# GRAPH THEORY WITH APPLICATIONS

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*To our parents*

# Preface

This book is intended as an introduction to graph theory. Our aim has been to present what we consider to be the basic material, together with a wide variety of applications, both to other branches of mathematics and to real-world problems. Included are simple new proofs of theorems of Brooks, Chvátal, Tutte and Vizing. The applications have been carefully selected, and are treated in some depth. We have chosen to omit all so-called 'applications' that employ just the language of graphs and no theory. The applications appearing at the end of each chapter actually make use of theory developed earlier in the same chapter. We have also stressed the importance of efficient methods of solving problems. Several good algorithms are included and their efficiencies are analysed. We do not, however, go into the computer implementation of these algorithms.

The exercises at the end of each section are of varying difficulty. The harder ones are starred (*) and, for these, hints are provided in appendix I. In some exercises, new definitions are introduced. The reader is recommended to acquaint himself with these definitions. Other exercises, whose numbers are indicated by bold type, are used in subsequent sections; these should all be attempted.

Appendix II consists of a table in which basic properties of four graphs are listed. When new definitions are introduced, the reader may find it helpful to check his understanding by referring to this table. Appendix III includes a selection of interesting graphs with special properties. These may prove to be useful in testing new conjectures. In appendix IV, we collect together a number of unsolved problems, some known to be very difficult, and others more hopeful. Suggestions for further reading are given in appendix V.

Many people have contributed, either directly or indirectly, to this book. We are particularly indebted to C. Berge and D. J. A. Welsh for introducing us to graph theory, to G. A. Dirac, J. Edmonds, L. Lovász and W. T. Tutte, whose works have influenced our treatment of the subject, to V. Chungphaisan and C. St. J. A. Nash-Williams for their careful reading of the

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J. A. Bondy
U. S. R. Murty

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# 1 Graphs and Subgraphs

## 1.1 GRAPHS AND SIMPLE GRAPHS

Many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, the points could represent people, with lines joining pairs of friends; or the points might be communication centres, with lines representing communication links. Notice that in such diagrams one is mainly interested in whether or not two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of a graph.

A *graph* $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an *incidence function* $\psi_G$ that associates with each edge of $G$ an unordered pair of (not necessarily distinct) vertices of $G$. If $e$ is an edge and $u$ and $v$ are vertices such that $\psi_G(e) = uv$, then $e$ is said to *join* $u$ and $v$; the vertices $u$ and $v$ are called the *ends* of $e$.

Two examples of graphs should serve to clarify the definition.

*Example 1*

$$G = (V(G), E(G), \psi_G)$$

where

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$
$$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

and $\psi_G$ is defined by

$$\psi_G(e_1) = v_1v_2,\ \psi_G(e_2) = v_2v_3,\ \psi_G(e_3) = v_3v_3,\ \psi_G(e_4) = v_3v_4$$
$$\psi_G(e_5) = v_2v_4,\ \psi_G(e_6) = v_4v_5,\ \psi_G(e_7) = v_2v_5,\ \psi_G(e_8) = v_2v_5$$

*Example 2*

$$H = (V(H), E(H), \psi_H)$$

where

$$V(H) = \{u, v, w, x, y\}$$
$$E(H) = \{a, b, c, d, e, f, g, h\}$$

and $\psi_H$ is defined by

$$\psi_H(a) = uv,\quad \psi_H(b) = uu,\quad \psi_H(c) = vw,\quad \psi_H(d) = wx$$
$$\psi_H(e) = vx,\quad \psi_H(f) = wx,\quad \psi_H(g) = ux,\quad \psi_H(h) = xy$$

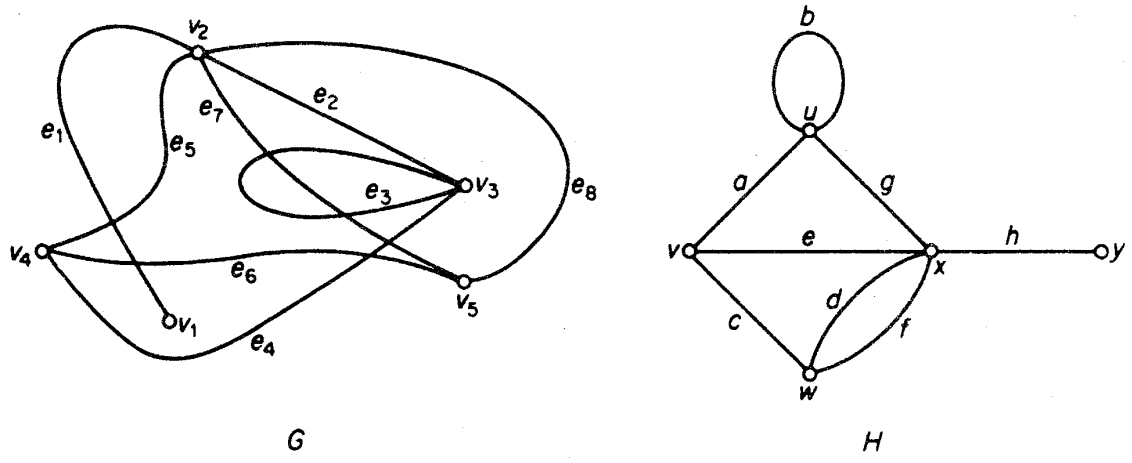



Figure 1.1. Diagrams of graphs $G$ and $H$

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points which represent its ends.† Diagrams of $G$ and $H$ are shown in figure 1.1. (For clarity, vertices are depicted here as small circles.)

There is no unique way of drawing a graph; the relative positions of points representing vertices and lines representing edges have no significance. Another diagram of $G$, for example, is given in figure 1.2. A diagram of a graph merely depicts the incidence relation holding between its vertices and edges. We shall, however, often draw a diagram of a graph and refer to it as the graph itself; in the same spirit, we shall call its points 'vertices' and its lines 'edges'.

Note that two edges in a diagram of a graph may intersect at a point that

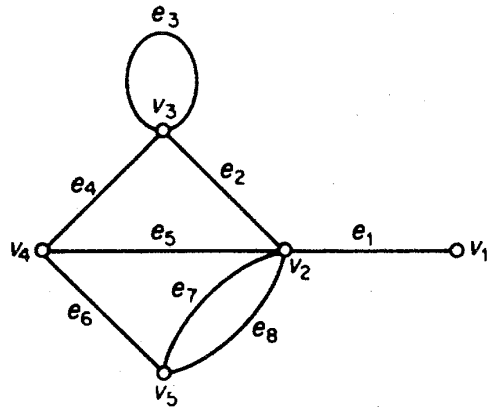



Figure 1.2. Another diagram of $G$

† In such a drawing it is understood that no line intersects itself or passes through a point representing a vertex which is not an end of the corresponding edge—this is clearly always possible.

is not a vertex (for example $e_1$ and $e_6$ of graph $G$ in figure 1.1). Those graphs that have a diagram whose edges intersect only at their ends are called *planar*, since such graphs can be represented in the plane in a simple manner. The graph of figure 1.3*a* is planar, even though this is not immediately clear from the particular representation shown (see exercise 1.1.2). The graph of figure 1.3*b*, on the other hand, is nonplanar. (This will be proved in chapter 9.)

Most of the definitions and concepts in graph theory are suggested by the graphical representation. The ends of an edge are said to be *incident* with the edge, and vice versa. Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. For example, the edge $e_3$ of $G$ (figure 1.2) is a loop; all other edges of $G$ are links.

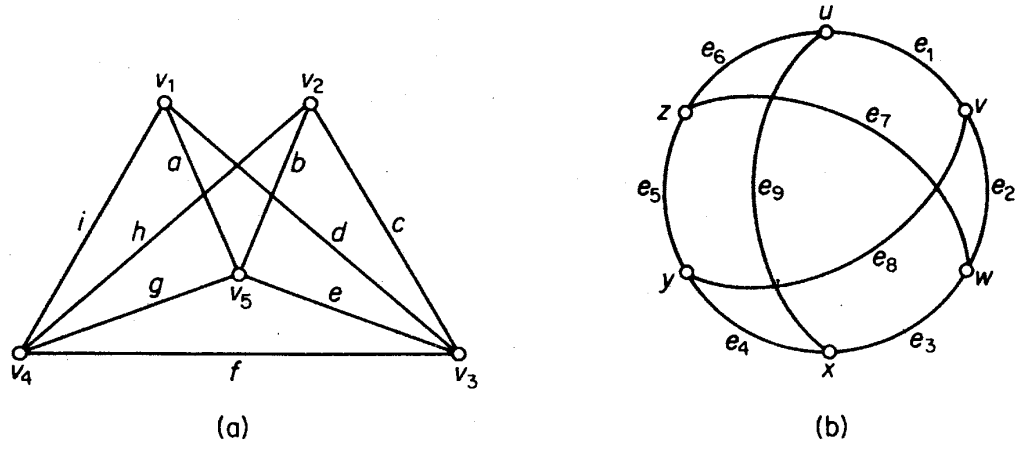



Figure 1.3. Planar and nonplanar graphs

A graph is *finite* if both its vertex set and edge set are finite. In this book we study only finite graphs, and so the term 'graph' always means 'finite graph'. We call a graph with just one vertex *trivial* and all other graphs *nontrivial.*

A graph is *simple* if it has no loops and no two of its links join the same pair of vertices. The graphs of figure 1.1 are not simple, whereas the graphs of figure 1.3 are. Much of graph theory is concerned with the study of simple graphs.

We use the symbols $\nu(G)$ and $\varepsilon(G)$ to denote the numbers of vertices and edges in graph $G$. Throughout the book the letter $G$ denotes a graph. Moreover, when just one graph is under discussion, we usually denote this graph by $G$. We then omit the letter $G$ from graph-theoretic symbols and write, for instance, $V$, $E$, $\nu$ and $\varepsilon$ instead of $V(G)$, $E(G)$, $\nu(G)$ and $\varepsilon(G)$.

*Exercises*

1.1.1 List five situations from everyday life in which graphs arise naturally.

1.1.2 Draw a different diagram of the graph of figure 1.3*a* to show that it is indeed planar.

1.1.3 Show that if $G$ is simple, then $\varepsilon \leq \binom{\nu}{2}$.

## 1.2 GRAPH ISOMORPHISM

Two graphs $G$ and $H$ are *identical* (written $G = H$) if $V(G) = V(H)$, $E(G) = E(H)$, and $\psi_G = \psi_H$. If two graphs are identical then they can clearly be represented by identical diagrams. However, it is also possible for graphs that are not identical to have essentially the same diagram. For example, the diagrams of $G$ in figure 1.2 and $H$ in figure 1.1 look exactly the same, with the exception that their vertices and edges have different labels. The graphs $G$ and $H$ are not identical, but isomorphic. In general, two graphs $G$ and $H$ are said to be *isomorphic* (written $G \cong H$) if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair $(\theta, \phi)$ of mappings is called an *isomorphism* between $G$ and $H$.

To show that two graphs are isomorphic, one must indicate an isomorphism between them. The pair of mappings $(\theta, \phi)$ defined by

$$\theta(v_1) = y, \qquad \theta(v_2) = x, \qquad \theta(v_3) = u, \qquad \theta(v_4) = v, \qquad \theta(v_5) = w$$

and

$$\phi(e_1) = h, \qquad \phi(e_2) = g, \qquad \phi(e_3) = b, \qquad \phi(e_4) = a$$
$$\phi(e_5) = e, \qquad \phi(e_6) = c, \qquad \phi(e_7) = d, \qquad \phi(e_8) = f$$

is an isomorphism between the graphs $G$ and $H$ of examples 1 and 2; $G$ and $H$ clearly have the same structure, and differ only in the names of vertices and edges. Since it is in structural properties that we shall primarily be interested, we shall often omit labels when drawing graphs; an unlabelled graph can be thought of as a representative of an equivalence class of isomorphic graphs. We assign labels to vertices and edges in a graph mainly for the purpose of referring to them. For instance, when dealing with simple graphs, it is often convenient to refer to the edge with ends $u$ and $v$ as 'the edge $uv$'. (This convention results in no ambiguity since, in a simple graph, at most one edge joins any pair of vertices.)

We conclude this section by introducing some special classes of graphs. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Up to isomorphism, there is just one complete graph on $n$ vertices; it is denoted by $K_n$. A drawing of $K_5$ is shown in figure 1.4*a*. An *empty graph*, on the other hand, is one with no edges. A *bipartite*

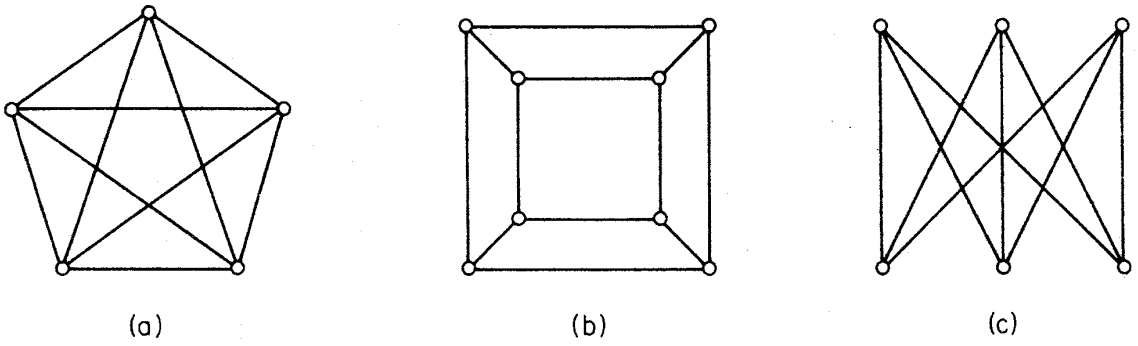



Figure 1.4. (*a*) $K_5$; (*b*) the cube; (*c*) $K_{3,3}$

*graph* is one whose vertex set can be partitioned into two subsets $X$ and $Y$, so that each edge has one end in $X$ and one end in $Y$; such a partition $(X, Y)$ is called a *bipartition* of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition $(X, Y)$ in which each vertex of $X$ is joined to each vertex of $Y$; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. The graph defined by the vertices and edges of a cube (figure 1.4*b*) is bipartite; the graph in figure 1.4*c* is the complete bipartite graph $K_{3,3}$.

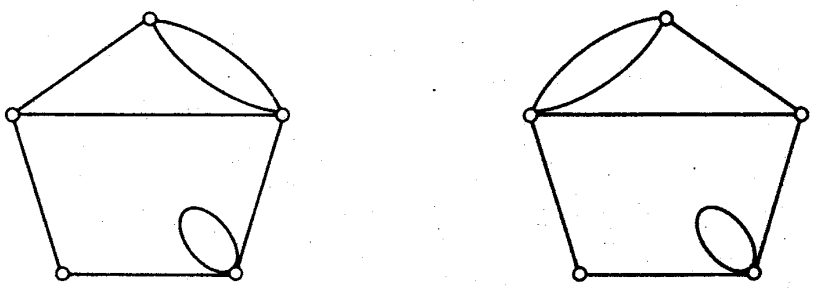
There are many other graphs whose structures are of special interest. Appendix III includes a selection of such graphs.

*Exercises*

1.2.1 Find an isomorphism between the graphs $G$ and $H$ of examples 1 and 2 different from the one given.

1.2.2 (*a*) Show that if $G \cong H$, then $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$.
(*b*) Give an example to show that the converse is false.

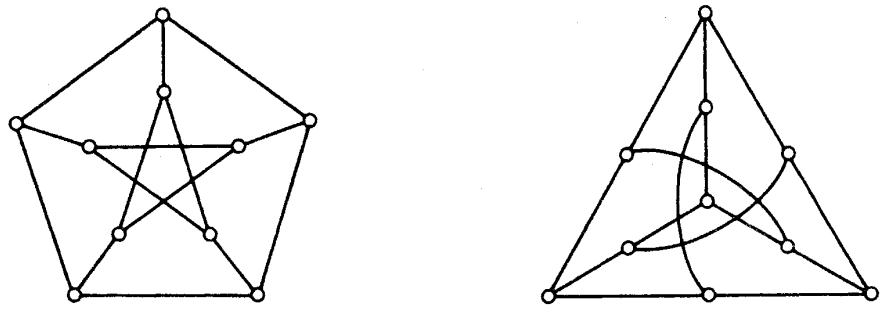
1.2.3 Show that the following graphs are not isomorphic:



1.2.4 Show that there are eleven nonisomorphic simple graphs on four vertices.

1.2.5 Show that two simple graphs $G$ and $H$ are isomorphic if and only if there is a bijection $\theta : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\theta(u)\theta(v) \in E(H)$.

1.2.6 Show that the following graphs are isomorphic:



1.2.7 Let $G$ be simple. Show that $\varepsilon = \binom{\nu}{2}$ if and only if $G$ is complete.

1.2.8 Show that

(*a*) $\varepsilon(K_{m,n}) = mn$;

(*b*) if $G$ is simple and bipartite, then $\varepsilon \leq \nu^2/4$.

**1.2.9** A *k-partite graph* is one whose vertex set can be partitioned into $k$ subsets so that no edge has both ends in any one subset; a *complete k-partite graph* is one that is simple and in which each vertex is joined to every vertex that is not in the same subset. The complete $m$-partite graph on $n$ vertices in which each part has either $[n/m]$ or $\{n/m\}$ vertices is denoted by $T_{m,n}$. Show that

(*a*) $\varepsilon(T_{m,n}) = \binom{n-k}{2} + (m-1)\binom{k+1}{2}$, where $k = [n/m]$;

(*b*)* if $G$ is a complete $m$-partite graph on $n$ vertices, then $\varepsilon(G) \leq \varepsilon(T_{m,n})$, with equality only if $G \cong T_{m,n}$.

1.2.10 The *k-cube* is the graph whose vertices are the ordered $k$-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. (The graph shown in figure 1.4*b* is just the 3-cube.) Show that the $k$-cube has $2^k$ vertices, $k2^{k-1}$ edges and is bipartite.

1.2.11 (*a*) The *complement* $G^c$ of a simple graph $G$ is the simple graph with vertex set $V$, two vertices being adjacent in $G^c$ if and only if they are not adjacent in $G$. Describe the graphs $K_n^c$ and $K_{m,n}^c$.

(*b*) A simple graph $G$ is *self-complementary* if $G \cong G^c$. Show that if $G$ is self-complementary, then $\nu \equiv 0, 1 \pmod 4$.

1.2.12 An *automorphism* of a graph is an isomorphism of the graph onto itself.

(*a*) Show, using exercise 1.2.5, that an automorphism of a simple graph $G$ can be regarded as a permutation on $V$ which preserves adjacency, and that the set of such permutations form a

group $\Gamma(G)$ (the *automorphism group* of $G$) under the usual operation of composition.

(*b*) Find $\Gamma(K_n)$ and $\Gamma(K_{m,n})$.

(*c*) Find a nontrivial simple graph whose automorphism group is the identity.

(*d*) Show that for any simple graph $G$, $\Gamma(G) = \Gamma(G^c)$.

(*e*) Consider the permutation group $\Lambda$ with elements (1)(2)(3), (1, 2, 3) and (1, 3, 2). Show that there is no simple graph $G$ with vertex set $\{1, 2, 3\}$ such that $\Gamma(G) = \Lambda$.

(*f*) Find a simple graph $G$ such that $\Gamma(G) \cong \Lambda$. (Frucht, 1939 has shown that every abstract group is isomorphic to the automorphism group of some graph.)

1.2.13 A simple graph $G$ is *vertex-transitive* if, for any two vertices $u$ and $v$, there is an element $g$ in $\Gamma(G)$ such that $g(u) = g(v)$; $G$ is *edge-transitive* if, for any two edges $u_1v_1$ and $u_2v_2$, there is an element $h$ in $\Gamma(G)$ such that $h(\{u_1, v_1\}) = \{u_2, v_2\}$. Find

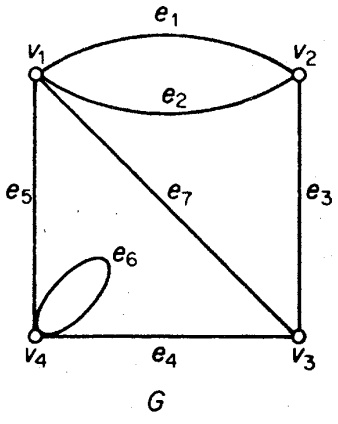
(*a*) a graph which is vertex-transitive but not edge-transitive;

(*b*) a graph which is edge-transitive but not vertex-transitive.

## 1.3 THE INCIDENCE AND ADJACENCY MATRICES

To any graph $G$ there corresponds a $\nu \times \varepsilon$ matrix called the incidence matrix of $G$. Let us denote the vertices of $G$ by $v_1, v_2, \ldots, v_\nu$ and the edges by $e_1, e_2, \ldots, e_\varepsilon$. Then the *incidence matrix* of $G$ is the matrix $\mathbf{M}(G) = [m_{ij}]$, where $m_{ij}$ is the number of times (0, 1 or 2) that $v_i$ and $e_j$ are incident. The incidence matrix of a graph is just a different way of specifying the graph.

Another matrix associated with $G$ is the *adjacency matrix;* this is the $\nu \times \nu$ matrix $\mathbf{A}(G) = [a_{ij}]$, in which $a_{ij}$ is the number of edges joining $v_i$ and $v_j$. A graph, its incidence matrix, and its adjacency matrix are shown in figure 1.5.




| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $v_1$ | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $v_2$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $v_4$ | 0 | 0 | 0 | 1 | 1 | 2 | 0 |

$\mathbf{M}(G)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $v_1$ | 0 | 2 | 1 | 1 |
| $v_2$ | 2 | 0 | 1 | 0 |
| $v_3$ | 1 | 1 | 0 | 1 |
| $v_4$ | 1 | 0 | 1 | 1 |

$\mathbf{A}(G)$

Figure 1.5

The adjacency matrix of a graph is generally considerably smaller than its incidence matrix, and it is in this form that graphs are commonly stored in computers.

*Exercises*

**1.3.1** Let **M** be the incidence matrix and **A** the adjacency matrix of a graph $G$.

(*a*) Show that every column sum of **M** is 2.

(*b*) What are the column sums of **A**?

1.3.2 Let $G$ be bipartite. Show that the vertices of $G$ can be enumerated so that the adjacency matrix of $G$ has the form

$$\left[\begin{array}{c|c} \mathbf{0} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{0} \end{array}\right]$$

where $\mathbf{A}_{21}$ is the transpose of $\mathbf{A}_{12}$.

1.3.3* Show that if $G$ is simple and the eigenvalues of **A** are distinct, then the automorphism group of $G$ is abelian

## 1.4 SUBGRAPHS

A graph $H$ is a *subgraph* of $G$ (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call $H$ a *proper subgraph* of $G$. If $H$ is a subgraph of $G$, $G$ is a *supergraph* of $H$. A *spanning subgraph* (or *spanning supergraph*) of $G$ is a subgraph (or supergraph) $H$ with $V(H) = V(G)$.

By deleting from $G$ all loops and, for every pair of adjacent vertices, all but one link joining them, we obtain a simple spanning subgraph of $G$, called the *underlying simple graph* of $G$. Figure 1.6 shows a graph and its underlying simple graph.

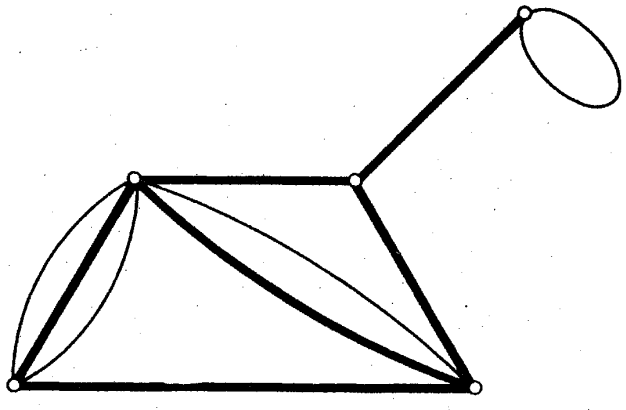


Figure 1.6. A graph and its underlying simple graph

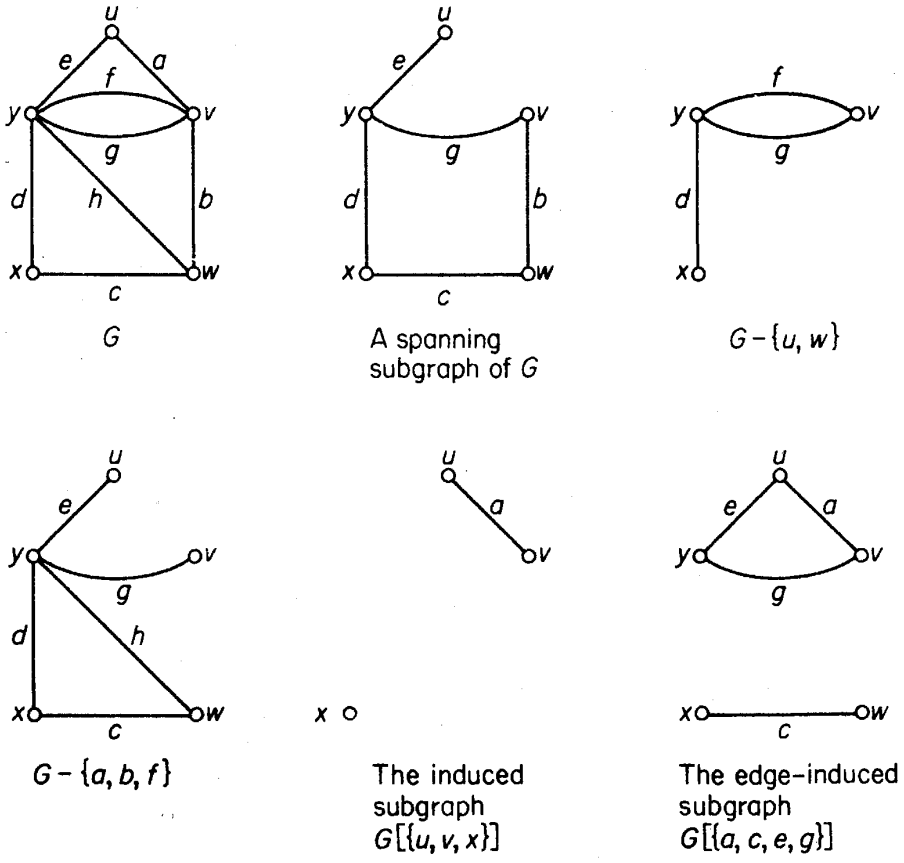



Figure 1.7

Suppose that $V'$ is a nonempty subset of $V$. The subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ *induced* by $V'$ and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of $G$. The induced subgraph $G[V \backslash V']$ is denoted by $G - V'$; it is the subgraph obtained from $G$ by deleting the vertices in $V'$ together with their incident edges. If $V' = \{v\}$ we write $G - v$ for $G - \{v\}$.

Now suppose that $E'$ is a nonempty subset of $E$. The subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$ is called the subgraph of $G$ *induced* by $E'$ and is denoted by $G[E']$; $G[E']$ is an *edge-induced subgraph* of $G$. The spanning subgraph of $G$ with edge set $E \backslash E'$ is written simply as $G - E'$; it is the subgraph obtained from $G$ by deleting the edges in $E'$. Similarly, the graph obtained from $G$ by adding a set of edges $E'$ is denoted by $G + E'$. If $E' = \{e\}$ we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$.

Subgraphs of these various types are depicted in figure 1.7.

Let $G_1$ and $G_2$ be subgraphs of $G$. We say that $G_1$ and $G_2$ are *disjoint* if they have no vertex in common, and *edge-disjoint* if they have no edge in common. The *union* $G_1 \cup G_2$ of $G_1$ and $G_2$ is the subgraph with vertex set

$V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$; if $G_1$ and $G_2$ are disjoint, we sometimes denote their union by $G_1 + G_2$. The *intersection* $G_1 \cap G_2$ of $G_1$ and $G_2$ is defined similarly, but in this case $G_1$ and $G_2$ must have at least one vertex in common.

*Exercises*

1.4.1 Show that every simple graph on $n$ vertices is isomorphic to a subgraph of $K_n$.

1.4.2 Show that

(*a*) every induced subgraph of a complete graph is complete;
(*b*) every subgraph of a bipartite graph is bipartite.

1.4.3 Describe how $\mathbf{M}(G-E')$ and $\mathbf{M}(G-V')$ can be obtained from $\mathbf{M}(G)$, and how $\mathbf{A}(G-V')$ can be obtained from $\mathbf{A}(G)$.

1.4.4 Find a bipartite graph that is not isomorphic to a subgraph of any $k$-cube.

1.4.5* Let $G$ be simple and let $n$ be an integer with $1 < n < \nu - 1$. Show that if $\nu \geq 4$ and all induced subgraphs of $G$ on $n$ vertices have the same number of edges, then either $G \cong K_\nu$ or $G \cong K_\nu^c$.

## 1.5 VERTEX DEGREES

The *degree* $d_G(v)$ of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$, each loop counting as two edges. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively, of vertices of $G$.

*Theorem 1.1*

$$\sum_{v \in V} d(v) = 2\varepsilon$$

*Proof* Consider the incidence matrix $\mathbf{M}$. The sum of the entries in the row corresponding to vertex $v$ is precisely $d(v)$, and therefore $\sum_{v \in V} d(v)$ is just the sum of all entries in $\mathbf{M}$. But this sum is also $2\varepsilon$, since (exercise 1.3.1*a*) each of the $\varepsilon$ column sums of $\mathbf{M}$ is 2 □

*Corollary 1.1* In any graph, the number of vertices of odd degree is even.

*Proof* Let $V_1$ and $V_2$ be the sets of vertices of odd and even degree in $G$, respectively. Then

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = \sum_{v \in V} d(v)$$

is even, by theorem 1.1. Since $\sum_{v \in V_2} d(v)$ is also even, it follows that $\sum_{v \in V_1} d(v)$ is even. Thus $|V_1|$ is even □

A graph $G$ is *$k$-regular* if $d(v)=k$ for all $v \in V$; a *regular graph* is one that is $k$-regular for some $k$. Complete graphs and complete bipartite graphs $K_{n,n}$ are regular; so, also, are the $k$-cubes.

*Exercises*

1.5.1 Show that $\delta \leq 2\varepsilon/\nu \leq \Delta$.

1.5.2 Show that if $G$ is simple, the entries on the diagonals of both $\mathbf{MM}'$ and $\mathbf{A}^2$ are the degrees of the vertices of $G$.

1.5.3 Show that if a $k$-regular bipartite graph with $k>0$ has bipartition $(X, Y)$, then $|X|=|Y|$.

1.5.4 Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.

1.5.5 If $G$ has vertices $v_1, v_2, \ldots, v_\nu$, the sequence $(d(v_1), d(v_2), \ldots, d(v_\nu))$ is called a *degree sequence* of $G$. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is a degree sequence of some graph if and only if $\sum_{i=1}^{n} d_i$ is even.

1.5.6 A sequence $\mathbf{d}=(d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Show that

(*a*) the sequences (7, 6, 5, 4, 3, 3, 2) and (6, 6, 5, 4, 3, 3, 1) are not graphic;

(*b*) if $\mathbf{d}$ is graphic and $d_1 \geq d_2 \geq \ldots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1)+\sum_{i=k+1}^{n} \min\{k, d_i\} \quad \text{for} \quad 1 \leq k \leq n$$

(Erdös and Gallai, 1960 have shown that this necessary condition is also sufficient for $\mathbf{d}$ to be graphic.)

1.5.7 Let $\mathbf{d}=(d_1, d_2, \ldots, d_n)$ be a nonincreasing sequence of non-negative integers, and denote the sequence $(d_2-1,\ d_3-1, \ldots, d_{d_1+1}-1, d_{d_1+2}, \ldots, d_n)$ by $\mathbf{d}'$.

(*a*)* Show that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'$ is graphic.

(*b*) Using (*a*), describe an algorithm for constructing a simple graph with degree sequence $\mathbf{d}$, if such a graph exists.

(V. Havel, S. Hakimi)

1.5.8* Show that a loopless graph $G$ contains a bipartite spanning subgraph $H$ such that $d_H(v) \geq \frac{1}{2} d_G(v)$ for all $v \in V$.

1.5.9* Let $S=\{x_1, x_2, \ldots, x_n\}$ be a set of points in the plane such that the distance between any two points is at least one. Show that there are at most $3n$ pairs of points at distance exactly one.

1.5.10 The *edge graph* of a graph $G$ is the graph with vertex set $E(G)$ in which two vertices are joined if and only if they are adjacent edges in

$G$. Show that, if $G$ is simple

(*a*) the edge graph of $G$ has $\varepsilon(G)$ vertices and $\sum_{v \in V(G)} \binom{d_G(v)}{2}$ edges;

(*b*) the edge graph of $K_5$ is isomorphic to the complement of the graph featured in exercise 1.2.6.

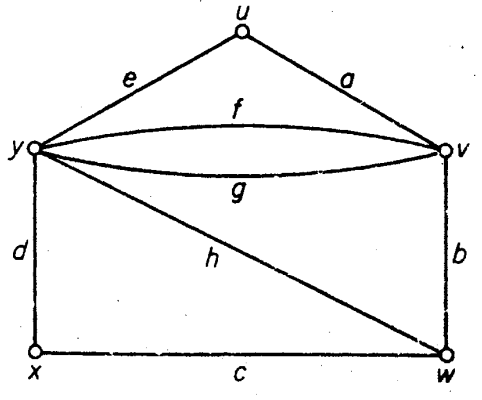
## 1.6 PATHS AND CONNECTION

A *walk* in $G$ is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of $e_i$ are $v_{i-1}$ and $v_i$. We say that $W$ is a walk *from* $v_0$ *to* $v_k$, or a $(v_0, v_k)$-*walk*. The vertices $v_0$ and $v_k$ are called the *origin* and *terminus* of $W$, respectively, and $v_1, v_2, \ldots, v_{k-1}$ its *internal vertices*. The integer $k$ is the *length* of $W$.

If $W = v_0 e_1 v_1 \ldots e_k v_k$ and $W' = v_k e_{k+1} v_{k+1} \ldots e_l v_l$ are walks, the walk $v_k e_k v_{k-1} \ldots e_1 v_0$, obtained by reversing $W$, is denoted by $W^{-1}$ and the walk $v_0 e_1 v_1 \ldots e_l v_l$, obtained by concatenating $W$ and $W'$ at $v_k$, is denoted by $WW'$. A *section* of a walk $W = v_0 e_1 v_1 \ldots e_k v_k$ is a walk that is a subsequence $v_i e_{i+1} v_{i+1} \ldots e_j v_j$ of consecutive terms of $W$; we refer to this subsequence as the $(v_i, v_j)$-*section* of $W$.

In a simple graph, a walk $v_0 e_1 v_1 \ldots e_k v_k$ is determined by the sequence $v_0 v_1 \ldots v_k$ of its vertices; hence a walk in a simple graph can be specified simply by its vertex sequence. Moreover, even in graphs that are not simple, we shall sometimes refer to a sequence of vertices in which consecutive terms are adjacent as a 'walk'. In such cases it should be understood that the discussion is valid for every walk with that vertex sequence.

If the edges $e_1, e_2, \ldots, e_k$ of a walk $W$ are distinct, $W$ is called a *trail*; in this case the length of $W$ is just $\varepsilon(W)$. If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct, $W$ is called a *path*. Figure 1.8 illustrates a walk, a trail and a path in a graph. We shall also use the word 'path' to denote a graph or subgraph whose vertices and edges are the terms of a path.




Walk: *uavfyfvgyhwbv*
Trail: *wcxdyhwbvgy*
Path: *xcwhyeuav*

Figure 1.8

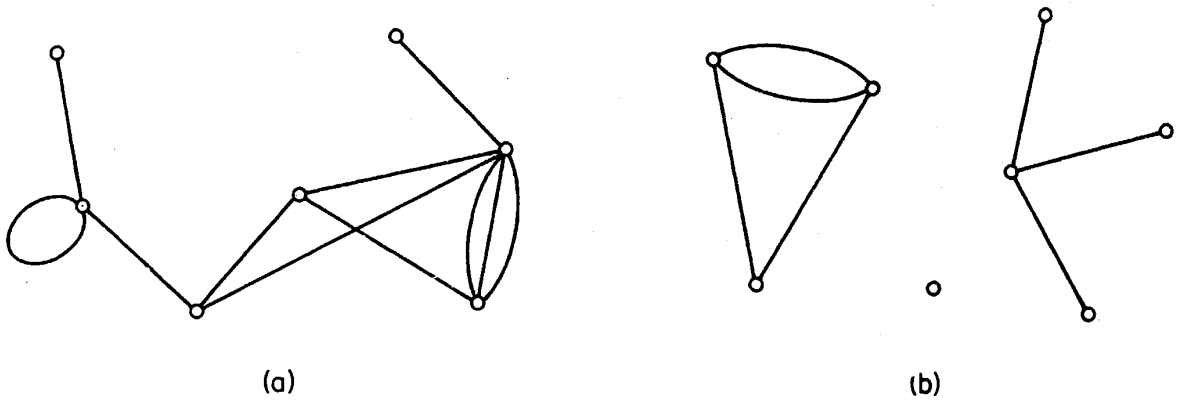



Figure 1.9. (*a*) A connected graph; (*b*) a disconnected graph with three components

Two vertices $u$ and $v$ of $G$ are said to be *connected* if there is a $(u, v)$-path in $G$. Connection is an equivalence relation on the vertex set $V$. Thus there is a partition of $V$ into nonempty subsets $V_1, V_2, \ldots, V_\omega$ such that two vertices $u$ and $v$ are connected if and only if both $u$ and $v$ belong to the same set $V_i$. The subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are called the *components* of $G$. If $G$ has exactly one component, $G$ is *connected*; otherwise $G$ is *disconnected*. We denote the number of components of $G$ by $\omega(G)$. Connected and disconnected graphs are depicted in figure 1.9.

*Exercises*

1.6.1 Show that if there is a $(u, v)$-walk in $G$, then there is also a $(u, v)$-path in $G$.

1.6.2 Show that the number of $(v_i, v_j)$-walks of length $k$ in $G$ is the $(i, j)$th entry of $\mathbf{A}^k$.

1.6.3 Show that if $G$ is simple and $\delta \geq k$, then $G$ has a path of length $k$.

1.6.4 Show that $G$ is connected if and only if, for every partition of $V$ into two nonempty sets $V_1$ and $V_2$, there is an edge with one end in $V_1$ and one end in $V_2$.

1.6.5 (*a*) Show that if $G$ is simple and $\varepsilon > \binom{\nu-1}{2}$, then $G$ is connected.

(*b*) For $\nu > 1$, find a disconnected simple graph $G$ with $\varepsilon = \binom{\nu-1}{2}$.

1.6.6 (*a*) Show that if $G$ is simple and $\delta > [\nu/2] - 1$, then $G$ is connected.

(*b*) Find a disconnected $([\nu/2]-1)$-regular simple graph for $\nu$ even.

1.6.7 Show that if $G$ is disconnected, then $G^c$ is connected.

**1.6.8** (*a*) Show that if $e \in E$, then $\omega(G) \leq \omega(G-e) \leq \omega(G)+1$.

(*b*) Let $v \in V$. Show that $G-e$ cannot, in general, be replaced by $G-v$ in the above inequality.

1.6.9 Show that if $G$ is connected and each degree in $G$ is even, then, for any $v \in V$, $\omega(G-v) \leq \frac{1}{2}d(v)$.

1.6.10 Show that any two longest paths in a connected graph have a vertex in common.

1.6.11 If vertices $u$ and $v$ are connected in $G$, the *distance* between $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length of a shortest $(u, v)$-path in $G$; if there is no path connecting $u$ and $v$ we define $d_G(u, v)$ to be infinite. Show that, for any three vertices $u$, $v$ and $w$, $d(u, v) + d(v, w) \geq d(u, w)$.

1.6.12 The *diameter* of $G$ is the maximum distance between two vertices of $G$. Show that if $G$ has diameter greater than three, then $G^c$ has diameter less than three.

1.6.13 Show that if $G$ is simple with diameter two and $\Delta = \nu - 2$, then $\varepsilon \geq 2\nu - 4$.

**1.6.14** Show that if $G$ is simple and connected but not complete, then $G$ has three vertices $u$, $v$ and $w$ such that $uv, vw \in E$ and $uw \notin E$.

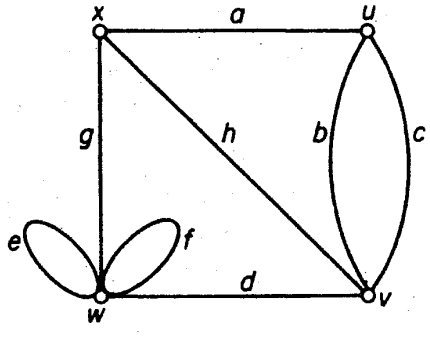
## 1.7 CYCLES

A walk is *closed* if it has positive length and its origin and terminus are the same. A closed trail whose origin and internal vertices are distinct is a *cycle*. Just as with paths we sometimes use the term 'cycle' to denote a graph corresponding to a cycle. A cycle of length $k$ is called a *$k$-cycle;* a $k$-cycle is *odd* or *even* according as $k$ is odd or even. A 3-cycle is often called a *triangle*. Examples of a closed trail and a cycle are given in figure 1.10.

Using the concept of a cycle, we can now present a characterisation of bipartite graphs.

*Theorem 1.2* A graph is bipartite if and only if it contains no odd cycle.

*Proof* Suppose that $G$ is bipartite with bipartition $(X, Y)$, and let $C = v_0v_1 \ldots v_kv_0$ be a cycle of $G$. Without loss of generality we may assume that $v_0 \in X$. Then, since $v_0v_1 \in E$ and $G$ is bipartite, $v_1 \in Y$. Similarly $v_2 \in X$ [illegible], in general, $v_{2i} \in X$ and $v_{2i+1} \in Y$. Since $v_0 \in X$, $v_k \in Y$. Thus $k = 2i + 1$, for some $i$, and it follows that $C$ is even.




Closed trail: *ucvhxgwfwdvbu*
Cycle: *xaubvhx*

Figure 1.10

It clearly suffices to prove the converse for connected graphs. Let $G$ be a connected graph that contains no odd cycles. We choose an arbitrary vertex $u$ and define a partition $(X, Y)$ of $V$ by setting

$$X = \{x \in V \mid d(u, x) \quad \text{is even}\}$$
$$Y = \{y \in V \mid d(u, y) \quad \text{is odd}\}$$

We shall show that $(X, Y)$ is a bipartition of $G$. Suppose that $v$ and $w$ are two vertices of $X$. Let $P$ be a shortest $(u, v)$-path and $Q$ be a shortest $(u, w)$-path. Denote by $u_1$ the last vertex common to $P$ and $Q$. Since $P$ and $Q$ are shortest paths, the $(u, u_1)$-sections of both $P$ and $Q$ are shortest $(u, u_1)$-paths and, therefore, have the same length. Now, since the lengths of both $P$ and $Q$ are even, the lengths of the $(u_1, v)$-section $P_1$ of $P$ and the $(u_1, w)$-section $Q_1$ of $Q$ must have the same parity. It follows that the $(v, w)$-path $P_1^{-1}Q_1$ is of even length. If $v$ were joined to $w$, $P_1^{-1}Q_1wv$ would be a cycle of odd length, contrary to the hypothesis. Therefore no two vertices in $X$ are adjacent; similarly, no two vertices in $Y$ are adjacent □

*Exercises*

1.7.1 Show that if an edge $e$ is in a closed trail of $G$, then $e$ is in a cycle of $G$.

**1.7.2** Show that if $\delta \geq 2$, then $G$ contains a cycle.

1.7.3* Show that if $G$ is simple and $\delta \geq 2$, then $G$ contains a cycle of length at least $\delta + 1$.

1.7.4 The *girth* of $G$ is the length of a shortest cycle in $G$; if $G$ has no cycles we define the girth of $G$ to be infinite. Show that

(*a*) a $k$-regular graph of girth four has at least $2k$ vertices, and (up to isomorphism) there exists exactly one such graph on $2k$ vertices;

(*b*) a $k$-regular graph of girth five has at least $k^2 + 1$ vertices.

1.7.5 Show that a $k$-regular graph of girth five and diameter two has exactly $k^2 + 1$ vertices, and find such a graph for $k = 2, 3$. (Hoffman and Singleton, 1960 have shown that such a graph can exist only if $k = 2, 3, 7$ and, possibly, 57.)

1.7.6 Show that

(*a*) if $\varepsilon \geq \nu$, $G$ contains a cycle;

(*b*)* if $\varepsilon \geq \nu + 4$, $G$ contains two edge-disjoint cycles. (L. Pósa)

## APPLICATIONS

### 1.8 THE SHORTEST PATH PROBLEM

With each edge $e$ of $G$ let there be associated a real number $w(e)$, called its *weight*. Then $G$, together with these weights on its edges, is called a *weighted*

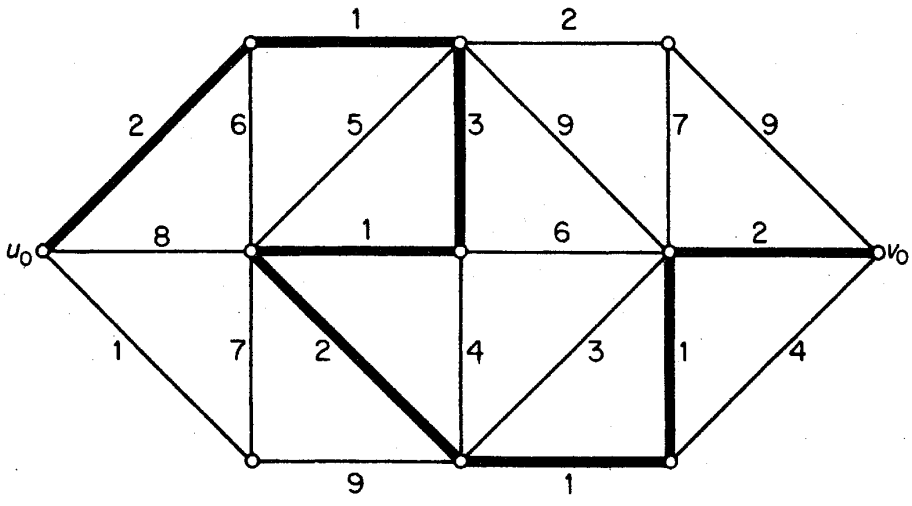



Figure 1.11. A $(u_0, v_0)$-path of minimum weight

*graph*. Weighted graphs occur frequently in applications of graph theory. In the friendship graph, for example, weights might indicate intensity of friendship; in the communications graph, they could represent the construction or maintenance costs of the various communication links.

If $H$ is a subgraph of a weighted graph, the *weight* $w(H)$ of $H$ is the sum of the weights $\sum_{e \in E(H)} w(e)$ on its edges. Many optimisation problems amount to finding, in a weighted graph, a subgraph of a certain type with minimum (or maximum) weight. One such is the *shortest path problem:* given a railway network connecting various towns, determine a shortest route between two specified towns in the network.

Here one must find, in a weighted graph, a path of minimum weight connecting two specified vertices $u_0$ and $v_0$; the weights represent distances by rail between directly-linked towns, and are therefore non-negative. The path indicated in the graph of figure 1.11 is a $(u_0, v_0)$-path of minimum weight (exercise 1.8.1).

We now present an algorithm for solving the shortest path problem. For clarity of exposition, we shall refer to the weight of a path in a weighted graph as its *length*; similarly the minimum weight of a $(u, v)$-path will be called the *distance* between $u$ and $v$ and denoted by $d(u, v)$. These definitions coincide with the usual notions of length and distance, as defined in section 1.6, when all the weights are equal to one.

It clearly suffices to deal with the shortest path problem for simple graphs; so we shall assume here that $G$ is simple. We shall also assume that all the weights are positive. This, again, is not a serious restriction because, if the weight of an edge is zero, then its ends can be identified. We adopt the convention that $w(uv) = \infty$ if $uv \notin E$.

The algorithm to be described was discovered by Dijkstra (1959) and, independently, by Whiting and Hillier (1960). It finds not only a shortest $(u_0, v_0)$-path, but shortest paths from $u_0$ to all other vertices of $G$. The basic idea is as follows.

Suppose that $S$ is a proper subset of $V$ such that $u_0 \in S$, and let $\bar{S}$ denote $V \backslash S$. If $P = u_0 \ldots \bar{u}\bar{v}$ is a shortest path from $u_0$ to $\bar{S}$ then clearly $\bar{u} \in S$ and the $(u_0, \bar{u})$-section of $P$ must be a shortest $(u_0, \bar{u})$-path. Therefore

$$d(u_0, \bar{v}) = d(u_0, \bar{u}) + w(\bar{u}\bar{v})$$

and the distance from $u_0$ to $\bar{S}$ is given by the formula

$$d(u_0, \bar{S}) = \min_{\substack{u \in S \\ v \in \bar{S}}} \{d(u_0, u) + w(uv)\} \tag{1.1}$$

This formula is the basis of Dijkstra's algorithm. Starting with the set $S_0 = \{u_0\}$, an increasing sequence $S_0, S_1, \ldots, S_{\nu-1}$ of subsets of $V$ is constructed, in such a way that, at the end of stage $i$, shortest paths from $u_0$ to all vertices in $S_i$ are known.

The first step is to determine a vertex nearest to $u_0$. This is achieved by computing $d(u_0, \bar{S}_0)$ and selecting a vertex $u_1 \in \bar{S}_0$ such that $d(u_0, u_1) = d(u_0, \bar{S}_0)$; by (1.1)

$$d(u_0, \bar{S}_0) = \min_{\substack{u \in S_0 \\ v \in \bar{S}_0}} \{d(u_0, u) + w(uv)\} = \min_{v \in \bar{S}_0} \{w(u_0 v)\}$$

and so $d(u_0, \bar{S}_0)$ is easily computed. We now set $S_1 = \{u_0, u_1\}$ and let $P_1$ denote the path $u_0 u_1$; this is clearly a shortest $(u_0, u_1)$-path. In general, if the set $S_k = \{u_0, u_1, \ldots, u_k\}$ and corresponding shortest paths $P_1, P_2, \ldots, P_k$ have already been determined, we compute $d(u_0, \bar{S}_k)$ using (1.1) and select a vertex $u_{k+1} \in \bar{S}_k$ such that $d(u_0, u_{k+1}) = d(u_0, \bar{S}_k)$. By (1.1), $d(u_0, u_{k+1}) = d(u_0, u_j) + w(u_j u_{k+1})$ for some $j \leq k$; we get a shortest $(u_0, u_{k+1})$-path by adjoining the edge $u_j u_{k+1}$ to the path $P_j$.

We illustrate this procedure by considering the weighted graph depicted in figure 1.12*a*. Shortest paths from $u_0$ to the remaining vertices are determined in seven stages. At each stage, the vertices to which shortest paths have been found are indicated by solid dots, and each is labelled by its distance from $u_0$; initially $u_0$ is labelled 0. The actual shortest paths are indicated by solid lines. Notice that, at each stage, these shortest paths together form a connected graph without cycles; such a graph is called a *tree*, and we can think of the algorithm as a 'tree-growing' procedure. The final tree, in figure 1.12*h*, has the property that, for each vertex $v$, the path connecting $u_0$ and $v$ is a shortest $(u_0, v)$-path.

Dijkstra's algorithm is a refinement of the above procedure. This refinement is motivated by the consideration that, if the minimum in (1.1) were to be computed from scratch at each stage, many comparisons would be

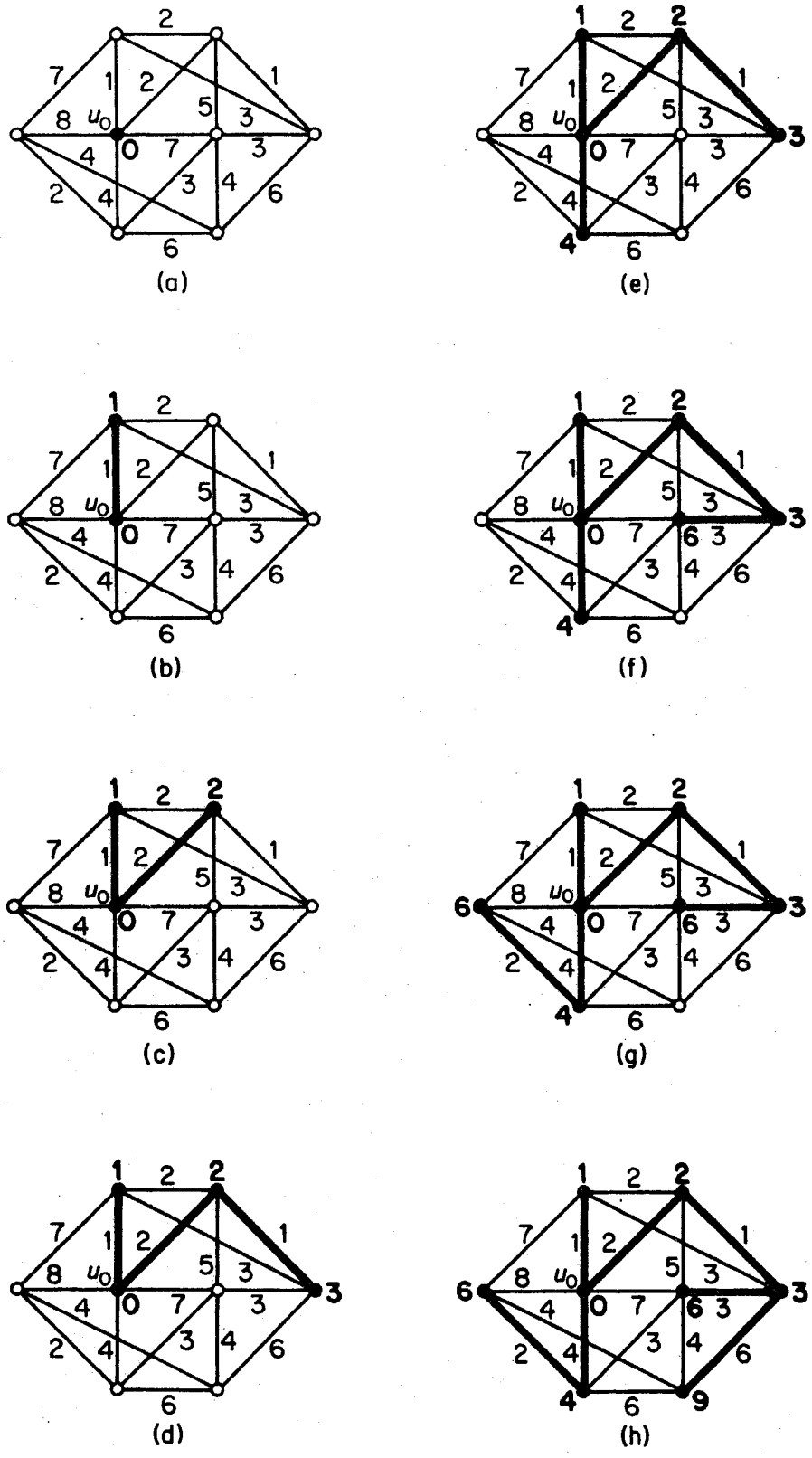



**Figure 1.12. Shortest path algorithm**

repeated unnecessarily. To avoid such repetitions, and to retain computational information from one stage to the next, we adopt the following labelling procedure. Throughout the algorithm, each vertex $v$ carries a label $l(v)$ which is an upper bound on $d(u_0, v)$. Initially $l(u_0)=0$ and $l(v)=\infty$ for $v \neq u_0$. (In actual computations $\infty$ is replaced by any sufficiently large number.) As the algorithm proceeds, these labels are modified so that, at the end of stage $i$,

$$l(u) = d(u_0, u) \quad \text{for} \quad u \in S_i$$

and

$$l(v) = \min_{u \in S_{i-1}} \{d(u_0, u) + w(uv)\} \quad \text{for} \quad v \in \bar{S}_i$$

*Dijkstra's Algorithm*

1. Set $l(u_0)=0$, $l(v)=\infty$ for $v \neq u_0$, $S_0=\{u_0\}$ and $i=0$.
2. For each $v \in \bar{S}_i$, replace $l(v)$ by $\min\{l(v), l(u_i)+w(u_i v)\}$. Compute $\min_{v \in \bar{S}_i}\{l(v)\}$ and let $u_{i+1}$ denote a vertex for which this minimum is attained. Set $S_{i+1} = S_i \cup \{u_{i+1}\}$.
3. If $i = \nu - 1$, stop. If $i < \nu - 1$, replace $i$ by $i+1$ and go to step 2.

When the algorithm terminates, the distance from $u_0$ to $v$ is given by the final value of the label $l(v)$. (If our interest is in determining the distance to one specific vertex $v_0$, we stop as soon as some $u_j$ equals $v_0$.) A flow diagram summarising this algorithm is shown in figure 1.13.

As described above, Dijkstra's algorithm determines only the distances from $u_0$ to all the other vertices, and not the actual shortest paths. These shortest paths can, however, be easily determined by keeping track of the predecessors of vertices in the tree (exercise 1.8.2).

Dijkstra's algorithm is an example of what Edmonds (1965) calls a good algorithm. A graph-theoretic algorithm is *good* if the number of computational steps required for its implementation on any graph $G$ is bounded above by a polynomial in $\nu$ and $\varepsilon$ (such as $3\nu^2\varepsilon$). An algorithm whose implementation may require an exponential number of steps (such as $2^\nu$) might be very inefficient for some large graphs.

To see that Dijkstra's algorithm is good, note that the computations involved in boxes 2 and 3 of the flow diagram, totalled over all iterations, require $\nu(\nu-1)/2$ additions and $\nu(\nu-1)$ comparisons. One of the questions that is not elaborated upon in the flow diagram is the matter of deciding whether a vertex belongs to $\bar{S}$ or not (box 1). Dreyfus (1969) reports a technique for doing this that requires a total of $(\nu-1)^2$ comparisons. Hence, if we regard either a comparison or an addition as a basic computational unit, the total number of computations required for this algorithm is approximately $5\nu^2/2$, and thus of order $\nu^2$. (A function $f(\nu, \varepsilon)$ is of *order*

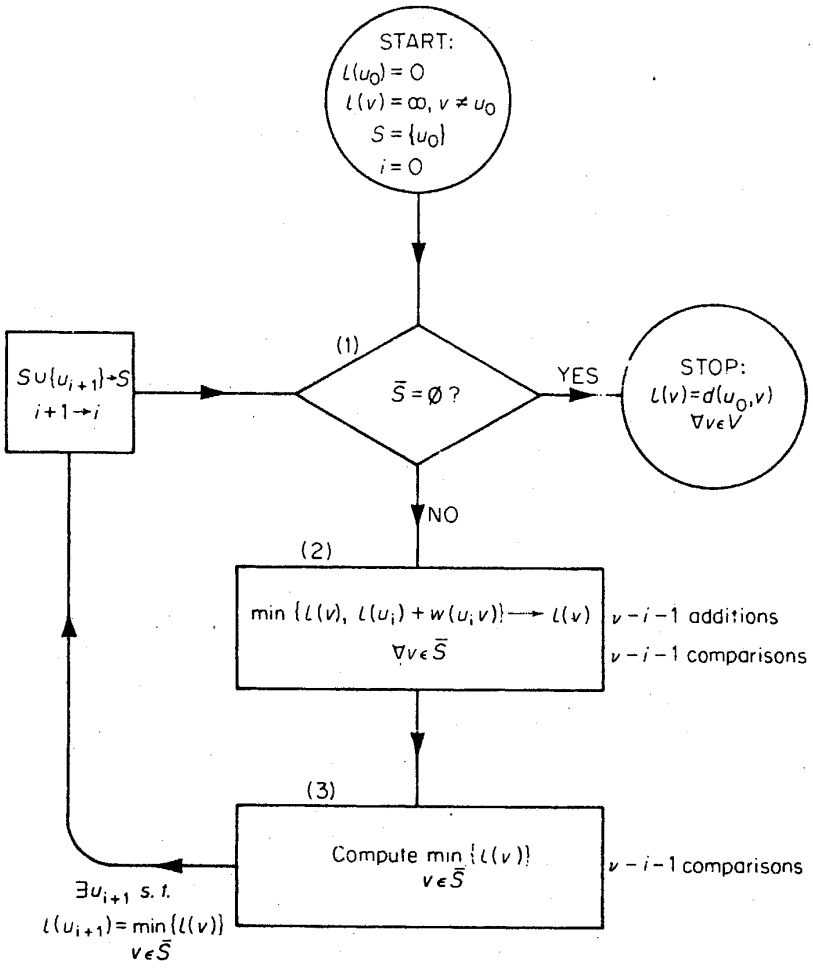



Figure 1.13. Dijkstra's algorithm

$g(\nu, \varepsilon)$ if there exists a positive constant $c$ such that $f(\nu, \varepsilon)/g(\nu, \varepsilon) \leq c$ for all $\nu$ and $\varepsilon$.)

Although the shortest path problem can be solved by a good algorithm, there are many problems in graph theory for which no good algorithm is known. We refer the reader to Aho, Hopcroft and Ullman (1974) for further details.

## *Exercises*

1.8.1 Find shortest paths from $u_0$ to all other vertices in the weighted graph of figure 1.11.

1.8.2 What additional instructions are needed in order that Dijkstra's algorithm determine shortest paths rather than merely distances?

1.8.3 A company has branches in each of six cities $C_1, C_2, \ldots, C_6$. The fare for a direct flight from $C_i$ to $C_j$ is given by the $(i, j)$th entry in the following matrix ($\infty$ indicates that there is no direct flight):

$$\begin{bmatrix} 0 & 50 & \infty & 40 & 25 & 10 \\ 50 & 0 & 15 & 20 & \infty & 25 \\ \infty & 15 & 0 & 10 & 20 & \infty \\ 40 & 20 & 10 & 0 & 10 & 25 \\ 25 & \infty & 20 & 10 & 0 & 55 \\ 10 & 25 & \infty & 25 & 55 & 0 \end{bmatrix}$$

The company is interested in computing a table of cheapest routes between pairs of cities. Prepare such a table.

1.8.4 A wolf, a goat and a cabbage are on one bank of a river. A ferryman wants to take them across, but, since his boat is small, he can take only one of them at a time. For obvious reasons, neither the wolf and the goat nor the goat and the cabbage can be left unguarded. How is the ferryman going to get them across the river?

1.8.5 Two men have a full eight-gallon jug of wine, and also two empty jugs of five and three gallons capacity, respectively. What is the simplest way for them to divide the wine equally?

1.8.6 Describe a good algorithm for determining

(*a*) the components of a graph;
(*b*) the girth of a graph.

How good are your algorithms?

## 1.9 SPERNER'S LEMMA

Every continuous mapping $f$ of a closed $n$-disc to itself has a fixed point (that is, a point $x$ such that $f(x)=x$). This powerful theorem, known as *Brouwer's fixed-point theorem*, has a wide range of applications in modern mathematics. Somewhat surprisingly, it is an easy consequence of a simple combinatorial lemma due to Sperner (1928). And, as we shall see in this section, Sperner's lemma is, in turn, an immediate consequence of corollary 1.1.

Sperner's lemma concerns the decomposition of a simplex (line segment, triangle, tetrahedron and so on) into smaller simplices. For the sake of simplicity we shall deal with the two-dimensional case.

Let $T$ be a closed triangle in the plane. A subdivision of $T$ into a finite number of smaller triangles is said to be *simplicial* if any two intersecting triangles have either a vertex or a whole side in common (see figure 1.14*a*).

Suppose that a simplicial subdivision of $T$ is given. Then a labelling of the vertices of triangles in the subdivision in three symbols 0, 1 and 2 is said to be *proper* if

(i) the three vertices of $T$ are labelled 0, 1 and 2 (in any order), and
(ii) for $0 \leq i < j \leq 2$, each vertex on the side of $T$ joining vertices labelled $i$ and $j$ is labelled either $i$ or $j$.

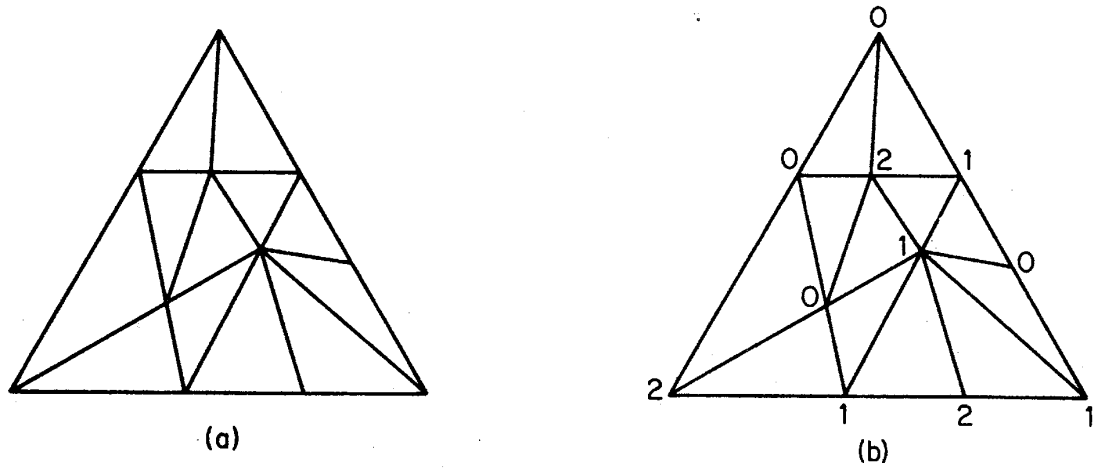



Figure 1.14. (*a*) A simplicial subdivision of a triangle; (*b*) a proper labelling of the subdivision

We call a triangle in the subdivision whose vertices receive all three labels a *distinguished* triangle. The proper labelling in figure 1.14*b* has three distinguished triangles.

*Theorem 1.3* (*Sperner's lemma*) Every properly labelled simplicial subdivision of a triangle has an odd number of distinguished triangles.

*Proof* Let $T_0$ denote the region outside $T$, and let $T_1, T_2, \ldots, T_n$ be the triangles of the subdivision. Construct a graph on the vertex set $\{v_0, v_1, \ldots, v_n\}$ by joining $v_i$ and $v_j$ whenever the common boundary of $T_i$ and $T_j$ is an edge with labels 0 and 1 (see figure 1.15).

In this graph, $v_0$ is clearly of odd degree (exercise 1.9.1). It follows from corollary 1.1 that an odd number of the vertices $v_1, v_2, \ldots, v_n$ are of odd degree. Now it is easily seen that none of these vertices can have degree

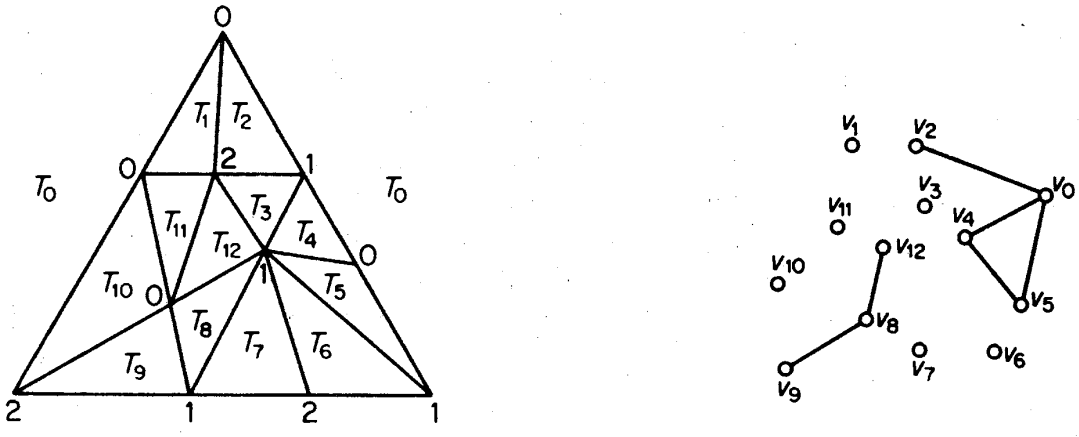


Figure 1.15

three, and so those with odd degree must have degree one. But a vertex $v_i$ is of degree one if and only if the triangle $T_i$ is distinguished □

We shall now briefly indicate how Sperner's lemma can be used to deduce Brouwer's fixed-point theorem. Again, for simplicity, we shall only deal with the two-dimensional case. Since a closed 2-disc is homeomorphic to a closed triangle, it suffices to prove that a continuous mapping of a closed triangle to itself has a fixed point.

Let $T$ be a given closed triangle with vertices $x_0$, $x_1$ and $x_2$. Then each point $x$ of $T$ can be written uniquely as $x = a_0x_0 + a_1x_1 + a_2x_2$, where each $a_i \geq 0$ and $\Sigma\, a_i = 1$, and we can represent $x$ by the vector $(a_0, a_1, a_2)$; the real numbers $a_0$, $a_1$ and $a_2$ are called the *barycentric coordinates* of $x$.

Now let $f$ be any continuous mapping of $T$ to itself, and suppose that

$$f(a_0, a_1, a_2) = (a_0', a_1', a_2')$$

Define $S_i$ as the set of points $(a_0, a_1, a_2)$ in $T$ for which $a_i' \leq a_i$. To show that $f$ has a fixed point, it is enough to show that $S_0 \cap S_1 \cap S_2 \neq \emptyset$. For suppose that $(a_0, a_1, a_2) \in S_0 \cap S_1 \cap S_2$. Then, by the definition of $S_i$, we have that $a_i' \leq a_i$ for each $i$, and this, coupled with the fact that $\Sigma\, a_i' = \Sigma\, a_i$, yields

$$(a_0', a_1', a_2') = (a_0, a_1, a_2)$$

In other words, $(a_0, a_1, a_2)$ is a fixed point of $f$.

So consider an arbitrary subdivision of $T$ and a proper labelling such that each vertex labelled $i$ belongs to $S_i$; the existence of such a labelling is easily seen (exercise 1.9.2*a*). It follows from Sperner's lemma that there is a triangle in the subdivision whose three vertices belong to $S_0$, $S_1$ and $S_2$. Now this holds for any subdivision of $T$ and, since it is possible to choose subdivisions in which each of the smaller triangles are of arbitrarily small diameter, we conclude that there exist three points of $S_0$, $S_1$ and $S_2$ which are arbitrarily close to one another. Because the sets $S_i$ are closed (exercise 1.9.2*b*), one may deduce that $S_0 \cap S_1 \cap S_2 \neq \emptyset$.

For details of the above proof and other applications of Sperner's lemma, the reader is referred to Tompkins (1964).

*Exercises*

1.9.1 In the proof of Sperner's lemma, show that the vertex $v_0$ is of odd degree.

1.9.2 In the proof of Brouwer's fixed-point theorem, show that

(*a*) there exists a proper labelling such that each vertex labelled $i$ belongs to $S_i$;

(*b*) the sets $S_i$ are closed.

1.9.3 State and prove Sperner's lemma for higher dimensional simplices.

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# 2 Trees

## 2.1 TREES

An *acyclic* graph is one that contains no cycles. A *tree* is a connected acyclic graph. The trees on six vertices are shown in figure 2.1.

*Theorem 2.1* In a tree, any two vertices are connected by a unique path.

*Proof* By contradiction. Let $G$ be a tree, and assume that there are two distinct $(u, v)$-paths $P_1$ and $P_2$ in $G$. Since $P_1 \neq P_2$, there is an edge $e = xy$ of $P_1$ that is not an edge of $P_2$. Clearly the graph $(P_1 \cup P_2) - e$ is connected. It therefore contains an $(x, y)$-path $P$. But then $P + e$ is a cycle in the acyclic graph $G$, a contradiction □

The converse of this theorem holds for graphs without loops (exercise 2.1.1).

Observe that all the trees on six vertices (figure 2.1) have five edges. In general we have:

*Theorem 2.2* If $G$ is a tree, then $\varepsilon = \nu - 1$.

*Proof* By induction on $\nu$. When $\nu = 1$, $G \cong K_1$ and $\varepsilon = 0 = \nu - 1$.

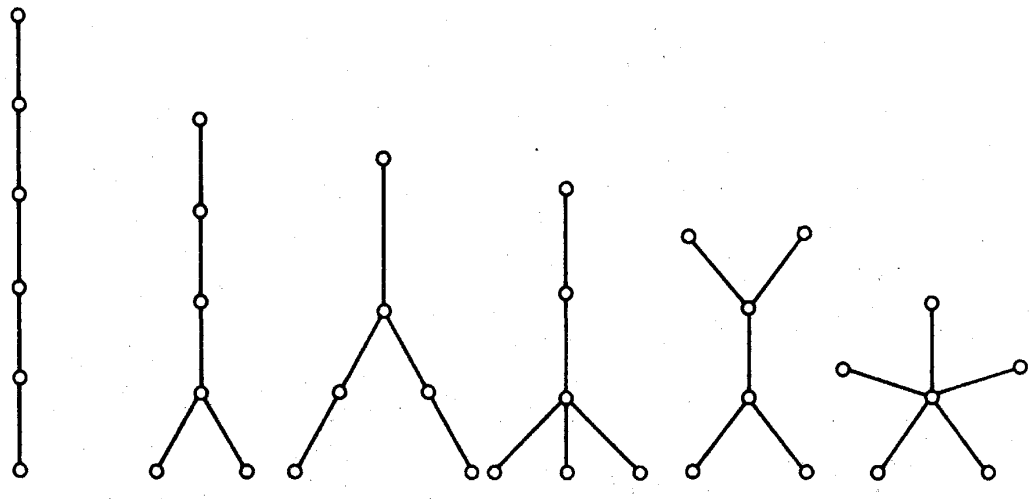


Figure 2.1. The trees on six vertices

Suppose the theorem true for all trees on fewer than $\nu$ vertices, and let $G$ be a tree on $\nu \geq 2$ vertices. Let $uv \in E$. Then $G - uv$ contains no $(u, v)$-path, since $uv$ is the unique $(u, v)$-path in $G$. Thus $G - uv$ is disconnected and so (exercise 1.6.8*a*) $\omega(G - uv) = 2$. The components $G_1$ and $G_2$ of $G - uv$, being acyclic, are trees. Moreover, each has fewer than $\nu$ vertices. Therefore, by the induction hypothesis

$$\varepsilon(G_i) = \nu(G_i) - 1 \quad \text{for} \quad i = 1, 2$$

Thus

$$\varepsilon(G) = \varepsilon(G_1) + \varepsilon(G_2) + 1 = \nu(G_1) + \nu(G_2) - 1 = \nu(G) - 1 \quad \square$$

*Corollary 2.2* Every nontrivial tree has at least two vertices of degree one.

*Proof* Let $G$ be a nontrivial tree. Then

$$d(v) \geq 1 \quad \text{for all} \quad v \in V$$

Also, by theorems 1.1 and 2.2, we have

$$\sum_{v \in V} d(v) = 2\varepsilon = 2\nu - 2$$

It now follows that $d(v) = 1$ for at least two vertices $v$ $\square$

Another, perhaps more illuminating, way of proving corollary 2.2 is to show that the origin and terminus of a longest path in a nontrivial tree both have degree one (see exercise 2.1.2).

*Exercises*

2.1.1 Show that if any two vertices of a loopless graph $G$ are connected by a unique path, then $G$ is a tree.

2.1.2 Prove corollary 2.2 by showing that the origin and terminus of a longest path in a nontrivial tree both have degree one.

2.1.3 Prove corollary 2.2 by using exercise 1.7.2.

2.1.4 Show that every tree with exactly two vertices of degree one is a path.

**2.1.5** Let $G$ be a graph with $\nu - 1$ edges. Show that the following three statements are equivalent:

(*a*) $G$ is connected;
(*b*) $G$ is acyclic;
(*c*) $G$ is a tree.

2.1.6 Show that if $G$ is a tree with $\Delta \geq k$, then $G$ has at least $k$ vertices of degree one.

2.1.7 An acyclic graph is also called a *forest*. Show that

(*a*) each component of a forest is a tree;
(*b*) $G$ is a forest if and only if $\varepsilon = \nu - \omega$.

2.1.8 A *centre* of $G$ is a vertex $u$ such that $\max_{v \in V} d(u, v)$ is as small as possible. Show that a tree has either exactly one centre or two, adjacent, centres.

2.1.9 Show that if $G$ is a forest with exactly $2k$ vertices of odd degree, then there are $k$ edge-disjoint paths $P_1, P_2, \ldots, P_k$ in $G$ such that $E(G) = E(P_1) \cup E(P_2) \cup \ldots \cup E(P_k)$.

2.1.10* Show that a sequence $(d_1, d_2, \ldots, d_\nu)$ of positive integers is a degree sequence of a tree if and only if $\sum_{i=1}^{\nu} d_i = 2(\nu - 1)$.

2.1.11 Let $T$ be an arbitrary tree on $k+1$ vertices. Show that if $G$ is simple and $\delta \geq k$ then $G$ has a subgraph isomorphic to $T$.

2.1.12 A saturated hydrocarbon is a molecule $C_mH_n$ in which every carbon atom has four bonds, every hydrogen atom has one bond, and no sequence of bonds forms a cycle. Show that, for every positive integer $m$, $C_mH_n$ can exist only if $n = 2m + 2$.

## 2.2 CUT EDGES AND BONDS

A *cut edge* of $G$ is an edge $e$ such that $\omega(G - e) > \omega(G)$. The graph of figure 2.2 has the three cut edges indicated.

*Theorem 2.3* An edge $e$ of $G$ is a cut edge of $G$ if and only if $e$ is contained in no cycle of $G$.

*Proof* Let $e$ be a cut edge of $G$. Since $\omega(G - e) > \omega(G)$, there exist vertices $u$ and $v$ of $G$ that are connected in $G$ but not in $G - e$. There is therefore some $(u, v)$-path $P$ in $G$ which, necessarily, traverses $e$. Suppose that $x$ and $y$ are the ends of $e$, and that $x$ precedes $y$ on $P$. In $G - e$, $u$ is connected to $x$ by a section of $P$ and $y$ is connected to $v$ by a section of $P$. If $e$ were in a cycle $C$, $x$ and $y$ would be connected in $G - e$ by the path $C - e$. Thus, $u$ and $v$ would be connected in $G - e$, a contradiction.

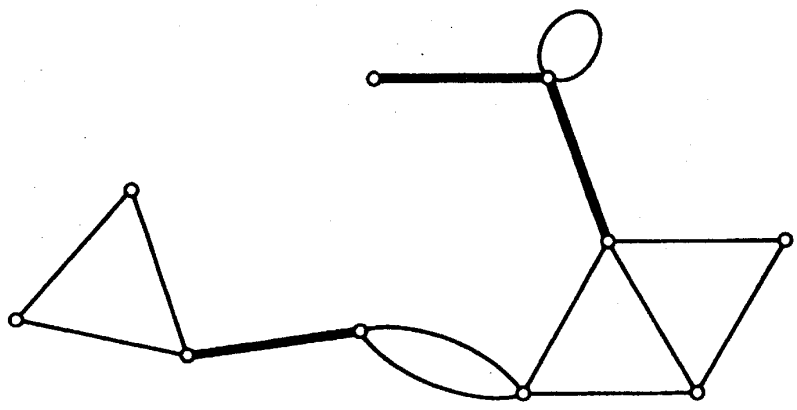


Figure 2.2. The cut edges of a graph

Conversely, suppose that $e = xy$ is not a cut edge of $G$; thus, $\omega(G-e) = \omega(G)$. Since there is an $(x, y)$-path (namely $xy$) in $G$, $x$ and $y$ are in the same component of $G$. It follows that $x$ and $y$ are in the same component of $G-e$, and hence that there is an $(x, y)$-path $P$ in $G-e$. But then $e$ is in the cycle $P+e$ of $G$ □

*Theorem 2.4* A connected graph is a tree if and only if every edge is a cut edge.

*Proof* Let $G$ be a tree and let $e$ be an edge of $G$. Since $G$ is acyclic, $e$ is contained in no cycle of $G$ and is therefore, by theorem 2.3, a cut edge of $G$.

Conversely, suppose that $G$ is connected but is not a tree. Then $G$ contains a cycle $C$. By theorem 2.3, no edge of $C$ can be a cut edge of $G$ □

A *spanning tree* of $G$ is a spanning subgraph of $G$ that is a tree.

*Corollary 2.4.1* Every connected graph contains a spanning tree.

*Proof* Let $G$ be connected and let $T$ be a minimal connected spanning subgraph of $G$. By definition $\omega(T) = 1$ and $\omega(T-e) > 1$ for each edge $e$ of $T$. It follows that each edge of $T$ is a cut edge and therefore, by theorem 2.4, that $T$, being connected, is a tree □

Figure 2.3 depicts a connected graph and one of its spanning trees.

*Corollary 2.4.2* If $G$ is connected, then $\varepsilon \geq \nu - 1$.

*Proof* Let $G$ be connected. By corollary 2.4.1, $G$ contains a spanning tree $T$. Therefore

$$\varepsilon(G) \geq \varepsilon(T) = \nu(T) - 1 = \nu(G) - 1 \quad \square$$

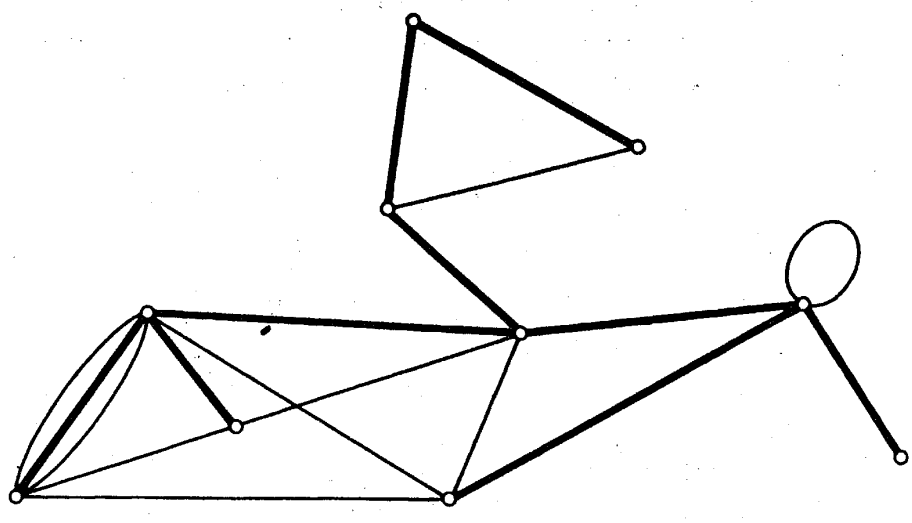
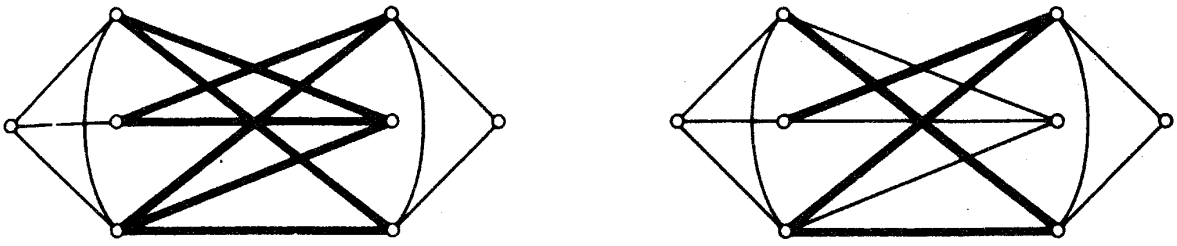


Figure 2.3. A spanning tree in a connected graph



(a) (b)

Figure 2.4. (*a*) An edge cut; (*b*) a bond

*Theorem 2.5* Let $T$ be a spanning tree of a connected graph $G$ and let $e$ be an edge of $G$ not in $T$. Then $T+e$ contains a unique cycle.

*Proof* Since $T$ is acyclic, each cycle of $T+e$ contains $e$. Moreover, $C$ is a cycle of $T+e$ if and only if $C-e$ is a path in $T$ connecting the ends of $e$. By theorem 2.1, $T$ has a unique such path; therefore $T+e$ contains a unique cycle □

For subsets $S$ and $S'$ of $V$, we denote by $[S, S']$ the set of edges with one end in $S$ and the other in $S'$. An *edge cut* of $G$ is a subset of $E$ of the form $[S, \bar{S}]$, where $S$ is a nonempty proper subset of $V$ and $\bar{S} = V \backslash S$. A minimal nonempty edge cut of $G$ is called a *bond*; each cut edge $e$, for instance, gives rise to a bond $\{e\}$. If $G$ is connected, then a bond $B$ of $G$ is a minimal subset of $E$ such that $G-B$ is disconnected. Figure 2.4 indicates an edge cut and a bond in a graph.

If $H$ is a subgraph of $G$, the *complement of H in G*, denoted by $\bar{H}(G)$, is the subgraph $G-E(H)$. If $G$ is connected, a subgraph of the form $\bar{T}$, where $T$ is a spanning tree, is called a *cotree* of $G$.

*Theorem 2.6* Let $T$ be a spanning tree of a connected graph $G$, and let $e$ be any edge of $T$. Then

(i) the cotree $\bar{T}$ contains no bond of $G$;
(ii) $\bar{T}+e$ contains a unique bond of $G$.

*Proof* (i) Let $B$ be a bond of $G$. Then $G-B$ is disconnected, and so cannot contain the spanning tree $T$. Therefore $B$ is not contained in $\bar{T}$. (ii) Denote by $S$ the vertex set of one of the two components of $T-e$. The edge cut $B=[S, \bar{S}]$ is clearly a bond of $G$, and is contained in $\bar{T}+e$. Now, for any $b \in B$, $T-e+b$ is a spanning tree of $G$. Therefore every bond of $G$ contained in $\bar{T}+e$ must include every such element $b$. It follows that $B$ is the only bond of $G$ contained in $\bar{T}+e$ □

The relationship between bonds and cotrees is analogous to that between cycles and spanning trees. Statement (i) of theorem 2.6 is the analogue for

bonds of the simple fact that a spanning tree is acyclic, and (ii) is the analogue of theorem 2.5. This 'duality' between cycles and bonds will be further explored in chapter 12 (see also exercise 2.2.10).

*Exercises*

2.2.1 Show that $G$ is a forest if and only if every edge of $G$ is a cut edge.

2.2.2 Let $G$ be connected and let $e \in E$. Show that

(*a*) $e$ is in every spanning tree of $G$ if and only if $e$ is a cut edge of $G$;

(*b*) $e$ is in no spanning tree of $G$ if and only if $e$ is a loop of $G$.

2.2.3 Show that if $G$ is loopless and has exactly one spanning tree $T$, then $G = T$.

**2.2.4** Let $F$ be a maximal forest of $G$. Show that

(*a*) for every component $H$ of $G$, $F \cap H$ is a spanning tree of $H$;

(*b*) $\varepsilon(F) = \nu(G) - \omega(G)$.

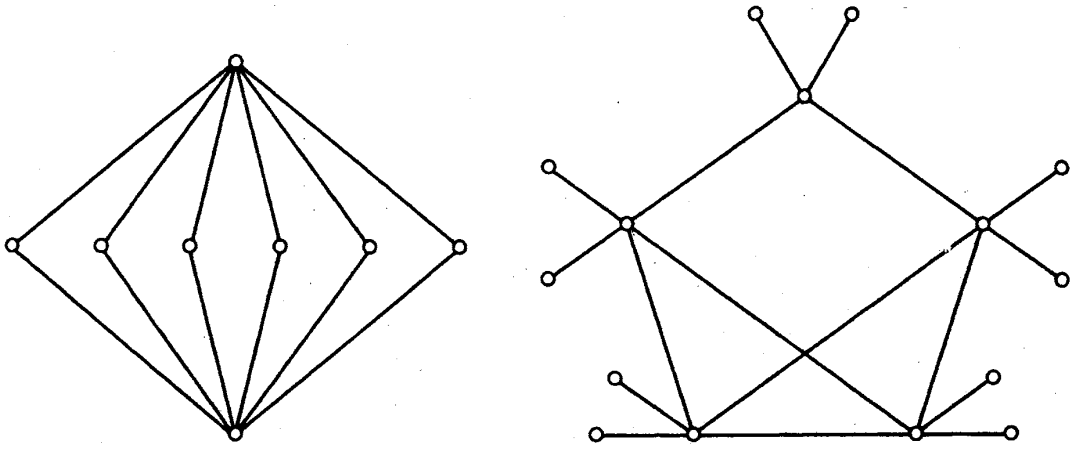
2.2.5 Show that $G$ contains at least $\varepsilon - \nu + \omega$ distinct cycles.

**2.2.6** Show that

(*a*) if each degree in $G$ is even, then $G$ has no cut edge;

(*b*) if $G$ is a $k$-regular bipartite graph with $k \geq 2$, then $G$ has no cut edge.

2.2.7 Find the number of nonisomorphic spanning trees in the following graphs:



2.2.8 Let $G$ be connected and let $S$ be a nonempty proper subset of $V$. Show that the edge cut $[S, \bar{S}]$ is a bond of $G$ if and only if both $G[S]$ and $G[\bar{S}]$ are connected.

2.2.9 Show that every edge cut is a disjoint union of bonds.

2.2.10 Let $B_1$ and $B_2$ be bonds and let $C_1$ and $C_2$ be cycles (regarded as

sets of edges) in a graph. Show that

(*a*) $B_1 \Delta B_2$ is a disjoint union of bonds;
(*b*) $C_1 \Delta C_2$ is a disjoint union of cycles,

where $\Delta$ denotes symmetric difference;

(*c*) for any edge $e$, $(B_1 \cup B_2)\backslash\{e\}$ contains a bond;
(*d*) for any edge $e$, $(C_1 \cup C_2)\backslash\{e\}$ contains a cycle.

2.2.11 Show that if a graph $G$ contains $k$ edge-disjoint spanning trees then, for each partition $(V_1, V_2, \ldots, V_n)$ of $V$, the number of edges which have ends in different parts of the partition is at least $k(n-1)$.

(Tutte, 1961 and Nash-Williams, 1961 have shown that this necessary condition for $G$ to contain $k$ edge-disjoint spanning trees is also sufficient.)

2.2.12* Let $S$ be an $n$-element set, and let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of $n$ distinct subsets of $S$. Show that there is an element $x \in S$ such that the sets $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_n \cup \{x\}$ are all distinct.

## 2.3 CUT VERTICES

A vertex $v$ of $G$ is a *cut vertex* if $E$ can be partitioned into two nonempty subsets $E_1$ and $E_2$ such that $G[E_1]$ and $G[E_2]$ have just the vertex $v$ in common. If $G$ is loopless and nontrivial, then $v$ is a cut vertex of $G$ if and only if $\omega(G-v) > \omega(G)$. The graph of figure 2.5 has the five cut vertices indicated.

*Theorem* 2.7 A vertex $v$ of a tree $G$ is a cut vertex of $G$ if and only if $d(v) > 1$.

*Proof* If $d(v) = 0$, $G \cong K_1$ and, clearly, $v$ is not a cut vertex.

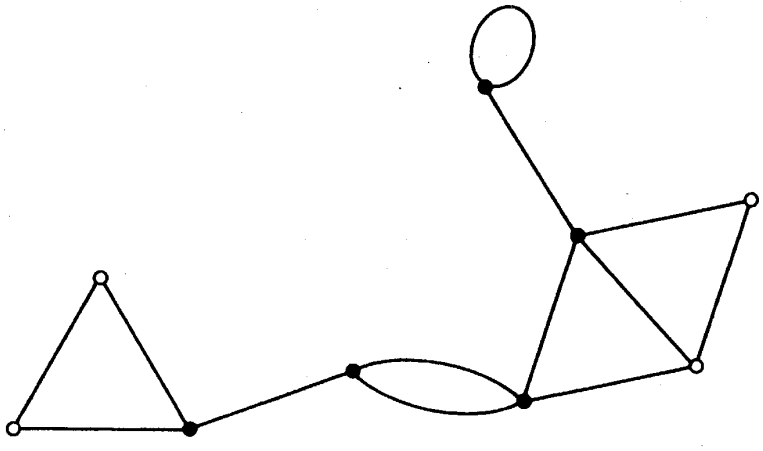


Figure 2.5. The cut vertices of a graph

If $d(v)=1$, $G-v$ is an acyclic graph with $\nu(G-v)-1$ edges, and thus (exercise 2.1.5) a tree. Hence $\omega(G-v)=1=\omega(G)$, and $v$ is not a cut vertex of $G$.

If $d(v)>1$, there are distinct vertices $u$ and $w$ adjacent to $v$. The path $uvw$ is a $(u, w)$-path in $G$. By theorem 2.1 $uvw$ is the unique $(u, w)$-path in $G$. It follows that there is no $(u, w)$-path in $G-v$, and therefore that $\omega(G-v)>1=\omega(G)$. Thus $v$ is a cut vertex of $G$ □

*Corollary 2.7* Every nontrivial loopless connected graph has at least two vertices that are not cut vertices.

