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Abstract

In this work we survey some of the basic analysis and synthesis results in network reliability theory. The model consists of an undirected simple graph with independent probabilities that edges operate. The reliability is then defined as the probability that the network has some path between each pair of nodes. The analysis problem is to calculate the reliability given the graph and the probabilities. We introduce a parameter known as the reliability domination of a graph and show its role in calculating the reliability. There are a number of interesting synthesis problems concerning the reliability model. These are illustrated by considering the case where the edge probabilities are all equal. Two specific synthesis problems are discussed in detail giving the known results and the outstanding questions. We conclude with a discussion of how the topology of the optimal network might change as a function of the edge probabilities.

Introduction

Many large-scale systems consist of an interconnection of many components. Examples include power grids, cable tv distribution networks, computer communication nets, and parallel processing computers. In such cases, the mathematical theory of network reliability plays an important role. For purposes of this presentation we consider the following model. A network is an undirected simple n node, e edge graph G with edge weights representing the probability that the edge is operational. We assume that these probabilities are independent. The network is said to be operational if a subgraph H which is connected and includes all the nodes of G has all of its edges operational; such a subgraph is called an operating state. The reliability $R(G)$ of the network is then defined as the probability that the network is in an operating state. Thus

$$R = \sum_{H \in \Theta} p(H)$$

where the summation extends over the set of all operating states Θ and $p(H)$ is the probability of operating state H . Now for simplicity of notation, let us assume that all edge weights have the same value p . Then if H has k edges

$$p(H) = p^k(1-p)^{e-k}$$

and

$$R = \sum_{k=0}^e Q_k p^k (1-p)^{e-k}$$

where Q_k is the total number of operating states having k edges. For example consider the graph G_0 shown in Figure 1.

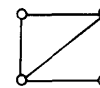


Figure 1- G_0 with $R(G_0) = 8p^3(1-p)^2 + 5p^4(1-p) + p^5$

Since each operating state is connected, there can be no operating states with fewer edges than a tree has. Thus $Q_1=Q_2=0$ and Q_3 is the number of trees which is 8.² Now each 4 edge subgraph is obtained by removing a single edge from G_0 . However the edge connectivity, defined as the minimum number of edges whose removal disconnects a connected graph, of G_0 is 2. Thus removing any edge from G produces an operating state, and $Q_4 = 5$ which is the number of ways to remove a single edge. Clearly there is only one operating state having 5 edges.

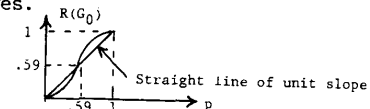


Figure 2 - A plot of $R(G_0)$

Notice that R exhibits an "S shape", viz., for $p < .59$ $R < p$ and for $p > .59$ $R > p$. This means that for edge reliabilities of less than .59 the network is less reliable than any single edge and visa versa. For any graph with edge connectivity at least two there will be some

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$p_0 (0 < p_0 < 1)$ such that $R(p) < p$ for $p < p_0$ and $R(p) > p$ for $p > p_0$. This property is known as Shannon's Reliability Theorem [1].

Suppose that we wish to express $R(G)$ as a polynomial in p . For example

$$R(G_0) = 8p^3(1-p)^2 + 5p^4(1-p) + p^5 = 8p^3 - 11p^4 + 4p^5.$$

In general we can obtain the coefficients of the polynomial by using the binomial theorem. Namely

$$R = \sum_{k=0}^e Q_k p^k (1-p)^{e-k} = \sum_{k=0}^e \sum_{j=0}^{e-k} Q_k p^k (-1)^j \binom{e-k}{j} p^j,$$

Hence

$$R = \sum_{i=0}^e r_i p^i$$

$$\text{where } r_i = \sum_{j=0}^i (-1)^j \binom{e-i+j}{j} Q_{i-1}$$

Notice that

$$r_e = (-1)^e$$

This parameter r_e which is an invariant of the graph is known as the reliability domination $d(G)$. It can be shown that if H is any n node subgraph with $|H|$ edges

$$R(G) = \sum_{H \subseteq G} d(H) p^{|H|},$$

and that

$$r_i = \sum_{\substack{H \subseteq G \\ |H|=i}} d(H).$$

Note that $r_i = 0$ for $i < n-1$,

$r_{n-1} = t(G)$ the number of spanning trees of G ,

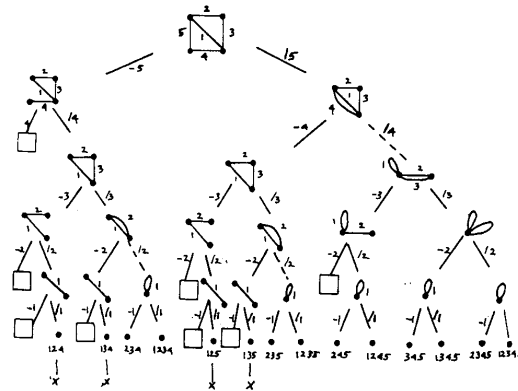
and $r_e = d(G)$.

