on different pixel sample spacings (see Hunt and Sementilli for details [16]). The Poisson MAP algorithm as modified for different sample spacings takes similar form to Equation (6) above:

$$f^{n+1}(x, y) = f^n(x, y) \times \exp \left[\left(\frac{g(x, y)}{[h(x, y) * f^n(x, y)] \downarrow} - 1 \right)^\uparrow \odot h(x, y) \right]. \quad (13)$$

The up ($\uparrow$) and down ($\downarrow$) arrows denote up- and downsampling, respectively. The quantities h and f are defined on a pixel grid of finer sample spacing, e.g., one half the spacing of the image data pixel grid. After the convolution of h and f in the denominator of Equation (13), we must downsample that result to the same grid spacing as the image data, g . This can be done without aliasing error, of course, because h limits the

bandwidth of f'' to that identical to g . After the division into g , we now upsample (interpolate) to a finer grid spacing for the remaining computations. Typically, the use of a bandlimited interpolator (sinc interpolator) is desirable, because it can be done by inserting zeros into the Fourier domain values in the computational implementation of the corresponding convolutions and correlations.

VI. PERFORMANCE OF SUPER-RESOLUTION ALGORITHMS

We have emphasized previously that super-resolution algorithms succeed at recovering information from beyond the diffraction limit because of the additional information that is added about the problem beyond the simple linear relationship of Equation (1). We wish to discuss this in further detail in this section, focusing on what the useful sources of information are that can be added, and how the effects of the same can be quantified.

A variety of useful information is usually available a priori about the object that gave rise to the image. Consider the following:

- Object size, shape, and location. Usually the image of an object is adequate to make an estimate of the approximate size, shape, and location of the object. The size and shape characteristics of the object can be inferred from measured size and shape of the image and the optical system PSF. This can then be used to estimate a "mask" that bounds wherein the object lies, based on the fact that the convolution operations of image formation make the image size to be the sum of object and PSF extent. In the limit, this estimation of object size and shape can be reduced to the location of a point source object by correlating the optical system PSF with the image. (This is, in fact, the basis of a venerable image processing algorithm used in both radio astronomy and optical astronomy, the CLEAN algorithm [18,19]. This can also be a super-resolution algorithm, although astronomers do not interpret it as such, because proper correlation procedures can give subpixel resolution.)
- Bounds on object intensity. The minimum intensity level of an object is zero, because negative light is meaningless in incoherent optical image formation. Often, however, the minimum intensity is greater than zero, as a result of atmospheric haze, dark-current in the focal plane detector, etc. In such cases, it is a trivial matter to subtract the equivalent of a constant bias from the image data before processing, so that the minimum intensity is zero. In any event, the object data estimated by the super-resolution algorithm must be positive, and enforcing the positivity in the algorithm as a lower bound becomes an essential source of bandwidth extension. A second bound on the object is the maximum intensity. This is usually more difficult to precisely fix, but can be estimated from the illumination source and crude knowledge of object properties.
- Object statistics. Modeling of the object as a type of statistical process is the heart of the Poisson MAP algorithm discussed earlier, and the object model contri-

butes to positivity by the exponential function seen in Equation (6). Note that by statistics we imply any relevant probabilistic information. A particularly useful set of models has been the Markov random field models, originally pioneered by Geman and Geman [20] and used in improvements to the Poisson super-resolution algorithm by Sementilli [21].

How useful can additional information, such as these items, be in recovering spatial frequencies from beyond the diffraction limit of the optical system? There are several ways of answering this question. First, we consider a related question: How much information is there in the image? or in the object? The concepts of information theory give us a direct way to answer this question. It was shown by Fellgett and Linfoot [22] that given a signal with power spectrum $\Phi_s(u, v)$ and noise (in measuring the signal) with power spectrum $\Phi_n(u, v)$, the total information in the measured signal is:

$$C = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log_2 \left(1 + \frac{\Phi_s(u, v)}{\Phi_n(u, v)} \right) du dv, \quad (14)$$

which can be recognized as a simple extension of one of the forms of Shannon's channel capacity theorem. Applying this expression to the image obtained by the optical system, g , we have the information content of the image as:

$$C_g = \frac{1}{2} \int \int \log_2 \left(1 + \frac{|H(u, v)|^2 \Phi_f(u, v)}{\Phi_{gn}(u, v)} \right) du dv, \quad (u, v) \in R \quad (15)$$

