



GRAPH THEORY APPLIED TO OPTIMAL CONNECTIVITY IN COMPUTER NETWORKS*

Dr. J. M. McQuillan
Bolt Beranek and Newman Inc.
Cambridge, MA USA

1. Graphs as Models

This is a report on some of the research that has been carried out in applying graph theoretical results to communications networks. The networks that we wish to investigate here consist of computers at various sites which are linked together by telecommunications circuits.

Many of the results of graph theory may be applied to such networks; we will restrict ourselves to the consideration of connectivity. In designing a computer communications network, it is desirable to provide good connectivity among all sites at reasonable cost. For this reason, extremes like the fully-connected network (too expensive) and the star network (only as reliable as its center) are usually not considered. The connectivity constraints or reliability measures can be stated in different ways, and analytic techniques have been developed for some of these measures. Further, procedures for the synthesis of well-connected networks have also been invented.

The reliability of communications networks is an important issue in their design and operation. Telecommunications circuits become noisy and unusable, and the communications computers may also fail. This report is a survey of that part of applied graph theory which is useful in the study of the connectivity of communications networks.

*This research was sponsored under Advanced Research Projects Agency contract DAHC-69-C-0179.

2. Definitions of Optimal Connectivity

Our model of the communications network will be a connected, undirected graph, $G=(X,E)$. The size of the node set, X , will be denoted as p , and the size of the line set, E , will be denoted as q . Following standard graph theoretic terminology, we state the following definitions:

Definition 1. The *degree* of a node x , denoted $\deg(x)$, is the number of lines incident on x .

Definition 2. The *minimum degree* of a graph G , $\delta(G)$, is the minimum of $\deg(x)$ over all x in X .

Definition 3. A graph is *regular* of degree r if $\deg(x)=r$ for all x in X .

We now introduce some elementary measures of the connectivity of a graph. In particular, we define parameters of the graph which pertain to partitioning the graph into 2 or more connected components:

Definition 4. The *connectivity* of a graph G , $\kappa(G)$, is the minimum size of a disconnecting set of nodes.

Definition 5. The *cohesion* of a graph G , $\lambda(G)$, is the minimum size of a disconnecting set of lines.

Definition 6. The *mixed connectivity* of a graph G , $\nu(G)$, is the minimum size of a disconnecting set of nodes and lines.

There are two well-known results of graph theory that we state without proof:

Whitney's Theorem. A graph G has $\kappa(G)=n$ iff each pair of nodes is joined by at least n node-disjoint paths [WHI 32].

An analogue of this result was discovered independently by Ford and Fulkerson [FOR 56] and Elias, Feinstein, and Shannon [ELI 56];

Line Form of Whitney's Theorem. A graph G has $\lambda(G)=n$ iff each pair of nodes is joined by at least n line-disjoint paths.

We now present a result which relates each of these measures of connectivity and the size of the graph in terms of lines and nodes [HAR 69].

Theorem 1. For a connected, non-trivial graph, G ,

$$0 < v(G) = \kappa(G) \leq \lambda(G) \leq \delta(G) \leq \lfloor 2q/p \rfloor$$

where $\lfloor n \rfloor$ is the largest integer less than or equal to n .

Proof:

i) $\delta(G) \leq \lfloor 2q/p \rfloor$: The sum of the degrees of all nodes is $2q$.

This means that the average degree is $2q/p$ and thus

$$\delta(G) \leq \lfloor 2q/p \rfloor.$$

ii) $\lambda(G) \leq \delta(G)$: We observe that a disconnected graph results from the removal of all the lines incident on a node of minimum degree; therefore, $\lambda(G) \leq \delta(G)$.

iii) $\kappa(G) \leq \lambda(G)$: If $\lambda = 1$, either there is a cutpoint or G is trivial. For $\lambda \geq 2$, removing some set of $\lambda - 1$ lines leaves a connected graph with a bridge $x = uv$. Remove a node incident on each of the $\lambda - 1$ lines, different from u and v . If the resulting graph is disconnected, $\kappa < \lambda$.

If not, remove u or v and $\kappa = \lambda$.

iv) $v(G) = \kappa(G)$: We define a connectivity pair (a, b) as a set of a nodes and b lines which disconnect G . Clearly, $(\kappa, 0)$ and $(0, \lambda)$ are connectivity pairs. For each a between 0 and κ there is a unique b . Further, if (a, b) is a connectivity pair, then there is clearly a set of $a + 1$ nodes and $b - 1$ lines which disconnects G . Thus $\kappa(G) \leq a + b \leq \lambda(G)$ for all connectivity pairs (a, b) .

Thus, $v(G) = \min(a + b) = \kappa(G)$.

v) $0 < v(G)$: By definition, since G is a connected graph.

The bounds provided by Theorem 1 are the best possible.

A construction procedure can be given for all integers a, b, c such that for $0 < a \leq b \leq c$ a graph G can be constructed with $\kappa(G) = a$, $\lambda(G) = b$, and $\delta(G) = c$ [CHA 68].

We now introduce some definitions suggested by Theorem 1:
Definition 7. A graph G is *pseudo-regular* if $\delta(G) = \lfloor 2q/p \rfloor$.
Definition 8. A graph G is λ -*optimal* if $\lambda(G) = \lfloor 2q/p \rfloor$.
Definition 9. A graph G is κ -*optimal* if $\kappa(G) = \lfloor 2q/p \rfloor$.
Clearly, a κ -optimal graph is also λ -optimal, and both kinds of optimal graphs are pseudo-regular. They are also regular if $2q/p$ is integral. Figure 1 illustrates a graph which is regular and λ -optimal but not κ -optimal. It is also clear that κ -optimal graphs are optimally connected over the class of graphs with p nodes and q lines, since for any such G_1 , we have $\kappa(G_1) \leq \delta(G_1) \leq \lfloor 2q/p \rfloor = \kappa(G)$ for G a κ -optimal graph.

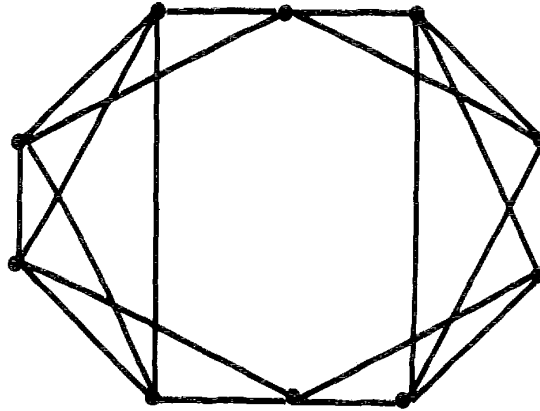


Figure 1. A graph with $\delta(G)=4$, $\lambda(G)=4$, $\kappa(G)=2$

3. Synthesis of Optimally Connected Graphs

The body of this report will be concerned with the construction of graphs with optimal connectivity properties. We will consider both observations on certain classes of graphs and synthesis procedures for constructing and modifying graphs.