*Proof* Let $G$ be a nontrivial loopless connected graph. By corollary 2.4.1, $G$ contains a spanning tree $T$. By corollary 2.2 and theorem 2.7, $T$ has at least two vertices that are not cut vertices. Let $v$ be any such vertex. Then

$$\omega(T-v)=1$$

Since $T$ is a spanning subgraph of $G$, $T-v$ is a spanning subgraph of $G-v$ and therefore

$$\omega(G-v)\leq\omega(T-v)$$

It follows that $\omega(G-v)=1$, and hence that $v$ is not a cut vertex of $G$. Since there are at least two such vertices $v$, the proof is complete □

*Exercises*

**2.3.1** Let $G$ be connected with $\nu\geq 3$. Show that

(*a*) if $G$ has a cut edge, then $G$ has a vertex $v$ such that $\omega(G-v)>\omega(G)$;

(*b*) the converse of (*a*) is not necessarily true.

2.3.2 Show that a simple connected graph that has exactly two vertices which are not cut vertices is a path.

## 2.4 CAYLEY'S FORMULA

There is a simple and elegant recursive formula for the number of spanning trees in a graph. It involves the operation of contraction of an edge, which we now introduce. An edge $e$ of $G$ is said to be *contracted* if it is deleted and its ends are identified; the resulting graph is denoted by $G\cdot e$. Figure 2.6 illustrates the effect of contracting an edge.

It is clear that if $e$ is a link of $G$, then

$$\nu(G\cdot e)=\nu(G)-1 \qquad \varepsilon(G\cdot e)=\varepsilon(G)-1 \quad \text{and} \quad \omega(G\cdot e)=\omega(G)$$

Therefore, if $T$ is a tree, so too is $T\cdot e$.

We denote the number of spanning trees of $G$ by $\tau(G)$.

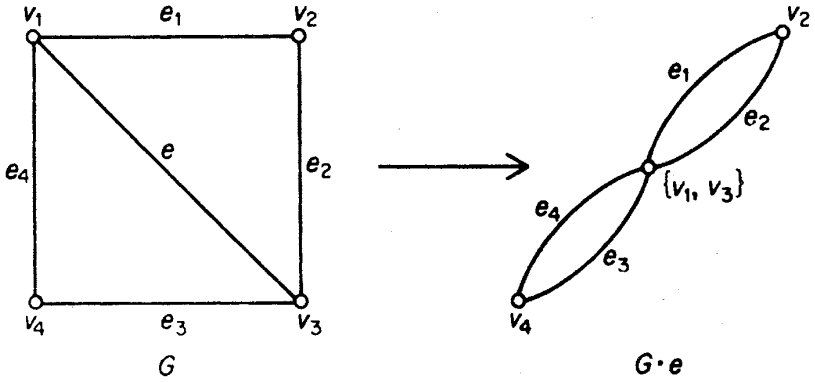



Figure 2.6. Contraction of an edge

*Theorem 2.8* If $e$ is a link of $G$, then $\tau(G) = \tau(G-e) + \tau(G \cdot e)$.

*Proof* Since every spanning tree of $G$ that does not contain $e$ is also a spanning tree of $G-e$, and conversely, $\tau(G-e)$ is the number of spanning trees of $G$ that do not contain $e$.

Now to each spanning tree $T$ of $G$ that contains $e$, there corresponds a spanning tree $T \cdot e$ of $G \cdot e$. This correspondence is clearly a bijection (see figure 2.7). Therefore $\tau(G \cdot e)$ is precisely the number of spanning trees of $G$ that contain $e$. It follows that $\tau(G) = \tau(G-e) + \tau(G \cdot e)$ □

Figure 2.8 illustrates the recursive calculation of $\tau(G)$ by means of theorem 2.8; the number of spanning trees in a graph is represented symbolically by the graph itself.

Although theorem 2.8 provides a method of calculating the number of spanning trees in a graph, this method is not suitable for large graphs. Fortunately, and rather surprisingly, there is a closed formula for $\tau(G)$ which expresses $\tau(G)$ as a determinant; we shall present this result in chapter 12. In the special case when $G$ is complete, a simple formula for $\tau(G)$ was discovered by Cayley (1889). The proof we give is due to Prüfer (1918).

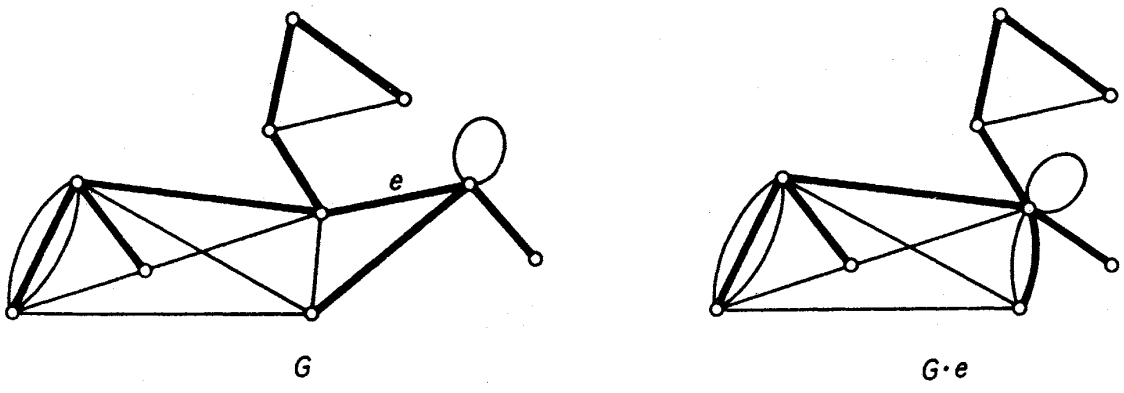



Figure 2.7

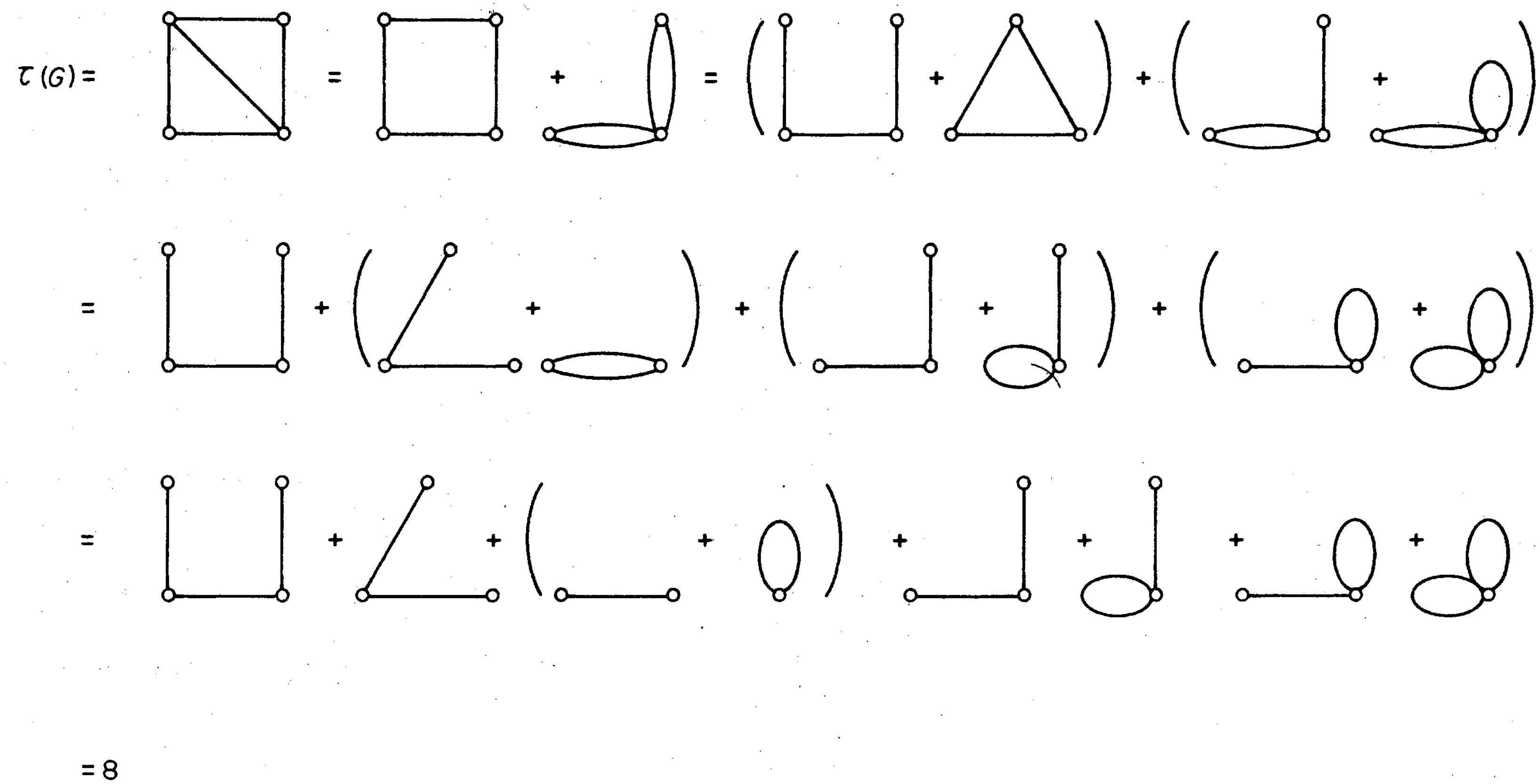



Figure 2.8. Recursive calculation of $\tau(G)$

*Theorem 2.9* $\tau(K_n) = n^{n-2}$.

*Proof* Let the vertex set of $K_n$ be $N = \{1, 2, \ldots, n\}$. We note that $n^{n-2}$ is the number of sequences of length $n-2$ that can be formed from $N$. Thus, to prove the theorem, it suffices to establish a one–one correspondence between the set of spanning trees of $K_n$ and the set of such sequences.

With each spanning tree $T$ of $K_n$, we associate a unique sequence $(t_1, t_2, \ldots, t_{n-2})$ as follows. Regarding $N$ as an ordered set, let $s_1$ be the first vertex of degree one in $T$; the vertex adjacent to $s_1$ is taken as $t_1$. We now delete $s_1$ from $T$, denote by $s_2$ the first vertex of degree one in $T - s_1$, and take the vertex adjacent to $s_2$ as $t_2$. This operation is repeated until $t_{n-2}$ has been defined and a tree with just two vertices remains; the tree in figure 2.9, for instance, gives rise to the sequence (4, 3, 5, 3, 4, 5). It can be seen that different spanning trees of $K_n$ determine difference sequences.

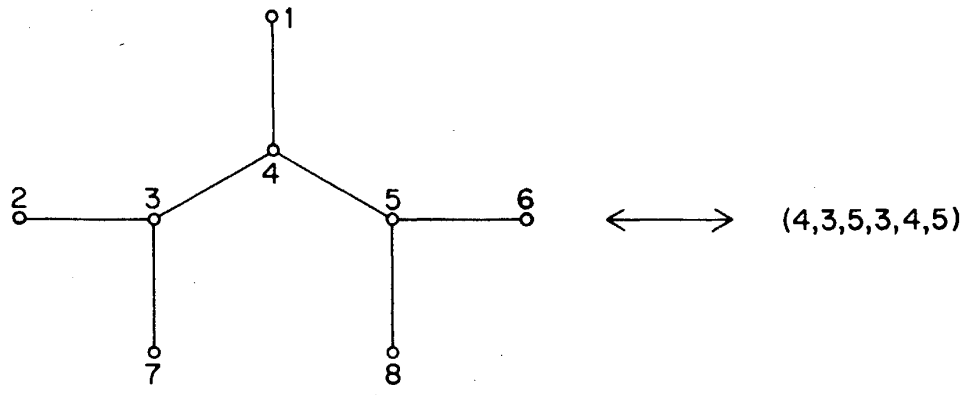



Figure 2.9

The reverse procedure is equally straightforward. Observe, first, that any vertex $v$ of $T$ occurs $d_T(v) - 1$ times in $(t_1, t_2, \ldots, t_{n-2})$. Thus the vertices of degree one in $T$ are precisely those that do not appear in this sequence. To reconstruct $T$ from $(t_1, t_2, \ldots, t_{n-2})$, we therefore proceed as follows. Let $s_1$ be the first vertex of $N$ not in $(t_1, t_2, \ldots, t_{n-2})$; join $s_1$ to $t_1$. Next, let $s_2$ be the first vertex of $N \backslash \{s_1\}$ not in $(t_2, \ldots, t_{n-2})$, and join $s_2$ to $t_2$. Continue in this way until the $n-2$ edges $s_1t_1, s_2t_2, \ldots, s_{n-2}t_{n-2}$ have been determined. $T$ is now obtained by adding the edge joining the two remaining vertices of $N \backslash \{s_1, s_2, \ldots, s_{n-2}\}$. It is easily verified that different sequences give rise to different spanning trees of $K_n$. We have thus established the desired one–one correspondence □

Note that $n^{n-2}$ is not the number of nonisomorphic spanning trees of $K_n$, but the number of distinct spanning trees of $K_n$; there are just six nonisomorphic spanning trees of $K_6$ (see figure 2.1), whereas there are $6^4 = 1296$ distinct spanning trees of $K_6$.

*Exercises*

2.4.1 Using the recursion formula of theorem 2.8, evaluate the number of spanning trees in $K_{3,3}$.

2.4.2* A *wheel* is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle; the new edges are called the *spokes* of the wheel. Obtain an expression for the number of spanning trees in a wheel with $n$ spokes.

2.4.3 Draw all sixteen spanning trees of $K_4$.

2.4.4 Show that if $e$ is an edge of $K_n$, then $\tau(K_n - e) = (n-2)n^{n-3}$.

2.4.5 (*a*) Let $H$ be a graph in which every two adjacent vertices are joined by $k$ edges and let $G$ be the underlying simple graph of $H$. Show that $\tau(H) = k^{\nu-1}\tau(G)$.

(*b*) Let $H$ be the graph obtained from a graph $G$ when each edge of $G$ is replaced by a path of length $k$. Show that $\tau(H) = k^{\epsilon-\nu+1}\tau(G)$.

(*c*) Deduce from (*b*) that $\tau(K_{2,n}) = n2^{n-1}$.

## APPLICATIONS

### 2.5 THE CONNECTOR PROBLEM

A railway network connecting a number of towns is to be set up. Given the cost $c_{ij}$ of constructing a direct link between towns $v_i$ and $v_j$, design such a network to minimise the total cost of construction. This is known as the *connector problem*.

By regarding each town as a vertex in a weighted graph with weights $w(v_iv_j) = c_{ij}$, it is clear that this problem is just that of finding, in a weighted graph $G$, a connected spanning subgraph of minimum weight. Moreover, since the weights represent costs, they are certainly non-negative, and we may therefore assume that such a minimum-weight spanning subgraph is a spanning tree $T$ of $G$. A minimum-weight spanning tree of a weighted graph will be called an *optimal tree*; the spanning tree indicated in the weighted graph of figure 2.10 is an optimal tree (exercise 2.5.1).

We shall now present a good algorithm for finding an optimal tree in a nontrivial weighted connected graph, thereby solving the connector problem.

Consider, first, the case when each weight $w(e) = 1$. An optimal tree is then a spanning tree with as few edges as possible. Since each spanning tree of a graph has the same number of edges (theorem 2.2), in this special case we merely need to construct some spanning tree of the graph. A simple

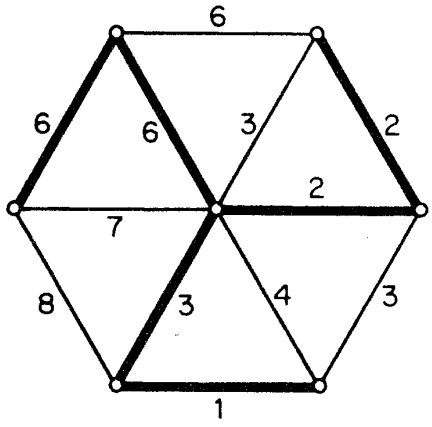



Figure 2.10. An optimal tree in a weighted graph

inductive algorithm for finding such a tree is the following:

1. Choose a link $e_1$.
2. If edges $e_1, e_2, \ldots, e_i$ have been chosen, then choose $e_{i+1}$ from $E \backslash \{e_1, e_2, \ldots, e_i\}$ in such a way that $G[\{e_1, e_2, \ldots, e_{i+1}\}]$ is acyclic.
3. Stop when step 2 cannot be implemented further.

This algorithm works because a maximal acyclic subgraph of a connected graph is necessarily a spanning tree. It was extended by Kruskal (1956) to solve the general problem; his algorithm is valid for arbitrary real weights.

*Kruskal's Algorithm*

1. Choose a link $e_1$ such that $w(e_1)$ is as small as possible.
2. If edges $e_1, e_2, \ldots, e_i$ have been chosen, then choose an edge $e_{i+1}$ from $E \backslash \{e_1, e_2, \ldots, e_i\}$ in such a way that
   (i) $G[\{e_1, e_2, \ldots, e_{i+1}\}]$ is acyclic;
   (ii) $w(e_{i+1})$ is as small as possible subject to (i).
3. Stop when step 2 cannot be implemented further.

As an example, consider the table of airline distances in miles between six of the largest cities in the world, London, Mexico City, New York, Paris, Peking and Tokyo:

| | L | MC | NY | Pa | Pe | T |
|---|---|---|---|---|---|---|
| L | — | 5558 | 3469 | 214 | 5074 | 5959 |
| MC | 5558 | — | 2090 | 5725 | 7753 | 7035 |
| NY | 3469 | 2090 | — | 3636 | 6844 | 6757 |
| Pa | 214 | 5725 | 3636 | — | 5120 | 6053 |
| Pe | 5074 | 7753 | 6844 | 5120 | — | 1307 |
| T | 5959 | 7035 | 6757 | 6053 | 1307 | — |

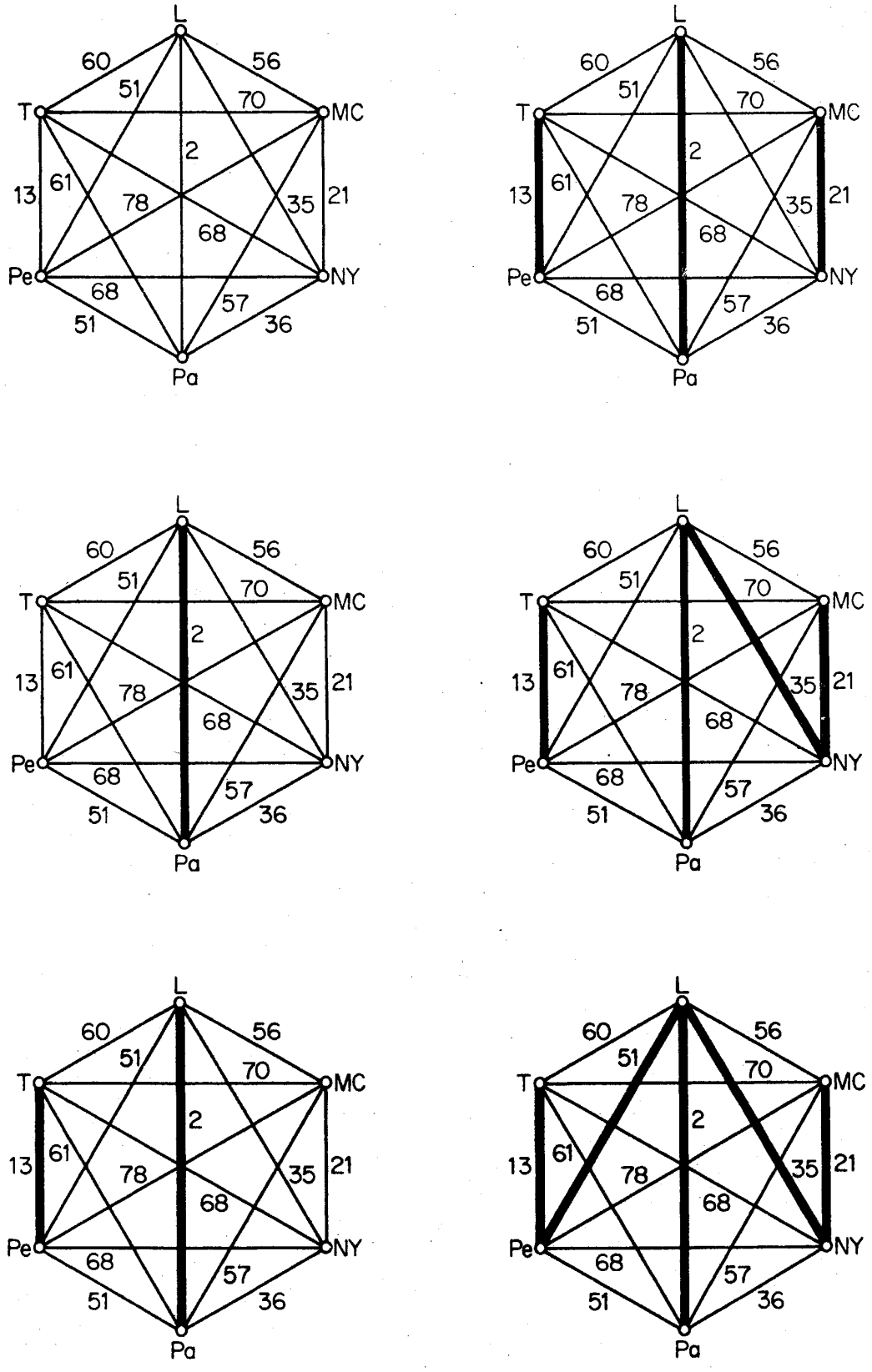



Figure 2.11

This table determines a weighted complete graph with vertices L, MC, NY, Pa, Pe and T. The construction of an optimal tree in this graph is shown in figure 2.11 (where, for convenience, distances are given in hundreds of miles).

Kruskal's algorithm clearly produces a spanning tree (for the same reason that the simpler algorithm above does). The following theorem ensures that such a tree will always be optimal.

*Theorem 2.10* Any spanning tree $T^* = G[\{e_1, e_2, \ldots, e_{\nu-1}\}]$ constructed by Kruskal's algorithm is an optimal tree.

*Proof* By contradiction. For any spanning tree $T$ of $G$ other than $T^*$, denote by $f(T)$ the smallest value of $i$ such that $e_i$ is not in $T$. Now assume that $T^*$ is not an optimal tree, and let $T$ be an optimal tree such that $f(T)$ is as large as possible.

Suppose that $f(T) = k$; this means that $e_1, e_2, \ldots, e_{k-1}$ are in both $T$ and $T^*$, but that $e_k$ is not in $T$. By theorem 2.5, $T + e_k$ contains a unique cycle $C$. Let $e'_k$ be an edge of $C$ that is in $T$ but not in $T^*$. By theorem 2.3, $e'_k$ is not a cut edge of $T + e_k$. Hence $T' = (T + e_k) - e'_k$ is a connected graph with $\nu - 1$ edges, and therefore (exercise 2.1.5) is another spanning tree of $G$. Clearly

$$w(T') = w(T) + w(e_k) - w(e'_k) \tag{2.1}$$

Now, in Kruskal's algorithm, $e_k$ was chosen as an edge with the smallest weight such that $G[\{e_1, e_2, \ldots, e_k\}]$ was acyclic. Since $G[\{e_1, e_2, \ldots, e_{k-1}, e'_k\}]$ is a subgraph of $T$, it is also acyclic. We conclude that

$$w(e'_k) \geq w(e_k) \tag{2.2}$$

Combining (2.1) and (2.2) we have

$$w(T') \leq w(T)$$

and so $T'$, too, is an optimal tree. However

$$f(T') > k = f(T)$$

contradicting the choice of $T$. Therefore $T = T^*$, and $T^*$ is indeed an optimal tree □

A flow diagram for Kruskal's algorithm is shown in figure 2.12. The edges are first sorted in order of increasing weight (box 1); this takes about $\varepsilon \log \varepsilon$ computations (see Knuth, 1973). Box 2 just checks to see how many edges have been chosen. ($S$ is the set of edges already chosen and $i$ is their number.) When $i = \nu - 1$, $S = \{e_1, e_2, \ldots, e_{\nu-1}\}$ is the edge set of an optimal tree $T^*$ of $G$. In box 3, to check if $G[S \cup \{a_j\}]$ is acyclic, one must ascertain whether the ends of $a_j$ are in different components of the forest $G[S]$ or not. This can be achieved in the following way. The vertices are labelled so that, at any stage, two vertices belong to the same component of $G[S]$ if and only

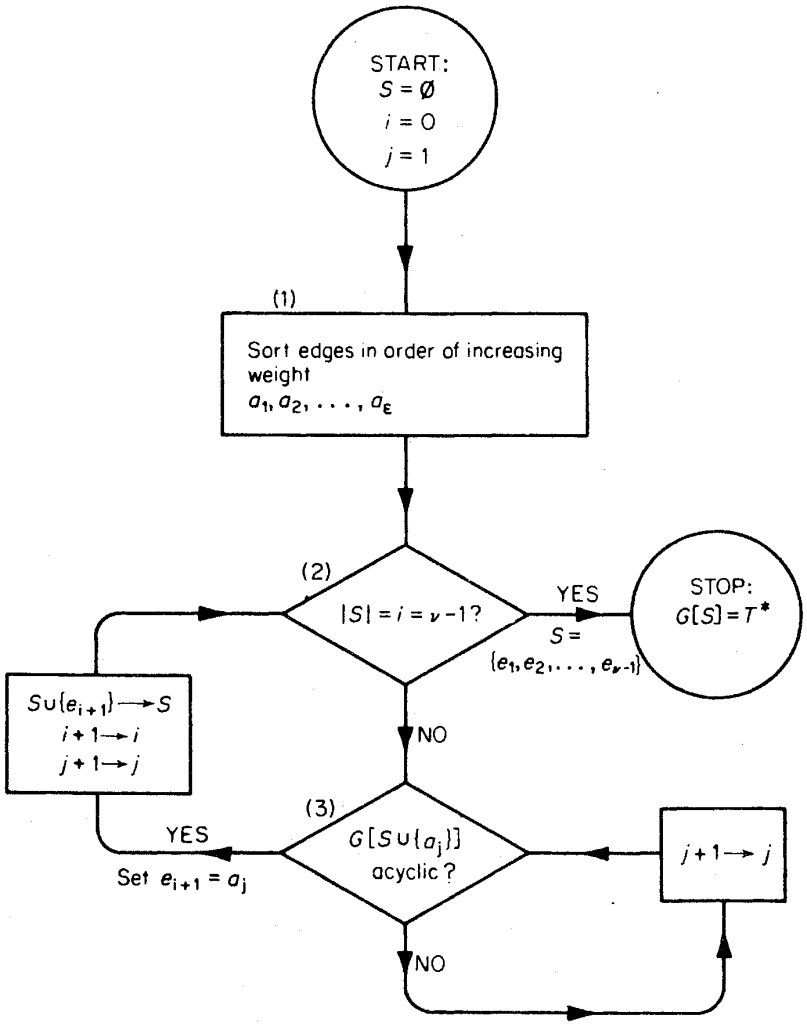



Figure 2.12. Kruskal's algorithm

if they have the same label; initially, vertex $v_l$ is assigned the label $l$, $1 \leq l \leq \nu$. With this labelling scheme, $G[S \cup \{a_j\}]$ is acyclic if and only if the ends of $a_j$ have different labels. If this is the case, $a_j$ is taken as $e_{i+1}$; otherwise, $a_j$ is discarded and $a_{j+1}$, the next candidate for $e_{i+1}$, is tested. Once $e_{i+1}$ has been added to $S$, the vertices in the two components of $G[S]$ that contain the ends of $e_{i+1}$ are relabelled with the smaller of their two labels. For each edge, one comparison suffices to check whether its ends have the same or different labels; this takes $\varepsilon$ computations. After edge $e_{i+1}$ has been added to $S$, the relabelling of vertices takes at most $\nu$ comparisons; hence, for all $\nu - 1$ edges $e_1, e_2, \ldots, e_{\nu-1}$ we need $\nu(\nu - 1)$ computations. Kruskal's algorithm is therefore a good algorithm.

*Exercises*

2.5.1 Show, by applying Kruskal's algorithm, that the tree indicated in figure 2.10 is indeed optimal.

2.5.2 Adapt Kruskal's algorithm to solve the *connector problem with preassignments*: construct, at minimum cost, a network linking a number of towns, with the additional requirement that certain selected pairs of towns be directly linked.

2.5.3 Can Kruskal's algorithm be adapted to find

(*a*) a *maximum*-weight tree in a weighted connected graph?
(*b*) a minimum-weight maximal *forest* in a weighted graph?

If so, how?

2.5.4 Show that the following Kruskal-type algorithm does not necessarily yield a minimum-weight spanning *path* in a weighted complete graph:

1. Choose a link $e_1$ such that $w(e_1)$ is as small as possible.
2. If edges $e_1, e_2, \ldots, e_i$ have been chosen, then choose an edge $e_{i+1}$ from $E \backslash \{e_1, e_2, \ldots, e_i\}$ in such a way that
   (i) $G[\{e_1, e_2, \ldots, e_{i+1}\}]$ is a union of disjoint paths;
   (ii) $w(e_{i+1})$ is as small as possible subject to (i).
3. Stop when step 2 cannot be implemented further.

2.5.5 The *tree graph* of a connected graph $G$ is the graph whose vertices are the spanning trees $T_1, T_2, \ldots, T_\tau$ of $G$, with $T_i$ and $T_j$ joined if and only if they have exactly $\nu - 2$ edges in common. Show that the tree graph of any connected graph is connected.

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# 3 Connectivity

## 3.1 CONNECTIVITY

In section 1.6 we introduced the concept of connection in graphs. Consider, now, the four connected graphs of figure 3.1.

$G_1$ is a tree, a minimal connected graph; deleting any edge disconnects it. $G_2$ cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in $G_3$, but even so $G_3$ is clearly not as well connected as $G_4$, the complete graph on five vertices. Thus, intuitively, each successive graph is more strongly connected than the previous one. We shall now define two parameters of a graph, its connectivity and edge connectivity, which measure the extent to which it is connected.

A *vertex cut* of $G$ is a subset $V'$ of $V$ such that $G-V'$ is disconnected. A *$k$-vertex cut* is a vertex cut of $k$ elements. A complete graph has no vertex cut; in fact, the only graphs which do not have vertex cuts are those that contain complete graphs as spanning subgraphs. If $G$ has at least one pair of distinct nonadjacent vertices, the *connectivity* $\kappa(G)$ of $G$ is the minimum $k$ for which $G$ has a $k$-vertex cut; otherwise, we define $\kappa(G)$ to be $\nu-1$. Thus $\kappa(G)=0$ if $G$ is either trivial or disconnected. $G$ is said to be *$k$-connected* if $\kappa(G)\geq k$. All nontrivial connected graphs are 1-connected.

Recall that an edge cut of $G$ is a subset of $E$ of the form $[S, \bar{S}]$, where $S$ is a nonempty proper subset of $V$. A *$k$-edge cut* is an edge cut of $k$ elements. If $G$ is nontrivial and $E'$ is an edge cut of $G$, then $G-E'$ is disconnected; we then define the *edge connectivity* $\kappa'(G)$ of $G$ to be the minimum $k$ for which $G$ has a $k$-edge cut. If $G$ is trivial, $\kappa'(G)$ is defined to be zero. Thus $\kappa'(G)=0$ if $G$ is either trivial or disconnected, and $\kappa'(G)=1$ if $G$ is a connected graph with a cut edge. $G$ is said to be *$k$-edge-connected* if $\kappa'(G)\geq k$. All nontrivial connected graphs are 1-edge-connected.

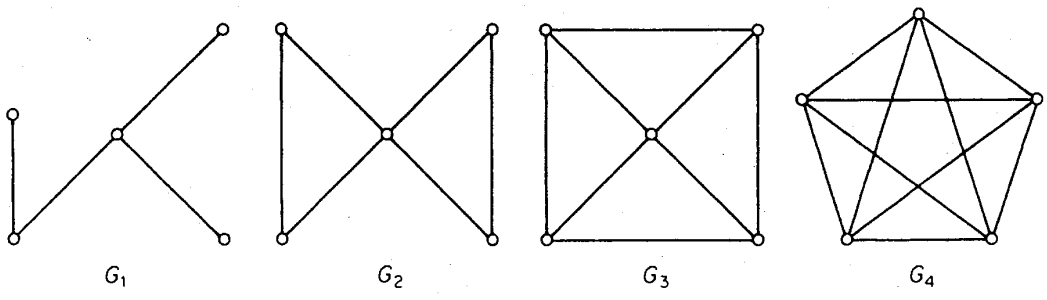


Figure 3.1

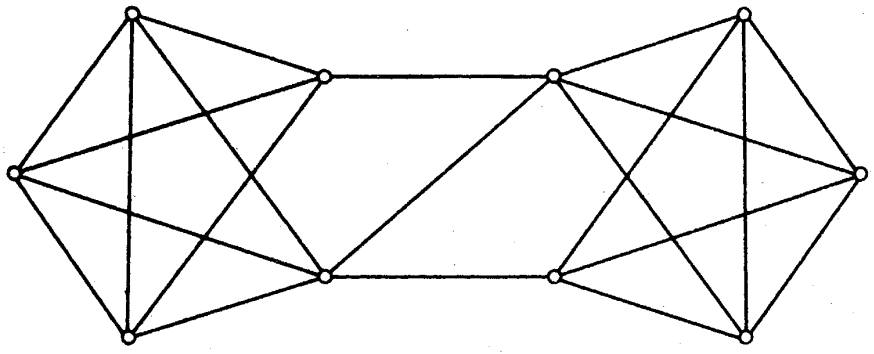

Figure 3.2

*Theorem 3.1* $\kappa \leq \kappa' \leq \delta$.

*Proof* If $G$ is trivial, then $\kappa' = 0 \leq \delta$. Otherwise, the set of links incident with a vertex of degree $\delta$ constitute a $\delta$-edge cut of $G$. It follows that $\kappa' \leq \delta$.

We prove that $\kappa \leq \kappa'$ by induction on $\kappa'$. The result is true if $\kappa' = 0$, since then $G$ must be either trivial or disconnected. Suppose that it holds for all graphs with edge connectivity less than $k$, let $G$ be a graph with $\kappa'(G) = k > 0$, and let $e$ be an edge in a $k$-edge cut of $G$. Setting $H = G - e$, we have $\kappa'(H) = k - 1$ and so, by the induction hypothesis, $\kappa(H) \leq k - 1$.

If $H$ contains a complete graph as a spanning subgraph, then so does $G$ and

$$\kappa(G) = \kappa(H) \leq k - 1$$

Otherwise, let $S$ be a vertex cut of $H$ with $\kappa(H)$ elements. Since $H - S$ is disconnected, either $G - S$ is disconnected, and then

$$\kappa(G) \leq \kappa(H) \leq k - 1$$

or else $G - S$ is connected and $e$ is a cut edge of $G - S$. In this latter case, either $\nu(G - S) = 2$ and

$$\kappa(G) \leq \nu(G) - 1 = \kappa(H) + 1 \leq k$$

or (exercise 2.3.1*a*) $G - S$ has a 1-vertex cut $\{v\}$, implying that $S \cup \{v\}$ is a vertex cut of $G$ and

$$\kappa(G) \leq \kappa(H) + 1 \leq k$$

Thus in each case we have $\kappa(G) \leq k = \kappa'(G)$. The result follows by the principle of induction □

The inequalities in theorem 3.1 are often strict. For example, the graph $G$ of figure 3.2 has $\kappa = 2$, $\kappa' = 3$ and $\delta = 4$.

*Exercises*

3.1.1 (*a*) Show that if $G$ is $k$-edge-connected, with $k>0$, and if $E'$ is a set of $k$ edges of $G$, then $\omega(G-E')\leq 2$.
(*b*) For $k>0$, find a $k$-connected graph $G$ and a set $V'$ of $k$ vertices of $G$ such that $\omega(G-V')>2$.

3.1.2 Show that if $G$ is $k$-edge-connected, then $\varepsilon \geq k\nu/2$.

3.1.3 (*a*) Show that if $G$ is simple and $\delta \geq \nu-2$, then $\kappa=\delta$.
(*b*) Find a simple graph $G$ with $\delta=\nu-3$ and $\kappa<\delta$.

3.1.4 (*a*) Show that if $G$ is simple and $\delta \geq \nu/2$, then $\kappa'=\delta$.
(*b*) Find a simple graph $G$ with $\delta=[(\nu/2)-1]$ and $\kappa'<\delta$.

3.1.5 Show that if $G$ is simple and $\delta \geq (\nu+k-2)/2$, then $G$ is $k$-connected.

3.1.6 Show that if $G$ is simple and 3-regular, then $\kappa=\kappa'$.

3.1.7 Show that if $l$, $m$ and $n$ are integers such that $0<l\leq m\leq n$, then there exists a simple graph $G$ with $\kappa=l$, $\kappa'=m$, and $\delta=n$.

(G. Chartrand and F. Harary)

## 3.2 BLOCKS

A connected graph that has no cut vertices is called a *block*. Every block with at least three vertices is 2-connected. A *block of a graph* is a subgraph that is a block and is maximal with respect to this property. Every graph is the union of its blocks; this is illustrated in figure 3.3.

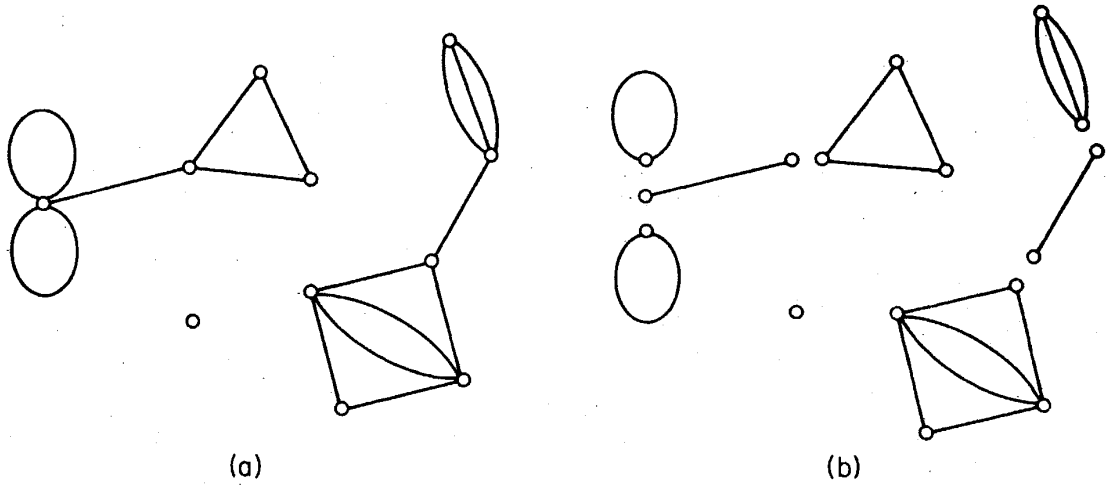


Figure 3.3. (*a*) $G$; (*b*) the blocks of $G$

A family of paths in $G$ is said to be *internally-disjoint* if no vertex of $G$ is an internal vertex of more than one path of the family. The following theorem is due to Whitney (1932).

*Theorem 3.2* A graph $G$ with $\nu \geq 3$ is 2-connected if and only if any two vertices of $G$ are connected by at least two internally-disjoint paths.

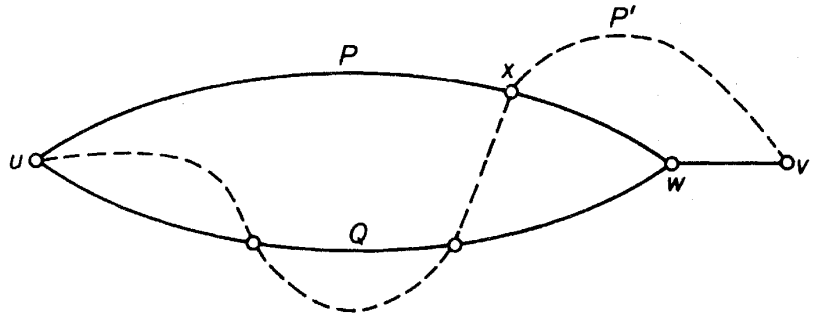


Figure 3.4

*Proof* If any two vertices of $G$ are connected by at least two internally-disjoint paths then, clearly, $G$ is connected and has no 1-vertex cut. Hence $G$ is 2-connected.

Conversely, let $G$ be a 2-connected graph. We shall prove, by induction on the distance $d(u, v)$ between $u$ and $v$, that any two vertices $u$ and $v$ are connected by at least two internally-disjoint paths.

Suppose, first, that $d(u, v) = 1$. Then, since $G$ is 2-connected, the edge $uv$ is not a cut edge and therefore, by theorem 2.3, it is contained in a cycle. It follows that $u$ and $v$ are connected by two internally-disjoint paths in $G$.

Now assume that the theorem holds for any two vertices at distance less than $k$, and let $d(u, v) = k \geq 2$. Consider a $(u, v)$-path of length $k$, and let $w$ be the vertex that precedes $v$ on this path. Since $d(u, w) = k - 1$, it follows from the induction hypothesis that there are two internally-disjoint $(u, w)$-paths $P$ and $Q$ in $G$. Also, since $G$ is 2-connected, $G - w$ is connected and so contains a $(u, v)$-path $P'$. Let $x$ be the last vertex of $P'$ that is also in $P \cup Q$ (see figure 3.4). Since $u$ is in $P \cup Q$, there is such an $x$; we do not exclude the possibility that $x = v$.

We may assume, without loss of generality, that $x$ is in $P$. Then $G$ has two internally-disjoint $(u, v)$-paths, one composed of the section of $P$ from $u$ to $x$ together with the section of $P'$ from $x$ to $v$, and the other composed of $Q$ together with the path $wv$ □

*Corollary 3.2.1* If $G$ is 2-connected, then any two vertices of $G$ lie on a common cycle.

*Proof* This follows immediately from theorem 3.2 since two vertices lie on a common cycle if and only if they are connected by two internally disjoint paths □

It is convenient, now, to introduce the operation of subdivision of an edge. An edge $e$ is said to be *subdivided* when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path being a new vertex. This is illustrated in figure 3.5.

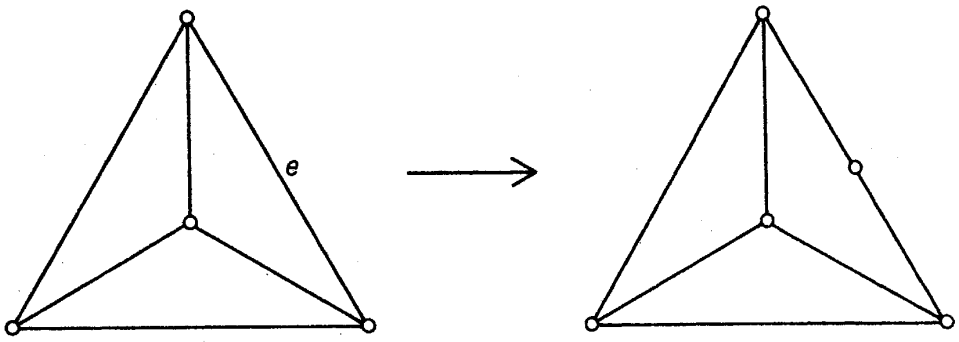


Figure 3.5. Subdivision of an edge

It can be seen that the class of blocks with at least three vertices is closed under the operation of subdivision. The proof of the next corollary uses this fact.

*Corollary 3.2.2* If $G$ is a block with $\nu \geq 3$, then any two edges of $G$ lie on a common cycle.

*Proof* Let $G$ be a block with $\nu \geq 3$, and let $e_1$ and $e_2$ be two edges of $G$. Form a new graph $G'$ by subdividing $e_1$ and $e_2$, and denote the new vertices by $v_1$ and $v_2$. Clearly, $G'$ is a block with at least five vertices, and hence is 2-connected. It follows from corollary 3.2.1 that $v_1$ and $v_2$ lie on a common cycle of $G'$. Thus $e_1$ and $e_2$ lie on a common cycle of $G$ (see figure 3.6) □

Theorem 3.2 has a generalisation to $k$-connected graphs, known as *Menger's theorem*: a graph $G$ with $\nu \geq k+1$ is $k$-connected if and only if any two distinct vertices of $G$ are connected by at least $k$ internally-disjoint paths. There is also an edge analogue of this theorem: a graph $G$ is $k$-edge-connected if and only if any two distinct vertices of $G$ are connected

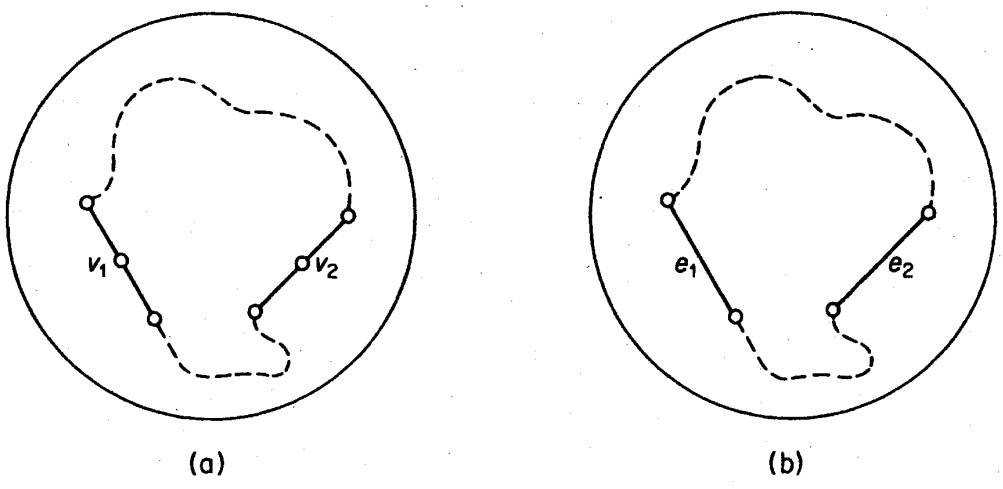


Figure 3.6. (*a*) $G'$; (*b*) $G$

by at least $k$ edge-disjoint paths. Proofs of these theorems will be given in chapter 11.

*Exercises*

**3.2.1** Show that a graph is 2-edge-connected if and only if any two vertices are connected by at least two edge-disjoint paths.

3.2.2 Give an example to show that if $P$ is a $(u, v)$-path in a 2-connected graph $G$, then $G$ does not necessarily contain a $(u, v)$-path $Q$ internally-disjoint from $P$.

3.2.3 Show that if $G$ has no even cycles, then each block of $G$ is either $K_1$ or $K_2$, or an odd cycle.

3.2.4 Show that a connected graph which is not a block has at least two blocks that each contain exactly one cut vertex.

3.2.5 Show that the number of blocks in $G$ is equal to $\omega + \sum_{v \in V} (b(v) - 1)$, where $b(v)$ denotes the number of blocks of $G$ containing $v$.

3.2.6* Let $G$ be a 2-connected graph and let $X$ and $Y$ be disjoint subsets of $V$, each containing at least two vertices. Show that $G$ contains disjoint paths $P$ and $Q$ such that

(i) the origins of $P$ and $Q$ belong to $X$,
(ii) the termini of $P$ and $Q$ belong to $Y$, and
(iii) no internal vertex of $P$ or $Q$ belongs to $X \cup Y$.

3.2.7* A nonempty graph $G$ is *$\kappa$-critical* if, for every edge $e$, $\kappa(G-e) < \kappa(G)$.

(*a*) Show that every $\kappa$-critical 2-connected graph has a vertex of degree two.
(Halin, 1969 has shown that, in general, every $\kappa$-critical $k$-connected graph has a vertex of degree $k$.)

(*b*) Show that if $G$ is a $\kappa$-critical 2-connected graph with $\nu \geq 4$, then $\varepsilon \leq 2\nu - 4$. (G. A. Dirac)

3.2.8 Describe a good algorithm for finding the blocks of a graph.

## APPLICATIONS

### 3.3 CONSTRUCTION OF RELIABLE COMMUNICATION NETWORKS

If we think of a graph as representing a communication network, the connectivity (or edge connectivity) becomes the smallest number of communication stations (or communication links) whose breakdown would jeopardise communication in the system. The higher the connectivity and edge connectivity, the more reliable the network. From this point of view, a

tree network, such as the one obtained by Kruskal's algorithm, is not very reliable, and one is led to consider the following generalisation of the connector problem.

Let $k$ be a given positive integer and let $G$ be a weighted graph. Determine a minimum-weight $k$-connected spanning subgraph of $G$.

For $k = 1$, this problem reduces to the connector problem, which can be solved by Kruskal's algorithm. For values of $k$ greater than one, the problem is unsolved and is known to be difficult. However, if $G$ is a complete graph in which each edge is assigned unit weight, then the problem has a simple solution which we now present.

Observe that, for a weighted complete graph on $n$ vertices in which each edge is assigned unit weight, a minimum-weight $m$-connected spanning subgraph is simply an $m$-connected graph on $n$ vertices with as few edges as possible. We shall denote by $f(m, n)$ the least number of edges that an $m$-connected graph on $n$ vertices can have. (It is, of course, assumed that $m < n$.) By theorems 3.1 and 1.1

$$f(m, n) \geq \{mn/2\} \tag{3.1}$$

We shall show that equality holds in (3.1) by constructing an $m$-connected graph $H_{m,n}$ on $n$ vertices that has exactly $\{mn/2\}$ edges. The structure of $H_{m,n}$ depends on the parities of $m$ and $n$; there are three cases.

*Case 1* $m$ even. Let $m = 2r$. Then $H_{2r,n}$ is constructed as follows. It has vertices $0, 1, \ldots, n-1$ and two vertices $i$ and $j$ are joined if $i - r \leq j \leq i + r$ (where addition is taken modulo $n$). $H_{4,8}$ is shown in figure 3.7*a*.

*Case 2* $m$ odd, $n$ even. Let $m = 2r + 1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex $i$ to vertex $i + (n/2)$ for $1 \leq i \leq n/2$. $H_{5,8}$ is shown in figure 3.7*b*.

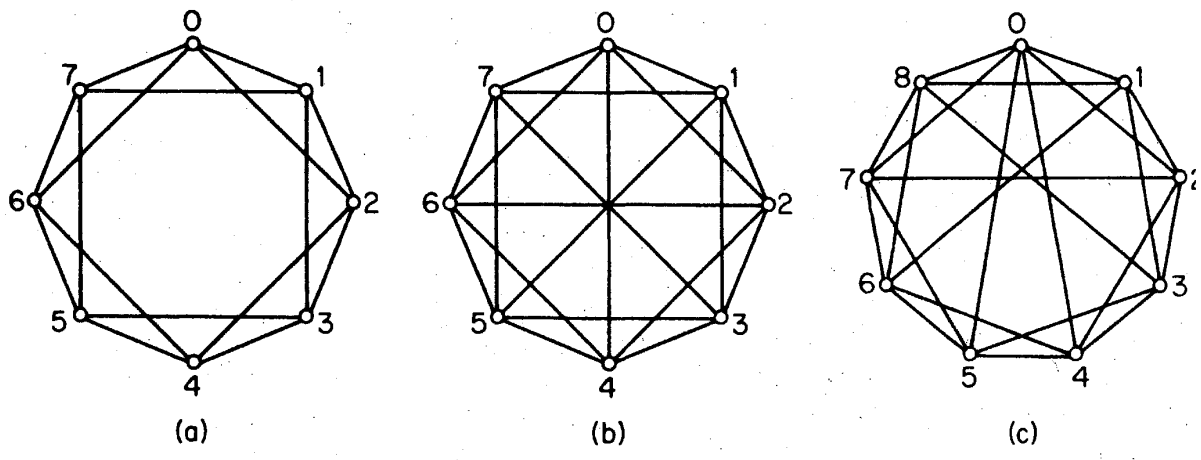


Figure 3.7. (*a*) $H_{4,8}$; (*b*) $H_{5,8}$; (*c*) $H_{5,9}$

*Case 3* $m$ odd, $n$ odd. Let $m = 2r + 1$. Then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex $i$ to vertex $i + (n+1)/2$ for $1 \leq i < (n-1)/2$. $H_{5,9}$ is shown in figure 3.7*c*.

*Theorem 3.3* (Harary, 1962) The graph $H_{m,n}$ is $m$-connected.

*Proof* Consider the case $m = 2r$. We shall show that $H_{2r,n}$ has no vertex cut of fewer than $2r$ vertices. If possible, let $V'$ be a vertex cut with $|V'| < 2r$. Let $i$ and $j$ be vertices belonging to different components of $H_{2r,n} - V'$. Consider the two sets of vertices

$$S = \{i, i+1, \ldots, j-1, j\}$$

and

$$T = \{j, j+1, \ldots, i-1, i\}$$

where addition is taken modulo $n$. Since $|V'| < 2r$, we may assume, without loss of generality, that $|V' \cap S| < r$. Then there is clearly a sequence of distinct vertices in $S \backslash V'$ which starts with $i$, ends with $j$, and is such that the difference between any two consecutive terms is at most $r$. But such a sequence is an $(i, j)$-path in $H_{2r,n} - V'$, a contradiction. Hence $H_{2r,n}$ is $2r$-connected.

The case $m = 2r + 1$ is left as an exercise (exercise 3.3.1) □

It is easy to see that $\varepsilon(H_{m,n}) = \{mn/2\}$. Thus, by theorem 3.3,

$$f(m, n) \leq \{mn/2\} \tag{3.2}$$

It now follows from (3.1) and (3.2) that

$$f(m, n) = \{mn/2\}$$

and that $H_{m,n}$ is an $m$-connected graph on $n$ vertices with as few edges as possible.

We note that since, for any graph $G$, $\kappa \leq \kappa'$ (theorem 3.1), $H_{m,n}$ is also $m$-edge-connected. Thus, denoting by $g(m, n)$ the least possible number of edges in an $m$-edge-connected graph on $n$ vertices, we have, for $1 < m < n$

$$g(m, n) = \{mn/2\} \tag{3.3}$$

*Exercises*

3.3.1 Show that $H_{2r+1,n}$ is $(2r+1)$-connected.

3.3.2 Show that $\kappa(H_{m,n}) = \kappa'(H_{m,n}) = m$.

3.3.3 Find a graph with nine vertices and 23 edges that is 5-connected but not isomorphic to the graph $H_{5,9}$ of figure 3.7*c*.

3.3.4 Show that (3.3) holds for all values of $m$ and $n$ with $m > 1$ and $n > 1$.

3.3.5 Find, for all $\nu \geq 5$, a 2-connected graph $G$ of diameter two with $\varepsilon = 2\nu - 5$.

(Murty, 1969 has shown that every such graph has at least this number of edges.)

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# 4 Euler Tours and Hamilton Cycles

## 4.1 EULER TOURS

A trail that traverses every edge of $G$ is called an *Euler trail* of $G$ because Euler was the first to investigate the existence of such trails in graphs. In the earliest known paper on graph theory (Euler, 1736), he showed that it was impossible to cross each of the seven bridges of Königsberg once and only once during a walk through the town. A plan of Königsberg and the river Pregel is shown in figure 4.1*a*. As can be seen, proving that such a walk is impossible amounts to showing that the graph of figure 4.1*b* contains no Euler trail.

A *tour* of $G$ is a closed walk that traverses each edge of $G$ at least once. An *Euler tour* is a tour which traverses each edge exactly once (in other words, a closed Euler trail). A graph is *eulerian* if it contains an Euler tour.

*Theorem 4.1* A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.

*Proof* Let $G$ be eulerian, and let $C$ be an Euler tour of $G$ with origin (and terminus) $u$. Each time a vertex $v$ occurs as an internal vertex of $C$, two of the edges incident with $v$ are accounted for. Since an Euler tour contains

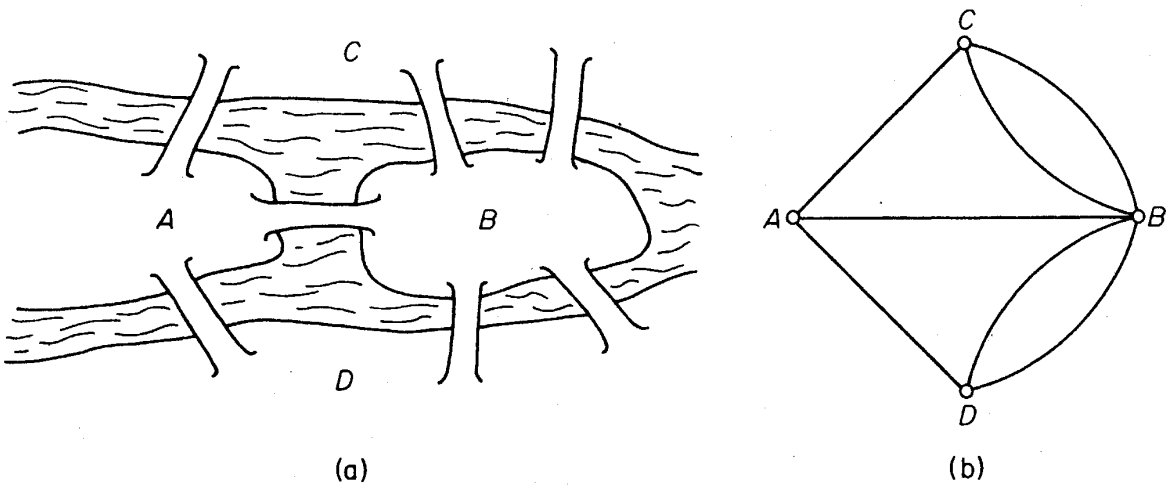



Figure 4.1. The bridges of Königsberg and their graph

every edge of $G$, $d(v)$ is even for all $v \neq u$. Similarly, since $C$ starts and ends at $u$, $d(u)$ is also even. Thus $G$ has no vertices of odd degree.

Conversely, suppose that $G$ is a noneulerian connected graph with at least one edge and no vertices of odd degree. Choose such a graph $G$ with as few edges as possible. Since each vertex of $G$ has degree at least two, $G$ contains a closed trail (exercise 1.7.2). Let $C$ be a closed trail of maximum possible length in $G$. By assumption, $C$ is not an Euler tour of $G$ and so $G - E(C)$ has some component $G'$ with $\varepsilon(G') > 0$. Since $C$ is itself eulerian, it has no vertices of odd degree; thus the connected graph $G'$ also has no vertices of odd degree. Since $\varepsilon(G') < \varepsilon(G)$, it follows from the choice of $G$ that $G'$ has an Euler tour $C'$. Now, because $G$ is connected, there is a vertex $v$ in $V(C) \cap V(C')$, and we may assume, without loss of generality, that $v$ is the origin and terminus of both $C$ and $C'$. But then $CC'$ is a closed trail of $G$ with $\varepsilon(CC') > \varepsilon(C)$, contradicting the choice of $C$ □

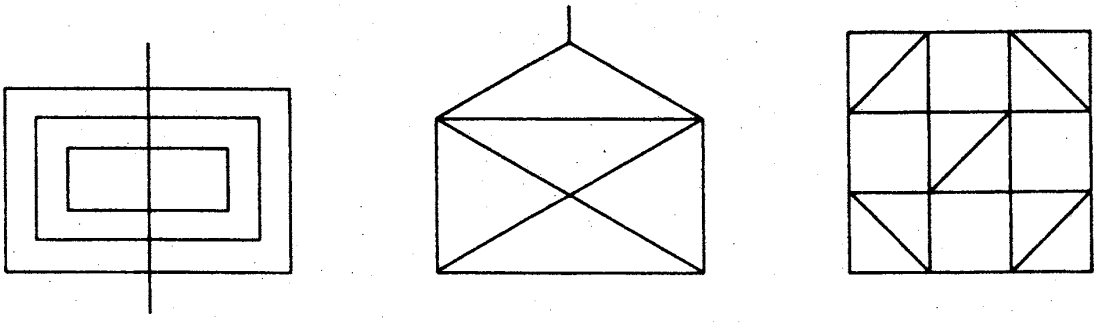
*Corollary 4.1* A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

*Proof* If $G$ has an Euler trail then, as in the proof of theorem 4.1, each vertex other than the origin and terminus of this trail has even degree.

Conversely, suppose that $G$ is a nontrivial connected graph with at most two vertices of odd degree. If $G$ has no such vertices then, by theorem 4.1, $G$ has a closed Euler trail. Otherwise, $G$ has exactly two vertices, $u$ and $v$, of odd degree. In this case, let $G + e$ denote the graph obtained from $G$ by the addition of a new edge $e$ joining $u$ and $v$. Clearly, each vertex of $G + e$ has even degree and so, by theorem 4.1, $G + e$ has an Euler tour $C = v_0 e_1 v_1 \ldots e_{\varepsilon+1} v_{\varepsilon+1}$, where $e_1 = e$. The trail $v_1 e_2 v_2 \ldots e_{\varepsilon+1} v_{\varepsilon+1}$ is an Euler trail of $G$ □

*Exercises*

4.1.1 Which of the following figures can be drawn without lifting one's pen from the paper or covering a line more than once?



4.1.2 If possible, draw an eulerian graph $G$ with $\nu$ even and $\varepsilon$ odd; otherwise, explain why there is no such graph.

4.1.3 Show that if $G$ is eulerian, then every block of $G$ is eulerian.

4.1.4 Show that if $G$ has no vertices of odd degree, then there are edge-disjoint cycles $C_1, C_2, \ldots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$.

4.1.5 Show that if a connected graph $G$ has $2k > 0$ vertices of odd degree, then there are $k$ edge-disjoint trails $Q_1, Q_2, \ldots, Q_k$ in $G$ such that $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$.

4.1.6* Let $G$ be nontrivial and eulerian, and let $v \in V$. Show that every trail of $G$ with origin $v$ can be extended to an Euler tour of $G$ if and only if $G - v$ is a forest. (O. Ore)

## 4.2 HAMILTON CYCLES

A path that contains every vertex of $G$ is called a *Hamilton path* of $G$; similarly, a *Hamilton cycle* of $G$ is a cycle that contains every vertex of $G$. Such paths and cycles are named after Hamilton (1856), who described, in a letter to his friend Graves, a mathematical game on the dodecahedron (figure 4.2*a*) in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a

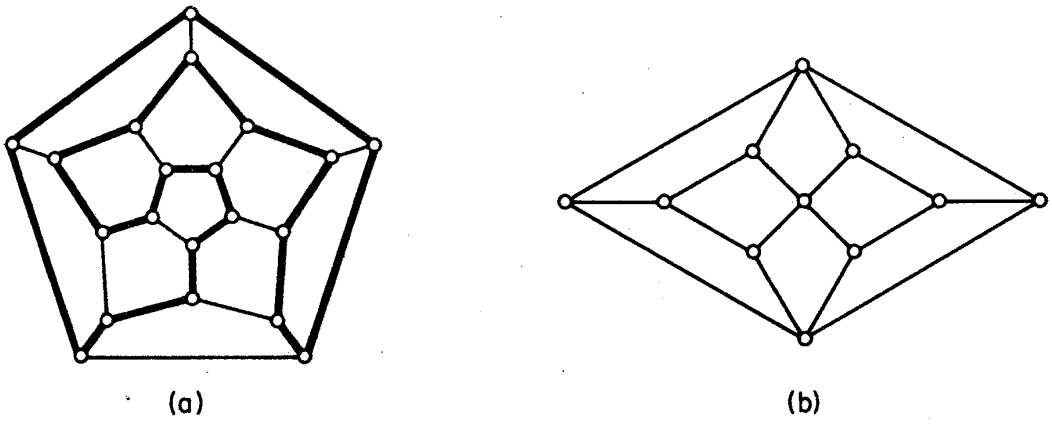



Figure 4.2. (*a*) The dodecahedron; (*b*) the Herschel graph

spanning cycle. A graph is *hamiltonian* if it contains a Hamilton cycle. The dodecahedron is hamiltonian (see figure 4.2*a*); the Herschel graph (figure 4.2*b*) is nonhamiltonian, because it is bipartite and has an odd number of vertices.

In contrast with the case of eulerian graphs, no nontrivial necessary and sufficient condition for a graph to be hamiltonian is known; in fact, the problem of finding such a condition is one of the main unsolved problems of graph theory.

We shall first present a simple, but useful, necessary condition.

*Theorem 4.2* If $G$ is hamiltonian then, for every nonempty proper subset $S$ of $V$

$$\omega(G - S) \leq |S| \tag{4.1}$$

*Proof* Let $C$ be a Hamilton cycle of $G$. Then, for every nonempty proper subset $S$ of $V$

$$\omega(C-S) \leq |S|$$

Also, $C-S$ is a spanning subgraph of $G-S$ and so

$$\omega(G-S) \leq \omega(C-S)$$

The theorem follows □

As an illustration of the above theorem, consider the graph of figure 4.3. This graph has nine vertices; on deleting the three indicated in black, four components remain. Therefore (4.1) is not satisfied and it follows from theorem 4.2 that the graph is nonhamiltonian.

We thus see that theorem 4.2 can sometimes be applied to show that a particular graph is nonhamiltonian. However, this method does not always

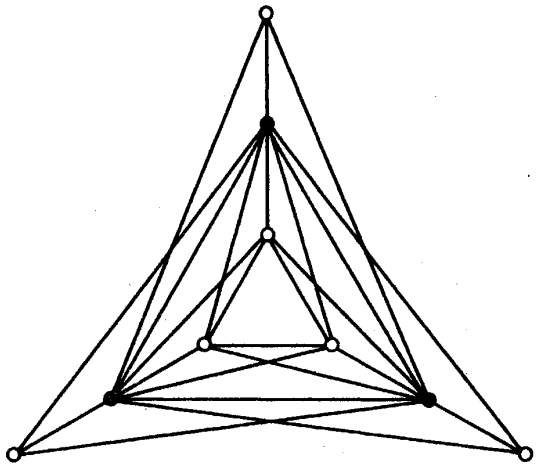


Figure 4.3

work; for instance, the Petersen graph (figure 4.4) is nonhamiltonian, but one cannot deduce this by using theorem 4.2.

We now discuss sufficient conditions for a graph $G$ to be hamiltonian; since a graph is hamiltonian if and only if its underlying simple graph is hamiltonian, it suffices to limit our discussion to simple graphs. We start with a result due to Dirac (1952).

*Theorem 4.3* If $G$ is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, then $G$ is hamiltonian.

*Proof* By contradiction. Suppose that the theorem is false, and let $G$ be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$. Since $\nu \geq 3$, $G$ cannot be complete. Let $u$ and $v$ be nonadjacent vertices in $G$. By the choice of $G$, $G+uv$ is hamiltonian. Moreover, since $G$ is nonhamiltonian,

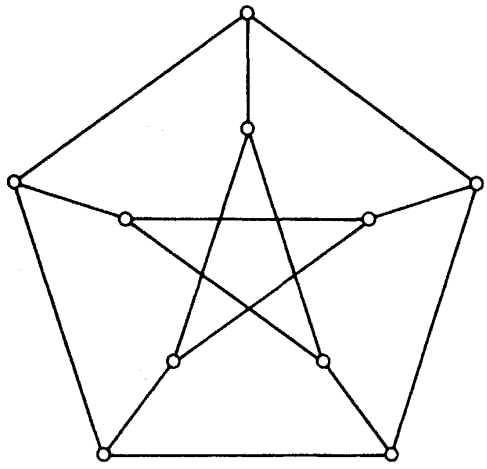


Figure 4.4. The Petersen graph

each Hamilton cycle of $G+uv$ must contain the edge $uv$. Thus there is a Hamilton path $v_1v_2\ldots v_\nu$ in $G$ with origin $u=v_1$ and terminus $v=v_\nu$. Set

$$S=\{v_i \mid uv_{i+1}\in E\} \quad \text{and} \quad T=\{v_i \mid v_iv\in E\}$$

Since $v_\nu \notin S\cup T$ we have

$$|S\cup T|<\nu \tag{4.2}$$

Furthermore

$$|S\cap T|=0 \tag{4.3}$$

since if $S\cap T$ contained some vertex $v_i$, then $G$ would have the Hamilton cycle $v_1v_2\ldots v_iv_\nu v_{\nu-1}\ldots v_{i+1}v_1$, contrary to assumption (see figure 4.5).

Using (4.2) and (4.3) we obtain

$$d(u)+d(v)=|S|+|T|=|S\cup T|+|S\cap T|<\nu \tag{4.4}$$

But this contradicts the hypothesis that $\delta\geq\nu/2$ □

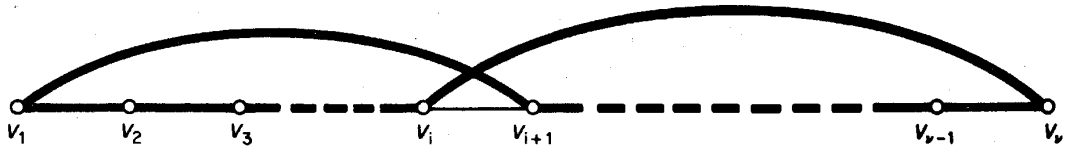



Figure 4.5

Bondy and Chvátal (1974) observed that the proof of theorem 4.3 can be modified to yield stronger sufficient conditions than that obtained by Dirac. The basis of their approach is the following lemma.

*Lemma 4.4.1* Let $G$ be a simple graph and let $u$ and $v$ be nonadjacent vertices in $G$ such that

$$d(u)+d(v)\geq\nu \tag{4.5}$$

Then $G$ is hamiltonian if and only if $G+uv$ is hamiltonian.

*Proof* If $G$ is hamiltonian then, trivially, so too is $G+uv$. Conversely, suppose that $G+uv$ is hamiltonian but $G$ is not. Then, as in the proof of theorem 4.3, we obtain (4.4). But this contradicts hypothesis (4.5) □

Lemma 4.4.1 motivates the following definition. The *closure* of $G$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $\nu$ until no such pair remains. We denote the closure of $G$ by $c(G)$.

*Lemma 4.4.2* $c(G)$ is well defined.

*Proof* Let $G_1$ and $G_2$ be two graphs obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $\nu$ until no such pair remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequences of edges added to $G$ in obtaining $G_1$ and $G_2$, respectively. We shall show that each $e_i$ is an edge of $G_2$ and each $f_j$ is an edge of $G_1$.

If possible, let $e_{k+1}=uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of $G_2$. Set $H=G+\{e_1, e_2, \ldots, e_k\}$. It follows from the definition of $G_1$ that

$$d_H(u)+d_H(v)\geq \nu$$

By the choice of $e_{k+1}$, $H$ is a subgraph of $G_2$. Therefore

$$d_{G_2}(u)+d_{G_2}(v)\geq \nu$$

This is a contradiction, since $u$ and $v$ are nonadjacent in $G_2$. Therefore each $e_i$ is an edge of $G_2$ and, similarly, each $f_j$ is an edge of $G_1$. Hence $G_1=G_2$, and $c(G)$ is well defined □

Figure 4.6 illustrates the construction of the closure of a graph $G$ on six vertices. It so happens that in this example $c(G)$ is complete; note, however, that this is by no means always the case.

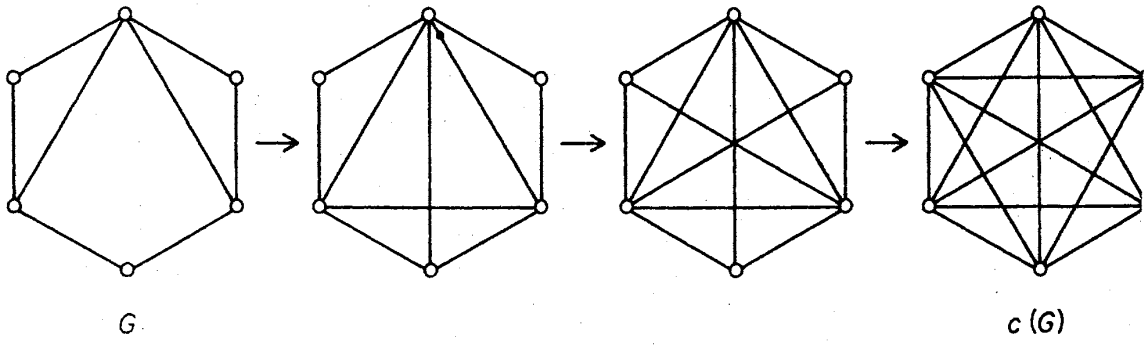



Figure 4.6. The closure of a graph

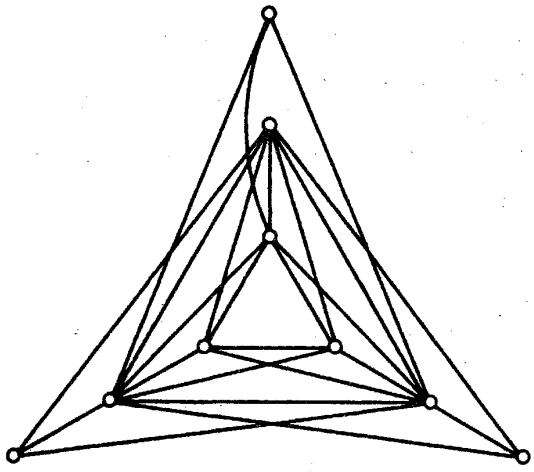


Figure 4.7. A hamiltonian graph

*Theorem 4.4* A simple graph is hamiltonian if and only if its closure is hamiltonian.

*Proof* Apply lemma 4.4.1 each time an edge is added in the formation of the closure □

Theorem 4.4 has a number of interesting consequences. First, upon making the trivial observation that all complete graphs on at least three vertices are hamiltonian, we obtain the following result.

*Corollary 4.4* Let $G$ be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then $G$ is hamiltonian.

Consider, for example, the graph of figure 4.7. One readily checks that its closure is complete. Therefore, by corollary 4.4, it is hamiltonian. It is perhaps interesting to note that the graph of figure 4.7 can be obtained from the graph of figure 4.3 by altering just one end of one edge, and yet we have results (corollary 4.4 and theorem 4.2) which tell us that this one is hamiltonian whereas the other is not.

Corollary 4.4 can be used to deduce various sufficient conditions for a graph to be hamiltonian in terms of its vertex degrees. For example, since $c(G)$ is clearly complete when $\delta \geq \nu/2$, Dirac's condition (theorem 4.3) is an immediate corollary. A more general condition than that of Dirac was obtained by Chvátal (1972).

*Theorem 4.5* Let $G$ be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of $m$ less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then $G$ is hamiltonian.

*Proof* Let $G$ satisfy the hypothesis of the theorem. We shall show that its closure $c(G)$ is complete, and the conclusion will then follow from corollary 4.4. We denote the degree of a vertex $v$ in $c(G)$ by $d'(v)$.

Assume that $c(G)$ is not complete, and let $u$ and $v$ be two nonadjacent vertices in $c(G)$ with

$$d'(u) \leq d'(v) \tag{4.6}$$

and $d'(u)+d'(v)$ as large as possible; since no two nonadjacent vertices in $c(G)$ can have degree sum $\nu$ or more, we have

$$d'(u)+d'(v) < \nu \tag{4.7}$$

Now denote by $S$ the set of vertices in $V\backslash\{v\}$ which are nonadjacent to $v$ in $c(G)$, and by $T$ the set of vertices in $V\backslash\{u\}$ which are nonadjacent to $u$ in $c(G)$. Clearly

$$|S| = \nu - 1 - d'(v) \quad \text{and} \quad |T| = \nu - 1 - d'(u) \tag{4.8}$$

Furthermore, by the choice of $u$ and $v$, each vertex in $S$ has degree at most $d'(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d'(v)$. Setting $d'(u) = m$ and using (4.7) and (4.8), we find that $c(G)$ has at least $m$ vertices of degree at most $m$ and at least $\nu - m$ vertices of degree less than $\nu - m$. Because $G$ is a spanning subgraph of $c(G)$, the same is true of $G$; therefore $d_m \leq m$ and $d_{\nu-m} < \nu - m$. But this is contrary to hypothesis since, by (4.6) and (4.7), $m < \nu/2$. We conclude that $c(G)$ is indeed complete and hence, by corollary 4.4, that $G$ is hamiltonian □

One can often deduce that a given graph is hamiltonian simply by computing its degree sequence and applying theorem 4.5. This method works with the graph of figure 4.7 but not with the graph $G$ of figure 4.6, even though the closure of the latter graph is complete. From these examples, we see that theorem 4.5 is stronger than theorem 4.3 but not as strong as corollary 4.4.

A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorised* by another such sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$. A graph $G$ is *degree-majorised* by a graph $H$ if $\nu(G) = \nu(H)$ and the nondecreasing degree sequence of $G$ is majorised by that of $H$. For instance, the 5-cycle is degree-majorised by $K_{2,3}$ because (2, 2, 2, 2, 2) is majorised by (2, 2, 2, 3, 3). The family of degree-maximal nonhamiltonian graphs (those that are degree-majorised by no others) admits of a simple description. We first introduce the notion of the join of two graphs. The *join* $G \vee H$ of disjoint graphs $G$ and $H$ is the graph obtained from $G+H$ by joining each vertex of $G$ to each vertex of $H$; it is represented diagrammatically as in figure 4.8.

Now, for $1 \leq m < n/2$, let $C_{m,n}$ denote the graph $K_m \vee (K_m^c + K_{n-2m})$, depicted in figure 4.9*a*; two specific examples, $C_{1,5}$ and $C_{2,5}$, are shown in figures 4.9*b* and 4.9*c*.

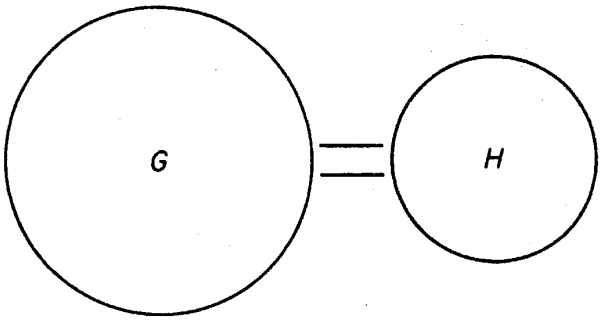



Figure 4.8. The join of $G$ and $H$

That $C_{m,n}$ is nonhamiltonian follows immediately from theorem 4.2; for if $S$ denotes the set of $m$ vertices of degree $n-1$ in $C_{m,n}$, we have $\omega(C_{m,n}-S)=m+1>|S|$.

*Theorem* 4.6 (Chvátal, 1972) If $G$ is a nonhamiltonian simple graph with $\nu \geq 3$, then $G$ is degree-majorised by some $C_{m,\nu}$.

*Proof* Let $G$ be a nonhamiltonian simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Then, by theorem 4.5, there exists $m < \nu/2$ such that $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Therefore $(d_1, d_2, \ldots, d_\nu)$ is majorised by the sequence

$$(m, \ldots, m, \nu-m-1, \ldots, \nu-m-1, \nu-1, \ldots, \nu-1)$$

with $m$ terms equal to $m$, $\nu-2m$ terms equal to $\nu-m-1$ and $m$ terms equal to $\nu-1$, and this latter sequence is the degree sequence of $C_{m,\nu}$ □

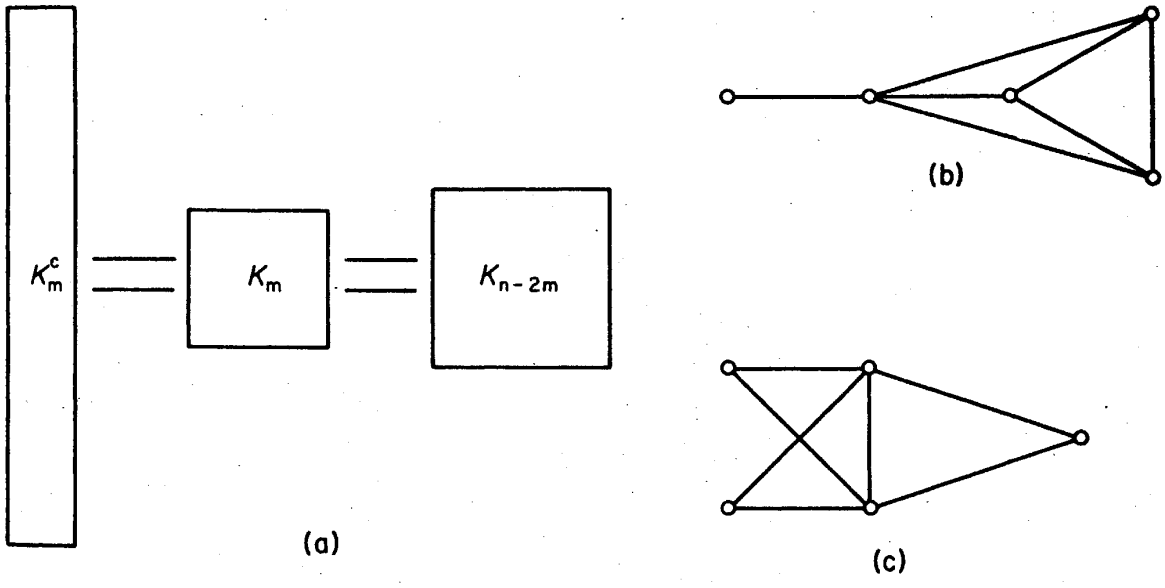



Figure 4.9. (*a*) $C_{m,n}$; (*b*) $C_{1,5}$; (*c*) $C_{2,5}$

From theorem 4.6 we can deduce a result due to Ore (1961) and Bondy (1972).

*Corollary 4.6* If $G$ is a simple graph with $\nu \geq 3$ and $\varepsilon > \binom{\nu-1}{2}+1$, then $G$ is hamiltonian. Moreover, the only nonhamiltonian simple graphs with $\nu$ vertices and $\binom{\nu-1}{2}+1$ edges are $C_{1,\nu}$ and, for $\nu = 5$, $C_{2,5}$.

*Proof* Let $G$ be a nonhamiltonian simple graph with $\nu \geq 3$. By theorem 4.6, $G$ is degree-majorised by $C_{m,\nu}$ for some positive integer $m < \nu/2$. Therefore, by theorem 1.1,

$$\begin{aligned}\varepsilon(G) &\leq \varepsilon(C_{m,\nu}) && (4.9)\\ &= \tfrac{1}{2}(m^2+(\nu-2m)(\nu-m-1)+m(\nu-1))\\ &= \binom{\nu-1}{2}+1-\tfrac{1}{2}(m-1)(m-2)-(m-1)(\nu-2m-1)\\ &\leq \binom{\nu-1}{2}+1 && (4.10)\end{aligned}$$

Furthermore, equality can only hold in (4.9) if $G$ has the same degree sequence as $C_{m,\nu}$; and equality can only hold in (4.10) if either $m = 2$ and $\nu = 5$, or $m = 1$. Hence $\varepsilon(G)$ can equal $\binom{\nu-1}{2}+1$ only if $G$ has the same degree sequence as $C_{1,\nu}$ or $C_{2,5}$, which is easily seen to imply that $G \cong C_{1,\nu}$ or $G \cong C_{2,5}$ □

*Exercises*

4.2.1 Show that if either

(*a*) $G$ is not 2-connected, or

(*b*) $G$ is bipartite with bipartition $(X, Y)$ where $|X| \neq |Y|$,

then $G$ is nonhamiltonian.

4.2.2 A mouse eats his way through a $3\times3\times3$ cube of cheese by tunnelling through all of the 27 $1\times1\times1$ subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the centre of the cube?

4.2.3 Show that if $G$ has a Hamilton path then, for every proper subset $S$ of $V$, $\omega(G-S) \leq |S|+1$.

4.2.4* Let $G$ be a nontrivial simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$. Show that if there is no

value of $m$ less than $(\nu+1)/2$ for which $d_m < m$ and $d_{\nu-m+1} < \nu - m$, then $G$ has a Hamilton path. (V. Chvátal)

4.2.5 (*a*) Let $G$ be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$ and let $G^c$ have degree sequence $(d'_1, d'_2, \ldots, d'_\nu)$ where $d_1 \le d_2 \le \ldots \le d_\nu$ and $d'_1 \le d'_2 \le \ldots \le d'_\nu$. Show that if $d_m \ge d'_m$ for all $m \le \nu/2$, then $G$ has a Hamilton path.

(*b*) Deduce that if $G$ is self-complementary, then $G$ has a Hamilton path. (C. R. J. Clapham)

4.2.6* Let $G$ be a simple bipartite graph with bipartition $(X, Y)$, where $|X| = |Y| \ge 2$, and let $G$ have degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \le d_2 \le \ldots \le d_\nu$. Show that if there is no value of $m$ less than or equal to $\nu/4$ for which $d_m \le m$ and $d_{\nu/2} \le \nu/2 - m$, then $G$ is hamiltonian. (V. Chvátal)

4.2.7 Prove corollary 4.6 directly from corollary 4.4.

4.2.8 Show that if $G$ is simple with $\nu \ge 6\delta$ and $\varepsilon > \binom{\nu-\delta}{2} + \delta^2$, then $G$ is hamiltonian. (P. Erdös)

4.2.9* Show that if $G$ is a connected graph with $\nu > 2\delta$, then $G$ has a path of length at least $2\delta$. (G. A. Dirac)

(Dirac, 1952 has also shown that if $G$ is a 2-connected simple graph with $\nu \ge 2\delta$, then $G$ has a cycle of length at least $2\delta$.)

4.2.10 Using the remark to exercise 4.2.9, show that every $2k$-regular simple graph on $4k+1$ vertices is hamiltonian $(k \ge 1)$. (C. St. J. A. Nash-Williams)

4.2.11 $G$ is *Hamilton-connected* if every two vertices of $G$ are connected by a Hamilton path.

(*a*) Show that if $G$ is Hamilton-connected and $\nu \ge 4$, then $\varepsilon \ge [\frac{1}{2}(3\nu+1)]$.

(*b*)* For $\nu \ge 4$, construct a Hamilton-connected graph $G$ with $\varepsilon = [\frac{1}{2}(3\nu+1)]$. (J. W. Moon)

4.2.12 $G$ is *hypohamiltonian* if $G$ is not hamiltonian but $G - v$ is hamiltonian for every $v \in V$. Show that the Petersen graph (figure 4.4) is hypohamiltonian.

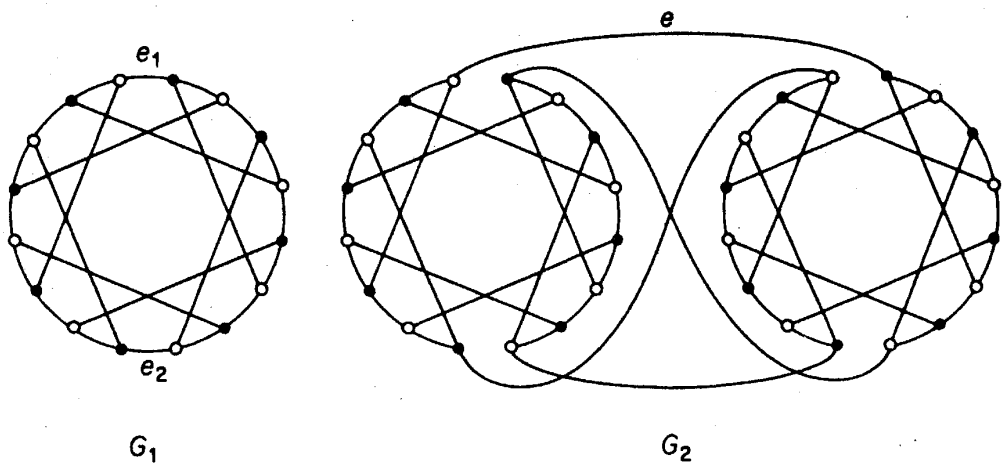
(Herz, Duby and Vigué, 1967 have shown that it is, in fact, the smallest such graph.)

4.2.13* $G$ is *hypotraceable* if $G$ has no Hamilton path but $G - v$ has a Hamilton path for every $v \in V$. Show that the Thomassen graph (p. 240) is hypotraceable.

4.2.14 (*a*) Show that there is no Hamilton cycle in the graph $G_1$ below which contains exactly one of the edges $e_1$ and $e_2$.

(*b*) Using (*a*), show that every Hamilton cycle in $G_2$ includes the edge $e$.

(*c*) Deduce that the Horton graph (p. 240) is nonhamiltonian.




4.2.15 Describe a good algorithm for

(*a*) constructing the closure of a graph;

(*b*) finding a Hamilton cycle if the closure is complete.

## APPLICATIONS

### 4.3 THE CHINESE POSTMAN PROBLEM

In his job, a postman picks up mail at the post office, delivers it, and then returns to the post office. He must, of course, cover each street in his area at least once. Subject to this condition, he wishes to choose his route in such a way that he walks as little as possible. This problem is known as the *Chinese postman problem*, since it was first considered by a Chinese mathematician, Kuan (1962).

In a weighted graph, we define the *weight* of a tour $v_0e_1v_1 \ldots e_nv_0$ to be $\sum_{i=1}^{n} w(e_i)$. Clearly, the Chinese postman problem is just that of finding a minimum-weight tour in a weighted connected graph with non-negative weights. We shall refer to such a tour as an *optimal tour.*

If $G$ is eulerian, then any Euler tour of $G$ is an optimal tour because an Euler tour is a tour that traverses each edge exactly once. The Chinese postman problem is easily solved in this case, since there exists a good algorithm for determining an Euler tour in an eulerian graph. The algorithm, due to Fleury (see Lucas, 1921), constructs an Euler tour by tracing out a trail, subject to the one condition that, at any stage, a cut edge of the untraced subgraph is taken only if there is no alternative.

*Fleury's Algorithm*

1. Choose an arbitrary vertex $v_0$, and set $W_0 = v_0$.
2. Suppose that the trail $W_i = v_0e_1v_1 \ldots e_iv_i$ has been chosen.

Then choose an edge $e_{i+1}$ from $E\backslash\{e_1, e_2, \ldots, e_i\}$ in such a way that

(i) $e_{i+1}$ is incident with $v_i$;

(ii) unless there is no alternative, $e_{i+1}$ is not a cut edge of

$$G_i = G - \{e_1, e_2, \ldots, e_i\}$$

3. Stop when step 2 can no longer be implemented.

By its definition, Fleury's algorithm constructs a trail in $G$.

*Theorem* 4.7 If $G$ is eulerian, then any trail in $G$ constructed by Fleury's algorithm is an Euler tour of $G$.

*Proof* Let $G$ be eulerian, and let $W_n = v_0e_1v_1 \ldots e_nv_n$ be a trail in $G$ constructed by Fleury's algorithm. Clearly, the terminus $v_n$ must be of degree zero in $G_n$. It follows that $v_n = v_0$; in other words, $W_n$ is a closed trail.

Suppose, now, that $W_n$ is not an Euler tour of $G$, and let $S$ be the set of vertices of positive degree in $G_n$. Then $S$ is nonempty and $v_n \in \bar{S}$, where $\bar{S} = V\backslash S$. Let $m$ be the largest integer such that $v_m \in S$ and $v_{m+1} \in \bar{S}$. Since $W_n$ terminates in $\bar{S}$, $e_{m+1}$ is the only edge of $[S, \bar{S}]$ in $G_m$, and hence is a cut edge of $G_m$ (see figure 4.10).

Let $e$ be any other edge of $G_m$ incident with $v_m$. It follows (step 2) that $e$ must also be a cut edge of $G_m$, and hence of $G_m[S]$. But since $G_m[S] = G_n[S]$, every vertex in $G_m[S]$ is of even degree. However, this implies (exercise 2.2.6*a*) that $G_m[S]$ has no cut edge, a contradiction □

The proof that Fleury's algorithm is a good algorithm is left as an exercise (exercise 4.3.2).

If $G$ is not eulerian, then any tour in $G$ and, in particular, an optimal tour in $G$, traverses some edges more than once. For example, in the graph of figure 4.11*a* $xuywvzwyxuwvxzyx$ is an optimal tour (exercise 4.3.1). Notice that the four edges $ux$, $xy$, $yw$ and $wv$ are traversed twice by this tour.