where Φ_f is the object power spectrum, Φ_{gn} is the power spectrum of noise measured in the image, and R is the region of support of the OTF. A similar calculation for the total information in the object gives:

$$C_f = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log_2 \left(1 + \frac{\Phi_f(u, v)}{\Phi_{fn}(u, v)} \right) du dv, \quad (16)$$

where Φ_f is the object power spectrum and Φ_{fn} is the power spectrum of noise measured in the object. From these two equations we can infer that:

$$C_A = C_f - C_g, \quad (17)$$

which is the additional information that lies outside the bandwidth of the OTF of the optical system. Thus, from Equation (17) we see that if we were given additional information, such that the object were completely specified, the additional information, over what is known in the image, could not exceed C_A , assuming that the additional information is subject to measurement noise of the same magnitude as described by Φ_{fn} . (Of course, additional information about the the object which is subject to no measurement noise would imply an infinite amount of extra information, but we consider this case to be unrealistic.)

A measure of the additional information provided to a super-resolution algorithm can now be constructed from these equations. First, assume that the additional information allows the reconstruction of object spatial frequencies in a region outside the boundary of the region of support of the OTF. For example, in the case of a circular limiting aperture,

the region of support is also circular, so we might describe the region wherein reconstruction of spatial frequencies occurs as another region that completely surrounds the circular region of support of the OTF. We label this reconstruction region R' . We now compute the information in the reconstructed region in a fashion identical to that previously:

$$C'_f = \frac{1}{2} \iint \log_2 \left(1 + \frac{\Phi_f(u, v)}{\Phi_{fn}(u, v)} \right) du dv, \quad (u, v) \in R' \quad (18)$$

which leads directly to the additional information:

$$C'_\Delta = C_f - C'_f, \quad (19)$$

a measure of how the various constraints placed on the object (such as positivity or compact support) injected information into the solution that extended the reconstruction of the object beyond the diffraction limit.

From this analysis, we quantify the additional information about the object by the properties of the region R' , which can be considered to lie beyond the region of support of the OTF as a result of diffraction. The direct question to confront us now is, How can the size of this new region of support be quantified? Research in this area is still not complete. However, there are two results that have been established that give a direct measure for the magnitude of the new region R' . Two cases have been analyzed that demonstrate the value of additional information in extending the bandwidth beyond the region of support for the OTF, R . We have the following:

- Coherent imaging system with object compact support. Matson used the prolate spheroidal wave functions to show that knowing the region of support of the object, say an interval of dimension L , leads to the extension of object spatial frequencies beyond the cutoff f_c characterized as:

$$f'_c \cong f_c + \frac{1}{2L}, \quad (20)$$

where f'_c is the effective cutoff from a super-resolution process [21]. The result in Equation (20) assumes the image is observed with an SNR of at least 10:1 in the higher spatial frequencies, which can be difficult to achieve in typical imaging operations. The main features of Matson's analysis are that support constraints by themselves lead to only modest prospects for super-resolution. Using the results of Equation (20) in conjunction with the analysis of Equations (14) through (19) imply that object support constraints add only minor amounts of information. Notice also that a larger object, for which the interval dimension L is larger, leads to less super-resolution than for a small object. The basic results associated with Equation (20) were verified in an extensive simulation study by Fried and Sherwood [23].

- Incoherent imaging system with object compact support and object positivity. Sementilli added positivity of the object to the support constraint, assuming the object was contained in an interval of dimension L and was everywhere positive [21]. He then showed that the

extension of object spatial frequencies could be expressed as:

$$f'_c \cong f_c + \frac{3}{4\pi L} \left[\text{sinc}^{-1}(K) - \text{sinc}^{-1}\left(\frac{KT}{\sigma_n}\right) \right], \quad (21)$$

where K is a numerical constant derived from the sinc function, T is a threshold of error by which the reconstructed spatial frequencies cannot exceed the actual spatial frequencies, σ_n is the standard deviation of image noise, and the notation $\text{sinc}^{-1}(\cdot)$ means the inverse to the sinc junction, but restricted to the main lobe of the function. Note the similarity to Matson's analysis in the additivity of the new spatial frequencies beyond f_c . Sementilli constructed the estimate in Equation (21) as a lower bound on the amount of new spatial frequencies that can be reconstructed beyond the diffraction limit, i.e., it is often possible to reconstruct significantly more spatial frequencies than expressed in Equation (21). Simulation of hundreds of super-resolution problems, using both the Gerchberg algorithm and the Poisson MAP algorithm, demonstrated the accuracy of Equation (21) as a lower bound on the super-resolution process [21].