3.1 Two Classes of λ -optimal Graphs

In this section we present two classes of graphs which can be shown to be λ -optimal. Neither class is necessarily κ -optimal, but it will be shown that the first class is related to a synthesis procedure for graphs which are κ -optimal. The first result is quite simple to prove [BOE 70].

Theorem 2. Any line-disjoint union of Hamiltonian cycles is a λ -optimal graph. (A Hamiltonian cycle is a connected regular graph of degree 2).

Proof: If there are n line-disjoint Hamiltonian cycles, then $q=np$ and $\lambda \leq 2q/p$ gives $\lambda \leq 2n$.

For any two nodes, each cycle provides 2 line-disjoint paths. Thus for n cycles, there are $2n$ line-disjoint paths between any two nodes. Then, by the line form of Menger's Theorem,

$$\lambda \geq 2n \quad \text{and therefore, } \lambda = 2n = 2q/p.$$

Such a graph is not necessarily optimal. Figure 1 illustrates a graph which is the line-disjoint union of two Hamiltonian cycles, which is λ -optimal but not κ -optimal.

The next class of λ -optimal graphs is interesting because it provides an upper bound for other techniques presented later. It is also a best-possible statement of the result [CHA 66].

Theorem 3. If a graph G has $\delta(G) \geq \lfloor p/2 \rfloor$, then G is connected, and $\lambda(G) = \delta(G)$. Thus if G is pseudo-regular, it is λ -optimal, that is $\lambda(G) = \lfloor 2q/p \rfloor$. Further, if G is regular of degree r , $\lambda(G) = r$.

Proof: Assume G is not connected. Then each component contains at least $\lfloor p/2 \rfloor + 1$ nodes, and there are at least two components. Thus there are at least $2(\lfloor p/2 \rfloor + 1)$ nodes in G . But this is more than p nodes, a contradiction, so G must be connected.

From Theorem 1 we know that $\lambda(G) \leq \delta(G)$. Here we show that $\lambda(G) \geq \delta(G)$. Assume this is not true, then there must be a partition $[X_1, X_2]$ such that exactly λ lines join X_1 and X_2 . Assume that these lines are incident on m nodes in X_1 , with $m \leq \lambda(G)$. Suppose that there are only m nodes in X_1 . Then the number of lines incident with two nodes in X_1 is at least

$$1/2[m\delta(G) - \lambda(G)] \text{ which is } > 1/2[m\delta(G) - \delta(G)] > 1/2m(m-1).$$

This is impossible; the maximum number of such edges is $1/2m(m-1)$. A similar argument holds for X_2 . The only remaining possibility is that both X_1 and X_2 have more than m nodes, and thus some nodes adjacent only to nodes in their respective subsets. Then X_1 and X_2 must each contain at least $\delta(G) + 1$ nodes, giving $2\delta(G) + 2$ nodes in total. But

$$2\delta(G) + 2 > 2\lfloor p/2 \rfloor + 2 \geq p + 1$$

This is a contradiction, so we have $\lambda(G) = \delta(G)$, and the proof is complete.

Note that Theorem 3 is the strongest possible statement since we can construct graphs with $\lambda(G) < \delta(G) < \lfloor p/2 \rfloor$. For even $p \geq 8$, let A_n and B_n be disjoint complete graphs with $p/2$ nodes. Now delete a line (a_1, a_2) from A_n and a line (b_1, b_2) from B_n . Add the lines (a_1, b_1) and (a_2, b_2) . This constructs a regular graph of degree $(p-2)/2$ with $\lambda(G) = 2$. Figure 2 shows the graph for $p=8$.

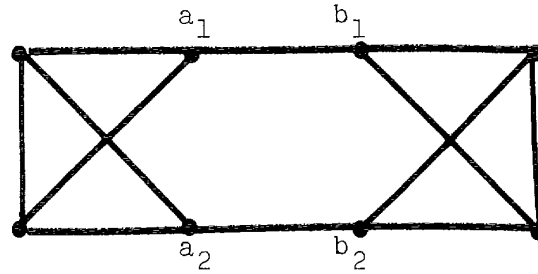


Figure 2. A graph with $\lambda(G)=2$, $\delta(G)=3$, and $\lfloor p/2 \rfloor=4$

3.2 Synthesis of λ -optimal Graphs

In this section we present a set of algorithms for constructing a graph with given node degrees and modifying it so that the final graph is λ -optimal. Almost all this research is due to S. L. Hakimi [HAK 62], [HAK 63], and some similar work has been done by Berge [BER 62].

We begin with three definitions:

Definition 10. A set of non-negative integers $k_1 \geq k_2 \geq \dots \geq k_n$, with r integers positive, is said to be *compressible* if $k_1 \leq r-1$.

Definition 11. A *compression* of such a set of integers consists of subtracting 1 from k_2 through k_{k_1+1} , letting $k_1=0$, and relabelling so that $k_1 \geq k_2 \geq \dots \geq k_n$.

Definition 12. Such a set of integers is *successively compressible* if it is compressible and every set resulting from a compression is again compressible, until all integers are zero.

With these definitions, we can give necessary and sufficient conditions for the existence of a graph with p nodes and node degrees $k_1, k_2, \dots, k_p$ [HAK 62], [HAK 63].

Theorem 4. A set of non-negative integers $k_1 \geq k_2 \geq \dots \geq k_p$ is realizable as the degrees of a p -node graph G iff the given set of integers is successively compressible.

Proof: i) Necessity. Suppose we have a graph G such that $\deg(x_1) = k_1$ for nodes x_i , $1 \leq i \leq p$. The k_1 lines incident with node x_1 must each be incident with a set of nodes among $x_2, \dots, x_r$ where r is the number of positive k_i . Since there are no parallel lines, $k_1 \leq r-1$. Hence the original set is compressible.

If x_1 is adjacent to $x_2, \dots, x_{k_1+1}$, the resulting compressed integers represent the degrees of nodes in the graph obtained from G by removing x_1 and all lines incident with it. If x_1 is not adjacent to $x_2, \dots, x_{k_1+1}$, we now show that G can be transformed into a graph G_1 with the same set of degrees as G and in G_1 , x_1 is adjacent to $x_2, \dots, x_{k_1+1}$. The resulting set of integers would again be compressible. The argument can then be repeated for every successive set of compressed integers.