It is convenient, at this stage, to introduce the operation of duplication of an edge. An edge $e$ is said to be *duplicated* when its ends are joined by a

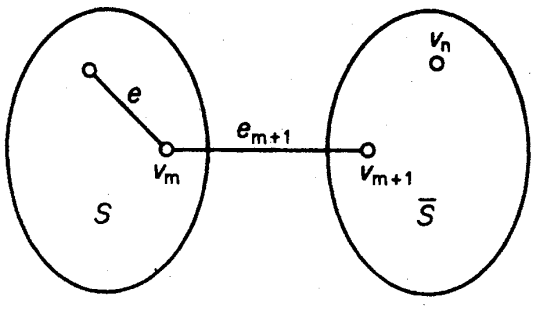



Figure 4.10

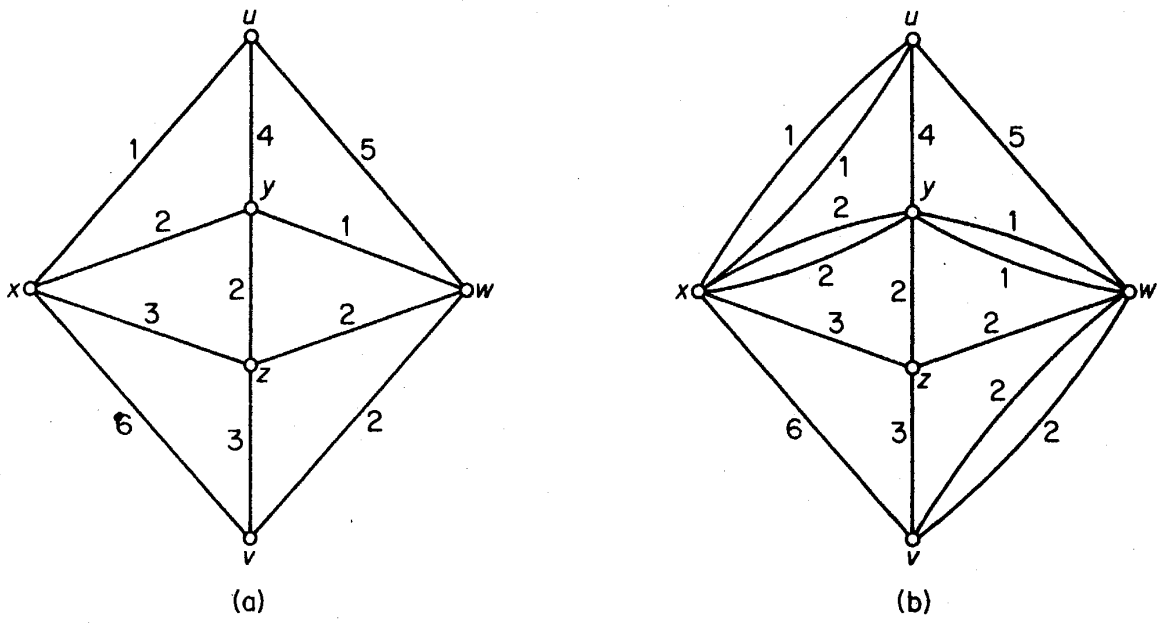



Figure 4.11

new edge of weight $w(e)$. By duplicating the edges $ux$, $xy$, $yw$ and $wv$ in the graph of figure 4.11*a*, we obtain the graph shown in figure 4.11*b*.

We may now rephrase the Chinese postman problem as follows: given a weighted graph $G$ with non-negative weights,

(i) find, by duplicating edges, an eulerian weighted supergraph $G^*$ of $G$ such that $\sum_{e \in E(G^*) \setminus E(G)} w(e)$ is as small as possible;

(ii) find an Euler tour in $G^*$.

That this is equivalent to the Chinese postman problem follows from the observation that a tour of $G$ in which edge $e$ is traversed $m(e)$ times corresponds to an Euler tour in the graph obtained from $G$ by duplicating $e$ $m(e)-1$ times, and vice versa.

We have already presented a good algorithm for solving (ii), namely Fleury's algorithm. A good algorithm for solving (i) has been given by Edmonds and Johnson (1973). Unfortunately, it is too involved to be presented here. However, we shall consider one special case which affords an easy solution. This is the case where $G$ has exactly two vertices of odd degree.

Suppose that $G$ has exactly two vertices $u$ and $v$ of odd degree; let $G^*$ be an eulerian spanning supergraph of $G$ obtained by duplicating edges, and write $E^*$ for $E(G^*)$. Clearly the subgraph $G^*[E^* \setminus E]$ of $G^*$ (induced by the edges of $G^*$ that are not in $G$) also has only the two vertices $u$ and $v$ of odd degree. It follows from corollary 1.1 that $u$ and $v$ are in the same component of $G^*[E^* \setminus E]$ and hence that they are connected by a $(u, v)$-path $P^*$.

Clearly

$$\sum_{e \in E^* \setminus E} w(e) \geq w(P^*) \geq w(P)$$

where $P$ is a minimum-weight $(u, v)$-path in $G$. Thus $\sum_{e \in E^* \setminus E} w(e)$ is a minimum when $G^*$ is obtained from $G$ by duplicating each of the edges on a minimum-weight $(u, v)$-path. A good algorithm for finding such a path was given in section 1.8.

*Exercises*

4.3.1 Show that $xuywvzwyxuwvxzyx$ is an optimal tour in the weighted graph of figure 4.11*a*.

4.3.2 Draw a flow diagram summarising Fleury's algorithm, and show that it is a good algorithm.

## 4.4 THE TRAVELLING SALESMAN PROBLEM

A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and travels in all for as short a time as possible? This is known as the *travelling salesman problem*. In graphical terms, the aim is to find a minimum-weight Hamilton cycle in a weighted complete graph. We shall call such a cycle an *optimal cycle*. In contrast with the shortest path problem and the connector problem, no efficient algorithm for solving the travelling salesman problem is known. It is therefore desirable to have a method for obtaining a reasonably good (but not necessarily optimal) solution. We shall show how some of our previous theory can be employed to this end.

One possible approach is to first find a Hamilton cycle $C$, and then search for another of smaller weight by suitably modifying $C$. Perhaps the simplest such modification is as follows.

Let $C = v_1 v_2 \ldots v_\nu v_1$. Then, for all $i$ and $j$ such that $1 < i+1 < j < \nu$, we can obtain a new Hamilton cycle

$$C_{ij} = v_1 v_2 \ldots v_i v_j v_{j-1} \ldots v_{i+1} v_{j+1} v_{j+2} \ldots v_\nu v_1$$

by deleting the edges $v_i v_{i+1}$ and $v_j v_{j+1}$ and adding the edges $v_i v_j$ and $v_{i+1} v_{j+1}$, as shown in figure 4.12.

If, for some $i$ and $j$

$$w(v_i v_j) + w(v_{i+1} v_{j+1}) < w(v_i v_{i+1}) + w(v_j v_{j+1})$$

the cycle $C_{ij}$ will be an improvement on $C$.

After performing a sequence of the above modifications, one is left with a cycle that can be improved no more by these methods. This final cycle will

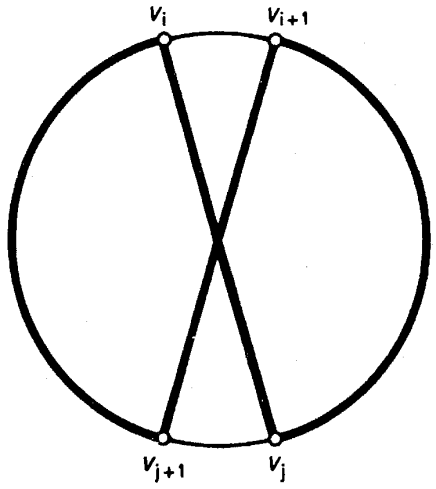



Figure 4.12

almost certainly not be optimal, but it is a reasonable assumption that it will often be fairly good; for greater accuracy, the procedure can be repeated several times, starting with a different cycle each time.

As an example, consider the weighted graph shown in figure 4.13; it is the same graph as was used in our illustration of Kruskal's algorithm in section 2.5.

Starting with the cycle L MC NY Pa Pe T L, we can apply a sequence of three modifications, as illustrated in figure 4.14, and end up with the cycle L NY MC T Pe Pa L of weight 192.

An indication of how good our solution is can sometimes be obtained by applying Kruskal's algorithm. Suppose that $C$ is an optimal cycle in $G$. Then, for any vertex $v$, $C-v$ is a Hamilton path in $G-v$, and is therefore a

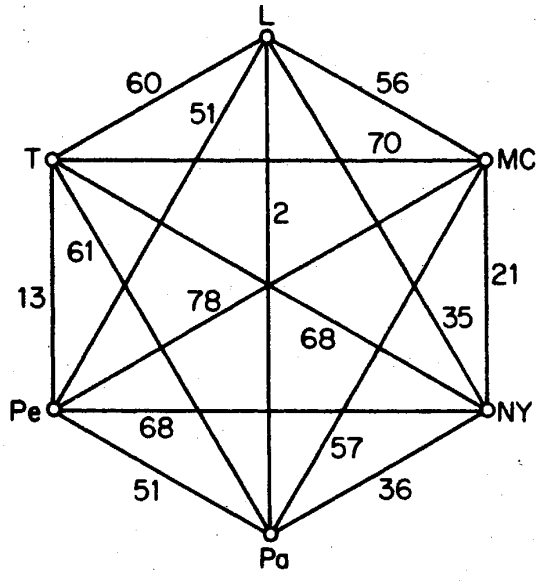



Figure 4.13

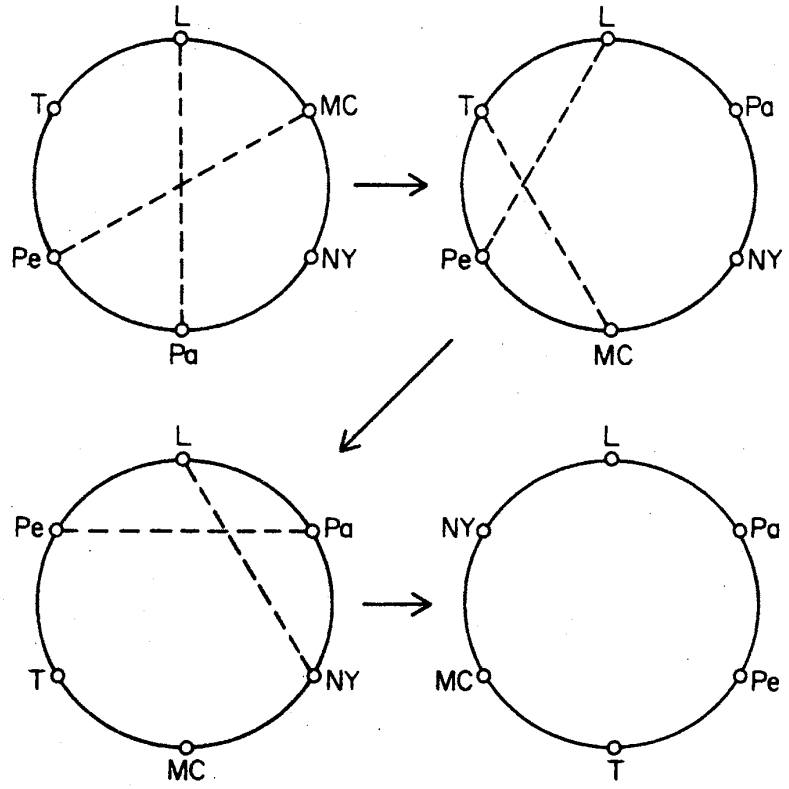



Figure 4.14

spanning tree of $G-v$. It follows that if $T$ is an optimal tree in $G-v$, and if $e$ and $f$ are two edges incident with $v$ such that $w(e)+w(f)$ is as small as possible, then $w(T)+w(e)+w(f)$ will be a lower bound on $w(C)$. In our example, taking NY as the vertex $v$, we find (see figure 4.15) that

$$w(T)=122 \qquad w(e)=21 \quad \text{and} \quad w(f)=35$$

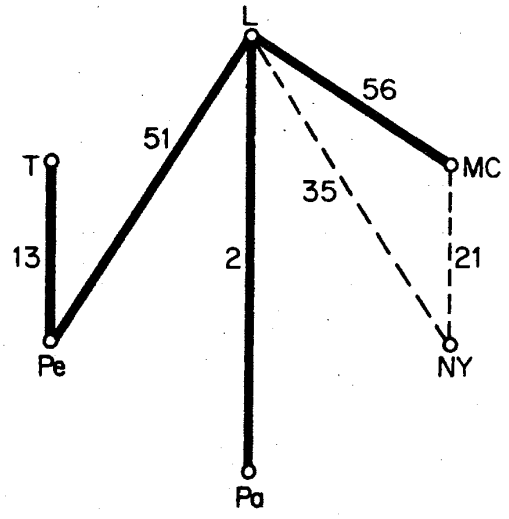



Figure 4.15

We can therefore conclude that the weight $w(C)$ of an optimal cycle in the graph of figure 4.13 satisfies

$$178 \leq w(C) \leq 192$$

The methods described here have been further developed by Lin (1965) and Held and Karp (1970; 1971). In particular, Lin has found that the cycle modification procedure can be made more efficient by replacing three edges at a time rather than just two; somewhat surprisingly, however, it is not advantageous to extend this same idea further. For a survey of the travelling salesman problem, see Bellmore and Nemhauser (1968).

*Exercise*

4.4.1* Let $G$ be a weighted complete graph in which the weights satisfy the triangle inequality: $w(xy)+w(yz) \geq w(xz)$ for all $x, y, z \in V$. Show that an optimal cycle in $G$ has weight at most $2w(T)$, where $T$ is an optimal tree in $G$.

(D. J. Rosencrantz, R. E. Stearns, P. M. Lewis)

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# 5 Matchings

## 5.1 MATCHINGS

A subset $M$ of $E$ is called a *matching* in $G$ if its elements are links and no two are adjacent in $G$; the two ends of an edge in $M$ are said to be *matched under M*. A matching $M$ *saturates* a vertex $v$, and $v$ is said to be *M-saturated*, if some edge of $M$ is incident with $v$; otherwise, $v$ is *M-unsaturated*. If every vertex of $G$ is $M$-saturated, the matching $M$ is *perfect*. $M$ is a *maximum matching* if $G$ has no matching $M'$ with $|M'|>|M|$; clearly, every perfect matching is maximum. Maximum and perfect matchings in graphs are indicated in figure 5.1.

Let $M$ be a matching in $G$. An *M-alternating path* in $G$ is a path whose edges are alternately in $E\backslash M$ and $M$. For example, the path $v_5v_8v_1v_7v_6$ in the graph of figure 5.1*a* is an $M$-alternating path. An *M-augmenting path* is an $M$-alternating path whose origin and terminus are $M$-unsaturated.

*Theorem 5.1* (Berge, 1957) A matching $M$ in $G$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.

*Proof* Let $M$ be a matching in $G$, and suppose that $G$ contains an $M$-augmenting path $v_0v_1 \ldots v_{2m+1}$. Define $M' \subseteq E$ by

$$M' = (M\backslash\{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{2m}v_{2m+1}\}$$

Then $M'$ is a matching in $G$, and $|M'| = |M| + 1$. Thus $M$ is not a maximum matching.

Conversely, suppose that $M$ is not a maximum matching, and let $M'$ be a maximum matching in $G$. Then

$$|M'|>|M| \tag{5.1}$$

Set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of $M$ and $M'$ (see figure 5.2).

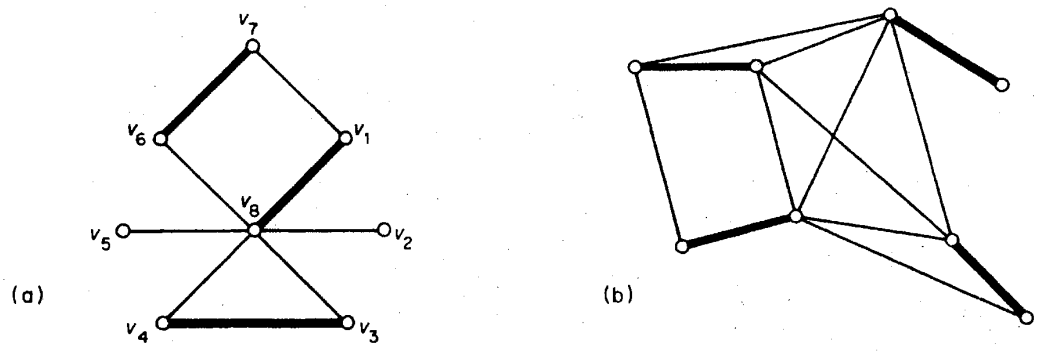



Figure 5.1. (*a*) A maximum matching; (*b*) a perfect matching

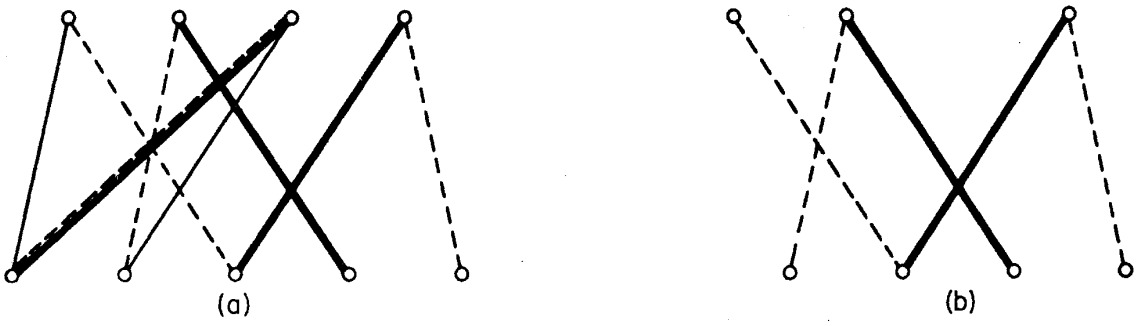



Figure 5.2. (*a*) $G$, with $M$ heavy and $M'$ broken; (*b*) $G[M \Delta M']$

Each vertex of $H$ has degree either one or two in $H$, since it can be incident with at most one edge of $M$ and one edge of $M'$. Thus each component of $H$ is either an even cycle with edges alternately in $M$ and $M'$, or else a path with edges alternately in $M$ and $M'$. By (5.1), $H$ contains more edges of $M'$ than of $M$, and therefore some path component $P$ of $H$ must start and end with edges of $M'$. The origin and terminus of $P$, being $M'$-saturated in $H$, are $M$-unsaturated in $G$. Thus $P$ is an $M$-augmenting path in $G$ □

*Exercises*

5.1.1 (*a*) Show that every $k$-cube has a perfect matching ($k \geq 2$).
(*b*) Find the number of different perfect matchings in $K_{2n}$ and $K_{n,n}$.

5.1.2 Show that a tree has at most one perfect matching.

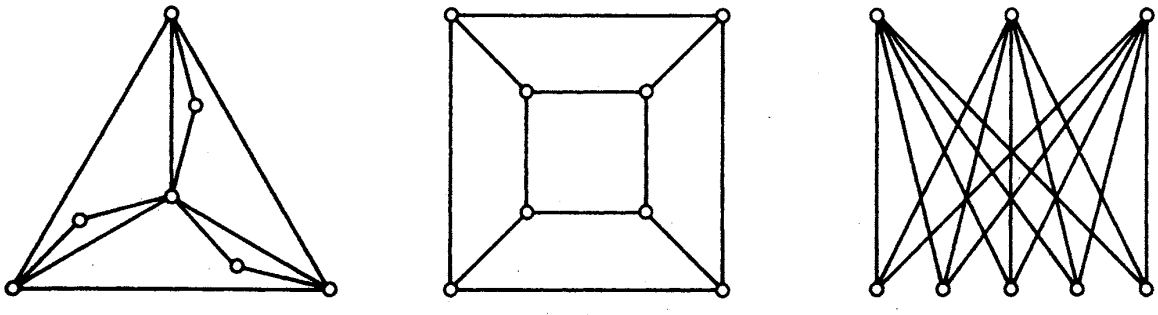
5.1.3 For each $k > 1$, find an example of a $k$-regular simple graph that has no perfect matching.

5.1.4 Two people play a game on a graph $G$ by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots$ such that, for $i > 0$, $v_i$ is adjacent to $v_{i-1}$. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if $G$ has no perfect matching.

5.1.5 A *k-factor* of $G$ is a $k$-regular spanning subgraph of $G$, and $G$ is *k-factorable* if there are edge-disjoint $k$-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_n$.

(*a*)* Show that
(i) $K_{n,n}$ and $K_{2n}$ are 1-factorable;
(ii) the Petersen graph is not 1-factorable.

(*b*) Which of the following graphs have 2-factors?



(c) Using Dirac's theorem (4.3), show that if $G$ is simple, with $\nu$ even and $\delta \geq (\nu/2)+1$, then $G$ has a 3-factor.

5.1.6* Show that $K_{2n+1}$ can be expressed as the union of $n$ connected 2-factors ($n \geq 1$).

## 5.2 MATCHINGS AND COVERINGS IN BIPARTITE GRAPHS

For any set $S$ of vertices in $G$, we define the *neighbour set* of $S$ in $G$ to be the set of all vertices adjacent to vertices in $S$; this set is denoted by $N_G(S)$. Suppose, now, that $G$ is a bipartite graph with bipartition $(X, Y)$. In many applications one wishes to find a matching of $G$ that saturates every vertex in $X$; an example is the personnel assignment problem, to be discussed in section 5.4. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935).

*Theorem 5.2* Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ contains a matching that saturates every vertex in $X$ if and only if

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq X \tag{5.2}$$

*Proof* Suppose that $G$ contains a matching $M$ which saturates every vertex in $X$, and let $S$ be a subset of $X$. Since the vertices in $S$ are matched under $M$ with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, suppose that $G$ is a bipartite graph satisfying (5.2), but that $G$ contains no matching saturating all the vertices in $X$. We shall obtain a contradiction. Let $M^*$ be a maximum matching in $G$. By our supposition, $M^*$ does not saturate all vertices in $X$. Let $u$ be an $M^*$-unsaturated vertex in $X$, and let $Z$ denote the set of all vertices connected to $u$ by $M^*$-alternating paths. Since $M^*$ is a maximum matching, it follows from theorem 5.1 that $u$ is the only $M^*$-unsaturated vertex in $Z$. Set $S = Z \cap X$ and $T = Z \cap Y$ (see figure 5.3).

Clearly, the vertices in $S \backslash \{u\}$ are matched under $M^*$ with the vertices in $T$. Therefore

$$|T| = |S| - 1 \tag{5.3}$$

and $N(S) \supseteq T$. In fact, we have

$$N(S) = T \tag{5.4}$$

since every vertex in $N(S)$ is connected to $u$ by an $M^*$-alternating path. But

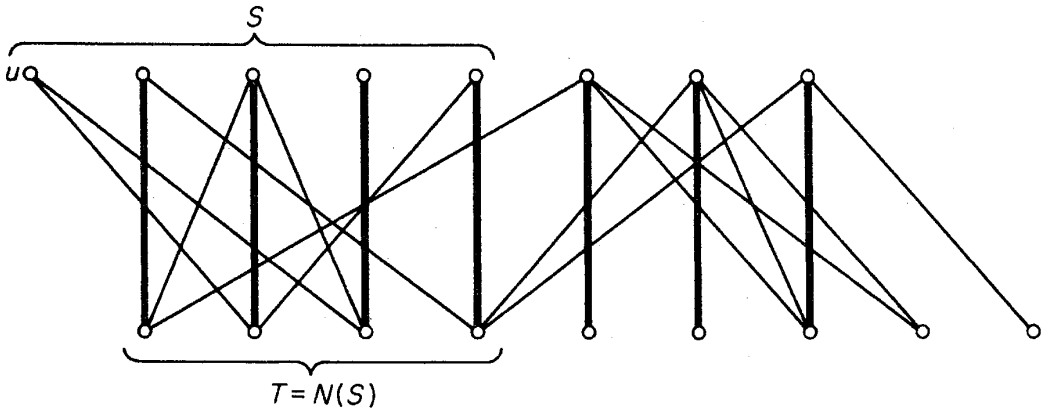



Figure 5.3

(5.3) and (5.4) imply that

$$|N(S)| = |S| - 1 < |S|$$

contradicting assumption (5.2) □

The above proof provides the basis of a good algorithm for finding a maximum matching in a bipartite graph. This algorithm will be presented in section 5.4.

*Corollary 5.2* If $G$ is a $k$-regular bipartite graph with $k > 0$, then $G$ has a perfect matching.

*Proof* Let $G$ be a $k$-regular bipartite graph with bipartition $(X, Y)$. Since $G$ is $k$-regular, $k\,|X| = |E| = k\,|Y|$ and so, since $k > 0$, $|X| = |Y|$. Now let $S$ be a subset of $X$ and denote by $E_1$ and $E_2$ the sets of edges incident with vertices in $S$ and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore

$$k\,|N(S)| = |E_2| \geq |E_1| = k\,|S|$$

It follows that $|N(S)| \geq |S|$ and hence, by theorem 5.2, that $G$ has a matching $M$ saturating every vertex in $X$. Since $|X| = |Y|$, $M$ is a perfect matching □

Corollary 5.2 is sometimes known as the *marriage theorem*, since it can be more colourfully restated as follows: if every girl in a village knows exactly $k$ boys, and every boy knows exactly $k$ girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.

A *covering* of a graph $G$ is a subset $K$ of $V$ such that every edge of $G$ has at least one end in $K$. A covering $K$ is a *minimum covering* if $G$ has no covering $K'$ with $|K'| < |K|$ (see figure 5.4).

If $K$ is a covering of $G$, and $M$ is a matching of $G$, then $K$ contains at

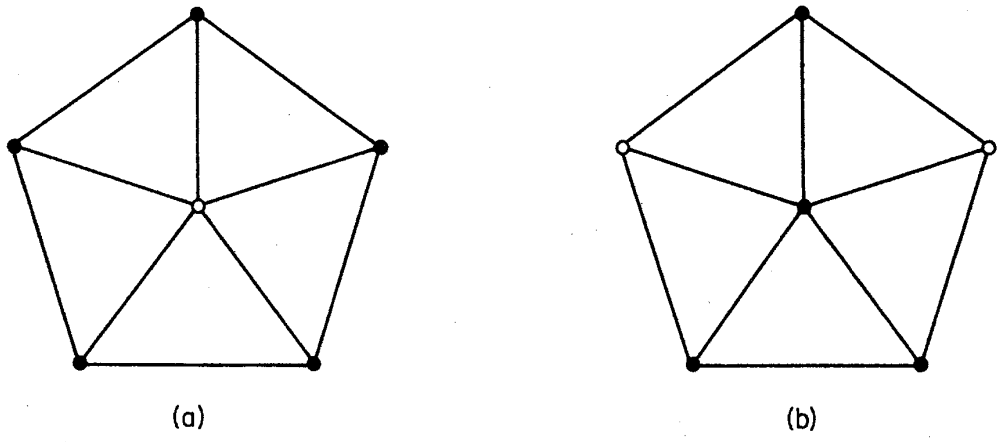



Figure 5.4. (*a*) A covering; (*b*) a minimum covering

least one end of each of the edges in $M$. Thus, for any matching $M$ and any covering $K$, $|M| \leq |K|$. Indeed, if $M^*$ is a maximum matching and $\tilde{K}$ is a minimum covering, then

$$|M^*| \leq |\tilde{K}| \tag{5.5}$$

In general, equality does not hold in (5.5) (see, for example, figure 5.4). However, if $G$ is bipartite we do have $|M^*| = |\tilde{K}|$. This result, due to König (1931), is closely related to Hall's theorem. Before presenting its proof, we make a simple, but important, observation.

*Lemma 5.3* Let $M$ be a matching and $K$ be a covering such that $|M| = |K|$. Then $M$ is a maximum matching and $K$ is a minimum covering.

*Proof* If $M^*$ is a maximum matching and $\tilde{K}$ is a minimum covering then, by (5.5),

$$|M| \leq |M^*| \leq |\tilde{K}| \leq |K|$$

Since $|M| = |K|$, it follows that $|M| = |M^*|$ and $|K| = |\tilde{K}|$ □

*Theorem 5.3* In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

*Proof* Let $G$ be a bipartite graph with bipartition $(X, Y)$, and let $M^*$ be a maximum matching of $G$. Denote by $U$ the set of $M^*$-unsaturated vertices in $X$, and by $Z$ the set of all vertices connected by $M^*$-alternating paths to vertices of $U$. Set $S = Z \cap X$ and $T = Z \cap Y$. Then, as in the proof of theorem 5.2, we have that every vertex in $T$ is $M^*$-saturated and $N(S) = T$. Define $\tilde{K} = (X \backslash S) \cup T$ (see figure 5.5). Every edge of $G$ must have at least one of its ends in $\tilde{K}$. For, otherwise, there would be an edge with one end in

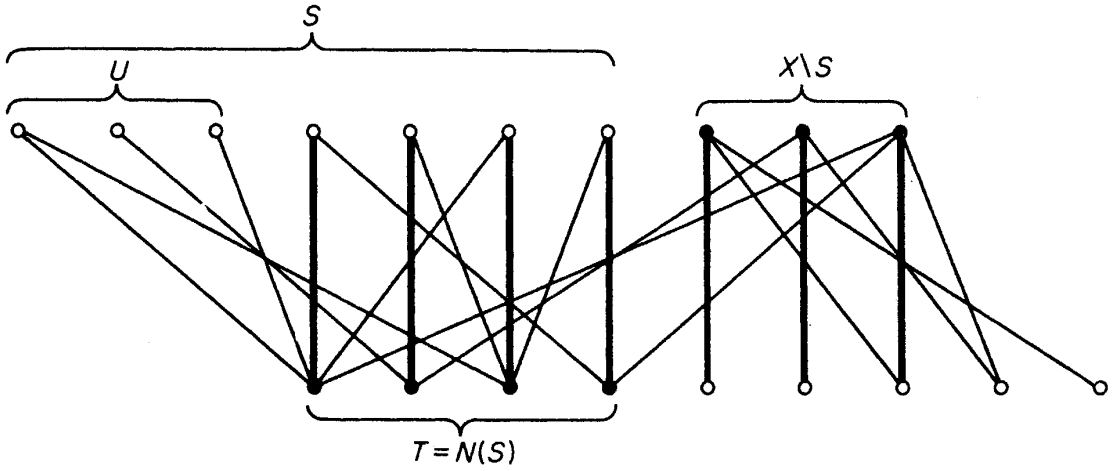



Figure 5.5

$S$ and one end in $Y \backslash T$, contradicting $N(S) = T$. Thus $\tilde{K}$ is a covering of $G$ and clearly

$$|M^*| = |\tilde{K}|$$

By lemma 5.3, $\tilde{K}$ is a minimum covering, and the theorem follows □

*Exercises*

5.2.1 Show that it is impossible, using $1 \times 2$ rectangles, to exactly cover an $8 \times 8$ square from which two opposite $1 \times 1$ corner squares have been removed.

5.2.2 (*a*) Show that a bipartite graph $G$ has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V$.

(*b*) Give an example to show that the above statement does not remain valid if the condition that $G$ be bipartite is dropped.

5.2.3 For $k > 0$, show that

(*a*) every $k$-regular bipartite graph is 1-factorable;

(*b*)* every $2k$-regular graph is 2-factorable. (J. Petersen)

5.2.4 Let $A_1, A_2, \ldots, A_m$ be subsets of a set $S$. A *system of distinct representatives* for the family $(A_1, A_2, \ldots, A_m)$ is a subset $\{a_1, a_2, \ldots, a_m\}$ of $S$ such that $a_i \in A_i$, $1 \leq i \leq m$, and $a_i \neq a_j$ for $i \neq j$. Show that $(A_1, A_2, \ldots, A_m)$ has a system of distinct representatives if and only if $\left| \bigcup_{i \in J} A_i \right| \geq |J|$ for all subsets $J$ of $\{1, 2, \ldots, m\}$. (P. Hall)

5.2.5 A *line* of a matrix is a row or a column of the matrix. Show that the minimum number of lines containing all the 1's of a (0, 1)-matrix is equal to the maximum number of 1's, no two of which are in the same line.

5.2.6 (*a*) Prove the following generalisation of Hall's theorem (5.2): if $G$ is a bipartite graph with bipartition $(X, Y)$, the number of edges in a maximum matching of $G$ is

$$|X| - \max_{S \subseteq X} \{|S| - |N(S)|\}$$

(D. König, O. Ore)

(*b*) Deduce that if $G$ is simple with $|X| = |Y| = n$ and $\varepsilon > (k-1)n$, then $G$ has a matching of cardinality $k$.

5.2.7 Deduce Hall's theorem (5.2) from König's theorem (5.3).

5.2.8* A non-negative real matrix $\mathbf{Q}$ is *doubly stochastic* if the sum of the entries in each row of $\mathbf{Q}$ is 1 and the sum of the entries in each column of $\mathbf{Q}$ is 1. A *permutation matrix* is a (0, 1)-matrix which has exactly one 1 in each row and each column. (Thus every permutation matrix is doubly stochastic.) Show that

(*a*) every doubly stochastic matrix is necessarily square;

(*b*) every doubly stochastic matrix $\mathbf{Q}$ can be expressed as a convex linear combination of permutation matrices; that is

$$\mathbf{Q} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + \ldots + c_k\mathbf{P}_k$$

where each $\mathbf{P}_i$ is a permutation matrix, each $c_i$ is a non-negative real number, and $\sum_1^k c_i = 1$. (G. Birkhoff, J. von Neumann)

5.2.9 Let $H$ be a finite group and let $K$ be a subgroup of $H$. Show that there exist elements $h_1, h_2, \ldots, h_n \in H$ such that $h_1K, h_2K, \ldots, h_nK$ are the left cosets of $K$ and $Kh_1, Kh_2, \ldots, Kh_n$ are the right cosets of $K$. (P. Hall)

## 5.3 PERFECT MATCHINGS

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte (1947). The proof given here is due to Lovász (1973).

A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of $G$.

*Theorem 5.4* $G$ has a perfect matching if and only if

$$o(G-S) \leq |S| \quad \text{for all} \quad S \subset V \tag{5.6}$$

*Proof* It clearly suffices to prove the theorem for simple graphs.

Suppose first that $G$ has a perfect matching $M$. Let $S$ be a proper subset of $V$, and let $G_1, G_2, \ldots, G_n$ be the odd components of $G - S$. Because $G_i$ is odd, some vertex $u_i$ of $G_i$ must be matched under $M$ with a vertex $v_i$ of $S$ (see figure 5.6). Therefore, since $\{v_1, v_2, \ldots, v_n\} \subseteq S$

$$o(G-S) = n = |\{v_1, v_2, \ldots, v_n\}| \leq |S|$$

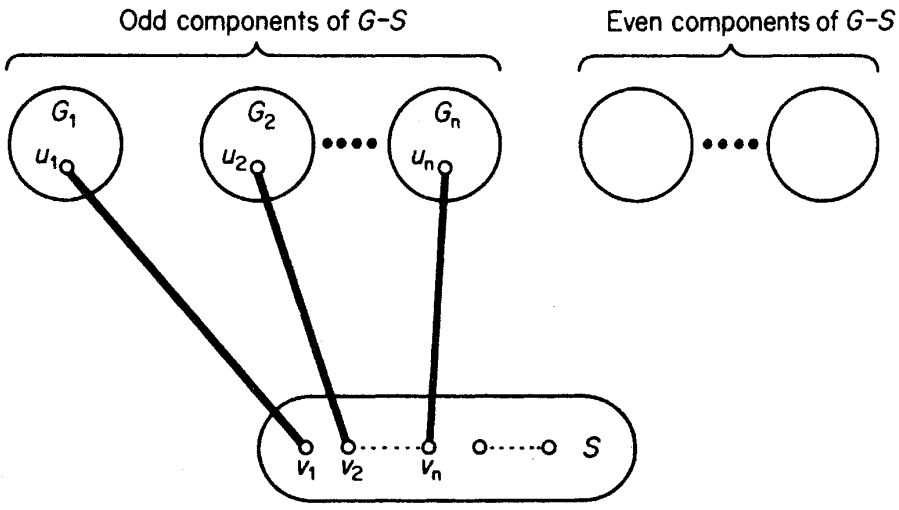



Figure 5.6

Conversely, suppose that $G$ satisfies (5.6) but has no perfect matching. Then $G$ is a spanning subgraph of a maximal graph $G^*$ having no perfect matching. Since $G-S$ is a spanning subgraph of $G^*-S$ we have $o(G^*-S) \leq o(G-S)$ and so, by (5.6),

$$o(G^*-S) \leq |S| \quad \text{for all} \quad S \subset V(G^*) \tag{5.7}$$

In particular, setting $S=\emptyset$, we see that $o(G^*)=0$, and so $\nu(G^*)$ is even.

Denote by $U$ the set of vertices of degree $\nu-1$ in $G^*$. Since $G^*$ clearly has a perfect matching if $U=V$, we may assume that $U \neq V$. We shall show that $G^*-U$ is a disjoint union of complete graphs. Suppose, to the contrary, that some component of $G^*-U$ is not complete. Then, in this component, there are vertices $x$, $y$ and $z$ such that $xy \in E(G^*)$, $yz \in E(G^*)$ and $xz \notin E(G^*)$ (exercise 1.6.14). Moreover, since $y \notin U$, there is a vertex $w$ in $G^*-U$ such that $yw \notin E(G^*)$. The situation is illustrated in figure 5.7.

Since $G^*$ is a maximal graph containing no perfect matching, $G^*+e$ has a perfect matching for all $e \notin E(G^*)$. Let $M_1$ and $M_2$ be perfect matchings in $G^*+xz$ and $G^*+yw$, respectively, and denote by $H$ the subgraph of

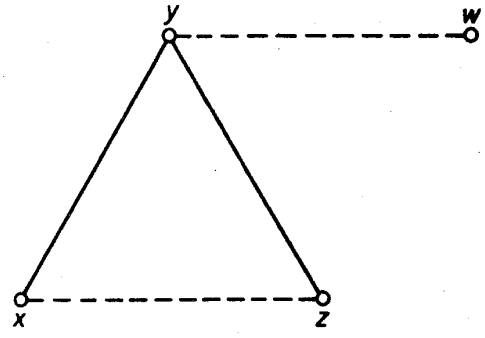



Figure 5.7

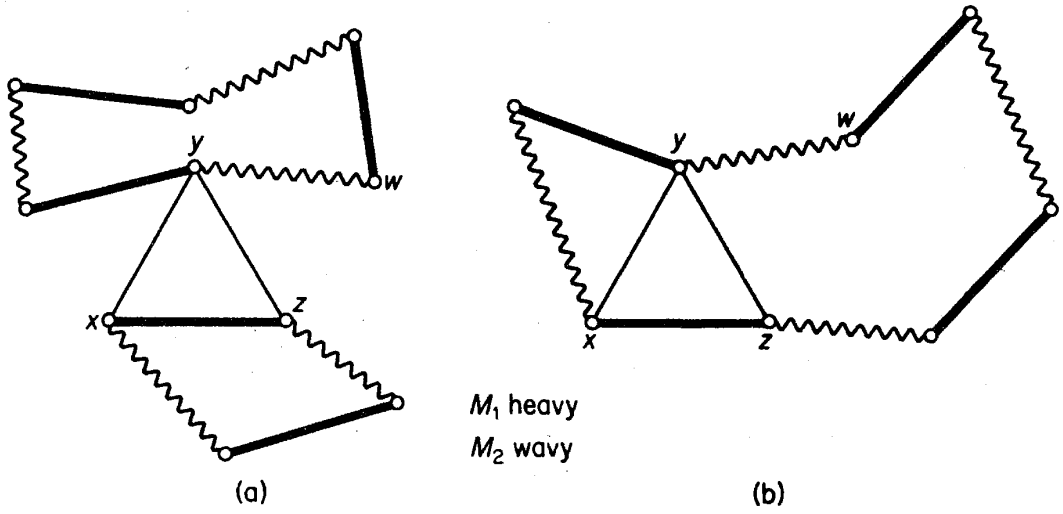



Figure 5.8

$G^* \cup \{xz, yw\}$ induced by $M_1 \Delta M_2$. Since each vertex of $H$ has degree two, $H$ is a disjoint union of cycles. Furthermore, all of these cycles are even, since edges of $M_1$ alternate with edges of $M_2$ around them. We distinguish two cases:

*Case 1* $xz$ and $yw$ are in different components of $H$ (figure 5.8*a*). Then, if $yw$ is in the cycle $C$ of $H$, the edges of $M_1$ in $C$, together with the edges of $M_2$ not in $C$, constitute a perfect matching in $G^*$, contradicting the definition of $G^*$.

*Case 2* $xz$ and $yw$ are in the same component $C$ of $H$. By symmetry of $x$ and $z$, we may assume that the vertices $x$, $y$, $w$ and $z$ occur in that order on $C$ (figure 5.8*b*). Then the edges of $M_1$ in the section $yw \ldots z$ of $C$, together with the edge $yz$ and the edges of $M_2$ not in the section $yw \ldots z$ of $C$,

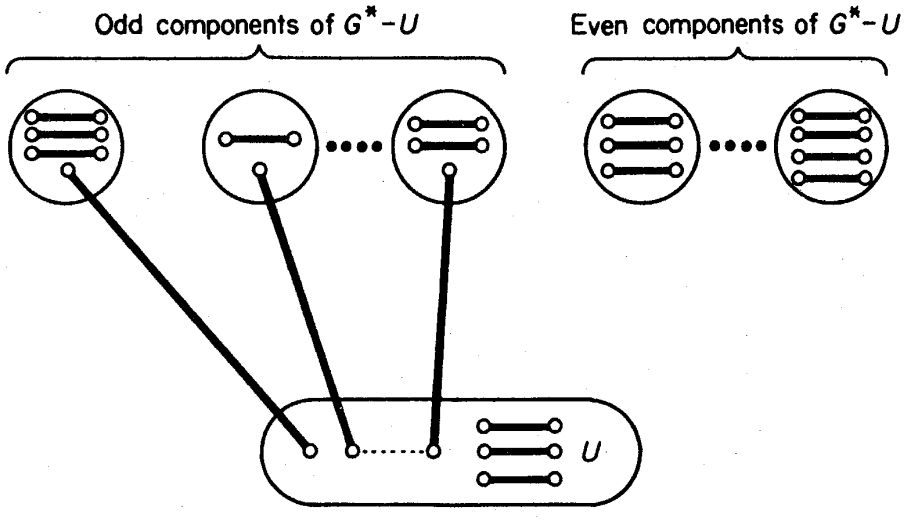



Figure 5.9

constitute a perfect matching in $G^*$, again contradicting the definition of $G^*$.

Since both case 1 and case 2 lead to contradictions, it follows that $G^* - U$ is indeed a disjoint union of complete graphs.

Now, by (5.7), $o(G^* - U) \leq |U|$. Thus at most $|U|$ of the components of $G^* - U$ are odd. But then $G^*$ clearly has a perfect matching: one vertex in each odd component of $G^* - U$ is matched with a vertex of $U$; the remaining vertices in $U$, and in components of $G^* - U$, are then matched as indicated in figure 5.9.

Since $G^*$ was assumed to have no perfect matching we have obtained the desired contradiction. Thus $G$ does indeed have a perfect matching □

The above theorem can also be proved with the aid of Hall's theorem (see Anderson, 1971).

From Tutte's theorem, we now deduce a result first obtained by Petersen (1891).

*Corollary 5.4* Every 3-regular graph without cut edges has a perfect matching.

*Proof* Let $G$ be a 3-regular graph without cut edges, and let $S$ be a proper subset of $V$. Denote by $G_1, G_2, \ldots, G_n$ the odd components of $G - S$, and let $m_i$ be the number of edges with one end in $G_i$ and one end in $S$, $1 \leq i \leq n$. Since $G$ is 3-regular

$$\sum_{v \in V(G_i)} d(v) = 3\nu(G_i) \quad \text{for} \quad 1 \leq i \leq n \tag{5.8}$$

and

$$\sum_{v \in S} d(v) = 3\,|S| \tag{5.9}$$

By (5.8), $m_i = \sum_{v \in V(G_i)} d(v) - 2\varepsilon(G_i)$ is odd. Now $m_i \neq 1$ since $G$ has no cut edge. Thus

$$m_i \geq 3 \quad \text{for} \quad 1 \leq i \leq n \tag{5.10}$$

It follows from (5.10) and (5.9) that

$$o(G - S) = n \leq \frac{1}{3} \sum_{i=1}^{n} m_i \leq \frac{1}{3} \sum_{v \in S} d(v) = |S|$$

Therefore, by theorem 5.4, $G$ has a perfect matching □

A 3-regular graph with cut edges need not have a perfect matching. For example, it follows from theorem 5.4 that the graph $G$ of figure 5.10 has no perfect matching, since $o(G - v) = 3$.

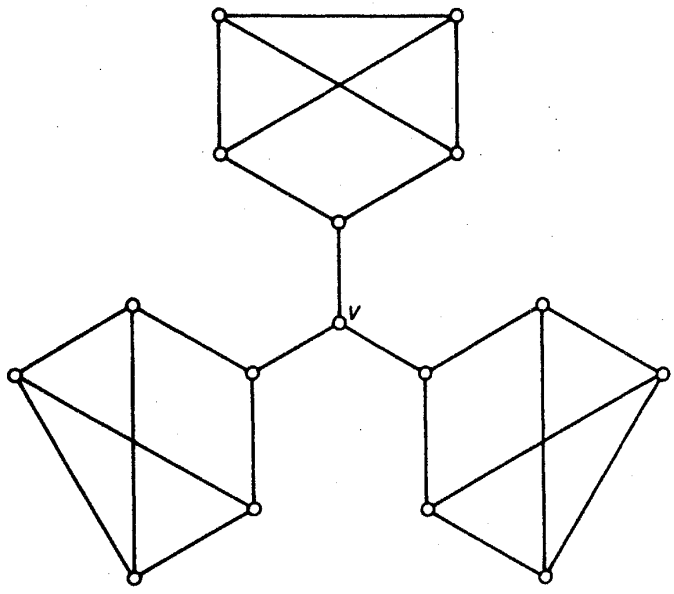



Figure 5.10

*Exercises*

5.3.1* Derive Hall's theorem (5.2) from Tutte's theorem (5.4).

5.3.2 Prove the following generalisation of corollary 5.4: if $G$ is a $(k-1)$-edge-connected $k$-regular graph with $\nu$ even, then $G$ has a perfect matching.

5.3.3 Show that a tree $G$ has a perfect matching if and only if $o(G-v)=1$ for all $v \in V$. (V. Chungphaisan)

5.3.4* Prove the following generalisation of Tutte's theorem (5.4): the number of edges in a maximum matching of $G$ is $\frac{1}{2}(\nu-d)$, where $d=\max_{S\subseteq V}\{o(G-S)-|S|\}$. (C. Berge)

5.3.5 (*a*) Using Tutte's theorem (5.4), characterise the maximal simple graphs which have no perfect matching.

(*b*) Let $G$ be simple, with $\nu$ even and $\delta<\nu/2$. Show that if $\varepsilon > \binom{\delta}{2}+\binom{\nu-2\delta-1}{2}+\delta(\nu-\delta)$, then $G$ has a perfect matching.

## APPLICATIONS

### 5.4 THE PERSONNEL ASSIGNMENT PROBLEM

In a certain company, $n$ workers $X_1, X_2, \ldots, X_n$ are available for $n$ jobs $Y_1, Y_2, \ldots, Y_n$, each worker being qualified for one or more of these jobs. Can all the men be assigned, one man per job, to jobs for which they are qualified? This is the *personnel assignment problem*.

We construct a bipartite graph $G$ with bipartition $(X, Y)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and $x_i$ is joined to $y_j$ if and only if worker $X_i$ is qualified for job $Y_j$. The problem becomes one of determining whether or not $G$ has a perfect matching. According to Hall's theorem (5.2), either $G$ has such a matching or there is a subset $S$ of $X$ such that $|N(S)| < |S|$. In the sequel, we shall present an algorithm to solve the personnel assignment problem. Given any bipartite graph $G$ with bipartition $(X, Y)$, the algorithm either finds a matching of $G$ that saturates every vertex in $X$ or, failing this, finds a subset $S$ of $X$ such that $|N(S)| < |S|$.

The basic idea behind the algorithm is very simple. We start with an arbitrary matching $M$. If $M$ saturates every vertex in $X$, then it is a matching of the required type. If not, we choose an $M$-unsaturated vertex $u$ in $X$ and systematically search for an $M$-augmenting path with origin $u$. Our method of search, to be described in detail below, finds such a path $P$ if one exists; in this case $\hat{M} = M \Delta E(P)$ is a larger matching than $M$, and hence saturates more vertices in $X$. We then repeat the procedure with $\hat{M}$ instead of $M$. If such a path does not exist, the set $Z$ of all vertices which are connected to $u$ by $M$-alternating paths is found. Then (as in the proof of theorem 5.2) $S = Z \cap X$ satisfies $|N(S)| < |S|$.

Let $M$ be a matching in $G$, and let $u$ be an $M$-unsaturated vertex in $X$. A tree $H \subseteq G$ is called an *$M$-alternating tree rooted at $u$* if (i) $u \in V(H)$, and (ii) for every vertex $v$ of $H$, the unique $(u, v)$-path in $H$ is an $M$-alternating path. An $M$-alternating tree in a graph is shown in figure 5.11.

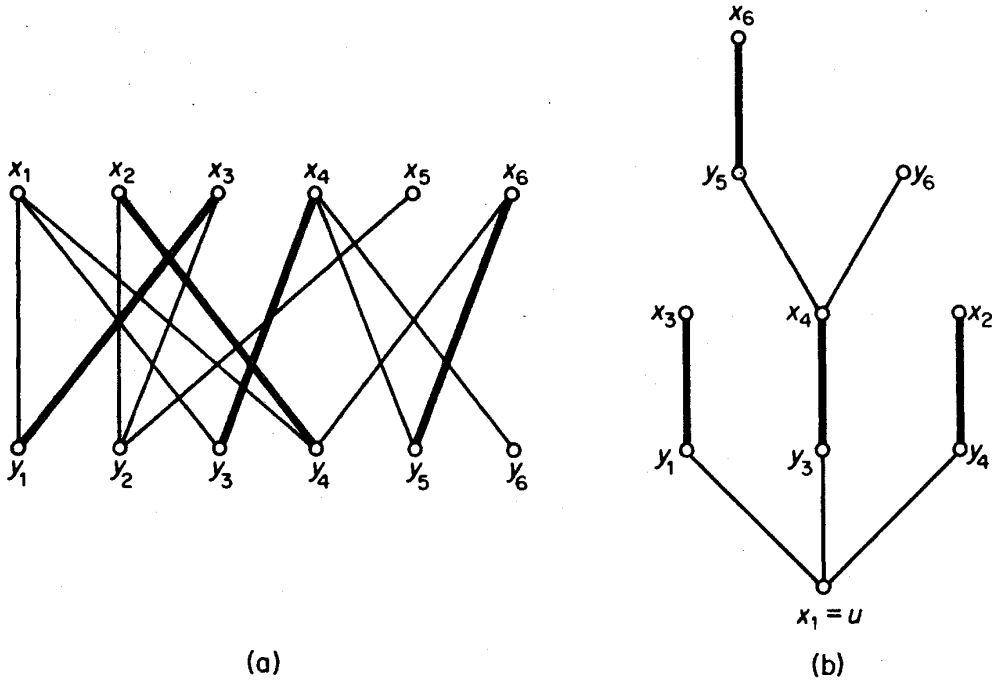



Figure 5.11. (*a*) A matching $M$ in $G$; (*b*) an $M$-alternating tree in $G$

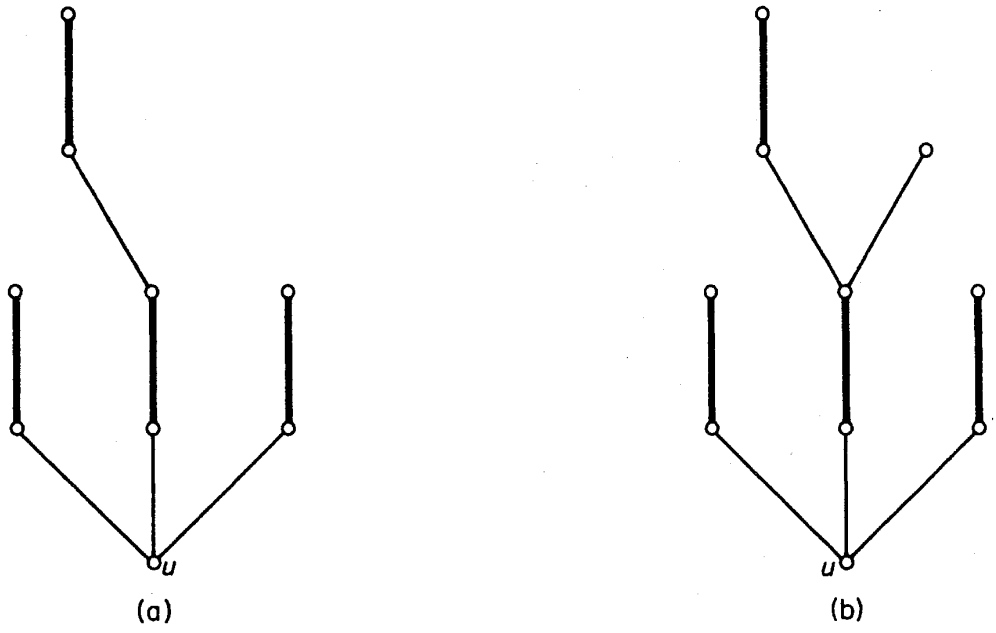



Figure 5.12. (*a*) Case (i); (*b*) case (ii)

The search for an $M$-augmenting path with origin $u$ involves 'growing' an $M$-alternating tree $H$ rooted at $u$. This procedure was first suggested by Edmonds (1965). Initially, $H$ consists of just the single vertex $u$. It is then grown in such a way that, at any stage, either

(i) all vertices of $H$ except $u$ are $M$-saturated and matched under $M$ (as in figure 5.12*a*), or
(ii) $H$ contains an $M$-unsaturated vertex different from $u$ (as in figure 5.12*b*).

If (i) is the case (as it is initially) then, setting $S = V(H) \cap X$ and $T = V(H) \cap Y$, we have $N(S) \supseteq T$; thus either $N(S) = T$ or $N(S) \supset T$.

(a) If $N(S) = T$ then, since the vertices in $S \backslash \{u\}$ are matched with the vertices in $T$, $|N(S)| = |S| - 1$, indicating that $G$ has no matching saturating all vertices in $X$.
(b) If $N(S) \supset T$, there is a vertex $y$ in $Y \backslash T$ adjacent to a vertex $x$ in $S$. Since all vertices of $H$ except $u$ are matched under $M$, either $x = u$ or else $x$ is matched with a vertex of $H$. Therefore $xy \notin M$. If $y$ is $M$-saturated, with $yz \in M$, we grow $H$ by adding the vertices $y$ and $z$ and the edges $xy$ and $yz$. We are then back in case (i). If $y$ is $M$-unsaturated, we grow $H$ by adding the vertex $y$ and the edge $xy$, resulting in case (ii). The $(u, y)$-path of $H$ is then an $M$-augmenting path with origin $u$, as required.

Figure 5.13 illustrates the above tree-growing procedure.

The algorithm described above is known as the *Hungarian method*, and

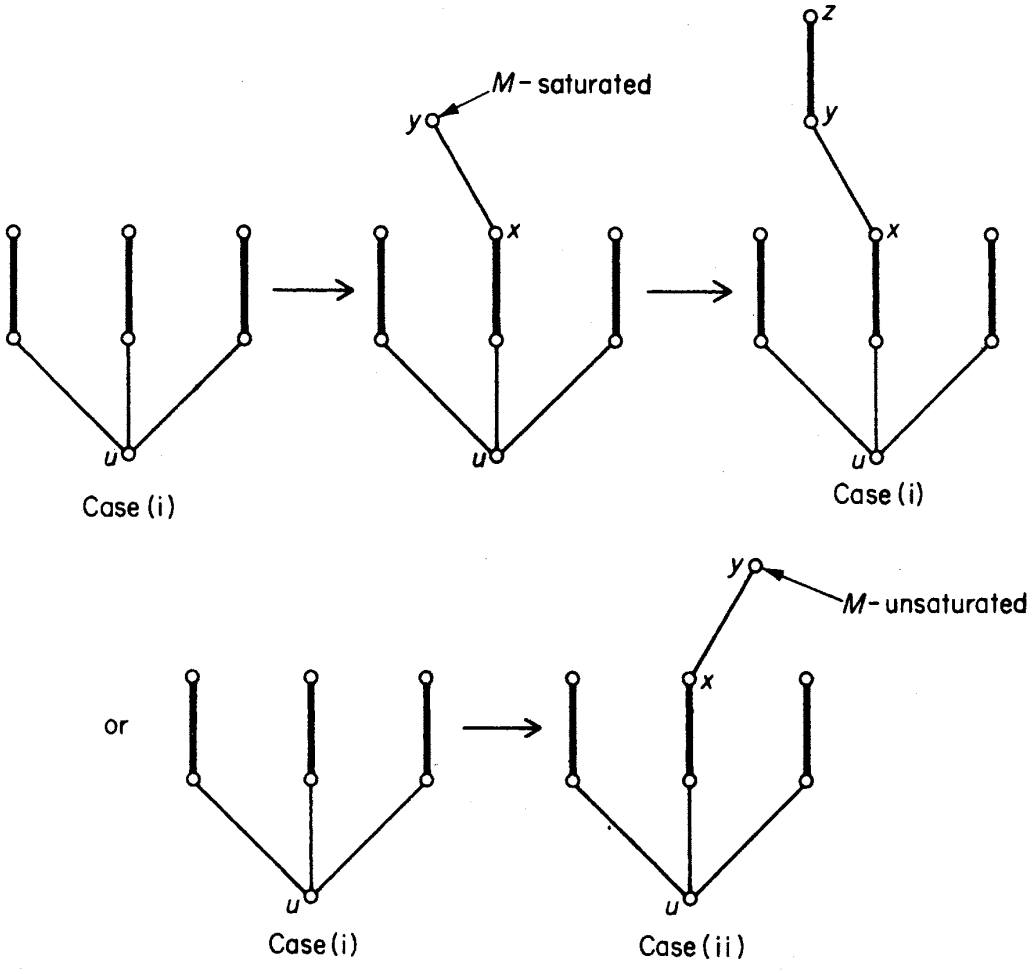



Figure 5.13. The tree-growing procedure

can be summarised as follows:

Start with an arbitrary matching $M$.

1. If $M$ saturates every vertex in $X$, stop. Otherwise, let $u$ be an $M$-unsaturated vertex in $X$. Set $S=\{u\}$ and $T=\emptyset$.
2. If $N(S)=T$ then $|N(S)|<|S|$, since $|T|=|S|-1$. Stop, since by Hall's theorem there is no matching that saturates every vertex in $X$. Otherwise, let $y \in N(S)\backslash T$.
3. If $y$ is $M$-saturated, let $yz \in M$. Replace $S$ by $S \cup \{z\}$ and $T$ by $T \cup \{y\}$ and go to step 2. (Observe that $|T|=|S|-1$ is maintained after this replacement.) Otherwise, let $P$ be an $M$-augmenting $(u, y)$-path. Replace $M$ by $\hat{M}=M \Delta E(P)$ and go to step 1.

Consider, for example, the graph $G$ in figure 5.14*a*, with initial matching $M=\{x_2y_2, x_3y_3, x_5y_5\}$. In figure 5.14*b* an $M$-alternating tree is grown, starting with $x_1$, and the $M$-augmenting path $x_1y_2x_2y_1$ found. This results in a new matching $\hat{M}=\{x_1y_2, x_2y_1, x_3y_3, x_5y_5\}$, and an $\hat{M}$-alternating tree is now grown from $x_4$ (figures 5.14*c* and 5.14*d*) Since there is no $\hat{M}$-augmenting

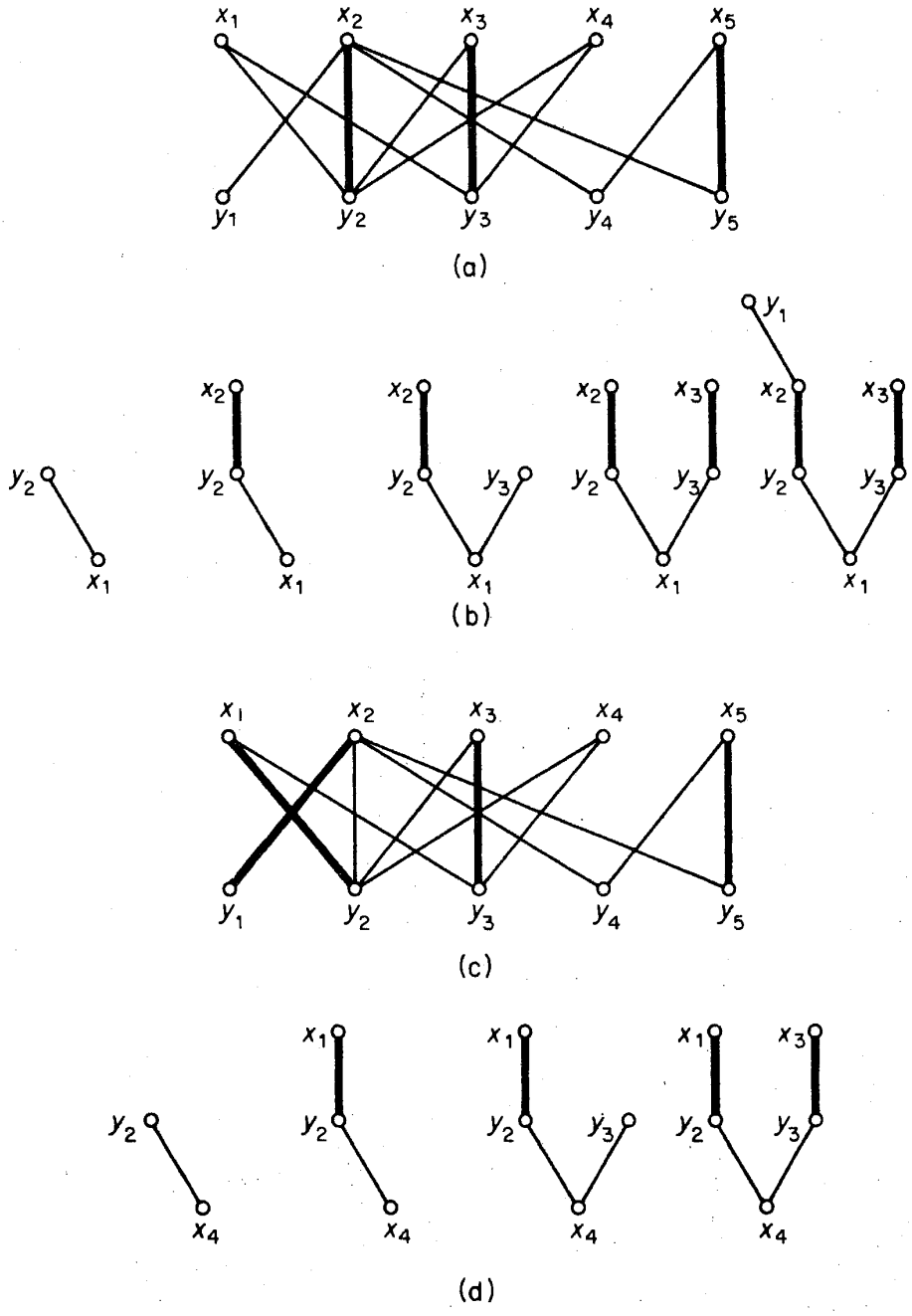



Figure 5.14. (*a*) Matching $M$; (*b*) an $M$-alternating tree; (*c*) matching $\hat{M}$; (*d*) an $\hat{M}$-alternating tree

path with origin $x_4$, the algorithm terminates. The set $S=\{x_1, x_3, x_4\}$, with neighbour set $N(S)=\{y_2, y_3\}$, shows that $G$ has no perfect matching.

A flow diagram of the Hungarian method is given in figure 5.15. Since the algorithm can cycle through the tree-growing procedure, I, at most $|X|$ times before finding either an $S \subseteq X$ such that $|N(S)|<|S|$ or an $M$-augmenting path, and since the initial matching can be augmented at most $|X|$ times

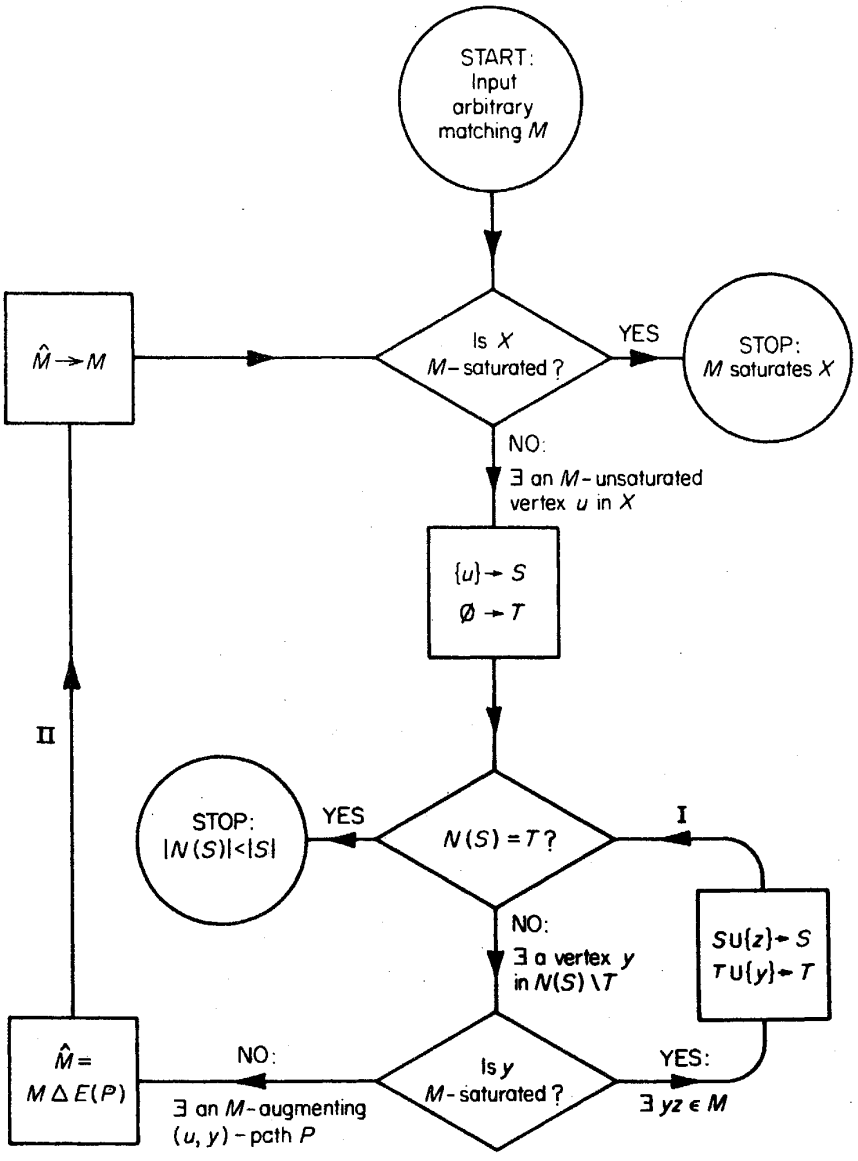



Figure 5.15. The Hungarian method

before a matching of the required type is found, it is clear that the Hungarian method is a good algorithm.

One can find a maximum matching in a bipartite graph by slightly modifying the above procedure (exercise 5.4.1). A good algorithm that determines such a matching in any graph has been given by Edmonds (1965).

*Exercise*

5.4.1 Describe how the Hungarian method can be used to find a maximum matching in a bipartite graph.

## 5.5 THE OPTIMAL ASSIGNMENT PROBLEM

The Hungarian method, described in section 5.4, is an efficient way of determining a feasible assignment of workers to jobs, if one exists. However one may, in addition, wish to take into account the effectiveness of the workers in their various jobs (measured, perhaps, by the profit to the company). In this case, one is interested in an assignment that maximises the total effectiveness of the workers. The problem of finding such an assignment is known as the *optimal assignment problem.*

Consider a weighted complete bipartite graph with bipartition $(X, Y)$, where $X=\{x_1, x_2, \ldots, x_n\}$, $Y=\{y_1, y_2, \ldots, y_n\}$ and edge $x_iy_j$ has weight $w_{ij}=w(x_iy_j)$, the effectiveness of worker $X_i$ in job $Y_j$. The optimal assignment problem is clearly equivalent to that of finding a maximum-weight perfect matching in this weighted graph. We shall refer to such a matching as an *optimal matching.*

To solve the optimal assignment problem it is, of course, possible to enumerate all $n!$ perfect matchings and find an optimal one among them. However, for large $n$, such a procedure would clearly be most inefficient. In this section we shall present a good algorithm for finding an optimal matching in a weighted complete bipartite graph.

We define a *feasible vertex labelling* as a real-valued function $l$ on the vertex set $X \cup Y$ such that, for all $x \in X$ and $y \in Y$

$$l(x)+l(y) \geq w(xy) \tag{5.11}$$

(The real number $l(v)$ is called the *label* of the vertex $v$.) A feasible vertex labelling is thus a labelling of the vertices such that the sum of the labels of the two ends of an edge is at least as large as the weight of the edge. No matter what the edge weights are, there always exists a feasible vertex labelling; one such is the function $l$ given by

$$\left.\begin{aligned} l(x) &= \max_{y \in Y} w(xy) \quad \text{if} \quad x \in X \\ l(y) &= 0 \qquad\qquad\quad \text{if} \quad y \in Y \end{aligned}\right\} \tag{5.12}$$

If $l$ is a feasible vertex labelling, we denote by $E_l$ the set of those edges for which equality holds in (5.11); that is

$$E_l=\{xy \in E \mid l(x)+l(y)=w(xy)\}$$

The spanning subgraph of $G$ with edge set $E_l$ is referred to as the *equality subgraph* corresponding to the feasible vertex labelling $l$, and is denoted by $G_l$. The connection between equality subgraphs and optimal matchings is provided by the following theorem.

*Theorem 5.5* Let $l$ be a feasible vertex labelling of $G$. If $G_l$ contains a perfect matching $M^*$, then $M^*$ is an optimal matching of $G$.

*Proof* Suppose that $G_l$ contains a perfect matching $M^*$. Since $G_l$ is a spanning subgraph of $G$, $M^*$ is also a perfect matching of $G$. Now

$$w(M^*) = \sum_{e \in M^*} w(e) = \sum_{v \in V} l(v) \tag{5.13}$$

since each $e \in M^*$ belongs to the equality subgraph and the ends of edges of $M^*$ cover each vertex exactly once. On the other hand, if $M$ is any perfect matching of $G$, then

$$w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in V} l(v) \tag{5.14}$$

It follows from (5.13) and (5.14) that $w(M^*) \geq w(M)$. Thus $M^*$ is an optimal matching □

The above theorem is the basis of an algorithm, due to Kuhn (1955) and Munkres (1957), for finding an optimal matching in a weighted complete bipartite graph. Our treatment closely follows Edmonds (1967).

Starting with an arbitrary feasible vertex labelling $l$ (for example, the one given in (5.12)), we determine $G_l$, choose an arbitrary matching $M$ in $G_l$ and apply the Hungarian method. If a perfect matching is found in $G_l$ then, by theorem 5.5, this matching is optimal. Otherwise, the Hungarian method terminates in a matching $M'$ that is not perfect, and an $M'$-alternating tree $H$ that contains no $M'$-augmenting path and cannot be grown further (in $G_l$). We then modify $l$ to a feasible vertex labelling $\hat{l}$ with the property that both $M'$ and $H$ are contained in $G_{\hat{l}}$ and $H$ can be extended in $G_{\hat{l}}$. Such modifications in the feasible vertex labelling are made whenever necessary, until a perfect matching is found in some equality subgraph.

*The Kuhn–Munkres Algorithm*

Start with an arbitrary feasible vertex labelling $l$, determine $G_l$, and choose an arbitrary matching $M$ in $G_l$.

1. If $X$ is $M$-saturated, then $M$ is a perfect matching (since $|X| = |Y|$) and hence, by theorem 5.5, an optimal matching; in this case, stop. Otherwise, let $u$ be an $M$-unsaturated vertex. Set $S = \{u\}$ and $T = \emptyset$.
2. If $N_{G_l}(S) \supset T$, go to step 3. Otherwise, $N_{G_l}(S) = T$. Compute

$$\alpha_l = \min_{\substack{x \in S \\ y \notin T}} \{l(x) + l(y) - w(xy)\}$$

and the feasible vertex labelling $\hat{l}$ given by

$$\hat{l}(v) = \begin{cases} l(v) - \alpha_l & \text{if} \quad v \in S \\ l(v) + \alpha_l & \text{if} \quad v \in T \\ l(v) & \text{otherwise} \end{cases}$$

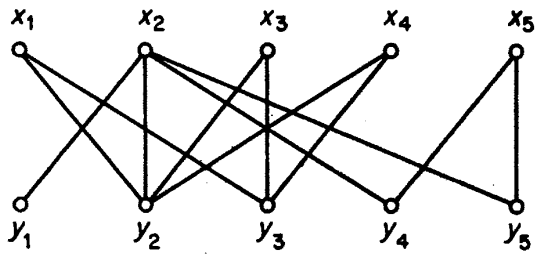
(Note that $\alpha_l > 0$ and that $N_{G_{\hat{l}}}(S) \supset T$.) Replace $l$ by $\hat{l}$ and $G_l$ by $G_{\hat{l}}$.

$$\begin{bmatrix} 3 & 5 & 5 & 4 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix}$$

(a)

$$\begin{array}{cc} \begin{bmatrix} 3 & 5 & 5 & 4 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix} & \begin{matrix} 5 \\ 2 \\ 4 \\ 1 \\ 3 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & \end{array}$$

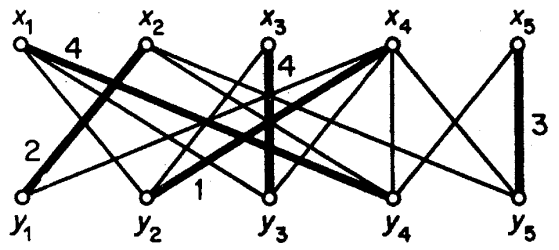
(b)




(c)

$$\begin{array}{cc} \begin{bmatrix} 3 & 5 & 5 & 4 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix} & \begin{matrix} 4 \\ 2 \\ 3 \\ 0 \\ 3 \end{matrix} \\ \begin{matrix} 0 & 1 & 1 & 0 & 0 \end{matrix} & \end{array}$$

(d)




(e)

Figure 5.16

3. Choose a vertex $y$ in $N_{G_l}(S)\backslash T$. As in the tree-growing procedure of section 5.4, consider whether or not $y$ is $M$-saturated. If $y$ is $M$-saturated, with $yz \in M$, replace $S$ by $S \cup \{z\}$ and $T$ by $T \cup \{y\}$, and go to step 2. Otherwise, let $P$ be an $M$-augmenting $(u, y)$-path in $G_l$, replace $M$ by $\hat{M} = M \,\Delta E(P)$, and go to step 1.

In illustrating the Kuhn–Munkres algorithm, it is convenient to represent a weighted complete bipartite graph $G$ by a matrix $\mathbf{W} = [w_{ij}]$, where $w_{ij}$ is the weight of edge $x_i y_j$ in $G$. We shall start with the matrix of figure 5.16*a*. In figure 5.16*b*, the feasible vertex labelling (5.12) is shown (by placing the label of $x_i$ to the right of row $i$ of the matrix and the label of $y_j$ below column $j$) and the entries corresponding to edges of the associated equality subgraph are indicated; the equality subgraph itself is depicted (without weights) in figure 5.16*c*. It was shown in the previous section that this graph has no perfect matching (the set $S = \{x_1, x_3, x_4\}$ has neighbour set $\{y_2, y_3\}$). We therefore modify our initial feasible vertex labelling to the one given in figure 5.16*d*. An application of the Hungarian method now shows that the associated equality subgraph (figure 5.16*e*) has the perfect matching $\{x_1y_4, x_2y_1, x_3y_3, x_4y_2, x_5y_5\}$. This is therefore an optimal matching of $G$.

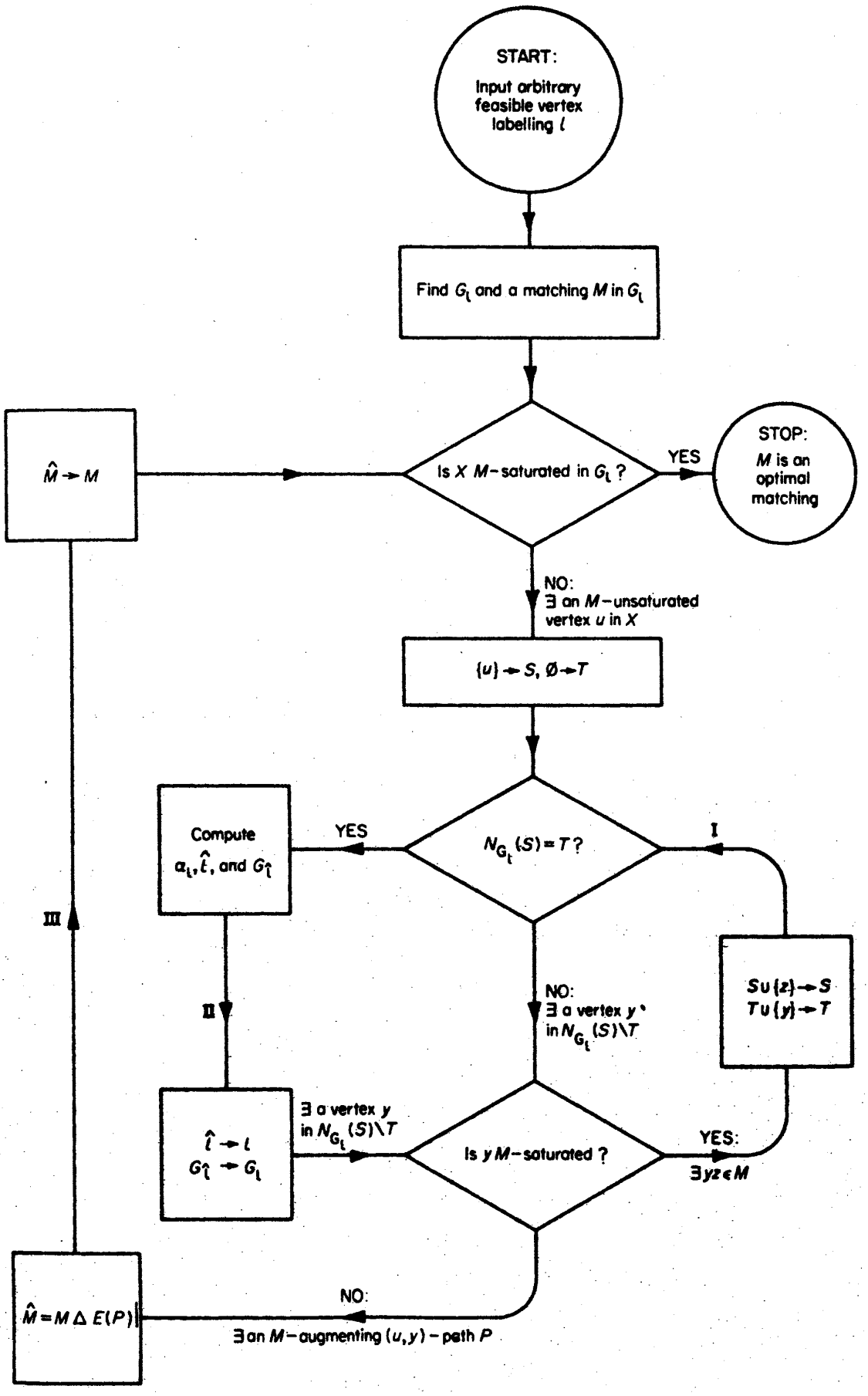



**Figure 5.17. The Kuhn–Munkres algorithm**

A flow diagram for the Kuhn–Munkres algorithm is given in figure 5.17. In cycle II, the number of computations required to compute $G_l$ is clearly of order $\nu^2$. Since the algorithm can cycle through I and II at most $|X|$ times before finding an $M$-augmenting path, and since the initial matching can be augmented at most $|X|$ times before an optimal matching is found, we see that the Kuhn–Munkres algorithm is a good algorithm.

*Exercise*

5.5.1 A *diagonal* of an $n \times n$ matrix is a set of $n$ entries no two of which belong to the same row or the same column. The weight of a diagonal is the sum of the entries in it. Find a minimum-weight diagonal in the following matrix:

$$\begin{bmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{bmatrix}$$

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# 6 Edge Colourings

## 6.1 EDGE CHROMATIC NUMBER

A *k-edge colouring* $\mathscr{C}$ of a loopless graph $G$ is an assignment of $k$ colours, $1, 2, \ldots, k$, to the edges of $G$. The colouring $\mathscr{C}$ is *proper* if no two adjacent edges have the same colour.

Alternatively, a $k$-edge colouring can be thought of as a partition $(E_1, E_2, \ldots, E_k)$ of $E$, where $E$ denotes the (possibly empty) subset of $E$ assigned colour $i$. A proper $k$-edge colouring is then a $k$-edge colouring $(E_1, E_2, \ldots, E_k)$ in which each subset $E_i$ is a matching. The graph of figure 6.1 has the proper 4-edge colouring $(\{a, g\}, \{b, e\}, \{c, f\}, \{d\})$.

$G$ is *k-edge colourable* if $G$ has a proper $k$-edge-colouring. Trivially, every loopless graph $G$ is $\varepsilon$-edge-colourable; and if $G$ is $k$-edge-colourable, then $G$ is also $l$-edge-colourable for every $l > k$. The ***edge chromatic number*** $\chi'(G)$, of a loopless graph $G$, is the minimum $k$ for which $G$ is $k$-edge-colourable. $G$ is *k-edge-chromatic* if $\chi'(G) = k$. It can be readily verified that the graph of figure 6.1 has no proper 3-edge colouring. This graph is therefore 4-edge-chromatic.

Clearly, in any proper edge colouring, the edges incident with any one vertex must be assigned different colours. It follows that

$$\chi' \geq \Delta \tag{6.1}$$

Referring to the example of figure 6.1, we see that inequality (6.1) may be strict. However, we shall show that, in the case when $G$ is bipartite, $\chi' = \Delta$. The following simple lemma is basic to our proof. We say that colour $i$ is *represented* at vertex $v$ if some edge incident with $v$ has colour $i$.

*Lemma 6.1.1* Let $G$ be a connected graph that is not an odd cycle. Then

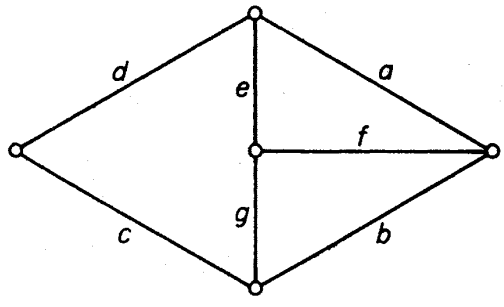



Figure 6.1

$G$ has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

*Proof* We may clearly assume that $G$ is nontrivial. Suppose, first, that $G$ is eulerian. If $G$ is an even cycle, the proper 2-edge colouring of $G$ has the required property. Otherwise, $G$ has a vertex $v_0$ of degree at least four. Let $v_0e_1v_1 \ldots e_\varepsilon v_0$ be an Euler tour of $G$, and set

$$E_1 = \{e_i \mid i \text{ odd}\} \quad \text{and} \quad E_2 = \{e_i \mid i \text{ even}\} \tag{6.2}$$

Then the 2-edge colouring $(E_1, E_2)$ of $G$ has the required property, since each vertex of $G$ is an internal vertex of $v_0e_1v_1 \ldots e_\varepsilon v_0$.

If $G$ is not eulerian, construct a new graph $G^*$ by adding a new vertex $v_0$ and joining it to each vertex of odd degree in $G$. Clearly $G^*$ is eulerian. Let $v_0e_1v_1 \ldots e_{\varepsilon^*} v_0$ be an Euler tour of $G^*$ and define $E_1$ and $E_2$ as in (6.2). It is then easily verified that the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of $G$ has the required property □

Given a $k$-edge colouring $\mathscr{C}$ of $G$ we shall denote by $c(v)$ the number of distinct colours represented at $v$. Clearly, we always have

$$c(v) \leq d(v) \tag{6.3}$$

Moreover, $\mathscr{C}$ is a proper $k$-edge colouring if and only if equality holds in (6.3) for all vertices $v$ of $G$. We shall call a $k$-edge colouring $\mathscr{C}'$ an *improvement* on $\mathscr{C}$ if

$$\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$$

where $c'(v)$ is the number of distinct colours represented at $v$ in the colouring $\mathscr{C}'$. An *optimal* $k$-edge colouring is one which cannot be improved.

*Lemma 6.1.2* Let $\mathscr{C} = (E_1, E_2, \ldots, E_k)$ be an optimal $k$-edge colouring of $G$. If there is a vertex $u$ in $G$ and colours $i$ and $j$ such that $i$ is not represented at $u$ and $j$ is represented at least twice at $u$, then the component of $G[E_i \cup E_j]$ that contains $u$ is an odd cycle.

*Proof* Let $u$ be a vertex that satisfies the hypothesis of the lemma, and denote by $H$ the component of $G[E_i \cup E_j]$ containing $u$. Suppose that $H$ is not an odd cycle. Then, by lemma 6.1.1, $H$ has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in $H$. When we recolour the edges of $H$ with colours $i$ and $j$ in this way, we obtain a new $k$-edge colouring $\mathscr{C}' = (E_1', E_2', \ldots, E_k')$ of $G$. Denoting by $c'(v)$ the number of distinct colours at $v$ in the colouring $\mathscr{C}'$, we have

$$c'(u) = c(u) + 1$$

since, now, both $i$ and $j$ are represented at $u$, and also

$$c'(v) \geq c(v) \quad \text{for} \quad v \neq u$$

Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathscr{C}$. It follows that $H$ is indeed an odd cycle □

*Theorem 6.1* If $G$ is bipartite, then $\chi' = \Delta$.

*Proof* Let $G$ be a graph with $\chi' > \Delta$, let $\mathscr{C} = (E_1, E_2, \ldots, E_\Delta)$ be an optimal $\Delta$-edge colouring of $G$, and let $u$ be a vertex such that $c(u) < d(u)$. Clearly, $u$ satisfies the hypothesis of lemma 6.1.2. Therefore $G$ contains an odd cycle and so is not bipartite. It follows from (6.1) that if $G$ is bipartite, then $\chi' = \Delta$ □

An alternative proof of theorem 6.1, using exercise 5.2.3*a*, is outlined in exercise 6.1.3.

*Exercises*

6.1.1 Show, by finding an appropriate edge colouring, that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.

6.1.2 Show that the Petersen graph is 4-edge-chromatic.

6.1.3 (*a*) Show that if $G$ is bipartite, then $G$ has a $\Delta$-regular bipartite supergraph.
(*b*) Using (*a*) and exercise 5.2.3*a*, give an alternative proof of theorem 6.1.

6.1.4 Describe a good algorithm for finding a proper $\Delta$-edge colouring of a bipartite graph $G$.

6.1.5 Using exercise 1.5.8 and theorem 6.1, show that if $G$ is loopless with $\Delta = 3$, then $\chi' \leq 4$.

6.1.6 Show that if $G$ is bipartite with $\delta > 0$, then $G$ has a $\delta$-edge colouring such that all $\delta$ colours are represented at each vertex.

(R. P. Gupta)

## 6.2 VIZING'S THEOREM

As has already been noted, if $G$ is not bipartite then we cannot necessarily conclude that $\chi' = \Delta$. An important theorem due to Vizing (1964) and, independently, Gupta (1966), asserts that, for any simple graph $G$, either $\chi' = \Delta$ or $\chi' = \Delta + 1$. The proof given here is by Fournier (1973).

*Theorem 6.2* If $G$ is simple, then either $\chi' = \Delta$ or $\chi' = \Delta + 1$.

*Proof* Let $G$ be a simple graph. By virtue of (6.1) we need only show that $\chi' \leq \Delta + 1$. Suppose, then, that $\chi' > \Delta + 1$. Let $\mathscr{C} = (E_1, E_2, \ldots, E_{\Delta+1})$ be

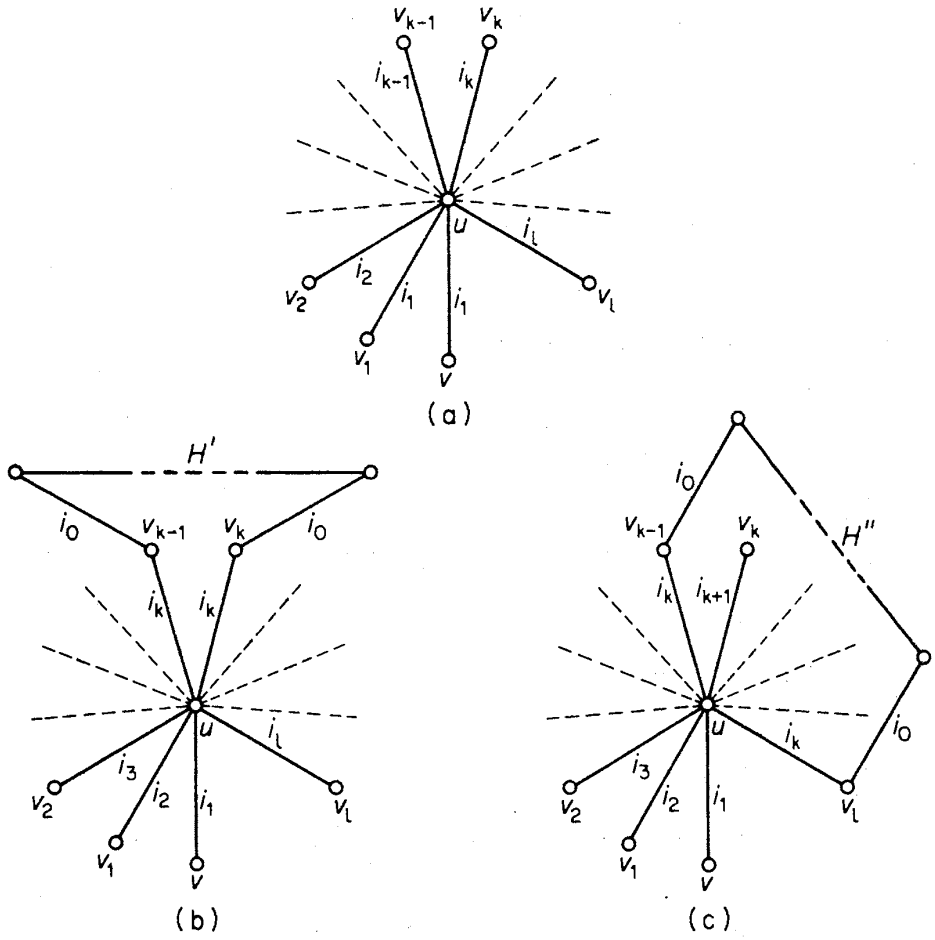



Figure 6.2

an optimal $(\Delta+1)$-edge colouring of $G$ and let $u$ be a vertex such that $c(u)<d(u)$. Then there exist colours $i_0$ and $i_1$ such that $i_0$ is not represented at $u$, and $i_1$ is represented at least twice at $u$. Let $uv_1$ have colour $i_1$, as in figure 6.2*a*.

Since $d(v_1)<\Delta+1$, some colour $i_2$ is not represented at $v_1$. Now $i_2$ must be represented at $u$ since otherwise, by recolouring $uv_1$ with $i_2$, we would obtain an improvement on $\mathscr{C}$. Thus some edge $uv_2$ has colour $i_2$. Again, since $d(v_2)<\Delta+1$, some colour $i_3$ is not represented at $v_2$; and $i_3$ must be represented at $u$ since otherwise, by recolouring $uv_1$ with $i_2$ and $uv_2$ with $i_3$, we would obtain an improved $(\Delta+1)$-edge colouring. Thus some edge $uv_3$ has colour $i_3$. Continuing this procedure we construct a sequence $v_1, v_2, \ldots$ of vertices and a sequence $i_1, i_2, \ldots$ of colours, such that

(i) $uv_j$ has colour $i_j$, and
(ii) $i_{j+1}$ is not represented at $v_j$.

Since the degree of $u$ is finite, there exists a smallest integer $l$ such that, for some $k<l$,

(iii) $i_{l+1}=i_k$.

The situation is depicted in figure 6.2*a*.

We now recolour $G$ as follows. For $1 \leq j \leq k-1$, recolour $uv_j$ with colour $i_{j+1}$, yielding a new $(\Delta+1)$-edge colouring $\mathscr{C}' = (E'_1, E'_2, \ldots, E'_{\Delta+1})$ (figure 6.2*b*). Clearly

$$c'(v) \geq c(v) \quad \text{for all} \quad v \in V$$

and therefore $\mathscr{C}'$ is also an optimal $(\Delta+1)$-edge colouring of $G$. By lemma 6.1.2, the component $H'$ of $G[E'_{i_0} \cup E'_{i_k}]$ that contains $u$ is an odd cycle.

Now, in addition, recolour $uv_j$ with colour $i_{j+1}$, $k \leq j \leq l-1$, and $uv_l$ with colour $i_k$, to obtain a $(\Delta+1)$-edge colouring $\mathscr{C}'' = (E''_1, E''_2, \ldots, E''_{\Delta+1})$ (figure 6.2*c*). As above

$$c''(v) \geq c(v) \quad \text{for all} \quad v \in V$$

and the component $H''$ of $G[E''_{i_0} \cup E''_{i_k}]$ that contains $u$ is an odd cycle. But, since $v_k$ has degree two in $H'$, $v_k$ clearly has degree one in $H''$. This contradiction establishes the theorem □

Actually, Vizing proved a more general theorem than that given above, one that is valid for all loopless graphs. The maximum number of edges joining two vertices in $G$ is called the *multiplicity* of $G$, and denoted by $\mu(G)$. We can now state Vizing's theorem in its full generality: if $G$ is loopless, then $\Delta \leq \chi' \leq \Delta + \mu$.