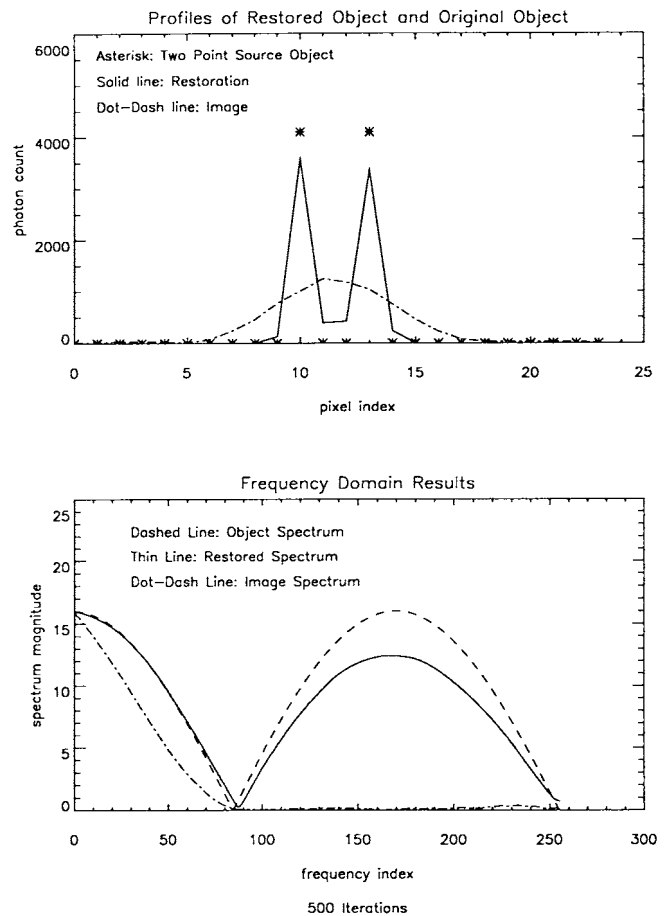


Figure 2. (a) Original object, diffracted object (image), and super-resolved image. (b) Fourier spectra of original object, diffracted object (image), and super-resolved image.

For either Equation (20) or (21) the value of f'_c can be used to establish the region R' and the information added by the assumptions on the object computed. It remains to be done to make comprehensive models of objects and their power spectra and construct closed-form evaluations of the information content of the support and positivity constraints. Probably closed-form expressions will not be possible, given the complexity of the expressions, but parametrically evaluated computer models could be successful.

VII. EXAMPLES OF SUPER-RESOLUTION

We conclude with a pair of examples to demonstrate the type of results that can be achieved with a super-resolution algorithm. Figure 2a shows two point source objects, the objects as blurred by a diffraction-limited (one-dimensional) OTF, and the reconstruction of those objects by the Poisson MAP super-resolution algorithm [Equation (16)]. The image values in Figure 2a have been converted to Poisson random variables (e.g., see Press et al. [24]), so that the SNR at a given pixel is the square root of the photon count, i.e., a maximum SNR at the peak of the image of approximately 30:1. We see in Figure 2b the corresponding amplitude spectra associated with Figure 1a. Note that the reconstructed spectrum deviates from the original at successively higher frequencies, an effect which can be used to calculate the error

in the sense of the discussion associated with Equation (21) in the previous section.

In Figure 3, we see the results of super-resolution processing on several simple geometric objects. In the top row are the original object (at left), the results of imaging through an incoherent optical system that cuts off spatial frequencies at one half the folding frequency of the sampled data used in the computations (at center), and the result of processing by the Poisson MAP algorithm (at right). The bottom row of pictures shows the Fourier spectra of the pictures in the row immediately above. Compare the lower right to lower left spectrum for a vivid illustration of the efficacy in recovering object spatial frequency information. The objects in the top of Figure 3 had no more than 250 photon counts per pixel, for maximum SNR of 16:1.