Suppose there is a node x_j in G with $\deg(x_j) > \deg(x_1)$ such that x_1 is adjacent to x_j and x_j is not. Let X_j be the set of all such nodes. Since $\deg(x_j) > \deg(x_1)$, there exists a node x_k in G adjacent to x_j but not to x_1 . Furthermore, $i \neq j$ since x_j and x_1 are not adjacent. Then remove the lines (x_1, x_j) and (x_k, x_j) and add the lines (x_1, x_k) and (x_j, x_1) . In the resulting graph the degrees of the nodes are the same as in G . Moreover the size of X_j has decreased. By repeating this operation we eventually obtain G_1 .

ii) Sufficiency. Assume that $k_1 \geq k_2 \geq \dots \geq k_p$ is a set of successively compressible integers. Let $(k_{\alpha_1}, k_{\alpha_2}, \dots, k_{\alpha_p})$ be any permutation of the p -tuple $(k_1, k_2, \dots, k_p)$ such that β is the map which takes $(1, 2, \dots, p)$ to $(\alpha_1, \alpha_2, \dots, \alpha_p)$. We give a

method for constructing a graph with nodes $x_1 \dots x_p$ such that $\deg(x_i) = k_{\alpha i}$ for $i=1, \dots, n$. Let G_1 consist of the nodes $x_1 \dots x_p$ and lines $(\beta(1), \beta(2)), (\beta(1), \beta(3)), \dots, (\beta(1), \beta(k_1+1))$; G_1 exists since $k_1 \leq r-1$. To construct graphs $G_2, G_3, \dots$, repeat this operation on each set of compressed integers. Each step is possible since each set of integers is compressible. Eventually all integers will be zero since otherwise there would be an incompressible set of integers. By taking the union of all the resulting graphs to obtain the graph G , the proof is completed.

We are now able to state an algorithm for constructing a graph with specified node degrees:

Algorithm 1. Number the nodes of the graph 1 through p , and add lines according to the following rules.

Step 1. Compress the integers in the given p -tuple $(k_{\alpha 1}, k_{\alpha 2}, \dots, k_{\alpha p})$ with $k_1 \geq k_2 \geq \dots \geq k_p$ where $(\alpha 1, \alpha 2, \dots, \alpha p)$ is a permutation of $(1, 2, \dots, p)$.

Step 2. Add lines $(\beta(1), \beta(2)), (\beta(1), \beta(3)), \dots, (\beta(1), \beta(k_1+1))$.

If all integers in the compressed set are zero, terminate.

Otherwise, return to Step 1.

Figure 3 shows an example of how this procedure operates.

We now give a second theoretical result and algorithm, relating the degree-realizing procedure to λ -optimal graphs. The proof of Theorem 5 is too long to include here; it can be found in [HAK 62] or [EDM 64].

Theorem 5. A set of integers k_1 through k_p , $p > 1$, is realizable as the set of node degrees of a p -node graph G with $\lambda(G) = n$ iff

- i) k_1 through k_p are successively compressible; and
- ii) $k_i \geq n$ for $1 \leq i \leq p$; and
- iii) if $n=1$, $\sum k_p \geq 2(p-1)$

The necessity is clear; the proof of sufficiency requires a detailed examination of cases, which is not relevant to optimal connectivity.

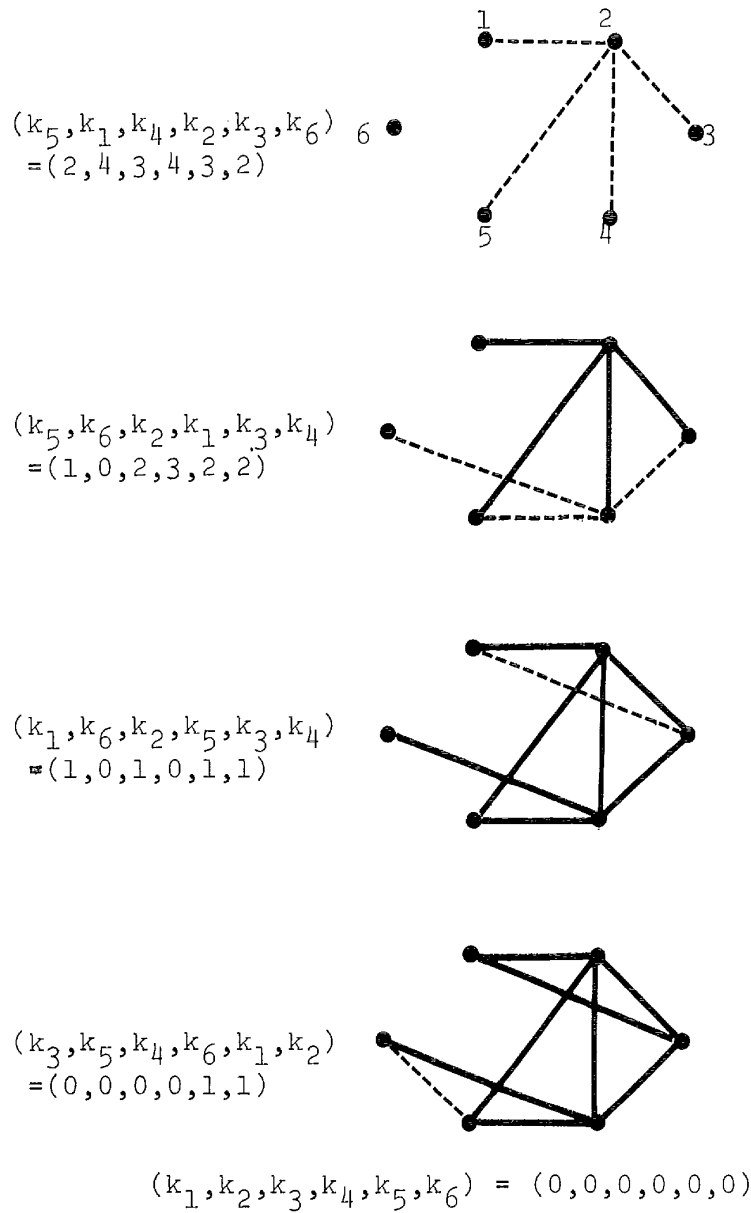


Figure 3. An example of the construction of a graph with specified node degrees, using Algorithm 1.

We are now in a position to present a synthesis procedure for λ -optimal graphs. The fundamental operation of this procedure is first defined:

Definition 13. If two lines are present in a graph, one between nodes a and b , and the other between nodes c and d , the operation of *line exchange* removes these two lines and introduces two new lines, one between a and c and the other between b and d .