This theorem is best possible in the sense that, for any $\mu$, there exists a graph $G$ such that $\chi' = \Delta + \mu$. For example, in the graph $G$ of figure 6.3, $\Delta = 2\mu$ and, since any two edges are adjacent, $\chi' = \varepsilon = 3\mu$.

Strong as theorem 6.2 is, it leaves open one interesting question: which simple graphs satisfy $\chi' = \Delta$? The significance of this question will become apparent in chapter 9, when we study edge colourings of planar graphs.

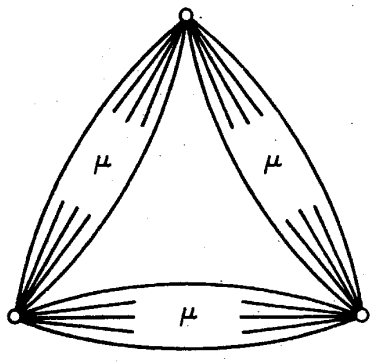



Figure 6.3. A graph $G$ with $\chi' = \Delta + \mu$

*Exercises*

6.2.1* Show, by finding appropriate edge colourings, that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n-1$.

6.2.2 Show that if $G$ is a nonempty regular simple graph with $\nu$ odd, then $\chi' = \Delta + 1$.

6.2.3 (*a*) Let $G$ be a simple graph. Show that if $\nu = 2n + 1$ and $\varepsilon > n\Delta$, then $\chi' = \Delta + 1$. (V. G. Vizing)

(*b*) Using (*a*), show that

(i) if $G$ is obtained from a simple regular graph with an even number of vertices by subdividing one edge, then $\chi' = \Delta + 1$;

(ii) if $G$ is obtained from a simple $k$-regular graph with an odd number of vertices by deleting fewer than $k/2$ edges, then $\chi' = \Delta + 1$. (L. W. Beineke and R. J. Wilson)

6.2.4 (*a*) Show that if $G$ is loopless, then $G$ has a $\Delta$-regular loopless supergraph.

(*b*) Using (*a*) and exercise 5.2.3*b*, show that if $G$ is loopless and $\Delta$ is even, then $\chi' \leq 3\Delta/2$.

(Shannon, 1949 has shown that this inequality also holds when $\Delta$ is odd.)

6.2.5 $G$ is called *uniquely k-edge-colourable* if any two proper $k$-edge colourings of $G$ induce the same partition of $E$. Show that every uniquely 3-edge-colourable 3-regular graph is hamiltonian. (D. L. Greenwell and H. V. Kronk)

6.2.6 The *product* of simple graphs $G$ and $H$ is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

(*a*) Using Vizing's theorem (6.2), show that $\chi'(G \times K_2) = \Delta(G \times K_2)$.

(*b*) Deduce that if $H$ is nontrivial with $\chi'(H) = \Delta(H)$, then $\chi'(G \times H) = \Delta(G \times H)$.

6.2.7 Describe a good algorithm for finding a proper $(\Delta + 1)$-edge colouring of a simple graph $G$.

6.2.8* Show that if $G$ is simple with $\delta > 1$, then $G$ has a $(\delta - 1)$-edge colouring such that all $\delta - 1$ colours are represented at each vertex. (R. P. Gupta)

## APPLICATIONS

### 6.3 THE TIMETABLING PROBLEM

In a school, there are $m$ teachers $X_1, X_2, \ldots, X_m$, and $n$ classes $Y_1, Y_2, \ldots, Y_n$. Given that teacher $X_i$ is required to teach class $Y_j$ for $p_{ij}$ periods, schedule a complete timetable in the minimum possible number of periods.

The above problem is known as the *timetabling problem*, and can be solved completely using the theory of edge colourings developed in this chapter. We represent the teaching requirements by a bipartite graph $G$ with bipartition $(X, Y)$, where $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and vertices $x_i$ and $y_j$ are joined by $p_{ij}$ edges. Now, in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher—this, at least, is our assumption. Thus a teaching schedule for one period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore, is to partition the edges of $G$ into as few matchings as possible or, equivalently, to properly colour the edges of $G$ with as few colours as possible. Since $G$ is bipartite, we know, by theorem 6.1, that $\chi' = \Delta$. Hence, if no teacher teaches for more than $p$ periods, and if no class is taught for more than $p$ periods, the teaching requirements can be scheduled in a $p$-period timetable. Furthermore, there is a good algorithm for constructing such a timetable, as is indicated in exercise 6.1.4. We thus have a complete solution to the timetabling problem.

However, the situation might not be so straightforward. Let us assume that only a limited number of classrooms are available. With this additional constraint, how many periods are now needed to schedule a complete timetable?

Suppose that altogether there are $l$ lessons to be given, and that they have been scheduled in a $p$-period timetable. Since this timetable requires an average of $l/p$ lessons to be given per period, it is clear that at least $\{l/p\}$ rooms will be needed in some one period. It turns out that one can always arrange $l$ lessons in a $p$-period timetable so that at most $\{l/p\}$ rooms are occupied in any one period. This follows from theorem 6.3 below. We first have a lemma.

*Lemma 6.3* Let $M$ and $N$ be disjoint matchings of $G$ with $|M| > |N|$. Then there are disjoint matchings $M'$ and $N'$ of $G$ such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and $M' \cup N' = M \cup N$.

*Proof* Consider the graph $H = G[M \cup N]$. As in the proof of theorem 5.1, each component of $H$ is either an even cycle, with edges alternately in $M$ and $N$, or else a path with edges alternately in $M$ and $N$. Since $|M| > |N|$, some path component $P$ of $H$ must start and end with edges of $M$. Let $P = v_0 e_1 v_1 \ldots e_{2n+1} v_{2n+1}$, and set

$$M' = (M \backslash \{e_1, e_3, \ldots, e_{2n+1}\}) \cup \{e_2, e_4, \ldots, e_{2n}\}$$
$$N' = (N \backslash \{e_2, e_4, \ldots, e_{2n}\}) \cup \{e_1, e_3, \ldots, e_{2n+1}\}$$

Then $M'$ and $N'$ are matchings of $G$ that satisfy the conditions of the lemma □

$$P = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{ccccc} Y_1 & Y_2 & Y_3 & Y_4 & Y_5 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(a)

| | Period 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $X_1$ | $Y_1$ | $Y_1$ | $Y_3$ | $Y_4$ |
| $X_2$ | $Y_2$ | – | $Y_4$ | – |
| $X_3$ | $Y_3$ | $Y_4$ | – | $Y_2$ |
| $X_4$ | $Y_4$ | $Y_5$ | – | – |

(b)

Figure 6.4

*Theorem 6.3* If $G$ is bipartite, and if $p \geq \Delta$, then there exist $p$ disjoint matchings $M_1, M_2, \ldots, M_p$ of $G$ such that

$$E = M_1 \cup M_2 \cup \ldots \cup M_p \tag{6.4}$$

and, for $1 \leq i \leq p$

$$[\varepsilon/p] \leq |M_i| \leq \{\varepsilon/p\} \tag{6.5}$$

(Note: condition (6.5) says that any two matchings $M_i$ and $M_j$ differ in size by at most one.)

*Proof* Let $G$ be a bipartite graph. By theorem 6.1, the edges of $G$ can be partitioned into $\Delta$ matchings $M'_1, M'_2, \ldots, M'_\Delta$. Therefore, for any $p \geq \Delta$, there exist $p$ disjoint matchings $M'_1, M'_2, \ldots, M'_p$ (with $M'_i = \emptyset$ for $i > \Delta$) such that

$$E = M'_1 \cup M'_2 \cup \ldots \cup M'_p$$

By repeatedly applying lemma 6.3 to pairs of these matchings that differ in size by more than one, we eventually obtain $p$ disjoint matchings $M_1, M_2, \ldots, M_p$ of $G$ satisfying (6.4) and (6.5), as required □

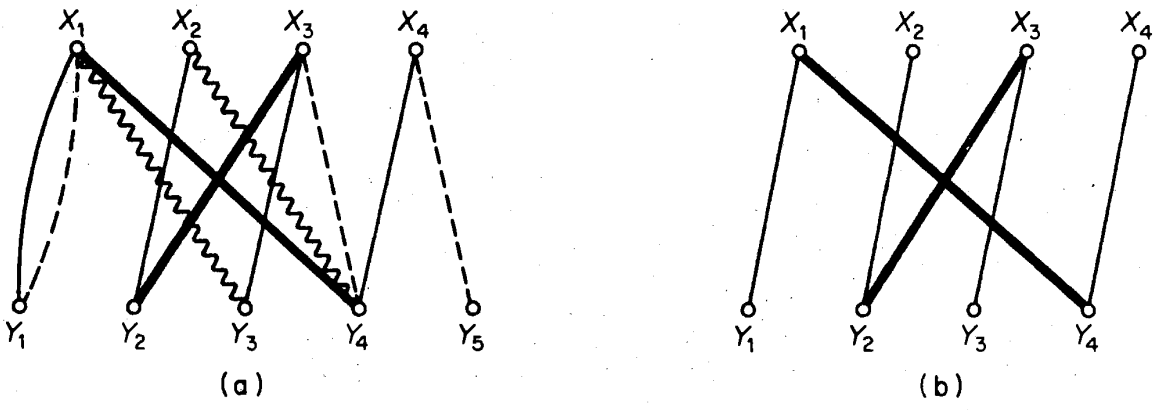
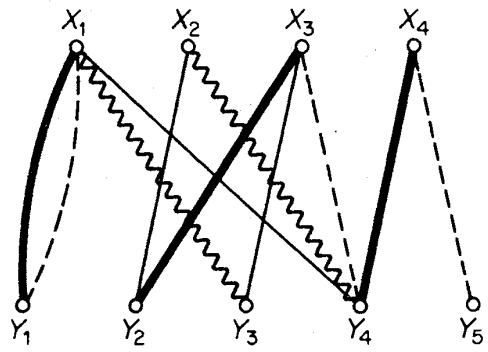



Figure 6.5




(a)

| | Period 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $X_1$ | $Y_4$ | $Y_1$ | $Y_3$ | $Y_1$ |
| $X_2$ | $Y_2$ | – | $Y_4$ | – |
| $X_3$ | $Y_3$ | $Y_4$ | – | $Y_2$ |
| $X_4$ | – | $Y_5$ | – | $Y_4$ |

(b)

Figure 6.6

As an example, suppose that there are four teachers and five classes, and that the teaching requirement matrix $\mathbf{P} = [p_{ij}]$ is as given in figure 6.4*a*. One possible 4-period timetable is shown in figure 6.4*b*.

We can represent the above timetable by a decomposition into matchings of the edge set of the bipartite graph $G$ corresponding to $\mathbf{P}$, as shown in figure 6.5*a*. (Normal edges correspond to period 1, broken edges to period 2, wavy edges to period 3, and heavy edges to period 4.)

From the timetable we see that four classes are taught in period 1, and so four rooms are needed. However $\varepsilon = 11$ and so, by theorem 6.4, a 4-period timetable can be arranged so that in each period either $2(=[11/4])$ or $3(=\{11/4\})$ classes are taught. Let $M_1$ denote the normal matching and $M_4$ the heavy matching; notice that $|M_1| = 4$ and $|M_4| = 2$. We can now find a 4-period 3-room timetable by considering $G[M_1 \cup M_4]$ (figure 6.5*b*). $G[M_1 \cup M_4]$ has two components, each consisting of a path of length three. Both paths start and end with normal edges and so, by interchanging the matchings on one of the two paths, we shall reduce the normal matching to one of three edges, and at the same time increase the heavy matching to one of three edges. If we choose the path $y_1x_1y_4x_4$, making the edges $y_1x_1$ and $y_4x_4$ heavy and the edge $x_1y_4$ normal, we obtain the decomposition of $E$ shown in figure 6.6*a*. This then gives the revised timetable shown in figure 6.6*b*; here, only three rooms are needed at any one time.

| | Period 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $X_1$ | $Y_4$ | $Y_3$ | $Y_1$ | – | $Y_1$ | – |
| $X_2$ | $Y_2$ | $Y_4$ | – | – | – | – |
| $X_3$ | – | – | $Y_4$ | $Y_3$ | $Y_2$ | – |
| $X_4$ | – | – | – | $Y_4$ | – | $Y_5$ |

Figure 6.7

However, suppose that there are just two rooms available. Theorem 6.4 tells us that there must be a 6-period timetable that satisfies our requirements (since $\{11/6\} = 2$). Such a timetable is given in figure 6.7.

In practice, most problems on timetabling are complicated by preassignments (that is, conditions specifying the periods during which certain teachers and classes must meet). This generalisation of the timetabling problem has been studied by Dempster (1971) and de Werra (1970).

*Exercise*

6.3.1 In a school there are seven teachers and twelve classes. The teaching requirements for a five-day week are given by the matrix

$$\mathbf{P} = \begin{array}{c|cccccccccccc} & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 & Y_{10} & Y_{11} & Y_{12} \\ \hline X_1 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ X_2 & 1 & 3 & 6 & 0 & 4 & 2 & 5 & 1 & 3 & 3 & 0 & 4 \\ X_3 & 5 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 5 \\ X_4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 \\ X_5 & 3 & 5 & 2 & 2 & 0 & 3 & 1 & 4 & 4 & 3 & 2 & 5 \\ X_6 & 5 & 5 & 0 & 0 & 5 & 5 & 0 & 5 & 0 & 5 & 5 & 0 \\ X_7 & 0 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 3 & 0 \end{array}$$

where $p_{ij}$ is the number of periods that teacher $X_i$ must teach class $Y_j$.

(*a*) Into how many periods must a day be divided so that the requirements can be satisfied?

(*b*) If an eight-period/day timetable is drawn up, how many classrooms will be needed?

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# 7 Independent Sets and Cliques

## 7.1 INDEPENDENT SETS

A subset $S$ of $V$ is called an *independent set* of $G$ if no two vertices of $S$ are adjacent in $G$. An independent set is *maximum* if $G$ has no independent set $S'$ with $|S'|>|S|$. Examples of independent sets are shown in figure 7.1.

Recall that a subset $K$ of $V$ such that every edge of $G$ has at least one end in $K$ is called a covering of $G$. The two examples of independent sets given in figure 7.1 are both complements of coverings. It is not difficult to see that this is always the case.

*Theorem 7.1* A set $S \subseteq V$ is an independent set of $G$ if and only if $V \backslash S$ is a covering of $G$.

*Proof* By definition, $S$ is an independent set of $G$ if and only if no edge of $G$ has both ends in $S$ or, equivalently, if and only if each edge has at least one end in $V \backslash S$. But this is so if and only if $V \backslash S$ is a covering of $G$ □

The number of vertices in a maximum independent set of $G$ is called the *independence number* of $G$ and is denoted by $\alpha(G)$; similarly, the number of vertices in a minimum covering of $G$ is the *covering number* of $G$ and is denoted by $\beta(G)$.

*Corollary 7.1* $\alpha+\beta=\nu$.

*Proof* Let $S$ be a maximum independent set of $G$, and let $K$ be a minimum covering of $G$. Then, by theorem 7.1, $V \backslash K$ is an independent set

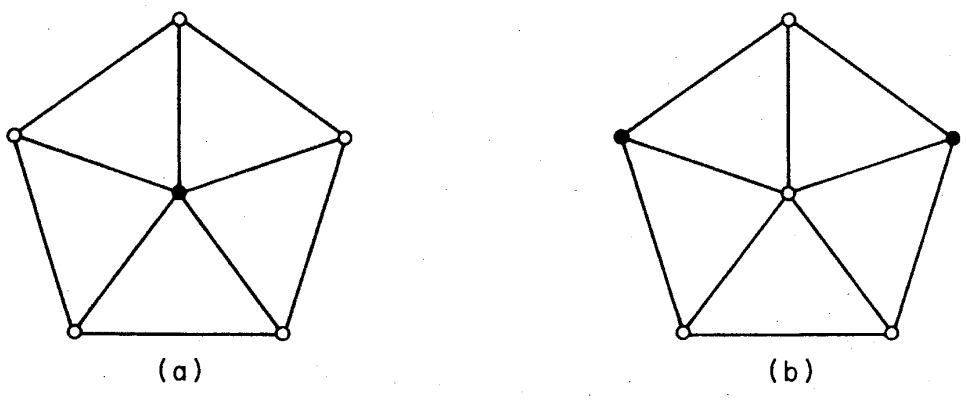


Figure 7.1. (*a*) An independent set; (*b*) a maximum independent set

and $V\backslash S$ is a covering. Therefore

$$\nu - \beta = |V\backslash K| \leq \alpha \tag{7.1}$$

and

$$\nu - \alpha = |V\backslash S| \geq \beta \tag{7.2}$$

Combining (7.1) and (7.2) we have $\alpha + \beta = \nu$ □

The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An *edge covering* of $G$ is a subset $L$ of $E$ such that each vertex of $G$ is an end of some edge in $L$. Note that edge coverings do not always exist; a graph $G$ has an edge covering if and only if $\delta > 0$. We denote the number of edges in a maximum matching of $G$ by $\alpha'(G)$, and the number of edges in a minimum edge covering of $G$ by $\beta'(G)$; the numbers $\alpha'(G)$ and $\beta'(G)$ are the *edge independence number* and *edge covering number* of $G$, respectively.

Matchings and edge coverings are not related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters $\alpha'$ and $\beta'$ are related in precisely the same manner as are $\alpha$ and $\beta$.

*Theorem 7.2* (Gallai, 1959) If $\delta > 0$, then $\alpha' + \beta' = \nu$.

*Proof* Let $M$ be a maximum matching in $G$ and let $U$ be the set of $M$-unsaturated vertices. Since $\delta > 0$ and $M$ is maximum, there exists a set $E'$ of $|U|$ edges, one incident with each vertex in $U$. Clearly, $M \cup E'$ is an edge covering of $G$, and so

$$\beta' \leq |M \cup E'| = \alpha' + (\nu - 2\alpha') = \nu - \alpha'$$

or

$$\alpha' + \beta' \leq \nu \tag{7.3}$$

Now let $L$ be a minimum edge covering of $G$, set $H = G[L]$ and let $M$ be a maximum matching in $H$. Denote the set of $M$-unsaturated vertices in $H$ by $U$. Since $M$ is maximum, $H[U]$ has no links and therefore

$$|L| - |M| = |L\backslash M| \geq |U| = \nu - 2|M|$$

Because $H$ is a subgraph of $G$, $M$ is a matching in $G$ and so

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \tag{7.4}$$

Combining (7.3) and (7.4), we have $\alpha' + \beta' = \nu$ □

We can now prove a theorem that bears a striking formal resemblance to König's theorem (5.3).

*Theorem 7.3* In a bipartite graph $G$ with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

*Proof* Let $G$ be a bipartite graph with $\delta > 0$. By corollary 7.1 and theorem 7.2, we have

$$\alpha + \beta = \alpha' + \beta'$$

and, since $G$ is bipartite, it follows from theorem 5.3 that $\alpha' = \beta$. Thus $\alpha = \beta'$ □

Even though the concept of an independent set is analogous to that of a matching, there exists no theory of independent sets comparable to the theory of matchings presented in chapter 5; for example, no good algorithm for finding a maximum independent set in a graph is known. However, there are two interesting theorems that relate the number of vertices in a maximum independent set of a graph to various other parameters of the graph. These theorems will be discussed in sections 7.2 and 7.3.

*Exercises*

7.1.1 (*a*) Show that $G$ is bipartite if and only if $\alpha(H) \geq \frac{1}{2}\nu(H)$ for every subgraph $H$ of $G$.

(*b*) Show that $G$ is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph $H$ of $G$ such that $\delta(H) > 0$.

7.1.2 A graph is *$\alpha$-critical* if $\alpha(G-e) > \alpha(G)$ for all $e \in E$. Show that a connected $\alpha$-critical graph has no cut vertices.

7.1.3 A graph $G$ is *$\beta$-critical* if $\beta(G-e) < \beta(G)$ for all $e \in E$. Show that

(*a*) a connected $\beta$-critical graph has no cut vertices;

(*b*)* if $G$ is connected, then $\beta \leq \frac{1}{2}(\varepsilon + 1)$.

## 7.2 RAMSEY'S THEOREM

In this section we deal only with simple graphs. A *clique* of a simple graph $G$ is a subset $S$ of $V$ such that $G[S]$ is complete. Clearly, $S$ is a clique of $G$ if and only if $S$ is an independent set of $G^c$, and so the two concepts are complementary.

If $G$ has no large cliques, then one might expect $G$ to have a large independent set. That this is indeed the case was first proved by Ramsey (1930). He showed that, given any positive integers $k$ and $l$, there exists a smallest integer $r(k, l)$ such that every graph on $r(k, l)$ vertices contains either a clique of $k$ vertices or an independent set of $l$ vertices. For example, it is easy to see that

$$r(1, l) = r(k, 1) = 1 \tag{7.5}$$

and

$$r(2, l) = l, \qquad r(k, 2) = k \tag{7.6}$$

The numbers $r(k, l)$ are known as the *Ramsey numbers*. The following theorem on Ramsey numbers is due to Erdös and Szekeres (1935) and Greenwood and Gleason (1955).

*Theorem 7.4* For any two integers $k \geq 2$ and $l \geq 2$

$$r(k, l) \leq r(k, l-1) + r(k-1, l) \tag{7.7}$$

Furthermore, if $r(k, l-1)$ and $r(k-1, l)$ are both even, then strict inequality holds in (7.7).

*Proof* Let $G$ be a graph on $r(k, l-1) + r(k-1, l)$ vertices, and let $v \in V$. We distinguish two cases:

(i) $v$ is nonadjacent to a set $S$ of at least $r(k, l-1)$ vertices, or
(ii) $v$ is adjacent to a set $T$ of at least $r(k-1, l)$ vertices.

Note that either case (i) or case (ii) must hold because the number of vertices to which $v$ is nonadjacent plus the number of vertices to which $v$ is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$.

In case (i), $G[S]$ contains either a clique of $k$ vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Since one of case (i) and case (ii) must hold, it follows that $G$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. This proves (7.7).

Now suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even, and let $G$ be a graph on $r(k, l-1) + r(k-1, l) - 1$ vertices. Since $G$ has an odd number of vertices, it follows from corollary 1.1 that some vertex $v$ is of even degree; in particular, $v$ cannot be adjacent to precisely $r(k-1, l) - 1$ vertices. Consequently, either case (i) or case (ii) above holds, and therefore $G$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Thus

$$r(k, l) \leq r(k, l-1) + r(k-1, l) - 1$$

as stated □

The determination of the Ramsey numbers in general is a very difficult unsolved problem. Lower bounds can be obtained by the construction of suitable graphs. Consider, for example, the four graphs in figure 7.2.

The 5-cycle (figure 7.2*a*) contains no clique of three vertices and no independent set of three vertices. It shows, therefore, that

$$r(3, 3) \geq 6 \tag{7.8}$$

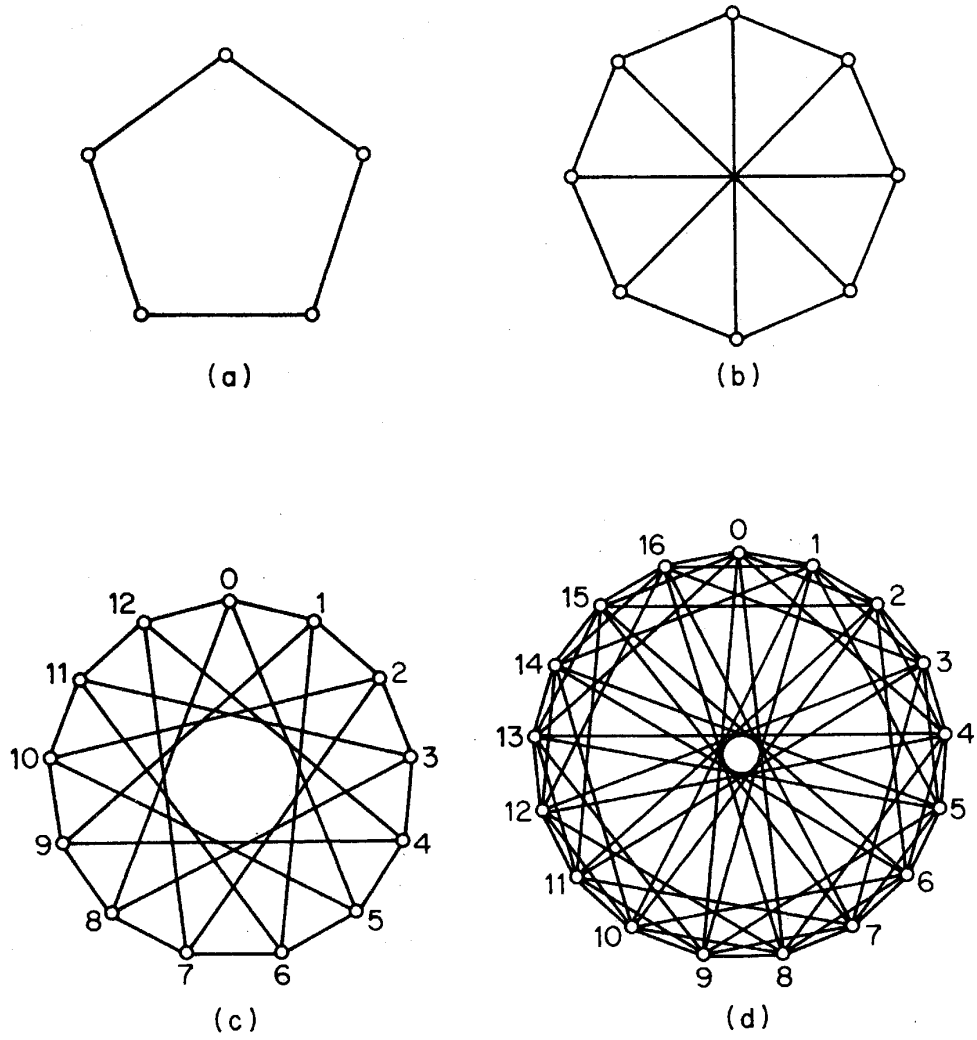


Figure 7.2. (*a*) A (3,3)-Ramsey graph; (*b*) a (3,4)-Ramsey graph; (*c*) a (3,5)-Ramsey graph; (*d*) a (4,4)-Ramsey graph

The graph of figure 7.2*b* contains no clique of three vertices and no independent set of four vertices. Hence

$$r(3, 4) \geq 9 \tag{7.9}$$

Similarly, the graph of figure 7.2*c* shows that

$$r(3, 5) \geq 14 \tag{7.10}$$

and the graph of figure 7.2*d* yields

$$r(4, 4) \geq 18 \tag{7.11}$$

With the aid of theorem 7.4 and equations (7.6) we can now show that equality in fact holds in (7.8), (7.9), (7.10) and (7.11). Firstly, by (7.7) and (7.6)

$$r(3, 3) \leq r(3, 2) + r(2, 3) = 6$$

and therefore, using (7.8), we have $r(3, 3) = 6$. Noting that $r(3, 3)$ and $r(2, 4)$ are both even, we apply theorem 7.4 and (7.6) to obtain

$$r(3, 4) \leq r(3, 3) + r(2, 4) - 1 = 9$$

With (7.9) this gives $r(3, 4) = 9$. Now we again apply (7.7) and (7.6) to obtain

$$r(3, 5) \leq r(3, 4) + r(2, 5) = 14$$

and

$$r(4, 4) \leq r(4, 3) + r(3, 4) = 18$$

which, together with (7.10) and (7.11), respectively, yield $r(3, 5) = 14$ and $r(4, 4) = 18$.

The following table shows all Ramsey numbers $r(k, l)$ known to date.

| $k \backslash l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 1 | 3 | 6 | 9 | 14 | 18 | 23 |
| 4 | 1 | 4 | 9 | 18 | | | |

A $(k, l)$-*Ramsey graph* is a graph on $r(k, l) - 1$ vertices that contains neither a clique of $k$ vertices nor an independent set of $l$ vertices. By definition of $r(k, l)$ such graphs exist for all $k \geq 2$ and $l \geq 2$. Ramsey graphs often seem to possess interesting structures. All of the graphs in figure 7.2 are Ramsey graphs; the last two can be obtained from finite fields in the following way. We get the (3, 5)-Ramsey graph by regarding the thirteen vertices as elements of the field of integers modulo 13, and joining two vertices by an edge if their difference is a cubic residue of 13 (either 1, 5, 8 or 12); the (4, 4)-Ramsey graph is obtained by regarding the vertices as elements of the field of integers modulo 17, and joining two vertices if their difference is a quadratic residue of 17 (either 1, 2, 4, 8, 9, 13, 15 or 16). It has been conjectured that the $(k, k)$-Ramsey graphs are always self-complementary (that is, isomorphic to their complements); this is true for $k = 2$, 3 and 4.

In general, theorem 7.4 yields the following upper bound for $r(k, l)$.

*Theorem 7.5* $r(k, l) \leq \binom{k+l-2}{k-1}$

*Proof* By induction on $k + l$. Using (7.5) and (7.6) we see that the theorem holds when $k + l \leq 5$. Let $m$ and $n$ be positive integers, and assume that the theorem is valid for all positive integers $k$ and $l$ such that

$5 \leq k+l < m+n$. Then, by theorem 7.4 and the induction hypothesis

$$\begin{aligned} r(m, n) &\leq r(m, n-1) + r(m-1, n) \\ &\leq \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} = \binom{m+n-2}{m-1} \end{aligned}$$

Thus the theorem holds for all values of $k$ and $l$ □

A lower bound for $r(k, k)$ is given in the next theorem. It is obtained by means of a powerful technique known as the *probabilistic method* (see Erdös and Spencer, 1974). The probabilistic method is essentially a crude counting argument. Although nonconstructive, it can often be applied to assert the existence of a graph with certain specified properties.

*Theorem 7.6* (Erdös, 1947) $r(k, k) \geq 2^{k/2}$

*Proof.* Since $r(1, 1) = 1$ and $r(2, 2) = 2$, we may assume that $k \geq 3$. Denote by $\mathcal{G}_n$ the set of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$, and by $\mathcal{G}_n^k$ the set of those graphs in $\mathcal{G}_n$ that have a clique of $k$ vertices. Clearly

$$|\mathcal{G}_n| = 2^{\binom{n}{2}} \tag{7.12}$$

since each subset of the $\binom{n}{2}$ possible edges $v_i v_j$ determines a graph in $\mathcal{G}_n$. Similarly, the number of graphs in $\mathcal{G}_n$ having a particular set of $k$ vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct $k$-element subsets of $\{v_1, v_2, \ldots, v_n\}$, we have

$$|\mathcal{G}_n^k| \leq \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} \tag{7.13}$$

By (7.12) and (7.13)

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \leq \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!} \tag{7.14}$$

Suppose, now, that $n < 2^{k/2}$. From (7.14) it follows that

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} < \frac{2^{k^2/2} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \tfrac{1}{2}$$

Therefore, fewer than half of the graphs in $\mathcal{G}_n$ contain a clique of $k$ vertices. Also, because $\mathcal{G}_n = \{G \mid G^c \in \mathcal{G}_n\}$, fewer than half of the graphs in $\mathcal{G}_n$ contain an independent set of $k$ vertices. Hence some graph in $\mathcal{G}_n$ contains neither a clique of $k$ vertices nor an independent set of $k$ vertices. Because this holds for any $n < 2^{k/2}$, we have $r(k, k) \geq 2^{k/2}$ □

From theorem 7.6 we can immediately deduce a lower bound for $r(k, l)$.

*Corollary 7.6* If $m = \min\{k, l\}$, then $r(k, l) \geq 2^{m/2}$

All known lower bounds for $r(k, l)$ obtained by constructive arguments are much weaker than that given in corollary 7.6; the best is due to Abbott (1972), who shows that $r(2^n+1, 2^n+1) \geq 5^n+1$ (exercise 7.2.4).

The Ramsey numbers $r(k, l)$ are sometimes defined in a slightly different way from that given at the beginning of this section. One easily sees that $r(k, l)$ can be thought of as the smallest integer $n$ such that every 2-edge colouring $(E_1, E_2)$ of $K_n$ contains either a complete subgraph on $k$ vertices, all of whose edges are in colour 1, or a complete subgraph on $l$ vertices, all of whose edges are in colour 2. Expressed in this form, the Ramsey numbers have a natural generalisation. We define $r(k_1, k_2, \ldots, k_m)$ to be the smallest integer $n$ such that every $m$-edge colouring $(E_1, E_2, \ldots, E_m)$ of $K_n$ contains, for some $i$, a complete subgraph on $k_i$ vertices, all of whose edges are in colour $i$.

The following theorem and corollary generalise (7.7) and theorem 7.5, and can be proved in a similar manner. They are left as an exercise (7.2.2).

*Theorem 7.7* $r(k_1, k_2, \ldots, k_m) \leq r(k_1-1, k_2, \ldots, k_m) + r(k_1, k_2-1, \ldots, k_m) + \ldots + r(k_1, k_2, \ldots, k_m-1) - m + 2$

*Corollary 7.7* $r(k_1+1, k_2+1, \ldots, k_m+1) \leq \dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!\,k_2!\ldots k_m!}$

*Exercises*

7.2.1 Show that, for all $k$ and $l$, $r(k, l) = r(l, k)$.

7.2.2 Prove theorem 7.7 and corollary 7.7.

**7.2.3** Let $r_n$ denote the Ramsey number $r(k_1, k_2, \ldots, k_n)$ with $k_i = 3$ for all $i$.

(*a*) Show that $r_n \leq n(r_{n-1}-1)+2$.
(*b*) Noting that $r_2 = 6$, use (*a*) to show that $r_n \leq [n!\,e]+1$.
(*c*) Deduce that $r_3 \leq 17$.
(Greenwood and Gleason, 1955 have shown that $r_3 = 17$.)

7.2.4 The *composition* of simple graphs $G$ and $H$ is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

(*a*) Show that $\alpha(G[H]) \leq \alpha(G)\alpha(H)$.
(*b*) Using (*a*), show that

$$r(kl+1, kl+1) - 1 \geq (r(k+1, k+1)-1) \times (r(l+1, l+1)-1)$$

(*c*) Deduce that $r(2^n+1, 2^n+1) \geq 5^n+1$ for all $n \geq 0$.

(H. L. Abbott)

7.2.5 Show that the join of a 3-cycle and a 5-cycle contains no $K_6$, but that every 2-edge colouring yields a monochromatic triangle.

(R. L. Graham)

(Folkman, 1970 has constructed a graph containing no $K_4$ in which every 2-edge colouring yields a monochromatic triangle—this graph has a very large number of vertices.)

7.2.6 Let $G_1, G_2, \ldots, G_m$ be simple graphs. The *generalised Ramsey number* $r(G_1, G_2, \ldots, G_m)$ is the smallest integer $n$ such that every $m$-edge colouring $(E_1, E_2, \ldots, E_m)$ of $K_n$ contains, for some $i$, a subgraph isomorphic to $G_i$ in colour $i$. Show that

(*a*) if $G$ is a path of length three and $H$ is a 4-cycle, then $r(G, G) = 5$, $r(G, H) = 5$ and $r(H, H) = 6$;

(*b*)* if $T$ is any tree on $m$ vertices and if $m-1$ divides $n-1$, then $r(T, K_{1,n}) = m+n-1$;

(*c*)* if $T$ is any tree on $m$ vertices, then $r(T, K_n) = (m-1)(n-1)+1$.

(V. Chvátal)

## 7.3 TURÁN'S THEOREM

In this section, we shall prove a well-known theorem due to Turán (1941). It determines the maximum number of edges that a simple graph on $\nu$ vertices can have without containing a clique of size $m+1$. Turán's theorem has become the basis of a significant branch of graph theory known as *extremal graph theory* (see Erdös, 1967). We shall derive it from the following result of Erdös (1970).

*Theorem 7.8* If a simple graph $G$ contains no $K_{m+1}$, then $G$ is degree-majorised by some complete $m$-partite graph $H$. Moreover, if $G$ has the same degree sequence as $H$, then $G \cong H$.

*Proof* By induction on $m$. The theorem is trivial for $m = 1$. Assume that it holds for all $m < n$, and let $G$ be a simple graph which contains no $K_{n+1}$. Choose a vertex $u$ of degree $\Delta$ in $G$, and set $G_1 = G[N(u)]$. Since $G$ contains no $K_{n+1}$, $G_1$ contains no $K_n$ and therefore, by the induction hypothesis, is degree-majorised by some complete $(n-1)$-partite graph $H_1$.

Next, set $V_1 = N(u)$ and $V_2 = V \backslash V_1$, and denote by $G_2$ the graph whose vertex set is $V_2$ and whose edge set is empty. Consider the join $G_1 \vee G_2$ of $G_1$ and $G_2$. Since

$$N_G(v) \subseteq N_{G_1 \vee G_2}(v) \quad \text{for} \quad v \in V_1 \tag{7.15}$$

and since each vertex of $V_2$ has degree $\Delta$ in $G_1 \vee G_2$, $G$ is degree-majorised by $G_1 \vee G_2$. Therefore $G$ is also degree-majorised by the complete $n$-partite graph $H = H_1 \vee G_2$. (See figure 7.3 for illustration.)

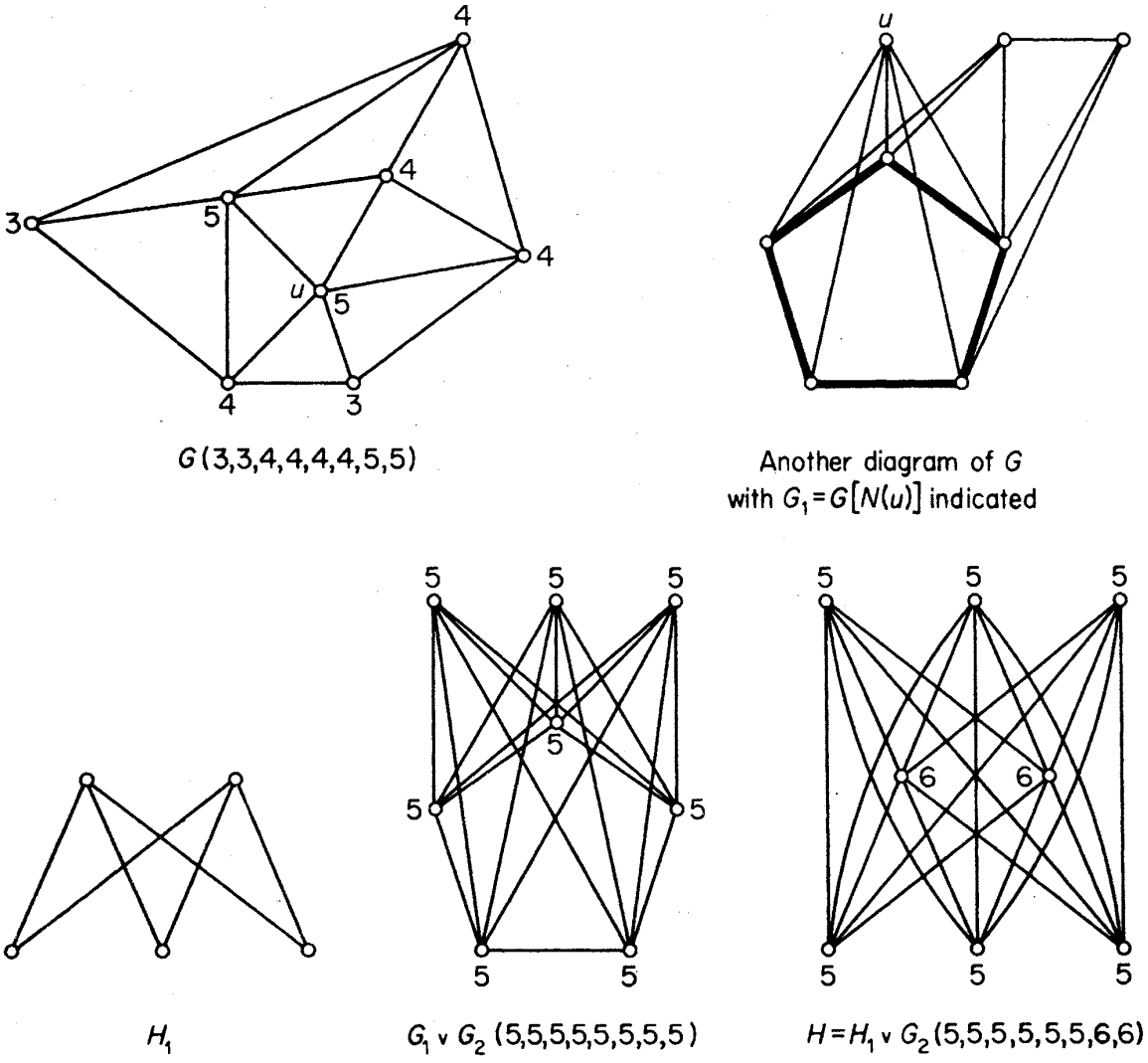


Figure 7.3

Suppose, now, that $G$ has the same degree sequence as $H$. Then $G$ has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in (7.15). Thus, in $G$, every vertex of $V_1$ must be joined to every vertex of $V_2$. It follows that $G = G_1 \vee G_2$. Since $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs $G_1$ and $H_1$ must have the same degree sequence and therefore, by the induction hypothesis, be isomorphic. We conclude that $G \cong H$ □

It is interesting to note that the above theorem bears a striking similarity to theorem 4.6.

Let $T_{m,n}$ denote the complete $m$-partite graph on $n$ vertices in which all parts are as equal in size as possible; the graph $H$ of figure 7.3 is $T_{3,8}$.

*Theorem 7.9* If $G$ is simple and contains no $K_{m+1}$, then $\varepsilon(G) \leq \varepsilon(T_{m,\nu})$. Moreover, $\varepsilon(G) = \varepsilon(T_{m,\nu})$ only if $G \cong T_{m,\nu}$.

*Proof* Let $G$ be a simple graph that contains no $K_{m+1}$. By theorem 7.8, $G$ is degree-majorised by some complete $m$-partite graph $H$. It follows from theorem 1.1 that

$$\varepsilon(G) \leq \varepsilon(H) \tag{7.16}$$

But (exercise 1.2.9)

$$\varepsilon(H) \leq \varepsilon(T_{m,\nu}) \tag{7.17}$$

Therefore, from (7.16) and (7.17)

$$\varepsilon(G) \leq \varepsilon(T_{m,\nu}) \tag{7.18}$$

proving the first assertion.

Suppose, now, that equality holds in (7.18). Then equality must hold in both (7.16) and (7.17). Since $\varepsilon(G) = \varepsilon(H)$ and $G$ is degree-majorised by $H$, $G$ must have the same degree sequence as $H$. Therefore, by theorem 7.8, $G \cong H$. Also, since $\varepsilon(H) = \varepsilon(T_{m,\nu})$, it follows (exercise 1.2.9) that $H \cong T_{m,\nu}$. We conclude that $G \cong T_{m,\nu}$ □

*Exercises*

7.3.1 In a group of nine people, one person knows two of the others, two people each know four others, four each know five others, and the remaining two each know six others. Show that there are three people who all know one another.

7.3.2 A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.

7.3.3 (*a*) Show that if $G$ is simple and $\varepsilon > \nu^2/4$, then $G$ contains a triangle.

(*b*) Find a simple graph $G$ with $\varepsilon = [\nu^2/4]$ that contains no triangle.

(*c*)* Show that if $G$ is simple and not bipartite with $\varepsilon > ((\nu-1)^2/4)+1$, then $G$ contains a triangle.

(*d*) Find a simple non-bipartite graph $G$ with $\varepsilon = [(\nu-1)^2/4]+1$ that contains no triangle. (P. Erdös)

7.3.4 (*a*)* Show that if $G$ is simple and $\sum_{v \in V} \binom{d(v)}{2} > (m-1)\binom{\nu}{2}$, then $G$ contains $K_{2,m} (m \geq 2)$.

(*b*) Deduce that if $G$ is simple and $\varepsilon > \frac{(m-1)^{\frac{1}{2}}\nu^{\frac{3}{2}}}{2} + \frac{\nu}{4}$, then $G$ contains $K_{2,m} (m \geq 2)$.

(c) Show that, given a set of $n$ points in the plane, the number of pairs of points at distance exactly 1 is at most $n^{\frac{3}{2}}/\sqrt{2}+n/4$.

7.3.5 Show that if $G$ is simple and $\varepsilon > \frac{(m-1)^{1/m}\nu^{2-1/m}}{2} + \frac{(m-1)\nu}{2}$ then $G$ contains $K_{m,m}$.

## APPLICATIONS

### 7.4 SCHUR'S THEOREM

Consider the partition $(\{1, 4, 10, 13\}, \{2, 3, 11, 12\}, \{5, 6, 7, 8, 9\})$ of the set of integers $\{1, 2, \ldots, 13\}$. We observe that in no subset of the partition are there integers $x$, $y$ and $z$ (not necessarily distinct) which satisfy the equation

$$x + y = z \tag{7.19}$$

Yet, no matter how we partition $\{1, 2, \ldots, 14\}$ into three subsets, there always exists a subset of the partition which contains a solution to (7.19). Schur (1916) proved that, in general, given any positive integer $n$, there exists an integer $f_n$ such that, in any partition of $\{1, 2, \ldots, f_n\}$ into $n$ subsets, there is a subset which contains a solution to (7.19). We shall show how Schur's theorem follows from the existence of the Ramsey numbers $r_n$ (defined in exercise 7.2.3).

*Theorem 7.10* Let $(S_1, S_2, \ldots, S_n)$ be any partition of the set of integers $\{1, 2, \ldots, r_n\}$. Then, for some $i$, $S_i$ contains three integers $x$, $y$ and $z$ satisfying the equation $x + y = z$.

*Proof* Consider the complete graph whose vertex set is $\{1, 2, \ldots, r_n\}$. Colour the edges of this graph in colours $1, 2, \ldots, n$ by the rule that the edge $uv$ is assigned colour $j$ if and only if $|u - v| \in S_j$. By Ramsey's theorem (7.7) there exists a monochromatic triangle; that is, there are three vertices $a$, $b$ and $c$ such that $ab$, $bc$ and $ca$ have the same colour, say $i$. Assume, without loss of generality that $a > b > c$ and write $x = a - b$, $y = b - c$ and $z = a - c$. Then $x, y, z \in S_i$ and $x + y = z$ □

Let $s_n$ denote the least integer such that, in any partition of $\{1, 2, \ldots, s_n\}$ into $n$ subsets, there is a subset which contains a solution to (7.19). It can be easily seen that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$ (exercise 7.4.1). Also, from theorem 7.10 and exercise 7.2.3 we have the upper bound

$$s_n \leq r_n \leq [n!\, e] + 1$$

Exercise 7.4.2*b* provides a lower bound for $s_n$.

*Exercises*

7.4.1 Show that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$.

7.4.2 (*a*) Show that $s_n \geq 3s_{n-1} - 1$.

(*b*) Using (*a*) and the fact that $s_3 = 14$, show that $s_n \geq \frac{1}{2}(27(3)^{n-3} + 1)$. (A better lower bound has been obtained by Abbott and Moser, 1966.)

## 7.5 A GEOMETRY PROBLEM

The *diameter* of a set $S$ of points in the plane is the maximum distance between two points of $S$. It should be noted that this is a purely geometric notion and is quite unrelated to the graph-theoretic concepts of diameter and distance.

We shall discuss sets of diameter 1. A set of $n$ points determines $\binom{n}{2}$ distances between pairs of these points. It is intuitively clear that if $n$ is 'large', then some of these distances must be 'small'. Therefore, for any $d$ between 0 and 1, we can ask how many pairs of points in a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 can be at distance greater than $d$. Here, we shall present a solution to one special case of this problem, namely when $d = 1/\sqrt{2}$.

As an illustration, consider the case $n = 6$. We then have six points $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$. If we place them at the vertices of a regular hexagon so that the pairs $(x_1, x_4)$, $(x_2, x_5)$ and $(x_3, x_6)$ are at distance 1, as shown in figure 7.4*a*, these six points constitute a set of diameter 1.

It is easily calculated that the pairs $(x_1, x_2)$, $(x_2, x_3)$, $(x_3, x_4)$, $(x_4, x_5)$, $(x_5, x_6)$ and $(x_6, x_1)$ are at distance 1/2, and the pairs $(x_1, x_3)$, $(x_2, x_4)$, $(x_3, x_5)$, $(x_4, x_6)$, $(x_5, x_1)$ and $(x_6, x_2)$ are at distance $\sqrt{3}/2$. Since $\sqrt{3}/2 > \sqrt{2}/2 = 1/\sqrt{2}$, there are nine pairs of points at distance greater than $1/\sqrt{2}$ in this set of diameter 1.

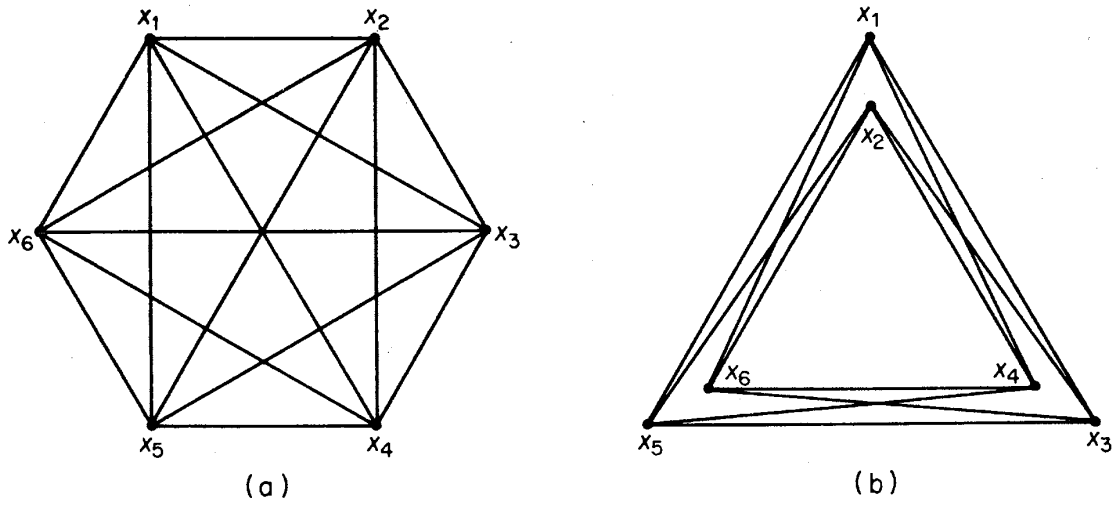


Figure 7.4

However, nine is not the best that we can do with six points. By placing the points in the configuration shown in figure 7.4*b*, all pairs of points except $(x_1, x_2)$, $(x_3, x_4)$ and $(x_5, x_6)$ are at distance greater than $1/\sqrt{2}$. Thus we have twelve pairs at distance greater than $1/\sqrt{2}$; this is, in fact, the best we can do. The solution to the problem in general is given by the following theorem.

*Theorem 7.11* If $\{x_1, x_2, \ldots, x_n\}$ is a set of diameter 1 in the plane, the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $[n^2/3]$. Moreover, for each $n$, there is a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 with exactly $[n^2/3]$ pairs of points at distance greater than $1/\sqrt{2}$.

*Proof* Let $G$ be the graph defined by

$$V(G) = \{x_1, x_2, \ldots, x_n\}$$

and

$$E(G) = \{x_i x_j \mid d(x_i, x_j) > 1/\sqrt{2}\}$$

where $d(x_i, x_j)$ here denotes the *euclidean* distance between $x_i$ and $x_j$. We shall show that $G$ cannot contain a $K_4$.

First, note that any four points in the plane must determine an angle of at least 90°. For the convex hull of the points is either (a) a line, (b) a triangle, or (c) a quadrilateral (see figure 7.5). Clearly, in each case there is an angle $x_i x_j x_k$ of at least 90°.

Now look at the three points $x_i$, $x_j$, $x_k$ which determine this angle. Not all the distances $d(x_i, x_j)$, $d(x_i, x_k)$ and $d(x_j, x_k)$ can be greater than $1/\sqrt{2}$ and less than or equal to 1. For, if $d(x_i, x_j) > 1/\sqrt{2}$ and $d(x_j, x_k) > 1/\sqrt{2}$, then $d(x_i, x_k) > 1$. Since the set $\{x_1, x_2, \ldots, x_n\}$ is assumed to have diameter 1, it follows that, of any four points in $G$, at least one pair cannot be joined by an edge, and hence that $G$ cannot contain a $K_4$. By Turán's theorem (7.9)

$$\varepsilon(G) \leq \varepsilon(T_{3,n}) = [n^2/3]$$

One can construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in which exactly

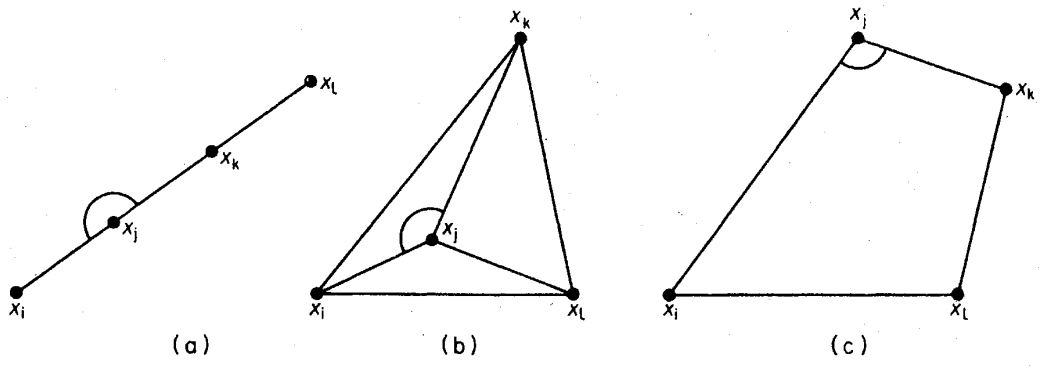


Figure 7.5

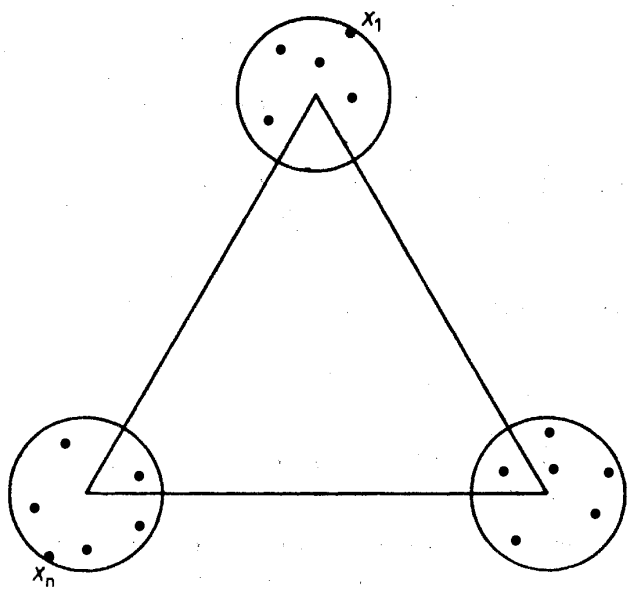


Figure 7.6

$[n^2/3]$ pairs of points are at distance greater than $1/\sqrt{2}$ as follows. Choose $r$ such that $0<r<(1-1/\sqrt{2})/4$, and draw three circles of radius $r$ whose centres are at a distance of $1-2r$ from one another (figure 7.6). Place $x_1, \ldots, x_{[n/3]}$ in one circle, $x_{[n/3]+1}, \ldots, x_{[2n/3]}$ in another, and $x_{[2n/3]+1}, \ldots, x_n$ in the third, in such a way that $d(x_1, x_n)=1$. This set clearly has diameter 1. Also, $d(x_i, x_j)>1/\sqrt{2}$ if and only if $x_i$ and $x_j$ are in different circles, and so there are exactly $[n^2/3]$ pairs $(x_i, x_j)$ for which $d(x_i, x_j)>1/\sqrt{2}$ □

*Exercises*

7.5.1* Let $\{x_1, x_2, \ldots, x_n\}$ be a set of diameter 1 in the plane.

(*a*) Show that the maximum possible number of pairs of points at distance 1 is $n$.

(*b*) Construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in the plane in which exactly $n$ pairs of points are at distance 1. (E. Pannwitz)

7.5.2 A flat circular city of radius six miles is patrolled by eighteen police cars, which communicate with one another by radio. If the range of a radio is nine miles, show that, at any time, there are always at least two cars each of which can communicate with at least five other cars.

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# 8 Vertex Colourings

## 8.1 CHROMATIC NUMBER

In chapter 6 we studied edge colourings of graphs. We now turn our attention to the analogous concept of vertex colouring.

A *k-vertex colouring* of $G$ is an assignment of $k$ colours, $1, 2, \ldots, k$, to the vertices of $G$; the colouring is *proper* if no two distinct adjacent vertices have the same colour. Thus a proper $k$-vertex colouring of a loopless graph $G$ is a partition $(V_1, V_2, \ldots, V_k)$ of $V$ into $k$ (possibly empty) independent sets. $G$ is *k-vertex-colourable* if $G$ has a proper $k$-vertex colouring. It will be convenient to refer to a 'proper vertex colouring' as, simply, a *colouring* and to a 'proper $k$-vertex colouring' as a *k-colouring;* we shall similarly abbreviate '$k$-vertex-colourable' to *k-colourable.* Clearly, a graph is $k$-colourable if and only if its underlying simple graph is $k$-colourable. Therefore, in discussing colourings, we shall restrict ourselves to simple graphs; a simple graph is 1-colourable if and only if it is empty, and 2-colourable if and only if it is bipartite. The *chromatic number,* $\chi(G)$, of $G$ is the minimum $k$ for which $G$ is $k$-colourable; if $\chi(G) = k$, $G$ is said to be *k-chromatic.* A 3-chromatic graph is shown in figure 8.1. It has the indicated 3-colouring, and is not 2-colourable since it is not bipartite.

It is helpful, when dealing with colourings, to study the properties of a special class of graphs called critical graphs. We say that a graph $G$ is *critical* if $\chi(H) < \chi(G)$ for every proper subgraph $H$ of $G$. Such graphs were first investigated by Dirac (1952). A *k-critical* graph is one that is $k$-chromatic and critical; every $k$-chromatic graph has a $k$-critical subgraph. A 4-critical graph, due to Grötzsch (1958), is shown in figure 8.2.

An easy consequence of the definition is that every critical graph is connected. The following theorems establish some of the basic properties of critical graphs.

*Theorem 8.1* If $G$ is $k$-critical, then $\delta \geq k-1$.

*Proof* By contradiction. If possible, let $G$ be a $k$-critical graph with $\delta < k-1$, and let $v$ be a vertex of degree $\delta$ in $G$. Since $G$ is $k$-critical, $G-v$ is $(k-1)$-colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$-colouring of $G-v$. By definition, $v$ is adjacent in $G$ to $\delta < k-1$ vertices, and therefore $v$ must be nonadjacent in $G$ to every vertex of some $V_j$. But then $(V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1})$ is a $(k-1)$-colouring of $G$, a contradiction. Thus $\delta \geq k-1$ □

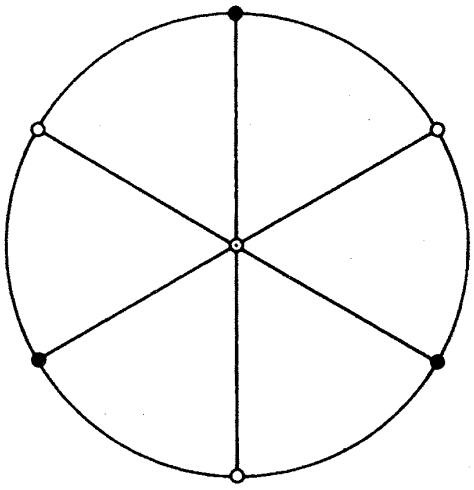

Figure 8.1. A 3-chromatic graph

*Corollary 8.1.1* Every $k$-chromatic graph has at least $k$ vertices of degree at least $k-1$.

*Proof* Let $G$ be a $k$-chromatic graph, and let $H$ be a $k$-critical subgraph of $G$. By theorem 8.1, each vertex of $H$ has degree at least $k-1$ in $H$, and hence also in $G$. The corollary now follows since $H$, being $k$-chromatic, clearly has at least $k$ vertices □

*Corollary 8.1.2* For any graph $G$,

$$\chi \leq \Delta + 1$$

*Proof* This is an immediate consequence of corollary 8.1.1 □

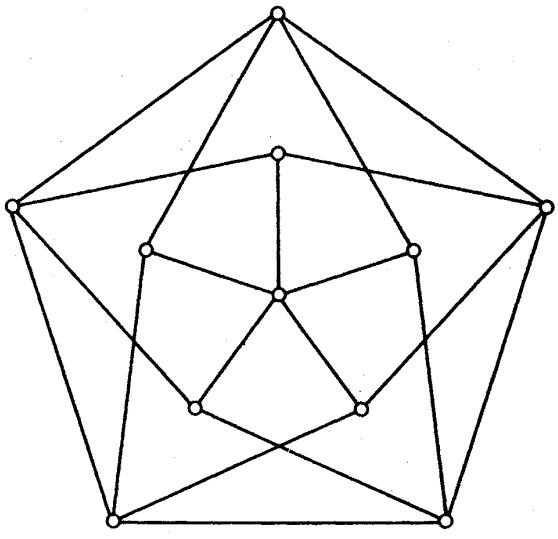

Figure 8.2. The Grötzsch graph—a 4-critical graph

Let $S$ be a vertex cut of a connected graph $G$, and let the components of $G-S$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the subgraphs $G_i = G[V \cup S]$ are called the *S-components* of $G$ (see figure 8.3). We say that colourings of $G_1, G_2, \ldots, G_n$ *agree* on $S$ if, for every $v \in S$, vertex $v$ is assigned the same colour in each of the colourings.

*Theorem 8.2* In a critical graph, no vertex cut is a clique.

*Proof* By contradiction. Let $G$ be a $k$-critical graph, and suppose that $G$ has a vertex cut $S$ that is a clique. Denote the $S$-components of $G$ b $G_1, G_2, \ldots, G_n$. Since $G$ is $k$-critical, each $G_i$ is $(k-1)$-colourable. Furth more, because $S$ is a clique, the vertices in $S$ must receive distinct colours in any $(k-1)$-colouring of $G_i$. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_n$ which agree on $S$. But these colourings together yield a $(k-1)$-colouring of $G$, a contradiction □

*Corollary 8.2* Every critical graph is a block.

*Proof* If $v$ is a cut vertex, then $\{v\}$ is a vertex cut which is also, trivially, a clique. It follows from theorem 8.2 that no critical graph has a cut vertex; equivalently, every critical graph is a block □

Another consequence of theorem 8.2 is that if a $k$-critical graph $G$ has a 2-vertex cut $\{u, v\}$, then $u$ and $v$ cannot be adjacent. We shall say that a $\{u, v\}$-component $G_i$ of $G$ is of *type 1* if every $(k-1)$-colouring of $G_i$ assigns the same colour to $u$ and $v$, and of *type 2* if every $(k-1)$-colouring of $G_i$ assigns different colours to $u$ and $v$ (see figure 8.4).

*Theorem 8.3* (Dirac, 1953) Let $G$ be a $k$-critical graph with a 2-vertex cut $\{u, v\}$. Then

(i) $G = G_1 \cup G_2$, where $G_i$ is a $\{u, v\}$-component of type $i$ $(i = 1, 2)$, and

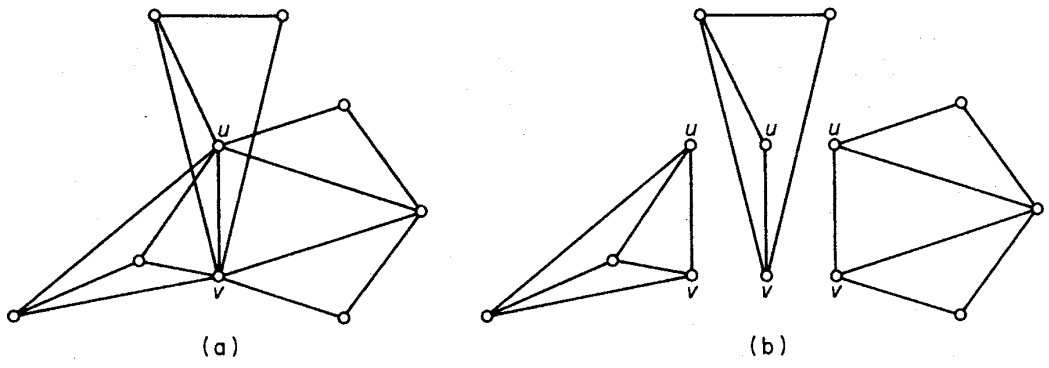


Figure 8.3. (*a*) $G$; (*b*) the $\{u, v\}$-components of $G$

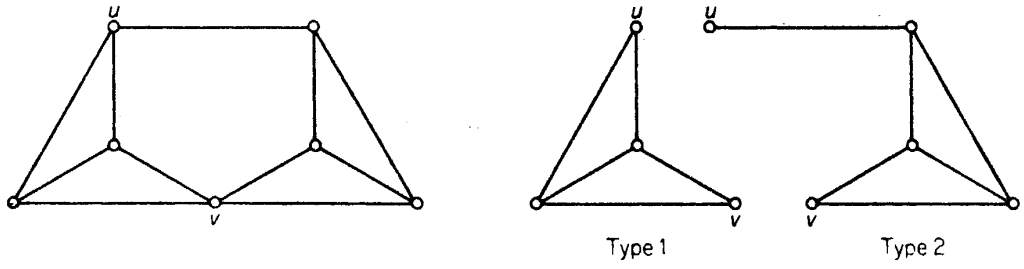


Figure 8.4

(ii) both $G_1 + uv$ and $G_2 \cdot uv$ are $k$-critical (where $G_2 \cdot uv$ denotes the graph obtained from $G_2$ by identifying $u$ and $v$).,

*Proof* (i) Since $G$ is critical, each $\{u, v\}$-component of $G$ is $(k-1)$-colourable. Now there cannot exist $(k-1)$-colourings of these $\{u, v\}$-components all of which agree on $\{u, v\}$, since such colourings would together yield a $(k-1)$-colouring of $G$. Therefore there are two $\{u, v\}$-components $G_1$ and $G_2$ such that no $(k-1)$-colouring of $G_1$ agrees with any $(k-1)$-colouring of $G_2$. Clearly one, say $G_1$, must be of type 1 and the other, $G_2$, of type 2. Since $G_1$ and $G_2$ are of different types, the subgraph $G_1 \cup G_2$ of $G$ is not $(k-1)$-colourable. Therefore, because $G$ is critical, we must have $G = G_1 \cup G_2$.

(ii) Set $H_1 = G_1 + uv$. Since $G_1$ is of type 1, $H_1$ is $k$-chromatic. We shall prove that $H_1$ is critical by showing that, for every edge $e$ of $H_1$, $H_1 - e$ is $(k-1)$-colourable. This is clearly so if $e = uv$, since then $H_1 - e = G_1$. Let $e$ be some other edge of $H_1$. In any $(k-1)$-colouring of $G - e$, the vertices $u$ and $v$ must receive different colours, since $G_2$ is a subgraph of $G - e$. The restriction of such a colouring to the vertices of $G_1$ is a $(k-1)$-colouring of $H_1 - e$. Thus $G_1 + uv$ is $k$-critical. An analogous argument shows that $G_2 \cdot uv$ is $k$-critical □

*Corollary 8.3* Let $G$ be a $k$-critical graph with a 2-vertex cut $\{u, v\}$. Then

$$d(u) + d(v) \geq 3k - 5 \tag{8.1}$$

*Proof* Let $G_1$ be the $\{u, v\}$-component of type 1 and $G_2$ the $\{u, v\}$-component of type 2. Set $H_1 = G_1 + uv$ and $H_2 = G_2 \cdot uv$. By theorems 8.3 and 8.1

$$d_{H_1}(u) + d_{H_1}(v) \geq 2k - 2$$

and

$$d_{H_2}(w) \geq k - 1$$

where $w$ is the new vertex obtained by identifying $u$ and $v$.
It follows that

$$d_{G_1}(u) + d_{G_1}(v) \geq 2k - 4$$

and

$$d_{G_2}(u)+d_{G_2}(v)\geq k-1$$

These two inequalities yield (8.1) □

*Exercises*

8.1.1 Show that if $G$ is simple, then $\chi\geq\nu^2/(\nu^2-2\varepsilon)$.

8.1.2 Show that if any two odd cycles of $G$ have a vertex in common, then $\chi\leq 5$.

8.1.3 Show that if $G$ has degree sequence $(d_1, d_2, \ldots, d_\nu)$ with $d_1\geq d_2\geq \ldots\geq d_\nu$, then $\chi\leq\max_i \min\{d_i+1, i\}$.

(D. J. A. Welsh and M. B. Powell)

8.1.4 Using exercise 8.1.3, show that

(*a*) $\chi\leq\{(2\varepsilon)^{\frac{1}{2}}\}$;

(*b*) $\chi(G)+\chi(G^c)\leq\nu+1$. (E. A. Nordhaus and J. W. Gaddum)

8.1.5 Show that $\chi(G)\leq 1+\max\delta(H)$, where the maximum is taken over all induced subgraphs $H$ of $G$. (G. Szekeres and H. S. Wilf)

8.1.6* If a $k$-chromatic graph $G$ has a colouring in which each colour is assigned to at least two vertices, show that $G$ has a $k$-colouring of this type. (T. Gallai)

**8.1.7** Show that the only 1-critical graph is $K_1$, the only 2-critical graph is $K_2$, and the only 3-critical graphs are the odd $k$-cycles with $k\geq 3$.

8.1.8 A graph $G$ is *uniquely $k$-colourable* if any two $k$-colourings of $G$ induce the same partition of $V$. Show that no vertex cut of a $k$-critical graph induces a uniquely $(k-1)$-colourable subgraph.

8.1.9 (*a*) Show that if $u$ and $v$ are two vertices of a critical graph $G$, then $N(u)\not\subseteq N(v)$.

(*b*) Deduce that no $k$-critical graph has exactly $k+1$ vertices.

8.1.10 Show that

(*a*) $\chi(G_1\vee G_2)=\chi(G_1)+\chi(G_2)$;

(*b*) $G_1\vee G_2$ is critical if and only if both $G_1$ and $G_2$ are critical.

8.1.11 Let $G_1$ and $G_2$ be two $k$-critical graphs with exactly one vertex $v$ in common, and let $vv_1$ and $vv_2$ be edges of $G_1$ and $G_2$. Show that the graph $(G_1-vv_1)\cup(G_2-vv_2)+v_1v_2$ is $k$-critical. (G. Hajós)

8.1.12 For $n=4$ and all $n\geq 6$, construct a 4-critical graph on $n$ vertices.

8.1.13 (*a*)* Let $(X, Y)$ be a partition of $V$ such that $G[X]$ and $G[Y]$ are both $n$-colourable. Show that, if the edge cut $[X, Y]$ has at most $n-1$ edges, then $G$ is also $n$-colourable.

(P. C. Kainen)

(*b*) Deduce that every $k$-critical graph is $(k-1)$-edge-connected.

(G. A. Dirac)

## 8.2 BROOKS' THEOREM

The upper bound on chromatic number given in corollary 8.1.2 is sometimes very much greater than the actual value. For example, bipartite graphs are 2-chromatic, but can have arbitrarily large maximum degree. In this sense corollary 8.1.2 is a considerably weaker result than Vizing's theorem (6.2). There is another sense in which Vizing's result is stronger. Many graphs $G$ satisfy $\chi' = \Delta + 1$ (see exercises 6.2.2 and 6.2.3). However, as is shown in the following theorem due to Brooks (1941), there are only two types of graph $G$ for which $\chi = \Delta + 1$. The proof of Brooks' theorem given here is by Lovász (1973).

*Theorem 8.4* If $G$ is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

*Proof* Let $G$ be a $k$-chromatic graph which satisfies the hypothesis of the theorem. Without loss of generality, we may assume that $G$ is $k$-critical. By corollary 8.2, $G$ is a block. Also, since 1-critical and 2-critical graphs are complete and 3-critical graphs are odd cycles (exercise 8.1.7), we have $k \geq 4$.

If $G$ has a 2-vertex cut $\{u, v\}$, corollary 8.3 gives

$$2\Delta \geq d(u) + d(v) \geq 3k - 5 \geq 2k - 1$$

This implies that $\chi = k \leq \Delta$, since $2\Delta$ is even.

Assume, then, that $G$ is 3-connected. Since $G$ is not complete, there are three vertices $u$, $v$ and $w$ in $G$ such that $uv$, $vw \in E$ and $uw \notin E$ (exercise 1.6.14). Set $u = v_1$ and $w = v_2$ and let $v_3, v_4, \ldots, v_\nu = v$ be any ordering of the vertices of $G - \{u, w\}$ such that each $v_i$ is adjacent to some $v_j$ with $j > i$. (This can be achieved by arranging the vertices of $G - \{u, w\}$ in nonincreasing order of their distance from $v$.) We can now describe a $\Delta$-colouring of $G$: assign colour 1 to $v_1 = u$ and $v_2 = w$; then successively colour $v_3, v_4, \ldots, v_\nu$, each with the first available colour in the list $1, 2, \ldots, \Delta$. By the construction of the sequence $v_1, v_2, \ldots, v_\nu$, each vertex $v_i$, $1 \leq i \leq \nu - 1$, is adjacent to some vertex $v_j$ with $j > i$, and therefore to at most $\Delta - 1$ vertices $v_j$ with $j < i$. It follows that, when its turn comes to be coloured, $v_i$ is adjacent to at most $\Delta - 1$ colours, and thus that one of the colours $1, 2, \ldots, \Delta$ will be available. Finally, since $v_\nu$ is adjacent to two vertices of colour 1 (namely $v_1$ and $v_2$), it is adjacent to at most $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta$ □

*Exercises*

8.2.1 Show that Brooks' theorem is equivalent to the following statement: if $G$ is $k$-critical ($k \geq 4$) and not complete, then $2\varepsilon \geq \nu(k-1) + 1$.

8.2.2 Use Brooks' theorem to show that if $G$ is loopless with $\Delta = 3$, then $\chi' \leq 4$.

## 8.3 HAJÓS' CONJECTURE

A *subdivision* of a graph $G$ is a graph that can be obtained from $G$ by a sequence of edge subdivisions. A subdivision of $K_4$ is shown in figure 8.5. Although no necessary and sufficient condition for a graph to be $k$-chromatic is known when $k \geq 3$, a plausible necessary condition has been proposed by Hajós (1961): if $G$ is $k$-chromatic, then $G$ contains a subdivision of $K_k$. This is known as *Hàjós' conjecture*. It should be noted that the condition is not sufficient; for example, a 4-cycle is a subdivision of $K_3$, but is not 3-chromatic.

For $k = 1$ and $k = 2$, the validity of Hajós' conjecture is obvious. It is also easily verified for $k = 3$, because a 3-chromatic graph necessarily contains an odd cycle, and every odd cycle is a subdivision of $K_3$. Dirac (1952) settled the case $k = 4$.

*Theorem 8.5* If $G$ is 4-chromatic, then $G$ contains a subdivision of $K_4$.

*Proof* Let $G$ be a 4-chromatic graph. Note that if some subgraph of $G$ contains a subdivision of $K_4$, then so, too, does $G$. Without loss of generality, therefore, we may assume that $G$ is critical, and hence that $G$ is a block with $\delta \geq 3$. If $\nu = 4$, then $G$ is $K_4$ and the theorem holds trivially. We proceed by induction on $\nu$. Assume the theorem true for all 4-chromatic graphs with fewer than $n$ vertices, and let $\nu(G) = n > 4$.

Suppose, first, that $G$ has a 2-vertex cut $\{u, v\}$. By theorem 8.3, $G$ has two $\{u, v\}$-components $G_1$ and $G_2$, where $G_1 + uv$ is 4-critical. Since $\nu(G_1 + uv) < \nu(G)$, we can apply the induction hypothesis and deduce that $G_1 + uv$

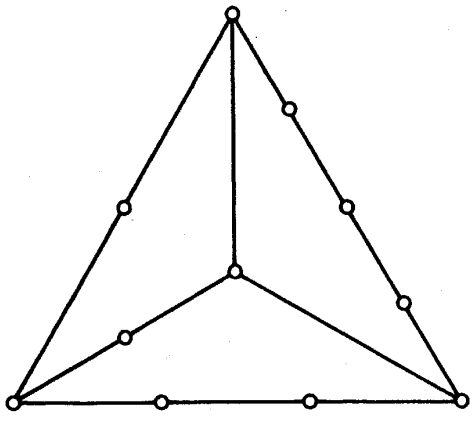

Figure 8.5. A subdivision of $K_4$

contains a subdivision of $K_4$. It follows that, if $P$ is a $(u, v)$-path in $G_2$, then $G_1 \cup P$ contains a subdivision of $K_4$. Hence so, too, does $G$, since $G_1 \cup P \subseteq G$.

Now suppose that $G$ is 3-connected. Since $\delta \geq 3$, $G$ has a cycle $C$ of length at least four. Let $u$ and $v$ be nonconsecutive vertices on $C$. Since $G-\{u, v\}$ is connected, there is a path $P$ in $G-\{u, v\}$ connecting the two components of $C-\{u, v\}$; we may assume that the origin $x$ and the terminus $y$ are the only vertices of $P$ on $C$. Similarly, there is a path $Q$ in $G-\{x, y\}$ (see figure 8.6).

If $P$ and $Q$ have no vertex in common, then $C \cup P \cup Q$ is a subdivision of $K_4$ (figure 8.6*a*). Otherwise, let $w$ be the first vertex of $P$ on $Q$, and let $P'$ denote the $(x, w)$-section of $P$. Then $C \cup P' \cup Q$ is a subdivision of $K_4$ (figure 8.6*b*). Hence, in both cases, $G$ contains a subdivision of $K_4$ □

Hajós' conjecture has not yet been settled in general, and its resolution is known to be a very difficult problem. There is a related conjecture due to Hadwiger (1943): if $G$ is $k$-chromatic, then $G$ is 'contractible' to a graph which contains $K_k$. Wagner (1964) has shown that the case $k=5$ of Hadwiger's conjecture is equivalent to the famous four-colour conjecture, to be discussed in chapter 9.

*Exercises*

8.3.1* Show that if $G$ is simple and has at most one vertex of degree less than three, then $G$ contains a subdivision of $K_4$.

8.3.2 (*a*)* Show that if $G$ is simple with $\nu \geq 4$ and $\varepsilon \geq 2\nu - 2$, then $G$ contains a subdivision of $K_4$.

(*b*) For $\nu \geq 4$, find a simple graph $G$ with $\varepsilon = 2\nu - 3$ that contains no subdivision of $K_4$.

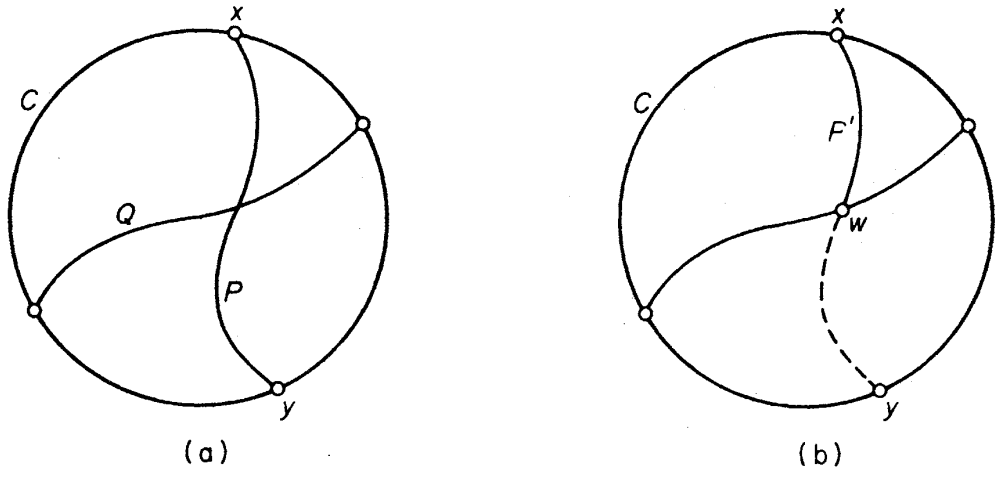


Figure 8.6

## 8.4 CHROMATIC POLYNOMIALS

In the study of colourings, some insight can be gained by considering not only the existence of colourings but the number of such colourings; this approach was developed by Birkhoff (1912) as a possible means of attacking the four-colour conjecture.

We shall denote the number of distinct $k$-colourings of $G$ by $\pi_k(G)$; thus $\pi_k(G)>0$ if and only if $G$ is $k$-colourable. Two colourings are to be regarded as distinct if some vertex is assigned different colours in the two colourings; in other words, if $(V_1, V_2, \ldots, V_k)$ and $(V_1', V_2', \ldots, V_k')$ are two colourings, then $(V_1, V_2, \ldots, V_k)=(V_1', V_2', \ldots, V_k')$ if and only if $V_i=V_i'$ for $1 \leq i \leq k$. For example, a triangle has the six distinct 3-colourings shown in figure 8.7. Note that even though there is exactly one vertex of each colour in each colouring, we still regard these six colourings as distinct.

If $G$ is empty, then each vertex can be independently assigned any one of the $k$ available colours. Therefore $\pi_k(G)=k^\nu$. On the other hand, if $G$ is complete, then there are $k$ choices of colour for the first vertex, $k-1$ choices for the second, $k-2$ for the third, and so on. Thus, in this case, $\pi_k(G)=k(k-1)\ldots(k-\nu+1)$. In general, there is a simple recursion formula for $\pi_k(G)$. It bears a close resemblance to the recursion formula for $\tau(G)$ (the number of spanning trees of $G$), given in theorem 2.8.

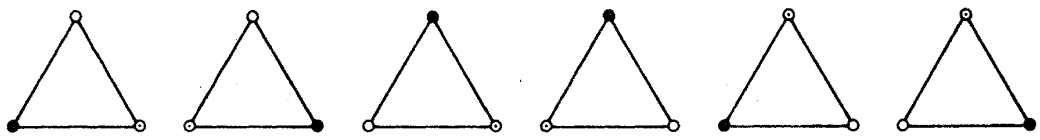

Figure 8.7

*Theorem 8.6* If $G$ is simple, then $\pi_k(G)=\pi_k(G-e)-\pi_k(G\cdot e)$ for any edge $e$ of $G$.

*Proof* Let $u$ and $v$ be the ends of $e$. To each $k$-colouring of $G-e$ that assigns the same colour to $u$ and $v$, there corresponds a $k$-colouring of $G\cdot e$ in which the vertex of $G\cdot e$ formed by identifying $u$ and $v$ is assigned the common colour of $u$ and $v$. This correspondence is clearly a bijection (see figure 8.8). Therefore $\pi_k(G\cdot e)$ is precisely the number of $k$-colourings of $G-e$ in which $u$ and $v$ are assigned the same colour.

Also, since each $k$-colouring of $G-e$ that assigns different colours to $u$ and $v$ is a $k$-colouring of $G$, and conversely, $\pi_k(G)$ is the number of $k$-colourings of $G-e$ in which $u$ and $v$ are assigned different colours. It follows that $\pi_k(G-e)=\pi_k(G)+\pi_k(G\cdot e)$ □

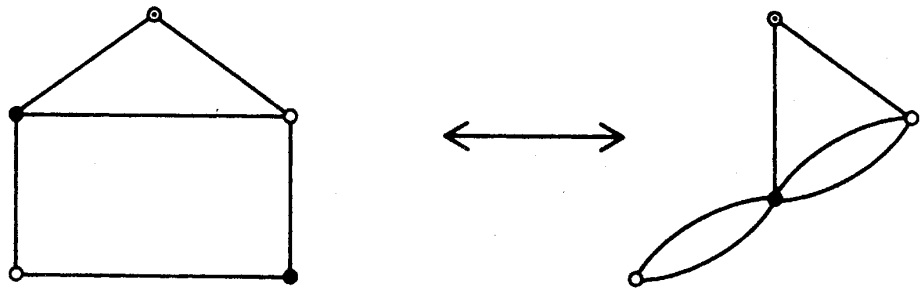

Figure 8.8

*Corollary 8.6* For any graph $G$, $\pi_k(G)$ is a polynomial in $k$ of degree $\nu$, with integer coefficients, leading term $k^\nu$ and constant term zero. Furthermore, the coefficients of $\pi_k(G)$ alternate in sign.

*Proof* By induction on $\varepsilon$. We may assume, without loss of generality, that $G$ is simple. If $\varepsilon = 0$ then, as has already been noted, $\pi_k(G) = k^\nu$, which trivially satisfies the conditions of the corollary. Suppose, now, that the corollary holds for all graphs with fewer than $m$ edges, and let $G$ be a graph with $m$ edges, where $m \geq 1$. Let $e$ be any edge of $G$. Then both $G - e$ and $G \cdot e$ have $m - 1$ edges, and it follows from the induction hypothesis that there are non-negative integers $a_1, a_2, \ldots, a_{\nu-1}$ and $b_1, b_2, \ldots, b_{\nu-2}$ such that

$$\pi_k(G - e) = \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i + k^\nu$$

and

$$\pi_k(G \cdot e) = \sum_{i=1}^{\nu-2} (-1)^{\nu-i-1} b_i k^i + k^{\nu-1}$$

By theorem 8.6

$$\begin{aligned}\pi_k(G) &= \pi_k(G - e) - \pi_k(G \cdot e) \\ &= \sum_{i=1}^{\nu-2} (-1)^{\nu-i}(a_i + b_i)k^i - (a_{\nu-1} + 1)k^{\nu-1} + k^\nu\end{aligned}$$

Thus $G$, too, satisfies the conditions of the corollary. The result follows by the principle of induction □

By virtue of corollary 8.6, we can now refer to the function $\pi_k(G)$ as the *chromatic polynomial* of $G$. Theorem 8.6 provides a means of calculating the chromatic polynomial of a graph recursively. It can be used in either of two ways:

(i) by repeatedly applying the recursion $\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e)$, and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of empty graphs, or
(ii) by repeatedly applying the recursion $\pi_k(G - e) = \pi_k(G) + \pi_k(G \cdot e)$, and

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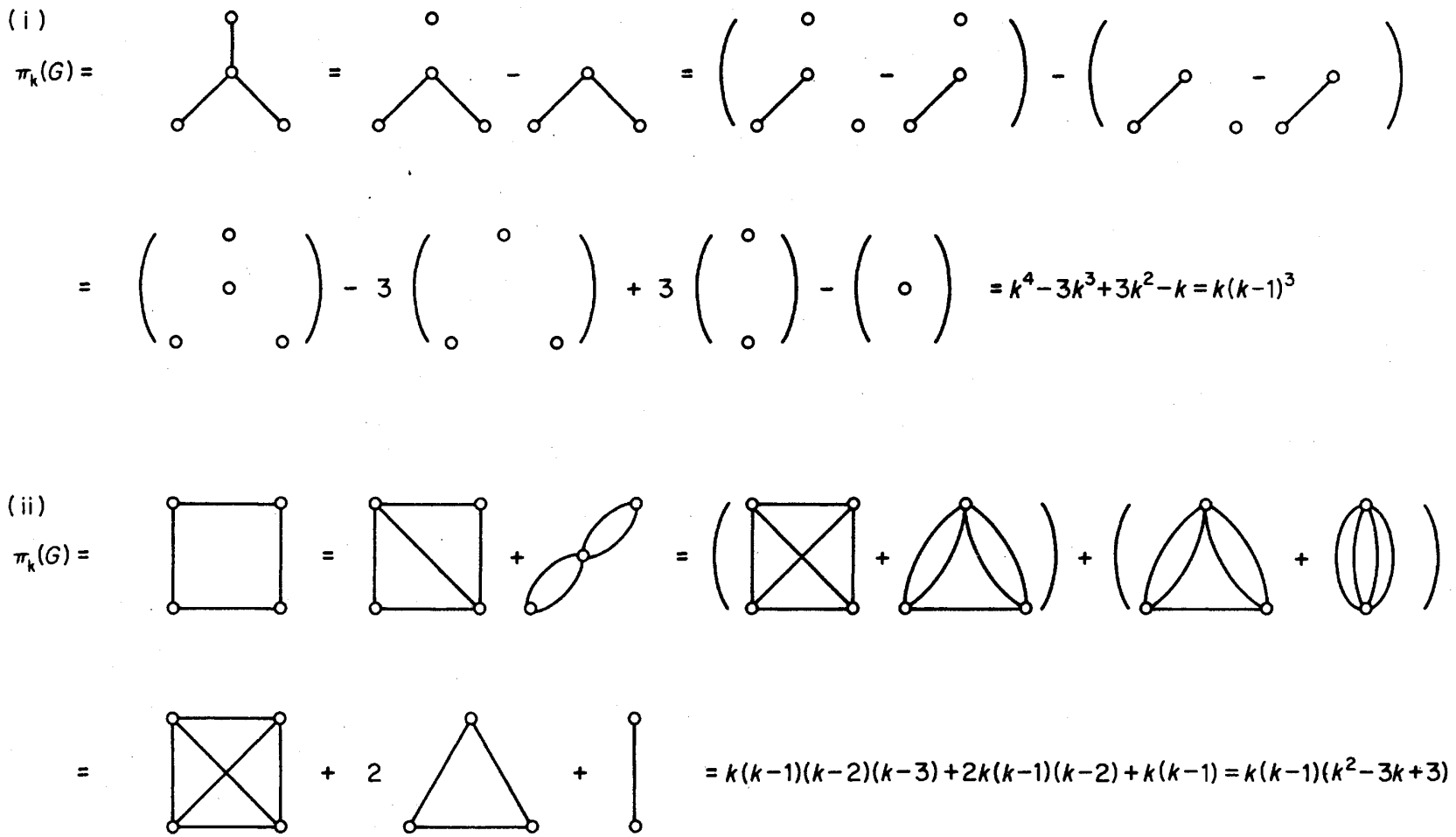



Figure 8.9. Recursive calculation of $\pi_k(G)$

thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of complete graphs.

Method (i) is more suited to graphs with few edges, whereas (ii) can be applied more efficiently to graphs with many edges. These two methods are illustrated in figure 8.9 (where the chromatic polynomial of a graph is represented symbolically by the graph itself).

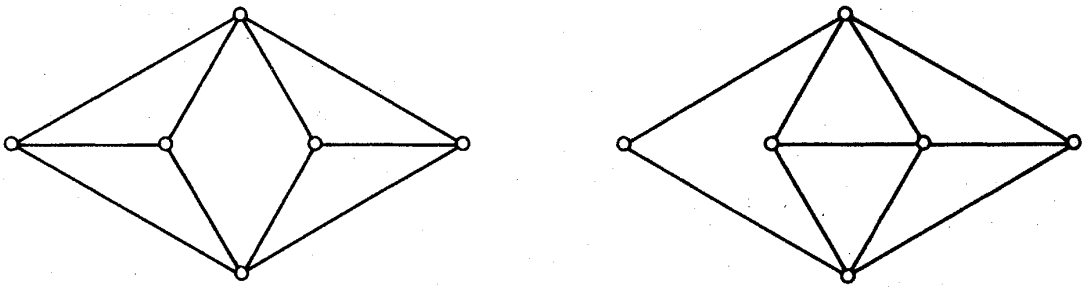
The calculation of chromatic polynomials can sometimes be facilitated by the use of a number of formulae relating the chromatic polynomial of $G$ to the chromatic polynomials of various subgraphs of $G$ (see exercises 8.4.5*a*, 8.4.6 and 8.4.7). However, no good algorithm is known for finding the chromatic polynomial of a graph. (Such an algorithm would clearly provide an efficient way to determine the chromatic number.)

Although many properties of chromatic polynomials are known, no one has yet discovered which polynomials are chromatic. It has been conjectured by Read (1968) that the sequence of coefficients of any chromatic polynomial must first rise in absolute value and then fall—in other words, that no coefficient may be flanked by two coefficients having greater absolute value. However, even if true, this condition, together with the conditions of corollary 8.6, would not be enough. The polynomial $k^4-3k^3+3k^2$, for example, satisfies all these conditions, but still is not the chromatic polynomial of any graph (exercise 8.4.2*b*).

Chromatic polynomials have been used with some success in the study of planar graphs, where their roots exhibit an unexpected regularity (see Tutte, 1970). Further results on chromatic polynomials can be found in the lucid survey article by Read (1968).