VIII. CLOSING REMARKS

The topic of super-resolution appears to be poised for a rapid expansion in both practical methods and fundamental knowledge. In all of the coming activity it will be necessary to exercise vigilance against an attitude that is easily associated with super-resolution, that "You can't get something for nothing," and that the super-resolution process is a sham. The author of this article was guilty of falling into this attitude 20 years ago. In this article, we have tried to emphasize that

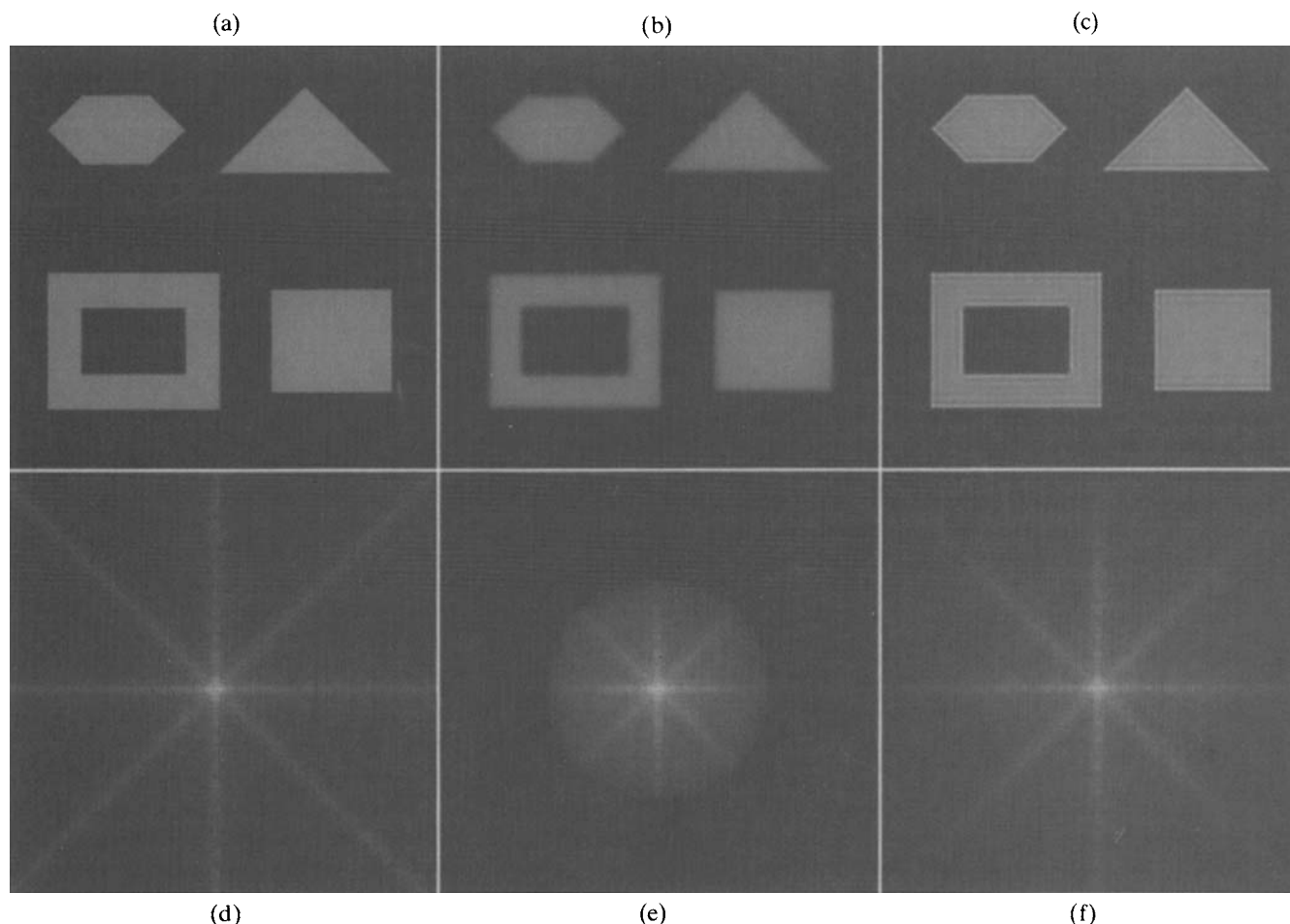


Figure 3. (a) Original object; (b) diffracted object (image); (c) super-resolved image; (d) Fourier spectrum of object; (e) Fourier spectrum of diffracted object (image); and (f) Fourier spectrum of super-resolved image.