We now give an algorithm for constructing a graph G with p nodes, q lines, and $\lambda(G)=n$. Thus, it can be used to construct sub-optimal or λ -optimal graphs [EDM 64].

Algorithm 2. Given a graph G with $\delta(G) \geq n$, repeat Step 1 until G is connected.

Step 1. Pick a component of G , say G_1 . Select a tree of G_1 and find a line of G_1 , say e_1 , not in the tree. Select any line in G not in G_1 , say e_2 . Exchange lines e_1 and e_2 .

Repeat Step 2 until $\lambda(G)=n$.

Step 2. Find a minimal line disconnecting set with fewer than n lines, which disconnects G into G_1 and G_2 , and pick one of its lines, say e_3 . If e_3 connects nodes a and b , find lines e_4 and e_5 so that e_4 connects a to a node in G_1 not adjacent to G_2 , and e_5 connects b to a node in G_2 not adjacent to G_1 . Exchange lines e_4 and e_5 .

Figure 4 illustrates the operation of Algorithm 2. We have now given procedures for realizing a graph with a given set of node degrees and for modifying the graph so that it is connected and that its $\lambda(G)$ is as large as possible, given the initial set of node degrees.

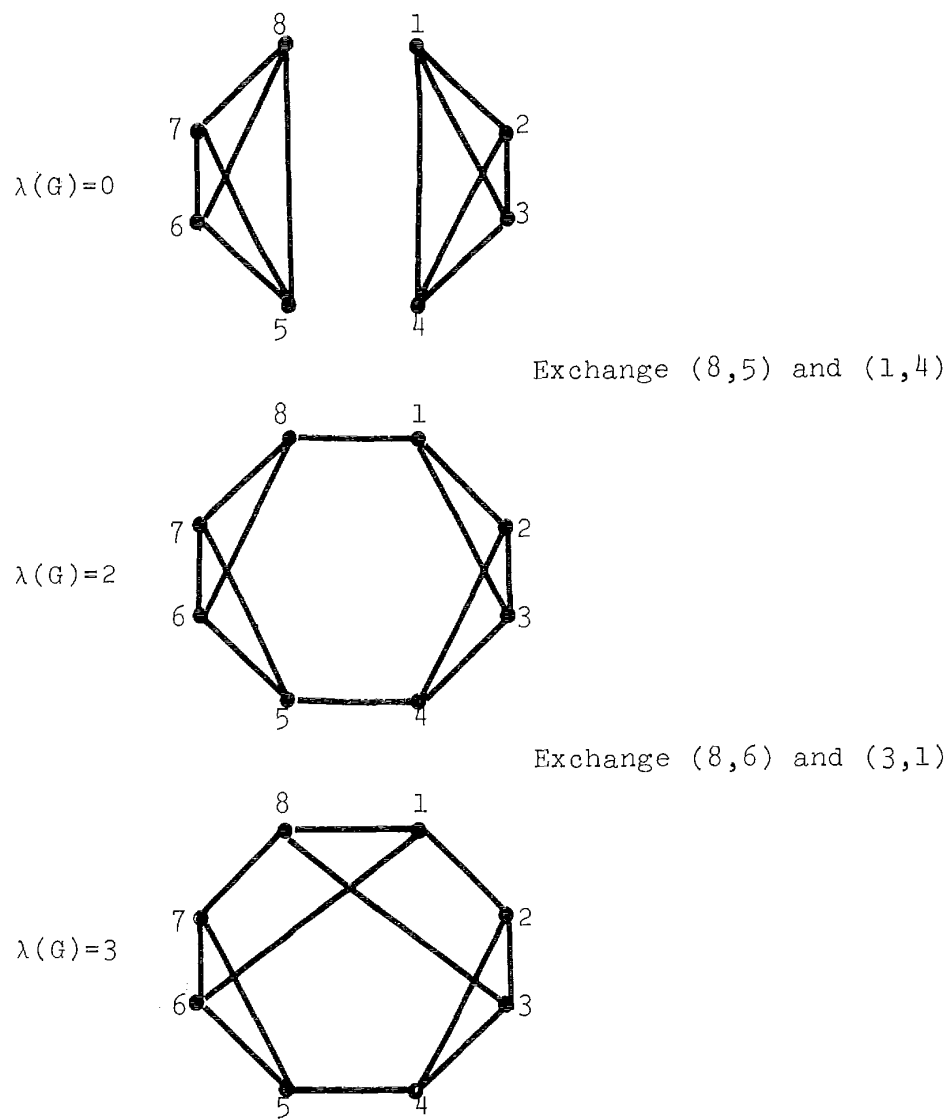


Figure 4. An example of line exchange to optimize $\lambda(G)$, using Algorithm 2.

3.3 Synthesis of κ -optimal Graphs (1)

We now begin consideration of κ -optimal graphs. We assume that $p \leq q \leq p(p-1)/2$ which gives $q = dp + r$ with $0 \leq r < p$ and $0 < d \leq [(p-1)/2]$. A construction procedure can be specified for any graph of p nodes and q lines which produces a κ -optimal graph G , with $\kappa(G) = \lfloor 2q/p \rfloor$. Several authors have contributed to this area, first in [ROS 61] and [HAR 62], and in this form primarily in [BOE 70] and [HAK 69].

Algorithm 3. Number the nodes of the graph 1 through p and add lines to the graph according to the following rules.

Step 1. To construct the first dp lines, add a line between node i and node j if

$$i - j \equiv h \pmod{p} \text{ for } 1 \leq j \leq d.$$

Step 2. To construct the remaining r lines, add a line between node i and node j

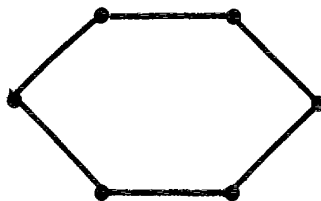
a) If $r \leq [(p+1)/2]$, when $j = i + \lfloor p/2 \rfloor$ for $1 \leq i \leq r$.

b) Otherwise, when $j = i + \lfloor p/2 \rfloor$ for $1 \leq i \leq [(p+1)/2]$.

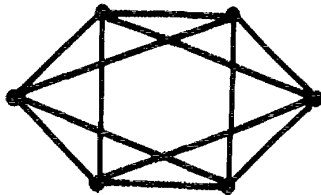
In this case, the other $r - [(p+1)/2]$ lines do not affect the connectivity and may be added arbitrarily.