*Exercises*

8.4.1 Calculate the chromatic polynomials of the following two graphs:



8.4.2 (*a*) Show, by means of theorem 8.6, that if $G$ is simple, then the coefficient of $k^{\nu-1}$ in $\pi_k(G)$ is $-\varepsilon$.

(*b*) Deduce that no graph has chromatic polynomial $k^4-3k^3+3k^2$.

8.4.3 (*a*) Show that if $G$ is a tree, then $\pi_k(G)=k(k-1)^{\nu-1}$.

(*b*) Deduce that if $G$ is connected, then $\pi_k(G)\leq k(k-1)^{\nu-1}$, and show that equality holds only when $G$ is a tree.

8.4.4 Show that if $G$ is a cycle of length $n$, then $\pi_k(G) = (k-1)^n + (-1)^n(k-1)$.

8.4.5 (*a*) Show that $\pi_k(G \vee K_1) = k\pi_{k-1}(G)$.

(*b*) Using (*a*) and exercise 8.4.4, show that if $G$ is a wheel with $n$ spokes, then $\pi_k(G) = k(k-2)^n + (-1)^n k(k-2)$.

8.4.6 Show that if $G_1, G_2, \ldots, G_\omega$ are the components of $G$, then $\pi_k(G) = \pi_k(G_1)\pi_k(G_2)\ldots\pi_k(G_\omega)$.

8.4.7 Show that if $G \cap H$ is complete, then $\pi_k(G \cup H)\pi_k(G \cap H) = \pi_k(G)\pi_k(H)$.

8.4.8* Show that no real root of $\pi_k(G)$ is greater than $\nu$. (L. Lovász)

## 8.5 GIRTH AND CHROMATIC NUMBER

In any colouring of a graph, the vertices in a clique must all be assigned different colours. Thus a graph with a large clique necessarily has a high chromatic number. What is perhaps surprising is that there exist triangle-free graphs with arbitrarily high chromatic number. A recursive construction for such graphs was first described by Blanches Descartes (1954). (Her method, in fact, yields graphs that possess no cycles of length less than six.) We describe here an easier construction due to Mycielski (1955).

*Theorem 8.7* For any positive integer $k$, there exists a $k$-chromatic graph containing no triangle.

*Proof* For $k = 1$ and $k = 2$, the graphs $K_1$ and $K_2$ have the required property. We proceed by induction on $k$. Suppose that we have already constructed a triangle-free graph $G_k$ with chromatic number $k \geq 2$. Let the vertices of $G_k$ be $v_1, v_2, \ldots, v_n$. Form a new graph $G_{k+1}$ from $G_k$ as follows: add $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$, and then, for $1 \leq i \leq n$, join $u_i$ to the neighbours of $v_i$ and to $v$. For example, if $G_2$ is $K_2$ then $G_3$ is the 5-cycle and $G_4$ the Grötzsch graph (see figure 8.10).

The graph $G_{k+1}$ clearly has no triangles. For, since $\{u_1, u_2, \ldots, u_n\}$ is an independent set in $G_{k+1}$, no triangles can contain more than one $u_i$; and if $u_iv_jv_ku_i$ were a triangle in $G_{k+1}$, then $v_iv_jv_kv_i$ would be a triangle in $G_k$, contrary to assumption.

We now show that $G_{k+1}$ is $(k+1)$-chromatic. Note, first, that $G_{k+1}$ is certainly $(k+1)$-colourable, since any $k$-colouring of $G_k$ can be extended to a $(k+1)$-colouring of $G_{k+1}$ by colouring $u_i$ the same as $v_i$, $1 \leq i \leq n$, and then assigning a new colour to $v$. Therefore it remains to show that $G_{k+1}$ is not $k$-colourable. If possible, consider a $k$-colouring of $G_{k+1}$ in which, without loss of generality, $v$ is assigned colour $k$. Clearly, no $u_i$ can also have colour $k$. Now recolour each vertex $v_i$ of colour $k$ with the colour assigned to $u_i$.

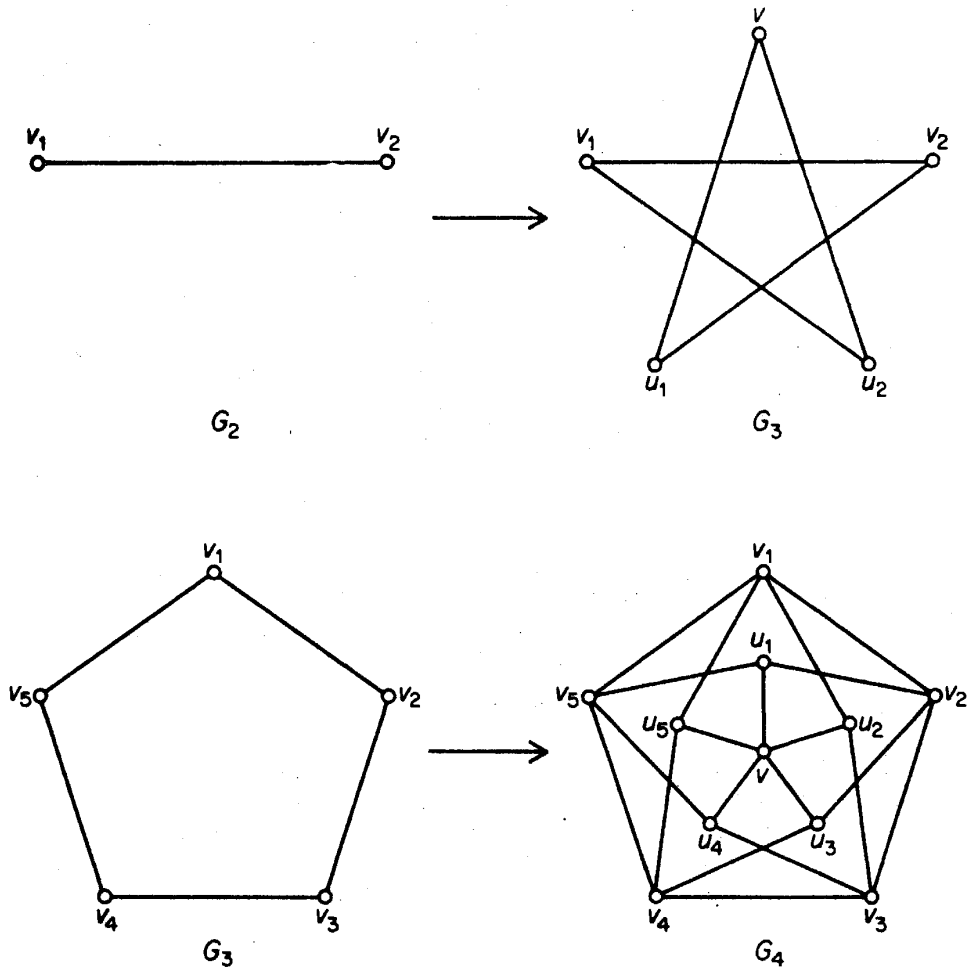



Figure 8.10. Mycielski's construction

This results in a $(k-1)$-colouring of the $k$-chromatic graph $G_k$. Therefore $G_{k+1}$ is indeed $(k+1)$-chromatic. The theorem follows from the principle of induction □

By starting with the 2-chromatic graph $K_2$, the above construction yields, for all $k \geq 2$, a triangle-free $k$-chromatic graph on $3.2^{k-2}-1$ vertices.

We have already noted that there are graphs with girth six and arbitrary chromatic number. Using the probabilistic method, Erdös (1961) has, in fact, shown that, given any two integers $k \geq 2$ and $l \geq 2$, there is a graph with girth $k$ and chromatic number $l$. Unfortunately, this application of the probabilistic method is not quite as straightforward as the one given in section 7.2, and we therefore choose to omit it. A constructive proof of Erdös' result has been given by Lovász (1968).

*Exercises*

8.5.1 Let $G_3, G_4, \ldots$ be the graphs obtained from $G_2 = K_2$, using Mycielski's construction. Show that each $G_k$ is $k$-critical.

8.5.2 (*a*)* Let $G$ be a $k$-chromatic graph of girth at least six ($k \geq 2$). Form a new graph $H$ as follows: Take $\binom{k\nu}{\nu}$ disjoint copies of $G$ and a set $S$ of $k\nu$ new vertices, and set up a one–one correspondence between the copies of $G$ and the $\nu$-element subsets of $S$. For each copy of $G$, join its vertices to the members of the corresponding $\nu$-element subset of $S$ by a matching. Show that $H$ has chromatic number at least $k+1$ and girth at least six.

(*b*) Deduce that, for any $k \geq 2$, there exists a $k$-chromatic graph of girth six. (B. Descartes)

## APPLICATIONS

### 8.6 A STORAGE PROBLEM

A company manufactures $n$ chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph $G$ on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices $v_i$ and $v_j$ if and only if the chemicals $C_i$ and $C_j$ are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of $G$.

The solution of many problems of practical interest (of which the storage problem is one instance) involves finding the chromatic number of a graph. Unfortunately, no good algorithm is known for determining the chromatic number. Here we describe a systematic procedure which is basically 'enumerative' in nature. It is not very efficient for large graphs.

Since the chromatic number of a graph is the least number of independent sets into which its vertex set can be partitioned, we begin by describing a method for listing all the independent sets in a graph. Because every independent set is a subset of a maximal independent set, it suffices to determine all the maximal independent sets. In fact, our procedure first determines complements of maximal independent sets, that is, minimal coverings.

Observe that a subset $K$ of $V$ is a minimal covering of $G$ if and only if, for each vertex $v$, either $v$ belongs to $K$ or all the neighbours of $v$ belong to $K$ (but not both). This provides us with a procedure for finding minimal coverings:

FOR EACH VERTEX $v$, CHOOSE EITHER $v$, OR ALL THE NEIGHBOURS OF $v$

(8.2)

To implement this procedure effectively, we make use of an algebraic device. First, we denote the instruction 'choose vertex $v$' simply by the symbol $v$. Then, given two instructions $X$ and $Y$, the instructions 'either $X$ or $Y$' and 'both $X$ and $Y$' are denoted by $X+Y$ (the *logical sum*) and $XY$ (the *logical product*), respectively. For example, the instruction 'choose either $u$ and $v$ or $v$ and $w$' is written $uv+vw$. Formally, the logical sum and logical product behave like $\cup$ and $\cap$ for sets, and the algebraic laws that hold with respect to $\cup$ and $\cap$ also hold with respect to these two operations (see exercise 8.6.1). By using these laws, we can often simplify logical expressions; thus

$$\begin{aligned}(uv+vw)(u+vx) &= uvu+uvvx+vwu+vwvx\\ &= uv+uvx+vwu+vwx\\ &= uv+vwx\end{aligned}$$

Consider, now, the graph $G$ of figure 8.11. Our prescription (8.2) for finding the minimal coverings in $G$ is

$$(a+bd)(b+aceg)(c+bdef)(d+aceg)(e+bcdf)(f+ceg)(g+bdf) \quad (8.3)$$

It can be checked (exercise 8.6.2) that, on simplification, (8.3) reduces to

$$aceg+bcdeg+bdef+bcdf$$

In other words, 'choose $a$, $c$, $e$ and $g$ or $b$, $c$, $d$, $e$ and $g$ or $b$, $d$, $e$ and $f$ or $b$, $c$, $d$ and $f$'. Thus $\{a, c, e, g\}$, $\{b, c, d, e, g\}$, $\{b, d, e, f\}$ and $\{b, c, d, f\}$ are the minimal coverings of $G$. On complementation, we obtain the list of all maximal independent sets of $G$: $\{b, d, f\}$, $\{a, f\}$, $\{a, c, g\}$ and $\{a, e, g\}$.

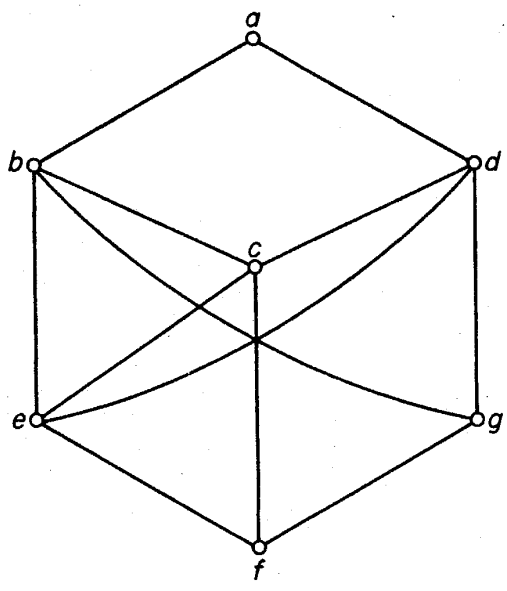



Figure 8.11

Now let us return to the problem of determining the chromatic number of a graph. A $k$-colouring $(V_1, V_2, \ldots, V_k)$ of $G$ is said to be *canonical* if $V_1$ is a maximal independent set of $G$, $V_2$ is a maximal independent set of $G - V_1$, $V_3$ is a maximal independent set of $G - (V_1 \cup V_2)$, and so on. It is easy to see (exercise 8.6.3) that if $G$ is $k$-colourable, then there exists a canonical $k$-colouring of $G$. By repeatedly using the above method for finding maximal independent sets, one can determine all the canonical colourings of $G$. The least number of colours used in such a colouring is then the chromatic number of $G$. For the graph $G$ of figure 8.11, $\chi = 3$; a corresponding canonical colouring is $(\{b, d, f\}, \{a, e, g\}, \{c\})$.

Christofides (1971) gives some improvements on this procedure.

*Exercises*

8.6.1 Verify the associative, commutative, distributive and absorption laws for the logical sum and logical product.

8.6.2 Reduce (8.3) to $aceg + bcdeg + bdef + bcdf$.

8.6.3 Show that if $G$ is $k$-vertex-colourable, then $G$ has a canonical $k$-vertex colouring.

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# 9 Planar Graphs

## 9.1 PLANE AND PLANAR GRAPHS

A graph is said to be *embeddable in the plane*, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph $G$ is called a *planar embedding* of $G$. A planar embedding $\tilde{G}$ of $G$ can itself be regarded as a graph isomorphic to $G$; the vertex set of $\tilde{G}$ is the set of points representing vertices of $G$, the edge set of $\tilde{G}$ is the set of lines representing edges of $G$, and a vertex of $\tilde{G}$ is incident with all the edges of $\tilde{G}$ that contain it. We therefore sometimes refer to a planar embedding of a planar graph as a *plane graph*. Figure 9.1*b* shows a planar embedding of the planar graph in figure 9.1*a*.

It is clear from the above definition that the study of planar graphs necessarily involves the topology of the plane. However, we shall not attempt here to be strictly rigorous in topological matters, and will be content to adopt a naive point of view toward them. This is done so as not to obscure the combinatorial aspect of the theory, which is our main interest.

The results of topology that are especially relevant in the study of planar graphs are those which deal with Jordan curves. (A *Jordan curve* is a continuous non-self-intersecting curve whose origin and terminus coincide.) The union of the edges in a cycle of a plane graph constitutes a Jordan curve; this is the reason why properties of Jordan curves come into play in planar graph theory. We shall recall a well-known theorem about Jordan curves and use it to demonstrate the nonplanarity of $K_5$.

Let $J$ be a Jordan curve in the plane. Then the rest of the plane is partitioned into two disjoint open sets called the *interior* and *exterior* of $J$. We shall denote the interior and exterior of $J$, respectively, by $\operatorname{int} J$ and $\operatorname{ext} J$, and their closures by $\operatorname{Int} J$ and $\operatorname{Ext} J$. Clearly $\operatorname{Int} J \cap \operatorname{Ext} J = J$. The *Jordan curve theorem* states that any line joining a point in $\operatorname{int} J$ to a point in $\operatorname{ext} J$ must meet $J$ in some point (see figure 9.2). Although this theorem is intuitively obvious, a formal proof of it is quite difficult.

*Theorem 9.1* $K_5$ is nonplanar.

*Proof* By contradiction. If possible let $G$ be a plane graph corresponding to $K_5$. Denote the vertices of $G$ by $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$. Since $G$ is complete, any two of its vertices are joined by an edge. Now the cycle $C = v_1v_2v_3v_1$ is a Jordan curve in the plane, and the point $v_4$ must lie either in $\operatorname{int} C$ or $\operatorname{ext} C$.

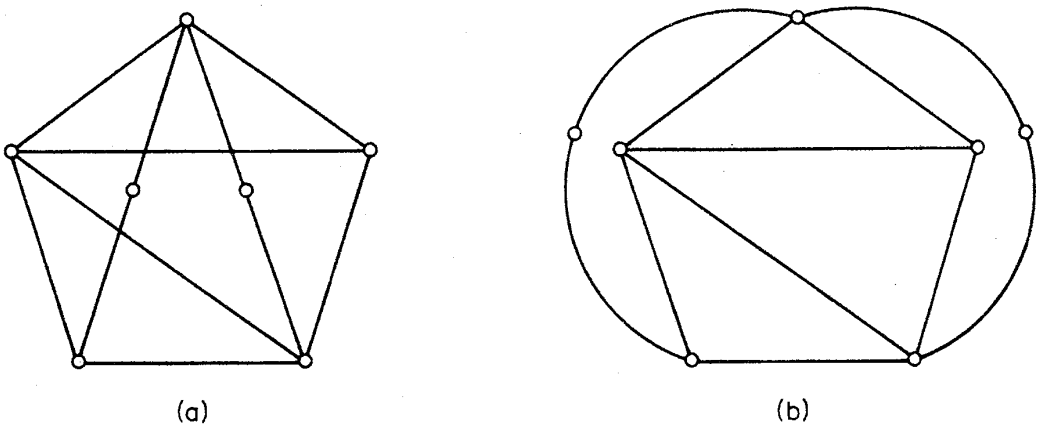



Figure 9.1. (*a*) A planar graph $G$; (*b*) a planar embedding of $G$

We shall suppose that $v_4 \in \text{int } C$. (The case where $v_4 \in \text{ext } C$ can be dealt with in a similar manner.) Then the edges $v_4v_1$, $v_4v_2$ and $v_4v_3$ divide int $C$ into the three regions int $C_1$, int $C_2$ and int $C_3$, where $C_1 = v_1v_4v_2v_1$, $C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$ (see figure 9.3).

Now $v_5$ must lie in one of the four regions ext $C$, int $C_1$, int $C_2$ and int $C_3$. If $v_5 \in \text{ext } C$ then, since $v_4 \in \text{int } C$, it follows from the Jordan curve theorem that the edge $v_4v_5$ must meet $C$ in some point. But this contradicts the assumption that $G$ is a plane graph. The cases $v_5 \in \text{int } C_i$, $i = 1, 2, 3$, can be disposed of in like manner □

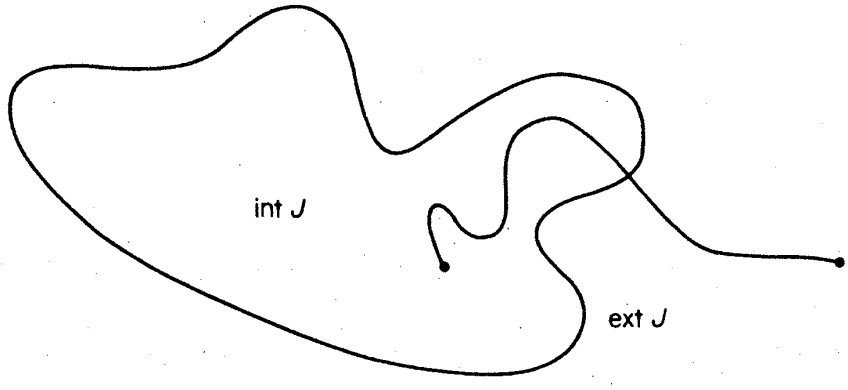



Figure 9.2

A similar argument can be used to establish that $K_{3,3}$, too, is nonplanar (exercise 9.1.1). We shall see in section 9.5 that, on the other hand, every nonplanar graph contains a subdivision of either $K_5$ or $K_{3,3}$.

The notion of a planar embedding extends to other surfaces.† A graph $G$ is said to be *embeddable* on a surface $S$ if it can be drawn in $S$ so that its

† A *surface* is a 2-dimensional manifold. Closed surfaces are divided into two classes, orientable and non-orientable. The sphere and the torus are examples of orientable surfaces; the projective plane and the Möbius band are non-orientable. For a detailed account of embeddings of graphs on surfaces the reader is referred to Fréchet and Fan (1967).

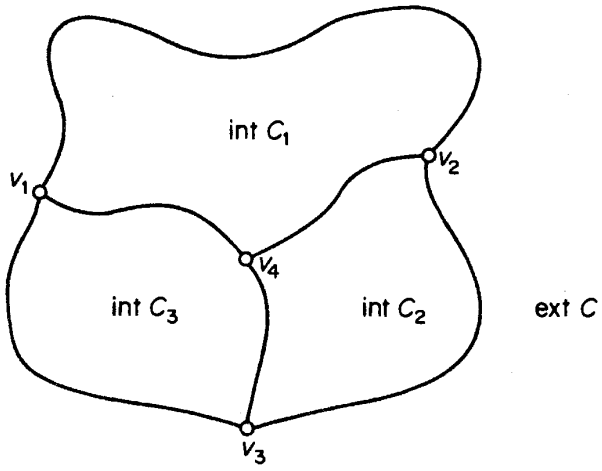



Figure 9.3

edges intersect only at their ends; such a drawing (if one exists) is called an *embedding* of $G$ on $S$. Figure 9.4*a* shows an embedding of $K_5$ on the torus, and figure 9.4*b* an embedding of $K_{3,3}$ on the Möbius band. The torus is represented as a rectangle in which opposite sides are identified, and the Möbius band as a rectangle whose two ends are identified after one half-twist.

We have seen that not all graphs can be embedded in the plane; this is also true of other surfaces. It can be shown (see, for example, Fréchet and Fan, 1967) that, for every surface $S$, there exist graphs which are not embeddable on $S$. Every graph can, however, be 'embedded' in 3-dimensional space $\mathscr{R}^3$ (exercise 9.1.3).

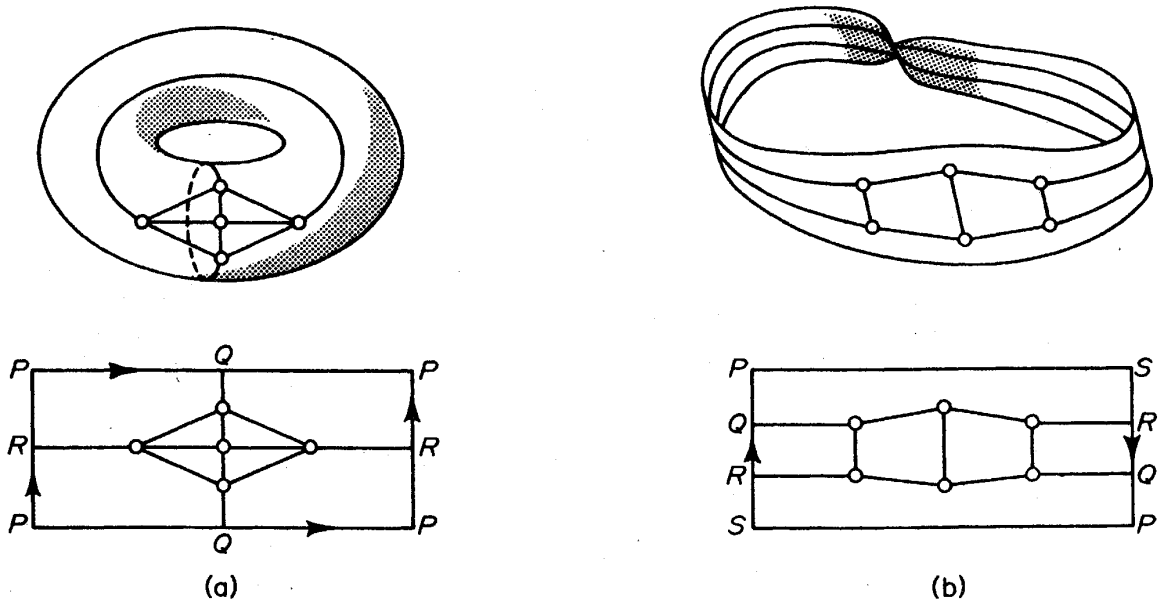



Figure 9.4. (*a*) An embedding of $K_5$ on the torus; (*b*) an embedding of $K_{3,3}$ on the Möbius band

Planar graphs and graphs embeddable on the sphere are one and the same. To show this we make use of a mapping known as stereographic projection. Consider a sphere $S$ resting on a plane $P$, and denote by $z$ the point of $S$ that is diagonally opposite the point of contact of $S$ and $P$. The mapping $\pi : S\backslash\{z\} \to P$, defined by $\pi(s) = p$ if and only if the points $z$, $s$ and $p$ are collinear, is called *stereographic projection* from $z$; it is illustrated in figure 9.5.

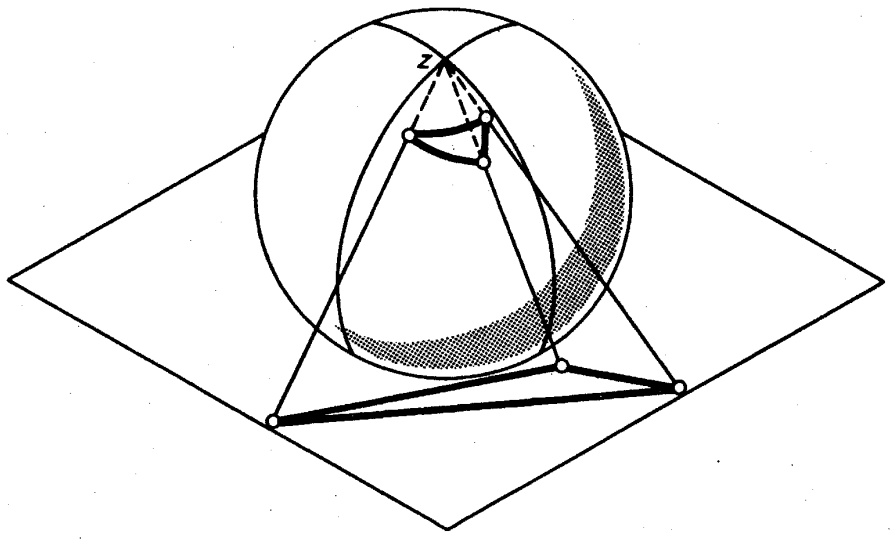



Figure 9.5. Stereographic projection

*Theorem 9.2* A graph $G$ is embeddable in the plane if and only if it is embeddable on the sphere.

*Proof* Suppose $G$ has an embedding $\tilde{G}$ on the sphere. Choose a point $z$ of the sphere not in $\tilde{G}$. Then the image of $\tilde{G}$ under stereographic projection from $z$ is an embedding of $G$ in the plane. The converse is proved similarly □

On many occasions it is advantageous to consider embeddings of planar graphs on the sphere; one instance is provided by the proof of theorem 9.3 in the next section.

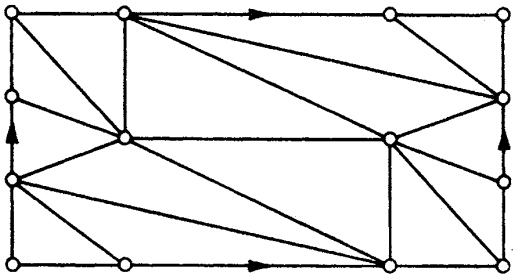
*Exercises*

**9.1.1** Show that $K_{3,3}$ is nonplanar.

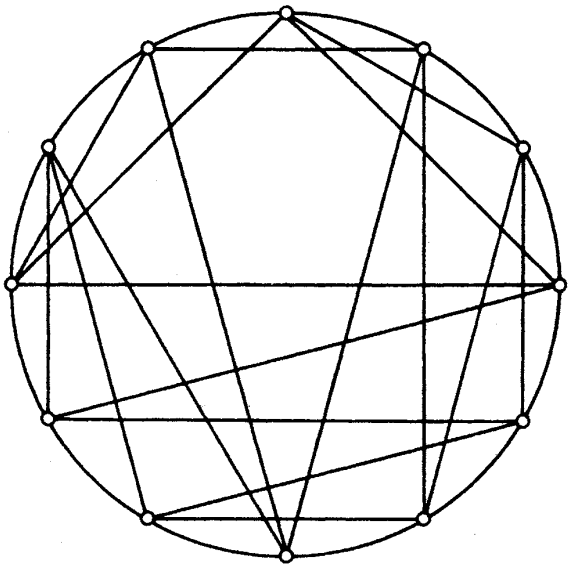
**9.1.2** (*a*) Show that $K_5 - e$ is planar for any edge $e$ of $K_5$.
(*b*) Show that $K_{3,3} - e$ is planar for any edge $e$ of $K_{3,3}$.

9.1.3 Show that all graphs are 'embeddable' in $\mathcal{R}^3$.

9.1.4 Verify that the following is an embedding of $K_7$ on the torus:



9.1.5 Find a planar embedding of the following graph in which each edge is a straight line.
(Fáry, 1948 has proved that every simple planar graph has such an embedding.)



## 9.2 DUAL GRAPHS

A plane graph $G$ partitions the rest of the plane into a number of connected regions; the closures of these regions are called the *faces* of $G$. Figure 9.6 shows a plane graph with six faces, $f_1$, $f_2$, $f_3$, $f_4$, $f_5$ and $f_6$. The notion of a face applies also to embeddings of graphs on other surfaces. We shall denote by $F(G)$ and $\phi(G)$, respectively, the set of faces and the number of faces of a plane graph $G$.

Each plane graph has exactly one unbounded face, called the *exterior face;* in the plane graph of figure 9.6, $f_1$ is the exterior face.

*Theorem 9.3* Let $v$ be a vertex of a planar graph $G$. Then $G$ can be embedded in the plane in such a way that $v$ is on the exterior face of the embedding.

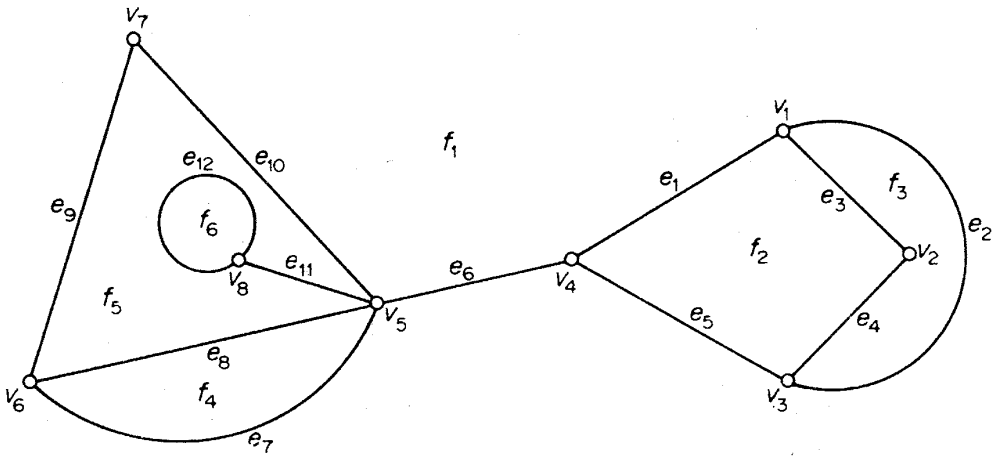



Figure 9.6. A plane graph with six faces

*Proof* Consider an embedding $\tilde{G}$ of $G$ on the sphere; such an embedding exists by virtue of theorem 9.2. Let $z$ be a point in the interior of some face containing $v$, and let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from $z$. Clearly $\pi(\tilde{G})$ is a planar embedding of $G$ of the desired type □

We denote the boundary of a face $f$ of a plane graph $G$ by $b(f)$. If $G$ is connected, then $b(f)$ can be regarded as a closed walk in which each cut edge of $G$ in $b(f)$ is traversed twice; when $b(f)$ contains no cut edges, it is a cycle of $G$. For example, in the plane graph of figure 9.6,

$$b(f_2) = v_1e_3v_2e_4v_3e_5v_4e_1v_1$$

and

$$b(f_5) = v_7e_{10}v_5e_{11}v_8e_{12}v_8e_{11}v_5e_8v_6e_9v_7$$

A face $f$ is said to be *incident* with the vertices and edges in its boundary. If $e$ is a cut edge in a plane graph, just one face is incident with $e$; otherwise, there are two faces incident with $e$. We say that an edge *separates* the faces incident with it. The *degree*, $d_G(f)$, of a face $f$ is the number of edges with which it is incident (that is, the number of edges in $b(f)$), cut edges being counted twice. In figure 9.6, $f_1$ is incident with the vertices $v_1$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ and the edges $e_1$, $e_2$, $e_5$, $e_6$, $e_7$, $e_9$, $e_{10}$; $e_1$ separates $f_1$ from $f_2$ and $e_{11}$ separates $f_5$ from $f_5$; $d(f_2) = 4$ and $d(f_5) = 6$.

Given a plane graph $G$, one can define another graph $G^*$ as follows: corresponding to each face $f$ of $G$ there is a vertex $f^*$ of $G^*$, and corresponding to each edge $e$ of $G$ there is an edge $e^*$ of $G^*$; two vertices $f^*$ and $g^*$ are joined by the edge $e^*$ in $G^*$ if and only if their corresponding faces $f$ and $g$ are separated by the edge $e$ in $G$. The graph $G^*$ is called the *dual* of $G$. A plane graph and its dual are shown in figures 9.7*a* and 9.7*b*.

It is easy to see that the dual $G^*$ of a plane graph $G$ is planar; in fact,

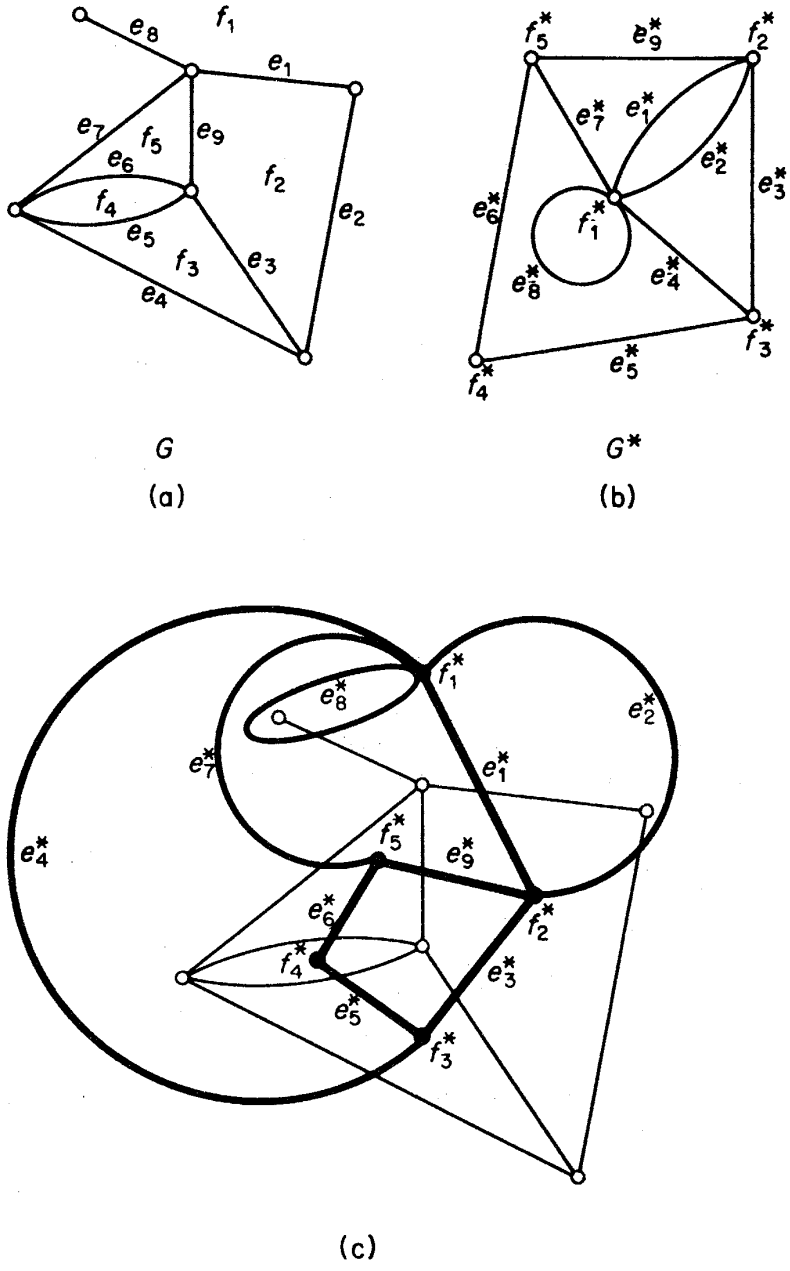



Figure 9.7. A plane graph and its dual

there is a natural way to embed $G^*$ in the plane. We place each vertex $f^*$ in the corresponding face $f$ of $G$, and then draw each edge $e^*$ in such a way that it crosses the corresponding edge $e$ of $G$ exactly once (and crosses no other edge of $G$). This procedure is illustrated in figure 9.7*c*, where the dual is indicated by heavy points and lines. It is intuitively clear that we can always draw the dual as a plane graph in this way, but we shall not prove this fact. Note that if $e$ is a loop of $G$, then $e^*$ is a cut edge of $G^*$, and vice versa.

Although defined abstractly, it is sometimes convenient to regard the dual

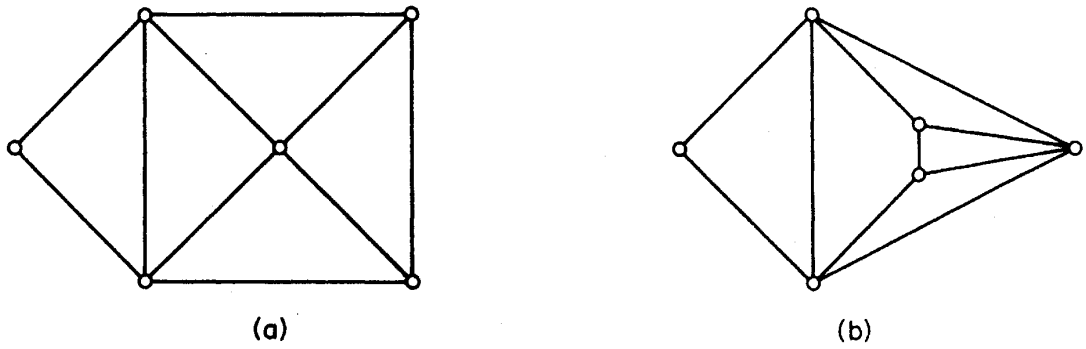



Figure 9.8. Isomorphic plane graphs with nonisomorphic duals

$G^*$ of a plane graph $G$ as a plane graph (embedded as described above). One can then consider the dual $G^{**}$ of $G^*$, and it is not difficult to prove that, when $G$ is connected, $G^{**} \cong G$ (exercise 9.2.4); a glance at figure 9.7*c* will indicate why this is so.

It should be noted that isomorphic plane graphs may have nonisomorphic duals. For example, the plane graphs in figure 9.8 are isomorphic, but their duals are not—the plane graph of figure 9.8*a* has a face of degree five, whereas the plane graph of figure 9.8*b* has no such face. Thus the notion of a dual is meaningful only for plane graphs, and cannot be extended to planar graphs in general.

The following relations are direct consequences of the definition of $G^*$:

$$\begin{aligned} \nu(G^*) &= \phi(G) \\ \varepsilon(G^*) &= \varepsilon(G) \\ d_{G^*}(f^*) &= d_G(f) \quad \text{for all} \quad f \in F(G) \end{aligned} \tag{9.1}$$

*Theorem 9.4* If $G$ is a plane graph, then

$$\sum_{f \in F} d(f) = 2\varepsilon$$

*Proof* Let $G^*$ be the dual of $G$. Then

$$\begin{aligned} \sum_{f \in F(G)} d(f) &= \sum_{f^* \in V(G^*)} d(f^*) && \text{by (9.1)} \\ &= 2\varepsilon(G^*) && \text{by theorem 1.1} \\ &= 2\varepsilon(G) && \text{by (9.1)} \quad \square \end{aligned}$$

*Exercises*

**9.2.1** (*a*) Show that a graph is planar if and only if each of its blocks is planar.

(*b*) Deduce that a minimal nonplanar graph is a simple block.

9.2.2 A plane graph is *self-dual* if it is isomorphic to its dual.

(*a*) Show that if $G$ is self-dual, then $\varepsilon = 2\nu - 2$.

(*b*) For each $n \geq 4$, find a self-dual plane graph on $n$ vertices.

9.2.3 (*a*) Show that $B$ is a bond of a plane graph $G$ if and only if $\{e^* \in E(G^*) \mid e \in B\}$ is a cycle of $G^*$.
(*b*) Deduce that the dual of an eulerian plane graph is bipartite.

9.2.4 Let $G$ be a plane graph. Show that
(*a*) $G^{**} \cong G$ if and only if $G$ is connected;
(*b*) $\chi(G^{**}) = \chi(G)$.

9.2.5 Let $T$ be a spanning tree of a connected plane graph $G$, and let $E^* = \{e^* \in E(G^*) \mid e \notin E(T)\}$. Show that $T^* = G^*[E^*]$ is a spanning tree of $G^*$.

**9.2.6** A *plane triangulation* is a plane graph in which each face has degree three. Show that every simple plane graph is a spanning subgraph of some simple plane triangulation ($\nu \geq 3$).

**9.2.7** Let $G$ be a simple plane triangulation with $\nu \geq 4$. Show that $G^*$ is a simple 2-edge-connected 3-regular planar graph.

9.2.8* Show that any plane triangulation $G$ contains a bipartite subgraph with $2\varepsilon(G)/3$ edges. (F. Harary, D. Matula)

## 9.3 EULER'S FORMULA

There is a simple formula relating the numbers of vertices, edges and faces in a connected plane graph. It is known as *Euler's formula* because Euler established it for those plane graphs defined by the vertices and edges of polyhedra.

*Theorem 9.5* If $G$ is a connected plane graph, then

$$\nu - \varepsilon + \phi = 2$$

*Proof* By induction on $\phi$, the number of faces of $G$. If $\phi = 1$, then each edge of $G$ is a cut edge and so $G$, being connected, is a tree. In this case $\varepsilon = \nu - 1$, by theorem 2.2, and the theorem clearly holds. Suppose that it is true for all connected plane graphs with fewer than $n$ faces, and let $G$ be a connected plane graph with $n \geq 2$ faces. Choose an edge $e$ of $G$ that is not a cut edge. Then $G - e$ is a connected plane graph and has $n - 1$ faces, since the two faces of $G$ separated by $e$ combine to form one face of $G - e$. By the induction hypothesis

$$\nu(G-e) - \varepsilon(G-e) + \phi(G-e) = 2$$

and, using the relations

$$\nu(G-e) = \nu(G) \qquad \varepsilon(G-e) = \varepsilon(G) - 1 \qquad \phi(G-e) = \phi(G) - 1$$

we obtain

$$\nu(G) - \varepsilon(G) + \phi(G) = 2$$

The theorem follows by the principle of induction □

*Corollary 9.5.1* All planar embeddings of a given connected planar graph have the same number of faces.

*Proof* Let $G$ and $H$ be two planar embeddings of a given connected planar graph. Since $G \cong H$, $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$. Applying theorem 9.5, we have

$$\phi(G) = \varepsilon(G) - \nu(G) + 2 = \varepsilon(H) - \nu(H) + 2 = \phi(H) \quad \square$$

*Corollary 9.5.2* If $G$ is a simple planar graph with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

*Proof* It clearly suffices to prove this for connected graphs. Let $G$ be a simple connected graph with $\nu \geq 3$. Then $d(f) \geq 3$ for all $f \in F$, and

$$\sum_{f \in F} d(f) \geq 3\phi$$

By theorem 9.4

$$2\varepsilon \geq 3\phi$$

Thus, from theorem 9.5

$$\nu - \varepsilon + 2\varepsilon/3 \geq 2$$

or

$$\varepsilon \leq 3\nu - 6 \quad \square$$

*Corollary 9.5.3* If $G$ is a simple planar graph, then $\delta \leq 5$.

*Proof* This is trivial for $\nu = 1$, 2. If $\nu \geq 3$, then, by theorem 1.1 and corollary 9.5.2,

$$\delta\nu \leq \sum_{v \in V} d(v) = 2\varepsilon \leq 6\nu - 12$$

It follows that $\delta \leq 5$ $\square$

We have already seen that $K_5$ and $K_{3,3}$ are nonplanar (theorem 9.1 and exercise 9.1.1). Here, we shall derive these two results as corollaries of theorem 9.5.

*Corollary 9.5.4* $K_5$ is nonplanar.

*Proof* If $K_5$ were planar then, by corollary 9.5.2, we would have

$$10 = \varepsilon(K_5) \leq 3\nu(K_5) - 6 = 9$$

Thus $K_5$ must be nonplanar $\square$

*Corollary 9.5.5* $K_{3,3}$ is nonplanar.

*Proof* Suppose that $K_{3,3}$ is planar and let $G$ be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ has no cycles of length less than four, every face of $G$ must

have degree at least four. Therefore, by theorem 9.4, we have

$$4\phi \leq \sum_{f \in F} d(f) = 2\varepsilon = 18$$

That is

$$\phi \leq 4$$

Theorem 9.5 now implies that

$$2 = \nu - \varepsilon + \phi \leq 6 - 9 + 4 = 1$$

which is absurd □

*Exercises*

9.3.1 (*a*) Show that if $G$ is a connected planar graph with girth $k \geq 3$, then $\varepsilon \leq k(\nu - 2)/(k - 2)$.

(*b*) Using (*a*), show that the Petersen graph is nonplanar.

**9.3.2** Show that every planar graph is 6-vertex-colourable.

9.3.3 (*a*) Show that if $G$ is a simple planar graph with $\nu \geq 11$, then $G^c$ is nonplanar.

(*b*) Find a simple planar graph $G$ with $\nu = 8$ such that $G^c$ is also planar.

9.3.4 The *thickness* $\theta(G)$ of $G$ is the minimum number of planar graphs whose union is $G$. (Thus $\theta(G) = 1$ if and only if $G$ is planar.)

(*a*) Show that $\theta(G) \geq \{\varepsilon/(3\nu - 6)\}$.

(*b*) Deduce that $\theta(K_\nu) \geq \{\nu(\nu - 1)/6(\nu - 2)\}$ and show, using exercise 9.3.3*b*, that equality holds for all $\nu \leq 8$.

9.3.5 Use the result of exercise 9.2.5 to deduce Euler's formula.

9.3.6 Show that if $G$ is a plane triangulation, then $\varepsilon = 3\nu - 6$.

9.3.7 Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least one. Show that there are at most $3n - 6$ pairs of points at distance exactly one.

## 9.4 BRIDGES

In the study of planar graphs, certain subgraphs, called bridges, play an important rôle. We shall discuss properties of these subgraphs in this section.

Let $H$ be a given subgraph of a graph $G$. We define a relation $\sim$ on $E(G) \backslash E(H)$ by the condition that $e_1 \sim e_2$ if there exists a walk $W$ such that

(i) the first and last edges of $W$ are $e_1$ and $e_2$, respectively, and
(ii) $W$ is internally-disjoint from $H$ (that is, no internal vertex of $W$ is a vertex of $H$).

It is easy to verify that $\sim$ is an equivalence relation on $E(G) \backslash E(H)$. A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$

is called a *bridge* of $H$ in $G$. It follows immediately from the definition that if $B$ is a bridge of $H$, then $B$ is a connected graph and, moreover, that any two vertices of $B$ are connected by a path that is internally-disjoint from $H$. It is also easy to see that two bridges of $H$ have no vertices in common except, possibly, for vertices of $H$. For a bridge $B$ of $H$, we write $V(B) \cap V(H) = V(B, H)$, and call the vertices in this set the *vertices of attachment* of $B$ to $H$. Figure 9.9 shows a variety of bridges of a cycle in a graph; edges of different bridges are represented by different kinds of lines.

In this section we are concerned with the study of bridges of a cycle $C$. Thus, to avoid repetition, we shall abbreviate 'bridge of $C$' to 'bridge' in the coming discussion; all bridges will be understood to be bridges of a given cycle $C$.

In a connected graph every bridge has at least one vertex of attachment, and in a block every bridge has at least two vertices of attachment. A bridge with $k$ vertices of attachment is called a *$k$-bridge*. Two $k$-bridges with the same vertices of attachment are *equivalent* $k$-bridges; for example, in figure 9.9, $B_1$ and $B_2$ are equivalent 3-bridges.

The vertices of attachment of a $k$-bridge $B$ with $k \geq 2$ effect a partition of $C$ into edge-disjoint paths, called the *segments* of $B$. Two bridges *avoid* one another if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise they *overlap*. In figure 9.9, $B_2$ and $B_3$ avoid one another, whereas $B_1$ and $B_2$ overlap. Two bridges $B$ and $B'$ are *skew* if there are four distinct vertices $u$, $v$, $u'$ and $v'$ of $C$ such that $u$ and $v$ are vertices of attachment of $B$, $u'$ and $v'$ are vertices of attachment of $B'$, and the four vertices appear in the cyclic order $u$, $u'$, $v$, $v'$ on $C$. In figure 9.9, $B_3$ and $B_4$ are skew, but $B_1$ and $B_2$ are not.

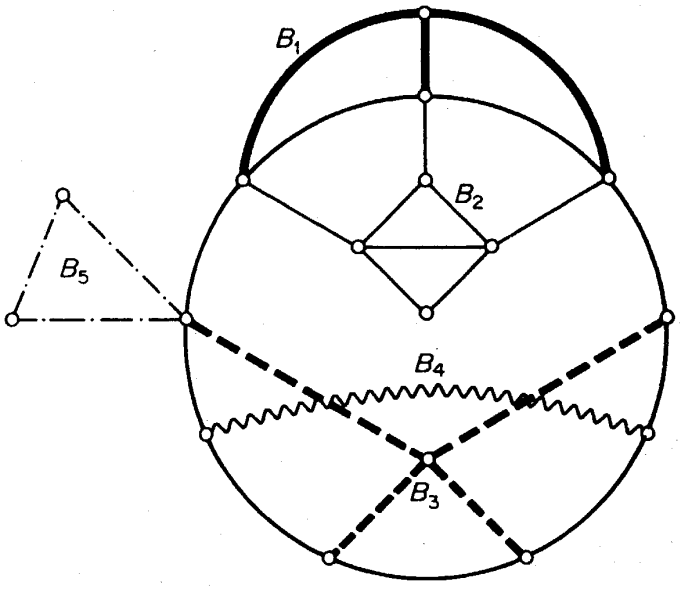



Figure 9.9. Bridges in a graph

*Theorem 9.6* If two bridges overlap, then either they are skew or else they are equivalent 3-bridges.

*Proof* Suppose that the bridges $B$ and $B'$ overlap. Clearly, each must have at least two vertices of attachment. Now if either $B$ or $B'$ is a 2-bridge, it is easily verified that they must be skew. We may therefore assume that both $B$ and $B'$ have at least three vertices of attachment. There are two cases.

*Case 1* $B$ and $B'$ are not equivalent bridges. Then $B'$ has a vertex of attachment $u'$ between two consecutive vertices of attachment $u$ and $v$ of $B$. Since $B$ and $B'$ overlap, some vertex of attachment $v'$ of $B'$ does not lie in the segment of $B$ connecting $u$ and $v$. It now follows that $B$ and $B'$ are skew.

*Case 2* $B$ and $B'$ are equivalent $k$-bridges, $k \geq 3$. If $k \geq 4$, then $B$ and $B'$ are clearly skew; if $k = 3$, they are equivalent 3-bridges □

*Theorem 9.7* If a bridge $B$ has three vertices of attachment $v_1$, $v_2$ and $v_3$, then there exists a vertex $v_0$ in $V(B)\backslash V(C)$ and three paths $P_1$, $P_2$ and $P_3$ in $B$ joining $v_0$ to $v_1$, $v_2$ and $v_3$, respectively, such that, for $i \neq j$, $P_i$ and $P_j$ have only the vertex $v_0$ in common (see figure 9.10).

*Proof* Let $P$ be a $(v_1, v_2)$-path in $B$, internally-disjoint from $C$. $P$ must have an internal vertex $v$, since otherwise the bridge $B$ would be just $P$, and would not contain a third vertex $v_3$. Let $Q$ be a $(v_3, v)$-path in $B$, internally-disjoint from $C$, and let $v_0$ be the first vertex of $Q$ on $P$. Denote by $P_1$ the $(v_0, v_1)$-section of $P^{-1}$, by $P_2$ the $(v_0, v_2)$-section of $P$, and by $P_3$ the $(v_0, v_3)$-section of $Q^{-1}$. Clearly $P_1$, $P_2$ and $P_3$ satisfy the required conditions □

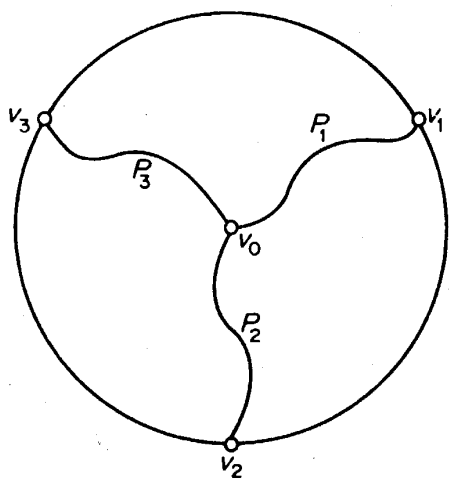



Figure 9.10

We shall now consider bridges in plane graphs. Suppose that $G$ is a plane graph and that $C$ is a cycle in $G$. Then $C$ is a Jordan curve in the plane, and each edge of $E(G)\backslash E(C)$ is contained in one of the two regions Int $C$ and Ext $C$. It follows that a bridge of $C$ is contained entirely in Int $C$ or Ext $C$. A bridge contained in Int $C$ is called an *inner bridge*, and a bridge contained in Ext $C$, an *outer bridge*. In figure 9.11 $B_1$ and $B_2$ are inner bridges, and $B_3$ and $B_4$ are outer bridges.

*Theorem 9.8* Inner (outer) bridges avoid one another.

*Proof* By contradiction. Let $B$ and $B'$ be two inner bridges that overlap. Then, by theorem 9.6, they must be either skew or equivalent 3-bridges.

*Case 1* $B$ and $B'$ are skew. By definition, there exist distinct vertices $u$ and $v$ in $B$ and $u'$ and $v'$ in $B'$, appearing in the cyclic order $u$, $u'$, $v$, $v'$ on $C$. Let $P$ be a $(u, v)$-path in $B$ and $P'$ a $(u', v')$-path in $B'$, both internally-disjoint from $C$. The two paths $P$ and $P'$ cannot have an internal vertex in common because they belong to different bridges. At the same time, both $P$ and $P'$ must be contained in Int $C$ because $B$ and $B'$ are inner bridges. By the Jordan curve theorem, $G$ cannot be a plane graph, contrary to hypothesis (see figure 9.12).

*Case 2* $B$ and $B'$ are equivalent 3-bridges. Let the common set of vertices of attachment be $\{v_1, v_2, v_3\}$. By theorem 9.7, there exist in $B$ a vertex $v_0$ and three paths $P_1$, $P_2$ and $P_3$ joining $v_0$ to $v_1$, $v_2$ and $v_3$, respectively, such that, for $i \neq j$, $P_i$ and $P_j$ have only the vertex $v_0$ in common. Similarly, $B'$ has a vertex $v_0'$ and three paths $P_1'$, $P_2'$ and $P_3'$ joining $v_0'$ to $v_1$, $v_2$ and $v_3$, respectively, such that, for $i \neq j$, $P_i'$ and $P_j'$ have only the vertex $v_0'$ in common (see figure 9.13).

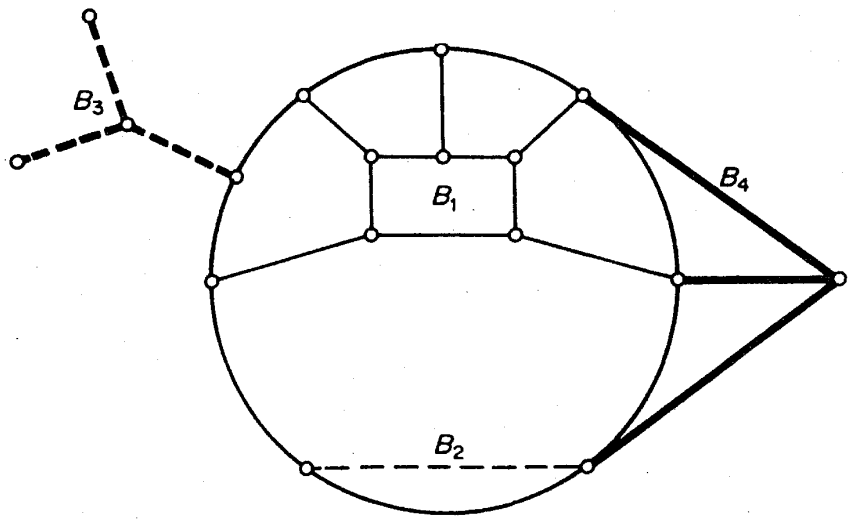



Figure 9.11. Bridges in a plane graph

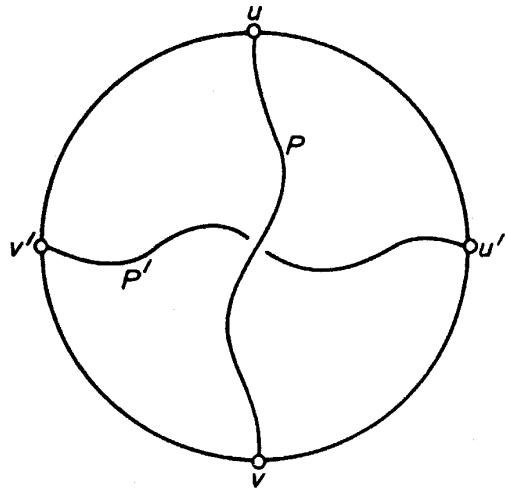



Figure 9.12

Now the paths $P_1$, $P_2$ and $P_3$ divide Int $C$ into three regions, and $v_0'$ must be in the interior of one of these regions. Since only two of the vertices $v_1$, $v_2$ and $v_3$ can lie on the boundary of the region containing $v_0'$, we may assume, by symmetry, that $v_3$ is not on the boundary of this region. By the Jordan curve theorem, the path $P_3'$ must cross either $P_1$, $P_2$ or $C$. But since $B$ and $B'$ are distinct inner bridges, this is clearly impossible.

We conclude that inner bridges avoid one another. Similarly, outer bridges avoid one another □

Let $G$ be a plane graph. An inner bridge $B$ of a cycle $C$ in $G$ is *transferable* if there exists a planar embedding $\tilde{G}$ of $G$ which is identical to $G$ itself, except that $B$ is an outer bridge of $C$ in $\tilde{G}$. The plane graph $\tilde{G}$ is said to be obtained from $G$ by *transferring* $B$. Figure 9.14 illustrates the transfer of a bridge.

*Theorem 9.9* An inner bridge that avoids every outer bridge is transferable.

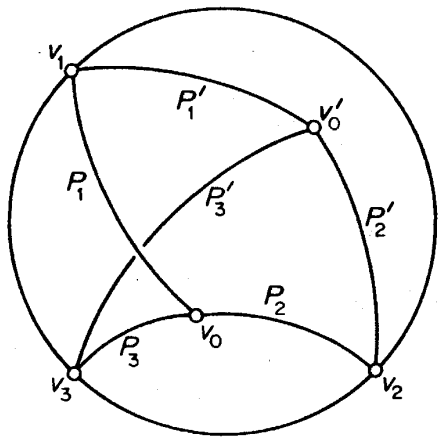



Figure 9.13

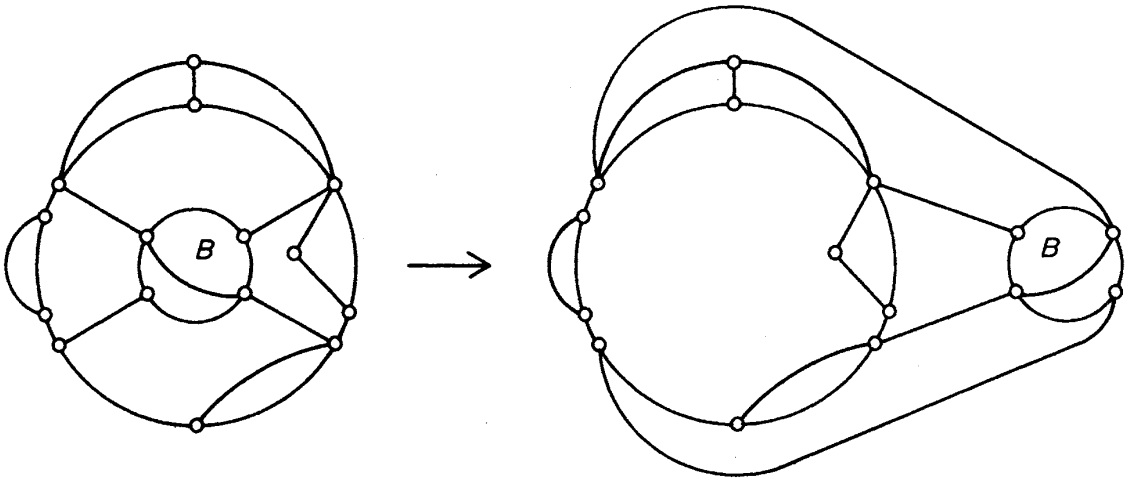



Figure 9.14. The transfer of a bridge

*Proof* Let $B$ be an inner bridge that avoids every outer bridge. Then the vertices of attachment of $B$ to $C$ all lie on the boundary of some face of $G$ contained in Ext $C$. $B$ can now be drawn in this face, as shown in figure 9.15 □

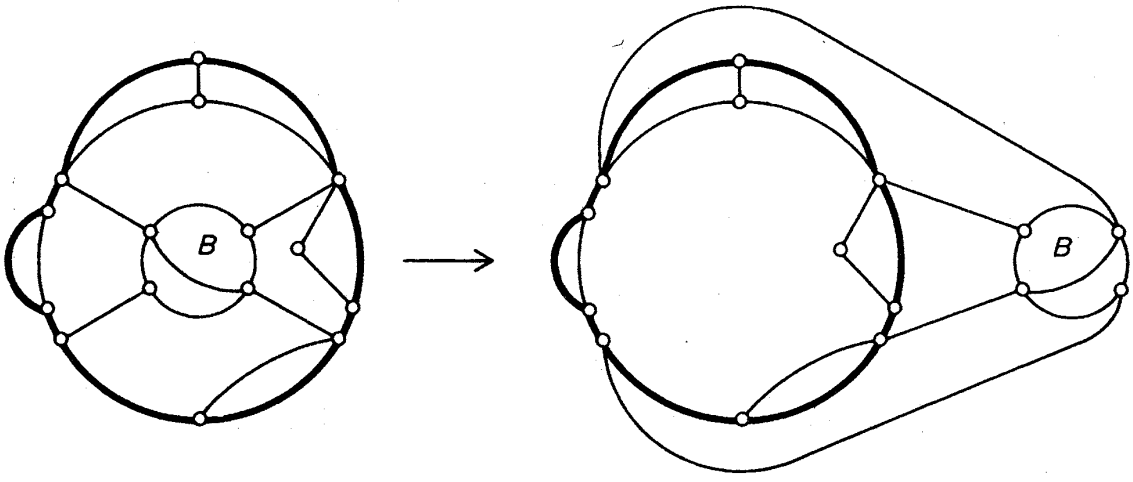



Figure 9.15

Theorem 9.9 is crucial to the proof of Kuratowski's theorem, which will be proved in the next section.

*Exercises*

9.4.1 Show that if $B$ and $B'$ are two distinct bridges, then $V(B) \cap V(B') \subseteq V(C)$.

**9.4.2** Let $u$, $x$, $v$ and $y$ (in that cyclic order) be four distinct vertices of attachment of a bridge $B$ to a cycle $C$ in a plane graph. Show that there is a $(u, v)$-path $P$ and an $(x, y)$-path $Q$ in $B$ such that (i) $P$ and $Q$ are internally-disjoint from $C$, and (ii) $|V(P) \cap V(Q)| \geq 1$.

9.4.3 (*a*) Let $C = v_1v_2 \ldots v_nv_1$ be a longest cycle in a nonhamiltonian connected graph $G$. Show that

(i) there exists a bridge $B$ such that $V(B) \backslash V(C) \neq \emptyset$;

(ii) if $v_i$ and $v_j$ are vertices of attachment of $B$, then $v_{i+1}v_{j+1} \notin E$.

(*b*) Deduce that if $\alpha \leqslant \kappa$, then $G$ is hamiltonian.

(V. Chvátal and P. Erdös)

## 9.5 KURATOWSKI'S THEOREM

Since planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not. We have already noted that, in particular, $K_5$ and $K_{3,3}$ are nonplanar and that any proper subgraph of either of these graphs is planar (exercise 9.1.2). A remarkably simple characterisation of planar graphs was given by Kuratowski (1930). This section is devoted to a proof of Kuratowski's theorem.

The following lemmas are simple observations, and we leave their proofs as an exercise (9.5.1).

*Lemma 9.10.1* If $G$ is nonplanar, then every subdivision of $G$ is nonplanar.

*Lemma 9.10.2* If $G$ is planar, then every subgraph of $G$ is planar.

Since $K_5$ and $K_{3,3}$ are nonplanar, we see from these two lemmas that if $G$ is planar, then $G$ cannot contain a subdivision of $K_5$ or of $K_{3,3}$ (figure 9.16). Kuratowski showed that this necessary condition is also sufficient.

Before proving Kuratowski's theorem, we need to establish two more simple lemmas.

Let $G$ be a graph with a 2-vertex cut $\{u, v\}$. Then there exist edge-disjoint subgraphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $G_1 \cup G_2 = G$. Consider such a separation of $G$ into subgraphs. In both $G_1$ and $G_2$ join $u$

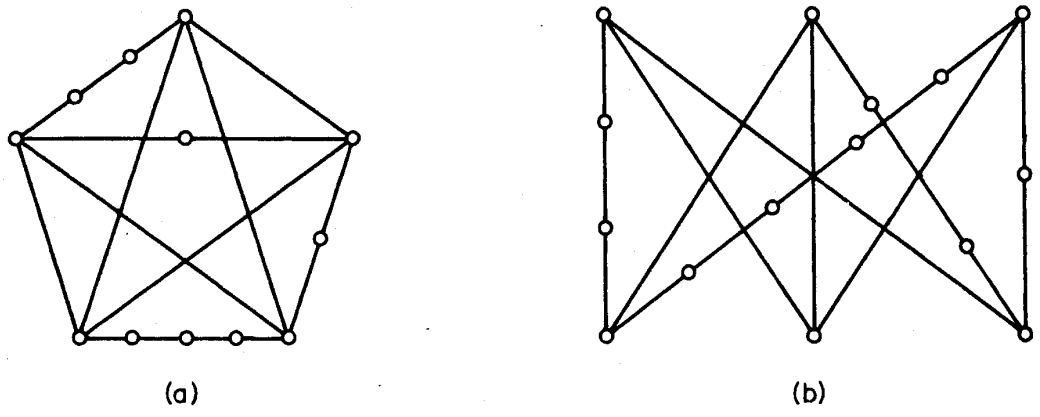



Figure 9.16. (*a*) A subdivision of $K_5$; (*b*) a subdivision of $K_{3,3}$

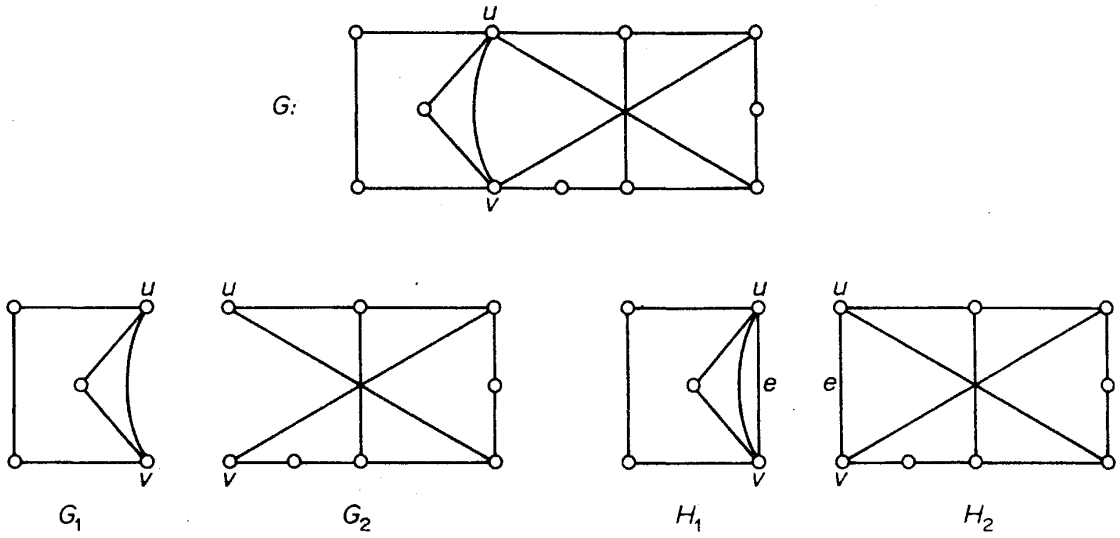



Figure 9.17

and $v$ by a new edge $e$ to obtain graphs $H_1$ and $H_2$, as in figure 9.17. Clearly $G = (H_1 \cup H_2) - e$. It is also easily seen that $\varepsilon(H_i) < \varepsilon(G)$ for $i = 1, 2$.

*Lemma 9.10.3* If $G$ is nonplanar, then at least one of $H_1$ and $H_2$ is also nonplanar.

*Proof* By contradiction. Suppose that both $H_1$ and $H_2$ are planar. Let $\tilde{H}_1$ be a planar embedding of $H_1$, and let $f$ be a face of $\tilde{H}_1$ incident with $e$. If $\tilde{H}_2$ is an embedding of $H_2$ in $f$ such that $\tilde{H}_1$ and $\tilde{H}_2$ have only the vertices $u$ and $v$ and the edge $e$ in common, then $(\tilde{H}_1 \cup \tilde{H}_2) - e$ is a planar embedding of $G$. This contradicts the hypothesis that $G$ is nonplanar □

*Lemma 9.10.4* Let $G$ be a nonplanar connected graph that contains no subdivision of $K_5$ or $K_{3,3}$ and has as few edges as possible. Then $G$ is simple and 3-connected.

*Proof* By contradiction. Let $G$ satisfy the hypotheses of the lemma. Then $G$ is clearly a minimal nonplanar graph, and therefore (exercise 9.2.1*b*) must be a simple block. If $G$ is not 3-connected, let $\{u, v\}$ be a 2-vertex cut of $G$ and let $H_1$ and $H_2$ be the graphs obtained from this cut as described above. By lemma 9.10.3, at least one of $H_1$ and $H_2$, say $H_1$, is nonplanar. Since $\varepsilon(H_1) < \varepsilon(G)$, $H_1$ must contain a subgraph $K$ which is a subdivision of $K_5$ or $K_{3,3}$; moreover $K \not\subseteq G$, and so the edge $e$ is in $K$. Let $P$ be a $(u, v)$-path in $H_2 - e$. Then $G$ contains the subgraph $(K \cup P) - e$, which is a subdivision of $K$ and hence a subdivision of $K_5$ or $K_{3,3}$. This contradiction establishes the lemma □

We shall find it convenient to adopt the following notation in the proof of Kuratowski's theorem. Suppose that $C$ is a cycle in a plane graph. Then we

can regard the two possible orientations of $C$ as 'clockwise' and 'anticlockwise'. For any two vertices, $u$ and $v$ of $C$, we shall denote by $C[u, v]$ the $(u, v)$-path which follows the clockwise orientation of $C$; similarly we shall use the symbols $C(u, v]$, $C[u, v)$ and $C(u, v)$ to denote the paths $C[u, v]-u$, $C[u, v]-v$ and $C[u, v]-\{u, v\}$. We are now ready to prove Kuratowski's theorem. Our proof is based on that of Dirac and Schuster (1954).

*Theorem 9.10* A graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

*Proof* We have already noted that the necessity follows from lemmas 9.10.1 and 9.10.2. We shall prove the sufficiency by contradiction.

If possible, choose a nonplanar graph $G$ that contains no subdivision of $K_5$ or $K_{3,3}$ and has as few edges as possible. From lemma 9.10.4 it follows that $G$ is simple and 3-connected. Clearly $G$ must also be a minimal nonplanar graph.

Let $uv$ be an edge of $G$, and let $H$ be a planar embedding of the planar graph $G-uv$. Since $G$ is 3-connected, $H$ is 2-connected and, by corollary 3.2.1, $u$ and $v$ are contained together in a cycle of $H$. Choose a cycle $C$ of $H$ that contains $u$ and $v$ and is such that the number of edges in $\operatorname{Int} C$ is as large as possible.

Since $H$ is simple and 2-connected, each bridge of $C$ in $H$ must have at least two vertices of attachment. Now all outer bridges of $C$ must be 2-bridges that overlap $uv$ because, if some outer bridge were a $k$-bridge for $k \geq 3$ or a 2-bridge that avoided $uv$, then there would be a cycle $C'$ containing $u$ and $v$ with more edges in its interior than $C$, contradicting the choice of $C$. These two cases are illustrated in figure 9.18 (with $C'$ indicated by heavy lines).

In fact, all outer bridges of $C$ in $H$ must be single edges. For if a 2-bridge with vertices of attachment $x$ and $y$ had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of $G$, contradicting the fact that $G$ is 3-connected.

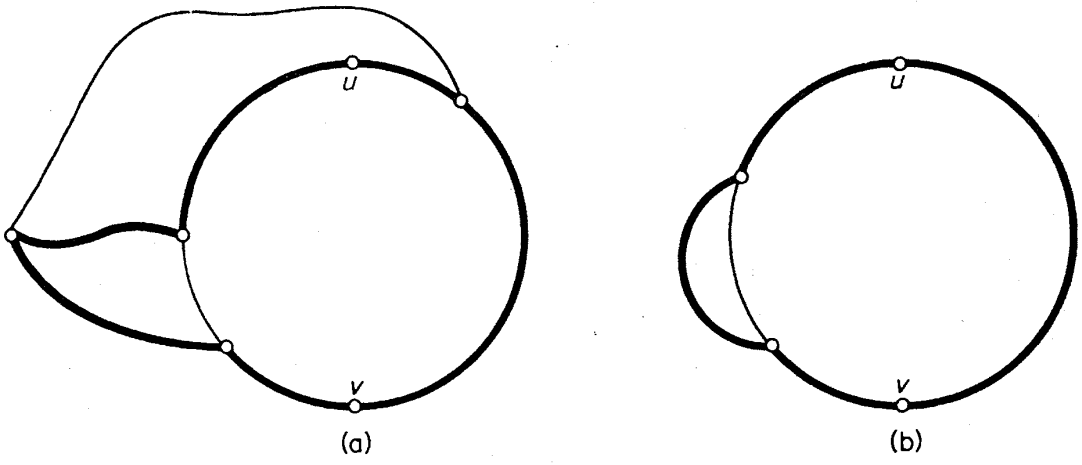



Figure 9.18

By theorem 9.8, no two inner bridges overlap. Therefore some inner bridge skew to $uv$ must overlap some outer bridge. For otherwise, by theorem 9.9, all such bridges could be transferred (one by one), and then the edge $uv$ could be drawn in Int $C$ to obtain a planar embedding of $G$; since $G$ is nonplanar, this is not possible. Therefore, there is an inner bridge $B$ that is both skew to $uv$ and skew to some outer bridge $xy$.

Two cases now arise, depending on whether $B$ has a vertex of attachment different from $u$, $v$, $x$ and $y$ or not.

*Case 1* $B$ has a vertex of attachment different from $u$, $v$, $x$ and $y$. We can choose the notation so that $B$ has a vertex of attachment $v_1$ in $C(x, u)$ (see figure 9.19). We consider two subcases, depending on whether $B$ has a vertex of attachment in $C(y, v)$ or not.

*Case 1a* $B$ has a vertex of attachment $v_2$ in $C(y, v)$. In this case there is a $(v_1, v_2)$-path $P$ in $B$ that is internally-disjoint from $C$. But then $(C \cup P) + \{uv, xy\}$ is a subdivision of $K_{3,3}$ in $G$, a contradiction (see figure 9.19).

*Case 1b* $B$ has no vertex of attachment in $C(y, v)$. Since $B$ is skew to $uv$ and to $xy$, $B$ must have vertices of attachment $v_2$ in $C(u, y]$ and $v_3$ in $C[v, x)$. Thus $B$ has three vertices of attachment $v_1$, $v_2$ and $v_3$. By theorem 9.7, there exists a vertex $v_0$ in $V(B) \backslash V(C)$ and three paths $P_1$, $P_2$ and $P_3$ in $B$ joining $v_0$ to $v_1$, $v_2$ and $v_3$, respectively, such that, for $i \neq j$, $P_i$ and $P_j$ have only the vertex $v_0$ in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$, a contradiction. This case is illustrated in figure 9.20. The subdivision of $K_{3,3}$ is indicated by heavy lines.

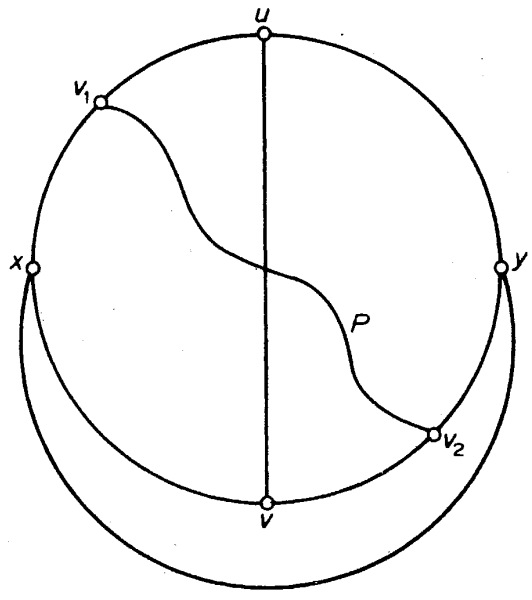



Figure 9.19

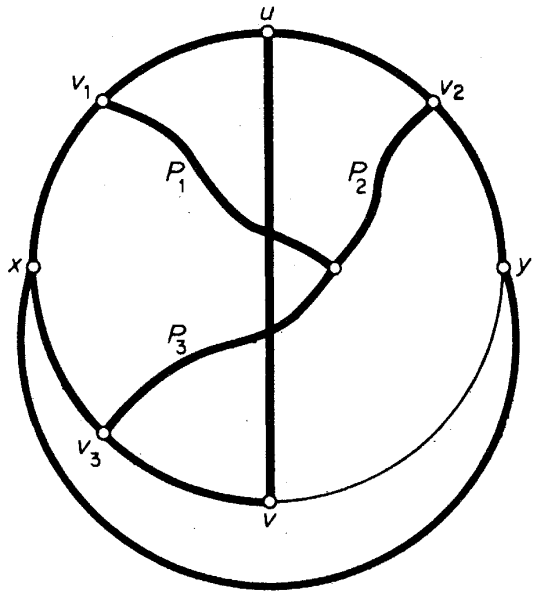



Figure 9.20

*Case 2* $B$ has no vertex of attachment other than $u$, $v$, $x$ and $y$. Since $B$ is skew to both $uv$ and $xy$, it follows that $u$, $v$, $x$ and $y$ must all be vertices of attachment of $B$. Therefore (exercise 9.4.2) there exists a $(u, v)$-path $P$ and an $(x, y)$-path $Q$ in $B$ such that (i) $P$ and $Q$ are internally-disjoint from $C$, and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two subcases, depending on whether $P$ and $Q$ have one or more vertices in common.

*Case 2a* $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of $K_5$ in $G$, again a contradiction (see figure 9.21).

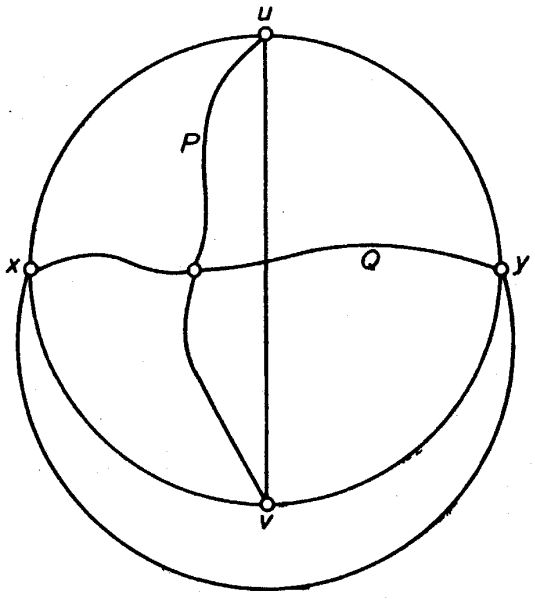



Figure 9.21

*Case 2b* $|V(P) \cap V(Q)| \geq 2$. Let $u'$ and $v'$ be the first and last vertices of $P$ on $Q$, and let $P_1$ and $P_2$ denote the $(u, u')$- and $(v', v)$-sections of $P$. Then $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ in $G$, once more a contradiction (see figure 9.22).

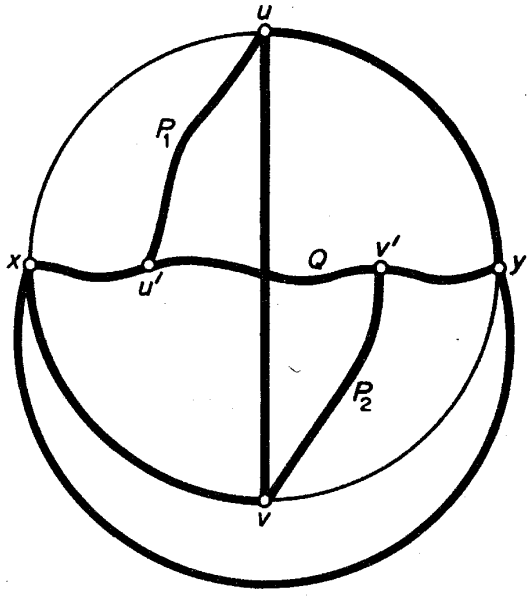



Figure 9.22

Thus all the possible cases lead to contradictions, and the proof is complete □

There are several other characterisations of planar graphs. For example, Wagner (1937) has shown that a graph is planar if and only if it contains no subgraph contractible to $K_5$ or $K_{3,3}$.

*Exercises*

9.5.1 Prove lemmas 9.10.1 and 9.10.2.

9.5.2 Show, using Kuratowski's theorem, that the Petersen graph is non-planar.

## 9.6 THE FIVE-COLOUR THEOREM AND THE FOUR-COLOUR CONJECTURE

As has already been noted (exercise 9.3.2), every planar graph is 6-vertex-colourable. Heawood (1890) improved upon this result by showing that one can always properly colour the vertices of a planar graph with at most five colours. This is known as the *five-colour theorem.*

*Theorem 9.11* Every planar graph is 5-vertex-colourable.

*Proof* By contradiction. Suppose that the theorem is false. Then there exists a 6-critical plane graph $G$. Since a critical graph is simple, we see from

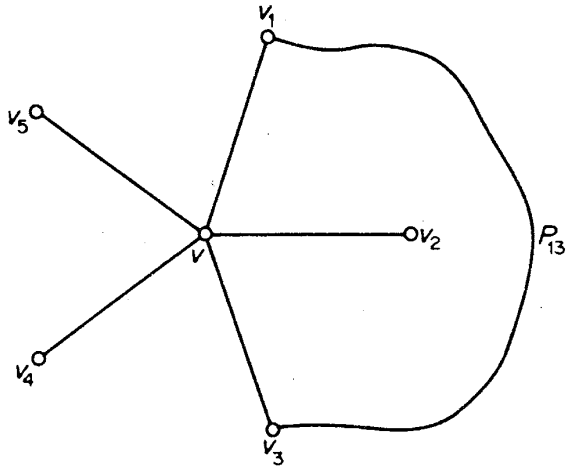



Figure 9.23

corollary 9.5.3 that $\delta \leq 5$. On the other hand we have, by theorem 8.1, that $\delta \geq 5$. Therefore $\delta = 5$. Let $v$ be a vertex of degree five in $G$, and let $(V_1, V_2, V_3, V_4, V_5)$ be a proper 5-vertex colouring of $G - v$; such a colouring exists because $G$ is 6-critical. Since $G$ itself is not 5-vertex-colourable, $v$ must be adjacent to a vertex of each of the five colours. Therefore we can assume that the neighbours of $v$ in clockwise order about $v$ are $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$, where $v_i \in V_i$ for $1 \leq i \leq 5$.

Denote by $G_{ij}$ the subgraph $G[V_i \cup V_j]$ induced by $V_i \cup V_j$. Now $v_i$ and $v_j$ must belong to the same component of $G_{ij}$. For, otherwise, consider the component of $G_{ij}$ that contains $v_i$. By interchanging the colours $i$ and $j$ in this component, we obtain a new proper 5-vertex colouring of $G - v$ in which only four colours (all but $i$) are assigned to the neighbours of $v$. We have already shown that this situation cannot arise. Therefore $v_i$ and $v_j$ must belong to the same component of $G_{ij}$. Let $P_{ij}$ be a $(v_i, v_j)$-path in $G_{ij}$, and let $C$ denote the cycle $vv_1P_{13}v_3v$ (see figure 9.23).

Since $C$ separates $v_2$ and $v_4$ (in figure 9.23, $v_2 \in \text{int}\, C$ and $v_4 \in \text{ext}\, C$), it follows from the Jordan curve theorem that the path $P_{24}$ must meet $C$ in some point. Because $G$ is a plane graph, this point must be a vertex. But this is impossible, since the vertices of $P_{24}$ have colours 2 and 4, whereas no vertex of $C$ has either of these colours □

The question now arises as to whether the five-colour theorem is best possible. It has been conjectured that every planar graph is 4-vertex-colourable; this is known as the *four-colour conjecture*. The four-colour conjecture has remained unsettled for more than a century, despite many attempts by major mathematicians to solve it. If it were true, then it would, of course, be best possible because there do exist planar graphs which are not 3-vertex-colourable ($K_4$ is the simplest such graph). For a history of the four-colour conjecture, see Ore (1967)†.

† The four-colour conjecture has now been settled in the affirmative by K. Appel and W. Haken; see page 253.

The problem of deciding whether the four-colour conjecture is true or false is called the *four-colour problem*.† There are several problems in graph theory that are equivalent to the four-colour problem; one of these is the case $n = 5$ of Hadwiger's conjecture (see section 8.3). We now establish the equivalence of certain problems concerning edge and face colourings with the four-colour problem. A *k-face colouring* of a plane graph $G$ is an assignment of $k$ colours $1, 2, \ldots, k$ to the faces of $G$; the colouring is *proper* if no two faces that are separated by an edge have the same colour. $G$ is *k-face-colourable* if it has a proper $k$-face colouring, and the minimum $k$ for which $G$ is $k$-face-colourable is the *face chromatic number* of $G$, denoted by $\chi^*(G)$. It follows immediately from these definitions that, for any plane graph $G$ with dual $G^*$,

$$\chi^*(G) = \chi(G^*) \tag{9.2}$$

*Theorem 9.12* The following three statements are equivalent:

(i) every planar graph is 4-vertex-colourable;
(ii) every plane graph is 4-face-colourable;
(iii) every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.

*Proof* We shall show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).

(*a*) (i)$\Rightarrow$(ii). This is a direct consequence of (9.2) and the fact that the dual of a plane graph is planar.

(*b*) (ii)$\Rightarrow$(iii). Suppose that (ii) holds, let $G$ be a simple 2-edge-connected 3-regular planar graph, and let $\tilde{G}$ be a planar embedding of $G$. By (ii), $\tilde{G}$ has a proper 4-face-colouring. It is, of course, immaterial which symbols are used as the 'colours', and in this case we shall denote the four colours by the vectors $c_0 = (0, 0)$, $c_1 = (1, 0)$, $c_2 = (0, 1)$ and $c_3 = (1, 1)$, over the field of integers modulo 2. We now obtain a 3-edge-colouring of $\tilde{G}$ by assigning to each edge the sum of the colours of the faces it separates (see figure 9.24). If $c_i$, $c_j$ and $c_k$ are the three colours assigned to the three faces incident with a vertex $v$, then $c_i + c_j$, $c_j + c_k$ and $c_k + c_i$ are the colours assigned to the three edges incident with $v$. Since $\tilde{G}$ is 2-edge-connected, each edge separates two distinct faces, and it follows that no edge is assigned the colour $c_0$ under this scheme. It is also clear that the three edges incident with a given vertex are assigned different colours. Thus we have a proper 3-edge-colouring of $\tilde{G}$, and hence of $G$.

† The four-colour problem is often posed in the following terms: can the countries of any map be coloured in four colours so that no two countries which have a common boundary are assigned the same colour? The equivalence of this problem with the four-colour problem follows from theorem 9.12 on observing that a map can be regarded as a plane graph with its countries as the faces.

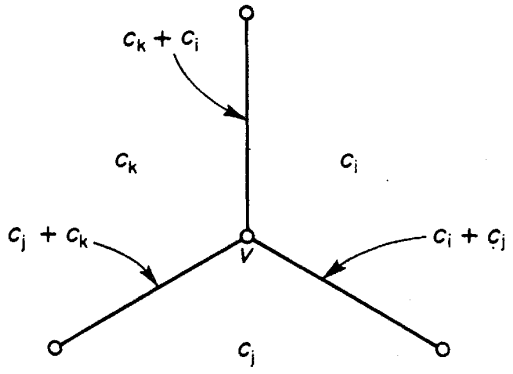



Figure 9.24

(c) (iii)$\Rightarrow$(i). Suppose that (iii) holds, but that (i) does not. Then there is a 5-critical planar graph $G$. Let $\tilde{G}$ be a planar embedding of $G$. Then (exercise 9.2.6) $\tilde{G}$ is a spanning subgraph of a simple plane triangulation $H$. The dual $H^*$ of $H$ is a simple 2-edge-connected 3-regular planar graph (exercise 9.2.7). By (iii), $H^*$ has a proper 3-edge colouring $(E_1, E_2, E_3)$. For $i \neq j$, let $H^*_{ij}$ denote the subgraph of $H^*$ induced by $E_i \cup E_j$. Since each vertex of $H^*$ is incident with one edge of $E_i$ and one edge of $E_j$, $H^*_{ij}$ is a union of disjoint cycles and is therefore (exercise 9.6.1) 2-face-colourable. Now each face of $H^*$ is the intersection of a face of $H^*_{12}$ and a face of $H^*_{23}$. Given proper 2-face colourings of $H^*_{12}$ and $H^*_{23}$ we can obtain a 4-face colouring of $H^*$ by assigning to each face $f$ the pair of colours assigned to the faces whose intersection is $f$. Since $H^* = H^*_{12} \cup H^*_{23}$ it is easily verified that this 4-face colouring of $H^*$ is proper. Since $H$ is a supergraph of $G$ we have

$$5 = \chi(G) \leq \chi(H) = \chi^*(H^*) \leq 4$$

This contradiction shows that (i) does, in fact, hold □

That statement (iii) of theorem 9.12 is equivalent to the four-colour problem was first observed by Tait (1880). A proper 3-edge colouring of a 3-regular graph is often called a *Tait colouring*. In the next section we shall discuss Tait's ill-fated approach to the four-colour conjecture. Grötzsch (1958) has verified the four-colour conjecture for planar graphs without triangles. In fact, he has shown that every such graph is 3-vertex-colourable.

*Exercises*

9.6.1 Show that a plane graph $G$ is 2-face-colourable if and only if $G$ is eulerian.

9.6.2 Show that a plane triangulation $G$ is 3-vertex colourable if and only if $G$ is eulerian.

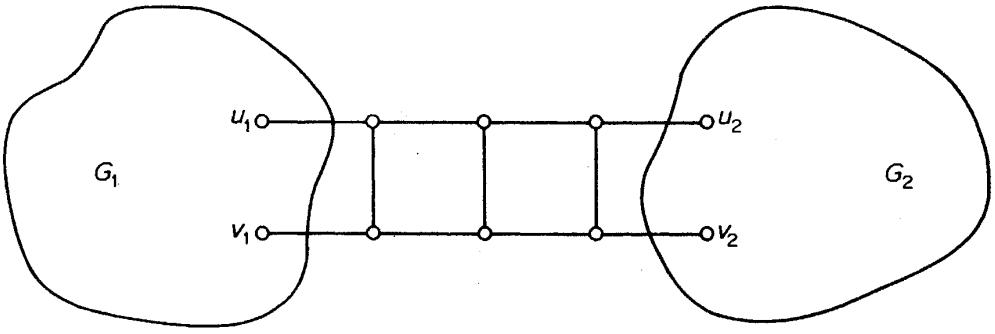
9.6.3 Show that every hamiltonian plane graph is 4-face-colourable.