The role of domination in network reliability was originally recognized by Satyanarayana [12] who introduced and studied $d(G)$. Incidentally it was not required for H to be connected in the above formula as $d(H) = 0$ for a disconnected subgraph H .

It is natural at this point to inquire about efficient algorithms to calculate $R(G)$. Unfortunately it has been shown by Provan and Ball [11] that calculating $R(G)$ belongs to the class of NP-Hard problems for which no efficient algorithm is known. Of course one could use a binary search procedure to find all the operating states of a network. Formally one would proceed as follows. Label

the e edges of G as $x_1, x_2, \dots, x_e$. Starting with x_e one could then consider all connected n node subgraphs that contain x_e and all those that do not contain x_e . Those that do not contain x_e are spanning (containing all the nodes) subgraphs of $G - x_e$ (G with x_e removed); those that contain x_e are spanning subgraphs of G/x_e (G with the end points of x_e coalesced). This process can then be repeated, abandoning any disconnected graphs, until only one node graphs are obtained. The binary decision tree for such a process is shown in Figure 3 for the graph G_0 . The final one node graphs are leaves of the decision tree, and the corresponding edges which were coalesced to arrive at a leaf is the operating state represented by this leaf. Notice that for G_0 there are 8 operating states with 3

edges, 5 operating states with 4 edges, and one operating state with 5 edges. Thus one could write an expression for R using the formula involving Q_k .



Labelled nodes at bottom level represent all 14 operating states

□ represents a disconnected subgraph
x represents leaves obtained by parallel reduction

Figure 3 - Binary search tree for $R(G_0)$

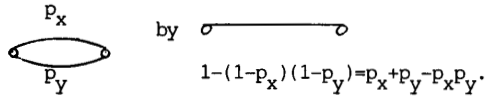
Alternatively one could use the fact that by the inclusion-exclusion principle

$$R(G) = (1-p_x) R(G-x) + p_x R(G/x)$$

where p_x is the probability that edge x operates. Now using this inclusion-exclusion approach a great deal of simplification could be introduced by recognizing that whenever parallel edges are encountered one could replace

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Using this replacement one could also abandon all tree branches that started with a self loop (the dotted lines in Figure 3.) Thus for G_0 one has reduced the search tree to one having only 4 leaves. These are marked by an x in Figure 3. The fact that the domination of G_0 is also 4 is no coincidence. Now as self-loops only result from a set of edges forming a cycle, it is clear that the reduced search tree leaves all represent trees. A characterization of exactly which trees are the search tree leaves is due to Satyanarayana [12] and Boesch, Satyanarayana and Suffel [6]. To explain his results we need some notation.

Let the e edges of a graph G be labelled $1, 2, \dots, e$. A spanning tree T of G is said to have a broken cycle if there is some non tree edge x such that the label on x is less than the labels on all tree edges in the fundamental cycle created by $T + x$. The results of Satyanarayana are that t_0 , the number of spanning trees with no broken cycles, is independent of the labelling; the number of leaves of the reduced search tree is t_0 ;

$$d(G) = (-1)^{e-n+1} t_0.$$

The fact that the reduced search tree leaves are trees with no broken cycles should be clear from the example as any time an edge x is removed it has the largest label in its subgraph at this stage. Thus the only way x could create a broken cycle is if all the tree edges in the fundamental cycle of $T + x$ had been coalesced at a previous stage. This is impossible as that would have forced x to be a self-loop.

A variety of other properties of $d(G)$ can be found in the survey paper [6].

Synthesis

There are a variety of questions that could be regarded as synthesis problems for reliable networks. One reasonable synthesis problem is the following: given the number of nodes n , the number of edges e , and the edge probability p ; find a graph G which maximizes $R(G)$ over the class of all n -node, e -edge graphs. Notice that this is a well defined question as there are a finite number of n -node, e -edge graphs and for each one $R(G)$ is a real number as p is given. However the difficulty with this problem is that it is

possible for the optimum graph G to change as p changes. Hence one might not be able to classify reliable graphs independent of p . One method of avoiding this difficulty was given in [3], where this synthesis problem is considered for p "small". In [3] it is shown that given any $\epsilon > 0$, n and e ($e \geq n-1 \geq 2$) there exists a $\hat{p}$ and a class T of n -node, e -edge graphs (known as t -optimal graphs) such that

$$R(G) \geq (1+\epsilon)R(G) \text{ for } G \in T \text{ all } G \notin T \text{ and all } p < \hat{p}.$$