We denote the resulting graph as $G_{p,d}$. The operation of Algorithm 3 can be illustrated by a series of examples. First, consider the case when $r=0$ and only step 1 is used. For $h=1$, p lines are constructed to form a Hamiltonian cycle (Fig 5a). For higher values of q , other regular graphs of even degree are generated (Fig 5b). When $r = [(p+1)/2]$, a regular graph of odd degree and connectivity $2d+1$ results (Fig 5c). When $r < [(p+1)/2]$, a graph with connectivity $2d$ is constructed (Fig 5d). When $r > [(p+1)/2]$, a graph with connectivity $2d+1$ is constructed (Fig 5e).

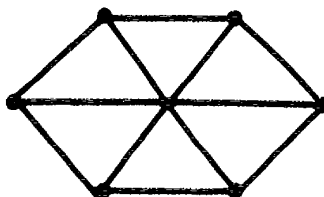
a) $G_{6,2}$ $p=6, q=6, \kappa=2$



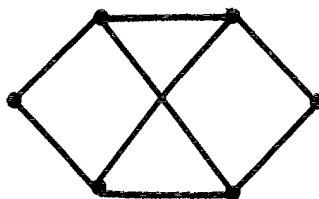
b) $G_{6,4}$ $p=6, q=12, \kappa=4$



c) $G_{6,3}$ $p=6, q=9, \kappa=3$



d) $G_{6,2}$ $p=6, q=8, \kappa=2$



e) $G_{5,3}$ $p=5, q=8, \kappa=3$

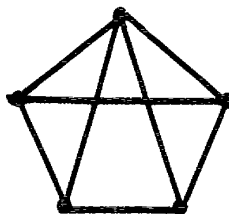


Figure 5. Some κ -optimal graphs constructed by Algorithm 3

We now present a result on the κ -optimality of graphs constructed by Algorithm 3, and indicate part of its proof [HAK 69].

Theorem 6. Let $G_{p,d}$ be constructed by Algorithm 3. Then $G_{p,d}$ is κ -optimal. That is,

$$\kappa(G_{p,d}) = \lfloor 2q/p \rfloor = 2d \text{ if } r < \lfloor (p+1)/2 \rfloor$$

$$= 2d+1 \text{ otherwise.}$$

Proof: We will prove the proposition for $r=0$ and indicate the rest of the proof. We can exhibit $2d$ node-disjoint paths from node 1 (which is chosen arbitrarily) to node x in $G_{p,d}$. By Whitney's Theorem, this means that $\kappa(G_{p,d}) \geq 2d$. We define node x by $x = yd + r_x$ $0 \leq r_x$. Then there are k paths from node 1 to node x for $x > d+1$:

$$1, i, i+k, \dots, i+yd, x \text{ for } 2 \leq i < r_x$$

$$1, i, i+k, \dots, i + \lfloor (x-i)/d \rfloor d, x \text{ for } r_x \leq i \leq d+1.$$

These paths have no nodes in common except 1 and x . All their lines are in $G_{p,d}$ by Algorithm 3. They only use nodes in the range $1 \leq i \leq x$. Thus, for $d < x \leq p$, there are d node-disjoint paths from 1 to x , containing node i for $1 \leq i \leq x$. Similarly, there are d node-disjoint paths for 1 to x containing node i for $x \leq i \leq p$, for a total of $2d$ paths. This means that $\kappa(G_{p,d}) \geq 2d \geq 2q/p$ since $d = q/p$ for $r=0$. By Theorem 1, $\kappa(G_{p,d}) \leq 2q/p$, so therefore $\kappa(G_{p,d}) = 2q/p$, and $G_{p,d}$ is κ -optimal.

We now sketch a proof of κ -optimality when $r > 0$. First, it can be shown that the graph produced by Step 1 only has $\kappa(G_{p,d}) = 2d = \lfloor 2q/p \rfloor$ for $r < \lfloor (p+1)/2 \rfloor$ and so is κ -optimal, and no more lines are necessary. For $r \geq \lfloor (p+1)/2 \rfloor$, we need to show that the smallest node disconnecting set produced by Step 1 has two sets of d consecutively numbered nodes. Then it can be shown that these two sets are not disconnecting in the graph $G_{p,d}$ when lines are added by Step 2. This gives the necessary increment of 1 to the connectivity to make the resulting graph κ -optimal.

3.4 A Note on Hamiltonian Cycles

We now reconsider the graphs produced by Algorithm 3 and examine the subgraphs of $G_{p,d}$. The graph $G_{p,d}$ is the union of d graphs which we will denote as $G_{p,d}^h$ for $1 \leq h \leq d$ in which a line from node i to node j is present if

$$i-j \equiv h \pmod{p} \quad \text{for } 1 \leq h \leq d.$$

(We ignore here the additional r lines which may be added to form $G_{p,d}$). We now state a result which links some of the results for λ -optimal and κ -optimal graphs [BOE 70].

Theorem 7. If p and h are relatively prime, then $G_{p,d}^h$ is a Hamiltonian cycle.

Proof. Since $G_{p,d}^h$ is always of degree 2, it is a Hamiltonian cycle iff it is connected. The $p+1$ term sequence for the graph is $1, h+1, 2h+1, \dots, (p-1)h+1, ph+1 \pmod{p}$.

Thus $G_{p,d}^h$ is a Hamiltonian cycle iff this sequence has only the first and last terms equal to $1 \pmod{p}$. Suppose this is not the case. Then we have a cycle with $g < p$ terms, and

$$gh+1 \equiv 1 \pmod{p} \quad \text{which implies that } gh \equiv 0 \pmod{p}.$$

Hence, for some integer z , $gh/p = z$. This means $h/p = z/g$.

Since p and h are relatively prime and $g < p$, this is a contradiction.

Therefore, $G_{p,d}^h$ is a Hamiltonian cycle. Note that this is equivalent to proving that the $p+1$ term sequence above, that is $i-j \equiv h \pmod{p}$, generates a complete system of residues mod p when p and h are relatively prime.

Note also that if n is prime, that graphs $G_{p,d}^i$ for $1 \leq i \leq (n-1)/2$ represent a decomposition of the complete graph of n nodes into line-disjoint Hamiltonian cycles. By Theorem 7, they are Hamiltonian, and by Algorithm 3 they are line-disjoint.

3.5 A Class of κ -optimal Graphs

In this section we give a class of optimally connected graphs which are related to a synthesis procedure exhibited below. These results first appeared in [BOE 70]. We begin with some standard definitions:

Definition 14. A *bipartite* graph is one in which the nodes fall into 2 disjoint sets, X_1 and X_2 , and all lines in the graph go between nodes in X_1 and nodes in X_2 . We will take the number of nodes in X_1 to be p_1 and the number of nodes in X_2 to be p_2 .