**9.6.4** Show that every hamiltonian 3-regular graph has a Tait colouring.

9.6.5 Prove theorem 9.12 by showing that (iii)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iii).

**9.6.6** Let $G$ be a 3-regular graph with $\kappa' = 2$.

(*a*) Show that there exist subgraphs $G_1$ and $G_2$ of $G$ and non-adjacent pairs of vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that $G$ consists of the graphs $G_1$ and $G_2$ joined by a 'ladder' at the vertices $u_1$, $v_1$, $u_2$ and $v_2$.




(*b*) Show that if $G_1 + u_1v_1$ and $G_2 + u_2v_2$ both have Tait colourings, then so does $G$.

(*c*) Deduce, using theorem 9.12, that the four-colour conjecture is equivalent to *Tait's conjecture:* every simple 3-regular 3-connected planar graph has a Tait colouring.

9.6.7 Give an example of

(*a*) a 3-regular planar graph with no Tait colouring;

(*b*) a 3-regular 2-connected graph with no Tait colouring.

## 9.7 NONHAMILTONIAN PLANAR GRAPHS

In his attempt to prove the four-colour conjecture, Tait (1880) observed that it would be enough to show that every 3-regular 3-connected planar graph has a Tait colouring (exercise 9.6.6). By mistakenly assuming that every such graph is hamiltonian, he gave a 'proof' of the four-colour conjecture (see exercise 9.6.4). Over half a century later, Tutte (1946) showed Tait's proof to be invalid by constructing a nonhamiltonian 3-regular 3-connected planar graph; it is depicted in figure 9.25.

Tutte proved that his graph is nonhamiltonian by using ingenious *ad hoc* arguments (exercise 9.7.1), and for many years the Tutte graph was the only known example of a nonhamiltonian 3-regular 3-connected planar graph. However, Grinberg (1968) then discovered a necessary condition for a plane graph to be hamiltonian. His discovery has led to the construction of many nonhamiltonian planar graphs.

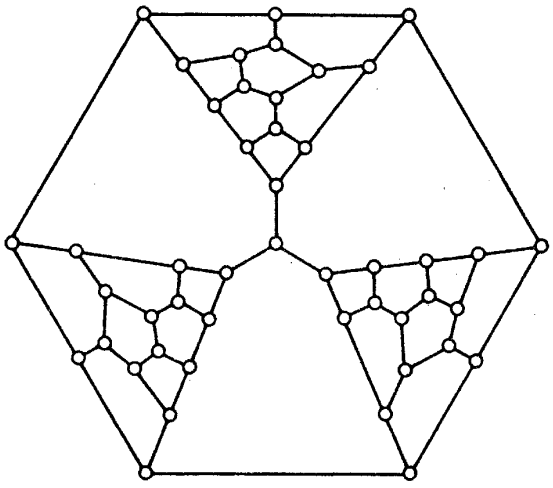


Figure 9.25. The Tutte graph

*Theorem 9.13* Let $G$ be a loopless plane graph with a Hamilton cycle $C$. Then

$$\sum_{i=1}^{\nu} (i-2)(\phi_i' - \phi_i'') = 0 \tag{9.3}$$

where $\phi_i'$ and $\phi_i''$ are the numbers of faces of degree $i$ contained in Int $C$ and Ext $C$, respectively.

*Proof* Denote by $E'$ the subset of $E(G)\backslash E(C)$ contained in Int $C$, and let $\varepsilon' = |E'|$. Then Int $C$ contains exactly $\varepsilon' + 1$ faces (see figure 9.26), and so

$$\sum_{i=1}^{\nu} \phi_i' = \varepsilon' + 1 \tag{9.4}$$

Now each edge in $E'$ is on the boundary of two faces in Int $C$, and each edge

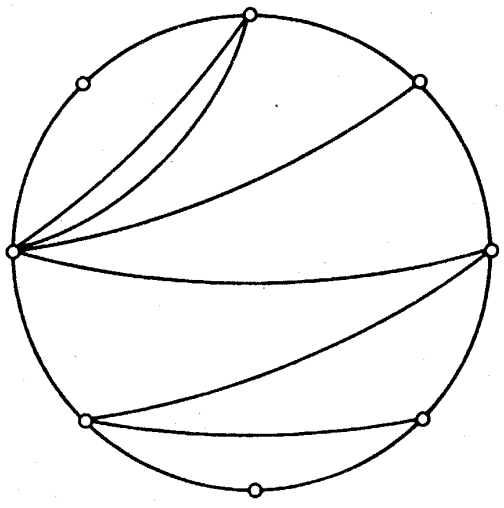


Figure 9.26

of $C$ is on the boundary of exactly one face in Int $C$. Therefore

$$\sum_{i=1}^{\nu} i\phi_i' = 2\varepsilon' + \nu \tag{9.5}$$

Using (9.4), we can eliminate $\varepsilon'$ from (9.5) to obtain

$$\sum_{i=1}^{\nu} (i-2)\phi_i' = \nu - 2 \tag{9.6}$$

Similarly

$$\sum_{i=1}^{\nu} (i-2)\phi_i'' = \nu - 2 \tag{9.7}$$

Equations (9.6) and (9.7) now yield (9.3) □

With the aid of theorem 9.13, it is a simple matter to show, for example, that the Grinberg graph (figure 9.27) is nonhamiltonian.

Suppose that this graph is hamiltonian. Then, noting that it only has faces of degrees five, eight and nine, condition (9.3) yields

$$3(\phi_5' - \phi_5'') + 6(\phi_8' - \phi_8'') + 7(\phi_9' - \phi_9'') = 0$$

We deduce that

$$7(\phi_9' - \phi_9'') \equiv 0 \quad \text{(modulo 3)}$$

But this is clearly impossible, since the value of the left-hand side is 7 or $-7$, depending on whether the face of degree nine is in Int $C$ or Ext $C$. Therefore the graph cannot be hamiltonian.

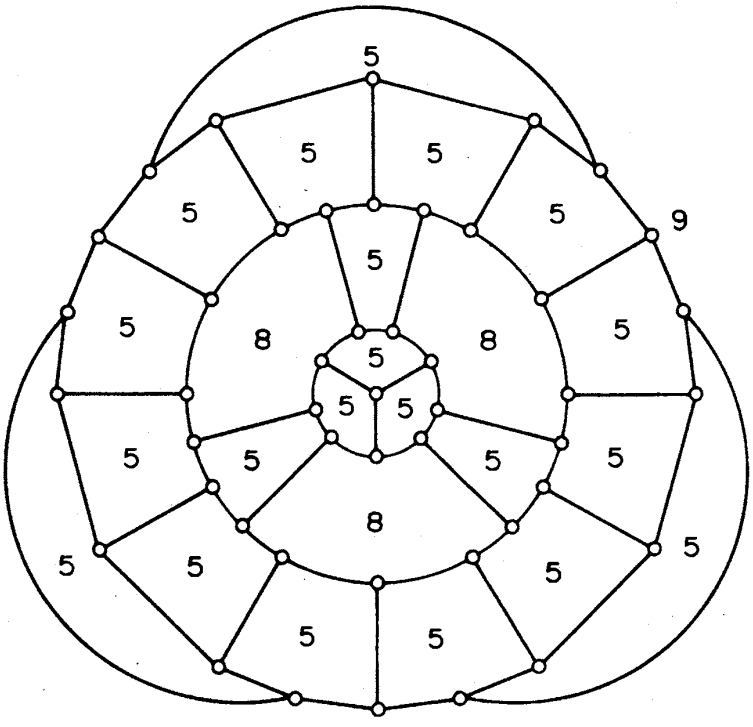



Figure 9.27. The Grinberg graph

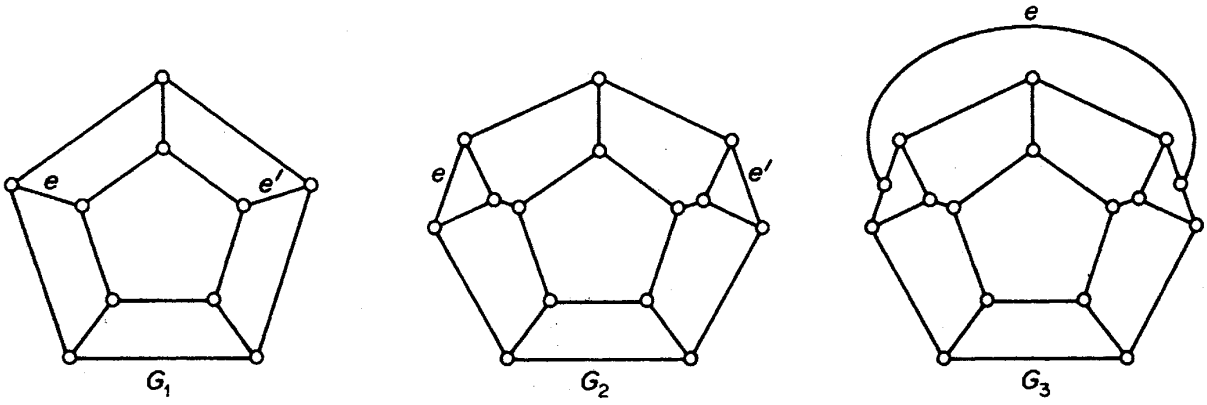
Although there exist nonhamiltonian 3-connected planar graphs, Tutte (1956) has shown that every 4-connected planar graph is hamiltonian.

*Exercises*

9.7.1 (*a*) Show that no Hamilton cycle in the graph $G_1$ below can contain both the edges $e$ and $e'$.

(*b*) Using (*a*), show that no Hamilton cycle in the graph $G_2$ can contain both the edges $e$ and $e'$.

(*c*) Using (*b*), show that every Hamilton cycle in the graph $G_3$ must contain the edge $e$.




(*d*) Deduce that the Tutte graph (figure 9.25) is nonhamiltonian.

9.7.2 Show, by applying theorem 9.13, that the Herschel graph (figure 4.2*b*) is nonhamiltonian. (It is, in fact, the smallest nonhamiltonian 3-connected planar graph.)

9.7.3 Give an example of a simple nonhamiltonian 3-regular planar graph with connectivity two.

## APPLICATIONS

### 9.8 A PLANARITY ALGORITHM

There are many practical situations in which it is important to decide whether a given graph is planar, and, if so, to then find a planar embedding of the graph. For example, in the layout of printed circuits one is interested in knowing if a particular electrical network is planar. In this section, we shall present an algorithm for solving this problem, due to Demoucron, Malgrange and Pertuiset (1964).

Let $H$ be a planar subgraph of a graph $G$ and let $\tilde{H}$ be an embedding of $H$ in the plane. We say that $\tilde{H}$ is *$G$-admissible* if $G$ is planar and there is a planar embedding $\tilde{G}$ of $G$ such that $\tilde{H} \subseteq \tilde{G}$. In figure 9.28, for example, two embeddings of a planar subgraph of $G$ are shown; one is $G$-admissible and the other is not.

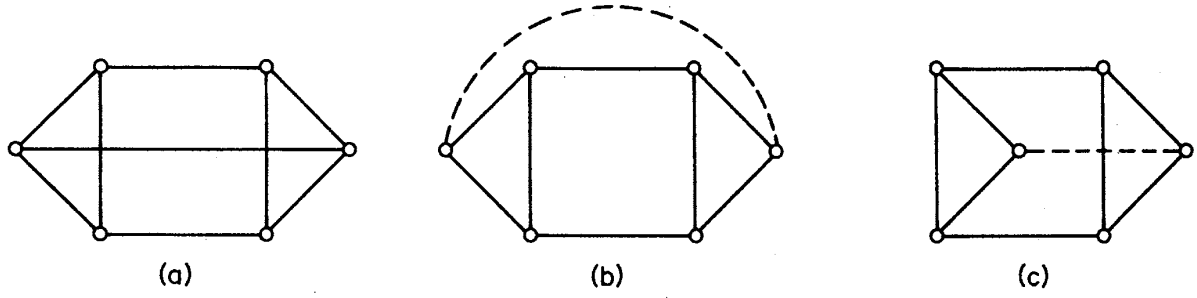



Figure 9.28. (*a*) $G$; (*b*) $G$-admissible; (*c*) $G$-inadmissible

If $B$ is any bridge of $H$ (in $G$), then $B$ is said to be *drawable* in a face $f$ of $\tilde{H}$ if the vertices of attachment of $B$ to $H$ are contained in the boundary of $f$. We write $F(B, \tilde{H})$ for the set of faces of $\tilde{H}$ in which $B$ is drawable. The following theorem provides a necessary condition for $G$ to be planar.

*Theorem 9.14* If $\tilde{H}$ is $G$-admissible then, for every bridge $B$ of $H$, $F(B, \tilde{H}) \neq \emptyset$.

*Proof* If $\tilde{H}$ is $G$-admissible then, by definition, there exists a planar embedding $\tilde{G}$ of $G$ such that $\tilde{H} \subseteq \tilde{G}$. Clearly, the subgraph of $\tilde{G}$ which corresponds to a bridge $B$ of $H$ must be confined to one face of $\tilde{H}$. Hence $F(B, \tilde{H}) \neq \emptyset$ □

Since a graph is planar if and only if each block of its underlying simple graph is planar, it suffices to consider simple blocks. Given such a graph $G$, the algorithm determines an increasing sequence $G_1, G_2, \ldots$ of planar subgraphs of $G$, and corresponding planar embeddings $\tilde{G}_1, \tilde{G}_2, \ldots$. When $G$ is planar, each $\tilde{G}_i$ is $G$-admissible and the sequence $\tilde{G}_1, \tilde{G}_2, \ldots$ terminates in a planar embedding of $G$. At each stage, the necessary condition in theorem 9.14 is used to test $G$ for nonplanarity.

*Planarity Algorithm*

1. Let $G_1$ be a cycle in $G$. Find a planar embedding $\tilde{G}_1$ of $G_1$. Set $i = 1$.
2. If $E(G) \backslash E(G_i) = \emptyset$, stop. Otherwise, determine all bridges of $G_i$ in $G$; for each such bridge $B$ find the set $F(B, \tilde{G}_i)$.
3. If there exists a bridge $B$ such that $F(B, \tilde{G}_i) = \emptyset$, stop; by theorem 9.14, $G$ is nonplanar. If there exists a bridge $B$ such that $|F(B, \tilde{G}_i)| = 1$, let $\{f\} = F(B, \tilde{G}_i)$. Otherwise, let $B$ be any bridge and $f$ any face such that $f \in F(B, \tilde{G}_i)$.
4. Choose a path $P_i \subseteq B$ connecting two vertices of attachment of $B$ to $G_i$. Set $G_{i+1} = G_i \cup P_i$ and obtain a planar embedding $\tilde{G}_{i+1}$ of $G_{i+1}$ by drawing $P_i$ in the face $f$ of $\tilde{G}_i$. Replace $i$ by $i+1$ and go to step 2.

To illustrate this algorithm, we shall consider the graph $G$ of figure 9.29. We start with the cycle $\tilde{G}_1 = 2345672$ and a list of its bridges (denoted, for

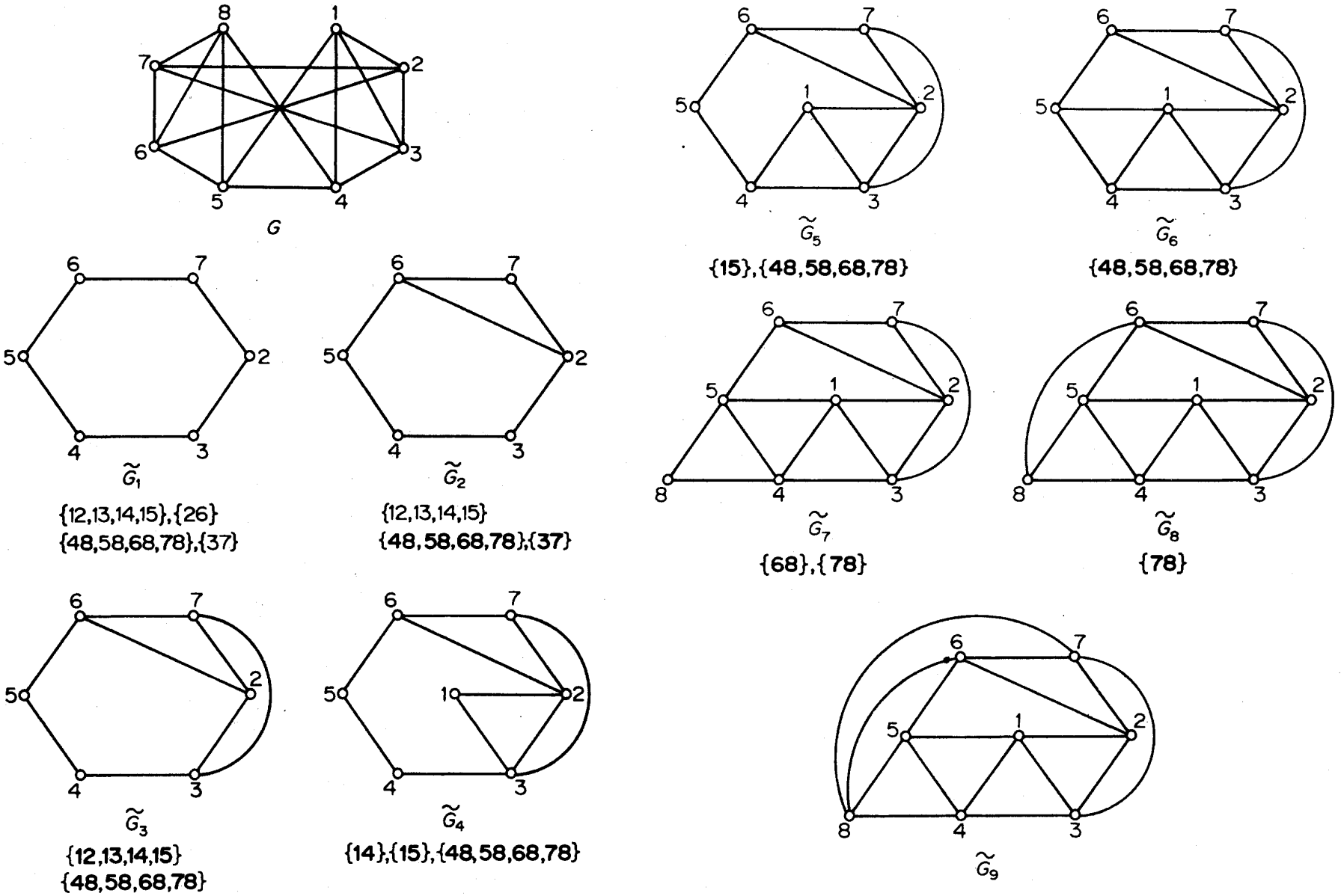

G
$\tilde{G}_1$
{12,13,14,15},{26}
{48,58,68,78},{37}
$\tilde{G}_2$
{12,13,14,15}
{48,58,68,78},{37}
$\tilde{G}_3$
{12,13,14,15}
{48,58,68,78}
$\tilde{G}_4$
{14},{15},{48,58,68,78}
$\tilde{G}_5$
{15},{48,58,68,78}
$\tilde{G}_6$
{48,58,68,78}
$\tilde{G}_7$
{68},{78}
$\tilde{G}_8$
{78}
$\tilde{G}_9$

Figure 9.29

brevity, by their edge sets); at each stage, the bridges $B$ for which $|F(B, \tilde{G}_i)| = 1$ are indicated in bold face. In this example, the algorithm terminates with a planar embedding $\tilde{G}_9$ of $G$. Thus $G$ is planar.

Now let us apply the algorithm to the graph $H$ obtained from $G$ by deleting edge 45 and adding edge 36 (figure 9.30). Starting with the cycle 23672, we proceed as shown in figure 9.30. It can be seen that, having constructed $\tilde{H}_3$, we find a bridge $B = \{12, 13, 14, 15, 34, 48, 56, 58, 68, 78\}$

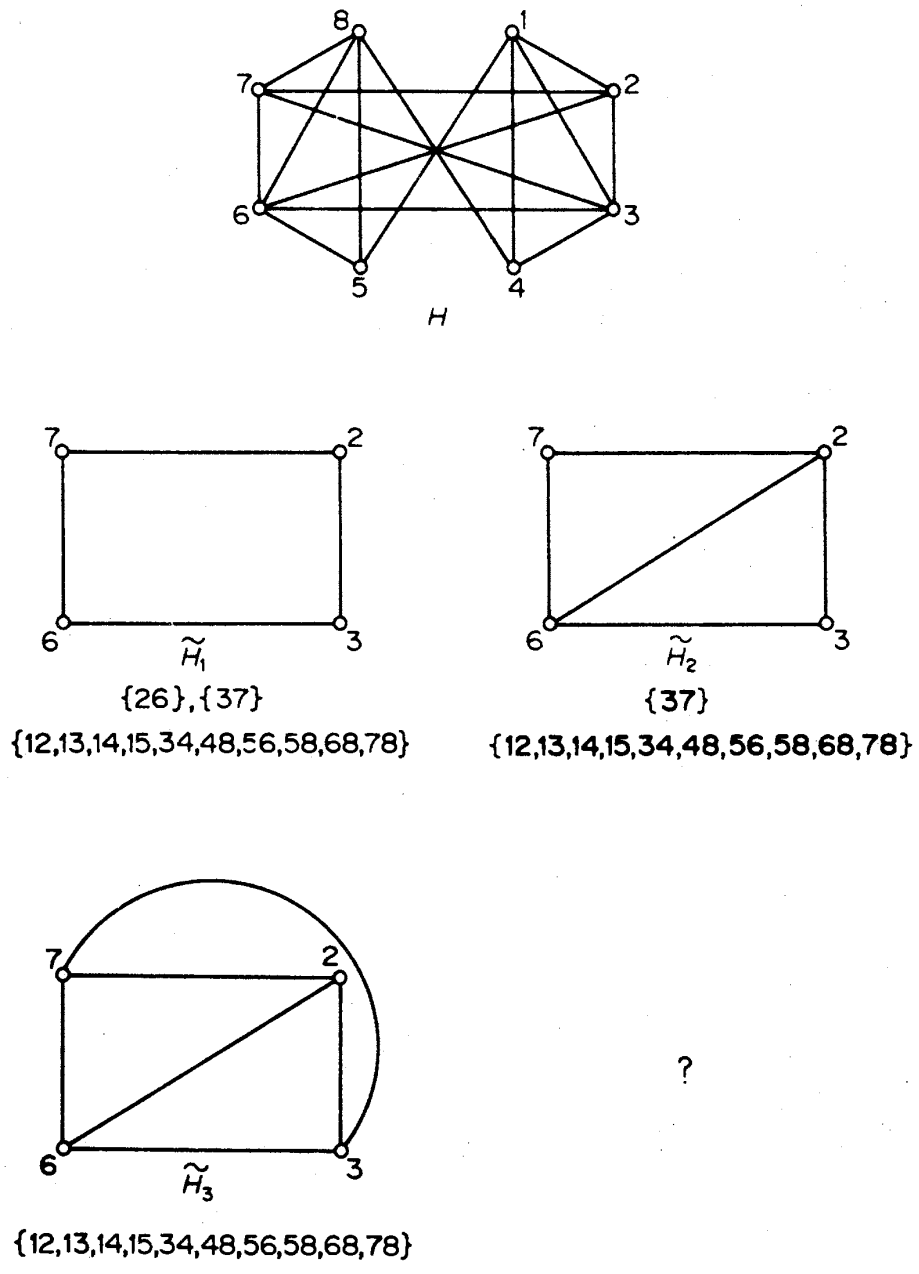



Figure 9.30

such that $F(B, \tilde{H}_3) = \emptyset$. At this point the algorithm stops (step 3), and we conclude that $H$ is nonplanar.

In order to establish the validity of the algorithm, one needs to show that if $G$ is planar, then each term of the sequence $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_{\varepsilon-\nu+1}$ is $G$-admissible. Demoucron, Malgrange and Pertuiset prove this by induction. We shall give a general outline of their proof.

Suppose that $G$ is planar. Clearly $\tilde{G}_1$ is $G$-admissible. Assume that $\tilde{G}_i$ is $G$-admissible for $1 \leq i \leq k < \varepsilon - \nu + 1$. By definition, there is a planar embedding $\tilde{G}$ of $G$ such that $\tilde{G}_k \subset \tilde{G}$. We wish to show that $\tilde{G}_{k+1}$ is $G$-admissible. Let $B$ and $f$ be as defined in step 3 of the algorithm. If, in $\tilde{G}$, $B$ is drawn in $f$, $\tilde{G}_{k+1}$ is clearly $G$-admissible. So assume that no bridge of $G_k$ is drawable in only one face of $\tilde{G}_k$, and that, in $\tilde{G}$, $B$ is drawn in some other face $f'$. Since no bridge is drawable in just one face, no bridge whose vertices of attachment are restricted to the common boundary of $f$ and $f'$ can be skew to a bridge not having this property. Hence we can interchange bridges across the common boundary of $f$ and $f'$ and thereby obtain a planar embedding of $G$ in which $B$ is drawn in $f$ (see figure 9.31). Thus, again, $\tilde{G}_{k+1}$ is $G$-admissible.

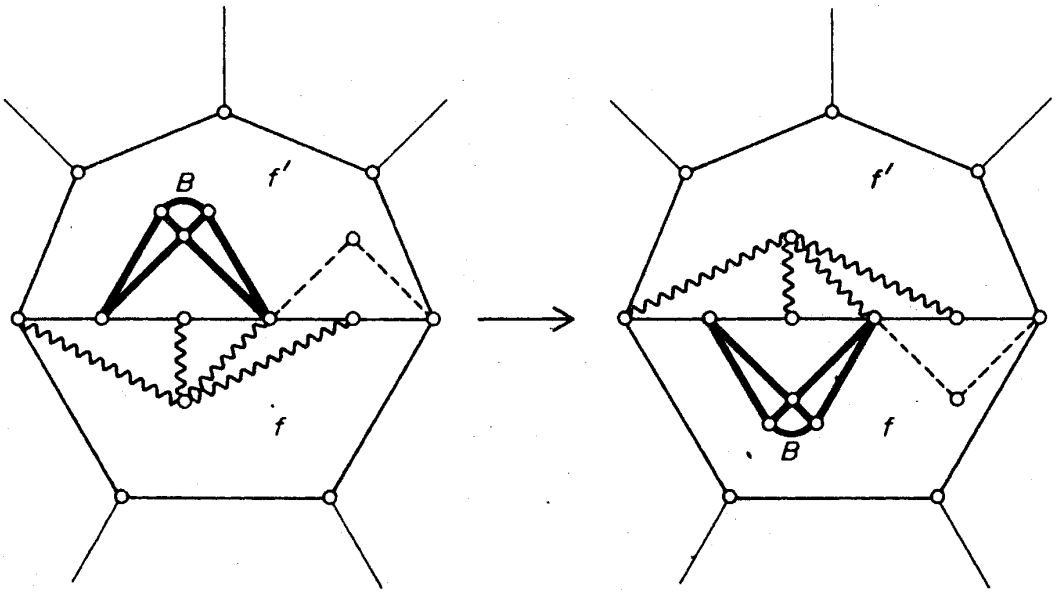



Figure 9.31

The algorithm that we have described is good. From the flow diagram (figure 9.32), one sees that the main operations involved are

(i) finding a cycle $G_1$ in the block $G$;
(ii) determining the bridges of $G_i$ in $G$ and their vertices of attachment to $G_i$;

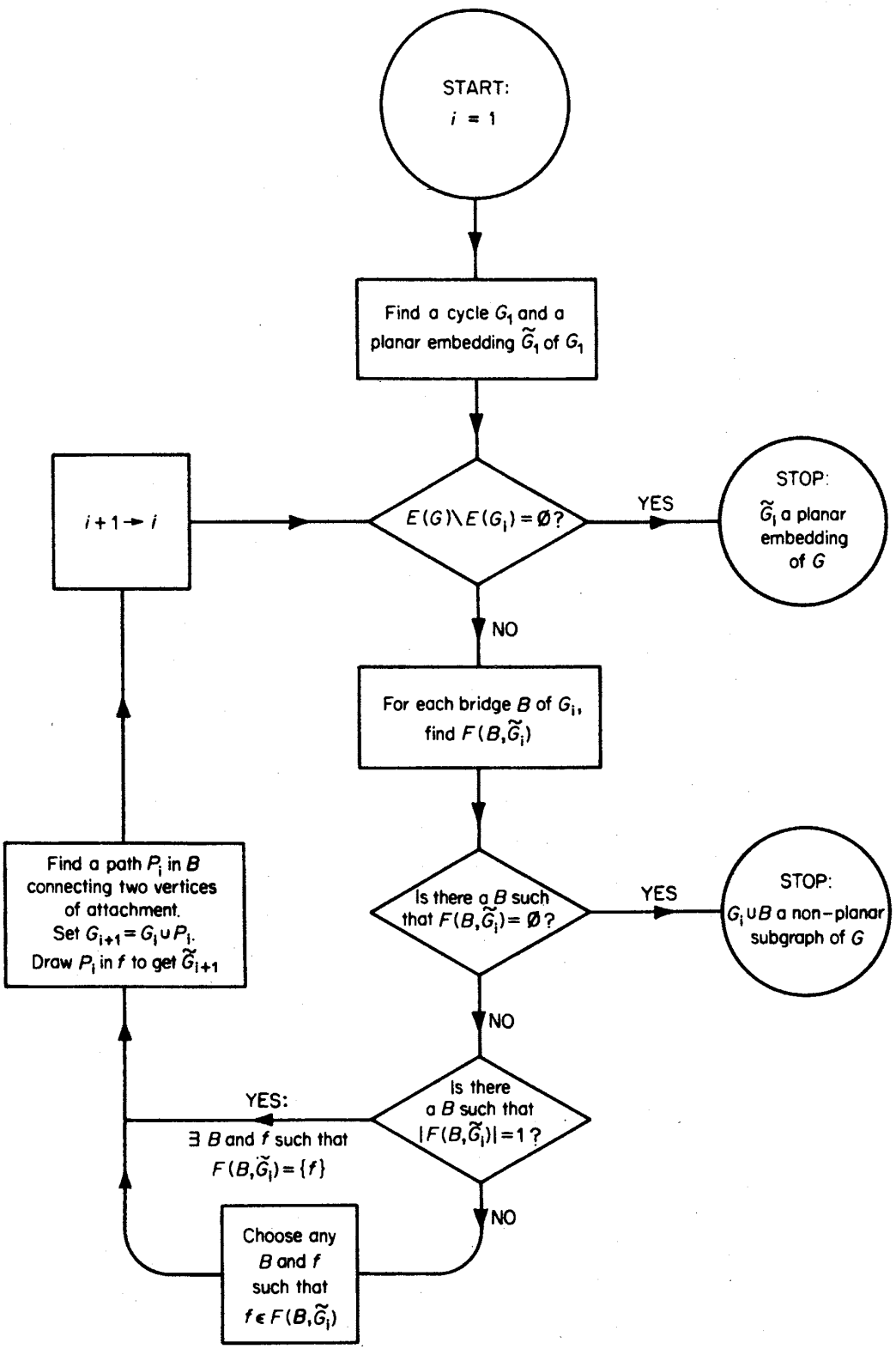



**Figure 9.32. Planarity algorithm**

(iii) determining $b(f)$ for each face $f$ of $\tilde{G}_i$;
(iv) determining $F(B, \tilde{G}_i)$ for each bridge $B$ of $G_i$;
(v) finding a path $P_i$ in some bridge $B$ of $G_i$ between two vertices of $V(B, G_i)$.

There exists a good algorithm for each of these operations; we leave the details as an exercise.

More sophisticated algorithms for testing planarity than the above have since been obtained. See, for example, Hopcroft and Tarjan (1974).

*Exercise*

9.8.1 Show that the Petersen graph is nonplanar by applying the above algorithm.

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# 10 Directed Graphs

## 10.1 DIRECTED GRAPHS

Although many problems lend themselves naturally to a graph-theoretic formulation, the concept of a graph is sometimes not quite adequate. When dealing with problems of traffic flow, for example, it is necessary to know which roads in the network are one-way, and in which direction traffic is permitted. Clearly, a graph of the network is not of much use in such a situation. What we need is a graph in which each link has an assigned orientation—a directed graph. Formally, a *directed graph* $D$ is an ordered triple $(V(D), A(D), \psi_D)$ consisting of a nonempty set $V(D)$ of *vertices*, a set $A(D)$, disjoint from $V(D)$, of *arcs*, and an *incidence function* $\psi_D$ that associates with each arc of $D$ an ordered pair of (not necessarily distinct) vertices of $D$. If $a$ is an arc and $u$ and $v$ are vertices such that $\psi_D(a) = (u, v)$, then $a$ is said to *join* $u$ *to* $v$; $u$ is the *tail* of $a$, and $v$ is its *head*. For convenience, we shall abbreviate 'directed graph' to *digraph*. A digraph $D'$ is a *subdigraph* of $D$ if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$ and $\psi_{D'}$ is the restriction of $\psi_D$ to $A(D')$. The terminology and notation for subdigraphs is similar to that used for subgraphs.

With each digraph $D$ we can associate a graph $G$ on the same vertex set; corresponding to each arc of $D$ there is an edge of $G$ with the same ends. This graph is the *underlying graph* of $D$. Conversely, given any graph $G$, we can obtain a digraph from $G$ by specifying, for each link, an order on its ends. Such a digraph is called an *orientation* of $G$.

Just as with graphs, digraphs have a simple pictorial representation. A digraph is represented by a diagram of its underlying graph together with arrows on its edges, each arrow pointing towards the head of the corresponding arc. A digraph and its underlying graph are shown in figure 10.1.

Every concept that is valid for graphs automatically applies to digraphs too. Thus the digraph of figure 10.1*a* is connected and has no cycle of length three because its underlying graph (figure 10.1*b*) has these properties. However, there are many concepts that involve the notion of orientation, and these apply only to digraphs.

A *directed walk* in $D$ is a finite non-null sequence $W = (v_0, a_1, v_1, \ldots, a_k, v_k)$, whose terms are alternately vertices and arcs, such that, for $i = 1, 2, \ldots, k$, the arc $a_i$ has head $v_i$ and tail $v_{i-1}$. As with walks in graphs, a directed walk $(v_0, a_1, v_1, \ldots, a_k, v_k)$ is often represented simply by

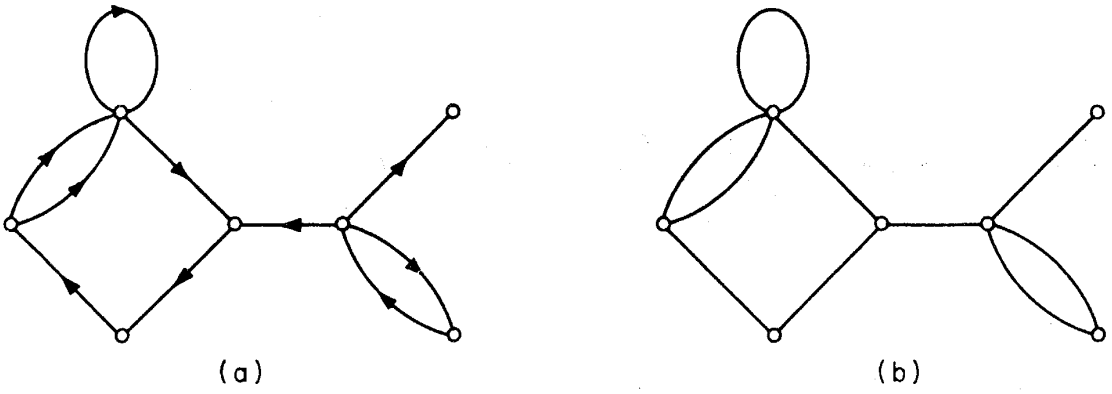



Figure 10.1. (*a*) A digraph $D$; (*b*) the underlying graph of $D$

its vertex sequence $(v_0, v_1, \ldots, v_k)$. A *directed trail* is a directed walk that is a trail; *directed paths*, *directed cycles* and *directed tours* are similarly defined.

If there is a directed $(u, v)$-path in $D$, vertex $v$ is said to be *reachable* from vertex $u$ in $D$. Two vertices are *diconnected* in $D$ if each is reachable from the other. As in the case of connection in graphs, diconnection is an equivalence relation on the vertex set of $D$. The subdigraphs $D[V_1], D[V_2], \ldots, D[V_m]$ induced by the resulting partition $(V_1, V_2, \ldots, V_m)$ of $V(D)$ are called the *dicomponents* of $D$. A digraph $D$ is *diconnected* if it has exactly one dicomponent. The digraph of figure 10.2*a* is not diconnected; it has the three dicomponents shown in figure 10.2*b*.

The *indegree* $d_D^-(v)$ of a vertex $v$ in $D$ is the number of arcs with head $v$; the *outdegree* $d_D^+(v)$ of $v$ is the number of arcs with tail $v$. We denote the minimum and maximum indegrees and outdegrees in $D$ by $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$ and $\Delta^+(D)$, respectively. A digraph is *strict* if it has no loops and no two arcs with the same ends have the same orientation.

Throughout this chapter, $D$ will denote a digraph and $G$ its underlying graph. This is a useful convention; it allows us, for example, to denote the vertex set of $D$ by $V$ (since $V = V(G)$), and the numbers of vertices and arcs in $D$ by $\nu$ and $\varepsilon$, respectively. Also, as with graphs, we shall drop the letter $D$ from our notation whenever possible; thus we write $A$ for $A(D)$, $d^+(v)$ for $d_D^+(v)$, $\delta^-$ for $\delta^-(D)$, and so on.

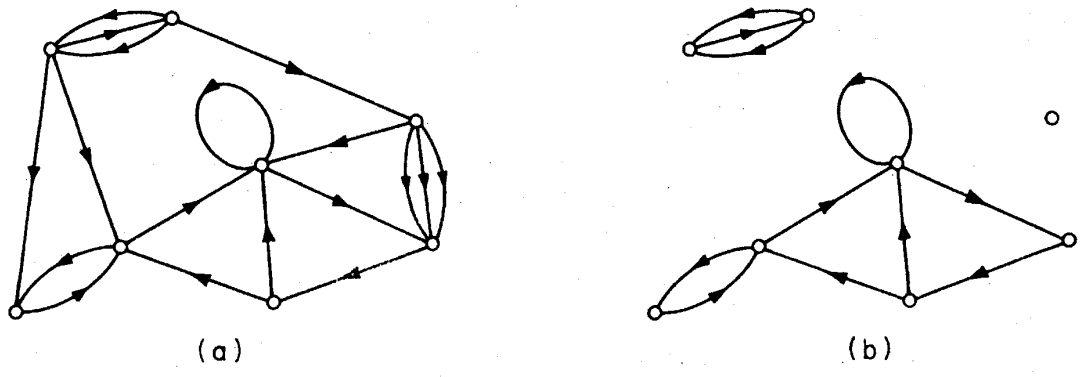



Figure 10.2. (*a*) A digraph $D$; (*b*) the three dicomponents of $D$

*Exercises*

10.1.1 How many orientations does a simple graph $G$ have?

10.1.2 Show that $\sum_{v \in V} d^-(v) = \varepsilon = \sum_{v \in V} d^+(v)$

**10.1.3** Let $D$ be a digraph with no directed cycle.

(*a*) Show that $\delta^- = 0$.

(*b*) Deduce that there is an ordering $v_1, v_2, \ldots, v_\nu$ of $V$ such that, for $1 \leq i \leq \nu$, every arc of $D$ with head $v_i$ has its tail in $\{v_1, v_2, \ldots, v_{i-1}\}$.

10.1.4 Show that $D$ is diconnected if and only if $D$ is connected and each block of $D$ is diconnected.

10.1.5 The *converse* $\overleftarrow{D}$ of $D$ is the digraph obtained from $D$ by reversing the orientation of each arc.

(*a*) Show that

(i) $\overleftarrow{\overleftarrow{D}} = D$;

(ii) $d^+_{\overleftarrow{D}}(v) = d^-_D(v)$;

(iii) $v$ is reachable from $u$ in $\overleftarrow{D}$ if and only if $u$ is reachable from $v$ in $D$.

(*b*) By using part (ii) of (*a*), deduce from exercise 10.1.3*a* that if $D$ is a digraph with no directed cycle, then $\delta^+ = 0$.

10.1.6 Show that if $D$ is strict, then $D$ contains a directed path of length at least $\max\{\delta^-, \delta^+\}$.

**10.1.7** Show that if $D$ is strict and $\max\{\delta^-, \delta^+\} = k > 0$, then $D$ contains a directed cycle of length at least $k+1$.

**10.1.8** Let $v_1, v_2, \ldots, v_\nu$ be the vertices of a digraph $D$. The *adjacency matrix* of $D$ is the $\nu \times \nu$ matrix $\mathbf{A} = [a_{ij}]$ in which $a_{ij}$ is the number of arcs of $D$ with tail $v_i$ and head $v_j$. Show that the $(i, j)$th entry of $\mathbf{A}^k$ is the number of directed $(v_i, v_j)$-walks of length $k$ in $D$.

**10.1.9** Let $D_1, D_2, \ldots, D_m$ be the dicomponents of $D$. The *condensation* $\hat{D}$ of $D$ is a directed graph with $m$ vertices $w_1, w_2, \ldots, w_m$; there is an arc in $\hat{D}$ with tail $w_i$ and head $w_j$ if and only if there is an arc in $D$ with tail in $D_i$ and head in $D_j$. Show that the condensation $\hat{D}$ of $D$ contains no directed cycle.

10.1.10 Show that $G$ has an orientation $D$ such that $|d^+(v) - d^-(v)| \leq 1$ for all $v \in V$.

## 10.2 DIRECTED PATHS

There is no close relationship between the lengths of paths and directed paths in a digraph. That this is so is clear from the digraph of figure 10.3, which has no directed path of length greater than one.

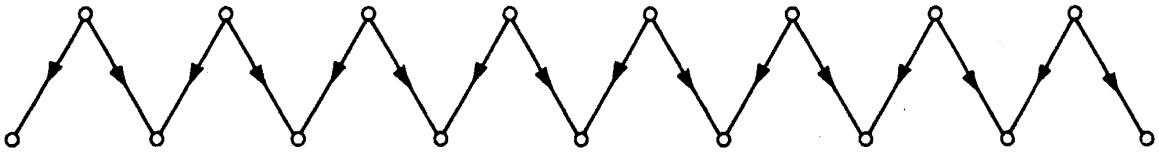


Figure 10.3

Surprisingly, some information about the lengths of directed paths in a digraph can be obtained by looking at its chromatic number. The following theorem, due to Roy (1967) and Gallai (1968), makes this precise.

*Theorem 10.1* A digraph $D$ contains a directed path of length $\chi - 1$.

*Proof* Let $A'$ be a minimal set of arcs of $D$ such that $D' = D - A'$ contains no directed cycle, and let the length of a longest directed path in $D'$ be $k$. Now assign colours $1, 2, \ldots, k+1$ to the vertices of $D'$ by assigning colour $i$ to vertex $v$ if the length of a longest directed path in $D'$ with origin $v$ is $i-1$. Denote by $V_i$ the set of vertices with colour $i$. We shall show that $(V_1, V_2, \ldots, V_{k+1})$ is a proper $(k+1)$-vertex colouring of $D$.

First, observe that the origin and terminus of any directed path in $D'$ have different colours. For let $P$ be a directed $(u, v)$-path of positive length in $D'$ and suppose $v \in V_i$. Then there is a directed path $Q = (v_1, v_2, \ldots, v_i)$ in $D'$, where $v_1 = v$. Since $D'$ contains no directed cycle, $PQ$ is a directed path with origin $u$ and length at least $i$. Thus $u \notin V_i$.

We can now show that the ends of any arc of $D$ have different colours. Suppose $(u, v) \in A(D)$. If $(u, v) \in A(D')$ then $(u, v)$ is a directed path in $D'$ and so $u$ and $v$ have different colours. Otherwise, $(u, v) \in A'$. By the minimality of $A'$, $D' + (u, v)$ contains a directed cycle $C$. $C - (u, v)$ is a directed $(v, u)$-path in $D'$ and hence in this case, too, $u$ and $v$ have different colours.

Thus $(V_1, V_2, \ldots, V_{k+1})$ is a proper vertex colouring of $D$. It follows that $\chi \leq k+1$, and so $D$ has a directed path of length $k \geq \chi - 1$ □

Theorem 10.1 is best possible in that every graph $G$ has an orientation in which the longest directed path is of length $\chi - 1$. Given a proper $\chi$-vertex colouring $(V_1, V_2, \ldots, V_\chi)$ of $G$, we orient $G$ by converting edge $uv$ to arc $(u, v)$ if $u \in V_i$ and $v \in V_j$ with $i < j$. Clearly, no directed path in this orientation of $G$ can contain more than $\chi$ vertices, since no two vertices of the path can have the same colour.

An orientation of a complete graph is called a *tournament*. The tournaments on four vertices are shown in figure 10.4. Each can be regarded as indicating the results of games in a round-robin tournament between four players; for example, the first tournament in figure 10.4 shows that one player has won all three games and that the other three have each won one.

A *directed Hamilton path* of $D$ is a directed path that includes every

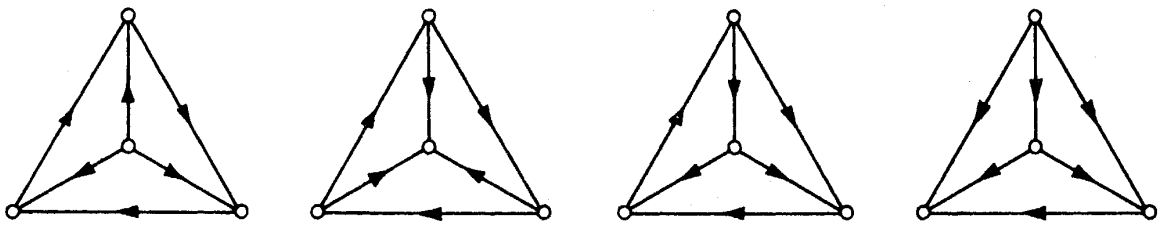


Figure 10.4. The tournaments on four vertices

vertex of $D$. An immediate corollary of theorem 10.1 is that every tournament has such a path. This was first proved by Rédei (1934).

*Corollary 10.1* Every tournament has a directed Hamilton path.

*Proof* If $D$ is a tournament, then $\chi = \nu$ □

Another interesting fact about tournaments is that there is always a vertex from which every other vertex can be reached in at most two steps. We shall obtain this as a special case of a theorem of Chvátal and Lovász (1974). An *in-neighbour* of a vertex $v$ in $D$ is a vertex $u$ such that $(u, v) \in A$; an *out-neighbour* of $v$ is a vertex $w$ such that $(v, w) \in A$. We denote the sets of in-neighbours and out-neighbours of $v$ in $D$ by $N_D^-(v)$ and $N_D^+(v)$, respectively.

*Theorem 10.2* A loopless digraph $D$ has an independent set $S$ such that each vertex of $D$ not in $S$ is reachable from a vertex in $S$ by a directed path of length at most two.

*Proof* By induction on $\nu$. The theorem holds trivially for $\nu = 1$. Assume that it is true for all digraphs with fewer than $\nu$ vertices, and let $v$ be an arbitrary vertex of $D$. By the induction hypothesis there exists in $D' = D - (\{v\} \cup N^+(v))$ an independent set $S'$ such that each vertex of $D'$ not in $S'$ is reachable from a vertex in $S'$ by a directed path of length at most two. If $v$ is an out-neighbour of some vertex $u$ of $S'$, then every vertex of $N^+(v)$ is reachable from $u$ by a directed path of length two. Hence, in this case, $S = S'$ satisfies the required property. If, on the other hand, $v$ is not an out-neighbour of any vertex of $S'$, then $v$ is joined to no vertex of $S'$ and the independent set $S = S' \cup \{v\}$ has the required property □

*Corollary 10.2* A tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two.

*Proof* If $D$ is a tournament, then $\alpha = 1$ □

*Exercises*

10.2.1 Show that every tournament is either diconnected or can be transformed into a diconnected tournament by the reorientation of just one arc.

10.2.2* A digraph $D$ is *unilateral* if, for any two vertices $u$ and $v$, either $v$ is reachable from $u$ or $u$ is reachable from $v$. Show that $D$ is unilateral if and only if $D$ has a spanning directed walk.

10.2.3 (*a*) Let $P=(v_1, v_2, \ldots, v_k)$ be a maximal directed path in a tournament $D$. Suppose that $P$ is not a directed Hamilton path and let $v$ be any vertex not on $P$. Show that, for some $i$, both $(v_i, v)$ and $(v, v_{i+1})$ are arcs of $D$.

(*b*) Deduce Rédei's theorem.

10.2.4 Prove corollary 10.2 by considering a vertex of maximum outdegree.

10.2.5* (*a*) Let $D$ be a digraph with $\chi > mn$, and let $f$ be a real-valued function defined on $V$. Show that $D$ has either a directed path $(u_0, u_1, \ldots, u_m)$ with $f(u_0) \leq f(u_1) \leq \ldots \leq f(u_m)$ or a directed path $(v_0, v_1, \ldots, v_n)$ with $f(v_0) > f(v_1) > \ldots > f(v_n)$.

(V. Chvátal and J. Komlós)

(*b*) Deduce that any sequence of $mn+1$ distinct integers contains either an increasing subsequence of $m$ terms or a decreasing subsequence of $n$ terms. (P. Erdös and G. Szekeres)

10.2.6 (*a*) Using theorem 10.1 and corollary 8.1.2, show that $G$ has an orientation in which each directed path is of length at most $\Delta$.

(*b*) Give a constructive proof of (*a*).

## 10.3 DIRECTED CYCLES

Corollary 10.1 tells us that every tournament contains a directed Hamilton path. Much stronger conclusions can be drawn, however, if the tournament is assumed to be diconnected. The following theorem is due to Moon (1966). If $S$ and $T$ are subsets of $V$, we denote by $(S, T)$ the set of arcs of $D$ that have their tails in $S$ and their heads in $T$.

*Theorem 10.3* Each vertex of a diconnected tournament $D$ with $\nu \geq 3$ is contained in a directed $k$-cycle, $3 \leq k \leq \nu$.

*Proof* Let $D$ be a diconnected tournament with $\nu \geq 3$, and let $u$ be any vertex of $D$. Set $S = N^+(u)$ and $T = N^-(u)$. We first show that $u$ is in a directed 3-cycle. Since $D$ is diconnected, neither $S$ nor $T$ can be empty; and, for the same reason, $(S, T)$ must be nonempty (see figure 10.5). There is thus some arc $(v, w)$ in $D$ with $v \in S$ and $w \in T$, and $u$ is in the directed 3-cycle $(u, v, w, u)$.

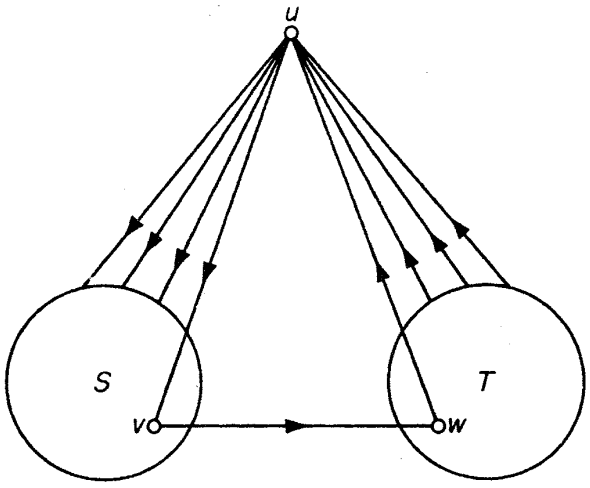



Figure 10.5

The theorem is now proved by induction on $k$. Suppose that $u$ is in directed cycles of all lengths between 3 and $n$, where $n < \nu$. We shall show that $u$ is in a directed $(n+1)$-cycle.

Let $C = (v_0, v_1, \ldots, v_n)$ be a directed $n$-cycle in which $v_0 = v_n = u$. If there is a vertex $v$ in $V(D)\backslash V(C)$ which is both the head of an arc with tail in $C$ and the tail of an arc with head in $C$, then there are adjacent vertices $v_i$ and $v_{i+1}$ on $C$ such that both $(v_i, v)$ and $(v, v_{i+1})$ are arcs of $D$. In this case $u$ is in the directed $(n+1)$-cycle $(v_0, v_1, \ldots, v_i, v, v_{i+1}, \ldots, v_n)$.

Otherwise, denote by $S$ the set of vertices in $V(D)\backslash V(C)$ which are heads of arcs joined to $C$, and by $T$ the set of vertices in $V(D)\backslash V(C)$ which are tails of arcs joined to $C$ (see figure 10.6).

As before, since $D$ is diconnected, $S$, $T$ and $(S, T)$ are all nonempty, and there is some arc $(v, w)$ in $D$ with $v \in S$ and $w \in T$. Hence $u$ is in the directed $(n+1)$-cycle $(v_0, v, w, v_2, \ldots, v_n)$ □

A *directed Hamilton cycle* of $D$ is a directed cycle that includes every vertex of $D$. It follows from theorem 10.3 (and was first proved by Camion, 1959) that every diconnected tournament contains such a cycle. The next

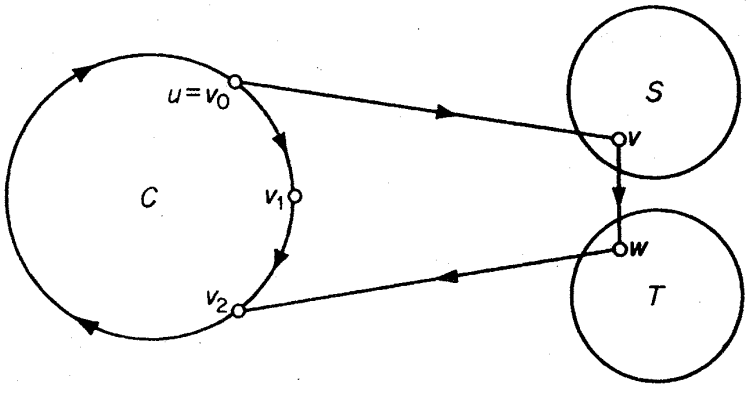



Figure 10.6

theorem extends Dirac's theorem (4.3) to digraphs. It is a special case of a theorem due to Ghouila-Houri (1960).

*Theorem 10.4* If $D$ is strict and $\min\{\delta^-, \delta^+\} \geq \nu/2 > 1$, then $D$ contains a directed Hamilton cycle.

*Proof* Suppose that $D$ satisfies the hypotheses of the theorem, but does not contain a directed Hamilton cycle. Denote the length of a longest directed cycle in $D$ by $l$, and let $C = (v_1, v_2, \ldots, v_l, v_1)$ be a directed cycle in $D$ of length $l$. We note that $l > \nu/2$ (exercise 10.1.7). Let $P$ be a longest directed path in $D - V(C)$ and suppose that $P$ has origin $u$, terminus $v$ and length $m$ (see figure 10.7). Clearly

$$\nu \geq l + m + 1 \tag{10.1}$$

and, since $l > \nu/2$,

$$m < \nu/2 \tag{10.2}$$

Set

$$S = \{i \mid (v_{i-1}, u) \in A\} \quad \text{and} \quad T = \{i \mid (v, v_i) \in A\}$$

We first show that $S$ and $T$ are disjoint. Let $C_{j,k}$ denote the section of $C$ with origin $v_j$ and terminus $v_k$. If some integer $i$ were in both $S$ and $T$, $D$ would contain the directed cycle $C_{i,i-1}(v_{i-1}, u)P(v, v_i)$ of length $l + m + 1$, contradicting the choice of $C$. Thus

$$S \cap T = \emptyset \tag{10.3}$$

Now, because $P$ is a maximal directed path in $D - V(C)$, $N^-(u) \subseteq V(P) \cup V(C)$. But the number of in-neighbours of $u$ in $C$ is precisely $|S|$ and so $d_D^-(u) = d_P^-(u) + |S|$. Since $d_D^-(u) \geq \delta^- \geq \nu/2$ and $d_P^-(u) \leq m$,

$$|S| \geq \nu/2 - m \tag{10.4}$$

A similar argument yields

$$|T| \geq \nu/2 - m \tag{10.5}$$

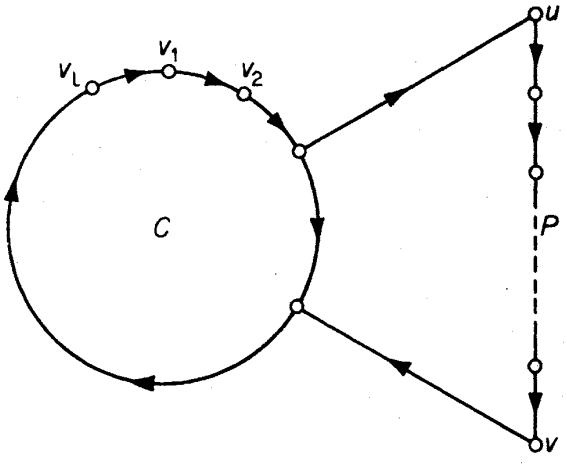



Figure 10.7

Note that, by (10.2), both $S$ and $T$ are nonempty. Adding (10.4) and (10.5) and using (10.1), we obtain

$$|S|+|T| \geq l-m+1$$

and therefore, by (10.3),

$$|S \cup T| \geq l-m+1 \tag{10.6}$$

Since $S$ and $T$ are disjoint and nonempty, there are positive integers $i$ and $k$ such that $i \in S$, $i+k \in T$ and

$$i+j \notin S \cup T \quad \text{for} \quad 1 \leq j < k \tag{10.7}$$

where addition is taken modulo $l$.

From (10.6) and (10.7) we see that $k \leq m$. Thus the directed cycle $C_{i+k,i-1}(v_{i-1}, u)P(v, v_{i+k})$, which has length $l+m+1-k$, is longer than $C$. This contradiction establishes the theorem □

*Exercises*

10.3.1 Show how theorem 4.3 can be deduced from theorem 10.4.

**10.3.2** A *directed Euler tour* of $D$ is a directed tour that traverses each arc of $D$ exactly once. Show that $D$ contains a directed Euler tour if and only if $D$ is connected and $d^+(v) = d^-(v)$ for all $v \in V$.

**10.3.3** Let $D$ be a digraph such that

(i) $d^+(x) - d^-(x) = l = d^-(y) - d^+(y)$;
(ii) $d^+(v) = d^-(v)$ for $v \in V \backslash \{x, y\}$.

Show, using exercise 10.3.2, that there exist $l$ arc-disjoint directed $(x, y)$-paths in $D$.

10.3.4* Show that a diconnected digraph which contains an odd cycle, also contains a directed odd cycle.

10.3.5 A nontrivial digraph $D$ is *$k$-arc-connected* if, for every nonempty proper subset $S$ of $V$, $|(S, \bar{S})| \geq k$. Show that a nontrivial digraph is diconnected if and only if it is 1-arc-connected.

**10.3.6** The *associated digraph* $D(G)$ of a graph $G$ is the digraph obtained when each edge $e$ of $G$ is replaced by two oppositely oriented arcs with the same ends as $e$. Show that

(*a*) there is a one-one correspondence between paths in $G$ and directed paths in $D(G)$;
(*b*) $D(G)$ is $k$-arc-connected if and only if $G$ is $k$-edge-connected.

## APPLICATIONS

### 10.4 A JOB SEQUENCING PROBLEM

A number of jobs $J_1, J_2, \ldots, J_n$, have to be processed on one machine; for example, each $J_i$ might be an order of bottles or jars in a glass factory. After

each job, the machine must be adjusted to fit the requirements of the next job. If the time of adaptation from job $J_i$ to job $J_j$ is $t_{ij}$, find a sequencing of the jobs that minimises the total machine adjustment time.

This problem is clearly related to the travelling salesman problem, and no efficient method for its solution is known. It is therefore desirable to have a method for obtaining a reasonably good (but not necessarily optimal) solution. Our method makes use of Rédei's theorem (corollary 10.1).

*Step 1* Construct a digraph $D$ with vertices $v_1, v_2, \ldots, v_n$, such that $(v_i, v_j) \in A$ if and only if $t_{ij} \leq t_{ji}$. By definition, $D$ contains a spanning tournament.

*Step 2* Find a directed Hamilton path $(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ of $D$ (exercise 10.4.1), and sequence the jobs accordingly.

Since step 1 discards the larger half of the adjustment matrix $[t_{ij}]$, it is a reasonable supposition that this method, in general, produces a fairly good job sequence. Note, however, that when the adjustment matrix is symmetric, the method is of no help whatsoever.

As an example, suppose that there are six jobs $J_1$, $J_2$, $J_3$, $J_4$, $J_5$ and $J_6$ and that the adjustment matrix is

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ |
|---|---|---|---|---|---|---|
| $J_1$ | 0 | 5 | 3 | 4 | 2 | 1 |
| $J_2$ | 1 | 0 | 1 | 2 | 3 | 2 |
| $J_3$ | 2 | 5 | 0 | 1 | 2 | 3 |
| $J_4$ | 1 | 4 | 4 | 0 | 1 | 2 |
| $J_5$ | 1 | 3 | 4 | 5 | 0 | 5 |
| $J_6$ | 4 | 4 | 2 | 3 | 1 | 0 |

The sequence $J_1 \to J_2 \to J_3 \to J_4 \to J_5 \to J_6$ requires 13 units in adjustment time. To find a better sequence, construct the digraph $D$ as in step 1 (figure 10.8).

$(v_1, v_6, v_3, v_4, v_5, v_2)$ is a directed Hamilton path of $D$, and yields the sequence

$$J_1 \to J_6 \to J_3 \to J_4 \to J_5 \to J_2$$

which requires only eight units of adjustment time. Note that the reverse sequence

$$J_2 \to J_5 \to J_4 \to J_3 \to J_6 \to J_1$$

is far worse, requiring 19 units of adjustment time.

*Exercises*

10.4.1 With the aid of exercise 10.2.3, describe a good algorithm for finding a directed Hamilton path in a tournament.

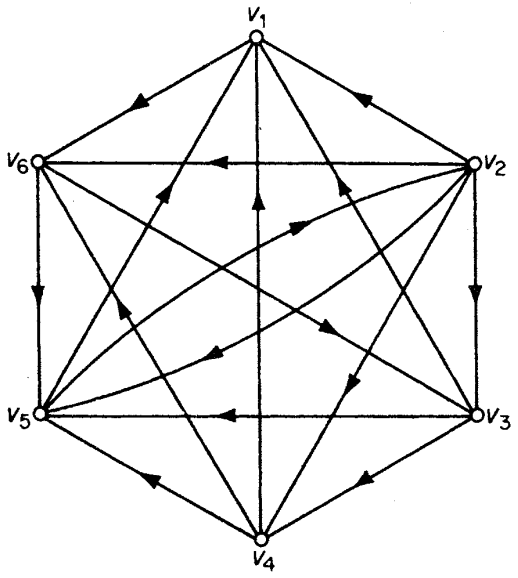



Figure 10.8

10.4.2 Show, by means of an example, that a sequencing of jobs obtained by the above method may be far from optimal.

## 10.5 DESIGNING AN EFFICIENT COMPUTER DRUM

The position of a rotating drum is to be recognised by means of binary signals produced at a number of electrical contacts at the surface of the drum. The surface is divided into $2^n$ sections, each consisting of either insulating or conducting material. An insulated section gives signal 0 (no current), whereas a conducting section gives signal 1 (current). For example, the position of the drum in figure 10.9 gives a reading 0010 at the four

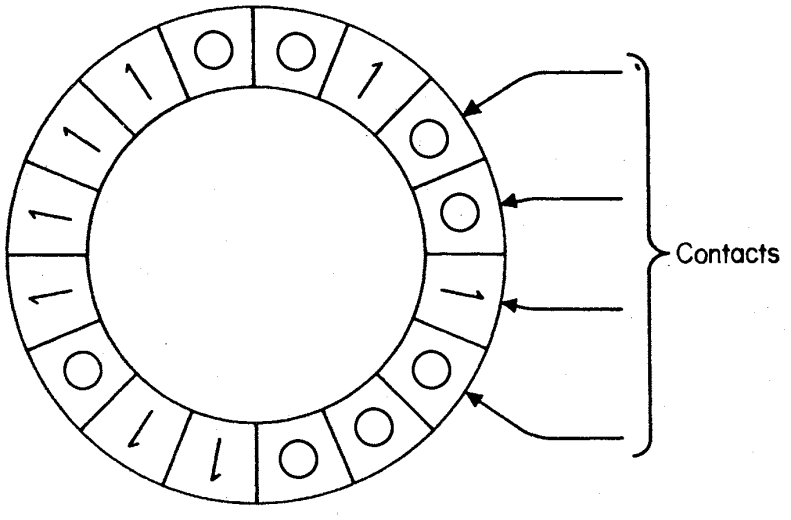



Figure 10.9. A computer drum

contacts. If the drum were rotated clockwise one section, the reading would be 1001. Thus these two positions can be distinguished, since they give different readings. However, a further rotation of two sections would result in another position with reading 0010, and therefore this latter position is indistinguishable from the initial one.

We wish to design the drum surface in such a way that the $2^n$ different positions of the drum can be distinguished by $k$ contacts placed consecutively around part of the drum, and we would like this number $k$ to be as small as possible. How can this be accomplished?

First note that $k$ contacts yield a $k$-digit binary number, and there are $2^k$ such numbers. Therefore, if all $2^n$ positions are to give different readings, we must have $2^k \geq 2^n$, that is, $k \geq n$. We shall show that the surface of the drum can be designed in such a way that $n$ contacts suffice to distinguish all $2^n$ positions.

We define a digraph $D_n$ as follows: the vertices of $D_n$ are the $(n-1)$-digit binary numbers $p_1p_2 \ldots p_{n-1}$ with $p_i = 0$ or 1. There is an arc with tail $p_1p_2 \ldots p_{n-1}$ and head $q_1q_2 \ldots q_{n-1}$ if and only if $p_{i+1} = q_i$ for $1 \leq i \leq n-2$; in other words, all arcs are of the form $(p_1p_2 \ldots p_{n-1}, p_2p_3 \ldots p_n)$. In addition, each arc $(p_1p_2 \ldots p_{n-1}, p_2p_3 \ldots p_n)$ of $D_n$ is assigned the label $p_1p_2 \ldots p_n$. $D_4$ is shown in figure 10.10.

Clearly, $D_n$ is connected and each vertex of $D_n$ has indegree two and outdegree two. Therefore (exercise 10.3.2) $D_n$ has a directed Euler tour. This directed Euler tour, regarded as a sequence of arcs of $D_n$, yields a binary sequence of length $2^n$ suitable for the design of the drum surface.

For example, the digraph $D_4$ of figure 10.10 has a directed Euler tour $(a_1, a_2, \ldots, a_{16})$, giving the 16-digit binary sequence 0000111100101101. (Just read off the first digits of the labels of the $a_i$.) A drum constructed from this sequence is shown in figure 10.11.

This application of directed Euler tours is due to Good (1946).

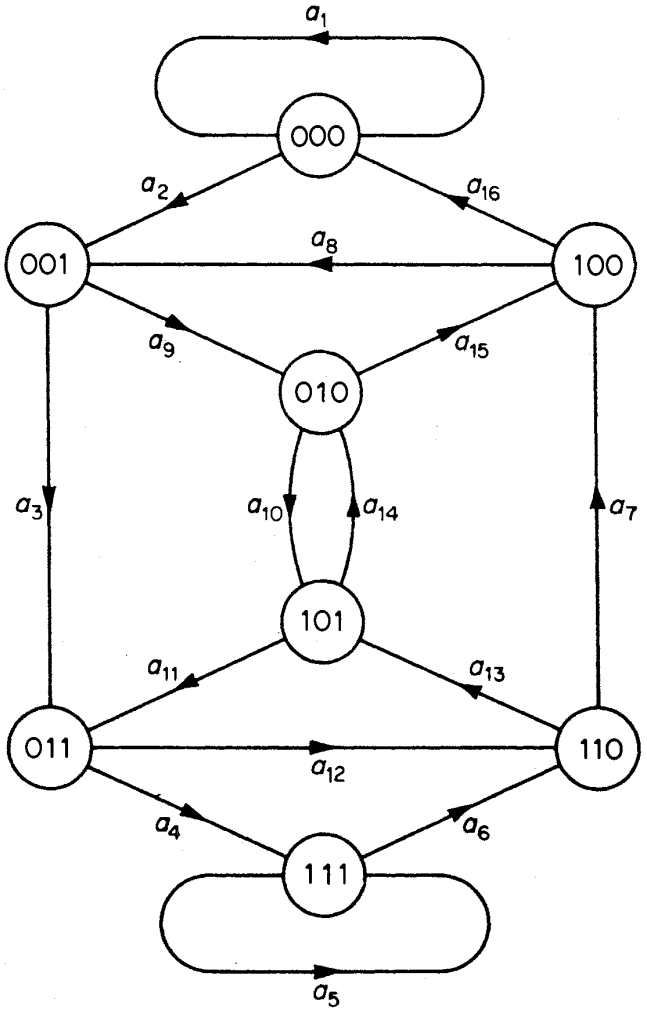
*Exercises*

10.5.1. Find a circular sequence of seven 0's and seven 1's such that all 4-digit binary numbers except 0000 and 1111 appear as blocks of the sequence.

10.5.2 Let $S$ be an alphabet of $n$ letters. Show that there is a circular sequence containing $n^3$ copies of each letter such that every four-letter 'word' formed from letters of $S$ appears as a block of the sequence.

## 10.6 MAKING A ROAD SYSTEM ONE-WAY

Given a road system, how can it be converted to one-way operation so that traffic may flow as smoothly as possible?




| Arc | Label |
|---|---|
| $a_1$ | 0000 |
| $a_2$ | 0001 |
| $a_3$ | 0011 |
| $a_4$ | 0111 |
| $a_5$ | 1111 |
| $a_6$ | 1110 |
| $a_7$ | 1100 |
| $a_8$ | 1001 |
| $a_9$ | 0010 |
| $a_{10}$ | 0101 |
| $a_{11}$ | 1011 |
| $a_{12}$ | 0110 |
| $a_{13}$ | 1101 |
| $a_{14}$ | 1010 |
| $a_{15}$ | 0100 |
| $a_{16}$ | 1000 |

Figure 10.10

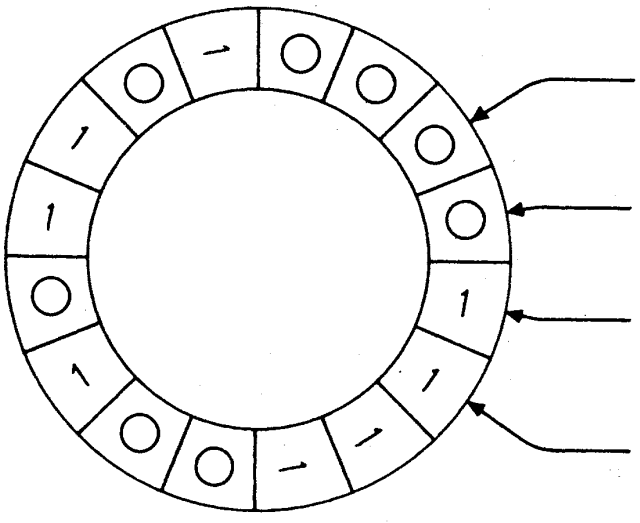


Figure 10.11

This is clearly a problem on orientations of graphs. Consider, for example, the two graphs, representing road networks, in figures 10.12*a* and 10.12*b*.

No matter how $G_1$ may be oriented, the resulting orientation cannot be diconnected—traffic will not be able to flow freely through the system. The trouble is that $G_1$ has a cut edge. On the other hand $G_2$ has the 'balanced' orientation $D_2$ (figure 10.12*c*), in which each vertex is reachable from each other vertex in at most two steps; in particular $D_2$ is diconnected.

Certainly, a necessary condition for $G$ to have a diconnected orientation is that $G$ be 2-edge-connected. Robbins (1939) showed that this condition is also sufficient.

*Theorem 10.5* If $G$ is 2-edge-connected, then $G$ has a diconnected orientation.

*Proof* Let $G$ be 2-edge-connected. Then $G$ contains a cycle $G_1$. We define inductively a sequence $G_1, G_2, \ldots$ of connected subgraphs of $G$ as follows: if $G_i$ $(i = 1, 2, \ldots)$ is not a spanning subgraph of $G$, let $v_i$ be a vertex of $G$ not in $G_i$. Then (exercise 3.2.1) there exist edge-disjoint paths $P_i$ and $Q_i$ from $v_i$ to $G_i$. Define

$$G_{i+1} = G_i \cup P_i \cup Q_i$$

Since $\nu(G_{i+1}) > \nu(G_i)$, this sequence must terminate in a spanning subgraph $G_n$ of $G$.

We now orient $G_n$ by orienting $G_1$ as a directed cycle, each path $P_i$ as a directed path with origin $v_i$, and each path $Q_i$ as a directed path with terminus $v_i$. Clearly every $G_i$, and hence in particular $G_n$, is thereby given a diconnected orientation. Since $G_n$ is a spanning subgraph of $G$ it follows that $G$, too, has a diconnected orientation □

Nash-Williams (1960) has generalised Robbins' theorem by showing that every $2k$-edge-connected graph $G$ has a $k$-arc-connected orientation. Although the proof of this theorem is difficult, the special case when $G$ has an Euler trail admits of a simple proof.

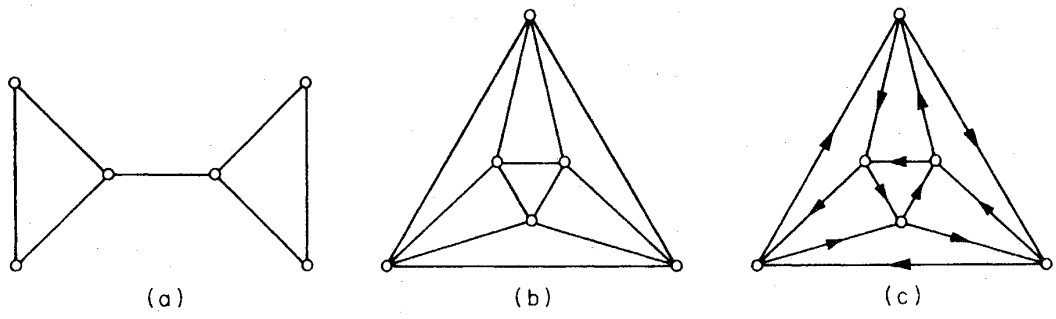



Figure 10.12. (*a*) $G_1$; (*b*) $G_2$; (*c*) $D_2$

*Theorem 10.6* Let $G$ be a $2k$-edge-connected graph with an Euler trail. Then $G$ has a $k$-arc-connected orientation.

*Proof* Let $v_0e_1v_1 \ldots e_\varepsilon v_\varepsilon$ be an Euler trail of $G$. Orient $G$ by converting the edge $e_i$ with ends $v_{i-1}$ and $v_i$ to an arc $a_i$ with tail $v_{i-1}$ and head $v_i$, for $1 \leq i \leq \varepsilon$. Now let $[S, \bar{S}]$ be an $m$-edge cut of $G$. The number of times the directed trail $(v_0, a_1, v_1, \ldots, a_\varepsilon, v_\varepsilon)$ crosses from $S$ to $\bar{S}$ differs from the number of times it crosses from $\bar{S}$ to $S$ by at most one. Since it includes all arcs of $D$, both $(S, \bar{S})$ and $(\bar{S}, S)$ must contain at least $[m/2]$ arcs. The result follows □

*Exercises*

10.6.1 Show, by considering the Petersen graph, that the following statement is false: every graph $G$ has an orientation in which, for every $S \subseteq V$, the cardinalities of $(S, \bar{S})$ and $(\bar{S}, S)$ differ by at most one.

10.6.2 (*a*) Show that Nash-Williams' theorem is equivalent to the following statement: if every bond of $G$ has at least $2k$ edges, then there is an orientation of $G$ in which every bond has at least $k$ arcs in each direction.

(*b*) Show, by considering the Grötzsch graph (figure 8.2), that the following analogue of Nash-Williams' theorem is false: if every cycle of $G$ has at least $2k$ edges, then there is an orientation of $G$ in which every cycle has at least $k$ arcs in each direction.

## 10.7 RANKING THE PARTICIPANTS IN A TOURNAMENT

A number of players each play one another in a tennis tournament. Given the outcomes of the games, how should the participants be ranked?

Consider, for example, the tournament of figure 10.13. This represents the result of a tournament between six players; we see that player 1 beat players 2, 4, 5 and 6 and lost to player 3, and so on.

One possible approach to ranking the participants would be to find a directed Hamilton path in the tournament (such a path exists by virtue of corollary 10.1), and then rank according to the position on the path. For instance, the directed Hamilton path (3, 1, 2, 4, 5, 6) would declare player 3 the winner, player 1 runner-up, and so on. This method of ranking, however, does not bear further examination, since a tournament generally has many directed Hamilton paths; our example has (1, 2, 4, 5, 6, 3), (1, 4, 6, 3, 2, 5) and several others.

Another approach would be to compute the scores (numbers of games won by each player) and compare them. If we do this we obtain the score vector

$$\mathbf{s}_1 = (4, 3, 3, 2, 2, 1)$$

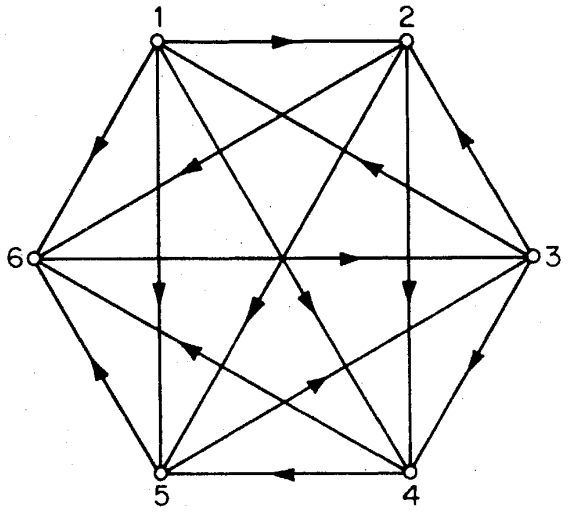



Figure 10.13

The drawback here is that this score vector does not distinguish between players 2 and 3 even though player 3 beat players with higher scores than did player 2. We are thus led to the second-level score vector

$$\mathbf{s}_2 = (8, 5, 9, 3, 4, 3)$$

in which each player's second-level score is the sum of the scores of the players he beat. Player 3 now ranks first. Continuing this procedure we obtain further vectors

$$\mathbf{s}_3 = (15, 10, 16, 7, 12, 9)$$
$$\mathbf{s}_4 = (38, 28, 32, 21, 25, 16)$$
$$\mathbf{s}_5 = (90, 62, 87, 41, 48, 32)$$
$$\mathbf{s}_6 = (183, 121, 193, 80, 119, 87)$$

The ranking of the players is seen to fluctuate a little, player 3 vying with player 1 for first place. We shall show that this procedure always converges to a fixed ranking when the tournament in question is diconnected and has at least four vertices. This will then lead to a method of ranking the players in any tournament.

In a diconnected digraph $D$, the length of a shortest directed $(u, v)$-path is denoted by $\vec{d}_D(u, v)$ and is called the *distance from u to v*; the *directed diameter* of $D$ is the maximum distance from any one vertex of $D$ to any other.

*Theorem 10.7* Let $D$ be a diconnected tournament with $\nu \geq 5$, and let $\mathbf{A}$ be the adjacency matrix of $D$. Then $\mathbf{A}^{d+3} > \mathbf{0}$ (every entry positive), where $d$ is the directed diameter of $D$.

*Proof* The $(i, j)$th entry of $\mathbf{A}^k$ is precisely the number of directed $(v_i, v_j)$-walks of length $k$ in $D$ (exercise 10.1.8). We must therefore show that, for any two vertices $v_i$ and $v_j$ (possibly identical), there is a directed $(v_i, v_j)$-walk of length $d+3$.

Let $d_{ij} = \vec{d}(v_i, v_j)$. Then $0 \leq d_{ij} \leq d \leq \nu - 1$ and therefore

$$3 \leq d - d_{ij} + 3 \leq \nu + 2$$

If $d - d_{ij} + 3 \leq \nu$ then, by theorem 10.3, there is a directed $(d - d_{ij} + 3)$-cycle $C$ containing $v_j$. A directed $(v_i, v_j)$-path $P$ of length $d_{ij}$ followed by the directed cycle $C$ together form a directed $(v_i, v_j)$-walk of length $d+3$, as desired.

There are two special cases. If $d - d_{ij} + 3 = \nu + 1$, then $P$ followed by a directed $(\nu - 2)$-cycle through $v_j$ followed by a directed 3-cycle through $v_j$ constitute a directed $(v_i, v_j)$-walk of length $d+3$ (the $(\nu - 2)$-cycle exists since $\nu \geq 5$); and if $d - d_{ij} + 3 = \nu + 2$, then $P$ followed by a directed $(\nu - 1)$-cycle through $v_j$ followed by a directed 3-cycle through $v_j$ constitute such a walk □

A real matrix $\mathbf{R}$ is called *primitive* if $\mathbf{R}^k > \mathbf{0}$ for some $k$.

*Corollary 10.7* The adjacency matrix $\mathbf{A}$ of a tournament $D$ is primitive if and only if $D$ is diconnected and $\nu \geq 4$.

*Proof* If $D$ is not diconnected, then there are vertices $v_i$ and $v_j$ in $D$ such that $v_j$ is not reachable from $v_i$. Thus there is no directed $(v_i, v_j)$-walk in $D$. It follows that the $(i, j)$th entry of $\mathbf{A}^k$ is zero for all $k$, and hence $\mathbf{A}$ is not primitive.

Conversely, suppose that $D$ is diconnected. If $\nu \geq 5$ then, by theorem 10.7, $\mathbf{A}^{d+3} > \mathbf{0}$ and so $\mathbf{A}$ is primitive. There is just one diconnected tournament on three vertices (figure 10.14*a*), and just one diconnected tournament on four vertices (figure 10.14*b*). It is readily checked that the adjacency

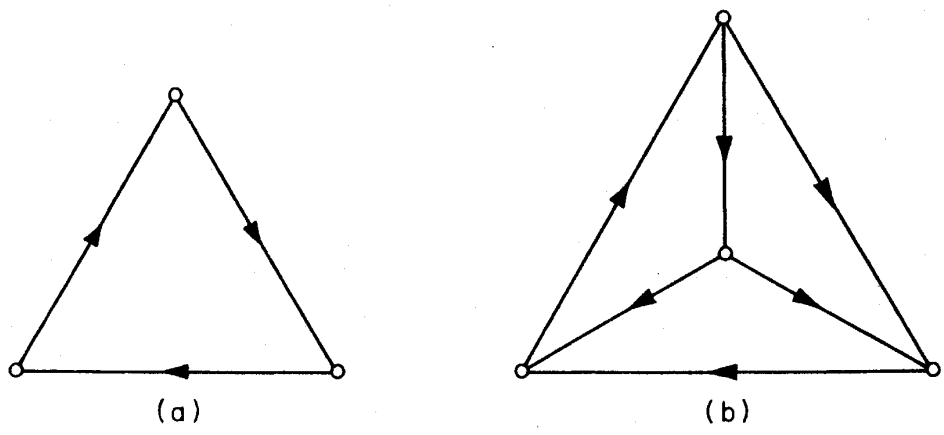



Figure 10.14

matrix of the 3-vertex tournament is not primitive, and it can be shown that the ninth power of the adjacency matrix of the 4-vertex tournament has all entries positive □

Returning now to the score vectors, we see that the $i$th-level score vector in a tournament $D$ is given by

$$\mathbf{s}_i = \mathbf{A}^i\mathbf{J}$$

where $\mathbf{A}$ is the adjacency matrix of $D$, and $\mathbf{J}$ is a column vector of 1's. If the matrix $\mathbf{A}$ is primitive then, by the Perron–Frobenius theorem (see Gantmacher, 1960), the eigenvalue of $\mathbf{A}$ with largest absolute value is a real positive number $r$ and, furthermore,

$$\lim_{i\to\infty}\left(\frac{\mathbf{A}}{r}\right)^i\mathbf{J} = \mathbf{s}$$

where $\mathbf{s}$ is a positive eigenvector of $\mathbf{A}$ corresponding to $r$. Therefore, by corollary 10.7, if $D$ is a diconnected tournament on at least four vertices, the normalised vector $\bar{\mathbf{s}}$ (with entries summing to one) can be taken as the vector of relative strengths of the players in $D$. In the example of figure 10.13, we find that (approximately)

$$r = 2.232 \quad \text{and} \quad \bar{\mathbf{s}} = (.238, .164, .231, .113, .150, .104)$$

Thus the ranking of the players given by this method is 1, 3, 2, 5, 4, 6.

If the tournament is not diconnected, then (exercises 10.1.9 and 10.1.3*b*) its dicomponents can be linearly ordered so that the ordering preserves dominance. The participants in a round-robin tournament can now be ranked according to the following procedure.

*Step 1* In each dicomponent on four or more vertices, rank the players using the eigenvector $\bar{\mathbf{s}}$; in a dicomponent on three vertices rank all three players equal.

*Step 2* Rank the dicomponents in their dominance-preserving linear order $D_1, D_2, \ldots, D_m$; that is, if $i < j$ then every arc with one end in $D_i$ and one end in $D_j$ has its head in $D_j$.

This method of ranking is due to Wei (1952) and Kendall (1955). For other ranking procedures, see Moon and Pullman (1970).

*Exercises*

10.7.1 Apply the method of ranking described in section 10.7 to

(*a*) the four tournaments shown in figure 10.4;

(*b*) the tournament with adjacency matrix

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| B | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| E | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| F | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| G | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| H | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| I | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| J | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 |

10.7.2 An alternative method of ranking is to consider 'loss vectors' instead of score vectors.

(*a*) Show that this amounts to ranking the converse tournament and then reversing the ranking so found.

(*b*) By considering the diconnected tournament on four vertices, show that the two methods of ranking do not necessarily yield the same result.

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# 11 Networks

## 11.1 FLOWS

Transportation networks, the means by which commodities are shipped from their production centres to their markets, can be most effectively analysed when they are viewed as digraphs that possess some additional structure. The resulting theory is the subject of this chapter. It has a wide range of important applications.

A *network* $N$ is a digraph $D$ (the *underlying digraph* of $N$) with two distinguished subsets of vertices, $X$ and $Y$, and a non-negative integer-valued function $c$ defined on its arc set $A$; the sets $X$ and $Y$ are assumed to be disjoint and nonempty. The vertices in $X$ are the *sources* of $N$ and those in $Y$ are the *sinks* of $N$. They correspond to production centres and markets, respectively. Vertices which are neither sources nor sinks are called *intermediate vertices;* the set of such vertices will be denoted by $I$. The function $c$ is the *capacity function* of $N$ and its value on an arc $a$ the *capacity* of $a$. The capacity of an arc can be thought of as representing the maximum rate at which a commodity can be transported along it.

We represent a network by drawing its underlying digraph and labelling each arc with its capacity. Figure 11.1 shows a network with two sources $x_1$ and $x_2$, three sinks $y_1$, $y_2$ and $y_3$, and four intermediate vertices $v_1$, $v_2$, $v_3$ and $v_4$.

If $S \subseteq V$, we denote $V \backslash S$ by $\bar{S}$. In addition, we shall find the following notation useful. If $f$ is a real-valued function defined on the arc set $A$ of $N$, and if $K \subseteq A$, we denote $\sum_{a \in K} f(a)$ by $f(K)$. Furthermore, if $K$ is a set of arcs of the form $(S, \bar{S})$, we shall write $f^+(S)$ for $f(S, \bar{S})$ and $f^-(S)$ for $f(\bar{S}, S)$.

A *flow* in a network $N$ is an integer-valued function $f$ defined on $A$ such that

$$0 \leq f(a) \leq c(a) \quad \text{for all} \quad a \in A \tag{11.1}$$

and

$$f^-(v) = f^+(v) \quad \text{for all} \quad v \in I \tag{11.2}$$

The value $f(a)$ of $f$ on an arc $a$ can be likened to the rate at which material is transported along $a$ under the flow $f$. The upper bound in condition (11.1) is called the *capacity constraint;* it imposes the natural restriction that the rate of flow along an arc cannot exceed the capacity of the arc. Condition (11.2), called the *conservation condition*, requires that, for any intermediate vertex $v$, the rate at which material is transported into $v$ is

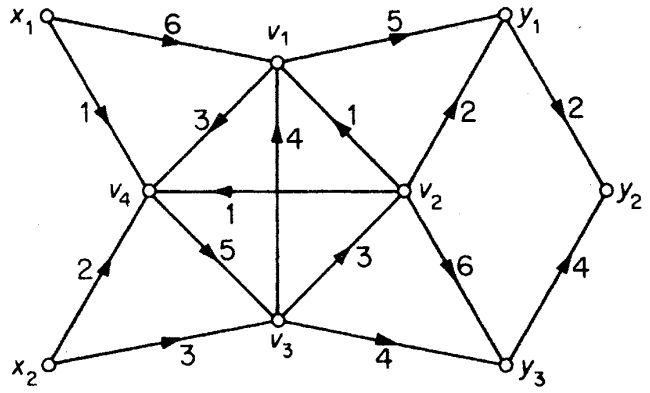



Figure 11.1. A network

equal to the rate at which it is transported out of $v$. Note that every network has at least one flow, since the function $f$ defined by $f(a)=0$, for all $a \in A$, clearly satisfies both (11.1) and (11.2); it is called the *zero flow*. A less trivial example of a flow is given in figure 11.2. The flow along each arc is indicated in bold type.

If $S$ is a subset of vertices in a network $N$ and $f$ is a flow in $N$, then $f^+(S)-f^-(S)$ is called the *resultant flow out of* $S$, and $f^-(S)-f^+(S)$ the *resultant flow into* $S$, relative to $f$. Since the conservation condition requires that the resultant flow out of any intermediate vertex is zero, it is intuitively clear and not difficult to show (exercise 11.1.3) that, relative to any flow $f$, the resultant flow out of $X$ is equal to the resultant flow into $Y$. This common quantity is called the *value* of $f$, and is denoted by val $f$; thus

$$\text{val}\, f = f^+(X) - f^-(X)$$

The value of the flow indicated in figure 11.2 is 6.

A flow $f$ in $N$ is a *maximum flow* if there is no flow $f'$ in $N$ such that val $f' >$ val $f$. Such flows are of obvious importance in the context of transportation networks. The problem of determining a maximum flow in an arbitrary network can be reduced to the case of networks that have just one

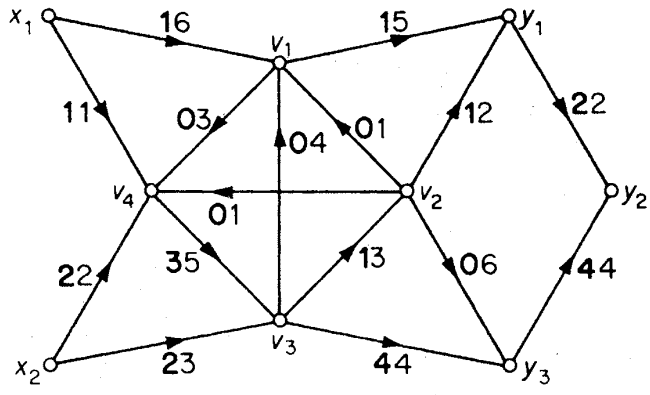



Figure 11.2. A flow in a network

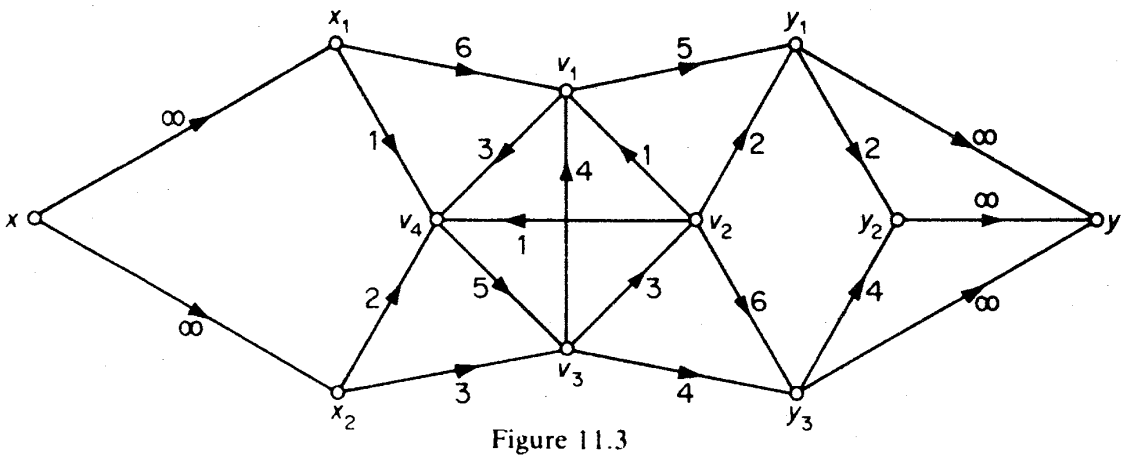



Figure 11.3

source and one sink by means of a simple device. Given a network $N$, construct a new network $N'$ as follows:

(i) adjoin two new vertices $x$ and $y$ to $N$;
(ii) join $x$ to each vertex in $X$ by an arc of capacity $\infty$;
(iii) join each vertex in $Y$ to $y$ by an arc of capacity $\infty$;
(iv) designate $x$ as the source and $y$ as the sink of $N'$.