“something for nothing” is not occurring. Super-resolution can take place because we impose constraints on the solution for the object that represent additional information beyond the simple linear Fourier descriptions associated with Equations (1) and (4). These constraints reduce the fundamental ambiguity associated with the OTF being zero when solving for the object as in Equation (4). This insight also gives guidance as to how to construct more powerful super-resolution concepts for the future, i.e., use more available information about the object in a form that can be readily embedded into a computational algorithm. This also illustrates a philosophic and practical tension in super-resolution: The more available information there exists about the object, the less valuable the super-resolved result may be. Obviously, complete knowledge of the object is not possible. The questions then becomes, What are realistic constraints on the object? What are constraints that we can know on the basis of fundamental physics or from reasonable observations of the image or a family of similar images? It is the construction of practical algorithms that recognize and answer these questions that constitute the interest and challenge for super-resolution.

REFERENCES

1. H. C. Andrews and B. R. Hunt, *Digital Image Restoration* (Prentice Hall, Englewood Cliffs, NJ), 1977.
2. W. K. Pratt, *Digital Image Processing*, 2nd Edition (Wiley, New York), 1990.
3. R. Gonzalez and J. Woods, *Digital Image Processing* (Addison-Wesley, Reading, MA), 1994.
4. P. Sementilli, B. Hunt, and M. Nadar, “Analysis of the limit to super-resolution in incoherent imaging,” *J. Opt. Soc. Amer., Series A* **10**, 2265–2276 (1993).
5. T. J. Holmes, “Maximum likelihood image restoration adapted for noncoherent optical imaging,” *J. Opt. Soc. Amer. Series A* **5**, 666–673 (1988).
6. R. W. Gerchberg, “Super-resolution through error energy reduction,” *Opt. Acta* **21**, 709–720 (1974).
7. J. W. Goodman, *Introduction to Fourier Optics* (McGraw-Hill, New York), 1968.
8. E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, New York), 1962.
9. J. L. Harris, “Diffraction and resolving power,” *J. Opt. Soc. Amer.* **54**, 931–936 (1964).
10. C. K. Rusfirth, in *Image Reconstruction, Theory and Application*, Henry Stark, Eds., Academic Press, New York, 1987, Chapter 1.
11. M. C. Jones, “The discrete Gerchberg algorithm,” *IEEE Trans. ASSP* **34**, 624–626 (1986).
12. D. O. Walsh and P. Nielsen-Delaney, “Direct method for super-resolution,” *J. Opt. Soc. Amer., series A* **11**, 572–579 (1994).
13. W. H. Richardson, “Bayesian-based iterative method of image restoration,” *J. Opt. Soc. Am.* **62**, 55–60 (1972).
14. L. B. Lucy, “An iterative technique for the rectification of observed distributions,” *Astron. J.* **79**, 745–759 (1974).
15. L. A. Shepp and Y. Vardi, “Maximum likelihood reconstruction for emission tomography,” *IEEE Trans. Med. Imaging* **MI-1**, 113–122 (1982).
16. B. R. Hunt and P. Sementilli, “Description of a Poisson imagery super-resolution algorithm,” in *Astronomical Data Analysis Software and Systems, I*, Worrall et al., Eds., vol. 25, Astronomical Society of the Pacific, San Francisco, 1992.
17. A. V. Oppenheim and R. Schaffer, *Discrete-Time Signal Processing* (Prentice-Hall, Englewood Cliffs, NJ), 1989.
18. J. A. Hogeboom, “The CLEAN algorithm,” *Astrophysics Astron. Sci.* **15**, 417–429 (1974).
19. W. C. Keel, “Applications of CLEAN to deconvolution of direct images,” *Astronomical Data Analysis Software and Systems, I*, Worrall et al., Eds., Vol. 25, Astronomical Society of the Pacific, San Francisco, 1992.
20. S. German and D. Geman, “Stochastic relaxation, Gibbs distribution, and the Bayesian restoration of images,” *IEEE Trans. Pattern Analysis and Machine Intelligence*, **PAMI-6**, 721–741 (1984).
21. P. J. Sementilli, *Suppression of Artifacts in Super-Resolved Images*, Ph.D. dissertation, Department of Electrical & Computer Engineering, University of Arizona, Tucson, 1993.
22. P. B. Fellgett and E. H. Linfoot, “On the assessment of optical images,” *Phil. Trans. Roy. Soc.*, **A247**, 369–381 (1955).
23. D. Fried and D. Sherwood, “Final report on ONR Contract no. N000-14-88-C-0692: Phased Array Imaging,” *Optical Sciences Co.*, Report no. DR-514 (1990).
24. W. Press et al., *Numerical Recipes, The Art of Scientific Computing* (Cambridge University Press, New York), 1986.