Definition 15. A *complete bipartite* graph is one in which each node in X_1 is adjacent to each node in X_2 , and thus it has $p_1 p_2$ lines.

We will begin consideration of the complete bipartite graph G^* on p nodes with $p_1 = \lfloor p/2 \rfloor$ and $p_2 = p - \lfloor p/2 \rfloor$.

Theorem 8. The complete bipartite graph G^* on p nodes is κ -optimal, that is, $\kappa(G^*) = \lfloor 2q/p \rfloor = \lfloor p/2 \rfloor$.

Proof: Any pair of nodes in X_1 are joined by p_2 node-disjoint paths. Any pair of nodes in X_2 are joined by p_1 node-disjoint paths. Any pair of nodes, one in X_1 and one in X_2 , are joined by p_1 node-disjoint paths. Therefore, the minimum node disconnecting set is of size $p_1 = \lfloor p/2 \rfloor$. And thus, by Whitney's Theorem, $\kappa(G^*) \geq \lfloor p/2 \rfloor$. We now show that $\lfloor p/2 \rfloor = \lfloor 2q/p \rfloor$ for G^* . For p even, $2q/p = 2(p/2)(p/2)/p = p/2$. Hence G^* is κ -optimal. For p odd, $\lfloor 2q/p \rfloor = \lfloor 2\lfloor p/2 \rfloor(p - \lfloor p/2 \rfloor)/p \rfloor = \lfloor p/2 \rfloor(2 - \lfloor p/2 \rfloor/p) = \lfloor p/2 \rfloor$. Here too, G^* is κ -optimal. This completes the proof of the κ -optimality of the class of complete bipartite graphs G^* on p nodes.

3.6 Another Note on Hamiltonian Cycles

In this section, we present a result which interconnects the concepts of bipartite graphs and Hamiltonian cycles, and provides the groundwork for a second synthesis procedure for κ -optimal graphs. We begin by defining a matrix associated with bipartite graphs:

Definition 16. For any bipartite graph, G , the *bipartite adjacency matrix* $B(G)$ is composed of ones when node i of X_1 is adjacent to node j of X_2 and zeros otherwise. For instance, $B(G^*)$ is a $[p/2]$ by $(p-[p/2])$ matrix with all ones. In this section and the next, we constrain p to be even. We now construct a square $p/2$ by $p/2$ bipartite adjacency matrix, B_1 , with

$b_{ii}=1$, $b_{ij}=1$ for $j-i \equiv 1 \pmod{p/2}$, and $b_{ij}=0$ otherwise. It is clear that the non-zero elements constitute a pair of adjacent diagonals in the bipartite adjacency matrix. Further, B_1 represents a Hamiltonian cycle on p nodes, since it passes through all the nodes in X_1 and X_2 in order, one from X_1 , then one from X_2 , and so on.

Now consider the square $p/2$ by $p/2$ bipartite adjacency matrix, B_n , with

$b_{ij}=1$ for $j \equiv i+2n-1 \pmod{p/2}$, and $b_{ij}=1$ for $j \equiv i+2n-2 \pmod{p/2}$ and $b_{ij}=0$ otherwise.

B_n is obtained from B_1 by cyclic row permutations, and thus B_n is also a representation of a Hamiltonian cycle. Note also that B_n and B_m for $n \neq m$ have no lines in common. We are now ready to present a result which relates the complete bipartite graph G^* and Hamiltonian cycles [BOE 70].

Theorem 9. The complete bipartite graph on p nodes, G^* , is the union of $\lfloor p/4 \rfloor$ line-disjoint Hamiltonian cycles, when p is even.
Proof: For $p \equiv 0 \pmod{4}$, the number of columns in $B(G^*)$ is $p/2$ and thus even. Therefore, $B(G^*)$ is simply the union of all B_n for $1 \leq n \leq p/4$. For $p \equiv 2 \pmod{4}$, the number of columns in $B(G^*)$ is odd. Therefore, the union of all B_n for $1 \leq n \leq \lfloor p/4 \rfloor$ gives a $p/2$ by $p/2$ square matrix with all elements equal to one except for a single zero entry in each row and column. If we define a *matching* as a regular bipartite graph of degree 1, and note that its bipartite adjacency matrix is any row permutation of the identity matrix, then the case of $p \equiv 2 \pmod{4}$ can be stated as: G^* consists of the union of $\lfloor p/4 \rfloor$ line-disjoint Hamiltonian cycles and one matching. This completes the proof. This further relation between Hamiltonian cycles and optimally connected graphs leads to another synthesis procedure, detailed in the next section.

3.7 Synthesis of κ -optimal Graphs (2)

Using the results developed above, we now consider another synthesis procedure for κ -optimal graphs, related to other methods examined earlier. We define $G_{p,c}$ to be constructed as follows, for p even:

$B(G_{p,c})$ is the union of all B_n for n between 1 and c . Thus $G_{p,c}$ is a bipartite graph with pc lines, and is defined in the range $1 \leq c \leq \lfloor p/4 \rfloor$. It is the line-disjoint union of c Hamiltonian cycles. We know from Theorem 2 that $G_{p,c}$ is λ -optimal with $\lambda(G_{p,c}) = 2c$. The following theorem strengthens that result; generally line-disjoint unions of Hamiltonian cycles are not κ -optimal [BOE 70], [BOE 71].

Theorem 10. The bipartite graph $G_{p,c}$ is κ -optimal with $\kappa(G_{p,c})=2c$.

Proof: Number the nodes in $G_{p,c}$ from 1 to p so that the rows of $B(G_{p,c})$ correspond to the $p/2$ consecutive odd integers and the columns correspond to the $p/2$ consecutive even integers. Then, if node i is odd, it is adjacent to $i+1, i+3, \dots, i+4c-1 \pmod{p}$; if node i is even, it is adjacent to $i-1, i-3, \dots, i-(4c-1) \pmod{p}$. Now assume there is a node disconnecting set, S , of size $2c-1$ or less. The remaining nodes can be numbered as

$$i_1, i_2, \dots, i_{p-(2c-1)} \quad \text{with } i_1 < i_2 < \dots < i_{p-(2c-1)}.$$

There must be an integer z such that i_z and i_{z+1} are separated by the removal of the set S . We now perform a cyclic permutation of the subscripts, preserving adjacency, which maps z to 1 and $z+1$ to j . (We assume z is odd, the proof follows in parallel for z even). In this numbering, i_n is in S for $1 < n < j$, since i_z and i_{z+1} are consecutive integers and incidence has been preserved. Define y by $j=y+2$ where $y \leq 2c-1$. This gives that $j \leq 2c+1$. We know that j cannot be even, since they are not connected, since $2c+1 < 4c$. Since j is odd, let $y=2r-1$ where $1 \leq r \leq c$. Now, since 1 and j are odd, they are adjacent to the even nodes greater than j : $2r+2, 2r+4, \dots, 4c$. There are $2c-r$ of these even nodes, which must be in S since they are adjacent to j . There are also $2r-1$ nodes between 1 and j . Thus S has at least

$$(2c-r) + (2r-1) = 2c+r-1 \text{ nodes.}$$

Since $r \geq 1$, this contradicts the assumed size for S , $2c-1$.