Figure 11.3 illustrates this procedure as applied to the network $N$ of figure 11.1.

Flows in $N$ and $N'$ correspond to one another in a simple way. If $f$ is a flow in $N$ such that the resultant flow out of each source and into each sink is non-negative (it suffices to restrict our attention to such flows) then the function $f'$ defined by

$$f'(a)=\begin{cases} f(a) & \text{if} \quad a \text{ is an arc of } N \\ f^+(v)-f^-(v) & \text{if} \quad a=(x,v) \\ f^-(v)-f^+(v) & \text{if} \quad a=(v,y) \end{cases} \tag{11.3}$$

is a flow in $N'$ such that $\operatorname{val} f' = \operatorname{val} f$ (exercise 11.1.4*a*). Conversely, the restriction to the arc set of $N$ of a flow in $N'$ is a flow in $N$ having the same value (exercise 11.1.4*b*). Therefore, throughout the next three sections, we shall confine our attention to networks that have a single source $x$ and a single sink $y$.

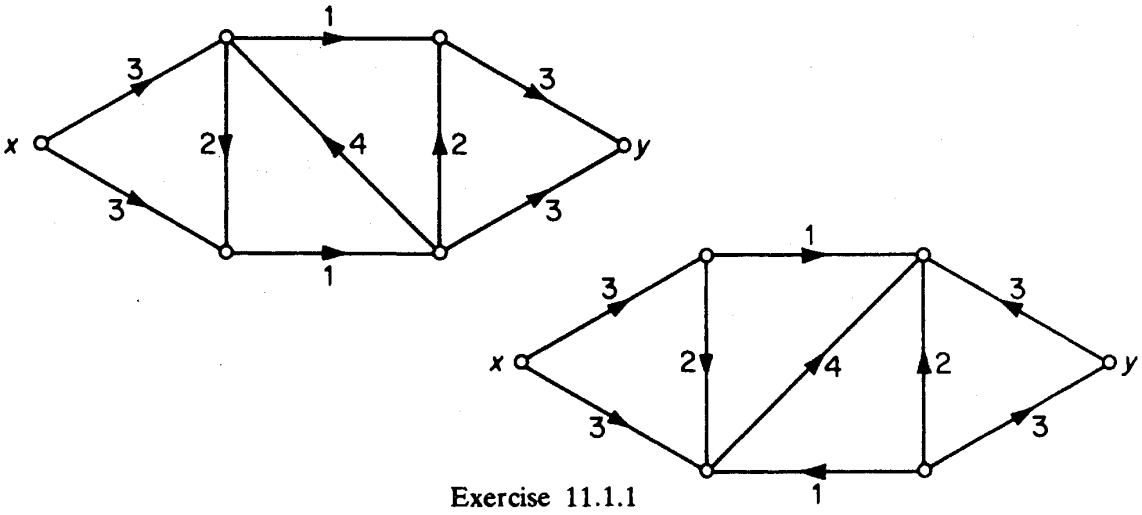
*Exercises*

11.1.1 For each of the following networks (see diagram, p. 194), determine all possible flows and the value of a maximum flow.

**11.1.2** Show that, for any flow $f$ in $N$ and any $S \subseteq V$,

$$\sum_{v \in S} (f^+(v)-f^-(v)) = f^+(S)-f^-(S)$$

(Note that, in general, $\sum_{v \in S} f^+(v) \neq f^+(S)$ and $\sum_{v \in S} f^-(v) \neq f^-(S)$).




Exercise 11.1.1

11.1.3 Show that, relative to any flow $f$ in $N$, the resultant flow out of $X$ is equal to the resultant flow into $Y$.

11.1.4 Show that

(*a*) the function $f'$ given by (11.3) is a flow in $N'$ and that $\text{val}\, f' = \text{val}\, f$;

(*b*) the restriction to the arc set of $N$ of a flow in $N'$ is a flow in $N$ having the same value.

## 11.2 CUTS

Let $N$ be a network with a single source $x$ and a single sink $y$. A *cut* in $N$ is a set of arcs of the form $(S, \bar{S})$, where $x \in S$ and $y \in \bar{S}$. In the network of figure 11.4, a cut is indicated by heavy lines.

The *capacity* of a cut $K$ is the sum of the capacities of its arcs. We denote the capacity of $K$ by cap $K$; thus

$$\text{cap}\, K = \sum_{a \in K} c(a)$$

The cut indicated in figure 11.4 has capacity 16.

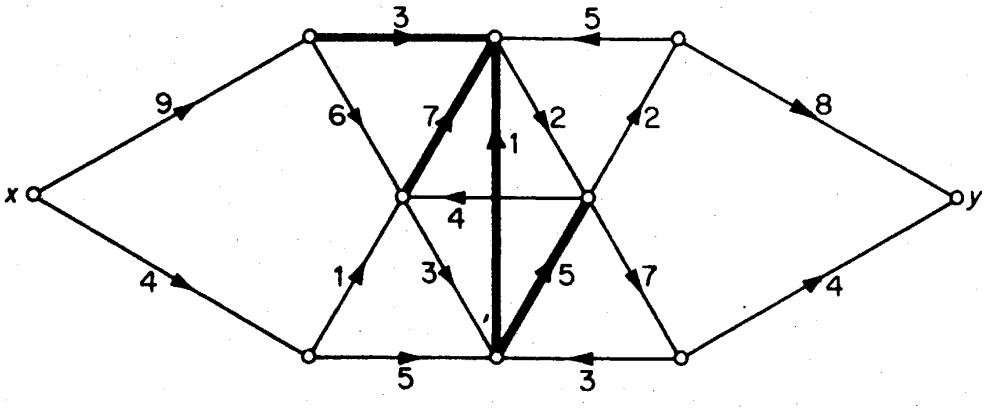



Figure 11.4. A cut in a network

*Lemma 11.1* For any flow $f$ and any cut $(S, \bar{S})$ in $N$

$$\text{val}\, f = f^+(S) - f^-(S) \tag{11.4}$$

*Proof* Let $f$ be a flow and $(S, \bar{S})$ a cut in $N$. From the definitions of flow and value of a flow, we have

$$f^+(v) - f^-(v) = \begin{cases} \text{val}\, f & \text{if} \quad v = x \\ 0 & \text{if} \quad v \in S \backslash \{x\} \end{cases}$$

Summing these equations over $S$ and simplifying (exercise 11.1.2), we obtain

$$\text{val}\, f = \sum_{v \in S} (f^+(v) - f^-(v)) = f^+(S) - f^-(S) \quad \square$$

It is convenient to call an arc *a f-zero* if $f(a) = 0$, *f-positive* if $f(a) > 0$, *f-unsaturated* if $f(a) < c(a)$ and *f-saturated* if $f(a) = c(a)$.

*Theorem 11.1* For any flow $f$ and any cut $K = (S, \bar{S})$ in $N$

$$\text{val}\, f \leq \text{cap}\, K \tag{11.5}$$

Furthermore, equality holds in (11.5) if and only if each arc in $(S, \bar{S})$ is $f$-saturated and each arc in $(\bar{S}, S)$ is $f$-zero.

*Proof* By (11.1)

$$f^+(S) \leq \text{cap}\, K \tag{11.6}$$

and

$$f^-(S) \geq 0 \tag{11.7}$$

We obtain (11.5) by substituting inequalities (11.6) and (11.7) in (11.4). The second statement follows, on noting that equality holds in (11.6) if and only if each arc in $(S, \bar{S})$ is $f$-saturated, and equality holds in (11.7) if and only if each arc in $(\bar{S}, S)$ is $f$-zero $\square$

A cut $K$ in $N$ is a *minimum cut* if there is no cut $K'$ in $N$ such that $\text{cap}\, K' < \text{cap}\, K$. If $f^*$ is a maximum flow and $\tilde{K}$ is a minimum cut, we have, as a special case of theorem 11.1, that

$$\text{val}\, f^* \leq \text{cap}\, \tilde{K} \tag{11.8}$$

*Corollary 11.1* Let $f$ be a flow and $K$ be a cut such that $\text{val}\, f = \text{cap}\, K$. Then $f$ is a maximum flow and $K$ is a minimum cut.

*Proof* Let $f^*$ be a maximum flow and $\tilde{K}$ a minimum cut. Then, by (11.8),

$$\text{val}\, f \leq \text{val}\, f^* \leq \text{cap}\, \tilde{K} \leq \text{cap}\, K$$

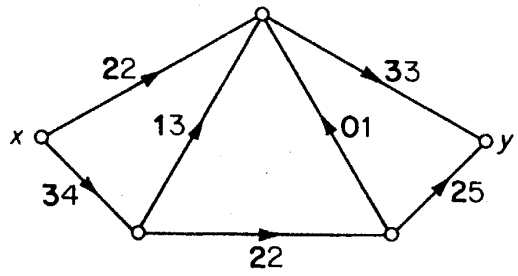
Since, by hypothesis, $\text{val}\, f = \text{cap}\, K$, it follows that $\text{val}\, f = \text{val}\, f^*$ and $\text{cap}\, K = \text{cap}\, \tilde{K}$. Thus $f$ is a maximum flow and $K$ is a minimum cut $\square$

In the next section, we shall prove the converse of corollary 11.1, namely that equality always holds in (11.8).

*Exercises*

11.2.1 In the following network:

(*a*) determine all cuts;
(*b*) find the capacity of a minimum cut;
(*c*) show that the flow indicated is a maximum flow.




11.2.2 Show that, if there exists no directed $(x, y)$-path in $N$, then the value of a maximum flow and the capacity of a minimum cut are both zero.

11.2.3 If $(S, \bar{S})$ and $(T, \bar{T})$ are minimum cuts in $N$, show that $(S \cup T, \overline{S \cup T})$ and $(S \cap T, \overline{S \cap T})$ are also minimum cuts in $N$.

## 11.3 THE MAX-FLOW MIN-CUT THEOREM

In this section we shall present an algorithm for determining a maximum flow in a network. Since a basic requirement of any such algorithm is that it be able to decide when a given flow is, in fact, a maximum flow, we first look at this question.

Let $f$ be a flow in a network $N$. With each path $P$ in $N$ we associate a non-negative integer $\iota(P)$ defined by

$$\iota(P) = \min_{a \in A(P)} \iota(a)$$

where

$$\iota(a) = \begin{cases} c(a) - f(a) & \text{if } a \text{ is a forward arc of } P \\ f(a) & \text{if } a \text{ is a reverse arc of } P \end{cases}$$

As may easily be seen, $\iota(P)$ is the largest amount by which the flow along $P$ can be increased (relative to $f$) without violating condition (11.1). The path $P$ is said to be *f-saturated* if $\iota(P) = 0$ and *f-unsaturated* if $\iota(P) > 0$ (or, equivalently, if each forward arc of $P$ is $f$-unsaturated and each reverse arc of $P$ is $f$-positive). Put simply, an $f$-unsaturated path is one that is not being used to its full capacity. An *f-incrementing* path is an $f$-unsaturated path

from the source $x$ to the sink $y$. For example, if $f$ is the flow indicated in the network of figure 11.5$a$, then one $f$-incrementing path is the path $P = xv_1v_2v_3y$. The forward arcs of $P$ are $(x, v_1)$ and $(v_3, y)$ and $\iota(P) = 2$.

The existence of an $f$-incrementing path $P$ in a network is significant since it implies that $f$ is not a maximum flow; in fact, by sending an additional flow of $\iota(P)$ along $P$, one obtains a new flow $\hat{f}$ defined by

$$\hat{f}(a) = \begin{cases} f(a) + \iota(P) & \text{if } a \text{ is a forward arc of } P \\ f(a) - \iota(P) & \text{if } a \text{ is a reverse arc of } P \\ f(a) & \text{otherwise} \end{cases} \tag{11.9}$$

for which $\operatorname{val} \hat{f} = \operatorname{val} f + \iota(P)$ (exercise 11.3.1). We shall refer to $\hat{f}$ as the *revised flow based on P*. Figure 11.5$b$ shows the revised flow in the network of figure 11.5$a$, based on the $f$-incrementing path $xv_1v_2v_3y$.

The rôle played by incrementing paths in flow theory is analogous to that of augmenting paths in matching theory, as the following theorem shows (compare theorem 5.1).

*Theorem 11.2* A flow $f$ in $N$ is a maximum flow if and only if $N$ contains no $f$-incrementing path.

*Proof* If $N$ contains an $f$-incrementing path $P$, then $f$ cannot be a maximum flow since $\hat{f}$, the revised flow based on $P$, has a larger value.

Conversely, suppose that $N$ contains no $f$-incrementing path. Our aim is to show that $f$ is a maximum flow. Let $S$ denote the set of all vertices to which $x$ is connected by $f$-unsaturated paths in $N$. Clearly $x \in S$. Also, since $N$ has no $f$-incrementing path, $y \in \bar{S}$. Thus $K = (S, \bar{S})$ is a cut in $N$. We shall show that each arc in $(S, \bar{S})$ is $f$-saturated and each arc in $(\bar{S}, S)$ is $f$-zero.

Consider an arc $a$ with tail $u \in S$ and head $v \in \bar{S}$. Since $u \in S$, there exists an $f$-unsaturated $(x, u)$-path $Q$. If $a$ were $f$-unsaturated, then $Q$ could be extended by the arc $a$ to yield an $f$-unsaturated $(x, v)$-path. But $v \in \bar{S}$, and so there is no such path. Therefore $a$ must be $f$-saturated. Similar reasoning shows that if $a \in (\bar{S}, S)$, then $a$ must be $f$-zero.

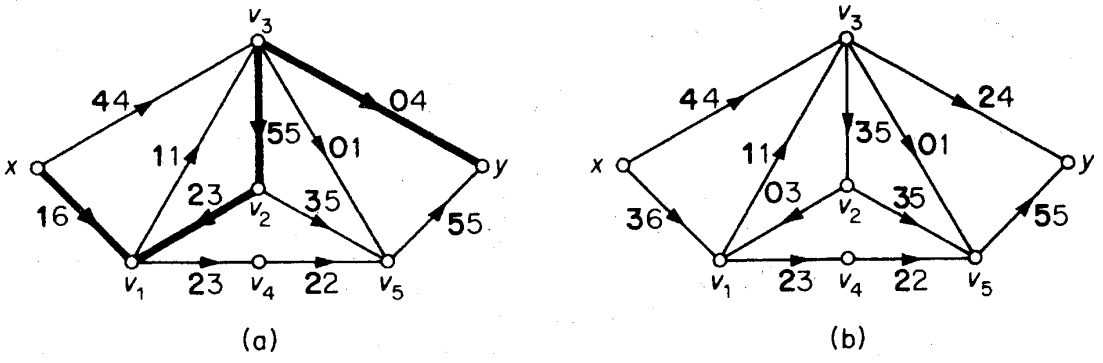



Figure 11.5. ($a$) An $f$-incrementing path $P$; ($b$) revised flow based on $P$

On applying theorem 11.1, we obtain

$$\text{val } f = \text{cap } K$$

It now follows from corollary 11.1 that $f$ is a maximum flow (and that $K$ is a minimum cut) □

In the course of the above proof, we established the existence of a maximum flow $f$ and a minimum cut $K$ such that val $f$ = cap $K$. We thus have the following theorem, due to Ford and Fulkerson (1956).

*Theorem 11.3* In any network, the value of a maximum flow is equal to the capacity of a minimum cut.

Theorem 11.3 is known as the *max-flow min-cut theorem*. It is of central importance in graph theory. Many results on graphs turn out to be easy consequences of this theorem as applied to suitably chosen networks. In sections 11.4 and 11.5 we shall demonstrate two such applications.

The proof of theorem 11.2 is constructive in nature. We extract from it an algorithm for finding a maximum flow in a network. This algorithm, also due to Ford and Fulkerson (1957), is known as the *labelling method*. Starting with a known flow, for instance the zero flow, it recursively constructs a sequence of flows of increasing value, and terminates with a maximum flow. After the construction of each new flow $f$, a subroutine called the *labelling procedure* is used to find an $f$-incrementing path, if one exists. If such a path $P$ is found, then $\hat{f}$, the revised flow based on $P$, is constructed and taken as the next flow in the sequence. If there is no such path, the algorithm terminates; by theorem 11.2, $f$ is a maximum flow.

To describe the labelling procedure we need the following definition. A tree $T$ in $N$ is an *$f$-unsaturated tree* if (i) $x \in V(T)$, and (ii) for every vertex $v$ of $T$, the unique $(x, v)$-path in $T$ is an $f$-unsaturated path. Such a tree is shown in the network of figure 11.6.

The search for an $f$-incrementing path involves growing an $f$-unsaturated tree $T$ in $N$. Initially, $T$ consists of just the source $x$. At any stage, there are two ways in which the tree may grow:

1. If there exists an $f$-unsaturated arc $a$ in $(S, \bar{S})$, where $S = V(T)$, then both $a$ and its head are adjoined to $T$.

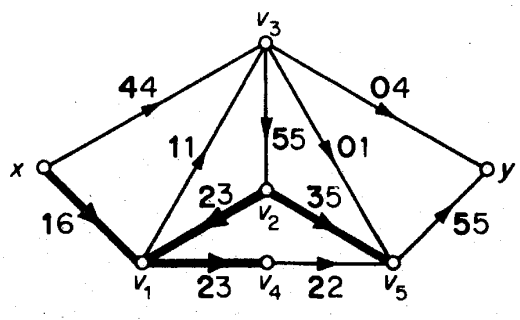



Figure 11.6. An $f$-unsaturated tree

2. If there exists an $f$-positive arc $a$ in $(\bar{S}, S)$, then both $a$ and its tail are adjoined to $T$.

Clearly, each of the above procedures results in an enlarged $f$-unsaturated tree.

Now either $T$ eventually reaches the sink y or it stops growing before reaching y. The former case is referred to as *breakthrough;* in the event of breakthrough, the $(x, y)$-path in $T$ is our desired $f$-incrementing path. If, however, $T$ stops growing before reaching y, we deduce from theorem 11.1 and corollary 11.1 that $f$ is a maximum flow. In figure 11.7, two iterations of this tree-growing procedure are illustrated. The first leads to breakthrough; the second shows that the resulting revised flow is a maximum flow.

The labelling procedure is a systematic way of growing an $f$-unsaturated tree $T$. In the process of growing $T$, it assigns to each vertex $v$ of $T$ the label $l(v) = \iota(P_v)$, where $P_v$ is the unique $(x, v)$-path in $T$. The advantage of this labelling is that, in the event of breakthrough, we not only have the $f$-incrementing path $P_y$, but also the quantity $\iota(P_y)$ with which to calculate the revised flow based on $P_y$. The labelling procedure begins by assigning to the source $x$ the label $l(x) = \infty$. It continues according to the following rules:

1. If $a$ is an $f$-unsaturated arc whose tail $u$ is already labelled but whose head $v$ is not, then $v$ is labelled $l(v) = \min\{l(u), c(a) - f(a)\}$.
2. If $a$ is an $f$-positive arc whose head $u$ is already labelled but whose tail $v$ is not, then $v$ is labelled $l(v) = \min\{l(u), f(a)\}$.

In each of the above cases, $v$ is said to be labelled *based on u.* To *scan* a labelled vertex $u$ is to label all unlabelled vertices that can be labelled based on $u$. The labelling procedure is continued until either the sink y is labelled (breakthrough) or all labelled vertices have been scanned and no more vertices can be labelled (implying that $f$ is a maximum flow).

A flow diagram summarising the labelling method is given in figure 11.8.

It is worth pointing out that the labelling method, as described above, is *not* a good algorithm. Consider, for example, the network $N$ in figure 11.9. Clearly, the value of a maximum flow in $N$ is $2m$. The labelling method will use the labelling procedure $2m + 1$ times if it starts with the zero flow and alternates between selecting *xpuvsy* and *xrvuqy* as an incrementing path; for, in each case, the flow value increases by exactly one. Since $m$ is arbitrary, the number of computational steps required to implement the labelling method in this instance can be bounded by no function of $\nu$ and $\varepsilon$. In other words, it is not a good algorithm.

However, Edmonds and Karp (1970) have shown that a slight refinement of the labelling procedure turns it into a good algorithm. The refinement suggested by them is the following: in the labelling procedure, scan on a 'first-labelled first-scanned' basis; that is, before scanning a labelled vertex

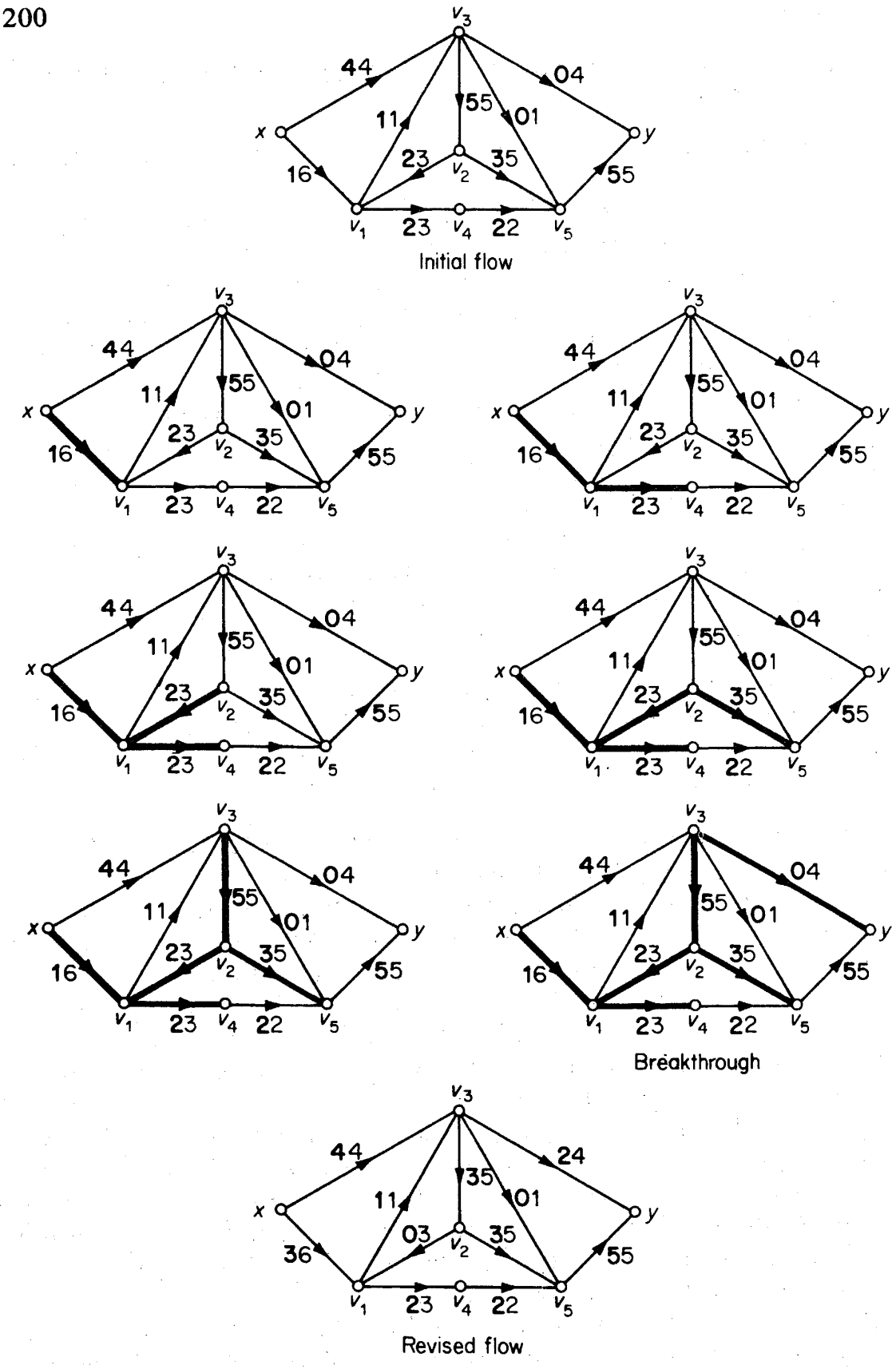



Figure 11.7

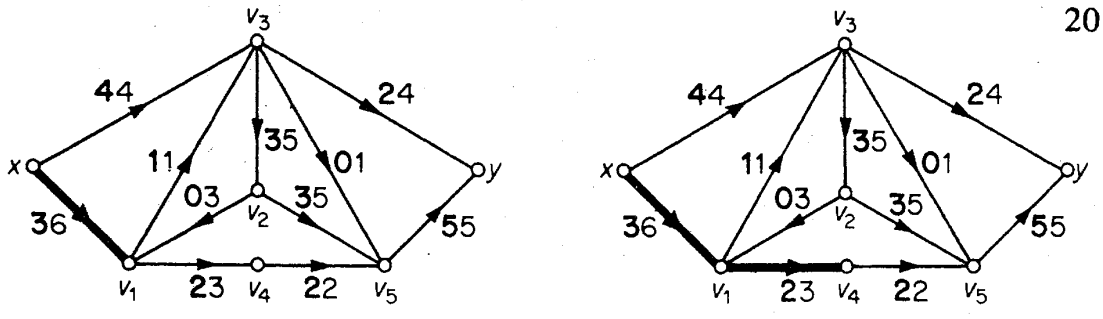



Figure 11.7. (Cont'd)

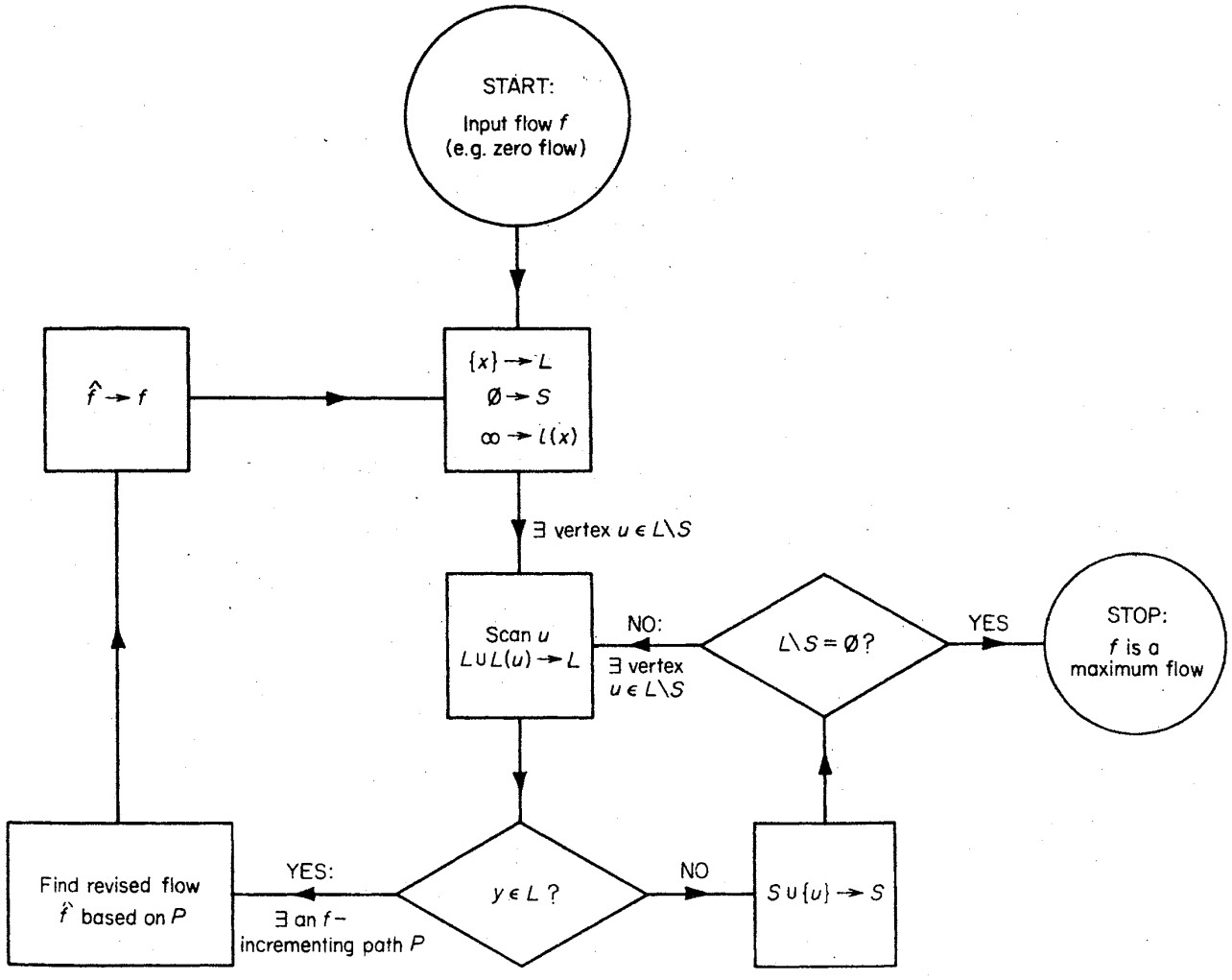



Figure 11.8. The labelling method ($L$, set of labelled vertices; $S$, set of scanned vertices; $L(u)$, set of vertices labelled during scanning of $u$)

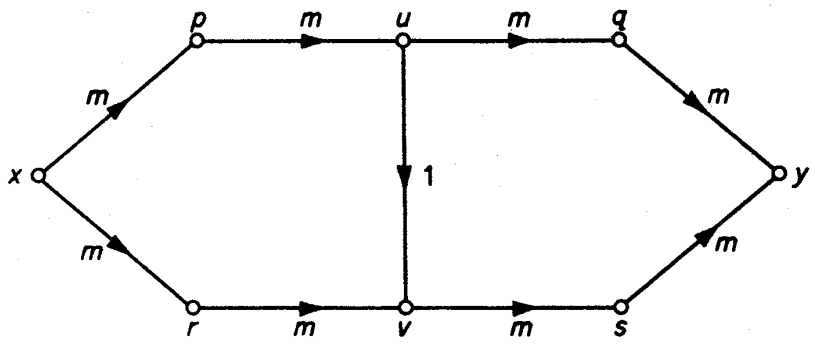


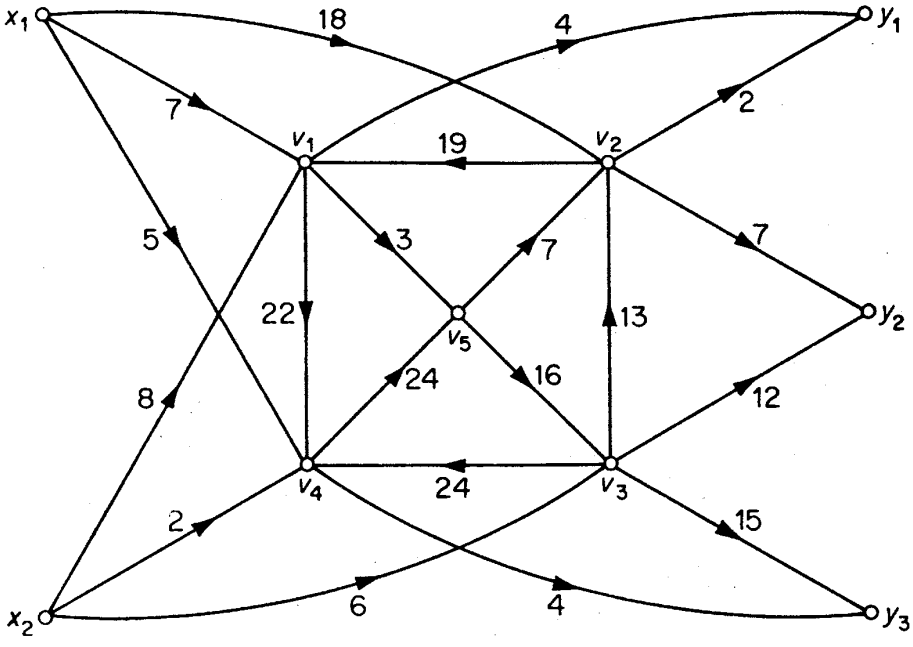

Figure 11.9

$u$, scan the vertices that were labelled before $u$. It can be seen that this amounts to selecting a shortest incrementing path. With this refinement, clearly, the maximum flow in the network of figure 11.9 would be found in just two iterations of the labelling procedure.

*Exercises*

11.3.1 Show that the function $\hat{f}$ given by (11.9) is a flow with $\operatorname{val} \hat{f} = \operatorname{val} f + \iota(P)$.

11.3.2 A certain commodity is produced at two factories $x_1$ and $x_2$. The commodity is to be shipped to markets $y_1$, $y_2$ and $y_3$ through the network shown below. Use the labelling method to determine the maximum amount that can be shipped from the factories to the markets.




11.3.3 Show that, in any network $N$ (with integer capacities), there is a maximum flow $f$ such that $f(a)$ is an integer for all $a \in A$.

11.3.4 Consider a network $N$ such that with each arc $a$ is associated an integer $b(a) \leq c(a)$. Modify the labelling method to find a maximum flow $f$ in $N$ subject to the constraint $f(a) \geq b(a)$ for all $a \in A$ (assuming that there is an initial flow satisfying this condition).

11.3.5* Consider a network $N$ such that with each intermediate vertex $v$ is associated a non-negative integer $m(v)$. Show how a maximum flow $f$ satisfying the constraint $f^-(v) \leq m(v)$ for all $v \in V \backslash \{x, y\}$ can be found by applying the labelling method to a modified network.

## APPLICATIONS

### 11.4 MENGER'S THEOREMS

In this section, we shall use the max-flow min-cut theorem to obtain a number of theorems due to Menger (1927); two of these have already been mentioned in section 3.2. The following lemma provides a basic link.

*Lemma 11.4* Let $N$ be a network with source $x$ and sink $y$ in which each arc has unit capacity. Then

(a) the value of a maximum flow in $N$ is equal to the maximum number $m$ of arc-disjoint directed $(x, y)$-paths in $N$; and
(b) the capacity of a minimum cut in $N$ is equal to the minimum number $n$ of arcs whose deletion destroys all directed $(x, y)$-paths in $N$.

*Proof* Let $f^*$ be a maximum flow in $N$ and let $D^*$ denote the digraph obtained from $D$ by deleting all $f^*$-zero arcs. Since each arc of $N$ has unit capacity, $f^*(a) = 1$ for all $a \in A(D^*)$. It follows that

(i) $d^+_{D^*}(x) - d^-_{D^*}(x) = \text{val}\, f^* = d^-_{D^*}(y) - d^+_{D^*}(y)$;
(ii) $d^+_{D^*}(v) = d^-_{D^*}(v)$ for all $v \in V \backslash \{x, y\}$.

Therefore (exercise 10.3.3) there exist $\text{val}\, f^*$ arc-disjoint directed $(x, y)$-paths in $D^*$, and hence also in $D$. Thus

$$\text{val}\, f^* \leq m \tag{11.10}$$

Now let $P_1, P_2, \ldots, P_m$ be any system of $m$ arc-disjoint directed $(x, y)$-paths in $N$, and define a function $f$ on $A$ by

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is an arc of } \bigcup_{i=1}^{m} P_i \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f$ is a flow in $N$ with value $m$. Since $f^*$ is a maximum flow, we have

$$\text{val}\, f^* \geq m \tag{11.11}$$

It now follows from (11.10) and (11.11) that

$$\text{val}\, f^* = m$$

Let $\tilde{K} = (S, \bar{S})$ be a minimum cut in $N$. Then, in $N - \tilde{K}$, no vertex of $\bar{S}$ is reachable from any vertex in $S$; in particular, $y$ is not reachable from $x$. Thus $\tilde{K}$ is a set of arcs whose deletion destroys all directed $(x, y)$-paths, and we have

$$\text{cap}\, \tilde{K} = |\tilde{K}| \geq n \tag{11.12}$$

Now let $Z$ be a set of $n$ arcs whose deletion destroys all directed $(x, y)$-paths, and denote by $S$ the set of all vertices reachable from $x$ in $N - Z$. Since $x \in S$ and $y \in \bar{S}$, $K = (S, \bar{S})$ is a cut in $N$. Moreover, by the definition of $S$, $N - Z$ can contain no arc of $(S, \bar{S})$, and so $K \subseteq Z$. Since $\tilde{K}$ is a minimum cut, we conclude that

$$\text{cap}\, \tilde{K} \leq \text{cap}\, K = |K| \leq |Z| = n \tag{11.13}$$

Together, (11.12) and (11.13) now yield

$$\text{cap}\, \tilde{K} = n \quad \square$$

*Theorem 11.4* Let $x$ and $y$ be two vertices of a digraph $D$. Then the maximum number of arc-disjoint directed $(x, y)$-paths in $D$ is equal to the minimum number of arcs whose deletion destroys all directed $(x, y)$-paths in $D$.

*Proof* We obtain a network $N$ with source $x$ and sink $y$ by assigning unit capacity to each arc of $D$. The theorem now follows from lemma 11.4 and the max-flow min-cut theorem (11.3) $\square$

A simple trick immediately yields the undirected version of theorem 11.4.

*Theorem 11.5* Let $x$ and $y$ be two vertices of a graph $G$. Then the maximum number of edge-disjoint $(x, y)$-paths in $G$ is equal to the minimum number of edges whose deletion destroys all $(x, y)$-paths in $G$.

*Proof* Apply theorem 11.4 to $D(G)$, the associated digraph of $G$ (exercise 10.3.6) $\square$

*Corollary 11.5* A graph $G$ is $k$-edge-connected if and only if any two distinct vertices of $G$ are connected by at least $k$ edge-disjoint paths.

*Proof* This follows directly from theorem 11.5 and the definition of $k$-edge-connectedness $\square$

We now turn to the vertex versions of the above theorems.

*Theorem 11.6* Let $x$ and $y$ be two vertices of a digraph $D$, such that $x$ is not joined to $y$. Then the maximum number of internally-disjoint directed $(x, y)$-paths in $D$ is equal to the minimum number of vertices whose deletion destroys all directed $(x, y)$-paths in $D$.

*Proof* Construct a new digraph $D'$ from $D$ as follows:

(i) split each vertex $v \in V\backslash\{x, y\}$ into two new vertices $v'$ and $v''$, and join them by an arc $(v', v'')$;
(ii) replace each arc of $D$ with head $v \in V\backslash\{x, y\}$ by a new arc with head $v'$, and each arc of $D$ with tail $v \in V\backslash\{x, y\}$ by a new arc with tail $v''$. This construction is illustrated in figure 11.10.

Now to each directed $(x, y)$-path in $D'$ there corresponds a directed $(x, y)$-path in $D$ obtained by contracting all arcs of type $(v', v'')$; and, conversely, to each directed $(x, y)$-path in $D$, there corresponds a directed $(x, y)$-path in $D'$ obtained by splitting each internal vertex of the path. Furthermore, two directed $(x, y)$-paths in $D'$ are arc-disjoint if and only if the corresponding paths in $D$ are internally-disjoint. It follows that the maximum number of arc-disjoint directed $(x, y)$-paths in $D'$ is equal to the maximum number of internally-disjoint directed $(x, y)$-paths in $D$. Similarly, the minimum number of arcs in $D'$ whose deletion destroys all directed $(x, y)$-paths is equal to the minimum number of vertices in $D$ whose deletion destroys all directed $(x, y)$-paths (exercise 11.4.1). The theorem now follows from theorem 11.4 □

*Theorem 11.7* Let $x$ and $y$ be two nonadjacent vertices of a graph $G$. Then the maximum number of internally-disjoint $(x, y)$-paths in $G$ is equal to the minimum number of vertices whose deletion destroys all $(x, y)$-paths.

*Proof* Apply theorem 11.6 to $D(G)$, the associated digraph of $G$ □

The following corollary is immediate.

*Corollary 11.7* A graph $G$ with $\nu \geq k+1$ is $k$-connected if and only if any two distinct vertices of $G$ are connected by at least $k$ internally-disjoint paths.

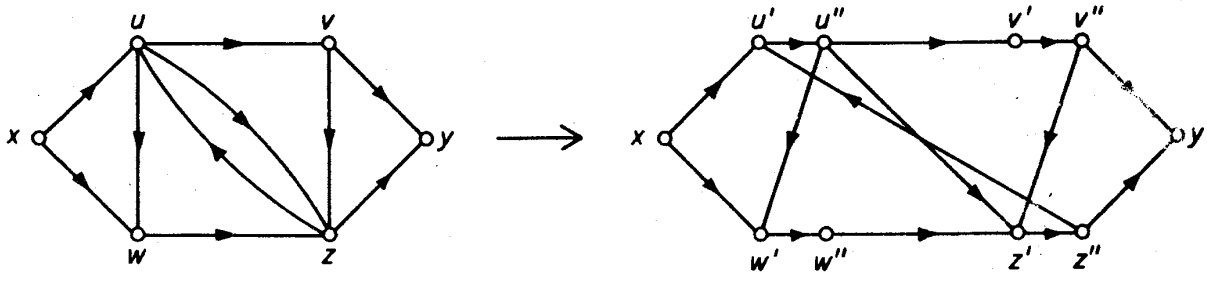



Figure 11.10

*Exercises*

11.4.1 Show that, in the proof of theorem 11.6, the minimum number of arcs in $D'$ whose deletion destroys all directed $(x, y)$-paths is equal to the minimum number of vertices in $D$ whose deletion destroys all directed $(x, y)$-paths.

11.4.2 Derive König's theorem (5.3) from theorem 11.7.

11.4.3 Let $G$ be a graph and let $S$ and $T$ be two disjoint subsets of $V$. Show that the maximum number of vertex-disjoint paths with one end in $S$ and one end in $T$ is equal to the minimum number of vertices whose deletion separates $S$ from $T$ (that is, after deletion no component contains a vertex of $S$ and a vertex of $T$).

11.4.4* Show that if $G$ is $k$-connected with $k \geq 2$, then any $k$ vertices of $G$ are contained together in some cycle. (G. A. Dirac)

## 11.5 FEASIBLE FLOWS

Let $N$ be a network. Suppose that to each source $x_i$ of $N$ is assigned a non-negative integer $\sigma(x_i)$, called the *supply* at $x_i$, and to each sink $y_j$ of $N$ is assigned a non-negative integer $\partial(y_j)$, called the *demand* at $y_j$. A flow $f$ in $N$ is said to be *feasible* if

$$f^+(x_i) - f^-(x_i) \leq \sigma(x_i) \quad \text{for all} \quad x_i \in X$$

and

$$f^-(y_j) - f^+(y_j) \geq \partial(y_j) \quad \text{for all} \quad y_j \in Y$$

In other words, a flow $f$ is feasible if the resultant flow out of each source $x_i$ relative to $f$ does not exceed the supply at $x_i$, and the resultant flow into each sink $y_j$ relative to $f$ is at least as large as the demand at $y_j$. A natural question, then, is to ask for necessary and sufficient conditions for the existence of a feasible flow in $N$. Theorem 11.8, due to Gale (1957), provides an answer to this question. It says that a feasible flow exists if and only if, for every subset $S$ of $V$, the total capacity of arcs from $S$ to $\bar{S}$ is at least as large as the net demand of $\bar{S}$.

For any subset $S$ of $V$, we shall denote $\sum_{v \in S} \sigma(v)$ by $\sigma(S)$ and $\sum_{v \in S} \partial(v)$ by $\partial(S)$.

*Theorem 11.8* There exists a feasible flow in $N$ if and only if, for all $S \subseteq V$

$$c(S, \bar{S}) \geq \partial(Y \cap \bar{S}) - \sigma(X \cap \bar{S}) \tag{11.14}$$

*Proof* Construct a new network $N'$ from $N$ as follows:

(i) adjoin two new vertices $x$ and $y$ to $N$;
(ii) join $x$ to each $x_i \in X$ by an arc of capacity $\sigma(x_i)$;

(iii) join each $y_j \in Y$ to $y$ by an arc of capacity $\partial(y_j)$;
(iv) designate $x$ as the source and $y$ as the sink of $N'$.

This construction is illustrated in figure 11.11.

It is not difficult to see that $N$ has a feasible flow if and only if $N'$ has a flow that saturates each arc of the cut $(Y, \{y\})$ (exercise 11.5.1). Now a flow in $N'$ that saturates each arc of $(Y, \{y\})$ clearly has value $\partial(Y) = \text{cap}(Y, \{y\})$, and is therefore, by corollary 11.1, a maximum flow. It follows that $N$ has a feasible flow if and only if, for each cut $(S \cup \{x\}, \bar{S} \cup \{y\})$ of $N'$

$$\text{cap}(S \cup \{x\}, \bar{S} \cup \{y\}) \geq \partial(Y) \tag{11.15}$$

But conditions (11.14) and (11.15) are precisely the same; for, denoting the capacity function in $N'$ by $c'$, we have

$$\begin{aligned}\text{cap}(S \cup \{x\}, \bar{S} \cup \{y\}) &= c'(S, \bar{S}) + c'(S, \{y\}) + c'(\{x\}, \bar{S}) \\ &= c(S, \bar{S}) + \partial(Y \cap S) + \sigma(X \cap \bar{S}) \quad \square\end{aligned}$$

There are many applications of theorem 11.8 to problems in graph theory. We shall discuss one such application.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two sequences of non-negative integers. We say that the pair $(\mathbf{p}, \mathbf{q})$ is *realisable by a simple bipartite graph* if there exists a simple bipartite graph $G$ with bipartition $(\{x_1, x_2, \ldots, x_m\}, \{y_1, y_2, \ldots, y_n\})$, such that

$$d(x_i) = p_i \quad \text{for} \quad 1 \leq i \leq m$$

and

$$d(y_j) = q_j \quad \text{for} \quad 1 \leq j \leq n$$

For example, the pair $(\mathbf{p}, \mathbf{q})$, where

$$\mathbf{p} = (3, 2, 2, 2, 1) \quad \text{and} \quad \mathbf{q} = (3, 3, 2, 1, 1)$$

is realisable by the bipartite graph of figure 11.12.

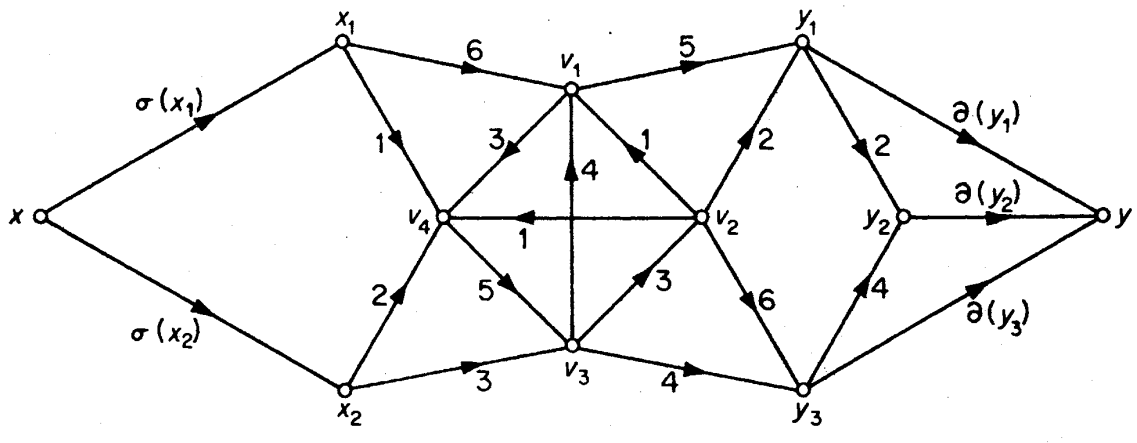



Figure 11.11

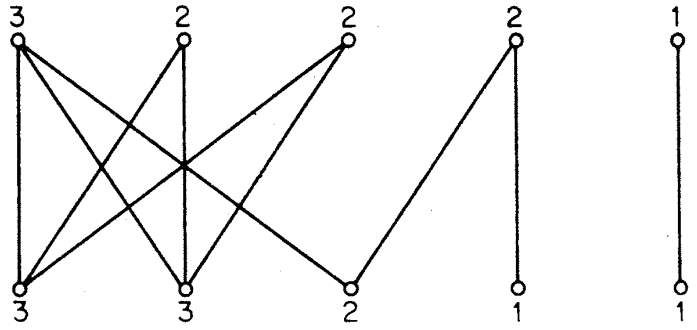



Figure 11.12

An obvious necessary condition for realisability is that

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \tag{11.16}$$

However, (11.16) is not in itself sufficient. For instance, the pair $(\mathbf{p}, \mathbf{q})$, where

$$\mathbf{p} = (5, 4, 4, 2, 1) \quad \text{and} \quad \mathbf{q} = (5, 4, 4, 2, 1)$$

is not realisable by any simple bipartite graph (exercise 11.5.2). In the following theorem we present necessary and sufficient conditions for the realisability of a pair of sequences by a simple bipartite graph. The order of the terms in the sequences clearly has no bearing on the question of realisability, and we shall find it convenient to assume that the terms of $\mathbf{q}$ are arranged in nonincreasing order

$$q_1 \geq q_2 \geq \ldots \geq q_n \tag{11.17}$$

*Theorem 11.9* Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two sequences of non-negative integers that satisfy (11.16) and (11.17). Then $(\mathbf{p}, \mathbf{q})$ is realisable by a simple bipartite graph if and only if

$$\sum_{i=1}^{m} \min\{p_i, k\} \geq \sum_{j=1}^{k} q_j \quad \text{for} \quad 1 \leq k \leq n \tag{11.18}$$

*Proof* Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two disjoint sets, and let $D$ be the digraph obtained from the complete bipartite graph with bipartition $(X, Y)$ by orienting each edge from $X$ to $Y$. We obtain a network $N$ by assigning unit capacity to each arc of $D$ and designating the vertices in $X$ and $Y$ as its sources and sinks, respectively. We shall assume, further, that the supply at source $x_i$ is $p_i$, $1 \leq i \leq m$, and that the demand at sink $y_j$ is $q_j$, $1 \leq j \leq n$.

Now, to each spanning subgraph of $D$, there corresponds a flow in $N$ which saturates precisely the arcs of the subgraph, and this correspondence is clearly one-one. In view of (11.16), it follows that $(\mathbf{p}, \mathbf{q})$ is realisable by a

simple bipartite graph if and only if the network $N$ has a feasible flow. We now use theorem 11.8.

For any set $S$ of vertices in $N$, write

$$I(S)=\{i \mid x_i \in S\} \quad \text{and} \quad J(S)=\{j \mid y_j \in S\}$$

Then, by definition,

$$\left.\begin{aligned} c(S, \bar{S}) &= |I(S)|\,|J(\bar{S})| \\ \sigma(X \cap \bar{S}) = \sum_{i \in I(\bar{S})} p_i \quad &\text{and} \quad \partial(Y \cap \bar{S}) = \sum_{j \in J(\bar{S})} q_j \end{aligned}\right\} \tag{11.19}$$

Suppose that $N$ has a feasible flow. By theorem 11.8 and (11.19)

$$|I(S)|\,|J(\bar{S})| \geq \sum_{j \in J(\bar{S})} q_j - \sum_{i \in I(\bar{S})} p_i$$

for any $S \subseteq X \cup Y$. Setting $S=\{x_i \mid p_i > k\} \cup \{y_j \mid j > k\}$, we have

$$\sum_{i \in I(\bar{S})} \min\{p_i, k\} \geq \sum_{j=1}^{k} q_j - \sum_{i \in I(\bar{S})} \min\{p_i, k\}$$

Since this holds for all values of $k$, (11.18) follows.

Conversely, suppose that (11.18) is satisfied. Let $S$ be any set of vertices in $N$. By (11.18) and (11.19)

$$c(S, \bar{S}) \geq \sum_{i \in I(\bar{S})} \min\{p_i, k\} \geq \sum_{j=1}^{k} q_j - \sum_{j \in I(\bar{S})} \min\{p_i, k\} \geq \partial(Y \cap \bar{S}) - \sigma(X \cap \bar{S})$$

where $k=|J(\bar{S})|$. It follows from theorem 11.8 that $N$ has a feasible flow □

We conclude by looking at theorem 11.9 from the viewpoint of matrices. With each simple bipartite graph $G$ having bipartition $(\{x_1, x_2, \ldots, x_m\}, \{y_1, y_2, \ldots, y_n\})$, we can associate an $m \times n$ matrix $\mathbf{B}$ in which $b_{ij}=1$ or 0, depending on whether $x_i y_j$ is an edge of $G$ or not. Conversely, every $m \times n$ (0, 1)-matrix corresponds in this way to a simple bipartite graph. Thus theorem 11.9 provides necessary and sufficient conditions for the existence of an $m \times n$ (0, 1)-matrix $\mathbf{B}$ with row sums $p_1, p_2, \ldots, p_m$ and column sums $q_1, q_2, \ldots, q_n$.

There is a simple way of visualising condition (11.18) in terms of matrices. Let $\mathbf{B}^*$ denote the (0, 1)-matrix in which the $p_i$ leading terms in each row $i$ are ones, and the remaining entries are zeros, and let $p_1^*, p_2^*, \ldots, p_n^*$ be the column sums of $\mathbf{B}^*$. The sequence $\mathbf{p}^*=(p_1^*, p_2^*, \ldots, p_n^*)$ is called the *conjugate* of $\mathbf{p}$. The conjugate of (5, 4, 4, 2, 1) is (5, 4, 3, 3, 1), for example (see figure 11.13).

Now consider the sum $\sum_{j=1}^{k} p_j^*$. Row $i$ of $\mathbf{B}^*$ contributes $\min\{p_i, k\}$ to this sum. Therefore the left-hand side of (11.18) is equal to $\sum_{j=1}^{k} p_j^*$, and (11.18) is

$$\begin{array}{cc} & \mathbf{p}^* \\ & \begin{array}{ccccc} 5 & 4 & 3 & 3 & 1 \end{array} \\ \mathbf{p} \begin{array}{c} 5 \\ 4 \\ 4 \\ 2 \\ 1 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 11.13

equivalent to the condition

$$\sum_{j=1}^{k} p_j^* \geq \sum_{j=1}^{k} q_j \quad \text{for} \quad 1 \leq k \leq n$$

This formulation of theorem 11.9 in terms of (0, 1)-matrices is due to Ryser (1957). For other applications of the theory of flows in networks, we refer the reader to Ford and Fulkerson (1962).

*Exercises*

11.5.1 Show that the network $N$ in the proof of theorem 11.8 has a feasible flow if and only if $N'$ has a flow that saturates each arc of the cut $(Y, \{y\})$.

11.5.2 Show that the pair $(\mathbf{p}, \mathbf{q})$, where

$$\mathbf{p} = (5, 4, 4, 2, 1) \quad \text{and} \quad \mathbf{q} = (5, 4, 4, 2, 1)$$

is not realisable by any simple bipartite graph.

11.5.3 Given two sequences, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, find necessary and sufficient conditions for the existence of a digraph $D$ on the vertex set $\{v_1, v_2, \ldots, v_n\}$, such that (i) $d^-(v_i) = p_i$ and $d^+(v_i) = q_i$, $1 \leq i \leq n$, and (ii) $D$ has a (0, 1) adjacency matrix.

11.5.4* Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be two nonincreasing sequences of non-negative integers, and denote the sequences $(p_2, p_3, \ldots, p_m)$ and $(q_1 - 1, q_2 - 1, \ldots, q_{p_1} - 1, q_{p_1+1}, \ldots, q_n)$ by $\mathbf{p}'$ and $\mathbf{q}'$, respectively.

(*a*) Show that $(\mathbf{p}, \mathbf{q})$ is realisable by a simple bipartite graph if and only if the same is true of $(\mathbf{p}', \mathbf{q}')$.

(*b*) Using (*a*), describe an algorithm for constructing a simple bipartite graph which realises $(\mathbf{p}, \mathbf{q})$, if such a realisation exists.

11.5.5 An $(m+n)$-regular graph $G$ is $(m,n)$-*orientable* if it can be oriented so that each indegree is either $m$ or $n$.

(*a*)* Show that $G$ is $(m,n)$-orientable if and only if there is a partition $(V_1, V_2)$ of $V$ such that, for every $S \subseteq V$,

$$|(m-n)(|V_1 \cap S| - |V_2 \cap S|)| \leq |[S, \bar{S}]|$$

(*b*) Deduce that if $G$ is $(m,n)$-orientable and $m > n$, then $G$ is also $(m-1, n+1)$-orientable.

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# 12 The Cycle Space and Bond Space

## 12.1 CIRCULATIONS AND POTENTIAL DIFFERENCES

Let $D$ be a digraph. A real-valued function $f$ on $A$ is called a *circulation* in $D$ if it satisfies the conservation condition at each vertex:

$$f^-(v) = f^+(v) \quad \text{for all} \quad v \in V \tag{12.1}$$

If we think of $D$ as an electrical network, then such a function $f$ represents a circulation of currents in $D$. Figure 12.1 shows a circulation in a digraph.

If $f$ and $g$ are any two circulations and $r$ is any real number, then it is easy to verify that both $f+g$ and $rf$ are also circulations. Thus the set of all circulations in $D$ is a vector space. We denote this space by $\mathscr{C}$. In what follows, we shall find it convenient to identify a subset $S$ of $A$ with $D[S]$, the subdigraph of $D$ induced by $S$.

There are certain circulations of special interest. These are associated with cycles in $D$. Let $C$ be a cycle in $D$ with an assigned orientation and let $C^+$ denote the set of arcs of $C$ whose direction agrees with this orientation. We associate with $C$ the function $f_C$ defined by

$$f_C(a) = \begin{cases} 1 & \text{if} \quad a \in C^+ \\ -1 & \text{if} \quad a \in C \backslash C^+ \\ 0 & \text{if} \quad a \notin C \end{cases}$$

Clearly, $f_C$ satisfies (12.1) and hence is a circulation. Figure 12.2 depicts a circulation associated with a cycle.

We shall see later on that each circulation is a linear combination of the circulations associated with cycles. For this reason we refer to $\mathscr{C}$ as the *cycle space* of $D$.

We now turn our attention to a related class of functions. Given a function $p$ on the vertex set $V$ of $D$, we define the function $\delta p$ on the arc set $A$ by the rule that, if an arc $a$ has tail $x$ and head $y$, then

$$\delta p(a) = p(x) - p(y) \tag{12.2}$$

If $D$ is thought of as an electrical network with potential $p(v)$ at $v$, then, by (12.2), $\delta p$ represents the potential difference along the wires of the network. For this reason a function $g$ on $A$ is called a *potential difference* in $D$ if

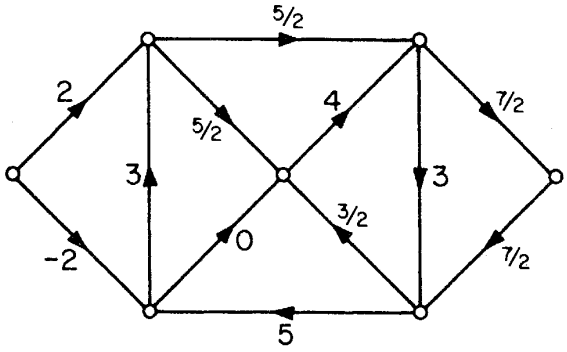



Figure 12.1. A circulation

$g = \delta p$ for some function $p$ on $V$. Figure 12.3 shows a digraph with an assignment of potentials to its vertices and the corresponding potential difference.

As with circulations, the set $\mathscr{B}$ of all potential differences in $D$ is closed under addition and scalar multiplication and, hence, is a vector space.

Analogous to the function $f_C$ associated with a cycle $C$, there is a function $g_B$ associated with a bond $B$. Let $B = [S, \bar{S}]$ be a bond of $D$. We define $g_B$ by

$$g_B(a) = \begin{cases} 1 & \text{if} \quad a \in (S, \bar{S}) \\ -1 & \text{if} \quad a \in (\bar{S}, S) \\ 0 & \text{if} \quad a \notin B \end{cases}$$

It can be verified that $g_B = \delta p$ where

$$p(v) = \begin{cases} 1 & \text{if} \quad v \in S \\ 0 & \text{if} \quad v \in \bar{S} \end{cases}$$

Figure 12.4 depicts the potential difference associated with a bond.

We shall see that each potential difference is a linear combination of potential differences associated with bonds. For this reason we refer to $\mathscr{B}$ as the *bond space* of $D$.

In studying the properties of the two vector spaces $\mathscr{B}$ and $\mathscr{C}$, we shall find

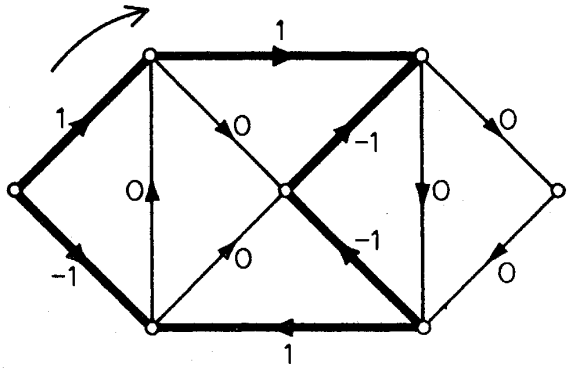



Figure 12.2

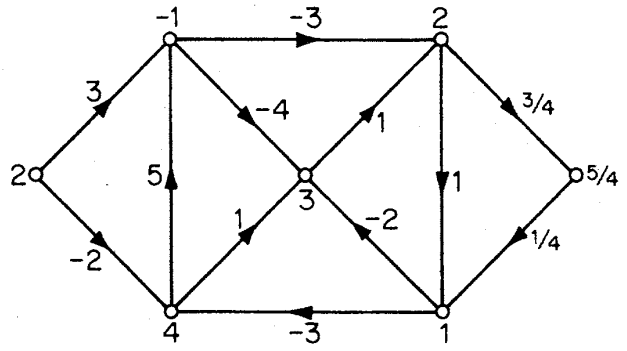



Figure 12.3. A potential difference

it convenient to regard a function on $A$ as a row vector whose coordinates are labelled with the elements of $A$. The relationship between $\mathscr{B}$ and $\mathscr{C}$ is best seen by introducing the incidence matrix of $D$. With each vertex $v$ of $D$ we associate the function $m_v$ on $A$ defined by

$$m_v(a) = \begin{cases} 1 & \text{if} \quad a \text{ is a link and } v \text{ is the tail of } a \\ -1 & \text{if} \quad a \text{ is a link and } v \text{ is the head of } a \\ 0 & \text{otherwise} \end{cases}$$

The *incidence matrix* of $D$ is the matrix $\boldsymbol{M}$ whose rows are the functions $m_v$. Figure 12.5 shows a digraph and its incidence matrix.

*Theorem 12.1* Let $\boldsymbol{M}$ be the incidence matrix of a digraph $D$. Then $\mathscr{B}$ is the row space of $\boldsymbol{M}$ and $\mathscr{C}$ is its orthogonal complement.

*Proof* Let $g = \delta p$ be a potential difference in $D$. It follows from (12.2) that

$$g(a) = \sum_{v \in V} p(v) m_v(a) \quad \text{for all} \quad a \in A$$

Thus $g$ is a linear combination of the rows of $\boldsymbol{M}$. Conversely, any linear combination of the rows of $\boldsymbol{M}$ is a potential difference. Hence $\mathscr{B}$ is the row space of $\boldsymbol{M}$.

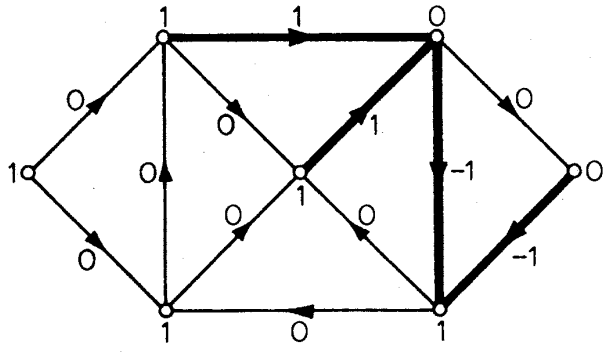
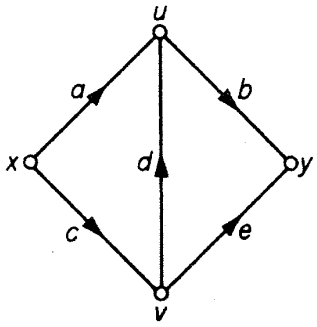



Figure 12.4




(a)

| | a | b | c | d | e |
|---|---|---|---|---|---|
| x | 1 | 0 | 1 | 0 | 0 |
| u | -1 | 1 | 0 | -1 | 0 |
| v | 0 | 0 | -1 | 1 | 1 |
| y | 0 | -1 | 0 | 0 | -1 |

(b)

Figure 12.5. (*a*) $D$; (*b*) the incidence matrix of $D$

Now let $f$ be a function on $A$. The condition (12.1) for $f$ to be a circulation can be rewritten as

$$\sum_{a \in A} m_v(a) f(a) = 0 \quad \text{for all} \quad v \in V$$

This implies that $f$ is a circulation if and only if it is orthogonal to each row of $\mathbf{M}$. Hence $\mathscr{C}$ is the orthogonal complement of $\mathscr{B}$ □

The *support* of a function $f$ on $A$ is the set of elements of $A$ at which the value of $f$ is nonzero. We denote the support of $f$ by $\|f\|$.

*Lemma 12.2.1* If $f$ is a nonzero circulation, then $\|f\|$ contains a cycle.

*Proof* This follows immediately, since $\|f\|$ clearly cannot contain a vertex of degree one □

*Lemma 12.2.2* If g is a nonzero potential difference, then $\|g\|$ contains a bond.

*Proof* Let $g = \delta p$ be a nonzero potential difference in $D$. Choose a vertex $u \in V$ which is incident with an arc of $\|g\|$ and set

$$U = \{v \in V \mid p(v) = p(u)\}$$

Clearly, $\|g\| \supseteq [U, \bar{U}]$ since $g(a) \neq 0$ for all $a \in [U, \bar{U}]$. But, by the choice of $u$, $[U, \bar{U}]$ is nonempty. Thus $\|g\|$ contains a bond □

A matrix $\mathbf{B}$ is called a *basis matrix* of $\mathscr{B}$ if the rows of $\mathbf{B}$ form a basis for $\mathscr{B}$; a basis matrix of $\mathscr{C}$ is similarly defined. We shall find the following notation convenient. If $\mathbf{R}$ is a matrix whose columns are labelled with the elements of $A$, and if $S \subseteq A$, we shall denote by $\mathbf{R} \mid S$ the submatrix of $\mathbf{R}$ consisting of those columns of $\mathbf{R}$ labelled with elements in $S$. If $\mathbf{R}$ has a single row, our notation is the same as the usual notation for the restriction of a function to a subset of its domain.

*Theorem 12.2* Let **B** and **C** be basis matrices of $\mathscr{B}$ and $\mathscr{C}$, respectively. Then, for any $S \subseteq A$

(i) the columns of $\mathbf{B} \mid S$ are linearly independent if and only if $S$ is acyclic, and
(ii) the columns of $\mathbf{C} \mid S$ are linearly independent if and only if $S$ contains no bond.

*Proof* Denote the column of **B** corresponding to arc $a$ by $\mathbf{B}(a)$. The columns of $\mathbf{B} \mid S$ are linearly dependent if and only if there exists a function $f$ on $A$ such that

$$f(a) \neq 0 \quad \text{for some} \quad a \in S$$
$$f(a) = 0 \quad \text{for all} \quad a \notin S$$

and

$$\sum_{a \in A} f(a)\mathbf{B}(a) = \mathbf{O}$$

We conclude that the columns of $\mathbf{B} \mid S$ are linearly dependent if and only if there exists a nonzero circulation $f$ such that $\|f\| \subseteq S$. Now if there is such an $f$ then, by lemma 12.2.1, $S$ contains a cycle. On the other hand, if $S$ contains a cycle $C$, then $f_C$ is a nonzero circulation with $\|f_C\| = C \subseteq S$. It follows that the columns of $\mathbf{B} \mid S$ are linearly independent if and only if $S$ is acyclic. A similar argument using lemma 12.2.2 yields a proof of (ii) □

*Corollary 12.2* The dimensions of $\mathscr{B}$ and $\mathscr{C}$ are given by

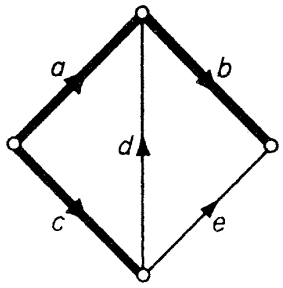
$$\dim \mathscr{B} = \nu - \omega \tag{12.3}$$
$$\dim \mathscr{C} = \varepsilon - \nu + \omega \tag{12.4}$$

*Proof* Consider a basis matrix **B** of $\mathscr{B}$. By theorem 12.2

$$\operatorname{rank} \mathbf{B} = \max\{|S| \mid S \subseteq A,\ S \text{ acyclic}\}$$

The above maximum is attained when $S$ is a maximal forest of $D$, and is therefore (exercise 2.2.4) equal to $\nu - \omega$. Since $\dim \mathscr{B} = \operatorname{rank} \mathbf{B}$, this establishes (12.3). Now (12.4) follows, since $\mathscr{C}$ is the orthogonal complement of $\mathscr{B}$ □

Let $T$ be a maximal forest of $D$. Associated with $T$ is a special basis matrix of $\mathscr{C}$. If $a$ is an arc of $\bar{T}$, then $T + a$ contains a unique cycle. Let $C_a$ denote this cycle and let $f_a$ denote the circulation corresponding to $C_a$, defined so that $f_a(a) = 1$. The $(\varepsilon - \nu + \omega) \times \varepsilon$ matrix **C** whose rows are $f_a$, $a \in \bar{T}$, is a basis matrix of $\mathscr{C}$. This follows from the fact that each row is a circulation and that $\operatorname{rank} \mathbf{C} = \varepsilon - \nu + \omega$ (because $\mathbf{C} \mid \bar{T}$ is an identity matrix). We refer to **C** as the basis matrix of $\mathscr{C}$ *corresponding to* $T$. Figure 12.6*b* shows the basis matrix of $\mathscr{C}$ corresponding to the tree indicated in figure 12.6*a*.




(a)

| | a | b | c | d | e |
|---|---|---|---|---|---|
| $f_d$ | −1 | 0 | 1 | 1 | 0 |
| $f_e$ | −1 | −1 | 1 | 0 | 1 |

(b)

| | a | b | c | d | e |
|---|---|---|---|---|---|
| $g_a$ | 1 | 0 | 0 | 1 | 1 |
| $g_b$ | 0 | 1 | 0 | 0 | 1 |
| $g_c$ | 0 | 0 | 1 | −1 | −1 |

(c)

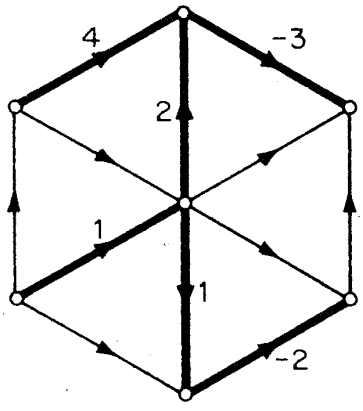
Figure 12.6

Analogously, if $a$ is an arc of $T$, then $\bar{T}+a$ contains a unique bond (see theorem 2.6). Let $B_a$ denote this bond and $g_a$ the potential difference corresponding to $B_a$, defined so that $g_a(a)=1$. The $(\nu-\omega)\times\varepsilon$ matrix **B** whose rows are $g_a$, $a \in T$, is a basis matrix of $\mathscr{B}$, called the basis matrix of $\mathscr{B}$ *corresponding to* $T$. Figure 12.6c gives an example of such a matrix.

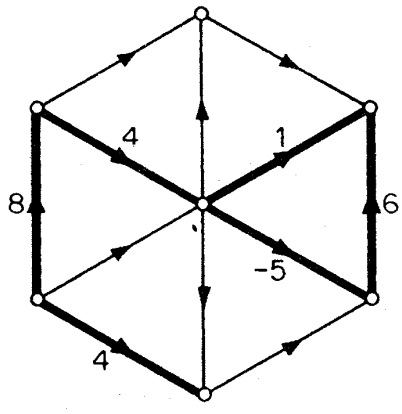
The relationship between cycles and bonds that has become apparent from the foregoing discussion finds its proper setting in the theory of matroids. The interested reader is referred to Tutte (1971).

*Exercises*

12.1.1 (*a*) In figure (i) below is indicated a function on a spanning tree and in figure (ii) a function on the complement of the tree. Extend the function in (i) to a potential difference and the function in (ii) to a circulation.




(i)




(ii)

(*b*) Let $f$ be a circulation and $g$ a potential difference in $D$, and let $T$ be a spanning tree of $D$. Show that $f$ is uniquely determined by $f \mid \bar{T}$ and $g$ by $g \mid T$.

**12.1.2** (*a*) Let **B** and **C** be basis matrices of $\mathscr{B}$ and $\mathscr{C}$ and let $T$ be any spanning tree of $D$. Show that **B** is uniquely determined by $\mathbf{B} \mid T$ and **C** is uniquely determined by $\mathbf{C} \mid \bar{T}$.

(*b*) Let $T$ and $T_1$ be two fixed spanning trees of $D$. Let $\mathbf{B}$ and $\mathbf{B}_1$ denote the basis matrices of $\mathscr{B}$, and $\mathbf{C}$ and $\mathbf{C}_1$ the basis matrices of $\mathscr{C}$, corresponding to the trees $T$ and $T_1$. Show that $\mathbf{B} = (\mathbf{B} \mid T_1)\mathbf{B}_1$ and $\mathbf{C} = (\mathbf{C} \mid \bar{T}_1)\mathbf{C}_1$.

**12.1.3** Let $\mathbf{K}$ denote the matrix obtained from the incidence matrix $\boldsymbol{M}$ of a connected digraph $D$ by deleting any one of its rows. Show that $\mathbf{K}$ is a basis matrix of $\mathscr{B}$.

12.1.4 Show that if $G$ is a plane graph, then $\mathscr{B}(G) \cong \mathscr{C}(G^*)$ and $\mathscr{C}(G) \cong \mathscr{B}(G^*)$.

12.1.5 A *circulation* of $D$ *over a field* $F$ is a function $f: A \to F$ which satisfies (12.1) in $F$; a *potential difference* of $D$ *over* $F$ is similarly defined. The vector spaces of these potential differences and circulations are denoted by $\mathscr{B}_F$ and $\mathscr{C}_F$. Show that theorem 12.2 remains valid if $\mathscr{B}$ and $\mathscr{C}$ are replaced by $\mathscr{B}_F$ and $\mathscr{C}_F$, respectively.

## 12.2 THE NUMBER OF SPANNING TREES

In this section we shall derive a formula for the number of spanning trees in a graph.

Let $G$ be a connected graph and let $T$ be a fixed spanning tree of $G$. Consider an arbitrary orientation $D$ of $G$ and let $\mathbf{B}$ be the basis matrix of $\mathscr{B}$ corresponding to $T$. It follows from theorem 12.2 that if $S$ is a subset of $A$ with $|S| = \nu - 1$ then the square submatrix $\mathbf{B} \mid S$ is nonsingular if and only if $S$ is a spanning tree of $G$. Thus the number of spanning trees of $G$ is equal to the number of nonsingular submatrices of $\mathbf{B}$ of order $\nu - 1$.

A matrix is said to be *unimodular* if all its full square submatrices have determinants 0, +1 or −1. The proof of the following theorem is due to Tutte (1965*b*).

*Theorem 12.3* The basis matrix $\mathbf{B}$ is unimodular.

*Proof* Let $\mathbf{P}$ be a full submatrix of $\mathbf{B}$ (one of order $\nu - 1$). Suppose that $\mathbf{P} = \mathbf{B} \mid T_1$. We may assume that $T_1$ is a spanning tree of $D$ since, otherwise, $\det \mathbf{P} = 0$ by theorem 12.2. Let $\mathbf{B}_1$ denote the basis matrix of $\mathscr{B}$ corresponding to $T_1$. Then (exercise 12.1.2*b*)

$$(\mathbf{B} \mid T_1)\mathbf{B}_1 = \mathbf{B}$$

Restricting both sides to $T$, we obtain

$$(\mathbf{B} \mid T_1)(\mathbf{B}_1 \mid T) = \mathbf{B} \mid T$$

Noting that $\mathbf{B} \mid T$ is an identity matrix, and taking determinants, we get

$$\det(\mathbf{B} \mid T_1)\det(\mathbf{B}_1 \mid T) = 1 \tag{12.5}$$

Both determinants in (12.5), being determinants of integer matrices, are themselves integers. It follows that $\det(\mathbf{B} \mid T_1) = \pm 1$ □

*Theorem 12.4* $\tau(G) = \det \mathbf{BB}'$ (12.6)

*Proof* Using the formula for the determinant of the product of two rectangular matrices (see Hadley, 1961), we obtain

$$\det \mathbf{BB}' = \sum_{\substack{S \subseteq A \\ |S| = \nu - 1}} (\det(\mathbf{B} \mid S))^2 \tag{12.7}$$

Now, by theorem 12.2, the number of nonzero terms in (12.7) is equal to $\tau(G)$. But, by theorem 12.3, each such term has value 1 □

One can similarly show that if $\mathbf{C}$ is a basis matrix of $\mathscr{C}$ corresponding to a tree, then $\mathbf{C}$ is unimodular and

$$\tau(G) = \det \mathbf{CC}' \tag{12.8}$$

*Corollary 12.4* $\tau(G) = \pm \det \begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix}$

*Proof* By (12.6) and (12.8)

$$(\tau(G))^2 = \det \mathbf{BB}' \det \mathbf{CC}' = \det \left[\begin{array}{c:c} \mathbf{BB}' & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{CC}' \end{array}\right]$$

Since $\mathscr{B}$ and $\mathscr{C}$ are orthogonal, $\mathbf{BC}' = \mathbf{CB}' = \mathbf{0}$. Thus

$$(\tau(G))^2 = \det \left[\begin{array}{c:c} \mathbf{BB}' & \mathbf{BC}' \\ \hdashline \mathbf{CB}' & \mathbf{CC}' \end{array}\right] = \det \left( \begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix} [\mathbf{B}' \mid \mathbf{C}'] \right)$$

$$= \det \begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix} \det[\mathbf{B}' \mid \mathbf{C}'] = \left( \det \begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix} \right)^2$$

The corollary follows on taking square roots □

Since theorem 12.2 is valid for all basis matrices of $\mathscr{B}$, (12.6) clearly holds for any such matrix $\mathbf{B}$ that is unimodular. In particular, a matrix $\mathbf{K}$ obtained by deleting any one row of the incidence matrix $\boldsymbol{M}$ is unimodular (exercise 12.2.1*a*). Thus

$$\tau(G) = \det \mathbf{KK}'$$

This expression for the number of spanning trees in a graph is implicit in the work of Kirchhoff (1847), and is known as the *matrix-tree theorem.*

*Exercises*

**12.2.1** Show that

(*a*)* a matrix **K** obtained from ***M*** by deleting any one row is unimodular;

(*b*) $\tau(G) = \pm\det\begin{bmatrix}\mathbf{K}\\ \cdots \\ \mathbf{C}\end{bmatrix}$

12.2.2 The *conductance matrix* $\boldsymbol{C} = [c_{ij}]$ of a loopless graph $G$ is the $\nu \times \nu$ matrix in which

$$c_{ii} = \sum_{j \neq i} a_{ij} \quad \text{for all} \quad i$$

$$c_{ij} = -a_{ij} \quad \text{for all} \quad i \quad \text{and} \quad j \quad \text{with} \quad i \neq j$$

where $\mathbf{A} = [a_{ij}]$ is the adjacency matrix of $G$. Show that

(*a*) $\boldsymbol{C} = \boldsymbol{MM}'$, where ***M*** is the incidence matrix of any orientation of $G$;

(*b*) all cofactors of $\boldsymbol{C}$ are equal to $\tau(G)$.

12.2.3 A matrix is *totally unimodular* if all square submatrices have determinants 0, +1 or −1. Show that

(*a*) any basis matrix of $\mathscr{B}$ or $\mathscr{C}$ corresponding to a tree is totally unimodular;

(*b*) the incidence matrix of a simple graph $G$ is totally unimodular if and only if $G$ is bipartite.

12.2.4 Let $F$ be a field of characteristic $p$. Show that

(*a*) if **B** and **C** are basis matrices of $\mathscr{B}_F$ and $\mathscr{C}_F$, respectively, corresponding to a tree, then $\det\begin{bmatrix}\mathbf{B}\\ \cdots \\ \mathbf{C}\end{bmatrix} = \pm\,\tau(G) (\bmod p)$;

(*b*) $\dim(\mathscr{B}_F \cap \mathscr{C}_F) > 0$ if and only if $p \mid \tau(G)$. (H. Shank)

## APPLICATIONS

### 12.3 PERFECT SQUARES

A *squared rectangle* is a rectangle dissected into at least two (but a finite number of) squares. If no two of the squares in the dissection have the same size, then the squared rectangle is *perfect*. The *order* of a squared rectangle is the number of squares into which it is dissected. Figure 12.7 shows a perfect rectangle of order 9. A squared rectangle is *simple* if it does not contain a rectangle which is itself squared. Clearly, every squared rectangle is composed of ones that are simple.

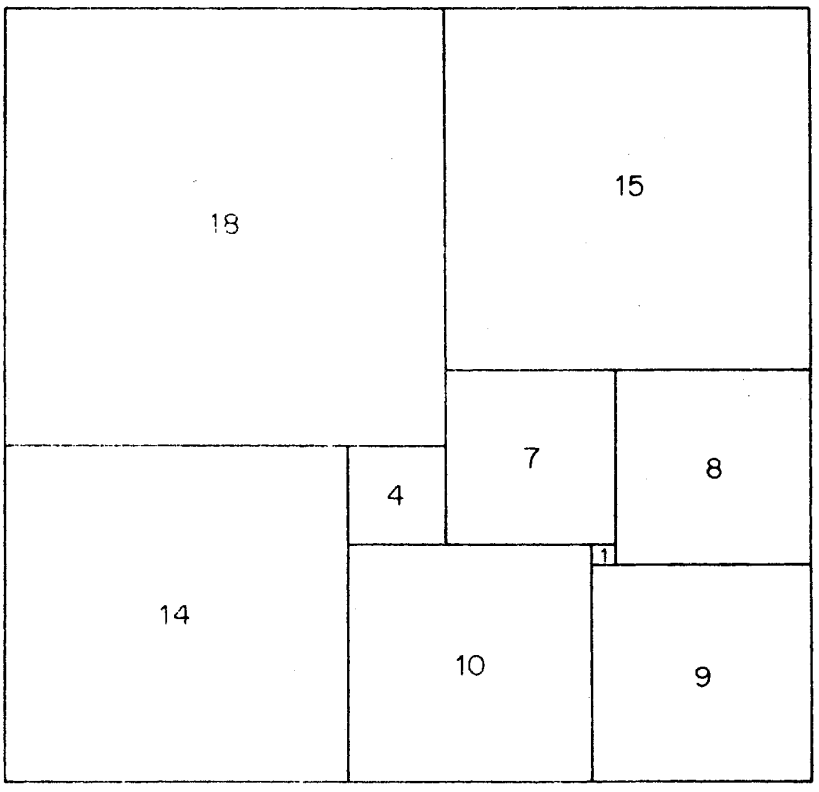



Figure 12.7. A perfect rectangle

For a long time no perfect squares were known, and it was conjectured that such squares did not exist. Sprague (1939) was the first to publish an example of a perfect square. About the same time, Brooks et al. (1940) developed systematic methods for their construction by using the theory of graphs. In this section, we shall present a brief discussion of their methods.

We first show how a digraph can be associated with a given squared rectangle $R$. The union of the horizontal sides of the constituent squares in the dissection consists of horizontal line segments; each such segment is called a *horizontal dissector* of $R$. In figure 12.8*a*, the horizontal dissectors are indicated by solid lines. We can now define the digraph $D$ associated with $R$. To each horizontal dissector of $R$ there corresponds a vertex of $D$; two vertices $v_i$ and $v_j$ of $D$ are joined by an arc $(v_i, v_j)$ if and only if their corresponding horizontal dissectors $H_i$ and $H_j$ flank some square of the dissection and $H_i$ lies above $H_j$ in $R$. Figure 12.8*b* shows the digraph associated with the squared rectangle in figure 12.8*a*. The vertices corresponding to the upper and lower sides of $R$ are called the *poles* of $D$ and are denoted by $x$ and $y$, respectively.

We now assign to each vertex $v$ of $D$ a potential $p(v)$ equal to the height (above the lower side of $R$) of the corresponding horizontal dissector. If we regard $D$ as an electrical network in which each wire has unit resistance, the potential difference $g = \delta p$ determines a flow of currents from $x$ to $y$ (see

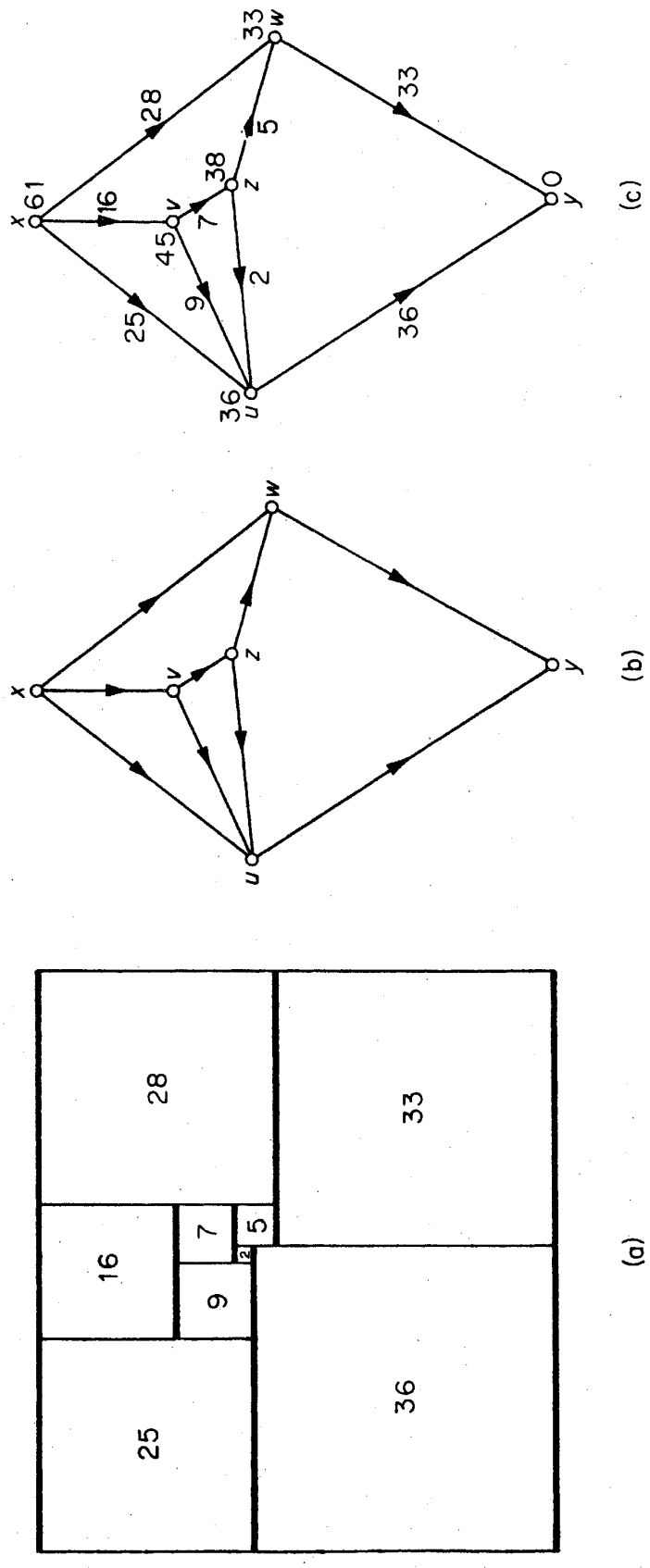

x
61
16
v
45
7
38
z
5
33
w
28
33
25
9
2
36
u
36
0
y
(c)
x
v
z
w
y
u
(b)
28
33
16
7
5
2
9
25
36
(a)

Figure 12.8

figure 12.8c). These currents satisfy *Kirchhoff's current law:* the total amount of current entering a vertex $v \in V\backslash\{x, y\}$ is equal to the total amount leaving it. For example, the total amount entering $u$ in figure 12.8c is $25+9+2=36$, and the same amount leaves this vertex.

Let $D$ be the digraph corresponding to a squared rectangle $R$, with poles $x$ and $y$, and let $G$ be the underlying graph of $D$. Then the graph $G+xy$ is called the *horizontal graph* of $R$. Brooks et al. (1940) showed that the horizontal graph of any simple squared rectangle is a 3-connected planar graph (their definition of connectivity differs slightly from the one used in this book). They also showed that, conversely, if $H$ is a 3-connected planar graph and $xy \in E(H)$, then any flow of currents from $x$ to $y$ in $H-xy$ determines a squared rectangle. Thus one possible way of searching for perfect rectangles of order $n$ is to

(i) list all 3-connected planar graphs with $n+1$ edges, and
(ii) for each such graph $H$ and each edge $xy$ of $H$, determine a flow of currents from $x$ to $y$ in $H-xy$.

Tutte (1961) showed that every 3-connected planar graph can be derived from a wheel by a sequence of operations involving face subdivisions and the taking of duals. Bouwkamp, Duijvestijn and Medema (1960) then applied Tutte's theorem to list all 3-connected planar graphs with at most 16 edges. Here we shall see how the theory developed in sections 12.1 and 12.2 can be used in computing a flow of currents from $x$ to $y$ in a digraph $D$.

Let $g(a)$ denote the current in arc $a$ of $D$, and suppose that the total current leaving $x$ is $\sigma$. Then

$$\sum_{a \in A} m_x(a)g(a) = \sigma \tag{12.9}$$

Kirchhoff's current law can be formulated as

$$\sum_{a \in A} m_v(a)g(a) = 0 \quad \text{for all} \quad v \in V\backslash\{x, y\} \tag{12.10}$$

Now, since g is a potential difference, it is orthogonal to every circulation. Therefore,

$$\mathbf{C}g' = \mathbf{0} \tag{12.11}$$

where $\mathbf{C}$ is a basis matrix of $\mathscr{C}$ corresponding to a tree $T$ of $D$ and $g'$ is the transpose of the vector g. Equations (12.9)–(12.11) together give the matrix equation

$$\begin{bmatrix} \mathbf{K} \\ \hline \mathbf{C} \end{bmatrix} g' = \begin{bmatrix} \sigma \\ \hline \mathbf{0} \end{bmatrix} \tag{12.12}$$

where $\mathbf{K}$ is the matrix obtained from $\boldsymbol{M}$ by deleting the row $m_y$. This

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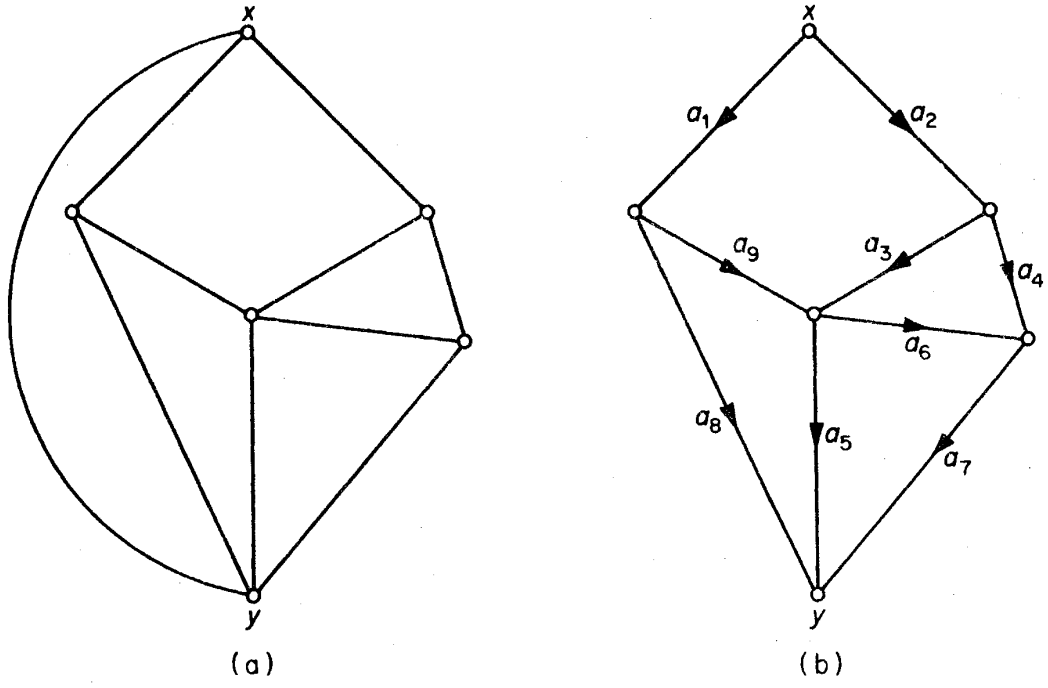


Figure 12.9

equation can be solved using Cramér's rule. Note that, since $\det\begin{bmatrix}\mathbf{K}\\ \hline \mathbf{C}\end{bmatrix} = \pm\tau(G)$ (exercise 12.2.1*b*), we obtain a solution in integers if $\sigma = \tau(G)$. Thus, in computing the currents, it is convenient to take the total current leaving $x$ to be equal to the number of spanning trees of $D$.