Furthermore for any two graphs $G_1, G_2 \in T$

$$\begin{aligned} R(G_1) - R(G_2) &< \epsilon, \text{ for all } p < \hat{p}. \\ R(G_1) \end{aligned}$$

This for all practical purposes any graph in T will serve as an optimal graph for p small. It has also been verified in [3] that the class T may be defined as those n -node, e -edge graphs that maximize $t(G)$ the number of spanning trees. It is rather surprising that although the study of spanning trees dates back to the earliest work in graph theory by Cayley and Kirchhoff [7], [10] the question of how to construct number of T is only answered for certain special cases. A discussion of the known result can be found in [4]. We repeat the known theorems here. To this end we define a set of edges as independent if they have no common endpoints. The complete graph K_n has an edge between each pair of nodes.

Theorem (Shier [13], Kelman and Chelkonov [9])

If a set of independent edges are deleted from a complete graph then the result is t -optimal.

Notice that connected graphs have $n-1 \leq e \leq n(n-1)/2 = \# \text{edges in } K_n$. Thus this theorem gives t -optimal graphs for large values of e , i.e., as the maximum number of independent edges is $\lfloor n/2 \rfloor$ the theorem applies for

$$n(n-1)/2 - \lfloor n/2 \rfloor \leq e \leq n(n-1)/2$$

The next result applies for the medium range of possible values for e . To state this result we need to define a special class of graphs. A graph whose node set can be partitioned into k disjoint sets called parts each having m nodes is called a uniform complete k -partite graph $K(m^k)$ if every pair of nodes from different parts are adjacent and no two nodes in the same part are adjacent. An example is shown in Figure 4.

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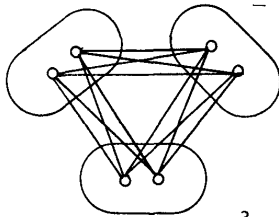


Figure 4 - $K(2^3)$

Theorem (Cheng [8])

The graphs $K(m^k)$ are unique t -optimal graphs.

Note that the number of edges in $K(m^k)$ is $n^2(k-1)/2k$ where $n = mk$. Finally for e small we have a few results. To state them we need to define subdivision of an edge x which means to replace x by two edges incident at a new node μ . An example is shown in Figure 5

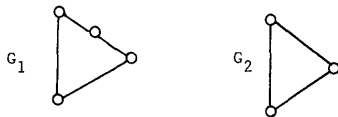


Figure 5 - G_1 a Subdivision of G_2

Theorem (Boesch and Li [unpublished])

For $e = n+1$ there is a unique t -optimal graph which is a subdivision of graph $3K_2$, shown in Figure 6. For $e = n+2$ there is a unique t -optimal graph which is a subdivision of K_4



Figure 6 - The graph $3K_2$

Consider now the case where p is "large". By a procedure similar to the one given above for p "small", we can show that there is some $\hat{p}'$ such that for $p > \hat{p}'$ graphs which maximize $R(G)$ belong to a special class Γ . As in the previous case, the percentage difference in the reliabilities of two graphs in Γ is small. The class Γ can be shown to be characterized by the following properties. For given n and e consider all graphs having the maximum possible value of edge connectivity λ . The class Γ is the subclass of these graphs having the minimum possible value for the number of min-cuts. We call graphs in Γ λ -optimal graphs. A complete discussion of methods to produce λ -optimal graphs for any n and e was given by Bauer, Boesch, Suffel, and Tindell [2].

Conclusions

It should be clear from the section on analysis that determining $R(G;p)$ for given G and p is a hard problem. Thus it is totally appropriate to search for good heuristics. The synthesis problem, however, is solved at least for certain special cases. Herein we defined the synthesis problem as finding graphs which maximize $R(G;p)$ for given values of p , n , and e . Since these are a finite number of graphs having prescribed values of n and e , there always exists an answer to the synthesis question. However it is an a priori possibility that this optimal graph may depend on p , i.e., as p changes it may be necessary to change the optimal graph. Now for large p the optimal graphs are λ -optimal and for small p they are t -optimal. It is interesting to note that there are no known examples of values of n and e for which no t -optimal graph is λ -optimal. If such an example existed then one could state that the optimal graph must change as p changes. Indeed there are many examples where it has been shown that a uniform optimal graph exists, viz., there is one graph which maximizes $R(G;p)$ for all p . We conjecture that such uniform optimal graphs always exist.

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