Therefore, $G_{p,c}$ is κ -optimal with $\kappa(G_{p,c})=2c$.

A property of the graphs $G_{p,c}$ which is quite important in applications is that any minimum size disconnecting set of nodes is adjacent to a single node of $G_{p,c}$ for $1 < c \leq \lfloor p/4 \rfloor$. This means that if the set is removed, a disconnected component and a single isolated node result, which may be the least costly disconnection for certain applications.

4. Methods for the Analysis of the Connectivity of a Graph

In this section we will consider analytic techniques for determining the connectivity of a graph. We will use two basic results, one is Whitney's Theorem stating that $\kappa(G)$ is equal to the minimum number of node-disjoint paths connecting any two nodes in G . The other technique we will assume without proof is the existence of efficient labelling algorithms for determining the number of node-disjoint paths between any pair of nodes [FRI 67b], which are related to the Max Flow-Min Cut Theorem.

4.1 General Reduction Techniques

If we were to apply a labelling algorithm to each pair of nodes, it would require $p(p-1)/2$ labellings. There are three results stated below which reduce that number of labellings required to ascertain the connectivity of the graph as a whole [STE 69], [KLE 69].

Theorem 11. If there are at least r node-disjoint paths from nodes a and b to a set, R , of r other nodes, then there are at least r node-disjoint paths from a to b .

Proof: Assume the theorem is false, and first that a and b are not adjacent. There is a set S of $s < r$ nodes which disconnects a and b , leaving a in G_a and b in G_b . Now pick an element of the set R . It cannot be in G_a , since b is r -connected to the set R . Likewise, it cannot be in G_b . Thus the entire set R must be in S , which contradicts $s < r$. For a and b adjacent, if the line between a and b is removed, the argument above can be repeated. Thus the proof is complete.

Theorem 12. To verify the existence of r node-disjoint paths between all pairs of nodes in G , it is sufficient to follow the following procedure:

Step 1. Choose a node, call it 1 , and verify the existence of r node-disjoint paths from node 1 to all other nodes in G .

Let $i=1$, then:

Step 2. Remove node i to form G_1 . Choose a node, call it $i+1$, and verify the existence of $r-1$ node-disjoint paths from node $i+1$ to all other nodes in G_1 .

Step 3. Stop if $i=r$. Otherwise increment i by 1 and return to Step 2.

Proof: If we wish to show that i and j are not r -connected, then we must exhibit a disconnecting set of $r-1$ or fewer elements. If there are r paths from 1 to i and also r paths from 1 to j , then node 1 must be in the disconnecting set. Thus, if we cannot find an i - j disconnecting set with $r-2$ elements in G_1 , we cannot find an $r-1$ size set in G . Hence verifying the existence of r node-disjoint paths in G means we have to verify only $r-1$ paths in G_1 , and so on, for r iterations. This completes the proof.

Theorem 13. Let G be a graph with m nodes, 1 through m , adjacent to node j . Then there are r node-disjoint paths from node i to node j if

i) there are r node-disjoint paths from node i to each node k , $1 \leq k \leq m$, and

ii) there are $r-1$ node-disjoint paths from node i to node j in the graph formed by removing nodes k , $1 \leq k \leq m$.

Proof: Assume there are r node-disjoint paths from node i to each of the m nodes. Then the m nodes, 1 through m , must each be in an i - j disconnecting set of $r-1$ or fewer elements.

To verify the existence of r i - j paths, we must show there is no i - j disconnecting set with $(r-m-1)$ elements in the graph with the m elements removed. But this can be done by finding the $r-m$ paths in the reduced graph. Therefore, the given conditions are sufficient.

Let us review the improvements resulting from these theorems. To verify the r -connectivity of a graph of p nodes by labelling each node pair requires $rp(p-1)/2$ labellings. The techniques presented here reduce this to $r(r-1)/2 + r(p-r)$ labellings, which is linear in p . An alternative method is to examine each group of r nodes for r -connectivity with all other nodes, which requires r^2p labellings.

4.2 Special Techniques for Low Connectivities

For completeness, we now list without proof a selection of more efficient techniques for verifying that a graph has a specific low connectivity.

Connectivity 1. Tarjan [TAR 72] has shown that a depth-first search procedure (DFS) applied to a graph results in the construction of a spanning tree. DFS is a procedure which travels along the lines of a graph in random order, marking nodes as they are visited, and employing backtracking when old nodes are encountered. It has been shown that DFS visits all nodes and lines exactly twice and is thus linear in the number of nodes, p , and the number of lines, q . Since it constructs a spanning tree of all nodes which can be reached from a random starting node, it can be used for the verification of the simple connectivity of a graph.

Connectivity 2. Tarjan [TAR 27] also developed a modification to simple DFS which verifies the biconnectivity of a graph by identifying all of its biconnected components. This procedure is also linear in p and q .

5. Conclusion

We have presented a detailed survey of some of the applied research that has been conducted in the area of connectivity of graphs. There are useful results for building networks with optimal damage-resistant properties, both in terms of line loss and node loss. We also discussed efficient methods for determining the connectivity of a given network. At this point it is appropriate to indicate some of the other directions of research in this field.

One extension to the simple measures of cohesion discussed here has been introduced in the form of a redundancy matrix giving the required number of line-disjoint paths between each pair of nodes. Analogous results to Algorithm 1 (for degree realization) and Algorithm 2 (for line exchange to optimize cohesion) have been obtained in this more general case [FRA 70], [CHO 70]. In addition, the results presented in Section 4.1 for analyzing the connectivity of a graph can clearly be extended to a pair-wise notion of connectivity [KLE 69].

A second area of interest is the problem of synthesis of very large networks which are approximately optimal for some measure of connectivity. These approaches use linear programming techniques for finding local optima and for generating a wide range of starting feasible solutions. The basic operations employed have typically been the addition of Hamiltonian cycles and the successive line exchange to achieve local optima [LIN 65], [FRI 67c], [STE 69].