We illustrate the above procedure with an example. Consider the 3-connected planar graph in figure 12.9*a*. On deleting the edge $xy$ and orienting each edge we obtain the digraph $D$ of figure 12.9*b*.

It can be checked that the number of spanning trees in $D$ is 66. By considering the tree $T = \{a_1, a_2, a_3, a_4, a_5\}$ we obtain the following nine equations, as in (12.12), (with $g(a_i)$ written simply as $g_i$).

$$
\begin{aligned}
g_1 + g_2 \qquad\qquad\qquad\qquad\qquad\qquad &= 66\\
g_1 \qquad\qquad\qquad\qquad\qquad - g_8 - g_9 &= 0\\
g_2 - g_3 - g_4 \qquad\qquad\qquad\qquad\qquad &= 0\\
g_3 \qquad - g_5 - g_6 \qquad\qquad + g_9 &= 0\\
g_4 \qquad + g_6 - g_7 \qquad\qquad &= 0\\
g_3 - g_4 \qquad + g_6 \qquad\qquad\qquad &= 0\\
- g_3 + g_4 - g_5 \qquad + g_7 \qquad\qquad &= 0\\
g_1 - g_2 - g_3 \qquad - g_5 \qquad\qquad + g_8 \qquad &= 0\\
g_1 - g_2 - g_3 \qquad\qquad\qquad\qquad + g_9 &= 0
\end{aligned}
$$

The solution to this system of equations is given by

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9) = (36, 30, 14, 16, 20, 2, 18, 28, 8)$$

The squared rectangle based on this flow of currents is just the one in figure 12.7 with all dimensions doubled.

Figure 12.10 shows a simple perfect square of order 25. It was discovered by Wilson (1967), and is the smallest (least order) such square known.

Further results on perfect squares can be found in the survey article by Tutte (1965*a*).

*Exercises*

12.3.1 Show that the constituent squares in a squared rectangle have commensurable sides.

12.3.2 The *vertical graph* of a squared rectangle $R$ is the horizontal graph of the squared rectangle obtained by rotating $R$ through a right angle. If no point of $R$ is the corner of four constituent squares, show that the horizontal and vertical graphs of $R$ are duals.

12.3.3* A *perfect cube* is a cube dissected into a finite number of smaller cubes, no two of the same size. Show that there exists no perfect cube.

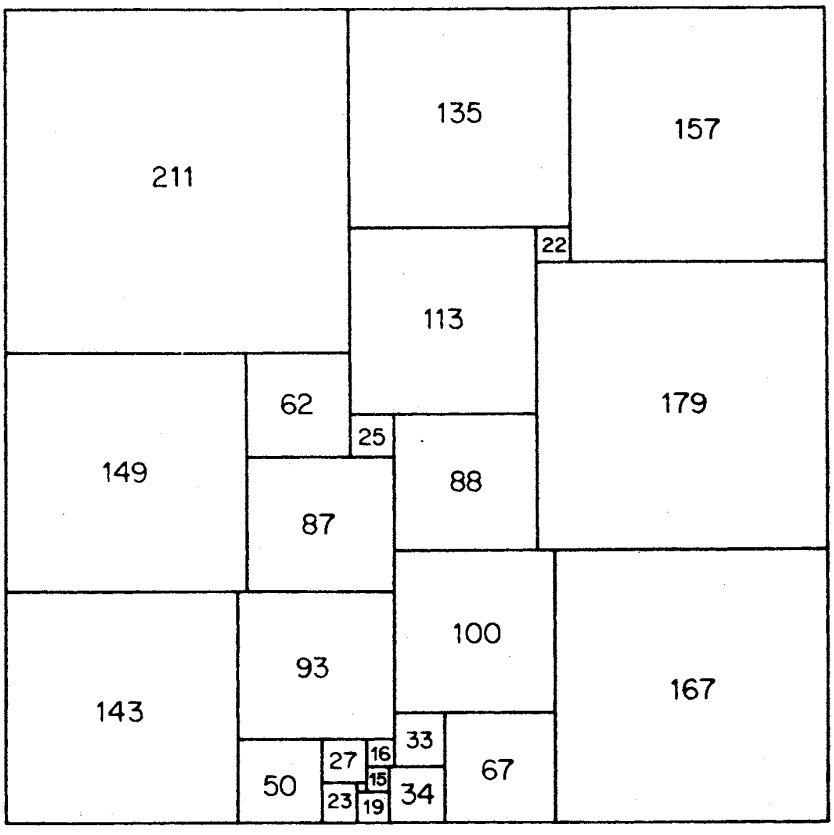


Figure 12.10. A simple perfect square of order 25

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# Appendix I
# Hints to Starred Exercises

1.2.9(*b*) If $G \not\cong T_{m,n}$, then $G$ has parts of size $n_1, n_2, \ldots, n_m$, with $n_i - n_j > 1$ for some $i$ and $j$. Show that the complete $m$-partite graph with parts of size $n_1, n_2, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_m$ has more edges than $G$.

1.3.3 In terms of the adjacency matrix $\mathbf{A}$, an automorphism of $G$ is a permutation matrix $\mathbf{P}$ such that $\mathbf{PAP}' = \mathbf{A}$ or, equivalently, $\mathbf{PA} = \mathbf{AP}$ (since $\mathbf{P}' = \mathbf{P}^{-1}$). Show that if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ belonging to an eigenvalue $\lambda$, then, for any automorphism $\mathbf{P}$ of $G$, so is $\mathbf{Px}$. Since the eigenvalues of $\mathbf{A}$ are distinct and $\mathbf{P}$ is orthogonal, $\mathbf{P}^2\mathbf{x} = \mathbf{x}$ for all eigenvectors $\mathbf{x}$.

1.4.5 Suppose that all induced subgraphs of $G$ on $n$ vertices have $m$ edges. Show that, for any two vertices $v_i$ and $v_j$,

$$\varepsilon(G) - d(v_i) = \varepsilon(G - v_i) = m\binom{\nu-1}{n}\Big/\binom{\nu-3}{n-2}$$

$$\varepsilon(G) - d(v_i) - d(v_j) + a_{ij} = \varepsilon(G - v_i - v_j) = m\binom{\nu-2}{n}\Big/\binom{\nu-4}{n-2}$$

where $a_{ij} = 1$ or $0$ according as $v_i$ and $v_j$ are adjacent or not. Deduce that $a_{ij}$ is independent of $i$ and $j$.

1.5.7(*a*) To prove the necessity, first show that if $G$ is simple with $u_1v_1$, $u_2v_2 \in E$ and $u_1v_2$, $u_2v_1 \notin E$, then $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$ has the same degree sequence as $G$. Using this, show that if $\mathbf{d}$ is graphic, then there is a simple graph $G$ with $V = \{v_1, v_2, \ldots, v_n\}$ such that (i) $d(v_i) = d_i$ for $1 \le i \le n$, and (ii) $v_1$ is joined to $v_2, v_3, \ldots, v_{d_1+1}$. The graph $G - v_1$ has degree sequence $\mathbf{d}'$.

1.5.8 Show that a bipartite subgraph with the largest possible number of edges has this property.

1.5.9 Define a graph on $S$ in which $x_i$ and $x_j$ are adjacent if and only if they are at distance one. Show that in this graph each vertex has degree at most six.

1.7.3 Consider a longest path and the vertices adjacent to the origin of this path.

1.7.6(*b*) By contradiction. Let $G$ be a smallest counter-example. Show that (i) the girth of $G$ is at least five, and (ii) $\delta \ge 3$. Deduce that $\nu \le 8$ and show that no such graph exists.

2.1.10 To prove the sufficiency, consider a graph $G$ with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_\nu)$ and as few components as possible. If

$G$ is not connected, show that, by a suitable exchange of edges (as in the hint to exercise 1.5.7$a$), there is a graph with degree sequence $\mathbf{d}$ and fewer components than $G$.

2.2.12 Define a labelled graph $G$ as follows: the vertices of $G$ are the subsets $A_1, A_2, \ldots, A_n$, and $A_i$ is joined to $A_j$ $(i \neq j)$ by an edge labelled $a$ if either $A_i = A_j \cup \{a\}$ or $A_j = A_i \cup \{a\}$. For any subgraph $H$ of $G$, let $L(H)$ be the set of labels on edges of $H$. Show that if $F$ is a maximal forest of $G$, then $L(F) = L(G)$. Any element $x$ in $S \backslash L(F)$ has the required property.

2.4.2 Several applications of theorem 2.8 yield the recurrence relation

$$w_n - 4w_{n-1} + 4w_{n-2} - 1 = 0$$

where $w_n$ is the number of spanning trees in the wheel with $n$ spokes. Solve this recurrence relation.

3.2.6 Form a new graph $G'$ by adding two vertices $x$ and $y$, and joining $x$ to all vertices in $X$ and $y$ to all vertices in $Y$. Show that $G'$ is 2-connected and apply theorem 3.2.

3.2.7($a$) Use induction on $\varepsilon$. Let $e_1 \in E$. If $G \cdot e_1$ is a critical block, then $G \cdot e_1$ has a vertex of degree two and, hence, so does $G$. If $G \cdot e_1$ is not critical, there is an $e_2 \in E \backslash \{e_1\}$ such that $(G \cdot e_1) - e_2$ is a block. Using the fact that $(G \cdot e_1) - e_2 = (G - e_2) \cdot e_1$, show that $e_1$ and $e_2$ are incident with a vertex of degree two in $G$.

($b$) Use ($a$) and induction on $\nu$.

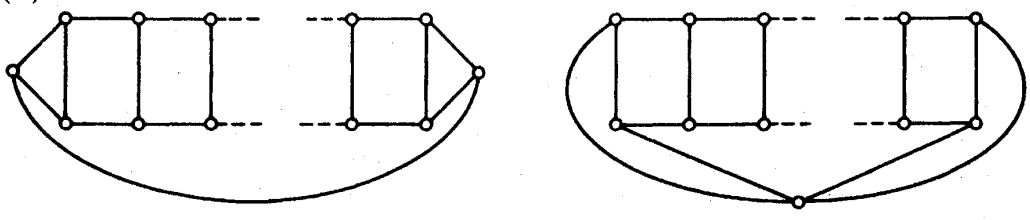
4.1.6 Necessity: if $G - v$ contains a cycle $C$, consider an Euler tour (with origin $v$) of the component of $G - E(C)$ that contains $v$. Sufficiency: let $Q$ be a $(v, w)$-trail of $G$ which is not an Euler tour. Show that $G - E(Q)$ has exactly one nontrivial component.

4.2.4 Form a new graph $G'$ by adding a new vertex and joining it to every vertex of $G$. Show that $G$ has a Hamilton path if and only if $G'$ has a Hamilton cycle, and apply theorem 4.5.

4.2.6 Form a new graph $G'$ by adding edges so that $G'[X]$ is complete. Show that $G$ is hamiltonian if and only if $G'$ is hamiltonian, and apply theorem 4.5.

4.2.9 Let $P$ be a longest path in $G$. If $P$ has length $l < 2\delta$, show, using the proof technique of theorem 4.3, that $G$ has a cycle of length $l + 1$. Now use the fact that $G$ is connected to obtain a contradiction.

4.2.11($b$)



$\nu$ even $\qquad$ $\nu$ odd

4.2.13 Use the fact that the Petersen graph is hypohamiltonian (exercise 4.2.12).

4.4.1 Consider an Euler tour $Q$ in the weighted graph formed from $T$ by duplicating each of its edges. Now make use of triangle inequalities to obtain from $Q$ a Hamilton cycle in $G$ of weight at most $w(Q)$.

5.1.5($a$) To show that $K_{2n}$ is 1-factorable, arrange the vertices in the form of a regular $(2n-1)$-gon with one vertex in the centre. A radial edge together with the edges perpendicular to it is a perfect matching.

5.1.6 Label the vertices $0, 1, 2, \ldots, 2n$ and arrange the vertices $1, 2, \ldots, 2n$ in a circle with 0 at the centre. Let $C = (0, 1, 2, 2n, 3, 2n-1, 4, 2n-2, \ldots, n+2, n+1, 0)$ and consider the rotations of $C$.

5.2.3($b$) Let $G$ be a $2k$-regular graph with $V = \{v_1, v_2, \ldots, v_\nu\}$; without loss of generality, assume that $G$ is connected. Let $C$ be an Euler tour in $G$. Form a bipartite graph $G'$ with bipartition $(X, Y)$, where $X = \{x_1, x_2, \ldots, x_\nu\}$ and $Y = \{y_1, y_2, \ldots, y_\nu\}$ by joining $x_i$ to $y_j$ whenever $v_i$ immediately precedes $v_j$ on $C$. Show that $G'$ is 1-factorable and hence that $G$ is 2-factorable.

5.2.8 Construct a bipartite graph $G$ with bipartition $(X, Y)$ in which $X$ is the set of rows of $\mathbf{Q}$, $Y$ is the set of columns of $\mathbf{Q}$, and row $i$ is joined to column $j$ if and only if the entry $q_{ij}$ is positive. Show that $G$ has a perfect matching, and then use induction on the number of nonzero entries of $\mathbf{Q}$.

5.3.1 Let $G$ be a bipartite graph with bipartition $(X, Y)$. Assume that $\nu$ is even (the case when $\nu$ is odd requires a little modification). Obtain a graph $H$ from $G$ by joining all pairs of vertices in $Y$. $G$ has a matching that saturates every vertex in $X$ if and only if $H$ has a perfect matching.

5.3.4 Let $G^*$ be a maximal spanning supergraph of $G$ such that the number of edges in a maximum matching of $G^*$ is the same as for $G$. Show, using the proof technique ot theorem 5.4, that if $U$ is the set of vertices of degree $\nu - 1$ in $G^*$ then $G^* - U$ is a disjoint union of complete graphs.

6.2.1 See the hint to exercise 5.1.5$a$.

6.2.8 Use the proof technique of theorem 6.2.

7.1.3($b$) Let $v_1 v_2 \ldots v_n$ be a longest path in $G$. Show that $G - v_2$ has at most one nontrivial component, and use induction on $\varepsilon$.

7.2.6($b$) Let $p(m-1) = n-1$. The complete $(p+1)$-partite graph with $m-1$ vertices in each part shows that $r(T, K_{1,n}) > (p+1)(m-1) = m+n-2$. To prove that $r(T, K_{1,n}) \leq m+n-1$, show that any simple graph $G$ with $\delta \geq m-1$ contains every tree $T$ on $m$ vertices.

(*c*) The complete $(n-1)$-partite graph with $m-1$ vertices in each part shows that $r(T, K_n) > (m-1)(n-1)$. To prove that $r(T, K_n) \leq (m-1)(n-1)+1$, use induction on $n$ and the fact that any simple graph with $\delta \geq m-1$ contains every tree $T$ on $m$ vertices.

7.3.3(*c*) Assume $G$ contains no triangle. Choose a shortest odd cycle $C$ in $G$. Show that each vertex in $V(G)\backslash V(C)$ can be joined to at most two vertices of $C$. Apply exercise 7.3.3*a* to $G - V(C)$, and obtain a contradiction.

7.3.4(*a*) $G$ contains $K_{2,m}$ if and only if there are $m$ vertices with a pair of common neighbours. Any vertex $v$ has $\binom{d(v)}{2}$ pairs of neighbours. Therefore if $\sum_{v \in V} \binom{d(v)}{2} > (m-1)\binom{\nu}{2}$, $G$ contains $K_{2,m}$.

7.5.1 Define a graph $G$ by $V(G) = \{x_1, \ldots, x_n\}$, and $E(G) = \{x_i x_j \mid d(x_i, x_j) = 1\}$, and show that if all edges of $G$ are drawn as straight line segments, then (i) any two edges of $G$ are either adjacent or cross, and (ii) if some vertex of $G$ has degree greater than two, it is adjacent to a vertex of degree one. Then prove (a) by induction on $n$.

8.1.6 Let $\mathscr{C} = (V_1, V_2, \ldots, V_k)$ be a $k$-colouring of $G$, and let $\mathscr{C}'$ be a colouring of $G$ in which each colour class contains at least two vertices. If $|V_i| \geq 2$ for all $i$, there is nothing to prove, so assume that $V_1 = \{v_1\}$. Let $u_2 \in V_2$ be a vertex of the same colour as $v_1$ in $\mathscr{C}'$. Clearly $|V_2| \geq 2$. If $|V_2| > 2$, transfer $u_2$ to $V_1$. Otherwise, let $v_2$ be the other vertex in $V_2$. In $\mathscr{C}'$, $v_1$ and $v_2$ must be assigned different colours. Let $u_3 \in V_3$ be a vertex of the same colour as $v_2$ in $\mathscr{C}'$. As before, $|V_3| \geq 2$. Proceeding in this way, one must eventually find a set $V_i$ with $|V_i| > 2$. $G$ can now be recoloured so that fewer colour classes contain only one vertex.

8.1.13(*a*) Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be $n$-colourings of $G[X]$ and $G[Y]$, respectively. Construct a bipartite graph $H$ with bipartition $(\{x_1, x_2, \ldots, x_n\}, \{y_1, y_2, \ldots, y_n\})$ by joining $x_i$ and $y_j$ if and only if the edge cut $[X_i, Y_j]$ is empty in $G$. Using exercise 5.2.6*b*, show that $H$ has a perfect matching. If $x_i$ is matched with $y_{f(i)}$ under this matching, let $V_i = X_i \cup Y_{f(i)}$. Show that $(V_1, V_2, \ldots, V_n)$ is an $n$-colouring of $G$.

8.3.1 Show that it suffices to consider 2-connected graphs. Choose a longest cycle $C$ in $G$ and show that there are two paths across $C$ as in theorem 8.5.

8.3.2(*a*) If $\delta \geq 3$, use exercise 8.3.1. If there is a vertex of degree less than three, delete it and use induction.

8.4.8 Consider the expansion of $\pi_k(G)$ in terms of chromatic polynomials of complete graphs.

8.5.2(*a*) It is easily verified that $H$ has girth at least six. If $H$ is $k$-colourable, there is a $\nu$-element subset of $S$ all of whose members receive the same colour. Consider the corresponding copy of $G$ and obtain a contradiction.

9.2.8 The dual $G^*$ is 2-edge-connected and 3-regular and, hence (corollary 5.4), has a perfect matching $M$. $(G^* \cdot M)^*$ is a bipartite subgraph of $G$.

10.2.2 Form a new digraph on the same vertex set joining $u$ to $v$ if $v$ is reachable from $u$, and apply corollary 10.1.

10.2.5 Let $D_1$ and $D_2$ be the spanning subdigraphs of $D$ such that the arcs of $D_1$ are the arcs $(u, v)$ of $D$ for which $f(u) \leq f(v)$, and the arcs of $D_2$ are the arcs $(u, v)$ for which $f(u) > f(v)$. Show that either $\chi(D_1) > m$ or $\chi(D_2) > n$, and apply theorem 10.1.

10.3.4 Let $v_1 v_2 \ldots v_{2n+1} v_1$ be an odd cycle. If $(v_i, v_{i+1}) \in A$, set $P_i = (v_i, v_{i+1})$; if $(v_i, v_{i+1}) \notin A$, let $P_i$ be a directed $(v_i, v_{i+1})$-path. If some $P_i$ is of even length, $P_i + (v_{i+1}, v_i)$ is a directed odd cycle; otherwise, $P_1 P_2 \ldots P_{2n+1}$ is a closed directed trail of odd length, and therefore contains a directed odd cycle.

11.3.5 Use the construction given in the proof of theorem 11.6, and assign capacity $m(v)$ to arc $(v', v'')$.

11.4.4 Use induction on $k$ and exercise 11.4.3.

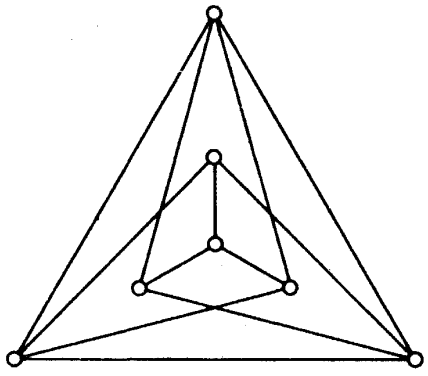
11.5.4 Use an argument similar to that in exercise 1.5.7.

11.5.5(*a*) Necessity follows on taking $V_1$ as the set of vertices with indegree $m$ and $V_2$ as the set of vertices with indegree $n$. To prove sufficiency, construct a network $N$ by forming the associated digraph of $G$, assigning unit capacity to each arc, and regarding the elements of $V_1$ as sources and the elements of $V_2$ as sinks. By theorem 11.8, there is a flow $f$ in $N$ (which can be assumed integral) in which the supply at each source and demand at each sink is $|m-n|$. The $f$-saturated arcs induce an $(m,n)$-orientation on a subgraph $H$ of $G$. An $(m,n)$-orientation of $G$ can now be obtained by giving the remaining edges an eulerian orientation.

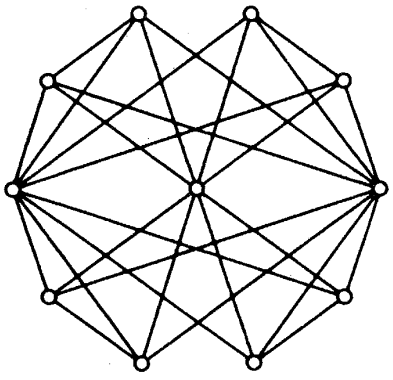
12.2.1(*a*) Use induction on the order of the submatrix. Let $\mathbf{P}$ be a square submatrix. If each column of $\mathbf{P}$ contains two nonzero entries, then $\det \mathbf{P} = 0$. Otherwise, expand $\det \mathbf{P}$ about a column with exactly one nonzero entry, and apply the induction hypothesis.

12.3.3 Show, first, that in any perfect rectangle the smallest constituent square is not on the boundary of the rectangle. Now suppose that there is a perfect cube and consider the perfect square induced on the base of this cube by the constituent cubes.

# Appendix II
# Four Graphs and a Table of their Properties

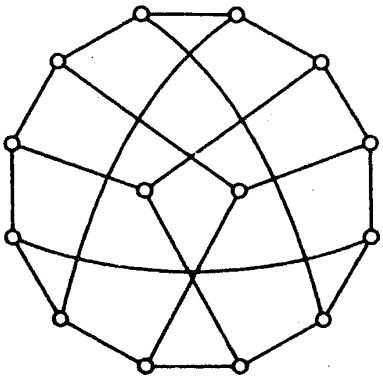


$G_1$

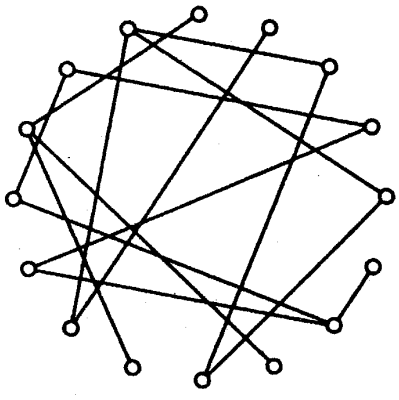


$G_2$

| | $\nu$ | $\varepsilon$ | $\delta$ | $\Delta$ | $\omega$ | $\kappa$ | $\kappa$ | $\alpha$ | $\alpha'$ | $\beta$ | $\beta'$ | $\chi$ | $\chi'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G_1$ | 7 | 12 | 3 | 4 | 1 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
| $G_2$ | 11 | 28 | 4 | 8 | 1 | 3 | 4 | 4 | 5 | 7 | 6 | 3 | 8 |
| $G_3$ | 14 | 21 | 3 | 3 | 1 | 3 | 3 | 7 | 7 | 7 | 7 | 2 | 3 |
| $G_4$ | 16 | 15 | 1 | 3 | 3 | 0 | 0 | 9 | 7 | 7 | 9 | 3 | 3 |



$G_3$



$G_4$

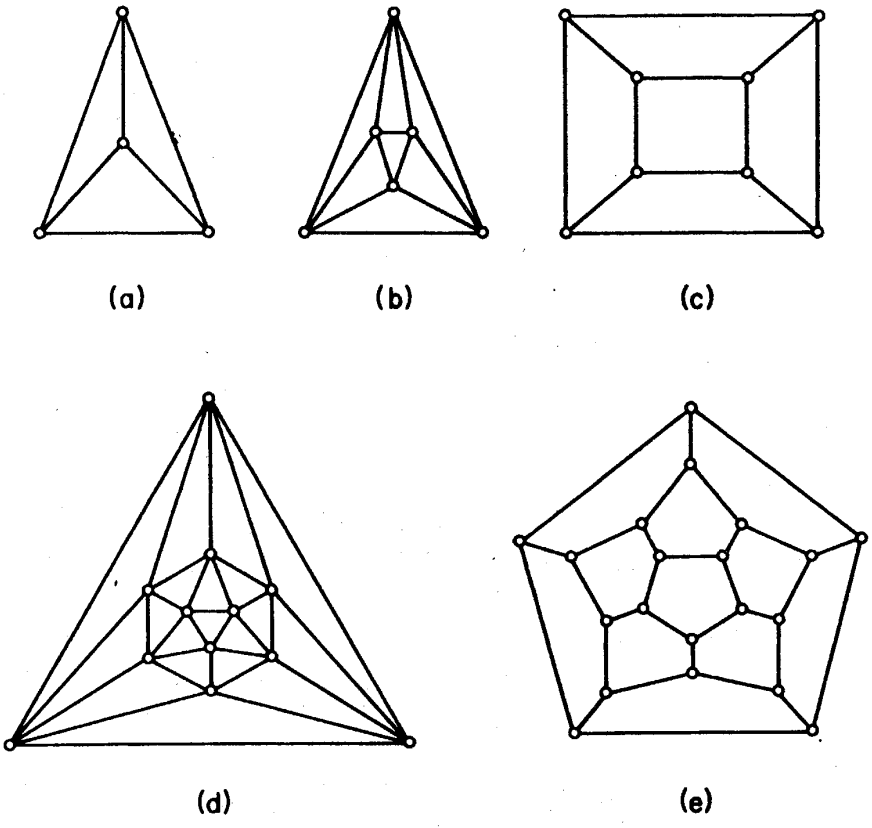
| diameter | girth | bipartite? | eulerian? | hamiltonian? | critical? | planar? |
|---|---|---|---|---|---|---|
| 2 | 3 | No | No | Yes | Yes | Yes |
| 2 | 3 | No | Yes | No | No | No |
| 3 | 6 | Yes | No | Yes | No | No |
| ∞ | 4 | No | No | No | No | Yes |

# Appendix III
# Some Interesting Graphs

There are a number of graphs which are interesting because of their special structure. We have already met some of these (for example, the Grinberg graph, the Grötzsch graph, the Herschel graph and the Ramsey graphs). Here we present a selection of other interesting graphs and families of graphs.

### THE PLATONIC GRAPHS

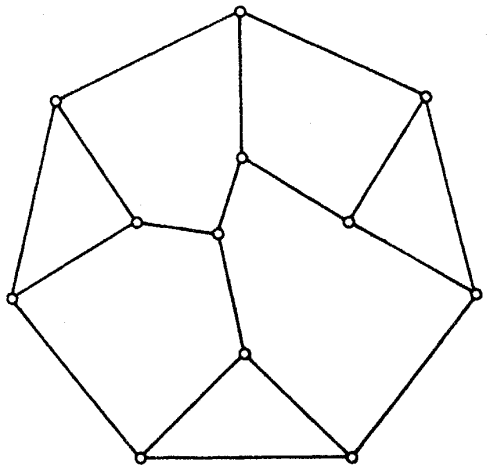
These are the graphs whose vertices and edges are the vertices and edges of the platonic solids (see Fréchet and Fan, 1967).



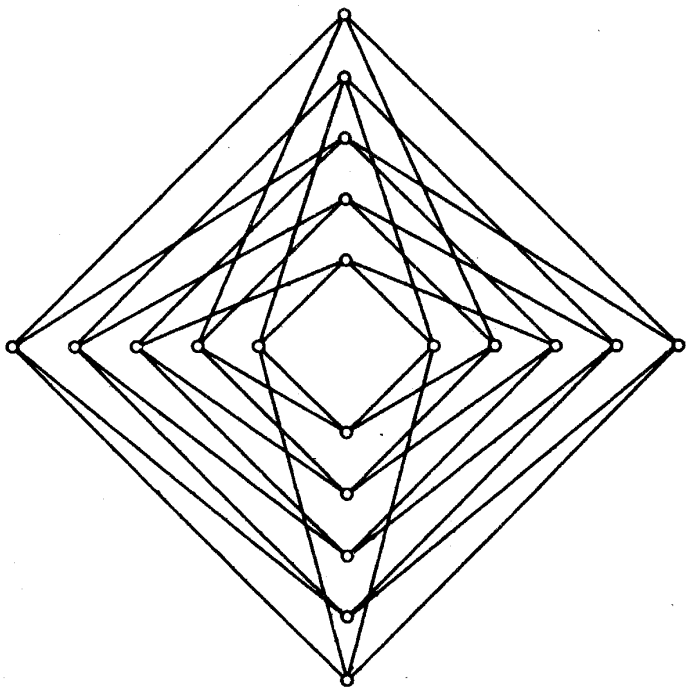

(*a*) The tetrahedron; (*b*) the octahedron; (*c*) the cube; (*d*) the icosahedron; (*e*) the dodecahedron

AUTOMORPHISM GROUPS

(i) As has already been noted (exercise 1.2.12), every group is isomorphic to the automorphism group of some graph. Frucht (1949) showed, in fact, that for any group there is a 3-regular graph with that group. The smallest 3-regular graph whose group is the identity is the following:



(ii) Folkman (1967) proved that every edge- but not vertex-transitive regular graph has at least 20 vertices. This result is best possible:



The Folkman graph

The *Gray graph* (see Bouwer, 1972) is a 3-regular edge- but not vertex-transitive graph on 54 vertices. It has the following description: take three copies of $K_{3,3}$. For a particular edge $e$, subdivide $e$ in each of the three

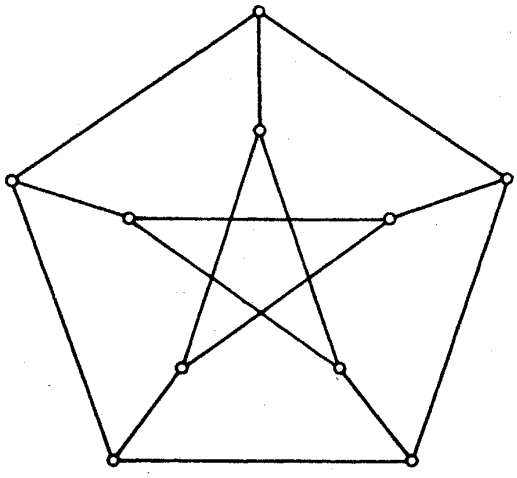
copies and join the resulting three vertices to a new vertex. Repeat this with each edge.

CAGES

An $m$-regular graph of girth $n$ with the least possible number of vertices is called an $(m, n)$-*cage*. If we denote by $f(m, n)$ the number of vertices in an $(m, n)$-cage, it is easy to see that $f(2, n) = n$ and for $m \geq 3$,

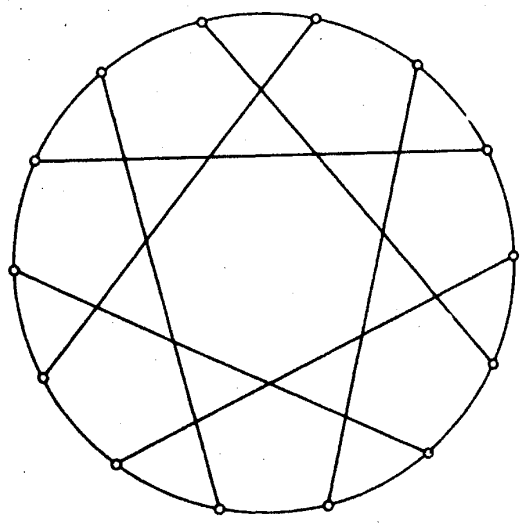
$$f(m, n) \geq \begin{cases} \dfrac{m(m-1)^r - 2}{m-2} & \text{if} \quad n = 2r+1 \\ \dfrac{2(m-1)^r - 2}{m-2} & \text{if} \quad n = 2r \end{cases} \tag{III.1}$$

The $(2, n)$-cage is the $n$-cycle, the $(m, 3)$-cage is $K_{m+1}$, and the $(m, 4)$-cage is $K_{m,m}$. In each of these cases, equality holds in (III.1). It has been shown by Hoffman and Singleton (1960) that, for $m \geq 3$ and $n \geq 5$, equality can hold in (III.1) only if $n = 5$ and $m = 3$, 7 or 57, or $n = 6$, 8 or 12. When $m - 1$ is a prime power, the $(m, 6)$-cage is the point-line incidence graph of the projective plane of order $m - 1$; the $(m, 8)$- and $(m, 12)$-cages are also obtained from projective geometries (see Biggs, 1974 for further details). Some of the smaller $(m, n)$-cages are depicted below:



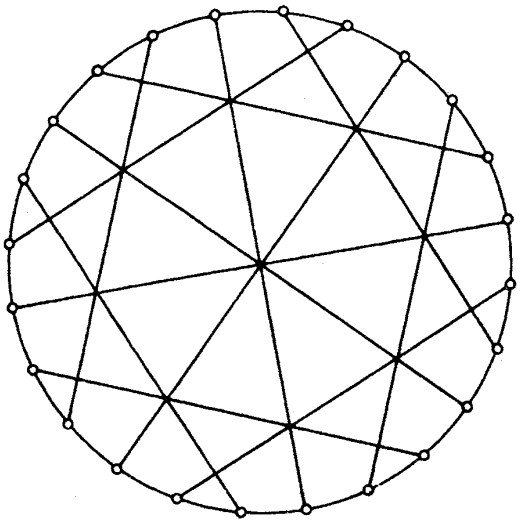
(3,5) - cage

The Petersen graph



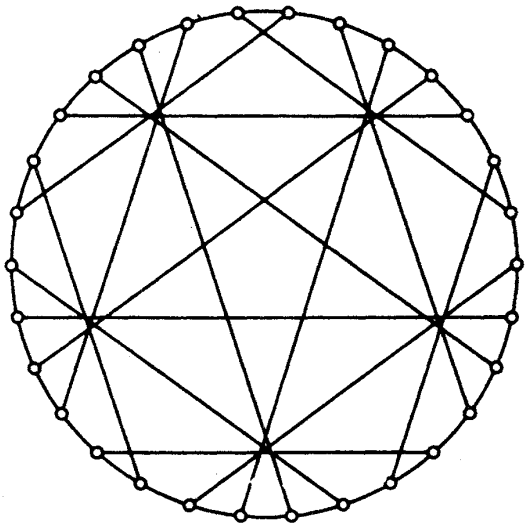
(3,6) - cage

The Heawood graph



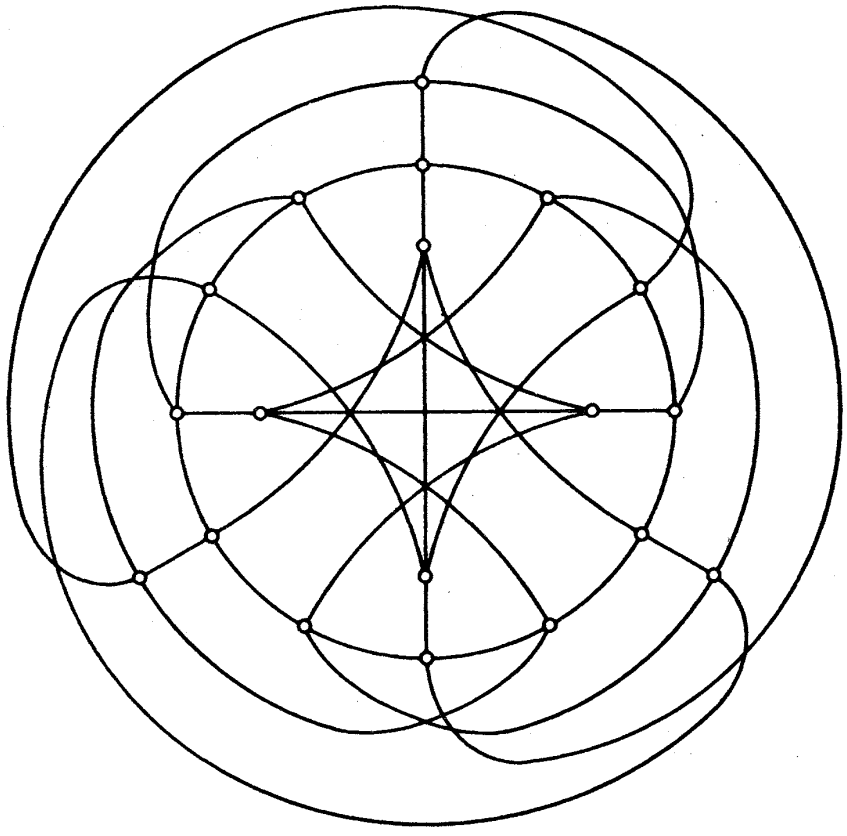
(3,7)-cage

The McGee graph



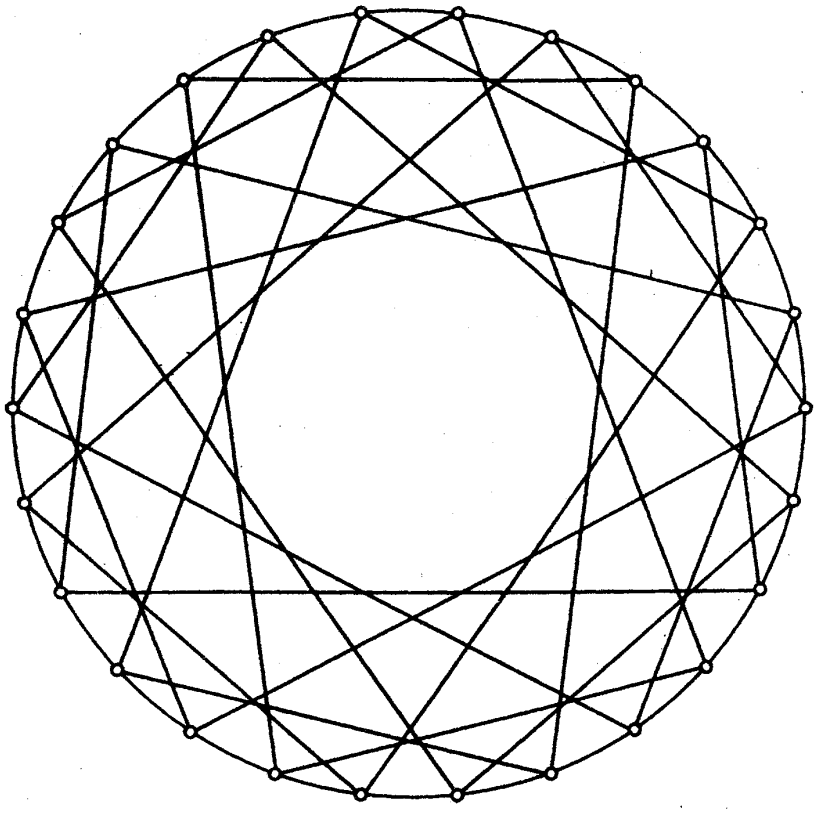
(3,8)-cage

The Tutte–Coxeter graph

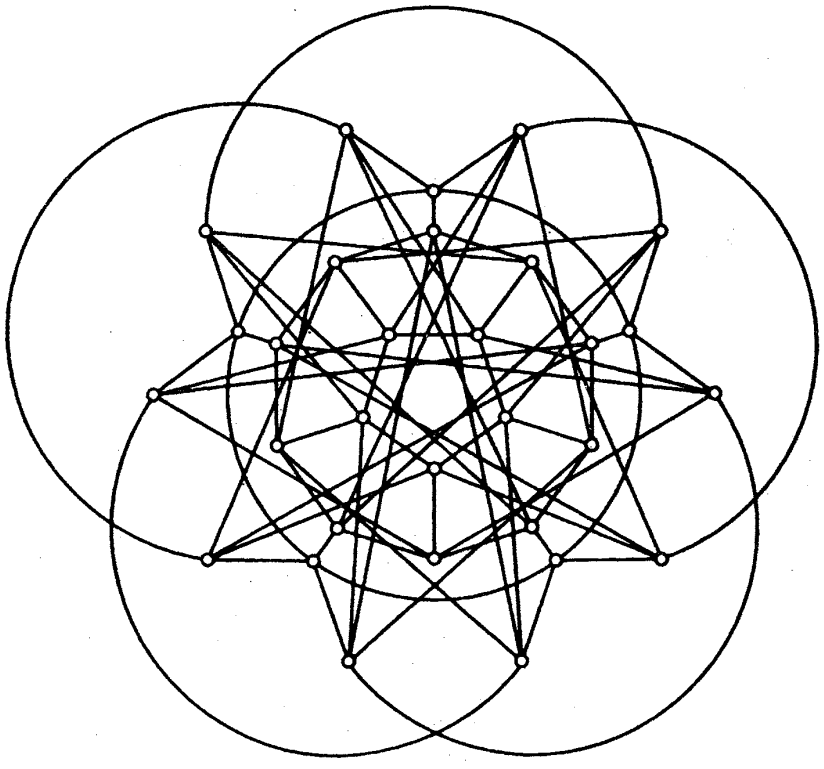


(4,5)-cage

The Robertson graph



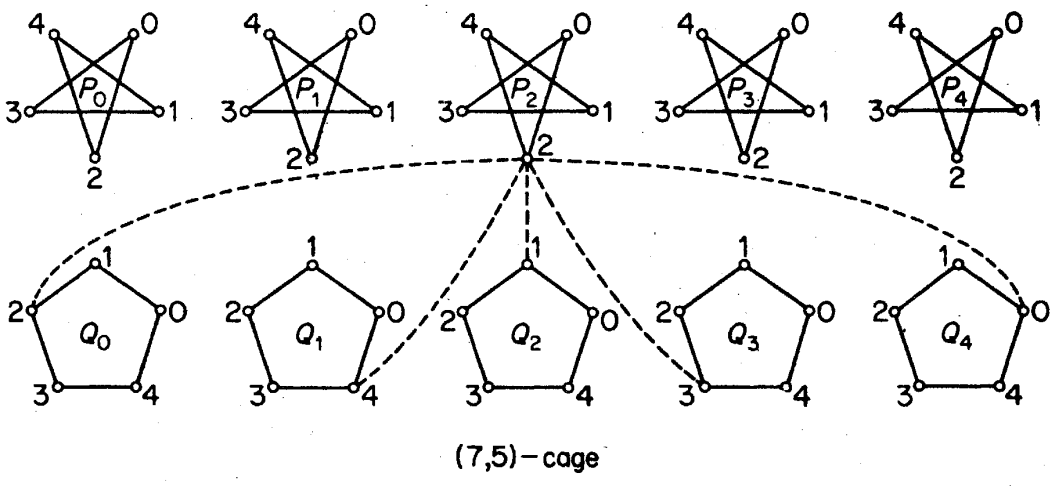
(4,6) – cage



(5,5) – cage

The Robertson–Wegner graph

The (7, 5)-cage (the *Hoffman–Singleton graph*) can be described as follows: it has ten 5-cycles $P_0$, $P_1$, $P_2$, $P_3$, $P_4$, $Q_0$, $Q_1$, $Q_2$, $Q_3$, $Q_4$, labelled as shown below; vertex $i$ of $P_j$ is joined to vertex $i+jk$ (mod 5) of $Q_k$. (For example, vertex 2 of $P_2$ is connected as indicated.)

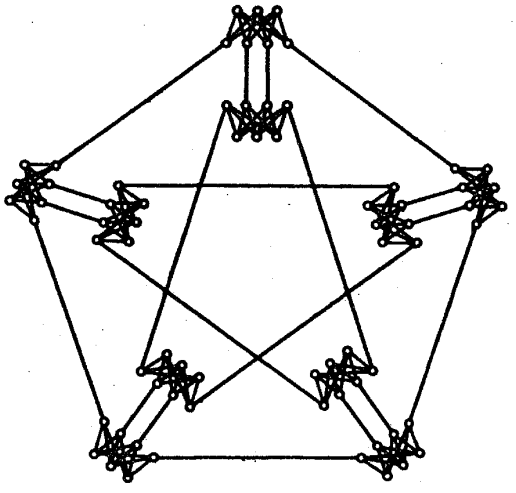



The Hoffman–Singleton graph

NONHAMILTONIAN GRAPHS

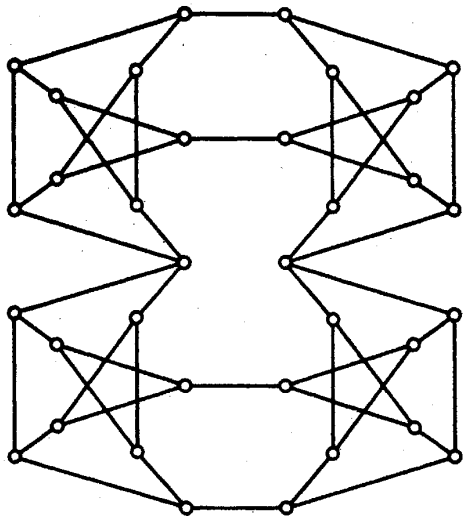
(i) Conditions for a graph to be hamiltonian have been sought ever since Tait made his conjecture on planar graphs. Listed here are counter-examples to several conjectured results.

(a) Every 4-regular 4-connected graph is hamiltonian (C. St. J. A. Nash-Williams).



The Meredith graph

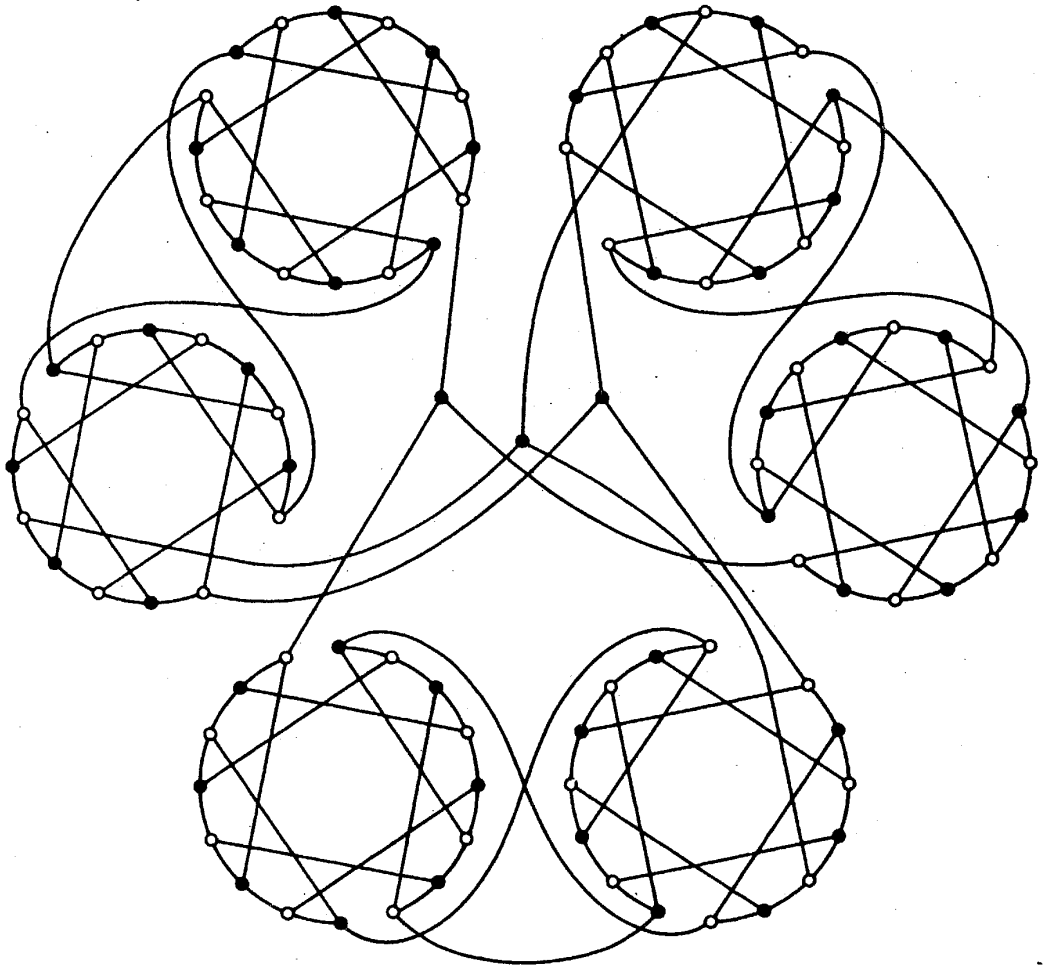
(b) There is no hypotraceable graph (T. Gallai).



The Thomassen graph

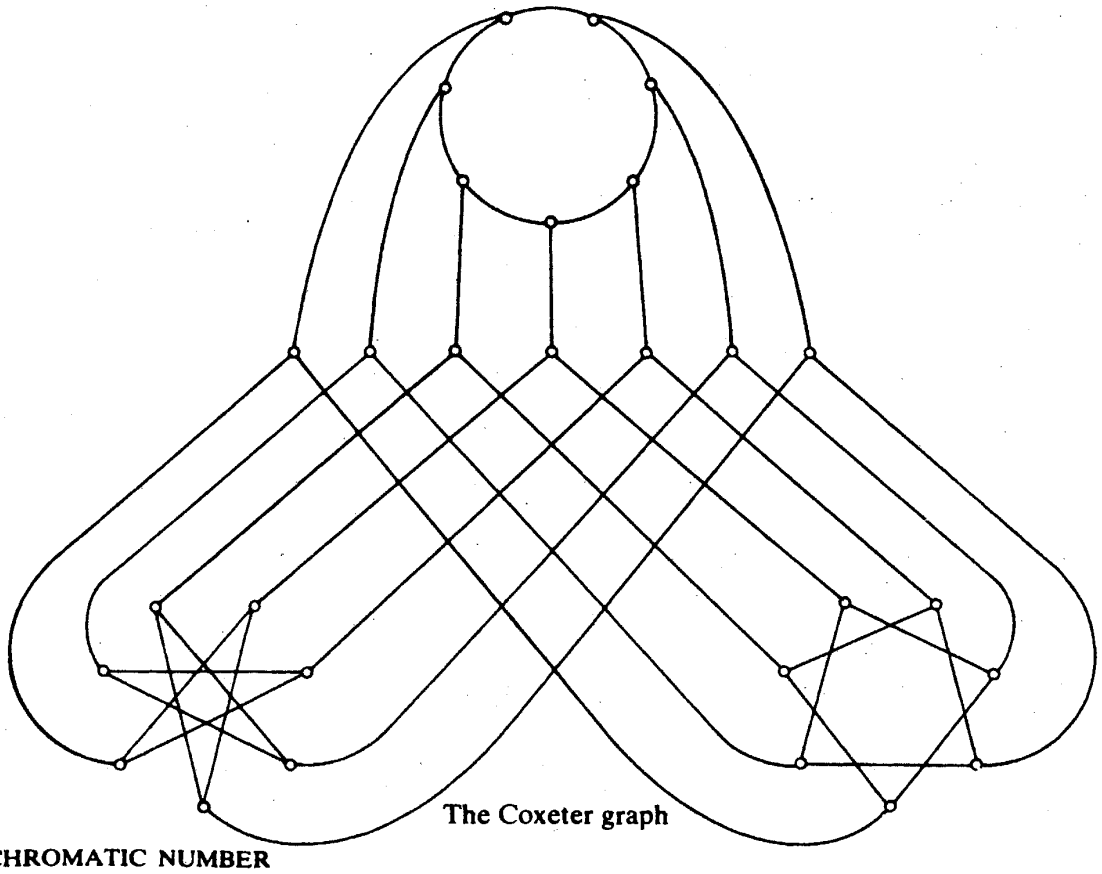
(The first hypotraceable graph was discovered by J. D. Horton.)

(c) Every 3-regular 3-connected bipartite graph is hamiltonian (W. T. Tutte).



The Horton graph

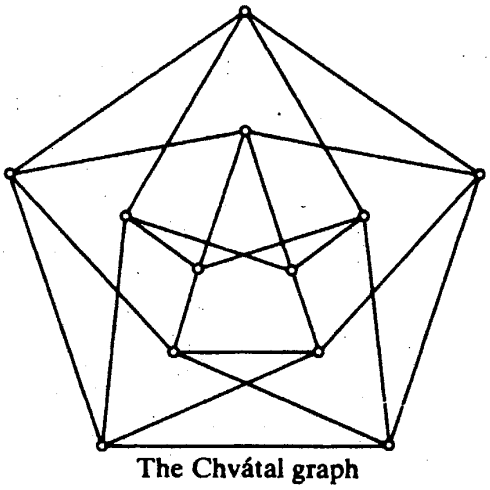
(ii) An example of a nonhamiltonian graph with a high degree of symmetry—there is an automorphism taking any path of length three into any other. (The Petersen graph also has this property.) See Tutte (1960).



The Coxeter graph

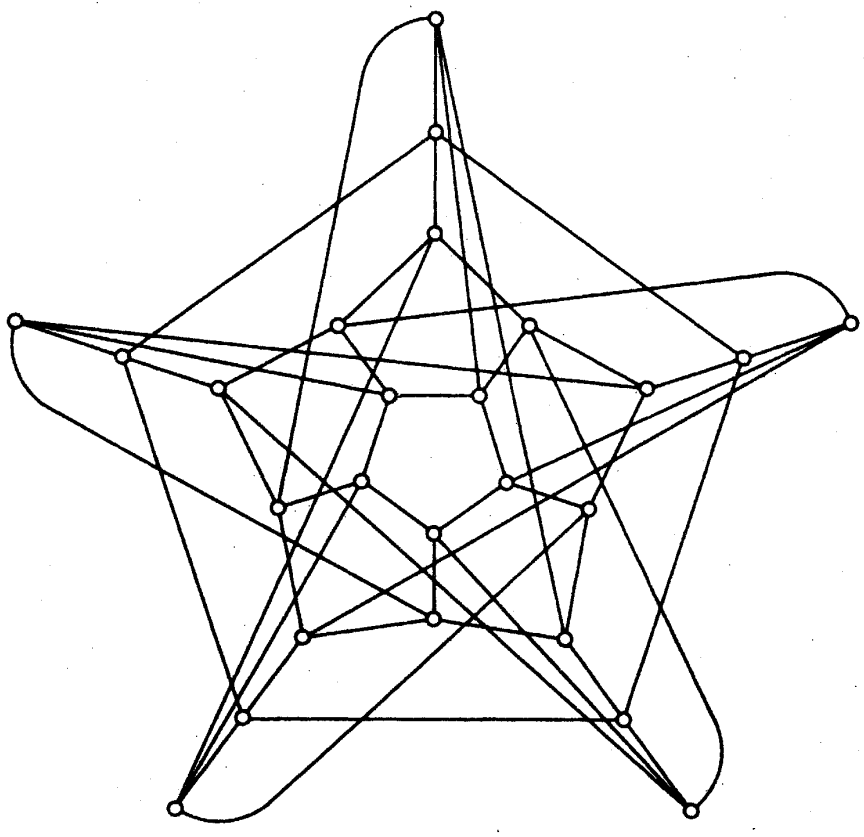
CHROMATIC NUMBER

(i) Grünbaum (1970) has conjectured that, for every $m > 1$ and $n > 2$, there exists an $m$-regular, $m$-chromatic graph of girth at least $n$. For $n = 3$, this is trivial, and for $m = 2$ and 3, the validity of the conjecture follows from the existence of the cages†. Apart from this, only two such graphs are known:



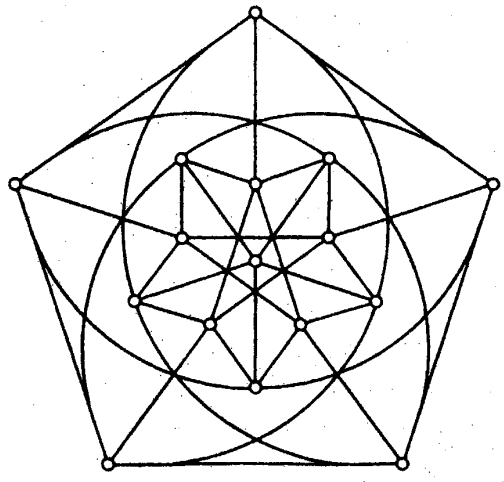
The Chvátal graph

† This conjecture has now been disproved: (Borodin, O. V. and Kostochka, A. V. (1976). On an upper bound of the graph's chromatic number depending on graph's degree and density. *Inst. Maths.*, Novosibirsk, preprint GT-7).



The Grünbaum graph

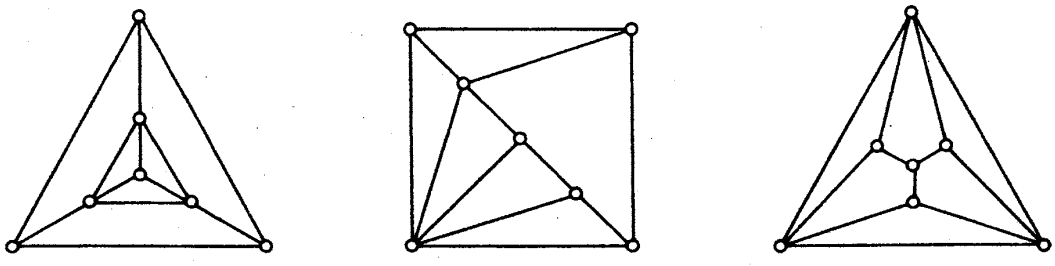
(ii) Since $r(3, 3, 3) = 17$ (see exercise 7.2.3), there is a 3-edge colouring of $K_{16}$ without monochromatic triangles. Kalbfleisch and Stanton (1968) showed that, in such a colouring, the subgraph induced by the edges of any one colour is isomorphic to the following graph:



The Greenwood–Gleason graph

EMBEDDINGS

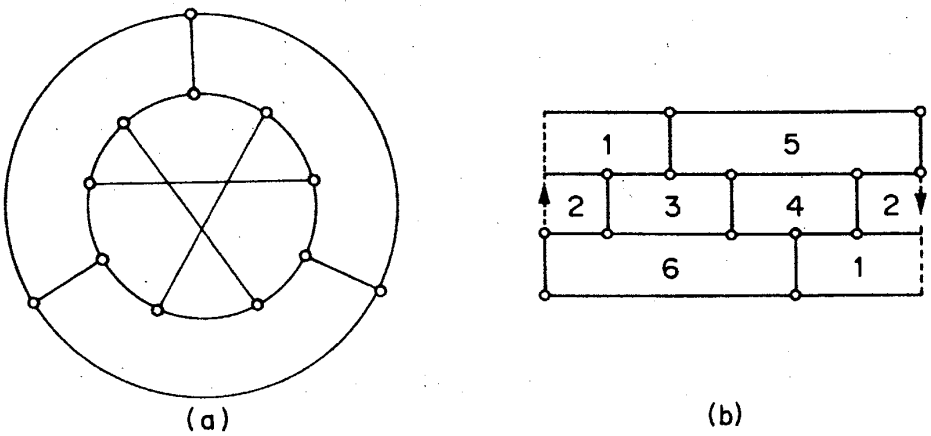
(i) Simple examples of self-dual plane graphs are the wheels. Some more interesting plane graphs with this property are depicted below (see, for example, Smith and Tutte, 1950).



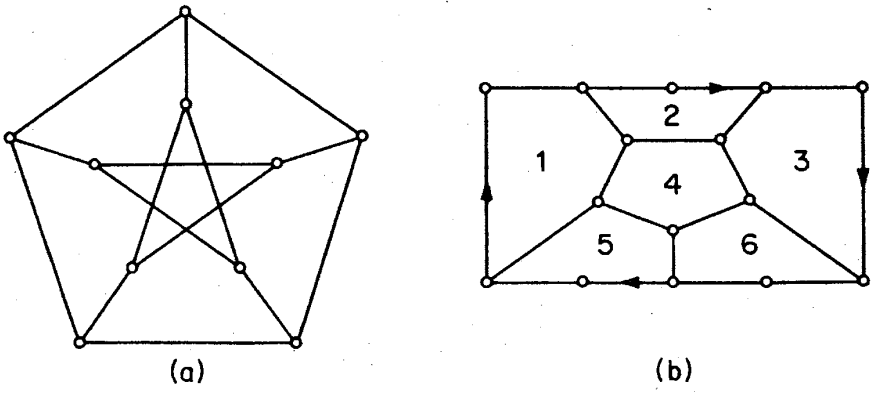
(ii) The *chromatic number* $\chi(S)$ of a surface $S$ is the maximum number of colours required to properly colour the faces of any map on $S$. (The four-colour conjecture claims that the sphere is 4-chromatic.) Heawood (1890) proved that if $S$ has characteristic $n<2$, then

$$\chi(S) \leq [\tfrac{1}{2}(7+\sqrt{49-24n})] \tag{III.2}$$

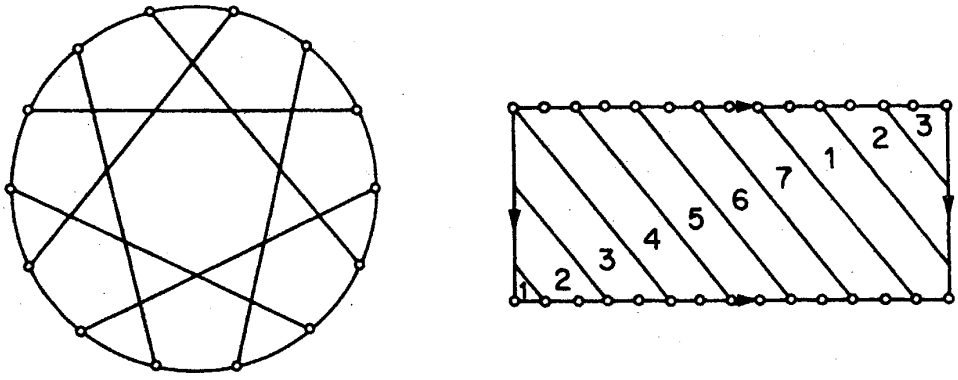
For the projective plane and Möbius band (characteristic 1) and for the torus (characteristic 0), the bound given in (III.2) is attained, as is shown by the following graphs and their embeddings:




(*a*) The Tietze graph; (*b*) an embedding on the Möbius band



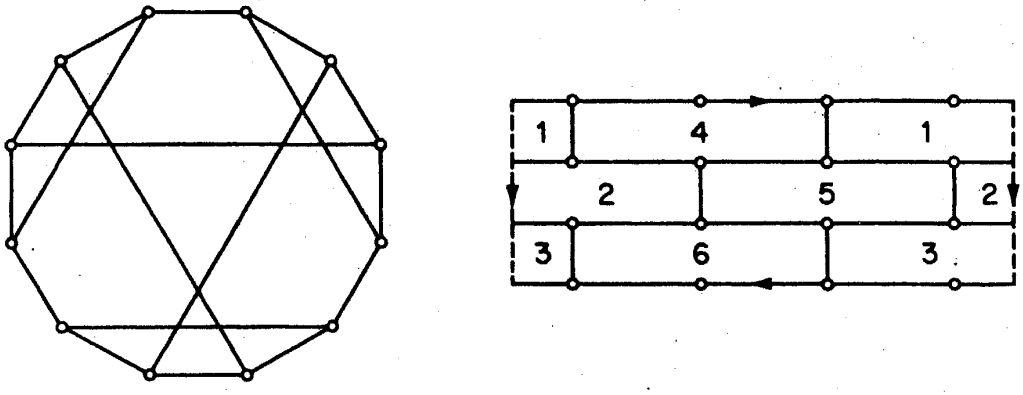

(*a*) The Petersen graph; (*b*) an embedding on the projective plane




(a) (b)

(*a*) The Heawood graph; (*b*) an embedding on the torus

Although the Klein bottle has characteristic 0, Franklin (1934) proved that it is only 6-chromatic, and found the following 6-chromatic map on the Klein bottle:




(a) (b)

(*a*) The Franklin graph; (*b*) an embedding on the Klein bottle

It has been shown that, with the sole exception of the Klein bottle, equality holds in (III.2) for every surface $S$ of characteristic $n<2$. This result is known as the *map colour theorem* (see Ringel, 1974).

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# Appendix IV
# Unsolved Problems

Collected here are a number of unsolved problems of varying difficulty, with originators, dates and relevant bibliography. Conjectures are displayed in bold type. Problems marked † have now been solved; see page 253.

1. Two graphs $G$ and $H$ are *hypomorphic* (written $G \equiv H$) if there is a bijection $\sigma: V(G) \to V(H)$ such that $G - v \cong H - \sigma(v)$ for all $v \in V(G)$. A graph $G$ is *reconstructible* if $G \equiv H$ implies $G \cong H$. The *reconstruction conjecture* claims that **every graph $G$ with $\nu > 2$ is reconstructible** (S. M. Ulam, 1929). This has been verified for disconnected graphs, trees and a few other classes of graphs (see Harary, 1974).

   There is a corresponding *edge reconstruction conjecture:* **every graph $G$ with $\varepsilon > 3$ is edge reconstructible.** Lovász (1972) has shown that every simple graph $G$ with $\varepsilon > \binom{\nu}{2}\Big/2$ is edge reconstructible.

   P. K. Stockmeyer has found an infinite family of counterexamples to the analogous reconstruction conjecture for digraphs.

   Bondy, J. A. and Hemminger, R. L. (1976). Graph reconstruction—a survey. *J. Graph Theory*, to be published

   Lovász, L. (1972). A note on the line reconstruction problem. *J. Combinatorial Theory* B, **13,** 309–10

2. A graph $G$ is *embeddable* in a graph $H$ if $G$ is isomorphic to a subgraph of $H$. Characterise the graphs embeddable in the $k$-cube (V. V. Firsov, 1965).

   Garey, M. R. and Graham, R. L. (1975). On cubical graphs. *J. Combinatorial Theory* (B), **18,** 84–95

3. **Every 4-regular simple graph contains a 3-regular subgraph** (N. Sauer, 1973).
4. **If $k > 2$, there exists no graph with the property that every pair of vertices is connected by a unique path of length $k$** (A. Kotzig, 1974). Kotzig has verified his conjecture for $k < 9$.
5. **Every connected graph $G$ is the union of at most $[(\nu+1)/2]$ edge-disjoint paths** (T. Gallai, 1962). Lovász (1968) has shown that every graph $G$ is the union of at most $[\nu/2]$ edge-disjoint paths and cycles.

Lovász, L. (1968). On coverings of graphs, in *Theory of Graphs* (eds. P. Erdös and G. Katona), Academic Press, New York, pp. 231–36

6. **Every 2-edge-connected simple graph $G$ is the union of $\nu-1$ cycles** (P. Erdös, A. W. Goodman and L. Pósa, 1966).

Erdös, P., Goodman, A. W. and Pósa, L. (1966). The representation of a graph by set intersections. *Canad. J. Math.*, **18,** 106–12

7. **If $G$ is a simple block with at least $\nu/2+k$ vertices of degree at least $k$, then $G$ has a cycle of length at least $2k$** (D. R. Woodall, 1975).
8. Let $f(m, n)$ be the maximum possible number of edges in a simple graph on $n$ vertices which contains no $m$-cycle. It is known that

$$f(m,n)=\begin{cases}[n^2/4] & \text{if } m \text{ is odd}, \quad m\leq\tfrac{1}{2}(n+3)\\ \binom{n-m+2}{2}+\binom{m-1}{2} & \text{if } m\geq\tfrac{1}{2}(n+3)\end{cases}$$

Determine $f(m, n)$ for the remaining cases (P. Erdös, 1963).

Bondy, J. A. and Simonovits, M. (1974). Cycles of even length in graphs. *J. Combinatorial Theory* (B), **16,** 97–105

Woodall, D. R. (1972). Sufficient conditions for circuits in graphs. *Proc. London Math. Soc.*, **24,** 739–55

9. Let $f(n)$ be the maximum possible number of edges in a simple graph on $n$ vertices which contains no 3-regular subgraph. Determine $f(n)$ (P. Erdös and N. Sauer, 1974). Since there is a constant $c$ such that every simple graph $G$ with $\varepsilon\geq c\nu^{8/5}$ contains the 3-cube (Erdös and Simonovits, 1970), clearly $f(n)<cn^{8/5}$.

Erdös, P. and Simonovits, M. (1970). Somé extremal problems in graph theory, in *Combinatorial Theory and its Applications I* (eds. P. Erdös, A. Rényi and V. T. Sós), North-Holland, Amsterdam, pp. 378–92

10. Determine which simple graphs $G$ have exactly one cycle of each length $l$, $3\leq l\leq\nu$ (R. C. Entringer, 1973).
11. Let $f(n)$ be the maximum possible number of edges in a graph on $n$ vertices in which no two cycles have the same length. Determine $f(n)$ (P. Erdös, 1975).
12. **If $G$ is simple and $\varepsilon>\nu(k-1)/2$, then $G$ contains every tree with $k$ edges** (P. Erdös and V. T. Sós, 1963). It is known that every such graph contains a path of length $k$ (Erdös and Gallai, 1959).

Erdös, P. and Gallai, T. (1959). On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, **10,** 337–56

13. Find a (6, 5)-cage (see appendix III).

14. The *bandwidth* of $G$ is defined to be

$$\min_{l} \max_{uv \in E} |l(u) - l(v)|$$

where the minimum is taken over all labellings $l$ of $V$ in distinct integers. Find bounds for the bandwidth of a graph (L. H. Harper, 1964). The bandwidth of the $k$-cube has been determined by Harper (1966).

Chvátalová, J. (1975). Optimal labelling of a product of two paths. *Discrete Math.*, **11**, 249–53

Harper, L. H. (1966). Optimal numberings and isoperimetric problems on graphs. *J. Combinatorial Theory*, **1**, 385–93

15. A simple graph $G$ is *graceful* if there is a labelling $l$ of its vertices with distinct integers from the set $\{0, 1, \ldots, \varepsilon\}$, so that the induced edge labelling $l'$ defined by

$$l'(uv) = |l(u) - l(v)|$$

assigns each edge a different label. Characterise the graceful graphs (S. Golomb, 1972). It has been conjectured that, in particular, **every tree is graceful** (A. Rosa, 1966).

Golomb, S. (1972). How to number a graph, in *Graph Theory and Computing* (ed. R. C. Read), Academic Press, New York, pp. 23–37

†16. **The 3-connected planar graph on $2m$ edges with the least possible number of spanning trees is the wheel with $m$ spokes** (W. T. Tutte, 1940).

Kelmans, A. K. and Chelnokov, V. M. (1974). A certain polynomial of a graph and graphs with an extremal number of trees. *J. Combinatorial Theory* (B), **16**, 197–214

17. Let $u$ and $v$ be two vertices in a graph $G$. Denote the minimum number of vertices whose deletion destroys all $(u, v)$-paths of length at most $n$ by $a_n$, and the maximum number of internally disjoint $(u, v)$-paths of length at most $n$ by $b_n$. Let $f(n)$ denote the maximum possible value of $a_n/b_n$. Determine $f(n)$ (V. Neumann, 1974). L. Lovász has conjectured that $\boldsymbol{f(n) \leq \sqrt{n}}$. It is known that

$$[\sqrt{n/2}] \leq f(n) \leq [n/2]$$

18. **Every 3-regular 3-connected bipartite planar graph is hamiltonian** (D. Barnette, 1970). P. Goodey has verified this conjecture for plane graphs whose faces are all of degree four or six. Note that if the planarity condition is dropped, the conjecture is no longer valid (see appendix III).

19. A graphic sequence $\mathbf{d}$ is *forcibly hamiltonian* if every simple graph with degree sequence $\mathbf{d}$ is hamiltonian. Characterise the forcibly hamiltonian

sequences (C. St. J. A. Nash-Williams, 1970). (Theorem 4.5 gives a partial solution.)

Nash-Williams, C. St. J. A. (1970). Valency sequences which force graphs to have Hamiltonian circuits: interim report, University of Waterloo preprint

20. **Every connected vertex-transitive graph has a Hamilton path** (L. Lovász, 1968). L. Babai has verified this conjecture for graphs with a prime number of vertices.

21. A graph $G$ is *t-tough* if, for every vertex cut $S$, $\omega(G-S) \leq |S|/t$. (Thus theorem 4.2 says that every hamiltonian graph is 1-tough.)

(*a*) **If $G$ is 2-tough, then $G$ is hamiltonian** (V. Chvátal, 1971). C. Thomassen has obtained an example of a nonhamiltonian $t$-tough graph with $t > 3/2$

(*b*) **If $G$ is 3/2-tough, then $G$ has a 2-factor** (V. Chvátal, 1971).

Chvátal, V. (1973). Tough graphs and hamiltonian circuits. *Discrete Math.*, **5,** 215–28

22. The *binding number* of $G$ is defined by

$$\text{bind } G = \min_{\substack{\emptyset \neq S \subseteq V \\ N(S) \neq V}} |N(S)|/|S|$$

(*a*) **If bind $G \geq 3/2$, then $G$ contains a triangle** (D. R. Woodall, 1973).

(*b*) **If bind $G \geq 3/2$, then $G$ is pancyclic** (contains cycles of all lengths $l$, $3 \leq l \leq \nu$) (D. R. Woodall, 1973).

Woodall (1973) has shown that $G$ is hamiltonian if bind $G \geq 3/2$, and that $G$ contains a triangle if bind $G \geq \frac{1}{2}(1+\sqrt{5})$.

Woodall, D. R. (1973). The binding number of a graph and its Anderson number. *J. Combinatorial Theory* (B), **15,** 225–55

23. **Every nonempty regular simple graph contains two disjoint maximal independent sets** (C. Payan, 1973)

24. Find the Ramsey number $r(3, 3, 3, 3)$. It is known that

$$51 \leq r(3, 3, 3, 3) \leq 65$$

Chung, F. R. K. (1973). On the Ramsey numbers $N(3, 3, \ldots, 3; 2)$, *Discrete Math.*, **5,** 317–21

Folkman, J. (1974). Notes on the Ramsey number $N(3, 3, 3, 3)$. *J. Combinatorial Theory* (A), **16,** 371–79

25. For $m < n$, let $f(m, n)$ denote the least possible number of vertices in a graph which contains no $K_n$ but has the property that in every 2-edge colouring there is a monochromatic $K_m$. (Folkman, 1970 has established the existence of such graphs.) Determine bounds for $f(m, n)$. It is

known that

$$f(3, n) = 6 \quad \text{for} \quad n \geq 7$$
$$f(3, 6) = 8 \quad \text{(see exercise 7.2.5)}$$
$$10 \leq f(3, 5) \leq 18$$

Folkman, J. (1970). Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.*, **18,** 19–24

Irving, R. W. (1973). On a bound of Graham and Spencer for a graph-colouring constant. *J. Combinatorial Theory* (B), **15,** 200–203

Lin, S. On Ramsey numbers and $K_r$-coloring of graphs. *J. Combinatorial Theory* (B), **12,** 82–92

26. **If $G$ is $n$-chromatic, then $r(G, G) \geq r(n, n)$** (P. Erdös, 1973). ($r(G, G)$ is defined in exercise 7.2.6.)
27. What is the maximum possible chromatic number of a graph which can be drawn in the plane so that each edge is a straight line segment of unit length? (L. Moser, 1958).

Erdös. P., Harary, F. and Tutte, W. T. (1965). On the dimension of a graph. *Mathematika*, **12,** 118–22

28. **The absolute values of the coefficients of any chromatic polynomial form a unimodal sequence** (that is, no term is flanked by terms of greater value) (R. C. Read, 1968).

Chvátal, V. (1970). A note on coefficients of chromatic polynomials. *J. Combinatorial Theory*, **9,** 95–96

29. **If $G$ is not complete and $\chi = m + n - 1$, where $m \geq 2$ and $n \geq 2$, then there exist disjoint subgraphs $G_1$ and $G_2$ of $G$ such that $\chi(G_1) = m$ and $\chi(G_2) = n$** (L. Lovász, 1968).
30. A simple graph $G$ is *perfect* if, for every induced subgraph $H$ of $G$, the number of vertices in a maximum clique is $\chi(H)$. **$G$ is perfect if and only if no induced subgraph of $G$ or $G^c$ is an odd cycle of length greater than three** (C. Berge, 1961). This is the *strong perfect graph conjecture.* Lovász (1972) has shown that the complement of any perfect graph is perfect.

Lovász, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, **2,** 253–67

Parthasarathy, K. R. and Ravindra, G. (to be published). The strong perfect-graph conjecture is true for $K_{1,3}$-free graphs. *J. Combinatorial Theory*

31. **If $G$ is a 3-regular simple block and $H$ is obtained from $G$ by duplicating each edge, then $\chi'(H) = 6$** (D. R. Fulkerson, 1971).
32. **If $G$ is simple, with $\nu$ even and $\chi'(G) = \Delta(G) + 1$, then $\chi'(G - v) = \chi'(G)$**

**for some $v \in V$** (I. T. Jakobsen, L. W. Beineke and R. J. Wilson, 1973). This has been verified for all graphs $G$ with $\nu \leq 10$ and all 3-regular graphs $G$ with $\nu = 12$.

Beineke, L. W. and Wilson, R. J. (1973). On the edge-chromatic number of a graph. *Discrete Math.*, **5**, 15–20

33. **For any simple graph $G$, the elements of $V \cup E$ can be coloured in $\Delta + 2$ colours so that no two adjacent or incident elements receive the same colour** (M. Behzad, 1965). This is known as the *total colouring conjecture.* M. Rosenfeld and N. Vijayaditya have verified it for all graphs $G$ with $\Delta \leq 3$.

Vijayaditya, N. (1971). On total chromatic number of a graph. *J. London Math. Soc.*, **3**, 405–408

34. **If $G$ is simple and $\varepsilon > 3\nu - 6$, then $G$ contains a subdivision of $K_5$** (G. A. Dirac, 1964). Thomassen (1975) has shown that $G$ contains a subdivision of $K_5$ if $\varepsilon \geq 4\nu - 10$.

Dirac, G. A. (1964). Homomorphism theorems for graphs. *Math. Ann.*, **153**, 69–80

Thomassen, C. (1974). Some homeomorphism properties of graphs, *Math. Nachr.*, **64**, 119–33

35. A sequence **d** of non-negative integers is *potentially planar* if there is a simple planar graph with degree sequence **d**. Characterise the potentially planar sequences (S. L. Hakimi, 1963).

Owens, A. B. (1971). On the planarity of regular incidence sequences. *J. Combinatorial Theory* (B), **11**, 201–12

†36. **If $G$ is a loopless planar graph, then $\alpha \gneq \nu/4$** (P. Erdös, 1968). Albertson (1974) has shown that every such graph satisfies $\alpha > 2\nu/9$.

Albertson, M. O. (1974). Finding an independent set in a planar graph, in *Graphs and Combinatorics* (eds. R. A. Bari and F. Harary), Springer-Verlag, New York, pp. 173–79

†37. **Every planar graph is 4-colourable** (F. Guthrie, 1852).

Ore, O. (1969). *The Four-Color Problem*, Academic Press, New York

38. **Every $k$-chromatic graph contains a subgraph contractible to $K_k$** (H. Hadwiger, 1943). Dirac (1964) has proved that every 6-chromatic graph contains a subgraph contractible to $K_6$ less one edge.

Dirac, G. A. (1964). Generalizations of the five colour theorem, in *Theory of Graphs and its Applications* (ed. M. Fiedler), Academic Press, New York, pp. 21–27

39. **Every $k$-chromatic graph contains a subdivision of $K_k$** (G. Hajós, 1961). Pelikán (1969) has shown that every 5-chromatic graph contains a subdivision of $K_5$ less one edge.

Pelikán, J. (1969). Valency conditions for the existence of certain subgraphs, in *Theory of Graphs* (eds. P. Erdös and G. Katona), Academic Press, New York, pp. 251–58

40. **Every 2-edge-connected 3-regular simple graph which has no Tait colouring contains a subgraph contractible to the Petersen graph** (W. T. Tutte, 1966).

Isaacs, R. (1975). Infinite families of nontrivial trivalent graphs which are not Tait colourable. *Amer. Math. Monthly*, **82**, 221–39

Tutte, W. T. (1966). On the algebraic theory of graph colorings. *J. Combinatorial Theory*, **1**, 15–50

41. **For every surface $S$, there exists a finite number of graphs which have minimum degree at least three and are minimally nonembeddable on $S$.**

†42. **If $D$ is diconnected, then $D$ has a directed cycle of length at least $\chi$** (M. Las Vergnas, 1974).

43. **If $D$ is a tournament with $v$ odd and every indegree and outdegree equal to $(v-1)/2$, then $D$ is the union of $(v-1)/2$ arc-disjoint directed Hamilton cycles** (P. Kelly, 1966).

44. **If $D$ is a tournament with $v$ even, then $D$ is the union of $\sum_{v \in V} \max\{0, d^+(v) - d^-(v)\}$ arc-disjoint directed paths** (R. O'Brien, 1974). This would imply the truth of conjecture 43.

45. Characterise the tournaments $D$ with the property that all subtournaments $D - v$ are isomorphic (A. Kotzig, 1973).

46. **If $D$ is a digraph which contains a directed cycle, then there is some arc whose reversal decreases the number of directed cycles in $D$** (A. Adám, 1963).

47. **Given a positive integer $n$, there exists a least integer $f(n)$ such that in any digraph with at most $n$ arc-disjoint directed cycles there are $f(n)$ arcs whose deletion destroys all directed cycles** (T. Gallai, 1964; D. H. Younger, 1968).

Erdös. P. and Pósa, L. (1962). On the maximal number of disjoint circuits of a graph. *Publ. Math. Debrecen*, **9**, 3–12

Younger, D. H. (1973). Graphs with interlinked directed circuits, in *Proceedings of Midwest Symposium on Circuit Theory*

48. An $(m+n)$-regular graph is $(m, n)$-*orientable* if it can be oriented so that each indegree is either $m$ or $n$. **Every 5-regular simple graph with no 1-edge cut or 3-edge cut is (4, 1)-orientable** (W. T. Tutte, 1972). Tutte has shown that this would imply Grötzsch's theorem

49. Obtain an algorithm to find a maximum flow in a network with two sources $x_1$ and $x_2$, two sinks $y_1$ and $y_2$, and two commodities, the requirement being to ship commodity 1 from $x_1$ to $y_1$ and commodity 2 from $x_2$ to $y_2$ (L. R. Ford and D. R. Fulkerson, 1962).

Rothschild, B. and Whinston, A. (1966). On two commodity network flows. *Operations Res.*, **14,** 377–87

50. **Every 2-edge-connected digraph $D$ has a circulation $f$ over the field of integers modulo 5 in which $f(a) \neq 0$ for all arcs $a$** (W. T. Tutte, 1949). Tutte has shown that this would imply the five-colour theorem.

References for problems solved since first printing:

16. Göbel, F. and Jagers, A. A. (1976). On a conjecture of Tutte concerning minimal tree numbers. *J. Combinatorial Theory* (B), to be published

36 and 37. Appel, K. and Haken, W. (1976). Every planar map is four colorable. *Bull. Amer. Math. Soc.*, **82,** 711–2

42. Bondy, J. A. (1976). Diconnected orientations and a conjecture of Las Vergnas. *J. London Math. Soc.*, to be published

# Appendix V
# Suggestions for Further Reading

BOOKS OF A GENERAL NATURE, LISTED ACCORDING TO LEVEL OF TREATMENT


Ore, O. (1963). *Graphs and Their Uses*, Random House, New York

Rouse Ball, W. W. and Coxeter, H. S. M. (1974). *Mathematical Recreations and Essays*, University of Toronto Press, Toronto

Liu, C. L. (1968). *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York

Wilson, R. J. (1972). *Introduction to Graph Theory*, Oliver and Boyd, Edinburgh

Deo, N. (1974). *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, Englewood Cliffs, N.J.

Behzad, M. and Chartrand, G. (1971). *Introduction to the Theory of Graphs*, Allyn and Bacon, Boston

Harary, F. (ed.) (1967). *A Seminar on Graph Theory*, Holt, Rinehart and Winston, New York

Ore, O. (1962). *Theory of Graphs*, American Mathematical Society, Providence, R.I.

König, D. (1950). *Theorie der Endlichen und Unendlichen Graphen*, Chelsea, New York

Sachs, H. (1970). *Einführung in die Theorie der Endlichen Graphen*, Teubner Verlagsgesellschaft, Leipzig

Harary, F. (1969). *Graph Theory*, Addison-Wesley, Reading, Mass.

Berge, C. (1973). *Graphs and Hypergraphs*, North Holland, Amsterdam


SPECIAL TOPICS


Biggs, N. (1974). *Algebraic Graph Theory*, Cambridge University Press, Cambridge

Tutte, W. T. (1966). *Connectivity in Graphs*, University of Toronto Press, Toronto

Ore, O. (1967). *The Four-Color Problem*, Academic Press, New York

Ringel, G. (1974). *Map Color Theorem*, Springer-Verlag, Berlin

Moon, J. W. (1968). *Topics on Tournaments*, Holt, Rinehart and Winston, New York

Ford, L. R. Jr. and Fulkerson, D. R. (1962). *Flows in Networks*, Princeton University Press, Princeton

Berge, C. and Ghouila-Houri, A. (1965). *Programming, Games, and Transportation Networks*, John Wiley, New York

Seshu, S. and Reed, M. B. (1961). *Linear Graphs and Electrical Networks*, Addison-Wesley, Reading, Mass.

Tutte, W. T. (1971). *Introduction to the Theory of Matroids*, American Elsevier, New York

Harary, F. and Palmer, E. (1973). *Graphical Enumeration*, Academic Press, New York

Aho, A. V., Hopcroft, J. E. and Ullman, J. D. (1974). *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, Mass.

Welsh, D. J. A. (1976). *Matroid Theory*, Academic Press, New York

Biggs, N., Lloyd, E. K. and Wilson, R. J. (1976). *Graph Theory 1736–1936*, Clarendon Press, Oxford

# Glossary of Symbols

GENERAL MATHEMATICAL SYMBOLS

| | | Page |
|---|---|---|
| $\cup$ | union | |
| $\cap$ | intersection | |
| $\subseteq$ | subset | |
| $\subset$ | proper subset | |
| $\setminus$ | set-theoretic difference | |
| $\Delta$ | symmetric difference | |
| $[x]$ | greatest integer $\leq x$ | |
| $\{x\}$ | least integer $\geq x$ | |
| $\lVert f \rVert$ | support of $f$ | 215 |
| $\mathbf{R} \mid S$ | restriction of $\mathbf{R}$ to $S$ | 215 |
| $\mathbf{R}'$ | transpose of $\mathbf{R}$ | |

GRAPH-THEORETIC SYMBOLS

| | | |
|---|---|---|
| $A$ | arc set | 171 |
| $\mathbf{A}$ | adjacency matrix of a graph | 7 |
| $\mathbf{A}$ | adjacency matrix of a digraph | 173 |
| $b(f)$ | boundary of $f$ | 140 |
| $\mathscr{B}$ | bond space | 213 |
| $c(G)$ | closure of $G$ | 56 |
| cap $K$ | capacity of cut $K$ | 194 |
| $\mathscr{C}$ | cycle space | 212 |
| $d_G(v)$ | degree of vertex $v$ in $G$ | 10 |
| $d_G(f)$ | degree of face $f$ in $G$ | 140 |
| $d_D^-(v)$ | indegree of $v$ in $D$ | 172 |
| $d_D^+(v)$ | outdegree of $v$ in $D$ | 172 |
| $d_G(u, v)$ | distance between $u$ and $v$ in $G$ | 14 |
| $D$ | directed graph | 171 |
| $D(G)$ | associated digraph of $G$ | 179 |
| ext $J$ | exterior of $J$ | 135 |
| Ext $J$ | closure of ext $J$ | 135 |
| $E$ | edge set | 1 |
| $f^-(S)$ | flow into $S$ | 191 |
| $f^+(S)$ | flow out of $S$ | 191 |
| $F$ | face set | 139 |
| $F(B, \tilde{H})$ | set of faces of $\tilde{H}$ in which $B$ is drawable | 164 |

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| | | Page |
|---|---|---|
| $G$ | graph | 1 |
| $G[S]$ | subgraph of $G$ induced by $S$ | 9 |
| int $J$ | interior of $J$ | 135 |
| Int $J$ | closure of int $J$ | 135 |
| $K_n$ | complete graph | 4 |
| $K_{m,n}$ | complete bipartite graph | 5 |
| $\mathbf{M}$ | incidence matrix of a graph | 7 |
| $\boldsymbol{M}$ | incidence matrix of a digraph | 214 |
| $N$ | network | 191 |
| $N_G(S)$ | neighbour set of $S$ in $G$ | 72 |
| $N_D^-(v)$ | in-neighbour set of $v$ in $D$ | 175 |
| $N_D^+(v)$ | out-neighbour set of $v$ in $D$ | 175 |
| $r(k, l)$ | Ramsey number | 103 |
| $r(k_1, k_2, \ldots, k_m)$ | Ramsey number | 108 |
| $r_n$ | $r(3, 3, \ldots, 3)$ | 108 |
| val $f$ | value of flow $f$ | 192 |
| $V$ | vertex set | 1 |
| $V(B, H)$ | set of vertices of attachment of $B$ to $H$ | 146 |
| $\alpha$ | independence number | 101 |
| $\alpha'$ | edge independence number | 102 |
| $\beta$ | covering number | 101 |
| $\beta'$ | edge covering number | 102 |
| $\delta$ | minimum degree | 10 |
| $\delta^-$ | minimum indegree | 172 |
| $\delta^+$ | minimum outdegree | 172 |
| $\Delta$ | maximum degree | 10 |
| $\Delta^-$ | maximum indegree | 172 |
| $\Delta^+$ | maximum outdegree | 172 |
| $\varepsilon$ | number of edges | 3 |
| $\kappa$ | connectivity | 42 |
| $\kappa'$ | edge connectivity | 42 |
| $\nu$ | number of vertices | 3 |
| $o$ | number of odd components | 76 |
| $\pi_k$ | chromatic polynomial | 125 |
| $\tau$ | number of spanning trees | 32 |
| $\phi$ | number of faces | 139 |
| $\chi$ | chromatic number | 117 |
| $\chi'$ | edge chromatic number | 91 |
| $\chi^*$ | face chromatic number | 158 |
| $\omega$ | number of components | 13 |
| $\overleftarrow{D}$ | converse of $D$ | 173 |
| $\hat{D}$ | condensation of $D$ | 173 |

| | | *Page* |
|---|---|---|
| $G^c$ | complement of $G$ | 6 |
| $G^*$ | dual of $G$ | 140 |
| $\bar{G}$ | planar embedding of $G$ | 135 |
| $W^{-1}$ | reverse of walk $W$ | 12 |
| $G \cdot e$ | contraction of $e$ | 32 |
| $G - e$ | deletion of $e$ | 9 |
| $G + e$ | addition of $e$ | 9 |
| $G - v$ | deletion of $v$ | 9 |
| $G + E'$ | addition of $E'$ | 9 |
| $G - S$ | deletion of $S$ | 9 |
| $G \cong H$ | isomorphism | 4 |
| $H \subseteq G$ | subgraph | 8 |
| $H \subset G$ | proper subgraph | 8 |
| $G \cup H$ | union | 9 |
| $G \cap H$ | intersection | 10 |
| $G + H$ | disjoint union | 10 |
| $G \times H$ | product | 96 |
| $G \vee H$ | join | 58 |
| $\bar{H}(G)$ | complement of $H$ in $G$ | 29 |
| $[S, T]$ | set of edges between $S$ and $T$ | 29 |
| $(S, T)$ | set of arcs from $S$ to $T$ | 176 |
| $WW'$ | concatenation of walks | 12 |

# Index

This index is arranged strictly in alphabetical order according to the first significant word. Thus, 'edge connectivity' is listed under E and '$k$-chromatic graph' under C.


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