Finally, there is a generalized form of connectivity and cohesion developed in [BOE 70] and [BOE 71]. By analogy with $\lambda(G)$ and $\kappa(G)$, they define $\lambda_m(G)$ and $\kappa_m(G)$ as the minimum size line and node sets, respectively, which isolate a set of m nodes from the rest of the network. Then the concepts of λ_m -optimality and κ_m -optimality can be defined over the class of pseudo-regular graphs. It can be shown that λ_m -optimal graphs are λ -optimal, and κ_m -optimal graphs are κ -optimal; the converse relations do not hold. The complete bipartite graphs are the unique λ_m -optimal graphs of their class. Lastly, the complete regular k -partite graph, defined in a similar fashion to the bipartite graph, is λ_m -optimal. It is also κ -optimal, but not κ_m -optimal.

This research field is still quite active. The reader is encouraged to see Wilkov's excellent survey [WIL 72] for further results and an extensive bibliography. There are many problems, both theoretical and practical, in the study of the connectivity of communications networks.

REFERENCES

- AYO 70 J.N.Ayoub and I.T.Frisch, "Optimally Invulnerable Directed Communication Networks", IEEE Trans. Commun. Technol. COM-18,484-489 (1970).
- BEH 71 M.Behzad and G.Chartrand, *Introduction to the Theory of Graphs*, 118-130, Allyn and Bacon, 1971.
- BER 62 C.Berge, *The Theory of Graphs and its Applications*, Chapter 9, Wiley, New York, 1962.
- BOE 68 F.T.Boesch and I.T.Frisch, "On the Smallest Disconnecting Set in a Graph", IEEE Trans. Circuit Theory CT-15,286-288(1968).
- BOE 70 F.T.Boesch and R.E.Thomas, "Optimal Damage Resistant Communication Nets", IEEE Trans. Circuit Theory CT-17, 183-192(1970).
- BOE 71 F.T.Boesch and A.Felzer, "On the Invulnerability of the Regular k-partite Graphs", SIAM J. Appl. Math, 20 176-182(1971).
- BUT 67 D.E.Butler, "Communication Nets with Specified Survivability", Coordinated Science Laboratory Report r-359, University of Illinois, Urbana, July 1967.
- CHA 66 G.Chartrand, "A Graph Theoretic Approach to the Communication Problem", SIAM J. Appl. Math 14,778-781(1966).
- CHA 68 G.Chartrand and F.Harary, "Graphs with Prescribed Connectivities", in *Theory of Graphs*, 61-63 (P. Erdos and G.Katona, eds.) Budapest, 1968
- CHO 70 W.Chou and H.Frank, "Survivable Communication Networks and the Terminal Capacity Matrix" IEEE Trans. Circuit Theory CT-17, 192-197(1970).
- EDM 64 J.Edmonds, "Existence of k-edge Connected Ordinary Graphs with Prescribed Degrees", J. Res. Natl. Bur. Std. 68B 73-74(1964).

- ELI 56 P.Elias, A.Feinstein, and C.E.Shannon, "A Note on the Maximum Flow Through a Network", IRE Trans. Inform. Theory IT-2, 117-119 (1956).
- FOR 56 L.R.Ford and D.R.Fulkerson, "Maximal Flow Through a Network", Canad J. Math. 8, 399-404(1956).
- FRA 70 H.Frank and W.Chou, "Connectivity Considerations in the Design of Survivable Networks",IEEE Trans. Circuit Theory CT-17, 486-490(1970).
- FRI 63 I.T.Frisch, "Optimization of Communications Nets with Switching", J. Franklin Inst. 275, 405-430(1963).
- FRI 67a I.T.Frisch, "A Computer Technique for Design and Analysis of Hard Communication Systems", Digest of the 1967 IEEE Intl. Conf. on Commun. p.96 (1967).
- FRI 67b I.T.Frisch, "An Algorithm for Vertex Pair Connectivity", Intern. J. Control, 6, 579-593(1967).
- FRI 67c I.T.Frisch, "Analysis of the Vulnerability of Communications Nets", Proc. of First Annual Princeton Conf. on Systems Science, 188-192(1967).
- FRI 69 I.T.Frisch, "Flow Variation in Multiterminal Min-Cut Calculations", J. Franklin Inst. 287, 61-72(1969).
- FUL 61 D.R.Fulkerson and L.S.Shapley, "Minimal k-arc Connected Graphs", RAND Report p-2371(1961).
- FUL 65 D.R.Fulkerson and A.J.Hoffman, and H.McAndrew, "Some Properties of Graphs with Multiple Edges", Canad. J. Math. 17,166-177(1965).
- HAK 62 S.L.Hakimi, "On the Realizability of Integers as the Degrees of the Vertices of a Linear Graph-I", J. Soc. Ind. Appl. Math. 10, 496-506(1962).
- HAK 63 S.L.Hakimi, "On the Realizability of a Set of Integers as the Degrees of the Vertices of a linear Graph-II Uniqueness",J.Soc.Ind.Appl.Math, 11, 135-147(1963).

- HAK 69 S.L.Hakimi, "An Algorithm for Construction of the Least Vulnerable Communication Network", IEEE Trans. Circuit Theory CT-16, 229-230(1969).
- HAR 62 F.Harary, "The Maximum Connectivity of a Graph", Proc. Natl. Acad. Sci. US. 48, 1142-1146(1962).
- HAR 69 F.Harary, *Graph Theory*, 43-56, Addison Wesley, 1969
- KLE 69 D.J.Kleitman, "Methods for Investigating Connectivity of Large Graphs", IEEE Trans. Circuit Theory, CT-16, 232-233(1969).
- LIN 65 S.Lin, "Computer Solutions of the Travelling Salesman Problem", Bell System Tech. J. 44, 2245-2269(1965).
- ROS 61 D.M.Rosenbaum and J.B.Freidman, "Redundant Networks", MITRE Tech. Memo. TM-3195 Oct. 1961.
- STE 69 K.Steiglitz, P.Weiner, and D.J.Kleitman, "The Design of Minimum-Cost Survivable Networks", IEEE Trans. Circuit Theory CT-16, 455-460(1969).
- TAR 72 R.Tarjan, "Depth-First Search and Linear Graph Algorithms", SIAM J. Comput. 1, 146-160(1972).
- WHI 32 H.Whitney, "Congruent Graphs and the Connectivity of Graphs", Amer. J. Math. 54, 150-168(1932).
- WIL 72 R.S.Wilkov, "Analysis and Design of Reliable Computer Networks", IEEE Trans. on Comm., Vol Com-20, No. 3, 660-678(1972).